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Special Issue
Discrete and Dynamic Optimization Problems in Operation Management

Guest Editors: Xiang Li, Chen Zhou, and Xiaochen Sun

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Editorial Board

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Editorial

Discrete and Dynamic Optimization Problems in Operation Management

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1. Introduction

Discrete and dynamic optimization is a significant methodology that has been widely applied to operation management fields. This issue on discrete and dynamic optimization problems in operation management aims at an all-around research and the state-of-the-art theoretical, numerical, and practical achievements that contribute to this field. This issue contains 21 papers, with the following features.

2. On Financial Engineering

C. Zhang and X. Rong study the optimal investment strategies of defined contribution pension, with the stochastic interest rate and stochastic salary. H. Zhao et al. consider the optimal investment problem for an insurer in a complete market. Optimal strategies are obtained via martingale approach, and computational results can provide practical advice to insurers. Y. Li and G. Liu shed light on the dynamic proportional reinsurance in a two-dimensional compound Poissons' risk model.

3. On Production and Distribution Optimization

Y. Zhou and J. Sun investigate an inventory replenishment problem with component substitution and product substitution simultaneously in the product-updated system. X. Sun et al. study an inventory replenishment and production planning problem for a two-period inventory system with dependent returns and demands. L. Liao et al. develop mathematical models of inventory-distribution routing problem for two-echelon agriculture products distribution network. J. Ma and G. Sun use a mutation Ant Colony Algorithm to solve the milk-run VRP with fastest completion time. W. Xue et al. study the joint inventory and sales-effort management problems of a retailer in a broad context and investigate the optimal policies for a single-item, periodic-review system.

4. On Supply Chain Optimization

L. Xu et al. introduce the reference effect into the sea-cargo supply chain and study a multiple-period contract problem between the carrier and forwarder. G. Sun develops a dynamic model in a one-supplier—one-retailer fresh agricultural products supply chain that experiences supply disruptions during the planning horizon. W. Hu and J. Li integrate a retailer's return policy and a supplier's buy-back policy within a supply chain with return logistics modeling framework. F. Hu et al. analyze a coordination problem with random demands and random supplies with disruptions in a supply chain system with one retailer and two suppliers. W. Liu et al. built a time scheduling model of logistics service supply chain, which is constrained by the service order time requirement. Y. Luo et al. study a process of product renewal in a supply chain with one manufacturer and one retailer, with optimal product pricing strategies derived. J. Wei et al. study the pricing decisions of a two-echelon supply chain with one manufacturer and duopolistic retailers in fuzzy environment. J. Zhao et al. develop the optimal pricing and remanufacturing decisions problem of a fuzzy closed-loop supply chain.

5. On Scheduling and Queuing Systems

X. Li and J. Li consider the workload process of the M/G/1 queuing system. Under different cases, they analyze the explicit criteria of the geometric rate of convergence and the geometric decay of stationary tail. R. Zhang considers a parallel machine scheduling problem with random processing/setup times and adjustable production rates. S. Lv and J. Li shed light on the multiserver repairable queuing system with variable breakdown rates, which widely exists in practical life.

6. Others

Z. Liao et al. study the innovation diffusion problem with the affection of urbanization, propose a dynamical innovation diffusion model with fuzzy coefficient, and use the shifting rate of people from rural areas stepping into urban areas to show the process of urbanization. L. Fu et al. use universal combinatorial operation model to describe the logic relationship and gave a flexible logic control method, which is useful to realize an effective control for complex system.

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Research Article

Mutation Ant Colony Algorithm of Milk-Run Vehicle Routing Problem with Fastest Completion Time Based on Dynamic Optimization

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The objective of vehicle routing problem is usually to minimize the total traveling distance or cost. But in practice, there are a lot of problems needed to minimize the fastest completion time. The milk-run vehicle routing problem (MRVRP) is widely used in milk-run distribution. The mutation ACO is given to solve MRVRP with fastest completion time in this paper. The milk-run VRP with fastest completion time is introduced first, and then the customer division method based on dynamic optimization and split algorithm is given to transform this problem into finding the optimal customer order. At last the mutation ACO is given and the numerical examples verify the effectiveness of the algorithm.

1. Introduction

The vehicle routing problem (VRP) was firstly brought forward by Dantzig and Ramser in 1960 [1]. With the development of VRP, there exist several variations and specializations. Mostly VRP aims to minimize the total travel distance (or travel time) and total cost. But for distribution of fast foods, express delivery, and emergency supplies, these objectives are not suitable and the completion time is more important.

Though the vehicle routing problem with time windows takes the service time into consideration [2], it cannot solve the problem when all customers want to be served as early as possible. Nikolakopoulou et al. solved a vehicle routing problem by balancing the vehicles time utilization [3]. But it maybe takes a long time for some vehicles to complete the distribution tasks when the vehicles time utilizations are balanced. The VRP with fastest completion time was studied to minimize the fastest completion time [4–6].

Traditional vehicle routing problem assumes that the total distribution can be completed by one vehicle in a round trip. In practice, the number of vehicles is limited and there are many customers to be served. So a round trip by one vehicle is impossible and more round trips and vehicles are needed, such as the tobacco distribution with thousands of customers. As JIT production, small-scaled and multiple-batch distributions are more popular, in which one vehicle is needed to collect goods on multiple round trips. This kind of vehicle routing problem is called milk-run vehicle routing problem (MRVRP).

For problem aiming to minimize total travel distance or total cost, MRVRP can be transferred into the traditional VPR by increasing the routes number and allowing a vehicle serving customers in different routes. But for problem to minimize the fastest completion time, the completion time in the first period affects that in the second period. It is multiperiod optimization problem, so the algorithm of the MRVRP with fastest completion time will be studied based on dynamic optimization.

VRP is an NP-hard problem [7]; heuristics and evolutionary algorithms are used to solve VRP. In this paper, mutation ant colony algorithm is used to solve MRVRP with fastest completion time. In the next section, we will give the description of the MRVRP with fastest completion time. In Section 3, optimal division methods of customer orders based on dynamic optimization are given. In Section 4, mutation

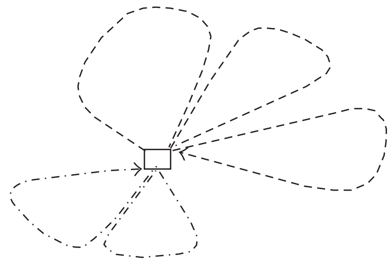

Figure 1: The milk-run vehicle routes of two vehicles.

ant colony algorithm is given to solve the problem. In the last section, a numerical example is given.

2. MRVRP with Fastest Completion Time

Suppose that there is one depot serving n customers, which have m vehicles with capacity w. The demand of customer i is q_i $(i = 1,\dots,n)$ and satisfies $(k-1)w \leq \sum_{i=1}^{n} q_i < kw$, which means that for every vehicle, it needs k times to finish the distribution tasks on average.

Because the distances between the depot and the customers and the distances between the customers are different, the travel time between any two nodes is different. Supposing that the traveling time between node i and node j is t_{ij}, $i = 0,1,\dots,n$, $j = 0,1,\dots,n$, node 0 is depot and the times satisfy the triangle inequality.

The vehicle routes satisfy the following.

(i) Each customer is served by a certain vehicle.

(ii) One vehicle can serve many customers. When the demands of the customers exceed the vehicle capacity, the vehicle returns to the depot to unload and goes back to serve the next customer.

(iii) The demands of the customers served by a vehicle cannot exceed k times vehicle capacity.

(iv) All vehicles should depart from the depot and return to the depot.

When the demands of the customers exceed the vehicle capacity, the customers should be divided into different groups. The customers in the first group are served first, then the second group, until all the customers are served, which is shown in Figure 1.

For the MRVRP aiming to minimize total travel distance or total cost, because the traveling distance of a vehicle is the sum of the distances of each round and the total distance is the sum of all vehicle traveling distances, the MRVRP can be transferred as the VRP with km vehicles. The customers are divided into km routes and each vehicle serves k routes. The total traveling distance of m vehicles equals the total distance of km routes. Then if the total distance of km routes is minimized, the total traveling distance of m vehicles is also minimized.

But for the MRVRP with fastest completion time, the completion time of the whole distribution task is the completion time of the last vehicle. Suppose that the completion time of vehicle l is T_l, $l = 1,2,\dots,m$, and the completion time of the whole distribution task is T; then we have,

$$T = \max_{l=1,2,\dots m} T_l. \quad (1)$$

For each vehicle, the distribution task needs to complete several routes and the completion time is determined by the time when last customer served. Because there is order of each route, the completion time of vehicle i is the sum of every route completion time. In the last route, the time from the customer to the depot is not considered.

If this problem is transferred to the problem with km routes in which each vehicle is in charge of k routes, the completion time is the sum of the travel times of previous $k-1$ routes and the completion time of the last routes. Supposing that the lth vehicle is in charge of the distribution task in k routes, the traveling time of the jth route is a_j, $j = 1,2,\dots,k$, and the traveling time from the last served customer to the depot is $\bar{t}$, the completion time of this vehicle is

$$T_l = \sum_{j=1}^{k} a_j - \bar{t}. \quad (2)$$

The difficulty of solving VRP lies in too many arrays of customer service order. It is hard to solve VRP by dynamic programming since there are too many states. For a given array of customers, the problem is transferred to how to divide the customers into groups and the customers in the same group are served by one vehicle. The feasible groups' division scheme is much less and it is easy to be solved by dynamic programming. In this paper, we will fix the serving array first and then give the optimal division scheme and calculate the fastest completion time. Taking this completion time as the objective, we will determine the final customer serving order by ACO.

3. Customer Division Method Based on Dynamic Programming

The problem of dividing customers into groups is how to divide a given serving array $i_1, i_2, \dots, i_n$ into m groups to ensure that the completion time of all the distribution tasks is minimized and the total demands of the same group do not exceed kw. The customers belonging to the same group are served by one vehicle on the array order.

Usually there are a lot of division schemes and their completion times are different. Nikolakopoulou gave the split method to minimize the total traveling distances by transferring the dividing problem into the shortest-path problem [3]. We gave the improved split method to minimize the fastest completion time by transferring the division problem into the longest-edge shortest problem [5]. The improved split method spends a long calculation time, as it repeats to find the shortest paths. Then the customer division method based on dynamic programming is given which spends less time [6].

For the MRVRP, there are two divisions: the first is to divide the customers into m groups and the second is to divide the customers in the same groups into k routes. The aim of the first division is to minimize the longest completion time and is solved with dynamic programming. The aim of the second division is to minimize the total traveling time and is solved with split method.

3.1. The First Division of Customers. The problem of dividing customers into m groups is to determine $m-1$ cut points and can be looked as at m-period decision problem. In a period, the number of customers served by a vehicle is determined. So the state variable is the number of customers who have not assigned, denoted as s, and the number of vehicles which have not assigned, denoted as x. Then the state variable is (s, x), and the initial state is (n, m). The decision variable is the number z of customers served by the next vehicle.

In the next period, the numbers of unassigned customers and vehicles are $s-z$ and $x-1$. The state variable is $(s-z,\ x-1)$. The state variable and decision variable are both confined by the vehicle capacity. For a given (s, x), the demands of assigned customer cannot exceed k times vehicle capacity. Supposing that d_i $(i = 1, \ldots, n)$ is the total demand from the first customer to the ith customer, we have

$$d_{n-s} \leq (m-x)\, kw. \tag{3}$$

When the decision variable in the present period is z and the unassigned customers need $x-1$ vehicles to distribute, the total demand of the z customers cannot exceed kw; that is,

$$\begin{gathered} d_n - d_{n-s+z} \leq (x-1)\, kw, \\ d_{n-s+z} - d_{n-s} \leq kw. \end{gathered} \tag{4}$$

Denote decision variable set as Ω; then

$$\Omega = \left\{ z \in \mathrm{Z} \mid d_n - d_{n-s+z} \leq (x-1)\, kw,\ d_{n-s+z} - d_{n-s} \leq kw \right\}. \tag{5}$$

Supposing that $f(s,x)$ is the fastest completion time of serving the s customers with x vehicles and $h(s,z)$ is the completion time if the former z customers are served by a vehicle, we have

$$f(s,x) = \min_{z \in \Omega} \max \left(h(s,z),\, f(s-z, x-1) \right). \tag{6}$$

When there is a vehicle left, if the total demands of s customers do not exceed kw, the s customers are served by this vehicle and the completion time is $h(s,s)$; otherwise the completion time is infinite; that is,

$$f(s,1) = \begin{cases} h(s,s), & \text{if } q_{n-s+1} + \cdots q_n \leq w, \\ +\infty, & \text{if } q_{n-s+1} + \cdots q_n > w. \end{cases} \tag{7}$$

The state variable and decision variable are discrete. It can be solved by enumeration method. In a period, the vehicle number decreases one unit and the unassigned vehicle is determined by the periods. Then the number of states is determined by the number of unassigned customers, the number of states of each period does not exceed n. Since the upper bound of vehicle capacity is kw, there is also an upper bound of the number of customers to be served, denoted as g. The maximum iteration time of the method is mng. In practice, since the state variable should satisfy the inequality (3) and the decision variable should satisfy the inequality (4), the actual iteration time is much less than mng.

In each period, the completion time of every vehicle is needed to calculate and the second division of customers is needed.

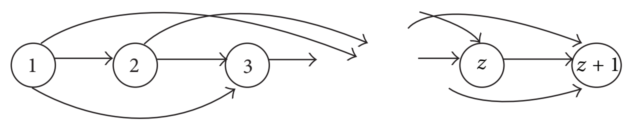

Figure 2: Split graph.

3.2. The Second Division of Customers. Suppose that the customers served by a given vehicle are $i_1, i_2, \ldots, i_z$. Because the total demands exceed the vehicle capacity w, the route should be cut into smaller groups to ensure that the customers in the same group can be served by the same vehicle. This is also a division problem which is different from that in Section 3.1 in the scale. The scale is smaller and the objective is not to minimize the maximal completion but to minimize the total milk run completion time.

To get the division schemes, a directed graph is constructed. The vertex set has the depot and the customers served by the same vehicle. For nodes j_1 and j_2, supposing that i_{j_1} is in front of i_{j_2}, the total demand of all customers between i_{j_1} and i_{j_2} (Including customer i_{j_1} but not i_{j_2}) is calculated. If the total demand is less than w, an arc from j_1 to j_2 is drawn. The weight is the traveling time from the depot in turn reach all customers between i_{j_1} and i_{j_2} (including customer i_{j_1} but not i_{j_2}), and then return. The weighted direct graph is called split graph which is shown in Figure 2.

The shortest distance from 1 to $z+1$ is the shortest travel time of all distribution tasks and the fastest completion time is this time minus the return time from customer i_z to depot.

Suppose the capacity of a vehicle is 15 and the order of customers to be served by the vehicle is 1-2-5-3-4-7; the traveling times between the nodes are shown in Figure 3, where the numbers in the brackets are the demands.

The total demand of customers 1, 2, and 5 is no more than 15; then an edge is drawn between node 1 and 3 with the weight 90. The other weights are got similarly. After calculation, the directed weighted graph is got in Figure 4.

The shortest path is from node 1 to node 3 plus from node 3 to node 0. The optimal distance is 180. Customers 1, 2, and 5 construct the first route. Customers 3, 4, and 7 construct the second route. The completion time is 155 (= 180 − 25).

The algorithm to solve the shortest path problem such as Dijkstra algorithm is polynomial. Its complexity is $O(n^2)$, where n is the number of nodes. For this problem, the maximum number of nodes is g and the complexity is $O(g^2)$. Because for every iteration the shortest path problem should be solved to calculate the completion time of a vehicle, the complexity of the whole algorithm is $O(mng^3)$.

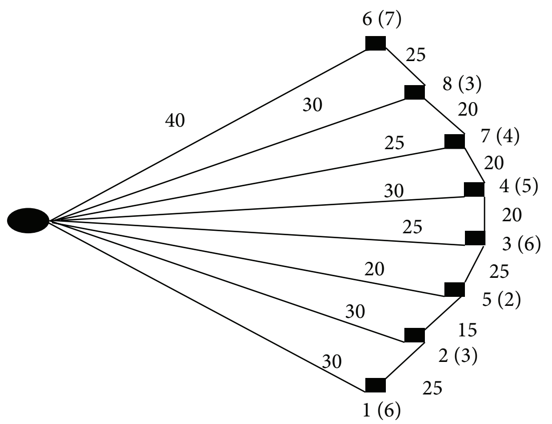

Figure 3: The distribution of customers.

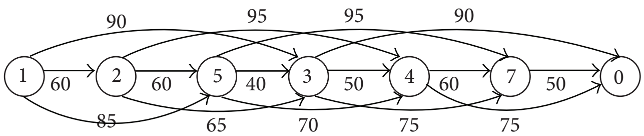

Figure 4: Split graph.

4. Mutation ACO Algorithm

In this section, we will give an improved ACO to solve the optimal customers array.

4.1. ACO Algorithm. The main idea of ant colony algorithms is to mimic the pheromone trail used by real ants as a medium for communication and feedback among ants. Basically, the ACO algorithm is a population-based, cooperative search procedure that is derived from the behavior of real ants. ACO algorithms make use of simple agents called ants that iteratively construct solutions to combinatorial optimization problems. The key problem to solve VRP with ACO is how an individual ant constructs a complete solution by starting with a null solution and iteratively adding solution components until a complete solution is constructed. The key problem of ACO is to determine the pheromone matrix. The pheromone matrix is $(n+1)\times n$ when there are n customers, where the last row stands for the information from the depot to the customer and the ith row stands for the information from customer i to other customers. Initially, since for a given customer there is the same possibility following other customers, the pheromone matrix starts with equal probable matrix. When $t=0$, the pheromone matrix is

$$B_{ij}(0)=\begin{Bmatrix} 0 & \dfrac{1}{n-1} & \cdots & \dfrac{1}{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{1}{n-1} & \dfrac{1}{n-1} & \cdots & 0 \\ \dfrac{1}{n} & \dfrac{1}{n} & \cdots & \dfrac{1}{n} \end{Bmatrix}_{(n+1)\times n}, \qquad (8)$$

where b_{ij}, $i=1,2,\dots,n+1$, $j=1,2,\dots,n$ is the pheromone from i to j.

In the tth iteration, there are m ants and m customer arrays can be got. Supposing that the fastest completion time of each array is $L_k(t)$; $k=1,2,\dots,m$, the increased value of pheromone in the tth iteration is $\Delta b_{ij}(t)=\sum_{k=1}^{m}\Delta b_{ij}^{k}(t)$, where

$$\Delta b_{ij}^{k}(t)=\begin{cases}\dfrac{Q}{L_k(t)}; & \text{if } j \text{ follows } i \text{ closely, where } Q \text{ is a constant,}\\ 0; & \text{otherwise.}\end{cases} \qquad (9)$$

Local update is performed during the ant constructive procedure in the following way:

$$b_{ij}(t+1)=(1-\rho)\,b_{ij}(t)+\Delta b_{ij}(t), \qquad (10)$$

where ρ is the evaporation coefficient.

The heuristic information is in an $(n+1)\times n$ matrix, with the last row $\eta_{n+1,j}=A/t_j$ and the other rows $\eta_{ij}=A/t_{ij}$, where t_{ij} is the traveling time between two nodes, t_j is the average traveling time from customer j to the depot, and A is a constant depending on the situations.

In a given service order, the first served customer is determined by probability. The next served customer is selected from the allowing set allowed$_i$, which is the customers set that can be served after i. The customer j is chosen with the probability p_{ij}, where

$$p_{ij}(t)=\begin{cases}\dfrac{(b_{ij}(t))^{\alpha}(\eta_{ij})^{\beta}}{\sum_{s\in \text{allowed}_i}(b_{is}(t))^{\alpha}(\eta_{is})^{\beta}}; & j\in \text{allowed}_i,\\ 0; & j\notin \text{allowed}_i.\end{cases} \qquad (11)$$

α is the importance of pheromone information, and β is the importance of heuristics information. After calculation of the probability matrix, the node is selected by the probability p_{ij} and the node is deleted from the allowing set. Then a customer service order is got with the pheromone.

4.2. Mutation Operator. ACO algorithm possibly runs into prematurity just as other evolutionary algorithms. The main reason is the concentrations of pheromone which makes the same solutions be got. In reality, the pheromone may be changed by rain and other factors and this change may help the ants find a new route. A mutation operator will be introduced into the pheromone of ACO to escape from local optima and strengthen its global search ability.

The initial pheromone matrix's factors are the same. As time goes on, some elements become large but other elements become small in the same row. The pheromone is concentrated which obstructs the ability to find more optimal solutions. The concentration of a row is defined as the ratio of the maximum factor and the sum of the factors in the row. The ith row concentration $\mu_i(t)$ is

$$\mu_i(t)=\frac{\max_{j=1,2,\dots,n}\tau_{ij}}{\sum_{j=1}^{n}\tau_{ij}}, \qquad i=1,2,\dots,n+1. \qquad (12)$$

The concentrations of different row are not the same. Whether to mutate a row should depend on the concentration of the row. When the concentrations of a row exceed the threshold value, the factors in the row should be mutated.

For the row mutation, the maximum factor decreases randomly and the decreased value is assigned to other positions. Suppose j_0th factor is the maximum factor in the ith row, and given a random number r $(0 \le r \le 1)$ and a random vector $R_{1\times n}$. A new row is got with the j_0th factor multiplied by r, and the decreased value is assigned to other positions; that is,

$$\tau_i' = \tau_i' + \frac{\tau_{ij_0}(1-r)}{\sum_{j=1}^{n} R_j} R. \tag{13}$$

Row mutation is a local mutation and only affects the choice of routes partly. When the matrix has high concentration, local mutation cannot ensure escaping the local optima and matrix mutation is necessary. The minimum concentration of the row is the matrix concentration, which is $\nu(t)$:

$$\nu(t) = \min_{i=1,2,\ldots,n+1} \mu_i(t). \tag{14}$$

Whether to mutate a matrix should also depend on the concentration of the matrix. When the concentrations of a matrix exceed the threshold value, the matrix should be mutated. For the matrix mutation, every element in the matrix decreases randomly and the decreased value is again randomly assigned. A random number $\overline{r}$ $(0 \le \overline{r} \le 1)$ and a random vector $R_{n+1\times n}$ are got. New values are got with every factor multiplied by $\overline{r}$. Then the decreased value is assigned randomly' that is,

$$\tau' = \overline{r} * \tau,$$
$$\tau' = \tau' + \frac{\sum_{i=1}^{n+1}\sum_{j=1}^{n}\left(\tau_{ij} - \tau_{ij}'\right)}{\sum_{i=1}^{n+1}\sum_{j=1}^{n} R_{ij}} R. \tag{15}$$

Then a new pheromone matrix is got and the mutation can ensure that the total pheromones do not change.

The mutation ACO algorithm is as follows.

Step 1. Initiation: Determine the parameters m, T, α, β, ε, ϕ, and p. Input the initial pheromone $\tau_{ij}(0)$ and heuristics matrix η_{ij} and get the initial ants. Give m customers arrays randomly as

$$A(0) = \{A_1(0), A_2(0), \ldots, A_m(0)\}. \tag{16}$$

Step 2. For a given array, assign the customers to the vehicle and get the corresponding completion time by dynamic programming and split algorithm. Record the present best route *besttrip* and the fastest completion time *finishtime*.

Step 3. The local pheromone update is performed by all the ants after each construction step on the formulas (9) and (10).

Step 4. Row Mutation: Inspect whether the pheromone concentration of the row in the pheromone matrix is more than ε; if yes, a random number r is got. If $r \le p$, a new row is got by (13).

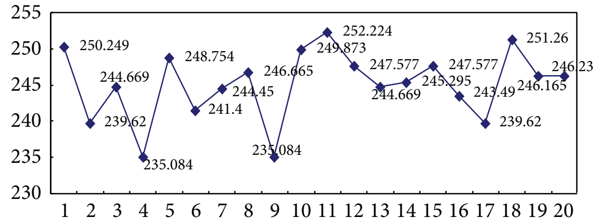

Figure 5: 20 times calculation results.

Step 5. Matrix Mutation: Inspect whether the pheromone matrix concentration is more than ϕ; if yes, a random number $\overline{r}$ is got. If $\overline{r} \le p$, a new pheromone matrix is got by the formula (15).

Step 6. According to the new pheromone matrix $B_{ij}(t)$ and the heuristics information matrix η_{ij}, Give m customers arrays randomly as

$$A(t+1) = \{A_1(t+1), A_2(t+1), \ldots, A_m(t+1)\}. \tag{17}$$

Step 7. Let $t = t+1$. ACO procedure stops if $t = T$ and output the *finishtime* and besttrip; otherwise return to Step 2.

5. Numerical Examples

In this section, we will consider the emergency supplies distribution after the earthquake in Wenchuan. After the earthquake, the roads on earth are not fluent. The emergency supplies are mainly transported by helicopter. As the number of helicopters is limited comparED to the broad place, the helicopter s is needed for distribution many times. The problem is a typical MRVRP with fastest completion time.

Suppose there are 3 helicopters in charge of the distribution of emergency supplies to 20 settlements. The location of material distribution center is (30, 40), the demands and location of the settlements are given in Table 1.

The capacity of a helicopter is $12t$. The total demand is $92t$. Every plane is needed to fly 3 times on average and distribute at most 5 settlements every time.

The algorithm is realized by Scilab. The parameters are as follows: the ants number is 10 and $\alpha = 1.2$, $\beta = 0.5$, $P = 0.15$, $\varepsilon = 0.8$, $\phi = 0.75$. The iteration time is 400. Run the program 20 times for the same problem, and get the fastest completion which times are as shown in Figure 5.

Average fastest completion time is 244.9978, their maximum gap is 7.291, and the gap is 3.1014% of the best scheme. It shows that the algorithm has good convergence.

Best distribution route is as shown in Figure 6.

Its fastest completion time is 235.08428 and total travel time is 865.66435. A new scheme to minimize total travel time is as shown in Figure 7.

Its shortest total travel time is 845.89676, the fastest completion time of this customer array is 273.379, and it is 38.295 minutes more than the first scheme. And the total

Table 1: Locations and demands of settlement.

Settlement	1	2	3	4	5	6	7	8	9	10
Location	(74, 29)	(64, 26)	(67, 80)	(88, 15)	(21, 65)	(72, 42)	(92, 80)	(46, 38)	(76, 86)	(30, 46)
Demand	3	7	5	6	2	9	7	4	4	3.5
Settlement	11	12	13	14	15	16	17	18	19	20
Location	(63, 48)	(23, 11)	(36, 72)	(29, 54)	(66, 16)	(10, 10)	(15, 50)	(10, 10)	(20, 70)	(70, 12)
Demand	4	3	8	6	5	1.5	5	2.5	3	3.5

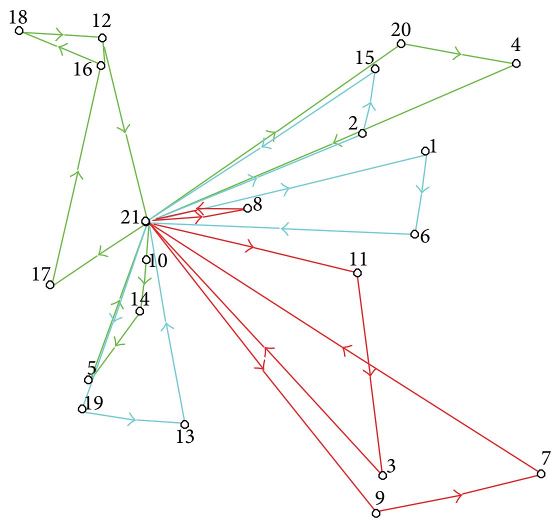

Figure 6: Optimal scheme of MRVRP with fastest completion time.

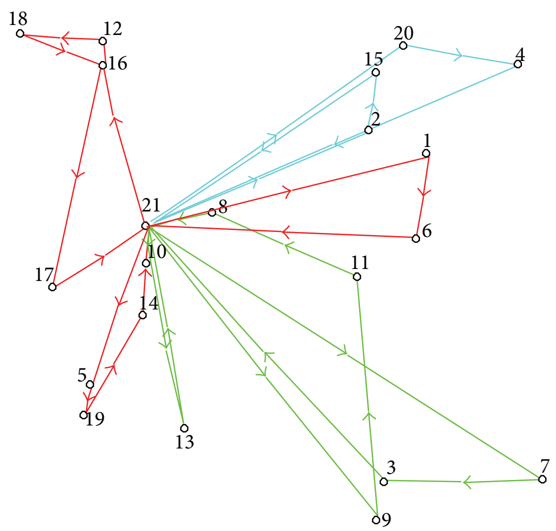

Figure 7: Optimal scheme.

flight time is reduced by 19.76759 minutes. In emergency management, effectively shortening the completion time is very necessary, and therefore the first scheme is more reasonable.

6. Conclusions

In this paper, the MRVRP with fastest completion time is proposed, which has many applications in fast foods distribution, express delivery, and emergency supplies. Solving the problem is more difficult than the general VRP. The key problem to solve MRVRP with fastest completion time is to give the division method for customer array. The customer division method based on dynamic programming and split method is given in this paper, which can transfer MRVRP with fastest completion time into the problem of finding the optimal customer service order. Then the problem is solved with mutation ACO.

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Research Article

Research on the Fresh Agricultural Product Supply Chain Coordination with Supply Disruptions

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This paper develops a dynamic model in a one-supplier-one-retailer fresh agricultural product supply chain that experiences supply disruptions during the planning horizon. The optimal solutions in the centralized and decentralized supply chains are studied. It is found that the retailer's optimal order quantity and the maximum total supply chain profit in the decentralized supply chain with wholesale price contract are less than that in the centralized supply chain. A two-part tariff contract is proposed to coordinate the decentralized supply chain with which the maximum profit can be achieved. It is found that the optimal wholesale price should be a decreasing piecewise function of the final output. To ensure that the supplier and the retailer both have incentives to accept the coordination contract, a lump-sum fee is offered. The interval of lump-sum fee is given leaving both the supplier and the retailer better off with the two-part tariff contract.

1. Introduction

Agriculture plays a vital role in the world economy. However, the production of most agricultural products is affected by a lot of external factors, such as the weather changes, seeds quality, and culture methods, which are not in full control by the supply chain members. The situation is further complicated by the fact that there is a long lead time in the production of agricultural product. It means that it is impossible to adjust the production plan when the environment changes. For the agricultural product producers, they lack the market information and are not certain of the final output when going into production. They are more blindfold to choose what to produce and how much to produce, especially in the uncertain environment. Then oversupply and shortage of the agricultural product are quite popular in the agricultural product market, which reduce the profit of the supply chain and hurt the enthusiasms of the supply chain members. How to reduce the effects of the fluctuations and share the risks facing the supply chain members is an important topic in the supply chain management.

Coordinating supply chain has been a major issue in supply chain management research. Supply chain contracts are contractual agreements governing the pricing and exchange of goods or services between independent members in a supply chain. Properly designed supply contracts are an effective means to share the demand and supply risk and better coordinate the decentralized supply chain. It is widely recognized that the supplier and retailer can both benefit from coordination and thereby improve the overall performance of the supply chain as a whole. Many well-known contract forms such as buy-back, revenue-sharing, quantity flexibility, sales rebate, two-part tariff, and quantity discount have shown to coordinate the supply chain. In this paper, a dynamic model in a one-supplier-one-retailer fresh agricultural product supply chain that experiences supply disruptions during the planning horizon is studied. The two-part tariff contract that can coordinate the fresh agricultural product supply chain with supply disruptions effectively is determined.

The remainder of the paper is organized as follows. In the next section, a brief literature review and a summary of the contributions of this research are provided. Section 3 introduces the notations and formulates the decision models. In Section 4, the centralized supply chain model is discussed. In Section 5, the fresh agricultural product supply chain in the decentralized case is studied. In Section 6, the design of coordination contract is given. The supply chain can be coordinated when the supplier and the retailer make decisions independently. A numerical example is given in Section 7. The work is summarized, and topics for future study are discussed in Section 8.

