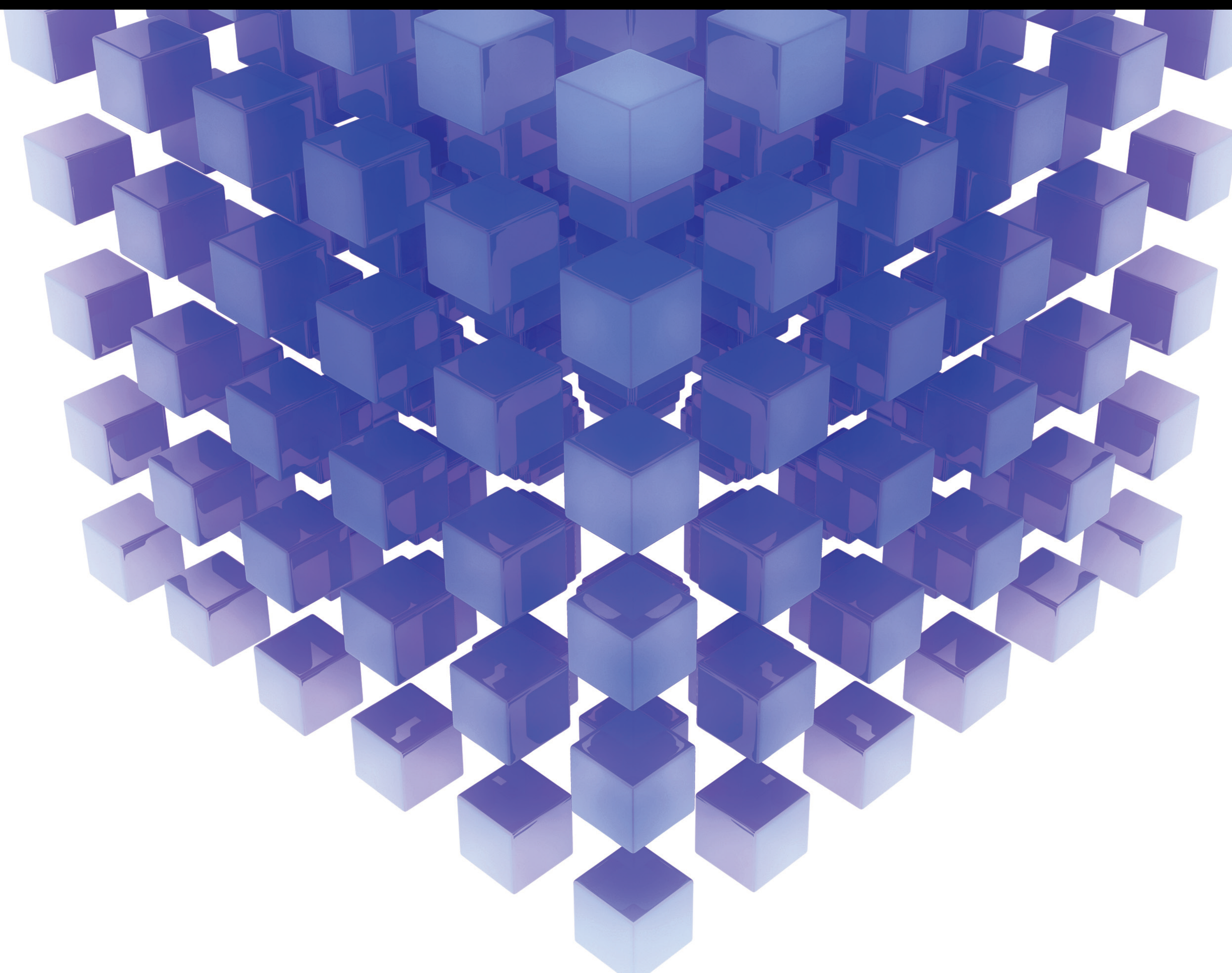
Mathematical Problems in Engineering

Machine Learning and Computational Intelligence in Supply Chains

Lead Guest Editor: Dragan Pamučar
Guest Editors: Fatih Ecer and Fausto Cavallaro



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Research Article

Cubic Intuitionistic Fuzzy Topology with Application to Uncertain Supply Chain Management

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The concept of the cubic intuitionistic fuzzy set is an effective hybrid model for modeling uncertainties with an intuitionistic fuzzy set and an interval-valued intuitionistic fuzzy set, simultaneously. The primary objective of this study is to develop a topological structure on cubic intuitionistic fuzzy sets with P-order and R-order as well as to define some fundamental characteristics and significant results with illustrations. Taking advantage of topological data analysis with cubic intuitionistic information, novel multicriteria group decision-making methods are developed for an uncertain supply chain management. Algorithms 1 and 2 are proposed for extensions of the weighted product model and the choice value method towards a cubic intuitionistic fuzzy environment, respectively. A comparative analysis is also given to discuss the validity and advantages of the proposed techniques.

1. Introduction

Topological data analysis (TDA) methods are rapidly growing approaches to inferring persistent key features from possibly complex data [1]. We deal with complex issues in our daily lives due to vague and uncertain information, and if we do not use the proper modeling techniques for them, we eventually wind up with vague and unclear reasoning. For this reason, making rational and logical conclusions in the face of such imprecise and inexplicit facts is a difficult task for decision-makers. As a result, dealing with vagueness and uncertainty is a necessary part of dealing with such challenges and difficulties. Zadeh [2] initiated the notion of fuzzy set (FS) theory, which is an instantaneous extension of a crisp set. Various sets of theories and models have been developed by researchers to manage the complexity of daily life problems that include vague and uncertain information. Atanassov [3] presented the idea of an intuitionistic fuzzy set (IFS), and Atanassov [4] further initiated the notion of circular intuitionistic fuzzy sets. Yager [5, 6] introduced the concept of a Pythagorean fuzzy set (PFS), and further Yager [7] developed the notion of a q-rung orthopair fuzzy set (q-ROFS). Molodtsov [8] was the first who proposed the idea of a soft set (SS), and Zhang [9, 10] originally introduced the notion of a bipolar fuzzy set (BFS) to address bipolarity and bipolar information. Smarandache [11, 12] initiated the concept of a neutrosophic set (NS). Cuong [13] introduced the idea of a picture fuzzy set (PiFS). Gundogdu and Kahraman [14], Mahmood et al. [15], and Ashraf et al. [16] independently introduced the notion of a spherical fuzzy set (SFS). These models have a strong foothold when it comes to modeling uncertainty in a real-life complex challenges. Atanassov and Gargov [17] introduced interval-valued intuitionistic fuzzy sets. Cagman and Enginoglu [18] proposed decision-making applications based on soft-set theory. Karaaslan and Cagman [19] introduce the parameter trees based on soft set theory and their similarity measures. Chen [20] proposed m-polar fuzzy sets (mPFS) with m membership values to address the multipolarity of objects.

Jun et al. [21] developed the cubic set (CS) and its internal and external environment. But CS has some limitations, as it does not convert membership degree grades into nonmembership grades. Riaz and Hashmi [22] proposed cubic m-polar fuzzy sets and cubic m-polar fuzzy averaging aggregation operators for MAGDM. So, for this, Kaur and Garg [23, 24] presented the concept of a cubic intuitionistic fuzzy set by combining the concepts of IFSs, CFSs, and IVIFSs. So, CIFS, rather than IFSs or IVIFSs, is a handy technique to address information more precisely throughout the DMP. Young et al. [25] proposed cubic interval-valued intuitionistic fuzzy sets. Senapati et al. [26] introduced a cubic intuitionistic WASPAS technique. Garg and Kaur [27] suggested cubic intuitionistic fuzzy Bonferroni mean operators. Garg and Kaur [28] proposed cubic intuitionistic fuzzy TOPSIS for nonlinear programming.

Classical topology derives its inspiration from classical analysis and has a wide range of scientific applications. In 1968, Chang [29] proposed the concept of fuzzy topology. Coker [30] pioneered intuitionistic fuzzy topology. Olgun [31] expanded on this concept by introducing Pythagorean fuzzy topology. Topological structures on fuzzy soft sets [32] and cubic m-polar fuzzy sets [33] have robust applications in decision-making. Xu and Yager [34] and Xu [35] originated the notion of an intuitionistic fuzzy number (IFN) and their aggregation operators. Zhang and Xu [36] developed an extension of TOPSIS for Pythagorean fuzzy numbers (PyFNs). They also suggested a domestic airline MCDM application to examine the service quality of airlines. Feng et al. [37] proposed the MADM application by using a new score function for ranking alternatives with generalized orthopair fuzzy membership grades. Akram [38] initiated the concept of BFS graphs, and Akram et al. [39] suggested a hybrid decision-making framework by using aggregation operators under a complex spherical fuzzy prioritization approach. Alghamdi et al. [40] proposed some MCDM methods for bipolar fuzzy environments. Liu and Wang [41] proposed some basic operational laws of q-ROFNs and q-ROF aggregation operators. Ye [42] proposed MADM with new similarity measures based on the generalized distance of neutrosophic Z-number sets. Senapati and Yager [43] proposed WPM for Fermatean fuzzy numbers. Kahraman and Alkan [44] developed the TOPSIS method for circular intuitionistic fuzzy sets. Sinha and Sarmah [45] developed supply chain coordination using fuzzy set theory. Alshurideh et al. [46] proposed supply chain management with fuzzy-assisted human resource management.

Seikh et al. [47, 48] proposed the solution of matrix games with rough interval pay-offs and a defuzzification approach of type-2 fuzzy variables to solving matrix games. They developed applications of matrix games to the telecom market share problem and the plastic ban problem. Ruidas et al. [49] developed an EPQ model with stock and selling price-dependent demand and a variable production rate in an interval environment. Ruidas et al. [50] suggested an interval environment with price revision using a single-period production inventory model. Ruidas et al. [51] introduced a production-repairing inventory model considering demand and the proportion of defective items as rough intervals. Seikh and Mandal [52] proposed q-rung orthopair fuzzy Frank aggregation operators and their application in MADM with unknown attribute weights. Seikh and Mandal [53] introduced the MADM method based on $3, 4$-quasirung fuzzy sets. Riaz and Farid [54] proposed the picture fuzzy aggregation approach and application to third-party logistic provider selection. Ashraf et al. [55] introduced the Maclaurin symmetric mean operator with an interval-valued picture fuzzy model. Baig et al. [56] developed new methods for enhancing resilience in developing countries for oil supply chains. Chattopadhyay et al. [57] proposed the idea of the development of a rough-MABAC-DoE-based metamodel for iron and steel supplier selection. Karamasa et al. [58] studied weighting the factors affecting logistics outsourcing. Bairagi [59] developed a novel MCDM model for warehouse location selection in supply chain management. Recently, some applications of fuzzy modeling have been developed, such as uncertain supply chains [60], medical tourism supply chains [61], sustainable plastic recycling processes [62], and pattern recognition [63].

Multicriteria group decision-making (MCGDM) is a branch of operation research in which the alternatives are evaluated by the group of decision-makers (DMs) under multiple criteria to find a ranking of alternatives and an optimal decision. It is an important aspect of MCGDM to evaluate alternatives based on their characteristics. It is extremely difficult for an individual to choose an option in a variety of situations due to inconsistencies in the data caused by human errors or a lack of knowledge. Dealing with vagueness and uncertainties in MCGDM problems is very crucial to dealing with daily life problems. For this purpose, a variety of strategies have been utilized to evaluate the stability of human decision-making by weighing a set of options against a set of criteria. The weighted product model and choice value method are well-known methods and are often utilized to rank the alternatives according to certain criteria.

The main objectives of this research work are given as follows:

(1) To develop a topological structure on cubic intuitionistic fuzzy sets (CIFSs) with P-order (P-CIFT) as well as R-order (R-CIFT) and to validate some significant results and fundamental characteristics with examples. The concept of the CIFS is a strong hybrid model for modeling uncertainties with an IFS and an interval-valued IFS, simultaneously.

(2) To examine various properties of the cubic intuitionistic fuzzy topology (CIFT) under P-order (R-order), such as open sets of CIFT, closed sets of CIFT, interior in CIFT, closure in CIFT, subspace of CIFT, exterior in CIFT, a frontier in CIFT, and a basis of CIFT.

(3) Taking advantage of topological data analysis with cubic intuitionistic fuzzy information, we proposed two multicriteria group decision-making (MCGDM) methods.

(4) To develop Algorithm 1 for a weighted product model (WPM) and Algorithm 2 for a choice value method (CVM). An application of the proposed techniques is also designed for the uncertain supply chain management.

(5) ranking of feasible alternatives is computed, and a comparative analysis of proposed methods with existing methods is also given to discuss the validity and advantage of the proposed techniques.

The remaining sections of this paper are organized as follows. In Section 2, we reviewed some fundamental concepts such as IFS, IVIFS, cubic sets, CIFS, operations on CIFSs, and some essential results on CIFSs. The idea of cubic intuitionistic fuzzy set topology with P-order is introduced in Section 3. We also investigated some key results on CIFSs in p-order. In Section 4, we discuss the major results of cubic intuitionistic fuzzy set topology with R-order. In Section 5, we discuss a useful application that employs the weighted product model and choice value method. The conclusion of the paper is given in Section 6.

2. Preliminaries

In this section, we study some basic concepts of IFSs, IVIFSs, CSs, and CIFSs. We also review some fundamental properties of CIFSs that are necessary to understand the topological structures of CIFSs.

Definition 1 (see [3]). An intuitionistic fuzzy set (IFS) in a set $\Bbbk$ is described as

$$\mathbb{I} = \{(\ell, \zeta(\ell), \eta(\ell)) \colon 0 \le \zeta(\ell) + \eta(\ell) \le 1, \ell \in \Bbbk\}, \quad (1)$$

where, $\zeta\colon \Bbbk \longrightarrow [0,1]$ represents the membership function, and the nonmembership function is denoted by $\eta\colon \Bbbk \longrightarrow [0,1]$.

Definition 2 (see [34, 35]). Let $\mathbb{I}_1 = (\zeta_1, \eta_1)$ and $\mathbb{I}_2 = (\zeta_2, \eta_2)$ be two IFNs. Then, we have the following operations on IFNs.

(i) $\mathbb{I}_1 \subseteq \mathbb{I}_2$ if $\zeta_1 \le \zeta_2$ and $\eta_1 \ge \eta_2$ for all $\ell \in \Bbbk$
(ii) $\mathbb{I}_1 = \mathbb{I}_2$ if $\mathbb{I}_1 \subseteq \mathbb{I}_2$ and $\mathbb{I}_2 \subseteq \mathbb{I}_1$
(iii) $\mathbb{I}_1^c = \{(\ell, \eta_1(\ell), \zeta_1(\ell)); \ell \in \Bbbk\}$
(iv) $\mathbb{I}_1 \bigcup \mathbb{I}_2 = \{(\ell, \vee\{\zeta_1, \zeta_2\}, \wedge\{\eta_1, \eta_2\}) \colon \ell \in \Bbbk\}$
(v) $\mathbb{I}_1 \cap \mathbb{I}_2 = \{(\ell, \wedge\{\zeta_1, \zeta_2\}, \vee\{\eta_1, \eta_2\}) \colon \ell \in \Bbbk\}$

In reality, it is difficult to determine the exact membership and nonmembership degrees of an element in a set. In this situation, a range of values may be a better measurement to accommodate the uncertainty. For this, Atanassov and Gargov [17] introduce the idea of an interval-valued intuitionistic fuzzy set (IVIFS).

Definition 3 (see [17]). Let $\Bbbk$ be a nonempty universal set. An interval-valued intuitionistic fuzzy set (IVIFS) on $\Bbbk$ is defined as

$$\mathbb{I} = \left\{\left(\ell, \left[\zeta^L(\ell), \zeta^U(\ell)\right], \left[\eta^L(\ell), \eta^U(\ell)\right]\right); \ell \in \Bbbk\right\}, \quad (2)$$

where, $[\zeta^L(\ell), \zeta^U(\ell)]$ and $[\eta^L(\ell), \eta^U(\ell)]$ are the closed subintervals of $[0,1]$ for every $\ell \in \Bbbk$. For simplicity, the pair $\mathbb{I} = ([\zeta^L(\ell), \zeta^U(\ell)], [\eta^L(\ell), \eta^U(\ell)])$ is called interval-valued intuitionistic fuzzy number (IVIFN).

By fusing the concept of IFS and IVIFS, Jun et al. [21] defined the cubic intuitionistic set as follows:

Definition 4 (see [21]). A cubic set $\mathfrak{C}$ on a universal set $\Bbbk$ is expressed as

$$\mathfrak{C} = \{\ell, C(\ell), \sigma(\ell) \colon \ell \in \Bbbk\}, \quad (3)$$

in which $C(\ell)$ is interval-valued fuzzy set and $\sigma(\ell)$ is fuzzy set on $\Bbbk$. For use of ease, this pair is referred as $\mathfrak{C} = \langle C, \sigma \rangle$

Definition 5 (see [21]). For any cubic fuzzy sets $\mathfrak{C}_i = \langle C_i, \sigma_i \rangle$, $i \in \Lambda$, we have

(i) P-union $\cup_p \mathfrak{C}_i = \langle \vee_{i\in\Lambda} C_i, \vee_{i\in\Lambda} \sigma_i \rangle$
(ii) P-intersection $\underset{p}{\cap} \mathfrak{C}_i = < \underset{i\in\Lambda}{\wedge} C_i, \underset{i\in\Lambda}{\wedge} \sigma_i >$
(iii) R-union $\cup_R \mathfrak{C}_i = \langle \vee_{i\in\Lambda} C_i, \vee_{i\in\Lambda} \sigma_i \rangle$
(iv) R-intersection $\cup_p \mathfrak{C}_i = \langle \vee_{i\in\Lambda} C_i, \vee_{i\in\Lambda} \sigma_i \rangle$

Definition 6 (see [23, 24]). Let $\Bbbk$ be a universal set of discourse. A cubic intuitionistic fuzzy set (CIFS) on universal set $\Bbbk$ is described as

$$\mathbb{C}_{\mathbb{I}} = \left\{\left(\ell, \left[\zeta^L(\ell), \zeta^U(\ell)\right], \left[\eta^L(\ell), \eta^U(\ell)\right], (\zeta, \eta)\right); \ell \in \Bbbk\right\}, \quad (4)$$

in which $([\zeta^L(\ell), \zeta^U(\ell)], [\eta^L(\ell), \eta^U(\ell)])$ is an IVIFS and (ζ, η) is an IFS in $\Bbbk$. For ease of use, we denote these pairs as $\mathbb{C}_{\mathbb{I}} = (C, \sigma)$, where $C = ([\zeta^L, \zeta^U], [\eta^L, \eta^U])$ and $\sigma = (\zeta, \eta)$ is known as cubic intuitionistic fuzzy number (CIFN) with the condition that $[\zeta^L, \zeta^U], [\eta^L, \eta^U] \subseteq [0,1]$, $\zeta, \eta \in [0,1]$ and $\zeta + \eta \le 1$.

That is why the CIFS has the advantage of being capable to contain a lot more data to represent both the IVIFN and the IFN at the same time.

2.1. Operations on CIFSs. Now we review some fundamental operations of CIFSs, which have been explored in [23, 24].

Definition 7. The complement of the CIFS $\mathbb{C}_{\mathbb{I}} = (C, \sigma)$ is defined as $\mathbb{C}_{\mathbb{I}}^c = (C^c, \sigma^c)$ where $C^c = ([\eta^L(\ell), \eta^U(\ell)], [\zeta^L(\ell), \zeta^U(\ell)])$ be the complement of the IVIFS, $C = ([\zeta^L(\ell), \zeta^U(\ell)], [\eta^L(\ell), \eta^U(\ell)])$ and $\sigma^c = (\eta, \zeta)$ be the complement of the IFS, $\sigma = (\zeta, \eta)$. Thus, the complement of CIFS is expressed as

$$\mathbb{C}_{\mathbb{I}}^c = \left\{\left(\ell, \left[\eta^L(\ell), \eta^U(\ell)\right], \left[\zeta^L(\ell), \zeta^U(\ell)\right], (\eta, \zeta)\right); \ell \in \Bbbk\right\}. \quad (5)$$

Definition 8. Consider two CIFSs on a universal set $\Bbbk$ is given as follow:

$$\mathbb{C}_{\mathbb{I}}^1 = \left\{\left(\ell, \left[\zeta_1^L, \zeta_1^U\right], \left[\eta_1^L, \eta_1^U\right], (\zeta_1, \eta_1)\right); \ell \in \Bbbk\right\}, \quad (6)$$

and

$$\mathbb{C}_{\mathbb{I}}^2 = \left\{\left(\ell, \left[\zeta_2^L, \zeta_2^U\right], \left[\eta_2^L, \eta_2^U\right], (\zeta_2, \eta_2)\right); \ell \in \Bbbk\right\}, \quad (7)$$

we define

Step 1. Obtain the decision matrix from the decision-makers, which indicates the alternative's $\mathbb{X}_j$, $(j = 1 \ldots m)$ evaluation values on the basis of criterion $\mathbb{C}_i$, $(i = 1, \ldots n)$ by $\mathbb{T}_{ji} = (C_{ji}, \sigma_{ji})$, where $C_{ji} = ([\zeta_{ji}^L, \zeta_{ji}^U], [\eta_{ji}^L, \eta_{ji}^U])$ an IVIFN and $\sigma_{ji} = (\zeta_{ji}, \eta_{ji})$ is known as a cubic intuitionistic fuzzy number. The decision-makers provide the decision matrix $M = (\mathbb{T}_{ji})_{m\times n}$ of the form.

$$\begin{matrix} \\ \mathbb{X}_1 \\ \mathbb{X}_2 \\ \mathbb{X}_m \\ \\ \end{matrix} \begin{bmatrix} \mathbb{C}_1 & \mathbb{C}_2 & \ldots\ldots & \mathbb{C}_n \\ (C_{11}, \sigma_{11}) & (C_{12}, \sigma_{12}) & \ldots\ldots & (C_{1n}, \sigma_{1n}) \\ (C_{21}, \sigma_{21}) & (C_{22}, \sigma_{22}) & \ldots\ldots & (C_{2n}, \sigma_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (C_{m1}, \sigma_{m1}) & (C_{m2}, \sigma_{m2}) & \ldots\ldots & (C_{mn}, \sigma_{mn}) \end{bmatrix}.$$

Step 2. Then, the decision matrix $M = (\mathbb{T}_{ji})_{m\times n}$ is made normalized by a linear approach. Assume the criteria are categorized into benefit criteria $\mathbb{B}$ and cost criteria $\mathbb{K}$. The normalization of every $i \in \mathbb{B}$ is defined as
$\overline{\mathbb{T}}_{ji} = \mathbb{T}_{ji}/\max_j \mathbb{T}_{ji}$,
where $\max_j \mathbb{T}_{ji}$ is defined as
$\max_j \mathbb{T}_{ji} = ([(\max \zeta_{ji}^L, \max \zeta_{ji}^U)], ([\min \eta_{ji}^L, \min \eta_{ji}^U]), (\min \zeta_{ji}, \max \eta_{ji}))$.
Similarly, the normalization of every $i \in \mathbb{K}$ is defined as
$\overline{\mathbb{T}}_{ji} = \min_j \mathbb{T}_{ji}/\mathbb{T}_{ji}$,
where $\min_j \mathbb{T}_{ji}$ is defined as
$\min_j \mathbb{T}_{ji} = ([(\min \zeta_{ji}^L, \min \zeta_{ji}^U)], ([\max \eta_{ji}^L, \max \eta_{ji}^U]), (\max \zeta_{ji}, \min \eta_{ji}))$.
The decision matrix $M = (\mathbb{T}_{ji})_{m\times n}$ is then transformed into normalized decision matrix $\overline{M} = (\overline{\mathbb{T}}_{ji})_{m\times n}$ and is given as

$$\begin{matrix} \\ \mathbb{X}_1 \\ \mathbb{X}_2 \\ \mathbb{X}_m \\ \\ \end{matrix} \begin{bmatrix} \mathbb{C}_1 & \mathbb{C}_2 & \ldots\ldots & \mathbb{C}_n \\ (\overline{C}_{11}, \overline{\sigma}_{11}) & (\overline{C}_{12}, \overline{\sigma}_{12}) & \ldots\ldots & (\overline{C}_{1n}, \overline{\sigma}_{1n}) \\ (\overline{C}_{21}, \overline{\sigma}_{21}) & (C_{22}, \sigma_{22}) & \ldots\ldots & (\overline{C}_{2n}, \overline{\sigma}_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\overline{C}_{m1}, \overline{\sigma}_{m1}) & (\overline{C}_{m2}, \overline{\sigma}_{m2}) & \ldots\ldots & (\overline{C}_{mn}, \overline{\sigma}_{mn}) \end{bmatrix}.$$

Step 3. According to CIFS-WPM, the relative importance of j alternatives is denoted as $\mathbb{Y}_j$ and is defined as
$\mathbb{Y}_j = \prod i = 1n(\overline{\mathbb{T}}_{ji})^{wi}$,
Here, we use the operation of power rule of CIFNs and also the product operation of CIFNs.
Step 4. Find the score function of all vales $\mathbb{Y}_j$.
Step 5. Ranking of alternatives according to the score functions of $\mathbb{Y}_j$.

Algorithm 1: Weighted product model (WPM).

Step 1. Obtain the decision matrix from the decision-makers, with alternative's $\mathbb{X}_j$ evaluate on the basis of criterion $\mathbb{C}_i$ by $\mathbb{T}_{ji} = (C_{ji}, \sigma_{ji})$, where $C_{ji} = ([\zeta_{ji}^L, \zeta_{ji}^U], [\eta_{ji}^L, \eta_{ji}^U])$ an IVIFN and $\sigma_{ji} = (\zeta_{ji}, \eta_{ji})$ is known as cubic intuitionistic fuzzy number. The decision-makers provide the decision matrix $M = (\mathbb{T}_{ji})_{m\times n}$ of the form.

$$\begin{matrix} \\ \mathbb{X}_1 \\ \mathbb{X}_2 \\ \mathbb{X}_m \\ \\ \end{matrix} \begin{bmatrix} \mathbb{C}_1 & \mathbb{C}_2 & \ldots\ldots & \mathbb{C}_n \\ (C_{11}, \sigma_{11}) & (C_{12}, \sigma_{12}) & \ldots\ldots & (C_{1n}, \sigma_{1n}) \\ (C_{21}, \sigma_{21}) & (C_{22}, \sigma_{22}) & \ldots\ldots & (C_{2n}, \sigma_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (C_{m1}, \sigma_{m1}) & (C_{m2}, \sigma_{m2}) & \ldots\ldots & (C_{mn}, \sigma_{mn}) \end{bmatrix}.$$

Step 2. The decision-makers also give weightage to the criteria, with the condition that the sum of the weights must be equal to unity. We compute the multiplication of the decision matrix with criteria weights.

$$\begin{bmatrix} (C_{11}, \sigma_{11}) & (C_{12}, \sigma_{12}) & \ldots\ldots & (C_{1n}, \sigma_{1n}) \\ (C_{21}, \sigma_{21}) & (C_{22}, \sigma_{22}) & \ldots\ldots & (C_{2n}, \sigma_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (C_{m1}, \sigma_{m1}) & (C_{m2}, \sigma_{m2}) & \ldots\ldots & (C_{mn}, \sigma_{mn}) \end{bmatrix} \begin{bmatrix} W1 \\ W2 \\ Wn \end{bmatrix}.$$

Step 3. We find the score function of each value.
Step 4. Compute the ranking of the alternatives according to their score function values.

Algorithm 2: Choice value method (CVM).

(i) (P-order) $\mathbb{C}_{\mathbb{I}}^1 \subseteq_P \mathbb{C}_{\mathbb{I}}^2$ if $[\zeta_1^L, \zeta_1^U] \subseteq [\zeta_2^L, \zeta_2^U]$, $[\eta_1^L, \eta_1^U] \supseteq [\eta_2^L, \eta_2^U]$, $\zeta_1 \leq \zeta_2$ and $\eta_1 \geq \eta_2$

(ii) (R-order) $\mathbb{C}_{\mathbb{I}}^1 \subseteq_R \mathbb{C}_{\mathbb{I}}^2$ if $[\zeta_1^L, \zeta_1^U] \subseteq [\zeta_2^L, \zeta_2^U]$, $[\eta_1^L, \eta_1^U] \supseteq [\eta_2^L, \eta_2^U]$, $\zeta_1 \geq \zeta_2$ and $\eta_1 \leq \eta_2$

(iii) (Equality) $\mathbb{C}_{\mathbb{I}}^1 = \mathbb{C}_{\mathbb{I}}^2$ if $[\zeta_1^L, \zeta_1^U] = [\zeta_2^L, \zeta_2^U]$, $[\eta_1^L, \eta_1^U] = [\eta_2^L, \eta_2^U]$, $\zeta_1 = \zeta_2$ and $\eta_1 = \eta_2$

Definition 9. For any CIFSs

$$\mathbb{C}_{\mathbb{I}}^i = \left(\left[\left[\zeta_i^L, \zeta_i^U\right], \left[\eta_i^L, \eta_i^U\right], (\zeta_i, \eta_i)\right); \ell \in \mathbb{k}\right\} i \in \Lambda, \quad (8)$$

the operations listed have been defined as follows:

(i) (P-union) $\cup_P \mathbb{C}_{\mathbb{I}}^i = \left\{([\vee_{i\in\Lambda} \zeta_i^L, \vee_{i\in\Lambda} \zeta_i^U], [\wedge_{i\in\Lambda} \eta_i^L, \wedge_{i\in\Lambda} \eta_i^U]), (\vee_{i\in\Lambda} \zeta_i, \wedge_{i\in\Lambda} \eta_i)\right\}$

(ii) (P-intersection) $\cap_P \mathbb{C}_{\mathbb{I}}^i = \left\{([\wedge_{i\in\Lambda} \zeta_i^L, \wedge_{i\in\Lambda} \zeta_i^U], [\vee_{i\in\Lambda} \eta_i^L, \vee_{i\in\Lambda} \eta_i^U]), (\wedge_{i\in\Lambda} \zeta_i, \vee_{i\in\Lambda} \eta_i)\right\}$

(iii) (R-union) $\cup_R \mathbb{C}_{\mathbb{I}}^i = \left\{([\vee_{i\in\Lambda} \zeta_i^L, \vee_{i\in\Lambda} \zeta_i^U], [\wedge_{i\in\Lambda} \eta_i^L, \wedge_{i\in\Lambda} \eta_i^U]), (\wedge_{i\in\Lambda} \zeta_i, \vee_{i\in\Lambda} \eta_i)\right\}$

(iv) (R-intersection) $\cap_R \mathbb{C}_{\mathbb{I}}^i = \left\{([\wedge_{i\in\Lambda} \zeta_i^L, \wedge_{i\in\Lambda} \zeta_i^U], [\vee_{i\in\Lambda} \eta_i^L, \vee_{i\in\Lambda} \eta_i^U]), (\vee_{i\in\Lambda} \zeta_i, \wedge_{i\in\Lambda} \eta_i)\right\}$

2.2. Some Results on CIFSs. Now we review some essential properties and results of CIFSs, which have been explored in [23, 24].

Definition 10. A CIFS

$$\mathbb{C}_{\mathbb{I}} = \left\{\left(\ell, \left[\zeta^L(\ell), \zeta^U(\ell)\right], \left[\eta^L(\ell), \eta^U(\ell)\right], (\zeta, \eta)\right); \ell \in \mathbb{k}\right\}, \quad (9)$$

for which $([\zeta^L(\ell), \zeta^U(\ell)], [\eta^L(\ell), \eta^U(\ell)]) = ([0,0], [1,1])$ and $(\zeta, \eta) = (1, 0)$ for all $\ell \in \mathbb{k}$ is denoted by ${}^0\mathbb{C}_{\mathbb{I}}$

Definition 11. A CIFS

$$\mathbb{C}_{\mathbb{I}} = \left\{\left(\ell, \left[\zeta^L(\ell), \zeta^U(\ell)\right], \left[\eta^L(\ell), \eta^U(\ell)\right], (\zeta, \eta)\right); \ell \in \mathbb{k}\right\}, \quad (10)$$

for which $([\zeta^L(\ell), \zeta^U(\ell)], [\eta^L(\ell), \eta^U(\ell)]) = ([1,1], [0,0])$ and $(\zeta, \eta) = (0, 1)$ for all $\ell \in \mathbb{k}$ is denoted by ${}^1\mathbb{C}_{\mathbb{I}}$

Definition 12. A CIFS

$$\mathbb{C}_{\mathbb{I}} = \left\{\left(\ell, \left[\zeta^L(\ell), \zeta^U(\ell)\right], \left[\eta^L(\ell), \eta^U(\ell)\right], (\zeta, \eta)\right); \ell \in \mathbb{k}\right\}, \quad (11)$$

for which $([\zeta^L(\ell), \zeta^U(\ell)], [\eta^L(\ell), \eta^U(\ell)]) = ([1,1], [0,0])$ and $(\zeta, \eta) = (0, 1)$ for all $\ell \in \mathbb{k}$ is denoted by ${}^0\mathbb{C}_{\mathbb{I}}$

Definition 13. A CIFS

$$\mathbb{C}_{\mathbb{I}} = \left\{\left(\ell, \left[\zeta^L(\ell), \zeta^U(\ell)\right], \left[\eta^L(\ell), \eta^U(\ell)\right], (\zeta, \eta)\right); \ell \in \mathbb{k}\right\}, \quad (12)$$

for which $([\zeta^L(\ell), \zeta^U(\ell)], [\eta^L(\ell), \eta^U(\ell)]) = ([1,1], [0,0])$ and $(\zeta, \eta) = (1, 0)$ for all $\ell \in \mathbb{k}$ is denoted by ${}^1\mathbb{C}_{\mathbb{I}}$

Definition 14. Let $\mathbb{C}_{\mathbb{I}} = \{(\ell, [\zeta^L(\ell), \zeta^U(\ell)], [\eta^L(\ell), \eta^U(\ell)], (\zeta, \eta))\}$ be a CIFN. The score function $\mathbb{S}(\mathbb{C}_{\mathbb{I}})$ and the accuracy function $\mathbb{A}(\mathbb{C}_{\mathbb{I}})$ on for CIFNs are defined as

For P-order

$$\mathbb{S}(\mathbb{C}_{\mathbb{I}}) = \frac{\zeta^L + \zeta^U - \eta^L - \eta^U}{2} + \zeta - \eta. \quad (13)$$

For R-order

$$\mathbb{S}(\mathbb{C}_{\mathbb{I}}) = \frac{\zeta^L + \zeta^U - \eta^L - \eta^U}{2} + \eta - \zeta,$$
$$\mathbb{A}(\mathbb{C}_{\mathbb{I}}) = \frac{\zeta^L + \zeta^U + \eta^L + \eta^U}{2} + \zeta + \eta. \quad (14)$$

The ranking of CIFNs in relation to the proposed scoring function and accuracy function is determined as.

(i) $\mathbb{C}_{\mathbb{I}} < \mathbb{C}_{\mathbb{I}}^1$ if $\mathbb{S}(\mathbb{C}^{\mathbb{I}}) < \mathbb{S}(\mathbb{C}_{\mathbb{I}}^1)$,

(ii) If $\mathbb{S}(\mathbb{C}_{\mathbb{I}}) = \mathbb{S}(\mathbb{C}_{\mathbb{I}}^1)$, then $\mathbb{C}_{\mathbb{I}} < \mathbb{C}_{\mathbb{I}}^1$ if $\mathbb{A}(\mathbb{C}_{\mathbb{I}}) < \mathbb{A}(\mathbb{C}_{\mathbb{I}}^1)$

(iii) If $\mathbb{S}(\mathbb{C}_{\mathbb{I}}) = \mathbb{S}(\mathbb{C}_{\mathbb{I}}^1)$ and $\mathbb{A}(\mathbb{C}_{\mathbb{I}}) = \mathbb{A}(\mathbb{C}_{\mathbb{I}}^1)$, then $\mathbb{C}_{\mathbb{I}} = \mathbb{C}_{\mathbb{I}}^1$

Definition 15. Let $\mathbb{C}_{\mathbb{I}} = \{(\ell, [\zeta^L, \zeta^U], [\eta^L, \eta^U], (\zeta, \eta)); \ell \in \mathbb{k}\}$ and

$$\mathbb{C}_{\mathbb{I}}^i = \left\{\left(\ell, \left[\zeta_i^L, \zeta_i^U\right], \left[\eta_i^L, \eta_i^U\right], (\zeta_i, \eta_i)\right); \ell \in \mathbb{k}\right\}, (i = 1, 2), \quad (15)$$

be the CIFNs and let $p > 0$ be any real number. The basic operations on CIFs are given as

(i) $\mathbb{C}_{\mathbb{I}}^1 + \mathbb{C}_{\mathbb{I}}^2 = (([1 - \prod_{i=1}^2 (1 - \zeta_i^L), 1 - \prod_{i=1}^2 (1 - \zeta_i^U)], [\prod_{i=1}^2 \eta_i^L, \prod_{i=1}^2 \eta_i^U]), (\prod_{i=1}^2 \zeta_i, 1 - \prod_{i=1}^2 \eta_i))$

(ii) $\mathbb{C}_{\mathbb{I}}^1 \times \mathbb{C}_{\mathbb{I}}^2 = ([\prod i = 12\zeta_i^L, \prod i = 12\zeta_i^U], [1 - \prod i = 12(1 - \eta_i^L), 1 - \prod i = 12(1 - 1 - \eta_i^U)])t, n(1 - \prod i = 12\zeta_i, \prod i = 12\eta_i))$

(iii) $p\mathbb{C}_{\mathbb{I}} = (([1 - (1 - \zeta^L)^p, 1 - (1 - \zeta^U)^p], [(\eta^L)^p, (\eta^U)^p]), ((\zeta)^p, 1 - (1 - (\eta)^p))$

(iv) $\mathbb{C}_{\mathbb{I}}^p = (([(\zeta^L)^p, (\zeta^U)^p], [1 - (1 - \eta^L)^p, 1 - (1 - \eta^U)^p]), (1 - \{1 - (\zeta)^p, (\eta)^p))$

Definition 16. Let

$$\mathbb{C}_{\mathbb{I}}^i = \left\{\left(\ell, \left[\zeta_i^L, \zeta_i^U\right], \left[\eta_i^L, \eta_i^U\right], (\zeta_i, \eta_i)\right); \ell \in \mathbb{k}\right\}, (i = 1, 2), \quad (16)$$

be the CIFNs. Then, the division operator on CIFN is given as

$$\frac{\mathbb{C}_{\mathbb{I}}^1}{\mathbb{C}_{\mathbb{I}}^2} = \left(\left(\left[\min \zeta_1^L, \zeta_2^L, \min \zeta_1^U, \zeta_2^U\right], \left[\max \eta_1^L, \eta_2^L, \max \eta_1^U, \eta_2^U\right]\right), \left(\max \zeta_1, \zeta_2, \min \eta_1^U, \eta_2^U\right)\right). \quad (17)$$

3. Cubic Intuitionistic Topology under P-Order

In this section, we introduce the concept of a P-cubic intuitionistic fuzzy topology (P-CIFT) or a cubic intuitionistic fuzzy topology with P-order.

Definition 17. Consider $\mathbb{k}$ to be a nonempty universal set, and let $ci(\mathbb{k})$ to be the accumulation of all CIFSs in $\mathbb{k}$. If the collection $\mathbb{T}_{\mathbb{C}_{\mathbb{I}\mathbb{P}}}$ containing CIFSs satisfies the following conditions, it is termed as a cubic intuitionistic fuzzy topology with a P-order (P-CIFT).

(1) ${}^{0}\mathbb{C}_{\mathbb{I}}$, ${}^{1}\mathbb{C}_{\mathbb{I}}$, $\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$ and $\overline{{}^{0}\mathbb{C}_{\mathbb{I}}} \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$
(2) If $(\mathbb{C}_{\mathbb{IP}})_i \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}} \forall i \in \Lambda$ then $\cup_p (\mathbb{C}_{\mathbb{IP}})_i \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$
(3) If $\mathbb{C}^1_{\mathbb{IP}}, \mathbb{C}^2_{\mathbb{IP}} \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$ then $\mathbb{C}^1_{\mathbb{IP}} \cap_p \mathbb{C}^2_{\mathbb{IP}} \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$

Then, the pair $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$ is called cubic intuitionistic fuzzy topological space with a P-order (P-CIFT).

Example 1. Let $\mathbb{k}$ be a universal set. Then, $ci(\mathbb{k})$ be the assemblage of all P-cubic intuitionistic fuzzy sets PCIFSs in $\mathbb{k}$. Consider P-order fuzzy subsets of $ci(\mathbb{k})$ given as

$$\begin{aligned}
\mathbb{C}^1_{\mathbb{IP}} &= \{[0.20, 0.31], [0.41, 0.52], (0.32, 0.44)\},\\
\mathbb{C}^2_{\mathbb{IP}} &= \{[0.20, 0.31], [0.41, 0.52], (1, 0)\},\\
\mathbb{C}^3_{\mathbb{IP}} &= \{[0.20, 0.31], [0.41, 0.52], (0, 1)\},\\
\mathbb{C}^4_{\mathbb{IP}} &= \{[1, 1], [0, 0], (0.32, 0.44)\},\\
\mathbb{C}^5_{\mathbb{IP}} &= \{[0, 0], [1, 1], (0.32, 0.44)\}.
\end{aligned} \quad (18)$$

The union and intersection with a P-order for the above CIFSs are given in Tables 1 and 2,, respectively.

Clearly,

$$\mathbb{T}_{\mathbb{C}^1_{\mathbb{IP}}} = \left\{{}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}\right\}, \quad (19)$$

and

$$\mathbb{T}^2_{\mathbb{C}_{\mathbb{IP}}} = \left\{{}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}, \mathbb{C}^1_{\mathbb{IP}}, \mathbb{C}^2_{\mathbb{IP}}, \mathbb{C}^3_{\mathbb{IP}}, \mathbb{C}^4_{\mathbb{IP}}, \mathbb{C}^5_{\mathbb{IP}}\right\}, \quad (20)$$

are cubic intuitionistic topology with a P-order.

Definition 18. Let $\mathbb{k}$ be a nonempty set and $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}} = \left\{\mathbb{C}^{\mathbb{k}}_{\mathbb{IP}}\right\}$ where $\mathbb{C}^{\mathbb{k}}_{\mathbb{IP}}$ represent the cubic intuitionistic fuzzy subsets of universal set $\mathbb{k}$. Then, $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$ is termed as a P-cubic intuitionistic fuzzy topology on $\mathbb{k}$ and it is the largest P-cubic intuitionistic fuzzy topology on $\mathbb{k}$ and is entitled as P-discrete cubic intuitionistic fuzzy topology.

Definition 19. Let $\mathbb{k}$ be a universal set and $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}} = \left\{{}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}\right\}$ be the assemblage of cubic intuitionistic fuzzy sets. Then, $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$ is termed as a P-cubic intuitionistic fuzzy topology on universal set $\mathbb{k}$ and is the smallest P-cubic intuitionistic fuzzy topology on $\mathbb{k}$ and is entitled as P-indiscrete cubic intuitionistic fuzzy topology.

Definition 20. The elements of a P-cubic intuitionistic fuzzy topology $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$ is termed as P-cubic intuitionistic fuzzy open sets PCIFOS in $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$.

Theorem 1. *If $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$ is any P-cubic intuitionistic fuzzy topological space. Then,*

(1) ${}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$ and $\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$ are PCIFOSs
(2) The P-union of any number of PCIFOSs is PCIFOS
(3) The P-intersection of finite PCIFOSs is PCIFOS

Proof

(1) By the Definition 4.2 of a P-cubic intuitionistic fuzzy topology (P-CIFT), ${}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$ and $\overline{{}^{1}\mathbb{C}_{\mathbb{I}}} \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$. Hence, ${}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$ and $\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$ are PCIFOSs.
(2) Let $\{(\mathbb{C}_{\mathbb{IP}})_i | i \in \Lambda\}$ be PCIFOSs. Then, $(\mathbb{C}_{\mathbb{IP}})_i \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$. From the definition of P-CIFT

$$\bigcup_p (\mathbb{C}_{\mathbb{IP}})_i \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}. \quad (21)$$

Hence, $\cup_p (\mathbb{C}_{\mathbb{IP}})_i \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$ is PCIFOSs.
(3) Let $\mathbb{C}^1_{\mathbb{IP}}, \mathbb{C}^2_{\mathbb{IP}}, \ldots, \mathbb{C}^n_{\mathbb{IP}}$ be PCIOSs. Then, from definition of P-CIFT

$$\bigcap_p (\mathbb{C}_{\mathbb{IP}})_i \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}. \quad (22)$$

Hence, $\cap_p (\mathbb{C}_{\mathbb{IP}})_i$ is PCIFOSs. □

Definition 21. The complement of elements of P-cubic intuitionistic fuzzy open sets is termed as P-cubic intuitionistic fuzzy closed sets PCIFCSs in $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$.

Theorem 2. *If $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$ is any P-cubic intuitionistic fuzzy topological space. Then,*

(1) ${}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$ and $\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$ are PCIFCSs
(2) The P-intersection of any number of PCIFCSs is PCIFCS
(3) The P-union of finite PCIFCSs is PCIFCS

Proof

(1) ${}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$ and $\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$ are PCIFOSs. From the definition of P-CIFT

$${}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}} \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}. \quad (23)$$

Since the complement of ${}^{0}\mathbb{C}_{\mathbb{I}} = {}^{1}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}} = {}^{0}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}} = \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$ and $\overline{{}^{1}\mathbb{C}_{\mathbb{I}}} = \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$. So, ${}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$ and $\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$ are PCIFCSs.
(2) Let $\{(\mathbb{C}_{\mathbb{IP}})_i | i \in \Lambda\}$ be PCIFCSs. Then,

$$((\mathbb{C}_{\mathbb{IP}})_i)^c \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}. \quad (24)$$

From the definition of P-CIFT,

$$\bigcup_p ((\mathbb{C}_{\mathbb{IP}})_i)^c \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}. \quad (25)$$

Hence, $\cup_p ((\mathbb{C}_{\mathbb{IP}})_i)^c$ is PCIFOSs, but

$$\left(\bigcup_p ((\mathbb{C}_{\mathbb{IP}})_i)^c\right) = \left(\bigcap_p ((\mathbb{C}_{\mathbb{IP}})_i)\right)^c. \quad (26)$$

So, $\cap_p (\mathbb{C}_{\mathbb{IP}})_i$ is PCIFCSs.
(3) Let $\mathbb{C}^1_{\mathbb{IP}}, \mathbb{C}^2_{\mathbb{IP}}, \ldots, \mathbb{C}^n_{\mathbb{IP}}$ be PCmPCSs. Then, $(\mathbb{C}^1_{\mathbb{IP}})^c, (\mathbb{C}^2_{\mathbb{IP}})^c, \ldots, (\mathbb{C}^n_{\mathbb{IP}})^c$ are PCIFOSs. So,

Table 1: Union under P-order.

$\cup_p$	${}^0\mathbb{C}_{\mathbb{I}}$	${}^1\mathbb{C}_{\mathbb{I}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$
${}^0\mathbb{C}_{\mathbb{I}}$	${}^0\mathbb{C}_{\mathbb{I}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	${}^0\mathbb{C}_{\mathbb{I}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	${}^0\mathbb{C}_{\mathbb{I}}$
${}^1\mathbb{C}_{\mathbb{I}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	${}^1\mathbb{C}_{\mathbb{I}}$	${}^1\mathbb{C}_{\mathbb{I}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	${}^1\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^4_{\mathbb{IP}}$
$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	${}^0\mathbb{C}_{\mathbb{I}}$	${}^1\mathbb{C}_{\mathbb{I}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$
$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$
$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^1_{\mathbb{IP}}$
$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^2_{\mathbb{IP}}$
$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	${}^1\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^1_{\mathbb{IP}}$
$\mathbb{C}^4_{\mathbb{IP}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^4_{\mathbb{IP}}$
$\mathbb{C}^5_{\mathbb{IP}}$	${}^0\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$

Table 2: Intersection under P-order.

$\cap_p$	${}^0\mathbb{C}_{\mathbb{I}}$	${}^1\mathbb{C}_{\mathbb{I}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$
${}^0\mathbb{C}_{\mathbb{I}}$	${}^0\mathbb{C}_{\mathbb{I}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	${}^0\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}^5_{\mathbb{IP}}$	${}^0\mathbb{C}_{\mathbb{I}}$	${}^0\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}^5_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$
${}^1\mathbb{C}_{\mathbb{I}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	${}^1\mathbb{C}_{\mathbb{I}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	${}^1\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	${}^1\mathbb{C}_{\mathbb{I}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$
$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	${}^0\mathbb{C}_{\mathbb{I}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$
$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	${}^0\mathbb{C}_{\mathbb{I}}$	${}^1\mathbb{C}_{\mathbb{I}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^1\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$
$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$
$\mathbb{C}^2_{\mathbb{IP}}$	${}^0\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^2_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$
$\mathbb{C}^3_{\mathbb{IP}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$
$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$	${}^1\mathbb{C}_{\mathbb{I}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^1_{\mathbb{IP}}$	$\mathbb{C}^3_{\mathbb{IP}}$	$\mathbb{C}^4_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$
$\mathbb{C}^5_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^5_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$	$\overline{{}^0\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}^5_{\mathbb{IP}}$	$\mathbb{C}^5_{\mathbb{IP}}$

$$\left(\mathbb{C}^1_{\mathbb{IP}}\right)^c, \left(\mathbb{C}^2_{\mathbb{IP}}\right)^c, \ldots, \left(\mathbb{C}^n_{\mathbb{IP}}\right)^c \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}. \quad (27)$$

From the definition of P-CIFT,

$$\underset{p}{\cap} \left((\mathbb{C}_{\mathbb{IP}})_i\right)^c \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}. \quad (28)$$

This gives $\cap_p \left((\mathbb{C}_{\mathbb{IP}})_i\right)^c \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$ is PCIFOSs, but

$$\left(\bigcap_p \left((\mathbb{C}_{\mathbb{IP}})_i\right)^c\right) = \left(\bigcup_p (\mathbb{C}_{\mathbb{IP}})_i\right)^c. \quad (29)$$

Hence, $\cup_p (\mathbb{C}_{\mathbb{IP}})_i$ is PCIFCSs. □

Definition 22. The P-cubic intuitionistic fuzzy sets PCIFSs, which are PCIFOSs and PCIFCSs, are entitled as P-cubic intuitionistic fuzzy clopen sets in $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$.

Proposition 1

(1) *For every* $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$, ${}^0\mathbb{C}_{\mathbb{I}}, {}^1\mathbb{C}_{\mathbb{I}}, \overline{{}^0\mathbb{C}_{\mathbb{I}}}$ *and* $\overline{{}^1\mathbb{C}_{\mathbb{I}}}$ *are P-cubic intuitionistic fuzzy clopen sets*

(2) *For discrete P-order cubic intuitionistic fuzzy topology, all the cubic intuitionistic subsets of* $\Bbbk$ *are P-cubic intuitionistic fuzzy clopen sets*

(3) *For in-discrete P-order cubic intuitionistic fuzzy topology,* ${}^0\mathbb{C}_{\mathbb{I}}, {}^1\mathbb{C}_{\mathbb{I}}, \overline{{}^0\mathbb{C}_{\mathbb{I}}}$ *and* $\overline{{}^1\mathbb{C}_{\mathbb{I}}}$ *are only P-cubic intuitionistic fuzzy clopen sets*

Definition 23. Let $(\Bbbk, \mathbb{T}^1_{\mathbb{C}_{\mathbb{IP}}})$ and $(\Bbbk, \mathbb{T}^2_{\mathbb{C}_{\mathbb{IP}}})$ be two P-CIFTs in $\Bbbk$. Two P-CIFTs are called comparable if

$$\mathbb{T}^1_{\mathbb{C}_{\mathbb{IP}}} \subseteq_P \mathbb{T}^2_{\mathbb{C}_{\mathbb{IP}}}, \quad (30)$$

or

$$\mathbb{T}^2_{\mathbb{C}_{\mathbb{IP}}} \subseteq_P \mathbb{T}^1_{\mathbb{C}_{\mathbb{IP}}}. \quad (31)$$

If $\mathbb{T}^1_{\mathbb{C}_{\mathbb{IP}}} \subseteq_P \mathbb{T}^2_{\mathbb{C}_{\mathbb{IP}}}$ then, $\mathbb{T}^1_{\mathbb{C}_{\mathbb{IP}}}$ is called P-cubic intuitionistic fuzzy coarser than $\mathbb{T}^2_{\mathbb{C}_{\mathbb{IP}}}$ and $\mathbb{T}^2_{\mathbb{C}_{\mathbb{IP}}}$ is called P-cubic intuitionistic fuzzy finer than. $\mathbb{T}^1_{\mathbb{C}_{\mathbb{IP}}}$

Example 2. Let $\Bbbk$ be a nonempty set and from Example 1

$$\mathbb{T}_{\mathbb{C}^1_{\mathbb{IP}}} = \left\{{}^0\mathbb{C}_{\mathbb{I}}, {}^1\mathbb{C}_{\mathbb{I}}, \overline{{}^0\mathbb{C}_{\mathbb{I}}}, \overline{{}^1\mathbb{C}_{\mathbb{I}}}\right\}, \quad (32)$$

and

$$\mathbb{T}^2_{\mathbb{C}_{\mathbb{IP}}} = \left\{{}^0\mathbb{C}_{\mathbb{I}}, {}^1\mathbb{C}_{\mathbb{I}}, \overline{{}^0\mathbb{C}_{\mathbb{I}}}, \overline{{}^1\mathbb{C}_{\mathbb{I}}}, \mathbb{C}^1_{\mathbb{IP}}, \mathbb{C}^2_{\mathbb{IP}}, \mathbb{C}^3_{\mathbb{IP}}, \mathbb{C}^4_{\mathbb{IP}}, \mathbb{C}^5_{\mathbb{IP}}\right\}, \quad (33)$$

are P-cubic intuitionistic fuzzy topologies on universal set. Then, $\mathbb{T}_{\mathbb{C}^1_{\mathbb{IP}}} \subseteq_P \mathbb{T}_{\mathbb{C}^2_{\mathbb{IP}}}$. Hence, $\mathbb{T}_{\mathbb{C}^1_{\mathbb{IP}}}$ is called a P-cubic intuitionistic fuzzy coarser then, $\mathbb{T}_{\mathbb{C}^2_{\mathbb{IP}}}$.

3.1. Subspace of $\mathbb{CIFT}p$

Definition 24. Let $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}\Bbbk})$ be a $\mathbb{CIFT}p$. Let $\mathbb{Y} \subseteq \Bbbk$ and $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}\mathbb{Y}}$ is a $\mathbb{CIFT}p$ on $\mathbb{Y}$ and whose PCIFOSs are

$$\mathbb{C}_{\mathbb{IP}\mathbb{Y}} = \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}\Bbbk} \underset{p}{\cap} \mathbb{Y}, \quad (34)$$

where $\mathbb{C}_{\mathbb{IP}_{\Bbbk}}$ are PCIFOSs of $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}\Bbbk}$, $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}\mathbb{Y}}$ are PCIFOSs of $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}\mathbb{Y}}$ and $\widetilde{\mathbb{Y}}$ is any P-cubic subset of PCIFS on $\mathbb{Y}$. Then, $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}\mathbb{Y}}$ is called a P-cubic intuitionistic fuzzy subspace of $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}\Bbbk}$, i.e.,

$$\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}\mathbb{Y}} = \left\{ \mathbb{C}_{\mathbb{IP}_{\mathbb{Y}}} : \mathbb{C}_{\mathbb{IP}_{\mathbb{Y}}} = \mathbb{C}_{\mathbb{IP}_{\mathbb{k}}} \underset{p}{\cap} \mathbb{Y}, \mathbb{C}_{\mathbb{IP}_{\mathbb{k}}} \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}\mathbb{k}} \right\}. \quad (35)$$

Example 3. Let $\mathbb{k}$ be a nonempty set. From Example 1,

$$\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}} = \left\{ {}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}, \mathbb{C}^{1}_{\mathbb{IP}}, \mathbb{C}^{2}_{\mathbb{IP}}, \mathbb{C}^{3}_{\mathbb{IP}}, \mathbb{C}^{4}_{\mathbb{IP}}, \mathbb{C}^{5}_{\mathbb{IP}} \right\}, \quad (36)$$

is a P-cubic intuitionistic fuzzy topology on $\mathbb{k}$.

Now, consider any P-cubic fuzzy subset on $\mathbb{k}$ such that $\mathbb{Y} \subseteq \mathbb{k}$ is

$$\mathbb{Y} = \{[0.98, 0.23], [0.46, 0.61], (0.27, 0.49)\}. \quad (37)$$

Since,

$$\begin{aligned}
\mathbb{Y} \overset{0}{\underset{p}{\cap}} \mathbb{C}_{\mathbb{I}} &= \{[0,0], [1,1], (0.27, 0.49)\} \\
&= \overrightarrow{\mathbb{C}_{\mathbb{IP}}}, \\
\mathbb{Y} \overset{1}{\underset{p}{\cap}} \mathbb{C}_{\mathbb{I}} &= \{[0.98, 0.23], [0.46, 0.61], (0,1)\} \\
&= \widetilde{\mathbb{C}}_{\mathbb{IP}}, \\
\mathbb{Y} \cap_p \overline{{}^{0}\mathbb{C}_{\mathbb{I}}} &= \{[0,0], [1,1], (0,1)\} \\
&= \mathbb{C}'_{\mathbb{IP}}, \\
\mathbb{Y} \cap_p \overline{{}^{1}\mathbb{C}_{\mathbb{I}}} &= \{[0.98, 0.23], [0.46, 0.61], (0.27, 0.49)\} \\
&= \mathbb{Y}, \\
\mathbb{Y} \underset{p}{\cap} \mathbb{C}^{1}_{\mathbb{IP}} &= \{[0.98, 0.23], [0.46, 0.61], (0.27, 0.49)\} \\
&= \mathbb{Y}, \\
\mathbb{Y} \underset{p}{\cap} \mathbb{C}^{2}_{\mathbb{IP}} &= \{[0.98, 0.23], [0.46, 0.61], (0.27, 0.49)\} \\
&= \mathbb{Y}, \\
\mathbb{Y} \underset{p}{\cap} \mathbb{C}^{3}_{\mathbb{IP}} &= \{[0.98, 0.23], [0.46, 0.61], (0,1)\} \\
&= \widetilde{\mathbb{C}}_{\mathbb{IP}}, \\
\mathbb{Y} \underset{p}{\cap} \mathbb{C}^{4}_{\mathbb{IP}} &= \{[0.98, 0.23], [0.46, 0.61], (0.27, 0.49)\} \\
&= \mathbb{Y}, \\
\mathbb{Y} \underset{p}{\cap} \mathbb{C}^{5}_{\mathbb{IP}} &= \{[0,0], [1,1], (0.27, 0.49)\} \\
&= \overrightarrow{\mathbb{C}}_{\mathbb{IP}}.
\end{aligned} \quad (38)$$

Then,

$$\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}\mathbb{Y}} = \left\{ \overrightarrow{\mathbb{C}_{\mathbb{IP}}}, \widetilde{\mathbb{C}_{\mathbb{IP}}}, {}'\mathbb{C}_{\mathbb{IP}}, \mathbb{Y} \right\}, \quad (39)$$

is a P-cubic intuitionistic fuzzy relative topology of $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}\mathbb{k}}$

3.2. Interior, Closure, Frontier and Exterior of PCIFSs

Definition 25. let $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$ be $\mathbb{CIFT}p$ and $\mathbb{C}_{\mathbb{IP}} \in ci(\mathbb{k})$, the interior of $\mathbb{C}_{\mathbb{IP}}$ is expressed as $\mathbb{C}^{0}_{\mathbb{IP}}$ and is described as union of all P-cubic intuitionistic fuzzy open subsets contained in $\mathbb{C}_{\mathbb{IP}}$. It is the greatest P-cubic intuitionistic fuzzy open set contained in $\mathbb{C}_{\mathbb{IP}}$.

Example 4. Consider a P-cubic intuitionistic fuzzy topological space as constructed in Example 1. Let $\mathbb{C}^{6}_{\mathbb{IP}} \in ci(\mathbb{k})$ given as

$$\mathbb{C}^{6}_{\mathbb{IP}} = \{[0.23, 0.39], [0.37, 0.48], (0.46, 0.33)\}. \quad (40)$$

Then,

$$\left(\mathbb{C}^{6}_{\mathbb{IP}}\right)^{0} = \overline{{}^{0}\mathbb{C}_{\mathbb{I}}} \underset{p}{\cup} \mathbb{C}^{1}_{\mathbb{IP}} \underset{p}{\cup} \mathbb{C}^{3}_{\mathbb{IP}} \underset{p}{\cup} \mathbb{C}^{5}_{\mathbb{IP}} = \mathbb{C}^{1}_{\mathbb{IP}}. \quad (41)$$

Theorem 3. *Let $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$ be $\mathbb{CIFT}$ p and $\mathbb{C}_{\mathbb{IP}} \in ci(\mathbb{k})$. Then, $\mathbb{C}_{\mathbb{IP}}$ is open CIFS iff $\mathbb{C}^{0}_{\mathbb{IP}} = \mathbb{C}_{\mathbb{IP}}$.*

Proof. If $\mathbb{C}_{\mathbb{IP}}$ is open CIFS, then we say that the greatest open CIFS contained in $\mathbb{C}_{\mathbb{IP}}$ is $\mathbb{C}_{\mathbb{IP}}$ itself. Thus,

$$\mathbb{C}^{0}_{\mathbb{IP}} = \mathbb{C}_{\mathbb{IP}}. \quad (42)$$

Conversely, if $\mathbb{C}^{0}_{\mathbb{IP}} = \mathbb{C}_{\mathbb{IP}}$, then $\mathbb{C}^{0}_{\mathbb{IP}}$ is open CIFS. This implies $\mathbb{C}_{\mathbb{IP}}$ is open CIFS.

Theorem 4. *Let $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$ be $\mathbb{CIFT}p$ and $\mathbb{C}^{1}_{\mathbb{IP}}, \mathbb{C}^{2}_{\mathbb{IP}} \in ci(\mathbb{k})$. Then,*

(i) $((\mathbb{C}_{\mathbb{IP}})^{0})^{0} = (\mathbb{C}_{\mathbb{IP}})^{0}$
(ii) $\mathbb{C}^{1}_{\mathbb{IP}} \subseteq_{p} \mathbb{C}^{2}_{\mathbb{IP}} \Rightarrow (\mathbb{C}^{1}_{\mathbb{IP}})^{0} \subseteq_{p} (\mathbb{C}^{2}_{\mathbb{IP}})^{0}$
(iii) $(\mathbb{C}^{1}_{\mathbb{IP}} \cap_{p} \mathbb{C}^{2}_{\mathbb{IP}})^{0} = (\mathbb{C}^{1}_{\mathbb{IP}})^{0} \subseteq_{p} (\mathbb{C}^{2}_{\mathbb{IP}})^{0}$
(iv) $(\mathbb{C}^{1}_{\mathbb{IP}} \cup_{p} \mathbb{C}^{2}_{\mathbb{IP}})^{0} \supseteq_{p} (\mathbb{C}^{1}_{\mathbb{IP}})^{0} \cup_{p} (\mathbb{C}^{2}_{\mathbb{IP}})^{0}$

Proof. Proof is trivial.

Definition 26. let $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$ be $\underline{\mathbb{CIFT}}p$ and $\mathbb{C}_{\mathbb{IP}} \in ci(\mathbb{k})$, the closure of $\overline{\mathbb{C}_{\mathbb{IP}}}$ is expressed as $\overline{\mathbb{C}_{\mathbb{IP}}}$ and is described as the intersection of all the P-cubic intuitionistic fuzzy closed supersets of $\mathbb{C}_{\mathbb{IP}}$. It is the smallest P-cubic intuitionistic fuzzy closed superset of $\mathbb{C}_{\mathbb{IP}}$.

Example 5. Let us consider a P-cubic intuitionistic topological space as constructed in Example 1. Then, the closed CIFSs are given as

$$\begin{aligned}
\left({}^{0}\mathbb{C}_{\mathbb{I}}\right)^{c} &= \{[1,1],[0,0],(0,1)\},\\
\left({}^{1}\mathbb{C}_{\mathbb{I}}\right)^{c} &= \{[0,0],[1,1],(1,0)\},\\
\left(\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}\right)^{c} &= \{[1,1],[0,0],(1,0)\},\\
\left(\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}\right)^{c} &= \{[0,0],[1,1],(0,1)\},\\
\left(\mathbb{C}^{1}_{\mathbb{IP}}\right)^{c} &= \{[0.41,0.52],[0.20,0.31],(0.44,0.32)\},\\
\left(\mathbb{C}^{2}_{\mathbb{IP}}\right)^{c} &= \{[0.41,0.52],[0.20,0.31],(0,1)\},\\
\left(\mathbb{C}^{3}_{\mathbb{IP}}\right)^{c} &= \{[0.41,0.52],[0.20,0.31],(1,0)\},\\
\left(\mathbb{C}^{4}_{\mathbb{IP}}\right)^{c} &= \{[0,0],[1,1],(0.44,0.32)\},\\
\left(\mathbb{C}^{5}_{\mathbb{IP}}\right)^{c} &= \{[1,1],[0,0],(0.44,0.32)\}.
\end{aligned} \tag{43}$$

Let $\mathbb{C}^{7}_{\mathbb{IP}} \in ci(\mathbb{k})$ given as

$$\mathbb{C}^{7}_{\mathbb{IP}} = \{[0.34,0.50],[0.27,0.38],(0.33,0.41)\}. \tag{44}$$

Then,

$$\begin{aligned}
\overline{\mathbb{C}^{7}_{\mathbb{IP}}} &= \left(\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}\right)^{c} \underset{p}{\cap} \left(\mathbb{C}^{1}_{\mathbb{IP}}\right)^{c} \underset{p}{\cap} \left(\mathbb{C}^{3}_{\mathbb{IP}}\right)^{c} \underset{p}{\cap} \left(\mathbb{C}^{5}_{\mathbb{IP}}\right)^{c}\\
&= \left(\mathbb{C}^{1}_{\mathbb{IP}}\right)^{c}.
\end{aligned} \tag{45}$$

Theorem 5. *Let* $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$ *be* $\mathbb{CIFT}\ p$ *and* $\mathbb{C}_{\mathbb{IP}} \in ci(\mathbb{k})$. *Then* $\overline{\mathbb{C}_{\mathbb{IP}}}$ *is closed CIFS iff* $\overline{\mathbb{C}_{\mathbb{IP}}} = \mathbb{C}_{\mathbb{IP}}$.

Proof. If $\mathbb{C}_{\mathbb{IP}}$ is closed CIFS, then we can say that the smallest closed CIFS superset of $\mathbb{C}_{\mathbb{IP}}$ is $\mathbb{C}_{\mathbb{IP}}$ itself. Thus,

$$\overline{\mathbb{C}_{\mathbb{IP}}} = \mathbb{C}_{\mathbb{IP}}. \tag{46}$$

Conversely, if $\overline{\mathbb{C}_{\mathbb{IP}}} = \mathbb{C}_{\mathbb{IP}}$, then $\overline{\mathbb{C}_{\mathbb{IP}}}$ is closed CIFS. This implies $\mathbb{C}_{\mathbb{IP}}$ is closed CIFS. □ □

Definition 27. Let $\mathbb{C}_{\mathbb{IP}}$ be a P-cubic intuitionistic fuzzy subset of $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$, then its boundary or frontier is defined as

$$Fr\left(\mathbb{C}_{\mathbb{IP}}\right) = \overline{\mathbb{C}_{\mathbb{IP}}} \underset{p}{\cap} \overline{\left(\mathbb{C}_{\mathbb{IP}}\right)^{c}}. \tag{47}$$

Definition 28. Let $\mathbb{C}_{\mathbb{IP}}$ be a P-cubic intuitionistic fuzzy subset of $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$, then the exterior is defined as

$$\mathrm{Ext}\left(\mathbb{C}_{\mathbb{IP}}\right) = \left(\overline{\mathbb{C}_{\mathbb{IP}}}\right)^{c} = \left(\mathbb{C}^{c}_{\mathbb{IP}}\right)^{0}. \tag{48}$$

Example 6. Consider a P-cubic intuitionistic topological space as constructed in Example 1 and $\mathbb{C}^{6}_{\mathbb{IP}}$ and $\mathbb{C}^{7}_{\mathbb{IP}}$ from Examples 4 and 5. Then,

$$\begin{aligned}
\left(\mathbb{C}^{6}_{\mathbb{IP}}\right)^{0} &= \mathbb{C}^{1}_{\mathbb{IP}},\\
\overline{\mathbb{C}^{6}_{\mathbb{IP}}} &= \left(\mathbb{C}^{3}_{\mathbb{IP}}\right)^{c},\\
Fr\left(\mathbb{C}^{6}_{\mathbb{IP}}\right) &= \left(\mathbb{C}^{1}_{\mathbb{IP}}\right)^{c},\\
Ext\left(\mathbb{C}^{6}_{\mathbb{IP}}\right) &= \mathbb{C}^{3}_{\mathbb{IP}},\\
\left(\mathbb{C}^{7}_{\mathbb{IP}}\right)^{0} &= \mathbb{C}^{1}_{\mathbb{IP}},\\
\overline{\mathbb{C}^{7}_{\mathbb{IP}}} &= \left(\mathbb{C}^{1}_{\mathbb{IP}}\right)^{c},\\
Fr\left(\mathbb{C}^{7}_{\mathbb{IP}}\right) &= \left(\mathbb{C}^{1}_{\mathbb{IP}}\right)^{c},\\
Ext\left(\mathbb{C}^{7}_{\mathbb{IP}}\right) &= \mathbb{C}^{1}_{\mathbb{IP}}.
\end{aligned} \tag{49}$$

Theorem 6. *Let* $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$ *be* $\mathbb{CIFT}p$ *and* $\mathbb{C}_{\mathbb{IP}} \in ci(\mathbb{k})$. *Then,*

(1) $(\mathbb{C}^{0}_{\mathbb{IP}})^{c} = \overline{(\mathbb{C}^{c}_{\mathbb{IP}})}$

(2) $(\overline{\mathbb{C}_{\mathbb{IP}}})^{c} = (\mathbb{C}^{c}_{\mathbb{IP}})^{0}$

(3) $Ext\,(\mathbb{C}^{c}_{\mathbb{IP}}) = \mathbb{C}^{0}_{\mathbb{IP}}$

(4) $Ext\,(\mathbb{C}_{\mathbb{IP}}) = (\mathbb{C}^{c}_{\mathbb{IP}})^{0}$

(5) $Ext\,(\mathbb{C}_{\mathbb{IP}}) \cup_{p} Fr\,(\mathbb{C}_{\mathbb{IP}}) \cup_{p} \mathbb{C}^{0}_{\mathbb{IP}} \neq {}^{1}\mathbb{C}_{\mathbb{IP}}$

(6) $Fr\,(\mathbb{C}_{\mathbb{IP}}) = Fr\,(\mathbb{C}^{c}_{\mathbb{IP}})$

(7) $Fr\,(\mathbb{C}_{\mathbb{IP}}) \cap_{p} \mathbb{C}^{0}_{\mathbb{IP}} \neq {}^{0}\mathbb{C}_{\mathbb{IP}}$

Proof

(1) The proof is obvious.

(2) The proof is obvious.

(3) $\mathrm{Ext}\,(\mathbb{C}^{c}_{\mathbb{IP}}) = (\overline{\mathbb{C}^{c}_{\mathbb{IP}}})^{c}$.
$\mathrm{Ext}\,(\mathbb{C}^{c}_{\mathbb{IP}}) = ((\mathbb{C}^{c}_{\mathbb{IP}})^{c})^{0}$.
$\mathrm{Ext}\,(\mathbb{C}^{c}_{\mathbb{IP}}) = \mathbb{C}^{0}_{\mathbb{IP}}$.

(4) $\mathrm{Ext}\,(\mathbb{C}_{\mathbb{IP}}) = (\overline{\mathbb{C}_{\mathbb{IP}}})^{c}$.
$\mathrm{Ext}\,(\mathbb{C}_{\mathbb{IP}}) = (\mathbb{C}^{c}_{\mathbb{IP}})^{0}$.

(5) $\mathrm{Ext}\,(\mathbb{C}_{\mathbb{IP}}) \cup_{p} Fr\,(\mathbb{C}_{\mathbb{IP}}) \cup_{p} \mathbb{C}^{0}_{\mathbb{IP}} \neq {}^{1}\mathbb{C}_{\mathbb{IP}}$. By Example 13, we can see that $\mathrm{Ext}\,(\mathbb{C}^{6}_{\mathbb{IP}}) \cup_{p} Fr\,(\mathbb{C}^{6}_{\mathbb{IP}}) \cup_{p} \mathbb{C}^{0}_{\mathbb{IP}} \neq {}^{1}\mathbb{C}_{\mathbb{I}}$.

(6) $Fr\,(\mathbb{C}^{c}_{\mathbb{IP}}) = \overline{(\mathbb{C}^{c}_{\mathbb{IP}})} \cap_{p} \overline{((\mathbb{C}^{c}_{\mathbb{IP}})^{c})} Fr\,(\mathbb{C}^{c}_{\mathbb{IP}}) = \overline{(\mathbb{C}^{c}_{\mathbb{IP}})} \cap_{p} \overline{(\mathbb{C}_{\mathbb{IP}})} = Fr\,(\mathbb{C}_{\mathbb{IP}})$.

(7) $Fr\,(\mathbb{C}_{\mathbb{IP}}) \cap_{p} \mathbb{C}^{0}_{\mathbb{IP}} \neq \mathbb{C}_{\mathbb{IP}}$. From Example 13, we can see that $Fr\,(\mathbb{C}^{6}_{\mathbb{IP}}) \cap_{p} \mathbb{C}^{0}_{\mathbb{IP}} \neq {}^{0}\mathbb{C}_{\mathbb{I}}$. □

3.3. P-Cubic Intuitionistic Fuzzy Basis

Definition 29. Let $(\mathbb{k}, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$ be $\mathbb{CIFT}p$. Then, $\mathbb{B} \subseteq \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$ is called P-cubic intuitionistic fuzzy basis for $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$ if for every $\mathbb{C}_{\mathbb{IP}} \in \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$, $\exists \mathscr{B} \in \mathbb{B}$ such that

$$\mathbb{C}_{\mathbb{IP}} = \underset{p}{\cup} \mathscr{B}. \tag{50}$$

Example 7. From Example 1,

$$\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}} = \left\{ {}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}, \mathbb{C}^{1}_{\mathbb{IP}}, \mathbb{C}^{2}_{\mathbb{IP}}, \mathbb{C}^{3}_{\mathbb{IP}}, \mathbb{C}^{4}_{\mathbb{IP}}, \mathbb{C}^{5}_{\mathbb{IP}} \right\}, \quad (51)$$

is a P-cubic intuitionistic fuzzy topology of $\Bbbk$. Then,

$$\mathbb{B} = \left\{ {}^{0}\mathbb{C}_{\mathbb{I}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}, \mathbb{C}^{1}_{\mathbb{IP}}, \mathbb{C}^{2}_{\mathbb{IP}}, \mathbb{C}^{3}_{\mathbb{IP}}, \mathbb{C}^{4}_{\mathbb{IP}}, \mathbb{C}^{5}_{\mathbb{IP}} \right\}, \quad (52)$$

Is a P-cubic intuitionistic fuzzy basis for $\mathbb{T}_{\mathbb{C}_{\mathbb{IP}}}$.

4. Cubic Intuitionistic Topology under R-Order

In this section, we introduce the concept of an R-cubic intuitionistic fuzzy topology (R-CIFT) or a cubic intuitionistic fuzzy topology with an R-order.

Definition 30. Consider $\Bbbk$ to be a nonempty universal set, and let $ci(\Bbbk)$ to be the collection of all CIFSs in $\Bbbk$. If the collection $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}}$ containing CIFSs satisfies the following conditions, it is termed a cubic intuitionistic fuzzy topology with an R-order (R-CIFT).

(1) ${}^{0}\mathbb{C}_{\mathbb{I}}$, ${}^{1}\mathbb{C}_{\mathbb{I}}$, $\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$ and $\overline{{}^{0}\mathbb{C}_{\mathbb{I}}} \in \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}}$

(2) If $(\mathbb{C}_{\mathbb{IR}})_i \in \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}} \forall i \in \Lambda$ then $\cup_R (\mathbb{C}_{\mathbb{IR}})_i \in \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}}$

(3) If $\mathbb{C}^{1}_{\mathbb{IR}}, \mathbb{C}^{2}_{\mathbb{IR}} \in \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}}$ then $\mathbb{C}^{1}_{\mathbb{IR}} \cap_R \mathbb{C}^{2}_{\mathbb{IR}} \in \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}}$

Then, the pair $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$ is termed as a cubic intuitionistic fuzzy topological space with an R-order (R-CIFT).

Example 8. Let $\Bbbk$ be a nonempty universal set. Then, $ci(\Bbbk)$ be the accumulation of all R-cubic intuitionistic fuzzy sets RCIFSs in $\Bbbk$. Consider the R-order fuzzy subsets of $ci(\Bbbk)$ given as

$$\begin{aligned}
\mathbb{C}^{1}_{\mathbb{IR}} &= \{[0.31, 0.42], [0.47, 0.56], (0.29, 0.39)\},\\
\mathbb{C}^{2}_{\mathbb{IR}} &= \{[0.31, 0.42], [0.47, 0.56], (0, 1)\},\\
\mathbb{C}^{3}_{\mathbb{IR}} &= \{[1, 1], [0, 0], (0.29, 0.39)\},\\
\mathbb{C}^{4}_{\mathbb{IR}} &= \{[0, 0], [1, 1], (0.29, 0.39)\},\\
\mathbb{C}^{5}_{\mathbb{IR}} &= \{[0.31, 0.42], [0.47, 0.56], (1, 0)\}.
\end{aligned} \quad (53)$$

The union and intersection with a P-order for the above CIFSs are given in Tables 3 and 4,, respectively.

Clearly,

$$\mathbb{T}_{\mathbb{C}^{1}_{\mathbb{IR}}} = \left\{ {}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}} \right\}, \quad (54)$$

and

$$\mathbb{T}^{2}_{\mathbb{C}_{\mathbb{IR}}} = \left\{ {}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}, \mathbb{C}^{1}_{\mathbb{IR}}, \mathbb{C}^{2}_{\mathbb{IR}}, \mathbb{C}^{3}_{\mathbb{IR}}, \mathbb{C}^{4}_{\mathbb{IR}}, \mathbb{C}^{5}_{\mathbb{IR}} \right\}, \quad (55)$$

are the cubic intuitionistic topology with an R-order.

Definition 31. Let $\Bbbk$ be a nonempty set, and $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}} = \left\{ \mathbb{C}^{\Bbbk}_{\mathbb{IR}} \right\}$ where $\mathbb{C}^{\Bbbk}_{\mathbb{IR}}$ represent the cubic intuitionistic fuzzy subsets of the universal set $\Bbbk$. Then, $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}}$ is termed as an R-cubic intuitionistic fuzzy topology on $\Bbbk$ and it is the largest R-cubic intuitionistic fuzzy topology on $\Bbbk$ and is entitled as an R-discrete cubic intuitionistic fuzzy topology.

Definition 32. Let $\Bbbk$ be a universal set and $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}} = \left\{ {}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}} \right\}$ be the assemblage of cubic intuitionistic fuzzy sets. Then, $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}}$ is termed as an R-cubic intuitionistic fuzzy topology on the universal set $\Bbbk$ and is the smallest R-cubic intuitionistic fuzzy topology on $\Bbbk$ and is entitled as an R-indiscrete cubic intuitionistic fuzzy topology.

Definition 33. The elements of an R-cubic intuitionistic fuzzy topology $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}}$ is termed as the R-cubic intuitionistic fuzzy open sets RCIFOS in $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$.

Theorem 7. *If $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$ is any R-cubic intuitionistic fuzzy topological space. Then,*

(1) ${}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$ and $\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$ are RCIFOSs.

(2) The R-union of any number of RCIFOSs is RCIFOS.

(3) The R-intersection of finite RCIFOSs is RCIFOS.

Proof. The proof is trivial. □

Definition 34. The complement of elements of an R-cubic intuitionistic fuzzy open sets is termed as the R-cubic intuitionistic fuzzy closed sets RCIFCSs in $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$.

Theorem 8. *If $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$ is any R-cubic intuitionistic fuzzy topological space. Then,*

(1) ${}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$ and $\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$ are RCIFCSs.

(2) The R-intersection of any number of RCIFCSs is RCIFCS.

(3) The R-union of finite RCIFCSs is RCIFCS.

Proof. The proof is trivial. □

Definition 35. The R-cubic intuitionistic fuzzy sets RCIFSs, which are RCIFOSs and RCIFCSs, are entitled as the R-cubic intuitionistic fuzzy clopen sets in $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$.

Proposition 2

(1) For every $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}}$, ${}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$ and $\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$ are R-cubic intuitionistic fuzzy clopen sets.

(2) For the discrete R-order cubic intuitionistic fuzzy topology, all the cubic intuitionistic subsets of $\Bbbk$ are R-cubic intuitionistic fuzzy clopen sets.

(3) For the in-discrete R-order cubic intuitionistic fuzzy topology, ${}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$ and $\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$ are only R-cubic intuitionistic fuzzy clopen sets.

TABLE 3: Union under R-order.

$\cup_R$	${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{5}$
${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{5}$
${}^{1}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$
$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$
$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$
$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{1}$
$\mathbb{C}_{\mathbb{IR}}^{2}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	$\mathbb{C}_{\mathbb{IR}}^{2}$
$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{3}$
$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{1}$
$\mathbb{C}_{\mathbb{IR}}^{5}$	$\mathbb{C}_{\mathbb{IR}}^{5}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{5}$

TABLE 4: Intersection under R-order.

$\cap_R$	${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{5}$
${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$
${}^{1}\mathbb{C}_{\mathbb{I}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{1}\mathbb{C}_{\mathbb{I}}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{5}$
$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	${}^{0}\mathbb{C}_{\mathbb{I}}$
$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}_{\mathbb{IR}}^{5}$	$\mathbb{C}_{\mathbb{IR}}^{5}$	$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{5}$
$\mathbb{C}_{\mathbb{IR}}^{1}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{5}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{5}$	$\mathbb{C}_{\mathbb{IR}}^{5}$
$\mathbb{C}_{\mathbb{IR}}^{2}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	$\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}_{\mathbb{IR}}^{5}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{2}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{5}$
$\mathbb{C}_{\mathbb{IR}}^{3}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{1}$	$\mathbb{C}_{\mathbb{IR}}^{3}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{5}$
$\mathbb{C}_{\mathbb{IR}}^{4}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	$\mathbb{C}_{\mathbb{IR}}^{4}$	${}^{0}\mathbb{C}_{\mathbb{I}}$
$\mathbb{C}_{\mathbb{IR}}^{5}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{5}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	$\mathbb{C}_{\mathbb{IR}}^{5}$	$\mathbb{C}_{\mathbb{IR}}^{5}$	$\mathbb{C}_{\mathbb{IR}}^{5}$	$\mathbb{C}_{\mathbb{IR}}^{5}$	${}^{0}\mathbb{C}_{\mathbb{I}}$	${}^{0}\mathbb{C}_{\mathbb{I}}$

Definition 36. Let $(\Bbbk, \mathbb{T}^{1}_{\mathbb{C}_{\mathbb{IR}}})$ and $(\Bbbk, \mathbb{T}^{2}_{\mathbb{C}_{\mathbb{IR}}})$ be two R-CIFTs in $\Bbbk$. Two R-CIFTs are called comparable if

$$\mathbb{T}^{1}_{\mathbb{C}_{\mathbb{IR}}} \subseteq_R \mathbb{T}^{2}_{\mathbb{C}_{\mathbb{IR}}}, \tag{56}$$

or

$$\mathbb{T}^{2}_{\mathbb{C}_{\mathbb{IR}}} \subseteq_R \mathbb{T}^{1}_{\mathbb{C}_{\mathbb{IR}}}. \tag{57}$$

If $\mathbb{T}^{2}_{\mathbb{C}_{\mathbb{IR}}} \subseteq_R \mathbb{T}^{2}_{\mathbb{C}_{\mathbb{IR}}}$ then, $\mathbb{T}^{1}_{\mathbb{C}_{\mathbb{IR}}}$ is called R-cubic intuitionistic fuzzy coarser than $\mathbb{T}^{2}_{\mathbb{C}_{\mathbb{IR}}}$ and $\mathbb{T}^{2}_{\mathbb{C}_{\mathbb{IR}}}$ is called the R-cubic intuitionistic fuzzy finer than. $\mathbb{T}^{1}_{\mathbb{C}_{\mathbb{IR}}}$

Example 9. Let $\Bbbk$ be a nonempty set and from Example 8,

$$\mathbb{T}_{\mathbb{C}^{1}_{\mathbb{IR}}} = \left\{{}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}\right\}, \tag{58}$$

and

$$\mathbb{T}^{2}_{\mathbb{C}_{\mathbb{IR}}} = \left\{{}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}, \mathbb{C}^{1}_{\mathbb{IR}}, \mathbb{C}^{2}_{\mathbb{IR}}, \mathbb{C}^{3}_{\mathbb{IR}}, \mathbb{C}^{4}_{\mathbb{IR}}, \mathbb{C}^{5}_{\mathbb{IR}}\right\}, \tag{59}$$

are R-cubic intuitionistic fuzzy topologies on the universal set. Then, $\mathbb{T}_{\mathbb{C}^{1}_{\mathbb{IR}}} \subseteq_R \mathbb{T}^{2}_{\mathbb{C}_{\mathbb{IR}}}$. Hence, $\mathbb{T}_{\mathbb{C}^{1}_{\mathbb{IR}}}$ is called the R-cubic intuitionistic fuzzy coarser then, $\mathbb{T}^{2}_{\mathbb{C}_{\mathbb{IR}}}$.

4.1. Subspace of $\mathbb{CIFT}r$

Definition 37. Let $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}\Bbbk})$ be a $\mathbb{CIFT}r$. Let $\mathbb{Y} \subseteq \Bbbk$ and $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}\mathbb{Y}}$ is a $\mathbb{CIFT}r$ on $\mathbb{Y}$ and whose RCIFOSs are

$$\mathbb{C}_{\mathbb{IR}\mathbb{Y}} = \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}\Bbbk} \underset{R}{\cap} \mathbb{Y}, \tag{60}$$

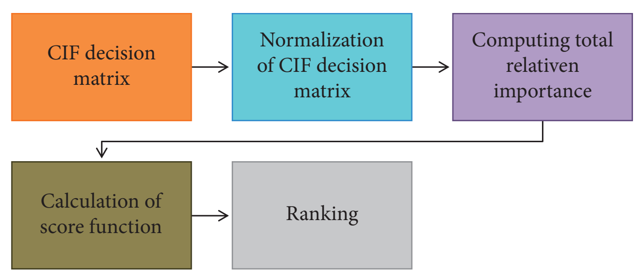


FIGURE 1: Flow chart of CIF WPM.

where $\mathbb{C}_{\mathbb{IR}_k}$ are RCIFOSs of $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}\Bbbk}$, $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}\mathbb{Y}}$ are RCIFOSs of $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}\mathbb{Y}}$ and $\widetilde{\mathbb{Y}}$ is any R-cubic subset of RCIFS on $\mathbb{Y}$. Then, $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}\mathbb{Y}}$ is called the R-cubic intuitionistic fuzzy subspace of $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}\Bbbk}$ i.e.,

$$\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}\mathbb{Y}} = \left\{\mathbb{C}_{\mathbb{IR}\mathbb{Y}}\colon \mathbb{C}_{\mathbb{IR}\mathbb{Y}} = \mathbb{C}_{\mathbb{IR}\Bbbk} \underset{R}{\cap} \mathbb{Y}, \mathbb{C}_{\mathbb{IR}\Bbbk} \in \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}\Bbbk}\right\}. \tag{61}$$

Example 10. Let $\Bbbk$ be a nonempty set. From Example 8,

$$\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}} = \left\{{}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}, \mathbb{C}^{1}_{\mathbb{IR}}, \mathbb{C}^{2}_{\mathbb{IR}}, \mathbb{C}^{3}_{\mathbb{IR}}, \mathbb{C}^{4}_{\mathbb{IR}}, \mathbb{C}^{5}_{\mathbb{IR}}\right\}, \tag{62}$$

is an R-cubic intuitionistic fuzzy topology on $\Bbbk$.

Now, consider any R-cubic fuzzy subset on $\Bbbk$ such that $\mathbb{Y} \subseteq \Bbbk$ is

$$\mathbb{Y} = \{[0.27, 0.38], [0.52, 0.67], (0.34, 0.28)\}. \tag{63}$$

Also,

Table 5: Cubic intuitionistic decision matrix from DMs.

Criteria	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$
$\mathbb{X}_1$	([0.17, 0.24], [0.36, 0.43], (0.56, 0.32))	([0.20, 0.28], [0.29, 0.31], (0.27, 0.20))	([0.18, 0.21], [0.21, 0.32], (0.39, 0.22))	([0.20, 0.37], [0.21, 0.43], (0.54, 0.23))
$\mathbb{X}_2$	([0.19, 0.22], [0.39, 0.42], (0.59, 0.40))	([0.27, 0.34], [0.33, 0.40], (0.43, 0.21))	([0.24, 0.30], [0.30, 0.39], (0.50, 0.20))	([0.32, 0.40], [0.19, 0.51], (0.52, 0.30))
$\mathbb{X}_3$	([0.20, 0.29], [0.40, 0.51], (0.81, 0.13))	([0.31, 0.52], [0.42, 0.50], (0.72, 0.17))	([0.31, 0.39], [0.18, 0.40], (0.67, 0.14))	([0.14, 0.63], [0.24, 0.50], (0.70, 0.18))
$\mathbb{X}_4$	([0.31, 0.37], [0.36, 0.49], (0.50, 0.36))	([0.18, 0.33], [0.28, 0.52], (0.40, 0.32))	([0.23, 0.40], [0.24, 0.51], (0.50, 0.33))	([0.08, 0.74], [0.32, 0.40], (0.46, 0.42))
$\mathbb{X}_5$	([0.40, 0.48], [0.51, 0.60], (0.52, 0.30))	([0.29, 0.41], [0.30, 0.39], (0.48, 0.40))	([0.40, 0.47], [0.38, 0.60], (0.56, 0.29))	([0.13, 0.64], [0.40, 0.47], (0.53, 0.37))
$\mathbb{X}_6$	([0.29, 0.38], [0.27, 0.42], (0.60, 0.27))	([0.40, 0.51], [0.41, 0.50], (0.47, 0.38))	([0.32, 0.38], [0.24, 0.72], (0.43, 0.32))	([0.42, 0.50], [0.37, 0.53], (0.53, 0.27))

Table 6: $\max_j \mathbb{T}_{ji}$ and $\min_j \mathbb{T}_{ji}$ values.

	$\max_j \mathbb{T}_{ji}$	$\min_j \mathbb{T}_{ji}$
$\mathbb{X}_1$	([0.20, 0.37], [0.21, 0.31], (0.50, 0.20))	([0.17, 0.21], [0.36, 0.43], (0.27, 0.32))
$\mathbb{X}_2$	([0.32, 0.40], [0.19, 0.39], (0.59, 0.20))	([0.19, 0.22], [0.39, 0.51], (0.43, 0.40))
$\mathbb{X}_3$	([0.31, 0.63], [0.18, 0.40], (0.81, 0.13))	([0.14, 0.29], [0.42, 0.50], (0.67, 0.18))
$\mathbb{X}_4$	([0.31, 0.74], [0.24, 0.40], (0.50, 0.32))	([0.08, 0.33], [0.36, 0.52], (0.40, 0.42))
$\mathbb{X}_5$	([0.40, 0.64], [0.30, 0.39], (0.56, 0.29))	([0.13, 0.41], [0.51, 0.60], (0.48, 0.40))
$\mathbb{X}_6$	([0.42, 0.51], [0.24, 0.42], (0.60, 0.27))	([0.29, 0.38], [0.41, 0.72], (0.43, 0.38))

$$\begin{aligned}
\mathbb{Y} \overset{0}{\underset{R}{\cap}} \mathbb{C}_{\mathbb{I}} &= \{[0,0], [1,1], (1,0)\} \\
&= \overrightarrow{\mathbb{C}_{\mathbb{IR}}}, \\
\mathbb{Y} \overset{1}{\underset{R}{\cap}} \mathbb{C}_{\mathbb{I}} &= \{[0.27,0.38], [0.52,0.67], (1,0)\} \\
&= \widetilde{\mathbb{C}_{\mathbb{IR}}}, \\
\mathbb{Y} \cap_R \overline{{}^{0}\mathbb{C}_{\mathbb{I}}} &= \{[0,0], [1,1], (0.34,0.28)\} \\
&= \mathbb{C}'_{\mathbb{IR}}, \\
\mathbb{Y} \cap_R \overline{{}^{1}\mathbb{C}_{\mathbb{I}}} &= \{[0.27,0.38], [0.52,0.67], (1,0)\} \\
&= \widetilde{\mathbb{C}_{\mathbb{IR}}}, \\
\mathbb{Y} \underset{R}{\cap} \mathbb{C}^1_{\mathbb{IR}} &= \{[0.27,0.38], [0.52,0.67], (0.34,0.28)\} \\
&= \mathbb{Y}, \\
\mathbb{Y} \underset{R}{\cap} \mathbb{C}^2_{\mathbb{IR}} &= \{[0.27,0.38], [0.52,0.67], (0.34,0.28)\} \\
&= \mathbb{Y}, \\
\mathbb{Y} \underset{R}{\cap} \mathbb{C}^3_{\mathbb{IR}} &= \{[0.27,0.38], [0.52,0.67], (0.34,0.28)\} \\
&= \mathbb{Y}, \\
\mathbb{Y} \underset{R}{\cap} \mathbb{C}^4_{\mathbb{IR}} &= \{[0,0], [1,1], (0.34,0.28)\} \\
&= \mathbb{C}'_{\mathbb{IR}}, \\
\mathbb{Y} \underset{R}{\cap} \mathbb{C}^5_{\mathbb{IR}} &= \{[0.27,0.38], [0.52,0.67], (1,0)\} \\
&= \widetilde{\mathbb{C}_{\mathbb{IR}}}.
\end{aligned} \tag{64}$$

Then,

$$\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}\mathbb{Y}} = \left\{\overrightarrow{\mathbb{C}_{\mathbb{IR}}}, \widetilde{\mathbb{C}_{\mathbb{IR}}}, {}'\mathbb{C}_{\mathbb{IR}}, \mathbb{Y}\right\}, \tag{65}$$

is an R-cubic intuitionistic fuzzy relative topology of $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}\Bbbk}$

4.2. Interior, Closure, Frontier, and Exterior of RCIFSs

Definition 38. let $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$ be $\mathbb{CIFT}r$ and $\mathbb{C}_{\mathbb{IR}} \in ci(\Bbbk)$, the interior of $\mathbb{C}_{\mathbb{IR}}$ is expressed as $\mathbb{C}^0_{\mathbb{IR}}$ and is described as a union of all the R-cubic intuitionistic fuzzy open subsets contained in $\mathbb{C}_{\mathbb{IR}}$. It is the greatest R-cubic intuitionistic fuzzy open set contained in $\mathbb{C}_{\mathbb{IR}}$.