2. Literature Review

One stream of the literature related to the research is on fresh agricultural product supply chain. Samuel et al. [1] examined contract practices between suppliers and retailers in the agricultural seed industry. Xiao et al. [2] researched on the optimization and coordination of fresh-product supply chains under the Cost Insurance and Freight business model with uncertain long distance transportation delays and devised a simple cost sharing mechanism to coordinate the supply chain under consideration. Wang and Chen [3] introduced the options contracts into the fresh produce supply chain and took the huge circulation wastages both from quantity and quality into account. Cai et al. [4] considered a supply chain in which a fresh-product producer supplies the product to a distant market, via a specialized third-party logistics (3PL) provider, where a distributor purchases and sells it to end customers. An incentive scheme is proposed to coordinate the supply chain. Yu and Nagurney [5] developed a network-based food supply chain model under oligopolistic competition and perishability with a focus on fresh produce and proposed an algorithm with elegant features for computation.

This paper is also closely related to supply chain coordination management and disruption management. In a decentralized decision-making setting, the optimal supply chain profit is usually not achieved due to double marginalization. Double marginalization means the fact that each supply chain member's relative cost structure is distorted when a transfer price is introduced within a supply chain. Designing coordination contract is an important issue which aimed at reconciling conflicts and achieves a better profit among supply chain members. Lariviere [6], Tsay et al. [7], and Cachon [8] provided excellent introduction and summaries on coordination contracts. Our coordination contract is closely related to Jeuland and Shugan [9], Moorthy [10], and Georges [11]. Georges [11] investigated under which conditions the manufacturer can reach the vertically integrated channel solution through the use of a two-part wholesale price in a static marketing channel where demand also depends on players' advertising.

For the literature on disruption management, Qi et al. [12] first introduced the disruption management into supply chain management. They investigate a one-supplier-one-retailer supply chain that experienced a disruption in demand during the planning horizon. They examined how the original production plan should be adjusted after demand disruptions occurred and how to coordinate the supply chain using wholesale quantity discount policies. Xiao et al. [13] further studied the coordination of the supply chain with demand disruptions and considered a price-subsidy rate contract to coordinate the investments of the competing retailers with sales promotion opportunities and demand disruptions. Xiao and Yu [14] developed an indirect evolutionary game model with two vertically integrated channels to study evolutionarily stable strategies (ESS) of retailers in the quantity-setting duopoly situation with homogeneous goods and analyzed the effects of the demand and raw material supply disruptions on the retailers' strategies. Xiao and Qi [15] studied the disruption management of the supply chain with two competing retailers, where the manufacturer faces a production cost disruption. Chen and Xiao [16] developed two coordination models of a supply chain consisting of one manufacturer, one dominant retailer, and multiple fringe retailers to investigate how to coordinate the supply chain after demand disruption. They considered two coordination schedules, linear quantity discount schedule and Groves wholesale price schedule. Li et al. [17] investigated the sourcing strategy of a retailer and the pricing strategies of two suppliers in a supply chain under an environment of supply disruption. They characterized the sourcing strategies of the retailer in a centralized and a decentralized system. They derived a sufficient condition for the existence of an equilibrium price in the decentralized system when the suppliers were competitive. Huang et al. [18] developed a two-period pricing and production decision model in a one-manufacturer-one-retailer dual-channel supply chain that experienced a disruption in demand during the planning horizon. They studied the scenarios where the manufacturer and the retailer were in a vertically integrated setting and in a decentralized decision-making setting. They derived conditions under which the maximum profit can be achieved. Anastasios et al. [19] proposed generic single period inventory models for capturing the tradeoff between inventory policies and disruption risks in a dual-sourcing supply chain network both unconstrained and under service level constraints, where both supply channels were susceptible to disruption risks. The models were developed for both risk-neutral and risk-averse decision-makers and can be applicable for different types of disruptions.

There are two main differences between the works cited and the paper. First, most previous results are dependent on the assumption that the supply is deterministic or obeys a certain distribution. Only few studies examine the supply chain with supply disruptions. Second, most research related to the supply chain disruptions focuses on the demand disruptions. In this paper, a dynamic model in a one-supplier-one-retailer fresh agricultural product supply chain that experiences supply disruptions during the planning horizon is proposed, and the agricultural product supply chain with supply disruptions is coordinated.

3. Problem Description

A fresh agricultural product supply chain composed of a supplier and a retailer is studied in the paper. The supplier produces fresh agricultural product with a long production lead time and a short lifecycle. The supplier sells the fresh agricultural product to the retailer, and the retailer sells the product to the customers in a single selling season. The supplier and the retailer are assumed to be risk neutral and pursue profit maximization.

The demand of the fresh agricultural product is $d = D - kp$, where D is the market scale, k is the price-sensitive coefficient, and d is the real demand under the unit retail price p.

Since the supplier produces fresh agricultural product with a long production lead time, the supplier must make the production plan before the retailer makes the order decision. While making the production plan, the supplier does not consider the supply disruptions, since the supply disruptions cannot be anticipated. Usually there is no supply disruption, that is, the supplier puts q_s units into production, and the final output is exactly q_s units. If there are supply disruptions, the supplier still puts q_s units into production and the final output is uncertain to be q_s.

When the agricultural product harvests and the selling season comes, the final output of the supplier is found to be $Q(= q_s + \Delta q_s)$, where the supply disruptions are captured by the term Δq_s. To be reasonable, there is an upper bound $\Delta \overline{q}_s$ of Δq_s and a lower bound $\Delta \underline{q}_s$ of Δq_s, where $\Delta \overline{q}_s \geq 0$ and $\Delta \underline{q}_s \leq 0$. The upper bound of final output is $\overline{Q}(= q_s+\Delta \overline{q}_s)$, and the lower bound of final output is $\underline{Q}(= q_s+\Delta \underline{q}_s)$. After the final output is realized, the supplier decides the wholesale price, and the retailer makes the order decision according to the wholesale price. The paper focuses on the decision-making problem after the supply disruptions happen.

c is the unit distributing cost of the fresh agricultural product from the supplier to the retailer. p_s is the unit supplying cost from the spot market incurred by the supplier when the retailer's demand cannot be satisfied by the product produced by the supplier, and v_s is unit salvage cost of the supplier when there are surplus products. To be reasonable, the following assumption is given.

Assumption 1 ($c > v_s$). $c > v_s$ is assumed, otherwise the supplier can earn infinite profit by distributing infinite products.

Assumption 2 ($p_s > v_s$). $p_s > v_s$ is assumed since the fresh agricultural product is of little salvage value.

The following mathematical notation is used. π_j^i denotes the profit function for channel member j in supply chain model i. Superscript i takes values ID, D, and C, which denote the centralized supply chain, decentralized supply chain and supply chain with coordination contract, respectively. The subscript j takes values r and s, which denotes the retailer and the supplier.

4. The Centralized Supply Chain with Supply Disruptions

It is obvious that the supply chain performs best if the channel is centrally controlled. The decision variable in the centralized supply chain is only the order quantity q.

When the supplier's final output is Q, the total supply chain profit is

$$\pi^{ID}(q) = q\left(\frac{D-q}{k} - c\right) + v_s(Q-q)^{+} - p_s(q-Q)^{+} \quad (1)$$

The following conclusions about the optimal order quantity in the centralized agricultural product supply chain are got.

Theorem 1. *When the final output is Q, the optimal order quantity q^{ID^*} of the retailer is*

(i) *when $\underline{Q} \leq Q \leq (D-k(c+p_s))/2$, $q^{ID^*} = (D-k(c+p_s))/2$;*

(ii) *when $(D-k(c+p_s))/2 \leq Q \leq (D-k(c+v_s))/2$, $q^{ID^*} = Q$;*

(iii) *when $(D-k(c+v_s))/2 \leq Q \leq \overline{Q}$, $q^{ID^*} = (D-k(c+v_s))/2$.*

From Theorem 1, the optimal profit in the centralized agricultural product supply chain can be got.

Theorem 2. *When the final output is Q, the maximum total profit in the centralized supply chain π^{ID^*} is*

(i) *when $\underline{Q} \leq Q \leq (D-k(c+p_s))/2$, $\pi^{ID^*} = (1/k)[(D-k(c+p_s))/2]^2 + p_sQ$;*

(ii) *when $(D-k(c+p_s))/2 \leq Q \leq (D-k(c+v_s))/2$, $\pi^{ID^*} = Q(((D-Q)/k)-c)$;*

(iii) *when $(D-k(c+v_s))/2 \leq Q \leq \overline{Q}$, $\pi^{ID^*} = (1/k)[(D-k(c+v_s))/2]^2 + v_sQ$.*

5. The Decentralized Supply Chain with Supply Disruptions

In this section, the problem that the supplier and the retailer make decisions with wholesale price contract in a decentralized way under the disrupted output is discussed. In this case, the supplier acts as the leader, and the retailer acts as the follower. When the agricultural product harvests, the supplier first decides on the wholesale price ω, and the retailer decides the order quantity q accordingly. The supplier distributes q units fresh agricultural product to the retailer. If the final output cannot satisfy the retailer, the supplier buys the remaining products from the spot market. If there is surplus after satisfying the retailer, the residual products are salvaged.

Because the supplier is the leader, the best-response function of the retailer should be got at first. For a given ω, the retailer's profit is

$$\pi_r^D(q) = q\left(\frac{D-q}{k} - \omega\right). \quad (2)$$

The retailer aims to maximize his profit. The objective function is concave in ω, and the retailer's first-order conditions characterize the unique best response: $q_r^{D^*}(\omega) = (D-k\omega)/2$.

The supplier's optimization problem can be stated as

$$\pi_s^D(q) = (\omega - c)\, q_r^{D^*} + v_s\left(Q - q_r^{D^*}\right)^{+} - p_s\left(q_r^{D^*} - Q\right)^{+}. \quad (3)$$

Lemma 3. *When the final output is Q, the optimal wholesale price ω^{D^*} of the supplier is*

(i) *when $\underline{Q} \le Q \le (D - k(c + p_s))/4$, $\omega^{D^*} = (D + k(c + p_s))/2k$;*

(ii) *when $(D - k(c + p_s))/4 \le Q \le (D - k(c + v_s))/4$, $\omega^{D^*} = (D - 2Q)/k$;*

(iii) *when $(D - k(c + v_s))/4 \le Q \le \overline{Q}$, $\omega^{D^*} = (D + k(c + v_s))/2k$.*

It is not hard to get Theorems 4 and 5 from Lemma 3.

Theorem 4. *When the final output is Q, the optimal order quantity $q_r^{D^*}$ of the retailer is*

(i) *when $\underline{Q} \le Q \le (D - k(c + p_s))/4$, $q_r^{D^*} = (D - k(p_s + c))/4$;*

(ii) *when $(D - k(c + p_s))/4 \le Q \le (D - k(c + v_s))/4$, $q_r^{D^*} = Q$;*

(iii) *when $(D - k(c + v_s))/4 \le Q \le \overline{Q}$, $q_r^{D^*} = (D - k(v_s + c))/4$.*

Theorem 5. *When the final output is Q, the maximum supplier profit $\pi_s^{D^*}$, the maximum retailer profit $\pi_r^{D^*}$, and the maximum total profit π^{D^*} in the decentralized supply chain are*

(i) *when $\underline{Q} \le Q \le (D - k(c + p_s))/4$, $\pi_s^{D^*} = (2/k)[(D - k(p_s + c))/4]^2 + p_s Q$, $\pi_r^{D^*} = (1/k)[(D - k(p_s + c))/4]^2$, $\pi^{D^*} = (3/k)[(D - k(p_s + c))/4]^2 + p_s Q$;*

(ii) *when $(D - k(c + p_s))/4 \le Q \le (D - k(c + v_s))/4$, $\pi_s^{D^*} = (((D - 2Q)/k) - c)Q$, $\pi_r^{D^*} = (1/k)Q^2$, $\pi^{D^*} = (((D - Q)/k) - c)Q$;*

(iii) *when $(D - k(c + v_s))/4 \le Q \le \overline{Q}$, $\pi_s^{D^*} = (2/k)[(D - k(v_s + c))/4]^2 + v_s Q$, $\pi_r^{D^*} = (1/k)[(D - k(v_s + c))/4]^2$, $\pi^{D^*} = (3/k)[(D - k(v_s + c))/4]^2 + v_s Q$.*

Theorem 6. *The optimal order quantity in the centralized and decentralized supply chains with supply disruptions satisfies $q_r^{D^*} \le q^{ID^*}$.*

Let $\overline{q}_1^{D^*} = (D - k(c + v_s))/4$, $\overline{q}_2^{D^*} = (D - k(c + p_s))/4$, $\overline{q}_1^{ID^*} = (D - k(c + v_s))/2$, and $\overline{q}_2^{ID^*} = (D - k(c + p_s))/2$; the relation between the optimal order quantity and the final output is shown in Figure 1. The optimal order quantity in the decentralized supply chain is always less than that in the centralized supply chain.

From Theorems 2 and 5, Theorem 7 can be got.

Theorem 7. *The maximum total supply chain profits in the centralized and decentralized supply chains with supply disruptions satisfy $\pi^{ID^*} \ge \pi^{D^*}$.*

From Theorem 7, the maximum total supply chain profit in the decentralized supply chain is less than that in the centralized supply chain. In the next section, it is shown that the supplier can further improve the profits in the decentralized supply chain by offering the coordination contract.

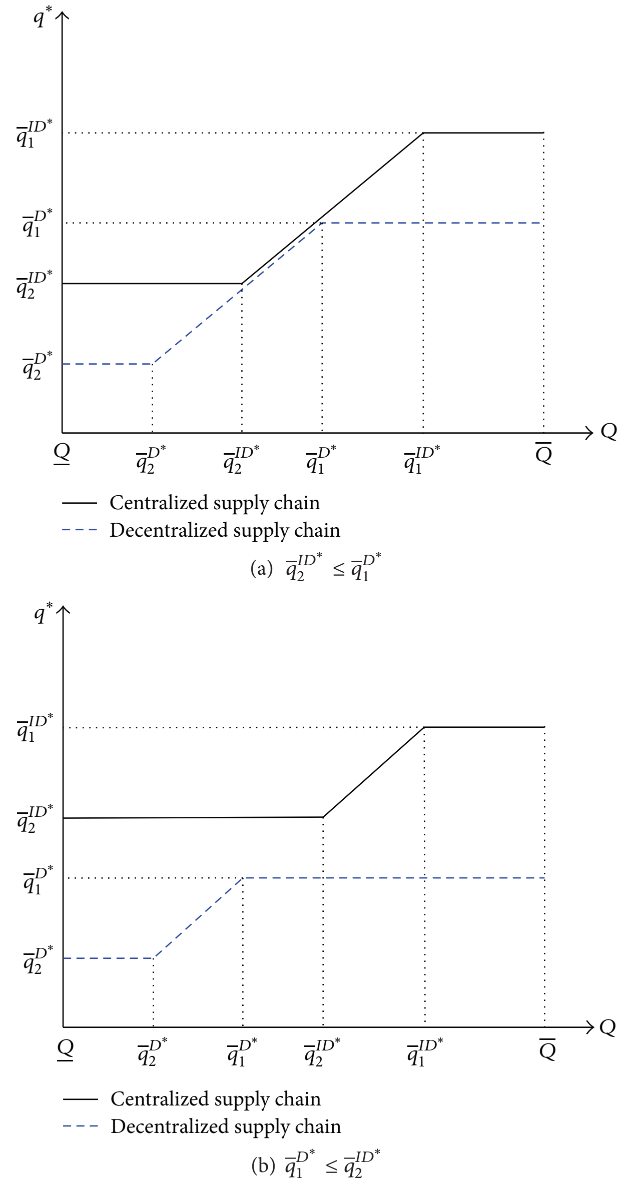

FIGURE 1: Relation between optimal order quantity and final output.

6. Design of Coordination Contract

In Section 5, it is shown that when the supplier and the retailer make decisions in a decentralized way, the wholesale price contract cannot coordinate the supply chain and must be modified to achieve the optimal total supply chain profit.

Theorem 8. *When the final output is Q, to ensure that the retailer's order quantity equals the optimal order quantity in the centralized supply chain, the optimal wholesale price ω^{C^*} is*

(i) *when $\underline{Q} \le Q \le (D - k(c + p_s))/2$, $\omega^{C^*} = c + p_s$;*

(ii) *when* $(D - k(c + p_s))/2 \le Q \le (D - k(c + v_s))/2$, $\omega^{C^*} = (D - 2Q)/k$;

(iii) *when* $(D - k(c + v_s))/2 \le Q \le \overline{Q}$, $\omega^{C^*} = c + v_s$.

From Theorem 8, Theorem 9 can be obtained.

Theorem 9. *When the wholesale price* $\omega^* = \omega^{C^*}$, *the decentralized supply chain can be coordinated.*

When the wholesale price $\omega^* = \omega^{C^*}$, the profits of the supplier and the retailer are

(i) when $\underline{Q} \le Q \le (D - k(c + p_s))/2$, $\pi_s^{C^*} = p_s Q$, $\pi_r^{C^*} = (1/k)[(D - k(p_s + c))/2]^2$;

(ii) when $(D - k(c + p_s))/2 \le Q \le (D - k(c + v_s))/2$, $\pi_s^{C^*} = (((D - 2Q)/k) - c)Q$, $\pi_r^{C^*} = Q^2/k$;

(iii) when $(D - k(c + v_s))/2 \le Q \le \overline{Q}$, $\pi_s^{C^*} = v_s Q$, $\pi_r^{C^*} = (1/k)[(D - k(c + v_s))/2]^2$.

It can be verified that $\pi_s^{C^*} + \pi_r^{C^*} = \pi^{ID^*}$.

Theorem 8 indicates that, in a decentralized supply chain, to coordinate the fresh agricultural product supply chain with supply disruptions, the optimal wholesale price depends on the final output. The optimal wholesale price is a decreasing piecewise function of final output. To ensure that the supplier and the retailer both have incentives to accept the coordination contract, the profits of the supplier and the retailer should satisfy $\pi_s^C \ge \pi_s^{D^*}$, $\pi_r^C \ge \pi_r^{D^*}$. This problem can be easily solved by offering a lump-sum fee $F(\underline{F} \le F \le \overline{F})$, where $\underline{F} = \max(\pi_s^{D^*} - \pi_s^{C^*}, \pi_r^{D^*} - \pi_r^{C^*}, 0)$, $\overline{F} = \max(\pi_s^{C^*} - \pi_s^{D^*}, \pi_r^{C^*} - \pi_r^{D^*})$. (Since $\pi_s^{C^*} + \pi_r^{C^*} \ge \pi_s^{D^*} + \pi_r^{D^*}$, $\underline{F}$ and $\overline{F}$ are different.) When $\pi_s^{D^*} \ge \pi_s^{C^*}$, that is, the supplier earns less with the coordination contract, the retailer should pay the lump-sum fee F to the supplier. The profits of the supplier and the retailer are $\pi_s^C = \pi_s^{C^*} + F$, $\pi_r^C = \pi_r^{C^*} - F$. Otherwise, the supplier should pay the lump-sum fee F to the retailer. The profits of the supplier and the retailer are $\pi_s^C = \pi_s^{C^*} - F$, $\pi_r^C = \pi_r^{C^*} + F$. Then $\pi_s^C \ge \pi_s^{D^*}$ and $\pi_r^C \ge \pi_r^{D^*}$ are satisfied.

7. Numerical Example

A numerical example is given to illustrate some of results derived throughout the paper.

Suppose the supplier's unit distribution cost $c = 2$, the unit buying cost $p_s = 5$, and the unit salvage cost $v_s = 1$. The demand function is $q = 40 - p$.

When there are supply disruptions and the supplier's production is q_s, the final output Q is not certain. The supply disruptions are captured by the term Δq_s, where Δq_s lies in the interval $[-q_s, 0.5q_s]$. That is, the final output Q lies in the interval $[\underline{Q}, \overline{Q}]$, where $\underline{Q} = 0$, $\overline{Q} = 1.5q_s$.

When the supply disruptions happen and the final output is found to be Q, in the centralized supply chain, the retailer's optimal order quantity q^{ID^*} is (i) when $0 \le Q \le 16.5$, $q^{ID^*} = 16.5$; (ii) when $16.5 \le Q \le 18.5$, $q^{ID^*} = Q$; (iii) when $18.5 \le Q \le \overline{Q}$, $q^{ID^*} = 18.5$.

Correspondingly, the maximum supply chain profit π^{ID^*} is

(i) when $0 \le Q \le 16.5$, $\pi^{ID^*} = 272.25 + 5Q$;

(ii) when $16.5 \le Q \le 18.5$, $\pi^{ID^*} = Q(38 - Q)$;

(iii) when $18.5 \le Q \le \overline{Q}$, $\pi^{ID^*} = 342.25 + Q$.

In the decentralized agricultural product supply chain, when the supply disruptions happen and the final output is found to be Q, the optimal wholesale price ω^{D^*} of the supplier is (i) when $0 \le Q \le 8.25$, $\omega^{D^*} = 23.5$; (ii) when $8.25 \le Q \le 9.25$, $\omega^{D^*} = 40 - 2Q$; (iii) when $9.25 \le Q \le \overline{Q}$, $\omega^{D^*} = 21.5$.

Correspondingly, the optimal order quantity $q_r^{D^*}$ of the retailer is (i) when $0 \le Q \le 8.25$, $q_r^{D^*} = 8.25$; (ii) when $8.25 \le Q \le 9.25$, $q_r^{D^*} = Q$; (iii) when $9.25 \le Q \le \overline{Q}$, $q_r^{D^*} = 9.25$.

The supplier profit $\pi_s^{D^*}$, the retailer profit $\pi_r^{D^*}$, and total supply chain profit π^{D^*} are

(i) when $0 \le Q \le 8.25$, $\pi_s^{D^*} = 136.125 + 5Q$, $\pi_r^{D^*} = 68.0625$, $\pi^{D^*} = 204.1875 + 5Q$;

(ii) when $8.25 \le Q \le 9.25$, $\pi_s^{D^*} = (38 - 2Q)Q$, $\pi_r^{D^*} = Q^2$, $\pi^{D^*} = (38 - Q)Q$;

(iii) when $9.25 \le Q \le \overline{Q}$, $\pi_s^{D^*} = 171.125 + Q$, $\pi_r^{D^*} = 85.5625$, $\pi^{D^*} = 256.6875 + Q$.

For a given final output Q, it can be verified that $q_r^{D^*} \le q^{ID^*}$, $\pi^{D^*} \le \pi^{ID^*}$. The decentralized supply chain should be coordinated to achieve the optimal total supply chain profit. To ensure that the retailer's order quantity in the decentralized supply chain equals that in the centralized supply chain, the optimal wholesale price ω^{C^*} is (i) when $0 \le Q \le 16.5$, $\omega^{C^*} = 7$; (ii) when $16.5 \le Q \le 18.5$, $\omega^{C^*} = 40 - 2Q$; (iii) when $18.5 \le Q \le \overline{Q}$, $\omega^{C^*} = 3$.

The retailer's profit is (i) when $0 \le Q \le 16.5$, $\pi_r^C(q) = q(33 - q)$; (ii) when $16.5 \le Q \le 18.5$, $\pi_r^C(q) = 2qQ$; (iii) when $18.5 \le Q \le \overline{Q}$, $\pi_r^C(q) = q(37 - q)$.

Then the retailer's optimal order quantity is (i) when $0 \le Q \le 16.5$, $q_r^{C^*} = 16.5$; (ii) when $16.5 \le Q \le 18.5$, $q_r^{C^*} = Q$; (iii) when $18.5 \le Q \le \overline{Q}$, $q_r^{C^*} = 18.5$.

Correspondingly, the maximum supplier profit $\pi_s^{D^*}$, the maximum retailer profit $\pi_r^{D^*}$, and the maximum total profit π^{D^*} in the decentralized supply chain are

(i) when $0 \le Q \le 16.5$, $\pi_s^{C^*} = 5Q$, $\pi_r^{C^*} = 272.25$, $\pi^{C^*} = 272.25 + 5Q$;

(ii) when $16.5 \le Q \le 18.5$, $\pi_s^{C^*} = Q(38 - 2Q)$, $\pi_r^{C^*} = Q^2$, $\pi^{C^*} = Q(38 - Q)$;

(iii) when $18.5 \le Q \le \overline{Q}$, $\pi_s^{C^*} = Q$, $\pi_r^{C^*} = 342.25$, $\pi^{C^*} = 342.25 + Q$.

It is obvious that $\pi^{C^*} = \pi^{ID^*}$, and the supply chain is coordinated.

To ensure that the supplier and the retailer both have incentives to accept the coordination contract, the profits of the supplier and the retailer should satisfy $\pi_s^C \geq \pi_s^{D^*}$, $\pi_r^C \geq \pi_r^{D^*}$.

(i) When $0 \leq Q \leq 8.25$, $\pi_s^{D^*} \geq \pi_s^{C^*}$, $\pi_r^{D^*} \leq \pi_r^{C^*}$. In this case, $\underline{F} = \pi_s^{D^*} - \pi_s^{C^*} = 136.125$, $\overline{F} = \pi_r^{C^*} - \pi_r^{D^*} = 204.1875$, and the retailer should pay the supplier a lump-sum fee $F(\underline{F} \leq F \leq \overline{F})$. The profits of the supplier and the retailer are $\pi_s^C = \pi_s^{C^*} + F$, $\pi_r^C = \pi_r^{C^*} - F$.

(ii) When $8.25 \leq Q \leq 9.25$, $\pi_s^{D^*} \geq \pi_s^{C^*}$, $\pi_r^{D^*} \leq \pi_r^{C^*}$. In this case, $\underline{F} = \pi_s^{D^*} - \pi_s^{C^*} = Q(33 - 2Q)$, $\overline{F} = \pi_r^{C^*} - \pi_r^{D^*} = 272.25 - Q^2$, and the retailer should pay the supplier a lump-sum fee $F(\underline{F} \leq F \leq \overline{F})$. The profits of the supplier and the retailer are $\pi_s^C = \pi_s^{C^*} + F$, $\pi_r^C = \pi_r^{C^*} - F$.

(iii) When $9.25 \leq Q \leq 16.5$, $\pi_s^{D^*} \geq \pi_s^{C^*}$, $\pi_r^{D^*} \leq \pi_r^{C^*}$. In this case, $\underline{F} = \pi_s^{D^*} - \pi_s^{C^*} = 171.125 - 4Q$, $\overline{F} = \pi_r^{C^*} - \pi_r^{D^*} = 186.6875$, and the retailer should pay the supplier a lump-sum fee $F(\underline{F} \leq F \leq \overline{F})$. The profits of the supplier and the retailer are $\pi_s^C = \pi_s^{C^*} + F$, $\pi_r^C = \pi_r^{C^*} - F$.

(iv) When $16.5 \leq Q \leq 18.5$, $\pi_s^{D^*} \geq \pi_s^{C^*}$, $\pi_r^{D^*} \leq \pi_r^{C^*}$. In this case, $\underline{F} = \pi_s^{D^*} - \pi_s^{C^*} = 171.125 - 37Q + 2Q^2$, $\overline{F} = \pi_r^{C^*} - \pi_r^{D^*} = 38Q - Q^2 - 85.5625$. The retailer should pay the supplier a lump-sum fee $F(\underline{F} \leq F \leq \overline{F})$. The profits of the supplier and the retailer are $\pi_s^C = \pi_s^{C^*} + F$, $\pi_r^C = \pi_r^{C^*} - F$.

(v) When $18.5 \leq Q \leq \overline{Q}$, $\pi_s^{D^*} \geq \pi_s^{C^*}$, $\pi_r^{D^*} \leq \pi_r^{C^*}$. In this case, $\underline{F} = \pi_s^{D^*} - \pi_s^{C^*} = 171.125$, $\overline{F} = \pi_r^{C^*} - \pi_r^{D^*} = 256.6875$, and the retailer should pay the supplier a lump-sum fee $F(\underline{F} \leq F \leq \overline{F})$. The profits of the supplier and the retailer are $\pi_s^C = \pi_s^{C^*} + F$, $\pi_r^C = \pi_r^{C^*} - F$. Then $\pi_s^C \geq \pi_s^{D^*}$ and $\pi_r^C \geq \pi_r^{D^*}$ is satisfied.

The supplier and the retailer both benefit from the coordination contract and have incentives to accept the contract.

8. Summary and Conclusions

In this paper, supply disruptions are introduced in the analysis of a one-supplier-one-retailer fresh agricultural product supply chain. The optimal decisions in the centralized and decentralized supply chain are analyzed. It is found that the retailer's optimal order quantity and the maximum total supply chain profit in the decentralized supply chain are less than that in the centralized supply chain. A two-part tariff contract is proposed. It shows that the supply chain can be coordinated leaving both the supplier and the retailer better off with a two-part tariff contract.

The aim of the paper is to develop a supply chain coordination scheme for adjusting the sale plan after supply disruptions occur, rather than making decisions considering all possible uncertainties in the planning stage. Of course, formulating a good plan based on certain probability assumptions is important, but, realistically, it is not possible for the decision-maker to anticipate all contingencies. In practice, for most agricultural products, the final output cannot be estimated precisely, so providing guidance for adjusting a predetermined plan can be as important as making the plan itself.

In the paper, one-supplier-one-retailer fresh agricultural product supply chain is studied. There are abundant opportunities for research on extensions ranging from multiple suppliers, multiple periods, and longer supply chains.

Appendix

Proof of Theorem 1. (i) When $q \leq Q$, the supply chain profit is $\pi_1^{ID}(q) = q(((D-q)/k)-c)+v_s(Q-q)$, where $\pi_1^{ID}(q)$ is concave in q. The optimal solution without constraint is $\overline{q}_1^{ID^*} = (D - k(c + v_s))/2$. Then if $\overline{q}_1^* \leq Q$, the optimal order quantity is $q_1^{ID^*} = \overline{q}_1^{ID^*}$, and the maximum supply chain profit is $\pi_1^{ID^*} = (1/k)\overline{q}_1^{ID^*2} + v_s Q$. Otherwise, the optimal solution is $q_1^{ID^*} = Q$. Correspondingly, the maximum supply chain profit is $\pi_1^{ID^*} = Q(((D - Q)/k) - c)$.

(ii) When $q \geq Q$, the supply chain profit is: $\pi_2^{ID}(q) = q(((D - q)/k) - c) - p_s(q - Q)$, where $\pi_2^{ID}(q)$ is concave in q. The optimal solution without constraint is $\overline{q}_2^{ID^*} = (D - k(c + p_s))/2$. Then if $\overline{q}_2^{ID^*} \geq Q$, the optimal order quantity is $q_2^{ID^*} = \overline{q}_2^{ID^*}$, and the maximum supply chain profit is $\pi_2^{ID^*} = (1/k)\overline{q}_2^{ID^*2} - p_s Q$. Otherwise, the optimal solution is: $q_2^{ID^*} = Q$. Correspondingly, the maximum supply chain profit is $\pi_2^{ID^*} = Q(((D - Q)/k) - c)$.

With Assumption 1, it is not hard to verify $\overline{q}_1^{ID^*} \geq \overline{q}_2^{ID^*}$.

(i) When $Q \leq \overline{q}_2^{ID^*}$, if the retailer chooses to order more than Q unit products, the maximum supply chain profit is $\pi^{ID^*} = (1/k)\overline{q}_2^{ID^*2} - p_s Q$; otherwise, if the retailer orders Q unit products from the supplier, the maximum supply chain profit is $\pi^{ID^*} = Q(((D-Q)/k) - c)$. As $\pi_2^{ID}(Q) = \pi_1^{ID}(Q)$ and $\pi_2^{ID}(\overline{q}_2^{ID^*}) \geq \pi_2^{ID}(Q)$, the optimal order quantity is $q^{ID^*} = (D - k(c + p_s))/2$.

(ii) When $\overline{q}_2^{ID^*} \leq Q \leq \overline{q}_1^{ID^*}$, if the retailer chooses to order less than Q unit products, the supplier salvages some products. Since $\pi_1^{ID}(q)$ is increasing in q, the more the retailer orders, the more supply chain profit is got. Otherwise, if the retailer chooses to order more than Q unit products, the supplier buys some products from the spot market. Since $\pi_2^{ID}(q)$ is decreasing in q, the more the retailer orders, the less supply chain profit is got. Then the optimal order quantity is Q.

(iii) When $Q \geq \overline{q}_1^{ID^*}$, if the retailer chooses to order less than Q unit products, the maximum supply chain profit is $\pi^{ID^*} = (1/k)\overline{q}_1^{ID^*2} + v_s Q$; otherwise if the retailer orders Q unit products from the supplier, the maximum supply chain profit is $\pi^{ID^*} = Q(((D-Q)/k)-c)$. As $\pi_2^{ID}(Q) = \pi_1^{ID}(Q)$ and $\pi_1^{ID}(\overline{q}_1^{ID^*}) \geq \pi_1^{ID}(Q)$, the optimal order quantity is $q^{ID^*} = (D-k(c+v_s))/2$.