Example 11. Consider an R-cubic intuitionistic fuzzy topological space as constructed in Example 8. Let $\mathbb{C}^6_{\mathbb{IR}} \in ci(\Bbbk)$ given as

$$\mathbb{C}^6_{\mathbb{IR}} = \{[0.38,0.46], [0.45,0.51], (0.26,0.40)\}. \tag{66}$$

Then,

$$\begin{aligned}
\left(\mathbb{C}^6_{\mathbb{IR}}\right)^0 &= {}^0\mathbb{C}_{\mathbb{I}} \underset{R}{\cup} \mathbb{C}^1_{\mathbb{IR}} \underset{R}{\cup} \mathbb{C}^4_{\mathbb{IP}} \underset{R}{\cup} \mathbb{C}^5_{\mathbb{IR}} \\
&= \mathbb{C}^1_{\mathbb{IR}}.
\end{aligned} \tag{67}$$

Theorem 9. *Let $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$ be $\mathbb{CIFT}$ r and $\mathbb{C}_{\mathbb{IR}} \in ci(\Bbbk)$. Then, $\mathbb{C}_{\mathbb{IR}}$ is open CIFS iff $\mathbb{C}^0_{\mathbb{IR}} = \mathbb{C}_{\mathbb{IR}}$.*

Proof. The proof is trivial. □

Theorem 10. *Let $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IP}}})$ be $\mathbb{CIFT}p$ and $\mathbb{C}^1_{\mathbb{IP}}, \mathbb{C}^2_{\mathbb{IP}} \in ci(\Bbbk)$. Then,*

(i) $((\mathbb{C}_{\mathbb{IR}})^0)^0 = (\mathbb{C}_{\mathbb{IR}})^0$
(ii) $\mathbb{C}^1_{\mathbb{IR}} \subseteq_R \mathbb{C}^2_{\mathbb{IR}} \Rightarrow (\mathbb{C}^1_{\mathbb{IR}})^0 \subseteq_R (\mathbb{C}^2_{\mathbb{IR}})^0$
(iii) $(\mathbb{C}^1_{\mathbb{IR}} \cap_R \mathbb{C}^2_{\mathbb{IR}})^0 = (\mathbb{C}^1_{\mathbb{IR}})^0 \subseteq_R (\mathbb{C}^2_{\mathbb{IR}})^0$
(iv) $(\mathbb{C}^1_{\mathbb{IR}} \cup_R \mathbb{C}^2_{\mathbb{IR}})^0 \supseteq_R (\mathbb{C}^1_{\mathbb{IR}})^0 \cup_R (\mathbb{C}^2_{\mathbb{IR}})^0$

Proof. Proof is trivial. □

Definition 39. let $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$ be $\underline{\mathbb{CIFT}}r$ and $\mathbb{C}_{\mathbb{IR}} \in ci(\Bbbk)$, the closure of $\mathbb{C}_{\mathbb{IR}}$ is expressed as $\overline{\mathbb{C}_{\mathbb{IR}}}$ and is described as the intersection of all the R-cubic intuitionistic fuzzy closed supersets of $\mathbb{C}_{\mathbb{IR}}$. It is the smallest R-cubic intuitionistic fuzzy closed superset of $\mathbb{C}_{\mathbb{IR}}$.

Example 12. Let us consider an R-cubic intuitionistic topological space as constructed in Example 8. Then, the closed CIFSs are given as

Table 7: Normalized decision matrix from DMs.

Criteria	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$
$\mathbb{X}_1$	$([0.17, 0.21], [0.36, 0.43], (0.56, 0.32))$	$([0.17, 0.21], [0.36, 0.43], (0.27, 0.20))$	$([0.17, 0.21], [0.36, 0.43], (0.39, 0.22))$	$([0.17, 0.21], [0.36, 0.43], (0.54, 0.23))$
$\mathbb{X}_2$	$([0.19, 0.22], [0.39, 0.51], (0.59, 0.40))$	$([0.19, 0.22], [0.39, 0.51], (0.43, 0.21))$	$([0.19, 0.22], [0.39, 0.51], (0.50, 0.20))$	$([0.19, 0.22], [0.39, 0.51], (0.52, 0.30))$
$\mathbb{X}_3$	$([0.20, 0.29], [0.40, 0.51], (0.81, 0.13))$	$([0.31, 0.52], [0.42, 0.50], (0.81, 0.13))$	$([0.31, 0.39], [0.18, 0.40], (0.81, 0.13))$	$([0.14, 0.63], [0.24, 0.50], (0.81, 0.13))$
$\mathbb{X}_4$	$([0.08, 0.33], [0.36, 0.52], (0.50, 0.36))$	$([0.08, 0.33], [0.36, 0.52], (0.40, 0.32))$	$([0.08, 0.33], [0.36, 0.52], (0.50, 0.33))$	$([0.08, 0.33], [0.36, 0.52], (0.46, 0.42))$
$\mathbb{X}_5$	$([0.40, 0.48], [0.51, 0.60], (0.56, 0.29))$	$([0.29, 0.41], [0.30, 0.39], (0.56, 0.29))$	$([0.40, 0.47], [0.38, 0.60], (0.56, 0.29))$	$([0.13, 0.64], [0.40, 0.47], (0.56, 0.29))$
$\mathbb{X}_6$	$([0.29, 0.38], [0.41, 0.72], (0.60, 0.27))$	$([0.29, 0.38], [0.41, 0.72], (0.47, 0.38))$	$([0.29, 0.38], [0.41, 0.72], (0.43, 0.32))$	$([0.29, 0.38], [0.41, 0.72], (0.53, 0.27))$

Table 8: Relative importance and score function.

Alternatives	CIFN-WPM values	Score values
$\mathbb{X}_1$	([0.16, 0.20], [0.34, 0.41], (0.9980, 0.00001))	0.8029
$\mathbb{X}_2$	([0.18, 0.21], [0.38, 0.49], (0.9993, 0.00002))	0.7592
$\mathbb{X}_3$	([0.21, 0.45], [0.29, 0.46], (0.9999, 0.0000007))	0.9548
$\mathbb{X}_4$	([0.07, 0.31], [0.34, 0.50], (0.9990, 0.00007))	0.6294
$\mathbb{X}_5$	([0.25, 0.49], [0.39, 0.50], (0.9996, 0.00002))	0.9245
$\mathbb{X}_6$	([0.28, 0.34], [0.39, 0.71], (0.9994, 0.00004))	0.7593

$$\begin{aligned}
\left({}^{0}\mathbb{C}_{\mathbb{I}}\right)^{c} &= \{[1,1],[0,0],(0,1)\},\\
\left({}^{1}\mathbb{C}_{\mathbb{I}}\right)^{c} &= \{[0,0],[1,1],(1,0)\},\\
\left(\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}\right)^{c} &= \{[1,1],[0,0],(1,0)\},\\
\left(\overline{{}^{1}\mathbb{C}_{\mathbb{I}}}\right)^{c} &= \{[0,0],[1,1],(0,1)\},\\
\left(\mathbb{C}^{1}_{\mathbb{IP}}\right)^{c} &= \{[0.47,0.56],[0.31,0.42],(0.39,0.29)\},\\
\left(\mathbb{C}^{2}_{\mathbb{IP}}\right)^{c} &= \{[0.47,0.56],[0.31,0.42],(1,0)\},\\
\left(\mathbb{C}^{3}_{\mathbb{IP}}\right)^{c} &= \{[0,0],[1,1],(0.39,0.29)\},\\
\left(\mathbb{C}^{4}_{\mathbb{IP}}\right)^{c} &= \{[1,1],[0,0],(0.39,0.29)\},\\
\left(\mathbb{C}^{5}_{\mathbb{IP}}\right)^{c} &= \{[0.47,0.56],[0.31,0.42],(0,1)\}.
\end{aligned} \tag{68}$$

Let $\mathbb{C}^{7}_{\mathbb{IR}} \in ci(\Bbbk)$ given as

$$\mathbb{C}^{7}_{\mathbb{IR}} = \{[0.30,0.37],[0.48,0.61],(0.38,0.24)\}. \tag{69}$$

Then,

$$\begin{aligned}
\overline{\mathbb{C}^{7}_{\mathbb{IR}}} &= \left(\overline{{}^{0}\mathbb{C}_{\mathbb{I}}}\right)^{c} \underset{R}{\cap} \left(\mathbb{C}^{5}_{\mathbb{IR}}\right)^{c}\\
&= \left(\mathbb{C}^{5}_{\mathbb{IR}}\right)^{c}.
\end{aligned} \tag{70}$$

Theorem 11. *Let $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$ be $\mathbb{CIFT}$ r and $\mathbb{C}_{\mathbb{IR}} \in ci(\Bbbk)$. Then, $\mathbb{C}_{\mathbb{IR}}$ is closed CIFS iff $\overline{\mathbb{C}_{\mathbb{IR}}} = \mathbb{C}_{\mathbb{IR}}$..*

Proof. The proof is trivial. ☐

Definition 40. Let $\mathbb{C}_{\mathbb{IR}}$ be an R-cubic intuitionistic fuzzy subset of $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$, then its boundary or frontier is defined as

$$Fr\left(\mathbb{C}_{\mathbb{IR}}\right) = \overline{\mathbb{C}_{\mathbb{IR}}} \underset{R}{\cap} \overline{\left(\mathbb{C}_{\mathbb{IR}}\right)^{c}}. \tag{71}$$

Definition 41. Let $\mathbb{C}_{\mathbb{IR}}$ be an R-cubic intuitionistic fuzzy subset of $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$, then the exterior is defined as

$$\begin{aligned}
\operatorname{Ext}\left(\mathbb{C}_{\mathbb{IR}}\right) &= \left(\overline{\mathbb{C}_{\mathbb{IR}}}\right)^{c}\\
&= \left(\mathbb{C}^{c}_{\mathbb{IR}}\right)^{0}.
\end{aligned} \tag{72}$$

Example 4.13. Consider an R-cubic intuitionistic topological space as constructed in Example 8 and $\mathbb{C}^{6}_{\mathbb{IR}}$ and $\mathbb{C}^{7}_{\mathbb{IR}}$ from Examples 11 and 12. Then,

$$\begin{aligned}
\left(\mathbb{C}^{6}_{\mathbb{IR}}\right)^{0} &= \mathbb{C}^{1}_{\mathbb{IR}},\\
\overline{\mathbb{C}^{6}_{\mathbb{IP}}} &= \left(\mathbb{C}^{5}_{\mathbb{IR}}\right)^{c},\\
Fr\left(\mathbb{C}^{6}_{\mathbb{IR}}\right) &= \left(\mathbb{C}^{5}_{\mathbb{IR}}\right)^{c},\\
\operatorname{Ext}\left(\mathbb{C}^{6}_{\mathbb{IR}}\right) &= \mathbb{C}^{5}_{\mathbb{IR}},\\
\left(\mathbb{C}^{7}_{\mathbb{IR}}\right)^{0} &= \mathbb{C}^{4}_{\mathbb{IR}},\\
\overline{\mathbb{C}^{7}_{\mathbb{IR}}} &= \left(\mathbb{C}^{5}_{\mathbb{IR}}\right)^{c},\\
Fr\left(\mathbb{C}^{7}_{\mathbb{IR}}\right) &= \left(\mathbb{C}^{5}_{\mathbb{IR}}\right)^{c},\\
\operatorname{Ext}\left(\mathbb{C}^{7}_{\mathbb{IR}}\right) &= \mathbb{C}^{5}_{\mathbb{IR}}.
\end{aligned} \tag{73}$$

Theorem 12. *Let $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$ be $\mathbb{CIFT}r$ and $\mathbb{C}_{\mathbb{IR}} \in ci(\Bbbk)$. Then,*

(1) $(\mathbb{C}^{0}_{\mathbb{IR}})^{c} = \overline{(\mathbb{C}^{c}_{\mathbb{IR}})}$
(2) $(\overline{\mathbb{C}_{\mathbb{IR}}})^{c} = (\mathbb{C}^{c}_{\mathbb{IR}})^{0}$
(3) $Ext(\mathbb{C}^{c}_{\mathbb{IR}}) = \mathbb{C}^{0}_{\mathbb{IR}}$
(4) $Ext(\mathbb{C}_{\mathbb{IR}}) = (\mathbb{C}^{c}_{\mathbb{IR}})^{0}$
(5) $Ext(\mathbb{C}_{\mathbb{IR}}) \cup_{R} Fr(\mathbb{C}_{\mathbb{IR}}) \cup_{R} \mathbb{C}^{0}_{\mathbb{IR}} \neq {}^{1}\mathbb{C}_{\mathbb{IR}}$
(6) $Fr(\mathbb{C}_{\mathbb{IR}}) = Fr(\mathbb{C}^{c}_{\mathbb{IR}})$
(7) $Fr(\mathbb{C}_{\mathbb{IR}}) \cap_{R} \mathbb{C}^{0}_{\mathbb{IR}} \neq {}^{0}\mathbb{C}_{\mathbb{IR}}$

Proof. The proof is trivial. ☐

4.3. R-Cubic Intuitionistic Fuzzy Basis

Definition 42. Let $(\Bbbk, \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}})$ be $\mathbb{CIFT}r$. Then $\mathbb{B} \subseteq \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}}$ is called an R-cubic intuitionistic fuzzy basis for $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}}$ if for every $\mathbb{C}_{\mathbb{IR}} \in \mathbb{T}_{\mathbb{C}_{\mathbb{IR}}}$, $\exists \mathscr{B} \in \mathbb{B}$ such that

$$\mathbb{C}_{\mathbb{IR}} = \underset{R}{\cup} \mathscr{B}. \tag{74}$$

Example 14. From Example 8,

$$\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}} = \left\{{}^{0}\mathbb{C}_{\mathbb{I}}, {}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{0}\mathbb{C}_{\mathbb{I}}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}, \mathbb{C}^{1}_{\mathbb{IR}}, \mathbb{C}^{2}_{\mathbb{IR}}, \mathbb{C}^{3}_{\mathbb{IR}}, \mathbb{C}^{4}_{\mathbb{IR}}, \mathbb{C}^{5}_{\mathbb{IR}}\right\}, \tag{75}$$

is an R-cubic intuitionistic fuzzy topology of $\Bbbk$. Then,

$$\mathbb{B} = \left\{{}^{1}\mathbb{C}_{\mathbb{I}}, \overline{{}^{1}\mathbb{C}_{\mathbb{I}}}, \mathbb{C}^{1}_{\mathbb{IR}}, \mathbb{C}^{2}_{\mathbb{IR}}, \mathbb{C}^{3}_{\mathbb{IR}}, \mathbb{C}^{4}_{\mathbb{IR}}, \mathbb{C}^{5}_{\mathbb{IR}}\right\}, \tag{76}$$

is an R-cubic intuitionistic fuzzy basis for $\mathbb{T}_{\mathbb{C}_{\mathbb{IR}}}$.

Table 9: Cubic intuitionistic decision matrix from DMs.

Criteria	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$
$\mathbb{X}_1$	([0.17, 0.24], [0.36, 0.43], (0.56, 0.32))	([0.20, 0.28], [0.29, 0.31], (0.27, 0.20))	([0.18, 0.21], [0.21, 0.32], (0.39, 0.22))	([0.20, 0.37], [0.21, 0.43], (0.54, 0.23))
$\mathbb{X}_2$	([0.19, 0.22], [0.39, 0.42], (0.59, 0.40))	([0.27, 0.34], [0.33, 0.40], (0.43, 0.21))	([0.24, 0.30], [0.30, 0.39], (0.50, 0.20))	([0.32, 0.40], [0.19, 0.51], (0.52, 0.30))
$\mathbb{X}_3$	([0.20, 0.29], [0.40, 0.51], (0.81, 0.13))	([0.31, 0.52], [0.42, 0.50], (0.72, 0.17))	([0.31, 0.39], [0.18, 0.40], (0.67, 0.14))	([0.14, 0.63], [0.24, 0.50], (0.70, 0.18))
$\mathbb{X}_4$	([0.31, 0.37], [0.36, 0.49], (0.50, 0.36))	([0.18, 0.33], [0.28, 0.52], (0.40, 0.32))	([0.23, 0.40], [0.24, 0.51], (0.50, 0.33))	([0.08, 0.74], [0.32, 0.40], (0.46, 0.42))
$\mathbb{X}_5$	([0.40, 0.48], [0.51, 0.60], (0.52, 0.30))	([0.29, 0.41], [0.30, 0.39], (0.48, 0.40))	([0.40, 0.47], [0.38, 0.60], (0.56, 0.29))	([0.13, 0.64], [0.40, 0.47], (0.53, 0.37))
$\mathbb{X}_6$	([0.29, 0.38], [0.27, 0.42], (0.60, 0.27))	([0.40, 0.51], [0.41, 0.50], (0.47, 0.38))	([0.32, 0.38], [0.24, 0.72], (0.43, 0.32))	([0.42, 0.50], [0.37, 0.53], (0.53, 0.27))

Table 10: Score values.

Alternatives	Score values
$\mathbb{X}_1$	0.1125
$\mathbb{X}_2$	0.1705
$\mathbb{X}_3$	0.5490
$\mathbb{X}_4$	0.0725
$\mathbb{X}_5$	0.1435
$\mathbb{X}_6$	0.204

5. Multicriteria Group Decision-Making

The weighted product model (WPM) is a renowned and widely used MCGDM approach for evaluating a set of choices using a set of criteria. Each choice is contrasted to the others by calculating a number of ratios, one per choice criterion. Every ratio is multiplied by the proportional weight of the criterion in consideration. For the selection of one or more options from the set of alternatives based on a number of criteria is a fundamental task in MCGDM problems. Let us consider m alternatives, n criteria with weighted vectors, with the condition that the sum of the weights will be one, for an MCGDM problem in a cubic intuitionistic fuzzy set domain.

Figure 1 shows the flow chart of WPM.

5.1. Application to Uncertain Supply Chain Management. Communication and information technologies are affecting every area of the industrial sector at a rapid pace. In reality, it would be difficult to pinpoint an organization that does not use or is not touched by information and communications technologies in some way. In many cases, if technology is not employed appropriately, the firm's survival is jeopardized. Companies nowadays use technology to boost productivity, streamline operations, and form electronic conglomerates. Advanced technologies and electronic systems are radically altering how businesses operate and stay competitive. Many businesses are making strategic technology investments to obtain and maintain a competitive advantage in their industry. Management teams must use technology throughout the organization to enhance information flow, reduce cost, streamline operations, provide product variety, formulate connections with suppliers, and reduce response times to customers' needs to gain a competitive advantage through the use of information and communications technology.

Administrators and top executives should be associated with the development of enterprise-wide information systems (EIS), which should take into account such matters as computer hardware and software and infrastructure facilities, online systems, digital applications, electronic commerce, and alterations to current processes and practices. Managers can integrate data and telecommunications technologies throughout the corporation and connect all business areas by developing an enterprise broad information systems plan. Enterprise-wide integration of technology enables firms to allow consumers to get timely access to the information they need to make informed decisions. Recent research has looked at information systems as useful tools for integrating systems like enterprise resource planning, knowledge management, e-commerce, electronic markets, and supply chain management (SCM) to enhance organizational profit and efficiency.

Companies must analyze both internal and external processes for the production and exchange of products and services to be more efficient and competitive. The managers will be able to evaluate the value of actions for each process to determine how to boost the value among these operations that form a supply chain from supplier to business to dealer to customer through the evaluation of these processes. The level of integration among suppliers, business associates, and buyers, independent of their geographical location, determines the value chain's or supply chain's effectiveness.

The construction of an integrated organizational system capable of information sharing, resources, and services in the supply chain is central to the digital supply chain management paradigm. To gain and maintain competitive advantages, companies use digital information and communications networks to standardize manufacturing processes, reduce cycle time, increase the effectiveness of procurement procedures and logistical support, reduce production costs, and increase customer satisfaction, among other things. Supply chain management based on the Internet allows a company to streamline its supply chain, increase speed, reduce costs, and be more adaptable. It can also increase consumer and supplier communications as well as smooth the ongoing flow of goods along the supply chain.

Supplier selection is highly essential in supply chain management. The objective is to locate a supplier who can offer the best products and services for the lowest price. Proper supplier selection delivers a high profit and quality level. In this strategic collaboration, the supplier is viewed as a significant element of the business. Because of the increasing focus on sustainability, identifying these providers has become more challenging. Environmental studies, often known as sustainability studies, have become increasingly popular around the world. Identifying these suppliers has become increasingly difficult as a result of the rapidly increasing emphasis on sustainability. Many methodologies for sustainable supply chain selection have been developed.

To determine the most suitable supplier selection, MCGDM techniques can be used successfully. In this section, the suggested model is used to determine the selection of appropriate suppliers for fast-moving consumer products, with the goal of selecting the best supplier among various possibilities. Several criteria have been established based on expert opinions to evaluate supplier choices. In this study, $\mathbb{X}1$, $\mathbb{X}2$, $\mathbb{X}3$, $\mathbb{X}4$, $\mathbb{X}5$ and $\mathbb{X}6$ are examined as possible fast-moving customers goods suppliers using the four criteria established.

5.2. CIF Weighted Product Model. The proposed method is used to choose the best supplier among six alternatives. These alternatives are weighed against four criteria $\mathbb{C}1$ = price, $\mathbb{C}2$ = quality, $\mathbb{C}3$ = performance, and $\mathbb{C}4$ = delivery, derived from thorough expert opinions. A group of decision-makers has been assembled to assess the suppliers using the recommended methodology. Six decision-makers $\mathbb{D}1$, $\mathbb{D}2$, $\mathbb{D}3$, $\mathbb{D}4$, $\mathbb{D}5$, and $\mathbb{D}6$ were chosen, consisting of supplier experts and expert academics on multicriteria decision-making in a fuzzy environment:,,,

Step 1. Consider the decision matrix $M = (\mathbb{T}_{ji})_{m\times n}$ given by the decision-makers in the form of CIFNs on the basis of the cubic intuitionistic fuzzy linguistic scale to evaluate suppliers in accordance with established objectives and criteria is given in Table 5.

Step 2. With the help of the linear approach, we normalize the matrix $M = (\mathbb{T}_{ji})_{m\times n}$. We divide the criteria into two subsets, benefit criteria $\mathbb{B}$ and cost criteria $\mathbb{K}$. Here, $\mathbb{X}_3$ and $\mathbb{X}_6$ belong to benefit criteria $\mathbb{B}$, and the remaining others belong to cost criteria $\mathbb{K}$. For this, first, we find $\max_j \mathbb{T}_{ji}$ and $\min_j \mathbb{T}_{ji}$, which are given in Table 5. We normalized the decision matrix by utilizing the 1st and 2nd equations in Algorithm 1 and this is given in Table 6.

Normalized decision matrixes from DMs are expressed in Table 7.

Steps 3 and 4. We find the relative importance of all alternatives by utilizing the 3rd equation in Algorithm 1, and then, we calculate their score function as given in Table 8.

Step 5. Rank the alternatives according to the score function, and the final ranking is

$$\mathbb{X}_3 \succ \mathbb{X}_5 \succ \mathbb{X}_1 \succ \mathbb{X}_6 \succ \mathbb{X}_2 \succ \mathbb{X}_4. \quad (77)$$

As we can see that $\mathbb{X}_3$ is the most appropriate supplier among the six alternatives with the best of qualities of all criteria.

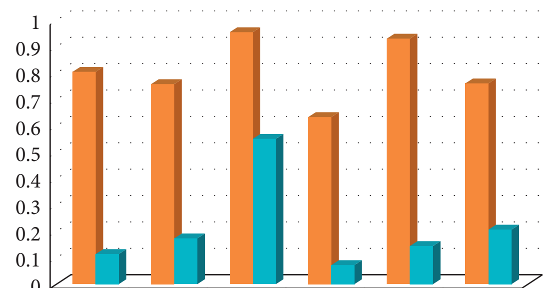
5.3. CIF Choice Value Method. The choice value method is a renowned and widely used MCGDM basis for evaluating a set of choices using a set of criteria. Each choice is contrasted to the others by calculating a number of ratios, one per choice criterion. Every ratio is multiplied by the proportional weight of the criterion in consideration. A fundamental task in MCGDM problems is the selection of one or more options from the set of alternatives based on a number of criteria. Let us consider m alternatives, n criteria with weighted vectors, with the condition that the sum of weights will be one, for an MCGDM problem in a cubic intuitionistic fuzzy set domain.,

5.4. MCDGM Application

Step 1. Consider the decision matrix $M = (\mathbb{T}_{ji})_{m\times n}$ given by the decision-makers in the form of CIFNs given in Table 9.

Step 2. Decision-makers gives the weights to the four criteria as $\mathbb{W}1 = 0.18$, $\mathbb{W}2 = 0.24$, $\mathbb{W}3 = 0.26$, and $\mathbb{W}4 = 0.32$ with $\sum \mathbb{W}i = 1$

$$\begin{pmatrix}
([0.17,0.24],[0.36,0.43],(0.56,0.32))\,([0.20,0.28],[0.29,0.31],(0.27,0.20)) \\
([0.18,0.21],[0.21,0.32],(0.39,0.22))\,([0.20,0.37],[0.21,0.43],(0.54,0.23)) \\
([0.19,0.22],[0.39,0.42],(0.59,0.40))\,([0.27,0.34],[0.33,0.40],(0.43,0.21)) \\
([0.24,0.30],[0.30,0.39],(0.50,0.20))\,([0.32,0.40],[0.19,0.51],(0.52,0.30)) \\
([0.20,0.29],[0.40,0.51],(0.81,0.13))\,([0.31,0.52],[0.42,0.50],(0.72,0.17)) \\
([0.31,0.39],[0.18,0.40],(0.67,0.14))\,([0.14,0.63],[0.24,0.50],(0.70,0.18)) \\
([0.31,0.37],[0.36,0.49],(0.50,0.36))\,([0.18,0.33],[0.28,0.52],(0.40,0.32)) \\
([0.23,0.40],[0.24,0.51],(0.50,0.33)) \\
([0.08,0.74],[0.32,0.40],(0.46,0.42)) \\
([0.40,0.48],[0.51,0.60],(0.52,0.30)) \\
([0.29,0.41],[0.30,0.39],(0.48,0.40))\,([0.40,0.47],[0.38,0.60],(0.56,0.29)) \\
([0.13,0.64],[0.40,0.47],(0.53,0.37)) \\
([0.29,0.38],[0.27,0.42],(0.60,0.27))\,([0.40,0.51],[0.41,0.50],(0.47,0.38)) \\
([0.32,0.38],[0.24,0.72],(0.43,0.32))\,([0.42,0.50],[0.37,0.53],(0.53,0.27))
\end{pmatrix}
\begin{pmatrix} 0.18 \\ 0.24 \\ 0.26 \\ 0.32 \end{pmatrix}$$



	X1	X2	X3	X4	X5	X6
CIF-WPM	0.8029	0.7592	0.9548	0.6294	0.9245	0.7593
CIF-CVM	0.1125	0.1705	0.549	0.0725	0.1435	0.204

Figure 2: Ranking of feasible alternatives.

$$\begin{pmatrix} ([0.032,0.048],[0.832,0.859],(0.900,0.067))+([0.052,0.075],[0.742,0.754],(0.730,0.052)) \\ +([0.050,0.059],[0.666,0.743],(0.782,0.062)) \\ +([0.068,0.137],[0.606,0.763],(0.821,0.080)) \\ ([0.037,0.043],[0.844,0.855],(0.909,0.087))+([0.072,0.094],[0.766,0.802],(0.816,0.055)) \\ +([0.068,0.088],[0.731,0.782],(0.835,0.056))+([0.116,0.150],[0.587,0.806],(0.811,0.107)) \\ ([0.039,0.059],[[0.847,0.885]],(0.962,0.024))+([0.085,0.161],[0.812,0.846],(0.924,0.043)) \\ +([0.091,0.120],[0.640,0.788],(0.901,0.038))+([0.047,0.272],[0.633,0.801],(0.892,0.061)) \\ ([0.064,0.079],[0.832,0.879],(0.882,0.077))+([0.046,0.091],[0.736,0.854],(0.802,0.088)) \\ +([0.065,0.124],[0.690,0.839],(0.835,0.098))+([0.026,0.350],[0.694,0.745],(0.779,0.159)) \\ ([0.087,0.111],[0.885,0.912],(0.888,0.062))+([0.078,0.118],[0.749,0.797],(0.888,0.115)) \\ +([0.124,0.152],[0.777,0.875],(0.860,0.085))+([0.043,0.278],[0.745,0.785],(0.816,0.137)) \\ ([0.059,0.082],[0.790,0.855],(0.912,0.055))+([0.115,0.157],[0.807,0.846],(0.834,0.108)) \\ +([0.095,0.116],[0.690,0.918],(0.802,0.095))+([0.159,0.198],[0.727,0.816],(0.840,0.095)) \end{pmatrix} \tag{78}$$

$$\begin{pmatrix} ([0.187,0.284],[0.249,0.367],(0.421,0.236)) \\ ([0.263,0.327],[0.277,0.432],(0.502,0.272)) \\ ([0.238,0.494],[0.278,0.472],(0.714,0.156)) \\ ([0.186,0.523],[0.293,0.469],(0.460,0.361)) \\ ([0.294,0.519],[0.383,0.499],(0.522,0.344)) \\ ([0.366,0.451],[0.278,0.537],(0.512,0.309)) \end{pmatrix}.$$

Step 3. We compute the score values for each alternative. The score values are expressed in Table 10.

Step 4. Rank the alternatives according to their score values.

$$\mathbb{X}_3 \succ \mathbb{X}_6 \succ \mathbb{X}_2 \succ \mathbb{X}_5 \succ \mathbb{X}_1 \succ \mathbb{X}_4. \tag{79}$$

As a result, $\mathbb{X}_3$ is best supplier among six alternatives with qualities of all criteria.

5.5. Comparative Analysis. This paper describes techniques for dealing with the cubic intuitionistic situation. We compare our two cubic intuitionistic strategies that are

Table 11: Comparative analysis and ranking of alternatives.

Methods	Ranking of alternatives	Top alternative
CIF-TOPSIS (Garg and Kaur [23])	$\mathbb{X}_3 \succ \mathbb{X}_6 \succ \mathbb{X}_2 \succ \mathbb{X}_5 \succ \mathbb{X}_1 \succ \mathbb{X}_4$	$\mathbb{X}_3$
CIF-WASPAS (Senapati et al. [26])	$\mathbb{X}_3 \succ \mathbb{X}_6 \succ \mathbb{X}_1 \succ \mathbb{X}_2 \succ \mathbb{X}_5 \succ \mathbb{X}_4$	$\mathbb{X}_3$
Frank AO (Seikh and Mandal [52])	$\mathbb{X}_3 \succ \mathbb{X}_6 \succ \mathbb{X}_2 \succ \mathbb{X}_5 \succ \mathbb{X}_1 \succ \mathbb{X}_4$	$\mathbb{X}_3$
CIF-WPM (Algorithm 1)	$\mathbb{X}_3 \succ \mathbb{X}_6 \succ \mathbb{X}_2 \succ \mathbb{X}_5 \succ \mathbb{X}_1 \succ \mathbb{X}_4$	$\mathbb{X}_3$
CIF-CVM (Algorithm 2)	$\mathbb{X}_3 \succ \mathbb{X}_5 \succ \mathbb{X}_1 \succ \mathbb{X}_6 \succ \mathbb{X}_2 \succ \mathbb{X}_4$	$\mathbb{X}_3$

already in use. If we use CIF-WPM to assemble the alternatives, they are ranked as

$$\mathbb{X}_3 \succ \mathbb{X}_5 \succ \mathbb{X}_1 \succ \mathbb{X}_6 \succ \mathbb{X}_2 \succ \mathbb{X}_4. \tag{80}$$

On the other side, when we use the technique of choice value method, the ranking of alternatives is

$$\mathbb{X}_3 \succ \mathbb{X}_6 \succ \mathbb{X}_2 \succ \mathbb{X}_5 \succ \mathbb{X}_1 \succ \mathbb{X}_4. \tag{81}$$

Based on these findings, it seemed that the ranking of the $\mathbb{X}_3$ alternative was the same as that produced by the suggested cubic intuitionistic procedures. The rest of the alternatives have been altered, as can be seen. As a result, we concluded that in the case of only IVIFSs, the best choice matches with the indicated one; however, the other alternatives are altered, resulting in numerous decisions. As a result, this CIS condition improves the application range of the membership and nonmembership intervals by considering IFS membership values in line with it.

Figure 2 shows the bar chart of ranking of feasible alternatives by using the WPM and CVM methods.

The comparison analysis of the proposed CIF-WPM and CIF-CVM with other existing techniques is expressed in Table 11.

6. Conclusion

A cubic intuitionistic fuzzy set is an effective method for dealing with various uncertainties in multicriteria group decision-making (MCGDM) settings. A cubic set is a two-component system that would be used to describe data with a fuzzy interval and a fuzzy number. The notion of cubic intuitionistic fuzzy sets (CIFS) is a strong hybrid model of IFSs and IVIFSs. A CIFS has two components, one indicating the IVIFS and the other indicating the IFS. A CIFS is a new fuzzy model for data analysis, computational intelligence, neural computing, soft computing, and others. The idea of cubic hesitant fuzzy topology defined on CIFS can be utilized to seek solutions to various problems of information analysis, information fusion, big data, and decision analysis.

Main findings in this manuscript are as follows:

(1) We introduced the concepts of "P-cubic intuitionistic fuzzy topology" as well as "R-cubic intuitionistic fuzzy topology." Topological structures provide robust approaches for data analysis and decision analysis under an uncertain environment.
(2) Certain properties of CIF topology under P(R)-order are explored, and their related results are elaborated with illustrations.
(3) The notions of CIF-open set, CIF-closed set, CIF-closure, CIF-interior, CIF-exterior, as well as CIF-frontier, CIF-dense set, and CIF-basis are investigated with a corresponding example.
(4) Algorithms 1 and 2 are proposed for extension of the weighted product model and the choice value method, respectively.
(5) The symmetry of optimal decisions is analyzed by computations with Algorithms 1 and 2. The numerical values of alternatives are very close by using Algorithm 1. However, the numerical values of alternatives have a clear difference when using Algorithm 2.
(6) An application of proposed methods named CIF-WPM and CIF-CVM towards uncertain supply chain management is presented.
(7) To discuss the advantages, flexibility, and validity of proposed methods, a comparison analysis is also expressed.

For forthcoming analysis, due to the flexibility of CIF topology towards data analysis and information analysis, one can extend this work to develop new MCDM techniques with CIF-VIKOR, CIF-AHP, and CIF-aggregation operators.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Muhammad Riaz was responsible for conceptualization, formal analysis, and supervision. Khadija Akmal conducted methodology, formal analysis, review, and editing. Yahya Almalki performed investigation, supervision, and funding acquisition. Daud Ahmad took part in investigation, methodology, review, and editing. All authors made a significant scientific contribution to the research in the manuscript. All authors read and approved the final manuscript.

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References

[1] F. Chazal and B. Michel, "An introduction to topological data analysis: fundamental and practical aspects for data scientists," *Frontiers in Artificial Intelligence*, vol. 4, pp. 1–28, 2021.

[2] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[3] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[4] K. T. Atanassov, "Circular intuitionistic fuzzy sets," *Journal of Intelligent and Fuzzy Systems*, vol. 39, no. 5, pp. 5981–5986, 2020.

[5] R. R. Yager, "Pythagorean fuzzy subsets," *Proceedings of the Joint IFSA World Congress and NAFIPS Annual Meeting*, pp. 57–61, 2013.

[6] R. R. Yager, "Pythagorean membership grades in multi-criteria decision making," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 4, pp. 958–965, 2014.

[7] R. R. Yager, "Generalized orthopair fuzzy sets," *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 5, pp. 1222–1230, 2017.

[8] D. Molodtsov, "Soft set theory-first results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[9] W. R. Zhang, "Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis," in *Proceedings of the NAFIPS/IFIS/NASA94. Proceedings of the 1st International Joint Conference of the North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intellige*, pp. 305–309, San Antonio, TX, USA, December 1994.

[10] W. R. Zhang, "Bipolar fuzzy sets," *Proceedings of IEEE Inrenational Conference on Fuzzy Systems*, pp. 835–840, 1998.

[11] F. Smarandache, *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic*, American Research Press, Rehoboth, DE, USA, 1999.

[12] F. Smarandache, "Neutrosophic set- a generalization of the intuitionistic fuzzy set," *International Journal of Pure and Applied Mathematics*, vol. 24, no. 3, pp. 287–297, 2005.

[13] B. C. Cuong, "Picture fuzzy sets," *Journal of Computer Science and Cybernetics*, vol. 30, no. 4, pp. 409–420, 2014.

[14] F. Kutlu Gündoğdu and C. Kahraman, "Spherical fuzzy sets and spherical fuzzy TOPSIS method," *Journal of Intelligent and Fuzzy Systems*, vol. 36, no. 1, pp. 337–352, 2019.

[15] T. Mahmood, K. Ullah, Q. Khan, and N. Jan, "An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets," *Neural Computing & Applications*, vol. 31, no. 11, pp. 7041–7053, 2019.

[16] S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, and T. Mahmood, "Spherical fuzzy sets and their applications in multi-attribute decision making problems," *Journal of Intelligent and Fuzzy Systems*, vol. 36, no. 3, pp. 2829–2844, 2019.

[17] K. Atanassov and G. Gargov, "Interval-valued intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 31, no. 3, pp. 343–349, 1989.

[18] N. Cagman and S. Enginoglu, "Soft set theory and uni–int decision making," *European Journal of Operational Research*, vol. 207, no. 2, pp. 848–855, 2010.

[19] F. Karaaslan and N. Cagman, "Parameter trees based on soft set theory and their similarity measures," *Soft Computing*, vol. 26, no. 10, pp. 4629–4639, 2022.

[20] J. Chen, S. Li, S. Ma, and X. Wang, "m-m-Polar fuzzy sets: an extension of bipolar fuzzy sets," *The Scientific World Journal*, vol. 2014, Article ID 416530, 8 pages, 2014.

[21] Y. B. Jun, C. S. Kim, and K. O. Yang, "Cubic sets," *Annals of Fuzzy Mathematics and Informatics*, vol. 4, no. 1, pp. 83–98, 2012.

[22] M. Riaz and M. R. Hashmi, "MAGDM for agribusiness in the environment of various cubic m-polar fuzzy averaging aggregation operators," *Journal of Intelligent and Fuzzy Systems*, vol. 37, no. 3, pp. 3671–3691, 2019.

[23] H. Garg and G. Kaur, "Extended TOPSIS method for multi-criteria group decision-making problems under cubic intuitionistic fuzzy environment," *Scientia Iranica*, vol. 27, pp. 1–18, 2018.

[24] H. Garg and G. Kaur, "Novel distance measures for cubic intuitionistic fuzzy sets and their applications to pattern recognitions and medical diagnosis," *Granular Computing*, vol. 5, pp. 1–16, 2018.

[25] B. J. Young, S. Seok-Zun, and K. J. Seon, "Cubic interval-valued intuitionistic fuzzy sets and their application in BCK/BCI-Algebras," *Axioms*, vol. 7, pp. 1–17, 2018.

[26] T. Senapati, R. R. Yager, and G. Chen, "Cubic intuitionistic WASPAS technique and its application in multi.criteria decision.making," *Journal of Ambient Intelligence and Humanized Computing*, vol. 12, no. 9, pp. 8823–8833, 2021.

[27] G. Kaur and H. Garg, "Multi-attribute decision-making based on Bonferroni mean operators under cubic intuitionistic fuzzy set environment," *Entropy*, vol. 20, pp. 65–16, 2018.

[28] H. Garg and G. Kaur, "TOPSIS based on nonlinear-programming methodology for solving decision-making problems under cubic intuitionistic fuzzy set environment," *Computational and Applied Mathematics*, vol. 38, no. 3, pp. 114–119, 2019.

[29] C. L. Chang, "Fuzzy topological spaces," *Journal of Mathematical Analysis and Applications*, vol. 24, no. 1, pp. 182–190, 1968.

[30] D. Coker, "An introduction to intuitionistic fuzzy topological spaces," *Fuzzy Sets and Systems*, vol. 88, no. 1, pp. 81–89, 1997.

[31] M. Olgun, M. Ünver, and S. Yardimci, "Pythagorean fuzzy topological spaces," *Complex & Intelligent Systems*, vol. 5, no. 2, pp. 177–183, 2019.

[32] N. Cagman, S. Karatas, and S. Enginoglu, "Soft topology," *Computers & Mathematics with Applications*, vol. 62, no. 1, pp. 351–358, 2011.

[33] M. Riaz, K. Akmal, Y. Almalki, and S. A. Alblowi, "Cubic m-polar fuzzy topology with multi-criteria group decision-making," *AIMS Mathematics*, vol. 7, no. 7, pp. 13019–13052, 2022.

[34] Z. S. Xu and R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *International Journal of General Systems*, vol. 35, no. 4, pp. 417–433, 2006.

[35] Z. S. Xu, "Intuitionistic fuzzy aggregation operators," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 6, pp. 1179–1187, 2007.

[36] X. L. Zhang and Z. S. Xu, "Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets," *International Journal of Intelligent Systems*, vol. 29, no. 12, pp. 1061–1078, 2014.

[37] F. Feng, Y. Zheng, B. Sun, and M. Akram, "Novel score functions of generalized orthopair fuzzy membership grades with application to multiple attribute decision making," *Granular Computing*, vol. 7, no. 1, pp. 95–111.

[38] M. Akram, "Bipolar fuzzy graphs," *Information Sciences*, vol. 181, no. 24, pp. 5548–5564, 2011.

[39] M. Akram, A. Khan, J. C. R. Alcantud, and G. Santos-Garcia, "A hybrid decision-making framework under complex spherical fuzzy prioritized weighted aggregation operators," *Expert Systems*, vol. 38, no. 6, pp. 1–24.
[40] M. A. Alghamdi, N. O. Alshehri, and M. Akram, "Multi-criteria decision-making methods in bipolar fuzzy environment," *International Journal of Fuzzy Systems*, vol. 20, no. 6, pp. 2057–2064, 2018.
[41] P. Liu and P. Wang, "Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making," *International Journal of Intelligent Systems*, vol. 33, no. 2, pp. 259–280, 2018.
[42] J. Ye, "Similarity measures based on the generalized distance of neutrosophic Z-number sets and their multi-attribute decision making method," *Soft Computing*, vol. 25, no. 22, pp. 13975–13985.
[43] T. Senapati and R. R. Yager, "Some new operations over f fuzzy numbers and application of f fuzzy WPM in multiple criteria decision making," *Informatica*, vol. 30, no. 2, pp. 391–412, 2019.
[44] C. Kahraman and N. Alkan, "Circular intuitionistic fuzzy TOPSIS method with vague membership functions: supplier selection application context," *Notes on Intuitionistic Fuzzy Sets*, vol. 27, no. 1, pp. 24–52, 2021.
[45] S. Sinha and S. P. Sarmah, "An application of fuzzy set theory for supply chain coordination," *International Journal of Management Science and Engineering Management*, vol. 3, no. 1, pp. 19–32, 2008.
[46] M. T. Alshurideh, B. Al Kurdi, H. M. Alzoubi et al., "Fuzzy assisted human resource management for supply chain management issues," *Annals of Operations Research*, pp. 1–19, 2021.
[47] M. R. Seikh, S. Dutta, and D. F. Li, "Solution of matrix games with rough interval pay-offs and its application in the telecom market share problem," *International Journal of Intelligent Systems*, vol. 36, no. 10, pp. 6066–6100, 2021.
[48] M. R. Seikh, S. Karmakar, and O. Castillo, "A novel defuzzification approach of type-2 fuzzy variable to solving matrix games: an application to plastic ban problem," *Iranian Journal of Fuzzy Systems*, vol. 18, no. 5, pp. 155–172, 2021.
[49] S. Ruidas, M. R. Seikh, and P. K. Nayak, "An EPQ model with stock and selling price dependent demand and variable production rate in interval environment," *International Journal of System Assurance Engineering and Management*, vol. 11, no. 2, pp. 385–399, 2020.
[50] S. Ruidas, M. R. Seikh, P. K. Nayak, and B. Sarkar, "A single period production inventory model in interval environment with price revision," *International Journal of Algorithms, Computing and Mathematics*, vol. 5, no. 1, pp. 7–20, 2019.
[51] S. Ruidas, M. R. Seikh, and P. K. Nayak, "A production-repairing inventory model considering demand and the proportion of defective items as rough intervals," *Operational Research*, vol. 22, no. 3, pp. 2803–2829.
[52] M. R. Seikh and U. Mandal, "Q-rung orthopair fuzzy Frank aggregation operators and its application in multiple attribute decision-making with unknown attribute weights," *Granular Computing*, vol. 7, no. 3, pp. 709–730, 2022.
[53] M. R. Seikh and U. Mandal, "Multiple attribute decision-making based on 3, 4-quasirung fuzzy sets," *Granular Computing*, pp. 1–14, 2022.
[54] M. Riaz and H. M. A. Farid, "Picture fuzzy aggregation approach with application to third-party logistic provider selection process," *Reports in Mechanical Engineering*, vol. 3, no. 1, pp. 318–327, 2022.
[55] A. Ashraf, K. Ullah, A. Hussain, and M. Bari, "Interval-valued picture fuzzy Maclaurin symmetric mean operator with application in multiple attribute decision-making," *Reports in Mechanical Engineering*, vol. 3, no. 1, pp. 301–317, 2022.
[56] M. M. U. Baig, Y. Ali, and O. U. Rehman, "Enhancing resilience of oil supply chains in context of developing countries," *Operational Research in Engineering Sciences: Theory and Applications*, vol. 5, no. 1, pp. 69–89, 2022.
[57] R. Chattopadhyay, P. P. Das, and S. Chakraborty, "Development of a rough-MABAC-DoE-based Metamodel for supplier selection in an iron and steel industry," *Operational Research in Engineering Sciences: Theory and Applications*, vol. 5, no. 1, pp. 20–40, 2022.
[58] C. Karamasa, E. Demir, S. Memis, and S. Korucuk, "Weighting the factors affecting logistics outsourcing," *Decision Making: Applications in Management and Engineering*, vol. 4, no. 1, pp. 19–33, 2021.
[59] B. Bairagi, "A novel MCDM model for warehouse location selection in supply chain management," *Decision Making: Applications in Management and Engineering*, vol. 5, no. 1, pp. 194–207, 2022.
[60] M. Riaz, Y. Almalki, S. Batool, and S. Tanveer, "Topological structure of single-valued neutrosophic hesitant fuzzy sets and data analysis for uncertain supply chains," *Symmetry*, vol. 14, no. 7, pp. 1382–1424, 2022.
[61] M. Riaz, D. Pamucar, A. Habib, and N. Jamil, "Innovative bipolar fuzzy sine trigonometric aggregation operators and SIR method for medical tourism supply chain," *Mathematical Problems in Engineering*, vol. 2022, Article ID 4182740, 17 pages, 2022.
[62] M. Riaz, D. Pamucar, A. Habib, and M. Riaz, "A new TOPSIS approach using cosine similarity measures and cubic bipolar fuzzy information for sustainable plastic recycling process," *Mathematical Problems in Engineering*, vol. 2021, Article ID 4309544, 18 pages, 2021.
[63] M. Ali, I. Deli, and F. Smarandache, "The theory of neutrosophic cubic sets and their applications in pattern recognition," *Journal of Intelligent and Fuzzy Systems*, vol. 30, no. 4, pp. 1957–1963, 2016.

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Research Article

Spherical Fuzzy Information Aggregation Based on Aczel–Alsina Operations and Data Analysis for Supply Chain

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Spherical fuzzy sets (SFSs) are often made up of membership, nonmembership, and hesitancy grades, and also have the advantage of accurately representing decision makers (DMs) preferences. This article proposes novel spherical fuzzy aggregation operators (AOs) based on Aczel–Alsina (AA) operations, which offer a lot of advantages when tackling real-world situations. We begin by introducing some new SFS operations, such as the Aczel–Alsina product, the Aczel–Alsina sum, the Aczel–Alsina exponent, and the Aczel–Alsina scalar multiplication. We developed many AOs namely, the "spherical fuzzy Aczel–Alsina weighted averaging (SFAAWA) operator," "spherical fuzzy Aczel–Alsina ordered weighted averaging (SFAAOWA) operator," "spherical fuzzy Aczel–Alsina hybrid averaging (SFAAHA) operator," "spherical fuzzy Aczel–Alsina weighted geometric (SFAAWG) operator," "spherical fuzzy Aczel–Alsina ordered weighted geometric (SFAAOWG) operator," and "spherical fuzzy Aczel–Alsina hybrid geometric (SFAAHG) operator." Different attributes of these operators have been defined. The idempotency, boundary, monotonicity, and commutativity of suggested averaging and geometric operators are demonstrated. Then, based on these operators, we propose a novel approach for tackling the "multi-criteria decision-making" (MCDM) problems. We use a agriculture land selection scenario to demonstrate the efficacy of our proposed approach. The outcome confirms the new technique's applicability and viability. Furthermore, sensitivity analysis and a comparison analysis between the existing approaches and the recommended technique have been provided.

1. Introduction

Decision-making problems are common in a wide range of fields, including technology, finance, and marketing. Traditionally, it has been assumed that all data on alternate access is kept as discrete integers. Because managing the imprecision and uncertainty inherent in data is crucial in real-world circumstances. There are three alternative reactions or attitudes when it comes to selection: yes, no, and refusal. However, first, the most sophisticated response is "refusal," which conventional "fuzzy sets" [1] and "intuitionistic fuzzy sets" (IFSs) [2] may not accurately represent. To address such problems, Cuong proposed the idea of "picture fuzzy set" (PFS) [3, 4]. In PFS, each component in the universe of discourse set has varying degrees of "positive membership degree" (PMD), "neutral membership degree" (N_u MD), and "negative membership degree" (N_g MD) with values ranging from $[0, 1]$.

Fuzzy clustering is a useful technique for pattern detection and information extraction from databases, and it has been used to a wide range of practical issues. Son [5] defined "distributed picture fuzzy clustering method" for PFSs. Singh [6] proposed some "correlation coefficients." The correlation coefficient is applied to clustering in PFS in this article. The benefits of proposed correlation coefficients as well as the disadvantages of existing correlation coefficients have been examined. Wei [7] proposed some new similarity measures (SMs) between PFSs, such as set-theoretic SMs, weighted set-theoretic cosine SMs, cosine SMs, weighted cosine SMs, grey SMs, and weighted grey SMs. Wei and Gao [8] introduced various innovative dice SMs for PFSs and generalized dice SMs for PFSs, indicating that dice SMs and asymmetric measures are special cases of generalized dice SMs for certain parameter values. Wei et al. [9] proposed some results about "projection model" under the PFSs, the modules of picture fuzzy number (PFN), and PF-ideal point.

Over the last few decades, there has been a strong emphasis on information fusion and the development of new AOs. AO's effectiveness and limitations have been entrenched in decision-making. AO obviously includes a number of operating rules for concatenating a finite set of fuzzy numbers into a single fuzzy number. Data aggregation is essential in decision-making, economy, administration, healthcare, technology, and intelligence areas. In terms of their functions and operating laws, numerous AOs have been established for PFSs. Wei et al. [10] proposed "picture 2-tuple linguistic AOs" with MCDM. Garg [11] proposed some weighted averaging and ordered weighted averaging operators for the aggregation of PFNs. Wei [12] presented "Hamacher" AOs for the PFS, and a realistic example of selecting an enterprise system is provided to validate the established approach and to illustrate its feasibility and efficiency. Jana et al. [13] gave the notion of picture fuzzy Dombi AOs for PFNs with MCDM applications. Tian et al. [14] defined some "picture fuzzy power Choquet ordered geometric AOs" and "picture fuzzy power shapley Choquet ordered geometric AOs" with shapley fuzzy measures-based MCDM. Wang et al. [15] proposed hotel building energy efficiency retrofit project selection under PFSs. Wang et al. [16] introduced "Muirhead mean AOs" for PFNs. Wei established TODIM method for PFSs [17], "picture 2-tuple linguistic Bonferroni mean AOs" [18], and "picture uncertain linguistic Bonferroni mean AOs" [19]. Abdullah et al. [20] proposed some new AOs based on sine trigonometric function with application. Qiyas et al. [21] defined some novel picture fuzzy AOs under the linguistic environment. Farid and Riaz [22] developed several new Einstein interactive geometric AOs for q-rung orthopair fuzzy numbers. Riaz and Farid [23] proposed some proportional distribution based spherical AOs. Farid et al. [24] introduced some AOs for the thermal power equipment supplier selection. Saha et al. [25] introduced the new hybrid hesitant fuzzy weighted AOs for MCDM that are based on Archimedean and Dombi operations. Feng et al. [26] proposed the idea of score functions related to generalized orthopair fuzzy membership grades with application. Akram et al. [27] introduced the idea of prioritized AOs for complex spherical fuzzy sets. Riaz and Farid [28] developed some fairly AOs for PFSs. Riaz et al. [29] proposed some Frank AOs for interval-valued linear Diophantine fuzzy set.

Menger [30] introduced the concept of triangle norms in his hypothesis of probabilistic metric spaces. It has been discovered that t-norms and their corresponding t-conorms are fundamental operations in fuzzy sets and structures, such as the product t-norm and probabilistic sum t-conorm [31], Einstein t-norm and t-conorm [32], and the Hamacher t-norm and t-conorm [33]. Klement et al. [34] carried out a thorough analysis of the characteristics and associated elements of triangular norms in recent years. In 1982, Aczél and Alsina [35] introduced new operations named Aczel–Alsina t-norm and Aczel–Alsina t-conorm, which place a high premium on parameter changeability. Based on the Aczel–Alsina triangular norm (AA t-norm), Wang et al. [36] devised a score level fusion technique that simultaneously increases the distance between imposters. Senapati et al. proposed Aczel–Alsina AOs for IFSs [37] and Aczel–Alsina AOs for interval-valued IFSs [38].

In everyday life, we might confront a variety of circumstances that PFS cannot resolve, such as when the sum of PMD, N_u MD, and N_g MD>1. PFS is incapable of producing an appropriate conclusion in such a case. Mahmood et al. [39], Gundogdu and Kahraman [40], and Ashraf et al. [41] separately developed the concept of SFSs in their works. SFS provides the DM with extra freedom when confronted with uncertainty in decision-making situations. Gundogdu and Kahraman developed the SF-TOPSIS [42], SF-WASPAS [43], and SF-VIKOR methods [44]. Ashraf et al. presented AOs for SFSs and the GRA method for the SF-linguistic set [45]. Zeng et al. [46] proposed a TOPSIS-based hybrid covering-based SF-rough set model. The cosine similarity measures for SFSs were proposed by Rafiq et al. [47]. Jin et al. [48] developed AOs for SFSs based on logarithmic functions. Additionally, Ashraf et al. [49] presented several Dombi AOs for SFSs with implementation to group MCDM. Jaller and Otay used [50] when they suggested SF AHP and TOPSIS for assessing efficient vehicle technology for cargo handling. Ali et al. [51] and Ashraf et al. [52] proposed some AOs for interval-valued picture fuzzy set. Kazemitash et al. [53] and Bozanic et al. [54] gave some ideas related to some different extensions of fuzzy set. For other terminologies not discussed in the paper, the readers are referred to [55–58].

In light of the foregoing, we recognize that decision-making concerns are becoming increasingly complex in reality. To select the superior alternatives to the MCDM concerns, it is significant to communicate the uncertain information in a far more beneficial approach. Furthermore, it is critical to control the relationship between input contentions. Taking each of these characteristics into account, the primary objective of this informative article is to demonstrate numerous aggregation operators under SF circumstances, referred as SF Aczel–Alsina AOs. Despite the established inventive approaches that have developed previously in this field, we have thoroughly researched every possibility to demonstrate our provided strategy, for it to outperform all previous efforts to comprehend the actual global presented concern.

The following information is included in the paper: the next section discusses some fundamental concepts relating to Aczel–Alsina triangular norms and SFSs. Section 3 summarizes the Aczel–Alsina operation laws for SFNs. In Section 4, we discuss the "spherical fuzzy Aczel–Alsina weighted averaging (SFAAWA) operator," "spherical fuzzy Aczel–Alsina ordered weighted averaging (SFAAOWA) operator," "spherical fuzzy Aczel–Alsina hybrid averaging (SFAAHA) operator," "spherical fuzzy Aczel–Alsina weighted geometric (SFAAWG) operator," "spherical fuzzy Aczel–Alsina ordered weighted geometric (SFAAOWG) operator," and "spherical fuzzy Aczel–Alsina hybrid geometric (SFAAHG) operator" as well as a few advantageous properties. In Section 5, we employ the proposed operators to develop a set of techniques for resolving MCDM difficulties in which the characteristic values are represented as SF data. Section 6 illustrates how to choose agriculture land to showcase the proposed technique. In Section 7, we analyze the effect of a parameter on the alternate raking order. In Section 8, we compare the developed method to current methods to determine the proposed technique's appropriateness. Finally, Section 9 discusses a few conclusions for future study.

2. Preliminaries

Several fundamental concepts associated with SFSs have been addressed in this section of the article.

Definition 1 (see [39–41]). A "spherical fuzzy set" (SFS) in X is defined as follows:

$$\chi = \left\{ \left\langle \breve{Y}, \kappa_{\chi}(\breve{Y}), \upsilon_{\chi}(\breve{Y}), \tau_{\chi}(\breve{Y}) \middle| \breve{Y} \in X \right\rangle \right\}, \tag{1}$$

where $\kappa_{\chi}(\breve{Y}), \upsilon_{\chi}(\breve{Y}), \tau_{\chi}(\breve{Y}) \in [0,1]$, such that $0 \leq \kappa_{\chi}^{2}(\breve{Y}) + \upsilon_{\chi}^{2}(\breve{Y}) + \tau^{2}\chi(\breve{Y}) \leq 1$ for all $\breve{Y} \in X$ and $\kappa_{\chi}(\breve{Y}), \upsilon_{\chi}(\breve{Y}), \tau_{\chi}(\breve{Y})$ denote degree of membership, nonmembership and hesitancy, respectively, for some $\breve{Y} \in X$.

We denote this pair as $\mathfrak{R}^{\gamma} = (\kappa_{\mathfrak{R}^{\gamma}}, \upsilon_{\mathfrak{R}^{\gamma}}, \tau_{\mathfrak{R}^{\gamma}})$, throughout this article, and called as SFN with the conditions $\kappa_{\mathfrak{R}^{\gamma}}, \upsilon_{\mathfrak{R}^{\gamma}}, \tau_{\mathfrak{R}^{\gamma}} \in [0,1]$ and $\kappa_{\mathfrak{R}^{\gamma}}^{2} + \upsilon_{\mathfrak{R}^{\gamma}}^{2} + \tau_{\mathfrak{R}^{\gamma}}^{2} \leq 1$.

Definition 2 (see [40]). When implementing SFNs to real-world situations, it is critical to prioritize them. For this, "score function" (SF) for SFN $\mathfrak{R}^{\gamma} = (\kappa_{\mathfrak{R}^{\gamma}}, \upsilon_{\mathfrak{R}^{\gamma}}, \tau_{\mathfrak{R}^{\gamma}})$ can be defined as follows:

$$S(\mathfrak{R}^{\gamma}) = (\kappa_{\mathfrak{R}^{\gamma}} - \tau_{\mathfrak{R}^{\gamma}})^{2} - (\upsilon_{\mathfrak{R}^{\gamma}} - \tau_{\mathfrak{R}^{\gamma}})^{2}. \tag{2}$$

Example 1. Consider two SFNs $\mathfrak{R}_{1}^{\gamma} = \langle 0.236, 0.126, 0.175 \rangle$ and $\mathfrak{R}_{2}^{\gamma} = \langle 0.308, 0.228, 0.482 \rangle$ then by using equation (2), we get $S(\mathfrak{R}_{1}^{\gamma}) = 0.00119646$ and $S(\mathfrak{R}_{2}^{\gamma}) = -0.0341963$. As $S(\mathfrak{R}_{1}^{\gamma}) > S(\mathfrak{R}_{2}^{\gamma})$ so, we have $\mathfrak{R}_{1}^{\gamma} > \mathfrak{R}_{2}^{\gamma}$.

However, because the aforementioned function appears incapable of classifying the SFNs in a variety of conditions, it is hard to determine which one is larger $S(\mathfrak{R}_{1}^{\gamma}) = S(\mathfrak{R}_{2}^{\gamma})$. For this, an "accuracy function H" of $\mathfrak{R}^{\gamma}$ is defined as follows:

$$H(\mathfrak{R}^{\gamma}) = \kappa_{\mathfrak{R}^{\gamma}}^{2} + \upsilon_{\mathfrak{R}^{\gamma}}^{2} + \tau_{\mathfrak{R}^{\gamma}}^{2}. \tag{3}$$

Based on [40], we presented some operational rules to aggregate the SFNs.

Definition 3. Let $\mathfrak{R}_{1}^{\gamma} = \langle \kappa_{1}, \upsilon_{1}, \tau_{1} \rangle$ and $\mathfrak{R}_{2}^{\gamma} = \langle \kappa_{1}, \upsilon_{1}, \tau_{1} \rangle$ be two SFNs, then

$$\begin{aligned}
\mathfrak{R}_{1}^{\gamma^{c}} &= \langle \upsilon_{1}, \kappa_{1}, \tau_{1} \rangle, \\
\mathfrak{R}_{1}^{\gamma} \vee \mathfrak{R}_{2}^{\gamma} &= \left\{ \begin{array}{c} \max\{\kappa_{1}, \kappa_{2}\}, \min\{\upsilon_{1}, \upsilon_{2}\}, \\ \min\left\{ \left(1 - \left((\max\{\kappa_{1}, \kappa_{2}\})^{2} + (\min\{\upsilon_{1}, \upsilon_{2}\})^{2}\right)\right)^{\frac{1}{2}}, \max\{\tau_{1}, \tau_{2}\} \right\} \end{array} \right\}, \\
\mathfrak{R}_{1}^{\gamma} \wedge \mathfrak{R}_{2}^{\gamma} &= \left\{ \begin{array}{c} \min\{\kappa_{1}, \kappa_{2}\}, \max\{\upsilon_{1}, \upsilon_{2}\}, \\ \max\left\{ \left(1 - \left((\min\{\kappa_{1}, \kappa_{2}\})^{2} + (\max\{\upsilon_{1}, \upsilon_{2}\})^{2}\right)\right)^{\frac{1}{2}}, \min\{\tau_{1}, \tau_{2}\} \right\} \end{array} \right\}, \\
\mathfrak{R}_{1}^{\gamma} \oplus \mathfrak{R}_{2}^{\gamma} &= \left\{ \begin{array}{c} \left(\kappa_{1}^{2} + \kappa_{2}^{2} - \kappa_{1}^{2}\kappa_{2}^{2}\right)^{1/2}, \upsilon_{1}\upsilon_{2}, \\ \left(\left(1 - \kappa_{2}^{2}\right)\tau_{1}^{2} + \left(1 - \kappa_{1}^{2}\right)\tau_{2}^{2} - \tau_{1}^{2}\tau_{2}^{2}\right)^{1/2} \end{array} \right\}, \\
\mathfrak{R}_{1}^{\gamma} \otimes \mathfrak{R}_{2}^{\gamma} &= \left\{ \begin{array}{c} \kappa_{1}\kappa_{2}, \left(\upsilon_{1}^{2} + \upsilon_{2}^{2} - \upsilon_{1}^{2}\upsilon_{2}^{2}\right)^{1/2}, \\ \left(\left(1 - \upsilon_{2}^{2}\right)\tau_{1}^{2} + \left(1 - \upsilon_{1}^{2}\right)\tau_{2}^{2} - \tau_{1}^{2}\tau_{2}^{2}\right)^{1/2} \end{array} \right\}, \\
\mathfrak{R}_{1}^{\gamma^{\sigma}} &= \left\{ \begin{array}{c} \mu_{A_{s}}^{\sigma}, \left(1 - \left(1 - \upsilon_{1}^{2}\right)^{\sigma}\right)^{1/2}, \\ \left(\left(1 - \upsilon_{1}^{2}\right)^{\sigma} - \left(1 - \upsilon_{1}^{2} - \pi_{A_{s}}^{2}\right)^{\sigma}\right)^{1/2}, \sigma > 0 \end{array} \right\}, \text{ and} \\
\sigma \cdot \mathfrak{R}_{1}^{\gamma} &= \left\{ \begin{array}{c} \left(1 - \left(1 - \kappa_{1}^{2}\right)^{\sigma}\right)^{1/2}, \upsilon_{1}^{\sigma}, \\ \left(\left(1 - \kappa_{1}^{2}\right)^{\sigma} - \left(1 - \kappa_{1}^{2} - \tau_{1}^{2}\right)^{\sigma}\right)^{1/2} \Big\}, \sigma > 0 \end{array} \right\}.
\end{aligned} \tag{4}$$

Definition 4 (see [40]). Let $\mathfrak{R}_1^\gamma = \langle \kappa_1, \upsilon_1, \tau_1 \rangle$ and $\mathfrak{R}_2^\gamma = \langle \kappa_2, \upsilon_2, \tau_2 \rangle$ be two SFNs and $\sigma, \sigma_1, \sigma_2 > 0$ be the real numbers, then we have,

(1) $\mathfrak{R}_1^\gamma \oplus \mathfrak{R}_2^\gamma = \mathfrak{R}_2^\gamma \oplus \mathfrak{R}_1^\gamma$
(2) $\mathfrak{R}_1^\gamma \otimes \mathfrak{R}_2^\gamma = \mathfrak{R}_2^\gamma \otimes \mathfrak{R}_1^\gamma$
(3) $\sigma(\mathfrak{R}_1^\gamma \oplus \mathfrak{R}_2^\gamma) = (\sigma\mathfrak{R}_1^\gamma) \oplus (\sigma\mathfrak{R}_2^\gamma)$
(4) $(\mathfrak{R}_1^\gamma \otimes \mathfrak{R}_2^\gamma)^\sigma = \mathfrak{R}_1^{\gamma^\sigma} \otimes \mathfrak{R}_2^{\gamma^\sigma}$
(5) $(\sigma_1 + \sigma_2)\mathfrak{R}_1^\gamma = (\sigma_1\mathfrak{R}_1^\gamma) \oplus (\sigma_2\mathfrak{R}_2^\gamma)$
(6) $\mathfrak{R}_1^{\gamma^{\sigma_1+\sigma_2}} = \mathfrak{R}_1^{\gamma^{\sigma_1}} \otimes \mathfrak{R}_1^{\gamma^{\sigma_1}}$

2.1. Basics about t-Norm, t-Conorm, and Aczel–Alsina t-Norm

Definition 5 (see [34]). A function $\breve{\mho}^\lambda$: $[0,1]^2 \longrightarrow [0,1]$ is a t-norm, if for all $g, h, u \in [0,1]$, the consecutive axioms are fulfilled.

(1) $\breve{\mho}^\lambda(g,h) = \breve{\mho}^\lambda(h,g)$
(2) $\breve{\mho}^\lambda(g,h) \le \breve{\mho}^\lambda(g,u)$ if $h \le u$
(3) $\breve{\mho}^\lambda(g, \breve{\mho}^\lambda(h,u)) = \breve{\mho}^\lambda(\breve{\mho}^\lambda(g,h),u)$
(4) $\breve{\mho}^\lambda(g,1) = g$

These axioms are called, symmetry, monotonicity, associativity, and "1" as identity, respectively.

Definition 6 (see [34]). A function $\overline{\mathfrak{I}}$: $[0,1]^2 \longrightarrow [0,1]$ is a t-conorm, if for all $g, h, u \in [0,1]$, the consecutive axioms are fulfilled.

(1) $\overline{\mathfrak{I}}(g,h) = \overline{\mathfrak{I}}(h,g)$
(2) $\overline{\mathfrak{I}}(g,h) \le \overline{\mathfrak{I}}(g,u)$ if $h \le u$
(3) $\overline{\mathfrak{I}}(g, \overline{\mathfrak{I}}(h,u)) = \overline{\mathfrak{I}}(\overline{\mathfrak{I}}(g,h),u)$
(4) $\overline{\mathfrak{I}}(g,0) = g$

These axioms are called, symmetry, monotonicity, associativity, and "0" as identity, respectively.

Example 2. Some famous t-norms are given as follows:

(i) $\breve{\mho}^\lambda P(g,h) = g.\, h$ (Product t-norm)
(ii) $\breve{\mho}^\lambda M(g,h) = \min(g,h)$ (Minimum t-norm)
(iii) $\breve{\mho}^\lambda L(g,h) = \max(g+h-1, 0)$ (Lukasiewicz t-norm)
(iv) $\breve{\mho}^\lambda D(g,h) = \begin{cases} f, & \text{if } h = 1 \\ h, & \text{if } g = 1 \\ 0, & \text{otherwise} \end{cases}$ for all $g, h \in [0,1]$
(Drastic t-norm)

Example 3. Some famous t-conorms are given as follows:

(i) $\overline{\mathfrak{I}}P(g,h) = g + h - g.\, h$ (Probabilistic sum)
(ii) $\overline{\mathfrak{I}}M(g,h) = \max(g,h)$ (Maximum t-conorm)
(iii) $\overline{\mathfrak{I}}L(g,h) = \min(g+h, 1)$ (Lukasiewicz t-conorm)
(iv) $\overline{\mathfrak{I}}D(g,h) = \begin{cases} g, & \text{if } h = 0 \\ h, & \text{if } g = 0 \\ 1, & \text{otherwise} \end{cases}$ for all $g, h \in [0,1]$
(Drastic t-conorm)

Definition 7 (see [35]). This class of t-norm is originally proposed by Aczel–Alsina in mid-1980s under the condition of functional equations.

The category $(\breve{\mho}^{\lambda^\beth}_A)_{\beth \in [0,\infty]}$ of Aczel–Alsina t-norms is stated by the following equation:

$$\breve{\mho}^{\lambda^\beth}_A(g,h) = \begin{cases} \breve{\mho}^\lambda D(g,h), & \text{if } \beth = 0, \\ \min(g,h), & \text{if } \beth = \infty, \\ e^{-\left((-\log g)^\beth + (-\log h)^\beth\right)^{1/\beth}}, & \text{otherwise.} \end{cases} \quad (5)$$

The category $(\overline{\mathfrak{I}}^\beth_A)_{\beth \in [0,\infty]}$ of Aczel–Alsina t-conorms is stated by the following equation:

$$\overline{\mathfrak{I}}^\beth_A(g,h) = \begin{cases} \overline{\mathfrak{I}}D(g,h), & \text{if } \beth = 0, \\ \max(g,h), & \text{if } \beth = \infty, \\ 1 - e^{-\left((-\log(1-g))^\beth + (-\log(1-h))^\beth\right)^{1/\beth}}, & \text{otherwise.} \end{cases} \quad (6)$$

Limiting Cases: $\breve{\mho}^{\lambda^0}_A = \breve{\mho}^\lambda{}_D$, $\breve{\mho}^{\lambda^1}_A = \breve{\mho}^\lambda{}_P$, $\breve{\mho}^{\lambda^\infty}_A = \min$, $\overline{\mathfrak{I}}^0_A = \overline{\mathfrak{I}}_D$, $\overline{\mathfrak{I}}^1_A = \overline{\mathfrak{I}}_P$, and $\overline{\mathfrak{I}}^\infty_A = \max$.

For every $\beth \in [0,\infty]$ the t-norm $\breve{\mho}^{\lambda^\beth}_A$ and t-conorm $\overline{\mathfrak{I}}^\beth_A$ are dual to each other. The class of Aczel–Alsina t-norms is strictly increasing and the class of Aczel–Alsina t-conorms is strictly decreasing.

3. Aczel–Alsina Operations for SFNs

In this section, we will introduce the Aczel–Alsina operations for SFNs and look at some of their basic properties.

Definition 8. Let $\mathfrak{R}^\gamma = (\kappa_{\mathfrak{R}^\gamma}, \upsilon_{\mathfrak{R}^\gamma}, \tau_{\mathfrak{R}^\gamma})$, $\mathfrak{R}_1^\gamma = (\kappa_{\mathfrak{R}_1^\gamma}, \upsilon_{\mathfrak{R}_1^\gamma}, \tau_{\mathfrak{R}_1^\gamma})$, and $\mathfrak{R}_2^\gamma = (\kappa_{\mathfrak{R}_2^\gamma}, \upsilon_{\mathfrak{R}_2^\gamma}, \tau_{\mathfrak{R}_2^\gamma})$ be three SFNs, $\aleph \ge 1$ and $\beth > 0$. Then, the Aczel–Alsina t-norm and t-conorm operations of SFNs are defined as follows:

(1) $$\mathfrak{R}_1^\gamma \oplus \mathfrak{R}_2^\gamma = \left\langle \sqrt{1 - e^{-\left((-\log(1-\kappa^2_{\mathfrak{R}_1^\gamma}))^\aleph + (-\log(1-\kappa^2_{\mathfrak{R}_2^\gamma}))^\aleph\right)^{1/\aleph}}}, \sqrt{e^{-\left((-\log\upsilon^2_{\mathfrak{R}_1^\gamma})^\aleph + (-\log\upsilon^2_{\mathfrak{R}_2^\gamma})^\aleph\right)^{1/\aleph}}}, \sqrt{e^{-\left((-\log\tau^2_{\mathfrak{R}_1^\gamma})^\aleph + (-\log\tau^2_{\mathfrak{R}_2^\gamma})^\aleph\right)^{1/\aleph}}} \right\rangle$$

(2) $$\mathfrak{R}_1^\gamma \otimes \mathfrak{R}_2^\gamma = \left\langle \sqrt{e^{-\left((-\log\kappa^2_{\mathfrak{R}_1^\gamma})^\aleph + (-\log\kappa^2_{\mathfrak{R}_2^\gamma})^\aleph\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left((-\log(1-\upsilon^2_{\mathfrak{R}_1^\gamma}))^\aleph + (-\log(1-\upsilon^2_{\mathfrak{R}_2^\gamma}))^\aleph\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left((-\log(1-\tau^2_{\mathfrak{R}_1^\gamma}))^\aleph + (-\log(1-\tau^2_{\mathfrak{R}_2^\gamma}))^\aleph\right)^{1/\aleph}}} \right\rangle$$

(3) $$\beth\mathfrak{R}^\gamma = \left\langle \sqrt{1 - e^{-\left(\beth(-\log(1-\kappa^2_{\mathfrak{R}^\gamma}))^\aleph\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth(-\log\upsilon^2_{\mathfrak{R}^\gamma})^\aleph\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth(-\log\tau^2_{\mathfrak{R}^\gamma})^\aleph\right)^{1/\aleph}}} \right\rangle$$

(4) $$\mathfrak{R}^{\gamma^\beth} = \left\langle \sqrt{e^{-\left(\beth(-\log\kappa^2_{\mathfrak{R}^\gamma})^\aleph\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left(\beth(-\log(1-\upsilon^2_{\mathfrak{R}^\gamma}))^\aleph\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left(\beth(-\log(1-\tau^2_{\mathfrak{R}^\gamma}))^\aleph\right)^{1/\aleph}}} \right\rangle$$

Theorem 1. *Let* $\mathfrak{R}^\gamma = (\kappa_{\mathfrak{R}^\gamma}, \upsilon_{\mathfrak{R}^\gamma}, \tau_{\mathfrak{R}^\gamma})$, $\mathfrak{R}_1^\gamma = (\kappa_{\mathfrak{R}_1^\gamma}, \upsilon_{\mathfrak{R}_1^\gamma}, \tau_{\mathfrak{R}_1^\gamma})$, *and* $\mathfrak{R}_2^\gamma = (\kappa_{\mathfrak{R}_2^\gamma}, \upsilon_{\mathfrak{R}_2^\gamma}, \tau_{\mathfrak{R}_2^\gamma})$ *be three SFNs, then we have*

(i) $\mathfrak{R}_1^{\gamma} \oplus \mathfrak{R}_2^{\gamma} = \mathfrak{R}_2^{\gamma} \oplus \mathfrak{R}_1^{\gamma}$

(ii) $\mathfrak{R}_1^{\gamma} \otimes \mathfrak{R}_2^{\gamma} = \mathfrak{R}_2^{\gamma} \otimes \mathfrak{R}_1^{\gamma}$

(iii) $\beth(\mathfrak{R}_1^{\gamma} \oplus \mathfrak{R}_2^{\gamma}) = \beth\mathfrak{R}_1^{\gamma} \oplus \beth\mathfrak{R}_2^{\gamma}, \beth > 0$

(iv) $(\beth_1 + \beth_2)\mathfrak{R}^{\gamma} = \beth_1\mathfrak{R}^{\gamma} \oplus \beth_2\mathfrak{R}^{\gamma}, \beth_1, \beth_2 > 0$

(v) $(\mathfrak{R}_1^{\gamma} \otimes \mathfrak{R}_2^{\gamma})^{\beth} = \mathfrak{R}_1^{\gamma^{\beth}} \otimes \mathfrak{R}_2^{\gamma^{\beth}}, \beth > 0$

(vi) $\mathfrak{R}^{\gamma^{\beth_1}} \otimes \mathfrak{R}^{\gamma^{\beth_2}} = \mathfrak{R}^{\gamma^{(\beth_1+\beth_2)}}, \beth_1, \beth_2 > 0$

Proof. For the three SFNs $\mathfrak{R}^{\gamma}$, $\mathfrak{R}_1^{\gamma}$, and $\mathfrak{R}_2^{\gamma}$, and $\beth, \beth_1, \beth_2 > 0$, we can get the following equations:

(i)

$$\begin{aligned}
\mathfrak{R}_1^{\gamma} \oplus \mathfrak{R}_2^{\gamma} &= \left\langle \sqrt{1 - e^{-\left(\left(-\log\left(1-\kappa^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph} + \left(-\log\left(1-\kappa^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\left(-\log \upsilon^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph} + \left(-\log \upsilon^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\left(-\log \tau^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph} + \left(-\log \tau^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&= \left\langle \sqrt{1 - e^{-\left(\left(-\log\left(1-\kappa^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph} + \left(-\log\left(1-\kappa^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\left(-\log \upsilon^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph} + \left(-\log \upsilon^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\left(-\log \tau^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph} + \left(-\log \tau^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&= \mathfrak{R}_2^{\gamma} \oplus \mathfrak{R}_1^{\gamma}
\end{aligned} \tag{7}$$

(ii)

$$\begin{aligned}
\mathfrak{R}_1^{\gamma} \otimes \mathfrak{R}_2^{\gamma} &= \left\langle \sqrt{e^{-\left(\left(-\log \kappa^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph} + \left(-\log \kappa^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left(\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph} + \left(-\log\left(1-\upsilon^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left(\left(-\log\left(1-\tau^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph} + \left(-\log\left(1-\tau^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&= \left\langle \sqrt{e^{-\left(\left(-\log \kappa^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph} + \left(-\log \kappa^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left(\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph} + \left(-\log\left(1-\upsilon^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left(\left(-\log\left(1-\tau^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph} + \left(-\log\left(1-\tau^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&= \mathfrak{R}_2^{\gamma} \otimes \mathfrak{R}_1^{\gamma}
\end{aligned} \tag{8}$$

(iii) Using this, we get

$$\begin{aligned}
\beth(\mathfrak{R}_1^{\gamma} \oplus \mathfrak{R}_2^{\gamma}) &= \beth\left\langle \sqrt{1 - e^{-\left(\left(-\log\left(1-\kappa^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph} + \left(-\log\left(1-\kappa^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\left(-\log \upsilon^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph} + \left(-\log \upsilon^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\left(-\log \tau^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph} + \left(-\log \tau^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&= \left\langle \sqrt{1 - e^{-\left(\beth\left(-\log\left(1-\kappa^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph} + \left(-\log\left(1-\kappa^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth\left(-\log \upsilon^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph} + \left(-\log \upsilon^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth\left(-\log \tau^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph} + \left(-\log \tau^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&= \left\langle \sqrt{1 - e^{-\left(\beth\left(-\log\left(1-\kappa^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth\left(-\log \upsilon^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth\left(-\log \tau^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&\oplus \left\langle \sqrt{1 - e^{-\left(\beth\left(-\log\left(1-\kappa^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth\left(-\log \upsilon^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth\left(-\log \tau^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&= \beth\mathfrak{R}_1^{\gamma} \oplus \beth\mathfrak{R}_2^{\gamma}
\end{aligned} \tag{9}$$

(iv)

$$
\begin{aligned}
\beth_1 \mathfrak{R}^{\gamma} \oplus \beth_2 \mathfrak{R}^{\gamma} &= \left\langle \sqrt{1 - e^{-\left(\beth_1\left(-\log\left(1-\kappa^2_{\mathfrak{R}^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_1\left(-\log \upsilon^2_{\mathfrak{R}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_1\left(-\log \tau^2_{\mathfrak{R}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&\oplus \left\langle \sqrt{1 - e^{-\left(\beth_2\left(-\log\left(1-\kappa^2_{\mathfrak{R}^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_2\left(-\log \upsilon^2_{\mathfrak{R}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_2\left(-\log \tau^2_{\mathfrak{R}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&= \left\langle \sqrt{1 - e^{-\left(\left(\beth_1+\beth_2\left(-\log\left(1-\kappa^2_{\mathfrak{R}^{\gamma}}\right)\right)^{\aleph}\right)\right)^{1/\aleph}}}, \sqrt{e^{-\left(\left(\beth_1+\beth_2\right)\left(-\log \upsilon^2_{\mathfrak{R}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\left(\beth_1+\beth_2\right)\left(-\log \tau^2_{\mathfrak{R}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&= \left(\beth_1 + \beth_2\right)\mathfrak{R}^{\gamma}
\end{aligned}
\tag{10}
$$

(v)

$$
\begin{aligned}
\left(\mathfrak{R}_1^{\gamma} \otimes \mathfrak{R}_2^{\gamma}\right)^{\beth} &= \left\langle \sqrt{e^{-\left(\left(-\log\kappa^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph}+\left(-\log\kappa^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left(\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}+\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \right. \\
&\left. \sqrt{1 - e^{-\left(\left(-\log\left(1-\tau^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}+\left(-\log\left(1-\tau^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}\,\beth} \right\rangle, \\
&= \left\langle \sqrt{e^{-\left(\beth\left(-\log\kappa^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph}+\left(-\log\kappa^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left(\beth\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}+\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \right. \\
&\left. \sqrt{1 - e^{-\left(\beth\left(-\log\left(1-\tau^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}+\left(-\log\left(1-\tau^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&= \left\langle \sqrt{e^{-\left(\beth\left(-\log\kappa^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left(\beth\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left(\beth\left(-\log\left(1-\tau^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&\otimes \left\langle \sqrt{e^{-\left(\beth\left(-\log\kappa^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left(\beth\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1 - e^{-\left(\beth\left(-\log\left(1-\tau^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&= \mathfrak{R}_1^{\gamma^{\beth}} \otimes \mathfrak{R}_2^{\gamma^{\beth}}
\end{aligned}
\tag{11}
$$

(vi)

$$
\begin{aligned}
\mathfrak{R}^{\gamma^{\beth_1}} \otimes \mathfrak{R}^{\gamma^{\beth_2}} &= \left\langle \sqrt{e^{-\left(\beth_1\left(-\log\kappa^2_{\mathfrak{R}^{\gamma}_1}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\beth_1\left(-\log\left(1-\upsilon^2_{\mathfrak{R}^{\gamma}_1}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\beth_1\left(-\log\left(1-\tau^2_{\mathfrak{R}^{\gamma}_1}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&\otimes \left\langle \sqrt{e^{-\left(\beth_2\left(-\log\kappa^2_{\mathfrak{R}^{\gamma}_2}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\beth_2\left(-\log\left(1-\upsilon^2_{\mathfrak{R}^{\gamma}_2}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\beth_2\left(-\log\left(1-\tau^2_{\mathfrak{R}^{\gamma}_2}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&= \left\langle \sqrt{e^{-\left(\left(\beth_1+\beth_2\right)\left(-\log\kappa^2_{\mathfrak{R}^{\gamma}_2}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\left(\beth_1+\beth_2\right)\left(-\log\left(1-\upsilon^2_{\mathfrak{R}^{\gamma}_2}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\left(\beth_1+\beth_2\right)\left(-\log\left(1-\tau^2_{\mathfrak{R}^{\gamma}_2}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\
&= \mathfrak{R}^{\gamma^{\left(\beth_1+\beth_2\right)}}
\end{aligned}
\tag{12}
$$

□

4. Spherical Fuzzy Aczel–Alsina Aggregation Operators

In this section, we present a few SF aggregation operators by means of the Aczel–Alsina operations.

4.1. Spherical Fuzzy Aczel–Alsina Averaging AOs

Definition 9. Let $\mathfrak{R}^{\gamma}_{\phi} = (\kappa_{\mathfrak{R}^{\gamma}_{\phi}}, \upsilon_{\mathfrak{R}^{\gamma}_{\phi}}, \tau_{\mathfrak{R}^{\gamma}_{\phi}})$, $(\phi = 1, 2, \ldots, \beth)$ be an accumulation of SFNs and $\xi^{\zeta} = (\xi^{\zeta}_1, \xi^{\zeta}_1, \ldots, \xi^{\zeta}_{\beth})^T$ be the weight vector (WV) of $\mathfrak{R}^{\gamma}_{\phi}$, with $\xi^{\zeta}_{\phi} > 0$ and $\sum_{\phi=1}^{\beth} \xi^{\zeta}_{\phi} = 1$. Then, "spherical fuzzy Aczel–Alsina weighted average (SFAAWA) operator" is a mapping SFAAWA: $(L^*)^{\beth} \longrightarrow L^*$, where

$$
\text{SFAAWA}\left(\mathfrak{R}^{\gamma}_1, \mathfrak{R}^{\gamma}_2, \ldots, \mathfrak{R}^{\gamma}_{\beth}\right) = \left(\xi^{\zeta}_1\mathfrak{R}^{\gamma}_1 \oplus \xi^{\zeta}_2\mathfrak{R}^{\gamma}_2 \oplus \ldots, \oplus \xi^{\zeta}_{\beth}\mathfrak{R}^{\gamma}_{\beth}\right). \tag{13}
$$

The following theorem is obtained using Aczel–Alsina operations on SFNs.

Theorem 2. *Let $\mathfrak{R}^{\gamma}_{\phi} = (\kappa_{\mathfrak{R}^{\gamma}_{\phi}}, \upsilon_{\mathfrak{R}^{\gamma}_{\phi}}, \tau_{\mathfrak{R}^{\gamma}_{\phi}})$ be an accumulation of SFNs, then aggregated value of them utilizing the SFAAWA operation is also an SFNs, and*

$$
\begin{aligned}
\text{SFAAWA}\left(\mathfrak{R}^{\gamma}_1, \mathfrak{R}^{\gamma}_2, \ldots, \mathfrak{R}^{\gamma}_{\beth}\right) &= \overset{\beth}{\underset{\phi=1}{\oplus}} \left(\xi^{\zeta}_{\phi}\mathfrak{R}^{\gamma}_{\phi}\right) \\
&= \left\langle \sqrt{1-e^{-\left(\sum_{\phi=1}^{\beth}\xi^{\zeta}_{\phi}\left(-\log\left(1-\kappa_{\mathfrak{R}^{\gamma}_{\phi}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{\beth}\xi^{\zeta}_{\phi}\left(-\log\upsilon^2_{\mathfrak{R}^{\gamma}_{\phi}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{\beth}\xi^{\zeta}_{\phi}\left(-\log\tau_{\mathfrak{R}^{\gamma}_{\phi}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle,
\end{aligned}
\tag{14}
$$

where $\xi^{\zeta} = (\xi^{\zeta}_1, \xi^{\zeta}_2, \ldots, \xi^{\zeta}_{\beth})$ is the WV of $\mathfrak{R}^{\gamma}_{\phi}$ s.t $\xi^{\zeta}_{\phi} > 0$ and $\sum_{\phi=1}^{\beth} \xi^{\zeta}_{\phi} = 1$.

Proof. We can derive Theorem 2 in the following way using the mathematical induction technique.

For $\beth = 2$, depend on Aczel–Alsina operations of SFNs, we obtain the following equation:

$$
\begin{aligned}
\xi^{\zeta}_1\mathfrak{R}^{\gamma}_1 &= \left\langle \sqrt{1-e^{-\left(\beth_1\left(-\log\left(1-\kappa^2_{\mathfrak{R}^{\gamma}_1}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_1\left(-\log\upsilon^2_{\mathfrak{R}^{\gamma}_1}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_1\left(-\log\tau^2_{\mathfrak{R}^{\gamma}_1}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle \text{ and} \\
\xi^{\zeta}_2\mathfrak{R}^{\gamma}_2 &= \left\langle \sqrt{1-e^{-\left(\beth_2\left(-\log\left(1-\kappa^2_{\mathfrak{R}^{\gamma}_2}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_2\left(-\log\upsilon^2_{\mathfrak{R}^{\gamma}_2}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_2\left(-\log\tau^2_{\mathfrak{R}^{\gamma}_2}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle.
\end{aligned}
\tag{15}
$$

Based on Aczel–Alsina operations of SFNs, we obtain the following equation:

$$
\begin{aligned}
\mathrm{SFAAWA}\left(\mathfrak{R}_1^{\gamma},\mathfrak{R}_2^{\gamma}\right) &= \left\langle \sqrt{1-e^{-\left(\beth_1\left(-\log\left(1-\kappa^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_1\left(-\log\upsilon^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_1\left(-\log\tau^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle,\\
&\oplus \left\langle \sqrt{1-e^{-\left(\beth_2\left(-\log\left(1-\kappa^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_2\left(-\log\upsilon^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_2\left(-\log\tau^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle,\\
&= \sqrt{1-e^{\left(-\beth_1\left(-\log\left(1-\kappa^2_{\mathfrak{R}_1^{\gamma}}\right)\right)^{\aleph}+\beth_2\left(-\log\left(1-\kappa^2_{\mathfrak{R}_2^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}},\\
&\cdot \sqrt{e^{-\left(\beth_1\left(-\log\upsilon^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph}+\beth_2\left(-\log\upsilon^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_1\left(-\log\tau^2_{\mathfrak{R}_1^{\gamma}}\right)^{\aleph}+\beth_2\left(-\log\tau^2_{\mathfrak{R}_2^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \text{ and}\\
&= \left\langle \sqrt{1-e^{-\left(\sum_{\phi=1}^{2}\xi_{\phi}^{\zeta}\left(-\log\left(1-\kappa^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{2}\xi_{\phi}^{\zeta}\left(-\log\upsilon^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{2}\xi_{\phi}^{\zeta}\left(-\log\tau^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle.
\end{aligned}
\tag{16}
$$

Thus, it is true for $\beth = 2$.

Consider equation (14) is true for $\beth = k$, then we have the following equation:

$$
\begin{aligned}
&\mathrm{SFAAWA}\left(\mathfrak{R}_1^{\gamma},\mathfrak{R}_2^{\gamma},\ldots,\mathfrak{R}_k^{\gamma}\right) =\\
&\left\langle \sqrt{1-e^{-\left(\sum_{\phi=1}^{k}\xi_{\phi}^{\zeta}\left(-\log\left(1-\kappa^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{k}\xi_{\phi}^{\zeta}\left(-\log\upsilon^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{k}\xi_{\phi}^{\zeta}\left(-\log\tau^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle,
\end{aligned}
\tag{17}
$$

we will prove that equation (14) holds for $\beth = k + 1$.

$$
\begin{aligned}
\mathrm{SFAAWA}\left(\mathfrak{R}_1^{\gamma},\mathfrak{R}_2^{\gamma},\ldots,\mathfrak{R}_k^{\gamma},\mathfrak{R}_{k+1}^{\gamma}\right) &= \overset{k}{\oplus}\left(\xi_{\phi}^{\zeta}\mathfrak{R}_{\phi}^{\gamma}\right)\oplus\left(\xi_{k+1}^{\zeta}\mathfrak{R}_{k+1}^{\gamma}\right),\\
&= \left\langle \sqrt{1-e^{-\left(\sum_{\phi=1}^{k}\xi_{\phi}^{\zeta}\left(-\log\left(1-\kappa^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{k}\xi_{\phi}^{\zeta}\left(-\log\upsilon^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{k}\xi_{\phi}^{\zeta}\left(-\log\tau^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle,\\
&\oplus \left\langle \sqrt{1-e^{-\left(\beth_{k+1}\left(-\log\left(1-\kappa^2_{\mathfrak{R}_{k+1}^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_{k+1}\left(-\log\upsilon^2_{\mathfrak{R}_{k+1}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\beth_{k+1}\left(-\log\tau^2_{\mathfrak{R}_{k+1}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle,\\
&= \left\langle \sqrt{1-e^{-\left(\sum_{\phi=1}^{k+1}\xi_{\phi}^{\zeta}\left(-\log\left(1-\kappa^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{k+1}\xi_{\phi}^{\zeta}\left(-\log\upsilon^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{k+1}\xi_{\phi}^{\zeta}\left(-\log\tau^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle.
\end{aligned}
\tag{18}
$$

As a result, we can conclude that equation (4.2) stands true for any $\beth$.