Proof of Theorem 2. The total supply chain profit is $\pi^{ID}(q) = q(((D-q)/k)-c) + v_s(Q-q)^+ - p_s(q-Q)^+$. From Theorem 1, the optimal profit of the centralized supply chain can be got.

Proof of Lemma 1. (i) When $Q \geq q$, the supply chain profit is $\pi_{s1}^{D}(q) = (\omega - c)q + v_s(Q-q)$, where $\pi_{s1}^{D}(q)$ is concave in q. The optimal solution without constraint is $\omega_1^{D^*} = (D + k(c+v_s))/2k$. The retailer's order quantity is: $\overline{q}_1^{D^*} = (D - k(c+v_s))/4$. Then if $Q \geq (D-k(c+v_s))/4$, the optimal wholesale price is $\omega_1^{D^*} = (D+k(c+v_s))/2k$. The supplier's profit is $\pi_{s1}^{D^*} = (2/k)[(D-k(c+v_s))/4]^2 + v_s Q$. Otherwise, the optimal solution is $\omega_1^{D^*} = (D-2Q)/k$. In this case, the retailer's order quantity is $q_1^{D^*} = Q$. Correspondingly, the maximum supply chain profit is: $\pi_{s1}^{D^*} = Q(((D-2Q)/k)-c)$.

(ii) When $Q \leq q$, the supply chain profit is $\pi_{s2}^{D}(q) = (\omega-c)q - p_s(q-Q)$, where $\pi_{s2}^{D}(q)$ is concave in q. The optimal solution without constraint is $\omega_2^{D^*} = (D+k(c+p_s))/2k$. The retailer's order quantity is: $\overline{q}_2^{D^*} = (D-k(c+p_s))/4$. Then if $Q \geq (D-k(c+p_s))/4$, the optimal wholesale price is $\omega_2^{D^*} = (D+k(c+p_s))/2k$. The supplier's profit is $\pi_{s1}^{D^*} = (2/k)[(D-k(c+p_s))/4]^2 + p_s Q$. Otherwise, the optimal solution is $\omega_2^{D^*} = (D-2Q)/k$. In this case, the retailer's order quantity is $q_2^{D^*} = Q$. Correspondingly, the maximum supply chain profit is $\pi_{s2}^{D^*} = Q(((D-2Q)/k)-c)$.

With Assumption 1, it is not hard to verify $\overline{q}_1^{D^*} \geq \overline{q}_2^{D^*}$.

(i) When $Q \leq \overline{q}_2^{*}$, if the retailer chooses to order more than Q unit products, the maximum supply chain profit is $\pi^{D^*} = (1/k)\overline{q}_2^{D^*2} - p_s Q$; otherwise if the retailer orders Q unit products from the supplier, the maximum supply chain profit is $\pi^{D^*} = Q(((D-Q)/k)-c)$. As $\pi_2^{D}(Q) = \pi_1^{D}(Q)$ and $\pi_2^{D}(\overline{q}_2^{D^*}) \geq \pi_2^{D}(Q)$; the optimal wholesale price is $\omega^{D^*} = (D+k(c+p_s))/2k$.

(ii) When $\overline{q}_2^{D^*} \leq Q \leq \overline{q}_1^{D^*}$, if the retailer chooses to order less than Q unit products, the supplier salvages some products. Since $\pi_1^{D}(q)$ is increasing in q, the more the retailer orders, the more supply chain profit is got. Otherwise, if the retailer chooses to order more than Q unit products, the supplier buys some products from the spot market. Since $\pi_2^{D}(q)$ is decreasing in q, the more the retailer orders, the less supply chain profit is got. Then the optimal order quantity is Q.

(iii) When $Q \geq \overline{q}_1^{D^*}$, if the retailer chooses to order less than Q unit products, the maximum supply chain profit is $\pi^{D^*} = (1/k)\overline{q}_1^{*2} + v_s Q$; otherwise, if the retailer orders Q unit products from the supplier, the maximum supply chain profit is $\pi^{D^*} = Q(((D-Q)/k)-c)$. As $\pi_2^{D}(Q) = \pi_1^{D}(Q)$ and $\pi_1^{D}(\overline{q}_1^{D^*}) \geq \pi_1^{D}(Q)$, the optimal wholesale price is: $\omega^{D^*} = (D+k(c+v_s))/2k$.

Proof of Theorem 3. From the retailer's best response $q_r^{D^*}(\omega) = (D-k\omega)/2$ and the optimal wholesale price ω^{D^*} given in Lemma 3, the results in Theorem 4 are obtained.

Proof of Theorem 4. By substituting the optimal wholesale price ω^{D^*} given in Lemma 3 and the optimal order quantity given in Theorem 4 in $\pi_r^{D}(q)$, $\pi_s^{D}(q)$, and $\pi_r^{D}(q) + \pi_s^{D}(q)$, the results in Theorem 5 are obtained.

Proof of Theorem 5. It is not hard to verify $\overline{q}_1^{D^*} \geq \overline{q}_2^{D^*}$ and $\overline{q}_1^{ID^*} \geq \overline{q}_2^{ID^*}$, where $\overline{q}_1^{ID^*} = (D-k(c+v_s))/2$, $\overline{q}_2^{ID^*} = (D-k(c+p_s))/2$, $\overline{q}_1^{D^*} = (D-k(c+v_s))/4$, and $\overline{q}_2^{D^*} = (D-k(c+p_s))/4$. It is not hard to verify $\overline{q}_1^{ID^*} \geq \overline{q}_1^{D^*}$ and $\overline{q}_2^{ID^*} \geq \overline{q}_2^{D^*}$. As the relation between $\overline{q}_1^{D^*}$ and $\overline{q}_2^{ID^*}$ is not known, the following two cases are discussed.

(i) When $\overline{q}_2^{ID^*} \leq \overline{q}_1^{D^*}$, $\overline{q}_2^{D^*} \leq \overline{q}_2^{ID^*} \leq \overline{q}_1^{D^*} \leq \overline{q}_1^{ID^*}$ is got.

In this case, when $0 \leq Q \leq \overline{q}_2^{D^*}$, $q^{ID^*} = \overline{q}_2^{ID^*}$, $q_r^{D^*} = \overline{q}_2^{D^*}$; when $\overline{q}_2^{D^*} \leq Q \leq \overline{q}_2^{ID^*}$, $q^{ID^*} = \overline{q}_2^{ID^*}$, $q_r^{D^*} = Q$; when $\overline{q}_2^{ID^*} \leq Q \leq \overline{q}_1^{D^*}$, $q^{ID^*} = Q$, $q_r^{D^*} = Q$; when $\overline{q}_1^{D^*} \leq Q \leq \overline{q}_1^{ID^*}$, $q^{ID^*} = Q$, $q_r^{D^*} = \overline{q}_1^{D^*}$; when $\overline{q}_1^{ID^*} \leq Q \leq \overline{Q}$, $q^{ID^*} = \overline{q}_1^{ID^*}$, $q_r^{D^*} = \overline{q}_1^{D^*}$. It is not hard to get $q_r^{D^*} \leq q^{ID^*}$, when $\overline{q}_2^{ID^*} \leq \overline{q}_1^{D^*}$.

(ii) When $\overline{q}_2^{ID^*} \geq \overline{q}_1^{D^*}$, $\overline{q}_2^{D^*} \leq \overline{q}_1^{D^*} \leq \overline{q}_2^{ID^*} \leq \overline{q}_1^{ID^*}$ is got.

In this case, when $0 \leq Q \leq \overline{q}_2^{D^*}$, $q^{ID^*} = \overline{q}_2^{ID^*}$, $q_r^{D^*} = \overline{q}_2^{D^*}$; when $\overline{q}_2^{D^*} \leq Q \leq \overline{q}_1^{D^*}$, $q^{ID^*} = \overline{q}_2^{ID^*}$, $q_r^{D^*} = Q$; when $\overline{q}_1^{D^*} \leq Q \leq \overline{q}_2^{ID^*}$, $q^{ID^*} = \overline{q}_2^{ID^*}$, $q_r^{D^*} = \overline{q}_1^{D^*}$; when $\overline{q}_2^{ID^*} \leq Q \leq \overline{q}_1^{ID^*}$, $q^{ID^*} = Q$, $q_r^{D^*} = \overline{q}_1^{D^*}$; when $\overline{q}_1^{ID^*} \leq Q \leq \overline{Q}$, $q^{ID^*} = \overline{q}_1^{ID^*}$, $q_r^{D^*} = \overline{q}_1^{D^*}$.

It is not hard to get $q_r^{D^*} \leq q^{ID^*}$, when $\overline{q}_2^{ID^*} \geq \overline{q}_1^{D^*}$.

Proof of Theorem 6. Because the relation between $(D-k(c+p_s))/2$, $(D-k(c+v_s))/4$ is not known, the following two cases are discussed.

Case 1. When $(D-k(c+p_s))/2 \leq (D-k(c+v_s))/4$, The following five cases are discussed.

(i) When $\underline{Q} \leq Q \leq (D-k(c+p_s)/4)$, $\pi^{ID^*} = (1/k)[(D-k(c+p_s))/2]^2 + p_s Q$, $\pi^{D^*} = (3/k)[(D-k(p_s+c))/4]^2 + p_s Q$. It is obvious that $\pi^{ID^*} \geq \pi^{D^*}$.

(ii) When $(D-k(c+p_s))/4 \leq Q \leq (D-k(c+p_s))/2$, $\pi^{ID^*} = (1/k)[(D-k(c+p_s))/2]^2 + p_s Q$, $\pi^{D^*} = Q(((D-Q)/k)-c)$. Because π^{D^*} is an increasing function of Q and $\pi^{D^*}((D-k(c+p_s)/2)) = \pi^{ID^*}$, it's obvious that $\pi^{ID^*} \geq \pi^{D^*}$.

(iii) When $(D - k(c + p_s))/2 \le Q \le (D - k(c + v_s))/4$, $\pi^{ID^*} = Q(((D-Q)/k) - c)$, $\pi^{D^*} = Q(((D-Q)/k) - c)$. It is obvious that $\pi^{ID^*} = \pi^{D^*}$.

(iv) When $(D-k(c+v_s))/4 \le Q \le (D-k(c+v_s))/2$, $\pi^{ID^*} = Q((((D-Q))/k)-c)$, $\pi^{D^*} = (3/k)[(D - k(v_s + c))/4]^2 + v_sQ$. Because π^{ID^*} is an increasing function of Q and $\pi^{ID^*}((D-k(c+v_s))/4) = \pi^{ID^*}$, it is obvious that $\pi^{ID^*} \ge \pi^{D^*}$.

(v) When $(D - k(c + v_s))/2 \le Q \le \overline{Q}$, $\pi^{ID^*} = (1/k)[(D-k(c+v_s))/2]^2 + v_sQ$, $\pi^{D^*} = (3/k)[(D - k(v_s + c))/4]^2 + v_sQ$. It is obvious that $\pi^{ID^*} \ge \pi^{D^*}$.

Case 2. When $(D - k(c + v_s))/4 \le (D - k(c + p_s))/2$, The following five cases are discussed.

(i) When $\underline{Q} \le Q \le (D - k(c + p_s))/4$, $\pi^{ID^*} = (1/k)[(D-k(c+p_s))/2]^2 + p_sQ$, $\pi^{D^*} = (3/k)[(D - k(p_s + c))/4]^2 + p_sQ$. It is obvious that $\pi^{ID^*} \ge \pi^{D^*}$.

(ii) When $(D-k(c+p_s))/4 \le Q \le (D-k(c+v_s))/4$, $\pi^{ID^*} = (1/k)[(D - k(c + p_s))/2]^2 + p_sQ$, $\pi^{D^*} = Q(((D - Q)/k) - c)$. Because π^{D^*} is an increasing function of Q and $\pi^{D^*}((D - k(c + v_s))/4) < \pi^{ID^*}$, it's obvious that $\pi^{ID^*} \ge \pi^{D^*}$.

(iii) When $(D - k(c + v_s))/4 \le Q \le (D - k(c + p_s))/2$, $\pi^{ID^*} = (1/k)[(D - k(c + p_s))/2]^2 + p_sQ$, $\pi^{D^*} = (3/k)[(D - k(v_s + c))/4]^2 + v_sQ$. It is obvious that $\pi^{ID^*} \ge \pi^{D^*}$.

(iv) When $(D-k(c+p_s))/2 \le Q \le (D-k(c+v_s))/2$, $\pi^{ID^*} = Q(((D-Q)/k) - c)$, $\pi^{D^*} = (3/k)[(D - k(v_s + c))/4]^2 + v_sQ$. Because π^{ID^*} is an increasing function of Q and $\pi^{ID^*}((D-k(c+p_s))/2) > \pi^{D^*}$, it is obvious that $\pi^{ID^*} \ge \pi^{D^*}$.

(v) When $(D - k(c + v_s))/2 \le Q \le \overline{Q}$, $\pi^{ID^*} = (1/k)[(D-k(c+v_s))/2]^2 + v_sQ$, $\pi^{D^*} = (3/k)[(D - k(v_s + c))/4]^2 + v_sQ$. It is obvious that $\pi^{ID^*} \ge \pi^{D^*}$.

In summary, $\pi^{ID^*} \ge \pi^{D^*}$ is always true.

Proof of Theorem 7. From the retailer's profit in the decentralized supply chain given in (2), the best response $q_r^{C^*}(\omega) = (D - k\omega)/2$ is got. To ensure that the retailer's order quantity equals the optimal order quantity in the centralized supply chain, that is, $q_r^{C^*}(\omega) = q^{ID^*}$, the results in Theorem 8 are obtained.

Proof of Theorem 8. (i) When $0 \le Q \le (D - k(c + p_s))/2$, the supplier and retailer's profits are $\pi_s^{C^*} = p_sQ$, $\pi_r^{C^*} = (1/k)[(D - k(p_s + c))/2]^2$. The total supply chain profit is: $\pi^{C^*} = (1/k)[(D - k(c + p_s))/2]^2 + p_sQ$;

(ii) when $(D - k(c + p_s))/2 \le Q \le (D - k(c + v_s))/2$, The supplier and retailer's profits are $\pi_s^{C^*} = Q(((D - 2Q)/k) - c)$, $\pi_r^{C^*} = Q^2/k$. The total supply chain profit is $\pi^{C^*} = Q(((D-Q)/k) - c)$;

(iii) when $(D - k(c + v_s))/2 \le Q \le \overline{Q}$, the supplier and retailer's profits are $\pi_s^{C^*} = v_sQ$, $\pi_r^{C^*} = (1/k)[(D - k(v_s + c))/2]^2$. The total supply chain profit is $\pi^{C^*} = (1/k)[(D - k(c + v_s))/2]^2 + v_sQ$.

Summarizing the above results, $\pi^{C^*} = \pi^{ID^*}$ is got; that is, the decentralized supply chain is coordinated.

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Research Article

Optimal Investment with Multiple Risky Assets for an Insurer in an Incomplete Market

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This paper studies the optimal investment problem for an insurer in an incomplete market. The insurer's risk process is modeled by a Lévy process and the insurer is supposed to have the option of investing in multiple risky assets whose price processes are described by the standard Black-Scholes model. The insurer aims to maximize the expected utility of terminal wealth. After the market is completed, we obtain the optimal strategies for quadratic utility and constant absolute risk aversion (CARA) utility explicitly via the martingale approach. Finally, computational results are presented for given raw market data.

1. Introduction

Recently, the problem of optimal investment for an insurer has attracted a lot of attention, due to the fact that the insurer is allowed to invest in financial markets in practice. In the meantime, this is also a very interesting portfolio selection problem in the finance theory. Some important early work is done by Browne [1], where the risk process is approximated by a Brownian motion with drift and the stock price is modeled by a geometric Brownian motion. Under these assumptions, when the interest rate of a risk-free bond is zero, Browne [1] shows that minimizing the ruin probability and maximizing the expected constant absolute risk aversion (CARA) utility of the terminal wealth produce the same type of strategy. In Hipp and Plum [2], the classical Cramér-Lundberg model is adopted for the risk reserve and the insurer can invest in a risky asset to minimize the ruin probability. However, the interest rate of the bond in their model is implicitly assumed to be zero. Liu and Yang [3] extend the model of Hipp and Plum [2] to incorporate a nonzero interest rate. But in this case, a closed-form solution cannot be obtained. Azcue and Muler [4] consider the borrowing constraints for an optimal investment problem of an insurer, and some numerical solutions to minimize the ruin probability are obtained. In the literature, the optimal investment problem for an insurer is usually studied via the stochastic control theory; see [5] for details. By this approach, Yang and Zhang [6] use a jump-diffusion process to model the risk process and obtain the optimal investment strategy for CARA utility maximization. The martingale approach, which has been widely used in mathematical finance, is applied to investigate the optimal investment problem for an insurer by Wang et al. [7]. In their paper, the risk process is modeled by a Lévy process, the security market containing a bond and a stock is described by the standard Black-Scholes model, and the optimal strategies are worked out explicitly for the mean-variance criteria and the CARA utility. Zhou [8] studies a model similar to that in [7], where the prices of assets are described by the Lévy processes, and the optimal strategy is obtained explicitly for the CARA utility.

In most of these papers, the optimal investment for an insurer is considered in a complete market. But the real market is always incomplete; that is, the number of risky assets (stocks) is strictly less than the dimension of the underlying Brownian motion. Hence, it is necessary to find an optimal strategy in an incomplete market. However, the traditional martingale method cannot be used directly in an incomplete market. To overcome the problem of incompleteness, many researchers have developed different ways to handle the problem. For instance, Karatzas et al. [9] complete the market by introducing additional fictitious stocks and then making them untradable. Zhang [10] provides an easier method by directly making the dimension of the underlying Brownian motion equal to the number of available stocks.

The optimal investment problem for an insurer in an incomplete market is studied in the present paper. For this case, the traditional martingale method is problematic. Thus, we first complete the market via the approach proposed by Zhang [10] so that the martingale method can be used. In addition, only one stock is considered in [7, 8]. But an insurer will invest in multiple stocks to decrease risk. Thus in order to make our model more practical, we consider a financial market of one bond and m stocks. Moreover, the insurer's objective in this paper is to maximize the utility of the discounted value of his/her terminal wealth, which is different from that in most works. When the risk process is modeled by a Lévy process and the security market is described by the standard Black-Scholes model, the optimal strategies are obtained explicitly for the quadratic and CARA utilities.

An optimal strategy of CARA utility maximizing in an incomplete market is also obtained in Wang [11]. The result is general but not unique and cannot be expressed concretely in [11]. Whereas our corresponding optimal strategy is explicit and concrete.

This paper is structured as follows. In Section 2 the problem is formulated. The incomplete market is transformed into a complete one in Section 3. The optimal portfolios for quadratic and CARA utility functions are worked out explicitly in Section 4. Section 5 presents some remarks and computational results for optimal strategies in a complete market and an incomplete market. Section 6 contains some conclusions.

2. Formulation of the Problem

The insurer can invest in a financial market consisting of one riskless bond with price $S_t^{(0)}$ given by

$$\mathrm{d}S_t^{(0)} = rS_t^{(0)}\mathrm{d}t, \quad S_0^{(0)} = 1, \tag{1}$$

and m stocks with prices $S_t^{(i)}$ satisfying

$$\mathrm{d}S_t^{(i)} = S_t^{(i)}\left[b_i\mathrm{d}t + \sum_{j=1}^{d}\sigma_{ij}\mathrm{d}W_t^{(j)}\right], \quad i = 1, \dots, m, \tag{2}$$

where r is the interest rate and $W_t := (W_t^{(1)}, \dots, W_t^{(d)})^T$ is a d-dimensional Brownian motion defined on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with the component Brownian motions $W_t^{(j)}$, $j = 1, \dots, d$ being independent. $b := (b_1, \dots, b_m)^T$ denotes the appreciation rate vector, $\sigma := (\sigma_{ij})_{m\times d}$ is the volatility matrix, and define $T > 0$ as a fixed and finite time horizon with $0 \le t \le T$. In addition, the matrix σ is assumed to have the full row rank. As in [10], under these assumptions, we restrict ourselves to the situation where $m \le d$ that is, the number of stocks in the market is smaller than or equal to the dimension of the underlying Brownian motion. This financial market is complete when $m = d$, while incompleteness arises if $m < d$. In this paper, we consider the insurer's investment problem in an incomplete market.

Suppose the initial reserve is x_0 and the premium rate is a constant α; the risk process (R_t) of the insurer is modeled by

$$R_t = x_0 + \alpha t - \sum_{i=1}^{N(t)} Y_i, \tag{3}$$

where $\sum_{i=1}^{N(t)} Y_i$ is a compound Poisson process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, $N(t)$ is a homogeneous Poisson process with intensity λ and represents the number of claims occurring in the time interval $[0, t]$, Y_i is the size of the ith claim. Thus the compound Poisson process $\sum_{i=1}^{N(t)} Y_i$ represents the cumulative amount of claims in the time interval $[0, t]$. The claims' sizes $Y = \{Y_i, i \ge 1\}$ are assumed to be an i.i.d. sequence with a common cumulative distribution function F satisfying $F(0) = 0$ and $\int_0^\infty x^2\mathrm{d}F(x) < \infty$. Furthermore, it is usually assumed that N, $\{Y_i, i \ge 1\}$, and the d-dimensional Brownian motion W are mutually independent. If L_t denotes the compensated compound Poisson process $\sum_{i=1}^{N(t)} Y_i - \lambda m_F t$, where m_F is the mean of F, then the risk process (3) can be rewritten as

$$R_t = x_0 + (\alpha - \lambda m_F)\, t - L_t. \tag{4}$$

As in [7], we generalize the above risk process model to a stochastic cash flow, which is still denoted by (R_t) and satisfies

$$R_t = x_0 + ct - L_t, \tag{5}$$

where L is a 1-dimensional compensated pure jump Lévy process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, c is a constant, $(\mathcal{F}_t)$ is the usual augmentation of the natural filtration of (W, L) with $\mathcal{F} = \mathcal{F}_T$, and W and L are mutually independent. Let μ denote the jump measure of L. Its dual predictable projection ν has the form $\nu(\mathrm{d}t, \mathrm{d}x) = \mathrm{d}t \times m(\mathrm{d}x)$ with $m(\{0\}) = 0$ and $\int_{\mathbb{R}}(x^2 \wedge 1)m(\mathrm{d}x) < \infty$. To avoid some tedious technical arguments, we assume $\int_{\mathbb{R}} x^2 m(\mathrm{d}x) < \infty$ throughout this paper. Under this assumption, it is well known that L is square integrable and has the following Lévy decomposition property:

$$L_t = \int_0^t \int_{\mathbb{R}} x\left(\mu(\mathrm{d}s, \mathrm{d}x) - \nu(\mathrm{d}s, \mathrm{d}x)\right). \tag{6}$$

For details, refer to Cont and Tankov [12]. Further, note that for the risk process model (3), we have $m(\mathrm{d}x) = \lambda F(\mathrm{d}x)$.

The insurer is allowed to invest in those m stocks as well as in the bond. A trading strategy can be expressed in terms of the dollar amount invested in each asset. Let $\pi_t^{(i)}$, $i = 1, \dots, m$, be the dollar amount of the ith stock in the investment portfolio at time t, and let $\pi_t := (\pi_t^{(1)}, \dots, \pi_t^{(m)})^T$ and $\pi := (\pi_t)$. To be mathematically rigorous, for each $1 \le i \le m$, $\pi_t^{(i)}$ is an $(\mathcal{F}_t)$-predictable process.

Definition 1. A trading strategy π is called admissible if $\mathbb{E}[\int_0^T \pi_t^T \pi_t \mathrm{d}t] < \infty$.

The set of all admissible trading strategies is denoted by Π. For each $\pi \in \Pi$ and an initial capital x_0, the wealth process $X^{x_0,\pi}$ of the insurer satisfies the following dynamics:

$$\begin{aligned} \mathrm{d}X_t^{x_0,\pi} &= \pi_t^T \left[b\mathrm{d}t + \sigma \mathrm{d}W_t\right] + \left(X_t^{x_0,\pi} - \pi_t^T \mathbf{1}_m\right) r\mathrm{d}t \\ &\quad + c\mathrm{d}t - \mathrm{d}L_t \\ X_0^{x_0,\pi} &= x_0, \end{aligned} \tag{7}$$

where $\mathbf{1}_m := (1,\dots,1)^T$ is an $m \times 1$ vector. Wang et al. [7] consider an optimal investment problem similar to (7) in the one stock case and solve it elegantly by the martingale method in a complete market. However, in an incomplete market, the martingale method will be problematic. Thus, we will use the method proposed by Zhang [10] to transform our incomplete market into a complete one.

3. Transformation from an Incomplete Market to a Complete One

In this section, we "complete" the incomplete market described in (1) and (2) in order to use the martingale method in the sequel.

Step 1. Reducing the dimension of the Brownian motion.

For each $i = 1,\dots,m,$ let $\sigma_i := (\sigma_{i1}, \sigma_{i2}, \dots, \sigma_{id})$. Then the volatility matrix in the incomplete market can be written as $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_m)^T$. Define

$$B_t^{(i)} := \sum_{j=1}^{d} \frac{\sigma_{ij}}{\|\sigma_i\|} W_t^{(j)}, \quad i = 1,\dots,m, \tag{8}$$

where $\|\cdot\|$ denotes the Euclidean norm. Expressing the stock prices in terms of $B_t^{(i)}$, it gives

$$\mathrm{d}S_t^{(i)} = S_t^{(i)} \left(b_i \mathrm{d}t + \|\sigma_i\| \mathrm{d}B_t^{(i)}\right), \quad i = 1,\dots,m, \tag{9}$$

where $(B_t^{(1)},\dots,B_t^{(m)})^T$ is an m-dimensional Brownian motion. But the component Brownian motions are no longer independent. Specifically, one can consult Exercise 4.15 in [13] for the following result:

$$\mathrm{d}B_t^{(i)} \mathrm{d}B_t^{(k)} = \rho_{ik} \mathrm{d}t, \quad \forall i \neq k, \tag{10}$$

where

$$\rho_{ik} = \frac{1}{\|\sigma_i\| \|\sigma_k\|} \sum_{j=1}^{d} \sigma_{ij}\sigma_{kj} = \frac{\langle \sigma_i, \sigma_k \rangle}{\|\sigma_i\| \|\sigma_k\|}, \tag{11}$$

and $\langle \sigma_i, \sigma_k \rangle$ represents the inner product of σ_i and σ_k. It follows from the Cauchy-Schwarz inequality that $|\rho_{ik}| \leq 1$. Under the assumption that the volatility matrix has the full row rank, we have

$$|\rho_{ik}| < 1 \quad \text{for } i \neq k. \tag{12}$$

Step 2. Creating independent component Brownian motions from the correlated ones.

Let

$$\Psi := \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1m} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{m1} & \rho_{m2} & \cdots & \rho_{mm} \end{bmatrix} \tag{13}$$

be the matrix generated by the correlation coefficients of the correlated m-dimensional Brownian motion $(B_t^{(1)},\dots,B_t^{(m)})^T$ with

$$\rho_{ik} \begin{cases} = 1, & \text{if } i = k, \\ < 1, & \text{if } i \neq k. \end{cases} \tag{14}$$

The matrix Ψ is nonsingular, symmetric, and positively semi-definite. So, there exists a nonsingular matrix $\overline{A} := (\overline{a}_{ij})_{m\times m}$ such that $\Psi = \overline{A}\,\overline{A}^T$. It can be shown that there exist m independent Brownian motions $\widetilde{W}_t^{(1)},\dots,\widetilde{W}_t^{(m)}$ such that

$$B_t^{(i)} = \sum_{j=1}^{m} \overline{a}_{ij} \widetilde{W}_t^{(j)}, \quad \forall i = 1,\dots,m. \tag{15}$$

For details, refer to Exercise 4.16 in [13]. So far, we have derived a complete market with m stocks and m-independent component Brownian motions. The m-stocks in the incomplete market can now be re-written as

$$\mathrm{d}S_t^{(i)} = S_t^{(i)} \left(b_i \mathrm{d}t + \|\sigma_i\| \sum_{j=1}^{m} \overline{a}_{ij} \mathrm{d}\widetilde{W}_t^{(j)} \right), \quad \forall i = 1,\dots,m. \tag{16}$$

The volatility matrix, denoted by $\widetilde{\sigma}$, under the completed market is given by

$$\begin{aligned} \widetilde{\sigma} &= \begin{bmatrix} \|\sigma_1\| \overline{a}_{11} & \|\sigma_1\| \overline{a}_{12} & \cdots & \|\sigma_1\| \overline{a}_{1m} \\ \|\sigma_2\| \overline{a}_{21} & \|\sigma_2\| \overline{a}_{22} & \cdots & \|\sigma_2\| \overline{a}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \|\sigma_m\| \overline{a}_{m1} & \|\sigma_m\| \overline{a}_{m2} & \cdots & \|\sigma_m\| \overline{a}_{mm} \end{bmatrix} \\ &= \begin{bmatrix} \|\sigma_1\| & 0 & \cdots & 0 \\ 0 & \|\sigma_2\| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|\sigma_m\| \end{bmatrix} \cdot \overline{A}, \end{aligned} \tag{17}$$

Put

$$\Sigma := \begin{bmatrix} \|\sigma_1\| & 0 & \cdots & 0 \\ 0 & \|\sigma_2\| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|\sigma_m\| \end{bmatrix}. \tag{18}$$

Then

$$\widetilde{\sigma} = \Sigma \overline{A}. \tag{19}$$

With the help of $\widetilde{\sigma}$ and $\widetilde{W}_t = (\widetilde{W}_t^{(1)}, \ldots, \widetilde{W}_t^{(m)})$, we transfer the model (7) to the following one in the completed market

$$\begin{aligned} \mathrm{d}X_t^{x_0,\pi} &= \pi_t^T \left[b\mathrm{d}t + \widetilde{\sigma}\mathrm{d}\widetilde{W}_t \right] + \left(X_t^{x_0,\pi} - \pi_t^T \mathbf{1}_m \right) r\mathrm{d}t \\ &\quad + c\mathrm{d}t - \mathrm{d}L_t \\ X_0^{x_0,\pi} &= x_0, \end{aligned} \tag{20}$$

which is equivalent to

$$\begin{aligned} X_t^{x_0,\pi} &= \mathrm{e}^{rt} \left(x_0 + \int_0^t \mathrm{e}^{-rs} \left(\pi_s^T (b - r\mathbf{1}_m) + c \right) \mathrm{d}s \right. \\ &\quad \left. + \int_0^t \mathrm{e}^{-rs} \pi_s^T \widetilde{\sigma} \mathrm{d}\widetilde{W}_s - \int_0^t \mathrm{e}^{-rs} \mathrm{d}L_s \right) \\ &= \mathrm{e}^{rt} x_0 + \frac{c(\mathrm{e}^{rt} - 1)}{r} + \int_0^t \mathrm{e}^{r(t-s)} \left(\pi_s^T (b - r\mathbf{1}_m) \right) \mathrm{d}s \\ &\quad + \int_0^t \mathrm{e}^{r(t-s)} \pi_s^T \widetilde{\sigma} \mathrm{d}\widetilde{W}_s - \int_0^t \mathrm{e}^{r(t-s)} \mathrm{d}L_s. \end{aligned} \tag{21}$$

Define

$$\begin{aligned} \widehat{X}_t^{x_0,\pi} &= \mathrm{e}^{-rt} X_t^{x_0,\pi} \\ &= x_0 + \frac{c(1 - \mathrm{e}^{-rt})}{r} + \int_0^t \mathrm{e}^{-rs} \left(\pi_s^T (b - r\mathbf{1}_m) \right) \mathrm{d}s \\ &\quad + \int_0^t \mathrm{e}^{-rs} \pi_s^T \widetilde{\sigma} \mathrm{d}\widetilde{W}_s - \int_0^t \mathrm{e}^{-rs} \mathrm{d}L_s, \end{aligned} \tag{22}$$

which represents the discounted value of $X_t^{x_0,\pi}$.