By applying the SFAAWA operator, we can illustrate the following features efficiently. □

Theorem 3. *If all* $\mathfrak{R}_{\phi}^{\gamma} = (\kappa_{\mathfrak{R}_{\phi}^{\gamma}}, \upsilon_{\mathfrak{R}_{\phi}^{\gamma}}, \tau_{\mathfrak{R}_{\phi}^{\gamma}})$ *are equal, that is,* $\mathfrak{R}_{\phi}^{\gamma} = \mathfrak{R}^{\gamma} \forall \phi$, *then*

$$\mathrm{SFAAWA}(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\beth}^{\gamma}) = \mathfrak{R}^{\gamma}. \tag{19}$$

Proof. Given that $\mathfrak{R}_{\phi}^{\gamma} = (\kappa_{\mathfrak{R}_{\phi}^{\gamma}}, \upsilon_{\mathfrak{R}_{\phi}^{\gamma}}, \tau_{\mathfrak{R}_{\phi}^{\gamma}})$, by equation (4.2) we get the following equation:

$$\begin{aligned}
\mathrm{SFAAWA}(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\beth}^{\gamma}) &= \left\langle \sqrt{1 - e^{-\left(\sum_{\phi=1}^{\beth} \xi_{\phi}^{\zeta}\left(-\log\left(1-\kappa_{\mathfrak{R}_{\phi}^{\gamma}}^2\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{\beth} \xi_{\phi}^{\zeta}\left(-\log \upsilon_{\mathfrak{R}_{\phi}^{\gamma}}^2\right)\right)^{\aleph \, 1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{\beth} \xi_{\phi}^{\zeta}\left(-\log \tau_{\mathfrak{R}_{\phi}^{\gamma}}^2\right)\right)^{\aleph \, 1/\aleph}}} \right\rangle, \\
&= \left\langle \sqrt{1 - e^{-\left(\left(-\log\left(1-\kappa_{\mathfrak{R}^{\gamma}}^2\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\left(-\log \upsilon_{\mathfrak{R}^{\gamma}}^2\right)\right)^{\aleph \, 1/\aleph}}}, \sqrt{e^{-\left(\left(-\log \tau_{\mathfrak{R}^{\gamma}}^2\right)\right)^{\aleph \, 1/\aleph}}} \right\rangle, \\
&= \left\langle \sqrt{1 - e^{\log\left(1-\kappa_{\mathfrak{R}^{\gamma}}^2\right)}}, \sqrt{e^{\log \upsilon_{\mathfrak{R}^{\gamma}}^2}}, \sqrt{e^{\log \tau_{\mathfrak{R}^{\gamma}}^2}} \right\rangle, \\
&= \left(\sqrt{\kappa_{\mathfrak{R}^{\gamma}}^2}, \sqrt{\upsilon_{\mathfrak{R}^{\gamma}}^2}, \sqrt{\tau_{\mathfrak{R}^{\gamma}}^2} \right) = \mathfrak{R}^{\gamma}.
\end{aligned} \tag{20}$$

□

Theorem 4. *Let* $\mathfrak{R}^{\gamma} = (\kappa_{\mathfrak{R}_{\phi}^{\gamma}}, \upsilon_{\mathfrak{R}_{\phi}^{\gamma}}, \tau_{\mathfrak{R}_{\phi}^{\gamma}})$ *be an accumulation of SFNs. Let* $\mathfrak{R}^{\gamma^-} = min(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_1^{\gamma}, \ldots, \mathfrak{R}_{\beth}^{\gamma})$ *and* $\mathfrak{R}^{\gamma^+} = max(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\beth}^{\gamma})$. *Then,*

$$\mathfrak{R}^{\gamma^-} \leq \mathrm{SFAAWA}(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\beth}^{\gamma}) \leq \mathfrak{R}^{\gamma^+}. \tag{21}$$

Proof. Let $\mathfrak{R}_{\phi}^{\gamma} = (\kappa_{\mathfrak{R}_{\phi}^{\gamma}}, \upsilon_{\mathfrak{R}_{\phi}^{\gamma}})$ be an accumulation of SFNs. Let $\mathfrak{R}^{\gamma^-} = \min(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\beth}^{\gamma}) = (\kappa_{\mathfrak{R}^{\gamma}}^-, \upsilon_{\mathfrak{R}^{\gamma}}^-, \tau_{\mathfrak{R}^{\gamma}}^-)$ and $\mathfrak{R}^{\gamma^+} = \max(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\beth}^{\gamma}) = (\kappa_{\mathfrak{R}^{\gamma}}^+, \upsilon_{\mathfrak{R}^{\gamma}}^+, \tau_{\mathfrak{R}^{\gamma}}^+)$. We have, $\kappa_{\mathfrak{R}^{\gamma}}^- = \min_{\phi}\left\{\kappa_{\mathfrak{R}_{\phi}^{\gamma}}\right\}$, $\upsilon_{\mathfrak{R}^{\gamma}}^- = \max_{\phi}\left\{\upsilon_{\mathfrak{R}_{\phi}^{\gamma}}\right\}$,

$$\tau_{\mathfrak{R}^{\gamma}}^- = \max_{\phi}\left\{\tau_{\mathfrak{R}_{\phi}^{\gamma}}\right\}, \kappa_{\mathfrak{R}^{\gamma}}^+ = \max_{\phi}\left\{\kappa_{\mathfrak{R}_{\phi}^{\gamma}}\right\},$$

$\upsilon_{\mathfrak{R}^{\gamma}}^+ = \min_{\phi}\left\{\upsilon_{\mathfrak{R}_{\phi}^{\gamma}}\right\}$, and $\tau_{\mathfrak{R}^{\gamma}}^+ = \min_{\phi}\left\{\tau_{\mathfrak{R}_{\phi}^{\gamma}}\right\}$ Hence, here we have the subsequent inequalities,

$$\begin{aligned}
\sqrt{1 - e^{-\left(\sum_{\phi=1}^{\beth} \xi_{\phi}^{\zeta}\left(-\log\left(1-\kappa_{\mathfrak{R}^{\gamma^-}}^2\right)\right)^{\aleph}\right)^{1/\aleph}}} &\leq \sqrt{1 - e^{-\left(\sum_{\phi=1}^{\beth} \xi_{\phi}^{\zeta}\left(-\log\left(1-\kappa_{\mathfrak{R}_{\phi}^{\gamma}}^2\right)\right)^{\aleph}\right)^{1/\aleph}}}, \\
&\leq \sqrt{1 - e^{-\left(\sum_{\phi=1}^{\beth} \xi_{\phi}^{\zeta}\left(-\log\left(1-\kappa_{\mathfrak{R}^{\gamma^+}}^2\right)\right)^{\aleph}\right)^{1/\aleph}}}, \\
\sqrt{e^{-\left(\sum_{\phi=1}^{\beth} \xi_{\phi}^{\zeta}\left(-\log \upsilon_{\mathfrak{R}^{\gamma^+}}^2\right)\right)^{\aleph \, 1/\aleph}}} &\leq \sqrt{e^{-\left(\sum_{\phi=1}^{\beth} \xi_{\phi}^{\zeta}\left(-\log \upsilon_{\mathfrak{R}_{\phi}^{\gamma}}^2\right)\right)^{\aleph \, 1/\aleph}}}, \\
&\leq \sqrt{e^{-\left(\sum_{\phi=1}^{\beth} \xi_{\phi}^{\zeta}\left(-\log \upsilon_{\mathfrak{R}^{\gamma^-}}^2\right)^{\aleph}\right)^{1/\aleph}}},
\end{aligned} \tag{22}$$

$$\begin{aligned}
\sqrt{e^{-\left(\sum_{\phi=1}^{\beth} \xi_{\phi}^{\zeta}\left(-\log \tau_{\mathfrak{R}^{\gamma^+}}^2\right)^{\aleph}\right)^{1/\aleph}}} &\leq \sqrt{e^{-\left(\sum_{\phi=1}^{\beth} \xi_{\phi}^{\zeta}\left(-\log \tau_{\mathfrak{R}_{\phi}^{\gamma}}^2\right)\right)^{\aleph \, 1/\aleph}}}, \\
&\leq \sqrt{e^{-\left(\sum_{\phi=1}^{\beth} \xi_{\phi}^{\zeta}\left(-\log \tau_{\mathfrak{R}^{\gamma^-}}^2\right)^{\aleph}\right)^{1/\aleph}}}.
\end{aligned} \tag{23}$$

Therefore, $\mathfrak{R}^{\gamma^-} \leq \mathrm{SFAAWA}(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\beth}^{\gamma}) \leq \mathfrak{R}^{\gamma^+}$ □ □

Theorem 5. *Let* $\mathfrak{R}_{\phi}^{\gamma}$ *and* $\mathfrak{R}_{\phi}^{\gamma'}$ *be two sets of SFNs, if* $\mathfrak{R}_{\phi}^{\gamma} \leq \mathfrak{R}_{\phi}^{\gamma'} \forall \phi$, *then*

$$\text{SFAAWA}\left(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\sqsupset}^{\gamma}\right) \leq \text{SFAAWA}\left(\mathfrak{R}_1^{\gamma'}, \mathfrak{R}_2^{\gamma'}, \ldots, \mathfrak{R}_{\sqsupset}^{\gamma'}\right), \tag{24}$$

Now, we present "spherical fuzzy Aczel–Alsina ordered weighted averaging (SFAAOWA) operator."

Definition 10. Let $\mathfrak{R}_\phi^\gamma = (\kappa_{\mathfrak{R}_\phi^\gamma}, \upsilon_{\mathfrak{R}_\phi^\gamma}, \tau_{\mathfrak{R}_\phi^\gamma})$ be an accumulation of SFNs. SFAAOWA operator is a mapping SFAAOWA: $(L^*)^{\sqsupset} \longrightarrow L^*$ with the corresponding WV $\xi^\zeta = (\xi_1^\zeta, \xi_2^\zeta, \ldots, \xi_{\sqsupset}^\zeta)^T$ such that $\xi_\phi^\zeta > 0$, and $\sum_{\phi=1}^{\sqsupset} \xi_\phi^\zeta = 1$, as follows:

$$\text{SFAAOWA}\left(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\sqsupset}^{\gamma}\right) = \overset{\sqsupset}{\underset{\phi=1}{\oplus}}\left(\xi_\phi^\zeta \mathfrak{R}_{\Upsilon(\phi)}^\gamma\right) = \xi_1^\zeta \mathfrak{R}_{\Upsilon(1)}^\gamma \oplus \xi_2^\zeta \mathfrak{R}_{\Upsilon(2)}^\gamma \oplus, \ldots, \oplus \xi_{\sqsupset}^\zeta \mathfrak{R}_{\Upsilon(\sqsupset)}^\gamma, \tag{25}$$

where $(\Upsilon(1), \Upsilon(2), \ldots, \Upsilon(\sqsupset))$ are the permutation of $(\phi = 1, 2, \ldots, \sqsupset)$, including $\mathfrak{R}_{\Upsilon(\phi-1)}^\gamma \geq \mathfrak{R}_{\Upsilon(\phi)}^\gamma \forall \phi = 1, 2, \ldots, \sqsupset$.

Thus, the following theorem is obtained using Aczel–Alsina operations on SFNs.

Theorem 6. *Let $\mathfrak{R}_\phi^\gamma = (\kappa_{\mathfrak{R}_\phi^\gamma}, \upsilon_{\mathfrak{R}_\phi^\gamma}, \tau_{\mathfrak{R}_\phi^\gamma})$ be an accumulation of SFNs. SFAAOWA operator is a mapping SFAAOWA: $(L^*)^{\sqsupset} \longrightarrow L^*$ with the corresponding vector $\xi^\zeta = (\xi_1^\zeta, \xi_2^\zeta, \ldots, \xi_{\sqsupset}^\zeta)^T$ such that $\xi_\phi^\zeta > 0$, and $\sum_{\phi=1}^{\sqsupset} \xi_\phi^\zeta = 1$. Then,*

$$\begin{aligned}\text{SFAAOWA}\left(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\sqsupset}^{\gamma}\right) &= \overset{\sqsupset}{\underset{\phi=1}{\oplus}}\left(\xi_\phi^\zeta \mathfrak{R}_\phi^\gamma\right) \\ &= \left\langle \sqrt{1 - e^{-\left(\sum_{\phi=1}^{\sqsupset} \xi_\phi^\zeta \left(-\log\left(1-\kappa^2_{\mathfrak{R}^\gamma_{\Upsilon(\phi)}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{\sqsupset} \xi_\phi^\zeta \left(-\log \upsilon^2_{\mathfrak{R}^\gamma_{\Upsilon(\phi)}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{\sqsupset} \xi_\phi^\zeta \left(-\log \tau^2_{\mathfrak{R}^\gamma_{\Upsilon(\phi)}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle,\end{aligned} \tag{26}$$

where $(\Upsilon(1), \Upsilon(2), \ldots, \Upsilon(\sqsupset))$ are the permutation of $(\phi = 1, 2, \ldots, \sqsupset)$, including $\mathfrak{R}_{\Upsilon(\phi-1)}^\gamma \geq \mathfrak{R}_{\Upsilon(\phi)}^\gamma \forall \phi = 1, 2, \ldots, \sqsupset$.

Proof. Same as Theorem 2.

By applying the SFAAOWA operator, we can illustrate the following features efficiently. □

Theorem 7. *If all $\mathfrak{R}_\phi^\gamma = (\kappa_{\mathfrak{R}_\phi^\gamma}, \upsilon_{\mathfrak{R}_\phi^\gamma}, \tau_{\mathfrak{R}_\phi^\gamma})$ are equal, that is, $\mathfrak{R}_\phi^\gamma = \mathfrak{R}^\gamma \forall \phi$, then*

$$\text{SFAAOWA}\left(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\sqsupset}^{\gamma}\right) = \mathfrak{R}^\gamma \tag{27}$$

Proof. Same as Theorem 3. □

Theorem 8. *Let $\mathfrak{R}_\phi^\gamma = (\kappa_{\mathfrak{R}_\phi^\gamma}, \upsilon_{\mathfrak{R}_\phi^\gamma}, \tau_{\mathfrak{R}_\phi^\gamma})$ be an accumulation of SFNs. Let$\mathfrak{R}^{\gamma^-} = min(\mathfrak{R}_1^\gamma, \mathfrak{R}_2^\gamma, \ldots, \mathfrak{R}_{\sqsupset}^\gamma)$ and $\mathfrak{R}^{\gamma^+} = min(\mathfrak{R}_1^\gamma, \mathfrak{R}_2^\gamma, \ldots, \mathfrak{R}_{\sqsupset}^\gamma)$. Then,*

$$\mathfrak{R}^{\gamma^-} \leq \text{SFAAOWA}\left(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\sqsupset}^{\gamma}\right) \leq \mathfrak{R}^{\gamma^+}. \tag{28}$$

Proof. Same as Theorem 4. □

Theorem 9. *Let $\mathfrak{R}_\phi^\gamma$ and $\mathfrak{R}_\phi^{\gamma'}$ be two sets of SFNs, if $\mathfrak{R}_\phi^\gamma \leq \mathfrak{R}_\phi^{\gamma'} \forall \phi$, then*

$$\text{SFAAOWA}\left(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\sqsupset}^{\gamma}\right) \leq \text{SFAAOWA}\left(\mathfrak{R}_1^{\gamma'}, \mathfrak{R}_2^{\gamma'}, \ldots, \mathfrak{R}_{\sqsupset}^{\gamma'}\right). \tag{29}$$

It is self-evident that the SFAAWA operator weights only the SFNs, and that the SFAAOWA operator weights only the SFN's ordered locations. Following that, weights are used to indicate various elements of the SFAAWA and SFAAOWA operators. Nonetheless, both one and the other operators consider only one of these. To address this shortcoming, we will also demonstrate the "spherical fuzzy Aczel–Alsina hybrid averaging (SFAAHA) operator," which weights all of the given SFN and their appropriate ordered position.

Definition 11. Let $\mathfrak{R}_\phi^\gamma$ be an accumulation of SFNs. SFAAHA operator is a mapping SFAAHA: $(L^*)^{\sqsupset} \longrightarrow L^*$, s.t.

$$\text{SFAAOWA}\left(\mathfrak{R}_1^{\gamma}, \mathfrak{R}_2^{\gamma}, \ldots, \mathfrak{R}_{\sqsupset}^{\gamma}\right) = \overset{\sqsupset}{\underset{\phi=1}{\oplus}}\left(\xi_\phi^\zeta \mathfrak{R}_{\Upsilon(\phi)}^{\ddot{\gamma}}\right) = \xi_1^\zeta \mathfrak{R}_{\Upsilon(1)}^{\ddot{\gamma}} \oplus \xi_2^\zeta \mathfrak{R}_{\Upsilon(2)}^{\ddot{\gamma}} \oplus, \ldots, \oplus \xi_{\sqsupset}^\zeta \mathfrak{R}_{\Upsilon(\sqsupset)}^{\ddot{\gamma}}, \tag{30}$$

where $\xi^\zeta = (\xi_1^\zeta, \xi_2^\zeta, \ldots, \xi_{\sqsupset}^\zeta)^T$ is the weighting vector associated with the SFAAHA operator, with $\xi_\phi^\zeta \in [0, 1]$ and $\sum_{\phi=1}^{\sqsupset} \xi_\phi^\zeta = 1; \ddot{\mathfrak{R}}_\phi^\gamma = \sqsupset \delta_\phi \mathfrak{R}_\phi^\gamma, \phi = 1, 2, \ldots, \sqsupset$, $(\mathfrak{R}_{\Upsilon(1)}^{\ddot{\gamma}}, \mathfrak{R}_{\Upsilon(2)}^{\ddot{\gamma}}, \ldots, \mathfrak{R}_{\Upsilon(\sqsupset)}^{\ddot{\gamma}})$ is any permutation of a collection of the weighted SFNs $(\ddot{\mathfrak{R}}_1^\gamma, \ddot{\mathfrak{R}}_2^\gamma, \ldots \ddot{\mathfrak{R}}_{\sqsupset}^\gamma)$, s.t. $\mathfrak{R}_{Y(\phi-1)}^{\ddot{\gamma}} \geq \mathfrak{R}_{Y(\phi)}^{\ddot{\gamma}} \cdot \delta = (\delta_1, \delta_2, \ldots \delta_{\sqsupset})^T$ is the weight vector of $\mathfrak{R}^{\gamma}_{\phi}$, with $\delta_{\phi} \in [0,1]$ and $\sum_{\phi=1}^{\beth} \delta_{\phi} = 1$, and $\beth$ is the balancing coefficient, which plays a role of balance.

Theorem 10. *Let $\mathfrak{R}^{\gamma}_{\phi}$ be the collection of SFNs. Their aggregated value by SFAAHA operator is still an SFN, and*

$$\begin{aligned}\mathrm{SFAAHA}\left(\mathfrak{R}^{\gamma}_{1}, \mathfrak{R}^{\gamma}_{2}, \ldots, \mathfrak{R}^{\gamma}_{\beth}\right) &= \overset{\beth}{\underset{\phi=1}{\oplus}}\left(\xi^{\zeta}_{\phi}\mathfrak{R}^{\gamma}_{\phi}\right)\\ &= \left\langle \sqrt{1-e^{-\left(\sum_{\phi=1}^{\beth}\xi^{\zeta}_{\phi}\left(-\log\left(1-\kappa^{2}_{\mathfrak{R}^{\ddot{\gamma}}_{\Upsilon(\phi)}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{\beth}\xi^{\zeta}_{\phi}\left(-\log\upsilon^{2}_{\mathfrak{R}^{\ddot{\gamma}}_{\Upsilon(\phi)}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{e^{-\left(\sum_{\phi=1}^{\beth}\xi^{\zeta}_{\phi}\left(-\log\tau^{2}_{\mathfrak{R}^{\ddot{\gamma}}_{\Upsilon(\phi)}}\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle.\end{aligned} \tag{31}$$

Proof. Same as Theorem 2. □

Theorem 11. *The SFAAWA and SFAAOWA operators are special cases of the SFAAHA operator.*

Proof. (1) Let $\xi^{\zeta} = (1/\beth, 1/\beth, \ldots, 1/\beth)^{T}$. Then,

$$\begin{aligned}\mathrm{SFAAHA}\left(\mathfrak{R}^{\gamma}_{1}, \mathfrak{R}^{\gamma}_{2}, \ldots, \mathfrak{R}^{\gamma}_{\beth}\right) &= \xi^{\zeta}_{1}\mathfrak{R}^{\dot{\gamma}}_{\beth(1)} \oplus \xi^{\zeta}_{2}\mathfrak{R}^{\dot{\gamma}}_{\beth(2)} \oplus \cdots \oplus \xi^{\zeta}_{\beth}\mathfrak{R}^{\dot{\gamma}}_{\beth(\beth)},\\ &= \frac{1}{\beth}\left(\mathfrak{R}^{\dot{\gamma}}_{\beth(1)} \oplus \mathfrak{R}^{\dot{\gamma}}_{\beth(2)} \oplus \cdots \oplus \mathfrak{R}^{\dot{\gamma}}_{\beth(\beth)}\right),\\ &= \xi^{\zeta}_{1}\mathfrak{R}^{\gamma}_{1} \oplus \xi^{\zeta}_{2}\mathfrak{R}^{\gamma}_{2} \oplus \cdots \oplus \xi^{\zeta}_{\beth}\mathfrak{R}^{\gamma}_{\beth},\\ &= \mathrm{SFAAWA}\left(\mathfrak{R}^{\gamma}_{1}, \mathfrak{R}^{\gamma}_{2}, \ldots, \mathfrak{R}^{\gamma}_{\beth}\right).\end{aligned} \tag{32}$$

(2) Let $\xi^{\zeta} = (1/\beth, 1/\beth, \ldots, 1/\beth)^{T}$. Then, $\dot{\mathfrak{R}}^{\gamma}_{\phi} = \mathfrak{R}^{\gamma}_{\phi}$ and

$$\begin{aligned}\mathrm{IFAAHA}_{\xi^{\zeta},\xi^{\zeta}}\left(\mathfrak{R}^{\gamma}_{1}, \mathfrak{R}^{\gamma}_{2}, \ldots, \mathfrak{R}^{\gamma}_{\beth}\right) &= \xi^{\zeta}_{1}\mathfrak{R}^{\dot{\gamma}}_{\beth(1)} \oplus \xi^{\zeta}_{2}\mathfrak{R}^{\dot{\gamma}}_{\beth(2)} \oplus \cdots \oplus \xi^{\zeta}_{\beth}\mathfrak{R}^{\dot{\gamma}}_{\beth(\beth)},\\ &\xi^{\zeta}_{1}\mathfrak{R}^{\dot{\gamma}}_{\beth(1)} \oplus \xi^{\zeta}_{2}\mathfrak{R}^{\dot{\gamma}}_{\beth(2)} \oplus \cdots \oplus \xi^{\zeta}_{\beth}\mathfrak{R}^{\dot{\gamma}}_{\beth\beth(\beth)},\\ &\mathrm{SFAAOWA}\left(\mathfrak{R}^{\gamma}_{1}, \mathfrak{R}^{\gamma}_{2}, \ldots, \mathfrak{R}^{\gamma}_{\beth}\right).\end{aligned} \tag{33}$$

□

4.2. Spherical Fuzzy Aczel–Alsina Geometric AOs

Definition 12. Let $\mathfrak{R}^{\gamma}_{\phi} = (\kappa_{\mathfrak{R}^{\gamma}_{\phi}}, \upsilon_{\mathfrak{R}^{\gamma}_{\phi}}, \tau_{\mathfrak{R}^{\gamma}_{\phi}})$ be an accumulation of SFNs and $\xi^{\zeta} = (\xi^{\zeta}_{1}, \xi^{\zeta}_{2}, \ldots, \xi^{\zeta}_{\beth})^{T}$ be the weight vector (WV) of $\mathfrak{R}^{\gamma}_{\phi}$, with $\xi^{\zeta}_{\phi} > 0$ and $\sum_{\phi=1}^{\beth} \xi^{\zeta}_{\phi} = 1$. Then, "spherical fuzzy Aczel–Alsina weighted geometric (SFAAWG) operator" is a mapping SFAAWG: $(L^{*})^{\beth} \longrightarrow L^{*}$, where

$$\mathrm{SFAAWG}\left(\mathfrak{R}^{\gamma}_{1}, \mathfrak{R}^{\gamma}_{2}, \ldots, \mathfrak{R}^{\gamma}_{\beth}\right) = \left(\mathfrak{R}^{\gamma^{\xi^{\zeta}_{1}}}_{1} \otimes \mathfrak{R}^{\gamma^{\xi^{\zeta}_{2}}}_{2} \otimes \ldots, \otimes \mathfrak{R}^{\gamma^{\xi^{\zeta}_{\beth}}}_{\beth}\right). \tag{34}$$

Thus, the following theorem is obtained using Aczel–Alsina operations on SFNs.

Theorem 12. *Let $\mathfrak{R}^{\gamma}_{\phi} = (\kappa_{\mathfrak{R}^{\gamma}_{\phi}}, \upsilon_{\mathfrak{R}^{\gamma}_{\phi}}, \tau_{\mathfrak{R}^{\gamma}_{\phi}})$ be an accumulation of SFNs, then aggregated value of them utilizing the SFAAWG operation is also an SFNs, and*

$$\begin{aligned}\mathrm{SFAAWG}\left(\mathfrak{R}^{\gamma}_{1}, \mathfrak{R}^{\gamma}_{2}, \ldots, \mathfrak{R}^{\gamma}_{\beth}\right) &= \overset{\beth}{\underset{\phi=1}{\otimes}}\left(\xi^{\zeta}_{\phi}\mathfrak{R}^{\gamma}_{\phi}\right),\\ &= \left\langle \sqrt{e^{-\left(\sum_{\phi=1}^{\beth}\xi^{\zeta}_{\phi}\left(-\log\kappa^{2}_{\mathfrak{R}^{\gamma}_{\phi}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{\beth}\xi^{\zeta}_{\phi}\left(-\log\left(1-\upsilon_{\mathfrak{R}^{\gamma}_{\phi}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{\beth}\xi^{\zeta}_{\phi}\left(-\log\left(1-\tau_{\mathfrak{R}^{\gamma}_{\phi}}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle,\end{aligned} \tag{35}$$

where $\xi^\zeta = (\xi_1^\zeta, \xi_2^\zeta, \ldots, \xi_\sqsupset^\zeta)^T$ *be the WV of* $\mathfrak{R}_\phi^\gamma$ *s.t* $\xi_\phi^\zeta > 0$ *and* $\sum_{\phi=1}^{\sqsupset} \xi_\phi^\zeta = 1$.

Proof. We can derive Theorem 12 in the following way using the mathematical induction technique.

For $\sqsupset = 2$, depend on Aczel–Alsina operations of SFNs, we obtain the following equation:

$$\mathfrak{R}_1^{\gamma^{\xi_1^\zeta}} = \left\langle \sqrt{e^{-\left(\gimel_1\left(-\log\kappa^2_{\mathfrak{R}_1^\gamma}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\gimel_1\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_1^\gamma}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\gimel_1\left(-\log\left(1-\tau^2_{\mathfrak{R}_1^\gamma}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle \text{ and}$$

$$\mathfrak{R}_2^{\gamma^{\xi_2^\zeta}} = \left\langle \sqrt{e^{-\left(\gimel_2\left(-\log\kappa^2_{\mathfrak{R}_2^\gamma}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\gimel_2\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_2^\gamma}\right)\right)^{\aleph}\right)^{2/\aleph}}}, \sqrt{1-e^{-\left(\gimel_2\left(-\log\left(1-\tau^2_{\mathfrak{R}_2^\gamma}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \tag{36}$$

Based on Aczel–Alsina operations of SFNs, we obtain the following equation:

$$\begin{aligned}\text{SFAAWG}\left(\mathfrak{R}_1^\gamma, \mathfrak{R}_2^\gamma\right) &= \left\langle \sqrt{e^{-\left(\gimel_1\left(-\log\kappa^2_{\mathfrak{R}_1^\gamma}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\gimel_1\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_1^\gamma}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\gimel_1\left(-\log\left(1-\tau^2_{\mathfrak{R}_1^\gamma}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\ &\oplus \left\langle \sqrt{e^{-\left(\gimel_2\left(-\log\kappa^2_{\mathfrak{R}_2^\gamma}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\gimel_2\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_2^\gamma}\right)\right)^{\aleph}\right)^{2/\aleph}}}, \sqrt{1-e^{-\left(\gimel_2\left(-\log\left(1-\tau^2_{\mathfrak{R}_2^\gamma}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle, \\ &= \left\langle \sqrt{e^{-\left(\sum_{\phi=1}^{2}\xi_\phi^\zeta\left(-\log\kappa^2_{\mathfrak{R}_\phi^\gamma}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{2}\xi_\phi^\zeta\left(-\log\left(1-\upsilon_{\mathfrak{R}_\phi^\gamma}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{2}\xi_\phi^\zeta\left(-\log\left(1-\tau_{\mathfrak{R}_\phi^\gamma}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle.\end{aligned} \tag{37}$$

Thus, it is true for $\sqsupset = 2$.

Consider equation (4.15) is true for $\sqsupset = k$, then we have the following equation:

$$\begin{aligned}&\text{SFAAWG}\left(\mathfrak{R}_1^\gamma, \mathfrak{R}_2^\gamma, \ldots, \mathfrak{R}_\sqsupset^\gamma\right) \\ &= \left\langle \sqrt{e^{-\left(\sum_{\phi=1}^{k}\xi_\phi^\zeta\left(-\log\kappa^2_{\mathfrak{R}_\phi^\gamma}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{k}\xi_\phi^\zeta\left(-\log\left(1-\upsilon_{\mathfrak{R}_\phi^\gamma}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{k}\xi_\phi^\zeta\left(-\log\left(1-\tau_{\mathfrak{R}_\phi^\gamma}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle,\end{aligned} \tag{38}$$

we will prove that equation (4.15) holds for $\sqsupset = k + 1$.

$$\begin{aligned}
\text{SFAAWG}\left(\mathfrak{R}_1^{\gamma},\mathfrak{R}_2^{\gamma},\ldots,\mathfrak{R}_k^{\gamma},\mathfrak{R}_{k+1}^{\gamma}\right) &= \overset{k}{\oplus}\left(\xi_{\phi}^{\zeta}\mathfrak{R}_{\phi}^{\gamma}\right)\oplus\left(\xi_{k+1}^{\zeta}\mathfrak{R}_{k+1}^{\gamma}\right)\\
&= \left\langle \sqrt{e^{-\left(\sum_{\phi=1}^{k}\xi_{\phi}^{\zeta}\left(-\log \kappa^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)^{\aleph}\right)^{\frac{1}{\aleph}}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{k}\xi_{\phi}^{\zeta}\left(-\log\left(1-\upsilon_{\mathfrak{R}_{\phi}^{\gamma}}\right)\right)^{\aleph}\right)^{\frac{1}{\aleph}}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{k}\xi_{\phi}^{\zeta}\left(-\log\left(1-\tau_{\mathfrak{R}_{\phi}^{\gamma}}\right)\right)^{\aleph}\right)^{\frac{1}{\aleph}}}} \right\rangle,\\
&\oplus\left\langle \sqrt{e^{-\left(\beth_{k+1}\left(-\log\left(\kappa^2_{\mathfrak{R}_{k+1}^{\gamma}}\right)\right)\right)^{\frac{1}{\aleph}}}}, \sqrt{1-e^{-\left(\beth_{k+1}\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_{k+1}^{\gamma}}\right)\right)^{\aleph}\right)^{\frac{1}{\aleph}}}}, \sqrt{1-e^{-\left(\beth_{k+1}\left(-\log\left(1-\tau_{k+1}^{\gamma}\right)^{\aleph}\right)\frac{1}{\aleph}\right)}} \right\rangle,\\
&= \left\langle \sqrt{e^{-\left(\sum_{\phi=1}^{k+1}\xi_{\phi}^{\zeta}\left(-\log \kappa^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)^{\aleph}\right)^{\frac{1}{\aleph}}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{k+1}\xi_{\phi}^{\zeta}\left(-\log\left(1-\upsilon_{\mathfrak{R}_{\phi}^{\gamma}}\right)\right)^{\aleph}\right)^{\frac{1}{\aleph}}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{k+1}\xi_{\phi}^{\zeta}\left(-\log\left(1-\tau_{\mathfrak{R}_{\phi}^{\gamma}}\right)\right)^{\aleph}\right)^{\frac{1}{\aleph}}}} \right\rangle.
\end{aligned} \tag{39}$$

As a result, we can conclude that equation (4.15) stands true for any $\beth$.

By applying the SFAAWG operator, we can illustrate the following features efficiently. □

Theorem 13. *If all* $\mathfrak{R}_{\phi}^{\gamma} = (\kappa_{\mathfrak{R}_{\phi}^{\gamma}}, \upsilon_{\mathfrak{R}_{\phi}^{\gamma}}, \tau_{\mathfrak{R}_{\phi}^{\gamma}})$ *are equal, that is,* $\mathfrak{R}_{\phi}^{\gamma} = \mathfrak{R}^{\gamma}\forall\phi$, *then*

$$\text{SFAAWG}\left(\mathfrak{R}_1^{\gamma},\mathfrak{R}_2^{\gamma},\ldots,\mathfrak{R}_{\beth}^{\gamma}\right) = \mathfrak{R}^{\gamma}. \tag{40}$$

Proof. Given that $\mathfrak{R}_{\phi}^{\gamma} = (\kappa_{\mathfrak{R}_{\phi}^{\gamma}}, \upsilon_{\mathfrak{R}_{\phi}^{\gamma}}, \tau_{\mathfrak{R}_{\phi}^{\gamma}})$, by equation (4.15) we get the following equation:

$$\begin{aligned}
\text{SFAAWG}\left(\mathfrak{R}_1^{\gamma},\mathfrak{R}_2^{\gamma},\ldots,\mathfrak{R}_{\beth}^{\gamma}\right) &= \left\langle \sqrt{e^{-\left(\sum_{\phi=1}^{\beth}\xi_{\phi}^{\zeta}\left(-\log\kappa^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{\beth}\xi_{\phi}^{\zeta}\left(-\log\left(1-\upsilon^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{\beth}\xi_{\phi}^{\zeta}\left(-\log\left(1-\tau^2_{\mathfrak{R}_{\phi}^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle,\\
&= \left\langle \sqrt{e^{-\left(\left(-\log\left(\kappa^2_{\mathfrak{R}^{\gamma}}\right)^{\aleph}\right)\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\left(-\log\left(1-\upsilon^2_{\mathfrak{R}^{\gamma}}\right)\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\left(-\log\left(1-\tau^2_{\mathfrak{R}^{\gamma}}\right)\right)^{\aleph}\right)^{\frac{1}{\aleph}}}} \right\rangle,\\
&= \left\langle \sqrt{e^{\log\kappa^2_{\mathfrak{R}^{\gamma}}}}\, \sqrt{1-e^{\log\left(1-\upsilon^2_{\mathfrak{R}^{\gamma}}\right)}}, \sqrt{1-e^{\log\left(1-\tau^2_{\mathfrak{R}^{\gamma}}\right)}} \right\rangle,\\
&= \left(\sqrt{\kappa^2_{\mathfrak{R}^{\gamma}}}, \sqrt{\upsilon^2_{\mathfrak{R}^{\gamma}}}, \sqrt{\tau^2_{\mathfrak{R}^{\gamma}}}\right) = \mathfrak{R}^{\gamma}.
\end{aligned} \tag{41}$$

□

Theorem 14. *Let* $\mathfrak{R}_{\phi}^{\gamma} = (\kappa_{\mathfrak{R}_{\phi}^{\gamma}}, \upsilon_{\mathfrak{R}_{\phi}^{\gamma}}, \tau_{\mathfrak{R}_{\phi}^{\gamma}})$ *be an accumulation of SFNs. Let* $\mathfrak{R}^{\gamma^-} = min(\mathfrak{R}_1^{\gamma},\mathfrak{R}_2^{\gamma},\ldots,\mathfrak{R}_{\beth}^{\gamma})$ *and* $\mathfrak{R}^{\gamma^+} = max(\mathfrak{R}_1^{\gamma},\mathfrak{R}_2^{\gamma},\ldots,\mathfrak{R}_{\beth}^{\gamma})$. *Then,*

$$\mathfrak{R}^{\gamma^-} \leq \text{SFAAWG}(\mathfrak{R}_1^\gamma, \mathfrak{R}_2^\gamma, \ldots, \mathfrak{R}_\sqsupset^\gamma) \leq \mathfrak{R}^{\gamma^+}. \quad (42)$$

Proof. Let $\mathfrak{R}_\phi^\gamma = (\kappa_{\mathfrak{R}_\phi^\gamma}, \upsilon_{\mathfrak{R}_\phi^\gamma})$ be an accumulation of SFNs. Let $\mathfrak{R}^{\gamma^-} = \min(\mathfrak{R}_1^\gamma, \mathfrak{R}_2^\gamma, \ldots, \mathfrak{R}_\sqsupset^\gamma) = (\kappa^-_{\mathfrak{R}^\gamma}, \upsilon^-_{\mathfrak{R}^\gamma}, \tau^-_{\mathfrak{R}^\gamma})$ and $\mathfrak{R}^{\gamma^+} = \max(\mathfrak{R}_1^\gamma, \mathfrak{R}_2^\gamma, \ldots, \mathfrak{R}_\sqsupset^\gamma) = (\kappa^+_{\mathfrak{R}^\gamma}, \upsilon^+_{\mathfrak{R}^\gamma}, \tau^+_{\mathfrak{R}^\gamma})$. We have, $\kappa^-_{\mathfrak{R}^\gamma} = \min_\phi\{\kappa_{\mathfrak{R}_\phi^\gamma}\}$, $\upsilon^-_{\mathfrak{R}^\gamma} = \max_\phi\{\upsilon_{\mathfrak{R}_\phi^\gamma}\}$, $\tau^-_{\mathfrak{R}^\gamma} = \max_\phi\{\tau_{\mathfrak{R}_\phi^\gamma}\}$, $\kappa^+_{\mathfrak{R}^\gamma} = \max_\phi\{\kappa_{\mathfrak{R}_\phi^\gamma}\}$, $\upsilon^+_{\mathfrak{R}^\gamma} = \min_\phi\{\upsilon_{\mathfrak{R}_\phi^\gamma}\}$, and $\tau^+_{\mathfrak{R}^\gamma} = \min_\phi\{\tau_{\mathfrak{R}_\phi^\gamma}\}$ Hence, here, we have the subsequent inequalities,

$$\sqrt{e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta\left(-\log\kappa^2_{\mathfrak{R}^{\gamma^+}}\right)^\aleph\right)^{1/\aleph}}} \leq \sqrt{e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta\left(-\log\kappa^2_{\mathfrak{R}^{\gamma^\phi}}\right)^\aleph\right)^{1/\aleph}}} \leq \sqrt{e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta\left(-\log\kappa^2_{\mathfrak{R}^{\gamma^-}}\right)^\aleph\right)^{1/\aleph}}},$$
$$\sqrt{1-e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta\left(-\log\left(1-\upsilon^2_{\mathfrak{R}^{\gamma^-}}\right)\right)^\aleph\right)^{1/\aleph}}} \leq \sqrt{1-e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta\left(-\log\left(1-\upsilon^2_{\mathfrak{R}^{\gamma^\phi}}\right)\right)^\aleph\right)^{1/\aleph}}} \leq \sqrt{1-e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta\left(-\log\left(1-\upsilon^2_{\mathfrak{R}^{\gamma^+}}\right)\right)^\aleph\right)^{1/\aleph}}}, \quad (43)$$

$$\sqrt{1-e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta\left(-\log\left(1-\tau^2_{\mathfrak{R}^{\gamma^-}}\right)\right)^\aleph\right)^{1/\aleph}}} \leq \sqrt{1-e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta\left(-\log\left(1-\tau^2_{\mathfrak{R}^{\gamma^\phi}}\right)\right)^\aleph\right)^{1/\aleph}}} \leq \sqrt{1-e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta\left(-\log\left(1-\tau^2_{\mathfrak{R}^{\gamma^+}}\right)\right)^\aleph\right)^{1/\aleph}}}. \quad (44)$$

Therefore,
$\mathfrak{R}^{\gamma^-} \leq \text{SFAAWG}(\mathfrak{R}_1^\gamma, \mathfrak{R}_2^\gamma, \ldots, \mathfrak{R}_\sqsupset^\gamma) \leq \mathfrak{R}^{\gamma^+}$ □

Theorem 15. *Let $\mathfrak{R}_\phi^\gamma$ and $\mathfrak{R}_\phi^{\gamma'}$ be two sets of SFNs, if $\mathfrak{R}_\phi^\gamma \leq \mathfrak{R}_\phi^{\gamma'} \forall\phi$, then*

$$\text{SFAAWG}(\mathfrak{R}_1^\gamma, \mathfrak{R}_2^\gamma, \ldots, \mathfrak{R}_\sqsupset^\gamma) \leq \text{SFAAWG}\left(\mathfrak{R}_1^{\gamma'}, \mathfrak{R}_2^{\gamma'}, \ldots, \mathfrak{R}_\sqsupset^{\gamma'}\right), \quad (45)$$

Now, we present "spherical fuzzy Aczel–Alsina ordered weighted geometric (SFAAOWG) operator."

Definition 13. Let $\mathfrak{R}_\phi^\gamma = (\kappa_{\mathfrak{R}_\phi^\gamma}, \upsilon_{\mathfrak{R}_\phi^\gamma}, \tau_{\mathfrak{R}_\phi^\gamma})$ be an accumulation of SFNs. SFAAOWG operator is a mapping SFAAOWG: $(L^*)^\sqsupset \longrightarrow L^*$ with the corresponding WV $\xi^\zeta = (\xi_1^\zeta, \xi_2^\zeta, \ldots, \xi_\sqsupset^\zeta)^T$ such that $\xi_\phi^\zeta > 0$ and $\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta = 1$, as follows:

$$\text{SFAAOWG}(\mathfrak{R}_1^\gamma, \mathfrak{R}_2^\gamma, \ldots, \mathfrak{R}_\sqsupset^\gamma) = \bigotimes_{\phi=1}^{\sqsupset}\left(\mathfrak{R}_{\Upsilon(\phi)}^{\gamma^{\xi_\phi^\zeta}}\right) = \mathfrak{R}_{\Upsilon(1)}^{\gamma^{\xi_1^\zeta}} \oplus \mathfrak{R}_{\Upsilon(2)}^{\gamma^{\xi_2^\zeta}} \oplus, \ldots, \oplus\mathfrak{R}_{\Upsilon(\sqsupset)}^{\gamma^{\xi_\sqsupset^\zeta}}, \quad (46)$$

where $(\Upsilon(1), \Upsilon(2), \ldots, \Upsilon(\sqsupset))$ are the permutation of $(\phi = 1, 2, \ldots, \sqsupset)$, including $\mathfrak{R}_{\Upsilon(\phi-1)}^\gamma \geq \mathfrak{R}_{\Upsilon(\phi)}^\gamma \forall\phi = 1, 2, \ldots, \sqsupset$.

Thus, the following theorem is obtained using Aczel–Alsina operations on SFNs.

Theorem 16. *Let $\mathfrak{R}_\phi^\gamma = (\kappa_{\mathfrak{R}_\phi^\gamma}, \upsilon_{\mathfrak{R}_\phi^\gamma}, \tau_{\mathfrak{R}_\phi^\gamma})$ be an accumulation of SFNs. SFAAOWG operator is a mapping SFAAOWG: $(L^*)^\sqsupset \longrightarrow L^*$ with the corresponding vector $\xi^\zeta = (\xi_1^\zeta > 0, \xi_2^\zeta > 0, \ldots, \xi_\sqsupset^\zeta)^T$ such that $\xi_\phi^\zeta > 0$ and $\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta = 1$. Then,*

$$\begin{aligned}\text{SFAAOWG}(\mathfrak{R}_1^\gamma, \mathfrak{R}_2^\gamma, \ldots, \mathfrak{R}_\sqsupset^\gamma) &= \bigotimes_{\phi=1}^{\sqsupset}\left(\mathfrak{R}_\phi^{\gamma^{\xi_\phi^\zeta}}\right),\\ &= \left\langle \sqrt{e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta\left(-\log\kappa^2_{\mathfrak{R}^\gamma_{\Upsilon(\phi)}}\right)^\aleph\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta\left(-\log\left(1-\upsilon^2_{\mathfrak{R}^\gamma_{\Upsilon(\phi)}}\right)\right)^\aleph\right)^{1/\aleph}}},\right.\\ &\left.\sqrt{1-e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi_\phi^\zeta\left(-\log\left(1-\tau^2_{\mathfrak{R}^\gamma_{\Upsilon(\phi)}}\right)\right)^\aleph\right)^{1/\aleph}}}\right\rangle,\end{aligned} \quad (47)$$

where $(\Upsilon(1), \Upsilon(2), \ldots, \Upsilon(\sqsupset))$ *are the permutation of* $(\phi = 1, 2, \ldots, \sqsupset)$, *including* $\mathfrak{R}^{\gamma}_{\Upsilon(\phi-1)} \geq \mathfrak{R}^{\gamma}_{\Upsilon(\phi)} \forall \phi = 1, 2, \ldots, \sqsupset$.

Proof. Same as Theorem 12.

By applying the SFAAOWG operator, we can illustrate the following features efficiently. □

Theorem 17. *If all* $\mathfrak{R}^{\gamma}_{\phi} = (\kappa_{\mathfrak{R}^{\gamma}_{\phi}}, \upsilon_{\mathfrak{R}^{\gamma}_{\phi}}, \tau_{\mathfrak{R}^{\gamma}_{\phi}})$ *are equal, that is,* $\mathfrak{R}^{\gamma}_{\phi} = \mathfrak{R}^{\gamma} \forall \phi$, *then*

$$\mathrm{SFAAOWG}(\mathfrak{R}^{\gamma}_1, \mathfrak{R}^{\gamma}_2, \ldots, \mathfrak{R}^{\gamma}_{\sqsupset}) = \mathfrak{R}^{\gamma}. \quad (48)$$

Proof. Same as Theorem 13. □

Theorem 18. *Let* $\mathfrak{R}^{\gamma}_{\phi} = (\kappa_{\mathfrak{R}^{\gamma}_{\phi}}, \upsilon_{\mathfrak{R}^{\gamma}_{\phi}}, \tau_{\mathfrak{R}^{\gamma}_{\phi}})$ *be an accumulation of SFNs. Let* $\mathfrak{R}^{\gamma^-} = min(\mathfrak{R}^{\gamma}_1, \mathfrak{R}^{\gamma}_2, \ldots, \mathfrak{R}^{\gamma}_{\sqsupset})$ *and* $\mathfrak{R}^{\gamma^+} = max(\mathfrak{R}^{\gamma}_1, \mathfrak{R}^{\gamma}_2, \ldots, \mathfrak{R}^{\gamma}_{\sqsupset})$. *Then,*

$$\mathfrak{R}^{\gamma^-} \leq \mathrm{SFAAOWG}(\mathfrak{R}^{\gamma}_1, \mathfrak{R}^{\gamma}_2, \ldots, \mathfrak{R}^{\gamma}_{\sqsupset}) \leq \mathfrak{R}^{\gamma^+}. \quad (49)$$

Proof. Same as Theorem 14. □

Theorem 19. *Let* $\mathfrak{R}^{\gamma}_{\phi}$ *and* $\mathfrak{R}^{\gamma'}_{\phi}$ *be two sets of SFNs, if* $\mathfrak{R}^{\gamma}_{\phi} \leq \mathfrak{R}^{\gamma'}_{\phi} \forall \phi$, *then*

$$\mathrm{SFAAOWG}(\mathfrak{R}^{\gamma}_1, \mathfrak{R}^{\gamma}_2, \ldots, \mathfrak{R}^{\gamma}_{\sqsupset}) \leq \mathrm{SFAAOWG}\left(\mathfrak{R}^{\gamma'}_1, \mathfrak{R}^{\gamma'}_2, \ldots, \mathfrak{R}^{\gamma'}_{\sqsupset}\right). \quad (50)$$

It is self-evident that the SFAAWG operator weights only the SFNs, and that the SFAAOWG operator weights only the SFN's ordered locations. Following that, weights are used to indicate various elements of the SFAAWG and SFAAOWG operators. Nonetheless, both one and the other operators consider only one of these. To address this shortcoming, we will also demonstrate the "spherical fuzzy Aczel–Alsina hybrid geometric (SFAAHG) operator," which weights all of the given SFN and their appropriate ordered position.

Definition 14. Let $\mathfrak{R}^{\gamma}_{\phi}$ be an accumulation of SFNs. SFAAHG operator is a mapping SFAAHG: $(L^*)^{\sqsupset} \longrightarrow L^*$, s.t.

$$\begin{aligned}\mathrm{SFAAHG}(\mathfrak{R}^{\gamma}_1, \mathfrak{R}^{\gamma}_2, \ldots, \mathfrak{R}^{\gamma}_{\sqsupset}) &= \mathop{\oplus}_{\phi=1}^{\sqsupset}\left(\mathfrak{R}\gamma^{\ddot{\xi^{\zeta}_{\phi}}}_{\Upsilon(\phi)}\right)\\ &= \xi^{\zeta}_1 \mathfrak{R}^{\ddot{\gamma}}_{\Upsilon(1)} \oplus \xi^{\zeta}_2 \mathfrak{R}^{\ddot{\gamma}}_{\Upsilon(2)} \oplus, \ldots, \oplus \xi^{\zeta}_{\sqsupset} \mathfrak{R}^{\ddot{\gamma}}_{\Upsilon(\sqsupset)}\end{aligned} \quad (51)$$

where $\xi^{\zeta} = (\xi^{\zeta}_1, \xi^{\zeta}_1, \ldots, \xi^{\zeta}_{\sqsupset})^T$ is the weighting vector associated with the SFAAHG operator, with $\xi^{\zeta}_{\phi} \in [0, 1]$ and $\sum_{\phi=1}^{\sqsupset} \xi^{\zeta}_{\phi} = 1$; $\ddot{\mathfrak{R}}^{\gamma}_{\phi} = \sqsupset \delta_{\phi} \mathfrak{R}^{\gamma}_{\phi}, \phi = 1, 2, \ldots, \sqsupset$, $(\mathfrak{R}^{\ddot{\gamma}}_{\Upsilon(1)}, \mathfrak{R}^{\ddot{\gamma}}_{\Upsilon(2)}, \ldots, \mathfrak{R}^{\ddot{\gamma}}_{\Upsilon(\sqsupset)})$ is any permutation of a collection of the weighted SFNs $(\ddot{\mathfrak{R}}^{\gamma}_1, \ddot{\mathfrak{R}}^{\gamma}_2, \ldots \ddot{\mathfrak{R}}^{\gamma}_{\sqsupset})$, s.t. $\mathfrak{R}^{\ddot{\gamma}}_{y(\phi-1)} \geq \mathfrak{R}^{\ddot{\gamma}}_{y(\phi)}$. $\delta = (\delta_1, \delta_2, \ldots \delta_{\sqsupset})^T$ is the weight vector of $\mathfrak{R}^{\gamma}_{\phi}$, with $\delta_{\phi} \in [0, 1]$ and $\sum_{\phi=1}^{\sqsupset} \delta_{\phi} = 1$, and $\sqsupset$ is the balancing coefficient, which plays a role of balance.

Theorem 20. *Let* $\mathfrak{R}^{\gamma}_{\phi}$ *be the collection of SFNs. Their aggregated value by SFAAHG operator is still an SFN, and*

$$\begin{aligned}\mathrm{SFAAHG}(\mathfrak{R}^{\gamma}_1, \mathfrak{R}^{\gamma}_2, \ldots, \mathfrak{R}^{\gamma}_{\sqsupset}) &= \mathop{\otimes}_{\phi=1}^{\sqsupset}\left(\mathfrak{R}^{\gamma^{\xi^{\zeta}_{\phi}}}_{\phi}\right),\\ &= \left\langle \sqrt{e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi^{\zeta}_{\phi}\left(-\log\kappa^2_{\mathfrak{R}^{\ddot{\gamma}}_{\Upsilon(\phi)}}\right)^{\aleph}\right)^{1/\aleph}}}, \sqrt{1-e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi^{\zeta}_{\phi}\left(-\log\left(1-\upsilon^2_{\mathfrak{R}^{\ddot{\gamma}}_{\Upsilon(\phi)}}\right)\right)^{\aleph}\right)^{1/\aleph}}} \sqrt{1-e^{-\left(\sum_{\phi=1}^{\sqsupset}\xi^{\zeta}_{\phi}\left(-\log\left(1-\tau^2_{\mathfrak{R}^{\ddot{\gamma}}_{\Upsilon(\phi)}}\right)\right)^{\aleph}\right)^{1/\aleph}}} \right\rangle.\end{aligned} \quad (52)$$

Proof. Same as Theorem 12. □

Theorem 21. *The SFAAWG and SFAAOWG operators are special cases of the SFAAHG operator.*

5. MCDM Approach Based on Proposed Aczel–Alsina AOs

With the assistance of suggested AOs, we investigate MCDM problems. Consider the set of possible alternatives $\Lambda^{\delta} = \{\Lambda^{\delta}_1, \Lambda^{\delta}_2, \ldots, \Lambda^{\delta}_m\}$ and $\sqsupset^{\daleth} = \{\sqsupset^{\daleth}_1, \sqsupset^{\daleth}_2, \ldots, \sqsupset^{\daleth}_n\}$ is the collection of criterion. DMs gave their opinion matrix $D = (\mathscr{P}_{ij})_{m\times n}$, where $\mathscr{P}_{ij}$ is given for the alternatives $\Lambda^{\delta}_i \in \Lambda^{\delta}$ with respect to the criteria $\sqsupset^{\daleth}_j \in \sqsupset^{\daleth}$ in the form of SFNs. SFN decision matrix denoted by $D = (\mathscr{P}_{ij})_{m\times n}$.The proposed operators will be applied to the MCDM, which will include the points listed in Algorithm 1.

6. Numerical Example

This section provides an illustration of how to apply the proposed strategy to the land selection for agriculture purpose.

6.1. Explanation of the Problem. Due to the COVID-19 pandemic's consequences, the E-commerce phenomenon is accelerating, having a huge influence on global supply chains. Thus, logistics management tasks have been elevated in importance in practically every organization that transports physical commodities. There are several methods for

Step 1: acquire the declaration matrix $D = (\mathscr{P}_{ij})_{m\times n}$ in the format of SFNs from the DM.
Step 2: the declaration matrix's attributes are classified into two types, such as the cost form parameter (τ_c) and the benefit form parameter (τ_b). If all parameters are of the same form, no normalization is required. However, because the MCDM contains parameters of multiple forms, the D matrix has been converted to a normalization matrix using the normalization formula, $Y(\breve{F}_{ij})$.

$$\breve{F}_{ij} = \begin{cases} \mathscr{P}^c_{ij}; & j \in \tau_c, \\ \mathscr{P}_{ij}; & j \in \tau_b. \end{cases}$$

where $\mathscr{P}^c_{ij}$ show the compliment of $\mathscr{P}_{ij}$.
Step 3: using one of the proposed operators to evaluate combined evaluations of the alternatives. One can use geometric operators also instead of averaging operators.
$\breve{F}_i = \text{SFAAWA}(\breve{F}_{i1}, \breve{F}_{i2}, \ldots, \breve{F}_{in})$,
$\breve{F}_i = \text{SFAAOWA}(\breve{F}_{i1}, \breve{F}_{i2}, \ldots, \breve{F}_{in})$,
or $\breve{F}_i = \text{SFAAHA}(\breve{F}_{i1}, \breve{F}_{i2}, \ldots, \breve{F}_{in})$.
Step 4: calculate the combined score for all alternative assessments.
Step 5: alternatives are ranked first by their scoring function, and the best one can be chosen.

Algorithm 1: A decision making process based on SFAAWA, SFAAOWA, SFAAHA, is proposed in Algorithm 1.

businesses to acquire a comparative edge via the out scoured of logistics management processes in today's diversified and incredibly quickly world. Exporters, distributors, and businesses with distribution networks have all demonstrated that turning to third-party logistics (3PL) providers benefits them. 3PL is a term that refers to the process through which a company outsources its warehouse and transportation activities. A 3PL organization can provide stock control, cross-docking, the door of the house distribution, and packaging material. The market for third-party logistics services has accelerated its expansion as a result of the E-commerce boom and expanded reverse logistics activities. The E-commerce trend includes faster, more dependable delivery, increased inventory turnover, and goods staged in forwarding sites near clients. There has been a large surge of 3PL firms offering a range of services to assist in maintaining this very sophisticated supply chain. 3PLs are frequently requested for assistance with E-commerce fulfilment, warehousing, and delivery facilities, and 3PLs invest in technology for both client service and internal usage. Due to the current worldwide problem, the COVID-19 pandemic, the function of E-commerce has been enhanced and expedited.

Due to the features of multidimensional decision-making difficulties, 3PL selection may well be considered a complex MCDM challenge, given the presence of statistical, interpersonal, and numerous factors in the natural decision-making phase. Given the critical nature of sustainable third-party logistics providers, there is a dearth of studies on the 3PL selection challenge in emerging economies. The 3PL sector is growing at a breakneck pace due to the rise of the e-commerce sector. Indeed, the requirement for 3PL services is projected to grow as brands and distributors seek to focus exclusively on their core industries. As a result, they frequently outsource logistical services. In a nutshell, analyzing and choosing optimum third-party logistics providers is a critical component of any business's long-term goals.

Consider a corporation that is looking for the best 3PL provider. Following prescreening, five 3PLs have been identified for further consideration $\Lambda^{\delta}_{\eta}(\eta = 1, 2, \ldots, 5)$. You must choose between the following four characteristics: (1) $\beth^{\daleth}_1$ = financial stability, (2) $\beth^{\daleth}_2$ = reliability and delivery time, (3) $\beth^{\daleth}_3$ = reputation, and (4) $\beth^{\daleth}_4$ = green operation. The DM distributes the attribute weight in the following way: $\xi^{\zeta} = (0.20, 0.30, 0.25, 0.25)^T$. Table 1 evaluates the five candidates $\Lambda^{\delta}_{\eta}(\eta = 1, 2, \ldots, 5)$.

6.2. Proposed Method Based on SFAAWA Operator. To select the best agriculture land $\Lambda^{\delta}\eta(\eta = 1, 2, \ldots, \upsilon)$, we utilize the SFAAWA operator to construct an MCDM premise with SF information, which is usually evaluated as follows:

Step 1: declaration matrix $D = (\mathscr{P}_{ij})_{m\times n}$ in the format of SFNs from the DM given in Table 1.

Step 2: there is no cost type attributes in the four characteristics: (1) $\beth^{\daleth}_1$ = financial stability, (2) $\beth^{\daleth}_2$ = reliability and delivery time, (3) $\beth^{\daleth}_3$ = reputation, and (4) $\beth^{\daleth}_4$ = green operation. So, there is no need of normalization in Table 1.

Step 3: consider that $\aleph = 4$. We take the SFAAWA operator to calculate the alternative values K_{η} of five applicants $\Lambda^{\delta}_{\eta}(\eta = 1, 2, \ldots, 5)$, $\breve{F}_1 = (0.23595, 0.0331817, 0.0329636)$, $\breve{F}_2 = (0.27865, 0.214317, 0.234073)$ $\breve{F}_3 = (0.308017, 0.19699, 0.300857)$, $\breve{F}_4 = (0.352767, 0.189726, 0.152533)$, and $\breve{F}_5 = (0.23513, 0.181907, 0.237248)$.

Step 4: evaluate the score functions $S(\breve{F}_{\eta})(\eta = 1, 2, \ldots, 5)$ of the SFNs $\breve{F}_{\eta}(\eta = 1, 2, \ldots, 5)$ as $S(\breve{F}_1) = 0.0412034$, $S(\breve{F}_2) = 0.00159675$, $S(\breve{F}_3) = -0.0107371$, $S(\breve{F}_4) = 0.0387103$, and $S(\breve{F}_5) = -0.00305816$.

Step 5: classify all five candidates according to the value of the SFN's score functions as $\Lambda^{\delta}_1 > \Lambda^{\delta}_4 > \Lambda^{\delta}_2 > \Lambda^{\delta}_5 > \Lambda^{\delta}_3$. Λ^{δ}_5 is chosen as the most suitable agriculture land.

Table 1: Spherical fuzzy decision matrix.

	$\beth_1^{\gimel}$	$\beth_2^{\gimel}$	$\beth_3^{\gimel}$	$\beth_4^{\gimel}$
Λ_1^{δ}	(0.173, 0.144, 0.108)	(0.243, 0.050, 0.203)	(0.253, 0.011, 0.113)	(0.223, 0.133, 0.009)
Λ_2^{δ}	(0.333, 0.244, 0.143)	(0.143, 0.223, 0.333)	(0.228, 0.151, 0.418)	(0.233, 0.433, 0.243)
Λ_3^{δ}	(0.368, 0.263, 0.273)	(0.133, 0.208, 0.543)	(0.233, 0.150, 0.218)	(0.263, 0.223, 0.431)
Λ_4^{δ}	(0.218, 0.463, 0.133)	(0.248, 0.413, 0.243)	(0.413, 0.150, 0.138)	(0.163, 0.443, 0.133)
Λ_5^{δ}	(0.141, 0.152, 0.463)	(0.268, 0.456, 0.163)	(0.118, 0.413, 0.258)	(0.213, 0.113, 0.433)

Table 2: Score functions for different values of $\aleph$.

$\aleph$	$S(\breve{F}_1)$	$S(\breve{F}_2)$	$S(\breve{F}_3)$	$S(\breve{F}_4)$	$S(\breve{F}_5)$	Ordering
4	0.0412034	0.00159675	-0.0107371	0.0387103	-0.00305816	$\Lambda_1^{\delta} > \Lambda_4^{\delta} > \Lambda_2^{\delta} > \Lambda_3^{\delta} > \Lambda_5^{\delta}$
10	0.0508129	0.0142563	0.00071527	0.059049	0.0000415248	$\Lambda_1^{\delta} > \Lambda_4^{\delta} > \Lambda_2^{\delta} > \Lambda_3^{\delta} > \Lambda_5^{\delta}$
15	0.053227	0.020284	0.00521721	0.0648227	0.00229417	$\Lambda_1^{\delta} > \Lambda_4^{\delta} > \Lambda_2^{\delta} > \Lambda_3^{\delta} > \Lambda_5^{\delta}$
20	0.0545411	0.023817	0.00785135	0.0678921	0.00366547	$\Lambda_1^{\delta} > \Lambda_4^{\delta} > \Lambda_2^{\delta} > \Lambda_3^{\delta} > \Lambda_5^{\delta}$
40	0.0568068	0.0296565	0.0124697	0.0727274	0.00595754	$\Lambda_1^{\delta} > \Lambda_4^{\delta} > \Lambda_2^{\delta} > \Lambda_3^{\delta} > \Lambda_5^{\delta}$
60	0.0576824	0.0317262	0.0141957	0.0744342	0.00677951	$\Lambda_1^{\delta} > \Lambda_4^{\delta} > \Lambda_2^{\delta} > \Lambda_3^{\delta} > \Lambda_5^{\delta}$
80	0.0581399	0.0327826	0.0150882	0.0753273	0.00720298	$\Lambda_1^{\delta} > \Lambda_4^{\delta} > \Lambda_2^{\delta} > \Lambda_3^{\delta} > \Lambda_5^{\delta}$
100	0.0584175	0.0334233	0.0156325	0.0758855	0.0074612	$\Lambda_1^{\delta} > \Lambda_4^{\delta} > \Lambda_2^{\delta} > \Lambda_3^{\delta} > \Lambda_5^{\delta}$

Table 3: Comparison of proposed AOs with some exiting operators.

Authors	AOs	Ranking of alternatives	The optimal alternative
Ashraf and Abdullah [59]	SFEWA	$\Lambda_1^{\delta} > \Lambda_2^{\delta} > \Lambda_5^{\delta} > \Lambda_4^{\delta} > \Lambda_3^{\delta}$	Λ_1^{δ}
	SFEWG	$\Lambda_1^{\delta} > \Lambda_2^{\delta} > \Lambda_5^{\delta} > \Lambda_3^{\delta} > \Lambda_4^{\delta}$	Λ_1^{δ}
Jin et al. [48]	LSFWA	$\Lambda_1^{\delta} > \Lambda_4^{\delta} > \Lambda_2^{\delta} > \Lambda_3^{\delta} > \Lambda_5^{\delta}$	Λ_1^{δ}
	LSFWG	$\Lambda_1^{\delta} > \Lambda_2^{\delta} > \Lambda_3^{\delta} > \Lambda_4^{\delta} > \Lambda_5^{\delta}$	Λ_1^{δ}
Ashraf et al. [49]	SFDWA	$\Lambda_1^{\delta} > \Lambda_4^{\delta} > \Lambda_2^{\delta} > \Lambda_3^{\delta} > \Lambda_5^{\delta}$	Λ_1^{δ}
	SFDWG	$\Lambda_1^{\delta} > \Lambda_4^{\delta} > \Lambda_5^{\delta} > \Lambda_2^{\delta} > \Lambda_3^{\delta}$	Λ_1^{δ}
Ashraf et al. [41]	SFNWAA	$\Lambda_1^{\delta} > \Lambda_2^{\delta} > \Lambda_4^{\delta} > \Lambda_5^{\delta} > \Lambda_3^{\delta}$	Λ_1^{δ}
	SFNWGA	$\Lambda_1^{\delta} > \Lambda_5^{\delta} > \Lambda_2^{\delta} > \Lambda_4^{\delta} > \Lambda_3^{\delta}$	Λ_1^{δ}
Proposed	SFAAWA	$\Lambda_1^{\delta} > \Lambda_4^{\delta} > \Lambda_2^{\delta} > \Lambda_4^{\delta} > \Lambda_3^{\delta}$	Λ_1^{δ}
	SFAAWG	$\Lambda_1^{\delta} > \Lambda_4^{\delta} > \Lambda_2^{\delta} > \Lambda_4^{\delta} > \Lambda_3^{\delta}$	Λ_1^{δ}

7. Sensitively Analysis

To highlight the influence of different parameter $\aleph$ magnitudes, we use different parameter $\aleph$ inside our discussed methodologies to classify the options. The ordering effects of the options based on the SFAAWA operator are reported in Table 2. It is clear that as the magnitude of $\aleph$ for the SFAAWA operator increases, the score values of the alternatives gradually increase, but the optimum alternative remains constant, implying that the proposed techniques have the property of isotonicity, and the DMs can choose the appropriate value based on their preferences.

Additionally, we can observe that even though the value of $\aleph$ is varied throughout the demonstration, the results produced from the alternatives appear to be consistent, confirming the proposed operator's robustness.

8. Comparative Analysis

This section compares proposed AOs to several existing AOs. We equate our findings by solving the data with preexisting AOs and get a comparable optimum solution. This demonstrates the AO's long-term viability and efficacy. The technique outlined here is more practical and superior to a number of previously published AOs. We validate our optimal solution by running it through numerous current operators. We receive the same optimal decision, demonstrating the validity of our proposed AOs. Comparison of proposed AOs with some exiting operators is given in Table 3.

9. Conclusion

The main contribution of the work is outlined as follows:

(1) The aim of this paper is to present a novel idea about the operational laws and the AOs for the aggregation of SFNs. For this we first extended the Aczel–Alsina t-norm and t-conorm to SF contexts, then established and assessed a number of innovative operational principles for SFNs. The fundamental properties of the proposed laws are discussed in detail.

(2) Based on the proposed laws, we define several AOs to aggregate the SF information, namely the "spherical fuzzy Aczel–Alsina weighted averaging (SFAAWA) operator," "spherical fuzzy Aczel–Alsina ordered

weighted averaging (SFAAOWA) operator," "spherical fuzzy Aczel–Alsina hybrid averaging (SFAAHA) operator," "spherical fuzzy Aczel–Alsina weighted geometric (SFAAWG) operator," "spherical fuzzy Aczel–Alsina ordered weighted geometric (SFAAOWG) operator," and "spherical fuzzy Aczel–Alsina hybrid geometric (SFAAHG) operator." The basic axioms of the operators are also satisfied with the proposed work.

(3) The proposed operators have been applied to MCDM approach with SF data, and a numerical model illustrating the decision-making technique has been provided.

(4) To highlight the influence of parameter $\aleph$ in the decision-making process, we also added the sensitivity analysis. Moreover, we equate our findings by solving the data with preexisting AOs and get a comparable optimum solution. This demonstrates the AO's long-term viability and efficacy.

We will gradually apply the aforementioned operators and techniques to a variety of multiple applications, including network analysis, risk assessment, cognitive science, reinforcement learning, signal processing, and many domains in ambiguous contexts. Furthermore, the current study does not take into account the interrelationships between the pairs of attributes during the aggregation process, which we will do in future projects. Furthermore, we will attempt to develop some more generalized information metrics in order to better realize the information in our everyday lives.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[3] B. C. Cuong, *Picture fuzzy sets-first results. part 1, seminar neuro-fuzzy systems with applications, Tech. rep*, Institute of Mathematics, Hanoi, Vietnam, 2013.

[4] B. C. Cuong, *Picture fuzzy sets-first results. part 2, seminar neuro-fuzzy systems with applications, Tech. rep*, Institute of Mathematics, Hanoi, Vietnam, 2013.

[5] L. H. Son, "DPFCM: a novel distributed picture fuzzy clustering method on picture fuzzy sets," *Expert Systems with Applications*, vol. 42, no. 1, pp. 51–66, 2015.

[6] P. Singh, "Correlation coefficients for picture fuzzy sets," *Journal of Intelligent & Fuzzy Systems*, vol. 27, pp. 2857–2868, 2014.

[7] G. W. Wei, "Some similarity measures for picture fuzzy sets and their applications," *Iranian Journal of Fuzzy Systems*, vol. 15, no. 1, pp. 77–89, 2018.

[8] G. W. Wei and H. Gao, "The generalized dice similarity measures for picture fuzzy sets and their applications," *Informatica*, vol. 29, no. 1, pp. 1–18, 2018.

[9] G. W. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Projection models for multiple attribute decision making with picture fuzzy information," *International Journal of Machine Learning and Cybernetics*, vol. 9, no. 4, pp. 713–719, 2018.

[10] G. W. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Picture 2-tuple linguistic aggregation operators in multiple attribute decision making," *Soft Computing*, vol. 22, no. 3, pp. 989–1002, 2018.

[11] H. Garg, "Some picture fuzzy aggregation operators and their applications to multicriteria decision-making," *Arabian Journal for Science and Engineering*, vol. 42, no. 12, pp. 5275–5290, 2017.

[12] G. W. Wei, "Picture fuzzy Hamacher aggregation operators and their application to multiple attribute decision making," *Fundamenta Informaticae*, vol. 157, no. 3, pp. 271–320, 2018.

[13] C. Jana, T. Senapati, M. Pal, and R. R. Yager, "Picture fuzzy Dombi aggregation operators: application to MADM process," *Applied Soft Computing*, vol. 74, pp. 99–109, 2019.

[14] C. Tian, J. J. Peng, S. Zhang, W. Y. Zhang, and J. Q. Wang, "Weighted picture fuzzy aggregation operators and their applications to multi-criteria decision-making problems," *Computers & Industrial Engineering*, vol. 137, Article ID 106037, 2019.

[15] L. Wang, H. Y. Zhang, J. Q. Wang, and G. F. Wu, "Picture fuzzy multi-criteria group decision-making method to hotel building energy efficiency retrofit project selection," *RAIRO - Operations Research*, vol. 54, no. 1, pp. 211–229, 2020.

[16] R. Wang, J. Wang, H. Gao, and G. Wei, "Methods for MADM with picture fuzzy muirhead mean operators and their application for evaluating the financial investment risk," *Symmetry*, vol. 11, no. 1, p. 6, 2019.

[17] G. W. Wei, "TODIM Method for Picture fuzzy multiple attribute decision making," *Informatica*, vol. 29, no. 3, pp. 555–566, 2018.

[18] G. W. Wei, "Picture 2-tuple linguistic Bonferroni mean operators and their application to multiple attribute decision making," *International Journal of Fuzzy Systems*, vol. 19, no. 4, pp. 997–1010, 2017.

[19] G. W. Wei, "Picture uncertain linguistic Bonferroni mean operators and their application to multiple attribute decision making," *Kybernetes*, vol. 46, no. 10, pp. 1777–1800, 2017.

[20] S. Abdullah, S. Khan, M. Qiyas, and R. Chinram, "A novel approach based on sine trigonometric picture fuzzy aggregation operators and their application in decision support system," *Journal of Mathematics*, vol. 2021, Article ID 8819517, 19 pages, 2021.

[21] M. Qiyas, S. Abdullah, S. Ashraf, and M. Aslam, "Utilizing linguistic picture fuzzy aggregation operators for multiple-attribute decision-making problems," *International Journal of Fuzzy Systems*, vol. 22, pp. 310–320.

[22] H. M. A. Farid and M. Riaz, "Some generalized q-rung orthopair fuzzy Einstein interactive geometric aggregation operators with improved operational laws," *International Journal of Intelligent Systems*, vol. 36, pp. 7239–7273, 2021.

[23] M. Riaz and H. M. A. Farid, "Multicriteria decision-making with proportional distribution based spherical fuzzy fairly aggregation operators," *International Journal of Intelligent Systems*, vol. 37, no. 10, pp. 7079–7109, 2022.

[24] H. M. A. Farid, M Riaz, M. J. Khan, P. Kumam, and K. Sitthithakerngkiet, "Sustainable thermal power equipment supplier selection by Einstein prioritized linear Diophantine

fuzzy aggregation operators," *AIMS Mathematics*, vol. 7, no. 6, pp. 11201–11242, 2022.
[25] A. Saha, D. Dutta, and S. Kar, "Some new hybrid hesitant fuzzy weighted aggregation operators based on Archimedean and Dombi operations for multi-attribute decision making," *Neural Computing & Application*, vol. 33, no. 14, pp. 8753–8776, 2021.
[26] F. Feng, Y. Zheng, B. Sun, and M. Akram, "Novel score functions of generalized orthopair fuzzy membership grades with application to multiple attribute decision making," *Granular Computing*, vol. 7, pp. 95–111, 2022.
[27] M. Akram, A. Khan, J. C. R. Alcantud, and G. Santos-Garcia, "A hybrid decision-making framework under complex spherical fuzzy prioritized weighted aggregation operators," *Expert Systems*, vol. 38, no. 6, Article ID e12712, 2021.
[28] M. Riaz and H. M. A. Farid, "Hierarchical Medical Diagnosis Approach for COVID-19 Based on Picture Fuzzy Fairly Aggregation Operators," *International Journal of Biomathematics*, 2022.
[29] M. Riaz, H. M. A. Farid, W. Wang, and D. Pamucar, "Interval-Valued Linear Diophantine Fuzzy Frank Aggregation Operators with Multi-Criteria Decision-Making," *Mathematics*, vol. 10, p. 1811, 2022.
[30] K. Menger, "Statistical metrics," *Proceedings of the National Academy of Sciences*, vol. 28, no. 12, pp. 535–537, 1942.
[31] M. Xia, Z. Xu, and B. Zhu, "Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm," *Knowledge-Based Systems*, vol. 31, pp. 78–88, 2012.
[32] W. Wang and X. Liu, "Intuitionistic fuzzy information aggregation using Einstein operations," *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 5, pp. 923–938, 2012.
[33] P. Liu, "Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 1, pp. 83–97, 2014.
[34] E. P. Klement, R. Mesiar, and E. Pap, *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, The netherlands, 2000.
[35] J. Aczél and C. Alsina, "Characterizations of some classes of quasilinear functions with applications to triangular norms and to synthesizing judgements," *Aequationes Mathematicae*, vol. 25, no. 1, pp. 313–315, 1982.
[36] N. Wang, Q. Li, A. A. A. El-Latif, X. Yan, and X. Niu, "A novel hybrid multibiometrics based on the fusion of dual Iris, visible and thermal face images," in *Proceedings of the International Symposium on Biometrics and Security Technologies*, pp. 217–223, Chengdu, China, July, 2013.
[37] T. Senapati, G. Chen, and R. R. Yager, "Aczel–alsina aggregation operators and their application to intuitionistic fuzzy multiple attribute decision making," *International Journal of Intelligent Systems*, vol. 37, no. 2022, pp. 1529–1551.
[38] T. Senapati, G. Chen, R. Mesiar, and R. R. Yager, "Novel Aczel.Alsina operations-based interval-valued intuitionistic fuzzy aggregation operatorsandtheir applications in multiple attribute decision-making process," *International Journal of Intelligent Systems*, vol. 37, no. 8, pp. 5059–5081, 2021.
[39] T. Mahmood, K. Ullah, Q. Khan, and N. Jan, "An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets," *Neural Computing and Applications*, vol. 31, no. 11, pp. 7041–7053, 2019.
[40] F. K. Gundogdu and C. Kahraman, "Spherical fuzzy sets and spherical fuzzy TOPSIS method," *Journal of Intelligent & Fuzzy Systems*, vol. 36, no. 1, pp. 337–352, 2019.
[41] S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, and T. Mahmood, "Spherical fuzzy sets and their applications in multi-attribute decision making problems," *Journal of Intelligent & Fuzzy Systems*, vol. 36, no. 3, pp. 2829–2844, 2019.
[42] F. K. Gundogdu and C. Kahraman, "A novel fuzzy TOPSIS method using emerging interval-valued spherical fuzzy sets," *Engineering Applications of Artificial Intelligence*, vol. 85, pp. 307–323, 2019.
[43] F. K. Gundogdu and C. Kahraman, "Extension of WASPAS with spherical fuzzy sets," *Informatica*, vol. 30, no. 2, pp. 269–292, 2019.
[44] F. K. Gundogdu and C. Kahraman, "A novel VIKOR method using spherical fuzzy sets and its application to warehouse site selection," *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 1, pp. 1197–1211, 2019.
[45] S. Ashraf, S. Abdullah, and T. Mahmood, "GRA method based on spherical linguistic fuzzy Choquet integral environment and its application in multi-attribute decision-making problems," *Mathematical Sciences*, vol. 12, no. 4, pp. 263–275, 2018.
[46] S. Zeng, A. Hussain, T. Mahmood, M. I. Irfan Ali, S. Ashraf, and M. Munir, "Covering-based spherical fuzzy rough set model hybrid with TOPSIS for multi-attribute decision-Making," *Symmetry*, vol. 11, no. 4, p. 547, 2019.
[47] M. Rafiq, S. Ashraf, S. Abdullah, T. Mahmood, and S. Muhammad, "The cosine similarity measures of spherical fuzzy sets and their applications in decision making," *Journal of Intelligent & Fuzzy Systems*, vol. 36, no. 6, pp. 6059–6073, 2019.
[48] Y. Jin, S. Ashraf, and S. Abdullah, "Spherical fuzzy logarithmic aggregation operators based on entropy and their application in decision support systems," *Entropy*, vol. 21, no. 7, p. 628, 2019.
[49] S. Ashraf, S. Abdullah, and T. Mahmood, "Spherical fuzzy Dombi aggregation operators and their application in group decision making problems," *Journal of Ambient Intelligence and Humanized Computing*, vol. 11, no. 7, pp. 2731–2749, 2020.
[50] M. Jaller and I. Otay, "Evaluating sustainable vehicle technologies for freight transportation using spherical fuzzy AHP and TOPSIS," in *Intelligent and Fuzzy Techniques: Smart and Innovative Solutions. INFUS 2020*, C. Kahraman, S. Cevik Onar, B. Oztaysi, I. Sari, S. Cebi, and A. Tolga, Eds., , vol. 1197, Cham, Switzerland, Springer.
[51] Z. Ali, T. Mahmood, K. Mahmood, and Q. Khan, "Einstein Geometric Aggregation Operators using a Novel Complex Interval-valued Pythagorean Fuzzy Setting with Application in Green Supplier Chain Management," *Reports in Mechanical Engineering*, vol. 2, no. 1, pp. 105–134, 2021.
[52] A. Ashraf, K. Ullah, A. Hussain, and M. Bari, "Interval-Valued Picture Fuzzy Maclaurin Symmetric Mean Operator with application in Multiple Attribute Decision-Making," *Reports in Mechanical Engineering*, vol. 3, no. 1, pp. 301–317, 2022.
[53] N. Kazemitash, H. Fazlollahtabar, and M. Abbaspour, "Rough Best-Worst Method for Supplier Selection in Biofuel Companies based on Green criteria," *Operational Research in Engineering Sciences: Theory and Applications*, vol. 4, no. 2, pp. 1–12, 2021.
[54] D. Bozanic, A. Milic, D. Tešic, W. Salabun, and D. Pamucar, "D numbers – FUCOM – Fuzzy RAFSI model for selecting the group of construction machines for enabling mobility," *Facta Universitatis, Series: Mechanical Engineering*, vol. 19, no. 3, pp. 447–471, 2021.

[55] I. Mukhametzyanov, "Specific character of objective methods for determining weights of criteria in MCDM problems: Entropy, CRITIC and SD," *Decision Making: Applications in Management and Engineering*, vol. 4, no. 2, pp. 76–105, 2021.

[56] R. Sahu, S. R. Dash, and S. Das, "Career selection of students using hybridized distance measure based on picture fuzzy set and rough set theory," *Decision Making: Applications in Management and Engineering*, vol. 4, no. 1, pp. 104–126, 2021.

[57] Ç. Karamasa, D. Karabasevic, D. Stanujkic, A. Kookhdan, A. Mishra, and M. Ertürk, "An extended single-valued neutrosophic AHP and MULTIMOORA method to evaluate the optimal training aircraft for flight training organizations," *Facta Universitatis, Series: Mechanical Engineering*, vol. 19, no. 3, pp. 555–578, 2021.

[58] M. Riaz, N. Cagman, N. Wali, and A. Mushtaq, "Certain Properties of Soft multi-set topology with applications in multi-criteria decision making," *Decision Making Applications in Management and Engineering*, vol. 3, no. 2, pp. 70–96, 2020.

[59] S. Ashraf and S. Abdullah, "Emergency decision support modeling for COVID-19 based on spherical fuzzy information," *International Journal of Intelligent Systems*, vol. 35, no. 11, pp. 1–45, 2020.

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Research Article

Application of Hamacher Aggregation Operators in the Selection of the Cite for Pilot Health Project based on Complex T-spherical Fuzzy Information

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The framework of complex T-spherical fuzzy set (CTSFS) deals with unclear and imprecise information with the help of membership degree (MD), abstinence degree (AD), nonmembership degree (NMD), and refusal degree (RD). Due to this characteristic, the CTSFSs can be applied to any phenomenon having the involvement of human opinions. This article aims to familiarize some Hamacher aggregation operators (HAOs) grounded on CTSFSs. To do so, we define some Hamacher operational laws in the environment of CTSFS by using Hamacher t-norm (HTNM) and Hamacher t-conorm (HTCNM). A few numbers of AOs are developed with the help of defined operational laws based on HTNM and HTCNM including the complex T-spherical fuzzy (CTSF), Hamacher weighted averaging (HWA) (CTSFHWA), CTSF Hamacher ordered weighted averaging (CTSFHOWA) operator, CTSF Hamacher hybrid weighted averaging (CTSFHHWA) operator, CTSF Hamacher weighted geometric (CTSFHWG) operator, CTSF Hamacher ordered weighted geometric (CTSFHOWG) operator, and CTSF Hamacher hybrid weighted geometric (CTSFHHWG) operator. Some interesting properties of developed HAOs are investigated and then these HAOs are applied to the multi-attribute decision making (MADM) problem. For the significance of these HAOs, the results obtained from these HAOs are compared with existing aggregation operators (AOs).

1. Introduction

The phenomenon of uncertainty and imperfect information has always perturbed mathematicians during the analysis of data. To rectify this problem, many theories have been presented. These theories strive to correct the inaccuracies prevalent in real-life problems. All these theories are composed of certain characteristics. Furthermore, they have their own merits and demerits but among these Zadeh's [1] theory of fuzzy set (FS) has a distinguished place. This theory deals with certain conceptions that are a pivotal part of our everyday lives, such as decision making, clustering, recognition of patterns, and various fields of computer and engineering. In this remarkable theory, Zadeh presented a concept of FS, which deals with uncertainties by comprehending them in terms of MD, which range on a scale of zero to one. This kind of mathematical representation has enabled mathematicians to describe the uncertainty of any given object or event in a numerical form. But the only demerit of Zadeh's theory was the lack of the notion of nonmembership for an object. To overcome this lack-ness, Atanassov [2] evolved Zadeh's concept of fuzzy set and presented a proposition of intuitionistic fuzzy set (IFS) by introducing MD and NMD. These concepts described the vagueness with some restrictions. This concept imposed a constraint on the sum value of both MD and NMD that restricted the value not to exceed 1. Yager [3] strengthen this concept by infusing an idea of the Pythagorean fuzzy set (PyFS), which expands the range of assigning values of MD and NMD. Moreover, another remarkable contribution has done by Yager [4] by introducing the model of q-rung Orthopair fuzzy set (q-ROFS). This unique framework assists to increase the value of the Atanassov intuitionistic fuzzy set (IFS) to infinity by inculcating the parameter q in them.

These concepts of IFS, PyFS, and q-ROFS work in unison to cater the real-life problems involving vagueness and uncertainty. But the only deficiency was their deficiency of degrees to express the phenomenon of favor and disfavor. As the human opinion is not simply constricted to a well said yes or no, rather it consists of multiple variations. A human opinion has also some sort of abstinence and refusal degree as well. Cuong and Kreinovich [5] tried to take into account this phenomenon. He stated that dealing with IFS and its generalized forms of MD and NMD, AD, and RD get ignored, which resultantly leads to the considerable loss of information. To resolve this loss, Cuong suggested the idea of a picture fuzzy set (PFS) in a form of a triplet that incorporates MD, NMD, AD, and RD with a restriction that their subsequent sum should not exceed the value of 1. To reduce this restriction, Mahmood et al. [6] extended this concept to a border level by delineating unique spherical fuzzy set (SFS) and T-spherical fuzzy set (TSFS) sets.

Ramot et al. [7] realized that the generalized frameworks aforementioned do not cover the information from the complex plane. Ramot et al. [7] studied to involve the complex plane FS and gave the idea of complex FS (CFS) by taking the complex numbers instead of the real numbers. This concept of the CFS extended the FS to the complex plane, but many complex numbers in the unit circle could not be part of the CFS. Consequently, the idea of complex IFS (CIFS) was presented by Alkouri and Salleh [8] to provide a huge platform for decision-makers to extract the larger information as compared to the CFS. To develop CIFSs, Alkouri and Salleh [8] used the MD and NMD in the form of a complex number from the unit circle in a complex plane. But the sum of the real and imaginary parts of the MD and NMD of the numbers in the CIFS was restricted within a unit circle. But the problem occurred when decision-makers choose the degrees of both real and imaginary parts whose sum exceeded a unit circle. Ullah et al. [9] covered the larger information than the CIFSs by extending the sum of degrees to the sum of their squares by introducing the complex PyFSs (CPyFSs). Liu et al. [10] improved the concept of CPyFSs by taking any integer to the power of MD and NMD and introduced the notion of complex q-ROFS (Cq-ROFS).

The theory of CIFS and CPyFSs has been used with applications in different fields of daily life. However, these frameworks do not entertain the phenomenon when there are four aspects to describe an object, especially whenever the human opinion is involved. For an instance, with $0.7e^{2\pi i(0.5)}, 0.3e^{2\pi i(0.4)}, 0.6e^{2\pi i(0.5)}$, CIFS, or CPFS fails to handle such situations because the decision-makers are restricted. To cope with this problem, Ali et al. [11] introduced the theory of complex spherical fuzzy sets (CSFSs) and CTSFS with some more flexible restrictions that the degrees of both real and imaginary parts of MD, NMD, and AD whose sum of a square and q power up to infinity cannot exceed from a unit circle.

The most advanced technique is to find the best alternative from a set of some specific alternatives based on the multiple criteria that often contrast each other. After the development of the above-mentioned frameworks, the MADM has become a very popular technique because the results obtained by MADM are based on the most reliable aggregation operators (AOs). Xu [12] emphasized AOs on IFS and applied them in MADM. Wei and Lu [13] used the PyFSs to develop AOs and then applied these AOs in MADM. Liu and Wang [14] improved the MADM by developing AOs for the basis of q-ROFS. Garg [15] applied some AOs in MADM based on PFSs. Zhou et al. [16] introduced power AOs for the enhancement of MADM by using the TSFS information. Some interesting work on the AOs can be found in references [17–20]. Interestingly, some basic operational laws are involved in the formation of these AOs. These laws are based on some triangular norms [21] to obtain flexibility. Wu et al. [22] developed Dombi AOs by using the Dombi t-norm (TNM) and t-conorm (TCNM) based on IFS and applied in the MADM. Akram et al. [23] formed the Dombi AOs and applied them to PyFS to solve the MADM problem. Wang and Liu [24] used the Einstein TNM and TCNM to develop AOs for the environment of IFS and applied them in MADM. Riaz et al. [25] presented AOs by using Einstein TNM and TCNM for the environment of q-ROFSs and gave a desirable application in supply chain management. Fahmi et al. [26] applied Einstein TNM and TCNM for the development of the AOs for application MADM. Senapati et al. [27] introduced Aczel-Alsina AOs in the environment of IFSs. In reference [28] Yang et al. developed some interval-valued PyFS AOs based on Frank TNM and TCNM. A few of AOs that are based on some other TNMs and TCNMs are referred to in references [29, 30].

The TNMs and TCNMs aforementioned play an important role in the development of the AOs that have a great impact on the application in MADM. Among these TNMs and TCNMs, the HTNM and HTCNM [31] are very impressive and have been used widely by researchers in almost all of the developed models of the FS theory. Garg [32] applied HTNM and HTCNM to IFS and formalized AOs. The HTNM and HTCNM have also been used by Wu and Wei [33] in the formalization of the AOs for the PyFS. Darko and Liang [34] introduced some AOs by using HTNM and HTCNM for the q-ROFS. Ullah et al. [35] evaluated an investment by introducing AOs based on HTNM and HTCNM for the TSFS. Zhao et al. [36] established generalized AOs for IFSs. Wu and Wei [33] developed and applied PyFS AOs based on HTNM and HTCNM in decision making. The remarkable literature can be found in reference [37–39].

It has been justified above that the CTSFS covers the huge loss of information while we extract information from any real-life phenomenon to perform decision making. Especially it is very effective to extract the most possible information whenever the human's opinion is involved. Hence, the use of CTSFS in MADM has a great chance to improve the results in the MADM. We also have noted the significance of HTNM and HTCNM in [40] where Klement and Navara did a survey on different types of TNs and TCNs and got different rankings and found the significant results for the HTNM and HTCNM. The main motivations for this article are (i) the significance of the HTNM and HTCNM while applying in frameworks of FS and (ii) the reduction of loss of information with the help of CTSFS. The main aspects of this article are to introduce some Hamacher operational laws grounded on CTSFSs and then to apply these operations to develop CTSFHWA and CTFSHWG AOs.

There are 6 further sections as we stated some elementary notions supportive of this article in Section 2. We stated the basic definition of CTSFS, the core function for ranking, HTNM, and HTCNM in this section. In Section 3, we developed some operations for the complex T-spherical fuzzy numbers (CTSFNs), which include the Hamacher sum and product of two CTSFNs, scalar multiplication, and the power operation for the CTSFNs. We developed some average AOs based on HTNM and HTCNM, i.e., CTSFHWA, CTSFOWA, and CTSFHHWA operators, and investigated their properties in Section 4. In Section 5, we formalized geometric AOs based on HTNM and HTCNM, i.e., CTSFHWG, CTSFHOWG, and CTSFHHWG operators, and studied some basic properties of these AOs. We stated the procedure to apply the proposed approach to the MADM problem in Section 6, then we applied it to a MADM problem. We also compared our proposed HAOs with some existing AOs and gave their graphical representation. In Section 7, we concluded our study.

2. Preliminaries

In this section, we have defined necessary preliminary concepts linked to CTSFS introduced over set X, some remarks are also explained to clear the concept. HTNM and HTCNM are also defined in this section.

Definition 1. [11] A CTSFS on a set X is defined by $I = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}\colon x \in X)$, where $r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}$, and $r_n(x).e^{2\pi i\theta_n(x)}$ are complex numbers in a unit circle denoting a complex MD, complex AD, and complex NMD with the conditions $0 \leq r_m^q(x) + r_i^q(x) + r_n^q(x) \leq 1$ and $0 \leq \theta_m^q(x) + \theta_i^q(x) + \theta_n^q(x) \leq 1$ for $q \in \mathbb{Z}^+$. The complex RD is defined by $r_h(x).e^{2\pi i\theta_h(x)}$, where $r_h(x) = \sqrt[q]{1 - (r_m^q(x) + r_i^q(x) + r_n^q(x))}$, $\theta_h(x) = \sqrt[q]{1 - (\theta_m^q(x) + \theta_i^q(x) + \theta_n^q(x))}$, and triplet $(r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$ is known as CTSFN.

Definition 2. [11] For a CTSF $I = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}\colon x \in X)$, the score function is defined by.

$$SC(I) = \frac{((r_m^q - r_i^q - r_n^q) + (\theta_m^q - \theta_i^q - \theta_n^q))}{2}, \quad SC(I) \in [-1, 1]. \tag{1}$$

Definition 3. [31] The HTNM and HTCNM are defined as

$$T_{hn}(l,m) = \frac{l.m}{\mathscr{L} + (1-\mathscr{L})(l+m-lm)}, \quad \mathscr{L} > 0, (l,m) \in [0,1]^2,$$
$$T_{hcn}(l,m) = \frac{l+m-lm-(1-\mathscr{L})lm}{1-(1-\mathscr{L})lm}, \quad \mathscr{L} > 0, (l,m) \in [0,1]^2. \tag{2}$$

Further, the Hamacher product and Hamacher sum are denoted as $T_{hn}(l,m)$ and $T_{hcn}(l,m)$ respectively, which are given below:

$$l \otimes m = \frac{l.m}{\mathscr{L} + (1-\mathscr{L})(l+m-lm)}, \quad \mathscr{L} > 0\,(l,m) \in [0,1]^2,$$
$$l \oplus m = \frac{l+m-lm-(1-\mathscr{L})lm}{1-(1-\mathscr{L})lm}, \quad \mathscr{L} > 0\,(l,m) \in [0,1]^2. \tag{3}$$

3. Hamacher Operations based on CTSFNs

In this section, we will define sum, products, scalar multiplication, and the power operation for two or more two CTSFNs based on HTNM and HTCNM.