Suppose that the insurer has a utility function U of his wealth and he/she aims to maximize the expected utility of his/her discounted terminal wealth, that is,

$$\text{maximize}\, \mathbb{E}\left[U\left(\widehat{X}_T^{x_0,\pi} \right) \right] \quad \text{over } \pi \in \Pi. \tag{23}$$

The utility function U is strictly concave and continuously differentiable on $(-\infty, \infty)$; thus there exists at most a unique optimal discounted terminal wealth for the company.

Furthermore, for any $\pi^* \in \Pi$ and $\pi \in \Pi$, we have

$$\begin{aligned} &\mathbb{E}\left[U\left(\widehat{X}_T^{x_0,\pi^*} \right) \right] - \mathbb{E}\left[U\left(\widehat{X}_T^{x_0,\pi} \right) \right] \\ &\quad \geq \mathbb{E}\left[U'\left(\widehat{X}_T^{x_0,\pi^*} \right) \left(\widehat{X}_T^{x_0,\pi^*} - \widehat{X}_T^{x_0,\pi} \right) \right], \end{aligned} \tag{24}$$

from the concavity of U. according to (24) we claim the following proposition.

Proposition 2. *If there exists a strategy $\pi^* \in \Pi$ such that*

$$\mathbb{E}\left[U'\left(\widehat{X}_T^{x_0,\pi^*} \right) \widehat{X}_T^{x_0,\pi} \right] \quad \text{is constant over } \pi \in \Pi, \tag{25}$$

then π^* is the optimal trading strategy.

Remark 3. Such sufficient conditions to the optimal investment are well known in the martingale approach; refer to [9] and others.

By (22), (25) is equivalent to that

$$\mathbb{E}\left[U'\left(\widehat{X}_T^{x_0,\pi^*} \right) \int_0^T \left(\mathrm{e}^{-rs} \pi_s^T (b - r\mathbf{1}_m) \mathrm{d}s + \mathrm{e}^{-rs} \pi_s^T \widetilde{\sigma} \mathrm{d}\widetilde{W}_s \right) \right] \\ \text{is constant over } \pi \in \Pi. \tag{26}$$

Thus, problem (23) can be solved by applying (26) in terms of the concrete utility functions.

It is well known that maximizing the expected quadratic utility is equivalent to finding a mean-variance efficient strategy while CARA utility is commonly used. Thus, we will work out the optimal strategies explicitly for these two utilities in the next section.

4. Optimal Strategies for Different Utility Functions

First, we introduce some notations and a martingale representation theorem that will be used. Let $\mathscr{P}$ be the predictable σ-algebra on $\Omega \times [0, T]$, which is generated by all left-continuous and $(\mathscr{F}_t)$-adapted processes, and $\widetilde{\mathscr{P}} := \mathscr{P} \otimes \mathscr{B}(\mathbb{R})$. Let $L(\mathscr{P})$ (resp., $L^2(\mathscr{P})$) be the set of all $(\mathscr{F}_t)$-predictable, $\mathbb{R}$-valued processes θ such that $\int_0^T |\theta(t)|^2 \mathrm{d}t < \infty$ a.s. (resp., $\mathbb{E}[\int_0^T |\theta(t)|^2 \mathrm{d}t] < \infty$). Let $L(\widetilde{\mathscr{P}})$ be the set of all $\widetilde{\mathscr{P}}$-measurable, $\mathbb{R}$-valued functions θ defined on $\Omega \times [0, T] \times \mathbb{R}$ such that

$$\sqrt{\sum_{0 < s \leq t} \left| \theta(s, \Delta L_s) \right|^2 I_{\{\Delta L_s \neq 0\}}} \tag{27}$$

is locally integrable and increasing, and also $\int_{\mathbb{R}} |\theta(t, x)| m(\mathrm{d}x) < \infty$ a.s. for all $t \in [0, T]$, where $I_{\{\cdots\}}$ is the indicator function. Moreover, let $L^2(\widetilde{\mathscr{P}})$ be the set of all $\widetilde{\mathscr{P}}$-measurable, $\mathbb{R}$-valued functions θ defined on $\Omega \times [0, T] \times \mathbb{R}$ such that $\mathbb{E}[\int_0^T \int_{\mathbb{R}} |\theta(t, x)|^2 m(\mathrm{d}x) \mathrm{d}t] < \infty$. Finally, $L_{\mathscr{F}}^2$ denotes the set of all $(\mathscr{F}_t)$-adapted processes (X_t) with Càdlàg paths such that $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t|^2] < \infty$. It is well known that every square-integrable martingale belongs to $L_{\mathscr{F}}^2$, and it is easy to see that if $\pi \in \Pi$ then $\widehat{X}^{x_0,\pi} \in L_{\mathscr{F}}^2$.

One can consult Proposition 9.4 in [12] for the following result.

Lemma 4 (Martingale Representation). *For any local (resp., square-integrable) martingale (Z_t), there exist some $\theta_1 = (\theta_1^{(1)}, \ldots, \theta_1^{(m)})^T$ with $\theta_1^T \in \underbrace{L(\mathscr{P}) \times \cdots \times L(\mathscr{P})}_{m}$,* *and* $\theta_2 \in L(\widetilde{\mathscr{P}})$ *(resp.,* $\theta_1^T \in \underbrace{L^2(\mathscr{P}) \times \cdots \times L^2(\mathscr{P})}_{m}$*, and* $\theta_2 \in L^2(\widetilde{\mathscr{P}})$*) such that*

$$\begin{aligned} Z_t = Z_0 &+ \int_0^t [\theta_1(s)]^T \, d\widetilde{W}_s \\ &+ \int_0^t \int_{\mathbb{R}} \theta_2(s,x) \left(\mu(ds, dx) - \nu(ds, dx)\right), \end{aligned} \tag{28}$$

for all $t \in [0,T]$.

4.1. Quadratic Utility. In this subsection, we consider problem (23) for the quadratic utility function $U(x) = x - (\gamma/2)x^2$, where $\gamma > 0$ is a parameter. In case that there is only one stock and one bond in a complete market, a closed-form solution is obtained by Wang et al. [7]. Motivated by their work, we show that a closed-form solution can also be obtained when there are several stocks in an incomplete market. Since $U'(x) = 1 - \gamma x$, condition (26) can be written as

$$\mathbb{E}\left[\left(1 - \gamma \widehat{X}_T^{x_0,\pi^*}\right) \int_0^T \left(e^{-rs} \pi_s^T (b - r\mathbf{1}_m) \, ds + e^{-rs} \pi_s^T \widetilde{\sigma} d\widetilde{W}_s\right)\right] \quad \text{is constant over } \pi \in \Pi. \tag{29}$$

Put

$$Z_t^* = \mathbb{E}\left[1 - \gamma \widehat{X}_T^{x_0,\pi^*} \mid \mathscr{F}_t\right], \quad t \in [0,T]. \tag{30}$$

Then $Z_T^* = 1 - \gamma \widehat{X}_T^{x_0,\pi^*}$ and $Z_\tau^* = \mathbb{E}[Z_T^* \mid \mathscr{F}_\tau]$ a.s. for any stopping time $\tau \le T$ a.s.

Lemma 5. *Let* $\pi^* \in \Pi$*. Then* π^* *satisfies condition* (29) *if and only if there exists a* $\theta_2^* \in L^2(\widetilde{\mathscr{P}})$ *such that* $(\widehat{X}^{x_0,\pi^*}, \pi^*, Z^*, \theta_2^*)$ *solves the following forward-backward stochastic differential equation (FBSDE)*

$$\begin{aligned} d\widehat{X}_t &= e^{-rt} \pi_t^T (b - r\mathbf{1}_m) \, dt + e^{-rt} \pi_t^T \widetilde{\sigma} \, d\widetilde{W}_t + c e^{-rt} \, dt \\ &\quad - e^{-rt} dL_t, \\ \widehat{X}_0 &= x_0, \\ dZ_t &= -Z_{t-} (b - r\mathbf{1}_m)^T \left(\widetilde{\sigma}^{-1}\right)^T d\widetilde{W}_t \\ &\quad + d\left(\int_0^t \int_{\mathbb{R}} \theta_2(s,x) \left(\mu(ds, dx) - \nu(ds, dx)\right)\right), \\ Z_T &= 1 - \gamma \widehat{X}_T, \end{aligned} \tag{31}$$

for $(\widehat{X}, \pi, Z, \theta_2) \in L_{\mathscr{F}}^2 \times \Pi \times L_{\mathscr{F}}^2 \times L^2(\widetilde{\mathscr{P}})$.

Proof. Suppose π^* satisfies condition (29). From (22) we have $(\widehat{X}^{x_0,\pi^*}, \pi^*) \in L_{\mathscr{F}}^2 \times \Pi$ solving the forward SDE in (31) for $(\widehat{X}, \pi)$ and it is clear that (Z_t^*) is a square-integrable martingale. For any stopping time $\tau \le T$, let $\pi_t^{\tau,i} = (0, \ldots, 0, I_{[t \le \tau]}, 0, \ldots, 0,)^T$, $i = 1, \ldots, m$, whose ith component is $I_{[t\le\tau]}$ and the other components are all 0. Then $\pi^{\tau,i} \in \Pi$, $i = 1, \ldots, m$. Denote by $\widetilde{\sigma}_i$ the ith row vector of $\widetilde{\sigma}$ for $i = 1, \ldots, m$; that is, $\widetilde{\sigma}_i := (\widetilde{\sigma}_{i1}, \ldots, \widetilde{\sigma}_{im})$. For any positive integer $1 \le i \le m$, substituting $\pi^{\tau,i}$ into (29), we have that

$$\begin{aligned} &\mathbb{E}\left[Z_T^* \int_0^\tau \left(e^{-rs}(b_i - r)\, ds + e^{-rs} \widetilde{\sigma}_i d\widetilde{W}_s\right)\right] \\ &\quad = \mathbb{E}\left[\mathbb{E}\left[Z_T^* \mid \mathscr{F}_\tau\right] \int_0^\tau \left(e^{-rs}(b_i - r)\, ds + e^{-rs} \widetilde{\sigma}_i d\widetilde{W}_s\right)\right] \\ &\quad = \mathbb{E}\left[Z_\tau^* \int_0^\tau \left(e^{-rs}(b_i - r)\, ds + e^{-rs} \widetilde{\sigma}_i d\widetilde{W}_s\right)\right] \end{aligned} \tag{32}$$

is constant over all stopping times $\tau \le T$ a.s., which implies that

$$Z_t^* \int_0^t \left(e^{-rs}(b_i - r)\, ds + e^{-rs} \widetilde{\sigma}_i d\widetilde{W}_s\right) \quad \text{is a martingale}, \quad i = 1, \ldots, m. \tag{33}$$

Since (Z_t^*) is a square-integrable martingale, by Lemma 4, there are $\theta_1^* = (\theta_1^{(1),*}, \ldots, \theta_1^{(m),*})^T$ with $(\theta_1^*)^T \in \underbrace{L^2(\mathscr{P}) \times \cdots \times L^2(\mathscr{P})}_{m}$ and $\theta_2^* \in L^2(\widetilde{\mathscr{P}})$ such that

$$\begin{aligned} dZ_t^* &= [\theta_1^*(t)]^T d\widetilde{W}_t \\ &\quad + d\left(\int_0^t \int_{\mathbb{R}} \theta_2^*(s,x) \left(\mu(ds, dx) - \nu(ds, dx)\right)\right). \end{aligned} \tag{34}$$

By Itô's formula,

$$\begin{aligned} &d\left(Z_t^* \int_0^t \left(e^{-rs}(b_i - r)\, ds + e^{-rs} \widetilde{\sigma}_i d\widetilde{W}_s\right)\right) \\ &\quad = \left((b_i - r) Z_{t-}^* + \widetilde{\sigma}_i \theta_1^*(t)\right) e^{-rt} dt + \text{a local martingale}, \end{aligned} \tag{35}$$

which together with (33) implies $(b_i - r) Z_{t-}^* + \widetilde{\sigma}_i \theta_1^*(t) = 0$ for each $i = 1, \ldots, m$. Thus, it is easy to see that

$$\widetilde{\sigma} \theta_1^*(t) = -(b - r\mathbf{1}_m) Z_{t-}^*, \tag{36}$$

that is,

$$\theta_1^*(t) = -\widetilde{\sigma}^{-1}(b - r\mathbf{1}_m) Z_{t-}^*. \tag{37}$$

By (34), (Z^*, θ_2^*) solves the backward SDE in (31) for (Z, θ_2), and hence $(\widehat{X}^{x_0,\pi^*}, \pi^*, Z^*, \theta_2^*)$ solves FBSDE (31).

Conversely, suppose that there exists $(Z^*, \theta_2^*) \in L_{\mathscr{F}}^2 \times L^2(\widetilde{\mathscr{P}})$ such that $(\widehat{X}^{x_0,\pi^*}, \pi^*, Z^*, \theta_2^*)$ solves FBSDE (31). It is easy to verify that for any $\pi \in \Pi$, by Itô's formula, $(Z_t^* M_t^\pi)$ is a local martingale, where

$$M_t^\pi := \int_0^t e^{-rs} \pi_s^T (b - r\mathbf{1}_m)\, ds + e^{-rs} \pi_s^T \widetilde{\sigma} d\widetilde{W}_s. \tag{38}$$

Furthermore, for any $\pi \in \Pi$, we have $M^{\pi} \in L^2_{\mathcal{F}}$. Hence,

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|Z_t^* M_t^{\pi}\right|\right] \le \sqrt{\mathbb{E}\left[\sup_{0\le t\le T}\left|Z_t^*\right|^2\right]\cdot\mathbb{E}\left[\sup_{0\le t\le T}\left|M_t^{\pi}\right|^2\right]} < \infty, \tag{39}$$

which implies that the family

$$\{Z_\tau^* M_\tau^{\pi} : \tau \text{ is a stopping time and } \tau \le T\} \tag{40}$$

is uniformly integrable and hence $Z^* M^{\pi}$ is a martingale. Thus we have $\mathbb{E}[Z_T^* M_T^{\pi}] = 0$ for any $\pi \in \Pi$, implying that π^* satisfies condition (29). □

In what follows, we will solve FBSDE (31) by two steps.

Step 1. Conjecturing the form of solution.

Put

$$A_t = \exp\left\{\int_0^t a_s \mathrm{d}s\right\}, \quad t \in [0, T], \tag{41}$$

where (a_t) is a nonrandom Lebesgue-integrable function to be determined. If $(\widehat{X}, \pi, Z, \theta_2)$ solves FBSDE (31), then by Itô's formula,

$$\begin{aligned} A_T Z_T &= Z_0 + \int_0^T A_s \mathrm{d}Z_s + \int_0^T Z_{s-}\mathrm{d}A_s \\ &= Z_0 - \int_0^T A_s Z_{s-}(b - r\mathbf{1}_m)^T\left(\widetilde{\sigma}^{-1}\right)^T \mathrm{d}\widetilde{W}_s \\ &\quad + \int_0^T\int_{\mathbb{R}} A_s\theta_2(s,x)\left(\mu(\mathrm{d}s,\mathrm{d}x) - \nu(\mathrm{d}s,\mathrm{d}x)\right) \\ &\quad + \int_0^T Z_{s-}A_s a_s \mathrm{d}s, \end{aligned} \tag{42}$$

which implies

$$\begin{aligned} \frac{1 - Z_T}{\gamma} &= \frac{1}{\gamma} - \frac{1}{\gamma A_T}A_T Z_T \\ &= \frac{1}{\gamma} - \frac{Z_0}{\gamma A_T} + \frac{1}{\gamma A_T}\int_0^T A_s Z_{s-}(b - r\mathbf{1}_m)^T\left(\widetilde{\sigma}^{-1}\right)^T \mathrm{d}\widetilde{W}_s \\ &\quad - \frac{1}{\gamma A_T}\int_0^T\int_{\mathbb{R}} A_s\theta_2(s,x)\left(\mu(\mathrm{d}s,\mathrm{d}x) - \nu(\mathrm{d}s,\mathrm{d}x)\right) \\ &\quad - \frac{1}{\gamma A_T}\int_0^T Z_{s-}A_s a_s \mathrm{d}s. \end{aligned} \tag{43}$$

Let $(\widehat{X}, \pi, Z, \theta_2)$ solve FBSDE (31). Then $\widehat{X}_T = (1 - Z_T)/\gamma$. Comparing $\mathrm{d}\widetilde{W}_t$-term $\mathrm{d}(\mu-\nu)$-term, respectively, in (43) with those in (22) and noting (6), it is reasonable to conjecture

$$\begin{aligned} \frac{A_t Z_{t-}}{\gamma A_T}\widetilde{\sigma}^{-1}(b - r\mathbf{1}_m) &= \mathrm{e}^{-rt}\widetilde{\sigma}^T\pi_t, \\ \frac{A_t}{\gamma A_T}\theta_2(t,x) &= \mathrm{e}^{-rt}x, \end{aligned} \tag{44}$$

that is,

$$\begin{aligned} \pi_t &= \frac{A_t Z_{t-}\mathrm{e}^{rt}}{\gamma A_T}\left(\widetilde{\sigma}^T\right)^{-1}\widetilde{\sigma}^{-1}(b - r\mathbf{1}_m), \\ \theta_2(t,x) &= \frac{\gamma \mathrm{e}^{-rt}x A_T}{A_t}. \end{aligned} \tag{45}$$

Note that by (19), Σ and A are both nonsingular, $\left(\widetilde{\sigma}^T\right)^{-1}\widetilde{\sigma}^{-1} = \left(\widetilde{\sigma}\widetilde{\sigma}^T\right)^{-1} = \left(\Sigma\overline{A}\,\overline{A}^T\Sigma^T\right)^{-1} = (\Sigma\Psi\Sigma)^{-1}$. Hence, (45) can be rewritten as

$$\begin{aligned} \pi_t &= \frac{A_t Z_{t-}\mathrm{e}^{rt}}{\gamma A_T}(\Sigma\Psi\Sigma)^{-1}(b - r\mathbf{1}_m), \\ \theta_2(t,x) &= \frac{\gamma \mathrm{e}^{-rt}x A_T}{A_t}. \end{aligned} \tag{46}$$

Substituting (44) and (46) into (43), we have

$$\begin{aligned} \frac{1 - Z_T}{\gamma} &= \frac{1}{\gamma} - \frac{Z_0}{\gamma A_T} + \int_0^T \mathrm{e}^{-rs}\pi_s^T\widetilde{\sigma}\mathrm{d}\widetilde{W}_s - \int_0^T \mathrm{e}^{-rs}\mathrm{d}L_s \\ &\quad - \frac{1}{\gamma A_T}\int_0^T Z_{s-}A_s a_s \mathrm{d}s \\ &= \frac{1}{\gamma} - \frac{Z_0}{\gamma A_T} + \widehat{X}_T - x_0 - \frac{c\left(1 - \mathrm{e}^{-rT}\right)}{r} \\ &\quad - \int_0^T \mathrm{e}^{-rs}\pi_s^T(b - r\mathbf{1}_m)\mathrm{d}s - \frac{1}{\gamma A_T}\int_0^T Z_{s-}A_s a_s \mathrm{d}s \\ &= \widehat{X}_T + \frac{1}{\gamma} - \frac{Z_0}{\gamma A_T} - x_0 - \frac{c\left(1 - \mathrm{e}^{-rT}\right)}{r} \\ &\quad - \int_0^T \frac{A_s Z_{s-}}{\gamma A_T}(b - r\mathbf{1}_m)^T(\Sigma\Psi\Sigma)^{-1}(b - r\mathbf{1}_m)\mathrm{d}s \\ &\quad - \frac{1}{\gamma A_T}\int_0^T Z_{s-}A_s a_s \mathrm{d}s, \end{aligned} \tag{47}$$

where the second equality follows from the forward SDE in (31) and the third equality holds owing to $[(\Sigma\Psi\Sigma)^{-1}]^T = (\Sigma\Psi\Sigma)^{-1}$. If we take

$$\begin{aligned} a_t &= -(b - r\mathbf{1}_m)^T(\Sigma\Psi\Sigma)^{-1}(b - r\mathbf{1}_m), \\ Z_0 &= A_T - \gamma A_T x_0 - \gamma A_T \frac{c\left(1 - \mathrm{e}^{-rT}\right)}{r} \\ &= \left(1 - \gamma x_0 - \frac{\gamma c\left(1 - \mathrm{e}^{-rT}\right)}{r}\right)\mathrm{e}^{-(b-r\mathbf{1}_m)^T(\Sigma\Psi\Sigma)^{-1}(b-r\mathbf{1}_m)T}, \end{aligned} \tag{48}$$

then (47) is reduced as

$$\frac{1 - Z_T}{\gamma} = \widehat{X}_T \tag{49}$$

and FBSDE (31) is solved.

Step 2. Verifying the conjecture in Step 1.

Let

$$\begin{aligned} &\theta_2^*(t, x) \\ &\quad = \gamma \exp\left\{-rt - (b - r\mathbf{1}_m)^T(\Sigma\Psi\Sigma)^{-1}(b - r\mathbf{1}_m)(T - t)\right\} x, \end{aligned} \tag{50}$$

$$Z_0^* = \left(1 - \gamma x_0 - \frac{\gamma c\left(1 - \mathrm{e}^{-rT}\right)}{r}\right)\mathrm{e}^{-(b-r\mathbf{1}_m)^T(\Sigma\Psi\Sigma)^{-1}(b-r\mathbf{1}_m)T}. \tag{51}$$

Then the following SDE

$$\begin{aligned} \mathrm{d}Z_t = &-Z_{t-}(b - r\mathbf{1}_m)^T\left(\tilde{\sigma}^{-1}\right)^T \mathrm{d}\widetilde{W}_t \\ &+ \mathrm{d}\left(\int_0^t \int_{\mathbb{R}} \theta_2^*(s, x)\left(\mu(\mathrm{d}s, \mathrm{d}x) - \nu(\mathrm{d}s, \mathrm{d}x)\right)\right) \end{aligned} \tag{52}$$

has a solution of the form

$$\begin{aligned} &Z_t^* \\ &= M_t\left(Z_0^* + \int_0^t \int_{\mathbb{R}} M_s^{-1}\theta_2^*(s, x)\left(\mu(\mathrm{d}s, \mathrm{d}x) - \nu(\mathrm{d}s, \mathrm{d}x)\right)\right) \\ &= M_t\left(Z_0^* + \gamma\int_0^t M_s^{-1}\exp\left\{-rs - (b - r\mathbf{1}_m)^T(\Sigma\Psi\Sigma)^{-1}\right.\right. \\ &\qquad\qquad \left.\left. \times (b - r\mathbf{1}_m)(T - s)\right\}\mathrm{d}L_s\right), \end{aligned} \tag{53}$$

where

$$\begin{aligned} M_t = \exp\Big\{ &-(b - r\mathbf{1}_m)^T\left(\tilde{\sigma}^{-1}\right)^T\widetilde{W}_t \\ &-\frac{1}{2}(b - r\mathbf{1}_m)^T(\Sigma\Psi\Sigma)^{-1}(b - r\mathbf{1}_m)t\Big\}. \end{aligned} \tag{54}$$

Furthermore, let

$$\begin{aligned} \pi_t^* = &\frac{1}{\gamma}\exp\left\{(b - r\mathbf{1}_m)^T(\Sigma\Psi\Sigma)^{-1}(b - r\mathbf{1}_m)(T - t) + rt\right\} \\ &\times Z_{t-}^*(\Sigma\Psi\Sigma)^{-1}(b - r\mathbf{1}_m), \end{aligned} \tag{55}$$

and $\widehat{X}_T^{x_0,\pi^*}$ is obtained through the forward SDE in (31). Then by the procedure same as that in (47), it is easy to verify that

$$\frac{1 - Z_T^*}{\gamma} = \widehat{X}_T^{x_0,\pi^*}, \qquad \text{that is, } 1 - \gamma\widehat{X}_T^{x_0,\pi^*} = Z_T^*. \tag{56}$$

Thus, $(\widehat{X}^{x_0,\pi^*}, \pi^*, Z^*, \theta_2^*)$ solves FBSDE (31).

Finally, by Proposition 2, Lemma 5, and arguments in Steps 1–2, we have the following theorem.

Theorem 6. *Let π^* be defined as in* (51)–(55). *Then π^* is the optimal trading strategy of problem* (23) *for the quadratic utility function $U(x) = x - (\gamma/2)x^2$.*

4.2. CARA Utility. In this subsection, we consider problem (23) for the CARA utility function $U(x) = -(1/\gamma)\mathrm{e}^{-\gamma x}$, where $\gamma > 0$. The procedure is similar to what is done in [7]. Since $U'(x) = \mathrm{e}^{-\gamma x}$, condition (26) can be written as

$$\mathbb{E}\left[\mathrm{e}^{-\gamma\widehat{X}_T^{x_0,\pi^*}}\int_0^T\left(\mathrm{e}^{-rs}\pi_s^T(b - r\mathbf{1}_m)\mathrm{d}s + \mathrm{e}^{-rs}\pi_s^T\tilde{\sigma}\mathrm{d}\widetilde{W}_s\right)\right] \quad \text{is constant over } \pi \in \Pi. \tag{57}$$

Similar to the case of quadratic utility, in what follows, we split the procedure into three steps.

Step 1. Conjecturing the form of π^* that satisfies condition (57).

Put

$$Z_T^* := \frac{\mathrm{e}^{-\gamma\widehat{X}_T^{x_0,\pi^*}}}{\mathbb{E}\left[\mathrm{e}^{-\gamma\widehat{X}_T^{x_0,\pi^*}}\right]}, \tag{58}$$

and $Z_t^* := \mathbb{E}[Z_T^* \mid \mathcal{F}_t]$ for all $t \in [0, T]$. Then $Z_\tau^* := \mathbb{E}[Z_T^* \mid \mathcal{F}_\tau]$ a.s. for any stopping time $\tau \le T$ a.s. Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$ such that $\mathrm{d}Q/\mathrm{d}P = Z_T^*$.

For any stopping time $\tau \le T$ a.s., let $\pi_t^{\tau,i} = (0, \ldots, 0, I_{[t\le\tau]}, 0, \ldots, 0,)^T$, $i = 1, \ldots, m$, whose ith component is $I_{[t\le\tau]}$ and the other components are all 0. Then $\pi^{\tau,i} \in \Pi$, $i = 1, \ldots, m$. As in Section 4.1, denote by $\tilde{\sigma}_i$ the ith row vector of $\tilde{\sigma}$ for $i = 1, \ldots, m$; that is, $\tilde{\sigma}_i := (\tilde{\sigma}_{i1}, \ldots, \tilde{\sigma}_{im})$. For any positive integer $1 \le i \le m$, substituting $\pi^{\tau,i}$ into (57), we have

$$\begin{aligned} &\mathbb{E}\left[Z_T^*\int_0^\tau\left(\mathrm{e}^{-rs}(b_i - r)\mathrm{d}s + \mathrm{e}^{-rs}\tilde{\sigma}_i\mathrm{d}\widetilde{W}_s\right)\right] \\ &\quad = \mathbb{E}_{\mathbb{Q}}\left[\int_0^\tau \mathrm{e}^{-rs}(b_i - r)\mathrm{d}s + \mathrm{e}^{-rs}\tilde{\sigma}_i\mathrm{d}\widetilde{W}_s\right] \end{aligned} \tag{59}$$

being constant over all stopping times $\tau \leq T$ a.s., which implies that

$$\int_0^t e^{-rs}(b_i - r)\,ds + e^{-rs}\widetilde{\sigma}_i d\widetilde{W}_s \text{ is a martingale under } \mathbb{Q}, \quad i = 1, \ldots, m. \tag{60}$$

Since (Z_t^*) is a martingale, then

$$K_t := \int_0^t \frac{1}{Z_{s-}^*} dZ_s^*, \quad t \in [0, T], \tag{61}$$

is a local martingale. By Lemma 4, there exist some $\theta_1 = (\theta_1^{(1)}, \ldots, \theta_1^{(m)})^T$ satisfying $\theta_1^T \in \underbrace{L(\mathscr{P}) \times \cdots \times L(\mathscr{P})}_{m}$ and $\theta_2 \in L(\widetilde{\mathscr{P}})$ such that

$$\begin{aligned} dK_t &= (\theta_1(t))^T d\widetilde{W}_t \\ &\quad + d\left(\int_0^t \int_{\mathbb{R}} \theta_2(s, x)(\mu(ds, dx) - \nu(ds, dx))\right), \end{aligned} \tag{62}$$

that is,

$$\begin{aligned} dZ_t^* = Z_{t-}^* \Big[&(\theta_1(t))^T d\widetilde{W}_t \\ &+ d\left(\int_0^t \int_{\mathbb{R}} \theta_2(s, x)(\mu(ds, dx) - \nu(ds, dx))\right)\Big]. \end{aligned} \tag{63}$$

Furthermore, by the Doléans-Dade exponential formula, we have

$$\begin{aligned} Z_t^* = \exp\Bigg\{ &\int_0^t (\theta_1(s))^T d\widetilde{W}_s - \frac{1}{2}\int_0^t (\theta_1(s))^T \theta_1(s)\,ds \\ &+ \int_0^t \int_{\mathbb{R}} \theta_2(s, x)(\mu(ds, dx) - \nu(ds, dx)) \\ &+ \int_0^t \int_{\mathbb{R}} (\log(1 + \theta_2(s, x)) - \theta_2(s, x))\,\mu(ds, dx)\Bigg\}. \end{aligned} \tag{64}$$

By Girsanov's Theorem, we know that $\widetilde{W}_t - \int_0^t \theta_1(s)ds$ is an m-dimensional Brownian motion under $\mathbb{Q}$, which together with (60) implies $\widetilde{\sigma}_i \theta_1(t) = -(b_i - r)$, $i = 1, \ldots, m$. Thus, it is easy to see that

$$\widetilde{\sigma}\theta_1(t) = -(b - r\mathbf{1}_m), \tag{65}$$

that is,

$$\theta_1(t) = -\widetilde{\sigma}^{-1}(b - r\mathbf{1}_m). \tag{66}$$

On the other hand, by (22), we have

$$\begin{aligned} e^{-\gamma \widehat{X}_T^{x_0,\pi^*}} = \exp\Bigg\{ &-\gamma x_0 - \frac{\gamma c(1 - e^{-rT})}{r} \\ &- \gamma \int_0^T e^{-rs}(\pi_s^*)^T (b - r\mathbf{1}_m)\,ds \\ &- \gamma \int_0^T e^{-rs}(\pi_s^*)^T \widetilde{\sigma} d\widetilde{W}_s + \gamma \int_0^T e^{-rs} dL_s \Bigg\}. \end{aligned} \tag{67}$$

In order to have (58), comparing $d\widetilde{W}_t$-term and $d\mu$-term, respectively, in (64) with those in (67) and taking (66) into account, it is reasonable to conjecture

$$\begin{aligned} -\widetilde{\sigma}^{-1}(b - r\mathbf{1}_m) &= -\gamma e^{-rt} \widetilde{\sigma}^T \pi_t^*, \\ \log(1 + \theta_2(t, x)) &= \gamma e^{-rt} x, \end{aligned} \tag{68}$$

that is,

$$\begin{aligned} \pi_t^* &= \frac{e^{rt}(\widetilde{\sigma}^T)^{-1}\widetilde{\sigma}^{-1}(b - r\mathbf{1}_m)}{\gamma} = \frac{e^{rt}(\Sigma\Psi\Sigma)^{-1}(b - r\mathbf{1}_m)}{\gamma} \\ \theta_2(t, x) &= e^{\gamma x e^{-rt}} - 1. \end{aligned} \tag{69}$$

Step 2. Verifying that Z_T^* in (64) satisfies (58), where π^* and θ_2 are defined in (69), and θ_1 is defined in (66).