Definition 4. Let

$$A = \left(r_{m_A}(x).e^{2\pi i\theta_{m_A}(x)}, r_{i_A}(x).e^{2\pi i\theta_{i_A}(x)}, r_{n_A}(x).e^{2\pi i\theta_{n_A}(x)}\right), \tag{4}$$

and

$$B = \left(r_{m_B}(x).e^{2\pi i\theta_{m_B}(x)}, r_{i_B}(x).e^{2\pi i\theta_{i_B}(x)}, r_{n_B}(x).e^{2\pi i\theta_{n_B}(x)}\right), \tag{5}$$

be the two CTSFNs, where $\lambda, \mathscr{L} > 0$. The Complex T-spherical fuzzy Hamacher (CTSFH) operations are defined as

$A \oplus B$,

$$\left(\begin{array}{c} \dfrac{r_{mA}(x) \cdot r_{mB}(x)}{\sqrt[q]{\mathscr{L} + (1-\mathscr{L})\left(r_{mA}^q(x) + r_{mB}^q(x) - r_{mA}^q(x) \cdot r_{mB}^q(x)\right)}} \cdot e^{\theta_{mA}^q(x) \cdot \theta_{mA}^q(x) / \sqrt[q]{\mathscr{L} + (1-\mathscr{L})\left(\theta_{mA}^q(x) + \theta_{mB}^q(x) - \theta_{mA}^q(x) \cdot \theta_{mB}^q(x)\right)}} \\ \dfrac{r_{iA}(x) \cdot r_{iB}(x)}{\sqrt[q]{\mathscr{L} + (1-\mathscr{L})\left(r_{iA}^q(x) + r_{iB}^q(x) - r_{iA}^q(x) \cdot r_{iB}^q(x)\right)}} e^{\theta_{iA}^q(x) \cdot \theta_{iA}^q(x) / \sqrt[2\pi i q]{\theta_{iA}^q(x) + \theta_{iA}^q(x) - \theta_{iA}^q(x) \cdot \theta_{iB}^q(x)}} \\ \dfrac{r_{nA}(x) \cdot r_{nB}(x)}{\sqrt[q]{\mathscr{L} + (1-\mathscr{L})\left(r_{nA}^q(x) + r_{nB}^q(x) - r_{nA}^q(x) \cdot r_{nB}^q(x)\right)}} \cdot e^{\theta_{nA}(x) \cdot \theta_{nB}(x) / \sqrt[q]{\mathscr{L} + (1-\mathscr{L})\left(\theta_{nA}^q(x) + \theta_{nB}^q(x) - \theta_{nA}^q(x) \cdot \theta_{nB}^q(x)\right)}} \end{array}\right), \tag{6}$$

$A \otimes B$,

$$\left(\begin{array}{c} \dfrac{r_{mA}(x) \cdot r_{mB}(x)}{\sqrt[q]{\mathscr{L} + (1-\mathscr{L})\left(r_{mA}^q(x) + r_{mB}^q(x) - r_{mA}^q(x) \cdot r_{mB}^q(x)\right)}} \cdot e^{\theta_{mA}^q(x) \cdot \theta_{mA}^q(x) / \sqrt[q]{\mathscr{L} + (1-\mathscr{L})\left(\theta_{mA}^q(x) + \theta_{mB}^q(x) - \theta_{mA}^q(x) \cdot \theta_{mB}^q(x)\right)}} \\ \sqrt[q]{\dfrac{r_{i_A}^q(x) + r_{i_B}^q(x) - r_{i_A}^q(x) \cdot r_{i_B}^q - (1-\mathscr{L}) \cdot r_{i_A}^q(x) \cdot r_{i_B}^q(x)}{1 - \left((1-\mathscr{L}) \cdot r_{i_A}^q(x) \cdot r_{i_B}^q\right)}} \cdot e^{\sqrt[2\pi i q]{2\theta_{i_A}^q(x) + \theta_{i_B}^q(x) - \theta_{i_A}^q(x) \cdot \theta_{i_B}^q - (1-\mathscr{L}) \cdot \theta_{i_A}^q(x) \cdot \theta_{i_B}^q(x2)/1 - \left((1-\mathscr{L}) \cdot \theta_{i_A}^q(x) \cdot \theta_{i_B}^q(x)\right)}} \\ \sqrt[q]{\dfrac{r_{n_A}^q(x) + r_{n_B}^q(x) - r_{n_A}^q(x) \cdot r_{n_B}^q(x) - (1-\mathscr{L}) \cdot r_{n_A}^q(x) \cdot r_{n_B}^q(x)}{1 - \left((1-\mathscr{L}) \cdot r_{n_A}^q(x) \cdot r_{n_B}^q(x)\right)}} \cdot e^{\sqrt[2\pi i q]{2\theta_{n_A}^q(x) + \theta_{n_B}^q(x) - \theta_{n_A}^q(x) \cdot \theta_{n_B}^q - (1-\mathscr{L}) \cdot \theta_{n_A}^q(x) \cdot \theta_{n_B}^q(x)/1 - \left((1-\mathscr{L}) \cdot \theta_{n_A}^q(x) \cdot \theta_{n_B}^q(x)\right)}} \end{array}\right), \tag{7}$$

$$\lambda A = \left(\begin{array}{c} \sqrt[q]{\dfrac{\left(1 + (\mathscr{L}-1) r_{mA}^q(x)\right)^\lambda - \left(1 - r_{mA}^q(x)\right)^\lambda}{\left(1 + (\mathscr{L}-1) r_{mA}^q(x)\right)^\lambda + (\mathscr{L}-1)\left(1 - r_{mA}^q(x)\right)^\lambda}} . e^{2\pi i \sqrt[q]{\left(1 + (\mathscr{L}-1)\theta_{mA}^q(x)\right)^\lambda - \left(1 - \theta_{mA}^q(x)\right)^\lambda / \left(1 + (\mathscr{L}-1)\theta_{mA}^q(x)\right)^\lambda + (\mathscr{L}-1)\left(1 - \theta_{mA}^q(x)\right)^\lambda}} \\ \dfrac{\sqrt[q]{\mathscr{L}}\left(r_{iA}(x)\right)^\lambda}{\sqrt[q]{\left(1 + (\mathscr{L}-1)\left(1 - r_{i_A}^q(x)\right)\right)^\lambda + (\mathscr{L}-1)\left(r_{i_A}^q(x)\right)^\lambda}} . e^{2\pi i \left(\sqrt[q]{\mathscr{L}}\left(\theta_{iA}(x)\right)^\lambda / \sqrt[q]{\left(1 + (\mathscr{L}-1)\left(1 - \theta_{i_A}^q(x)\right)\right)^\lambda + (\mathscr{L}-1)\left(\theta_{i_A}^q(x)\right)^\lambda}\right)} \\ \dfrac{\sqrt[q]{\mathscr{L}}\left(r_{nA}(x)\right)^\lambda}{\sqrt[q]{\left(1 + (\mathscr{L}-1)\left(1 - r_{n_A}^q(x)\right)\right)^\lambda + (\mathscr{L}-1)\left(r_{n_A}^q(x)\right)^\lambda}} . e^{2\pi i \left(\sqrt[q]{\mathscr{L}}\left(\theta_{nA}(x)\right)^\lambda / \sqrt[q]{\left(1 + (\mathscr{L}-1)\left(1 - \theta_{nA}^q(x)\right)\right)^\lambda + (\mathscr{L}-1)\left(\theta_{nA}^q(x)\right)^\lambda}\right)} \end{array}\right), \tag{8}$$

$$A^\lambda = \left(\begin{array}{c} \dfrac{\sqrt[q]{\mathscr{L}}\left(r_{mA}(x)\right)^\lambda}{\sqrt[q]{\left(1 + (\mathscr{L}-1)\left(1 - r_{m_A}^q(x)\right)\right)^\lambda + (\mathscr{L}-1)\left(r_{m_A}^q(x)\right)^\lambda}} . e^{2\pi i \left(\sqrt[q]{\mathscr{L}}\left(\theta_{mA}(x)\right)^\lambda / \sqrt[q]{\left(1 + (\mathscr{L}-1)\left(1 - \theta_{mA}^q(x)\right)\right)^\lambda + (\mathscr{L}-1)\left(\theta_{mA}^q(x)\right)^\lambda}\right)} \\ \sqrt[q]{\dfrac{\left(1 + (\mathscr{L}-1) r_{i_A}^q(x)\right)^\lambda - \left(1 - r_{i_A}^q(x)\right)^\lambda}{\left(1 + (\mathscr{L}-1) r_{i_A}^q(x)\right)^\lambda + (\mathscr{L}-1)\left(1 - r_{i_A}^q(x)\right)^\lambda}} . e^{2\pi i \sqrt[q]{\left(\left(1 + (\mathscr{L}-1)\theta_{iA}^q(x)\right)^\lambda - \left(1 - \theta_{iA}^q(x)\right)^\lambda\right) / \left(\left(1 + (\mathscr{L}-1)\theta_{i_A}^q(x)\right)^\lambda + (\mathscr{L}-1)\left(1 - \theta_{i_A}^q(x)\right)^\lambda\right)}} \\ \sqrt[q]{\dfrac{\left(1 + (\mathscr{L}-1) r_{n_A}^q(x)\right)^\lambda - \left(1 - r_{n_A}^q(x)\right)^\lambda}{\left(1 + (\mathscr{L}-1) r_{n_A}^q(x)\right)^\lambda + (\mathscr{L}-1)\left(1 - r_{n_A}^q(x)\right)^\lambda}} . e^{2\pi i \sqrt[q]{\left(\left(1 + (\mathscr{L}-1)\theta_{nA}^q(x)\right)^\lambda - \left(1 - \theta_{nA}^q(x)\right)^\lambda\right) / \left(\left(1 + (\mathscr{L}-1)\theta_{nA}^q(x)\right)^\lambda + (\mathscr{L}-1)\left(1 - \theta_{nA}^q(x)\right)^\lambda\right)}} \end{array}\right). \tag{9}$$

It is noted that the developed operations in equations (6) to (9) based on TNM and TCNM are the generalized form of the existing operations and they give comparatively better results.

4. Weighted Average Operator Based on CTSFS

This section consists of new developed CTSFHWA, CTSFHOWA, and CTSFHHWA operators and their properties. Note that we will use only w_j for the weight vector for $w_j = (w_1, w_2, \ldots, w_l)^T$ having $w_j > 0$ and $\sum_1^l w_j = 1$, where $j = \{1, 2, 3, \ldots, l\}$.

Definition 5. Suppose $T_j = (r_m(x).e^{2\pi i \theta_m(x)}, r_i(x).e^{2\pi i \theta_i(x)}, r_n(x).e^{2\pi i \theta_n(x)})$, $\forall j = 1, 2, 3, \ldots l.$ be some CTSFNs, then CTSFHWA operator $T^l \longrightarrow T$ is defined as

$$\text{CTSFHWA}(T_1, T_2, T_3, \ldots T_l) = \sum_{j=1}^{l} w_j T_j. \tag{10}$$

In Theorem 1, we give an interesting result by using Definition 5, as following.

Theorem 1. *Let $T_j = (r_m(x).e^{2\pi i \theta_m(x)}, r_i(x).e^{2\pi i \theta_i(x)}, r_n(x).e^{2\pi i \theta_n(x)})$ $\forall j = 1, 2, 3, \ldots l.$ be some CTSFNs. Then, CTSFHWA operator is a CTSFN and given by.*

$$\text{CTSFHWA}(T_1, T_2, T_3, \ldots T_l) = \begin{pmatrix} \sqrt[q]{\dfrac{\prod_{j=1}^{l}\left(1+(L-1)r_{m_j}^q\right)^{w_j} - \prod_{j=1}^{l}\left(1-r_{m_j}^q\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(L-1)r_{m_j}^q\right)^{w_j} + (L-1)\prod_{j=1}^{l}\left(1-r_{m_j}^q\right)^{w_j}}} \cdot e^{\left(2\pi i \sqrt[q]{\prod_{j=1}^{l}\left(1+(L-1)\theta_{m_j}^q\right)^{w_j} - \prod_{j=1}^{l}\left(1-\theta_{m_j}^q\right)^{w_j} / \prod_{j=1}^{l}\left(1+(L-1)\theta_{m_j}^q\right)^{w_j} + (L-1)\prod_{j=1}^{l}\left(1-\theta_{m_j}^q\right)^{w_j}}\right)} \\ \dfrac{\sqrt[q]{L}\prod_{j=1}^{l}\left(r_{i_j}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(L-1)\left(1-r_{i_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{l}\left(1-r_{i_j}^q\right)^{w_j}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{l}\left(\theta_{i_j}\right)^{w_j} / \sqrt[q]{\prod_{j=1}^{l}\left(1+(L-1)\left(1-\theta_{i_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{l}\left(1-\theta_{i_j}^q\right)^{w_j}}\right)} \\ \dfrac{\sqrt[q]{L}\prod_{j=1}^{l}\left(r_{n_j}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(L-1)\left(1-r_{n_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{l}\left(1-r_{n_j}^q\right)^{w_j}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{l}\left(\theta_{n_j}\right)^{w_j} / \sqrt[q]{\prod_{j=1}^{l}\left(1+(L-1)\left(1-\theta_{n_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{l}\left(1-\theta_{n_j}^q\right)^{w_j}}\right)} \end{pmatrix} \tag{11}$$

Proof. Here we have used mathematical induction method to prove.

Suppose $l = 2$.

$$w_1T_1 \oplus w_2T_2 = \begin{pmatrix} \sqrt[q]{\dfrac{\left(1+(L-1)r_{m_1}^q\right)^{w_1} - \left(1-r_{m_1}^q\right)^{w_1}}{\left(1+(L-1)r_{m_1}^q\right)^{w_1} + (L-1)\left(1-r_{m_1}^q\right)^{w_1}}} \cdot e^{\left(\sqrt[q]{\left(1+(L-1)\theta_{m_1}^q\right)^{w_1} - \left(1-\theta_{m_1}^q\right)^{w_1} / \left(1+(L-1)\theta_{m_1}^q\right)^{w_1} + (L-1)\left(1-\theta_{m_1}^q\right)^{w_1}}\right)} \\ \dfrac{\sqrt[q]{L}\left(r_{i_1}\right)^{w_1}}{\sqrt[q]{\left(1+(L-1)\left(1-r_{i_1}^q\right)\right)^{w_1} + (L-1)\left(1-r_{i_1}^q\right)^{w_1}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\left(\theta_{i_1}\right)^{w_1} / \sqrt[q]{\left(1+(L-1)\left(1-\theta_{i_1}^q\right)\right)^{w_1} + (L-1)\left(1-\theta_{i_1}^q\right)^{w_1}}\right)} \\ \dfrac{\sqrt[q]{L}\left(r_{n_1}\right)^{w_1}}{\sqrt[q]{\left(1+(L-1)\left(1-r_{n_1}^q\right)\right)^{w_1} + (L-1)\left(1-r_{n_1}^q\right)^{w_1}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\left(\theta_{n_1}\right)^{w_1} / \sqrt[q]{\left(1+(L-1)\left(1-\theta_{n_1}^q\right)\right)^{w_1} + (L-1)\left(1-\theta_{n_1}^q\right)^{w_1}}\right)} \end{pmatrix}$$

$$\oplus \begin{pmatrix} \sqrt[q]{\dfrac{\left(1+(L-1)r_{m_2}^q\right)^{w_2} - \left(1-r_{m_2}^q\right)^{w_2}}{\left(1+(L-1)r_{m_2}^q\right)^{w_2} + (L-1)\left(1-r_{m_2}^q\right)^{w_2}}} \cdot e^{\left(\sqrt[q]{\left(1+(L-1)\theta_{m_2}^q\right)^{w_2} - \left(1-\theta_{m_2}^q\right)^{w_2} / \left(1+(L-1)\theta_{m_2}^q\right)^{w_2} + (L-1)\left(1-\theta_{m_2}^q\right)^{w_2}}\right)} \\ \dfrac{\sqrt[q]{L}\left(r_{i_2}\right)^{w_2}}{\sqrt[q]{\left(1+(L-1)\left(1-r_{i_2}^q\right)\right)^{w_2} + (L-1)\left(1-r_{i_2}^q\right)^{w_2}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\left(\theta_{i_2}\right)^{w_2} / \sqrt[q]{\left(1+(L-1)\left(1-\theta_{i_2}^q\right)\right)^{w_2} + (L-1)\left(1-\theta_{i_2}^q\right)^{w_2}}\right)} \\ \dfrac{\sqrt[q]{L}\left(r_{n_2}\right)^{w_2}}{\sqrt[q]{\left(1+(L-1)\left(1-r_{n_2}^q\right)\right)^{w_2} + (L-1)\left(1-r_{n_2}^q\right)^{w_2}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\left(\theta_{n_2}\right)^{w_2} / \sqrt[q]{\left(1+(L-1)\left(1-\theta_{n_2}^q\right)\right)^{w_2} + (L-1)\left(1-\theta_{n_2}^q\right)^{w_2}}\right)} \end{pmatrix}$$

Table 1: CTSF information (decision matrix).

	G_1	G_2	G_3	G_4
A_1	$\left(0.5e^{2\pi i(0.3)}, 0.6e^{2\pi i(0.5)}, 0.4e^{2\pi i(0.2)}\right)$	$\left(0.4e^{2\pi i(0.3)}, 0.5e^{2\pi i(0.3)}, 0.3e^{2\pi i(0.2)}\right)$	$\left(0.4e^{2\pi i(0.7)}, 0.4e^{2\pi i(0.5)}, 0.5e^{2\pi i(0.6)}\right)$	$\left(0.8e^{2\pi i(0.5)}, 0.1e^{2\pi i(0.2)}, 0.8e^{2\pi i(0.1)}\right)$
A_2	$\left(0.6e^{2\pi i(0.6)}, 0.3e^{2\pi i(0.2)}, 0.5e^{2\pi i(0.5)}\right)$	$\left(0.5e^{2\pi i(0.2)}, 0.4e^{2\pi i(0.5)}, 0.5e^{2\pi i(0.7)}\right)$	$\left(0.5e^{2\pi i(0.6)}, 0.3e^{2\pi i(0.5)}, 0.6e^{2\pi i(0.3)}\right)$	$\left(0.4e^{2\pi i(0.6)}, 0.6e^{2\pi i(0.5)}, 0.5e^{2\pi i(0.4)}\right)$
A_3	$\left(0.8e^{2\pi i(0.4)}, 0.4e^{2\pi i(0.4)}, 0.3e^{2\pi i(0.5)}\right)$	$\left(0.7e^{2\pi i(0.9)}, 0.3e^{2\pi i(0.2)}, 0.9e^{2\pi i(0.3)}\right)$	$\left(0.3e^{2\pi i(0.4)}, 0.5e^{2\pi i(0.4)}, 0.4e^{2\pi i(0.4)}\right)$	$\left(0.5e^{2\pi i(0.3)}, 0.3e^{2\pi i(0.4)}, 0.2e^{2\pi i(0.3)}\right)$
A_4	$\left(0.7e^{2\pi i(0.5)}, 0.3e^{2\pi i(0.5)}, 0.5e^{2\pi i(0.6)}\right)$	$\left(0.5e^{2\pi i(0.4)}, 0.3e^{2\pi i(0.5)}, 0.4e^{2\pi i(0.3)}\right)$	$\left(0.1e^{2\pi i(0.3)}, 0.4e^{2\pi i(0.3)}, 0.1e^{2\pi i(0.2)}\right)$	$\left(0.4e^{2\pi i(0.3)}, 0.5e^{2\pi i(0.1)}, 0.3e^{2\pi i(0.5)}\right)$

$$= \begin{pmatrix} \sqrt[q]{\dfrac{\prod_{j=1}^{2}\left(1+(L-1)r_{m_j}^{q}\right)^{w_j}-\prod_{j=1}^{2}\left(1-r_{m_j}^{q}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+(L-1)r_{m_j}^{q}\right)^{w_j}+(L-1)\prod_{j=1}^{2}\left(1-r_{m_j}^{q}\right)^{w_j}}}\cdot e^{\left(2\pi i\sqrt[q]{\prod_{j=1}^{l}\left(1+(L-1)\theta_{m_j}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-\theta_{m_j}^{q}\right)^{w_j}/\prod_{j=1}^{l}\left(1+(L-1)\theta_{m_j}^{q}\right)^{w_j}+(L-1)\prod_{j=1}^{l}\left(1-\theta_{m_j}^{q}\right)^{w_j}}\right)} \\ \dfrac{\sqrt[q]{L}\prod_{j=1}^{2}\left(r_{i_j}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+(L-1)\left(1-r_{i_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{2}\left(1-r_{i_j}^{q}\right)^{w_j}}\cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{2}\left(\theta_{i_j}\right)^{w_j}/\prod_{j=1}^{2}\left(1+(L-1)\left(1-\theta_{i_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{2}\left(1-\theta_{i_j}^{q}\right)^{w_j}\right)} \\ \dfrac{\sqrt[q]{L}\prod_{j=1}^{2}\left(r_{n_j}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+(L-1)\left(1-r_{n_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{2}\left(1-r_{n_j}^{q}\right)^{w_j}}\cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{2}\left(\theta_{n_j}\right)^{w_j}/\prod_{j=1}^{2}\left(1+(L-1)\left(1-\theta_{n_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{2}\left(1-\theta_{n_j}^{q}\right)^{w_j}\right)} \end{pmatrix}. \tag{12}$$

It is satisfied for $l = 2$.

Now, we have to prove true for $l = k \oplus 1$ by assuming $l = k$, then we have

$$\text{CTSFHWA}\left(T_1, T_2, T_3, \ldots T_k\right) \oplus T_{k+1} = \begin{pmatrix} \sqrt[q]{\dfrac{\prod_{j=1}^{k}\left(1+(L-1)r_{m_j}^{q}\right)^{w_j}-\prod_{j=1}^{k}\left(1-r_{m_j}^{q}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(L-1)r_{m_j}^{q}\right)^{w_j}+(L-1)\prod_{j=1}^{k}\left(1-r_{m_j}^{q}\right)^{w_j}}}\cdot e^{\left(\sqrt[q]{\prod_{j=1}^{k}\left(1+(L-1)\theta_{m_j}^{q}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\theta_{m_j}^{q}\right)^{w_j}/\prod_{j=1}^{k}\left(1+(L-1)\theta_{m_j}^{q}\right)^{w_j}+(L-1)\prod_{j=1}^{k}\left(1-\theta_{m_j}^{q}\right)^{w_j}}\right)} \\ \dfrac{\sqrt[q]{L}\prod_{j=1}^{k}\left(r_{i_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(L-1)\left(1-r_{i_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{k}\left(1-r_{i_j}^{q}\right)^{w_j}}\cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{k}\left(\theta_{i_j}\right)^{w_j}/\prod_{j=1}^{k}\left(1+(L-1)\left(1-\theta_{i_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{k}\left(1-\theta_{i_j}^{q}\right)^{w_j}\right)} \\ \dfrac{\sqrt[q]{L}\prod_{j=1}^{k}\left(r_{n_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(L-1)\left(1-r_{n_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{k}\left(1-r_{n_j}^{q}\right)^{w_j}}\cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{k}\left(\theta_{n_j}\right)^{w_j}/\prod_{j=1}^{k}\left(1+(L-1)\left(1-\theta_{n_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{k}\left(1-\theta_{n_j}^{q}\right)^{w_j}\right)} \end{pmatrix}$$

$$\oplus \begin{pmatrix} \sqrt[q]{\dfrac{\left(1+(L-1)r_{m_{k+1}}^{q}\right)^{w_{k+1}}-\left(1-r_{m_{k+1}}^{q}\right)^{w_{k+1}}}{\left(1+(L-1)r_{m_{k+1}}^{q}\right)^{w_{k+1}}+(L-1)\left(1-r_{m_{k+1}}^{q}\right)^{w_{k+1}}}}\cdot e^{\left(\sqrt[q]{\left(1+(L-1)\theta_{m_{k+1}}^{q}\right)^{w_{k+1}}-\left(1-\theta_{m_{k+1}}^{q}\right)^{w_{k+1}}/\left(1+(L-1)\theta_{m_{k+1}}^{q}\right)^{w_{k+1}}+(L-1)\left(1-\theta_{m_{k+1}}^{q}\right)^{w_{k+1}}}\right)} \\ \dfrac{\sqrt[q]{L}\left(r_{i_{k+1}}\right)^{w_{k+1}}}{\sqrt[q]{\left(1+(L-1)\left(1-r_{i_{k+1}}^{q}\right)\right)^{w_{k+1}}+(L-1)\left(1-r_{i_{k+1}}^{q}\right)^{w_{k+1}}}}\cdot e^{2\pi i\left(\sqrt[q]{L}\left(\theta_{i_{k+1}}\right)^{w_{k+1}}/\sqrt[q]{\left(1+(L-1)\left(1-\theta_{i_{k+1}}^{q}\right)\right)^{w_{k+1}}+(L-1)\left(1-\theta_{i_{k+1}}^{q}\right)^{w_{k+1}}}\right)} \\ \dfrac{\sqrt[q]{L}\left(r_{n_{k+1}}\right)^{w_{k+1}}}{\sqrt[q]{\left(1+(L-1)\left(1-r_{n_{k+1}}^{q}\right)\right)^{w_{k+1}}+(L-1)\left(1-r_{n_{k+1}}^{q}\right)^{w_{k+1}}}}\cdot e^{2\pi i\left(\sqrt[q]{L}\left(\theta_{n_{k+1}}\right)^{w_{k+1}}/\sqrt[q]{\left(1+(L-1)\left(1-\theta_{n_{k+1}}^{q}\right)\right)^{w_{k+1}}+(L-1)\left(1-\theta_{n_{k+1}}^{q}\right)^{w_{k+1}}}\right)} \end{pmatrix},$$

Table 2: CTSFHWA and CTSFHWG operators.

	CTSFHWA operator	CTSFHWG operator
A_1	$\begin{pmatrix} 0.4808e^{2\pi i(0.5142)} \\ 0.4538e^{2\pi i(0.3706)}, 0.3805e^{2\pi i(0.2553)} \end{pmatrix}$	$\begin{pmatrix} 0.4340e^{2\pi i(0.3828)} \\ 0.107e^{2\pi i(0.0543)}, 0.078e^{2\pi i(0.0566)} \end{pmatrix}$
A_2	$\begin{pmatrix} 0.5218e^{2\pi i(0.5066)} \\ 0.3589e^{2\pi i(0.4173)}, 0.5236e^{2\pi i(0.5219)} \end{pmatrix}$	$\begin{pmatrix} 0.5131e^{2\pi i(0.3487)} \\ 0.0386e^{2\pi i(0.0846)}, 0.1334e^{2\pi i(0.2301)} \end{pmatrix}$
A_3	$\begin{pmatrix} 0.6781e^{2\pi i(0.7873)} \\ 0.3614e^{2\pi i(0.2634)}, 0.568e^{2\pi i(0.3581)} \end{pmatrix}$	$\begin{pmatrix} 0.5798e^{2\pi i(0.6113)} \\ 0.0424e^{2\pi i(0.0155)}, 0.6112e^{2\pi i(0.0393)} \end{pmatrix}$
A_4	$\begin{pmatrix} 0.5341e^{2\pi i(0.4081)} \\ 0.3308e^{2\pi i(0.407)}, 0.2922e^{2\pi i(0.3208)} \end{pmatrix}$	$\begin{pmatrix} 0.3567e^{2\pi i(0.384)} \\ 0.0256e^{2\pi i(0.077)}, 0.0433e^{2\pi i(0.0565)} \end{pmatrix}$

Table 3: Score values.

	CTSFHWA operator	CTSFHWG operator
A_1	0.0184	0.02839
A_2	−0.028	0.04048
A_3	0.2266	0.05656
A_4	0.0259	0.0189

Table 4: Impact of $\mathscr{L}$.

$\mathscr{L}$	Operators	Resulting pattern
2	CTSFHWA	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
4	CTSFHWA	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
5	CTSFHWA	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
7	CTSFHWA	$\rho_3 \succ \rho_1 \succ \rho_4 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
8	CTSFHWA	$\rho_1 \succ \rho_3 \succ \rho_4 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
9	CTSFHWA	$\rho_1 \succ \rho_4 \succ \rho_3 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
12	CTSFHWA	$\rho_1 \succ \rho_4 \succ \rho_3 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$

Table 5: Variation of q.

q	Operators resulting	Pattern
4	CTSFHWA	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
5	CTSFHWA	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
6	CTSFHWA	$\rho_3 \succ \rho_1 \succ \rho_4 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
8	CTSFHWA	$\rho_3 \succ \rho_1 \succ \rho_4 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$

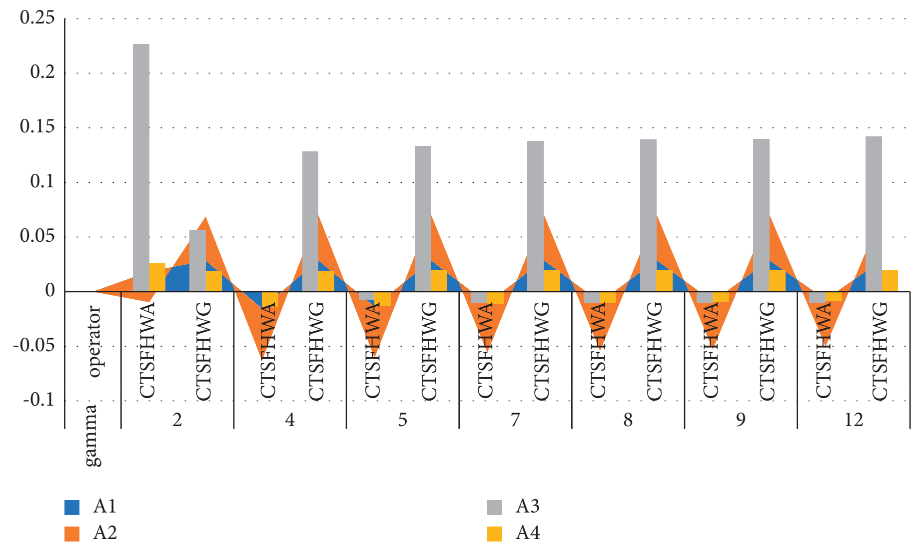

Figure 1: Variation of $\mathscr{L}$.

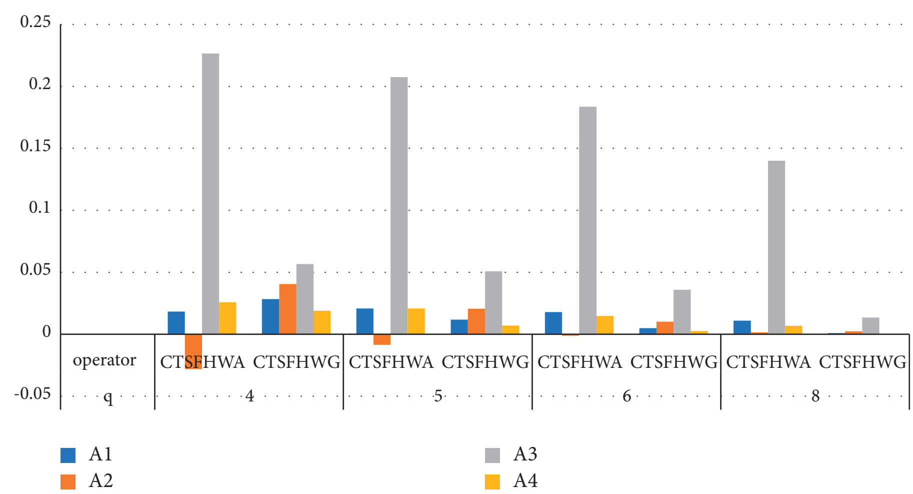

Figure 2: Variation of q.

Table 6: Ranking of alternatives.

Method	Ranking result
CTSFWA [11]	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
CTSFWG [11]	$\rho_4 \succ \rho_1 \succ \rho_2 \succ \rho_3$
CTSFHWA	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
CIF HAOs [8]	Cannot be quantified
CPyFS HAOs [9]	Cannot be quantified
Cq-ROFS HAOs [20]	Cannot be quantified
CPFS HAOs [40]	Cannot be quantified
CTSFS HAOs [35]	Cannot be quantified

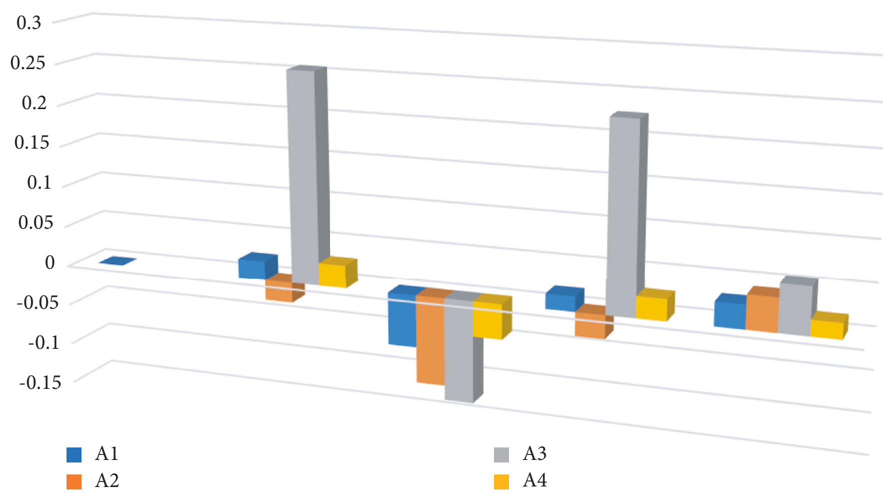

FIGURE 3: Interpretation of the information of Table 6.

$$\mathrm{CTSFHWA}(T_1, T_2, T_3, \ldots T_{k+1}) = \begin{pmatrix} \sqrt[q]{\dfrac{\prod_{j=1}^{k+1}\left(1+(L-1)r_{m_j}^q\right)^{w_j} - \prod_{j=1}^{k+1}\left(1-r_{m_j}^q\right)^{w_j}}{\prod_{j=1}^{k+1}\left(1+(L-1)r_{m_j}^q\right)^{w_j} + (L-1)\prod_{j=1}^{k+1}\left(1-r_{m_j}^q\right)^{w_j}}} \cdot e^{\left(2\pi i \sqrt[q]{\prod_{j=1}^{k+1}\left(1+(L-1)\theta_{m_j}^q\right)^{w_j} - \prod_{j=1}^{k+1}\left(1-\theta_{m_j}^q\right)^{w_j} / \prod_{j=1}^{k+1}\left(1+(L-1)\theta_{m_j}^q\right)^{w_j} + (L-1)\prod_{j=1}^{k+1}\left(1-\theta_{m_j}^q\right)^{w_j}}\right)} \\ \dfrac{\sqrt[q]{L}\prod_{j=1}^{k+1}\left(r_{i_j}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k+1}\left(1+(L-1)\left(1-r_{i_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{k+1}\left(1-r_{i_j}^q\right)^{w_j}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{k+1}\left(\theta_{i_j}\right)^{w_j} / \sqrt[q]{\prod_{j=1}^{k+1}\left(1+(L-1)\left(1-\theta_{i_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{k+1}\left(1-\theta_{i_j}^q\right)^{w_j}}\right)} \\ \dfrac{\sqrt[q]{L}\prod_{j=1}^{k+1}\left(r_{n_j}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k+1}\left(1+(L-1)\left(1-r_{n_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{k+1}\left(1-r_{n_j}^q\right)^{w_j}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{k+1}\left(\theta_{n_j}\right)^{w_j} / \sqrt[q]{\prod_{j=1}^{k+1}\left(1+(L-1)\left(1-\theta_{n_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{k+1}\left(1-\theta_{n_j}^q\right)^{w_j}}\right)} \end{pmatrix} \tag{13}$$

Hence proved for $l = k \oplus 1$. Thus, proof is completed. □

Theorem 2. *The HAOs defined for CTSFNs satisfy the subsequent properties.*

(i) iIdempotency If $T_j = T = (r_m(x).\ e^{2\pi i\theta_m(x)}, r_i(x).\ e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$, $\forall j = 1, 2, 3, \ldots l.$, then $\mathrm{CTSFHWA}(T_1, T_2, T_3, \ldots T_l) = T$.

(ii) Boundedness If $T^- = T = (\min_j r_m(x).e^{2\pi i\theta_m(x)}, \max_j r_i(x).e^{2\pi i\theta_i(x)}, \max_j r_n(x).e^{2\pi i\theta_n(x)})$ and $T^+ = T = (\max_j r_m(x).e^{2\pi i\theta_m(x)}, \min_j r_i(x).e^{2\pi i\theta_i(x)}, \min_j r_n(x).\ e^{2\pi i\theta_n(x)})$, then

$$T^- \le \mathrm{CTSFHWA}(T_1, T_2, T_3, \ldots, T_l) \le T^+. \tag{14}$$

(iii) Monotonicity Let T_j and P_j be two CTSFNs, such that $T_j \le P_j \forall j.$, then

$$\begin{aligned} &\mathrm{CTSFHWA}(T_1, T_2, T_3, \ldots, T_l) \\ &\le \mathrm{CTSFHWA}(P_1, P_2, P_3, \ldots, P_l). \end{aligned} \tag{15}$$

The CTSFHWA operator only evaluates CTSFN. In order to discuss conditions where we have a need to discuss the ranking orders of CTSFNs in MADM problems, CTSFHOWA operator is defined as follows:

Definition 6. Suppose $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$, $\forall j = 1, 2, 3, \ldots l.$ be CTSFNs. Then, the complex T-spherical fuzzy Hamacher ordered weighted average (CTSFHOWA) operator from $T^l \longrightarrow T$ is defined as

$$\mathrm{CTSFHOWA}(T_1, T_2, T_3, \ldots, T_l) = \sum_{j=1}^{T} w_j T_{\sigma(j)}, \tag{16}$$

where $T_{\sigma(j-1)} \ge T_{\sigma(j)} \forall j$ is satisfied.

Theorem 3. *Consider $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).\ e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$, $\forall j = 1, 2, 3, \ldots l.$ to be CTSFNs. Then, CTSFHOWA operator is a CTSFN and given by.*

$$\text{CTSFHHA}(T_1, T_2, T_3, \ldots, T_l) = \sum_{j=1}^{L} w_j T_{\sigma(j)}. \tag{17}$$

Definition 7. Suppose$T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$, $\forall j = 1, 2, 3, \ldots l.$ be CTSFNs. Then, complex T-spherical Hamacher hybrid aggregation (CTSFHHA) operator from $T^l \longrightarrow T$ is defined as

$$\text{CTSFHHA}(T_1, T_2, T_3, \ldots, T_l) = \sum_{j=1}^{L} w_j T_{\sigma(j)}. \tag{18}$$

Theorem 4. *Consider* $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots, l.$ *be CTSFNs. Then, CTSFHHA operator is a CTSFN and given by.*

$$\text{CTSFHHA}(T_1, T_2, T_3, \ldots T_l) =$$

$$= \left(\begin{array}{c} \sqrt[q]{\dfrac{\prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\dot{r}^q_{m_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{l}\left(1-\dot{r}^q_{m_{\sigma(j)}}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\dot{r}^q_{m_{\sigma(j)}}\right)^{w_j} - (\mathscr{L}-1)\prod_{j=1}^{l}\left(1-\dot{r}^q_{m_{\sigma(j)}}\right)^{w_j}}}\, e^{2\pi i \sqrt[q]{\prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\dot{\theta}^q_{m_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{l}\left(1-\dot{\theta}^q_{m_{\sigma(j)}}\right)^{w_j} / \prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\dot{\theta}^q_{m_{\sigma(j)}}\right)^{w_j} + (\mathscr{L}-1)\prod_{j=1}^{l}\left(1-\dot{\theta}^q_{m_{\sigma(j)}}\right)^{w_j}}} \\ \dfrac{\sqrt[q]{\mathscr{L}}\prod_{j=1}^{l}\left(\dot{r}_{i_{\sigma(j)}}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\left(1-\dot{r}^q_{i_{\sigma(j)}}\right)\right)^{w_j} + (\mathscr{L}-1)\prod_{j=1}^{l}\left(\dot{r}^q_{i_{\sigma(j)}}\right)^{w_j}}}\, e^{2\pi i \sqrt[q]{\mathscr{L}}\prod_{j=1}^{l}\left(\dot{\theta}_{i_{\sigma(j)}}\right)^{w_j} / \prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\left(1-\dot{\theta}^q_{i_{\sigma(j)}}\right)\right)^{w_j} + (\mathscr{L}-1)\prod_{j=1}^{l}\left(1-\dot{\theta}^q_{i_{\sigma(j)}}\right)^{w_j}} \\ \dfrac{\sqrt[q]{\mathscr{L}}\prod_{j=1}^{l}\left(\dot{r}_{n_{\sigma(j)}}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\left(1-\dot{r}^q_{n_{\sigma(j)}}\right)\right)^{w_j} + (\mathscr{L}-1)\prod_{j=1}^{l}\left(\dot{r}^q_{n_{\sigma(j)}}\right)^{w_j}}}\, e^{2\pi i \sqrt[q]{\mathscr{L}}\prod_{j=1}^{l}\left(\dot{\theta}_{n_{\sigma(j)}}\right)^{w_j} / \prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\left(1-\dot{\theta}^q_{n_{\sigma(j)}}\right)\right)^{w_j} + (\mathscr{L}-1)\prod_{j=1}^{l}\left(1-\dot{\theta}^q_{n_{\sigma(j)}}\right)^{w_j}} \end{array} \right) \tag{19}$$

5. CTSFHWG Aggregation Operator

This section contains the development of the CTSFHWG operator and their basic properties based on the operations defined in equations (6)-(9).

Definition 8. Let $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots, l.$ be some CTSFNs. Then, CTSFHWG operator from $T^l \longrightarrow T$ is defined as

$$\text{CTSFHWG}(T_1, T_2, T_3, \ldots, T_l) = \sum_{j=1}^{T} T_j^{w_j}. \tag{20}$$

By the Definition 8, we have

Theorem 5. *Consider* $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots, l$ *be CTSFNs. Then CTSFHWG operator is a CTSFN and given by.*

$$\text{CTSFHOWG}(T_1, T_2, T_3, \ldots, T_l) = \sum_{j=1}^{T} T_{\sigma(j)}^{w_j}, \tag{21}$$

Proof. Similar to Theorem 1.

The CTSFHWG operator also satisfies the boundedness, idem potency, and monotonicity like other operators. □

Definition 9. Let $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots, l$ be CTSFNs. Then, CTSFHOWG operator from $T^l \longrightarrow T$ is defined as

$$\text{CTSFHOWG}(T_1, T_2, T_3, \ldots, T_l) = \sum_{j=1}^{T} T_{\sigma(j)}^{w_j}, \tag{22}$$

where $T_{\sigma(j-1)} \geq T_{\sigma(j)} \forall j$ is satisfied.

Theorem 6. *Let* $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots l$ *be some CTSFNs. Then, CTSFHOWG operator is a CTSFN and given by.*

$$\text{CTSFHHG}(T_1, T_2, T_3, \ldots, T_l) = \sum_{j=1}^{T} T_{\sigma(j)}^{w_j}. \tag{23}$$

Definition 10. Suppose $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots, l$ be CTSFNs. Then, CTSFHHG operator from $T^l \longrightarrow T$ is defined as

$$\text{CTSFHHG}(T_1, T_2, T_3, \ldots, T_l) = \sum_{j=1}^{T} T_{\sigma(j)}^{w_j}. \tag{24}$$

Theorem 7. *Consider* $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots, l$ *be CTSFNs. Then, CTSFHHG operator is a CTSFN and given by.*

$$SC(I) = \frac{(r_m^q - r_i^q - r_n^q) + (\theta_m^q - \theta_i^q - \theta_n^q)}{2}, \quad SC(I) \in [-1, 1]. \tag{25}$$

6. Application of CTSFHAOs

In this section, we study the developed AOs in the MADM with the help of a real-life example. We structure a procedure to apply these AOs in MADM first. Secondly, we study the effects of the variation in the included parameters. Finally, we studied the comparison of proposed AOs with the existing AOs and plotted them graphically.

We have to select the best one from the list of a few alternatives in MADM with the help of the AOs and then the score function. In the case of this article, we want to apply the HAOs to select the best alternative. We can select the best alternative by the following approach. We use the information in the form of CTSFS. Let there be $\rho = \{\rho_1, \rho_2, \ldots, \rho_k\}$ alternatives and $G = \{G_1, G_2, \ldots, G_j\}$ be the attributes with weight vector w_j. The $D_{k\times j} = (T)_{k\times j} = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$ represents the decision matrix in the form of CTSFNs. The procedure to select the best alternative is as follows.

Step 1: form the decision matrix of the given information by taking the CTSFS. Also, investigate the data for the value of q.

Step 2: use the AOs to aggregate the CTSF information.

Step 3: calculate the score values of CTSFNs with the help of the following equation:

$$SC(I) = \frac{((r_m^q - r_i^q - r_n^q) + (\theta_m^q - \theta_i^q - \theta_n^q))}{2}, \quad SC(I) \in [-1, 1]. \tag{26}$$

Step 4: the greater the score value, the best the alternative is.

Example 1. with the help of this example, we apply our proposed approach to the MADM problem. The government wants to start a pilot health project in one city of a few major cities. After some basic screening, the department of health shortlisted four major cities. The department of health wants to select one city $\rho_j (1 \le j \le 4)$ based on some attributes i.e., the population of the city (G_1), number of hospitals in that city (G_2), the living standard of people of that city (G_3) and the number of NGOs working in that city (G_4), which have some weights $w_j = (0.2, 0.5, 0.25, 0.05)^T$. The information obtained in terms of CTFN is presented in Table 1. To select the best alternative, we apply for the proposed work as follows:

Step 1: the data given by the experts are in the form of CTSFNs to find the best enterprise shown in Table 1 for the value $q = 4$.

Step 2: the aggregated values of CTSFHWA and CTSFHWG operators are given in Table 2. We take $q = 4$, $\mathscr{L} = 2$, and $w = (0.2, 0.5, 0.25, 0.05)^T$ while calculating CTSFHWA and CTSFHWG operators.

Step 3: in Table 2, the values obtained by the AOs are presented. Table 3 represents the score value.

Step 4: by the help of score values obtained, we order the alternatives in descending order. We obtain $\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$ and $\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$. Hence, the city ρ_3 is the most suitable city to start the project, while we use the CTSFHWA operator and ρ_4 as the best city to start the project by using the CTSFHWG operator (Table 3).

6.1. Ranking Variation by "$\mathscr{L}$". As we note that the results in the previous sections depend upon the values of parameters q and $\mathscr{L}$. This Section contains the study of the effects of the variation in their values. The effects of the variation of $\mathscr{L}$ and q are presented in Tables 4 and 5, respectively.

From Table 4, we do not notice any significant change in ranking while we use the CTSFHWA operator for various values of $\mathscr{L} = 2, 3, \ldots, 12$. However, a fluctuation can be seen in the ranking pattern of alternatives using the CTSFHWG operator for various values of $\mathscr{L}$. We can see that the fluctuation occurs at $\mathscr{L} = 2, 7, 8, 9$. However, after $\mathscr{L} = 9$, the ranking results get stability and there does not occur any further change in the ranking pattern by varying the parameter $\mathscr{L}$ above 9. Figure 1 shows the whole scenario.

6.2. Ranking Variation by 'q'. In Section 6.1, we have observed the variation in $\mathscr{L}$ and its impact on the ranking result. Here we want to examine the effect on ranking results of CTSFHWA and CTSFHWG due to variation in q . The problem discussed in Section 6.2; Table 5 represents the variation of the values of "q" from 4 onward.

In the case of the CTSFHWA operator, it is clear from the above table that the ranking results order changed when $q = 6$, but in the case of the CTSFHWG operator the final ranking order does not change. However, when $q \ge 6$ the ranking results do not change in both cases. This result shows that both operators become consistent when $q = 6$. This phenomenon can be observed in Figure 2.

6.3. Comparative Study. To check the significance of our developed AOs, we do a comparative analysis with some existing approaches. The proposed approach is compared with AOs developed by Alkouri et al., such as [8] CPyFS AOs [9], CTSF AOs [11], Cq-ROFS HAOs [20], CTSFS HAOs [35], and CPFS HAOs [40]. The comparison is given in Table 6. Some AOs among these could not be applied to such information. Hence, the proposed approach is significantly improved. The results obtained by these AOs are stated in Table 6.

From Table 6, we observe that the proposed work is the comparative significant from the existing AOs. Some of the AOs fail to give the ranking of the alternatives. It is cleared from Table 6, except for CTSFWA, CTSFWG, CTFHWA, and CTSFHWG AOs, all other AOs have failed to rank the alternatives. In the following, Figure 3 shows the comparison of the proposed operator with the pre-existing operators.

7. Conclusion

In this manuscript, we developed some fundamental operations for CTSFNs by using HTNM and HTCNM, and then by applying these operations, we develop

CTSFHWA, CTSFHOWA, CTSFHHWA, CTSFHWG, CTSFOWG, and CTSFHHWG operators. Secondly, we inspected the fundamental properties of these operators. Then, we applied the developed CTSF HAOs in MADM, studied the variation of results by the changing values of the parameters involved, and finally compared these AOs with existing AOs. Some key points of the article are given as follows:

(1) We considered an example of having the information in the form of CTSFNs and applied the HAOs proposed. We obtained ρ_3 as the best alternative for both CTSFHWA and CTSFHWG operators as the best alternative.

(2) We obtained different ranking patterns of the score values by changing the values of parameters q and $\mathscr{L}$. (See Table 5).

(3) Then, we compared the obtained results with some existing AOs and obtained interesting results in Table 6. We found some AOs, which could not provide the ranking of alternatives.

(4) Lastly, we plotted the comparison in Figure 2.

The developed approaches with the help of HTNM and HTCNM are very useful techniques to aggregate the multiple values and we can obtain the ranking of the alternatives as done in the example above. However, the HTNM and HTCNM can be applied to other defined frameworks. Hence, we aim to apply these operations to the type 2 fuzzy set [41–43] and trapezoidal intuitionistic fuzzy set [44–46]. We also aim to extend this proposed work to the best-worst method [47–48].

Data Availability

The data used to support the findings of this study are included within this article.

Conflicts of Interest

The authors declared that they have no conflicts of interest.

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References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[3] R. R. Yager, "Pythagorean fuzzy subsets," in *Proceedings of the 2013 Joint IFSA World congress and NAFIPS Annual Meeting (IFSA/NAFIPS)*, pp. 57–61, IEEE, Edmonton, AB, Canada, 2013 June.

[4] R. R. Yager, "Generalized Orthopair fuzzy sets," *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 5, pp. 1222–1230, 2016.

[5] B. C. Cuong and V. Kreinovich, "Picture fuzzy sets," *Journal of Computer Science and Cybernetics*, vol. 30, no. 4, pp. 409–420, 2014.

[6] T. Mahmood, K. Ullah, Q. Khan, and N. Jan, "An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets," *Neural Computing & Applications*, vol. 31, no. 11, pp. 7041–7053, 2019.

[7] D. Ramot, R. Milo, M. Friedman, and A. Kandel, "Complex fuzzy sets," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 2, pp. 171–186, 2002.

[8] A. M. D. J. S. Alkouri and A. R. Salleh, "September. Complex intuitionistic fuzzy sets," *AIP Conference Proceedings*, vol. 1482, no. 1, pp. 464–470, 2012.

[9] K. Ullah, T. Mahmood, Z. Ali, T. Mahmood, Z. Ali, and N. Jan, "On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition," *Complex & Intelligent Systems*, vol. 6, no. 1, pp. 15–27, 2020.

[10] P. Liu, M. Akram, and A. Sattar, "Extensions of prioritized weighted aggregation operators for decision-making under complex q-rung orthopair fuzzy information," *Journal of Intelligent and Fuzzy Systems*, vol. 39, no. 5, pp. 7469–7493, 2020.

[11] Z. Ali, T. Mahmood, and M.-S. Yang, "Complex T-spherical fuzzy aggregation operators with application to multi-attribute decision making," *Symmetry*, vol. 12, no. 8, p. 1311, 2020.

[12] Z. Zeshui Xu, "Intuitionistic fuzzy aggregation operators," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 6, pp. 1179–1187, 2007.

[13] G. Wei and M. Lu, "Pythagorean fuzzy power aggregation operators in multiple attribute decision making," *International Journal of Intelligent Systems*, vol. 33, no. 1, pp. 169–186, 2018.

[14] P. Liu and P. Wang, "Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making," *International Journal of Intelligent Systems*, vol. 33, no. 2, pp. 259–280, 2018.

[15] H. Garg, "Some picture fuzzy aggregation operators and their applications to multicriteria decision-making," *Arabian Journal for Science and Engineering*, vol. 42, no. 12, pp. 5275–5290, 2017.

[16] L. Zhou, H. Chen, and J. Liu, "Generalized power aggregation operators and their applications in group decision making," *Computers & Industrial Engineering*, vol. 62, no. 4, pp. 989–999, 2012.

[17] O. F. Gorcun, S. Senthil, S. Senthil, and H. Küçükönder, "Evaluation of tanker vehicle selection using a novel hybrid fuzzy MCDM technique," *Decision Making: Applications in Management and Engineering*, vol. 4, no. 2, pp. 140–162, 2021.

[18] F. Sinani, Z. Erceg, and M. Vasiljević, "An evaluation of a third-party logistics provider: the application of the rough Dombi-Hamy mean operator," *Decision Making: Applications in Management and Engineering*, vol. 3, no. 1, pp. 92–107, 2020.

[19] D. Bozanic, A. Milic, D. Tešic, W. Salabun, and D. Pamucar, "D numbers – FUCOM – fuzzy RAFSI model for selecting the group of construction machines for enabling mobility," *Facta Universitatis – Series: Mechanical Engineering*, vol. 19, no. 3, pp. 447–471, 2021.

[20] D. Pamučar and A. Janković, "The application of the hybrid interval rough weighted Power-Heronian operator in multi-criteria decision making," *Operational Research in*

Engineering Sciences: Theory and Applications, vol. 3, no. 2, pp. 54–73, 2020.

[21] E. P. Klement, R. Mesiar, and E. Pap, "Triangular norms. Position paper I: basic analytical and algebraic properties," *Fuzzy Sets and Systems*, vol. 143, no. 1, pp. 5–26, 2004.

[22] L. Wu, G. Wei, H. Gao, and Y. Wei, "Some interval-valued intuitionistic fuzzy Dombi Hamy mean operators and their application for evaluating the elderly tourism service quality in tourism destination," *Mathematics*, vol. 6, no. 12, p. 294, 2018 Dec.

[23] M. Akram, W. A. Dudek, and J. M. Dar, "Pythagorean Dombi fuzzy aggregation operators with application in multicriteria decision-making," *International Journal of Intelligent Systems*, vol. 34, no. 11, pp. 3000–3019, 2019.

[24] W. Wang and X. Liu, "Intuitionistic fuzzy geometric aggregation operators based on Einstein operations," *International Journal of Intelligent Systems*, vol. 26, no. 11, pp. 1049–1075, 2011.

[25] M. Riaz, H. M. Athar Farid, H. Kalsoom, D. Pamučar, and Y.-M. Chu, "A robust q-rung orthopair fuzzy Einstein prioritized aggregation operators with application towards MCGDM," *Symmetry*, vol. 12, no. 6, p. 1058, 2020.

[26] A. Fahmi, F. Amin, S. Abdullah, and A. Ali, "Cubic fuzzy Einstein aggregation operators and its application to decision-making," *International Journal of Systems Science*, vol. 49, no. 11, pp. 2385–2397, 2018.

[27] T. Senapati, G. Chen, and R. R. Yager, "Aczel-Alsina aggregation operators and their application to intuitionistic fuzzy multiple attribute decision making," *International Journal of Intelligent Systems*, vol. 37, no. 2, pp. 1529–1551, 2022.

[28] Y. Yang, Z.-S. Chen, Y.-H. Chen, and K.-S. Chin, "Interval-valued Pythagorean fuzzy frank power aggregation operators based on an isomorphic frank dual triple," *International Journal of Computational Intelligence Systems*, vol. 11, no. 1, pp. 1091–1110, 2018.

[29] A. Singh, R. K. Ghadai, K. Kalita, P. Chatterjee, and D. Pamučar, "EDM process parameter optimization for efficient machining of Inconel-718," *Facta Universitatis – Series: Mechanical Engineering*, vol. 18, no. 3, pp. 473–490, 2020.

[30] G. Beliakov, H. Bustince, D. P. Goswami, U. K. Mukherjee, and N. R. Pal, "On averaging operators for Atanassov's intuitionistic fuzzy sets," *Information Sciences*, vol. 181, no. 6, pp. 1116–1124, 2011.

[31] E. P. Klement, R. Mesiar, and E. Pap, "Triangular norms. Position paper II: general constructions and parameterized families," *Fuzzy Sets and Systems*, vol. 145, no. 3, pp. 411–438, 2004.

[32] H. Garg, "Intuitionistic fuzzy Hamacher aggregation operators with entropy weight and their applications to multi-criteria decision-making problems," *Iranian Journal of Science and Technology, Transactions of Electrical Engineering*, vol. 43, no. 3, pp. 597–613, 2019.

[33] S.-J. Wu and G.-W. Wei, "Pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making," *International Journal of Knowledge-Based and Intelligent Engineering Systems*, vol. 21, no. 3, pp. 189–201, 2017.

[34] A. P. Darko and D. Liang, "Some q-rung orthopair fuzzy Hamacher aggregation operators and their application to multiple attribute group decision making with modified EDAS method," *Engineering Applications of Artificial Intelligence*, vol. 87, Article ID 103259, 2020.

[35] K. Ullah, T. Mahmood, and H. Garg, "Evaluation of the performance of search and rescue robots using T-spherical fuzzy Hamacher aggregation operators," *International Journal of Fuzzy Systems*, vol. 22, no. 2, pp. 570–582, 2020.

[36] H. Zhao, Z. Xu, M. Ni, and S. Liu, "Generalized aggregation operators for intuitionistic fuzzy sets," *International Journal of Intelligent Systems*, vol. 25, no. 1, pp. 1–30, 2010.

[37] K. Ullah, T. Mahmood, and N. Jan, "Similarity measures for T-spherical fuzzy sets with applications in pattern recognition," *Symmetry*, vol. 10, no. 6, p. 193, 2018.

[38] A. Ashraf, K. Ullah, K. Ullah, A. Hussain, and M. Bari, "Interval-valued picture fuzzy maclaurin symmetric mean operator with application in multiple attribute decision-making," *Reports in Mechanical Engineering*, vol. 3, no. 1, pp. 301–317, 2022.

[39] Q. Wu, P. Wu, L. Zhou, H. Chen, and X. Guan, "Some new Hamacher aggregation operators under single-valued neutrosophic 2-tuple linguistic environment and their applications to multi-attribute group decision making," *Computers & Industrial Engineering*, vol. 116, pp. 144–162, 2018.

[40] E. P. Klement and M. Navara, "A survey on different triangular norm-based fuzzy logics," *Fuzzy Sets and Systems*, vol. 101, no. 2, pp. 241–251, 1999.

[41] Q. Wu, L. Zhou, Y. Chen, and H. Chen, "An integrated approach to green supplier selection based on the interval type-2 fuzzy best-worst and extended VIKOR methods," *Information Sciences*, vol. 502, pp. 394–417, 2019.

[42] Q. Wu, X. Liu, L. Zhou, Z. Tao, and J. Qin, "A quantum framework for modeling interference effects in linguistic distribution multiple criteria group decision making," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 21, 2021.

[43] D. Bozanic, D. Tešić, and A. Milić, "Multicriteria decision making model with Z-numbers based on FUCOM and MABAC model," *Decision Making: Applications in Management and Engineering*, vol. 3, no. 2, pp. 19–36, 2020.

[44] X.-B. Mao, M. Wu, J.-Y. Dong, S.-P. Wan, and Z. Jin, "A new method for probabilistic linguistic multi-attribute group decision making: application to the selection of financial technologies," *Applied Soft Computing*, vol. 77, pp. 155–175, 2019.

[45] S. P. Wan and Z. H. Yi, "Power average of trapezoidal intuitionistic fuzzy numbers using strict t-norms and t-conorms," *IEEE Transactions on Fuzzy Systems*, vol. 24, no. 5, pp. 1035–1047, 2015.

[46] Ç. Karamaşa, D. Karabasevic, D. Stanujkic, A. Kookhdan, A. Mishra, and M. Ertürk, "An extended single-valued neutrosophic AHP and MULTIMOORA method to evaluate the optimal training aircraft for flight training organizations," *Facta Universitatis – Series: Mechanical Engineering*, vol. 19, no. 3, pp. 555–578, 2021.

[47] V. R. Milovanović, A. V. Aleksić, V. S. Sokolović, and M. A. Milenkov, "Uncertainty modeling using intuitionistic fuzzy numbers," *Military Technical Courier*, vol. 69, no. 4, pp. 905–929, 2021.

[48] S. Wan and J. Dong, "A novel extension of best-worst method with intuitionistic fuzzy reference comparisons," *IEEE Transactions on Fuzzy Systems*, vol. 9, 2021.

[49] K. Ullah, "Picture fuzzy maclaurin symmetric mean operators and their applications in solving multiattribute decision-making problems," *Mathematical Problems in Engineering*, vol. 2021, Article ID 1098631, 13 pages, 2021.

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Research Article

Identification of Encrypted Traffic Using Advanced Mathematical Modeling and Computational Intelligence

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This paper proposed a hybrid approach for the identification of encrypted traffic based on advanced mathematical modeling and computational intelligence. Network traffic identification is the premise and foundation of improving network management, service quality, and application security. It is also the focus of network behavior analysis, network planning and construction, network anomaly detection, and network traffic model research. With the increase in user and service requirements, many applications use encryption algorithms to encrypt traffic during data transmission. As a result, traditional traffic classification methods classify encrypted traffic on the network, which brings great difficulties and challenges to network monitoring and data mining. In our article, a nonlinear modified DBN method is proposed and applied to encrypted traffic identification. Firstly, based on Deep Belief Networks (DBN), this paper introduces the proposed Eodified Elliott (ME)-DBN model, analyzes the function image, and presents the ME-DBN learning algorithm. Secondly, this article designs an encrypted traffic recognition model based on the ME-DBN model. Feature extraction is carried out by training the ME-DBN model, and finally, classification and recognition are carried out by the classifier. The experimental results on the ISCX VPN-non-VPN database show that the MEDBN method proposed in this article can enhance the classification and recognition rate and has better robustness to encrypt traffic recognition from different software.

1. Introduction

Network traffic classification is a basic step for managing and controlling network resources. Previous traffic classification methods, such as the traffic classification method [1] based on port number and deep Packet Inspection (DPI), cannot deal with encrypted traffic and can hardly adapt to the current traffic environment. The method based on traffic statistics and machine learning (ML) is popular in current, which not only can deal with encrypted traffic but also regular traffic, for example, decision tree (DT) and KNN algorithm. Nevertheless, the performance based on the ML depends largely on artificially designed features and private information in traffic. Therefore, there is a limitation on the generality and accuracy of the method. In addition, the method t requires a mass of storage and computing resources, which limits its implementation in resource-constrained nodes [2], such as vehicles, home gateways, and mobile phones. Real-time and accurate network traffic classification is the basis of network management tasks and intrusion detection systems, so a new traffic identification method is urgently needed.

The development of mobile traffic identification technology has experienced three stages based on port, based on payload and based on traffic statistical characteristics. Nevertheless, the advent of port spoofing, random ports, and tunneling quickly rendered these models ineffective. As users become more aware of privacy protection and security, technologies such as SSL, SSH, VPN, and Tor have become more widely used, resulting in an increasing proportion of encrypted traffic in network traffic. The payload-based approach, known as Deep Packet Inspection (DPI) technology, cannot handle encrypted traffic because it requires matching Packet content and is computationally expensive. Therefore, to handle the problem of encrypted traffic classification, the method based on data traffic appears. Its generality depends on statistical or time series properties and uses ML algorithms, like some tree-based methods, classical model as SVM, and KNN, etc. Furthermore, some statistical methods such as GMM [3] and HMM [4] are also employed to identify the encrypted traffic. Classical machine learning methods could handle some problems that port and payload-based methods cannot solve, but they still have certain limitations: (1) The characteristics of data traffic need to be extracted manually, which often depends on expert experience and is very time-consuming and labor-intensive. (2) The characteristics of traffic change rapidly and need to be updated frequently. (3) For traffic identification tasks, category imbalance is a major problem. Category imbalance refers to the fact that the data volume of some samples in the data set is several times or even higher than that of others. Using such data set to train the model, a high recognition rate can be obtained as long as all the small samples are classified into large samples, which is not meaningful in actual production. The method to solve this problem is to expand the data amount of small samples through different ways, but the current data expansion method cannot accurately generate samples as close as possible to the original data. (4) In model training, marked samples are mostly relied on. How to combine a large number of easily obtained unlabeled data sets with some difficult-to-obtain labeled data sets for traffic classification in order to reduce the need for labeled data is a very key research topic. Different from most traditional ML algorithms, DL automatically extracts features without human intervention, which is undoubtedly an ideal traffic classification method, especially for mobile service encrypted traffic. Recent research work proves the superiority of the DL method in traffic classification [5–9]. Therefore, it is of great importance and far-reaching significance to study the application of DL in traffic classification and how to improve the recognition rate of small sample traffic in unbalance data sets so as to more effectively and conveniently encrypt traffic and improve the accuracy of application identification.

Recently, DL-based methods have been employed in many fields and achieved good results, such as image recognition, speech recognition, and natural language processing. Owning to the Deep Learning, this article proposes a frame of classification and detection based on Deep Learning (DL), which can construct feature space through the deep structure of multiple hidden layers and discover data features through autonomous learning of a large number of data. It solves the difficulty of feature subset selection and improves the classification efficiency, which lays a foundation for the real-time classification of network traffic. In the second part, we summarize the existing research. In the third part, we introduce the identification model of encrypted traffic; in the fourth part, we introduce the evaluation process of the model in detail; the fifth part describes the data collection and processing in detail; the sixth part introduces the detailed process of experiment and simulation; the summary and discussion are arranged in the last part.

2. Research Overview

As early as 1995, Claffy et al. [10] used the traditional classification method based on service host attributes to identify network traffic. Almost all communication protocol packets, including encrypted packets, have their own unique traffic characteristics, which can be analyzed and distinguished from a large number of traffic samples. Therefore, Gu et al. [11] made an in-depth analysis of the classification method of traffic load content characteristics. Yeganeh et al. [12] summarized the relative positional word set carried in the session flow payloads of each protocol and then detected the payloads by the deep packet detection method of word sequence matching to identify the protocol types as smart computing continues to evolve.

Due to the quantitative limitations of current technologies, new methods have been found that rely on the statistical characteristics of traffic to classify applications. In recent years, stream classification methods based on statistical features have attracted extensive attention. Common statistical features such as packet interval statistics, flow arrival time statistics, flow duration, packet length, traffic idle time, packet arrival interval, packet length, and other statistical characteristics of the network. With the explosive growth of traffic in the current network environment, the simple traffic statistics method has been unable to achieve the ideal classification effect of network traffic, and the method based on machine learning came into being. Machine learning mainly includes supervised learning, semi-supervised learning, and unsupervised learning.

Recognition models based on ML and DL are widely used. Ibrahim et al. [13] designed a classifier for online traffic classification (SSPC) that combines three identification methods: port-based, payload-based, and statistically based. The classification results based on payload are preferred for identification, followed by the same results based on port and statistical characteristics. Conti et al. [14] used the method of RF to identify the actions of users on mobile phones through the IP, packet size, port, direction and other characteristics of the encrypted traffic generated by marked users when using the application mobile phone client. Compared with the ML-based methods, in 2004, literature [15] used packet length, packet interval, and stream duration as statistical features and used an expectation maximization algorithm to classify traffic types by unsupervised learning. Literature [16] uses an unsupervised machine learning algorithm to carry out unsupervised machine learning training on long-term and short-term memory recurrent neural networks so that the network can distinguish a group of time series and group them. The results show that the neural network has a strong time series learning ability and clustering ability based on multiple features. Literature [17] proposed a method of malicious traffic detection using representation learning. This method does not need to manually design traffic characteristics but directly classifies the original data as input data. This is the first time that the representation learning method is applied to the detection and classification of malicious traffic. When the three classifiers are verified in two cases, the results meet the requirements of practical application accuracy. This document proves that the efficiency of representation learning in malicious traffic detection is high, but there are also shortcomings. The tuning parameters of the convolutional neural network are not studied, and the time factor and unknown malicious traffic are not considered [18]. Literature [19] classifies more than 20 kinds of fine-grained network traffic based on hierarchical learning. The results of large data sets show that the average accuracy of traffic classification of hierarchical classifier can reach 90%, and the accuracy and recall of commonly used traffic categories are higher than 95%. Wang et al. used the long-term and short-term memory recurrent neural network to automatically learn the timing characteristics in the traffic, solved the problem of manually designing the characteristics, and achieved a high detection rate and low false alarm rate. In the literature, a cyclic neural network is used to learn the timing characteristics of encrypted traffic to realize the mobile application type recognition of the Android platform, and a high recognition rate and recall rate are achieved. Literature [5] uses the improved RNN and density clustering method to detect network abnormal traffic, which has achieved better results than the current method. Document [20] introduces a deep packet, which is an algorithm that uses deep learning [21] to automatically extract features from network traffic to classify traffic. The deep packet is the first traffic classification system using a deep learning algorithm, namely SAE and a one-dimensional convolutional neural network, which can identify applications and handle traffic characterization tasks. The automatic feature extraction process of network traffic can save the cost of using experts to identify traffic and extract manual features and reduce the overhead of traffic classification. The classification method based on machine learning has high classification accuracy and can be used for the identification of encrypted traffic, but the cost is high, and the data set needs to be understood and preprocessed in advance. Different business types have different requirements for the packet size of data flow. For example, the flow media data is small, and the packet downloaded from the file can be the maximum message segment length. Therefore, there are differences in the distribution of packet sizes for different business types. The method based on packet size distribution is not affected by encryption and has good applicability. Qin et al. propose a new method based on packet size distribution signature, which can reduce the amount of packet processing and realize the accurate identification of P2P and VolP applications. Renyi cross-entropy is used to identify by calculating the similarity between the two-way flow and the message size distribution of specific applications [22]. Wang et al. [5] simultaneously used CNN and LSTM to learn and classify data packet headers and loads, showing good performance in real-time intrusion detection. Aceto et al. [23, 24] designed and implemented a recognition model based on MLP in order to track which APP the data stream came from. They used some features in the first N bytes of payload and original data, and some features in the first 20 packets before interactive communication, including source port, payload bytes, size of TCP slide window, Sequential packet arrival interval, and direction, which were used as input, and the experiments were compared with random forest, stack automatic encoder SAE, CNN, and LSTM. Martin et al. [25] conducted a group of controlled experiments, respectively, using the combination of RNN, CNN and recursive neural network RNN and CNN to identify the traffic of the Internet of things. The results showed that the combination of RNN and CNN had the best effect. Hochst et al. [26] designed and implemented an autoencoder SAE in order to find out actions such as web browsing interaction, game download, online play, and upload in network traffic, which achieved good results. It can be seen that using deep learning to classify encrypted traffic is a good research direction.

3. Encrypted Traffic Classification Model Design

3.1. Restricted Boltzmann Machine. The constrained Boltzmann machine (RBM) is a deformed structure of the Boltzmann machine (BM). Based on statistical mechanics, the sample of BM follows the Boltzmann machine distribution. The probability distribution of the energy-based probability model is defined by the energy function $E(x)$:

$$P(x) = \frac{e^{-E(x)}}{Z}, \tag{1}$$

where x is the input sample, $Z = \sum_x e^{-E(x)}$ is the normalized function, the commonly used method to solve $P(x)$ is gradient descent, and the negative logarithm of the training set D is its cost function:

$$l(\theta, D) = -L(\theta, D) = -\frac{1}{N} \sum_{x^{(i)} \in D} \log, \tag{2}$$

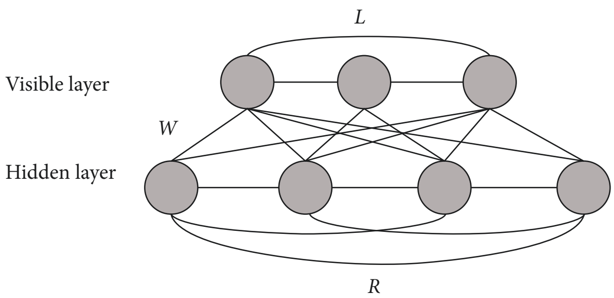
where θ is the parameter space of the model, the partial derivative of θ is obtained through the optimization algorithm so as to get the optimal solution of the cost function:

$$\Delta = \frac{\partial l(\theta, D)}{\partial \theta} = -\frac{1}{N} \frac{\partial \sum \log p\left(x^{(i)}\right)}{\partial \theta}. \tag{3}$$

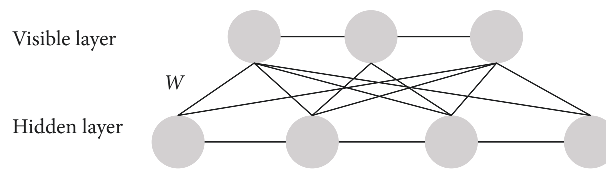
The boltzmann machine is a random NN defined by the above energy function, which consists of a visible layer and a hidden layer, as introduced in Figure 1(a). As can be seen from the figure, both intralayer nodes and interlayer nodes have connection weights, and there are only two states of the output node: activated and inactive. 1 means activated, and 0 means inactive. Therefore, we can see unit vector $v = \{0, 1\}^D$ and implicit unit vector $h = \{0, 1\}^k$, and their learning mode belongs to unsupervised learning. The energy function between the visible layer neuron and hidden layer neuron of the BM model is defined as follows:

$$E(v, h; \theta) = -v^T W h - \frac{1}{2} v^T L v - \frac{1}{2} h^T L h - v^T a - h^T b, \tag{4}$$

where $\theta = \{W, L, R, c, b\}$ is the parameter of the BM model; W, L, R are the connection weights between nodes respectively, and the diagonal elements of L and R are 0. a and b represent the bias of v and h. Through this energy function, the probability distribution can be obtained by formula (1), and the solution of the model can be obtained by further solving. Although BM has a strong self-learning ability and can learn complex internal features in data, BM has a



(a)



(b)

FIGURE 1: Structural comparison of BM and RBM models. (a) Structure of the BM model. (b) Structure of the RBM model.

complex structure, resulting in a very long training time. In addition, it is difficult to obtain random samples of the distribution represented by BM, so the practical value is relatively low.

The difference between RBM and BM lies in that the neurons at the same layer are independent of each other, that is, $L = 0$ and $R = 0$. Only interlayer neurons are connected, and their structure is shown in Figure 1(b). Similarly, it includes a visible layer v and a hidden layer h. The visible layer mainly describes the features of the observed data, while the hidden layer serves as the feature extraction layer. If v includes n nodes $v = \{v_1, \ldots v_n\}$, and h includes m nodes $h = \{h_1, \ldots h_m\}$, then the energy function under a given set of states can be expressed as follows:

$$E(v, h; \theta) = -v^T W h - v^T a - h^T b = -\sum_{i=1}^{n}\sum_{j=1}^{m} v_i W_{ij} h_j - \sum_{i=1}^{n} a_i v_i - \sum_{j=1}^{m} b_j h_j, \tag{5}$$

where $\theta = \{W_{ij}, a_i, b_j\}$, W_{ij} is the weight matrix among the visible layer and the hidden layer. The purpose of learning RBM is to fit the distribution of training data by finding the appropriate parameter θ. In order to get the optimal value of θ, we can use the stochastic gradient ascent method. Therefore, the key step is to find the partial derivatives of each parameter. The gradient of RBM logarithmic likelihood function is as follows:

$$\frac{\partial L(\theta)}{\partial \theta} = \sum_{t=1}^{T}\left(\langle \frac{\partial(-E(v^{(t)}))}{\partial \theta} \rangle_{P(h|v^{(t)},\theta)} - \langle \frac{\partial(-E(v,h;\theta))}{\partial \theta} \rangle_{P(v,h;\theta)} \right). \tag{6}$$

In the above formula, $L(\theta)$ is the likelihood function of the RBM model, and $\langle \cdot \rangle_P$ represents the expectation of distribution P. For the former term, the probability distribution of h under a given sample can be calculated; for the latter term, all possible values of v need to be searched before the joint probability distribution can be calculated. Therefore, a feasible sampling method is needed to obtain the value of the distribution.

3.2. Gibbs Sampling Method. Gibbs Sample [1] is a sampling method based on the Markov Chain Monte Carlo (MCMC) strategy that constructs random samples of probability distributions of multiple variables. For example, the joint distribution of more than two variables is constructed in order to work out integrals and expectations. The efficiency of the MCMC algorithm is low because the high-dimensional data has a certain reception rate. If the reception rate can be set to 1, the problem of slow convergence caused by the frequent rejection of transfer can be avoided, and Gibbs sampling can sample the joint distribution of high-dimensional random variables. The specific process mainly, assuming a kd random vector $X = \{x_1, x_2, \ldots, x_M\}$, cannot obtain the joint probability distribution $P(X)$ of X, but the rest of the components $x_{k^-} = \{x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_M\}$ of a given X, the conditional probability of the k-th component x_k is $P(x_k|x_{k^-})$, therefore can from an initial state of$X$ (such as $[x_1^{(0)}, x_2^{(0)}, \ldots, x_M^{(0)}]$), using the amount of conditional probability, iteratively to state the weight of samples, The distribution of the random variable converges geometrically to $P(X)$ as the number of samples increases.

Gibbs algorithm is employed to get random data conforming to the model distribution in the RBM model. The sampling process is introduced in Figure 2.

The specific steps of t-step Gibbs sampling in RBM are as three steps as follows:

Step 1: First, use the input sample to initialize the state v_0 of the visual node;

Step 2: Then, determine the sampling times t. Sampling is carried out according to the following conditional probability formula:

$h^{(s-1)}$ is obtained by sampling with conditional probability $P(h|v^{(s-1)})$;

Then the conditional probability $P(h|v^{(s-1)})$ is sampled to get v^s;

Among them, $s = 1, 2, \ldots, t$.

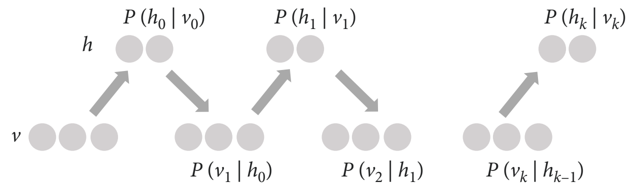


Figure 2: Gibbs sampling process.

Step 3: Step 2 is cyclically sampled for t times, and finally, when the sampling times t is large enough, v^t can be obtained.

3.3. Deep Belief Network Classification Method for Nonlinear Correction

3.3.1. DBN Model Based on Modified Elliott Function. DBN model is composed of multiple RBM stacked on top of each other, so in the training process of RBM, the activation function also determines the ability of feature extraction. RBM performs a step sampling by CD algorithm and Gibbs sampling. Firstly, the visible layer is mapped to the output of the hidden layer through the activation function, and then the output is taken as the input of the visible layer.

According to the analysis of the activation function in this article, it can be known that the activation function is a core position in network training. If the activation function is improperly selected, it is difficult to improve the accuracy of training learning no matter how to construct the model structure. However, if the activation function is properly selected, the feature extraction ability of the network can be significantly improved. Based on this, a DBN model based on the Modified Elliott function (ME-DBN) is proposed in our article. Elliott function [2] satisfies the generalized Logistic differential equation, so this paper introduces the Elliott function into the model to improve the traditional sigmoid activation function, as shown below:

$$f(x) = \frac{0.5x}{1 + |x|} + 0.5. \tag{7}$$

In order to ensure that all neurons are saturated in the pretraining stage, the activation function should have a high gradient zero value. Based on this analysis, formula (7) is revised in this paper:

$$f(x) = \frac{0.5x}{\sqrt{1 + x^2}} + 0.5. \tag{8}$$

Figure 3 shows the function graph of the modified Elliott function and sigmoid function.

As we can find from Figure 3, the modified Elliott function becomes steeper near zero, which causes more major features to fall into the middle region of the function, and at the same time, reaches the threshold at the lower value of its input, closer to the biological neuron than the sigmoid function.

Next, this paper compares the modified Elliott function with the sigmoid function, as shown in Figure 4:

As we can find from Figure 4 that ReLU has no gradient at the negative half-axis of input, and the modified function in this paper has a gradient, so the problem of failing to update the weight will not occur.

To better fit the distribution of input data in the network model in the pretraining stage, the activation function in the pretraining stage is improved in this paper. Therefore, after introducing the modified Elliott function into RBM, the conditional probability of the visible layer and hidden layer can be deduced as follows:

$$\begin{aligned} P(v_i = 1|h,\theta) &= \prod_i P(v_i|h)P(v_i = 1|h) = ME\left(\sum_j W_{ij}h_j + a_i\right) = \frac{\sum_j W_{ij}h_j + a_i}{2\sqrt{1 + \left(\sum_j W_{ij}h_j + a_i\right)^2}} + \frac{1}{2}, \\ P(h_i = 1|v,\theta) &= \prod_i P(h_i|v)P(h_i = 1|v) = ME\left(\sum_j W_{ij}v_j + b_i\right) = \frac{\sum_j W_{ij}v_j + b_i}{2\sqrt{1 + \left(\sum_j W_{ij}v_j + b_i\right)^2}} + \frac{1}{2}. \end{aligned} \tag{9}$$

In the pretraining stage, the training objective is still the maximized likelihood function. CD algorithm is used for sampling, so the parameter update formula is as follows:

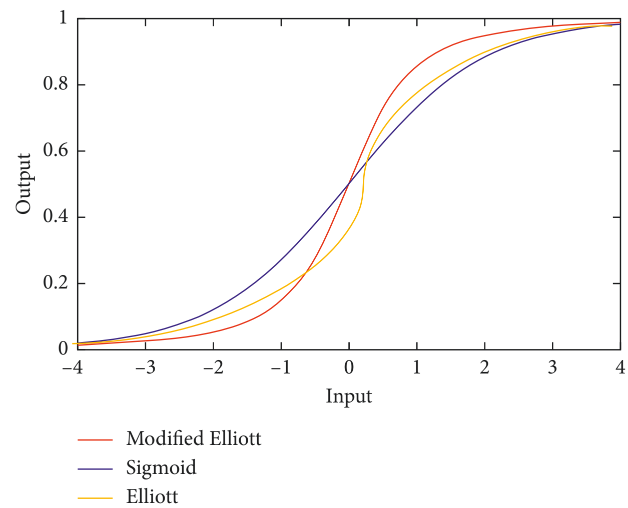


FIGURE 3: Comparison of modified Elliott function and sigmoid activation function.

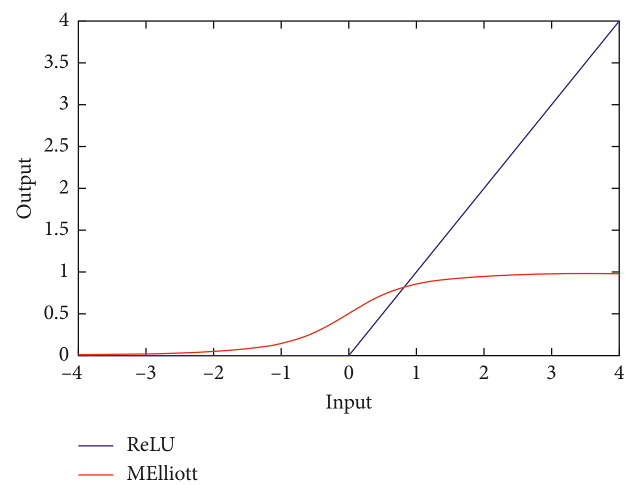


FIGURE 4: Comparison of modified Elliott function and ReLU activation function.

$$\begin{aligned}
\frac{\partial L(\theta)}{\partial W_{ij}} &= \sum_{t=1}^{T}\left[P\left(h_j = 1|v^{(t)}\right)v_i^{(t)} - \sum_{v} P(v)P\left(h_j = 1|v\right)v_i\right] \\
&= \sum_{t=1}^{T}\left[\left(\frac{\sum_j W_{ij}v_i^{(t)} + b_j}{2\sqrt{1 + \sum_j W_{ij}v_i^{(t)} + b_j^2}} + \frac{1}{2}\right)v_i^{(t)} - \sum_{v} P(v)\left(\frac{\sum_j W_{ij}v_i + b_j}{2\sqrt{1 + \sum_j W_{ij}v_i + b_j^2}} + \frac{1}{2}\right)v_i\right], \\
\frac{\partial L(\theta)}{\partial a_i} &= \sum_{t=1}^{T}\left[v_i^{(t)} - \sum_{v} P(v)v_i\right], \\
\frac{\partial L(\theta)}{\partial b_j} &= \sum_{t=1}^{T}\left[P\left(h_j = 1|v^{(t)}\right) - \sum_{v} P(v)\left(h_j = 1|v\right)\right] \\
&= \sum_{t=1}^{T}\left[\left(\frac{\sum_j W_{ij}v_i^{(t)} + b_j}{2\sqrt{1 + \sum_j W_{ij}v_i^{(t)} + b_j^2}} + \frac{1}{2}\right) - \sum_{v} P(v)\left(\frac{\sum_j W_{ij}v_i + b_j}{2\sqrt{1 + \sum_j W_{ij}v_i + b_j^2}} + \frac{1}{2}\right)\right].
\end{aligned} \tag{10}$$

The parameters trained in the pretraining stage are used as the initialization of the fine-tuning stage, and the whole MEDBN network is fine-tuned by using the gradient descent method.

4. Evaluation Processing

After the training of these three classification models and training data, the performance of these models is evaluated with test data. The classifier best suited to the current traffic environment is that it has the most accurate classification model. Accuracy is represented as follows:

$$\text{Accuracy} = \frac{TP + TN}{TP + FP + FN + TN}. \tag{11}$$

In the formula, TP is a true positive, indicating that the traffic that belonged to category C is classified in category C. FP is a false positive, showing that the traffic not belonging to category C is classified by mistake; FN is missing report, indicating that the traffic not belonging to category C is classified into others; TN is a true negative, indicating that the traffic of noncategory C that is classified as noncategory C.

The precision defined in formula (11) is used to select the optimal proposed model. At the same time, three indicators are used to evaluate the performance of the proposed model, which are Precision, Recall, and F1 score, The definition is as follows:

$$\begin{aligned} \text{Precision} &= \frac{TP}{TP + FP}, \\ \text{Recall} &= \frac{TP}{TP + FN}, \\ F1_{\text{score}} &= \frac{2 \times \text{Precision} \times \text{Recall}}{\text{Precision} + \text{Recall}}. \end{aligned} \tag{12}$$

5. Data Processing

ISCX VPn-NonVPN Traffic Dataset was selected in the experiment. As shown in Table 1, this data set consists of 15 applications, such as Facebook, Youtube, Netflix, and so on. The selected application uses various security protocols for encryption, including HTTPS, SSL, SSH, and proprietary protocols. The selected data set contains a total of 206,688 packets. Obviously, the data set is unbalanced. Some applications have a large number of traffic samples, such as Netflix, which accounts for 25.126% of the total data set. Meanwhile, some applications have a small number of traffic samples, such as Aim Chat and ICQ, which only account for 2-3% of the total data set.

6. Experiment and Simulation

To further distinguish the effectiveness of our proposed model for identifying encrypted traffic, we list a series of classic benchmark models that have been proven to achieve excellent prediction and classification results in various fields. Figure 5 shows the evaluation of classification recognition results with the benchmark models. They include XGboost algorithm and GBDT algorithm based on number model, Bayesian classification algorithm and SVM algorithm based on classical classification algorithm, LSTM model and RNN model based on the neural grid. At the same time, we also take a single DBN model as one of the benchmark models to compare the classification recognition results between the DBN model and our ME-DBN model. As can be seen from the figure above, (1) Among all benchmark models, the ME-DBN model achieves the best performance in five indicators, which indicates that our proposed model is effective in the classification of encrypted traffic; (2) compared with all benchmark models, DBN model achieves the best results in ACC and F1 indicators, and also ranks TOP3 in classification results of other indicators, which indicates that, on the whole, DBN model can effectively identify encrypted traffic; (3) although RNN model achieves the best result in FRrate index, its performance in other indicators is poor. We could find that classification results based on the RNN model are unstable. (4) Compared with the DBN model, the performance of ME-DBN proposed by us is superior to the DBN model in all five indicators, which indicates that the method proposed by us can effectively enhance the basic DBN model; (5) we can also know from the experimental results that neural grid based models such as CNN, RNN, and LSTM have significantly higher classification and identification effects on encrypted traffic than other benchmark models (including decision tree-based classification model and classical mathematical classification model).

Figures 6 and 7 show the comparison of training and prediction time of different algorithms on the ISCX VPN-nonVPN dataset. Among them, 70% samples are selected as training samples and 30% samples are selected as test samples. It can be seen from the figure that in the ISCX

Table 1: Description of sample dataset.

The application name	Unbalanced sample	
	Quantity	Ratio (%)
AIM_chat	4869	2.3
SCPdown	15390	7.4
Youtuebe	12738	6.1
Voipbuster	35469	17.1
E-mail	4417	2.1
Vimeo	18755	9
Facebook	5527	2.6
Gmail	7329	3.5
TorTwitter	14654	7
Hangout	7587	3.6
Spotify	14442	6.9
Skype	4607	2.2
ICQ	4243	2
SETPdown	4729	2.2
Netflix	51932	25.1
Total	206688	100

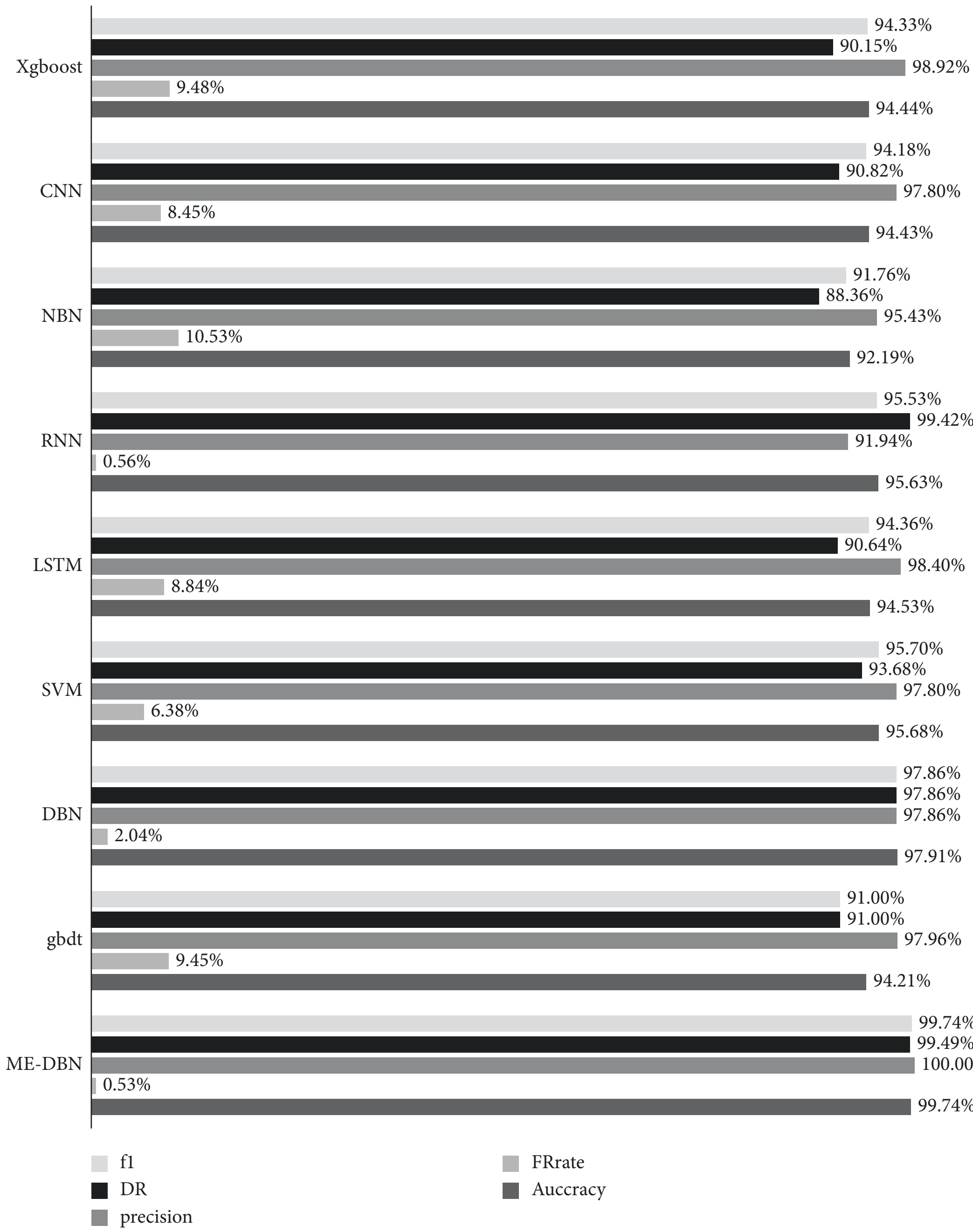


Figure 5: Evaluation of classification recognition results with the benchmark models.

VPN-NonVPN data set, the CNN model has the longest training time and the NBN model has the shortest prediction time, but at the same time, the CLASSIFICATION accuracy of the NBN model is not high. ME-DBN has more training time on the dataset than the DBN model because sparse regular terms are added to the likelihood function and the derivation process affects the training time. For deeper structure, it needs more training time than the traditional SVM and XGboost model because of the complicated calculation process, but the classification results have a certain increase, and the algorithm proposed in our article the ME - DBN and DBN algorithm forecast time differs only 4 s, demonstrate the algorithm based on sparse model time performance improves the classification accuracy. It has better classification performance.

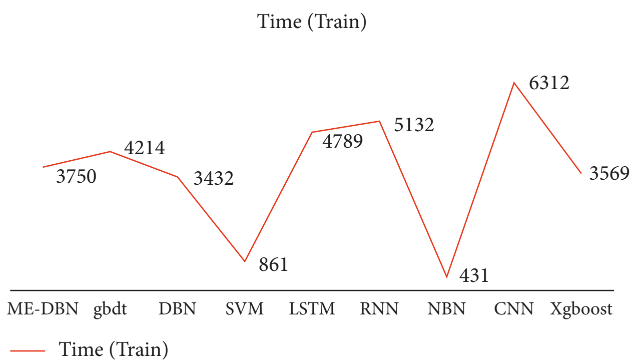


FIGURE 6: Comparison of training time of different algorithms.

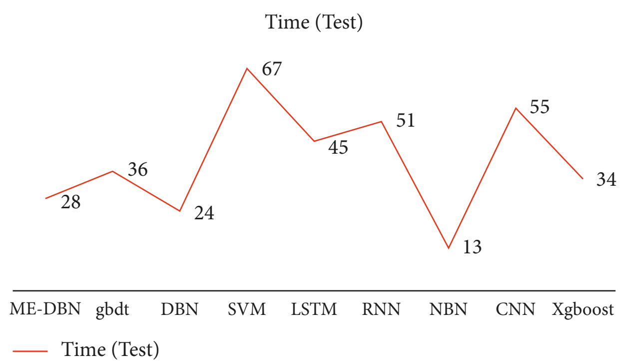


FIGURE 7: Comparison of prediction time of different algorithms.

7. Conclusions

At present, the deep neural network has become an important research content on machine learning. Feature extraction algorithm based on DNN mainly uses a deep neural network model to carry out feature extraction of data imitating the information processing mechanism of the human brain, so as to screen the important information in data. The deep neural network has an excellent performance in extracting images, sound, text, and other information. However, with the increasing scale of data sets, the network structure becomes more and more complex, making network training more difficult, which requires more effective training methods. Secondly, when the traditional sparse deep network model learns input data, all hidden layer nodes may have the same effect without completely changing the feature homogeneity phenomenon. In addition, the traditional Sigmoid activation function is nonzero mean, which is difficult to effectively train the network and is prone to the phenomenon of gradient disappearance. To handle the above problems, our article studies the feature extraction algorithm based on DNN, and proposes a DBN traffic classification method based on the nonlinear correction. In view of the phenomenon of gradient disappearance that the traditional Sigmoid function is prone to, the Elliott function satisfying the generalized Logistic differential equation is proposed to replace the Sigmoid activation function, and then the Elliott function is modified to meet the characteristics of RBM. The modified Elliott function can make the nodes in the saturation state, so it is not easy to cause the problem of gradient disappearance.

Data Availability

The support data can can be obtained from the author upon request.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] S. Kumar, S. Dharmapurikar, F. Yu, P. Crowley, and J. Turner, "Algorithms to accelerate multiple regular expressions matching for deep packet inspection," *ACM SIGCOMM - Computer Communication Review*, vol. 36, 2006.

[2] Y. Zeng, M. Qiu, Z. Ming, and M. Liu, "Senior2Local: a machine learning based intrusion detection method for VANETs," in *Proceedings of the International Conference on Smart Computing and Communication*, December 2018.

[3] M. Shen, M. Wei, L. Zhu, and M. Wang, "Classification of encrypted traffic with second-order Markov chains and application attribute bigrams," *IEEE Transactions on Information Forensics and Security*, vol. 12, no. 8, pp. 1830–1843, 2017.

[4] C. V. Wright, F. Monrose, and G. M. Masson, "On inferring application protocol behaviors in encrypted network traffic," *Journal of Machine Learning Research*, vol. 7, pp. 2745–2769, 2006.

[5] W. Wang, Y. Sheng, J. Wang et al., "HAST-IDS: learning hierarchical spatial-temporal features using deep neural networks to improve intrusion detection," *IEEE Access*, vol. 6, pp. 1792–1806, 2018.

[6] M. Lotfollahi, R. S. H. Zade, M. J. Siavoshani, and M. Saberian, "Deep packet: a novel approach for encrypted traffic classification using deep learning," 2017, https://arxiv.org/abs/1709.02656.

[7] J. Hochst, L. Baumgartner, M. Hollick, and B. Freisleben, "Unsupervised traffic flow classification using a neural autoencoder," in *Proceedings of the IEEE 42nd Conf. Local Comput. Netw. (LCN)*, pp. 523–526, Singapore, October 2017.

[8] P. Wang, F. Ye, X. Chen, and Y. Qian, "Datanet: deep learning based encrypted network traffic classification in sdn home gateway," *IEEE Access*, vol. 6, pp. 55380–55391, 2018.

[9] M. A. Qatf, Y. Lasheng, M. A. Habib, and K. A. Sabahi, "Deep learning approach combining sparse autoencoder with SVM for network intrusion detection," *IEEE Access*, vol. 6, pp. 52843–52856, 2018.

[10] K. Claffy, H. Braun, and G. Polyzos, "A parameterizable methodology for internet traffic flow profiling," *IEEE Journal on Selected Areas in Communications*, vol. 13, no. 8, pp. 1481–1494, 1995.

[11] R. Gu, H. Wang, Y. Sun, and Y. Ji, "Fast traffic classification using joint distribution of packet size and estimated protocol processing time," *IEICE - Transactions on Info and Systems*, vol. 93, no. 11, pp. 2944–2952, 2010.

[12] S. Yeganeh, M. Eftekhar, Y. Ganjali, R. Keralapura, and A. Nucci, "CUTE: traffic classification using terms," in *Proceedings of the 2012 21st International Conference on Computer Communications and Networks ICCCN*, pp. 1–9, Munich, Germany, July 2012.

[13] H. Ibrahim, S. Nor, and H. Jamil, "Online hybrid internet traffic classification algorithm based on signature statistical and port methods to identify internet applications," in *Proceedings of the 2013 IEEE International Conference on Control*

System, Computing and Engineering, pp. 185–190, Penang, Malaysia, November 2013.

[14] M. Conti, L. Mancini, R. Spolaor, and N. Verde, "Can't you hear me knocking: identification of user actions on android apps via traffic analysis," in *Proceedings of the 5th ACM Conference on Data and Application Security and Privacy*, pp. 297–304, New York, NY, U.S.A, March 2015.

[15] G. Xiong, W. Huang, Y. Zhao, M. Song, Z. Li, and L. Guo, *Real-time Detection of Encrypted Thunder Traffic Based on Trustworthy Behavior Association*, Springer, Berlin Germany, pp. 132–139, 2013.

[16] A. McGregor, M. Hall, P. Lorier, and J. Brunskill, *Flow clustering using machine learning techniques*, Springer, Berlin, Germany, pp. 205–214, 2004.