Substituting (69) into (67), we have

$$e^{-\gamma \widehat{X}_T^{x_0,\pi^*}} = I_T H_T, \tag{70}$$

where

$$\begin{aligned} I_T &= \exp\Bigg\{ -\gamma x_0 - \frac{\gamma c(1 - e^{-rT})}{r} \\ &\qquad - (b - r\mathbf{1}_m)^T (\Sigma\Psi\Sigma)^{-1}(b - r\mathbf{1}_m) T \Bigg\}, \\ H_T &= \exp\left\{ -(b - r\mathbf{1}_m)^T (\widetilde{\sigma}^T)^{-1} \widetilde{W}_T + \gamma \int_0^T e^{-rs} dL_s \right\}. \end{aligned} \tag{71}$$

For θ_1 and θ_2 defined as above, it is easy to see that Z^* is a martingale. Substituting (66) and (69) into (64), we have

$$Z_T^* = J_T H_T, \tag{72}$$

where

$$\begin{aligned} J_T = \exp\Bigg\{ &-\frac{(b - r\mathbf{1}_m)^T (\Sigma\Psi\Sigma)^{-1}(b - r\mathbf{1}_m)}{2} \\ &+ \int_0^T \int_{\mathbb{R}} \left(\gamma e^{-rs} x - e^{\gamma x e^{-rs}} + 1\right) \nu(ds, dxT) \Bigg\} \end{aligned} \tag{73}$$

is a constant. Since Z^* is a martingale, we have $\mathbb{E}[Z_T^*] = 1$ and hence

$$\mathbb{E}[H_T] = J_T^{-1}. \tag{74}$$

Finally, by (70)–(74), we have

$$\frac{e^{-\gamma \widehat{X}_T^{x_0,\pi^*}}}{\mathbb{E}\left[e^{-\gamma \widehat{X}_T^{x_0,\pi^*}}\right]} = \frac{I_T H_T}{I_T \mathbb{E}[H_T]} = J_T H_T = Z_T^*, \tag{75}$$

which is just what we desired.

Step 3. Verifying that π^* in (69) satisfies condition (57).

It is clear that $\mathbb{E}[(Z_T^*)^2] < \infty$. For any $\pi \in \Pi$, by Girsanov's Theorem, we can see that

$$M_t^{\pi} := \int_0^t e^{-rs} \pi_s^T (b - r\mathbf{1}_m)\, ds + e^{-rs} \pi_s^T \tilde{\sigma} d\widetilde{W}_s, \qquad t \in [0, T], \tag{76}$$

is a local martingale under $\mathbb{Q}$. Furthermore, for any $\pi \in \Pi$, we have $M^{\pi} \in L_{\mathcal{F}}^2$. Hence

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\sup_{0\le t\le T} |M_t^{\pi}|\right] &= \mathbb{E}\left[Z_T^* \sup_{0\le t\le T} |M_t^{\pi}|\right] \\ &\le \sqrt{\mathbb{E}\left[(Z_T^*)^2\right] \cdot \mathbb{E}\left[\sup_{0\le t\le T} |M_t^{\pi}|^2\right]} < \infty,\end{aligned} \tag{77}$$

which implies that the family

$$\{M_{\tau}^{\pi} : \ \tau \text{ is a stopping time and } \tau \le T\} \tag{78}$$

is uniformly integrable under $\mathbb{Q}$. Hence M^{π} is a martingale under $\mathbb{Q}$. Thus we have $\mathbb{E}_{\mathbb{Q}}[M_T^{\pi}] = 0$ for any $\pi \in \Pi$, which implies (57).

By Proposition 2 and arguments in steps 5–7, we can claim the following theorem.

Theorem 7. *Let π^* be defined as in* (69). *Then π^* is the optimal trading strategy of problem* (23) *for the CARA utility function $U(x) = -(1/\gamma)\, e^{-\gamma x}$.*

5. Remarks and Computational Results

In this section, we present some remarks and conduct computational experiments for the optimal investment problems of this paper according to the real market data. The objective of these computational tests is to contrast the insurer's optimal strategies in an incomplete market with those in a complete market.

According to Theorems 6 and 7, the optimal strategy of CARA utility is independent of the claim process, while that of quadratic utility is related to the claim process. This is due to the specific nature of the CARA utility.

Table 1: Stocks in a complete market.

Industry	Company	Code
Aerospace and defense	Honeywell Intl.	HON
Automanufactures	Toyota Motor Corp. ADR	TM
Biotechnology and drug manufacturers	Johnson & Johnson	JNJ
Chemicals	EI DuPont de Nemours & Co.	DD
Communication equipment	Qualcomm	QCOM
Computer software	Microsoft	MSFT
Discount	Wal-Mart Stores Inc.	WMT
Diversified computer systems	Hewlett-Packard Co.	HPQ
Major integrated oil and gas	BP Plc.	BP
Semiconductor-broad line	Intel Corp.	INTC
Telecom services	AT & T	T
Utilities (gas and electric)	Duke Energy Corporation	DUK

5.1. Quadratic Utility. For quadratic utility, as shown in (51)–(55), π^* is a left-continuous process with a jump

$$\pi_{t+}^* - \pi_t^* = (\Sigma\Psi\Sigma)^{-1} (b - r\mathbf{1}_m)\, \Delta L_t, \tag{79}$$

at any time t, where ΔL_t is the claim size at time t. From (79) the effect of claims on the efficient strategy of quadratic utility can be interpreted as follows.

(i) When no claim occurs at time t, the wealth process is continuous, and so is π_t^*.

(ii) When a claim occurs at time t, the wealth process has a jump $-\Delta L_t$, which is just the corresponding claim size; the optimal strategy also has a jump and the jump size on each stock is the corresponding element of $(\Sigma\Psi\Sigma)^{-1}(b - r\mathbf{1}_m)\Delta L_t$.

In terms of (79), we can consider the transaction cost of realizing the optimal strategy of quadratic utility. According to [14], the transaction cost is quantified by the portfolio weight turnover for a portfolio π, that is,

$$\|\pi_{t+} - \pi_t\|_1, \tag{80}$$

where $\|\cdot\|_1$ denotes the 1-norm. Since transaction cost is an important aspect of any investment strategy, we are interested in comparing the costs of implementing the optimal strategies in an incomplete market with those in a complete market.

Suppose there are 12 underline Brownian motions, 12 stocks for investment in a complete market, and 10 stocks in an incomplete market. The universe of assets are chosen from the 12 industry categories of finance.cn.yahoo.com in 2009 (see Tables 1 and 2). The appreciation rate vector b is estimated through the mean rate of return of each stock. The covariance matrix a of the rate of return on 12 stocks is estimated based

Table 2: Stocks in an incomplete market.

Industry	Company	Code
Aerospace and defense	Honeywell Intl.	HON
Automanufactures	Toyota Motor Corp. ADR	TM
Biotechnology and drug manufacturers	Johnson & Johnson	JNJ
Chemicals	EI DuPont de Nemours & Co.	DD
Communication equipment	Qualcomm	QCOM
Computer software	Microsoft	MSFT
Diversified computer systems	Hewlett-Packard Co.	HPQ
Major integrated oil & gas	BP Plc.	BP
Semiconductor-broad line	Intel Corp.	INTC
Utilities (gas & electric)	Duke Energy Corporation	DUK

on the real market data. In a complete market, the volatility matrix $\widehat{\sigma}$ is calculated from a such that $\widehat{\sigma}\widehat{\sigma}^T = a$. Next, the volatility matrix σ of the 10 stocks in an incomplete market is obtained by eliminating corresponding row of $\widehat{\sigma}$. Let the risk-free rate $r = 0.02$ and the claim size is distributed according to the exponential distribution with parameter 1; that is, $F(y) = 1 - e^{-y}$. Let $t = 0,\ 1/12,\ 1/6, \ldots, 2$. With exponential claim size ΔL_t randomly generated and Ψ, Σ calculated from (13) and (18), Figure 1 plots the transaction cost in an incomplete market and in a complete market, respectively.

We easily observe that the optimal strategies in an incomplete market incurs lower transaction cost than those in a complete market. Because there are fewer stocks in an incomplete market, than in a complete market, the number of stocks whose investment amount will be changed is smaller in an incomplete market. Thus the transaction cost for fewer stocks is low.

5.2. CARA Utility. For the optimal strategy of CARA utility, (69) shows that the discounted amount held in each stock is fixed. A similar investment strategy for CARA utility in an incomplete market is also obtained in Wang [11]. However, the expression of the optimal strategy in [11] contains a g-inverse of matrix $\sigma\sigma^T$, which is not unique. In our results, $(\Sigma\Psi\Sigma)^{-1}$ replaces the g-inverse matrix, which makes the optimal strategy explicit and deterministic. Due to the determinacy and nonrandomness of the optimal strategy for CARA utility, we can analyze the differences between complete market and incomplete market conveniently from the results of CARA utility.

The experimental procedure is the same as in Section 5.1 and all the parameters are set as in Section 5.1 except t. Let $T = 1$ represent one year and the insurer rebalances his portfolio once a month. Then $t = 0,\ 1/12,\ 1/6, \ldots, 11/12$ and there are 12 investment periods.

In Figure 2, we compare the average total dollar amount invested in stocks in an incomplete market with that in

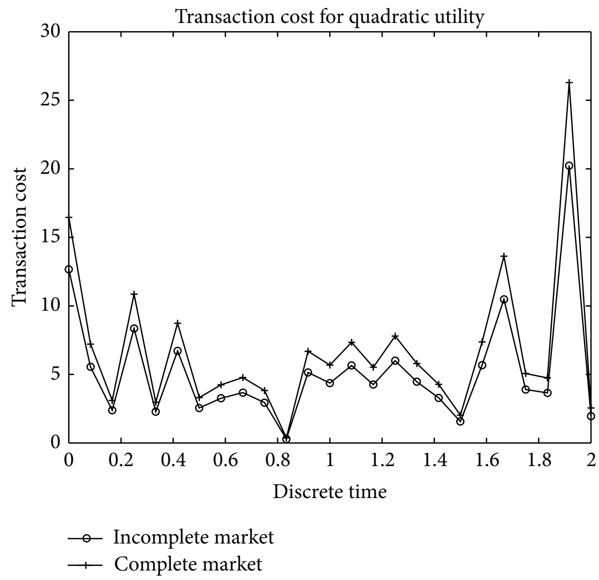

Figure 1: Transaction cost for quadratic utility.

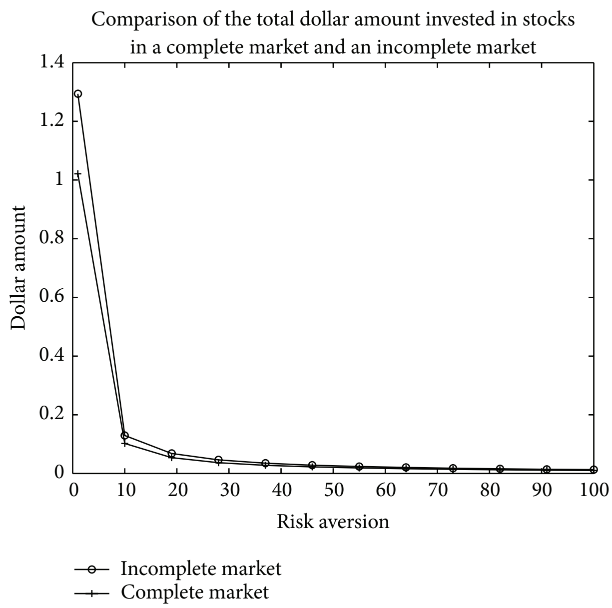

Figure 2: Comparison of the average total dollar amount invested in stocks in a complete market and an incomplete market.

a complete market as the risk aversion coefficient γ ranges from 1 to 100. The average total dollar amount invested in stocks for a sequence of portfolios $\{\pi_t\}_{t=0}^n$ with m stocks is defined as

$$\overline{\pi} = \frac{1}{n+1} \sum_{t=0}^{n} \pi_t^T \mathbf{1}_m. \tag{81}$$

From Figure 2 we find that in an incomplete market, the insurer will invest more money in stocks (risky assets). The practical number of stocks in an incomplete market is less than the ideal number of stocks. For example, in our experiments, there is only 10 stocks for investment while

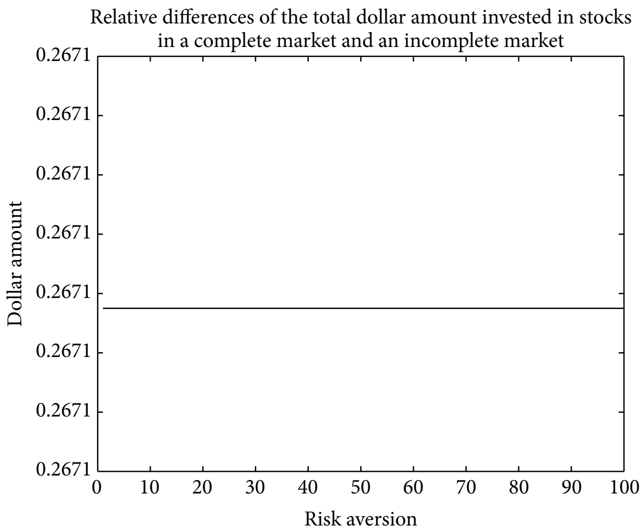

Figure 3: Relative differences of the average total dollar amount invested in stocks between the two markets.

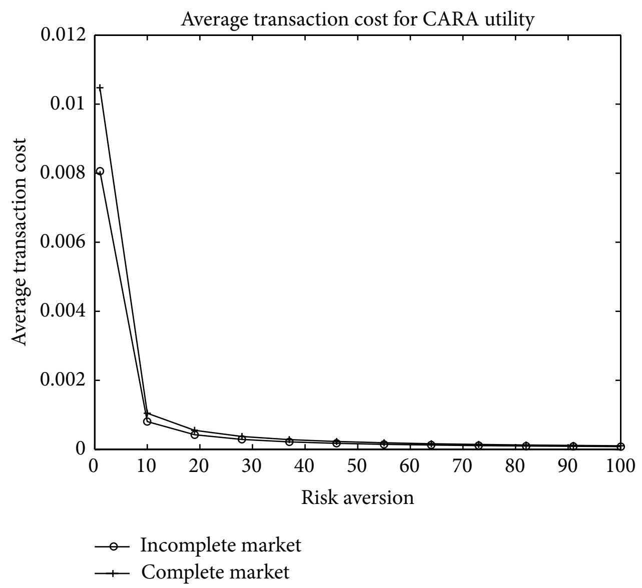

Figure 4: Average transaction cost for CARA utility in the two markets.

the ideal number of attainable stocks is 12 in the incomplete market. Thus an insurer will invest more in the only 10 stocks. An investor is venturesome in an incomplete market. Also we see that the total amount invested in stocks decreases as γ increases. Since γ is the absolute risk aversion coefficient, this result is consistent with intuition.

Figure 3 is a plot of the relative difference of the optimal strategy in an incomplete market with respect to that in a complete market; that is, it plots

$$\frac{(\overline{\pi}_{\text{in}} - \overline{\pi}_{\text{co}})}{\overline{\pi}_{\text{co}}} \tag{82}$$

as a function of γ. The above $\overline{\pi}_{\text{in}}$, $\overline{\pi}_{\text{co}}$ denote the average total dollar amount invested in stocks in an incomplete market and in a complete market, respectively. It is obvious that this difference is independent of the risk aversion coefficient γ but only related to the difference of the market structure. No matter what the risk aversion coefficients are, the effects of the incomplete market on the corresponding optimal strategies are the same.

Given a sequence of portfolios $\{\pi_t\}_{t=0}^{n}$, we quantify its average transaction cost by

$$\frac{1}{n}\sum_{t=0}^{n-1}\left\|\pi_{t+1} - \pi_t\right\|_1. \tag{83}$$

Figure 4 plots the average transaction costs of 12 periods optimal strategies for CARA utility in the two different markets. Same as the quadratic utility case, the cost of implementing the optimal investment in an incomplete market is lower than that in a complete market.

6. Conclusion

In this paper, we consider the optimal investment problem for an insurer in an incomplete market. After transforming the incomplete market into a complete one, we solve the problem via the martingale approach. From the computational experiments, we find that an insurer's optimal investment strategy in an incomplete market incurs lower transaction cost than the one in a complete market. The amount invested in stocks in an incomplete market is more than that in a complete market. For CARA utility, the computational results also show that the incomplete market produces the same effects on the optimal strategies of insurers with different risk aversion coefficients.

Appendix

See Tables 1 and 2.

Acknowledgments

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Research Article

Two-Period Inventory Control with Manufacturing and Remanufacturing under Return Compensation Policy

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As an effective way of decreasing production cost, remanufacturing has attracted more and more attention from firms. However, it also brings many difficulties to firms, especial when firms remanufacture products which they produce. A primary problem for the case is how to acquire the used product sold by the firm itself. In this paper, we consider a return compensation policy for acquiring used product from customers. Under this policy, the return quantity of used product is a proportion of demand. We study an inventory replenishment and production planning problem for a two-period inventory system with dependent return and demand. We formulate the problem into a three-stage stochastic programming problem, where the firm needs to make decisions on the replenishment quantity of new raw material inventory in each period and the production quantities of manufacturing and remanufacturing ways. We give the optimal production policy of manufacturing and remanufacturing ways for the realized demand and prove the objective function for each stage to be concave in the inventory replenishment quantity. Moreover, we prove that the basic inventory policy is still optimal for each period and give the analytical conditions of the optimal inventory levels which are unrelated to acquisition price. Finally, we investigate numerical studies to analyze managerial insights.

1. Introduction

With the development of global market competition, manufacturers constantly launch new products to substitute old products rapidly, which bring more and more end-of-use products into our life. Under the pressures of environmental protection and profit incentive, firms pay more and more attention to closed-loop supply chain (CLSC) management. Besides for the traditional production and selling process, CLSC also includes the process of taking back used products from customers, recovering their added value, and making recovered products reenter into the production system. In a closed-loop supply chain system, the decision-maker needs to consider more factors and decisions, and there are more uncertainties, for example, the uncertainties of used product returns on quantity, quality, and time, so managing closed-loop supply chain is more difficult than only managing forward supply chain or reverse supply chain, even though a large number of firms still join the ranks of operating closed-loop supply chain as profit or cost factor. However, the difficulty also cannot be ignored. Therefore, how should a firm operate closed-loop supply chain system to create more profits is important when the firm faces more factors and uncertainties in closed-loop supply chain system.

Remanufacturing is an important way of reusing used product, which can recover the function of used product but need not to change the original structure of the product. It is a low cost and high efficiency reusing way. However, remanufacturing also requests the firm to be more familiar with the production technology of the used product; otherwise, the firm will suffer a very high cost. Therefore, remanufacturing a product produced by the firm itself is more effective and profitable. The most primary problem when a firm remanufactures its products is how to acquire used product effectively. Pricing strategies have been adopted in many industries and are effective for controlling customer demand. Therefore, it may also be an effective method for controlling the return of used products. In this paper, we consider a policy of acquiring used product, where the firm pays a return compensation for the customers who return used products. It is obvious the customers will be stimulated to return their used products.

Under the return compensation policy, the random return depends on the random demand and it is influenced by the acquisition price of used products, and we consider an inventory replenishment and production planning problem in a two-period setting, where the firm's marketable products can be replenished by manufacturing way using new raw materials and remanufacturing way using used products. The firm needs to determine the inventory replenishment quantity of new raw material in each period, and the production quantities of manufacturing way and remanufacturing way. We formulate the problem into three-stage stochastic dynamic programming model and give all optimal decisions.

Our problem belongs to production planning and inventory control. However, most researchers assume that the return process is independent of the demand process, such as Fleischmann [1], Fleischmann and Kuik [2], Kiesmüller [3], Inderfurth [4], DeCroix and Zipkin [5], DeCroix [6], and Zhou et al. [7, 8]. Fleischmann et al. [9], and Dekker et al. [10] provide comprehensive reviews of production planning and inventory control.

Few papers consider the case that the return process is dependent on the demand process, except for the following research. Kiesmuller and van der Laan [11] investigated an inventory model for a single reusable product, where the random return depends on the demand based on the assumption that the selling quantity is approximately equal to the demand. The results show that it is necessary to consider the dependence between the demand process and the return process. Dobos and Richter [12] investigated a production/recycling system where customer demand and return rates are deterministic and stationary. They consider the EOQ environment with recovery and define return rate as a fraction of the constant demand rate. Atamer et al. [13] study an optimal pricing and production decision problem in utilizing reusable containers. They assume the return is a proportion of the demand under a single selling season.

Our work is mostly related to Atamer et al. [13], but they only considered a single-period setting. We consider a two-period setting, and another main difference from their research is that we consider a stochastic inventory problem, where the firm needs to make decisions on the replenishment quantities in each period and production decisions on remanufacturing and manufacturing. Moreover, we prove the existence of optimal inventory policy in each period and give the optimal policy structure.

The rest of this paper is organized as follows. In Section 2, we give the problem description and formulation. In Section 3, we provide the optimal analysis of the model and give the optimal policies. Numerical examples are provided in Section 4. Finally, we conclude our paper in Section 5.

2. Problem Description and Formulation

For a firm with manufacturing and remanufacturing production way, its marketable product inventory can be replenished by manufacturing way and remanufacturing way, and the products from different ways are homogeneous for the end customers, so we can assume that the selling prices of products from different ways are the same. Generally, the production cost of remanufacturing way is less than manufacturing way, and the firm has a motivation to collect used product actively. For acquiring more used products more effectively, the firm offers a return compensation for the customers who return used products. We name the return compensation as acquisition price, denoted by p_R.

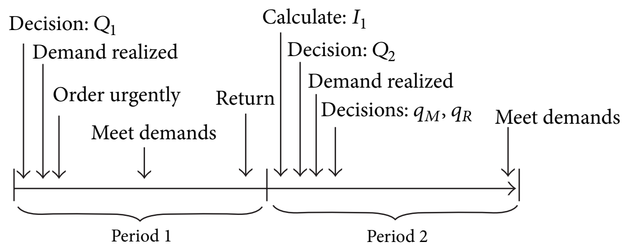

Figure 1: System event sequence.

The demand in period i for the marketable products is random, denoted by D_i, $i = 1, 2$. Under the return compensation policy, the return quantity of used products is related to the selling quantity. As only part of the customers return their used products, we assume it to be a proportion of selling quantity, and the proportion is affected by the acquisition price p_R and is the increasing function in p_R, denoted by $\theta(p_R)$.

We consider a two-period production decision and inventory control problem in a production-to-order system. In each period, the firm needs to make decisions on the replenishment quantity of new raw material inventory before realizing the demand and determines the production quantities of manufacturing and remanufacturing way after realizing the demand.

The sequence of system events is as follows. At the beginning of the first period, the firm replenishes new raw material inventory. Next, the demand is realized. If the raw material is enough, the firm manufactures products to meet the demand by its raw inventory; otherwise, the firm will urgently order the shortage by a higher cost. At the end of the period, customers return their used products. In the second period, based on new raw material inventory and the returned used products in the first period, the firm needs to make decision on the replenishment quantity of raw material inventory before realizing the demand. After realizing the demand, the firm needs to determine the production quantities of manufacturing and remanufacturing way. Here, we assume that the returned products in current period will be remanufactured in the next period. Therefore, the firm does not need to collect used products in the second period. We give the figure of event sequence in Figure 1.

The research aim of this paper is to determine the optimal replenishment quantity of new raw material inventory in each and the production quantities of manufacturing and remanufacturing way so as the expectation of firm's profit can be maximized.

Notation definitions:

p_s = Selling price of marketable product

C = Replenishment cost for the raw material of manufacturing per unit product

c_M = Manufacturing cost of per unit product

c_E = Urgent order cost of per unit shortage

c_R = Remanufacturing cost of per unit returned product

p_R = Acquisition price of per unit used product

s_M = Salvage value of the material of manufacturing per unit product

s_R = Salvage value of per unit returned product

h = Inventory holding cost of per unit marketable product

Q_i = Replenishment quantity of raw material inventory of period i, $i = 1, 2$

q_M = Assigned quantity for manufacturing way after demand of period i is realized

q_R = Assigned quantity for remanufacturing way after demand of period i is realized

D_i = Demand of period i, $i = 1, 2$

$\theta(p_R)$ = Proportion of returning used products in the demand, and is an increasing function in p_R

I_i = Initial inventory level of period i for raw material inventory, $i = 1, 2$.

From the sequence of system events, we know the returned quantity of used products at the end of the first period can be denoted by $\theta(p_R)D_1$. For parameters c_M, c_R, s_M, s_R, we assume $c_M \geq c_R$ and $s_M \geq s_R$, which mean the production cost of remanufacturing way is less than manufacturing way and the salvage value of per unit used products is less than new raw material of producing per unit new product.

Given the acquisition price p_R, the replenishment quantity of raw material inventory Q_i, $i = 1, 2$, and the initial inventory level I_i, the expectation of the revenue in the first period is as follows:

$$\begin{aligned}\pi_1(I_1, Q_1) = & -CQ_1 + p_s E[D_1] - c_M E[D_1] \\ & - hE(I_1 + Q_1 - D_1)^+ - C_E(D_1 - Q_1 - I_1)^+ \\ & - p_R E[\theta(p_R) D_1].\end{aligned} \tag{1}$$

In (1), the first term is the replenishment cost for new raw material inventory, the second term is the selling income, the third term is the manufacturing cost, the fourth term is the inventory holding cost, the fifth is the urgent order cost, and the last term is the acquisition cost for used products. The expectation of the revenue in the second period is

$$\pi_2(I_2, Q_2) = -CQ_2 + E_{D_2}[\max \pi(q_M, q_R)]. \tag{2}$$

In (2), the first term is the replenishment cost for raw material inventory, and the second term is as follows:

$$\begin{aligned}\max \pi(q_M, q_R) = & \ p_s(q_M + q_R) - c_M q_M - c_R q_R \\ & + s_M(I_2 + Q_2 - q_M) \\ & + s_R(\theta(p_R) D_1 - q_R) \\ \text{s.t.} & \begin{cases} q_M \leq I_2 + Q_2 \\ q_R \leq \theta(p_R) D_1 \\ q_M + q_R \leq D_2 \\ q_M, q_R \geq 0. \end{cases}\end{aligned} \tag{3}$$

In (3), the first term is the selling income, and the second term is the manufacturing cost and the third term is the remanufacturing cost, and the fourth term is the salvage value for surplus raw material and the fifth term is the salvage value for surplus used products.

Let $\Pi(I_1, p_R)$ denote system optimal expected revenue for given the initial inventory level I_1 and the acquisition price p_R. Therefore, our aim is to find the optimal replenishment quantities Q_i^*, $i = 1, 2$ and the optimal production quantities (q_M^*, q_R^*) so that

$$\begin{aligned}\Pi(I_1, p_R) = & \max_{Q_1, Q_2 \geq 0} \{\pi_1(I_1, Q_1) + E[\pi_2(I_2, Q_2)]\} \\ & \text{subject to } I_2 = (I_1 + Q_1 - D_1)^+.\end{aligned} \tag{4}$$

The above optimization model in (4) can be resolved by dynamic programming. In the following, we make the optimal analysis for the optimization model in (4).

3. Optimal Analysis for Optimization Model

By dynamic programming, we need first to solve the optimization problem in (3); for convenience, we name the problem as optimal assigning problem. Then we need to optimize the function in (2), and finally obtain the optimal system expectation revenue.

3.1. Optimal Decisions on Optimal Assigning Problem. When we make decisions on optimal manufacturing and remanufacturing quantities in the second period both demands and returns are realized, so we have the following proposition.

Proposition 1. *Given the realized demand and return in the second period, the optimal production decisions q_M^* and q_R^* are as follows:*

$$\begin{aligned} q_M^* &= \min\{(D_2 - \theta(p_R) D_1)^+, I_2 + Q_2\}, \\ q_R^* &= \min\{\theta(p_R) D_1, D_2\}.\end{aligned} \tag{5}$$

Proof. The optimal production decision problem in (3) is a linear programming problem. Because $c_M \geq c_R$ and $s_M \geq s_R$, that is, $c_M + s_M \geq c_R + s_R$, we have $p_s - c_M - s_M \leq p_s - c_R - s_R$. It is obvious that the optimal solution should make the remanufacturing quantity increase as possible. So when $\theta(p_R)D_1 \le D_2$, we have

$$q_R^* = \theta(p_R) D_1,$$
$$q_M^* = \min\{D_2 - \theta(p_R) D_1, I_2 + Q_2\}. \tag{6}$$

And when $\theta(p_R)D_1 > D_2$, we have

$$q_R^* = D_2, \qquad q_M^* = 0. \tag{7}$$

Therefore, $q_M^* = \min\{(D_2 - \theta(p_R)D_1)^+, I_2 + Q_2\}$ and $q_R^* = \min\{\theta(p_R)D_1, D_2\}$. □

Proposition 1 shows that the optimal production rule in the second period is to satisfy the demand by remanufacturing way as possible, only when the demand cannot be satisfied completely by remanufacturing way, the manufacturing way is considered.

3.2. Optimal Replenishment Decision in the Second Period. Let $y_2 = I_2 + Q_2$, it denotes the inventory level after replenishing the raw material inventory. And from Proposition 1, we can rewrite (2) as follows:

$$\begin{aligned}\pi_2(I_2, y_2) = & -C(y_2 - I_2) + (p_s - c_M - s_M) \\ & \times E\left[\min\left\{(D_2 - \theta(p_R) D_1)^+, y_2\right\}\right] \\ & + (p_s - c_R - s_R) E\left[\min\{\theta(p_R) D_1, D_2\}\right] \\ & + s_M y_2 + s_R E\left[\theta(p_R) D_1\right].\end{aligned} \tag{8}$$

Theorem 2. *For $\pi_2(I_2, y_2)$ in (8), there are the following.*

(a) *$\pi_2(I_2, y_2)$ is jointly concave in I_2 and y_2.*

(b) *The equation $\partial\pi_2(I_2, y_2)/\partial y_2 = 0$ has unique solution.*

Proof. It is obvious that $E[\min\{(D_2 - \theta(p_R)D_1)^+, y_2\}]$ is concave in y_2, and other parts in (8) are linear in I_2 and y_2. Therefore, $\pi_2(I_2, y_2)$ is jointly concave in I_2 and y_2.

The first-order derivative of $\pi_2(I_2, y_2)$ about y_2 is

$$\begin{aligned}\frac{\partial\pi_2(I_2, y_2)}{\partial y_2} = & -C + s_M + (p_s - c_M - s_M) \\ & \times \Pr\left\{(D_2 - \theta(p_R) D_1)^+ \ge y_2\right\}.\end{aligned} \tag{9}$$

Because $p_s - c_M - s_M - (C - s_M) = p_s - c_M - C > 0$ and $C - s_M > 0$, there must exist a certain y_2 satisfying the following equation:

$$\Pr\left\{(D_2 - \theta(p_R) D_1)^+ \ge y_2\right\} = \frac{C - s_M}{p_s - c_M - s_M}. \tag{10}$$

Therefore, $\partial\pi_2(I_2, y_2)/\partial y_2 = 0$ must exist. □

Theorem 2 shows that the optimal inventory level must exist and can be solved by $\partial\pi_2(I_2, y_2)/\partial y_2 = 0$. The optimal decision rule is given in the following proposition.

Proposition 3. *Given inventory level I_2, the optimal replenishment decision in the second period is as follows:*

$$Q_2^* = \begin{cases} S_2 - I_2 - \theta(p_R) D_1 & I_2 + \theta(p_R) D_1 < S_2 \\ 0 & I_2 + \theta(p_R) D_1 \ge S_2, \end{cases} \tag{11}$$

where S_2 satisfies

$$\Pr\{D_2 \ge S_2\} = \frac{C - s_M}{p_s - c_M - s_M}. \tag{12}$$

Proof. We know that

$$\begin{aligned}&\Pr\left\{(D_2 - \theta(p_R) D_1)^+ \ge y_2\right\} \\ &\quad = \Pr\{D_2 \ge \theta(p_R) D_1, D_2 - \theta(p_R) D_1 \ge y_2\} \\ &\quad = \Pr\{D_2 \ge \theta(p_R) D_1, D_2 - \theta(p_R) D_1 \ge y_2\} \\ &\quad = \Pr\{D_2 \ge y_2 + \theta(p_R) D_1\}.\end{aligned} \tag{13}$$

Let $z_2 = \theta(p_R)D_1 + y_2$; it can denote the whole inventory after the new raw material inventory is replenished, where the whole inventory includes new raw material inventory and used product inventory. Therefore, S_2 of satisfying $\Pr\{D_2 \ge z_2\} = (C - s_M)/(p_s - c_M - s_M)$ also must satisfy $\Pr\{(D_2 - \theta(p_R)D_1)^+ \ge y_2\} = (C - s_M)/(p_s - c_M - s_M)$, where $y_2 = S_2 - \theta(p_R)D_1$.

In the following, we consider two cases.