[17] W. Wang, M. Zhu, X. Zeng, X. Ye, and Y. Sheng, "Malware traffic classification using convolutional neural network for representation learning," in *Proceedings of the 2017 International Conference on Information Networking (ICOIN)*, pp. 712–717, IEEE, Da Nang, Vietnam, January 2017.

[18] M. Lotfollahi, M. J. Siavoshani, R. S. H. Zade, and M. Saberian, "Deep packet:A novel approach for encrypted traffic classification using deep learning," *Soft Computing*, vol. 24, no. 3, pp. 999–2012, 2020.

[19] L. Grimaudo, M. Mellia, and E. Baralis, "Hierarchical learning for fine grained intemet traffic classification," in *Proceedings of the Wireless Communications and Mobile Computing Conference (IWCMC)*, pp. 463–468, IEEE, Limassol, Cyprus, August 2012.

[20] M. Lotfoilahi, R. S. H. Zade, M. J. Siavoshani, and M. Saberian, "Deep packet:A novel approach for encrypted traffic classification using deep learning," *Soft Computing*, vol. 24, no. 3, pp. 1999–2012, 2017.

[21] Y. LeCun, Y. Bengio, and G. Hinton, "Deep learning," *Nature*, vol. 521, no. 7553, pp. 436–444, 2015.

[22] T. Qin, L. Wang, Z. Liu, and X. Guan, "Robust application identification methods for P2P and VoIP traffic classification in backbone networks," *Knowledge-Based Systems*, vol. 82, pp. 152–162, 2015.

[23] G. Aceto, D. Ciuonzo, A. Montieri, and A. Pescape, "Mobile encrypted traffic classification using deep learning," in *Proceedings of the 2018 Network Traffic Measurement and Analysis Conference (TMA)*, pp. 1–8, Vienna, Austria, June 2018.

[24] G. Aceto, D. Ciuonzo, A. Montieri, and A. Pescape, "Mobile encrypted traffic classification using deep learning: experimental evaluation, lessons learned, and challenges," *IEEE Transactions on Network and Service Management*, vol. 16, no. 2, pp. 445–458, 2019.

[25] M. Martín, B. Carro, A. S. Esguevillas, and J. Lloret, "Network traffic classifier with convolutional and recurrent neural networks for internet of things," *IEEE Access*, vol. 5, pp. 18042–18050, 2017.

[26] J. Hochst, L. Baumgartner, M. Hollick, and B. Freisleben, "Unsupervised traffic flow classification using a neural autoencoder," in *Proceedings of the 2017 IEEE 42nd Conference on Local Computer Networks (LCN)*, pp. 523–526, Singapore, October 2017.

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Research Article

Short-Term Electrical Load Demand Forecasting Based on LSTM and RNN Deep Neural Networks

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As the development of smart grids is increasing, accurate electric load demand forecasting is becoming more important for power systems, because it plays a vital role to improve the performance of power companies in terms of less operating cost and reliable operation. Short-term load forecasting (STLF), which focuses on the prediction of few hours to one week ahead predictions and is also beneficial for unit commitment and cost-effective operation of smart power grids, is receiving increasing attention these days. Development and selection of an accurate forecast model from different artificial intelligence (AI)-based techniques and meta-heuristic algorithms for better accuracy is a challenging task. Deep Neural Network (DNN) is a group of intelligent computational algorithms which have a viable approach for modeling across multiple hidden layers and complex nonlinear relationships between variables. In this paper, a model for STLF using deep learning neural network (DNN) with feature selection is proposed. A wide range of intelligent forecast models was designed and tested based on multiple activation functions, such as hyperbolic tangent (tanh), different variants of rectifier linear unit (ReLU), and sigmoid. Among the others, DNN with leaky ReLu produced the best forecast accuracy. Regarding the precision of the methods used in this research work, certain output measures, such as absolute percentage error (MAPE), mean square error (MSE), and root mean square error (RMSE) are used. There was also a reliance on multiple parametric and variable details to determine the capability of the smart load forecasting techniques.

1. Introduction

Load forecasting strengthens utility corporations' ability to model and anticipate power loads in order to maintain a balance between supply and demand, reduces manufacturing costs, estimates fair energy pricing, and regulates capacity scheduling and future planning. These forecasts are extremely important for energy suppliers and other power system stack holders, as well as for power generation, transmission, and distribution industries. There is also the precise projection of electric load magnitudes and geographic locations for various times of the planning horizon [1]. The main criterion is used to test the predictions of the model on the basis of lead-time horizon. Accurately predicting future load requirements is critical for proper generation planning. It additionally improves the performance of the power system and facilitates managerial decision making in the future. Inaccurate forecasts can be the reason for massive economic losses for housing and power system. Researchers have applied a number of techniques for electrical load demand forecasting using numerous statistical, mathematical, and artificial intelligence-based approaches to facilitate the supply chain of electrical load in a smooth manner. It is found that deep neural networks (DNN) and their hybrid combinations with other meta-heuristic optimization algorithms provided wonderful functionality in managing complicated nonlinear relationships, model complexity, and their prediction performance is found accurate as compared to other conventional methods [2]. The major objective of this research work is to enhance electrical load demand forecast accuracy by implementing the state-of-the-art deep neural networks using LSTM and RNN architectures. In particular, the impact of seasonal variation on forecast error has been explored and reported.

According to the literature, short-term electrical load demand forecasts are of considerable interest. This forecasting is important for power system control, unit input, security assessment, economic calculations, and power markets [3]. STLF is under high consideration for controlling and optimizing energy systems on a daily energy efficiency basis, exchange, and security checks. It is also useful for reliability considerations and mathematical calculations in the power system. However, STLF needs a great effort to produce reasonable forecast accuracy because of the lesser lead-time. For immediate and accurate future predictions on the basis of lesser lead-time, we need more parametric analysis and complex modeling techniques [4]. Choosing a good technique for STLF is most important for high accuracy in the results. One of an electric company's key jobs is to precisely estimate load demand at all times, which is especially important in the near term. Observing the behavior of near-future load demand may be highly useful for the assessment and operation of power systems in terms of a noninterrupted supply chain of power [5].

There are several load-forecasting techniques that are classified as parametric and nonparametric techniques in two major sections. Parametric techniques are based on mathematical and statistical equations such as time series and linear regression. Nonparametric techniques are artificial intelligence and machine learning-based techniques such as artificial neural networks (ANN), deep neural networks (DNN), fuzzy logic, and expert systems. In the category of nonlinear techniques, many hybrid combinations of ANN and DNN with nature inspired meta-heuristic techniques such as genetic algorithm (GA), particle swarm optimization (PSO), feature selection, and others have been reported frequently for STLF in the past decade. It is further reported by many researchers that these hybrid combinations of intelligent forecast methods produced highly efficient models in terms of accuracy and generalization.

The electric load forecasting is categorized into three classes including short-term forecasts, that is, from few minutes to few days ahead, medium-term forecasts from one week to few months ahead, and long-term forecasts of 1 year to 10 years ahead [6]. Short-term load forecasting (STLF) is useful for day-to-day decisions including fuel requirement and maintenance scheduling systems setup, whereas medium-term load forecasting (MTLF) is important for system maintenance, purchasing electricity, and pricing plans. This maintains the shutdown and maintenance scheduling, as well as load-switching operations. On the other hand, long-term load forecasting (LTLF) is highly beneficial for expansion plans and the development of new power plants.

In this study, intelligent computational models are designed and developed using a deep neural network integrated with feature selection and genetic algorithm using various activation functions, such as sigmoid, tanh, and ReLU to forecast short-term electrical energy demand. To make forecasts more trustworthy, all significant factors impacting future power usage must be included [7]. DNNs are always difficult to train, test, and validate, particularly when the dimensions of the input are very large. It is very critical to pick important features by evaluating a DNN-trained model's first-layer activation potential [8]. Moreover, a crucial factor in the DNN-based model for STLF is the availability of a small number of data samples for the training phase, which can cause the model to overfit. To avoid overfitting, we used 2 years of electricity load from FESCO, a company in Pakistan, to supply the electricity. In this investigation, there are one year of Australian electricity load data and other input parameters with feature selection to train the presented DNN models utilizing a single activation function. In the literature, [9], Denil et al. demonstrate that it is possible not only to forecast all the other weights but also to exclude some of the weights, providing a few weights to every element. It is shown for neurons with multiple layers, training 25% with parameters produces the same error as learning all weights. Sainath et al. [10] use low-rank matrix factorization to reduce the number of input parameters in the final layer of a DNN.

Hybrid load demand forecast model by integrating deep recurrent neural networks and LSTM architectures are designed, developed, and tested. Their performance is compared with the conventional ANN design. The performance of the developed hybrid model in different seasonal and load demand variations is examined on a day-ahead and week-ahead basis. The integration of various meta-heuristic techniques adds up the individual features of those methods to produce the summed up benefits. However, the hybridization of multiple methods leads to complexity and affects the transparency of these models. As the LSTM model keeps a track of the vibrant recent memory states, its strength in remembering the recent past states is considered superior as compared to other meta-heuristic methods.

The remainder of the paper is ordered as, Section 2 explains DNN's and RNNs importance for load forecasting. Section 3 explains the data description. Section 4 explains the methodology behind the hybridization of DNN and the feature selection-based model for STLF. The results and corresponding consequences are seen in Section 5, and finally, the conclusions regarding the proposed method are provided in Section 6.

2. Deep Neural Networks and Applications

Deep neural network is an advanced form of conventional ANN whose learning is typically carried out using the framework of complex architecture with multiple hidden layers and neurons. The word "deep" refers to the topological structure of NN with a number of layers in the network. A deep neural network (DNN) is just an ANN with some extra layers than the three standard layers of multiple-layer perceptron (MLP). A deep neural network integrates several nonlinear layers of computation, utilizing basic parallel operating components biologically inspired nervous systems. Deep learning is traditionally focused on using back propagation including gradient descent and a huge number of neurons and hidden layers [11]. The deep structure enhances the potential of neural networks for abstraction.

Currently, the advancement of the Internet of things (IoT) and big data allows the deployment of DNNs in a variety of ways. Moreover, recent findings for DNN have shown great promise in other fields, such as computer vision and voice recognition. However, there is much less work on applying DNN to short-term load forecasting in STLF is found in the literature [12]. DNNs allow higher precision to be achieved by detecting dominant factors that influence electricity consumption trends and can surely make a major contribution to next-generation energy systems and the recently launched Smart grid [13]. A typical DNN interconnection structure with one input and output layer and multiple hidden layers having neurons is shown in Figure 1. It can be seen that there are a number of hidden layers which convert this neural network to deep neural network. The results will also be more accurate by more layers.

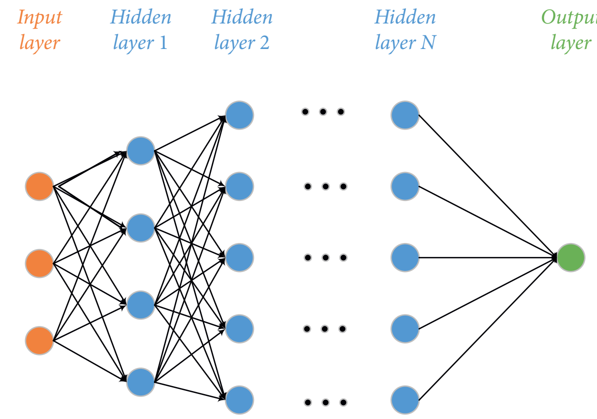

Figure 1: A typical deep neural network architecture.

2.1. DNN with Genetic Algorithm. Genetic algorithm is a method of programming which derives its foundation from biological evolution [14]. The Genetic Algorithm is generally used as a problem-solving technique to have the optimized value [15]. A hybrid model designed by integrating a genetic algorithm (GA) and deep neural network (DNN) is used to increase performance, cogency, and reduce the error [16]. Specifically, GA is used for selecting features and optimizing DNN design parameters [17]. A set of possible solutions is provided to GA as inputs and evaluation of the performance of each input is carried out with a metric called a fitness function, which allows each candidate to be quantitatively evaluated. The input to the GA is a series of feasible solutions to the problem, stored in some form, and a metric named as fitness function that allows every applicant to be concretely evaluated. The GA's functionality was proven by the creation of a DNN with more than four million parameters; the best infrastructure ever developed by an evolutionary algorithm [18].

2.2. DNN with Feature Selection. Selection of features distinguishes the relevant features from a collection of data and eliminates unrelated or less-significant features which do not lead most to our target variable in order to obtain optimal reliability for our model. It is commonly accepted that the performance of DNNs is because the relationship between the target value and the features of the input is very significant. It takes gradual and definite transformation to render useful features [19]. For a DNN, the measurement of sensitivity does not work far beyond one or two layers. Therefore, in order to better evaluate an input feature's contribution, we review its activation potential (averaged over all input training values and hidden neurons) relative to the full activation potential. The greater the possible activation involvement of an input factor, the more likely its inclusion in the hidden layers [20].

2.3. Hybrid LSTM and DNN Recurrent Neural Network. The ANN is referred to as recurrent neural networks when feed forward neural networks are expanded to provide feedback connections as shown in Figure 2. The input neurons are responsible to receive inputs, whereas relational ends receive the signals modified with an activation function from the current input layer and from the hidden nodes in the previous state of the network at each time-step of sending input through a recurrent network. Long short-term memory (LSTM) networks are a revamped variant of recurrent neural networks, allowing memory retrieval of previous data simpler. The RNN problem of the vanishing gradient is solved here. Given unpredictable time delays, LSTM is well suited for categorizing, analyzing, and forecasting time series. It trains the model using back-propagation. The performance is determined by the secret state of the hidden layers. The concept behind RNNs is to make use of the knowledge in sequence. It is generally assumed that in a typical neural network all inputs and outputs are distinct of one another.

3. Research Methodology

The main requirement for an accurate prediction model is careful analysis of the load data and its dynamics. A big quantity of data is being gathered with the aid of the intelligent meters on every day basis which is called raw data at initial stage as shown in Figure 3. Big statistics analytics can be helpful in reaching insights for smart grid energy management [4]. To achieve the good forecast results, variation and the behavior of the load data is of high consideration. Initial steps for treatment of data are data preprocessing which is also called the data normalization. These methods can be carried out according to simple load profile analysis. On the basis of traits of input data, it can be classified into distinctive clusters by which the network performance can be increased. To preprocess data, the first stage is the compilation of data from the different information systems, for instance, the equipment, customer and charging details, weather, and electrical load system [21]. As designing data management and smart electronics increases, data-oriented applications have been gaining far more interest in both academia and industry through power converters in power grid companies [22, 23]. In addition to the load demand data other factor are also important including metrological data and day type information.

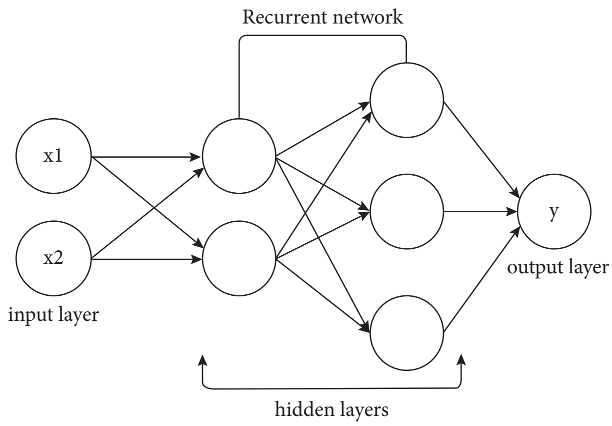

Figure 2: Recurrent neural network.

Long short-term memory (LSTM) networks are an improved variation of recurrent neural networks that make it easier to retrieve earlier data from memory. The declining gradient issue is handled here. Given the unpredictable nature of time delays, LSTM is ideal for categorizing, analyzing, and predicting time series. The performance of the LSTM-based intelligent forecast models has been proved in many smart systems such as smart grids.

3.1. Effect of Temperature. It is noticed that the electrical load demand increases with the rise in temperature during the summer season and it decreases in the cold season. Therefore, the seasonal variables should be included in the predicted model input to obtain accurate predictive results. A review of the literature shows that there is a strong correlation between seasonal variables and load demand. The results of human sensitivity test tell that dew point between 40 F and 60 F is considered comfortable for the humans [24]. The demand for load remains normal in this range of dew point. There is a more need of strength when the temperature falls below 10°C due to heating requirements in a family.

3.2. Working and Nonworking Days. Electricity usage is higher on weekdays while electricity consumption is low on Saturday and Sunday, and also on other public holidays. The "Working Day feature" is chosen based on these results to draw this impact.

3.3. Impact of Time. There is high impact of time on electricity usage. Energy usage values reflect an up-and-down trend, respectively, during both middays. To express the time dependence as hour and day of week, two functions are extracted.

3.4. External Factors. The external factors can also influence the power load behavior (we define data collected as external factors outside the energy database) such as season, climate, and holiday statistics [25].

3.5. Data Preprocessing. There is a process called preprocessing by which the input data is converted into normalized form to facilitate the NN for easy interpretation of input patterns for better results. The change between each input data point interval is between 0 and 1 throughout the normalization procedure. Each input's data can be transformed into normalized form independently or in groups. Preliminary work on entering the input data reduces the size of input space to DNN, which lowers the training time of the network. It shortens the input surface measurement and reduces the variety of parameters that need to be set for the training process.

3.6. Training and Test of Datasets. The model is trained before testing to forecast the input data at high accuracy. We also separated the input data into training and testing datasets in this model, utilizing two years of data for training and one year of data for testing. From the dataset testing, we used 24 and 168 hours forward records for day and week ahead prediction, respectively.

3.7. Training and Test of Datasets. In the design and development of hybrid forecasting models based on DNN and multiple meta-heuristic techniques, different activation functions were used which directly affect the behavior and ultimate DNN efficiency. DNNs typically need capacities for nonlinear activation. Because of their simplicity, rectified linear units (ReLU) are commonly used in modern-day DNNs. Arunadevi et. al. [26] researched the impact of activation function on classification accuracy using DNN. However, choosing an appropriate activation feature is a difficult task [27–29]. Some of the commonly employed activation functions are given in the subsequent sections.

3.8. Sigmoid Function. A sigmoid function is a type of activation function, more precisely a squashing function. Crushing functions, as seen in Figure 4, limit the output to a range of 0 to 1, making them effective in probability prediction.

3.9. Linear Rectified Unit (ReLU). The rectified linear activation function or ReLU for short is a linear piece-by-piece function that directly outputs the input if it is positive, otherwise, it will output zero as shown in Figure 5. For several forms of neural networks, it has become the default activation function, since a model that uses it is easier to train and often achieves better performance. We have, $f(x) = m(o,\ x)$.

3.10. Leaky ReLU. Leaky ReLUs are such method to overcome the "dying ReLU" problem. Rather than making the feature zero if x < 0. Instead, a leaky ReLU would have a slight negative slope (of 0.01, or so), as shown in Figure 6. It can be expressed mathematically:

$F(x) = 1\ (x < 0)\ (\alpha x) + 1\ (x \geq 0)\ (x)$.

Here the α is a constant of computation.

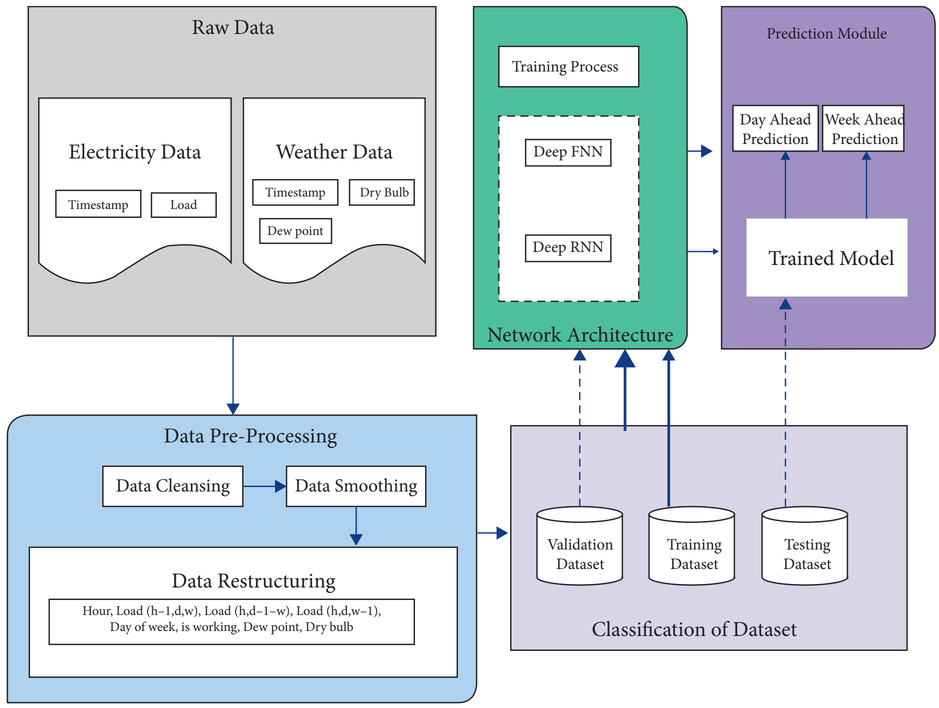

Figure 3: Modelling for load forecasting.

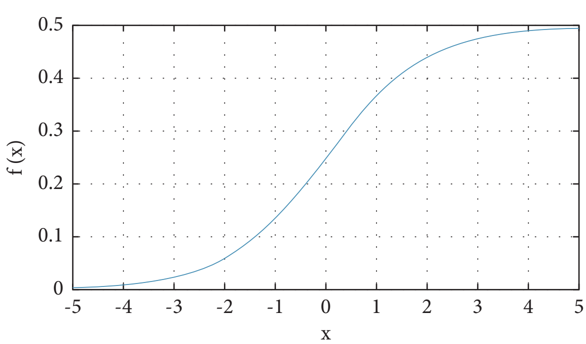

Figure 4: Sigmoid function.

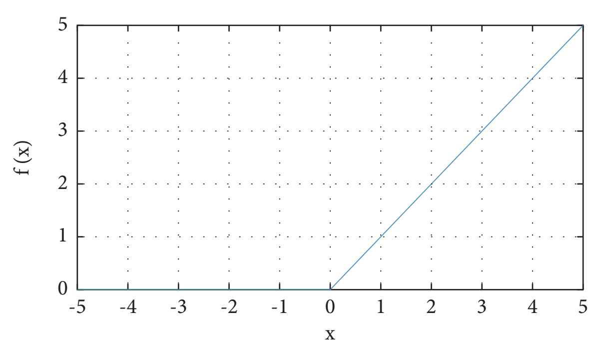

Figure 5: Linear rectified unit (ReLU).

4. Results and Discussion

This section presents the electrical load demand prediction results of benchmark combinational approaches using conventional ANN, RNN, and LSTM. The simulation results, as well as the pertinent discussion of the suggested prediction model for various forecast scenarios, are also presented. To guarantee that the model works successfully in different seasonal fluctuations, its forecast accuracy is validated using load demand and meteorological data for all four seasons of the year. Furthermore, to ensure that the model does not overfit, the forecast performance is validated under high variable load demand situations one day and one week ahead, as well as seasonal load changes.

The model's performance in the aforementioned varied situations demonstrates that it is capable of providing strong prediction results under the vivid and vibrant settings of load demand. To explore the influence of these tactics, a relative analysis of the aforesaid methodologies is performed with respect to the appropriateness of the input variables and ANN design optimization. On a one-hour sampling frequency, the data are collected at a rate of 24 samples per day and 168 samples per week, and it contains electrical load as well as four meteorological variables: dry bulb temperature, wet bulb temperature, dew point, and humidity. MSE and MAPE are performance measures employed to analyze and compare the efficacy of various methods.

The variations in load demand are analyzed w.r.t. seasons: the load demand in the spring and fall seasons is lesser than the load demand in the winter and summer seasons. Furthermore, the summer season's load is less consistent, having higher peaks than the winter season's load. This disparity is most likely due to the less frequent usage of air conditioning during extremely hot summer days, as opposed

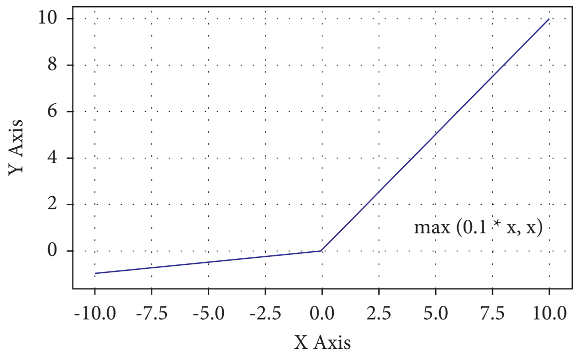

Figure 6: One day-ahead load forecast results of the ANN model for the summer season.

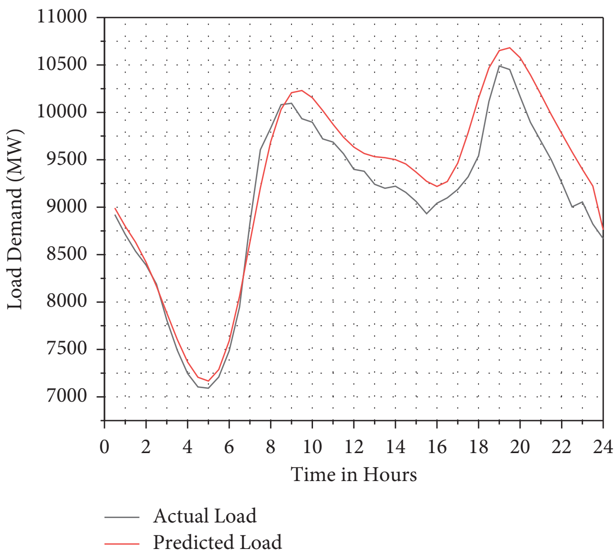

Figure 7: Leaky ReLU.

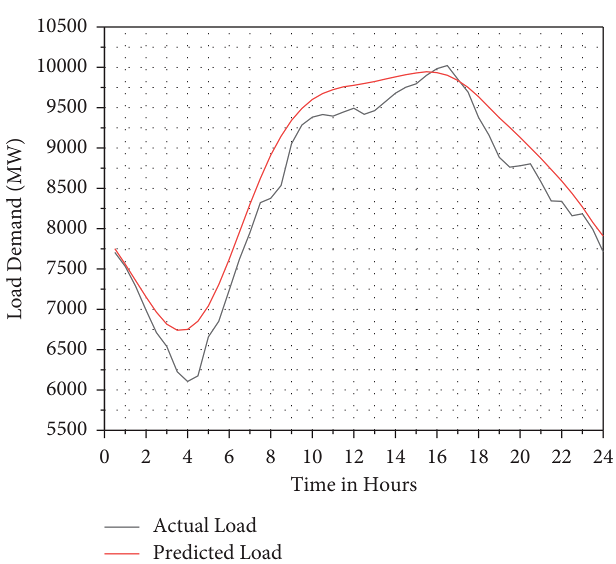

Figure 8: One day-ahead load forecast results of the ANN model for the winter season.

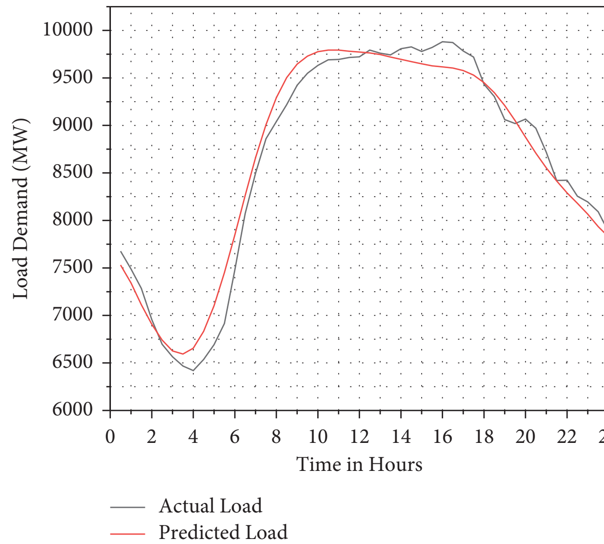

Figure 9: One day-ahead load forecast results of the ANN model for the autumn season.

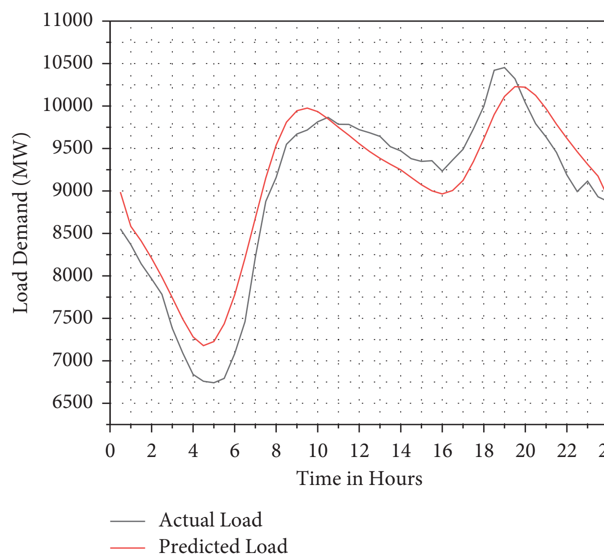

Figure 10: One day-ahead load forecast results of the ANN model for the spring season.

to the more consistent use of heaters throughout the winter. Based on this trend, the data have been divided into four seasons: summer is seen as lasting from November to January, autumn from February to April, winter from May to July and spring from August to October. In addition to the performance indicators, the number of repetitions for the same training error are also used to evaluate the prediction accuracy of the presented algorithms.

4.1. Prediction by ANN Model. A three-layer neural network with an 8-16-1 topology will be used in the tests. The transfer function for hidden layer neurons is logistic sigmoid; however, it is linear for output layer neurons. Figure 6 shows the projected load demand assessment of the ANN model for

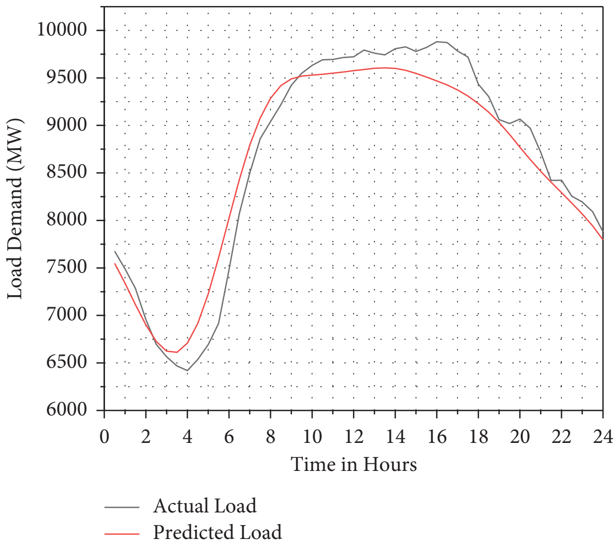

Figure 11: One day-ahead load forecast results of the LSTM model for autumn season.

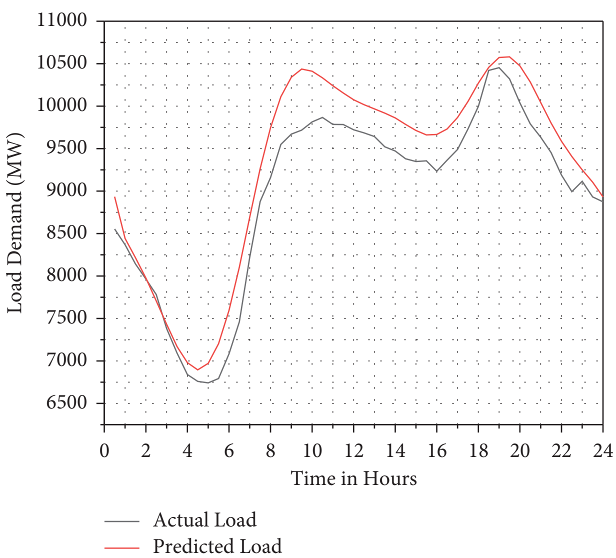

Figure 12: One day-ahead load forecast results of the LSTM model for the spring season.

twenty-four hours ahead load during the summer. The graph's X-axis indicates hourly time with an interval of one hour, while the actual and predicted load demand can be seen on Y-axis. It is apparent that load demand fluctuates depending on the time of day, starting from modest in the morning; however, it rises as the day activities are started. The results of summer season forecasting of one-day ahead of ANN model for other seasons including winter, autumn, and spring are shown in Figures 8–10, respectively. In this model, the best results are found in winter season, where MSE remained 0.10015 and MAPE is found 1.31% for day-ahead predictions.

4.2. Prediction by LSTM Model. The results of LSTM-based model are presented in Figures 11–14 for the autumn,

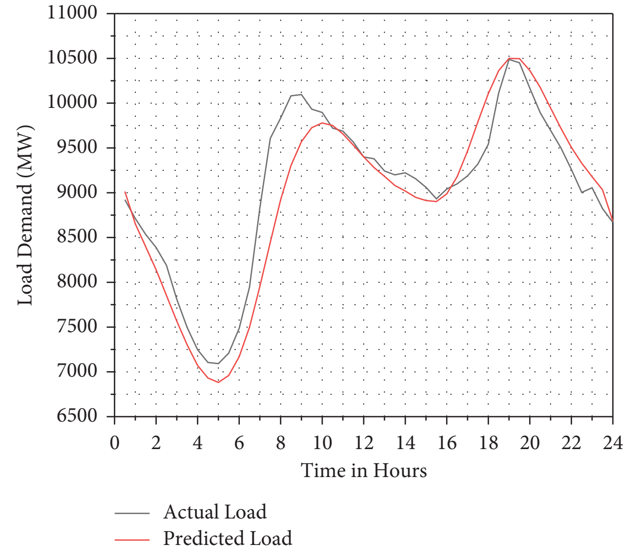

Figure 13: One day-ahead load forecast results of the LSTM model for the summer season.

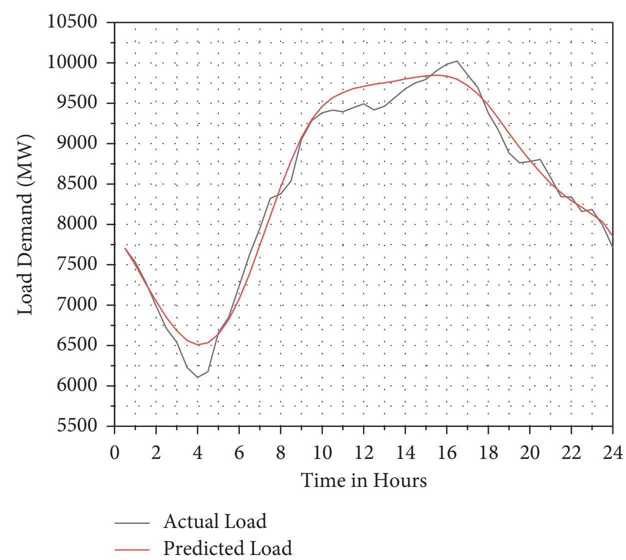

Figure 14: One day-ahead load forecast results of the LSTM model for the winter season.

spring, summer, and winter seasons, respectively. The red line shows the actual load and the green line shows the predicted values. The minimum forecast results for this model are observed in the summer season, where MSE is observed as 0.09153 and MAPE is 1.02% for 24 points per day. The predicted load is decreasing at the initial, but there is a difference between both lines giving better prediction results.

4.3. Prediction by RNN Model. The results of RNN-based hybrid model are presented in Figures 15–18 for the summer, winter, autumn, and spring seasons, respectively. The minimum forecast errors for this model are observed in the summer season, where MSE is observed as 0.09873 and

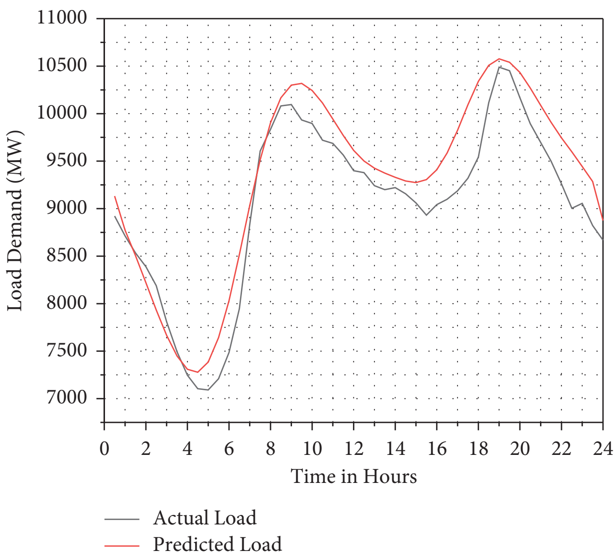

FIGURE 15: One day-ahead load forecast results of the RNN model for summer season.

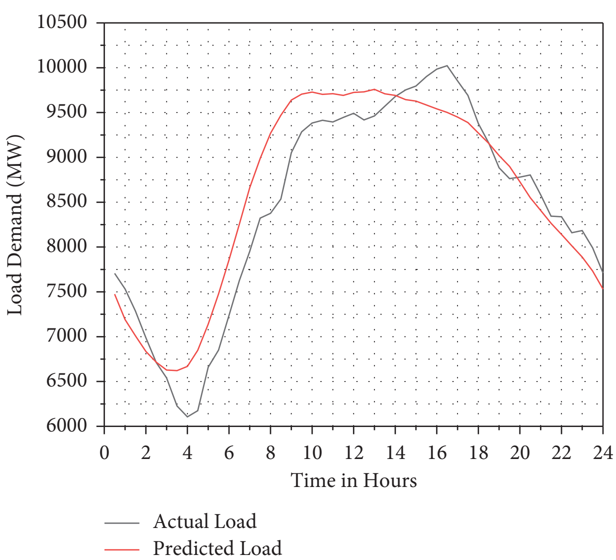

FIGURE 16: One day-ahead load forecast results of the RNN model for winter season.

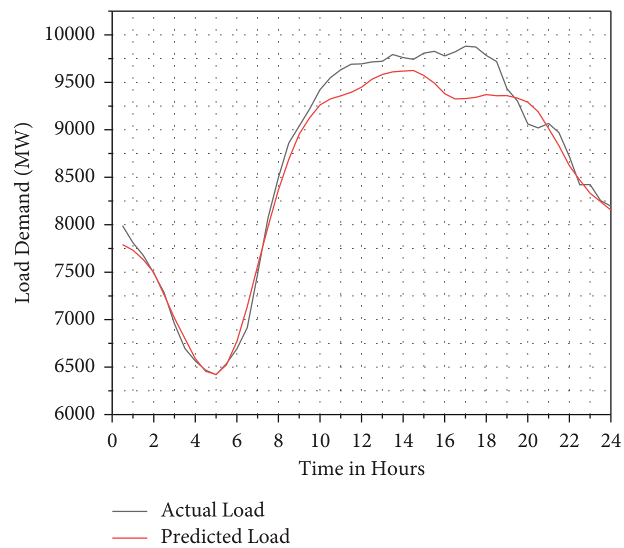

FIGURE 17: One day-ahead load forecast results of the RNN model for the autumn season.

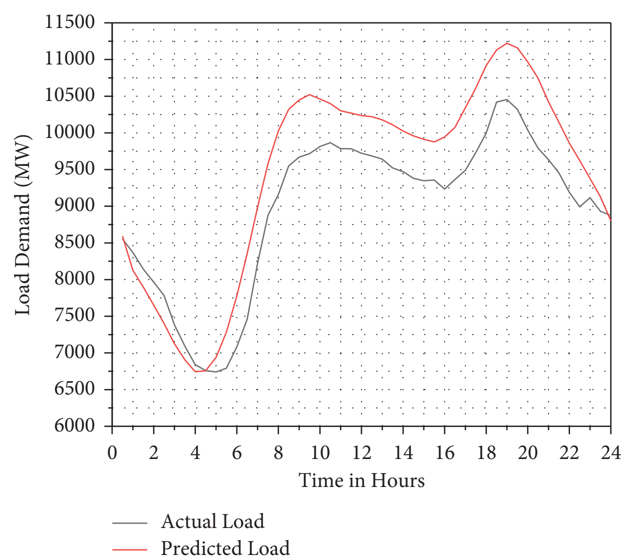

FIGURE 18: One day-ahead load forecast results of the RNN model for the spring season.

TABLE 1: Forecast error of the deployed models.

Model	Forecast error (MAPE) (%)
ANN	1.9
RNN	1.23
LSTM	1.01

MAPE is 1.23% for 24 points per day. The predicted load is decreasing at the initial, but there is a difference between both lines giving better prediction results. Figures 15–18 show the forecast error graphs of the RNN-based hybrid model for summer, winter, autumn, and spring seasons, respectively. The load pattern of all four seasons is different because of the changes in the meteorological parameters, such as temperature, humidity, and cloud cover. However, the proposed model shows reasonable forecast accuracy for all the seasons and demonstrates its generalized prediction strength throughout the year under different load demand conditions. The load forecast results in terms of MAPE are summarized in Table 1.

Among the three models deployed for the electrical load demand prediction, it is observed that the hybrid models based on the combination of LSTM and ANN and RNN performed better w.r.t. forecast accuracy for one-day ahead forecasts. Especially, the RNN model predicted the electrical load demand with an accuracy of 1.01% in terms of MAPE. The RNN model is applied for the prediction of one week ahead forecast, and the results for the summer season are depicted in Figure 19. The model predicted the load demand

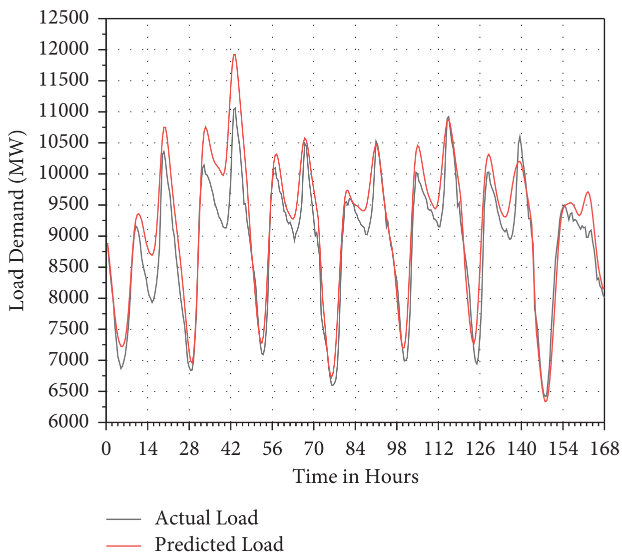

Figure 19: One day-ahead load forecast results of the RNN model for weekly summer.

with a reasonable forecast accuracy of 1.09% MAPE on week ahead bases. All these results show the superiority of the hybrid models in terms of forecast accuracy and generalization.

5. Conclusion

There are several feature descriptors currently available that provide high-dimensional features to identify the behavior in the video, but it takes detailed research to measure the impact of those features on classification. Although size reduction techniques are available to reduce the dimensions of items, their main focus is good reconstruction and the prejudicial information is lost in low-dimensional space. We have used the three types of modeling as LSTM modeling, RNN modeling, and NN modeling for one day forecasting of all the seasons. There is more accuracy using the leaky ReLU activation function with RNN. There are also good results for yearly forecasting data by using the above techniques. The training data is undertaking preprocessing step to predict the new features that will be more important for the use of electricity. The proposed hybrid forecast models have shown high forecast accuracy and generalization that would lead to less-operating costs and safe operation of the power utility companies.

Data Availability

No data are used in the study.

Disclosure

The statements made and views expressed are solely the responsibility of the authors.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

All authors equally contributed in this article.

References

[1] A. A. MamunMamun, M. Sohel, N. Mohammad, M. S. Haque SunnyHaque Sunny, D. R. Dipta, and E. Hossain, "A comprehensive review of the load forecasting techniques using single and hybrid predictive models," *IEEE Access*, vol. 8, pp. 134911–134939, 2020.

[2] M. Cai, M. Pipattanasomporn, and S. Rahman, "Day ahead building-level load forecasts using deep learning vs. traditional time series techniques," *Applied Energy*, vol. 236, pp. 1078–1088, 2019.

[3] S. Bouktif, A. Fiaz, A. Ouni, and M. A. Serhani, "Optimal deep learning LSTM model for electric load forecasting using feature selection and genetic algorithm: comparison with machine learning approaches," *Energies*, vol. 11, no. 7, p. 1636, 2018.

[4] M. Q. Raza and A. Khosravi, "A review on artificial intelligence based load demand forecasting techniques for smart grid and bui," *Renewable and Sustainable Energy Reviews*, vol. 50, pp. 1352–1372, 2015.

[5] L. Wu, C. Kong, X. Hao, and W. Chen, "A short-term load forecasting method based on GRU-CNN hybrid neural network model," *Mathematical Problems in Engineering*, vol. 2020, Article ID 1428104, pp. 1–10, 2020.

[6] H. Hahn, S. Meyer-Nieberg, and S. Pickl, "Electric load forecasting methods: tools for decision making," *European Journal of Operational Research*, vol. 199, no. 3, pp. 902–907, 2009.

[7] D. L. Marino, K. Amarasinghe, and M. Manic, "Building energy load forecasting using deep neural networks," in *Proceedngs of the IECON 2016-42nd Annual Conference of the IEEE Industrial Electronics Society*, pp. 7046–7051, IEEE, Florence, Italy, 2016 October.

[8] Z. Chen, M. Pang, Z. Zhao et al., "Feature selection may improve deep neural networks for the bioinformatics problems," *Bioinformatics*, vol. 36, no. 5, pp. 1542–1552, 2020.

[9] M. Denil, B. Shakibi, L. Dinh, M. A. Ranzato, and N. De Freitas, "Predicting parameters in deep learning," *Advances in Neural Information Processing Systems*, vol. 26, 2013.

[10] T. N. Sainath, K. Brian, S. Vikas, and A. Ebru, "Low rank matrix factorization for deep neural network training with high-dimensional output targets," in *Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing*, October 2013.

[11] K. R. Kareem Kamoona, C. Budayan, and C. Budayan, "Implementation of genetic algorithm integrated with the deep neural network for estimating at completion simulation," *Advances in Civil Engineering*, vol. 2019, pp. 1–15, 2019.

[12] S. Ryu, J. Noh, and H. Kim, "Deep neural network based demand side short term load forecasting," *Energies*, vol. 10, no. 1, p. 3, 2016.

[13] G. M U. Din and K. M Angelos, "Short term power load forecasting using deep neural networks," in *Proceedings of the International Conference on Computing, Networking and Communications (ICNC)*, IEEE, Silicon Valley, CA, USA, March 2017.

[14] L. Huang, S. Ding, S. Yu, J. Wang, and K. Lu, "Chaos enhanced Cuckoo search optimization algorithms for global

optimization," *Applied Mathematical Modelling*, vol. 40, no. 5-6, pp. 3860–3875, 2016.

[15] T. Ghosh, P. Dan, S. Sengupta, and M. Chattopadhyay, "Genetic rule based techniques in cellular manufacturing (1992-2010): a systematic survey," *International Journal of Engineering, Science and Technology*, vol. 2, no. 5, pp. 198–215, 2010.

[16] S. Sivapatham, R. Ramadoss, A. Kar, and B. Majhi, "Monaural speech separation using GA-DNN integration scheme," *Applied Acoustics*, vol. 160, Article ID 107140, 2020.

[17] E. Manzanarez Ozuna, D. L. Flores, E. Gutiérrez López, D. Cervantes, and P. Juárez, "Model based on GA and DNN for prediction of mRNA-Smad7 expression regulated by miRNAs in breast cancer," *Theoretical Biology and Medical Modelling*, vol. 15, no. 1, p. 24, 2018.

[18] X. Cui, W. Zhang, Z. Tüske, and M. Picheny, "Evolutionary stochastic gradient descent for optimization of deep neural networks," *Advances in Neural Information Processing Systems*, p. 31, 2018.

[19] Y. Fang, "Feature selection, deep neural network and trend prediction," *Journal of Shanghai Jiaotong University*, vol. 23, no. 2, pp. 297–307, 2018.

[20] R. K. Debaditya, S. Rama Murty, and C. Krishna Mohan, "Feature selection using deep neural networks," in *Proceedings of the International Joint Conference on Neural Networks (IJCNN)*, IEEE, Killarney, Ireland, October 2015.

[21] H. Li, Y. Li, and H. Dong, "A comprehensive learning-based model for power load forecasting in smart grid," *Computing and Informatics*, vol. 36, no. 2, pp. 470–492, 2017.

[22] L. Hernandez, C. Baladron, J. M. Aguiar et al., "A survey on electric power demand forecasting: future trends in smart grids, microgrids and smart buildings," *IEEE Communications Surveys & Tutorials*, vol. 16, no. 3, pp. 1460–1495, 2014.

[23] F. Javed, N. Arshad, F. Wallin, I. Vassileva, and E. Dahlquist, "Forecasting for demand response in smart grids: an analysis on use of anthropologic and structural data and short term multiple loads forecasting," *Applied Energy*, vol. 96, pp. 150–160, 2012.

[24] V. Gupta and S. Pal, "An overview of different types of load forecasting methods and the factors affecting the load forecasting," *International Journal for Research in Applied Science and Engineering Technology*, vol. 5, pp. 729–733, 2017.

[25] B. Rossi and S. Chren, "Smart grids data analysis: a systematic mapping study," *IEEE Transactions on Industrial Informatics*, vol. 16, no. 6, pp. 3619–3639, 2020.

[26] J. Arunadevi and M. Devaki, "Impact of Activation functions in Deep Neural net algorithm on Classification performance parameters," *International Journal of Pure and Applied Mathematics*, vol. 119, no. 12, pp. 16305–16312, 2018.

[27] Y. Deng, Y. Liu, and D. Zhou, "An Improved Genetic Algorithm with Initial Population Strategy for Symmetric TSP," *Mathematical Problems in Engineering*, vol. 2015, Article ID 212794, 2015.

[28] S. Mustafa, A. A. Bajwa, S. Bajwa, and S. Iqbal, "A new fuzzy grach model to forecast stock market technical analysis," *Operational Research in Engineering Sciences: Theory and Applications*, vol. 5, no. 1, pp. 185–204, 2022.

[29] B. C. Feijó, A. Pavlovic, L. A. O. Pavlovic et al., "Geometrical investigation of microchannel with two trapezoidal blocks subjected to laminar convective flows with and without boiling," *Reports in Mechanical Engineering*, vol. 3, no. 1, pp. 20–36, 2022.

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Research Article

Extended Transportation Models Based on Picture Fuzzy Sets

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Picture fuzzy set (PFS) is an extension of intuitionistic fuzzy set, and it is capable to analyze the transportation problems that contain uncertain and vague information. PFSs are applicable in situations where human decisions demand more variety of responses: yes, abstain, no, and rejection which cannot be addressed in the traditional FSs and IFSs. In this paper, we propose a technique to solve fully picture fuzzy transportation problems (FPFTPs) which are based on picture fuzzy linear programming formulation. Fully picture fuzzy transportation problems are developed by considering all the variables and parameters as nonnegative trapezoidal picture fuzzy numbers. A ranking function is used to transform picture fuzzy numbers into crisp numbers. A model is presented to explain the suggested scheme. Finally, comparison analysis of fully picture fuzzy transportation model with fully intuitionistic fuzzy transportation model and fully fuzzy transportation model is presented with pictorial illustrations.

1. Introduction

Zadeh [1] proposed the notion of fuzzy set (FS) theory to tackle the problems involving vague information. Atanassov [2] introduced intuitionistic fuzzy set (IFS), which is characterized by a membership function as well as a nonmembership function. Although IFSs have vast applications in many fields, it cannot provide all the information. In the voting process, we can vote in favor of someone, abstain, against someone, and even refuse to cost the vote, which cannot be handled by IFSs. To tackle such type of situations, Cuong [3] gave the idea of PFS, which is generalized structure of FS and IFS and further examined their basic properties and laws. In PFSs theory, we study about positive, neutral, and negative membership degrees of each element belonging to set. In practical life, PFSs theory contribute a significant role in medical diagnosis, career selection, decision making (DM), engineering, and networking.

Dubois and Prade [4] discussed basic arithmetic operations related to fuzzy numbers. Bellman and Zadeh [5] proposed the idea of decision making in fuzzy environment. Tanaka et al. [6] presented fuzzy linear programming (FLP) problems. Zimmerman [7] analyzed multiobjective functions in FLP problems. Ganesan and Veeramani [8] studied FLP problems by using trapezoidal fuzzy numbers. Lotfi et al. [9] suggested the lexicography method to solve fully FLP problems and obtained approximate solutions. Allahviranloo et al. [10] solved fully fuzzy linear system and achieved the general solutions. Kaur and Kumar [11] proposed Mehar's method to solve fully FLP problems by using *LR* fuzzy numbers. Pérez-Cañedo et al. [12] used lexicographical method and solved fully FLP problems having inequality constraints. Akram et al. [13] proposed Pythagorean FLP problems with equality constraints. Akram et al. [13] introduced *LR*-type Pythagorean fuzzy numbers and introduced a scheme to solve *LR*-type fully Pythagorean FLP problems with equality constraints. Mehmood et al. [14, 15] developed different techniques to find optimal solutions of fully bipolar FLP problems. Ahmed et al. [16] solved fully FLP problems in bipolar neutrosophic environment. Akram et al. [17] presented a scheme and obtained optimal solutions of fully FLP problems. For more information about FLP, the reader may study [18–20].

In real life, we have to minimize the transportation cost in transportation problems (TPs) because the companies have to deliver the products from different sources to numerous destinations [21]. Gani and Abbas [22] suggested a new technique to obtain optimal solution of TPs in intuitionistic fuzzy environment. TPs are being applied in many areas like networking [23], shipping [24], production [25], and shortest route problems [26]. Singh and Yadev [27] solved fuzzy transportation problems (FTPs) and obtained optimal solution by applying accuracy function. Kumar et al. [28] have proposed two algorithms to achieve initial basic solution and optimal cost of FTPs. Nagoorgani and Razak [29] developed a method to minimize the cost function by using trapezoidal fuzzy numbers (TrFNs). Kaur and Kumar [30] solved FTPs and determined optimal solution. Singh and Yadav [31] proposed a method to solve FTPs by using intuitionistic fuzzy numbers as supply and demand. Basirzadeh [32] worked on FTPs and presented three different kinds of problems. Kumar and Hussain [33] used a computational approach to tackle fully intuitionistic fuzzy real life TPs. Abhishekh and Nishad [34] solved LR-intuitionistic FTPs by using ranking approach. Kaur et al. [35] proposed different schemes to solve fully FTPs by using TrFNs. Wang et al. [36] introduced geometric operators in picture fuzzy environment and studied decision-making problems (DMPs).

PFSs are applied to handle real life TPs that involve uncertainty and have been effectively used in communication, management, and DM. Akram et al. [37] proposed the idea of complex PFS, which is generalized form of complex IFS and developed a DM model. Shit et al. [38] used harmonic operators with TrPFNs and showed its significance in multicriteria decision-making (MCDM) problems. Akram et al. [39] investigated shortest route problems by using trapezoidal picture fuzzy numbers (TrPFNs). Geetha and Selvakumari [40] solved picture fuzzy transportation problems and obtained minimum transportation cost. Mahmoodirad et al. [41] studied the existing shortcomings and proposed a method to handle TPs involving intuitionistic fuzzy numbers. Kane et al. [42] suggested a FLP scheme to solve TPs by using triangular fuzzy numbers. Veeramani et al. [43] proposed a technique regarding multiobjective fractional TP on the basis of NGP approach. Ali and Ansari [44] presented Fermatean fuzzy bipolar soft set (FFBSS) model and studied its basic properties. Tchier et al. [45] combined PFSs and soft expert sets and introduced a hybrid model which is used to analyze DMPs. Das [46] defined score function to get IBFS and solved neutrosophic TPs. Kané et al. [47] proposed two-step scheme to solve fully FTP by using all parameters as trapezoidal fuzzy numbers. Ali et al. [48] studied group DMPs and developed a hybrid model by considering bipolar soft expert sets. Ashraf et al. [49] explored MSM operator in the form of IVPFS and interpreted its applications. On the basis of CIVPFSs, Ali et al. [50] introduced Einstein operational laws by applying t-norm. Sahu et al. [51] analyzed student's career selection in PFS environment. Yildirim and Yıldırım [52] used picture fuzzy VIKOR technique and evaluated satisfaction level of people regarding municipality services. The motivation of this manuscript is described as follows:

(1) PFSs manage the problems involving uncertainty more expeditiously as compared with FSs and IFSs.
(2) No one has yet introduced this particular concept of FPFTPs which is based on picture fuzzy linear programming (PFLP) formulation.

The main contributions of this article are depicted as follows:

(1) We propose a scheme to solve FPFTPs based upon PFLP formulation
(2) We apply suggested technique to solve FPFTPs by considering all the variables as nonnegative TrPFNs
(3) We obtain picture fuzzy transportation cost/optimal value in the form of TrPFNs
(4) The supremacy of the proposed scheme is investigated by comparative analysis with existing approaches

The rest of the article is arranged as follows. Introductory concepts are depicted in Section 2. The proposed scheme is elucidated in Section 3. A model regarding FPFTPs is considered in Section 4. Comparison analysis is discussed in Section 5, and conclusion is given in the last section.

2. Preliminaries

In this section, we review some preliminary concepts regarding PFSs.

Definition 1 (see [3]). A PFS P on a universal set X is an object of the form as follows:

$$P = \{(x, \mu_P(x), \eta_P(x), \nu_P(x)) | x \in X\}, \quad (1)$$

where $\mu_P(x) \in [0,1], \eta_P(x) \in [0,1], \nu_P(x) \in [0,1]$ denote positive, neutral, and negative membership degrees, respectively, of element $x \in P$ with $0 \leq \mu_P(x) + \eta_P(x) + \nu_P(x) \leq 1, \forall x \in X$, and $\Pi_P(x) = 1 - \mu_P(x) - \eta_P(x) - \nu_P(x)$ is said to be refusal degree of x in set P.

Definition 2 (see [53]). A TrPFN $P = [(u_3, u_2, u_1, s, t, v_1, v_2, v_3); (\omega, \vartheta, \zeta)]$ is a PFS defined on $\mathbb{R}$, whose positive (μ_P), neutral (η_P), and negative (ν_P) membership functions are, respectively, defined as follows:

$$\mu_P(x) = \begin{cases} \dfrac{(x-u_1)\omega}{s-u_1}, & u_1 \le x \le s, \\ \omega, & s \le x \le t, \\ \dfrac{(v_1-x)\omega}{v_1-t}, & t \le x \le v_1, \\ 0, & otherwise, \end{cases}$$

$$\eta_P(x) = \begin{cases} \dfrac{(x-u_2)\vartheta}{s-u_2}, & u_2 \le x \le s, \\ \vartheta, & s \le x \le t, \\ \dfrac{(v_2-x)\vartheta}{v_2-t}, & t \le x \le v_2, \\ 0, & \text{otherwise}, \end{cases} \quad (2)$$

$$\nu_P(x) = \begin{cases} \dfrac{s-x+\zeta(x-u_3)}{s-u_3}, & u_3 \le x \le s, \\ \zeta, & s \le x \le t, \\ \dfrac{x-t+\zeta(v_3-x)}{v_3-t}, & t \le x \le v_3, \\ 1, & \text{otherwise}, \end{cases}$$

where $u_3 \le u_2 \le u_1 \le s \le t \le v_1 \le v_2 \le v_3$ and the values ω, ϑ, and ζ indicate maximum degree of (μ_P), maximum degree of (η_P), and minimum degree of (ν_P), respectively, such that $\omega, \vartheta, \zeta \in [0,1]$, with $0 \le \omega + \vartheta + \zeta \le 1$.

Definition 3 (see [53]). A TrPFN $P = [(u_3, u_2, u_1, s, t, v_1, v_2, v_3); (\omega, \vartheta, \zeta)]$ is said to be nonnegative (respectively, nonpositive) if $u_3 \ge 0$ (respectively, $v_3 \le 0$) and P is unrestricted TrPFN if u_3 is any real number.

Definition 4 (see [53]). Let $P_1 = [(u_3, u_2, u_1, s, t, v_1, v_2, v_3); (\omega, \vartheta, \zeta)]$ and $P_2 = [(u_3', u_2', u_1', s', t', v_1', v_2', v_3'); (\omega', \vartheta', \zeta')]$ be two TrPFNs and λ be real number, then

(1) $P_1 \oplus P_2 = [(u_3 + u_3', u_2 + u_2', u_1 + u_1', s + s', t + t', v_1 + v_1', v_2 + v_2', v_3 + v_3'); (\omega + \omega' - \omega\omega', \vartheta\vartheta', \zeta\zeta')]$

(2) $-P_1 = [(-v_3, -v_2, -v_1, -t, -s, -u_1, -u_2, -u_3); (\omega, \vartheta, \zeta)]$

(3) $P_1 \ominus P_2 = [(u_3 - v_3', u_2 - v_2', u_1 - v_1', s - t', t - s', v_1 - u_1', v_2 - u_2', v_3 - u_3'); (\omega + \omega' - \omega\omega', \vartheta\vartheta', \zeta\zeta')]$

(4) $\lambda P_1 = \begin{cases} [(\lambda u_3, \lambda u_2, \lambda u_1, \lambda s, \lambda t, \lambda v_1, \lambda v_2, \lambda v_3); (\omega^\lambda, \vartheta^\lambda, 1-(1-\zeta)^\lambda)], & \lambda \ge 0 \\ [(\lambda v_3, \lambda v_2, \lambda v_1, \lambda t, \lambda s, \lambda u_1, \lambda u_2, \lambda u_3); (\omega^{-\lambda}, \vartheta^{-\lambda}, 1-(1-\zeta)^{-\lambda})], & \lambda < 0 \end{cases}$

(5) $P_1 \otimes P_2 = [(U_3, U_2, U_1, S, T, V_1, V_2, V_3); (\omega\omega', \vartheta\vartheta', \zeta + \zeta' - \zeta\zeta')]$

where

$$U_1 = \begin{cases} \min\{u_1u_1', v_1u_1'\}, & u_1 \ge 0, \\ \min\{u_1v_1', v_1u_1'\}, & u_1 < 0, v_1 \ge 0, \\ \min\{u_1v_1', v_1v_1'\}, & v_1 < 0, \end{cases}$$
$$U_2 = \begin{cases} \min\{u_2u_2', v_2u_2'\}, & u_2 \ge 0, \\ \min\{u_2v_2', v_2u_2'\}, & u_2 < 0, v_2 \ge 0, \\ \min\{u_2v_2', v_2v_2'\}, & v_2 < 0, \end{cases}$$
$$U_3 = \begin{cases} \min\{u_3u_3', v_3u_3'\}, & u_3 \ge 0, \\ \min\{u_3v_3', v_3u_3'\}, & u_3 < 0, v_3 \ge 0, \\ \min\{u_3v_3', v_3v_3'\}, & v_3 < 0, \end{cases}$$
$$S = \begin{cases} \min\{ss', ts'\}, & s \ge 0, \\ \min\{st', ts'\}, & s < 0, t \ge 0, \\ \min\{st', tt'\}, & t < 0, \end{cases}$$
$$T = \begin{cases} \max\{st', tt'\}, & s \ge 0, \\ \max\{ss', tt'\}, & s < 0, t_1 \ge 0, \\ \max\{ss', ts'\}, & t_1 < 0, \end{cases} \quad (3)$$
$$V_1 = \begin{cases} \max\{u_1v_1', v_1v_1'\}, & u_1 \ge 0, \\ \max\{u_1u_1', v_1v_1'\}, & u_1 < 0, v_1 \ge 0, \\ \max\{u_1u_1', v_1u_1'\}, & v_1 < 0, \end{cases}$$
$$V_2 = \begin{cases} \max\{u_2v_2', v_2v_2'\}, & u_2 \ge 0, \\ \max\{u_2u_2', v_2v_2'\}, & u_2 < 0, v_2 \ge 0, \\ \max\{u_2u_2', v_2u_2'\}, & v_2 < 0, \end{cases}$$
$$V_3 = \begin{cases} \max\{u_3v_3', v_3v_3'\}, & u_3 \ge 0, \\ \max\{u_3u_3', v_3v_3'\}, & u_3 < 0, v_3 \ge 0, \\ \max\{u_3u_3', v_3u_3'\}, & v_3 < 0. \end{cases}$$

Definition 5 (see [53]). Two TrPFNs $P_1 = [(u_3, u_2, u_1, s, t, v_1, v_2, v_3); (\omega, \vartheta, \zeta)]$ and $P_2 = [(u_3', u_2', u_1', s', t', v_1', v_2', v_3'); (\omega', \vartheta', \zeta')]$ are said to be equal if $u_3 = u_3'$, $u_2 = u_2'$, $u_1 = u_1'$, $s = s'$, $t = t'$, $v_1 = v_1'$, $v_2 = v_2'$, $v_3 = v_3'$, $\omega = \omega'$, $\vartheta = \vartheta'$, and $\zeta = \zeta'$.

Definition 6 (see [53]). A TrPFN $P_1 = [(u_3, u_2, u_1, s, t, v_1, v_2, v_3); (\omega, \vartheta, \zeta)]$ is said to be zero if $u_3 = 0$, $u_2 = 0$, $u_1 = 0$, $s = 0$, $t = 0$, $v_1 = 0$, $v_2 = 0$, $v_3 = 0$, $\omega = 0$, $\vartheta = 0$, and $\zeta = 0$.

Definition 7 (see [53]). Let $p = [(u_3, u_2, u_1, s, t, v_1, v_2, v_3); (\omega, \vartheta, \zeta)]$ be a TrPFN, then ranking of P is symbolized as $\mathfrak{R}(P)$ and defined as

$$\Re(P) = \frac{\omega(s+t+u_1+v_1)+\vartheta(s+t+u_2+v_2)+(1-\zeta)(s+t+u_3+v_3)}{4}. \quad (4)$$

3. Fully Picture Fuzzy Transportation Problems

In this section, we present a scheme to solve FPFTPs based on PFLP formulation. The steps are explained as follows.

3.1. Steps to Find Picture Fuzzy Optimal Solution. Consider a FPFTP containing q sources and r destinations in which cost, supply, and demand are used as TrPFNs $\widetilde{C_{kl}^P}$, $\widetilde{A}_k^P$, and $\widetilde{B}_l^P$, respectively.

$$\text{Minimize } Z = \sum_{k=1}^{q}\sum_{l=1}^{r}\widetilde{C_{kl}^P}\otimes\widetilde{X_{kl}^P}, \quad (5)$$

subject to

$$\begin{aligned}&\sum_{l=1}^{r}\widetilde{X_{kl}^P} = \widetilde{A}_k^P, \forall k = 1,2,3,\ldots,q,\\ &\sum_{l=1}^{r}\widetilde{X_{kl}^P} = \widetilde{A}_k^P, \forall k = 1,2,3,\ldots,q,\\ &\widetilde{X}_{kl}^P \geq 0, \forall k = 1,2,3,\ldots,q, \quad \forall l = 1,2,3,\ldots,r,\end{aligned} \quad (6)$$

where $\widetilde{C_{kl}^P}$, $\widetilde{X_{kl}^P}$, $\widetilde{A}_k^P$, and $\widetilde{B}_l^P$ are all nonnegative TrPFNs.

To solve FPFTP equation (1), we give a criterion for picture fuzzy optimal solution (PFOS).

Definition 8. A PFOS of the FPFTP equation (1) with TrPFNs will be TrPFNs $\widetilde{X}_{kl}^P$ if

(i) $\widetilde{X}_{kl}^P$ are nonnegative TrPFNs.
(ii) $\Re(\sum_{l=1}^{r}\widetilde{X_{kl}^P}) = \Re(\widetilde{A}_k^P)$, $\forall k = 1,2,3,\ldots,q$.
(iii) $\Re(\sum_{k=1}^{q}\widetilde{X_{kl}^P}) = \Re(\widetilde{B}_l^P)$, $\forall l = 1,2,3,\ldots,r$.
(iv) If there exists any TrPFNs $\widetilde{X}_{kl'}^P$ satisfying the above three conditions, then

$$\Re\left(\sum_{k=1}^{q}\sum_{l=1}^{r}\widetilde{C_{kl}^P}\otimes\widetilde{X_{kl}^P}\right) < \Re\left(\sum_{k=1}^{q}\sum_{l=1}^{r}\widetilde{C_{kl}^P}\otimes\widetilde{X_{kl}^P}\right). \quad (7)$$

Now, we explain the steps to determine the PFOS of FPFTP as given in equation (2).

Step 1. Calculate total picture fuzzy supply and total picture fuzzy demand.

If

$$\sum_{l=1}^{r}\widetilde{B}_l^P = \sum_{k=1}^{q}\widetilde{A}_k^P, \quad (8)$$

it is a balanced FPFTP.

If

$$\sum_{l=1}^{r}\widetilde{B}_l^P \neq \sum_{k=1}^{q}\widetilde{A}_k^P, \quad (9)$$

it is an unbalanced FPFTP.

That is,

$$\begin{aligned}&[(a,b,c,d,e,f,g,h);(\alpha,\beta,\gamma)]\\ &\quad\neq [(a',b',c',d',e',f',g',h');(\alpha',\beta',\gamma')].\end{aligned} \quad (10)$$

Then, one of the following case arises:

Case (a):

$$\begin{aligned}&a\leq a', \quad b-a\leq b'-a', \quad c-b\leq c'-b', \quad d-c\leq d'-c', \quad e-d\leq e'-d', \quad f-e\leq f'-e',\\ &g-f\leq g'-f', \quad h-g\leq h'-g'.\end{aligned} \quad (11)$$

Case (b):

$$\begin{aligned}&a\geq a', \quad b-a\geq b'-a', \quad c-b\geq c'-b', \quad d-c\geq d'-c', \quad e-d\geq e'-d', \quad f-e\geq f'-e',\\ &g-f\geq g'-f', \quad h-g\geq h'-g'.\end{aligned} \quad (12)$$

Case (c): when the above two cases do not hold, then there may exists infinitely many nonnegative TrPFNs:

$$\begin{cases}[(a_1,b_1,c_1,d_1,e_1,f_1,g_1,h_1);(\alpha_1,\beta_1,\gamma_1)],\\ [(a_1',b_1',c_1',d_1',e_1',f_1',g_1',h_1');(\alpha_1',\beta_1',\gamma_1')],\end{cases} \quad (13)$$

such that

$$[(a,b,c,d,e,f,g,h);\ (\alpha,\beta,\gamma)]\oplus[(a_1,b_1,c_1,d_1,e_1,f_1,g_1,h_1);\ (\alpha_1,\beta_1,\gamma_1)] = [(a',b',c',d',e',f',g',h');\ (\alpha',\beta',\gamma')]\oplus[(a_1',b_1',c_1',d_1',e_1',f_1',g_1',h_1');\ (\alpha_1',\beta_1',\gamma_1')], \quad (14)$$

but we have to determine such nonnegative TrPFNs as

$$\begin{cases}[(a_1,b_1,c_1,d_1,e_1,f_1,g_1,h_1);\ (\alpha_1,\beta_1,\gamma_1)],\\ [(a_1',b_1',c_1',d_1',e_1',f_1',g_1',h_1');\ (\alpha_1',\beta_1',\gamma_1')],\end{cases} \quad (15)$$

which satisfy the following conditions.

(i) $[(a_1,b_1,c_1,d_1,e_1,f_1,g_1,h_1);\ (\alpha_1,\beta_1,\gamma_1)]$ and $[(a_1',b_1',c_1',d_1',e_1',f_1',g_1',h_1');\ (\alpha_1',\beta_1',\gamma_1')]$ are nonnegative TrPFNs.

(ii) It satisfies

$$[(a,b,c,d,e,f,g,h);\ (\alpha,\beta,\gamma)]\oplus[(a_1,b_1,c_1,d_1,e_1,f_1,g_1,h_1);\ (\alpha_1,\beta_1,\gamma_1)] = [(a',b',c',d',e',f',g',h');\ (\alpha',\beta',\gamma')]\oplus[(a_1',b_1',c_1',d_1',e_1',f_1',g_1',h_1');\ (\alpha_1',\beta_1',\gamma_1')]. \quad (16)$$

(iii) Further, if there exists two nonnegative TrPFNs,

$$\begin{aligned}&[(s_1,t_1,u_1,v_1,w_1,x_1,y_1,z_1);\ (\alpha_1,\beta_1,\gamma_1)],\\ &[(s_1',t_1',u_1',v_1',w_1',x_1',y_1',z_1');\ (\alpha_1',\beta_1',\gamma_1')],\end{aligned} \quad (17)$$

such that

$$[(a,b,c,d,e,f,g,h);\ (\alpha,\beta,\gamma)]\oplus[(s_1,t_1,u_1,v_1,w_1,x_1,y_1,z_1);\ (\alpha_1,\beta_1,\gamma_1)] = [(a',b',c',d',e',f',g',h');\ (\alpha',\beta',\gamma')]\oplus[(s_1',t_1',u_1',v_1',w_1',x_1',y_1',z_1');\ (\alpha_1',\beta_1',\gamma_1')], \quad (18)$$

then

$$\begin{aligned}\Re[(s_1,t_1,u_1,v_1,w_1,x_1,y_1,z_1);\ (\alpha_1,\beta_1,\gamma_1)] &\geq \Re[(a_1,b_1,c_1,d_1,e_1,f_1,g_1,h_1);\ (\alpha_1,\beta_1,\gamma_1)],\\ \Re[(s_1',t_1',u_1',v_1',w_1',x_1',y_1',z_1');\ (\alpha_1',\beta_1',\gamma_1')] &\geq \Re[(a_1',b_1',c_1',d_1',e_1',f_1',g_1',h_1');\ (\alpha_1',\beta_1',\gamma_1')].\end{aligned} \quad (19)$$

Step 2. Suppose

$$\begin{aligned}\widetilde{C_{kl}^{P}} &= \left[\left(c_{kl}^1,c_{kl}^2,c_{kl}^3,c_{kl}^4,c_{kl}^5,c_{kl}^6,c_{kl}^7,c_{kl}^8\right);\ \left(\xi_{kl},\psi_{kl},\omega_{kl}\right)\right],\\ \widetilde{X_{kl}^{P}} &= \left[\left(x_{kl}^1,x_{kl}^2,x_{kl}^3,x_{kl}^4,x_{kl}^5,x_{kl}^6,x_{kl}^7,x_{kl}^8\right);\ \left(\sigma_{kl},\tau_{kl},\upsilon_{kl}\right)\right],\\ \widetilde{A}_k^{P} &= \left[\left(a_k^1,a_k^2,a_k^3,a_k^4,a_k^5,a_k^6,a_k^7,a_k^8\right);\ \left(\kappa_k,\lambda_k,\theta_k\right)\right],\\ \widetilde{B}_l^{P} &= \left[\left(b_l^1,b_l^2,b_l^3,b_l^4,b_l^5,b_l^6,b_l^7,b_l^8\right);\ \left(\eta_l,\epsilon_l,\chi_l\right)\right],\end{aligned} \quad (20)$$

then FPFTP equation (1) can be transformed as follows:

$$\text{Minimize Z} = \sum_{k=1}^{q}\sum_{l=1}^{r}\left[\begin{matrix}\left(c_{kl}^1,c_{kl}^2,c_{kl}^3,c_{kl}^4,c_{kl}^5,c_{kl}^6,c_{kl}^7,c_{kl}^8\right);\\ \left(\xi_{kl},\psi_{kl},\omega_{kl}\right)\end{matrix}\right]\otimes\left[\begin{matrix}\left(x_{kl}^1,x_{kl}^2,x_{kl}^3,x_{kl}^4,x_{kl}^5,x_{kl}^6,x_{kl}^7,x_{kl}^8\right);\\ \left(\sigma_{kl},\tau_{kl},\upsilon_{kl}\right)\end{matrix}\right], \quad (21)$$

subject to

$$\begin{aligned}&\sum_{l=1}^{r}\left[\left(x_{kl}^{1},x_{kl}^{2},x_{kl}^{3},x_{kl}^{4},x_{kl}^{5},x_{kl}^{6},x_{kl}^{7},x_{kl}^{8}\right);\ \left(\sigma_{kl},\tau_{kl},\upsilon_{kl}\right)\right]=\left[\left(a_{k}^{1},a_{k}^{2},a_{k}^{3},a_{k}^{4},a_{k}^{5},a_{k}^{6},a_{k}^{7},a_{k}^{8}\right);\ \left(\kappa_{k},\lambda_{k},\theta_{k}\right)\right],\\&\sum_{k=1}^{q}\left[\left(x_{kl}^{1},x_{kl}^{2},x_{kl}^{3},x_{kl}^{4},x_{kl}^{5},x_{kl}^{6},x_{kl}^{7},x_{kl}^{8}\right);\ \left(\sigma_{kl},\tau_{kl},\upsilon_{kl}\right)\right]=\left[\left(b_{l}^{1},b_{l}^{2},b_{l}^{3},b_{l}^{4},b_{l}^{5},b_{l}^{6},b_{l}^{7},b_{l}^{8}\right);\ \left(\eta_{l},\epsilon_{l},\chi_{l}\right)\right],\end{aligned} \tag{22}$$

where $[(x_{kl}^{1},x_{kl}^{2},x_{kl}^{3},x_{kl}^{4},x_{kl}^{5},x_{kl}^{6},x_{kl}^{7},x_{kl}^{8});\ (\sigma_{kl},\tau_{kl},\upsilon_{kl})]$ are nonnegative TrPFNs.

Step 3. By applying arithmetic operations as described in Section 2 and putting

$$\begin{aligned}&\left[\left(c_{kl}^{1},c_{kl}^{2},c_{kl}^{3},c_{kl}^{4},c_{kl}^{5},c_{kl}^{6},c_{kl}^{7},c_{kl}^{8}\right)\right);t\left(\xi_{kl},\psi_{kl},\omega_{kl}\right)\right]\otimes\left[\left(x_{kl}^{1},x_{kl}^{2},x_{kl}^{3},x_{kl}^{4},x_{kl}^{5},x_{kl}^{6},x_{kl}^{7},x_{kl}^{8}\right);t\left(\sigma_{kl},\tau_{kl},\upsilon_{kl}\right)\right]\\&\quad=\left[\left(d_{kl}^{1},d_{kl}^{2},d_{kl}^{3},d_{kl}^{4},d_{kl}^{5},d_{kl}^{6},d_{kl}^{7},d_{kl}^{8}\right);\ \left(\mu_{kl},\delta_{kl},\eta_{kl}\right)\right],\end{aligned} \tag{23}$$

then the fully picture fuzzy linear programming problem (FPFLPP) equation (2) can be transformed as follows:

$$\text{Minimize } Z=\sum_{k=1}^{q}\sum_{l=1}^{r}\left[\left(d_{kl}^{1},d_{kl}^{2},d_{kl}^{3},d_{kl}^{4},d_{kl}^{5},d_{kl}^{6},d_{kl}^{7},d_{kl}^{8}\right);\ \left(\mu_{kl},\delta_{kl},\eta_{kl}\right)\right], \tag{24}$$

subject to

$$\begin{aligned}&\sum_{l=1}^{r}\left[\left(x_{kl}^{1},x_{kl}^{2},x_{kl}^{3},x_{kl}^{4},x_{kl}^{5},x_{kl}^{6},x_{kl}^{7},x_{kl}^{8}\right);\ \left(\sigma_{kl},\tau_{kl},\upsilon_{kl}\right)\right]=\left[\left(a_{k}^{1},a_{k}^{2},a_{k}^{3},a_{k}^{4},a_{k}^{5},a_{k}^{6},a_{k}^{7},a_{k}^{8}\right);\ \left(\kappa_{k},\lambda_{k},\theta_{k}\right)\right],\\&\sum_{k=1}^{q}\left[\left(x_{kl}^{1},x_{kl}^{2},x_{kl}^{3},x_{kl}^{4},x_{kl}^{5},x_{kl}^{6},x_{kl}^{7},x_{kl}^{8}\right)\right);t\left(\sigma_{kl},\tau_{kl},\upsilon_{kl}\right)\right]=\left[\left(b_{l}^{1},b_{l}^{2},b_{l}^{3},b_{l}^{4},b_{l}^{5},b_{l}^{6},b_{l}^{7},b_{l}^{8}\right);t\left(\eta_{l},\epsilon_{l},\chi_{l}\right)\right],\end{aligned} \tag{25}$$

where $[(x_{kl}^{1},x_{kl}^{2},x_{kl}^{3},x_{kl}^{4},x_{kl}^{5},x_{kl}^{6},x_{kl}^{7},x_{kl}^{8});\ (\sigma_{kl},\tau_{kl},\upsilon_{kl})]$ are nonnegative TrPFNs.

Step 4. Now, by applying Definitions 5 and 7, then the FPFLPP equation (3) can be transformed as follows:

$$\text{Minimize } Z=\sum_{k=1}^{q}\sum_{l=1}^{r}\Re\left[\left(d_{kl}^{1},d_{kl}^{2},d_{kl}^{3},d_{kl}^{4},d_{kl}^{5},d_{kl}^{6},d_{kl}^{7},d_{kl}^{8}\right);\ \left(\mu_{kl},\delta_{kl},\eta_{kl}\right)\right], \tag{26}$$

subject to

$$\sum_{l=1}^{r}x_{kl}^{1}=a_{k}^{1},\quad\forall k=1,2,3,\ldots,q,$$

$$\sum_{l=1}^{r}x_{kl}^{2}=a_{k}^{2},\quad\forall k=1,2,3,\ldots,q,$$

$$\sum_{l=1}^{r}x_{kl}^{3}=a_{k}^{3},\quad\forall k=1,2,3,\ldots,q,$$

$$\sum_{l=1}^{r}x_{kl}^{4}=a_{k}^{4},\quad\forall k=1,2,3,\ldots,q,$$

$$\sum_{l=1}^{r}x_{kl}^{5}=a_{k}^{5},\quad\forall k=1,2,3,\ldots,q,$$

$$\sum_{l=1}^{r}x_{kl}^{6}=a_{k}^{6},\quad\forall k=1,2,3,\ldots,q,$$

$$\sum_{l=1}^{r}x_{kl}^{7}=a_{k}^{7},\quad\forall k=1,2,3,\ldots,q,$$

$$\sum_{l=1}^{r}x_{kl}^{8}=a_{k}^{8},\quad\forall k=1,2,3,\ldots,q,$$

$$\sum_{l=1}^{r}\sigma_{kl}=\kappa_{k},\quad\forall k=1,2,3,\ldots,q,$$

$$\sum_{l=1}^{r}\tau_{kl}=\lambda_{k},\quad\forall k=1,2,3,\ldots,q,$$

$$\begin{aligned}
&\sum_{l=1}^{r} \upsilon_{kl} = \theta_k, \quad \forall k = 1,2,3,\ldots,q,\\
&\sum_{k=1}^{q} x_{kl}^1 = b_l^1, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^2 = b_l^2, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^3 = b_l^3, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^4 = b_l^4, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^5 = b_l^5, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^6 = b_l^6, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^7 = b_l^7, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^8 = b_l^8, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} \sigma_{kl} = \eta_l, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} \tau_{kl} = \epsilon_l, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} \upsilon_{kl} = \chi_l, \quad \forall l = 1,2,3,\ldots,r,
\end{aligned} \tag{27}$$

where $[((x_{kl}^1, x_{kl}^2, x_{kl}^3, x_{kl}^4, x_{kl}^5, x_{kl}^6, x_{kl}^7, x_{kl}^8)); (\sigma_{kl}, \tau_{kl}, \upsilon_{kl})]$ are nonnegative TrPFNs.

Step 5. To obtain PFOS, solve the following crisp linear/nonlinear programming problem (LPP):

$$\text{Minimize } Z = \sum_{k=1}^{q}\sum_{l=1}^{r}\frac{1}{4}\begin{bmatrix} \mu_{kl}\left(d_{kl}^3 + d_{kl}^4 + d_{kl}^5 + d_{kl}^6\right)+ \\ \delta_{kl}\left(d_{kl}^2 + d_{kl}^4 + d_{kl}^5 + d_{kl}^7\right)+ \\ (1-\eta_{kl})\left(d_{kl}^1 + d_{kl}^4 + d_{kl}^5 + d_{kl}^8\right) \end{bmatrix}, \tag{28}$$

subject to

$$\begin{aligned}
&\sum_{l=1}^{r} x_{kl}^1 = a_k^1, \quad \forall k = 1,2,3,\ldots,q,\\
&\sum_{l=1}^{r} x_{kl}^2 = a_k^2, \quad \forall k = 1,2,3,\ldots,q,\\
&\sum_{l=1}^{r} x_{kl}^3 = a_k^3, \quad \forall k = 1,2,3,\ldots,q,\\
&\sum_{l=1}^{r} x_{kl}^4 = a_k^4, \quad \forall k = 1,2,3,\ldots,q,\\
&\sum_{l=1}^{r} x_{kl}^5 = a_k^5, \quad \forall k = 1,2,3,\ldots,q,\\
&\sum_{l=1}^{r} x_{kl}^6 = a_k^6, \quad \forall k = 1,2,3,\ldots,q,\\
&\sum_{l=1}^{r} x_{kl}^7 = a_k^7, \quad \forall k = 1,2,3,\ldots,q,\\
&\sum_{l=1}^{r} x_{kl}^8 = a_k^8, \quad \forall k = 1,2,3,\ldots,q,\\
&\sum_{l=1}^{r} \sigma_{kl} = \kappa_k, \quad \forall k = 1,2,3,\ldots,q,\\
&\sum_{l=1}^{r} \tau_{kl} = \lambda_k, \quad \forall k = 1,2,3,\ldots,q,\\
&\sum_{l=1}^{r} \upsilon_{kl} = \theta_k, \quad \forall k = 1,2,3,\ldots,q,\\
&\sum_{k=1}^{q} x_{kl}^1 = b_l^1, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^2 = b_l^2, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^3 = b_l^3, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^4 = b_l^4, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^5 = b_l^5, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^6 = b_l^6, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^7 = b_l^7, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} x_{kl}^8 = b_l^8, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} \sigma_{kl} = \eta_l, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} \tau_{kl} = \epsilon_l, \quad \forall l = 1,2,3,\ldots,r,\\
&\sum_{k=1}^{q} \upsilon_{kl} = \chi_l, \quad \forall l = 1,2,3,\ldots,r,
\end{aligned}$$

$$x_{kl}^1 \geq 0, x_{kl}^2 - x_{kl}^1 \geq 0, x_{kl}^3 - x_{kl}^2 \geq 0, x_{kl}^4 - x_{kl}^3 \geq 0, x_{kl}^5 - x_{kl}^4 \geq 0, x_{kl}^6 - x_{kl}^5 \geq 0, x_{kl}^7 - x_{kl}^6 \geq 0, x_{kl}^8 - x_{kl}^7 \geq 0, \sigma_{kl} \geq 0, \tau_{kl} \geq 0, \upsilon_{kl} \geq 0, \sigma_{kl} + \tau_{kl} + \upsilon_{kl} \leq 1, \quad \forall k = 1, 2, \ldots, q, \ \forall l = 1, 2, \ldots, r. \tag{29}$$

Step 6. Solve crisp linear/non-LPP equation (28) to get optimal solution:

$$\left\{x^1 *_{kl}, x^2 *_{kl}, x^3 *_{kl}, x^4 *_{kl}, x^5 *_{kl}, x^6 *_{kl}, x^7 *_{kl}, x^8 *_{kl}, \sigma_{kl}^*, \tau_{kl}^*, \upsilon_{kl}^*\right\}. \tag{30}$$

Step 7. Find the PFOS $\widetilde{X}_{kl}^P$ of the FPFTP (1) by putting values of $x^1 *_{kl}, x^2 *_{kl}, x^3 *_{kl}, x^4 *_{kl}, x^5 *_{kl}, x^6 *_{kl}, x^7 *_{kl}, x^8 *_{kl}, \sigma_{kl}^*, \tau_{kl}^*$, and υ_{kl}^* in $\widetilde{X}_{kl}^P = [(x^1 *_{kl}, x^2 *_{kl}, x^3 *_{kl}, x^4 *_{kl}, x^5 *_{kl}, x^6 *_{kl}, x^7 *_{kl}, x^8 *_{kl}); \ (\sigma_{kl}^*, \tau_{kl}^*, \upsilon_{kl}^*)]$.

Step 8. Find picture fuzzy transportation cost/optimal value of FPFTP equation (1) by assigning values of $\widetilde{X}_{kl}^P$, as obtained in Step (7), in $\sum_{k=1}^q \sum_{l=1}^r \widetilde{C_{kl}^P} \otimes \widetilde{X_{kl}^P}$.

4. Numerical Examples

In this section, to explain the proposed methodology, we present a model related to FPFTPs.

Example 1. (FPFTP based on PFLP formulation). A company containing two plants produces urea fertilizer with picture fuzzy availabilities of

Proof. [(15, 35, 45, 60, 75, 95, 110, 200); (0.92, 0.02, 0.01)] *ton* and [(10, 25, 50, 70, 90, 120, 130, 190); (0.80, 0.01, 0.02)] *ton*, respectively, and supply it to two cities. The picture fuzzy demand at two cities is [(20, 40, 50, 65, 85, 100, 140, 160); (0.92, 0.01, 0.02)]*ton* and [(30, 35, 40, 50, 100, 110, 150, 170); (0.76, 0.02, 0.01)]*ton*. The price per ton by delivering the urea fertilizer at the two cities is presented in Table 1.

Find the minimum picture fuzzy transportation cost.

$$\text{Minimize}\begin{pmatrix} [(110, 150, 190, 210, 270, 300, 350, 390); (0.7, 0.1, 0.1)] \otimes \tilde{x}_{11} \oplus \\ [(130, 180, 220, 250, 290, 340, 370, 410); (0.8, 0.1, 0.1)] \otimes \tilde{x}_{12} \oplus \\ [(150, 250, 290, 350, 400, 440, 460, 500); (0.6, 0.1, 0.2)] \otimes \tilde{x}_{21} \oplus \\ [(190, 210, 250, 270, 310, 330, 380, 430); (0.6, 0.1, 0.2)] \otimes \tilde{x}_{22} \end{pmatrix}, \tag{31}$$

subject to

$$\begin{aligned} \tilde{x}_{11} \oplus \tilde{x}_{12} &= [(15, 35, 45, 60, 75, 95, 110, 200); (0.92, 0.02, 0.01)], \\ \tilde{x}_{21} \oplus \tilde{x}_{22} &= [(10, 25, 50, 70, 90, 120, 130, 190); (0.80, 0.01, 0.02)], \\ \tilde{x}_{11} \oplus \tilde{x}_{21} &= [(20, 40, 50, 65, 85, 100, 140, 160); (0.92, 0.01, 0.02)], \\ \tilde{x}_{12} \oplus \tilde{x}_{22} &= [(30, 35, 40, 50, 100, 110, 150, 170); (0.76, 0.02, 0.01)], \end{aligned} \tag{32}$$

where $\tilde{x}_{11}$, $\tilde{x}_{12}$, $\tilde{x}_{21}$, and $\tilde{x}_{22}$ are nonnegative TrPFNs.

Now,

Total supply = [(25, 60, 95, 130, 165, 215, 240, 390); (0.9840, 0.0002, 0.0002)]

Total demand = [(50, 75, 90, 115, 185, 210, 290, 330); (0.9808, 0.0002, 0.0002)]

For an unbalanced FPFTP, we add dummy row and dummy column to make a balanced picture fuzzy TP.

Dummy row = [(25, 25, 25, 25, 60, 60, 115, 115); (0.0000, 0.0000, 0.0000)]

Dummy column = [(0, 10, 30, 40, 40, 65, 65, 175); (0.1666, 0.0000, 0.0000)]

Therefore, by supposing picture fuzzy transportation cost of unit quantity of product from dummy source to all destinations and from all sources to dummy destination to be zero TrPFNs, then FPFLPP equation (31) can be transformed as follows:

Table 1: Input data for FPFTP.

	City (T_1)	City (T_2)
Plant (S_1)	[(110, 150, 190, 210, 270, 300, 350, 390); (0.7, 0.1, 0.1)]	[(130, 180, 220, 250, 290, 340, 370, 410); (0.8, 0.1, 0.1)]
Plant (S_2)	[(150, 250, 290, 350, 400, 440, 460, 500); (0.6, 0.1, 0.2)]	[(190, 210, 250, 270, 310, 330, 380, 430); (0.6, 0.1, 0.2)]

$$\text{Minimize}\begin{pmatrix} [(110,150,190,210,270,300,350,390);\ (0.7,0.1,0.1)]\otimes\tilde{x}_{11}\oplus \\ [(130,180,220,250,290,340,370,410);\ (0.8,0.1,0.1)]\otimes\tilde{x}_{12}\oplus \\ [(0,0,0,0,0,0,0,0);\ (0.0,0.0,0.0)]\otimes\tilde{x}_{13}\oplus \\ [(150,250,290,350,400,440,460,500);\ (0.6,0.1,0.2)]\otimes\tilde{x}_{21}\oplus \\ [(190,210,250,270,310,330,380,430);\ (0.6,0.1,0.2)]\otimes\tilde{x}_{22}\oplus \\ [(0,0,0,0,0,0,0,0);\ (0.0,0.0,0.0)]\otimes\tilde{x}_{23}\oplus \\ [(0,0,0,0,0,0,0,0);\ (0.0,0.0,0.0)]\otimes\tilde{x}_{31}\oplus \\ [(0,0,0,0,0,0,0,0);\ (0.0,0.0,0.0)]\otimes\tilde{x}_{32}\oplus \\ [(0,0,0,0,0,0,0,0);\ (0.0,0.0,0.0)]\otimes\tilde{x}_{33} \end{pmatrix} \tag{33}$$

$$\tilde{x}_{11}\oplus\tilde{x}_{12} = [(15,35,45,60,75,95,110,200);\ (0.92,0.02,0.01)],$$

subject to

$$\begin{aligned} \tilde{x}_{11}\oplus\tilde{x}_{12}\oplus\tilde{x}_{13} &= [(15,35,45,60,75,95,110,200);\ (0.92,0.02,0.01)],\\ \tilde{x}_{21}\oplus\tilde{x}_{22}\oplus\tilde{x}_{23} &= [(10,25,50,70,90,120,130,190);\ (0.80,0.01,0.02)],\\ \tilde{x}_{31}\oplus\tilde{x}_{32}\oplus\tilde{x}_{33} &= [(25,25,25,25,60,60,115,115);\ (0.0000,0.0000,0.0000)],\\ \tilde{x}_{11}\oplus\tilde{x}_{21}\oplus\tilde{x}_{31} &= [(20,40,50,65,85,100,140,160);\ (0.92,0.01,0.02)],\\ \tilde{x}_{12}\oplus\tilde{x}_{22}\oplus\tilde{x}_{32} &= [(30,35,40,50,100,110,150,170);\ (0.76,0.02,0.01)],\\ \tilde{x}_{13}\oplus\tilde{x}_{23}\oplus\tilde{x}_{33} &= [(0,10,30,40,40,65,65,175);\ (0.1666,0.0000,0.0000)], \end{aligned} \tag{34}$$

where $\tilde{x}_{11}$, $\tilde{x}_{12}$, $\tilde{x}_{13}$, $\tilde{x}_{21}$, $\tilde{x}_{22}$, $\tilde{x}_{23}$, $\tilde{x}_{31}$, $\tilde{x}_{32}$, and $\tilde{x}_{33}$, are nonnegative TrPFNs.

By supposing,

$$\begin{aligned} \tilde{x}_{11} &= [(\chi_{11},\delta_{11},\epsilon_{11},\eta_{11},\kappa_{11},\vartheta_{11},\omega_{11},\zeta_{11});\ (\alpha_{11},\beta_{11},\gamma_{11})],\\ \tilde{x}_{12} &= [(\chi_{12},\delta_{12},\epsilon_{12},\eta_{12},\kappa_{12},\vartheta_{12},\omega_{12},\zeta_{12});\ (\alpha_{12},\beta_{12},\gamma_{12})],\\ \tilde{x}_{13} &= [(\chi_{13},\delta_{13},\epsilon_{13},\eta_{13},\kappa_{13},\vartheta_{13},\omega_{13},\zeta_{13});\ (\alpha_{13},\beta_{13},\gamma_{13})],\\ \tilde{x}_{21} &= [(\chi_{21},\delta_{21},\epsilon_{21},\eta_{21},\kappa_{21},\vartheta_{21},\omega_{21},\zeta_{21});\ (\alpha_{21},\beta_{21},\gamma_{21})],\\ \tilde{x}_{22} &= [(\chi_{22},\delta_{22},\epsilon_{22},\eta_{22},\kappa_{22},\vartheta_{22},\omega_{22},\zeta_{22});\ (\alpha_{22},\beta_{22},\gamma_{22})],\\ \tilde{x}_{23} &= [(\chi_{23},\delta_{23},\epsilon_{23},\eta_{23},\kappa_{23},\vartheta_{23},\omega_{23},\zeta_{23});\ (\alpha_{23},\beta_{23},\gamma_{23})],\\ \tilde{x}_{31} &= [(\chi_{31},\delta_{31},\epsilon_{31},\eta_{31},\kappa_{31},\vartheta_{31},\omega_{31},\zeta_{31});\ (\alpha_{31},\beta_{31},\gamma_{31})],\\ \tilde{x}_{32} &= [(\chi_{32},\delta_{32},\epsilon_{32},\eta_{32},\kappa_{32},\vartheta_{32},\omega_{32},\zeta_{32});\ (\alpha_{32},\beta_{32},\gamma_{32})],\\ \tilde{x}_{33} &= [(\chi_{33},\delta_{33},\epsilon_{33},\eta_{33},\kappa_{33},\vartheta_{33},\omega_{33},\zeta_{33});\ (\alpha_{33},\beta_{33},\gamma_{33})], \end{aligned} \tag{35}$$

where $\tilde{x}_{11}$, $\tilde{x}_{12}$, $\tilde{x}_{13}$, $\tilde{x}_{21}$, $\tilde{x}_{22}$, $\tilde{x}_{23}$, $\tilde{x}_{31}$, $\tilde{x}_{32}$, and $\tilde{x}_{33}$ are nonnegative TrPFNs, then the FPFLPP equation (33) can be transformed as follows:

$$\text{Minimize}\left(\begin{array}{c}[(110,150,190,210,270,300,350,390);\ (0.7,0.1,0.1)]\otimes\\ [(\chi_{11},\delta_{11},\epsilon_{11},\eta_{11},\kappa_{11},\vartheta_{11},\omega_{11},\zeta_{11});\ (\alpha_{11},\beta_{11},\gamma_{11})]\oplus\\ [(130,180,220,250,290,340,370,410);\ (0.8,0.1,0.1)]\otimes\\ [(\chi_{12},\delta_{12},\epsilon_{12},\eta_{12},\kappa_{12},\vartheta_{12},\omega_{12},\zeta_{12});\ (\alpha_{12},\beta_{12},\gamma_{12})]\oplus\\ [(0,0,0,0,0,0,0,0);\ (0.0,0.0,0.0)]\otimes[(\chi_{13},\delta_{13},\epsilon_{13},\eta_{13},\kappa_{13},\vartheta_{13},\omega_{13},\zeta_{13});\ (\alpha_{13},\beta_{13},\gamma_{13})]\oplus\\ [(150,250,290,350,400,440,460,500);\ (0.6,0.1,0.2)]\otimes\\ [(\chi_{21},\delta_{21},\epsilon_{21},\eta_{21},\kappa_{21},\vartheta_{21},\omega_{21},\zeta_{21});\ (\alpha_{21},\beta_{21},\gamma_{21})]\oplus\\ [(190,210,250,270,310,330,380,430);\ (0.6,0.1,0.2)]\otimes\\ [(\chi_{22},\delta_{22},\epsilon_{22},\eta_{22},\kappa_{22},\vartheta_{22},\omega_{22},\zeta_{22});\ (\alpha_{22},\beta_{22},\gamma_{22})]\oplus\\ [(0,0,0,0,0,0,0,0);\ (0.0,0.0,0.0)]\otimes[(\chi_{23},\delta_{23},\epsilon_{23},\eta_{23},\kappa_{23},\vartheta_{23},\omega_{23},\zeta_{23});\ (\alpha_{23},\beta_{23},\gamma_{23})]\oplus\\ [(0,0,0,0,0,0,0,0);\ (0.0,0.0,0.0)]\otimes[(\chi_{31},\delta_{31},\epsilon_{31},\eta_{31},\kappa_{31},\vartheta_{31},\omega_{31},\zeta_{31});\ (\alpha_{31},\beta_{31},\gamma_{31})]\oplus\\ [(0,0,0,0,0,0,0,0);\ (0.0,0.0,0.0)]\otimes[(\chi_{32},\delta_{32},\epsilon_{32},\eta_{32},\kappa_{32},\vartheta_{32},\omega_{32},\zeta_{32});\ (\alpha_{32},\beta_{32},\gamma_{32})]\oplus\\ [(0,0,0,0,0,0,0,0);\ (0.0,0.0,0.0)]\otimes[(\chi_{33},\delta_{33},\epsilon_{33},\eta_{33},\kappa_{33},\vartheta_{33},\omega_{33},\zeta_{33});\ (\alpha_{33},\beta_{33},\gamma_{33})]\end{array}\right), \tag{36}$$

subject to

$$\begin{gathered}[(\chi_{11},\delta_{11},\epsilon_{11},\eta_{11},\kappa_{11},\vartheta_{11},\omega_{11},\zeta_{11});\ (\alpha_{11},\beta_{11},\gamma_{11})]\oplus[(\chi_{12},\delta_{12},\epsilon_{12},\eta_{12},\kappa_{12},\vartheta_{12},\omega_{12},\zeta_{12});\ (\alpha_{12},\beta_{12},\gamma_{12})]\\ \oplus[(\chi_{13},\delta_{13},\epsilon_{13},\eta_{13},\kappa_{13},\vartheta_{13},\omega_{13},\zeta_{13});\ (\alpha_{13},\beta_{13},\gamma_{13})]=[(15,35,45,60,75,95,110,200);\ (0.92,0.02,0.01)]\\ [(\chi_{21},\delta_{21},\epsilon_{21},\eta_{21},\kappa_{21},\vartheta_{21},\omega_{21},\zeta_{21});\ (\alpha_{21},\beta_{21},\gamma_{21})]\oplus[(\chi_{22},\delta_{22},\epsilon_{22},\eta_{22},\kappa_{22},\vartheta_{22},\omega_{22},\zeta_{22});\ (\alpha_{22},\beta_{22},\gamma_{22})]\\ \oplus[(\chi_{23},\delta_{23},\epsilon_{23},\eta_{23},\kappa_{23},\vartheta_{23},\omega_{23},\zeta_{23});\ (\alpha_{23},\beta_{23},\gamma_{23})]=[(10,25,50,70,90,120,130,190);\ (0.80,0.01,0.02)]\\ [(\chi_{31},\delta_{31},\epsilon_{31},\eta_{31},\kappa_{31},\vartheta_{31},\omega_{31},\zeta_{31});\ (\alpha_{31},\beta_{31},\gamma_{31})]\oplus[(\chi_{32},\delta_{32},\epsilon_{32},\eta_{32},\kappa_{32},\vartheta_{32},\omega_{32},\zeta_{32});\ (\alpha_{32},\beta_{32},\gamma_{32})]\\ \oplus[(\chi_{33},\delta_{33},\epsilon_{33},\eta_{33},\kappa_{33},\vartheta_{33},\omega_{33},\zeta_{33});\ (\alpha_{33},\beta_{33},\gamma_{33})]=[(25,25,25,25,60,60,115,115);\ (0.0000,0.0000,0.0000)]\\ [(\chi_{11},\delta_{11},\epsilon_{11},\eta_{11},\kappa_{11},\vartheta_{11},\omega_{11},\zeta_{11});\ (\alpha_{11},\beta_{11},\gamma_{11})]\oplus[(\chi_{21},\delta_{21},\epsilon_{21},\eta_{21},\kappa_{21},\vartheta_{21},\omega_{21},\zeta_{21});\ (\alpha_{21},\beta_{21},\gamma_{21})]\\ \oplus[(\chi_{31},\delta_{31},\epsilon_{31},\eta_{31},\kappa_{31},\vartheta_{31},\omega_{31},\zeta_{31});\ (\alpha_{31},\beta_{31},\gamma_{31})]=[(20,40,50,65,85,100,140,160);\ (0.92,0.01,0.02)]\\ [(\chi_{12},\delta_{12},\epsilon_{12},\eta_{12},\kappa_{12},\vartheta_{12},\omega_{12},\zeta_{12});\ (\alpha_{12},\beta_{12},\gamma_{12})]\oplus[(\chi_{22},\delta_{22},\epsilon_{22},\eta_{22},\kappa_{22},\vartheta_{22},\omega_{22},\zeta_{22});\ (\alpha_{22},\beta_{22},\gamma_{22})]\\ \oplus[(\chi_{32},\delta_{32},\epsilon_{32},\eta_{32},\kappa_{32},\vartheta_{32},\omega_{32},\zeta_{32});\ (\alpha_{32},\beta_{32},\gamma_{32})]=[(30,35,40,50,100,110,150,170);\ (0.76,0.02,0.01)]\\ [(\chi_{13},\delta_{13},\epsilon_{13},\eta_{13},\kappa_{13},\vartheta_{13},\omega_{13},\zeta_{13});\ (\alpha_{13},\beta_{13},\gamma_{13})]\oplus[(\chi_{23},\delta_{23},\epsilon_{23},\eta_{23},\kappa_{23},\vartheta_{23},\omega_{23},\zeta_{23});\ (\alpha_{23},\beta_{23},\gamma_{23})]\\ \oplus[(\chi_{33},\delta_{33},\epsilon_{33},\eta_{33},\kappa_{33},\vartheta_{33},\omega_{33},\zeta_{33});\ (\alpha_{33},\beta_{33},\gamma_{33})]=[(0,10,30,40,40,65,65,175);\ (0.1666,0.0000,0.0000)].\end{gathered} \tag{37}$$

By applying arithmetic operations as described in Section 2, the FPFLPP equation (37) can be transformed as follows:

$$\text{Minimize}\left(\begin{array}{c}
\left[\left(\begin{array}{c}(110\chi_{11},150\delta_{11},190\epsilon_{11},210\eta_{11},270\kappa_{11},300\vartheta_{11},350\omega_{11},390\zeta_{11});\\(0.7\alpha_{11},0.1\beta_{11},0.1+\gamma_{11}-0.1\gamma_{11})\end{array}\right)\right]\oplus\\
\left[\left(\begin{array}{c}(130\chi_{12},180\delta_{12},220\epsilon_{12},250\eta_{12},290\kappa_{12},340\vartheta_{12},370\omega_{12},410\zeta_{12});\\(0.8\alpha_{12},0.1\beta_{12},0.1+\gamma_{12}-0.1\gamma_{12})\end{array}\right)\right]\oplus\\
\left[\left(\begin{array}{c}(0\chi_{13},0\delta_{13},0\epsilon_{13},0\eta_{13},0\kappa_{13},0\vartheta_{13},0\omega_{13},0\zeta_{13});\\(0.0\alpha_{13},0.0\beta_{13},0.0+\gamma_{13}-0.0\gamma_{13})\end{array}\right)\right]\oplus\\
\left[\left(\begin{array}{c}(150\chi_{21},250\delta_{21},290\epsilon_{21},350\eta_{21},400\kappa_{21},440\vartheta_{21},460\omega_{21},500\zeta_{21});\\(0.6\alpha_{21},0.1\beta_{21},0.2+\gamma_{21}-0.2\gamma_{21})\end{array}\right)\right]\oplus\\
\left[\left(\begin{array}{c}(190\chi_{22},210\delta_{22},250\epsilon_{22},270\eta_{22},310\kappa_{22},330\vartheta_{22},380\omega_{22},430\zeta_{22});\\(0.6\alpha_{22},0.1\beta_{22},0.2+\gamma_{22}-0.2\gamma_{22})\end{array}\right)\right]\oplus\\
\left[\left(\begin{array}{c}(0\chi_{23},0\delta_{23},0\epsilon_{23},0\eta_{23},0\kappa_{23},0\vartheta_{23},0\omega_{23},0\zeta_{23});\\(0.0\alpha_{23},0.0\beta_{23},0.0+\gamma_{23}-0.0\gamma_{23})\end{array}\right)\right]\oplus\\
[((0\chi_{31},0\delta_{31},0\epsilon_{31},0\eta_{31},0\kappa_{31},0\vartheta_{31},0\omega_{31},0\zeta_{31});(0.0\alpha_{31},0.0\beta_{31},0.0+\gamma_{31}-0.0\gamma_{31}))]\oplus\\
[((0\chi_{32},0\delta_{32},0\epsilon_{32},0\eta_{32},0\kappa_{32},0\vartheta_{32},0\omega_{32},0\zeta_{32});(0.0\alpha_{32},0.0\beta_{32},0.0+\gamma_{32}-0.0\gamma_{32}))]\oplus\\
[((0\chi_{33},0\delta_{33},0\epsilon_{33},0\eta_{33},0\kappa_{33},0\vartheta_{33},0\omega_{33},0\zeta_{33});(0.0\alpha_{33},0.0\beta_{33},0.0+\gamma_{33}-0.0\gamma_{33}))]
\end{array}\right), \tag{38}$$

subject to

$$\left[\left(\begin{array}{c}\chi_{11}+\chi_{12}+\chi_{13},\delta_{11}+\delta_{12}+\delta_{13},\epsilon_{11}+\epsilon_{12}+\epsilon_{13},\eta_{11}+\eta_{12}+\eta_{13},\\\kappa_{11}+\kappa_{12}+\kappa_{13},\vartheta_{11}+\vartheta_{12}+\vartheta_{13},\omega_{11}+\omega_{12}+\omega_{13},\zeta_{11}+\zeta_{12}+\zeta_{13}\end{array}\right);\left(\begin{array}{c}\alpha_{11}+\alpha_{12}-\alpha_{11}\alpha_{12}\\+\alpha_{13}-\alpha_{11}\alpha_{13}-\alpha_{12}\alpha_{13}\\+\alpha_{11}\alpha_{12}\alpha_{13},\beta_{11}\beta_{12}\beta_{13},\\\gamma_{11}\gamma_{12}\gamma_{13}\end{array}\right)\right]$$
$$=[(15,35,45,60,75,95,110,200);(0.92,0.02,0.01)]$$

$$\left[\left(\begin{array}{c}\chi_{21}+\chi_{22}+\chi_{23},\delta_{21}+\delta_{22}+\delta_{23},\epsilon_{21}+\epsilon_{22}+\epsilon_{23},\eta_{21}+\eta_{22}+\eta_{23},\\\kappa_{21}+\kappa_{22}+\kappa_{23},\vartheta_{21}+\vartheta_{22}+\vartheta_{23},\omega_{21}+\omega_{22}+\omega_{23},\zeta_{21}+\zeta_{22}+\zeta_{23}\end{array}\right);\left(\begin{array}{c}\alpha_{21}+\alpha_{22}-\alpha_{21}\alpha_{22}\\+\alpha_{23}-\alpha_{21}\alpha_{23}-\alpha_{22}\alpha_{23}\\+\alpha_{21}\alpha_{22}\alpha_{23},\beta_{21}\beta_{22}\beta_{23},\\\gamma_{21}\gamma_{22}\gamma_{23}\end{array}\right)\right]$$
$$=[(10,25,50,70,90,120,130,190);(0.80,0.01,0.02)]$$

$$\left[\left(\begin{array}{c}\chi_{31}+\chi_{32}+\chi_{33},\delta_{31}+\delta_{32}+\delta_{33},\epsilon_{31}+\epsilon_{32}+\epsilon_{33},\eta_{31}+\eta_{32}+\eta_{33},\\\kappa_{31}+\kappa_{32}+\kappa_{33},\vartheta_{31}+\vartheta_{32}+\vartheta_{33},\omega_{31}+\omega_{32}+\omega_{33},\zeta_{31}+\zeta_{32}+\zeta_{33}\end{array}\right);\left(\begin{array}{c}\alpha_{31}+\alpha_{32}-\alpha_{31}\alpha_{32}\\+\alpha_{33}-\alpha_{31}\alpha_{33}-\alpha_{32}\alpha_{33}\\+\alpha_{31}\alpha_{32}\alpha_{33},\beta_{31}\beta_{32}\beta_{33},\\\gamma_{31}\gamma_{32}\gamma_{33}\end{array}\right)\right]$$
$$=[(25,25,25,25,60,60,115,115);(0.0000,0.0000,0.0000)]$$

$$\left[\left(\begin{array}{c}\chi_{11}+\chi_{21}+\chi_{31},\delta_{11}+\delta_{21}+\delta_{31},\epsilon_{11}+\epsilon_{21}+\epsilon_{31},\eta_{11}+\eta_{21}+\eta_{31},\\\kappa_{11}+\kappa_{21}+\kappa_{31},\vartheta_{11}+\vartheta_{21}+\vartheta_{31},\omega_{11}+\omega_{21}+\omega_{31},\zeta_{11}+\zeta_{21}+\zeta_{31}\end{array}\right);\left(\begin{array}{c}\alpha_{11}+\alpha_{21}-\alpha_{11}\alpha_{21}\\+\alpha_{31}-\alpha_{11}\alpha_{31}-\alpha_{21}\alpha_{31}\\+\alpha_{11}\alpha_{21}\alpha_{31},\beta_{11}\beta_{21}\beta_{31},\\\gamma_{11}\gamma_{21}\gamma_{31}\end{array}\right)\right]$$

$$
\begin{aligned}
&= [(20, 40, 50, 65, 85, 100, 140, 160);\ (0.92, 0.01, 0.02)]\\
&\left[\begin{pmatrix} \chi_{12} + \chi_{22} + \chi_{32}, \delta_{12} + \delta_{22} + \delta_{32}, \epsilon_{12} + \epsilon_{22} + \epsilon_{32}, \eta_{12} + \eta_{22} + \eta_{32},\\ \kappa_{12} + \kappa_{22} + \kappa_{32}, \vartheta_{12} + \vartheta_{22} + \vartheta_{32}, \omega_{12} + \omega_{22} + \omega_{32}, \zeta_{12} + \zeta_{22} + \zeta_{32} \end{pmatrix}; \begin{pmatrix} \alpha_{12} + \alpha_{22} - \alpha_{12}\alpha_{22}\\ +\alpha_{32} - \alpha_{12}\alpha_{32} - \alpha_{22}\alpha_{32}\\ +\alpha_{12}\alpha_{22}\alpha_{32}, \beta_{12}\beta_{22}\beta_{32},\\ \gamma_{12}\gamma_{22}\gamma_{32} \end{pmatrix}\right]\\
&= [(30, 35, 40, 50, 100, 110, 150, 170);\ (0.76, 0.02, 0.01)]\\
&\left[\begin{pmatrix} \chi_{13} + \chi_{23} + \chi_{33}, \delta_{13} + \delta_{23} + \delta_{33}, \epsilon_{13} + \epsilon_{23} + \epsilon_{33}, \eta_{13} + \eta_{23} + \eta_{33},\\ \kappa_{13} + \kappa_{23} + \kappa_{33}, \vartheta_{13} + \vartheta_{23} + \vartheta_{33}, \omega_{13} + \omega_{23} + \omega_{33}, \zeta_{13} + \zeta_{23} + \zeta_{33} \end{pmatrix}; \begin{pmatrix} \alpha_{13} + \alpha_{23} - \alpha_{13}\alpha_{23}\\ +\alpha_{33} - \alpha_{13}\alpha_{33} - \alpha_{23}\alpha_{33}\\ +\alpha_{13}\alpha_{23}\alpha_{33}, \beta_{13}\beta_{23}\beta_{33},\\ \gamma_{13}\gamma_{23}\gamma_{33} \end{pmatrix}\right]\\
&= [(0, 10, 30, 40, 40, 65, 65, 175);\ (0.1666, 0.0000, 0.0000)].
\end{aligned}
\tag{39}
$$

By applying Definition 7, the FPLPP equation (39) can be transformed as follows:

$$
\text{Minimize } \mathfrak{R}\left[\begin{array}{c}
\begin{pmatrix} 110\chi_{11} + 130\chi_{12} + 150\chi_{21} + 190\chi_{22}, 150\delta_{11} + 180\delta_{12} + 250\delta_{21} + 210\delta_{22},\\ 190\epsilon_{11} + 220\epsilon_{12} + 290\epsilon_{21} + 250\epsilon_{22}, 210\eta_{11} + 250\eta_{12} + 350\eta_{21} + 270\eta_{22},\\ 270\kappa_{11} + 290\kappa_{12} + 400\kappa_{21} + 310\kappa_{22}, 300\vartheta_{11} + 340\vartheta_{12} + 440\vartheta_{21} + 330\vartheta_{22},\\ 350\omega_{11} + 370\omega_{12} + 460\omega_{21} + 380\omega_{22}, 390\zeta_{11} + 410\zeta_{12} + 500\zeta_{21} + 430\zeta_{22} \end{pmatrix};\\
\begin{pmatrix} 0.7\alpha_{11} + 0.8\alpha_{12} - 0.56\alpha_{11}\alpha_{12} + 0.6\alpha_{21} - 0.42\alpha_{11}\alpha_{21}\\ -0.48\alpha_{12}\alpha_{21} + 0.336\alpha_{11}\alpha_{12}\alpha_{21} + 0.6\alpha_{22} - 0.42\alpha_{11}\alpha_{22} - 0.48\alpha_{12}\alpha_{22}\\ +0.336\alpha_{11}\alpha_{12}\alpha_{22} - 0.36\alpha_{21}\alpha_{22} + 0.252\alpha_{11}\alpha_{21}\alpha_{22}\\ +0.288\alpha_{12}\alpha_{21}\alpha_{22} - 0.2016\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}, 0.0001\beta_{11}\beta_{12}\beta_{21}\beta_{22},\\ \left(0.1 + \gamma_{11} - 0.1\gamma_{11}\right)\left(0.1 + \gamma_{12} - 0.1\gamma_{12}\right)\\ \left(0.2 + \gamma_{21} - 0.2\gamma_{21}\right)\left(0.2 + \gamma_{22} - 0.2\gamma_{22}\right) \end{pmatrix}
\end{array}\right],
\tag{40}
$$

subject to

$$\begin{aligned}
&\left[\begin{pmatrix} \chi_{11}+\chi_{12}+\chi_{13}, \delta_{11}+\delta_{12}+\delta_{13}, \epsilon_{11}+\epsilon_{12}+\epsilon_{13}, \eta_{11}+\eta_{12}+\eta_{13}, \\ \kappa_{11}+\kappa_{12}+\kappa_{13}, \vartheta_{11}+\vartheta_{12}+\vartheta_{13}, \omega_{11}+\omega_{12}+\omega_{13}, \zeta_{11}+\zeta_{12}+\zeta_{13} \end{pmatrix}; \begin{pmatrix} \alpha_{11}+\alpha_{12}-\alpha_{11}\alpha_{12} \\ +\alpha_{13}-\alpha_{11}\alpha_{13}-\alpha_{12}\alpha_{13} \\ +\alpha_{11}\alpha_{12}\alpha_{13}, \beta_{11}\beta_{12}\beta_{13}, \\ \gamma_{11}\gamma_{12}\gamma_{13} \end{pmatrix}\right] \\
&\quad = [(15, 35, 45, 60, 75, 95, 110, 200); (0.92, 0.02, 0.01)] \\
&\quad \left[\begin{pmatrix} \chi_{21}+\chi_{22}+\chi_{23}, \delta_{21}+\delta_{22}+\delta_{23}, \epsilon_{21}+\epsilon_{22}+\epsilon_{23}, \eta_{21}+\eta_{22}+\eta_{23}, \\ \kappa_{21}+\kappa_{22}+\kappa_{23}, \vartheta_{21}+\vartheta_{22}+\vartheta_{23}, \omega_{21}+\omega_{22}+\omega_{23}, \zeta_{21}+\zeta_{22}+\zeta_{23} \end{pmatrix}; \begin{pmatrix} \alpha_{21}+\alpha_{22}-\alpha_{21}\alpha_{22} \\ +\alpha_{23}-\alpha_{21}\alpha_{23}-\alpha_{22}\alpha_{23} \\ +\alpha_{21}\alpha_{22}\alpha_{23}, \beta_{21}\beta_{22}\beta_{23}, \\ \gamma_{21}\gamma_{22}\gamma_{23} \end{pmatrix}\right] \\
&\quad = [(10, 25, 50, 70, 90, 120, 130, 190); (0.80, 0.01, 0.02)] \\
&\quad \left[\begin{pmatrix} \chi_{31}+\chi_{32}+\chi_{33}, \delta_{31}+\delta_{32}+\delta_{33}, \epsilon_{31}+\epsilon_{32}+\epsilon_{33}, \eta_{31}+\eta_{32}+\eta_{33}, \\ \kappa_{31}+\kappa_{32}+\kappa_{33}, \vartheta_{31}+\vartheta_{32}+\vartheta_{33}, \omega_{31}+\omega_{32}+\omega_{33}, \zeta_{31}+\zeta_{32}+\zeta_{33} \end{pmatrix}; \begin{pmatrix} \alpha_{31}+\alpha_{32}-\alpha_{31}\alpha_{32} \\ +\alpha_{33}-\alpha_{31}\alpha_{33}-\alpha_{32}\alpha_{33} \\ +\alpha_{31}\alpha_{32}\alpha_{33}, \beta_{31}\beta_{32}\beta_{33}, \\ \gamma_{31}\gamma_{32}\gamma_{33} \end{pmatrix}\right] \\
&\quad = [(25, 25, 25, 25, 60, 60, 115, 115); (0.0000, 0.0000, 0.0000)] \\
&\quad \left[\begin{pmatrix} \chi_{11}+\chi_{21}+\chi_{31}, \delta_{11}+\delta_{21}+\delta_{31}, \epsilon_{11}+\epsilon_{21}+\epsilon_{31}, \eta_{11}+\eta_{21}+\eta_{31}, \\ \kappa_{11}+\kappa_{21}+\kappa_{31}, \vartheta_{11}+\vartheta_{21}+\vartheta_{31}, \omega_{11}+\omega_{21}+\omega_{31}, \zeta_{11}+\zeta_{21}+\zeta_{31} \end{pmatrix}; \begin{pmatrix} \alpha_{11}+\alpha_{21}-\alpha_{11}\alpha_{21} \\ +\alpha_{31}-\alpha_{11}\alpha_{31}-\alpha_{21}\alpha_{31} \\ +\alpha_{11}\alpha_{21}\alpha_{31}, \beta_{11}\beta_{21}\beta_{31}, \\ \gamma_{11}\gamma_{21}\gamma_{31} \end{pmatrix}\right] \\
&\quad = [(20, 40, 50, 65, 85, 100, 140, 160); (0.92, 0.01, 0.02)] \\
&\quad \left[\begin{pmatrix} \chi_{12}+\chi_{22}+\chi_{32}, \delta_{12}+\delta_{22}+\delta_{32}, \epsilon_{12}+\epsilon_{22}+\epsilon_{32}, \eta_{12}+\eta_{22}+\eta_{32}, \\ \kappa_{12}+\kappa_{22}+\kappa_{32}, \vartheta_{12}+\vartheta_{22}+\vartheta_{32}, \omega_{12}+\omega_{22}+\omega_{32}, \zeta_{12}+\zeta_{22}+\zeta_{32} \end{pmatrix}; \begin{pmatrix} \alpha_{12}+\alpha_{22}-\alpha_{12}\alpha_{22} \\ +\alpha_{32}-\alpha_{12}\alpha_{32}-\alpha_{22}\alpha_{32} \\ +\alpha_{12}\alpha_{22}\alpha_{32}, \beta_{12}\beta_{22}\beta_{32}, \\ \gamma_{12}\gamma_{22}\gamma_{32} \end{pmatrix}\right] \\
&\quad [(30, 35, 40, 50, 100, 110, 150, 170); (0.76, 0.02, 0.01)] \\
&\quad \left[\begin{pmatrix} \chi_{13}+\chi_{23}+\chi_{33}, \delta_{13}+\delta_{23}+\delta_{33}, \epsilon_{13}+\epsilon_{23}+\epsilon_{33}, \eta_{13}+\eta_{23}+\eta_{33}, \\ \kappa_{13}+\kappa_{23}+\kappa_{33}, \vartheta_{13}+\vartheta_{23}+\vartheta_{33}, \omega_{13}+\omega_{23}+\omega_{33}, \zeta_{13}+\zeta_{23}+\zeta_{33} \end{pmatrix}; \begin{pmatrix} \alpha_{13}+\alpha_{23}-\alpha_{13}\alpha_{23} \\ +\alpha_{33}-\alpha_{13}\alpha_{33}-\alpha_{23}\alpha_{33} \\ +\alpha_{13}\alpha_{23}\alpha_{33}, \beta_{13}\beta_{23}\beta_{33}, \\ \gamma_{13}\gamma_{23}\gamma_{33} \end{pmatrix}\right] \\
&\quad = [(0, 10, 30, 40, 40, 65, 65, 175); (0.1666, 0.0000, 0.0000)].
\end{aligned} \tag{41}$$

Now, we solve the following crisp LPP:

$$\text{Minimize}\frac{1}{4}\left(\begin{array}{c}(0.7\alpha_{11}+0.8\alpha_{12}-0.56\alpha_{11}\alpha_{12}+0.6\alpha_{21}-0.42\alpha_{11}\alpha_{21}\\ -0.48\alpha_{12}\alpha_{21}+0.336\alpha_{11}\alpha_{12}\alpha_{21}+0.6\alpha_{22}-0.42\alpha_{11}\alpha_{22}-0.48\alpha_{12}\alpha_{22}\\ +0.336\alpha_{11}\alpha_{12}\alpha_{22}-0.36\alpha_{21}\alpha_{22}+0.252\alpha_{11}\alpha_{21}\alpha_{22}\\ +0.288\alpha_{12}\alpha_{21}\alpha_{22}-0.2016\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22})(210\eta_{11}+250\eta_{12}\\ +350\eta_{21}+270\eta_{22}+270\kappa_{11}+290\kappa_{12}+400\kappa_{21}+310\kappa_{22})\\ +(190\epsilon_{11}+220\epsilon_{12}+290\epsilon_{21}+250\epsilon_{22})+(300\vartheta_{11}+340\vartheta_{12}\\ +440\vartheta_{21}+330\vartheta_{22})+(0.0001\beta_{11}\beta_{12}\beta_{21}\beta_{22})\\ ((210\eta_{11}+250\eta_{12}+350\eta_{21}+270\eta_{22}+270\kappa_{11}+290\kappa_{12}+400\kappa_{21}+310\kappa_{22})\\ +(150\delta_{11}+180\delta_{12}+250\delta_{21}+210\delta_{22})+(350\omega_{11}+370\omega_{12}\\ +460\omega_{21}+380\omega_{22})+(1-(0.1+\gamma_{11}-0.1\gamma_{11})\\ (0.1+\gamma_{12}-0.1\gamma_{12})(0.2+\gamma_{21}-0.2\gamma_{21})\\ (0.2+\gamma_{22}-0.2\gamma_{22}))(210\eta_{11}+250\eta_{12}+350\eta_{21}\\ +270\eta_{22}+270\kappa_{11}+290\kappa_{12}+400\kappa_{21}+310\kappa_{22})+(110\chi_{11}+130\chi_{12}\\ +150\chi_{21}+190\chi_{22})+(390\zeta_{11}+410\zeta_{12}+500\zeta_{21}+430\zeta_{22}))\end{array}\right), \tag{42}$$

subject to

$$\begin{array}{ll}
\chi_{11}+\chi_{12}+\chi_{13}=15, & \chi_{21}+\chi_{22}+\chi_{23}=10,\\
\delta_{11}+\delta_{12}+\delta_{13}=35, & \delta_{21}+\delta_{22}+\delta_{23}=25,\\
\epsilon_{11}+\epsilon_{12}+\epsilon_{13}=45, & \epsilon_{21}+\epsilon_{22}+\epsilon_{23}=50,\\
\eta_{11}+\eta_{12}+\eta_{13}=60, & \eta_{21}+\eta_{22}+\eta_{23}=70,\\
\kappa_{11}+\kappa_{12}+\kappa_{13}=75, & \kappa_{21}+\kappa_{22}+\kappa_{23}=90,\\
\vartheta_{11}+\vartheta_{12}+\vartheta_{13}=95, & \vartheta_{21}+\vartheta_{22}+\vartheta_{23}=120,\\
\omega_{11}+\omega_{12}+\omega_{13}=110, & \omega_{21}+\omega_{22}+\omega_{23}=130,\\
\zeta_{11}+\zeta_{12}+\zeta_{13}=200, & \zeta_{21}+\zeta_{22}+\zeta_{23}=190,\\
\chi_{31}+\chi_{32}+\chi_{33}=25, & \chi_{11}+\chi_{21}+\chi_{31}=20,\\
\delta_{31}+\delta_{32}+\delta_{33}=25, & \delta_{11}+\delta_{21}+\delta_{31}=40,\\
\epsilon_{31}+\epsilon_{32}+\epsilon_{33}=25, & \epsilon_{11}+\epsilon_{21}+\epsilon_{31}=50,\\
\eta_{31}+\eta_{32}+\eta_{33}=25, & \eta_{11}+\eta_{21}+\eta_{31}=65,\\
\kappa_{31}+\kappa_{32}+\kappa_{33}=60, & \kappa_{11}+\kappa_{21}+\kappa_{31}=85,\\
\vartheta_{31}+\vartheta_{32}+\vartheta_{33}=60, & \vartheta_{11}+\vartheta_{21}+\vartheta_{31}=100,\\
\omega_{31}+\omega_{32}+\omega_{33}=115, & \omega_{11}+\omega_{21}+\omega_{31}=140,\\
\zeta_{31}+\zeta_{32}+\zeta_{33}=115, & \zeta_{11}+\zeta_{21}+\zeta_{31}=160,\\
\chi_{12}+\chi_{22}+\chi_{32}=30, & \chi_{13}+\chi_{23}+\chi_{33}=0,\\
\delta_{12}+\delta_{22}+\delta_{32}=35, & \delta_{13}+\delta_{23}+\delta_{33}=10,\\
\epsilon_{12}+\epsilon_{22}+\epsilon_{32}=40, & \epsilon_{13}+\epsilon_{23}+\epsilon_{33}=30,
\end{array}$$

$$\eta_{12}+\eta_{22}+\eta_{32}=50,\quad \eta_{13}+\eta_{23}+\eta_{33}=40,$$
$$\kappa_{12}+\kappa_{22}+\kappa_{32}=100,\quad \kappa_{13}+\kappa_{23}+\kappa_{33}=40,$$
$$\vartheta_{12}+\vartheta_{22}+\vartheta_{32}=110,\quad \vartheta_{13}+\vartheta_{23}+\vartheta_{33}=65,$$
$$\omega_{12}+\omega_{22}+\omega_{32}=150,\quad \omega_{13}+\omega_{23}+\omega_{33}=65,$$
$$\zeta_{12}+\zeta_{22}+\zeta_{32}=170,\quad \zeta_{13}+\zeta_{23}+\zeta_{33}=175,$$
$$\alpha_{11}+\alpha_{12}-\alpha_{11}\alpha_{12}+\alpha_{13}-\alpha_{11}\alpha_{13}-\alpha_{12}\alpha_{13}+\alpha_{11}\alpha_{12}\alpha_{13}=0.92,\quad \alpha_{31}+\alpha_{32}-\alpha_{31}\alpha_{32}+\alpha_{33}-\alpha_{31}\alpha_{33}-\alpha_{32}\alpha_{33}+\alpha_{31}\alpha_{32}\alpha_{33}=0.00,$$
$$\beta_{11}\beta_{12}\beta_{13}=0.02,\quad \beta_{31}\beta_{32}\beta_{33}=0.00,$$
$$\gamma_{11}\gamma_{12}\gamma_{13}=0.01,\quad \gamma_{31}\gamma_{32}\gamma_{33}=0.00,$$
$$\alpha_{21}+\alpha_{22}-\alpha_{21}\alpha_{22}+\alpha_{23}-\alpha_{21}\alpha_{23}-\alpha_{22}\alpha_{23}+\alpha_{21}\alpha_{22}\alpha_{23}=0.80,\quad \alpha_{11}+\alpha_{21}-\alpha_{11}\alpha_{21}+\alpha_{31}-\alpha_{11}\alpha_{31}-\alpha_{21}\alpha_{31}+\alpha_{11}\alpha_{21}\alpha_{31}=0.92,$$
$$\beta_{21}\beta_{22}\beta_{23}=0.01,\quad \beta_{11}\beta_{21}\beta_{31}=0.01,$$
$$\gamma_{21}\gamma_{22}\gamma_{23}=0.02,\quad \gamma_{11}\gamma_{21}\gamma_{31}=0.02,$$
$$\alpha_{12}+\alpha_{22}-\alpha_{12}\alpha_{22}+\alpha_{32}-\alpha_{12}\alpha_{32}-\alpha_{22}\alpha_{32}+\alpha_{12}\alpha_{22}\alpha_{32}=0.76,\quad \alpha_{13}+\alpha_{23}-\alpha_{13}\alpha_{23}+\alpha_{33}-\alpha_{13}\alpha_{33}-\alpha_{23}\alpha_{33}+\alpha_{13}\alpha_{23}\alpha_{33}=0.1666,$$
$$\beta_{12}\beta_{22}\beta_{32}=0.02,\quad \beta_{13}\beta_{23}\beta_{33}=0.00,$$
$$\gamma_{12}\gamma_{22}\gamma_{32}=0.01,\quad \gamma_{13}\gamma_{23}\gamma_{33}=0.00,$$
$$\delta_{11}-\chi_{11}\geq 0,\quad \epsilon_{11}-\delta_{11}\geq 0,\quad \eta_{11}-\epsilon_{11}\geq 0,\quad \kappa_{11}-\eta_{11}\geq 0,\quad \vartheta_{11}-\kappa_{11}\geq 0,$$
$$\omega_{11}-\vartheta_{11}\geq 0,\quad \zeta_{11}-\omega_{11}\geq 0,\quad \delta_{12}-\chi_{12}\geq 0,\quad \epsilon_{12}-\delta_{12}\geq 0,\quad \eta_{12}-\epsilon_{12}\geq 0,$$
$$\kappa_{12}-\eta_{12}\geq 0, \vartheta_{12}-\kappa_{12}\geq 0,\quad \omega_{12}-\vartheta_{12}\geq 0,\quad \zeta_{12}-\omega_{12}\geq 0,\quad \delta_{21}-\chi_{21}\geq 0,$$
$$\epsilon_{21}-\delta_{21}\geq 0,\quad \delta_{13}-\chi_{13}\geq 0,\quad \epsilon_{13}-\delta_{13}\geq 0,\quad \eta_{13}-\epsilon_{13}\geq 0,\quad \kappa_{13}-\eta_{13}\geq 0,$$
$$\vartheta_{13}-\kappa_{13}\geq 0,\quad \omega_{13}-\vartheta_{13}\geq 0,\quad \zeta_{13}-\omega_{13}\geq 0,\quad \eta_{21}-\epsilon_{21}\geq 0,\quad \kappa_{21}-\eta_{21}\geq 0,$$
$$\vartheta_{21}-\kappa_{21}\geq 0,\quad \omega_{21}-\vartheta_{21}\geq 0,\quad \zeta_{21}-\omega_{21}\geq 0,\quad \delta_{22}-\chi_{22}\geq 0,\quad \epsilon_{22}-\delta_{22}\geq 0,$$
$$\eta_{22}-\epsilon_{22}\geq 0,\quad \kappa_{22}-\eta_{22}\geq 0,\quad \vartheta_{22}-\kappa_{22}\geq 0,\quad \omega_{22}-\vartheta_{22}\geq 0,\quad \zeta_{22}-\omega_{22}\geq 0,$$
$$\delta_{23}-\chi_{23}\geq 0,\quad \epsilon_{23}-\delta_{23}\geq 0,\quad \eta_{23}-\epsilon_{23}\geq 0,\quad \kappa_{23}-\eta_{23}\geq 0,\quad \vartheta_{23}-\kappa_{23}\geq 0,$$
$$\omega_{23}-\vartheta_{23}\geq 0,\quad \zeta_{23}-\omega_{23}\geq 0,\quad \delta_{31}-\chi_{31}\geq 0,\quad \epsilon_{31}-\delta_{31}\geq 0,\quad \eta_{31}-\epsilon_{31}\geq 0,$$
$$\kappa_{31}-\eta_{31}\geq 0,\quad \vartheta_{31}-\kappa_{31}\geq 0,\quad \omega_{31}-\vartheta_{31}\geq 0,\quad \zeta_{31}-\omega_{31}\geq 0,\quad \delta_{32}-\chi_{32}\geq 0,$$
$$\epsilon_{32}-\delta_{32}\geq 0,\quad \eta_{32}-\epsilon_{32}\geq 0,\quad \kappa_{32}-\eta_{32}\geq 0,\quad \vartheta_{32}-\kappa_{32}\geq 0,\quad \omega_{32}-\vartheta_{32}\geq 0,$$
$$\zeta_{32}-\omega_{32}\geq 0,\quad \delta_{33}-\chi_{33}\geq 0,\quad \epsilon_{33}-\delta_{33}\geq 0,\quad \eta_{33}-\epsilon_{33}\geq 0,\quad \kappa_{33}-\eta_{33}\geq 0,$$

$$\begin{aligned}
&\vartheta_{33}-\kappa_{33}\geq 0,\quad \omega_{33}-\vartheta_{33}\geq 0,\quad \zeta_{33}-\omega_{33}\geq 0,\quad \chi_{11}\geq 0,\quad \chi_{12}\geq 0,\quad \chi_{13}\geq 0,\\
&\chi_{21}\geq 0,\quad \chi_{22}\geq 0,\quad \chi_{23}\geq 0,\chi_{31}\geq 0,\quad \chi_{31}\geq 0,\quad \chi_{32}\geq 0,\quad \chi_{33}\geq 0,\alpha_{12}\geq 0,\\
&\alpha_{13}\geq 0,\quad \alpha_{21}\geq 0,\alpha_{22}\geq 0,\quad \alpha_{23}\geq 0,\quad \beta_{11}\geq 0,\quad \beta_{12}\geq 0,\beta_{21}\geq 0,\quad \beta_{22}\geq 0,\\
&\beta_{23}\geq 0,\quad \beta_{31}\geq 0,\quad \beta_{32}\geq 0,\quad \beta_{33}\geq 0,\quad \gamma_{11}\geq 0,\gamma_{13}\geq 0,\quad \gamma_{21}\geq 0,\quad \gamma_{22}\geq 0,\\
&\gamma_{23}\geq 0,\quad \gamma_{31}\geq 0,\quad \gamma_{32}\geq 0,\quad \gamma_{33}\geq 0,\quad \alpha_{11}\geq 0,\quad \beta_{13}\geq 0,\quad \gamma_{12}\geq 0,\\
&\alpha_{11}+\beta_{11}+\gamma_{11}\leq 1,\quad \alpha_{12}+\beta_{12}+\gamma_{12}\leq 1,\quad \alpha_{13}+\beta_{13}+\gamma_{13}\leq 1,\\
&\alpha_{21}+\beta_{21}+\gamma_{21}\leq 1,\quad \alpha_{22}+\beta_{22}+\gamma_{22}\leq 1,\quad \alpha_{23}+\beta_{23}+\gamma_{23}\leq 1,\\
&\alpha_{31}+\beta_{31}+\gamma_{31}\leq 1,\quad \alpha_{32}+\beta_{32}+\gamma_{32}\leq 1,\quad \alpha_{33}+\beta_{33}+\gamma_{33}\leq 1.
\end{aligned} \tag{43}$$

By using Software Maple, we get optimal solution.

$$\begin{aligned}
&\chi_{11}=15,\ \delta_{11}=35,\ \varepsilon_{11}=45,\ \eta_{11}=60,\ \kappa_{11}=75,\ \vartheta_{11}=90,\ \omega_{11}=105,\ \zeta_{11}=125,\ \alpha_{11}=0.8000,\ \beta_{11}=\\
&0.1000,\ \gamma_{11}=0.1000,\ \chi_{12}=0.0000,\ \delta_{12}=0.0000,\ \varepsilon_{12}=0.0000,\ \eta_{12}=0.0000,\ \kappa_{12}=0.0000,\ \vartheta_{12}=0.0000,\\
&\omega_{12}=0.0000,\ \zeta_{12}=20,\ \alpha_{12}=0.6000,\ \beta_{12}=0.2000,\ \gamma_{12}=0.1000,\ \chi_{13}=0.0000,\ \delta_{12}=0.0000,\ \varepsilon_{13}=0.0000,\\
&\eta_{13}=0.0000,\ \kappa_{13}=0.0000,\ \vartheta_{13}=5,\ \omega_{13}=5,\ \zeta_{13}=55,\ \alpha_{13}=0.1666,\ \beta_{13}=0.0000,\ \gamma_{13}=0.1000,\\
&\chi_{21}=0.0000,\ \delta_{21}=0.0000,\ \varepsilon_{21}=0.0000,\ \eta_{21}=0.0000,\ \kappa_{21}=0.0000,\ \vartheta_{21}=0.0000,\ \omega_{21}=0.0000\\
&,\ \zeta_{21}=0.0000,\ \alpha_{21}=0.6000,\ \beta_{21}=0.1000,\ \gamma_{21}=0.2000,\ \chi_{22}=10,\ \delta_{22}=15,\ \varepsilon_{22}=20,\ \eta_{22}=30,\ \kappa_{22}=50,\\
&\vartheta_{22}=60,\ \omega_{22}=70,\ \zeta_{22}=70,\ \alpha_{22}=0.5000,\ \beta_{22}=0.1000,\ \gamma_{22}=0.1000,\ \chi_{23}=0.0000,\ \delta_{23}=10,\ \varepsilon_{23}=30,\\
&\eta_{23}=40,\ \kappa_{23}=40,\ \vartheta_{23}=60,\ \omega_{23}=60,\ \zeta_{23}=120,\ \alpha_{23}=0.0000,\ \beta_{23}=0.0000,\ \gamma_{23}=0.0000,\ \chi_{31}=5,\\
&\delta_{31}=5,\ \varepsilon_{31}=5,\ \eta_{31}=5,\ \kappa_{31}=10,\ \vartheta_{31}=10,\ \omega_{31}=35,\ \zeta_{31}=35,\ \alpha_{31}=0.0000,\ \beta_{31}=0.0000,\ \gamma_{11}=0.0000,\\
&\chi_{32}=20,\ \delta_{32}=20,\ \varepsilon_{32}=20,\ \eta_{32}=20,\ \kappa_{32}=50,\ \vartheta_{32}=50,\ \omega_{32}=80,\ \zeta_{32}=80,\ \alpha_{32}=0.0000,\ \beta_{32}=\\
&0.0000,\ \gamma_{32}=0.0000,\ \chi_{33}=0.0000,\ \delta_{33}=0.0000,\ \varepsilon_{33}=0.0000\ \eta_{33}=0.0000,\ \kappa_{33}=0.0000,\ \vartheta_{33}=0.0000,\\
&\omega_{33}=0.0000,\ \zeta_{33}=0.0000,\ \alpha_{33}=0.0000,\ \beta_{33}=0.0000,\ \gamma_{33}=0.0000.
\end{aligned} \tag{44}$$

The PFOS is

$$\begin{aligned}
&\widetilde{x}_{11}=[(15,35,45,60,75,90,105,125);\ (0.8000,0.1000,0.1000)],\\
&\widetilde{x}_{12}=[(0.00,0.00,0.00,0.00,0.00,0.00,0.00,20);\ (0.6000,0.2000,0.1000)],\\
&\widetilde{x}_{13}=[(0.00,0.00,0.00,0.00,0.00,5,5,55);\ (0.1666,0.0000,0.0000)],\\
&\widetilde{x}_{21}=[(0.00,0.00,0.00,0.00,0.00,0.00,0.00,0.00);\ (0.6000,0.1000,0.2000)],\\
&\widetilde{x}_{22}=[(10,15,20,30,50,60,70,70);\ (0.5000,0.1000,0.1000)],\\
&\widetilde{x}_{23}=[(0.00,10,30,40,40,60,60,120);\ (0.0000,0.0000,0.0000)],\\
&\widetilde{x}_{31}=[(5,5,5,5,10,10,35,35);\ (0.0000,0.0000,0.0000)],\\
&\widetilde{x}_{32}=[(20,20,20,20,50,50,80,80);\ (0.0000,0.0000,0.0000)],\\
&\widetilde{x}_{33}=[(0.00,0.00,0.00,0.00,0.00,0.00,0.00,0.00);\ (0.0000,0.0000,0.0000)].
\end{aligned} \tag{45}$$

The picture fuzzy optimal value/transportation cost of FPFTP is

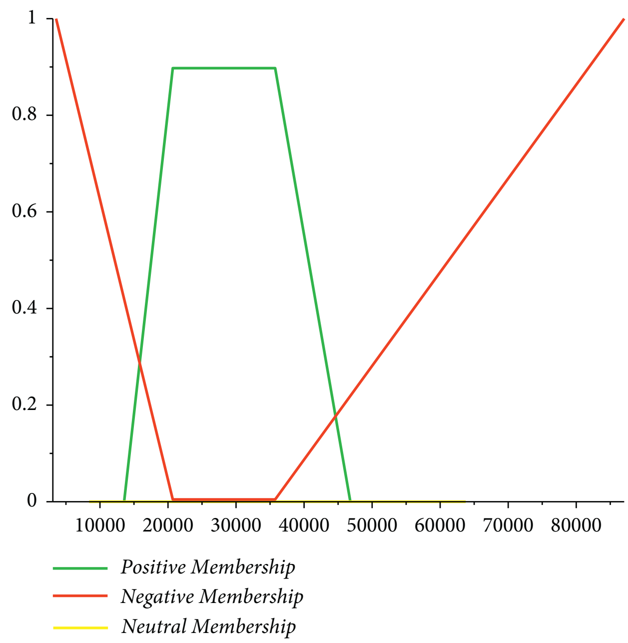

FIGURE 1: Graphical representation of picture fuzzy transportation cost.

[(3550, 8400, 13550, 20700, 35750, 46800, 63350, 87050); (0.8974976, 0.00000002, 0.00467856)] and is shown graphically in Figure 1. □

Example 2. (fully intuitionistic fuzzy transportation problem (FIFTP) based on FLP formulation).

$$\text{Minimize}\begin{pmatrix} [(190,210,270,300);\ (110,210,270,390)]\otimes\tilde{x}_{11}\oplus \\ [(220,250,290,340);\ (130,250,290,410)]\otimes\tilde{x}_{12}\oplus \\ [(0,0,0,0);\ (0,0,0,0)]\otimes\tilde{x}_{13}\oplus \\ [(290,350,400,440);\ (150,350,400,500)]\otimes\tilde{x}_{21}\oplus \\ [(250,270,310,330);\ (190,270,310,430)]\otimes\tilde{x}_{22}\oplus \\ [(0,0,0,0);\ (0,0,0,0)]\otimes\tilde{x}_{23}\oplus \\ [(0,0,0,0);\ (0,0,0,0)]\otimes\tilde{x}_{31}\oplus \\ [(0,0,0,0);\ (0,0,0,0)]\otimes\tilde{x}_{32}\oplus \\ [(0,0,0,0);\ (0,0,0,0)]\otimes\tilde{x}_{33} \end{pmatrix}, \quad (46)$$

subject to

$$\begin{aligned} \tilde{x}_{11}\oplus\tilde{x}_{12}\oplus\tilde{x}_{13} &= [(45,60,75,95);\ (15,60,75,200)], \\ \tilde{x}_{21}\oplus\tilde{x}_{22}\oplus\tilde{x}_{23} &= [(50,70,90,120);\ (10,70,90,190)], \\ \tilde{x}_{31}\oplus\tilde{x}_{32}\oplus\tilde{x}_{33} &= [(25,25,60,60);\ (25,25,60,115)], \\ \tilde{x}_{11}\oplus\tilde{x}_{21}\oplus\tilde{x}_{31} &= [(50,65,85,100);\ (20,65,85,160)], \\ \tilde{x}_{12}\oplus\tilde{x}_{22}\oplus\tilde{x}_{32} &= [(40,50,100,110);\ (30,50,100,170)], \\ \tilde{x}_{13}\oplus\tilde{x}_{23}\oplus\tilde{x}_{33} &= [(30,40,40,65);\ (0,40,40,175)], \end{aligned} \quad (47)$$

where $\tilde{x}_{11}$, $\tilde{x}_{12}$, $\tilde{x}_{13}$, $\tilde{x}_{21}$, $\tilde{x}_{22}$, $\tilde{x}_{23}$, $\tilde{x}_{31}$, $\tilde{x}_{32}$, and $\tilde{x}_{33}$, are nonnegative TrIFNs.

The intuitionistic fuzzy optimal value/transportation cost of FIFTP [41] is

[(13550, 20700, 35750, 46800); (3550, 20700, 35750, 87050)] and is shown graphically in Figure 2.

Example 3. (fully fuzzy transportation problem (FFTP) based on FLP formulation).

$$\text{Minimize}\begin{pmatrix} [(190,210,270,300)]\otimes\tilde{x}_{11}\oplus[(220,250,290,340)]\otimes\tilde{x}_{12}\oplus[(0,0,0,0)]\otimes\tilde{x}_{13}\oplus \\ [(290,350,400,440)]\otimes\tilde{x}_{21}\oplus[(250,270,310,330)]\otimes\tilde{x}_{22}\oplus[(0,0,0,0)]\otimes\tilde{x}_{23}\oplus \\ [(0,0,0,0)]\otimes\tilde{x}_{31}\oplus[(0,0,0,0)]\otimes\tilde{x}_{32}\oplus[(0,0,0,0)]\otimes\tilde{x}_{33} \end{pmatrix}, \quad (48)$$

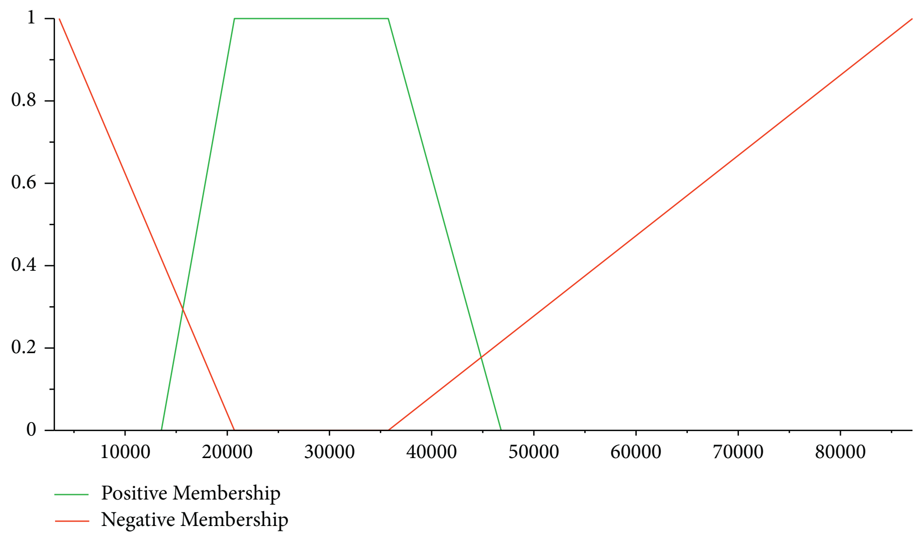

FIGURE 2: Graphical representation of intuitionistic fuzzy transportation cost.

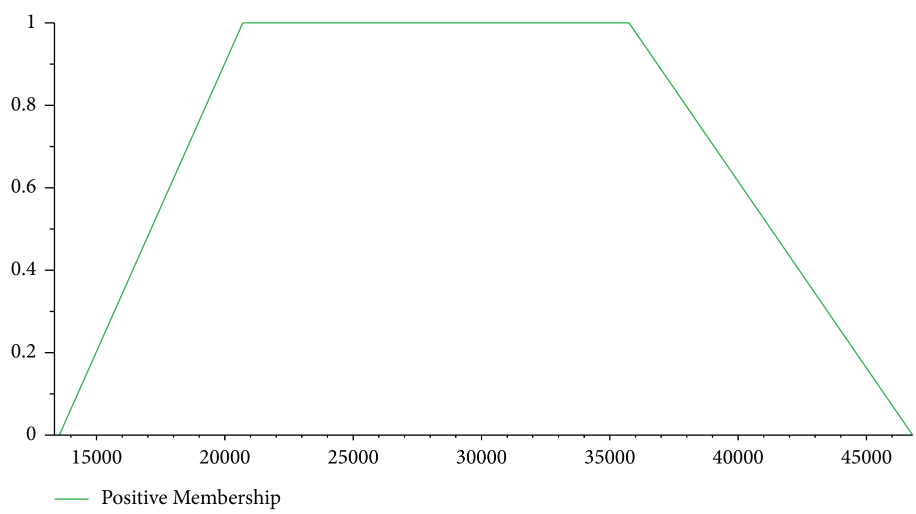

FIGURE 3: Graphical representation of fuzzy transportation cost.

TABLE 2: Comparison of optimal values.

FPFTP	[(3550, 8400, 13550, 20700, 35750, 46800, 63350, 87050)]
FIFTP [41]	[(3550, 13550, 20700, 35750, 46800, 87050)]
FFTP [35]	[(13550, 20700, 35750, 46800)]

subject to

$$\begin{aligned}
\tilde{x}_{11} \oplus \tilde{x}_{12} \oplus \tilde{x}_{13} &= [(45, 60, 75, 95)] \\
\tilde{x}_{21} \oplus \tilde{x}_{22} \oplus \tilde{x}_{23} &= [(50, 70, 90, 120)] \\
\tilde{x}_{31} \oplus \tilde{x}_{32} \oplus \tilde{x}_{33} &= [(25, 25, 60, 60)] \\
\tilde{x}_{11} \oplus \tilde{x}_{21} \oplus \tilde{x}_{31} &= [(50, 65, 85, 100)] \\
\tilde{x}_{12} \oplus \tilde{x}_{22} \oplus \tilde{x}_{32} &= [(40, 50, 100, 110)] \\
\tilde{x}_{13} \oplus \tilde{x}_{23} \oplus \tilde{x}_{33} &= [(30, 40, 40, 65)],
\end{aligned} \tag{49}$$

where $\tilde{x}_{11}$, $\tilde{x}_{12}$, $\tilde{x}_{13}$, $\tilde{x}_{21}$, $\tilde{x}_{22}$, $\tilde{x}_{23}$, $\tilde{x}_{31}$, $\tilde{x}_{32}$, and $\tilde{x}_{33}$ are nonnegative TrFNs.

The fuzzy optimal value/transportation cost of FFTP [35] is [(13550, 20700, 35750, 46800)] and is shown graphically in Figure 3.

5. Discussion

The picture fuzzy optimal value/transportation cost obtained in Example 1 by using the proposed method as discussed in

Section 3 is $[(3550, 8400, 13550, 20700, 35750, 46800, 63350, 87050)]$ and can be illustrated as follows:

(i) The lowest amount of picture fuzzy transportation cost is 3550 units
(ii) The feasible amount of picture fuzzy transportation cost lies in $[20700, 35750]$
(iii) The highest amount of picture fuzzy transportation cost is 87050 units

So, the least transportation cost will always be higher than 3550 units and lower than 87050 units, and the maximum chances of minimum transportation cost will belong to $[20700, 35750]$.

Furthermore, Examples 2 and 3 are considered in an intuitionistic fuzzy environment and fuzzy environment, respectively. The minimum intuitionistic fuzzy transportation cost and minimum fuzzy transportation along with picture fuzzy transportation cost are shown in Table 2.

Obviously, Table 2 demonstrates that the FPFTP gives the best optimal value/transportation cost as compared with FIFTP and FFTP.

6. Conclusion

PFS is the most comprehensive and generalized structure of FS and IFS because it is characterized by membership function, neutral membership function, and nonmembership function. In this paper, we have proposed a new scheme to discuss the fully picture fuzzy transportation problems. We have introduced FPFTPs by considering all the variables as nonnegative TrPFNs. The FPFTPs have been developed on the basis of picture fuzzy linear programming formulation. In order to transform the FPFTPs into crisp linear/non-LPPs, a ranking function is practised. The PFLP formulation technique has been applied to obtain the PFOSs in the form of TrPFNs. Further, we have investigated and compared the fully picture fuzzy transportation model with fully intuitionistic fuzzy transportation model and fully fuzzy transportation model to demonstrate that the proposed approach is more comprehensive and reliable as compared with the existing FIFTP approach [41] and FFTP approach [35].

In future, this work can be extended to

(1) *LR*-type fully picture fuzzy transportation problems
(2) *LR*-type bipolar single-valued neutrosophic transportation problems

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] L. A. Zadeh, "Information and Control," *Fuzzy sets*, vol. 8, no. 3, pp. 338–353, 1965.
[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.
[3] B. C. Cuong, "Picture fuzzy sets-first results, Part 1," in *Proceedings of the Seminar neuro-fuzzy systems with applications. Preprint 03/2013*, vol. 4, Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi-Vietnam, March 2013.
[4] D. Dubois and H. Prade, "Operations on fuzzy numbers," *International Journal of Systems Science*, vol. 9, no. 6, pp. 613–626, 1978.
[5] R. E. Bellman and L. A. Zadeh, "Decision making in a fuzzy environment," *Management Science*, vol. 17, pp. 141–164, 1970.
[6] H. Tanaka, T. Okuda, and K. Asai, "Fuzzy mathematical programming," *Transactions of the Society of Instrument and Control Engineers*, vol. 9, no. 5, pp. 607–613, 1973.
[7] H.-J. Zimmermann, "Fuzzy programming and linear programming with several objective functions," *Fuzzy Sets and Systems*, vol. 1, no. 1, pp. 45–55, 1978.
[8] K. Ganesan and P. Veeramani, "Fuzzy linear programs with trapezoidal fuzzy numbers," *Annals of Operations Research*, vol. 143, no. 1, pp. 305–315, 2006.
[9] F. H. Lotfi, T. Allahviranloo, M. Alimardani Jondabeh, and L. Alizadeh, "Solving a full fuzzy linear programming using lexicography method and fuzzy approximate solution," *Applied Mathematical Modelling*, vol. 33, no. 7, pp. 3151–3156, 2009.
[10] T. Allahviranloo, S. Salahshour, H. Nejad, and D. Baleanu, "General solutions of fully fuzzy linear systems," *Abstract and Applied Analysis*, vol. 2013, Article ID 593274, 2013.
[11] J. Kaur and A. Kumar, "Mehar's method for solving fully fuzzy linear programming problems with L - R fuzzy parameters," *Applied Mathematical Modelling*, vol. 37, no. 12-13, pp. 7142–7153, 2013.
[12] B. Pérez-Cañedo, E. R. Concepción-Morales, and S. A. Edalatpanah, "A revised version of a lexicographical-based method for solving fully fuzzy linear programming problems with inequality constraints," *Fuzzy Information and Engineering*, vol. 11, no. 4, pp. 474–493, 2019.
[13] M. Akram, I. Ullah, M. G. Alharbi, and S. A. Edalatpanah, "Methods for solving L R -type pythagorean fuzzy linear programming problems with mixed constraints," *Mathematical Problems in Engineering*, vol. 2021, no. 4, 29 pages, Article ID 4306058, 2021.
[14] M. A. Mehmood, M. Akram, M. G. Alharbi, and S. Bashir, "Solution of fully bipolar fuzzy linear programming models," *Mathematical Problems in Engineering*, vol. 2021, Article ID 9961891, 31 pages, 2021.
[15] M. A. Mehmood, M. Akram, M. G. Alharbi, and S. Bashir, "Optimization of LR-type fully bipolar fuzzy linear programming problems," *Mathematical Problems in Engineering*, vol. 2021, Article ID 1199336, 36 pages, 2021.
[16] J. Ahmed, M. G. Alharbi, M. Akram, and S. Bashir, "A new method to evaluate linear programming problem in bipolar single-valued neutrosophic environment," *Computer Modeling in Engineering and Sciences*, vol. 129, no. 2, pp. 881–906, 2021.
[17] M. Akram, I. Ullah, and T. Allahviranloo, "A new method for the solution of fully fuzzy linear programming models,"

Computational and Applied Mathematics, vol. 41, no. 1, pp. 1–25, 2022.
[18] T. Allahviranloo, *Uncertain Information and Linear Systems, Studies in Systems, Decision and Control*, Springer, USA, 2020.
[19] T. Allahviranloo, F. H. Lotfi, M. K. Kiasary, N. A. Kiani, and L. A. Zadeh, "Solving fully fuzzy linear programming problem by the ranking function," *Applied Mathematical Sciences*, vol. 2, no. 1, pp. 19–32, 2008.
[20] J. Kaur and A. Kumar, "An introduction to fuzzy linear programming problems," *Studies in Fuzziness and Soft Computing*, p. 340, 2016.
[21] F. L. Hitchcock, "The distribution of a product from several sources to numerous localities," *Journal of Mathematics and Physics*, vol. 20, no. 1-4, pp. 224–230, 1941.
[22] A. N. Gani and S. Abbas, "A new method for solving intuitionistic fuzzy transportation problem," *Applied Mathematical Sciences*, vol. 7, no. 28, pp. 1357–1365, 2013.
[23] L. R. Ford and D. R. Fulkerson, "A simple algorithm for finding maximal network flows and an application to the Hitchcock problem," *Canadian Journal of Mathematics*, vol. 9, pp. 210–218, 1957.
[24] Y. Li, H. Chen, and C. Prins, "Adaptive large neighborhood search for the pickup and delivery problem with time windows, profits, and reserved requests," *European Journal of Operational Research*, vol. 252, no. 1, pp. 27–38, 2016.
[25] M. Sakawa, I. Nishizaki, and Y. Uemura, "Fuzzy programming and profit and cost allocation for a production and transportation problem," *European Journal of Operational Research*, vol. 131, no. 1, pp. 1–15, 2001.
[26] R. Kumar, S. Jha, and R. Singh, "A different approach for solving the shortest path problem under mixed fuzzy environment," *International Journal of Fuzzy System Applications*, vol. 9, no. 2, pp. 132–161, 2020.
[27] S. K. Singh and S. P. Yadav, "A new approach for solving intuitionistic fuzzy transportation problem of type-2," *Annals of Operations Research*, vol. 243, no. 1, pp. 349–363, 2016.
[28] R. Kumar, S. A. Edalatpanah, S. Jha, and R. Singh, "A Pythagorean fuzzy approach to the transportation problem," *Complex & Intelligent Systems*, vol. 5, no. 2, pp. 255–263, 2019.
[29] A. Nagoorgani and K. A. Razak, "Two stage fuzzy transportation problem," *Journal of Physical Science*, vol. 10, pp. 63–69, 2006.
[30] A. Kaur and A. Kumar, "A new approach for solving fuzzy transportation problems using generalized trapezoidal fuzzy numbers," *Applied Soft Computing*, vol. 12, no. 3, pp. 1201–1213, 2012.
[31] S. K. Singh and S. P. Yadav, "Efficient approach for solving type-1 intuitionistic fuzzy transportation problem," *International Journal of System Assurance Engineering and Management*, vol. 6, no. 3, pp. 259–267, 2015.
[32] H. Basirzadeh, "An approach for solving fuzzy transportation problem," *Applied Mathematical Sciences*, vol. 5, no. 32, pp. 1549–1566, 2011.
[33] P. S. Kumar and R. J. Hussain, "Computationally simple approach for solving fully intuitionistic fuzzy real life transportation problems," *International Journal of System Assurance Engineering and Management*, vol. 7, no. 1, pp. 90–101, 2016.
[34] Abhishekh and A. K. Nishad, "A novel ranking approach to solving fully LR-intuitionistic fuzzy transportation problem," *New Mathematics and Natural Computation*, vol. 15, no. 1, pp. 95–112, 2019.
[35] A. Kaur, J. Kacprzyk, and A. Kumar, *Fuzzy Transportation and Transshipment Problems, Studies in Fuzziness and Soft Computing*, Springer, Cham, 2020.
[36] C. Wang, X. Zhou, H. Tu, and S. Tao, "Some geometric aggregation operators based on picture fuzzy sets and their application in multiple attribute decision making," *Italian Journal of Pure and Applied Mathematics*, vol. 37, pp. 477–492, 2017.
[37] M. Akram, A. Bashir, and H. Garg, "Decision-making model under complex picture fuzzy Hamacher aggregation operators," *Computational and Applied Mathematics*, vol. 39, no. 3, pp. 1–38, 2020.
[38] C. Shit, G. Ghorai, Q. Xin, and M. Gulzar, "Harmonic aggregation operator with trapezoidal picture fuzzy numbers and its application in a multiple-atribute decision making problem," *Symmetry*, vol. 14, no. 1, 2022.
[39] M. Akram, A. Habib, and J. C. R. Alcantud, "An optimization study based on Dijkstra algorithm for a network with trapezoidal picture fuzzy numbers," *Neural Computing & Applications*, vol. 33, no. 4, pp. 1329–1342, 2021.
[40] S. S. Geetha and K. Selvakumari, "A picture fuzzy approach to solving transportation problem," *European Journal of Molecular and Clinical Medicine*, vol. 7, no. 2, pp. 4982–4990, 2020.
[41] A. Mahmoodirad, T. Allahviranloo, and S. Niroomand, "A new effective solution method for fully intuitionistic fuzzy transportation problem," *Soft Computing*, vol. 23, no. 12, pp. 4521–4530, 2019.
[42] L. Kane, H. Sidibe, S. Kane et al., "A simplified new approach for solving fully fuzzy transportation problems with involving triangular fuzzy numbers," *Journal of fuzzy extension and applications*, vol. 2, no. 1, pp. 89–105, 2021.
[43] C. Veeramani, S. A. Edalatpanah, and S. Sharanya, "Solving the multiobjective fractional transportation problem through the neutrosophic goal programming approach," *Discrete Dynamics in Nature and Society*, vol. 2021, Article ID 7308042, 17 pages, 2021.
[44] G. Ali and M. N. Ansari, "Multiattribute decision-making under Fermatean fuzzy bipolar soft framework," *Granular Computing*, vol. 7, no. 2, pp. 337–352, 2022.
[45] F. Tchier, G. Ali, M. Gulzar, D. Pamučar, and G. Ghorai, "A new group decision-making technique under picture fuzzy soft expert information," *Entropy*, vol. 23, no. 9, 2021.
[46] S. K. Das, "Application of transportation problem under pentagonal neutrosophic environment," *Journal of Fuzzy Extension and Applications*, vol. 1, no. 1, pp. 27–40, 2020.
[47] L. Kané, M. Diakité, H. Bado, S. Kané, K. Moussa, and K. Traoré, "A new algorithm for fuzzy transportation problems with trapezoidal fuzzy numbers under fuzzy circumstances," *Journal of fuzzy extension and applications*, vol. 2, no. 3, pp. 204–225, 2021.
[48] G. Ali, M. Akram, S. Shahzadi, and M. Z. U. Abidin, "Group decision-making framework with bipolar soft expert sets," *Journal of Multiple-Valued Logic and Soft Computing*, vol. 37, pp. 211–246, 2021.
[49] A. Ashraf, K. Ullah, A. Hussain, and M. Bari, "Interval valued picture fuzzy Maclaurin symmetric mean operator with application in multiple attribute decision-making," *Reports in Mechanical Engineering*, vol. 3, no. 1, pp. 301–317, 2022.
[50] Z. Ali, T. Mahmood, T. Mahmood, K. Ullah, and Q. Khan, "Einstein geometric aggregation operators using a novel complex interval-valued Pythagorean fuzzy setting with application in green supplier chain management," *Reports in Mechanical Engineering*, vol. 2, no. 1, pp. 105–134, 2021.

[51] R. Sahu, S. R. Dash, S. R. Dash, and S. Das, "Career selection of students using hybridized distance measure based on picture fuzzy set and rough set theory," *Decision Making: Applications in Management and Engineering*, vol. 4, no. 1, pp. 104–126, 2021.

[52] B. F. Yildirim and S. K. Yıldırım, "Evaluating the satisfaction level of citizens in municipality services by using picture fuzzy VIKOR method: 2014-2019 period analysis," *Decision Making: Applications in Management and Engineering*, vol. 5, no. 1, pp. 50–66, 2022.

[53] M. Akram, I. Ullah, and T. Allahviranloo, "A new method to solve linear programming problems in the environment of picture fuzzy sets," *Iranian Journal of Fuzzy Systems*, vol. 61, 2022.

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Research Article

Innovative Bipolar Fuzzy Sine Trigonometric Aggregation Operators and SIR Method for Medical Tourism Supply Chain

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Bipolar fuzzy sets (BFSs) are effective tool for dealing with bipolarity and fuzziness. The sine trigonometric functions having two significant features, namely, periodicity and symmetry about the origin, are helping in decision analysis and information analysis. Taking the advantage of sine trigonometric functions and significance of BFSs, innovative sine trigonometric operational laws (STOLs) are proposed. New aggregation operators (AOs) are developed based on proposed operational laws to aggregate bipolar fuzzy information. Certain characteristics of these operators are also discussed, such as boundedness, monotonicity, and idempotency. Moreover, a modified superiority and inferiority ranking (SIR) method is proposed to cope with multicriteria group decision-making (MCGDM) with bipolar fuzzy (BF) information. To exhibit the relevance and feasibility of this methodology, a robust application of best medical tourism supply chain is presented. Finally, a comprehensive comparative and sensitivity analysis is evaluated to validate the efficiency of suggested methodology.

1. Introduction

Multicriteria group decision-making (MCGDM) is a process to seek an optimal alternative and ranking of feasible alternatives by a group of decision-experts under several stages and several criteria. However, this process is desperate with uncertainty due to data imprecision and vague perception. As a result, crisp theory is insufficient for dealing with MCGDM problems. To deal with these matters, Zadeh [1] initiated the conception of fuzzy set (FS) and membership function. Later on, different researchers presented different extensions of FSs including, intuitionistic fuzzy sets (IFSs) [2], Pythagorean fuzzy sets (PyFSs) [3, 4], q-rung orthopair fuzzy sets (q-ROFSs) [5], hesitant fuzzy sets (HFSs) [6], neutrosophic sets (NSs) [7], single-valued NSs [8], picture fuzzy sets (PFSs) [9], and spherical fuzzy sets (SFSs) [10–12].

The fuzzy models are extremely useful in dealing with uncertain MCGDM problems, and they have been widely used by decision makers. Nevertheless, they all have one flaw in common: they can only deal with one property and its not-property at a time. They are unable to cope with any property's counter property. It is quite common in decision analysis to have to consider both the positive and negative aspects of a specific object. Some well-known contradictory features in decision analysis include effects and side effects, profit and loss, health and sickness, and so on. Zhang [13, 14] propounded the abstraction of bipolar fuzzy sets (BFSs) which deal with both a property and its counter property. Lee [15] studied operations on bipolar-valued fuzzy sets. Tehrim and Riaz [16] introduced connection numbers of SPA theory for the decision support system by using the IVBF linguistic VIKOR method. Jana and Pal [17] proposed the BF-EDAS method for MCGDM problems. Liu et al. [18] suggested an integrated bipolar fuzzy SWARA-MABAC technique and utilized it for the safety risk and occupational health diagnosis. Jana et al. [19] introduced BF-Dombi AOs and Wei et al. [20] developed bipolar fuzzy Hamacher AOs.

Han et al. [21] proposed the TOPSIS method for YinYang bipolar fuzzy cognitive TOPSIS. Wei et al. [22] established MADM with IVBF information. Hamid et al.

[23] initiated weighted aggregation operators for q-rung orthopair m-polar fuzzy set. Akram et al. [24] proposed the notion of complex fermatean fuzzy N-soft sets. AOs are crucial in information aggregation and are subject to a variety of operational laws. Based on algebraic operational laws, Xu [25] and Xu and Yager [26] propounded weighted averaging and geometric AOs for IFSs. Garg [27] introduced interactive operators for IFSs. Huang [28] proposed intuitionistic fuzzy Hamacher aggregation operators. Gou and Xu [29] suggested exponential operational laws (EOLs) for IFSs.

Li and Wei [30] proposed logarithmic operational laws (LOLs) for IFSs. Peng et al. [31] proposed EOLs for q-ROFSs. Similarly, the LOLs for PFSs [32] are also defined. Aside from the exponential and logarithmic functions, sine trigonometric function is another suitable choice for information fusion. The two main characteristics are periodicity and symmetry about the origin which aid in meeting the decision makers' expectations during object evaluation. Abdullah et al. [33] developed STOLs for PFSs. Kabani [34] studied Pakistan as a medical tourism destination. Muzaffar and Hussain [35] investigated medical tourism to discuss the challenge: are we ready to take the challenge. Zhang and Xu [36] proposed TOPSIS for PFSs and PFNs with MCDM.

Mahmood et al. [37] proposed an innovative MCDM method with spherical fuzzy soft rough (SFSR) average aggregation operators. Ihsan et al. [38] presented the MADM support model based on bijective hypersoft expert set. Karaaslan and Karamaz [39] introduced an innovative decision-making approach with HFPHFS. Alcantud [40] introduced the novel concepts of soft topologies and fuzzy soft topologies and investigated their relationships. Liu et al. [41] introduced the idea of mining temporal association rules based on temporal soft sets. Riaz et al. [42] introduced a novel TOPSIS approach based on cosine similarity measures and CBF-information. Zararsiz and Riaz [43] introduced the notion of bipolar fuzzy metric spaces with application. Riaz et al. [44] proposed distance and similarity measures for bipolar fuzzy soft sets with application to pharmaceutical logistics and supply chain management.

In 2021, Gergin et al. [45] modified the TOPSIS method to deal the supplier selection for automotive industry. Karamasa et al. [46] introduced the weighting factors which affect the logistics out-sourcing decision-making problem. Ali et al. [47] introduced Einstein geometric AO to deal complex IVPFS, and its novel principles and its operational laws are defined. Muhammad et al. [48] and Biswas et al. [49] propounded multicriteria decision-making techniques to deal real world problems. Milovanovic et al. [50] developed uncertainty modeling using intuitionistic fuzzy numbers.

In 2021, Garg [51] introduced some robust STOLs, its operational laws for PFSs, and AOs and algorithms to interpret MCDM. In 2021, Mahmood et al. [52] interpreted BCFHWA, BCFHOWA, BCFHHA, BCFHWG, BCFHOWG, and BCFHHG operators. Palanikumar et al. [53] proposed some new methods to solve MCDM based on PNSNIVS. A notion of PNSNIVWA, PNSNIVWG, GPNSNIVWA, and GPNSNIVWG is also discussed in the article. In 2021, Jana et al. [54] applied IFDHWA and IFDHWG AO to evaluate enterprise financial performance. In 2021, Jana et al. [55] extended Dombi operations towards single-valued trapezoidal neutrosophic numbers (SVTrNNs). They also presented Dombi operation on SVTrNNs, and they proposed some new averaging and geometric averaging operators named as SVTrN Domi weighted averaging (SVTrNDWA) operator, SVTrN Dombi ordered weighted averaging (SVTrNDOWA) operator, SVTrN Dombi hybrid weighted averaging (SVTrNDHWA) operator, SVTrN Dombi weighted geometric (SVTrNDWGA) operator, SVTrN Dombi ordered weighted geometric (SVTrNDOWGA) operator, and SVTrN Dombi hybrid weighted geometric (SVTrNDHWGA) operator. In 2022, Ajay et al. [56] extended the STOLs for NSs and CNSs and defined the operational laws and their functionality. They also defined distance measures and ST-AOs. In 2022, Qiyas et al. [57] defined some reliable STOLs for SFNs and defined ST-OAs to deal real world problems.

The superiority and inferiority ranking (SIR) technique is a generalization of the eminent PROMETHEE method. This technique employs superiority and inferiority information to represent decision makers' behavior toward each criterion and to determine the degrees of domination and subordination of each alternative, from which superiority and inferiority flows are derived. It was introduced by Xu [58]. Chai and Liu [59] proposed the IF-SIR method to deal with MCGDM problems. Peng and Yang [60] extended the SIR technique to pythagorean fuzzy data. Zhu et al. [61] proposed the SIR approach for q-ROFSs.

Keeping in mind the importance of sine trigonometric function and SIR method, the aims and perks of this manuscript are as follows:

(1) To address bipolarity and uncertainty, innovative sine trigonometric operational laws (STOLs) are proposed for bipolar fuzzy sets (BFSs).

(2) Averaging AOs are developed named as sine trigonometric bipolar fuzzy weighted averaging (ST-BFWA) operator, sine trigonometric bipolar fuzzy ordered weighted averaging (ST-BFOWA) operator, and sine trigonometric bipolar fuzzy hybrid weighted averaging (ST-BFHWA) operator.

(3) Geometric AOs are proposed including sine trigonometric bipolar fuzzy weighted geometric (ST-BFWG) operator, sine trigonometric bipolar fuzzy ordered weighted geometric (ST-BFOWG) operator, and sine trigonometric bipolar fuzzy hybrid weighted geometric (ST-BFHWG) operator.

(4) Certain aspects of proposed operators are also discussed, such as idempotency, boundedness, and monotonicity.

(5) A modified SIR method by using features of proposed operators is proposed to cope with MCGDM problems.

(6) A robust application of best medical tourism supply chain is presented by using a modified SIR technique involving sine trigonometric AOs.

The layout of the remaining manuscript is as follows. In Section 2, some fundamental concepts about BFSs are reviewed. In Section 3, we define STOLs for BFSs and discuss their properties. In Sections 4 and 5, we introduce novel AOs based on BF-STOLs and explore their characteristics. Section 6 provides an extended version of the SIR technique for dealing with MCGDM problems using bipolar fuzzy data. A numerical illustration and a comparative analysis are also proffered to validate the efficaciousness of the propounded technique. Finally, in Section 7, there are some closing remarks.

2. Preliminaries

This section includes some rudimentary abstractions related to BFSs. Throughout this manuscript, we consider $\mathbb{Y}$ as universe of discourse.

Definition 1 (see [13]). A BFS $\mathfrak{B}$ on $\mathbb{Y}$ can be described as

$$\mathfrak{B} = \{\langle y, \aleph_{\mathfrak{B}}^{+}(y), \aleph_{\mathfrak{B}}^{-}(y)\rangle \colon y \in \mathbb{Y}\}, \tag{1}$$

where $\aleph_{\mathfrak{B}}^{+}(y) \in [0,1]$ denotes positive membership degree and $\aleph_{\mathfrak{B}}^{-}(y) \in [-1,0]$ denotes negative membership degree of an element $y \in \mathbb{Y}$. A bipolar fuzzy number (BFN) can be expressed as $\mathfrak{B} = \langle \aleph_{\mathfrak{B}}^{+}, \aleph_{\mathfrak{B}}^{-}\rangle$.

In 2015, Gul proposed operational laws of BFNs in his M.Phil Thesis.

Definition 2 [62]. Let $\mathfrak{B}_1 = \langle \aleph_{\mathfrak{B}_1}^{+}, \aleph_{\mathfrak{B}_1}^{-}\rangle$ and $\mathfrak{B}_2 = \langle \aleph_{\mathfrak{B}_2}^{+}, \aleph_{\mathfrak{B}_2}^{-}\rangle$ be two BFNs and $\sigma \in (0,\infty)$, then operational laws between them can be defined as

(i) $\mathfrak{B}_1 \oplus \mathfrak{B}_2 = \langle \aleph_{\mathfrak{B}_1}^{+} + \aleph_{\mathfrak{B}_2}^{+} - \aleph_{\mathfrak{B}_1}^{+}\aleph_{\mathfrak{B}_2}^{+}, -\aleph_{\mathfrak{B}_1}^{-}\aleph_{\mathfrak{B}_2}^{-}\rangle$

(ii) $\mathfrak{B}_1 \otimes \mathfrak{B}_2 = \langle \aleph_{\mathfrak{B}_1}^{+}\aleph_{\mathfrak{B}_2}^{+}, -((-\aleph_{\mathfrak{B}_1}^{-}) + (-\aleph_{\mathfrak{B}_2}^{-}) - \aleph_{\mathfrak{B}_1}^{-}\aleph_{\mathfrak{B}_2}^{-})\rangle$

(iii) $\sigma\mathfrak{B}_1 = \langle 1 - (1 - \aleph_{\mathfrak{B}_1}^{+})^{\sigma}, -(-\aleph_{\mathfrak{B}_1}^{-})^{\sigma}\rangle$

(iv) $\mathfrak{B}_1^{\sigma} = \langle (\aleph_{\mathfrak{B}_1}^{+})^{\sigma}, -(1 - (1 - (-\aleph_{\mathfrak{B}_1}^{-}))^{\sigma})\rangle$

(v) $\mathfrak{B}_1^{c} = \langle 1 - \aleph_{\mathfrak{B}_1}^{+}, -1 - \aleph_{\mathfrak{B}_1}^{-}\rangle$

(vi) $\mathfrak{B}_1 \prec \mathfrak{B}_2$ if $\aleph_{\mathfrak{B}_1}^{+} \le \aleph_{\mathfrak{B}_2}^{+}$ and $\aleph_{\mathfrak{B}_1}^{-} \ge \aleph_{\mathfrak{B}_2}^{-}$

(vii) $\mathfrak{B}_1 = \mathfrak{B}_2$ if $\mathfrak{B}_1 \prec \mathfrak{B}_2$ and $\mathfrak{B}_2 \prec \mathfrak{B}_1$

Definition 3 (see [20]). For a BFN $\mathfrak{B} = \langle \aleph_{\mathfrak{B}}^{+}, \aleph_{\mathfrak{B}}^{-}\rangle$, score and accuracy functions can be expressed as

$$\mathrm{Scr}(\mathfrak{B}) = \frac{1 + \aleph_{\mathfrak{B}}^{+} + \aleph_{\mathfrak{B}}^{-}}{2}, \tag{2}$$

$$\mathrm{Acr}(\mathfrak{B}) = \frac{\aleph_{\mathfrak{B}}^{+} - \aleph_{\mathfrak{B}}^{-}}{2}. \tag{3}$$

The values of score and accuracy functions are used to compare two BFNs. For two BFNs $\mathfrak{B}_1$ and $\mathfrak{B}_2$,

(i) If $\mathrm{Scr}(\mathfrak{B}_1) < \mathrm{Scr}(\mathfrak{B}_2)$, then $\mathfrak{B}_1 < \mathfrak{B}_2$

(ii) If $\mathrm{Scr}(\mathfrak{B}_1) > \mathrm{Scr}(\mathfrak{B}_2)$, then $\mathfrak{B}_1 > \mathfrak{B}_2$

(iii) If $\mathrm{Scr}(\mathfrak{B}_1) = \mathrm{Scr}(\mathfrak{B}_2)$, then $\mathfrak{B}_1 < \mathfrak{B}_2$ if $\mathrm{Acr}(\mathfrak{B}_1) < \mathrm{Acr}(\mathfrak{B}_2)$

(iv) If $\mathrm{Scr}(\mathfrak{B}_1) = \mathrm{Scr}(\mathfrak{B}_2)$, then $\mathfrak{B}_1 > \mathfrak{B}_2$ if $\mathrm{Acr}(\mathfrak{B}_1) > \mathrm{Acr}(\mathfrak{B}_2)$

(v) If $\mathrm{Scr}(\mathfrak{B}_1) = \mathrm{Scr}(\mathfrak{B}_2)$, then $\mathfrak{B}_1 = \mathfrak{B}_2$ if $\mathrm{Acr}(\mathfrak{B}_1) = \mathrm{Acr}(\mathfrak{B}_2)$

Definition 4 (see [21]). If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are two BFSs on $\mathbb{Y} = \{y_1, y_2, \ldots, y_n\}$, then the normalized Hamming distance between them is calculated as

$$d(\mathfrak{B}_1, \mathfrak{B}_2) = \frac{1}{2n}\sum_{i=1}^{n}\left(\left|\aleph_{\mathfrak{B}_1}^{+}(y_i) - \aleph_{\mathfrak{B}_2}^{+}(y_i)\right| + \left|\aleph_{\mathfrak{B}_1}^{-}(y_i) - \aleph_{\mathfrak{B}_2}^{-}(y_i)\right|\right). \tag{4}$$

3. Sine Trigonometric Operational Laws for BFSs

In this section, we suggest sine trigonometric operational laws (STOLs) for BFNs and investigate some useful results.

Definition 5. Let $\mathfrak{B} = \{\langle y, \aleph_{\mathfrak{B}}^{+}(y), \aleph_{\mathfrak{B}}^{-}(y)\rangle \colon y \in \mathbb{Y}\}$ be a BFS on $\mathbb{Y}$. A sine trigonometric operator on $\mathfrak{B}$ can be defined as

$$\sin\mathfrak{B} = \left\{\left\langle y,\ \sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}}^{+}(y)\right),\ \sin\left(\frac{\pi}{2}\left(1 + \aleph_{\mathfrak{B}}^{-}(y)\right)\right) - 1\right\rangle \colon y \in \mathbb{Y}\right\}. \tag{5}$$

Clearly, $\sin\mathfrak{B}$ is again a BFS on $\mathbb{Y}$ because $\sin((\pi/2)\aleph_{\mathfrak{B}}^{+}(y)) \in [0,1]$ and $\sin((\pi/2)(1 + \aleph_{\mathfrak{B}}^{-}(y))) - 1 \in [-1,0]$ serve as positive and negative membership degrees, respectively, for every element $y \in \mathbb{Y}$. The set $\sin\mathfrak{B}$ is called sine trigonometric-BFS (ST-BFS).

Definition 6. Let $\mathfrak{B} = \langle \aleph_{\mathfrak{B}}^{+}, \aleph_{\mathfrak{B}}^{-}\rangle$ be a BFN, then

$$\sin\mathfrak{B} = \left\langle \sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}}^{+}\right),\ \sin\left(\frac{\pi}{2}\left(1 + \aleph_{\mathfrak{B}}^{-}\right)\right) - 1\right\rangle, \tag{6}$$

is called ST-BFN.

Definition 7 For two BFNs $\mathfrak{B}_1 = \langle \aleph_{\mathfrak{B}_1}^{+}, \aleph_{\mathfrak{B}_1}^{-}\rangle$ and $\mathfrak{B}_2 = \langle \aleph_{\mathfrak{B}_2}^{+}, \aleph_{\mathfrak{B}_2}^{-}\rangle$, we propose STOLs as follows:

(i) $\sin\mathfrak{B}_1 \oplus \sin\mathfrak{B}_2 = 1 - (1 - \sin((\pi/2)\aleph_{\mathfrak{B}_1}^{+}))(1 - \sin((\pi/2)\aleph_{\mathfrak{B}_1}^{+})), -(\sin((\pi/2)(1 + \aleph_{\mathfrak{B}_1}^{-})) - 1)\rangle$ $\langle(\sin((\pi/2)(1 + \aleph_{\mathfrak{B}_2}^{-})) - 1)$

(ii) $\sin\mathfrak{B}_1 \otimes \sin\mathfrak{B}_2 = \langle \sin((\pi/2)\aleph_{\mathfrak{B}_1}^{+})\sin((\pi/2)\aleph_{\mathfrak{B}_1}^{+}), -(1 - \sin((\pi/2)(1 + \aleph_{\mathfrak{B}_1}^{-}))\sin((\pi/2)(1 + \aleph_{\mathfrak{B}_2}^{-})))\rangle$

(iii) $\sigma\sin\mathfrak{B}_1 = \langle 1 - (1 - \sin((\pi/2)\aleph_{\mathfrak{B}_1}^{+}))^{\sigma}, -(-(\sin((\pi/2)(1 + \aleph_{\mathfrak{B}_1}^{-}))) - 1)^{\sigma}\rangle;\ \sigma > 0$

(iv) $(\sin\mathfrak{B}_1)^{\sigma} = \langle(\sin((\pi/2)\aleph_{\mathfrak{B}_1}^{+}))^{\sigma}, -(1 - (\sin((\pi/2)(1 + \aleph_{\mathfrak{B}_1}^{-})))^{\sigma})\rangle;\ \sigma > 0$

Theorem 1. *Let* $\mathfrak{B}_1 = \langle \aleph_{\mathfrak{B}_1}^{+}, \aleph_{\mathfrak{B}_1}^{-}\rangle$ *and* $\mathfrak{B}_2 = \langle \aleph_{\mathfrak{B}_2}^{+}, \aleph_{\mathfrak{B}_2}^{-}\rangle$ *be two BFNs and* $\sigma > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$ *be three real numbers, then*

(i) $\sigma(\sin\mathfrak{B}_1 \oplus \sin\mathfrak{B}_2) = \sigma\sin\mathfrak{B}_1 \oplus \sigma\sin\mathfrak{B}_2$

(ii) $(\sin\mathfrak{B}_1 \otimes \sin\mathfrak{B}_2)^{\sigma} = (\sin\mathfrak{B}_1)^{\sigma} \otimes (\sin\mathfrak{B}_2)^{\sigma}$

(iii) $\sigma_1\sin\mathfrak{B}_1 \oplus \sigma_2\sin\mathfrak{B}_1 = (\sigma_1 + \sigma_2)\sin\mathfrak{B}_1$

(iv) $(\sin \mathfrak{B}_1)^{\sigma_1} \otimes (\sin \mathfrak{B}_1)^{\sigma_2} = (\sin \mathfrak{B}_1)^{\sigma_1+\sigma_2}$

Proof. We substantiate (i) and (iv), and others can be substantiated similarly.

(i) For $\sigma > 0$,

$$\begin{aligned}\sigma(\sin \mathfrak{B}_1 \oplus \sin \mathfrak{B}_2) &= \left\langle 1-\left(1-\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_1}\right)\right)^{\sigma}\left(1-\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_2}\right)\right)^{\sigma},\right.\\ &\left.-\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_1}\right)\right)-1\right)\right)^{\sigma}\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_2}\right)\right)-1\right)\right)^{\sigma}\right\rangle\\ &= \left\langle 1-\left(1-\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_1}\right)\right)^{\sigma},-\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_1}\right)\right)-1\right)\right)^{\sigma}\right\rangle\\ &\oplus\left\langle 1-\left(1-\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_2}\right)\right)^{\sigma},-\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_2}\right)\right)-1\right)\right)^{\sigma}\right\rangle = \sigma \sin \mathfrak{B}_1 \oplus \sigma \sin \mathfrak{B}_2.\end{aligned} \tag{7}$$

(iv) For $\sigma_1, \sigma_2 > 0$,

$$\begin{aligned}(\sin \mathfrak{B}_1)^{\sigma_1} \otimes (\sin \mathfrak{B}_1)^{\sigma_2} &= \left\langle\left(\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_1}\right)\right)^{\sigma_1},-\left(1-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_1}\right)\right)\right)^{\sigma_1}\right)\right\rangle\\ &\otimes\left\langle\left(\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_1}\right)\right)^{\sigma_2},-\left(1-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_1}\right)\right)\right)^{\sigma_2}\right)\right\rangle\\ &= \left(\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_1}\right)\right)^{\sigma_1}\left(\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_1}\right)\right)^{\sigma_2},\rangle\\ &\langle -\left(1-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_1}\right)\right)\right)^{\sigma_1}\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_1}\right)\right)\right)^{\sigma_2}\right)\rangle\\ &= \left\langle\left(\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_1}\right)\right)^{\sigma_1+\sigma_2},-\left(1-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_1}\right)\right)\right)^{\sigma_1+\sigma_2}\right)\right\rangle\\ &= (\sin \mathfrak{B}_1)^{\sigma_1+\sigma_2}.\end{aligned} \tag{8}$$

□

Definition 8. Let $\mathfrak{B} = \langle \aleph^+_{\mathfrak{B}}, \aleph^-_{\mathfrak{B}}\rangle$ be a BFN and $\sin \mathfrak{B}$ be the corresponding ST-BFN, then

$$(\sin \mathfrak{B})^c = \left\langle 1-\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}}\right),-\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}}\right)\right)\right\rangle, \tag{9}$$

is called complement of $\sin \mathfrak{B}$.