Case 1. When $I_2 + \theta(p_R)D_1 < S_2$, from Part (a) in Theorem 2, for any Q_2 satisfying $I_2 + Q_2 + \theta(p_R)D_1 \le S_2$, we have

$$\frac{\partial\pi_2(I_2, I_2 + Q_2 + \theta(p_R) D_1)}{\partial y_2} \ge \frac{\partial\pi_2(I_2, S_2)}{\partial y_2} = 0. \tag{14}$$

And for any Q_2 satisfying $I_2 + Q_2 + \theta(p_R)D_1 > S_2$,

$$\frac{\partial\pi_2(I_2, I_2 + Q_2 + \theta(p_R) D_1)}{\partial y_2} < \frac{\partial\pi_2(I_2, S_2)}{\partial y_2} = 0. \tag{15}$$

So Q_2^* should satisfy $I_2 + \theta(p_R)D_1 + Q_2^* = S_2$, that is, $Q_2^* = S_2 - I_2 - \theta(p_R)D_1$.

Case 2. When $I_2 + \theta(p_R)D_1 \ge S_2$, for any $Q_2 \ge 0$, we have

$$\begin{aligned}&\frac{\partial\pi_2(I_2, I_2 + Q_2 + \theta(p_R) D_1)}{\partial y_2} \\ &\quad \le \frac{\partial\pi_2(I_2, I_2 + \theta(p_R) D_1)}{\partial y_2} \le \frac{\partial\pi_2(I_2, S_2)}{\partial y_2} = 0\end{aligned} \tag{16}$$

so $Q_2^* = 0$.

In summary,

$$Q_2^* = \begin{cases} S_2 - I_2 - \theta(p_R) D_1 & I_2 + \theta(p_R) D_1 < S_2, \\ 0 & I_2 + \theta(p_R) D_1 \ge S_2. \end{cases} \tag{17}$$

Proposition 3 holds. □

Proposition 3 shows that the basic inventory policy is still optimal. We call S_2 as the optimal inventory level in the second period. It is obvious that it is unrelated to acquisition price.

3.3. Optimal Replenishment Decision in the First Period. For the convenience of description, we define

$$\begin{aligned}\Pi_2(I_2) = \max_{y_2\ge I_2}\{&-Cy_2 + (p_s - c_M - s_M)\\ &\times E\left[\min\left\{(D_2-\theta(p_R)D_1)^+, y_2\right\}\right]\\ &+ (p_s - c_R - s_R)\times E\left[\min\{\theta(p_R)D_1, D_2\}\right]\\ &+ s_M y_2 + s_R\theta(p_R)D_1\}.\end{aligned} \tag{18}$$

From (8), we know that

$$\Pi_2(I_2) = \max_{y_2\ge I_2}\{\pi_2(I_2,y_2) - CI_2\}, \tag{19}$$

or $\Pi_2(I_2) + CI_2 = \max_{y_2\ge I_2}\{\pi_2(I_2,y_2)\}$.

From dynamic programming, we have

$$\begin{aligned}\Pi(I_1,p_R,Q_1) = \max_{Q_1\ge 0}\ &\{\pi_1(I_1,Q_1)\\ &+ E\left[\Pi_2\left((I_1+Q_1-D_1)^+\right)\right.\\ &\left.+ C(I_1+Q_1-D_1)^+\right]\}.\end{aligned} \tag{20}$$

For analyzing the property of $\Pi(I_1,p_R,Q_1)$, we give the following lemma.

Lemma 4. *(a) If $f(x,y)$ is jointly concave in x and y and C is a convex set on R^2, then $g(x) = \max_{y\in C} f(x,y)$ is also concave in x.*

(b) If $h(x)$ is concave and nonincreasing and $f(x)$ is convex, then $g(x) = h(f(x))$ is also concave.

Proof. For the proof of (a) and (b), please see Pages 84 and 81 in Boyd and Vandenberghe [14]. □

Let $H(I_1,Q_1)$ denote the objective function of (20), that is,

$$\begin{aligned}H(I_1,Q_1) &= \pi_1(I_1,Q_1) + E\left[\Pi_2\left((I_1+Q_1-D_1)^+\right)\right.\\ &\quad\left.+ C(I_1+Q_1-D_1)^+\right].\end{aligned} \tag{21}$$

Let $y_1 = I_1 + Q_1$, it denotes the inventory level after the raw material inventory is replenished. And from (1), we have

$$\begin{aligned}H(I_1,y_1) = &-C(y_1-I_1)\\ &+ E\left[(p_s - c_M - p_R\theta(p_R))D_1\right.\\ &\quad\left. - h(y_1-D_1)^+ - C_E(D_1-y_1)^+\right]\\ &+ E\left[\Pi_2\left((y_1-D_1)^+\right) + C(y_1-D_1)^+\right].\end{aligned} \tag{22}$$

Theorem 5. *$H(I_1,y_1)$ in (22) is concave in y_1.*

Proof. We divide $H(I_1,y_1)$ into two parts $W_1(I_1,y_1)$ and $W_2(I_1,y_1)$, where

$$\begin{aligned}W_1(I_1,y_1) = &\ (p_s - c_M - p_R\theta(p_R))E[D_1]\\ &- hE\left[(y_1-D_1)^+\right] - C_E E\left[(D_1-y_1)^+\right]\\ &+ CE\left[(y_1-D_1)^+\right],\\ W_2(I_1,y_1) = &\ E\left[\Pi_2\left((y_1-D_1)^+\right)\right] + C(y_1-I_1).\end{aligned} \tag{23}$$

The first-order derivative of $W_1(I_1,y_1)$ about y_1 is

$$\frac{\partial W_1(I_1,y_1)}{\partial y_1} = (C-h)\Pr\{y_1\ge D_1\} + C_E\Pr\{y_1<D_1\}, \tag{24}$$

and the second-order derivative of $W_1(I_1,y_1)$ about y_1 is

$$\frac{\partial^2 W_1(I_1,y_1)}{\partial y_1^2} = (C-h-C_E)f_{D_1}(y_1) < 0, \tag{25}$$

so $W_1(I_1,y_1)$ is concave in y_1.

In the following, we will prove that $W_2(I_1,y_1)$ is still concave in y_1. From Part (a) in Theorem 2, we know that $\pi_2(I_2,y_2)$ is jointly concave in I_2 and y_2, so $\max_{y_2\ge I_2}\{\pi_2(I_2,y_2)\}$ is also concave in I_2 by Part (a) in Lemma 4. And From (19), we have

$$\Pi_2(I_2) = \max_{y_2\ge I_2}\{\pi_2(I_2,y_2)\} - CI_2, \tag{26}$$

so $\Pi_2(I_2)$ is also concave in I_2. Moreover, it is nonincreasing in I_2.

Let $\varphi(y) = (y)^+$; it is convex in y. From Part (b) in Lemma 4, we know that $\Pi_2(\varphi(y_1))$ is concave in y_1. Therefore, $E[\Pi_2(\varphi(y_1-D_1))] = E[\Pi_2((y_1-D_1)^+)]$ is also concave in y_1, and $W_2(I_1,y_1)$ is concave in y_1.

Therefore, $H(I_1,y_1)$ in (22) is concave in y_1 as follows: □

$$\begin{aligned}\frac{\partial H(I_1,y_1)}{\partial y_1} = &-C + (C-h)\Pr\{y_1\ge D_1\}\\ &+ C_E\Pr\{y_1<D_1\} + \frac{\partial E\left[\Pi_2\left((y_1-D_1)^+\right)\right]}{\partial y_1}.\end{aligned} \tag{27}$$

Let $S = \min\{y_1 \mid \partial H(0,y_1)/\partial y_1 \le 0,\ y_1\ge 0\}$, so we have the following theorem.

Theorem 6. *S must exist.*

Proof. From $E[\Pi_2((y_1-D_1)^+)] = \int_{-\infty}^{y_1}\Pi_2(0)\,dF(x) + \int_{y_1}^{+\infty}\Pi_2(y_1-x)\,dF(x)$, we have

$$\frac{dE\left[\Pi_2\left((y_1-D_1)^+\right)\right]}{dy_1} = \int_{y_1}^{+\infty}\frac{d\Pi_2(y_1-x)}{dy_1}\,dF(x). \tag{28}$$

Table 1: Basic parameter setting.

Parameter	p_s	C	C_E	h	c_M	c_r	s_M	s_R	a_i	σ	k	p_R
Range	3.5	1.8	2.2	0.2	0.6	0.5	0.8	0.6	1000	40~300	0.1~0.8	0.5~1.3

And because $\Pi_2(I_2)$ is nonincreasing in I_2, we have $dE[\Pi_2((y_1 - D_1)^+)]/dy_1 \leq 0$, and

$$\lim_{y_1 \to \infty} \frac{\partial H(I_1, y_1)}{\partial y_1} = -h + \lim_{y_1 \to \infty} \frac{\partial E\left[\Pi_2\left((y_1 - D_1)^+\right)\right]}{\partial y_1} \leq -h, \tag{29}$$

so S must exist. □

Similar to Proposition 3, we have Proposition 7.

Proposition 7. *Given the initial inventory level I_1, the optimal replenishment decision in the first period is given as follows:*

$$Q_1^* = \begin{cases} S_1 - I_1 & I_1 < S_1, \\ 0 & I_1 \geq S_1, \end{cases} \tag{30}$$

where S_1 satisfies

$$S_1 = \min\left\{y_1 \mid \frac{\partial H(0, y_1)}{\partial y_1} \leq 0, y_1 \geq 0\right\}. \tag{31}$$

Proof. The proof is similar to Proposition 3. □

4. Numerical Study

In this section, we study management insights by numerical examples and make the sensitivity analysis of optimal expected profit with respect to other parameters.

We assume the demand in each period to be $D_i = a_i + \varepsilon_i$. Random parts ε_1 and ε_2 are independent and identical distribution and are assumed to follow the normal distribution with the zero mean and the deviation σ. The proportion of returning used products in all demand for the first period is assumed to be $\theta(p_R) = 1 - e^{-kp_R}$. The parameter setting is provided in Table 1.

We analyze the following cases. (i) The effects of acquisition price and standard deviation of demand on system optimal expected profit. (ii) The effects of the return sensitive coefficient in acquisition price on system optimal expected profit. (iii) The effects of sensitive coefficient and standard deviation of demand on percent improvement in optimal expected profit.

Figure 2 shows that the system optimal expected profit is a concave function in acquisition price, which means that we can determine an optimal acquisition price for any set of parameter setting. It is obvious that a comfortable acquisition price can supply production material by a lower cost, but the profit of unit used product is decreasing as the acquisition price is increasing and it is possible that remanufacturing has no profit when the acquisition price is too large. Moreover, Figure 1 also shows that the system optimal expected profit is decreasing as the standard deviation of demand is increasing.

Figure 3 shows that the system optimal expected profit is increasing as different sensitive coefficients k are increasing. Because the return proportion is more sensitive on acquisition price, the firm can acquire used product more easily, the acquisition cost is lower, and the remanufacturing profit is larger.

To analyze the change of expected profit, we define a percent improvement in the expected profit by the following:

$$\begin{aligned} &\text{Percent improvement in expectation profit} \\ &= \frac{\text{Optimal expected profit } (p_R \text{ is nonzero}) - \text{Optimal expected profit } (p_R \text{ is zero})}{\text{Optimal expected profit } (p_R \text{ is zero})} \times 100\%. \end{aligned} \tag{32}$$

From Figure 4, we can know that the standard deviation is larger, the percent improvement in expectation profit is larger, the return process is more sensitive, and the percent improvement in expectation profit is larger. Return process can be viewed as another supply source, which may decrease the supply risk, and when the standard deviation of demand is larger, the action of multichannel for decreasing risk is larger. The return proportion is more sensitive on acquisition price, and the firm can control risk by a lower cost, so the profit improvement is larger.

5. Conclusion

In this paper, we study an inventory control and production planning problem when a firm with manufacturing and remanufacturing production way faces stochastic demand. In order to stimulate the return of used products, the firm offers a return compensation for the customers who return used products. Under the return stimulating policy, the return process depends on the demand process. Based on the setting, we investigate optimal policies on inventory replenishment

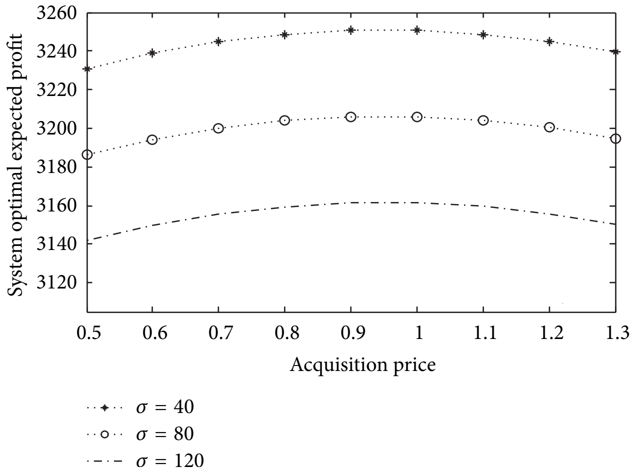

Figure 2: Optimal expected profits for different standard deviations.

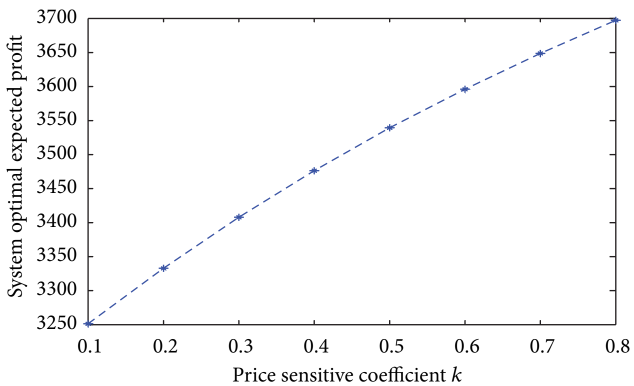

Figure 3: Optimal expected profits for different sensitive coefficients k.

and production planning problem for a single item two-period inventory system. Firstly, the problem is formulated into a three-stage stochastic programming problem. Secondly, the optimal production policies on manufacturing and remanufacturing for the realized demand are given. Next, we prove the objective function for each stage to be concave in decision variable and prove the existence and the uniqueness of optimal solutions. Moreover, the basic inventory policy is proved to be optimal for each period, and the optimal inventory levels are unrelated to acquisition price. Finally, we investigate numerical studies to analyze managerial insights.

There are some possible extensions in the future investigation. The current problem only considers inventory decisions and a single type of used product quality class. One extension would be to consider to make joint decisions on inventory replenishment and acquisition price. Another extension is to study the multitype product quality class setting.

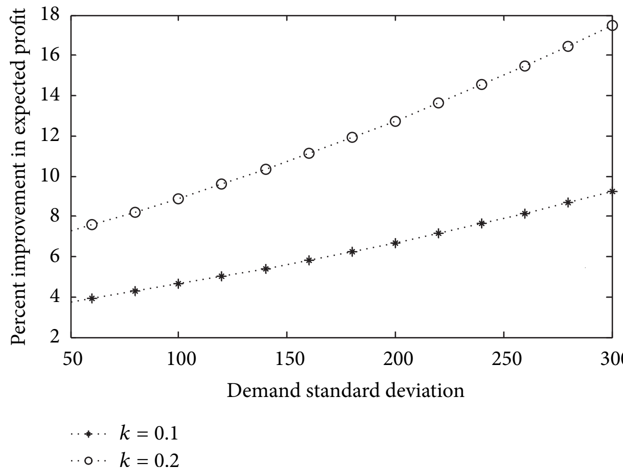

Figure 4: Percent improvement in the expected profit.

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Research Article

Pricing and Allotment in a Sea-Cargo Supply Chain with Reference Effect: A Dynamic Game Approach

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The contract between the carrier and forwarder is a long-term issue, and the repeated contract business makes the forwarder develop a reference point based on the contract prices, and this reference effect, to a large extent, affects the forwarder's contract purchasing decisions. Based on that, this paper introduces the reference effect in the sea-cargo supply chain and studies a multiple-period contract problem between the carrier and the forwarder. It is found that when the capacity price in the spot market is less than the forwarder's willingness-to-pay, the forwarder's contract purchasing decision is not affected by the reference effect, only by the capacity price in the spot market, and the multiple-period contract problem can be simplified into a single-period game. In addition, the carrier's optimal contract wholesale price approaches the capacity price in the spot market. Although, the forwarder's contract purchasing decision depends upon the reference effect, it is difficult to derive the closed-form solution. Moreover, because of the risk in the spot market, the carrier tends to sell his/her capacity in the contract market. Finally, we employ the numerical simulation to study the carrier's contract pricing decisions and the forwarder's capacity purchasing decisions in two cases.

1. Introduction

In the sea-cargo market, it is commonly seen that the spot market coexists with the contract market [1, 2]; the specific operational process is as follows. In the first stage, the carrier chooses the advanced-selling strategy to reduce the risk of capacity allotment, that is, to sell most capacity to forwarders or large shippers in advance in order to reduce the demand uncertainty. And then, in the second stage, the remaining space is sold by the carrier to direct-ship shippers on ad hoc or free-sale basis. As to the forwarders, in the first stage the forwarders procure large capacity, which qualifies them for a discount. The discount may depend upon the size of allotment as well as the actual volume tendered by the forwarder [3]. In the second stage, the forwarder drums up downstream shippers; forwarders can offer more services and often better prices to downstream shippers in comparison to the carriers' standard tariff.

The supply chain contracts on capacity allotment focused mostly on their applications in the air transport; for example, Hellermann [2] studied the application on the real option contract in air transportation, and Gupta [4] analyzed the flexible contract coordination between the carrier and forwarder in air transport. Spinler [1] studied the application of capacity reservation in capital-intensive industry and illustrated the applicability of option reservation in air transport, electronic, and tourism industry. However, because of the differences between air transport and liner transport [5, 6], liner transport is different from air transport in many aspects, such as products and services provided, market demand, and operation modes. And the supply chain contract in liner transportation is also different from that in air transport.

In the freight market, due to the long-term and repeated purchase behavior between the forwarder and carrier, the forwarder will develop reference price, through observing carrier's past contract prices or capacity prices in other channels. Comparing current price with the reference price, thereafter, the forwarder makes his capacity procurement decision. Therefore, the reference effect, to a large extent, affects the forwarder's capacity reservation decisions. As to the repeated purchasing case, the reference price theory indicates that economic agents are not completely rational, and they always consider the past selling price and make purchasing decisions based on this comparison [7]. To be specific, the forwarder will develop his expectation price or reference price based on the past contract prices and use this expectation price as a reference point to compare with the contract price. If the contract price is higher than the reference price, the forwarder perceives prices as loss and would like to purchase less capacity and turn to other channels to procure; otherwise the forwarder would purchase more on site. In this case, the forwarder's purchasing decisions are based not only on the contract price, but also on the reference price. This reference effect makes the forwarder's purchasing level much different [8].

The liner transport problem we addressed in this paper is consistent with the case where reference effect happens, generally the carrier and forwarder are in a long-term contract trading, and they always develop a long-term cooperation partnership, in each period the carrier determines the contract price, and the forwarder determines his contract purchasing quantity according to the contract price and freight demand. In this way, it makes the forwarder refer to the past contract prices to determine contract purchasing quantity in the long-term contract. And this makes the sea-cargo market much more complex.

The reminder of this paper is organized as follows. We begin by briefly formulating the problem and provide some notations in Section 2. In Section 3 we develop the mathematical model of the carrier and the forwarder, respectively. Section 4 conducts a numerical study and examines the sensitivity of contract price, spot price, and the demand variance to the carrier's expected profits. In Section 5, we summarize this paper and provide concluding remarks.

2. Problem Descriptions

We consider a two-echelon sea-cargo supply chain composed of one single carrier and one single forwarder. Due to the uncertainty of both the demand and price in the freight market, the carrier chooses to sell most capacity to large customers on the contract market, that is, the forwarders and large shippers, in advance. And the reminder capacity is sold on the spot market shortly. The forwarder determines his contract purchasing quantity according to the freight demand and his reference price. In the paper, we adopt the common supply chain contract wholesale contract. In a channel context, we observe in some cases that channel transactions are "governed by simple contracts defined only by a per unit wholesale price" [9]. As noted by Holmström and Milgrom [10], incentive contracts in the real world frequently take simpler forms than what the theory often predicts. This can also happen because firms have little to lose using a simpler contract [11]. We build a Von-Stackelberg model of carrier and forwarder where the carrier is game leader, and the forwarder is game follower. The carrier firstly announces the wholesale price according to the ordering quantity of the forwarder, and the forwarder decides the contract quantity. The mathematical representation of the interaction between carrier and forwarder is based on a stylized model in which certain real-life details have been either omitted or simplified to maintain tractability.

The chronology of events is described as follows. First, the carrier releases her wholesale price contract $\{k(t), q(t)\}$ in each period t, where $k(t)$ is the wholesale price per unit capacity. Next, the forwarder determines his contract purchasing quantity $q(t)$ in each period t based on wholesale prices announced by the carrier to achieve his/her long-term revenue maximization.

As described above, this paper studies a finite-period problem; the carrier and forwarder make decisions in multiple periods. Assume that the carrier's cost structure is $\widetilde{c}(W, z) = \widetilde{\beta}W + \widetilde{b}z$, where $\widetilde{\beta}$ is the capacity cost per unit, $\widetilde{b}$ is marginal sale cost per unit, and both $\widetilde{\beta}$ and $\widetilde{b}$ are constant. This assumption has been used widely (such as [12]). Constant variable cost satisfies economic of scale; in this situation, the production cost is the lowest. The two contract marginal costs of sales satisfy $\widetilde{b}^{s} \geq \widetilde{b}^{c}$, that is in accord with the real operation [1].

The demands on the spot market and contract market are independent; this assumption simplifies the revenue management problem [13, 14]. Moreover, because the difference of market structure and sensitivity of both the carrier and forwarder to the prices of both spot and contract markets, the spot and contract markets are completely separable [2]. In the spot market, it is assumed that the market price is the perfectly competitive equilibrium price $\widetilde{p}$, with mean $\mu_{\widetilde{p}}$ and standard deviation $\sigma_{\widetilde{p}}$, and is the normally distributed with p.d.f. $f(\widetilde{p})$ and c.d.f. $F(\widetilde{p})$. The carrier and forwarder are Price Takers. In the contract market, the forwarder faces a long-term freight demand $Q(p) = a - bp + \varepsilon$, and $a > 0$, $b \geq 0$, ε follows the standard normal distribution $\varepsilon \sim N(0, \sigma_{\varepsilon}^2)$ with mean 0 and standard deviation σ_{ε}. For linear demand function, a is the largest market demand scale, that is, the downstream customer's capacity demand when the contract price is zero; b means the sensitivity of downstream customer on the contract price; random variable ε characterizes the uncertainty of the downstream customer's demand on the capacity [15, 16]. Therefore, the downstream customer's capacity demand follows a normal distribution, with mean $\mu_Q = E[a - bp + \varepsilon] = a - bp$ and standard deviation $\sigma_Q = \sigma_{\varepsilon}$, respectively, and its p.d.f. $g(Q)$ and c.d.f. $G(Q)$.

In the contract and spot market, demand represents the sum of demands from a large number of customers. In the contract market, this is the (limited) number of end customers of the intermediary, in the spot market the (very large) number of spot market buyers. Following the argumentation in Porteus [17, page 613], aggregated demands then are, due to the Central Limit Theorem, approximately normally distributed. Though technically a normal distribution implies a small chance that demand is negative, demands are assumed to be nonnegative, so that $F(x) = 0$, as for $x < 0$. The probability of negative demand depends on the coefficient of variation σ/μ. By keeping σ/μ sufficiently small, the chance of negative demand can be kept negligibly small; for example, for $\sigma/\mu = 1/3$, the chance of negative demand is 0.135%.

Lau [18] suggests $\sigma/\mu < 0.3$ to keep the negative tail negligible. For the model at hand, let $\mu > 3\sigma$, $i = \tilde{p}, Q$. The assumption of normally distributed demand is widely used in supply contract (cf. [19]) and revenue management models (cf. [20]).

3. Model

3.1. The Forwarder's Decision Problem. The forwarder has a willingness-to-pay U and its form is $U = U(q(k))$, where U is a concave and strictly increasing function of $q(k)$, that is, $U_1 > 0$, $U_{11} \leq 0$, and k is the wholesale price in the contract market. According to the reference effect theory, the reference price is the expected price which economic agents make according to their own estimation on the capacity prices. The forwarder's willingness-to-pay is the expectation price plus a certain level of profit $p = r + \lambda$. According to the adaptation prospect theory framework of Nerlove [21], the reference price equation is $r_t = \alpha r_{t-1} + (1-\alpha)k_{t-1}$, and $\alpha \in [0, 1)$. The index smooth application form is most widely used, and the reference price form has empirically tested [22–24]. This paper adopts the cost-plus way between the carrier and forwarder based on Hellermann [2]; cost-plus ratio λ is constant. Thus the forwarder's willingness-to-pay is $r + \lambda$.

Without considering the discount factor, the forwarder makes decision by maximizing his expected utility in risk-neutral condition. Then the forwarder's utility function is given as a newsboy revenue function:

$$\begin{aligned} V(q, k, \tilde{p}) = U(q) - kq + U\left((Q-q)^{+}_{\tilde{p}<U_1}\right) \\ - \tilde{p}(Q-q)^{+}_{\tilde{p}<U_1}. \end{aligned} \tag{1}$$

When $\tilde{p} \geq U_1$, the forwarder's capacity purchasing quantity is zero in the spot market. Meanwhile, without loss of generality, we assume that residual value of the unsold capacity is zero at the flight departure time.

Since $U_1 = (r + \lambda)$, the forwarder's expected utility function can be rewritten into

$$\begin{aligned} &V(q) \\ &= \left\{(r+\lambda)\left[\min\{q, Q\} + (Q-q)^{+}\right] - kq - \tilde{p}(Q-q)^{+}\right\}_{\tilde{p}<U_1} \\ &\quad + \left\{(r+\lambda)\min\{q, Q\} - kq\right\}_{\tilde{p}\geq U_1}. \end{aligned} \tag{2}$$

According to the capacity contract provided by the carrier, the forwarder's optimal reservation quantity can be obtained by solving the following problem:

$$\begin{aligned} &\max_{q} \quad V(q) \\ &\text{s.t.} \quad q \geq 0. \end{aligned} \tag{3}$$

Theorem 1. *The forwarder's expected utility function $V(q, k, \tilde{p})$ is a concave function of contract purchasing quantity q.*

Proof. According to the characteristic of the concave function, the sum of two concave functions is still a concave function. Thus formula (2) can be rewritten into

$$V(q) = \begin{cases} \left\{(r+\lambda)\min\{q, Q\} - kq\right\}, \\ \qquad \tilde{p} \geq r + \lambda, \\ \left\{(r+\lambda)\left[\min\{q, Q\} + (Q-q)^{+}\right] \right. \\ \quad \left. -kq - \tilde{p}(Q-q)^{+}\right\}, \\ \qquad \tilde{p} < r + \lambda. \end{cases} \tag{4}$$

The forwarder's expected utility function can be further divided into

$$E[V(q)] = \begin{cases} (r+\lambda)E[Q] - kq \\ \quad -(r+\lambda)\displaystyle\int_q^{\infty}(Q-q)g(Q)\,dQ, \\ \qquad \tilde{p} \geq r + \lambda, \\ (r+\lambda)E[Q] - kq \\ \quad -\tilde{p}\displaystyle\int_q^{\infty}(Q-q)g(Q)\,dQ, \\ \qquad \tilde{p} < r + \lambda, \end{cases} \tag{5}$$

where in the above two cases, the first term is the forwarder's average profit, the second term indicates the capacity purchasing cost, and the last term is expected loss for capacity shortage.

From the newsboy model, the formula (2) is a basic newsboy model, and it is a concave function of q. The first term $(r+\lambda)E[Q]$ is constant and is also a concave function of q.

Therefore, we only need to prove that term

$$\Delta(q) = \begin{cases} -kq - (r+\lambda)\displaystyle\int_q^{\infty}(Q-q)g(Q)\,dQ, & \tilde{p} \geq r + \lambda, \\ -kq - \tilde{p}\displaystyle\int_q^{\infty}(Q-q)g(Q)\,dQ, & \tilde{p} < r + \lambda \end{cases} \tag{6}$$

is a concave function of q, to prove $E[V(q)]$ has the optimal purchasing quantity q^*. Proving that $E[V(q)]$ is a concave function of q can be obtained by proving that $\Delta(q)$ is a concave function of q. Taking the first-order derivative $\Delta(q)$ with respect to q, we can obtain that

$$\frac{\partial\Delta(q)}{\partial q} = \begin{cases} -k - (r+\lambda)[G(q) - 1], & \tilde{p} \geq r + \lambda, \\ -k - \tilde{p}[G(q) - 1], & \tilde{p} < r + \lambda. \end{cases} \tag{7}$$

By taking the second-order derivative, we can derive that

$$\frac{\partial^2\Delta(q)}{\partial q^2} = \begin{cases} -(r+\lambda)g(q) < 0, & \tilde{p} \geq r + \lambda, \\ -\tilde{p}g(q) < 0, & \tilde{p} < r + \lambda. \end{cases} \tag{8}$$

Therefore, $\Delta(q)$ is a concave function of q; this means that $E[V(q)]$ is a concave function of q. □

Theorem 1 indicates that there exists the optimal contract purchasing quantity for the forwarder to maximize his expected utility. Then we derive the forwarder's contract purchasing quantity in the following Theorem.

Theorem 2. *The forwarder's optimal contract purchasing quantity satisfies that*

$$q^* = \arg\max E[V(q)]. \quad (9)$$

Proof. According to the result of Theorem 1, the forwarder can determine his optimal contract purchasing quantity q^* to maximize his expected utility.

Equation $E[V(q)]$ can be rewritten as

$$E[V(q)] = \begin{cases} (r+\lambda)E[Q] - kq \\ \quad -(r+\lambda)\int_q^\infty (Q-q)\,g(Q)\,dQ, \\ \quad \tilde{p} \geq r+\lambda, \\ (r+\lambda)E[Q] \\ \quad -kq - \tilde{p}\int_q^\infty (Q-q)\,g(Q)\,dQ, \\ \quad \tilde{p} < r+\lambda. \end{cases} \quad (10)$$

By taking the first-order derivative of $E[V(q,k,\tilde{p})]$ with respect to q, we can obtain that

$$\frac{\partial\Delta(q)}{\partial q} = \begin{cases} -k-(r+\lambda)[G(q)-1], & \tilde{p} \geq r+\lambda, \\ -k-\tilde{p}[G(q)-1], & \tilde{p} < r+\lambda. \end{cases} \quad (11)$$

When $\partial\Delta(q)/\partial q$ is equal to zero, we can obtain that

$$G(q^*) = \begin{cases} \dfrac{r+\lambda-k}{r+\lambda}, & \tilde{p} \geq r+\lambda, \\ \dfrac{\tilde{p}-k}{\tilde{p}}, & \tilde{p} < r+\lambda. \end{cases} \quad (12)$$

Thus we can derive the forwarder's optimal contract purchasing quantity as follows:

$$q^* = \begin{cases} G^{-1}\left[\dfrac{r+\lambda-k}{r+\lambda}\right], & \tilde{p} \geq r+\lambda, \\ G^{-1}\left[\dfrac{\tilde{p}-k}{\tilde{p}}\right], & \tilde{p} < r+\lambda. \end{cases} \quad (13)$$

□

Theorem 2 is similar to the result of standard newsboy model, in the two cases: the shortage cost is $\tilde{p}-k$ $(\tilde{p} < r+\lambda)$ and $r+\lambda-k$ $(\tilde{p} \geq r+\lambda)$, both overage costs are k. $\tilde{p}$ is the forwarder's capacity purchasing cost in the spot market. If purchasing capacity in the contract market cannot meet the downstream customer's demand, when $\tilde{p} \geq r+\lambda$, the capacity shortage cost is $r+\lambda-k$; when $\tilde{p} < r+\lambda$, the capacity shortage cost is $\tilde{p}-k$ and is equal to the expected spot market premium. This shows that in the absence of the spot market, because the forwarder cannot procure capacity from the spot market once the downstream customer's demand is realized, this results in an opportunity loss; therefore the forwarder's shortage cost is higher than that when the spot market exists; then we have that $r+\lambda-k > \tilde{p}-k$. If the forwarder's contract purchasing quantity is more than the downstream customer's demand, because no secondary market considered, the overage cost is k per unit.

Corollary 3. *The carrier's contract wholesale price k must satisfy the following condition:*

$$\tilde{\beta} \leq k \leq \min\{\tilde{p}, r+\lambda\}. \quad (14)$$

Proof. By the assumptions, we have that k must satisfy that $k \leq r+\lambda$; only this condition holds; the forwarder choose to purchase the capacity in the contract market.

According to the result of Theorem 2, we have $0 \leq (\tilde{p}-k)/\tilde{p} \leq 1$, so $0 < k \leq \tilde{p}$.

Moreover, because the fixed cost per unit capacity is $\tilde{\beta}$, then the wholesale price satisfies $k \geq \tilde{\beta}$. In sum, we can obtain that $\tilde{\beta} \leq k \leq \min\{\tilde{p}, r+\lambda\}$. □

By the result of Corollary 3, the conditions $r+\lambda > k(\tilde{p} \geq r+\lambda)$ and $\tilde{p} > k(\tilde{p} < r+\lambda)$ are participation constraint for the forwarder to accept capacity contract. Only when $\tilde{p} \geq r+\lambda$, the forwarder's willingness-to-pay is greater than the out-of-pocket cost, that is, $\tilde{p} \geq r+\lambda$; or when $\tilde{p} < r+\lambda$, the capacity price in the spot market $\tilde{p}$ is greater than the contract wholesale price, that is, $\tilde{p} > k$.

By the result of Theorem 2, when determining contract wholesale price, the carrier needs to consider not only the forwarder's willingness-to-pay, but also the spot market price $\tilde{p}$. When $\tilde{p} < r+\lambda$, the forwarder purchases capacity in both the contract market and the spot market simultaneously. Otherwise, when $\tilde{p} \geq r+\lambda$, it is a classical newsboy model; the spot market is "nonexistent"; the forwarder does not consider the impact of spot market prices.

Because of the substitution between the contract market and spot market, the capacity price in the spot market will affect the carrier's pricing of the capacity contract and contract the forwarder's capacity purchasing decision. To be specific, the bigger the spot price $\tilde{p}$ is, the bigger $1-k/\tilde{p}$ is; then the carrier can make a higher contract wholesale price and vice versa. This is also consistent with the reality case. And the bigger the spot price is, the more the forwarder will purchase capacity in the contract market by the cross-price elasticity $\varepsilon_{q^*,\tilde{p}} = (\partial q^*/\partial\tilde{p}) \times (\tilde{p}/q^*)$. By the result of Theorem 2, the bigger the spot price $\tilde{p}$ is, the more the forwarder purchases contract q^*. When $\tilde{p} \geq r+\lambda$, it is the classical newsboy model; in this case, the carrier only needs to consider the demand uncertainty.

Meanwhile, Theorem 2 shows that when $\tilde{p} < r+\lambda$, $1-k/(r+\lambda) > 1-k/\tilde{p}$, that is, $\Phi^{-1}[1-k/(r+\lambda)] > \Phi^{-1}[1-k/\tilde{p}]$. The spot market makes the forwarder have more purchasing options; this results in a less contract capacity; the forwarder purchases more than when there is no spot market. To be specific, the forwarders can consider the spot market as an auxiliary capacity source when purchasing capacity from the contract market; when the contract purchasing quantity is less than the downstream customer's demand, the forwarder can turn to the spot market to procure more capacity and compare the spot price with his willingness-to-pay. Theoretically, the forwarder's contract purchasing quantity should fall in a range, that is, $\underline{q}^* \leq E[q^*] \leq \overline{q}^*$, where $E[q^*] = \int_0^{r+\lambda} \underline{q}^* f(\tilde{p})d\tilde{p} + \int_{r+\lambda}^{\infty} \overline{q}^* f(\tilde{p})d\tilde{p}$ and $\overline{q}^* = \mu_Q + \sigma_Q\Phi^{-1}[1-k/(r+\lambda)]$, $\underline{q}^* = \mu_Q + \sigma_Q\Phi^{-1}[1-k/p]$.

In order to solve the optimal purchasing quantity q^*, define $z_q = (q^* - \mu_Q)/\sigma_Q$. $\varphi(\cdot)$ and $\Phi(\cdot)$ denoting the probability density function and the cumulative distribution function of the standard normal distribution. Therefore, we have

$$q^* = \mu_Q + z_q\sigma_Q,$$
$$z_q = \begin{cases} \Phi^{-1}\left[1 - \dfrac{k}{r+\lambda}\right], & \widetilde{p} \geq r+\lambda, \\ \Phi^{-1}\left[1 - \dfrac{k}{\widetilde{p}}\right], & \widetilde{p} < r+\lambda. \end{cases} \tag{15}$$

Corollary 4. *When $\widetilde{p} \geq r + \lambda$, the contract purchasing quantity q^* is an increasing function of the reference price r, and the wholesale price k_{t-1} in the previous period is a decreasing function of the current price k_t; when $\widetilde{p} < r + \lambda$, q^* is an increasing function of the spot price $\widetilde{p}$ and is a decreasing function of the current price k_t.*

Proof. Solving the first-order derivative of (15), we obtain the following.

When $\widetilde{p} \geq r + \lambda$, we have

$$\begin{aligned}\frac{\partial q^*}{\partial r} &= \frac{\partial\left(\mu_Q + z_q\sigma_Q\right)}{\partial r} \\ &= b + \Phi^{-1'}\left[1 - \frac{k}{r+\lambda}\right]\cdot\frac{k}{(r+\lambda)^2} > 0, \\ \frac{\partial q^*}{\partial k} &= \frac{\partial\left(\mu_Q + z_q\sigma_Q\right)}{\partial k} \\ &= \Phi^{-1'}\left[1 - \frac{k}{r+\lambda}\right]\cdot\left(-\frac{1}{(r+\lambda)}\right) < 0, \\ \frac{\partial q^*}{\partial k_{t-1}} &= \frac{\partial\left(\mu_Q + z_q\sigma_Q\right)}{\partial k_{t-1}} \\ &= \Phi^{-1'}\left[1 - \frac{k}{\alpha r + (1-\alpha)k_{t-1} + \lambda}\right] \\ &\quad \cdot \frac{(1-\alpha)k}{\left[\alpha r + (1-\alpha)k_{t-1} + \lambda\right]^2} > 0.\end{aligned} \tag{16}$$

When $\widetilde{p} < r + \lambda$, we have

$$\begin{aligned}\frac{\partial q^*}{\partial k} &= \frac{\partial\left(\mu_Q + z_q\sigma_Q\right)}{\partial k} = -\left[b + \frac{1}{\widetilde{p}}\Phi^{-1'}\left[1 - \frac{k}{\widetilde{p}}\right]\right] < 0, \\ \frac{\partial q^*}{\partial \widetilde{p}} &= \frac{\partial\left(\mu_Q + z_q\sigma_Q\right)}{\partial \widetilde{p}} = \frac{k}{\widetilde{p}^2}\Phi^{-1'}\left[1 - \frac{k}{\widetilde{p}}\right] > 0.\end{aligned} \tag{17}$$

Thus this Corollary is derived. □

Corollary 4 shows that when $\widetilde{p} < r + \lambda$, the forwarder does not consider contract price in the previous period; this means that the forwarder's decisions are not affected by the reference effect, and it is only related to the wholesale price in the current period.

3.2. The Carrier's Decision Problem. The carrier sells her most capacity to the large customers, that is, forwarders and other large shippers, in advance with the form of capacity contract. In the second phase, the remainder capacity is sold in the spot market by the spot price. In the spot market, the carrier faces the risk of price uncertainty and capacity incompletely sold. In this case, advanced-selling strategy can hedge the risk from demand and price in the spot market, which is the main reason why the carrier chooses to sell the capacity in advance [1, 25]. Here we assume that the risk factor in the spot market is $\widetilde{m}$, and $0 \leq \widetilde{m} \leq 1$.

Moreover, the contract relationship between the carrier and forwarder is long term. Tsay [19] distinguished three effects of supply chain contract, where the supply chain contract coordination can help to form a long-term stable and cooperative partnership between businesses participates. Besides sharing the market risks, the contract coordination in practice between the carrier and forwarder focuses on the maintenance of long-term cooperative relationship. The carrier and forwarder will maintain such relationship through certain strategy in order to maximize the total profit of the supply chain [26].

First, we give the carrier's single-period decision-making problem and analyze the impact of the decision variables and state variables on the carrier expected profit function. One has

$$\begin{aligned}\Pi(k) &= \left(k - \widetilde{b}^c\right)q^* + \widetilde{m}\left(W - q^*\right)\left(\widetilde{p} - \widetilde{b}^s\right)^+ - \widetilde{\beta}W \\ &\text{s.t.}\quad 0 \leq k \leq \min\left\{\widetilde{p}, r+\lambda\right\},\end{aligned} \tag{18}$$

where the first term accounts for the profit generated through the long-term contract, the second term represents the profit derived from the spot market with a risk factor $\widetilde{m}$, and the last term displays the cost of holding capacity W.

Due to the impact of the forwarder's contract purchasing quantity, we consider two cases about the carrier's the single-period decision-making problem, that is, $E[\widetilde{p}] \geq r + \lambda$ and $E[\widetilde{p}] < r + \lambda$.

3.2.1. Two Cases on Spot Market Prices. Because of the price uncertainty in the spot market, the forwarder's contract purchasing quantity is different. When $\widetilde{p} \geq r+\lambda$, the capacity price in the spot market is higher than the forwarder's willingness-to-pay. Thus the forwarder procures capacity completely in the contract market. When $\widetilde{p} < r + \lambda$, the forwarder procures capacity in both markets, contract market and spot market, and capacity purchasing quantities are q^* and $E[Q] - q^*$, respectively.

Case 1 ($\widetilde{p} < r + \lambda$). According to Corollary 4 in Section 3.1, when $\widetilde{p} < r + \lambda$, the forwarder's contract purchasing quantity is unrelated to his reference price. Therefore, when $\widetilde{p} < r + \lambda$, the game problem between the carrier and forwarder becomes a single-period game problem.

Therefore, the carrier's decision problem is given as follows:

$$\Pi(k) = \left(k - \tilde{b}^c\right) q^* + \tilde{m}(W - q^*)\left(\tilde{p} - \tilde{b}^s\right)^+ - \tilde{\beta} W,$$
$$\text{s.t.} \quad q^* = \mu_Q + z_q \sigma_Q, \quad z_q = \Phi^{-1}\left[1 - \frac{k}{\tilde{p}}\right], \quad \tilde{p} < r + \lambda. \tag{19}$$

Theorem 5. *When $\tilde{p} < r + \lambda$, the carrier's optimal contract wholesale price satisfies*

$$k^* = E[\tilde{p}]. \tag{20}$$

Proof. Integrating (19),

$$\Pi(k) = \left[k - \tilde{b}^c - \tilde{m}\left(\tilde{p} - \tilde{b}^s\right)^+\right] q^* + \left[\tilde{m}\left(\tilde{p} - \tilde{b}^s\right)^+ - \tilde{\beta}\right] W. \tag{21}$$

By taking the first-order derivative, we have

$$\begin{aligned}
\frac{\partial \Pi(k)}{\partial k} &= \frac{\partial\left[k - \tilde{b}^c - \tilde{m}\left(\tilde{p} - \tilde{b}^s\right)^+\right] q^*}{\partial k} \\
&= \frac{\partial\left[k - \tilde{b}^c - \tilde{m}\left(\tilde{p} - \tilde{b}^s\right)^+\right]\left[\mu_Q + z_q \sigma_Q\right]}{\partial k} \\
&= \left(\mu_Q + z_q \sigma_Q\right) + \left[k - \tilde{b}^c - \tilde{m}\left(\tilde{p} - \tilde{b}^s\right)^+\right] \sigma_Q \frac{\partial z_q}{\partial k}, \\
\frac{\partial z_q}{\partial k} &= -\frac{1}{\tilde{p}} \Phi^{-1'}\left[1 - \frac{k}{\tilde{p}}\right].
\end{aligned} \tag{22}$$

Thus $\partial \Pi(k)/\partial k = (\mu_Q + z_q\sigma_Q) - (\sigma_Q/\tilde{p})\Phi^{-1'}[1 - k/\tilde{p}][k - \tilde{b}^c - \tilde{m}(\tilde{p} - \tilde{b}^s)^+]$. The term $k^* - \tilde{b}^c$ is the marginal profit from the contract market; the term $\tilde{m}(\tilde{p} - \tilde{b}^s)^+$ is the marginal profit from the spot market. The necessary condition that the carrier chooses to sell capacity in the contract market is that the marginal profit from the contract market is not less than that from the spot market, that is, $k^* - \tilde{b}^c \geq \tilde{m}(\tilde{p} - \tilde{b}^s)^+$.

By the result of Corollary 4, we have $\partial q^*/\partial k < 0$.

Thus, $(\mu_Q + z_q\sigma_Q) - (\sigma_Q/\tilde{p})\Phi^{-1'}[1 - k/\tilde{p}][k - \tilde{b}^c - \tilde{m}(\tilde{p} - \tilde{b}^s)^+] < 0$.

When $q^* > (\sigma_Q/\tilde{p})\Phi^{-1'}[1 - k^*/\tilde{p}][k - \tilde{b}^c - \tilde{m}(\tilde{p} - \tilde{b}^s)^+] > 0$, $\Pi(k)$ is monotonically increasing in k, and the carrier's optimal contract wholesale price is $k^* = E[\tilde{p}]$. □

This theorem shows that when $\tilde{p} < r + \lambda$, the reference effect does not affect the carrier's decisions, and her multiple-period decision problem can be considered as a single-period decision problem. And this Theorem also implies that the carrier's optimal wholesale price is the expectation of the capacity price in the spot market.

Theorem 6. *The carrier's profit function $\Pi(k)$ is increasing with respect to the wholesale price k.*

This theorem can be easily derived from Theorem 5.

Case 2 ($\tilde{p} \geq r + \lambda$). Because of the price volatility in the spot market, there is no clear relationship between the forwarder's willingness-to-pay and spot price, and it is always cross-sectional linked. Without loss of generality, we take the two-period decision-making problem as an example to analyze the carrier's decisions; in the first period, the carrier determines the contract wholesale price according to the forwarder's initial reference price, and the contract parameters will affect the decisions in the subsequent period. Therefore, the carrier's decision is to maximize his two-period profit. It is worth noting that this problem can straightforwardly extended to the multiple-period problem.

Assuming that the carrier is a risk-neutral economic agent, the carrier decision is a two-period profit function.

First, the carrier's single-period decision problem is given as follows:

$$\Pi(k) = \left(k - \tilde{b}^c\right) q^* + \tilde{m}(W - q^*)\left(\tilde{p} - \tilde{b}^s\right)^+ - \tilde{\beta} W$$
$$\text{s.t.} \quad q^* = \mu_Q + z_q \sigma_Q, \quad z_q = \Phi^{-1}\left[1 - \frac{k}{r + \lambda}\right], \tag{23}$$
$$\tilde{p} \geq r + \lambda.$$

We can obtain the carrier's optimal wholesale price in the following Theorem.

Theorem 7. *When $\tilde{p} \geq r + \lambda$, the carrier's optimal wholesale price satisfies*

$$k^* = r + \lambda. \tag{24}$$

The proof is similar to Theorem 5 (proof is abbreviated).

Theorem 8. *When $\tilde{p} \geq r + \lambda$, the carrier's profit function is increasing with respect to wholesale price k.*

This Theorem can be derived straightforwardly from Theorem 7.

4. Numerical Analysis

The above sections have analyzed the dynamic game between the carrier and forwarder. In this section, we mainly focus on examining the impact of the parameters on the carrier's and forwarder's decisions. The variables and parameters throughout this paper are given in Table 1.

4.1. The Forwarder's Optimal Reservation Strategy. Figures 1 and 2 show that when the capacity price in the spot market is low enough, even if the wholesale price in the contract market is low enough, the forwarder still will choose to procure capacity in the spot market. This is because the inflexibility of the contract prevents the forwarder's capacity purchasing option in the spot market; therefore the rational

Table 1: Simulation parameters.

Variable	Value	Variable description
W	400	The carrier's total capacity
r_0	25 \ 35	The forwarder's initial reference price
μ_Q	200	The mean of downstream customer's demand
σ_Q	20	The standard deviation of downstream customer's demand
$\widetilde{b}^c$	8	The operation cost in the contract market
$\widetilde{b}^s$	10	The operation cost in the spot market
$\mu_{\widetilde{p}}$	40	The mean of capacity price in the spot market
$\sigma_{\widetilde{p}}$	5	The standard deviation of capacity price in the spot market
$\widetilde{m}$	0.8	The risk factor in the spot market
λ	10	The forwarder's cost-plus ratio
α	0.4	The forwarder's memory effect
$\widetilde{\beta}$	5	The carrier capacity holding cost

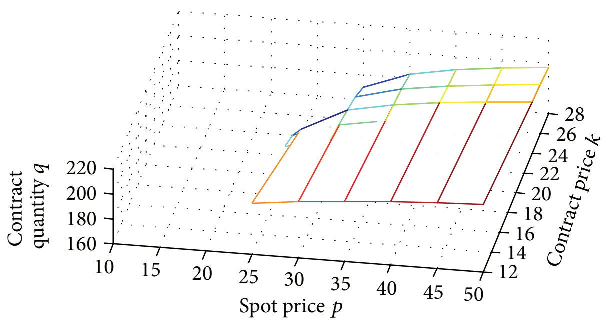

Figure 1: The relationship among the contract purchasing quantity, spot price, and contract price (observation chart in azimuth 10° and angle 65°).

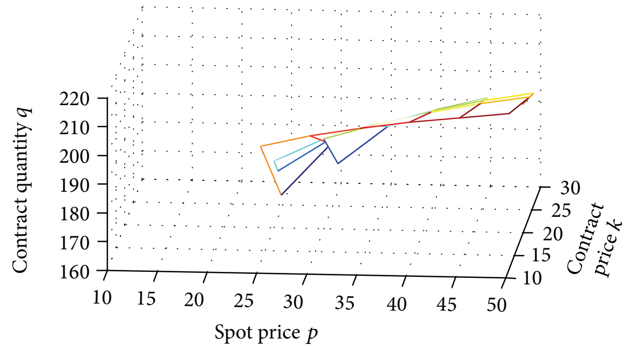

Figure 2: The relationship among the contract purchasing quantity, spot price, and contract price (observation chart in azimuth 10° and angle 25°).

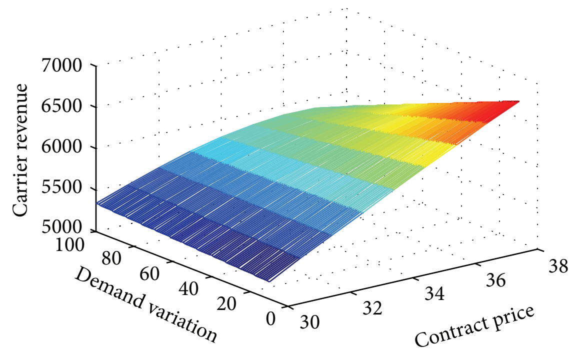

Figure 3: The relationship among the carrier's profit, contract price, and the demand variance in the spot market.

forwarder will choose to procure few enough capacity in the contract market, even abandon the contract market, and wait to procure capacity in the spot market.

Figure 2 describes the truncated phenomenon of the contract capacity; that is, when the capacity price in the spot market is lower than the forwarder's willingness-to-pay, the tiny increase of wholesale price will lead to a sharply change of the contract purchasing quantity, while when the capacity price in the spot market is close to or higher than the forwarder's willingness-to-pay, the change of capacity price in the contract market has less effect on the contract purchasing quantity. This is also consistent with the case in reality.

4.2. The Carrier's Optimal Pricing Strategy.

Case 1 ($\widetilde{p} < r + \lambda$). In this case, the forwarder's capacity purchasing decision is not affected by his reference price; his contract purchasing quantity is only related to the price $\widetilde{p}$ in the spot market and contract price k in the current period. The parameters in this case satisfy the following condition: $q^* = \mu_Q + z_q\sigma_Q$, $z_q = \Phi^{-1}[1 - k/\widetilde{p}]$, $\widetilde{p} < r + \lambda$, and $\beta \le k \le r + \lambda$. And the carrier's expected profit is a concave function of contract price k; the carrier's optimal contract price is a result of Theorem 5 (Figure 3).

As an intermediate agent, the forwarder's capacity demand is driven by the downstream shipper's freight transport demand. Generally, the forwarder gains the price difference by providing additional freight service, that is, the cost-plus pricing policy used in this paper. According to the analysis in Section 3.2.1, we cannot obtain the optimal contract wholesale price of the carrier, and it is difficult to analyze the nature of the carrier's profit function. In view of this problem, we employ a simulation analysis to investigate the carrier's profit function, and we draw the following conclusion: when the capacity demand of the downstream shippers satisfies $\mu_Q > 3\sigma_Q$, the carrier profit function is monotonically increasing with respect to the contract wholesale price k; otherwise when $\mu_Q \le 3\sigma_Q$, it is a concave function of the contract wholesale price.

If the carrier knows about the forwarder's capacity demand and the demand volatility is small, the forwarder will make the contract wholesale price equal to the capacity price in the spot market price to maximize his expected profit, so the forwarder does not procure capacity in the contract market. In the spot market, the forwarder will procure capacity equal to the capacity demand of the downstream shippers (we assume that the forwarder must provide enough capacity for the current capacity demand of the downstream shippers), in this case, the carrier must make the contract wholesale price equal to the capacity price in the spot market. Secondly, the carrier knows about the forwarder's capacity demand, but the demand volatility is very large, the carrier faces the risk of capacity overage, the contract wholesale price is not equal to the capacity price in the spot market, and the optimal contract wholesale price is less than the capacity price in the spot market, so the forwarder will procure capacity in the contract market. At the same time, the carrier can sell the remainder capacity to other forwarders or shippers in the spot market.

According to the assumption on the capacity demand of the downstream shippers, this capacity demand satisfies $\mu_Q > 3\sigma_Q$, so the problem discussed in this paper is the first case, and the carrier's optimal contract wholesale price is $k^* = \tilde{p}$. Intuitively, the wholesale pricing strategy equal to the capacity price in the spot market is inconsistent with the real operation. In practice, because a large capacity procurement, the forwarder can get a discount, which is similar to the result of Xiangzhi et al. [27]. By the result of Theorem 2 results, the forwarder's contract procurement is zero. On the contrary, due to the influence of the forwarders reference price and willingness-to-pay in this paper, when $\tilde{p} < r + \lambda$, the carrier's contract price can achieve the higher bound of the capacity price in the spot market. As a result, in consideration of the long-term relationship, the forwarder still chooses to procure capacity in the contract market.

Case 2 ($\tilde{p} \geq r + \lambda$). When $\tilde{p} \geq r + \lambda$, the forwarder's decision-making problem is a classical newsboy model; the forwarder does not consider the spot market when purchasing capacity; this case is similar to the multiple-period problems of Xiangzhi et al. [27]; the difference is that Xiangzhi et al. [27] do not consider the case that the carrier can sell the remainder capacity in the spot market.

Figure 4 shows that when the demand variance of the downstream shippers is bigger ($\mu_Q < 3\sigma_Q$), the carrier's profit function is monotonically increasing with respect to the contract price; otherwise when $\mu_Q > 3\sigma_Q$, the carrier's profit function is a concave function of the contract price. This is similar to the results of Section 3.2.1. According to the assumption of the capacity demand of the downstream shippers, the capacity demand satisfies $\mu_Q > 3\sigma_Q$. So the carrier's optimal contract wholesale price is $k_t^* = r_t + \lambda$.

The contract wholesale price of the carriers is the forwarder's willingness-to-pay. According to the results of Section 3.1, the forwarders does not procure capacity in the contract market, because the carrier earns all consumer surplus of the forwarder through the contract wholesale price. If the forwarder does not choose to procure capacity in the contract market, his capacity cost is the capacity price $\tilde{p} \geq r + \lambda$ in the spot market. Therefore, the carrier's multiple-period pricing problem is simplified into single-period problem; the forwarder's reference price is $r = r_0 + (1 - \alpha)\lambda$. Intuitively, this pricing strategy equal to the willingness-to-pay is not realistic, and the carrier's profit function is a concave function

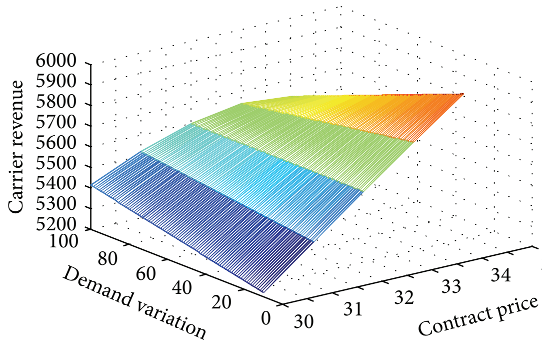

FIGURE 4: The relationship among the carrier's profit, contract price, and the demand variance in the spot market.

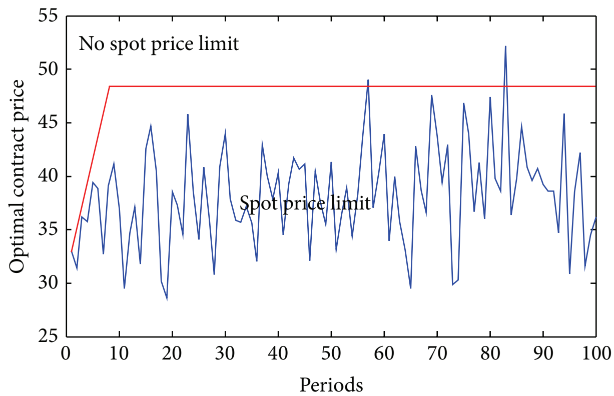

FIGURE 5: Simulation results of the optimal contract price.

of contract wholesale price; that is, there exists an optimal wholesale price less than the willingness-to-pay. By the result of Theorem 2, the forwarder's capacity procurement should be equal to zero. But the relationship among the forwarder's reference effect, capacity in the spot market, and the willingness-to-pay enables the carrier to charge a highest contract wholesale price equal to the forwarder's willingness-to-pay.

Case 3 (mixed decision analysis). According to analysis of Section 3.2.1, the carrier optimal contract wholesale price is $k^* \longrightarrow \min\{r + \lambda, \tilde{p}\}$.

In this section, we set $r_1 = 25$ (which satisfies $\tilde{p}_1 \geq r_1 + \lambda$) and use the matlab software to analyze the impact of the capacity price in the spot market randomly generated on the carrier's decisions, and the capacity price in the spot market follows the normal distribution $\tilde{p}_t \sim N(40, 10)$. The simulation results are as shown in Figure 5.

In the period $t = 1$, when the condition $\tilde{p}_1 \geq r_1 + \lambda$ holds, when there is no capacity price in the spot market influenced, the carrier's optimal contract wholesale price is $k = r + \lambda$; here we assume that the forwarder has a psychological limit value $\overline{k}$ when purchasing capacity, so the carrier's optimal contract wholesale price must satisfy $k \leq \overline{k}$. When the capacity price in the spot market satisfies $\widetilde{p}_1 \sim N(40, 10)$, the simulation shows that the carrier's optimal contract wholesale price fluctuates with the forwarder's willingness-to-pay $r_1 + \lambda$ in the period $t = 1$ as the mean. And in some periods, with the influence of the capacity price in the spot market, the carrier's contract wholesale price is beyond the forwarder's psychological limit value.

5. Conclusions and Future Research

The contract between the carrier and forwarder is a long-term issue, and the repeated contract business makes the forwarder develop a reference point on the contract price, and this reference effect, to a large extent, affects the forwarder's contract purchasing decisions. Based on that, this paper introduces the reference effect into the sea-cargo supply chain and studies a multiple-period contract problem between the carrier and forwarder. It is found that when the capacity price in the spot market is less than the forwarder's willingness-to-pay, the forwarder's contract purchasing decision is not affected by the reference effect, only by the capacity price in the spot market, and the multiple-period contract problem can be simplified into a single-period game. In addition, the carrier's optimal contract wholesale price approaches the capacity price in the spot market. Although, it is difficult to derive the closed-form solution, we employ the numerical simulation to study the carrier's contract pricing policy in two cases.

The paper only studies the contract business between the carrier and forwarder in the sea-cargo supply chain when the spot market and contract market coexist. However, we do not consider the information asymmetric and the specific mode of reference effect, for example, when the reference effect is asymmetric information between the carrier and forwarder, what happens on her contract pricing policy. Moreover, there are other reference effect modes in economics and psycology; it would be interesting to examine which reference effect mode is more realistic.

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Research Article

Inventory Decisions in a Product-Updated System with Component Substitution and Product Substitution

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Substitution behaviors happen frequently when demands are uncertain in a production inventory system, and it has attracted enough attention from firms. Related researches can be clearly classified into firm-driven substitution and customer-driven substitution. However, if production inventory is stock-out when a firm updates its product, the firm may use a new generation product to satisfy the customer's demand of old generation product or use updated component to substitute old component to satisfy production demand. Obviously, two cases of substitution exist simultaneously in the product-updated system when an emergent shortage happens. In this paper, we consider a component order problem with component substitution and product substitution simultaneously in a product-updated system, where the case of firm-driven substitution or customer-driven substitution can be reached by setting different values for two system parameters. Firstly, we formulate the problem into a two-stage dynamic programming. Secondly, we give the optimal decisions about assembled quantities of different types of products. Next, we prove that the expected profit function is jointly concave in order quantities and decrease the feasible domain by determining some bounds for decision variables. Finally, some management insights about component substitution and product substitution are investigated by theoretical analysis method.

1. Introduction

In an uncertainty environment, substitution is an effective way when planner incurs an emergent shortage, it can maximize the expected profit or minimize risk. For example, when a shortage happens for a supplier, he can choose to fill demands with the inventory of another product to decrease revenue loss; or for a manufacturer, once the shortage happens in manufacturing process, he may use another substitutable component to satisfy production demand. However, the substitution offered by the firm to hedge against uncertainty in future sales or production also increases management difficulty.

According to the current classification, the substitution problem mainly includes firm-driven substitution and customer-driven substitution. The former sources from the assortment problem has been studied adequately. Usually, this substitution happens when a lower grade component is stock-out, and the inventory of another updated component is surplus, which is a one-way substitution (see, e.g., [1], Pasternack and Drezner [2], [3–7]).

While for the latter, the firm only offers a substitution advice, the actual substitution behavior is determined by a large number of independently-minded and self-interested consumers. When the shortage case happens, to retain the original customer or decrease shortage penalty, firm may offer a type of substitutable product to the customers. Whether the customer accepts the substitution advice is affected by the variants in many aspects, such as cost, selling prices, and particular technical attributes.

Customer-driven substitution has also many researches and is more attractive in current issues. The correlated papers can be categorized according to two-product or multiproduct, the centralized or competitive decision, and partial or full substitution. Our paper is related to the case with the two product, centralized, and partial substitution (see, e.g., [8–11]).

Current research considered either firm-driven substitution or customer-driven substitution. It is possible that both two cases need to be considered in the same operation environment. For example, a manufacturer produces two products with an updated relation, replenishes the component inventory in advance, and assembles the components into end products according to the customer's order. Because manufacturer makes the replenishment decisions of component inventories before retailer's order arrivals, the shortage for component inventories are inevitable. Therefore, a manufacturer may fill the shortage demand using an updated component so that firm-driven substitution happens. At the mean time, the manufacturer also can stimulate the customer to buy the other product himself by offering a discount price. Certainly, the purchasing decision is made by the customer, so customer-driven substitution happens. However, there is no paper to consider the two cases simultaneously.

Our paper is mostly related to Hale [12]. The paper considers the optimal decision problem in an assemble-to-order system with only component substitution. However, product substitution is also considered in our paper, besides for component substitution. And we study a partial substitution case, and the proportion of substitution is related to a product substitution effort (it may represent an additional production, shipping costs, or loss in revenue, such as giving a price discount for a substitution action). In our problem, there are two important parameters: substitution effort and mark-up value. When substitution effort is zero, the problem can be realized as a pure firm-driven substitution problem. And when mark-up value is very high, the problem can be realized as a pure customer-driven substitution. To the best of our knowledge, our paper is the first paper of integrating product substitution and component substitution.