Theorem 2. *Let $\mathfrak{B}_1 = \langle \aleph^+_{\mathfrak{B}_1}, \aleph^-_{\mathfrak{B}_1}\rangle$ and $\mathfrak{B}_2 = \langle \aleph^+_{\mathfrak{B}_2}, \aleph^-_{\mathfrak{B}_2}\rangle$ be two BFNs and $\sigma > 0$, then*

(i) $\sigma(\sin \mathfrak{B}_1)^c = ((\sin \mathfrak{B}_1)^{\sigma})^c$

(ii) $((\sin \mathfrak{B}_1)^c)^{\sigma} = (\sigma \sin \mathfrak{B}_1)^c$

(iii) $(\sin \mathfrak{B}_1 \oplus \sin \mathfrak{B}_2)^c = (\sin \mathfrak{B}_1)^c \otimes (\sin \mathfrak{B}_2)^c$

(iv) $(\sin \mathfrak{B}_1 \otimes \sin \mathfrak{B}_2)^c = (\sin \mathfrak{B}_1)^c \oplus (\sin \mathfrak{B}_2)^c$

Proof. We substantiate (i) and (iv), and others can be substantiated similarly.

(i)

$$\begin{aligned}(\sin \mathfrak{B}_1)^c &= \left\langle 1-\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_1}\right),-\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_1}\right)\right)\right\rangle,\\ \sigma(\sin \mathfrak{B}_1)^c &= \left\langle 1-\left(\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_1}\right)\right)^{\sigma},-\left(-\left(-\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_1}\right)\right)\right)\right)^{\sigma}\right\rangle.\end{aligned} \tag{10}$$

Now, (iv)

$$(\sin \mathfrak{B}_1)^{\sigma} = \left\langle \left(\sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_1}\right)\right)^{\sigma}, -\left(1-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^{-}_{\mathfrak{B}_1}\right)\right)\right)^{\sigma}\right)\right\rangle,$$

$$\begin{aligned}((\sin \mathfrak{B}_1)^{\sigma})^{c} &= \left\langle 1-\left(\sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_1}\right)\right)^{\sigma}, -\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^{-}_{\mathfrak{B}_1}\right)\right)\right)^{\sigma}\right\rangle \\ &= \sigma(\sin \mathfrak{B}_1)^{c}.\end{aligned} \tag{11}$$

$$\begin{aligned}(\sin \mathfrak{B}_1 \otimes \sin \mathfrak{B}_2)^{c} &= \left\langle 1-\sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_1}\right)\sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_2}\right), -\sin\left(\frac{\pi}{2}\left(1+\aleph^{-}_{\mathfrak{B}_1}\right)\right)\sin\left(\frac{\pi}{2}\left(1+\aleph^{-}_{\mathfrak{B}_2}\right)\right)\right\rangle, \\ (\sin \mathfrak{B}_1)^{c} \oplus (\sin \mathfrak{B}_2)^{c} &= \left\langle 1-\sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_1}\right), -\sin\left(\frac{\pi}{2}\left(1+\aleph^{-}_{\mathfrak{B}_1}\right)\right)\right\rangle \oplus \left\langle 1-\sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_2}\right), -\sin\left(\frac{\pi}{2}\left(1+\aleph^{-}_{\mathfrak{B}_2}\right)\right)\right\rangle \\ &= \left\langle 1-\sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_1}\right)\sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_2}\right), -\left(-\sin\left(\frac{\pi}{2}\left(1+\aleph^{-}_{\mathfrak{B}_1}\right)\right)\right)\left(-\sin\left(\frac{\pi}{2}\left(1+\aleph^{-}_{\mathfrak{B}_2}\right)\right)\right)\right\rangle \\ &= (\sin \mathfrak{B}_1 \otimes \sin \mathfrak{B}_2)^{c}.\end{aligned} \tag{12}$$

□

Theorem 3. . *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two BFNs with $\mathfrak{B}_1 \prec \mathfrak{B}_2$, i.e., $\aleph^{+}_{\mathfrak{B}_1} \leq \aleph^{+}_{\mathfrak{B}_2}$ and $\aleph^{-}_{\mathfrak{B}_1} \geq \aleph^{-}_{\mathfrak{B}_2}$, then* $\sin \mathfrak{B}_1 \prec \sin \mathfrak{B}_2$.

Proof. Since sine is an increasing function on the interval $[0, (\pi/2)]$ so for $\aleph^{+}_{\mathfrak{B}_1} \leq \aleph^{+}_{\mathfrak{B}_2}$, we have $\sin((\pi/2)\aleph^{+}_{\mathfrak{B}_1}) \leq \sin((\pi/2)\aleph^{+}_{\mathfrak{B}_2})$. Likewise, for $\aleph^{-}_{\mathfrak{B}_1} \geq \aleph^{-}_{\mathfrak{B}_2}$, we obtain $1+\aleph^{-}_{\mathfrak{B}_1} \geq 1+\aleph^{-}_{\mathfrak{B}_2}$. This implicates that $\sin((\pi/2)(1+\aleph^{-}_{\mathfrak{B}_1})) \geq \sin((\pi/2)(1+\aleph^{-}_{\mathfrak{B}_2}))$ which further implicates that $\sin((\pi/2)(1+\aleph^{-}_{\mathfrak{B}_1}))-1 \geq \sin((\pi/2)(1+\aleph^{-}_{\mathfrak{B}_2}))-1$. Hence, by Definition 2 (part (vi)), we have $\sin \mathfrak{B}_1 = \langle \sin((\pi/2)\aleph^{+}_{\mathfrak{B}_1}), \sin((\pi/2)(1+\aleph^{-}_{\mathfrak{B}_1}))-1\rangle \prec \langle \sin((\pi/2)\aleph^{+}_{\mathfrak{B}_2}), \sin((\pi/2)(1+\aleph^{-}_{\mathfrak{B}_2}))-1\rangle = \sin \mathfrak{B}_2$. □

4. Bipolar Fuzzy Sine Trigonometric Averaging Aggregation Operators

In this section, some new averaging AOs have been proposed on the basis of STOLs of BFNs. These aggregation operators include (i) ST-BFWA operator, (ii) ST-BFOWA operator, and (iii) ST-BFHWA operator.

4.1. ST-BFWA Operator

Definition 9. . Let $\mathfrak{B}_i$, $i = 1, 2, \ldots, n$, be a compendium of BFNs and $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ be the weights of $\mathfrak{B}_i$, $i = 1, 2, \ldots, n$, with $\varphi_i > 0$ and $\sum_{i=1}^{n} \varphi_i = 1$. Then, ST-BFWA operator is described as

$$\text{ST}-\text{BFWA}(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n) = \varphi_1 \sin \mathfrak{B}_1 \oplus \varphi_2 \sin \mathfrak{B}_2 \oplus \cdots \oplus \varphi_n \sin \mathfrak{B}_n. \tag{13}$$

Theorem 4. . *Let $\mathfrak{B}_i = \langle \aleph^{+}_{\mathfrak{B}_i}, \aleph^{-}_{\mathfrak{B}_i}\rangle$ be n BFNs, then their cumulative value acquired by using (13) is again a BFN and is given by*

$$\text{ST}-\text{BFWA}(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n) = \left\langle 1-\prod_{i=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_i}\right)\right)^{\varphi_i}, -\prod_{i=1}^{n}\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^{-}_{\mathfrak{B}_i}\right)\right)-1\right)\right)^{\varphi_i}\right\rangle. \tag{14}$$

Proof. To prove the theorem, we employ mathematical induction on n. For $n = 2$, we have

$$\begin{aligned}
\mathrm{ST-BFWA}(\mathfrak{B}_1,\mathfrak{B}_2) &= \varphi_1 \sin \mathfrak{B}_1 \oplus \varphi_2 \sin \mathfrak{B}_2 \\
&= \left\langle 1-\left(1-\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_1}\right)\right)^{\varphi_1}, -\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_1}\right)\right)-1\right)\right)^{\varphi_1}\right\rangle \\
&\oplus \left\langle 1-\left(1-\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_2}\right)\right)^{\varphi_2}, -\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_2}\right)\right)-1\right)\right)^{\varphi_2}\right\rangle \\
&= \left\langle 1-\prod_{i=1}^{2}\left(1-\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_i}\right)\right)^{\varphi_i}, -\prod_{i=1}^{2}\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_i}\right)\right)-1\right)\right)^{\varphi_i}\right\rangle.
\end{aligned} \tag{15}$$

This shows that our assertion is correct for $n = 2$. Assume that the result holds true for $n = k$, i.e.,

$$\begin{aligned}
\mathrm{ST-BFWA}(\mathfrak{B}_1,\mathfrak{B}_2,\ldots,\mathfrak{B}_k) &= \varphi_1 \sin \mathfrak{B}_1 \oplus \varphi_2 \sin \mathfrak{B}_2 \oplus \cdots \oplus \varphi_k \sin \mathfrak{B}_k \\
&= \left\langle 1-\prod_{i=1}^{k}\left(1-\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_i}\right)\right)^{\varphi_i}, -\prod_{i=1}^{k}\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_i}\right)\right)-1\right)\right)^{\varphi_i}\right\rangle.
\end{aligned} \tag{16}$$

Now, for $n = k + 1$, we have

$$\begin{aligned}
ST-BFWA(\mathfrak{B}_1,\mathfrak{B}_2,\ldots,\mathfrak{B}_{k+1}) &= \varphi_1 \sin \mathfrak{B}_1 \oplus \varphi_2 \sin \mathfrak{B}_2 \oplus \cdots \oplus \varphi_k \sin \mathfrak{B}_k \oplus \varphi_{k+1} \sin \mathfrak{B}_{k+1} \\
&= \left\langle 1-\prod_{i=1}^{k}\left(1-\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_1}\right)\right)^{\varphi_i}, -\prod_{i=1}^{k}\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_1}\right)\right)-1\right)\right)^{\varphi_i}\right\rangle \\
&\oplus \left\langle 1-\left(1-\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_{k+1}}\right)\right)^{\varphi_{k+1}}, -\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_{k+1}}\right)\right)-1\right)\right)^{\varphi_{k+1}}\right\rangle \\
&= \left\langle 1-\prod_{i=1}^{k+1}\left(1-\sin\left(\frac{\pi}{2}\aleph^+_{\mathfrak{B}_i}\right)\right)^{\varphi_i}, -\prod_{i=1}^{k+1}\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph^-_{\mathfrak{B}_i}\right)\right)-1\right)\right)^{\varphi_i}\right\rangle.
\end{aligned} \tag{17}$$

Hence, the result holds $\forall n$. □

Example 1. . Let $\mathfrak{B}_1 = (0.41, -0.39)$, $\mathfrak{B}_2 = (0.66, -0.21)$, $\mathfrak{B}_3 = (0.59, -0.46)$, and $\mathfrak{B}_4 = (0.72, -0.56)$ be four BFNs and $\varphi = (0.23, 0.31, 0.27, 0.19)$ be the corresponding weight vector, then

$$\begin{aligned}
\prod_{i=1}^{4}\left(1-\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_i}^{+}\right)\right)^{\varphi_i} &= \left(1-\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_1}^{+}\right)\right)^{\varphi_1}\times\left(1-\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_2}^{+}\right)\right)^{\varphi_2}\times\left(1-\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_3}^{+}\right)\right)^{\varphi_3}\times\left(1-\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_4}^{+}\right)\right)^{\varphi_4}\\
&= \left(1-\sin\left(\frac{\pi}{2}(0.41)\right)\right)^{0.23}\times\left(1-\sin\left(\frac{\pi}{2}(0.66)\right)\right)^{0.31}\times\left(1-\sin\left(\frac{\pi}{2}(0.59)\right)\right)^{0.27}\times\left(1-\sin\left(\frac{\pi}{2}(0.72)\right)\right)^{0.19}\\
&= 0.1821,\\
\prod_{i=1}^{4}\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_i}^{-}\right)\right)-1\right)\right)^{\varphi_i} &= \left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_1}^{-}\right)\right)-1\right)\right)^{\varphi_1}\times\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_2}^{-}\right)\right)-1\right)\right)^{\varphi_2}\\
&\quad\times\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_3}^{-}\right)\right)-1\right)\right)^{\varphi_3}\times\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_4}^{-}\right)\right)-1\right)\right)^{\varphi_4}\\
&= \left(-\left(\sin\left(\frac{\pi}{2}(1-0.39)\right)-1\right)\right)^{0.23}\times\left(-\left(\sin\left(\frac{\pi}{2}(1-0.21)\right)-1\right)\right)^{0.31}\times\left(-\left(\sin\left(\frac{\pi}{2}(1-0.46)\right)-1\right)\right)^{0.27}\\
&\quad\times\left(-\left(\sin\left(\frac{\pi}{2}(1-0.56)\right)-1\right)\right)^{0.19}\\
&= 0.1550.
\end{aligned} \tag{18}$$

Now,

$$\begin{aligned}
\text{ST}-\text{BFWA}(\mathfrak{B}_1,\mathfrak{B}_2,\mathfrak{B}_3,\mathfrak{B}_4) &= \left\langle 1-\prod_{i=1}^{4}\left(1-\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_i}^{+}\right)\right)^{\varphi_i},\right.\\
&\quad\left.-\prod_{i=1}^{4}\left(-\left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_i}^{-}\right)\right)-1\right)\right)^{\varphi_i}\right\rangle\\
&= \langle 1-0.1821, -0.1550\rangle\\
&= \langle 0.8179, -0.1550\rangle.
\end{aligned} \tag{19}$$

Theorem 5. . *Let $\mathfrak{B}_i = \langle \aleph_{\mathfrak{B}_i}^{+}, \aleph_{\mathfrak{B}_i}^{-}\rangle$, $i = 1, 2, \ldots, n$, be a compendium of BFNs and $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ be the weight vector with $\varphi_i > 0$ and $\sum_{i=1}^{n}\varphi_i = 1$, then ST-BFWA operator holds the properties listed as follows:*

(i) *Idempotency. If all BFNs are equal, i.e., $\mathfrak{B}_i = \mathfrak{B} = \langle \aleph_{\mathfrak{B}}^{+}, \aleph_{\mathfrak{B}}^{-}\rangle$, then*

$$\text{ST}-\text{BFWA}(\mathfrak{B}_1,\mathfrak{B}_2,\ldots,\mathfrak{B}_n) = \sin\mathfrak{B}. \tag{20}$$

(ii) *Monotonicity. Let $\mathfrak{B}_i^{*} = \langle \aleph_{\mathfrak{B}_i^{*}}^{+}, \aleph_{\mathfrak{B}_i^{*}}^{-}\rangle$, $i = 1, 2, \ldots, n$, be another collection of BFNs such that $\mathfrak{B}_i \prec \mathfrak{B}_i^{*}$, $\forall i = 1, 2, \ldots, n$, then*

$$\begin{aligned}
&\text{ST}-\text{BFWA}(\mathfrak{B}_1,\mathfrak{B}_2,\ldots,\mathfrak{B}_n)\\
&\quad\preceq \text{ST}-\text{BFWA}(\mathfrak{B}_1^{*},\mathfrak{B}_2^{*},\ldots,\mathfrak{B}_n^{*}).
\end{aligned} \tag{21}$$

(iii) *Boundedness. Let $\mathfrak{B}\_ = \langle \min_i(\aleph_{\mathfrak{B}_i}^{+}), \max_i(\aleph_{\mathfrak{B}_i}^{-})\rangle$ and $\overline{\mathfrak{B}} = \langle \max_i(\aleph_{\mathfrak{B}_i}^{+}), \min_i(\aleph_{\mathfrak{B}_i}^{-})\rangle$, then*

$$\sin\mathfrak{B}\_ \prec \text{ST}-\text{BFWA}(\mathfrak{B}_1,\mathfrak{B}_2,\ldots,\mathfrak{B}_n) \prec \sin\overline{\mathfrak{B}}. \tag{22}$$

Proof

(i) Let $\mathfrak{B}_i = \mathfrak{B}\, \forall i = 1, 2, \ldots, n$. Then, by using (13), we have

$$\begin{aligned} ST - BFWA(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n) &= \left\langle 1 - \prod_{i=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_i}\right)\right)^{\varphi_i} - \prod_{i=1}^{n}\left(-\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\mathfrak{B}_i}\right)\right) - 1\right)\right)^{\varphi_i}\right\rangle \\ &= \left\langle 1 - \left(1 - \sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}}\right)\right)^{\sum_{i=1}^{n}\varphi_i} - \left(-\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\mathfrak{B}}\right)\right) - 1\right)\right)^{\sum_{i=1}^{n}\varphi_i}\right\rangle \\ &= \left\langle 1 - \left(1 - \sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}}\right)\right) - \left(-\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\mathfrak{B}}\right)\right) - 1\right)\right)\right\rangle \\ &= \left\langle \sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}}\right), \ \sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\mathfrak{B}}\right)\right) - 1\right\rangle \\ &= \sin \mathfrak{B}. \end{aligned} \tag{23}$$

(ii) Since $\mathfrak{B}_i \prec \mathfrak{B}^{*}_i$, this implies that $\aleph^{+}_{\mathfrak{B}_i} \leq \aleph^{+}_{\mathfrak{B}^{*}_i}$ and $\aleph^{-}_{\mathfrak{B}_i} \geq \aleph^{-}_{\mathfrak{B}^{*}_i}$, $\forall i = 1, 2, \ldots, n$. Suppose that $\mathrm{ST} - \mathrm{BFWA}(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n) = \langle \widetilde{\aleph}^{+}, \widetilde{\aleph}^{-}\rangle$ and $\mathrm{ST} - \mathrm{BFWA}(\mathfrak{B}^{*}_1, \mathfrak{B}^{*}_2, \ldots, \mathfrak{B}^{*}_n) = \langle \widetilde{\aleph}^{*^{+}}, \widetilde{\aleph}^{*^{-}}\rangle$. Due to the monotonicity of sine function, we get

$$\begin{aligned} &\sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_i}\right) \leq \sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}^{*}_i}\right), \\ &\Longrightarrow 1 - \sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_i}\right) \geq 1 - \sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}^{*}_i}\right), \\ &\Longrightarrow \left(1 - \sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_i}\right)\right)^{\varphi_i} \geq \left(1 - \sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}^{*}_i}\right)\right)^{\varphi_i}, \\ &\Longrightarrow \prod_{i=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_i}\right)\right)^{\varphi_i} \geq \prod_{i=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}^{*}_i}\right)\right)^{\varphi_i}, \\ &\Longrightarrow \widetilde{\aleph}^{+} = 1 - \prod_{i=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}_i}\right)\right)^{\varphi_i} \leq 1 - \prod_{i=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\aleph^{+}_{\mathfrak{B}^{*}_i}\right)\right)^{\varphi_i} = \widetilde{\aleph}^{*^{+}}. \end{aligned} \tag{24}$$

Similarly,

$$\begin{aligned} &\sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\mathfrak{B}_i}\right)\right) \geq \sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\mathfrak{B}^{*}_i}\right)\right), \\ &\Longrightarrow \sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\mathfrak{B}_i}\right)\right) - 1 \geq \sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\mathfrak{B}^{*}_i}\right)\right) - 1, \\ &\Longrightarrow \left(-\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\mathfrak{B}_i}\right)\right) - 1\right)\right)^{\varphi_i} \leq \left(-\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\mathfrak{B}^{*}_i}\right)\right) - 1\right)\right)^{\varphi_i}, \\ &\Longrightarrow \widetilde{\aleph}^{-} = -\prod_{i=1}^{n}\left(-\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\mathfrak{B}_i}\right)\right) - 1\right)\right)^{\varphi_i} \geq -\prod_{i=1}^{n}\left(-\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\mathfrak{B}^{*}_i}\right)\right) - 1\right)\right)^{\varphi_i} = \widetilde{\aleph}^{*^{-}}. \end{aligned} \tag{25}$$

Since $\widetilde{\aleph}^{+} \leq \widetilde{\aleph}^{*^{+}}$ and $\widetilde{\aleph}^{-} \geq \widetilde{\aleph}^{*^{-}}$, we have

$$\mathrm{ST-BFWA}\left(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\right) \leq \mathrm{ST-BFWA}\left(\mathfrak{B}_1^*, \mathfrak{B}_2^*, \ldots, \mathfrak{B}_n^*\right). \tag{26}$$

(iii) It is similar to the preceding proof, so we exclude it. □

4.2. ST-BFOWA Operator

Definition 10. Let $\mathfrak{B}_i$, $i = 1, 2, \ldots, n$, be a compendium of BFNs, then ST-BFOWA operator is explicated as

$$\begin{aligned}&\mathrm{ST-BFOWA}\left(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\right)\\&\quad = \varphi_1 \sin \mathfrak{B}_{\eta(1)} \oplus \varphi_2 \sin \mathfrak{B}_{\eta(2)} \oplus \cdots \oplus \varphi_n \sin \mathfrak{B}_{\eta(n)},\end{aligned} \tag{27}$$

where $(\eta(1), \eta(2), \ldots, \eta(n))$ is an arrangement of $(1, 2, \ldots, n)$ with the constraint that $\mathfrak{B}_{\eta(i-1)} \geq \mathfrak{B}_{\eta(i)}\ \forall i = 2, 3, \ldots, n$. It is noteworthy that the weights φ_i with $\varphi_i > 0$ and $\sum_{i=1}^{n} \varphi_i = 1$ are associated with the ordered positions of BFNs $\mathfrak{B}_i$.

Theorem 6. *The cumulative value of n BFNs $\mathfrak{B}_i = \langle \aleph_{\mathfrak{B}_i}^{+}, \aleph_{\mathfrak{B}_i}^{-} \rangle$ acquired by utilizing ST-BFOWA operator is still a BFN and is given by*

$$\begin{aligned}&\mathrm{ST-BFOWA}\left(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\right)\\&\quad = \left\langle 1 - \prod_{i=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_{\eta(i)}}^{+}\right)\right)^{\varphi_i},\right.\\&\qquad \left. - \prod_{i=1}^{n}\left(-\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph_{\mathfrak{B}_{\eta(i)}}^{-}\right)\right) - 1\right)\right)^{\varphi_i}\right\rangle.\end{aligned} \tag{28}$$

Proof. Straightforward. □

Theorem 7. *Let $\mathfrak{B}_i = \langle \aleph_{\mathfrak{B}_i}^{+}, \aleph_{\mathfrak{B}_i}^{-} \rangle$, $i = 1, 2, \ldots, n$, be a compendium of BFNs and $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ be the weight vector with $\varphi_i > 0$ and $\sum_{i=1}^{n} \varphi_i = 1$, then ST-BFOWA operator satisfies the following properties:*

(i) Idempotency. If $\mathfrak{B}_i = \mathfrak{B} = \langle \aleph_{\mathfrak{B}}^{+}, \aleph_{\mathfrak{B}}^{-} \rangle$, $\forall i = 1, 2, \ldots, n$, then

$$\mathrm{ST-BFOWA}\left(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\right) = \sin \mathfrak{B}. \tag{29}$$

(ii) Monotonicity. Let $\mathfrak{B}_i^* = \langle \aleph_{\mathfrak{B}_i^*}^{+}, \aleph_{\mathfrak{B}_i^*}^{-} \rangle$, $i = 1, 2, \ldots, n$, be another collection of BFNs such that $\mathfrak{B}_i \prec \mathfrak{B}_i^*$, $\forall i = 1, 2, \ldots, n$, then

$$\mathrm{ST-BFOWA}\left(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\right) \leq \mathrm{ST-BFOWA}\left(\mathfrak{B}_1^*, \mathfrak{B}_2^*, \ldots, \mathfrak{B}_n^*\right). \tag{30}$$

(iii) Boundedness. If $\underline{\mathfrak{B}} = \langle \min_i(\aleph_{\mathfrak{B}_i}^{+}), \max_i(\aleph_{\mathfrak{B}_i}^{-}) \rangle$ and $\overline{\mathfrak{B}} = \langle \max_i(\aleph_{\mathfrak{B}_i}^{+}), \min_i(\aleph_{\mathfrak{B}_i}^{-}) \rangle$, then

$$\sin \underline{\mathfrak{B}} \preceq \mathrm{ST-BFOWA}\left(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\right) \preceq \sin \overline{\mathfrak{B}}. \tag{31}$$

Proof. It is obvious. □

4.3. ST-BFHWA Operator

Definition 11. Let $\mathfrak{B}_i$, $i = 1, 2, \ldots, n$, be a compendium of BFNs and $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ be the weight vector of $\mathfrak{B}_i$ with $\varphi_i > 0$ and $\sum_{i=1}^{n} \varphi_i = 1$. A ST-BFHWA operator with associated weight vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ with $\gamma_i > 0$ and $\sum_{i=1}^{n} \gamma_i = 1$ can be described as

$$\begin{aligned}&\mathrm{ST-BFHWA}\left(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\right)\\&\quad = \gamma_1 \sin \dot{\mathfrak{B}}_{\eta(1)} \oplus \gamma_2 \sin \dot{\mathfrak{B}}_{\eta(2)} \oplus \cdots \oplus \gamma_n \sin \dot{\mathfrak{B}}_{\eta(n)},\end{aligned} \tag{32}$$

where $\dot{\mathfrak{B}}_i = n\varphi_i \mathfrak{B}_i$ and $(\eta(1), \eta(2), \ldots, \eta(n))$ is an arrangement of $(1, 2, \ldots, n)$ with the stipulation that $\dot{\mathfrak{B}}_{\eta(i-1)} \geq \dot{\mathfrak{B}}_{\eta(i)}\ \forall i = 2, 3, \ldots, n$.

Theorem 8. *The cumulative value of n BFNs $\mathfrak{B}_i = \langle \aleph_{\mathfrak{B}_i}^{+}, \aleph_{\mathfrak{B}_i}^{-} \rangle$ acquired by utilizing ST-BFHWA operator is still a BFN and is given by*

$$\begin{aligned}&\mathrm{ST-BFHWA}\left(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\right)\\&\quad = \left\langle 1 - \prod_{i=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\aleph_{\dot{\mathfrak{B}}_{\eta(i)}}^{+}\right)\right)^{\gamma_i},\right.\\&\qquad \left. - \prod_{i=1}^{n}\left(-\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph_{\dot{\mathfrak{B}}_{\eta(i)}}^{-}\right)\right) - 1\right)\right)^{\gamma_i}\right\rangle.\end{aligned} \tag{33}$$

Proof. Straightforward. □

5. Bipolar Fuzzy Sine Trigonometric Geometric Aggregation Operators

In this section, we propose geometric aggregation operators including (i) ST-BFWG operator, (ii) ST-BFOWG operator, and (iii) ST-BFHWG operator.

5.1. ST-BFWG Operator

Definition 12. For n BFNs $\mathfrak{B}_i$, a ST-BFWG operator is explicated as

$$\begin{aligned}\mathrm{ST-BFWG}\left(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\right) &= \left(\sin \mathfrak{B}_1\right)^{\varphi_1}\\&\quad \otimes \left(\sin \mathfrak{B}_2\right)^{\varphi_2} \otimes \cdots \otimes \left(\sin \mathfrak{B}_n\right)^{\varphi_n}.\end{aligned} \tag{34}$$

Theorem 9. *Let $\mathfrak{B}_i = \langle \aleph_{\mathfrak{B}_i}^{+}, \aleph_{\mathfrak{B}_i}^{-} \rangle$ be n BFNs, then their cumulative value obtained by utilizing ST-BFWG operator is expressed as*

$$\mathrm{ST-BFWG}(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n) = \left\langle \prod_{i=1}^{n}\left(\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_i}^{+}\right)\right)^{\varphi_i}, -\left(1-\prod_{i=1}^{n}\left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_i}^{-}\right)\right)\right)^{\varphi_i}\right)\right\rangle. \quad (35)$$

Proof. It is obvious. □

Example 2. Let $\mathfrak{B}_1 = (0.41, -0.39)$, $\mathfrak{B}_2 = (0.66, -0.21)$, $\mathfrak{B}_3 = (0.59, -0.46)$, and $\mathfrak{B}_4 = (0.72, -0.56)$ be four BFNs and $\varphi = (0.23, 0.31, 0.27, 0.19)$ be the corresponding weight vector, then

$$\begin{aligned}
\prod_{i=1}^{4}\left(\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_i}^{+}\right)\right)^{\varphi_i} &= \left(\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_1}^{+}\right)\right)^{\varphi_1}\times\left(\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_2}^{+}\right)\right)^{\varphi_2}\times\left(\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_3}^{+}\right)\right)^{\varphi_3}\times\left(\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_4}^{+}\right)\right)^{\varphi_4} \\
&= \left(\sin\left(\frac{\pi}{2}(0.41)\right)\right)^{0.23}\times\left(\sin\left(\frac{\pi}{2}(0.66)\right)\right)^{0.31}\times\left(\sin\left(\frac{\pi}{2}(0.59)\right)\right)^{0.27}\times\left(\sin\left(\frac{\pi}{2}(0.72)\right)\right)^{0.19} \\
&= 0.7841 \\
\prod_{i=1}^{4}\left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_i}^{-}\right)\right)\right)^{\varphi_i} &= \left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_1}^{-}\right)\right)\right)^{\varphi_1}\times\left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_2}^{-}\right)\right)\right)^{\varphi_2}\times\left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_3}^{-}\right)\right)\right)^{\varphi_3}\times\left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_4}^{-}\right)\right)\right)^{\varphi_4} \\
&= \left(\sin\left(\frac{\pi}{2}(1-0.39)\right)\right)^{0.23}\times\left(\left(\sin\left(\frac{\pi}{2}(1-0.21)\right)\right)^{0.31} \\
&\quad\times\left(\sin\left(\frac{\pi}{2}(1-0.46)\right)\right)^{0.27}\times\left(\left(\sin\left(\frac{\pi}{2}(1-0.56)\right)\right)^{0.19} = 0.7973.
\end{aligned} \quad (36)$$

Now,

$$\begin{aligned}
\mathrm{ST-BFWG}(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4) &= \left\langle \prod_{i=1}^{4}\left(\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_i}^{+}\right)\right)^{\varphi_i}, -\left(1-\prod_{i=1}^{4}\left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_i}^{-}\right)\right)\right)^{\varphi_i}\right)\right\rangle \\
&= \langle 0.7841, -(1-0.7973)\rangle \\
&= \langle 0.7841, -0.2027\rangle.
\end{aligned} \quad (37)$$

The properties mentioned in Theorem 5, namely, idempotency, monotonicity, and boundedness, also apply to the ST-BFWG operator.

5.2. ST-BFOWG Operator

Definition 13. A ST-BFOWG operator is defined as

$$\mathrm{ST-BFOWG}(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n) = \left(\sin\,\mathfrak{B}_{\eta(1)}\right)^{\varphi_1}\otimes\left(\sin\,\mathfrak{B}_{\eta(2)}\right)^{\varphi_2}\otimes\cdots\otimes\left(\sin\,\mathfrak{B}_{\eta(n)}\right)^{\varphi_n}, \quad (38)$$

where $(\eta(1), \eta(2), \ldots, \eta(n))$ is an arrangement of $(1, 2, \ldots, n)$ such that $\mathfrak{B}_{\eta(i-1)} \geq \mathfrak{B}_{\eta(i)}\ \forall i = 2, 3, \ldots, n$.

Theorem 10. *The cumulative value of n BFNs $\mathfrak{B}_i = \langle \aleph_{\mathfrak{B}_i}^{+}, \aleph_{\mathfrak{B}_i}^{-}\rangle$ acquired by utilizing ST-BFOWG operator is expressed as*

$$\mathrm{ST-BFOWG}(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n) = \left\langle \prod_{i=1}^{n}\left(\sin\left(\frac{\pi}{2}\aleph_{\mathfrak{B}_{\eta(i)}}^{+}\right)\right)^{\varphi_i}, -\left(1-\prod_{i=1}^{n}\left(\sin\left(\frac{\pi}{2}\left(1+\aleph_{\mathfrak{B}_{\eta(i)}}^{-}\right)\right)\right)^{\varphi_i}\right)\right\rangle. \quad (39)$$

Proof. It is obvious.

Idempotency, monotonicity, and boundedness are all satisfied by the ST-BFOWG operator. □

5.3. ST-BFHWG Operator

Definition 14. A ST-BFHWG operator with associated weight vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ with $\gamma_i > 0$ and $\sum_{i=1}^{n} \gamma_i = 1$ can be described as

$$\mathrm{ST-BFHWG}(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n) = \left(\sin \dot{\mathfrak{B}}_{\eta(1)}\right)^{\gamma_1} \otimes \left(\sin \dot{\mathfrak{B}}_{\eta(2)}\right)^{\gamma_2} \otimes \cdots \otimes \left(\sin \dot{\mathfrak{B}}_{\eta(n)}\right)^{\gamma_n}. \quad (40)$$

where $\dot{\mathfrak{B}}_i = (\mathfrak{B}_i)^{n\varphi_i}$ and $(\eta(1), \eta(2), \ldots, \eta(n))$ is an arrangement of $(1, 2, \ldots, n)$ such that $\dot{\mathfrak{B}}_{\eta(i-1)} \geq \dot{\mathfrak{B}}_{\eta(i)}\ \forall i = 2, 3, \ldots, n$.

Theorem 11. *The cumulative value of n BFNs $\mathfrak{B}_i = \langle \aleph^+_{\mathfrak{B}_i}, \aleph^-_{\mathfrak{B}_i} \rangle$ obtained by utilizing ST-BFHWG operator is expressed as*

$$\mathrm{ST-BFHWG}(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n) = \left\langle \prod_{i=1}^{n} \left(\sin\left(\frac{\pi}{2}\aleph^+_{\dot{\mathfrak{B}}_{\eta(i)}}\right)\right)^{\gamma_i}, -\left(1 - \prod_{i=1}^{n}\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph^-_{\dot{\mathfrak{B}}_{\eta(i)}}\right)\right)\right)^{\gamma_i}\right)\right\rangle. \quad (41)$$

Proof. Straightforward. □

6. Bipolar Fuzzy SIR Method

An MCGDM problem is made up of a finite number of alternatives, a set of criteria, and a set of decision makers. To solve an MCGDM problem, the most apposite alternative must be chosen among those available. Let $\mathbb{A} = \{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_m\}$ be a set of alternatives and $\mathbb{C} = \{\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n\}$ be a set of criteria. Suppose that the set of decision makers is $\mathbb{E} = \{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_l\}$ and their weight vector is $\vartheta = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_l\}$ where all the weights are BFNs. Construct the individual decision matrices $\mathbb{H}_k = (h^k_{ij})_{m\times n}$ in which h^k_{ij} denotes the evaluation information of the alternative $\hat{a}_i$ w.r.t the criterion $\hat{c}_j$ provided by the decision maker $\hat{e}_k$ in the form of BFNs. Assume that $\varphi = (\varphi^k_j)_{l\times n}$ is the criterion weight matrix in which φ^k_j is the weight of criterion $\hat{c}_j$ assigned by the decision maker $\hat{e}_k$ in the form of BFNs. In this section, we set up the BF-SIR technique to address this MCGDM problem. The steps in this technique are outlined as follows:

Step 1. Compute the relative propinquity coefficient of each ϑ_k, $k = 1, 2, \ldots, l$, by the equation

$$\delta_k = \frac{d(\vartheta_k, \underline{\vartheta})}{d(\vartheta_k, \underline{\vartheta}) + d(\vartheta_k, \overline{\vartheta})}. \quad (42)$$

where $\underline{\vartheta} = \langle \min_k(\aleph^+_{\vartheta_k}), \max_k(\aleph^-_{\vartheta_k})\rangle$ and $\overline{\vartheta} = \langle \max_k(\aleph^+_{\vartheta_k}), \min_k(\aleph^-_{\vartheta_k})\rangle$. It is evident that if $\vartheta_k \longrightarrow \underline{\vartheta}$, then $\delta_k \longrightarrow 0$, and if $\vartheta_k \longrightarrow \overline{\vartheta}$, then $\delta_k \longrightarrow 1$.

Step 2. Normalize δ_k, $k = 1, 2, \ldots, l$, by the equation

$$\zeta_k = \frac{\delta_k}{\sum_{k=1}^{l} \delta_k}. \quad (43)$$

In this way, we get a normalized vector $\zeta = \{\zeta_1, \zeta_2, \ldots, \zeta_l\}$ of relative propinquity coefficients.

Step 3. Acquire the accumulated bipolar fuzzy decision matrix and the criterion weight vector by utilizing ST-BFWA operator as follows:

$$\tilde{h}_{ij} = \mathrm{ST-BFWA}_{\zeta_k}\left(h^1_{ij}, h^2_{ij}, \ldots, h^l_{ij}\right) = \left\langle 1 - \prod_{k=1}^{l}\left(1 - \sin\left(\frac{\pi}{2}\aleph^+_{h^k_{ij}}\right)\right)^{\zeta_k}, -\prod_{k=1}^{l}\left(-\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph^-_{h^k_{ij}}\right)\right) - 1\right)\right)^{\zeta_k}\right\rangle. \quad (44)$$

$$\tilde{\varphi}_j = \mathrm{ST-BFWA}_{\zeta_k}\left(\varphi^1_j, \varphi^2_j, \ldots, \varphi^l_j\right) = \left\langle 1 - \prod_{k=1}^{l}\left(1 - \sin\left(\frac{\pi}{2}\aleph^+_{\varphi^k_j}\right)\right)^{\zeta_k}, -\prod_{k=1}^{l}\left(-\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph^-_{\varphi^k_j}\right)\right) - 1\right)\right)^{\zeta_k}\right\rangle. \quad (45)$$

Step 4. Obtain the relative efficiency function f_{ij} as follows:

$$f_{ij} = \frac{d\left(\tilde{h}_{ij}, \underline{\tilde{h}}\right)}{d\left(\tilde{h}_{ij}, \underline{\tilde{h}}\right) + d\left(\tilde{h}_{ij}, \overline{\tilde{h}}\right)}, \quad (46)$$

where $\underline{\tilde{h}} = \langle \min_i(\aleph^{\pm}_{\tilde{h}_{ij}}), \max_i(\aleph^{-}_{\tilde{h}_{ij}})\rangle$ and $\overline{\tilde{h}} = \langle \max_i(\aleph^{\pm}_{\tilde{h}_{ij}}), \min_i(\aleph^{-}_{\tilde{h}_{ij}})\rangle$. It follows that if $\tilde{h}_{ij} \longrightarrow \underline{\tilde{h}}$, then $f_{ij} \longrightarrow 0$, and if $\tilde{h}_{ij} \longrightarrow \overline{\tilde{h}}$, then $f_{ij} \longrightarrow 1$.

Step 5. Compute the preference intensity $PI_j(\widehat{a}_i, \widehat{a}_t)$ $(i, t = 1, 2, \ldots, m, i \neq t)$ which provides the degree of preference of alternative $\widehat{a}_i$ over alternative $\widehat{a}_t$ w.r.t the criterion $\widehat{c}_j$ and it can be defined as follows:

$$\mathrm{PI}_j(\widehat{a}_i, \widehat{a}_t) = \lambda_j\left(f_{ij} - f_{tj}\right), \tag{47}$$

where λ_j is a threshold function given by

$$\lambda_j(x) = \begin{cases} 0.01, & x > 0, \\ 0, & x \leq 0. \end{cases} \tag{48}$$

Step 6. Construct the superiority matrix $S = (S_{ij})_{m\times n}$ and inferiority matrix $I = (I_{ij})_{m\times n}$ by utilizing the following equations:

$$\begin{aligned}(S - \text{index})\, S_{ij} &= \sum_{t=1}^{m} \mathrm{PI}_j(\widehat{a}_i, \widehat{a}_t) \\ &= \sum_{t=1}^{m} \lambda_j\left(f_{ij} - f_{tj}\right),\end{aligned} \tag{49}$$

$$\begin{aligned}(I - \text{index})\, I_{ij} &= \sum_{t=1}^{m} \mathrm{PI}_j(\widehat{a}_t, \widehat{a}_i) \\ &= \sum_{t=1}^{m} \lambda_j\left(f_{tj} - f_{ij}\right).\end{aligned} \tag{50}$$

Step 7. Calculate the superiority flow (S-flow) and inferiority flow (I-flow) as follows:

$$\begin{aligned}\lambda^{>}(\widehat{a}_i) &= \mathrm{ST} - \mathrm{BFWA}_{S_{ij}}(\widetilde{\varphi}_1, \widetilde{\varphi}_2, \ldots, \widetilde{\varphi}_n) \\ &= \langle 1 - \prod_{j=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\aleph^{+}_{\varphi_j}\right)\right)^{S_{ij}}, \\ &\quad - \prod_{j=1}^{n}\left(-\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\varphi_j}\right)\right) - 1\right)\right)^{S_{ij}}\rangle.\end{aligned} \tag{51}$$

$$\begin{aligned}\lambda^{<}(\widehat{a}_i) &= \mathrm{ST} - \mathrm{BFWA}_{I_{ij}}(\widetilde{\varphi}_1, \widetilde{\varphi}_2, \ldots, \widetilde{\varphi}_n) \\ &= \left\langle 1 - \prod_{j=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\aleph^{+}_{\varphi_j}\right)\right)^{I_{ij}}, \right. \\ &\quad - \prod_{j=1}^{n}\left(-\left(\sin\left(\frac{\pi}{2}\left(1 + \aleph^{-}_{\varphi_j}\right)\right) - 1\right)\right)^{I_{ij}}\rangle.\end{aligned} \tag{52}$$

Step 8. Compute the score functions of $\lambda^{>}(\widehat{a}_i)$ and $\lambda^{<}(\widehat{a}_i)$, $i = 1, 2, \ldots, m$, by using (2).

Step 9. Apply the superiority ranking laws (SR-laws) and inferiority ranking laws (IR-laws) as follows:

SR-Law 1. If $\lambda^{>}(\widehat{a}_i) \succ \lambda^{>}(\widehat{a}_t)$ and $\lambda^{<}(\widehat{a}_i) \prec \lambda^{<}(\widehat{a}_t)$, then $\widehat{a}_i \succ \widehat{a}_t$

SR-Law 2. If $\lambda^{>}(\widehat{a}_i) \succ \lambda^{>}(\widehat{a}_t)$ and $\lambda^{<}(\widehat{a}_i) = \lambda^{<}(\widehat{a}_t)$, then $\widehat{a}_i \succ \widehat{a}_t$

SR-Law 3. If $\lambda^{>}(\widehat{a}_i) = \lambda^{>}(\widehat{a}_t)$ and $\lambda^{<}(\widehat{a}_i) \prec \lambda^{<}(\widehat{a}_t)$, then $\widehat{a}_i \succ \widehat{a}_t$

IR-Law 1. If $\lambda^{>}(\widehat{a}_i) \prec \lambda^{>}(\widehat{a}_t)$ and $\lambda^{<}(\widehat{a}_i) \succ \lambda^{<}(\widehat{a}_t)$, then $\widehat{a}_i \prec \widehat{a}_t$

IR-Law 2. If $\lambda^{>}(\widehat{a}_i) \prec \lambda^{>}(\widehat{a}_t)$ and $\lambda^{<}(\widehat{a}_i) = \lambda^{<}(\widehat{a}_t)$, then $\widehat{a}_i \prec \widehat{a}_t$

IR-Law 3. If $\lambda^{>}(\widehat{a}_i) = \lambda^{>}(\widehat{a}_t)$ and $\lambda^{<}(\widehat{a}_i) \succ \lambda^{<}(\widehat{a}_t)$, then $\widehat{a}_i \prec \widehat{a}_t$

Step 10. Integrate the SR-laws with the IR-laws to determine the optimal alternative.

6.1. Case Study. The process of seeking medical treatment supply chain from a foreign country is known as medical tourism. In the past, patients from underdeveloped parts of the world used to travel to Europe and America for diagnostics and treatment. However, in recent years, this trend has flipped as medical tourism, in which individuals from developed countries travel to developing countries for medical treatment. There are a variety of reasons why people from developed countries prefer less developed countries. The low cost of treatment is the main factor. Healthcare prices are dependent on a country's per capita gross domestic product (GDP), which serves as a procurator for income levels. The low cost of offshore medical care is indebted to low medicolegal and administrative costs. Second, people seek medical guidance from outside the country for the procedures for which health insurance is not provided, such as cosmetic surgery, fertility therapy, dental reconstruction, gender reassignment surgeries, and so on. Patients in countries where access to healthcare is regulated by the government, such as Canada and the United Kingdom, desire to avoid the delays that come with extensive waiting lists. Another factor could be the lack of availability of a certain operation in their home country, such as stem cell therapy, which may be inaccessible or limited in developed countries but widely available in emerging markets. Some patients believe that their privacy will be better protected in a remote location. Another motive for offshore treatment is the recreational aspect. As a result of these factors, medical tourism is expanding globally. Medical tourism was worth 54.4 billion US dollars in 2020, and by 2027, it was expected to be worth more than 200 billion US dollars (https://www.statista.com/statistics/1084720/medical-tourism-market-size-worldwide/). Figure 1 depicts the gradual expansion of the medical tourism industry from 2016 to 2020, with projections for 2027.

The medical tourism market in Asia-Pacific has a lot of room for expansion. Due to economic development, this region is expected to see rapid market expansion. Singapore, Japan, India, Thailand, and the Philippines are among the most popular medical tourism destinations. Singapore and India are well-known for their cardiac and orthopaedic surgery. Thailand is well-known for its dental procedures and gender reassignment surgeries. Japan has one of the best oncology treatment facilities in the world. The Philippines is famous for its cosmetic surgery, dentistry, and fertility treatment. The Medical Tourism Index (MTI) evaluates a country's suitability as a medical tourism destination by taking into account its overall environment, healthcare costs, tourist attractions, and the standard of medical amenities and services. The higher the MTI, the better the destination.

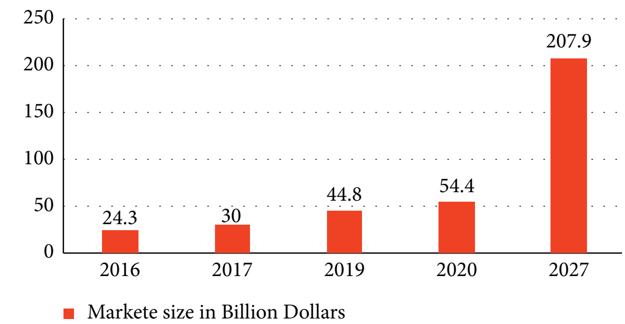

FIGURE 1: Medical tourism market size.

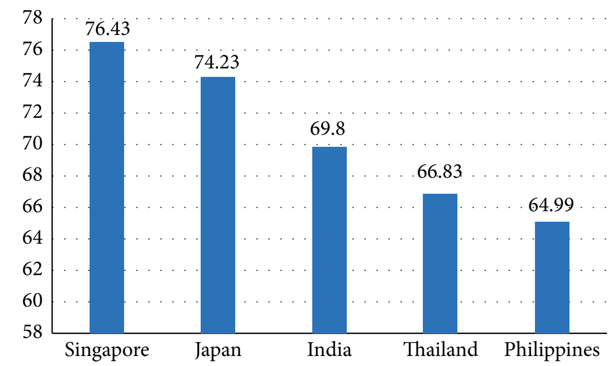

FIGURE 2: Medical tourism index (2020-2021).

Figure 2 depicts the medical tourism index for the aforementioned Asian countries (https://www.medicaltourism.com/mti/2020-2021/region/asia).

Medical tourism is seen as an unexplored sector in Pakistan that might be transformed into a lucrative potential if the government addresses some critical issues such as security, brain drain, and service quality. According to Pakistani medical professionals, Pakistan has "huge potential" to become a competitive medical tourism hub in Asia. In what follows, we will use the BF-SIR method to determine the best medical tourism destination in Pakistan.

6.2. Numerical Illustration. Suppose that ministry of health of a Pakistan needs to assess some true potential of medical tourism supply chain. For this purpose, the ministry hires three decision makers $\widehat{e}_1, \widehat{e}_2$, and $\widehat{e}_3$ and assigns them weights which are given in Table 1. Let $\mathbb{A} = \{\widehat{a}_1, \widehat{a}_2, \widehat{a}_3, \widehat{a}_4\}$ be the set of alternatives where $\widehat{a}_1 =$ Islamabad, $\widehat{a}_2 =$ Karachi, $\widehat{a}_3 =$ Lahore, and $\widehat{a}_4 =$ Peshawar. Table 2 lists the criteria for determining the best alternative. The weights of criteria $\widehat{c}_j$ given by the decision makers $\widehat{e}_k$ are given in Table 3. The decision makers evaluate each alternative $\widehat{a}_i$ w.r.t each criterion $\widehat{c}_j$ and give their assessment via BFNs. Three decision matrices are given in Tables 4–6.

Step 1. The relative propinquity coefficients δ_k $(k = 1, 2, 3)$ are computed using (42) as follows:

$$\delta = \{0.12, 0.72, 0.52\}. \tag{53}$$

Step 2. The normalized vector is obtained using (43)as follows:

$$\zeta = \{0.0882, 0.5294, 0.3824\}. \tag{54}$$

Step 3. The accumulated bipolar fuzzy decision matrix is acquired using (44), which is given in Table 7. Equation (45) is used to determine accumulated weights of criteria, which are as follows:

$$\begin{aligned} \widetilde{\varphi}_1 &= \langle 0.9357, -0.1744 \rangle, \\ \widetilde{\varphi}_2 &= \langle 0.9423, -0.0799 \rangle, \\ \widetilde{\varphi}_3 &= \langle 0.9183, -0.0632 \rangle. \end{aligned} \tag{55}$$

TABLE 1: Bipolar fuzzy weights of decision makers.

Decision makers	Weights
$\widehat{e}_1$	$\vartheta_1 = \langle 0.79, -0.28 \rangle$
$\widehat{e}_2$	$\vartheta_2 = \langle 0.85, -0.37 \rangle$
$\widehat{e}_3$	$\vartheta_3 = \langle 0.92, -0.25 \rangle$

Step 4. The relative efficiency function is calculated using (46)as follows:

$$f_{ij} = \begin{pmatrix} 0.3427 & 0.2234 & 0.4693 \\ 0.4320 & 0.4241 & 0.5963 \\ 1 & 0.8834 & 0.4820 \\ 0.7463 & 0.2807 & 0.0246 \end{pmatrix}. \tag{56}$$

Step 5, 6. The superiority and inferiority matrices are constructed using (49) and (50) as follows:

$$\begin{aligned} S &= \begin{pmatrix} 0 & 0 & 0.01 \\ 0.01 & 0.02 & 0.03 \\ 0.03 & 0.03 & 0.02 \\ 0.02 & 0.01 & 0 \end{pmatrix}, \\ I &= \begin{pmatrix} 0.03 & 0.03 & 0.02 \\ 0.02 & 0.01 & 0 \\ 0 & 0 & 0.01 \\ 0.01 & 0.02 & 0.03 \end{pmatrix}. \end{aligned} \tag{57}$$

Step 7, 8. The S-flow and I-flow are computed using (51) and (52), which are given in Table 8.

Step 9. Applying SR-laws to Table 8 yields the following ranking order:

$$\widehat{a}_3 \succ \widehat{a}_2 \succ \widehat{a}_4 \succ \widehat{a}_1. \tag{58}$$

Applying IR-laws to Table 8 yields the following ranking order:

$$\widehat{a}_3 \succ \widehat{a}_2 \succ \widehat{a}_4 \succ \widehat{a}_1. \tag{59}$$

Step 10. According to both SR and IR-laws, $\widehat{a}_3$ is the best alternative.

Table 2: Criteria for the selection of the best medical tourism destination.

Criteria	Description
(i) Service quality $(\hat{c}_1)$	This includes modern equipment, qualified staff, and variety of medical treatments.
(ii) Security $(\hat{c}_2)$	This includes life and fiscal security of the tourists.
(iii) Infrastructure facilities $(\hat{c}_3)$	This includes transportation and maintenance of hospitals and equipment.

Table 3: Bipolar fuzzy weights of criteria.

	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$
$\hat{e}_1$	$\langle 0.73, -0.26\rangle$	$\langle 0.65, -0.36\rangle$	$\langle 0.81, -0.29\rangle$
$\hat{e}_2$	$\langle 0.82, -0.38\rangle$	$\langle 0.76, -0.19\rangle$	$\langle 0.78, -0.27\rangle$
$\hat{e}_3$	$\langle 0.69, -0.42\rangle$	$\langle 0.83, -0.36\rangle$	$\langle 0.65, -0.17\rangle$

Table 4: BF decision matrix $\mathbb{H}_1$.

	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$
$\hat{a}_1$	$\langle 0.82, -0.21\rangle$	$\langle 0.92, -0.23\rangle$	$\langle 0.78, -0.26\rangle$
$\hat{a}_2$	$\langle 0.76, -0.19\rangle$	$\langle 0.52, -0.41\rangle$	$\langle 0.66, -0.24\rangle$
$\hat{a}_3$	$\langle 0.86, -0.17\rangle$	$\langle 0.87, -0.18\rangle$	$\langle 0.79, -0.34\rangle$
$\hat{a}_4$	$\langle 0.67, -0.31\rangle$	$\langle 0.42, -0.38\rangle$	$\langle 0.76, -0.12\rangle$

Table 5: BF decision matrix $\mathbb{H}_2$.

	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$
$\hat{a}_1$	$\langle 0.69, -0.33\rangle$	$\langle 0.82, -0.19\rangle$	$\langle 0.89, -0.17\rangle$
$\hat{a}_2$	$\langle 0.82, -0.21\rangle$	$\langle 0.66, -0.29\rangle$	$\langle 0.77, -0.32\rangle$
$\hat{a}_3$	$\langle 0.91, -0.36\rangle$	$\langle 0.79, -0.26\rangle$	$\langle 0.87, -0.29\rangle$
$\hat{a}_4$	$\langle 0.79, -0.29\rangle$	$\langle 0.56, -0.21\rangle$	$\langle 0.82, -0.26\rangle$

Table 6: BF decision matrix $\mathbb{H}_3$.

	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$
$\hat{a}_1$	$\langle 0.76, -0.22\rangle$	$\langle 0.88, -0.13\rangle$	$\langle 0.96, -0.41\rangle$
$\hat{a}_2$	$\langle 0.89, -0.16\rangle$	$\langle 0.62, -0.24\rangle$	$\langle 0.69, -0.56\rangle$
$\hat{a}_3$	$\langle 0.73, -0.29\rangle$	$\langle 0.92, -0.26\rangle$	$\langle 0.71, -0.31\rangle$
$\hat{a}_4$	$\langle 0.81, -0.32\rangle$	$\langle 0.63, -0.46\rangle$	$\langle 0.56, -0.21\rangle$

Table 7: Accumulated BF decision matrix.

	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$
$\hat{a}_1$	$\langle 0.9128, -0.0895\rangle$	$\langle 0.9747, -0.0342\rangle$	$\langle 0.9922, -0.0740\rangle$
$\hat{a}_2$	$\langle 0.9713, -0.0431\rangle$	$\langle 0.8396, -0.0938\rangle$	$\langle 0.9135, -0.1775\rangle$
$\hat{a}_3$	$\langle 0.9751, -0.1162\rangle$	$\langle 0.9762, -0.0771\rangle$	$\langle 0.9585, -0.1102\rangle$
$\hat{a}_4$	$\langle 0.9459, -0.1111\rangle$	$\langle 0.7886, -0.1074\rangle$	$\langle 0.9183, -0.0611\rangle$

Table 8: The BF-SIR flows.

Alternatives	$\lambda^{>}(\hat{a}_i)$	$Scr(\lambda^{>}(\hat{a}_i))$	$\lambda^{<}(\hat{a}_i)$	$Scr(\lambda^{<}(\hat{a}_i))$
$\hat{a}_1$	$\langle 0.0469, -0.9482\rangle$	0.0494	$\langle 0.3425, -0.7045\rangle$	0.319
$\hat{a}_2$	$\langle 0.2641, -0.7488\rangle$	0.2576	$\langle 0.1483, -0.8920\rangle$	0.1282
$\hat{a}_3$	$\langle 0.3425, -0.7045\rangle$	0.319	$\langle 0.0469, -0.9482\rangle$	0.0494
$\hat{a}_4$	$\langle 0.1483, -0.8920\rangle$	0.1282	$\langle 0.2641, -0.7488\rangle$	0.2576

Table 9: Comparative analysis of the suggested and existing methodologies.

Methods	Ranking of alternatives	Optimal alternative
Algorithm (Jana and Pal [17])	$\hat{a}_3 \succ \hat{a}_2 \succ \hat{a}_1 \succ \hat{a}_4$	$\hat{a}_3$
Algorithm (Wei et al. [20])	$\hat{a}_3 \succ \hat{a}_2 \succ \hat{a}_4 \succ \hat{a}_1$	$\hat{a}_3$
Algorithm (Hamid et al. [23])	$\hat{a}_3 \succ \hat{a}_4 \succ \hat{a}_2 \succ \hat{a}_1$	$\hat{a}_3$
Algorithm (Akram et al. [24])	$\hat{a}_3 \succ \hat{a}_2 \succ \hat{a}_4 \succ \hat{a}_1$	$\hat{a}_3$
Algorithm (Peng and Yang [60])	$\hat{a}_3 \succ \hat{a}_2 \succ \hat{a}_4 \succ \hat{a}_1$	$\hat{a}_3$
Algorithm (Zhang and Xu [36])	$\hat{a}_3 \succ \hat{a}_1 \succ \hat{a}_2 \succ \hat{a}_4$	$\hat{a}_3$
Algorithm (Proposed)	$\hat{a}_3 \succ \hat{a}_2 \succ \hat{a}_4 \succ \hat{a}_1$	$\hat{a}_3$

6.3. Comparative and Sensitivity Analysis. In this section, we compare our suggested BF-SIR technique to some existing approaches in order to evaluate its validity. Table 9 summarizes the comparative study of various techniques. It can be seen from Table 9 that our suggested approach is compatible with the existing techniques.

7. Conclusion

In daily life, we encounter many situations where we must deal with uncertainty as well as bipolarity when making a decision. Taking this into consideration, the bipolar fuzzy set (BFS) is a sophisticated model that can handle bipolarity and fuzziness at the same time. The main contributions of this manuscript are listed as follows:

(1) Since the sine trigonometric function is periodic and symmetric about the origin, it can accommodate the decision expert's choices during object appraisal. Therefore, we proposed sine trigonometric operational laws (STOLs) for BFSs. We explored some of their properties as well.

(2) Based on BF-STOLs, we suggested the following averaging AOs: bipolar fuzzy sine trigonometric weighted averaging (BF-STWA) operator; bipolar fuzzy sine trigonometric ordered weighted averaging (BF-STOWA) operator; and bipolar fuzzy sine trigonometric hybrid weighted averaging (BF-STHWA) operator.

(3) Based on BF-STOLs, we suggested the following geometric AOs: bipolar fuzzy sine trigonometric weighted geometric (BF-STWG) operator; bipolar fuzzy sine trigonometric ordered weighted geometric (BF-STOWG) operator; and bipolar fuzzy sine trigonometric hybrid weighted geometric (BF-STHWG) operator.

(4) We investigated some important characteristics of these operators, such as idempotency, monotonicity, and boundedness.

(5) We established an extended superiority and inferiority ranking (SIR) method to handle MCGDM problems in a bipolar fuzzy environment. We applied this technique to the selection of the best medical tourism supply chain.

(6) We compared our suggested model with some existing ones to exhibit its validity and efficiency.

In the future, we will develop bipolar fuzzy sine trigonometric power aggregation operators, bipolar fuzzy sine trigonometric Hamy mean operators, bipolar fuzzy sine trigonometric Bonferroni mean operators, bipolar fuzzy sine trigonometric prioritized operators, and bipolar fuzzy sine trigonometric Dombi operators.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[3] R. R. Yager, "Pythagorean fuzzy subsets," in *Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)*, pp. 57–61, Edmonton, AB, Canada, June 2013.

[4] R. R. Yager, "Pythagorean membership grades in m decision making," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 4, pp. 958–965, 2014.

[5] R. R. Yager, "Generalized orthopair fuzzy sets," *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 5, pp. 1222–1230, 2017.

[6] V. Torra, "Hesitant fuzzy sets," *International Journal of Intelligent Systems*, vol. 25, 2010.

[7] F. Smarandache, *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic*, American Research Press, Santa Fe , Mexico , USA, 1999.

[8] H. Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman, "Single valued neutrosophic sets," *Multispace and Multistructure*, vol. 4, pp. 410–413, 2010.

[9] B. C. Cuong, "Picture fuzzy sets," *Journal of Computer Science and Cybernetics*, vol. 30, no. 4, pp. 409–420, 2014.

[10] T. Mahmood, K. Ullah, Q. Khan, and N. Jan, "An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets," *Neural Computing & Applications*, vol. 31, no. 11, pp. 7041–7053, 2019.

[11] F. K. Gündoğdu and C. Kahraman, "Spherical fuzzy sets and spherical fuzzy TOPSIS method," *Journal of Intelligent and Fuzzy Systems*, vol. 36, no. 1, pp. 337–352, 2019.

[12] S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, and T. Mahmood, "Spherical fuzzy sets and their applications in

multi-attribute decision making problems," *Journal of Intelligent and Fuzzy Systems*, vol. 36, no. 3, pp. 2829–2844, 2019.

[13] W. R. Zhang, "Bipolar fuzzy sets and relations, A computational framework for cognitive modeling and multiagent decision analysis," in *Proceedings of the IEEE Conference Fuzzy Information Processing Society Biannual Conference*, pp. 305–309, San Antonio, TX, USA, December 1994.

[14] W. R. Zhang, "Bipolar fuzzy sets," in *Proceedings of the IEEE Irnenational Conference on Fuzzy Systems*, pp. 835–840, Anchorage, AK, USA, 1998.

[15] K. M. Lee, "Bipolar-valued fuzzy sets and their basic operations," in *Proceedings of the Internatinal Conference*, pp. 307–317, Bangkok, Thailand, 2000.

[16] S. T. Tehrim and M. Riaz, "An interval-valued bipolar fuzzy linguistic VIKOR method using connection numbers of SPA Theory and its application to decision support system," *Journal of Intelligent and Fuzzy Systems*, vol. 39, no. 3, pp. 3931–3948, 2020.

[17] C. Jana and M. Pal, "Extended bipolar fuzzy EDAS approach for multi-criteria group decision-making process," *Computational and Applied Mathematics*, vol. 40, no. 1, p. 9, 2021.

[18] R. Liu, L.-X. Hou, H.-C. Liu, and W. Lin, "Occupational health and safety risk assessment using an integrated SWARA-MABAC model under bipolar fuzzy environment," *Computational and Applied Mathematics*, vol. 39, no. 4, p. 276, 2020.

[19] C. Jana, M. Pal, and J.-q. Wang, "Bipolar fuzzy Dombi aggregation operators and its application in multiple-attribute decision-making process," *Journal of Ambient Intelligence and Humanized Computing*, vol. 10, no. 9, pp. 3533–3549, 2019.

[20] G. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Bipolar fuzzy Hamacher aggregation operators in multiple attribute decision making," *International Journal of Fuzzy Systems*, vol. 20, no. 1, pp. 1–12, 2017.

[21] Y. Han, Z. Lu, Z. Du, Q. Luo, and S. Chen, "A YinYang bipolar fuzzy cognitive TOPSIS method to bipolar disorder diagnosis," *Computer Methods and Programs in Biomedicine*, vol. 158, pp. 1–10, 2018.

[22] G. Wei, C. Wei, and H. Gao, "Multiple attribute decision making with interval-valued bipolar fuzzy information and their application to emerging technology commercialization evaluation," *IEEE Access*, vol. 6, Article ID 60955, 2018.

[23] M. T. Hamid, M. Riaz, and K. Naeem, "A study on weighted aggregation operators for q-rung orthopair m-polar fuzzy set with utility to multistage decision analysis," *International Journal of Intelligent Systems*, 2022.

[24] M. Akram, U. Amjad, J. C. R. Alcantud, and G. Santos-Garcia, "Complex fermatean fuzzy N-soft sets: a new hybrid model with applications," *Journal of Ambient Intelligence and Humanized Computing*, vol. 4, 2022.

[25] Z. S. Zeshui Xu, "Intuitionistic fuzzy aggregation operators," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 6, pp. 1179–1187, 2007.

[26] Z. Xu and R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *International Journal of General Systems*, vol. 35, no. 4, pp. 417–433, 2006.

[27] H. Garg, "Some series of intuitionistic fuzzy interactive averaging aggregation operators," *SpringerPlus*, vol. 5, no. 1, p. 999, 2016.

[28] J.-Y. Huang, "Intuitionistic fuzzy Hamacher aggregation operators and their application to multiple attribute decision making," *Journal of Intelligent and Fuzzy Systems*, vol. 27, no. 1, pp. 505–513, 2014.

[29] X. Gou and Z. Xu, "Exponential operations for intuitionistic fuzzy numbers and interval numbers in multi-attribute decision making," *Fuzzy Optimization and Decision Making*, vol. 16, no. 2, pp. 183–204, 2017.

[30] Z. Li and F. Wei, "The logarithmic operational laws of intuitionistic fuzzy sets and intuitionistic fuzzy numbers," *Journal of Intelligent and Fuzzy Systems*, vol. 33, no. 6, pp. 3241–3253, 2017.

[31] X. Peng, J. Dai, and H. Garg, "Exponential operation and aggregation operator for q-rung orthopair fuzzy set and their decision-making method with a new score function," *International Journal of Intelligent Systems*, vol. 33, no. 11, pp. 2255–2282, 2018.

[32] S. Khan, S. Abdullah, L. Abdullah, and S. Ashraf, "Logarithmic aggregation operators of picture fuzzy numbers for multi-attribute decision making problems," *Mathematics*, vol. 7, no. 7, p. 608, 2019.

[33] S. Abdullah, S. Khan, M. Qiyas, and R. Chinram, "A novel approach based on sine trigonometric picture fuzzy aggregation operators and their application in decision support system," *Journal of Mathematics*, vol. 2021, Article ID 8819517, 19 pages, 2021.

[34] O. Kabani, "Pakistan as a medical tourism destination. Just wishful thinking?" *Zdrowie Publiczne i Zarzadzanie*, vol. 13, no. 1, pp. 109–114, 2015.

[35] F. Muzaffar and I. Hussain, "Medical tourism: are we ready to take the challenge?" *Journal of Pakistan Association of Dermatologists*, vol. 17, pp. 215–218, 2007.

[36] X. Zhang and Z. Xu, "Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets," *International Journal of Intelligent Systems*, vol. 29, no. 12, pp. 1061–1078, 2014.

[37] T. Mahmood, Z. Ali, M. Aslam, and R. Chinram, "Spherical fuzzy soft rough average aggregation operators and their applications to multi-criteria decision making," *IEEE Access*, vol. 10, Article ID 126748, 2022.

[38] M. Ihsan, M. Saeed, A. U. Rahman, and F. Smarandache, "Multi-attribute Decision Support Model Based on Bijective Hypersoft Expert Set," *Punjab University Journal of Mathmatics*, vol. 54, 2022.

[39] F. Karaaslan and F. Karamaz, "Hesitant fuzzy parameterized hesitant fuzzy soft sets and their applications in decision-making," *International Journal of Computer Mathematics*, pp. 1–22, 2021.

[40] J. C. R. Alcantud, "The relationship between fuzzy soft and soft topologies," *International Journal of Fuzzy Systems*, 2022.

[41] X. Liu, F. Feng, Q. Wang, R. R. Yager, H. Fujita, and J. C. R. Alcantud, "Mining temporal association rules with temporal soft sets," *Journal of Mathematics*, vol. 2021, Article ID 7303720, 17 pages, 2021.

[42] M. Riaz, D. Pamucar, A. Habib, and M. Riaz, "A new TOPSIS approach using cosine similarity measures and cubic bipolar fuzzy information for sustainable plastic recycling process," *Mathematical Problems in Engineering*, vol. 2021, Article ID 4309544, 18 pages, 2021.

[43] Z. Zararsız and M. Riaz, "Bipolar fuzzy metric spaces with application," *Computational and Applied Mathematics*, vol. 41, no. 1, pp. 49–19, 2022.

[44] M. Riaz, M. Riaz, N. Jamil, and Z. Zararsiz, "Distance and similarity measures for bipolar fuzzy soft sets with application to pharmaceutical logistics and supply chain management," *Journal of Intelligent and Fuzzy Systems*, vol. 42, no. 4, pp. 3169–3188, 2022.

[45] R. E. Gergin, I. Peker, and A. C. G. Kisa, "Supplier Selection by Integrated IFDEMATEL-IFTOPSIS Method: A Case Study of Automotive Supply Industry," *Decision Making: Applications in Management and Engineering*, 2021.

[46] C. Karamasa, E. Demir, S. Memis, and S. Korucuk, "Weighting the Factors Affecting Logistics Outsourcing," *Infinite Study*, vol. 4, 2020.

[47] Z. Ali, T. Mahmood, K. Ullah, and Q. Khan, "Einstein geometric aggregation operators using a novel complex interval-valued pythagorean fuzzy setting with application in green supplier chain management," *Reports in Mechanical Engineering*, vol. 2, no. 1, pp. 105–134, 2021.

[48] L. J. Muhammad, I. Badi, A. A. Haruna, and I. A. Mohammed, "Selecting the best municipal solid waste management techniques in Nigeria using multi criteria decision making techniques," *Reports in Mechanical Engineering*, vol. 2, no. 1, pp. 180–189, 2021.

[49] T. K. Biswas, M. C. Das, and M. C. Das, "Selection of the barriers of supply chain management in Indian manufacturing sectors due to COVID-19 impacts," *Operational Research in Engineering Sciences: Theory and Applications*, vol. 3, no. 3, pp. 1–12, 2020.

[50] V. Milovanović, A. Aleksić, V. Sokolović, and M. Milenkov, "Uncertainty modeling using 929 intuitionistic fuzzy numbers," *Vojnotehnicki glasnik*, vol. 69, no. 4, pp. 905–929, 2021.

[51] H. Garg, "Sine trigonometric operational laws and its based Pythagorean fuzzy aggregation operators for group decision-making process," *Artificial Intelligence Review*, vol. 54, no. 6, pp. 4421–4447, 2021.

[52] T. Mahmood, U. U. Rehman, J. Ahmmad, and G. Santos-García, "Bipolar complex fuzzy h aggregation operators and their applications in multi-attribute decision making," *Mathematics*, vol. 10, no. 1, p. 23, 2021.

[53] M. Palanikumar, K. Arulmozhi, and C. Jana, "Multiple attribute decision-making approach for Pythagorean neutrosophic normal interval-valued fuzzy aggregation operators," *Computational and Applied Mathematics*, vol. 41, no. 3, pp. 90–27, 2022.

[54] C. Jana, G. Muhiuddin, M. Pal, and D. Al-Kadi, "Intuitionistic fuzzy dombi hybrid decision-making method and their applications to enterprise financial performance evaluation," *Mathematical Problems in Engineering*, vol. 2021, Article ID 3218133, 14 pages, 2021.

[55] C. Jana, G. Muhiuddin, and M. Pal, "Multi-criteria decision making approach based on SVTrN Dombi aggregation functions," *Artificial Intelligence Review*, vol. 54, no. 5, pp. 3685–3723, 2021.

[56] D. Ajay, J. Aldring, G. Rajchakit, P. Hammachukiattikul, and N. Boonsatit, "Sine trigonometry operational laws for complex neutrosophic sets and their aggregation operators in material selection," *Computer Modeling in Engineering and Sciences*, vol. 130, no. 2, pp. 1033–1076, 2022.

[57] M. Qiyas, S. Abdullah, S. Khan, and M. Naeem, "Multi-attribute group decision making based on sine trigonometric spherical fuzzy aggregation operators," *Granular Computing*, vol. 7, no. 1, pp. 141–162, 2022.

[58] X. Xu, "The SIR method: a superiority and inferiority ranking method for multiple criteria decision making," *European Journal of Operational Research*, vol. 131, no. 3, pp. 587–602, 2001.

[59] J. Chai and J. N. K. Liu, "A Novel Multicriteria Group Decision Making Approach with Intuitionistic Fuzzy Sir Method," in *Proceedings of the World Automation Congress*, pp. 1–6, Kobe, Japan, September 2010.

[60] X. Peng and Y. Yang, "Some results for pythagorean fuzzy sets," *International Journal of Intelligent Systems*, vol. 30, no. 11, pp. 1133–1160, 2015.

[61] H. Zhu, J. Zhao, and H. Li, "q-ROF-SIR methods and their applications to multiple attribute decision making," *International Journal of Machine Learning and Cybernetics*, vol. 13, no. 3, pp. 595–607, 2021.

[62] Z. Gul, *Some bipolar fuuzy aggregations operators and their applications in multicriteria group decision making*, M. Phil Thesis, 2005.

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Research Article

Fully Bipolar Single-Valued Neutrosophic Transportation Problems

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Transportation problem (TP) has its uses in real life because it has versatile applications. Real-life problems are often uncertain due to which it is difficult to find the accurate cost. The fuzzy set and intuitionistic fuzzy set are useful for handling the uncertainty, but these also have some limitations. For that reason, in this study, we worked on another set of values called bipolar single-valued neutrosophic set (BSNS) which is the generalization of crisp sets, fuzzy sets, and intuitionistic fuzzy sets to handle the uncertain, unpredictable, and insufficient information in real-life problems. In this study, we develop a new technique for solving transportation problems based on bipolar single-valued neutrosophic sets having nonnegative triangular bipolar single-valued neutrosophic numbers (TBSNNs). A score function is used to transform bipolar single-valued neutrosophic numbers (BSNNs) into crisp numbers. We compare our proposed model with fuzzy transportation and intuitionistic fuzzy transportation models and proved that bipolar single-valued neutrosophic transportation model is more admirable than the existing models. Furthermore, we apply the proposed technique to fully solve the bipolar single-valued neutrosophic transportation (FBSNT) model.

1. Introduction

Nowadays, there is great competition among organizations to find better methods to create and give better services to their customers. They need a cost-effective method to send the products to their customers. This is a more challenging task. The models for transportation provide an effective framework to encounter this challenge. They guarantee the effective movement of raw materials and find products.

1.1. Motivation. In the mathematical programming problem, the linear programming problem is very well-known and it has a wide scope to cover various fields. Among these, transportation problem is the commonly used field. The TP has its own importance in the field of logistics, supply chain management, supplier selection problem, etc. The TPs have great importance in many real-life applications. It works to maintain the supply from source to destination. It is generally considered that transversal costs of supply and demand are expressed in terms of crisp numbers. These values are often not precise. As a result, various researchers have been working on different TPs in fuzzy [1–3] and intuitionistic fuzzy [4, 5] environments, respectively. We proposed a new technique to solve TPs in the bipolar single-valued neutrosophic environment, which is a more generalized form.

1.2. Literature Review. Zadeh [6, 7] proposed the concept of fuzzy sets and fuzzy numbers to reduce uncertainty and incomplete information. Atanassove [8] gave the idea of intuitionistic fuzzy sets (IFSs), which is a generalization of fuzzy sets. In an IFS, we consider the problem in both angles' positive side and negative side to handle uncertainty. In an IFS, truth membership and falsity membership are independent, while indeterminacy-membership depends on their sum. Smarandache [9] proposed the notion of neutrosophic set theory. Smarandache [9] and Wang et al. [10] defined single-valued neutrosophic sets (SNSs), which are an extension of FSs and IFSs. In an SNS all membership functions truth, falsity, and indeterminacy are independent.

Deli et al. [11] suggested the idea of bipolar single-valued neutrosophic set, which is the generalization of single-valued neutrosophic sets [10].

The TPs are related to the transportation of raw material from different sources to different destinations in such a way that the total transportation cost is minimized. Hitchcock [12] was the first to develop a basic transportation problem. The transportation problem can be elaborated as a standard linear programming problem. This can be solved by the simplex method. It was found that the simplex method when applied to the transportation problem becomes more effective when evaluating the simplex method information. Basirzadeh [1] proposed a method to solve fuzzy transportation problems. Ladji et al. [13] proposed a two-step method for solving fuzzy transportation problems where all of the parameters are represented by triangular fuzzy numbers i.e., two interval transportation problems. Pratihar et al. [14] have modified the classical vogel's approximation method for solving the fuzzy transportation problem. They also worked on the interval type 2 fuzzy set and used it in a fuzzy transportation problem to represent the transportation cost, demand, and supply. Cam et al. [15] worked on the formulation of a linear programming model for the vehicle routing problem to minimize idle time. In fuzzy linear programming problems (FLPPs), many scholars have contributed. By using multiobjective function, Zimmerman [16] gave a technique to solve LP problem. After that, to solve the transportation problems, Zimmerman's fuzzy linear programming has developed into several fuzzy optimization methods. In a fuzzy environment, Bellman and Zadeh [6] introduced the different concepts of decision making. Lotfi et al. [17] considered FFLP problems in which all parameters and variables are triangular fuzzy numbers. Allahviranloo et al. [18] solved FFLP problem by using a kind of defuzzification approach. Behera et al. [19] suggested two new methods to solve fuzzy linear programming (FLP) problems. They solved two types of problems with two different methods. Kaur and Kumar [20] gave an introduction to FLP problems. Kaur and Kumar [21] suggested an approach to find the exact fuzzy optimal solution to FFLP problems by using unrestricted fuzzy variables. Kaur and Kumar [22] proposed a new method for finding the fuzzy optimal solution to fuzzy transportation problems in which the transportation cost are represented by generalized fuzzy numbers. Najafi and Edalatpanah [23] suggested a better technique to solve the FFLP problem than Kumar et al. [21].

Intuitionistic fuzzy linear programming problem is an extension of the fuzzy linear programming problem. Many researchers have worked on different methods to solve LP problems in an intuitionistic fuzzy environment by using intuitionistic fuzzy numbers (IFNs) and LR-type IFNs. Singh and Yadav [4, 5] suggested two different techniques to solve an intuitionistic fuzzy transportation problem (IFTPs) by using triangular intuitionistic fuzzy numbers (TIFNs). Abhishekh and Nishad [24] suggested a novel ranking function for finding an optimal solution to fully LR-intuitionistic fuzzy transportation problem. In an intuitionistic fuzzy environment, Edalatpanah [25] designed a model of data envelopment analysis with triangular intuitionistic fuzzy numbers (TIFNs) and established a strategy to solve it. Kabiraj et al. [26] solved IFLP problems by using a method based on a method suggested by Zimmerman [16]. Malathi and Umadevi [27] worked on IFLP problems in an intuitionistic fuzzy environment. Pythagorean fuzzy linear programming is an extension of intuitionistic fuzzy linear programming. Akram et al. [28, 29] suggested a technique to solve Pythagorean fuzzy linear programming problems by using Pythagorean fuzzy numbers and LR-type Pythagorean fuzzy numbers. Akram et al. [30] used two different techniques to solve Pythagorean fuzzy linear programming problems having mixed constraints.

In daily life routine, we meet a variety of situations depending on multiple factors like uncertainty in judgements. Often it becomes difficult to get relevant data for cost parameters. The data of this type cannot always be represented by random variables obtained from the probability distribution. These data may be represented by bipolar single-valued neutrosophic numbers. So, a bipolar single-valued neutrosophic method to make the decision is needed. Abdel-Basset et al. [31] suggested a technique to solve the fully neutrosophic linear programming (FNLP) problems. Hussian et al. [32] suggested a linear programming model based on neutrosophic environment. Bera and Mahapatra [33] developed the Big-M simplex method to solve the neutrosophic linear programming (NLP) problem. Das and Chakraborty [34] considered a pentagonal NLP problem to solve it. Das and Dash [35] solved NLP problems with mixed constraints. Edalatpanah [36] presented a direct algorithm to solve the linear programming problems. Khalifa et al. [37] solved the NLP problem with single-valued trapezoidal neutrosophic numbers. Ahmed [38] suggested a technique to solve LR-type single-valued neutrosophic linear programming problems by using unrestricted LR-type single-valued neutrosophic numbers. He proposed the ranking function to transform LR-type single-valued neutrosophic problems into crisp problems. Deli et al. [11] gave the idea of bipolar single-valued neutrosophic set. Akram et al. [39] suggested a technique to solve LR-bipolar fuzzy linear system. Mehmood et al. [40, 41] defined LR-type bipolar fuzzy numbers and their arithmetic operations. They also introduced the ranking for LR-type bipolar fuzzy numbers and solved numerical examples. Ahmed et al. [42] suggested a technique to solve bipolar single-valued neutrosophic linear programming problems in which all the coefficients, variables, and right-hand side are presented by bipolar single-valued neutrosophic numbers. Kumar [43] presented the cut of single-valued pentagonal neutrosophic numbers and also introduced the arithmetic operation of single-valued pentagonal neutrosophic numbers. By using two different objective functions, Singh et al. [44] formulated the journey of a vaccine from its manufacture to its delivery using bilevel transportation problems in a neutrosophic environment. Veeranani et al. [45] solved the multiobjective fractional transportation problem by using the neutrosophic goal programming approach [46–48].

1.3. Our Contribution. Considering all the available data, there are no methods in literature for TPs under the bipolar single-valued neutrosophic environment. So, there is a need to introduce a technique for BSNTPs. As a result of our facts, there are no optimization models available for TPs under bipolar single-valued neutrosophic environment. This has urged us to develop a new technique to solve TP with the bipolar single-valued neutrosophic environment, which is solved for the first time with the proposed technique. Bipolar single-valued neutrosophic set theory is a well-known technique to deal with uncertainty in the optimization problem. This study has been categorized as follows: In Section 2, basic concepts of BSNS, BSNN, TBSNNs, and their arithmetic operations are discussed. In Section 3, methodology for solving FBSNT problems are explained. In Section 4, a mathematical transportation model is solved. In Section 5, comparative analysis is discussed. In Section 6, the advantages of the proposed method are discussed. In Section 7, the limitations of the proposed method are given, and the conclusion is given in Section 8.

2. Preliminaries

Definition 1 (see [11]). Let Y be a nonempty set. A bipolar single-valued neutrosophic set (BSNS) $\widetilde{\lambda}$ in Y is an object having the form

$$\widetilde{\lambda} = \left\{ \prec y, T^{p}_{\widetilde{\lambda}}(y), I^{p}_{\widetilde{\lambda}}(y), F^{p}_{\widetilde{\lambda}}(y), T^{n}_{\widetilde{\lambda}}(y), I^{n}_{\widetilde{\lambda}}(y), F^{n}_{\widetilde{\lambda}}(y) \succ : y \in Y \right\}, \tag{1}$$

where $T^{p}_{\widetilde{\lambda}}(y), I^{p}_{\widetilde{\lambda}}(y), F^{p}_{\widetilde{\lambda}}(y): Y \longrightarrow [0,1]$ and $T^{n}_{\widetilde{\lambda}}(y), I^{n}_{\widetilde{\lambda}}(y), F^{n}_{\widetilde{\lambda}}(y): Y \longrightarrow [-1,0]$. The positive membership degree $T^{p}_{\widetilde{\lambda}}(y), I^{p}_{\widetilde{\lambda}}(y), F^{p}_{\widetilde{\lambda}}(y)$ denotes the truth membership, indeterminate membership, and falsity membership of an element $y \in Y$ corresponding to a bipolar neutrosophic set $\widetilde{\lambda}$ similarly negative membership degree $T^{n}_{\widetilde{\lambda}}(y), I^{n}_{\widetilde{\lambda}}(y), F^{n}_{\widetilde{\lambda}}(y)$ denotes the truth membership, indeterminate membership, and falsity membership of an element $y \in Y$ to some implicit counter-property corresponding to a bipolar neutrosophic set $\widetilde{\lambda}$.

Definition 2 (see [33]). A BSNN on $\mathbb{R}$ is a BSN set such that

$$\begin{aligned}\widetilde{\lambda} = \prec &\left([\mu_1, \nu_1, \pi_1, \theta_1]; \chi_p \right), \left([\mu_2, \nu_2, \pi_2, \theta_2]; \beta_p \right), \left([\mu_3, \nu_3, \pi_3, \theta_3]; \zeta_p \right), \left([\gamma_1, \eta_1, \iota_1, \kappa_1]; \alpha_n \right),\\ &\left([\gamma_2, \eta_2, \iota_2, \kappa_2]; \varphi_n \right), \left([\gamma_3, \eta_3, \iota_3, \kappa_3]; \nu_n \right) \succ\end{aligned} \tag{2}$$

where $\chi_p, \beta_p, \zeta_p \in [0,1]$ and $\alpha_n, \varphi_n, \nu_n \in [-1,0] \subset \mathbb{R}$

The truth membership values are given as

$$T^{p}_{\widetilde{\lambda}}(y) = \begin{cases} S^{L}_{T}(y), & \mu_1 \leq y \leq \nu_1,\\ \chi_p, & \nu_1 \leq y \leq \pi_1,\\ S^{R}_{T}(y), & \pi_1 \leq y \leq \theta_1,\\ 0, & \text{otherwise,} \end{cases} \quad T^{n}_{\widetilde{\lambda}}(y) = \begin{cases} U^{L}_{T}(y), & \gamma_1 \leq y \leq \eta_1,\\ \alpha_n, & \eta_1 \leq y \leq \iota_1,\\ U^{R}_{T}(y), & \iota_1 \leq y \leq \kappa_1,\\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

$S^{L}_{T}(y)$ and $U^{R}_{T}(y)$ are continuous and increasing functions satisfying the following conditions: $S^{L}_{T}(\mu_1) = 0, S^{L}_{T}(\nu_1) = \chi_p$, and $U^{R}_{T}(\gamma_1) = \alpha_n, U^{R}_{T}(\eta_1) = 0$, while $S^{R}_{T}(y)$ and $U^{L}_{T}(v)$ are continuous and decreasing functions and satisfying the following condition:

$$S^{R}_{T}(\pi_1) = \chi_p, S^{R}_{T}(\theta_1) = 0, U^{L}_{T}(\iota_1) = 0, U^{L}_{T}(\kappa_1) = \alpha_n, \tag{4}$$

where $\chi_p \in [0,1], \alpha_n \in [-1,0]$.

The indeterminacy-membership functions are given as

$$I^{p}_{\widetilde{\lambda}}(y) = \begin{cases} S^{L}_{I}(y), & \mu_2 \leq y \leq \nu_2,\\ \beta_p, & \nu_2 \leq y \leq \pi_2,\\ S^{R}_{I}(y), & \pi_2 \leq y \leq \theta_2,\\ 1, & \text{otherwise,} \end{cases} \quad I^{n}_{\widetilde{\lambda}}(y) = \begin{cases} U^{L}_{I}(y), & \gamma_2 \leq y \leq \eta_2,\\ \varphi_n, & \eta_2 \leq y \leq \iota_2,\\ U^{R}_{I}(y), & \iota_2 \leq y \leq \kappa_2,\\ -1, & \text{otherwise.} \end{cases} \tag{5}$$

$S_I^R(y)$ and $U_I^L(y)$ are continuous and increasing functions satisfying the following conditions: $S_I^R(\pi_2) = \beta_p, S_I^R(\theta_2) = 1$, $U_I^L(\gamma_2) = -1, U_I^L(\eta_2) = \varphi_n$, while $S_I^L(y)$ and $U_I^R(y)$ are continuous and decreasing functions and are satisfying the following conditions:

$$S_I^L(\mu_2) = 1, S_I^L(\nu_2) = \beta_p, U_I^R(\iota_2) = \varphi_n, U_I^R(\kappa_2) = -1, \quad (6)$$

where $\beta_p \in [0,1], \varphi_n \in [-1,0]$.The falsity membership functions are given as

$$F_{\tilde{\lambda}}^{p}(y) = \begin{cases} S_F^L(y), & \mu_3 \le y \le \nu_3, \\ \zeta_p, & \nu_3 \le y \le \pi_3, \\ S_F^R(y), & \pi_3 \le y \le \theta_3, \\ 1, & \text{otherwise}, \end{cases} F_{\tilde{\lambda}}^{n}(y) = \begin{cases} U_F^L(y), & \gamma_3 \le y \le \eta_3, \\ \nu_n, & \eta_3 \le y \le \iota_3, \\ U_F^R(y), & \iota_3 \le y \le \kappa_3, \\ -1, & \text{otherwise}. \end{cases} \quad (7)$$

$S_F^R(y)$ and $U_F^L(y)$ are continuous and increasing functions satisfying the following conditions:

$$S_F^R(\pi_3) = \zeta_p, S_F^R(\theta_3) = 1, U_F^L(\gamma_3) = -1, U_F^L(\eta_3) = \nu_n, \quad (8)$$

$S_F^L(y)$ and $U_F^R(y)$ are continuous and decreasing functions and are satisfying the following conditions:

$$S_F^L(\mu_3) = 1, S_F^L(\nu_3) = \zeta_p, U_F^R(\iota_3) = \nu_n, U_F^R(\kappa_3) = -1, \quad (9)$$

where $\zeta_p \in [0,1], \nu_n \in [-1,0]$.

Based on [49], we define some basic definitions.

Definition 3. We define a TBSNN defined on $\mathbb{R}$

$$\tilde{\lambda} = \prec\left([\mu_i, \nu_i, \pi_i]; \chi_p, \beta_p, \zeta_p\right), \left([\gamma_i, \eta_i, \iota_i]; \alpha_n, \varphi_n, \nu_n\right)\succ, \quad (10)$$

is said to be nonnegative TBSNN if and only if $\mu_i \ge 0$ and $\gamma_i \ge 0$, where $i = 1,2,3$ such that $\mu_i \le \nu_i \le \pi_i$ similarly $\gamma_i \le \eta_i \le \iota_i$ also $\chi_p, \beta_p, \zeta_p \in [0,1]$ and $\alpha_n, \varphi_n, \nu_n \in [-1,0] \subset \mathbb{R}$.

Definition 4. We define a $T_r BNN$ defined on $\mathbb{R}$

$$\begin{aligned}\tilde{\lambda} = \prec&\left([\mu_i, \nu_i, \pi_i, \theta_i]; \chi_p, \beta_p, \zeta_p\right), \\ &\cdot \left([\gamma_i, \eta_i, \iota_i, \kappa_i]; \alpha_n, \varphi_n, \nu_n\right)\succ,\end{aligned} \quad (11)$$

is said to be positive $T_r BSNN$ if and only if $\mu_i \ge 0$ and $\gamma_i \ge 0$, where $i = 1,2,3$ and $\mu_i \le \nu_i \le \pi_i \le \theta_i$. Similarly, $\gamma_i \le \eta_i \le \iota_i \le \kappa_i$,also $\chi_p, \beta_p, \zeta_p \in [0,1]$ and $\alpha_n, \varphi_n, \nu_n \in [-1,0] \subset \mathbb{R}$.

Definition 5. A TBSNN

$$\tilde{\lambda}_1 = \prec\left([\mu_1, \nu_1, \pi_1]; \chi_p, \beta_p, \zeta_p\right), \left([\gamma_1, \eta_1, \iota_1]; \alpha_n, \varphi_n, \nu_n\right)\succ, \quad (12)$$

is said to be zero if $\mu_1 = 0, \nu_1 = 0, \pi_1 = 0, \gamma_1 = 0, \eta_1 = 0, \iota_1 = 0, \chi_p = 0, \beta_p = 0, \zeta_p = 0, \alpha_n = 0, \varphi_n = 0$, and $\nu_n = 0$.

Definition 6. Two TBSNNs

$$\begin{aligned}\tilde{\lambda}_1 &= \prec\left([\mu_1, \nu_1, \pi_1]; \chi_p, \beta_p, \zeta_p\right), \left([\gamma_1, \eta_1, \iota_1]; \alpha_n, \varphi_n, \nu_n\right)\succ, \\ \tilde{\lambda}_2 &= \prec\left([\mu_2, \nu_2, \pi_2]; \chi_p', \beta_p', \zeta_p'\right), \left([\gamma_2, \eta_2, \iota_2]; \alpha_n', \varphi_n', \nu_n'\right)\succ,\end{aligned} \quad (13)$$

are said to be equal if $\mu_1 = \mu_2, \nu_1 = \nu_2, \pi_1 = \pi_2, \gamma_1 = \gamma_2, \eta_1 = \eta_2, \iota_1 = \iota_2, \chi_p = \chi_p', \beta_p = \beta_p', \zeta_p = \zeta_p', \alpha_n = \alpha_n', \varphi_n = \varphi_n'$, and $\nu_n = \nu_n'$.

Definition 7. Based on [50], we define $T_r BSNN$ defined $\mathbb{R}$ denoted by

$$\begin{aligned}\tilde{\lambda}(Y) = \prec&\left([\mu_1, \nu_1, \pi_1, \theta_1]; \chi_p\right), \left([\mu_2, \nu_2, \pi_2, \theta_2]; \beta_p\right), \\ &\cdot \left([\mu_3, \nu_3, \pi_3, \theta_3]; \zeta_p\right), \left([\gamma_1, \eta_1, \iota_1, \kappa_1]; \alpha_n\right), \\ &\left([\gamma_2, \eta_2, \iota_2, \kappa_2]; \varphi_n\right), \left([\gamma_3, \eta_3, \iota_3, \kappa_3]; \nu_n\right)\succ,\end{aligned} \quad (14)$$

whose truth, indeterminacy, and falsity membership functions are presented by

$$T_{\tilde{\lambda}}^{p}(y) = \begin{cases} \dfrac{y - \mu_1}{\nu_1 - \mu_1}\chi_p, & \mu_1 \le y \le \nu_1, \\ \chi_p, & \nu_1 \le y \le \pi_1, \\ \dfrac{\theta_1 - y}{\theta_1 - \pi_1}\chi_p, & \pi_1 \le y \le \theta_1, \\ 0, & \text{otherwise}, \end{cases} T_{\tilde{\lambda}}^{n}(y) = \begin{cases} \dfrac{y - \gamma_1}{\eta_1 - \gamma_1}\alpha_n, & \gamma_1 \le y \le \eta_1, \\ \alpha_n, & \eta_1 \le y \le \iota_1, \\ \dfrac{\kappa_1 - y}{\kappa_1 - \iota_1}\alpha_n, & \iota_1 \le y \le \kappa_1, \\ 0, & \text{otherwise}. \end{cases}$$

$$I_{\tilde{\lambda}}^{p}(y)=\begin{cases}\dfrac{(\nu_2-y)+\beta_p(x-\mu_2)}{\nu_2-\mu_2}, & \mu_2\le y\le \nu_2,\\ \beta_p, & \nu_2\le y\le \pi_2,\\ \dfrac{(y-\pi_2)+\beta_p(\theta_2-y)}{\theta_2-\pi_2}, & \pi_2\le y\le \theta_2,\\ 1, & \text{otherwise,}\end{cases}\quad I_{\tilde{\lambda}}^{n}(y)=\begin{cases}\dfrac{(\eta_2-y)+\varphi_n(y-\gamma_2)}{\eta_2-\gamma_2}, & \gamma_2\le y\le \eta_2,\\ \varphi_n, & \eta_2\le y\le \iota_2,\\ \dfrac{(y-\iota_2)+\varphi_n(\kappa_2-y)}{\kappa_2-\iota_2}, & \iota_2\le y\le \kappa_2,\\ -1, & \text{otherwise,}\end{cases}$$

$$F_{\tilde{\lambda}}^{p}(y)=\begin{cases}\dfrac{(\nu_3-y)+\zeta_p(y-\mu_3)}{\nu_3-\mu_3}, & \mu_3\le y\le \nu_3,\\ \zeta_p, & \nu_3\le y\le \pi_3,\\ \dfrac{(y-\pi_3)+\zeta_p(\theta_3-y)}{\theta_3-\pi_3}, & \pi_3\le y\le \theta_3,\\ 1, & \text{otherwise,}\end{cases}\quad F_{\tilde{\lambda}}^{n}(y)=\begin{cases}\dfrac{(\eta_3-y)+\nu_p(y-\gamma_3)}{\eta_3-\gamma_3}, & \gamma_3\le y\le \eta_3,\\ \nu_n, & \eta_3\le y\le \iota_3,\\ \dfrac{(y-\iota_3)+\nu_n(\kappa_3-y)}{\kappa_3-\iota_3}, & \iota_3\le y\le \kappa_3,\\ -1, & \text{otherwise,}\end{cases} \tag{15}$$

where $\chi_p,\beta_p,\zeta_p\in[0,1]$ and $\alpha_n,\varphi_n,\nu_n\in[-1,0]\subset\mathbb{R}$.

Remark 1. If we set $\nu_i=\pi_i$ and $\eta_i=\iota_i$ in Definition 7, the triangular bipolar single-valued neutrosophic number (TBSNN) is obtained, where $i=1,2,3$.

Definition 8 (see [51]). Let $\tilde{\lambda}=\prec(T^p(y),I^p(y),F^p(y),T^n(y),I^n(y),F^n(y))\succ$ be a BSNN then the score function is presented by

$$S(\tilde{\lambda})=\frac{(T^p(y)+1-I^p(y)+1-F^p(y)+1+T^n(y)-I^n(y)-F^n(y))}{6}. \tag{16}$$

Definition 9. Based on [52], let

$$\begin{aligned}\tilde{\lambda}_1&=\prec([\mu_1,\nu_1,\pi_1,\theta_1];\chi_p^1),([\mu_2,\nu_2,\pi_2,\theta_2];\beta_p^1),([\mu_3,\nu_3,\pi_3,\theta_3];\zeta_p^1),([\mu_4,\nu_4,\pi_4,\theta_4];\alpha_n^1),([\mu_5,\nu_5,\pi_5,\theta_5];\varphi_n^1),\\&\quad([\mu_6,\nu_6,\pi_6,\theta_6];\nu_n^1)\succ \text{ and}\\ \tilde{\lambda}_2&=\prec([\gamma_1,\eta_1,\iota_1,\kappa_1];\chi_p^2),([\gamma_2,\eta_2,\iota_2,\kappa_2];\beta_p^2),([\gamma_3,\eta_3,\iota_3,\kappa_3];\zeta_p^2),((\gamma_4,\eta_4,\iota_4,\kappa_4);\alpha_n^2),([\gamma_5,\eta_5,\iota_5,\kappa_5];\varphi_n^2),\\&\quad([\gamma_6,\eta_6,\iota_6,\kappa_6];\nu_n^2)\succ\end{aligned} \tag{17}$$

be two nonnegative T_rBSNNs, then,

(1) $\tilde{\lambda}_1\oplus\tilde{\lambda}_2=\prec([\mu_1+\gamma_1,\nu_1+\eta_1,\pi_1+\iota_1,\theta_1+\kappa_1];\chi_p^1\wedge\chi_p^2),([\mu_2+\gamma_2,\nu_2+\eta_2,\pi_2+\iota_2,\theta_2+\kappa_2];\beta_p^1\vee\beta_p^2),([\mu_3+\gamma_3,\nu_3+\eta_3,\pi_3+\iota_3,\theta_3+\kappa_3];\zeta_p^1\vee\zeta_p^2),([\mu_4+\gamma_4,\nu_4+\eta_4,\pi_4+\iota_4,\theta_4+\kappa_4];\alpha_n^1\vee\alpha_n^2),([\mu_5+\gamma_5,\nu_5+\eta_5,\pi_5+\iota_5,\theta_5+\kappa_5];\varphi_n^1\wedge\varphi_n^2),([\mu_6+\gamma_6,\nu_6+\eta_6,\pi_6+\iota_6,\theta_6+\kappa_6];\nu_n^1\wedge\nu_n^2)\succ$.

(2) $\tilde{\lambda}_1\ominus\tilde{\lambda}_2=\prec([\mu_1-\kappa_1,\nu_1-\iota_1,\pi_1-\eta_1,\theta_1-\gamma_1];\chi_p^1\wedge\chi_p^2),([\mu_2-\kappa_2,\nu_2-\iota_2,\pi_2-\eta_2,\theta_2-\gamma_2];\beta_p^1\vee\beta_p^2),([\mu_3-\kappa_3,\nu_3-\iota_3,\pi_3-\eta_3,\theta_3-\gamma_3];\zeta_p^1\vee\zeta_p^2),([\mu_4-\kappa_4,\nu_4-\iota_4,\pi_4-\eta_4,\theta_1-\gamma_1];\alpha_n^1\vee\alpha_n^2),([\mu_5-\kappa_5,\nu_5-\iota_5,\pi_5-\eta_5,\theta_5-\gamma_5];\varphi_n^1\wedge\varphi_n^2),([\mu_6-\kappa_6,\nu_6-\iota_6,\pi_6-\eta_6,\theta_6-\gamma_6];\nu_n^1\wedge\nu_n^2)\succ$

(3) $\tilde{\lambda}_1\otimes\tilde{\lambda}_2=\left\{\begin{array}{l}\prec([\mu_1\gamma_1,\nu_1\eta_1,\pi_1\iota_1,\theta_1\kappa_1];\chi_p^1\wedge\chi_p^2),([\mu_2\gamma_2,\nu_2\eta_2,\pi_2\iota_2,\theta_2\kappa_2];\beta_p^1\vee\beta_p^2),\\([\mu_3\gamma_3,\nu_3\eta_3,\pi_3\iota_3,\theta_3\kappa_3];\zeta_p^1\vee\zeta_p^2),([\mu_4\gamma_4,\nu_4\eta_4,\pi_4\iota_4,\theta_4\kappa_4];\alpha_n^1\vee\alpha_n^2),\\([\mu_5\gamma_5,\nu_5\eta_5,\pi_5\iota_5,\theta_5\kappa_5];\varphi_n^1\wedge\varphi_n^2),([\mu_6\gamma_6,\nu_6\eta_6,\pi_6\iota_6,\theta_6\kappa_6];\nu_n^1\wedge\nu_n^2)\succ\end{array}\right.$

3. Methodology to Solve FBSNT Problems

In this section, we present a new method that is based on the formulation of FBSNLP to solve FBSNT problems. We discuss the steps to calculate the bipolar single-valued neutrosophic optimal solution of FBSNT problems with nonnegative TBSNNs:

$$\text{Minimize Z}=\sum_{i=1}^{m}\sum_{j=1}^{n}\widetilde{C_{ij}^{W}}\otimes\widetilde{X_{ij}^{W}}, \tag{18}$$

subject to

$$\begin{aligned}&\sum_{j=1}^{n}\widetilde{X_{ij}^{W}}=\tilde{E}_i^{W}=\text{Supply},\forall i=1,2,3,\ldots,m,\\&\sum_{i=1}^{m}\widetilde{X_{ij}^{W}}=\tilde{F}_j^{W}=\text{Demand},\forall j=1,2,3,\ldots,n,\\&\tilde{X}_{ij}^{W}\ge 0,\forall i=1,2,3,\ldots,m,\forall j=1,2,3,\ldots,n,\end{aligned} \tag{19}$$

where $\widetilde{C_{ij}^{W}},\widetilde{X_{ij}^{W}},\tilde{E}_i^{W}$, and $\tilde{F}_j^{W}$ are all TBSNNs.

Step 1. Determine total bipolar single-valued neutrosophic availability and total bipolar single-valued neutrosophic demand.

If

$$\sum_{j=1}^{n} \widetilde{F}_j^W = \sum_{i=1}^{m} \widetilde{E}_i^W, \tag{20}$$

$\Rightarrow$ a balanced bipolar single-valued neutrosophic transportation problem (BSNTP).

If,

$$\sum_{j=1}^{n} \widetilde{F}_j^W \neq \sum_{i=1}^{m} \widetilde{E}_i^W. \tag{21}$$

An unbalanced BSNTP.

That is,

$$\begin{aligned} &\prec\left([s_k, t_k, u_k]; \chi_p, \beta_p, \zeta_p\right), \left([v_k, w_k, x_k]; \alpha_n, \varphi_n, \nu_n\right)\succ \\ &\quad \neq \prec\left([s'_k, t'_k, u'_k]; \chi'_p, \beta'_p, \zeta'_p\right), \left([v'_k, w'_k, x'_k]; \alpha'_n, \varphi'_n, \nu'_n\right)\succ \\ &k = 1, 2, 3. \end{aligned} \tag{22}$$

Then one of the following case arise.

Case 1. $s_k \leq s'_k$, $t_k - s_k \leq t'_k - s'_k$, $u_k - t_k \leq u'_k - t'_k, v_k \leq v'_k$, $w_k - v_k \leq w'_k - v'_k$, $x_k - w_k \leq x'_k - w'_k$.

Case 2. $s_k \geq s'_k$, $t_k - s_k \geq t'_k - s'_k$, $u_k - t_k \geq u'_k - t'_k, v_k \geq v'_k$, $w_k - v_k \geq w'_k - v'_k$, $x_k - w_k \geq x'_k - w'_k$.

Case 3. When the above two cases do not hold, then there may exist infinitely many TBSNNs,

$$\begin{cases} \prec\left(\left[s_k^1, t_k^1, u_k^1\right]; \chi_p^1, \beta_p^1, \zeta_p^1\right), \left(\left[v_k^1, w_k^1, x_k^1\right]; \alpha_n^1, \varphi_n^1, \nu_n^1\right)\succ \\ \prec\left(\left[s_k^{1'}, t_k^{1'}, u_k^{1'}\right]; \chi_p^{1'}, \beta_p^{1'}, \zeta_p^{1'}\right), \left(\left[v_k^{1'}, w_k^{1'}, x_k^{1'}\right]; \alpha_n^{1'}, \varphi_n^{1'}, \nu_n^{1'}\right)\succ, \\ k = 1, 2, 3, \end{cases} \tag{23}$$

such that

$$\begin{aligned} &\prec\left([s_k, t_k, u_k]; \chi_p, \beta_p, \zeta_p\right), \left([v_k, w_k, x_k]; \alpha_n, \varphi_n, \nu_n\right)\succ \oplus \prec\left(\left[s_k^1, t_k^1, u_k^1\right]; \chi_p^1, \beta_p^1, \zeta_p^1\right), \left(\left[v_k^1, w_k^1, x_k^1\right]; \alpha_n^1, \varphi_n^1, \nu_n^1\right)\succ = \\ &\prec\left([s'_k, t'_k, u'_k]; \chi'_p, \beta'_p, \zeta'_p\right), \left([v'_k, w'_k, x'_k]; \alpha'_n, \varphi'_n, \nu'_n\right)\succ \oplus \prec\left(\left[s_k^{1'}, t_k^{1'}, u_k^{1'}\right]; \chi_p^{1'}, \beta_p^{1'}, \zeta_p^{1'}\right), \left(\left[v_k^{1'}, w_k^{1'}, x_k^{1'}\right]; \alpha_n^{1'}, \varphi_n^{1'}, \nu_n^{1'}\right)\succ \end{aligned} \tag{24}$$

But we have to determine such TBSNNs

$$\begin{cases} \prec\left(\left[s_k^1, t_k^1, u_k^1\right]; \chi_p^1, \beta_p^1, \zeta_p^1\right), \left(\left[v_k^1, w_k^1, x_k^1\right]; \alpha_n^1, \varphi_n^1, \nu_n^1\right)\succ \\ \prec\left(\left[s_k^{1'}, t_k^{1'}, u_k^{1'}\right]; \chi_p^{1'}, \beta_p^{1'}, \zeta_p^{1'}\right), \left(\left[v_k^{1'}, w_k^{1'}, x_k^{1'}\right]; \alpha_n^{1'}, \varphi_n^{1'}, \nu_n^{1'}\right)\succ, \\ k = 1, 2, 3, \end{cases} \tag{25}$$

which satisfy the following conditions.

(i) $\prec([s_k^1, t_k^1, u_k^1]; \chi_p^1, \beta_p^1, \zeta_p^1), ([v_k^1, w_k^1, x_k^1]; \alpha_n^1, \varphi_n^1, \nu_n^1)\succ$ and $\prec([s_k^{1'}, t_k^{1'}, u_k^{1'}]; \chi_p^{1'}, \beta_p^{1'}, \zeta_p^{1'}), ([v_k^{1'}, w_k^{1'}, x_k^{1'}]; \alpha_n^{1'}, \varphi_n^{1'}, \nu_n^{1'})\succ$ are nonnegative TBSNNs.

(ii) It satisfies

$$\begin{aligned} &\prec\left([s_k, t_k, u_k]; \chi_p, \beta_p, \zeta_p\right), \left([v_k, w_k, x_k]; \alpha_n, \varphi_n, \nu_n\right)\succ \oplus \prec\left(\left[s_k^1, t_k^1, u_k^1\right]; \chi_p^1, \beta_p^1, \zeta_p^1\right), \left(\left[v_k^1, w_k^1, x_k^1\right]; \alpha_n^1, \varphi_n^1, \nu_n^1\right)\succ = \\ &\prec\left([s'_k, t'_k, u'_k]; \chi'_p, \beta'_p, \zeta'_p\right), \left([v'_k, w'_k, x'_k]; \alpha'_n, \varphi'_n, \nu'_n\right)\succ \oplus \prec\left(\left[s_k^{1'}, t_k^{1'}, u_k^{1'}\right]; \chi_p^{1'}, \beta_p^{1'}, \zeta_p^{1'}\right), \left(\left[v_k^{1'}, w_k^{1'}, x_k^{1'}\right]; \alpha_n^{1'}, \varphi_n^{1'}, \nu_n^{1'}\right)\succ. \end{aligned} \tag{26}$$

(iii) Further, if there exists two nonnegative TBSNNs $\prec([m_k, n_k, o_k]; \mu_p, \nu_p, \pi_p), ([p_k, q_k, r_k]; \eta_n, \zeta_n, \theta_n)\succ$ and $\prec([m'_k, n'_k, o'_k]; \mu'_p, \nu'_p, \pi'_p), ([p'_k, q'_k, r'_k]; \eta'_n, \zeta'_n, \theta'_n)\succ$ such that

$$\begin{aligned} &\prec\left([s_k, t_k, u_k]; \chi_p, \beta_p, \zeta_p\right), \left([v_k, w_k, x_k]; \alpha_n, \varphi_n, \nu_n\right)\succ \oplus \prec\left([m_k, n_k, o_k]; \mu_p, \nu_p, \pi_p\right), \left([p_k, q_k, r_k]; \eta_n, \zeta_n, \theta_n\right)\succ = \\ &\prec\left([s'_k, t'_k, u'_k]; \chi'_p, \beta'_p, \zeta'_p\right), \left([v'_k, w'_k, x'_k]; \alpha'_n, \varphi'_n, \nu'_n\right)\succ \oplus \prec\left([m'_k, n'_k, o'_k]; \mu'_p, \nu'_p, \pi'_p\right), \left([p'_k, q'_k, r'_k]; \eta'_n, \zeta'_n, \theta'_n\right)\succ, \end{aligned} \tag{27}$$

then

$$\Re\prec\left(\left[m_k, n_k, o_k\right]; \mu_p, \nu_p, \pi_p\right), \left(\left[p_k, q_k, r_k\right]; \eta_n, \zeta_n, \theta_n\right)\succ \geq \Re\prec\left(\left[s_k^1, t_k^1, u_k^1\right]; \chi_p^1, \beta_p^1, \zeta_p^1\right), \left(\left[v_k^1, w_k^1, x_k^1\right]; \alpha_n^1, \varphi_n^1, \nu_n^1\right)\succ$$
$$\Re\prec\left(\left[m_k', n_k', o_k'\right]; \mu_p', \nu_p', \pi_p'\right), \left(\left[p_k', q_k', r_k'\right]; \eta_n', \zeta_n', \theta_n'\right)\succ \geq \Re\prec\left(\left[s_k^{1'}, t_k^{1'}, u_k^{1'}\right]; \chi_p^{1'}, \beta_p^{1'}, \zeta_p^{1'}\right), \left(\left[v_k^{1'}, w_k^{1'}, x_k^{1'}\right]; \alpha_n^{1'}, \varphi_n^{1'}, \nu_n^{1'}\right)\succ. \quad (28)$$

Step 2. Suppose $\widetilde{C_{ij}}^W = \prec([c_{ij}^1, c_{ij}^2, c_{ij}^3]; \xi_{ij}, \psi_{ij}, \omega_{ij}), ([c_{ij}^4, c_{ij}^5, c_{ij}^6]; \xi_{ij}', \psi_{ij}', \omega_{ij}')\succ$,

$$\widetilde{X_{ij}^W} = \prec\left(\left[x_{ij}^1, x_{ij}^2, x_{ij}^3\right]; \sigma_{ij}, \tau_{ij}, \upsilon_{ij}\right), \left(\left[x_{ij}^4, x_{ij}^5, x_{ij}^6\right]; \sigma_{ij}', \tau_{ij}', \upsilon_{ij}'\right)\succ$$
$$\widetilde{E}_i^W = \prec\left(\left[e_i^1, e_i^2, e_i^3\right]; \kappa_i, \lambda_i, \theta_i\right), \left(\left[e_i^4, e_i^5, e_i^6\right]; \kappa_i', \lambda_i', \theta_i'\right)\succ \text{ and } \widetilde{F}_j^W = \prec\left(\left[f_j^1, f_j^2, f_j^3\right]; \eta_j, \epsilon_j, \chi_j\right), \left(\left[f_j^4, f_j^5, f_j^6\right]; \eta_j', \epsilon_j', \chi_j'\right)\succ. \quad (29)$$

Then FBSNTP (18) can be transformed as

$$\text{Min Z} = \sum_{i=1}^{m}\sum_{j=1}^{n} \prec\left(\left[c_{ij}^1, c_{ij}^2, c_{ij}^3\right]; \xi_{ij}, \psi_{ij}, \omega_{ij}\right), \left(\left[c_{ij}^4, c_{ij}^5, c_{ij}^6\right]; \xi_{ij}', \psi_{ij}', \omega_{ij}'\right)\succ \otimes \prec\left(\left[x_{ij}^1, x_{ij}^2, x_{ij}^3\right]; \sigma_{ij}, \tau_{ij}, \upsilon_{ij}\right), \left(\left[x_{ij}^4, x_{ij}^5, x_{ij}^6\right]; \sigma_{ij}', \tau_{ij}', \upsilon_{ij}'\right)\succ, \quad (30)$$

subject to

$$\sum_{j=1}^{n} \prec\left(\left[x_{ij}^1, x_{ij}^2, x_{ij}^3\right]; \sigma_{ij}, \tau_{ij}, \upsilon_{ij}\right), \left(\left[x_{ij}^4, x_{ij}^5, x_{ij}^6\right]; \sigma_{ij}', \tau_{ij}', \upsilon_{ij}'\right)\succ = \prec\left(\left[e_i^1, e_i^2, e_i^3\right]; \kappa_i, \lambda_i, \theta_i\right), \left(\left[e_i^4, e_i^5, e_i^6\right]; \kappa_i', \lambda_i', \theta_i'\right)\succ,$$
$$\sum_{i=1}^{m} \prec\left(\left[x_{ij}^1, x_{ij}^2, x_{ij}^3\right]; \sigma_{ij}, \tau_{ij}, \upsilon_{ij}\right), \left(\left[x_{ij}^4, x_{ij}^5, x_{ij}^6\right]; \sigma_{ij}', \tau_{ij}', \upsilon_{ij}'\right)\succ = \prec\left(\left[f_j^1, f_j^2, f_j^3\right]; \eta_j, \epsilon_j, \chi_j\right), \left(\left[f_j^4, f_j^5, f_j^6\right]; \eta_j', \epsilon_j', \chi_j'\right)\succ, \quad (31)$$

where $\prec([x_{ij}^1, x_{ij}^2, x_{ij}^3]; \sigma_{ij}, \tau_{ij}, \upsilon_{ij}), ([x_{ij}^4, x_{ij}^5, x_{ij}^6]; \sigma_{ij}', \tau_{ij}', \upsilon_{ij}')\succ$ are nonnegative TBSNNs.

Step 3. By applying arithmetic operations as described in Definition 9 and putting $\prec([c_{ij}^1, c_{ij}^2, c_{ij}^3]; \xi_{ij}, \psi_{ij}, \omega_{ij}), ([c_{ij}^4, c_{ij}^5, c_{ij}^6]; \xi_{ij}', \psi_{ij}', \omega_{ij}')\succ \otimes \prec([x_{ij}^1, x_{ij}^2, x_{ij}^3]; \sigma_{ij}, \tau_{ij}, \upsilon_{ij}), ([x_{ij}^4, x_{ij}^5, x_{ij}^6]; \sigma_{ij}', \tau_{ij}', \upsilon_{ij}')\succ = \prec([d_{ij}^1, d_{ij}^2, d_{ij}^3]; \mu_{ij}, \delta_{ij}, \eta_{ij}), ([d_{ij}^4, d_{ij}^5, d_{ij}^6]; \mu_{ij}', \delta_{ij}', \eta_{ij}')\succ$, then the FBSNTP (30) can be transformed as

$$\text{Min Z} = \sum_{i=1}^{m}\sum_{j=1}^{n} \prec\left(\left[d_{ij}^1, d_{ij}^2, d_{ij}^3\right]; \mu_{ij}, \delta_{ij}, \eta_{ij}\right), \left(\left[d_{ij}^4, d_{ij}^5, d_{ij}^6\right]; \mu_{ij}', \delta_{ij}', \eta_{ij}'\right)\succ, \quad (32)$$

subject to

$$\sum_{j=1}^{n} \prec\left(\left[x_{ij}^1, x_{ij}^2, x_{ij}^3\right]; \sigma_{ij}, \tau_{ij}, \upsilon_{ij}\right), \left(\left[x_{ij}^4, x_{ij}^5, x_{ij}^6\right]; \sigma_{ij}', \tau_{ij}', \upsilon_{ij}'\right)\succ = \prec\left(\left[e_i^1, e_i^2, e_i^3\right]; \kappa_i, \lambda_i, \theta_i\right), \left(\left[e_i^4, e_i^5, e_i^6\right]; \kappa_i', \lambda_i', \theta_i'\right)\succ,$$
$$\sum_{i=1}^{m} \prec\left(\left[x_{ij}^1, x_{ij}^2, x_{ij}^3\right]; \sigma_{ij}, \tau_{ij}, \upsilon_{ij}\right), \left(\left[x_{ij}^4, x_{ij}^5, x_{ij}^6\right]; \sigma_{ij}', \tau_{ij}', \upsilon_{ij}'\right)\succ = \prec\left(\left[f_j^1, f_j^2, f_j^3\right]; \eta_j, \epsilon_j, \chi_j\right), \left(\left[f_j^4, f_j^5, f_j^6\right]; \eta_j', \epsilon_j', \chi_j'\right)\succ, \quad (33)$$

where $\prec([x_{ij}^1, x_{ij}^2, x_{ij}^3]; \sigma_{ij}, \tau_{ij}, \upsilon_{ij}), ([x_{ij}^4, x_{ij}^5, x_{ij}^6]; \sigma_{ij}', \tau_{ij}', \upsilon_{ij}')\succ$ are nonnegative TBSNNs.

Step 4. Now applying the score function, the FBSNTP (32) can be transformed as

$$\text{Min Z} = \sum_{i=1}^{m}\sum_{j=1}^{n} S\prec\left(\left[d_{ij}^1, d_{ij}^2, d_{ij}^3\right]; \mu_{ij}, \delta_{ij}, \eta_{ij}\right), \left(\left[d_{ij}^4, d_{ij}^5, d_{ij}^6\right]; \mu_{ij}', \delta_{ij}', \eta_{ij}'\right)\succ, \quad (34)$$

subject to

$$\sum_{j=1}^{n} x_{ij}^{1} = e_i^{1}, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} x_{ij}^{2} = e_i^{2}, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} x_{ij}^{3} = e_i^{3}, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} x_{ij}^{4} = e_i^{4}, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} x_{ij}^{5} = e_i^{5}, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} x_{ij}^{6} = e_i^{6}, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} \sigma_{ij} \wedge \kappa_i, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} \tau_{ij} \vee \lambda_i, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} \upsilon_{ij} \vee \theta_i, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{i=1}^{m} x_{ij}^{1} = f_j^{1}, \forall j = 1,2,3,\ldots,n,$$
$$\sum_{i=1}^{m} x_{ij}^{2} = f_j^{2}, \forall j = 1,2,3,\ldots,n,$$
$$\sum_{i=1}^{m} x_{ij}^{3} = f_j^{3}, \forall j = 1,2,3,\ldots,n,$$
$$\sum_{i=1}^{m} x_{ij}^{4} = f_j^{4}, \forall j = 1,2,3,\ldots,n,$$
$$\sum_{i=1}^{m} x_{ij}^{5} = f_j^{5}, \forall j = 1,2,3,\ldots,n,$$
$$\sum_{i=1}^{m} x_{ij}^{6} = f_j^{6}, \forall j = 1,2,3,\ldots,n,$$
$$\sum_{j=1}^{n} \sigma_{ij}' \vee \kappa_i', \forall j = 1,2,3,\ldots,n,$$
$$\sum_{j=1}^{n} \tau_{ij}' \wedge \lambda_i', \forall j = 1,2,3,\ldots,n,$$
$$\sum_{j=1}^{n} \upsilon_{ij}' \wedge \theta_i', \forall j = 1,2,3,\ldots,n, \tag{35}$$

where $\prec([x_{ij}^1, x_{ij}^2, x_{ij}^3]; \sigma_{ij}, \tau_{ij}, \upsilon_{ij}), ([x_{ij}^4, x_{ij}^5, x_{ij}^6]; \sigma_{ij}', \tau_{ij}', \upsilon_{ij}')\succ$ are nonnegative TBSNNs.

Step 5. To obtain fuzzy optimal solution, the following crisp linear/nonlinear programming problem is solved:

$$\text{Min Z} = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{6}\left(3 + d_{ij}^{1} - d_{ij}^{2} - d_{ij}^{3} + d_{ij}^{4} - d_{ij}^{5} - d_{ij}^{6}\right), \tag{36}$$

subject to

$$\sum_{j=1}^{n} x_{ij}^{1} = e_i^{1}, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} x_{ij}^{2} = e_i^{2}, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} x_{ij}^{3} = e_i^{3}, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} x_{ij}^{4} = e_i^{4}, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} x_{ij}^{5} = e_i^{5}, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} x_{ij}^{6} = e_i^{6}, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} \sigma_{ij} \wedge \kappa_i, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} \tau_{ij} \vee \lambda_i, \forall i = 1,2,3,\ldots,m,$$
$$\sum_{j=1}^{n} \upsilon_{ij} \vee \theta_i, \forall i = 1,2,3,\ldots,m, \tag{37}$$
$$\sum_{i=1}^{m} x_{ij}^{1} = f_j^{1}, \forall j = 1,2,3,\ldots,n,$$
$$\sum_{i=1}^{m} x_{ij}^{2} = f_j^{2}, \forall j = 1,2,3,\ldots,n,$$
$$\sum_{i=1}^{m} x_{ij}^{3} = f_j^{3}, \forall j = 1,2,3,\ldots,n,$$
$$\sum_{i=1}^{m} x_{ij}^{4} = f_j^{4}, \forall j = 1,2,3,\ldots,n,$$
$$\sum_{i=1}^{m} x_{ij}^{5} = f_j^{5}, \forall j = 1,2,3,\ldots,n,$$
$$\sum_{i=1}^{m} x_{ij}^{6} = f_j^{6}, \forall j = 1,2,3,\ldots,n,$$
$$\sum_{j=1}^{n} \sigma_{ij}' \vee \kappa_i', \forall j = 1,2,3,\ldots,n,$$
$$\sum_{j=1}^{n} \tau_{ij}' \wedge \lambda_i', \forall j = 1,2,3,\ldots,n,$$
$$\sum_{j=1}^{n} \upsilon_{ij}' \wedge \theta_i', \forall j = 1,2,3,\ldots,n,$$

$x_{ij}^1 \geq 0,\ x_{ij}^2 - x_{ij}^1 \geq 0,\ x_{ij}^3 - x_{ij}^2 \geq 0,\ x_{ij}^4 \geq 0,\ x_{ij}^5 - x_{ij}^4 \geq 0,\ x_{ij}^6 - x_{ij}^5 \geq 0,\ \sigma_{ij},\ \tau_{ij},\ \upsilon_{ij} \in [0,1]$ and $\sigma_{,ij},\ \tau_{ij}',\ \upsilon_{ij}' \in [-1,0]$ $\forall i = 1,2,\ldots,m, \forall j = 1,2,\ldots,n.$

Step 6. An optimal solution is obtained by solving a crisp linear/nonlinear programming problem: $\{x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5, x_{ij}^6, \sigma_{ij}, \tau_{ij}, \upsilon *_{ij}, \sigma_{ij}', \tau_{ij}', \upsilon_{ij}'\}$.

Step 7. Find the bipolar single-valued neutrosophic optimal solution $\widetilde{X}_{ij}^W$ of the FBSNTP (18) by putting the values of $x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5, x_{ij}^6, \sigma_{ij}, \tau_{ij}, \upsilon_{ij}, \sigma_{ij}', \tau_{ij}'$ and υ_{ij}' in $\widetilde{X}_{ij}^W = \prec([x_{ij}^1, x_{ij}^2, x_{ij}^3]; \sigma_{ij}, \tau_{ij}, \upsilon_{ij}), ([x_{ij}^4, x_{ij}^5, x_{ij}^6]; \sigma_{ij}', \tau_{ij}', \upsilon_{ij}')\succ$.

Step 8. The minimum bipolar single-valued neutrosophic transportation cost/bipolar single-valued neutrosophic optimal value of the FBSNTP (18) are found by setting the values of $\widetilde{X}_{ij}^W$, as obtained in Step 7, in $\sum_{i=1}^m \sum_{j=1}^n \widetilde{C_{ij}^W} \otimes \widetilde{X_{ij}^W}$.

4. Numerical Example

Example 1. FFC Transportation Model

Fauji Fertilizer Company (FFC) has two plants in Gujranwala and Karachi and two main centers in Lahore and Peshawar. The capacities of producing urea at plants are $\prec([70,100,130];1,0,0.4),([55,100,155];-1,0,-0.3)\succ$ and $\prec([50,70,90];1,0,0.3),([35,70,115];-1,0,-0.2)\succ$ and the demands at the two delivery centers of urea for the same time are $\prec([50,60,70];1,0,0.2),([30,60,100];-1,0,-0.1)\succ$ and $\prec([40,70,100];1,0,0.1),([40,70,100];-1,0,-0.4)\succ$, respectively. The trucking association in charge of transporting the urea charge $\prec([5,6,7];1,0,0),([3,6,10];-1,0,-0.1)\succ$ Rs. per ton urea. Thus, the transporting costs per ton urea on different routes are given in Table 1.subject to

$$\begin{aligned} \tilde{x}_{11}\oplus\tilde{x}_{12} &= \prec([70,100,130];1,0,0.4),([55,100,155];-1,0,-0.3)\succ,\\ \tilde{x}_{21}\oplus\tilde{x}_{22} &= \prec([50,70,90];1,0,0.3),([35,70,115];-1,0,-0.2)\succ,\\ \tilde{x}_{11}\oplus\tilde{x}_{21} &= \prec([50,60,70];1,0,0.2),([30,60,100];-1,0,-0.1)\succ,\\ \tilde{x}_{12}\oplus\tilde{x}_{22} &= \prec([40,70,100];1,0,0.1),([40,70,100];-1,0,-0.4)\succ, \end{aligned} \tag{38}$$

where $\tilde{x}_{11}$, $\tilde{x}_{12}$, $\tilde{x}_{21}$, and $\tilde{x}_{22}$ are nonnegative TBSNNs.

$$\text{Minimize}\begin{pmatrix} \prec([15,17,18];1,0,0.2),([16,17,19];-1,0,-0.3)\succ\otimes\tilde{x}_{11}\oplus\\ \prec([20,22,24];1,0,0.3),([21,22,23];-1,0,-0.4)\succ\otimes\tilde{x}_{12}\oplus\\ \prec([25,28,30];1,0,0.4),([27,28,29];-1,0,-0.5)\succ\otimes\tilde{x}_{21}\oplus\\ \prec([30,40,50];1,0,0.1),([35,40,45];-1,0,-0.1)\succ\otimes\tilde{x}_{22} \end{pmatrix}, \tag{39}$$

4.1. Step 1.

Now as,

total supply $= \prec([120,170,220];1,0,0.3),([90,170,270];-1,0,-0.3)\succ$

Total demand $= \prec([90,130,170];1,0,0.2),([70,130,200];-1,0,-0.4)\succ$

$\Rightarrow$ an unbalanced FBSNTP. So, we add dummy rows and dummy columns to make a balanced TP.

Dummy row $= \prec([10,30,30];1,0,0.1),([0,30,40];-1,0,-0.2)\succ$

Dummy column $= \prec([40,70,80];1,0,0.2),([20,70,110];-1,0,-0.3)\succ$

So, by assuming bipolar single-valued neutrosophic transportation cost of unit quantity of product from dummy source to all destinations and from all sources to dummy destination to be zero bipolar single-valued neutrosophic numbers, then FBSNTP (38) can be transformed as follows.

Table 1: FFC transportation model.

Plants	Lahore	Peshawar	Supply
Gujranwala	$\prec([15,16,17];1,0,0.2),([16,17,19];-1,0,-0.3)\succ$	$\prec([20,22,24];1,0,0.3),([21,22,23];-1,0,-0.4)\succ$	$\prec([70,100,130];1,0,0.4),([55,100,155];-1,0,-0.3)\succ$
Karachi	$\prec([25,28,30];1,0,0.4),([27,28,29];-1,0,-0.5)\succ$	$\prec([30,40,50];1,0,0.1),([35,40,45];-1,0,-0.1)\succ$	$\prec([50,70,90];1,0,0.3),([35,70,115];-1,0,-0.2)\succ$
Demand	$\prec([50,60,70];1,0,0.2),([30,60,100];-1,0,-0.1)\succ$	$\prec([40,70,100];1,0,0.1),([40,70,100];-1,0,-0.4)\succ$	

4.2. Step 2.

$$\text{Minimize}\begin{pmatrix} \prec([15,17,18];1,0,0.2),([16,17,19];-1,0,-0.3)\succ\otimes\tilde{x}_{11}\oplus \\ \prec([20,22,24];1,0,0.3),([21,22,23];-1,0,-0.4)\succ\otimes\tilde{x}_{12}\oplus \\ \prec([0,0,0];1,0,0),([0,0,0];-1,0,0)\succ\otimes\tilde{x}_{13}\oplus \\ \prec([25,28,30];1,0,0.4),([27,28,29];-1,0,-0.5)\succ\otimes\tilde{x}_{21}\oplus \\ \prec([30,40,50];1,0,0.1),([35,40,45];-1,0,-0.1)\succ\otimes\tilde{x}_{22}\oplus \\ \prec([0,0,0];1,0,0),([0,0,0];-1,0,0)\succ\otimes\tilde{x}_{23}\oplus \\ \prec([0,0,0];1,0,0),([0,0,0];-1,0,0)\succ\otimes\tilde{x}_{31}\oplus \\ \prec([0,0,0];1,0,0),([0,0,0];-1,0,0)\succ\otimes\tilde{x}_{32}\oplus \\ \prec([0,0,0];1,0,0),([0,0,0];-1,0,0)\succ\otimes\tilde{x}_{33} \end{pmatrix}, \tag{40}$$

subject to

$$\begin{aligned} \tilde{x}_{11}\oplus\tilde{x}_{12}\oplus\tilde{x}_{13} &= \prec([70,100,130];1,0,0.4),([55,100,155];-1,0,-0.3)\succ, \\ \tilde{x}_{21}\oplus\tilde{x}_{22}\oplus\tilde{x}_{23} &= \prec([50,70,90];1,0,0.3),([35,70,115];-1,0,-0.2)\succ, \\ \tilde{x}_{31}\oplus\tilde{x}_{32}\oplus\tilde{x}_{33} &= \prec([10,30,30];1,0,0.1),([0,30,40];-1,0,-0.2)\succ, \\ \tilde{x}_{11}\oplus\tilde{x}_{21}\oplus\tilde{x}_{31} &= \prec([50,60,70];1,0,0.2),([30,60,100];-1,0,-0.1)\succ, \\ \tilde{x}_{12}\oplus\tilde{x}_{22}\oplus\tilde{x}_{32} &= \prec([40,70,100];1,0,0.1),([40,70,100];-1,0,-0.4)\succ, \\ \tilde{x}_{13}\oplus\tilde{x}_{23}\oplus\tilde{x}_{33} &= \prec([40,70,80];1,0,0.2),([20,70,110];-1,0,-0.3)\succ, \end{aligned} \tag{41}$$

where $\tilde{x}_{11}$, $\tilde{x}_{12}$, $\tilde{x}_{13}$, $\tilde{x}_{21}$, $\tilde{x}_{22}$, $\tilde{x}_{23}$, $\tilde{x}_{31}$, $\tilde{x}_{32}$, and $\tilde{x}_{33}$, are nonnegative TBSNNs.

4.3. Step 3. By assuming

$$\begin{aligned} \tilde{x}_{11} &= \prec([l_{11},m_{11},n_{11}];\pi_{11},\beta_{11},\phi_{11}),([l_{11}',m_{11}',n_{11}'];\pi_{11}',\beta_{11}',\phi_{11}')\succ, \\ \tilde{x}_{12} &= \prec([l_{12},m_{12},n_{12}];\pi_{12},\beta_{12},\phi_{12}),([l_{12}',m_{12}',n_{12}'];\pi_{12}',\beta_{12}',\phi_{12}')\succ, \\ \tilde{x}_{13} &= \prec([l_{13},m_{13},n_{13}];\pi_{13},\beta_{13},\phi_{13}),([l_{13}',m_{13}',n_{13}'];\pi_{13}',\beta_{13}',\phi_{13}')\succ, \\ \tilde{x}_{21} &= \prec([l_{21},m_{21},n_{21}];\pi_{21},\beta_{21},\phi_{21}),([l_{21}',m_{21}',n_{21}'];\pi_{21}',\beta_{21}',\phi_{21}')\succ, \\ \tilde{x}_{22} &= \prec([l_{22},m_{22},n_{22}];\pi_{22},\beta_{22},\phi_{22}),([l_{22}',m_{22}',n_{22}'];\pi_{22}',\beta_{22}',\phi_{22}')\succ, \\ \tilde{x}_{23} &= \prec([l_{23},m_{23},n_{23}];\pi_{23},\beta_{23},\phi_{23}),([l_{23}',m_{23}',n_{23}'];\pi_{23}',\beta_{23}',\phi_{23}')\succ, \\ \tilde{x}_{31} &= \prec([l_{31},m_{31},n_{31}];\pi_{31},\beta_{31},\phi_{31}),([l_{31}',m_{31}',n_{31}'];\pi_{31}',\beta_{31}',\phi_{31}')\succ, \\ \tilde{x}_{32} &= \prec([l_{32},m_{32},n_{32}];\pi_{32},\beta_{32},\phi_{32}),([l_{32}',m_{32}',n_{32}'];\pi_{32}',\beta_{32}',\phi_{32}')\succ, \\ \tilde{x}_{33} &= \prec([l_{33},m_{33},n_{33}];\pi_{33},\beta_{33},\phi_{33}),([l_{33}',m_{33}',n_{33}'];\pi_{33}',\beta_{33}',\phi_{33}')\succ, \end{aligned} \tag{42}$$

where $\tilde{x}_{11}$, $\tilde{x}_{12}$, $\tilde{x}_{13}$, $\tilde{x}_{21}$, $\tilde{x}_{22}$, $\tilde{x}_{23}$, $\tilde{x}_{31}$, $\tilde{x}_{32}$, and $\tilde{x}_{33}$ are nonnegative TBSNNs, then the FBSNPP (40) can be transformed as follows.

4.4. Step 4.

$$\text{Minimize}\begin{pmatrix} \prec([15,17,18];1,0,0.2),([16,17,19];-1,0,-0.3)\succ\otimes \\ \prec([l_{11},m_{11},n_{11}];\pi_{11},\beta_{11},\phi_{11}),([l_{11}',m_{11}',n_{11}'];\pi_{11}',\beta_{11}',\phi_{11}')\succ\oplus \\ \prec([20,22,24];1,0,0.3),([21,22,23];-1,0,-0.4)\succ\otimes \\ \prec([l_{12},m_{12},n_{12}];\pi_{12},\beta_{12},\phi_{12}),([l_{12}',m_{12}',n_{12}'];\pi_{12}',\beta_{12}',\phi_{12}')\succ\oplus \\ \prec([0,0,0];1,0,0),([0,0,0];-1,0,0)\succ\otimes \\ \prec([l_{13},m_{13},n_{13}];\pi_{13},\beta_{13},\phi_{13}),([l_{13}',m_{13}',n_{13}'];\pi_{13}',\beta_{13}',\phi_{13}')\succ\oplus \\ \prec([25,28,30];1,0,0.4),([27,28,29];-1,0,-0.5)\succ\otimes \\ \prec([l_{21},m_{21},n_{21}];\pi_{21},\beta_{21},\phi_{21}),([l_{21}',m_{21}',n_{21}'];\pi_{21}',\beta_{21}',\phi_{21}')\succ\oplus \\ \prec([30,40,50];1,0,0.1),([35,40,45];-1,0,-0.1)\succ\otimes \\ \prec([l_{22},m_{22},n_{22}];\pi_{22},\beta_{22},\phi_{22}),([l_{22}',m_{22}',n_{22}'];\pi_{22}',\beta_{22}',\phi_{22}')\succ\oplus \\ \prec([0,0,0];1,0,0),([0,0,0];-1,0,0)\succ\otimes \\ \prec([l_{23},m_{23},n_{23}];\pi_{23},\beta_{23},\phi_{23}),([l_{23}',m_{23}',n_{23}'];\pi_{23}',\beta_{23}',\phi_{23}')\succ\oplus \\ \prec([0,0,0];1,0,0),([0,0,0];-1,0,0)\succ\otimes \\ \prec([l_{31},m_{31},n_{31}];\pi_{31},\beta_{31},\phi_{31}),([l_{31}',m_{31}',n_{31}'];\pi_{31}',\beta_{31}',\phi_{31}')\succ\oplus \\ \prec([0,0,0];1,0,0),([0,0,0];-1,0,0)\succ\otimes \\ \prec([l_{32},m_{32},n_{32}];\pi_{32},\beta_{32},\phi_{32}),([l_{32}',m_{32}',n_{32}'];\pi_{32}',\beta_{32}',\phi_{32}')\succ\oplus \\ \prec([0,0,0];1,0,0),([0,0,0];-1,0,0)\succ\otimes \\ \prec([l_{33},m_{33},n_{33}];\pi_{33},\beta_{33},\phi_{33}),([l_{33}',m_{33}',n_{33}'];\pi_{33}',\beta_{33}',\phi_{33}')\succ \end{pmatrix}, \tag{43}$$

subject to

$$\begin{aligned}
&\prec([l_{11},m_{11},n_{11}];\pi_{11},\beta_{11},\phi_{11}),([l_{11}',m_{11}',n_{11}'];\pi_{11}',\beta_{11}',\phi_{11}')\succ\oplus\prec([l_{12},m_{12},n_{12}];\pi_{12},\beta_{12},\phi_{12}),\\
&([l_{12}',m_{12}',n_{12}'];\pi_{12}',\beta_{12}',\phi_{12}')\succ\oplus\prec([l_{13},m_{13},n_{13}];\pi_{13},\beta_{13},\phi_{13}),([l_{13}',m_{13}',n_{13}'];\pi_{13}',\beta_{13}',\phi_{13}')\succ\\
&\quad=\prec([70,100,130];1,0,0.4),([55,100,155];-1,0,-0.3)\succ\\
&\prec([l_{21},m_{21},n_{21}];\pi_{21},\beta_{21},\phi_{21}),([l_{21}',m_{21}',n_{21}'];\pi_{21}',\beta_{21}',\phi_{21}')\succ\oplus\prec([l_{22},m_{22},n_{22}];\pi_{22},\beta_{22},\phi_{22}),\\
&([l_{22}',m_{22}',n_{22}'];\pi_{22}',\beta_{22}',\phi_{22}')\succ\oplus\prec([l_{23},m_{23},n_{23}];\pi_{23},\beta_{23},\phi_{23}),([l_{23}',m_{23}',n_{23}'];\pi_{23}',\beta_{23}',\phi_{23}')\succ\\
&\quad=\prec([50,70,90];1,0,0.3),([35,70,115];-1,0,-0.2)\succ\\
&\prec([l_{31},m_{31},n_{31}];\pi_{31},\beta_{31},\phi_{31}),([l_{31}',m_{31}',n_{31}'];\pi_{31}',\beta_{31}',\phi_{31}')\succ\oplus\prec([l_{32},m_{32},n_{32}];\pi_{32},\beta_{32},\phi_{32}),\\
&([l_{32}',m_{32}',n_{32}'];\pi_{32}',\beta_{32}',\phi_{32}')\succ\oplus\prec([l_{33},m_{33},n_{33}];\pi_{33},\beta_{33},\phi_{33}),([l_{33}',m_{33}',n_{33}'];\pi_{33}',\beta_{33}',\phi_{33}')\succ\\
&\quad=\prec([10,30,30];1,0,0.1),([0,30,40];-1,0,-0.2)\succ\\
&\prec([l_{11},m_{11},n_{11}];\pi_{11},\beta_{11},\phi_{11}),([l_{11}',m_{11}',n_{11}'];\pi_{11}',\beta_{11}',\phi_{11}')\succ\oplus\prec([l_{21},m_{21},n_{21}];\pi_{21},\beta_{21},\phi_{21}),\\
&([l_{21}',m_{21}',n_{21}'];\pi_{21}',\beta_{21}',\phi_{21}')\succ\oplus\prec([l_{31},m_{31},n_{31}];\pi_{31},\beta_{31},\phi_{31}),([l_{31}',m_{31}',n_{31}'];\pi_{31}',\beta_{31}',\phi_{31}')\succ\\
&\quad=\prec([50,60,70];1,0,0.2),([30,60,100];-1,0,-0.1)\succ\\
&\prec([l_{12},m_{12},n_{12}];\pi_{12},\beta_{12},\phi_{12}),([l_{12}',m_{12}',n_{12}'];\pi_{12}',\beta_{12}',\phi_{12}')\succ\oplus\prec([l_{22},m_{22},n_{22}];\pi_{22},\beta_{22},\phi_{22}),\\
&([l_{22}',m_{22}',n_{22}'];\pi_{22}',\beta_{22}',\phi_{22}')\succ\oplus\prec([l_{32},m_{32},n_{32}];\pi_{32},\beta_{32},\phi_{32}),([l_{32}',m_{32}',n_{32}'];\pi_{32}',\beta_{32}',\phi_{32}')\succ\\
&\quad=\prec([40,70,100];1,0,0.1),([40,70,100];-1,0,-0.4)\succ\\
&\prec([l_{13},m_{13},n_{13}];\pi_{13},\beta_{13},\phi_{13}),([l_{13}',m_{13}',n_{13}'];\pi_{13}',\beta_{13}',\phi_{13}')\succ\oplus\prec([l_{23},m_{23},n_{23}];\pi_{23},\beta_{23},\phi_{23}),\\
&([l_{23}',m_{23}',n_{23}'];\pi_{23}',\beta_{23}',\phi_{23}')\succ\oplus\prec([l_{33},m_{33},n_{33}];\pi_{33},\beta_{33},\phi_{33}),([l_{33}',m_{33}',n_{33}'];\pi_{33}',\beta_{33}',\phi_{33}')\succ\\
&\quad=\prec([40,70,80];1,0,0.2),([20,70,110];-1,0,-0.3)\succ.
\end{aligned}\tag{44}$$

4.5. Step 5. By applying arithmetic operations, the FBSNTP (43) can be transformed as

$$\text{Minimize}\left(\begin{array}{c}
\prec\left(\left[15l_{11},17m_{11},18n_{11}\right];1\wedge\pi_{11},0\vee\beta_{11},0.2\vee\phi_{11}\right),\left(\left[16l_{11}',17m_{11}',19n_{11}'\right];\right.\\
\left.-1\vee\pi_{11}',0\wedge\beta_{11}',-0.3\wedge\phi_{11}'\right)\succ\oplus\prec\left(\left[20l_{12},22m_{12},24n_{12}\right];1\wedge\pi_{12},0\vee\beta_{12},0.3\vee\phi_{12}\right),\\
\left(\left[21l_{12}',22m_{12}',23n_{12}'\right];-1\vee\pi_{12}',0\wedge\beta_{12}',-0.4\wedge\phi_{12}'\right)\succ\oplus\prec\left(\begin{array}{c}\left[0l_{13},0m_{13},0n_{13}\right];\\1\wedge\pi_{13},0\vee\beta_{13},0\vee\phi_{13}\end{array}\right),\left(\left[0l_{13}',0m_{13}',0n_{13}'\right];-1\vee\pi_{13}',0\wedge\beta_{13}',0\wedge\phi_{13}'\right)\succ\oplus\\
\prec\left(\left[25l_{21},28m_{21},30n_{21}\right];1\wedge\pi_{21},0\vee\beta_{21},0.4\vee\phi_{21}\right),\left(\left[27l_{21}',28m_{21}',29n_{21}'\right];\right.\\
\left.-1\vee\pi_{21}',0\wedge\beta_{21}',-0.5\wedge\phi_{21}'\right)\succ\oplus\prec\left(\left[30l_{22},40m_{22},50n_{22}\right];1\wedge\pi_{22},0\vee\beta_{22},0.1\vee\phi_{22}\right),\\
\left(\left[35l_{22}',40m_{22}',45n_{22}'\right];-1\vee\pi_{22}',0\wedge\beta_{22}',-0.1\wedge\phi_{22}'\right)\succ\oplus\prec\left(\left[0l_{23},0m_{23},0n_{23}\right];\right.\\
\left.1\wedge\pi_{23},0\vee\beta_{23},0\vee\phi_{23}\right),\left(\left[0l_{23}',0m_{23}',0n_{23}'\right];-1\vee\pi_{23}',0\wedge\beta_{23}',0\wedge\phi_{23}'\right)\succ\oplus\\
\prec\left(\left[0l_{31},0m_{31},0n_{31}\right];1\wedge\pi_{31},0\vee\beta_{31},0\vee\phi_{31}\right),\left(\left[0l_{31}',0m_{31}',0n_{31}'\right];\right.\\
\left.-1\vee\pi_{31}',0\wedge\beta_{31}',0\wedge\phi_{31}'\right)\succ\oplus\prec\left(\left[0l_{32},0m_{32},0n_{32}\right];1\wedge\pi_{32},0\vee\beta_{32},0\vee\phi_{32}\right),\\
\left(\left[0l_{32}',0m_{32}',0n_{32}'\right];-1\vee\pi_{32}',0\wedge\beta_{32}',0\wedge\phi_{32}'\right)\succ\oplus\prec\left(\left[0l_{33},0m_{33},0n_{33}\right];\right.\\
\left.1\wedge\pi_{33},0\vee\beta_{33},0\vee\phi_{33}\right),\left(\left[0l_{33}',0m_{33}',0n_{33}'\right];-1\vee\pi_{33}',0\wedge\beta_{33}',0\wedge\phi_{33}'\right)\succ
\end{array}\right), \tag{45}$$

subject to

$$\left[\left(\sum_{j=1}^{3}l_{1j}=70,\sum_{j=1}^{3}m_{1j}=100,\sum_{j=1}^{3}n_{1j}=130,\ \sum_{j=1}^{3}l_{1j}'=55,\sum_{j=1}^{3}m_{1j}'=100,\sum_{j=1}^{3}n_{1j}'=155,\right);\left(\begin{array}{c}\wedge\left[\pi_{11}\wedge\pi_{12}\wedge\pi_{13}\right]=1\\\vee\left[\beta_{11}\vee\beta_{12}\vee\beta_{13}\right]=0\\\vee\left[\phi_{11}\vee\phi_{12}\vee\phi_{13}\right]=0.4\\\vee\left[\pi_{11}'\vee\pi_{12}'\vee\pi_{13}'\right]=-1\\\wedge\left[\beta_{11}'\wedge\beta_{12}'\wedge\beta_{13}'\right]=0\\\wedge\left[\phi_{11}'\wedge\phi_{12}'\wedge\phi_{13}'\right]=-0.3\end{array}\right)\right]$$

$$\left[\left(\sum_{j=1}^{3}l_{2j}=50,\sum_{j=1}^{3}m_{2j}=70,\sum_{j=1}^{3}n_{2j}=90,\ \sum_{j=1}^{3}l_{2j}'=35,\sum_{j=1}^{3}m_{2j}'=70,\sum_{j=1}^{3}n_{2j}'=115,\right);\left(\begin{array}{c}\wedge\left[\pi_{21}\wedge\pi_{22}\wedge\pi_{23}\right]=1\\\vee\left[\beta_{21}\vee\beta_{22}\vee\beta_{23}\right]=0\\\vee\left[\phi_{21}\vee\phi_{22}\vee\phi_{23}\right]=0.3\\\vee\left[\pi_{21}'\vee\pi_{22}'\vee\pi_{23}'\right]=-1\\\wedge\left[\beta_{21}'\wedge\beta_{22}'\wedge\beta_{23}'\right]=0\\\wedge\left[\phi_{21}'\wedge\phi_{22}'\wedge\phi_{23}'\right]=-0.2\end{array}\right)\right]$$

$$\left[\left(\sum_{j=1}^{3}l_{3j}=10,\sum_{j=1}^{3}m_{3j}=30,\sum_{j=1}^{3}n_{3j}=30,\ \sum_{j=1}^{3}l_{3j}'=0,\sum_{j=1}^{3}m_{3j}'=30,\sum_{j=1}^{3}n_{3j}'=40,\right);\left(\begin{array}{c}\wedge\left[\pi_{31}\wedge\pi_{32}\wedge\pi_{33}\right]=1\\\vee\left[\beta_{31}\vee\beta_{32}\vee\beta_{33}\right]=0\\\vee\left[\phi_{31}\vee\phi_{32}\vee\phi_{33}\right]=0.1\\\vee\left[\pi_{31}'\vee\pi_{32}'\vee\pi_{33}'\right]=-1\\\wedge\left[\beta_{31}'\wedge\beta_{32}'\wedge\beta_{33}'\right]=0\\\wedge\left[\phi_{31}'\wedge\phi_{32}'\wedge\phi_{33}'\right]=-0.2\end{array}\right)\right],$$

$$\left[\left(\sum_{k=1}^{3} l_{k1}=50, \sum_{k=1}^{3} m_{k1}=60, \sum_{k=1}^{3} n_{k1}=70, \ \sum_{k=1}^{3} l_{k1}'=30, \sum_{k=1}^{3} m_{k1}'=60, \sum_{k=1}^{3} n_{k1}'=100,\right); \begin{pmatrix} \wedge[\pi_{11}\wedge\pi_{21}\wedge\pi_{31}]=1 \\ \vee[\beta_{11}\vee\beta_{21}\vee\beta_{31}]=0 \\ \vee[\phi_{11}\vee\phi_{21}\vee\phi_{31}]=0.2 \\ \vee[\pi_{11}'\vee\pi_{21}'\vee\pi_{31}']=-1 \\ \wedge[\beta_{11}'\wedge\beta_{21}'\wedge\beta_{31}']=0 \\ \wedge[\phi_{11}'\wedge\phi_{21}'\wedge\phi_{31}']=-0.1 \end{pmatrix}\right]$$

$$\left[\left(\sum_{k=1}^{3} l_{k2}=40, \sum_{k=1}^{3} m_{k2}=70, \sum_{k=1}^{3} n_{k2}=100, \ \sum_{k=1}^{3} l_{k2}'=40, \sum_{k=1}^{3} m_{k2}'=70, \sum_{k=1}^{3} n_{k2}'=100,\right); \begin{pmatrix} \wedge[\pi_{12}\wedge\pi_{22}\wedge\pi_{32}]=1 \\ \vee[\beta_{12}\vee\beta_{22}\vee\beta_{32}]=0 \\ \vee[\phi_{12}\vee\phi_{22}\vee\phi_{32}]=0.1 \\ \vee[\pi_{12}'\vee\pi_{22}'\vee\pi_{32}']=-1 \\ \wedge[\beta_{12}'\wedge\beta_{22}'\wedge\beta_{32}']=0 \\ \wedge[\phi_{12}'\wedge\phi_{22}'\wedge\phi_{32}']=-0.4 \end{pmatrix}\right]$$

$$\left[\left(\sum_{k=1}^{3} l_{k3}=40, \sum_{k=1}^{3} m_{k3}=70, \sum_{k=1}^{3} n_{k3}=80, \ \sum_{k=1}^{3} l_{k3}'=20, \sum_{k=1}^{3} m_{k3}'=70, \sum_{k=1}^{3} n_{k3}'=110,\right); \begin{pmatrix} \wedge[\pi_{13}\wedge\pi_{23}\wedge\pi_{33}]=1 \\ \vee[\beta_{13}\vee\beta_{23}\vee\beta_{33}]=0 \\ \vee[\phi_{13}\vee\phi_{23}\vee\phi_{33}]=0.2 \\ \vee[\pi_{13}'\vee\pi_{23}'\vee\pi_{33}']=-1 \\ \wedge[\beta_{13}'\wedge\beta_{23}'\wedge\beta_{33}']=0 \\ \wedge[\phi_{13}'\wedge\phi_{23}'\wedge\phi_{33}']=-0.3 \end{pmatrix}\right], \quad (46)$$

where $\pi_{kj}, \beta_{kj}, \phi_{kj} \in [0,1]$, $\pi_{kj}', \beta_{kj}', \phi_{kj}' \in [-1,0]$ and l_{kj}, $m_{kj} - l_{kj}$, $n_{kj} - m_{kj} \geq 0, l_{kj}'$, $m_{kj}' - l_{kj}'$, $n_{kj}' - m_{kj}' \geq 0$; $k = 1,2,3$; $j = 1,2,3$.

4.6. Step 6. By applying score function Definition 8, the FBSNTP (45) can be transformed as

$$\text{Minimize } S\left[\begin{pmatrix} 15l_{11}+20l_{12}+0l_{13}+25l_{21}+30l_{22}+0l_{23}+0l_{31}+0l_{32}+0l_{33} \\ +17m_{11}+22m_{12}+0m_{13}+28m_{21}+40m_{22}+0m_{23}+0m_{31}+0m_{32}+0m_{33} \\ +18n_{11}+24n_{12}+0n_{13}+30n_{21}+50n_{22}+0n_{23}+0n_{31}+0n_{32}+0n_{33} \\ +16l_{11}'+21l_{12}'+0l_{13}'+27l_{21}'+35l_{22}'+0l_{23}'+0l_{31}'+0l_{32}'+0l_{33}'+ \\ 17m_{11}'+22m_{12}'+0m_{13}'+28m_{21}'+40m_{22}'+0m_{23}'+0m_{31}'+0m_{32}'+0m_{33}'+ \\ 19n_{11}'+23n_{12}'+0n_{13}'+29n_{21}'+45n_{22}'+0n_{23}'+0n_{31}'+0n_{32}'+0n_{33}' \end{pmatrix}\right], \quad (47)$$

subject to

$$\left[\left(\sum_{j=1}^{3} l_{1j}=70, \sum_{j=1}^{3} m_{1j}=100, \sum_{j=1}^{3} n_{1j}=130, \ \sum_{j=1}^{3} l_{1j}'=55, \sum_{j=1}^{3} m_{1j}'=100, \sum_{j=1}^{3} n_{1j}'=155,\right); \begin{pmatrix} \wedge[\pi_{11}\wedge\pi_{12}\wedge\pi_{13}]=1 \\ \vee[\beta_{11}\vee\beta_{12}\vee\beta_{13}]=0 \\ \vee[\phi_{11}\vee\phi_{12}\vee\phi_{13}]=0.4 \\ \vee[\pi_{11}'\vee\pi_{12}'\vee\pi_{13}']=-1 \\ \wedge[\beta_{11}'\wedge\beta_{12}'\wedge\beta_{13}']=0 \\ \wedge[\phi_{11}'\wedge\phi_{12}'\wedge\phi_{13}']=-0.3 \end{pmatrix}\right],$$

$$\left[\left(\sum_{j=1}^{3} l_{2j} = 50, \sum_{j=1}^{3} m_{2j} = 70, \sum_{j=1}^{3} n_{2j} = 90, \sum_{j=1}^{3} l_{2j}' = 35, \sum_{j=1}^{3} m_{2j}' = 70, \sum_{j=1}^{3} n_{2j}' = 115,\right); \begin{pmatrix} \wedge[\pi_{21}\wedge\pi_{22}\wedge\pi_{23}] = 1 \\ \vee[\beta_{21}\vee\beta_{22}\vee\beta_{23}] = 0 \\ \vee[\phi_{21}\vee\phi_{22}\vee\phi_{23}] = 0.3 \\ \vee[\pi_{21}'\vee\pi_{22}'\vee\pi_{23}'] = -1 \\ \wedge[\beta_{21}'\wedge\beta_{22}'\wedge\beta_{23}'] = 0 \\ \wedge[\phi_{21}'\wedge\phi_{22}'\wedge\phi_{23}'] = -0.2 \end{pmatrix}\right]$$

$$\left[\left(\sum_{j=1}^{3} l_{3j} = 10, \sum_{j=1}^{3} m_{3j} = 30, \sum_{j=1}^{3} n_{3j} = 30, \sum_{j=1}^{3} l_{3j}' = 0, \sum_{j=1}^{3} m_{3j}' = 30, \sum_{j=1}^{3} n_{3j}' = 40,\right); \begin{pmatrix} \wedge[\pi_{31}\wedge\pi_{32}\wedge\pi_{33}] = 1 \\ \vee[\beta_{31}\vee\beta_{32}\vee\beta_{33}] = 0 \\ \vee[\phi_{31}\vee\phi_{32}\vee\phi_{33}] = 0.1 \\ \vee[\pi_{31}'\vee\pi_{32}'\vee\pi_{33}'] = -1 \\ \wedge[\beta_{31}'\wedge\beta_{32}'\wedge\beta_{33}'] = 0 \\ \wedge[\phi_{31}'\wedge\phi_{32}'\wedge\phi_{33}'] = -0.2 \end{pmatrix}\right]$$

$$\left[\left(\sum_{k=1}^{3} l_{k1} = 50, \sum_{k=1}^{3} m_{k1} = 60, \sum_{k=1}^{3} n_{k1} = 70, \sum_{k=1}^{3} l_{k1}' = 30, \sum_{k=1}^{3} m_{k1}' = 60, \sum_{k=1}^{3} n_{k1}' = 100,\right); \begin{pmatrix} \wedge[\pi_{11}\wedge\pi_{21}\wedge\pi_{31}] = 1 \\ \vee[\beta_{11}\vee\beta_{21}\vee\beta_{31}] = 0 \\ \vee[\phi_{11}\vee\phi_{21}\vee\phi_{31}] = 0.2 \\ \vee[\pi_{11}'\vee\pi_{21}'\vee\pi_{31}'] = -1 \\ \wedge[\beta_{11}'\wedge\beta_{21}'\wedge\beta_{31}'] = 0 \\ \wedge[\phi_{11}'\wedge\phi_{21}'\wedge\phi_{31}'] = -0.1 \end{pmatrix}\right]$$

$$\left[\left(\sum_{k=1}^{3} l_{k2} = 40, \sum_{k=1}^{3} m_{k2} = 70, \sum_{k=1}^{3} n_{k2} = 100, \sum_{k=1}^{3} l_{k2}' = 40, \sum_{k=1}^{3} m_{k2}' = 70, \sum_{k=1}^{3} n_{k2}' = 100,\right); \begin{pmatrix} \wedge[\pi_{12}\wedge\pi_{22}\wedge\pi_{32}] = 1 \\ \vee[\beta_{12}\vee\beta_{22}\vee\beta_{32}] = 0 \\ \vee[\phi_{12}\vee\phi_{22}\vee\phi_{32}] = 0.1 \\ \vee[\pi_{12}'\vee\pi_{22}'\vee\pi_{32}'] = -1 \\ \wedge[\beta_{12}'\wedge\beta_{22}'\wedge\beta_{32}'] = 0 \\ \wedge[\phi_{12}'\wedge\phi_{22}'\wedge\phi_{32}'] = -0.4 \end{pmatrix}\right]$$

$$\left[\left(\sum_{k=1}^{3} l_{k3} = 40, \sum_{k=1}^{3} m_{k3} = 70, \sum_{k=1}^{3} n_{k3} = 80, \sum_{k=1}^{3} l_{k3}' = 20, \sum_{k=1}^{3} m_{k3}' = 70, \sum_{k=1}^{3} n_{k3}' = 110,\right); \begin{pmatrix} \wedge[\pi_{13}\wedge\pi_{23}\wedge\pi_{33}] = 1 \\ \vee[\beta_{13}\vee\beta_{23}\vee\beta_{33}] = 0 \\ \vee[\phi_{13}\vee\phi_{23}\vee\phi_{33}] = 0.2 \\ \vee[\pi_{13}'\vee\pi_{23}'\vee\pi_{33}'] = -1 \\ \wedge[\beta_{13}'\wedge\beta_{23}'\wedge\beta_{33}'] = 0 \\ \wedge[\phi_{13}'\wedge\phi_{23}'\wedge\phi_{33}'] = -0.3 \end{pmatrix}\right], \tag{48}$$

where $\pi_{kj}, \beta_{kj}, \phi_{kj} \in [0,1]$, $\pi_{kj}', \beta_{kj}', \phi_{kj}' \in [-1,0]$, and l_{kj}, $m_{kj} - l_{kj}$, $n_{kj} - m_{kj} \geq 0$, l_{kj}', $m_{kj}' - l_{kj}'$, $n_{kj}' - m_{kj}' \geq 0$; $k = 1,2,3$; $j = 1,2,3$.

4.7. Step 7. Now, we solve the following crisp linear programming problem as

$$\text{Minimize}\,\frac{1}{6}\begin{pmatrix} 15l_{11} + 20l_{12} + 0l_{13} + 25l_{21} + 30l_{22} + 0l_{23} + 0l_{31} + 0l_{32} + 0l_{33} \\ + 17m_{11} + 22m_{12} + 0m_{13} + 28m_{21} + 40m_{22} + 0m_{23} + 0m_{31} + 0m_{32} + 0m_{33} \\ + 18n_{11} + 24n_{12} + 0n_{13} + 30n_{21} + 50n_{22} + 0n_{23} + 0n_{31} + 0n_{32} + 0n_{33} \\ + 16l_{11}' + 21l_{12}' + 0l_{13}' + 27l_{21}' + 35l_{22}' + 0l_{23}' + 0l_{31}' + 0l_{32}' + 0l_{33}' + \\ 17m_{11}' + 22m_{12}' + 0m_{13}' + 28m_{21}' + 40m_{22}' + 0m_{23}' + 0m_{31}' + 0m_{32}' + 0m_{33}' + \\ 19n_{11}' + 23n_{12}' + 0n_{13}' + 29n_{21}' + 45n_{22}' + 0n_{23}' + 0n_{31}' + 0n_{32}' + 0n_{33}' \end{pmatrix}, \tag{49}$$

subject to

$$
\begin{aligned}
&\left[\left(\sum_{j=1}^{3} l_{1j}=70, \sum_{j=1}^{3} m_{1j}=100, \sum_{j=1}^{3} n_{1j}=130,\ \sum_{j=1}^{3} l_{1j}'=55, \sum_{j=1}^{3} m_{1j}'=100, \sum_{j=1}^{3} n_{1j}'=155,\right); \begin{pmatrix} \wedge[\pi_{11}\wedge\pi_{12}\wedge\pi_{13}]=1 \\ \vee[\beta_{11}\vee\beta_{12}\vee\beta_{13}]=0 \\ \vee[\phi_{11}\vee\phi_{12}\vee\phi_{13}]=0.4 \\ \vee[\pi_{11}'\vee\pi_{12}'\vee\pi_{13}']=-1 \\ \wedge[\beta_{11}'\wedge\beta_{12}'\wedge\beta_{13}']=0 \\ \wedge[\phi_{11}'\wedge\phi_{12}'\wedge\phi_{13}']=-0.3 \end{pmatrix}\right] \\
&\left[\left(\sum_{j=1}^{3} l_{2j}=50, \sum_{j=1}^{3} m_{2j}=70, \sum_{j=1}^{3} n_{2j}=90,\ \sum_{j=1}^{3} l_{2j}'=35, \sum_{j=1}^{3} m_{2j}'=70, \sum_{j=1}^{3} n_{2j}'=115,\right); \begin{pmatrix} \wedge[\pi_{21}\wedge\pi_{22}\wedge\pi_{23}]=1 \\ \vee[\beta_{21}\vee\beta_{22}\vee\beta_{23}]=0 \\ \vee[\phi_{21}\vee\phi_{22}\vee\phi_{23}]=0.3 \\ \vee[\pi_{21}'\vee\pi_{22}'\vee\pi_{23}']=-1 \\ \wedge[\beta_{21}'\wedge\beta_{22}'\wedge\beta_{23}']=0 \\ \wedge[\phi_{21}'\wedge\phi_{22}'\wedge\phi_{23}']=-0.2 \end{pmatrix}\right] \\
&\left[\left(\sum_{j=1}^{3} l_{3j}=10, \sum_{j=1}^{3} m_{3j}=30, \sum_{j=1}^{3} n_{3j}=30,\ \sum_{j=1}^{3} l_{3j}'=0, \sum_{j=1}^{3} m_{3j}'=30, \sum_{j=1}^{3} n_{3j}'=40,\right); \begin{pmatrix} \wedge[\pi_{31}\wedge\pi_{32}\wedge\pi_{33}]=1 \\ \vee[\beta_{31}\vee\beta_{32}\vee\beta_{33}]=0 \\ \vee[\phi_{31}\vee\phi_{32}\vee\phi_{33}]=0.1 \\ \vee[\pi_{31}'\vee\pi_{32}'\vee\pi_{33}']=-1 \\ \wedge[\beta_{31}'\wedge\beta_{32}'\wedge\beta_{33}']=0 \\ \wedge[\phi_{31}'\wedge\phi_{32}'\wedge\phi_{33}']=-0.2 \end{pmatrix}\right] \\
&\left[\left(\sum_{k=1}^{3} l_{k1}=50, \sum_{k=1}^{3} m_{k1}=60, \sum_{k=1}^{3} n_{k1}=70,\ \sum_{k=1}^{3} l_{k1}'=30, \sum_{k=1}^{3} m_{k1}'=60, \sum_{k=1}^{3} n_{k1}'=100,\right); \begin{pmatrix} \wedge[\pi_{11}\wedge\pi_{21}\wedge\pi_{31}]=1 \\ \vee[\beta_{11}\vee\beta_{21}\vee\beta_{31}]=0 \\ \vee[\phi_{11}\vee\phi_{21}\vee\phi_{31}]=0.2 \\ \vee[\pi_{11}'\vee\pi_{21}'\vee\pi_{31}']=-1 \\ \wedge[\beta_{11}'\wedge\beta_{21}'\wedge\beta_{31}']=0 \\ \wedge[\phi_{11}'\wedge\phi_{21}'\wedge\phi_{31}']=-0.1 \end{pmatrix}\right], \\
&\left[\left(\sum_{k=1}^{3} l_{k2}=40, \sum_{k=1}^{3} m_{k2}=70, \sum_{k=1}^{3} n_{k2}=100,\ \sum_{k=1}^{3} l_{k2}'=40, \sum_{k=1}^{3} m_{k2}'=70, \sum_{k=1}^{3} n_{k2}'=100,\right); \begin{pmatrix} \wedge[\pi_{12}\wedge\pi_{22}\wedge\pi_{32}]=1 \\ \vee[\beta_{12}\vee\beta_{22}\vee\beta_{32}]=0 \\ \vee[\phi_{12}\vee\phi_{22}\vee\phi_{32}]=0.1 \\ \vee[\pi_{12}'\vee\pi_{22}'\vee\pi_{32}']=-1 \\ \wedge[\beta_{12}'\wedge\beta_{22}'\wedge\beta_{32}']=0 \\ \wedge[\phi_{12}'\wedge\phi_{22}'\wedge\phi_{32}']=-0.4 \end{pmatrix}\right] \\
&\left[\left(\sum_{k=1}^{3} l_{k3}=40, \sum_{k=1}^{3} m_{k3}=70, \sum_{k=1}^{3} n_{k3}=80,\ \sum_{k=1}^{3} l_{k3}'=20, \sum_{k=1}^{3} m_{k3}'=70, \sum_{k=1}^{3} n_{k3}'=110,\right); \begin{pmatrix} \wedge[\pi_{13}\wedge\pi_{23}\wedge\pi_{33}]=1 \\ \vee[\beta_{13}\vee\beta_{23}\vee\beta_{33}]=0 \\ \vee[\phi_{13}\vee\phi_{23}\vee\phi_{33}]=0.2 \\ \vee[\pi_{13}'\vee\pi_{23}'\vee\pi_{33}']=-1 \\ \wedge[\beta_{13}'\wedge\beta_{23}'\wedge\beta_{33}']=0 \\ \wedge[\phi_{13}'\wedge\phi_{23}'\wedge\phi_{33}']=-0.3 \end{pmatrix}\right]
\end{aligned}
\tag{50}
$$

Table 2: Comparison of optimal values.

FBSNTP	$\prec$([2050, 3470, 5300]; 1, 0, 0.4), ([1810, 3915, 6490]; −1, 0, −0.5)$\succ$
FIFTP [5]	(2050, 3470, 5300); (2050, 3470, 5300)
FFTP [1]	(2050, 3470, 5300)

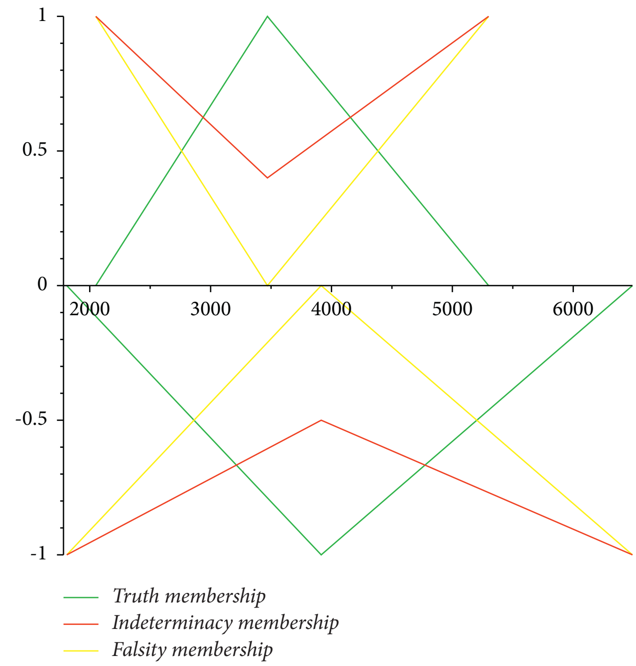

Figure 1: Graphical representation of bipolar single-valued neutrosophic transportation cost discussed in Example 1.

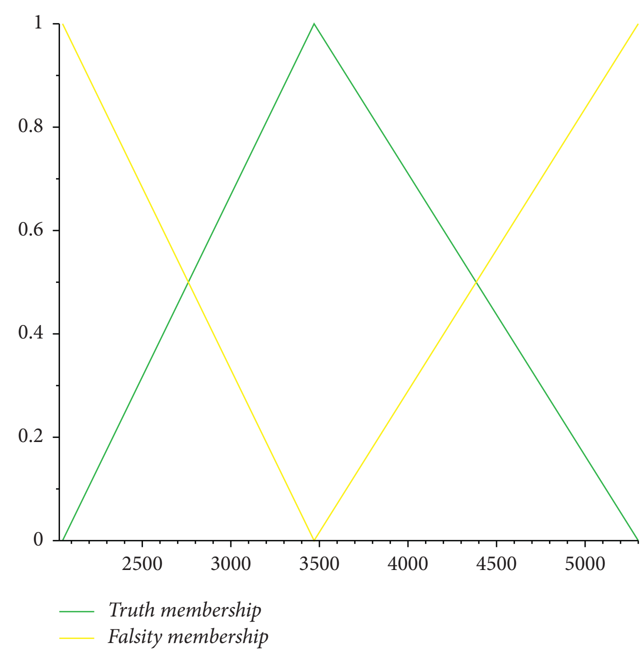

Figure 2: Graphical representation of intuitionistic transportation cost [5].

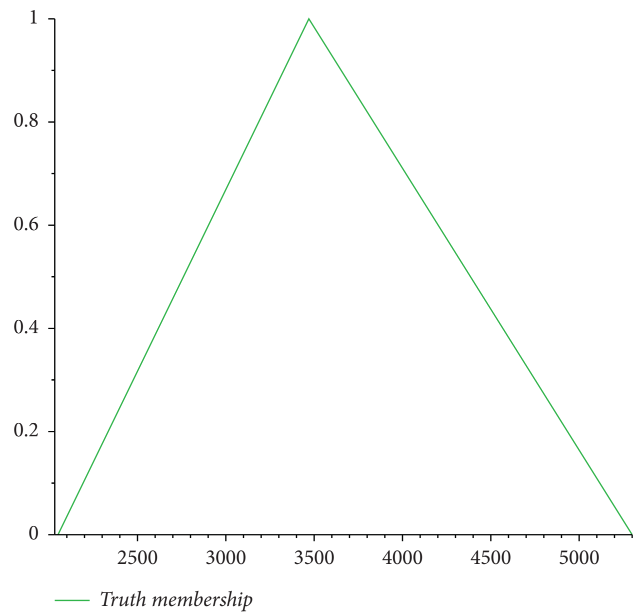

FIGURE 3: Graphical representation of fuzzy transportation cost [1].

where $\pi_{kj}, \beta_{kj}, \phi_{kj} \in [0,1]$, $\pi'_{kj}, \beta'_{kj}, \phi'_{kj} \in [-1,0]$, and l_{kj}, $m_{kj} - l_{kj}$, $n_{kj} - m_{kj} \geq 0$, l'_{kj}, $m'_{kj} - l'_{kj}$, $n'_{kj} - m'_{kj} \geq 0$; $k = 1,2,3$; $j = 1,2,3$.

4.8. Step 8. By using the software MAPLE, we get optimal solution

$$\begin{aligned}
&l_{11} = 0,\ l_{12} = 40,\ l_{13} = 30, l_{21} = 50,\ l_{22} = 0,\ l_{23} = 0,\ l_{31} = 0,\ l_{32} = 0,\ l_{33} = 10,\ m_{11} = 10,\\
&m_{12} = 50,\ m_{13} = 40,\ m_{21} = 50,\ m_{22} = 20,\ m_{23} = 0,\ m_{31} = 0,\ m_{32} = 0,\ m_{33} = 30,\ n_{11} = 20,\\
&n_{12} = 60,\ n_{13} = 50,\ n_{21} = 50,\ n_{22} = 40,\ n_{23} = 0,\ n_{31} = 0,\ n_{32} = 0,\ n_{33} = 30,\ l'_{11} = 30,\ l'_{12} = 5,\\
&l'_{13} = 20,\ l'_{21} = 0,\ l'_{22} = 35,\ l'_{23} = 0,\ l'_{31} = 0,\ l'_{32} = 0,\ l'_{33} = 0,\ m'_{11} = 55,\ m'_{12} = 5,\ m'_{13} = 40,\\
&m'_{21} = 5,\ m'_{22} = 65,\ m'_{23} = 0,\ m'_{31} = 0,\ m'_{32} = 0,\ m'_{33} = 30,\ n'_{11} = 80,\ n'_{12} = 5,\ n'_{13} = 70,\ n'_{21} = 20,\\
&n'_{22} = 95,\ n'_{23} = 0,\ n'_{31} = 0,\ n'_{32} = 0,\ n'_{33} = 40.
\end{aligned} \tag{51}$$

4.9. Step 9. The bipolar single-valued neutrosophic optimal solution is

$$\begin{aligned}
\tilde{x}_{11} &= \prec([0,10,20];1,0,0),\ ([30,55,80];-1,0,0),\\
\tilde{x}_{12} &= \prec([40,50,60];1,0,0),\ ([5,5,5];-1,0,0)\succ,\\
\tilde{x}_{13} &= \prec([30,40,50];1,0,0),\ ([20,40,70];-1,0,0)\succ,\\
\tilde{x}_{21} &= \prec([50,50,50];1,0,0),\ ([0,5,20];-1,0,0)\succ,\\
\tilde{x}_{22} &= \prec([0,20,40];1,0,0),\ ([35,65,95];-1,0,0)\succ,\\
\tilde{x}_{23} &= \prec([0,0,0];1,0,0),\ ([0,0,0];-1,0,0)\succ,\\
\tilde{x}_{31} &= \prec([0,0,0];1,0,0),\ ([0,0,0];-1,0,0)\succ,\\
\tilde{x}_{32} &= \prec([0,0,0];1,0,0),\ ([0,0,0];-1,0,0)\succ,\\
\tilde{x}_{33} &= \prec([10,30,30];1,0,0),\ ([0,30,40];-1,0,0)\succ.
\end{aligned} \tag{52}$$

4.10. Step 10. The minimum bipolar single-valued neutrosophic optimal value of FBSNTP is

$$\begin{aligned}
&\prec([2050,3470,5300];1,0,0.4),\\
&\quad([1810,3915,6490];-1,0,-0.5)\succ.
\end{aligned} \tag{53}$$

5. Comparison with Existing Transportation Model

Singh et al. [5] and Basirzadeh [1] suggested different techniques to solve intuitionistic fuzzy transportation problems and fuzzy transportation problems, respectively. We have proposed a method to solve an unbalanced FBSNTP. By using our proposed method to Example 1, which is discussed in Section 3, we have obtained the minimum total single-valued neutrosophic transportation cost $\prec([2050,3470,5300];1,0,0.4)$,

Table 3: FFC transportation model.

Plants	Lahore	Peshawar	Supply
Gujranwala	$\prec([15,16,17];1,0,0.2),([16,17,19];-1,0,-0.3)\succ$	$\prec([20,22,24];1,0,0.3),([21,22,23];-1,0,-0.4)\succ$	$\prec([70,100,130];1,0,0.4),([55,100,155];-1,0,-0.3)\succ$
Karachi	$\prec([25,28,30];1,0,0.4),([27,28,29];-1,0,-0.5)\succ$	$\prec([30,40,50];1,0,0.1),([35,40,45];-1,0,-0.1)\succ$	$\prec([50,70,90];1,0,0.3),([35,70,115];-1,0,-0.2)\succ$
Demand	$\prec([50,60,70];1,0,0.2),([30,60,100];-1,0,-0.1)\succ$	$\prec([40,70,100];1,0,0.1),([40,70,100];-1,0,-0.4)\succ$	

([1810, 3915, 6490]; − 1, 0, −0.5)>, which can be interpreted as follows:

(i) The smallest amount of the minimum total transportation cost is 2050 units for positive membership and 1810 units for negative membership

(ii) The achievable amount of the minimum total transportation cost is 3470 units for positive and 3915 units for negative membership respectively

(iii) The largest amount of the minimum total transportation cost is 5300 units for positive membership and 6490 units for negative membership

Thus, the minimum total transportation cost for positive and negative memberships will always be larger than 2050, 1810 units, and smaller than 5300, 6490 units respectively, while for both memberships most probably the minimum total transportation cost will be 3470, 3915 units.

Results of Example 1 and existing models [1, 5] are given in Table 2 and are shown graphically in Figures 1–3.

From Figures 1–3 it is proved that single-valued neutrosophic transportation model is the most generalized model.

6. Advantages of Proposed Method

The proposed transportation model is based on a bipolar single-valued neutrosophic environment. This method is comparatively better than other methods in terms of advantages.

(i) In literature there is no method to solve an unbalanced FBSNTP. So, this is a new and helpful approach for the decision makers.

(ii) A BSNT model is more powerful than an intuitionistic fuzzy model [5] and a fuzzy model [1]. Thus, this technique is more general than fuzzy and intuitionistic fuzzy environments respectively.

7. Limitations of the Proposed Method

In this section, the limitations of proposed method 3 are pointed out.

(i) The proposed method 3 can be used to find the minimum bipolar single-valued neutrosophic optimal value of the balanced and unbalanced FBSNTP by using nonnegative triangular and trapezoidal bipolar single-valued neutrosophic numbers

(ii) The unbalanced transportation problem 4.1 is given in Table 3

8. Conclusion

In this study, we have suggested a new technique to solve an unbalanced FBSNT problem by using nonnegative triangular bipolar single-valued neutrosophic numbers. A score function has been applied to transform TBSNNs into crisp numbers. We have solved the FBSNTP on the basis of bipolar single-valued neutrosophic linear programming formulation. Furthermore, we have compared our method with the existing fully intutionistic fuzzy transportation models [5] and fully fuzzy transportation models [1].

We aim to extend our study to include the following topic:

(1) bipolar single-valued neutrosophic rough transportation models.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] H. Basirzadeh, "An approach for solving fuzzy transportation problem," *Applied Mathematical Sciences*, vol. 5, no. 32, pp. 1549–1566, 2011.

[2] A. Kaur, J. Kacprzyk, and A. Kumar, "Fuzzy transportation and transshipment problems," *Studies in Fuzziness and Soft Computing*, vol. 16, p. 385, 2020.

[3] A. Nagoorgani and K. A. Razak, "Two stage fuzzy transportation problem," *Journal of Physical Science*, vol. 10, pp. 63–69, 2006.

[4] S. K. Singh and S. P. Yadav, "Efficient approach for solving type-1 intuitionistic fuzzy transportation problem," *International Journal of System Assurance Engineering and Management*, vol. 6, no. 3, pp. 259–267, 2015.

[5] S. K. Singh and S. P. Yadav, "A new approach for solving intuitionistic fuzzy transportation problem of type-2," *Annals of Operations Research*, vol. 243, no. 1-2, pp. 349–363, 2016.

[6] L. A. Zadeh, "Information and Control," *Fuzzy sets*, vol. 8, no. 3, pp. 338–353, 1965.

[7] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-I," *Information Sciences*, vol. 8, no. 3, pp. 199–249, 1965.

[8] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[9] F. Smarandache, *Neutrosophy, Neutrosophic Probability, Set and Logic*, American Research Press, Rehoboth, USA, 1998.

[10] H. Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman, "Single-valued neutrosophic sets," *Multispace and Multistructure*, vol. 4, pp. 410–413, 2010.

[11] I. Deli, M. Ali, and F. Smarandache, "Bipolar neutrosophic sets and their application based on multi-criteria decision making problems," in *Proceedings of the International Conference on Advanced Mechatronic System*, pp. 22–24, Beijing, China, August 2015.

[12] F. L. Hitchcock, "The distribution of a product from several sources to numerous localities," *Journal of Mathematics and Physics*, vol. 20, no. 1-4, pp. 224–230, 1941.

[13] K. Ladji, D. Moctar, K. Souleymane, B. Hawa, K. Moussa, and T. E. Koura, "A new algorithm for fuzzy transportation problems with trapezoidal fuzzy numbers under fuzzy circumstances," *Journal of Fuzzy Extension and Applications*, vol. 2, no. 3, pp. 204–225, 2021.

[14] J. Pratihar, R. Kumar, S. A. Edalatpanah, and A. Dey, "Modified Vogel's approximation method for transportation problem under uncertain environment," *Complex & Intelligent Systems*, vol. 7, no. 1, pp. 29–40, 2021.

[15] Ö. N. Çam and H. K. Sezen, "The formulation of a linear programming model for the vehicle routing problem in order to minimize idle time," *Decision Making: Applications in Management and Engineering*, vol. 3, no. 1, pp. 22–29, 2020.
[16] H.-J. Zimmermann, "Fuzzy programming and linear programming with several objective functions," *Fuzzy Sets and Systems*, vol. 1, no. 1, pp. 45–55, 1978.
[17] F. Hosseinzadeh Lotfi, T. Allahviranloo, M. Alimardani Jondabeh, and L. Alizadeh, "Solving a full fuzzy linear programming using lexicography method and fuzzy approximate solution," *Applied Mathematical Modelling*, vol. 33, no. 7, pp. 3151–3156, 2009.
[18] T. Allahviranloo, F. H. Lotfi, M. K. Kiasary, N. A. Kiani, and L. A. Zadeh, "Solving fully fuzzy linear programming problem by the ranking function," *Applied Mathematical Sciences*, vol. 2, no. 1, pp. 19–32, 2008.
[19] D. Behera, K. Peters, S. A. Edalatpanah, and D. Qiu, "New methods for solving imprecisely defined linear programming problem under trapezoidal fuzzy uncertainty," *Journal of Information and Optimization Sciences*, vol. 42, no. 7, pp. 603–629, 2020.
[20] J. Kaur and A. Kumar, *An Introduction to Fuzzy Linear Programming Problems*, Springer Science and Business Media LLC, Berlin, Germany, 2016.
[21] J. Kaur and A. Kumar, "Exact fuzzy optimal solution of fully fuzzy linear programming problems with unrestricted fuzzy variables," *Applied Intelligence*, vol. 37, no. 1, pp. 145–154, 2012.
[22] A. Kaur and A. Kumar, "A new approach for solving fuzzy transportation problems using generalized trapezoidal fuzzy numbers," *Applied Soft Computing*, vol. 12, no. 3, pp. 1201–1213, 2012.
[23] H. S. Najafi and S. A. Edalatpanah, "A note on a new method for solving fully fuzzy linear programming problems," *Applied Mathematical Modelling*, vol. 37, no. 14-15, pp. 7865–7867, 2013.
[24] A. Abhishekh and A. K. Nishad, "A novel ranking approach to solving fully LR-intuitionistic fuzzy transportation problem," *New Mathematics and Natural Computation*, vol. 15, no. 1, pp. 95–112, 2019.
[25] S. A. Edalatpanah, "A data envelopment analysis model with triangular intuitionistic fuzzy numbers," *International Journal of Data Envelopment Analysis*, vol. 7, no. 4, pp. 47–58, 2019.
[26] A. Kabiraj, P. K. Nayak, and S. Raha, "Solving intuitionistic fuzzy linear programming problem," *International Journal of Intelligence Science*, vol. 09, no. 01, pp. 44–58, 2019.
[27] C. Malathi and P. Umadevi, "A new procedure for solving linear programming problems in an intuitionistic fuzzy environment," *Journal of Physics: Conference Series*, vol. 1139, Article ID 012079, 2018.
[28] M. Akram, I. Ullah, M. G. Alharbi, and T. Allahviranloo, "Methods for solving L R -type pythagorean fuzzy linear programming problems with mixed constraints," *Mathematical Problems in Engineering*, vol. 2021, no. 4, pp. 1–29, 2021.
[29] M. Akram, I. Ullah, T. Allahviranloo, and S. A. Edalatpanah, "LR-type fully Pythagorean fuzzy linear programming problems with equality constraints," *Journal of Intelligent and Fuzzy Systems*, vol. 41, no. 1, pp. 1975–1992, 2021.
[30] M. Akram, I. Ullah, and M. G. Alharbi, "Methods for solving L R -type pythagorean fuzzy linear programming problems with mixed constraints," *Mathematical Problems in Engineering*, vol. 2021, pp. 1–29, Article ID 4306058, 2021.
[31] M. Abdel-Basset, M. Gunasekaran, M. Mohamed, and F. Smarandache, "A novel method for solving the fully neutrosophic linear programming problems, Neural Computing and Applications," *Spring*, vol. 31, no. 6, pp. 1595–1605, 2018.
[32] A.-N. Hussian, M. Mohamed, M. Abdel-Baset, and F. Smarandache, "Neutrosophic linear programming problem," *Mathematical Sciences Letters*, vol. 6, no. 3, pp. 319–324, 2017.
[33] T. Bera and N. K. Mahapatra, "Generalised single valued neutrosophic number and its application to neutrosophic linear programming," *Neutrosophic Sets and Systems*, vol. 25, no. 1, 2019.
[34] S. K. Das and A. Chakraborty, "A new approach to evaluate linear programming problem in pentagonal neutrosophic environment," *Complex Intell. Syst.*vol. 7, no. 2021, pp. 101–110, 2021.
[35] S. Das and J. K. Dash, "Modified solution for neutrosophic linear programming problems with mixed constraints," *International Journal of Research in Industrial Engineering*, vol. 9, no. 1, pp. 13–24, 2020.
[36] S. A. Edalatpanah, "A direct model for triangular neutrosophic linear programming," *International journal of neutrosophic science*, vol. 1, no. 1, pp. 19–28, 2020.
[37] H. A. E.-W. Khalifa, P. Kumar, and P. Kumar, "Solving fully neutrosophic linear programming problem with application to stock portfolio selection," *Croatian Operational Research Review*, vol. 11, no. 2, pp. 165–176, 2020.
[38] J. Ahmed, "LR-type fully single-valued neutrosophic linear programming problems," *Neutrosophic Sets and Systems*, vol. 46, no. 1, pp. 416–444, 2021.
[39] M. Akram, T. Allahviranloo, W. Pedrycz, and M. Ali, "Methods for solving LR-bipolar fuzzy linear systems," *Soft Computing*, vol. 25, no. 1, pp. 85–108, 2021.
[40] M. A. Mehmood, M. Akram, M. G. Alharbi, and S. Bashir, "Solution of fully bipolar fuzzy linear programming models," *Mathematical Problems in Engineering*, vol. 2021, pp. 1–31, Article ID 9961891, 2021.
[41] M. A. Mehmood, M. Akram, M. S. Alharbi, and S. Bashir, "Optimization of LR-type fully bipolar fuzzy linear programming problems," *Mathematical Problems in Engineering*, vol. 2021, p. 36, 2021.
[42] J. Ahmed, M. Alharbi, M. Akram, and S. Bashir, "A new method to evaluate linear programming problem in bipolar single-valued neutrosophic environment," *Computer Modeling in Engineering and Sciences*, vol. 129, no. 2, pp. 881–906, 2021.
[43] S. Kumar Das, "Application of transportation problem under pentagonal neutrosophic environment," *Journal of Fuzzy Extension and Applications*, vol. 1, no. 1, pp. 27–40, 2020.
[44] A. Singh, R. Arora, and S. Arora, "Bilevel transportation problem in neutrosophic environment," *Computational and Applied Mathematics*, vol. 41, no. 44, 2022.
[45] C. Veeramani, S. A. Edalatpanah, and S. Sharanya, "Solving the multiobjective fractional transportation problem through the neutrosophic goal programming approach," *Discrete Dynamics in Nature and Society*, vol. 2021, p. 17, Article ID 7308042, 2021.
[46] Z. Ali, T. Mahmood, T. Mahmood, K. Ullah, and Q. Khan, "Einstein geometric aggregation operators using a novel complex interval-valued Pythagorean fuzzy setting with application in green supplier chain management," *Reports in Mechanical Engineering*, vol. 2, no. 1, pp. 105–134, 2021.

[47] D. K. Kushwaha, D. Panchal, D. Panchal, and A. Sachdeva, "Risk analysis of cutting system under intuitionistic fuzzy environment," *Reports in Mechanical Engineering*, vol. 1, no. 1, pp. 162–173, 2020.
[48] R. Kumar, S. A. Edalatpanah, S. Jha, and R. Singh, "A Pythagorean fuzzy approach to the transportation problem," *Complex & Intelligent Systems*, vol. 5, no. 2, pp. 255–263, 2019.
[49] A. Kumar, J. Kaur, and P. Singh, "A new method for solving fully fuzzy linear programming problems," *Applied Mathematical Modelling*, vol. 35, no. 2, pp. 817–823, 2011.
[50] A. Chakraborty, P. S. Mondal, S. Alam, and A. Dey, "Classifcation of trapezoidal bipolar neutrosophic number, de bipolarization technique and its execution in cloud service based MCGDM problem," *Complex and Intelligent Systems*, vol. 7, no. 8, pp. 145–162, 2020.
[51] S. K. Bharati and S. R. Singh, "Solution of multiobjective linear programming problems in interval-valued intuitionistic fuzzy environment," *Soft Computing*, vol. 23, no. 1, pp. 77–84, 2019.
[52] Y. Subas, *Neutrosophic Numbers and Their Application to Multi-Attribute Decision Making Problems*, , Masters Thesis, Kilis 7 Aralik University, Graduate School of Natural and Applied Science, 2015.