The rest of this paper is organized as follows. Our model is formulated in Section 2. In Section 3, we provide optimal analysis, present the optimal policy of assembled quantities of different types of products, and give some bounds for ordering decisions. Some management insights are provided in Section 4. Finally, we conclude our paper in Section 5.

2. Problem Description and Formulation

A firm facing stochastic market demands produces new generation product and old generation product simultaneously. Each generation product is assembled by two types of components, one type is a specific component and the other is an updateable component. The specific component only can be used to produce a certain type of product alone. However, the updateable component of a new generation product also can be used to produce an old generation product, besides to produce itself. We call the updateable component of a new generation product as substitutable component and call the substitutable component of an old generation product as substituted component. The cost of substitutable component is higher than substituted component. Certainly, it is obvious that an old generation product assembled by its specific component and substitutable component has a higher performance than the products by its specific component and substituted component. We call this type of product as hybrid product, and we assume that its selling price is higher than a pure old generation. The price-increased value of hybrid product is called a mark-up value, denoted by c_{no}. Moreover, a new generation product has a better performance than an old generation product and a hybrid product, so its selling price is the highest. To stimulate a customer into accepting substitution product, the firm will offer a substitution effort, which may represent an additional production costs or shipping costs, or potential loss of customer's goodwill, or loss in revenue (such as giving a price discount for a substitution action), denoted by C_{no}. It means that the customers may not accept product substitution if the firm does not want to offer a satisfying effort level. Therefore, customer's quantity of accepting substitution product is affected by the substitution effort. Let $\theta(C_{no})$ denote substitution proportion of product substitution for given substitution effort. It is obvious that a larger C_{no} will result in a larger $\theta(C_{no})$. And we assume that $\theta(C_{no}) = 0$ for $C_{no} = 0$.

The research aim of this paper is to determine the optimal order quantities for all components and the optimal assembled quantities of different types of products in an assemble-to-order production system with component substitution and product substitution, so that the expectation of firm's profit is maximized.

The sequence of system events is as follows. Firstly, facing stochastic demands, the firm orders all components. Then, demands are realized. The firm makes decisions on the production quantities of all type of products. If the demands of old generation product cannot be satisfied totally, the firm will consider satisfying the shortage demand by using the surplus new generation product. If demands are still not be satisfied, the firm will consider producing a hybrid product. Finally, the firm assembles current components into end products.

Notation Definitions. For simplifying the following description, we use i and j as the subscript of notation. Let $i = n$ denote new generation product, $i = o$ denote old generation product, $j = 1$ denote specific component and $j = 2$ denote substitution component. Therefore, we may denote specific component and substitution component by the vector (i, j), for example, $(n, 2)$ denote the substitution component of new generation product, that is, the substitutable component.

C_{ij} = order cost of per unit component j of product i.

S_{ij} = salvage value of per unit component j of product i.

D_i = demand for product i and is a random variable.

p_i = selling price of product i.

c_{no} = mark-up value of per unit component substitution for old generation.

C_{no} = substitution effort of per unit product substitution.

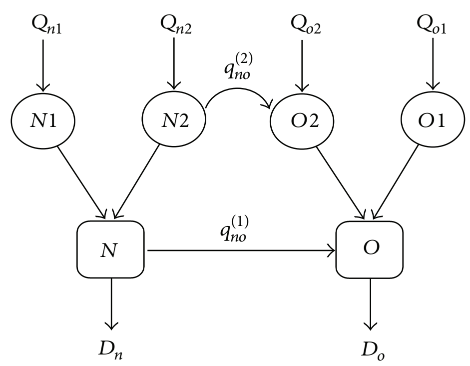

Figure 1: Notation sketch figure.

$\theta(C_{no})$ = substitution proportion of product substitution for a given substitution effort.

Q_{ij} = order quantity of component j of product i.

q_{nn} = assembled quantity of new generation product composed by its specific component and substitutable component.

q_{oo} = assembled quantity of old generation product composed by its specific component and substituted component.

$q_{no}^{(1)}$ = product quantity for satisfying product substitution.

$q_{no}^{(2)}$ = hybrid product quantity for satisfying component substitution.

We can figure a part of notations by Figure 1.

Firstly, we give some assumptions about system parameters.

Assumption 1. $p_n - C_{n1} - C_{n2} - C_{no} > p_o + c_{no} - C_{n2} - C_{o2}$. It means that the revenue of the case of product substitution is larger than the case of component substitution.

Assumption 2. $C_{o2} - S_{o2} < C_{n2} - S_{n2}$. It denotes that the cost loss of per unit surplus substitutable component is larger than per unit surplus substituted component.

Assumption 3. $c_{no} \leq C_{n2} - C_{o2}$. It means that the mark-up value should not be larger than the added cost for component substitution. Generally, the firm should bear some duties for the shortage as firm's reason.

Assumption 4. $p_n - C_{no} - S_{n1} - S_{n2} > p_o + c_{no} - S_{o1} - S_{n2} > 0$. It means that the selling revenue is larger than salvages, otherwise, the firm has no motivation to sell end products. It also means that the firm has a larger motivation to offer product substitution to the customer than to offer component substitution.

Let $Q = (Q_{n1}, Q_{n2}, Q_{o1}, Q_{n2})$ and $q = (q_{nn}, q_{oo}, q_{no}^{(1)}, q_{no}^{(2)})$, from the sequence of system events, the optimization problem is given as follows:

$$\max_{Q} \Pi = \max_{Q} \left\{ -\sum_{i=n,o} \sum_{j=1,2} C_{ij} Q_{ij} + E\left[\pi\left(Q, D_n, D_o\right)\right] \right\}, \quad (1)$$

where

$$\begin{aligned} \pi\left(Q, d_n, d_o\right) = \max_{q} \Big\{ & p_n q_{nn} + p_o q_{oo} + \left(p_n - C_{no}\right) q_{no}^{(1)} \\ & + \left(p_o + c_{no}\right) q_{no}^{(2)} \\ & + S_{n1}\left(Q_{n1} - q_{nn} - q_{no}^{(1)}\right) \\ & + S_{o1}\left(Q_{o1} - q_{oo} - q_{no}^{(2)}\right) \\ & + S_{n2}\left(Q_{n2} - q_{nn} - q_{no}^{(1)} - q_{no}^{(2)}\right) \\ & + S_{o2}\left(Q_{o2} - q_{oo}\right) \Big\} \end{aligned}$$

$$\text{s.t.} \quad \begin{cases} q_{nn} \leq d_n \\ q_{oo} + q_{no}^{(1)} + q_{no}^{(2)} \leq d_o \\ q_{nn} + q_{no}^{(1)} \leq Q_{n1} \\ q_{nn} + q_{no}^{(1)} + q_{no}^{(2)} \leq Q_{n2} \\ q_{oo} \leq Q_{o2} \\ q_{oo} + q_{no}^{(2)} \leq Q_{o1} \\ q_{nn}, q_{oo}, q_{no}^{(1)}, q_{no}^{(2)} \geq 0. \end{cases} \quad (2)$$

Let $Q^* = (Q_{n1}^*, Q_{n2}^*, Q_{o1}^*, Q_{o2}^*)$ denote the optimal solution in (1) and $q^* = (q_{nn}^*, q_{oo}^*, q_{no}^{(1*)}, q_{no}^{(2*)})$ denote the optimal solution of optimization problem in (2). In the following, we will make optimal analyses for the optimal solutions Q^* and q^*.

3. Optimal Analysis

The aforementioned optimization problem is a two-stage stochastic dynamic programming. We need to solve $\pi(Q, D_n, D_o)$ in (2), firstly, then solve the optimization problem in (1).

3.1. Optimal Assemble Decisions. To find the optimal solution q^*, we need to firstly give a property about optimal orders of several types of components.

Property 1. The optimal orders of several types of components satisfy

(a) $Q_{n1}^* \leq Q_{n2}^*$,

(b) $Q_{o2}^* \leq Q_{o1}^*$.

Proof. From the constraints $q_{nn} + q_{no}^{(1)} \leq Q_{n1}$ and $q_{nn} + q_{no}^{(1)} + q_{no}^{(2)} \leq Q_{n2}$, we know that if $Q_{n1}^{*} > Q_{n2}^{*}$, there must be $q_{nn}^{*} + q_{no}^{(1*)} < Q_{n1}$ for any realized demand, that is, the specific component of new generation product must be surplus, which means Q_{n1}^{*} is not optimal. Therefore, we have $Q_{n1}^{*} \leq Q_{n2}^{*}$. Similar to the process, we also can prove that part (b) holds. □

Property 1 means that the optimal order quantity of substitutable component is larger than the optimal order quantity of specific component of new generation product. However, for old generation product, the optimal order quantity of substituted component is less than the optimal order quantity of specific component of new generation product. Because the substitutable component needs to meet an additional demand except for the original demand, and the substituted component has an additional supply source, the property is obvious.

Property 1 not only give the bound constraints about the optimal order quantities of several types of components which is meaningful for shrinking the feasible domain by adding the constraints $Q_{n1} \leq Q_{n2}$ and $Q_{o2} \leq Q_{o1}$, but also important for analyzing the properties of optimization model. In the following, we will give the optimal decisions of assembled quantities.

Theorem 1. *Given the order quantity vector $(Q_{n1}, Q_{n2}, Q_{o1}, Q_{n2})$ and the realized demand (d_n, d_o), the optimal assembled quantities for all types of products are as follows:*

$$
\begin{aligned}
q_{nn}^{*} &= \min\{d_n, Q_{n1}\} \\
q_{oo}^{*} &= \min\{d_o, Q_{o2}\} \\
q_{no}^{(1*)} &= \min\{\max\{Q_{n1} - d_n, 0\}, \\
&\qquad \max\{\theta(C_{no})(d_o - Q_{o2}), 0\}\} \\
q_{no}^{(2*)} &= \min\{Q_{n2} - \min\{d_n, Q_{n1}\}, \\
&\qquad \max\{\min\{d_o, Q_{o1}\} - Q_{o2}, 0\}\} \\
&\quad - \min\{\max\{Q_{n1} - d_n, 0\}, \\
&\qquad \theta(C_{no}) \max\{d_o - Q_{o2}, 0\}\}.
\end{aligned} \tag{3}
$$

Proof. From Assumption 1, the optimal assemble rule is that firm produces products by the component itself as possible; and if old generation product is shortage, the firm should firstly consider product substitution and secondly consider component substitution. We analyze the optimal assemble decisions for different cases.

Case 1. When $d_n > \min\{Q_{n1}, Q_{n2}\}$ and $d_0 \leq \min\{Q_{o1}, Q_{o2}\}$, there are $d_n > Q_{n1}$ and $d_0 \leq Q_{o2}$ (by Property 1), that is, the demands of new generation product can not totally be satisfied, and there is no shortage for old generation product. Therefore,

$$q_{nn} = Q_{n1}, \qquad q_{no}^{(1)} = 0, \qquad q_{oo} = d_o, \qquad q_{no}^{(2)} = 0. \tag{4}$$

Case 2. When $d_n > \min\{Q_{n1}, Q_{n2}\}$ and $d_0 > \min\{Q_{o1}, Q_{o2}\}$, there are $d_n > Q_{n1}$ and $d_0 > Q_{o2}$ (by Property 1), that is, both demands of new and old generation product can not totally be satisfied by the components themselves. Therefore, there is no product substitution, but may exist the component substitution. The shortage quantity of substituted component is $\min\{d_o - Q_{o2}, Q_{o1} - Q_{o2}\}$, and the supply quantity of substitutable component is $Q_{n2} - Q_{n1}$. We have

$$
\begin{gathered}
q_{nn} = Q_{n1}, \qquad q_{no}^{(1)} = 0, \qquad q_{oo} = Q_{o2}, \\
q_{no}^{(2)} = \min\{Q_{n2} - Q_{n1}, \min\{d_o - Q_{o2}, Q_{o1} - Q_{o2}\}\}.
\end{gathered} \tag{5}
$$

Case 3. When $d_n \leq \min\{Q_{n1}, Q_{n2}\}$ and $d_o \leq \min\{Q_{o1}, Q_{o2}\}$, there are $d_n \leq Q_{n1}$ and $d_0 \leq Q_{o2}$ (by Property 1), that is, both demands of new and old generation product can totally be satisfied by the components themselves. Therefore, we have

$$q_{nn} = d_n, \qquad q_{oo} = d_o, \qquad q_{no}^{(1)} = 0, \qquad q_{no}^{(2)} = 0. \tag{6}$$

Case 4. When $d_n \leq \min\{Q_{n1}, Q_{n2}\}$ and $d_0 > \min\{Q_{o1}, Q_{o2}\}$, there are $d_n \leq Q_{n1}$ and $d_0 > Q_{o2}$ (by Property 1), that is, the demands of new generation product can totally be satisfied, and the demands of old generation product can not totally be satisfied by the components itself. So, we have $q_{nn} = d_n$ and $q_{oo} = Q_{o2}$. Product substitution needs to be considered firstly. The maximal supply quantity of new generation product is $Q_{n1} - d_n$, and the demand quantity of new generation product is $d_o - Q_{o2}$. We have

$$q_{no}^{(1)} = \min\{Q_{n1} - d_n, d_o - Q_{o2}\}. \tag{7}$$

Component substitution also may happen. If the demand shortage of old generation product is totally satisfied by product substitution, then $q_{no}^{(2)} = 0$; otherwise, component substitution happens. The maximal supply quantity of substitutable component is $Q_{n2} - d_n - q_{no}^{(1)}$, and the shortage of substituted component is $\min\{d_o, Q_{o1}\} - Q_{o2} - q_{no}^{(1)}$. Therefore, we have

$$
\begin{aligned}
q_{no}^{(2)} &= \min\left\{Q_{n2} - d_n - q_{no}^{(1)}, \min\{d_o, Q_{o1}\} - Q_{o2} - q_{no}^{(1)}\right\} \\
&= \min\{Q_{n2} - d_n, \min\{d_o - Q_{o2}, Q_{o1} - Q_{o2}\}\} - q_{no}^{(1)}.
\end{aligned} \tag{8}
$$

In summary, we can denote the optimal assembled quantities by a uniform form, that is, (3). The theorem holds. □

3.2. Bounds of Order Decisions. By Theorem 1, we can rewrite $\pi(Q, d_n, d_o)$ in (1) as follows:

$$\begin{aligned}\pi(Q,d_n,d_o) = {} & (p_n - S_{n1} - S_{n2})\min\{d_n,Q_{n1}\} \\ & + (p_o - S_{o1} - S_{o2})\min\{d_o,Q_{o2}\} \\ & + \sum_{j=i,2}\sum_{i=n,o} S_{ij}Q_{ij} \\ & + (p_n - C_{no} - S_{n1} - (p_o + c_{no} - S_{o1})) \\ & \times \min\{\max\{Q_{n1} - d_n, 0\}, \\ & \qquad \max\{\theta(C_{no})(d_o - Q_{o2}), 0\}\} \\ & + (p_o + c_{no} - S_{o1} - S_{n2}) \\ & \times \min\{Q_{n2} - \min\{d_n,Q_{n1}\}, \\ & \qquad \max\{\min\{d_o,Q_{o1}\} - Q_{o2}, 0\}\}.\end{aligned} \tag{9}$$

By (1), define

$$\Pi(Q) = E[\pi(Q,D_n,D_o)] - \sum_{i=n,o}\sum_{j=1,2} C_{ij}Q_{ij}. \tag{10}$$

We have the following property.

Property 2. $\Pi(Q)$ is jointly concave in the order quantity vector $(Q_{n1},Q_{n2},Q_{o1},Q_{n2})$.

Proof. From the theory of linear programming, the value of a linear maximization programming is concave in the right hand sides of the constraints ([13], page 438-439). Therefore, for the given realized demands d_n and d_o, $\pi(Q,d_n,d_o)$ is jointly concave in the order quantity vector $(Q_{n1},Q_{n2},Q_{o1},Q_{n2})$. Moreover, $E[\pi(Q,D_n,D_o)]$ is also jointly concave in the order quantity vector $(Q_{n1},Q_{n2},Q_{o1},Q_{n2})$. From (10), it is obvious that $\Pi(Q)$is concave. □

Property 2 shows that the optimal solution is unique. The following property will simplify our analysis.

Property 3. The optimal order quantity of substitutable component is equal to the optimal order quantity of substituted component, that is, $Q^*_{n2} = Q^*_{n1}$.

Proof. From Property 1, we know that the optimal solutions should satisfy $Q_{n2} \geq Q_{n1}$ and $Q_{o2} \leq Q_{o1}$. We only need to prove that the optimal solutions do not satisfy $Q_{n2} > Q_{n1}$ and $Q_{o2} \leq Q_{o1}$. We will prove that the system profit of decreasing per unit substitutable component and increasing per unit substituted component will be improved. Let

$$H(\Delta) = \Pi(Q_{n1},Q_{n2} - \Delta, Q_{o1}, Q_{o2} + \Delta), \tag{11}$$

where $Q_{n1} \leq Q_{n2} - \Delta$ and $Q_{o1} \geq Q_{o2} + \Delta$.

The first order condition is as follows:

$$\begin{aligned}\frac{dH(\Delta)}{d\Delta} = {} & (p_o - S_{o1} - S_{o2})\Pr\{D_o > Q_{o2} + \Delta\} - (S_{n2} - C_{n2}) \\ & + S_{o2} - C_{o2} + (p_n - C_{no} - S_{n1} - (p_o + c_{no} - S_{o1})) \\ & \times \Pr\{D_o > Q_{o2} + \Delta, Q_{n1} > D_n, Q_{n1} \\ & \qquad - D_n > \theta(C_{no})(D_o - Q_{o2} - \Delta)\} \\ & - (p_o + c_{no} - S_{o1} - S_{n2})\Pr\{D_o > Q_{o2} + \Delta\} \\ > {} & (S_{n2} - S_{o2} - c_{no})\Pr\{D_o > Q_{o2} + \Delta\} \\ & - (S_{n2} - C_{n2}) + S_{o2} - C_{o2} \\ \geq {} & (S_{n2} - S_{o2} - (C_{n2} - C_{o2}))\Pr\{D_o > Q_{o2} + \Delta\} \\ & - (S_{n2} - C_{n2}) + S_{o2} - C_{o2} \\ = {} & (C_{o2} - S_{o2} - (C_{n2} - S_{n2})) \\ & \times (\Pr\{D_o > Q_{o2} + \Delta\} - 1) \\ > {} & 0.\end{aligned} \tag{12}$$

For the aforementioned process, the first inequality in (12) is because of (13), and the second inequality is because of Assumption 3,

$$\begin{aligned} & p_n - C_{no} - S_{n1} - (p_o + c_{no} - S_{o1}) \\ & \quad = p_n - C_{no} - S_{n1} - S_{n2} \\ & \qquad - (p_o + c_{no} - S_{o1} - S_{n2}) > 0.\end{aligned} \tag{13}$$

The theorem holds. □

Therefore, we have $H(Q_{n2} - Q_{n1}) = \Pi(Q_{n1},Q_{n1},Q_{o1},Q_{o2} + Q_{n2} - Q_{n1}) > \Pi(Q_{n1},Q_{n2},Q_{o1},Q_{o2})$, that is, the optimal solution of $\max\{\Pi(Q_{n1},Q_{n2},Q_{o1},Q_{o2})\}$ must satisfy the constraint $Q_{n2} = Q_{n1}$.

By Property 3, we can rewrite $\Pi(Q)$ as follows:

$$\begin{aligned}\Pi(Q) = {} & \phi_n E[\min\{D_n,Q_{n1}\}] + \phi_o E[\min\{D_o,Q_{o2}\}] \\ & + \sum_{j=i,2}\sum_{i=n,o}(S_{ij} - C_{ij})Q_{ij} \\ & + \phi^{(1)}_{no}E[\min\{\max\{Q_{n1} - D_n, 0\}, \\ & \qquad \max\{\theta(C_{no})(D_o - Q_{o2}), 0\}\}] \\ & + \phi^{(2)}_{no}E[\min\{Q_{n1} - \min\{D_n,Q_{n1}\}, \\ & \qquad \max\{\min\{D_o,Q_{o1}\} - Q_{o2}, 0\}\}],\end{aligned} \tag{14}$$

where

$$\begin{aligned} & \phi_n = p_n - S_{n1} - S_{n2}, \qquad \phi_o = p_o - S_{o1} - S_{o2} \\ & \phi^{(1)}_{no} = p_n - C_{no} - S_{n1} - (p_o + c_{no} - S_{o1}) \\ & \phi^{(2)}_{no} = p_o + c_{no} - S_{o1} - S_{n2}.\end{aligned} \tag{15}$$

And, moreover, the original optimization problem in (1) is translated into an optimization problem with three decision variables. And $\Pi(Q)$ is concave in Q.

The first order conditions are as follows:

$$\frac{\partial \Pi(Q)}{\partial Q_{n1}} = \phi_n \Pr\{D_n > Q_{n1}\} + \phi_{no}^{(1)} \Pr\{Q_{n1} > D_n, D_o > Q_{o2}, Q_{n1} - D_n < \theta(C_{no})(D_o - Q_{o2})\} + S_{n1} - C_{n1} + S_{n2} - C_{n2} + \phi_{no}^{(2)} \Pr\{D_o > Q_{o2}, D_n < Q_{n1}, Q_{n1} - D_n < \min\{D_o, Q_{o1}\} - Q_{o2}\}, \tag{16}$$

$$\frac{\partial \Pi(Q)}{\partial Q_{o1}} = S_{o1} - C_{o1} + \phi_{no}^{(2)} \Pr\{D_o > Q_{o1}, D_n \le Q_{n1}, Q_{n1} - D_n > Q_{o1} - Q_{o2}\}, \tag{17}$$

$$\frac{\partial \Pi(Q)}{\partial Q_{o2}} = \phi_o \Pr\{D_o > Q_{o2}\} - \phi_{no}^{(2)} \Pr\{D_o > Q_{o2}, D_n \le Q_{n1}, Q_{n1} - D_n > \min\{D_o, Q_{o1}\} - Q_{o2}\} + S_{o2} - C_{o2} - \theta(C_{no}) \times \phi_{no}^{(1)} \Pr\{Q_{n1} > D_n, D_o > Q_{o2}, Q_{n1} - D_n > \theta(C_{no})(D_o - Q_{o2})\}. \tag{18}$$

Obviously, it is difficult to gain the analytical solutions by the first order conditions. Therefore, we will decrease the feasible domain by giving some bounds about decision variables.

Theorem 2. *A lower bound of Q_{n1}^* is the solution of* $\Pr\{D_n \le Q_{n1}\} = (p_n - C_{n1} - C_{n2})/(p_n - S_{n1} - S_{n2})$.

Proof. From (13), $\phi_{no}^{(1)} > 0$, and from Assumption 4, $\phi_{no}^{(2)} > 0$, we have

$$\frac{\partial \Pi(Q)}{\partial Q_{n1}} \ge \phi_n \Pr\{D_n > Q_{n1}\} + S_{n1} - C_{n1} + S_{n2} - C_{n2} = -(p_n - S_{n1} - S_{n2}) \Pr\{D_n \le Q_{n1}\} + p_n - C_{n1} - C_{n2}. \tag{19}$$

From Property 2, the solution of $\Pr\{D_n \le Q_{n1}\} = (p_n - C_{n1} - C_{n2})/(p_n - S_{n1} - S_{n2})$ is a lower bound of Q_{n1}^*. □

From Theorem 2, the optimal order quantity of specific component of new generation product has a lower solution of equaling to a news-vendor solution. And it is not affected by product substitution or component substitution.

Theorem 3. *An upper bound of Q_{n1}^* is the solution of*

$$\Pr\{Q_{n1} > D_n + D_o\} = \frac{p_n - C_{n1} - C_{n2}}{p_n + C_{no} - S_{n1} - S_{n2}}. \tag{20}$$

Proof. From (14),

$$\frac{\partial \Pi(Q)}{\partial Q_{n1}} \le \phi_n \Pr\{D_n > Q_{n1}\} + \phi_{no}^{(1)} \times \Pr\{Q_{n1} > D_n, Q_{n1} - D_n < D_o\} + S_{n1} - C_{n1} + S_{n2} - C_{n2} + \phi_{no}^{(2)} \Pr\{D_n < Q_{n1}, Q_{n1} - D_n < D_o\}$$
$$= -(p_n - S_{n1} - S_{n2}) \Pr\{D_n < Q_{n1}, Q_{n1} \ge D_n + D_o\} + (p_n - C_{n1} - C_{n2}) - C_{no} \Pr\{D_n < Q_{n1}, Q_{n1} < D_n + D_o\}$$
$$\le -(p_n + C_{no} - S_{n1} - S_{n2}) \Pr\{Q_{n1} > D_n + D_o\} + p_n - C_{n1} - C_{n2}. \tag{21}$$

Therefore, from Property 2, the solution of $\Pr\{Q_{n1} > D_n + D_o\} = (p_n - C_{n1} - C_{n2})/(p_n + C_{no} - S_{n1} - S_{n2})$ is an upper bound of Q_{n1}^*. □

Theorem 4. *An upper bound of Q_{o1}^* is the solution of* $\Pr\{D_o \le Q_{o1}\} = (p_o + c_{no} - C_{o1} - S_{n2})/(p_o + c_{no} - S_{o1} - S_{n2})$.

Proof. From (17), we have

$$\frac{\partial \Pi(Q)}{\partial Q_{o1}} = S_{o1} - C_{o1} + (p_o + c_{no} - S_{o1} - S_{n2}) \times \Pr\{D_o > Q_{o1}, Q_{n1} + Q_{o2} - Q_{o1} > D_n\} \le (p_o + c_{no} - C_{o1} - S_{n2}) - (p_o + c_{no} - S_{o1} - S_{n2}) \Pr\{D_o \le Q_{o1}\}. \tag{22}$$

Therefore, from Property 2, the solution of $\Pr\{D_o \le Q_{o1}\} = (p_o + c_{no} - C_{o1} - S_{n2})/(p_o + c_{no} - S_{o1} - S_{n2})$ is an upper bound of Q_{o1}^*. □

Theorem 5. *An upper bound of Q_{o2}^* is the solution of* $\Pr\{D_o < Q_{o2}\} = (p_o - C_{o2} - C_{o1})/(p_o - S_{o1} - S_{o2})$.

Proof. From $\partial \Pi(Q)/\partial Q_{o1} = 0$, we have

$$\Pr\{D_o > Q_{o1}, D_n < Q_{n1}, Q_{n1} - D_n > Q_{o1} - Q_{o2}\} = \frac{C_{o1} - S_{o1}}{p_o + c_{no} - S_{o1} - S_{n2}}. \tag{23}$$

Moreover, substituting the above equation into (18), we have

$$\begin{aligned}\frac{\partial \Pi(Q)}{\partial Q_{o2}} &= \phi_o \Pr\{D_o > Q_{o2}\} \\ &\quad - \phi_{no}^{(2)} \Pr\{D_o > Q_{o1}, D_n \le Q_{n1}, \\ &\qquad Q_{n1} - D_n > Q_{o1} - Q_{o2}\} \\ &\quad - \phi_{no}^{(2)} \Pr\{D_o > Q_{o2}, \\ &\qquad D_o \le Q_{o1}, D_n \le Q_{n1}, \\ &\qquad Q_{n1} - D_n > D_o - Q_{o2}\} \\ &\quad + S_{o2} - C_{o2} - \theta(C_{no})\phi_{no}^{(1)} \\ &\quad \times \Pr\{Q_{n1} > D_n, D_o > Q_{o2}, \\ &\qquad Q_{n1} - D_n > \theta(C_{no})(D_o - Q_{o2})\} \\ &\le -(p_o - S_{o1} - S_{o2}) \Pr\{D_o \le Q_{o2}\} \\ &\quad + p_o - C_{o2} - C_{o1}.\end{aligned} \tag{24}$$

Therefore, the solution of $\Pr\{D_o < Q_{o2}\} = (p_o - C_{o2} - C_{o1})/(p_o - S_{o1} - S_{o2})$ is an upper bound of Q_{o2}^*. □

From Theorem 5, the optimal order quantity of substituted component has an upper bound of equaling to a newsvendor solution. And it is not affected by product substitution or component substitution.

The aforementioned theorems about bounds of optimal decisions have two actions. One is to decrease the feasible domain of decision variables, which is very helpful for finding the optimal decisions. The other action is to assist us to make the sensitivity analysis.

4. Management Insights

In this section, we will investigate management insights about product substitution and component substitution by the first order conditions and the bounds in Theorems 2–5. For the single-period problem, we can give the following propositions by theoretical analysis rather than numerical analysis.

Proposition 6. *The optimal order quantity of any type of component for new generation product is larger for the case of considering product substitution and component product simultaneously than the case of only considering component substitution.*

Proof. When $C_{no} = 0$, $\theta(C_{no}) = 0$, which means that no customer accept product substitution, that is, there is no product substitution. From (16), we have

$$\begin{aligned}\frac{\partial \Pi(Q)}{\partial Q_{n1}}\Big|_{C_{no}>0, c_{no}>0} &= \phi_n \Pr\{D_n > Q_{n1}\} \\ &\quad + \phi_{no}^{(1)} \Pr\{Q_{n1} > D_n, D_o > Q_{o2}, \\ &\qquad Q_{n1} - D_n < \theta(C_{no})(D_o - Q_{o2})\} \\ &\quad + S_{n1} - C_{n1} + S_{n2} - C_{n2} \\ &\quad + \phi_{no}^{(2)} \Pr\{D_o > Q_{o2}, D_n < Q_{n1}, \\ &\qquad Q_{n1} - D_n < \min\{D_o, Q_{o1}\} - Q_{o2}\} \\ &> \phi_n \Pr\{D_n > Q_{n1}\} + S_{n1} - C_{n1} + S_{n2} - C_{n2} \\ &\quad + \phi_{no}^{(2)} \Pr\{D_o > Q_{o2}, D_n < Q_{n1}, \\ &\qquad Q_{n1} - D_n < \min\{D_o, Q_{o1}\} - Q_{o2}\} \\ &= \frac{\partial \Pi(Q)}{\partial Q_{n1}}\Big|_{C_{no}=0, c_{no}>0},\end{aligned} \tag{25}$$

so $Q_{n1}^*(C_{no}) > Q_{n1}^*(0)$. The proposition holds. □

Proposition 7. *The optimal order quantity of any type of component of new generation product is nonincreasing in substitution effort.*

Proof. The solution of $\Pr\{Q_{n1} > D_n + D_o\} = (p_n - C_{n1} - C_{n2})/(p_n + C_{no} - S_{n1} - S_{n2})$ is decreasing in substitution effort C_{no}. So, the feasible domain of Q_{n1} is decreasing in C_{no}. The proposition holds. □

Proposition 8. *The optimal order quantity of specific component of new generation product is larger for the case of considering mark-up value and substitution effort simultaneously than the case without mark-up value and substitution effort. And, for the case of considering product substitution, the optimal order quantity of specific component of new generation product is larger for the case of not considering mark-up value.*

Proof. From (16), we have

$$\begin{aligned}\frac{\partial \Pi(Q)}{\partial Q_{n1}}\Big|_{C_{no}>0, c_{no}>0} &= \phi_n \Pr\{D_n > Q_{n1}\} \\ &\quad + \phi_{no}^{(2)} \Pr\{D_o > Q_{o2}, D_n < Q_{n1}, \\ &\qquad Q_{n1} - D_n < \min\{D_o, Q_{o1}\} - Q_{o2}\} \\ &\quad + S_{n1} - C_{n1} + S_{n2} - C_{n2} \\ &\quad + \phi_{no}^{(1)} \Pr\{Q_{n1} > D_n, D_o > Q_{o2}, \\ &\qquad Q_{n1} - D_n < \theta(C_{no})(D_o - Q_{o2})\} \\ &> \phi_n \Pr\{D_n > Q_{n1}\} + S_{n1} - C_{n1} \\ &\quad + S_{n2} - C_{n2} + (p_o - S_{o1} - S_{n2}) \\ &\quad \times \Pr\{D_o > Q_{o2}, D_n < Q_{n1}, Q_{n1}\end{aligned}$$

$$-D_n < \min\{D_o, Q_{o1}\} - Q_{o2}\}$$
$$= \frac{\partial \Pi(Q)}{\partial Q_{n1}}\Big|_{C_{no}=0, c_{no}=0}. \tag{26}$$

Therefore, the front half part in this proposition holds. From the following inequality:

$$\begin{aligned}
\frac{\partial \Pi(Q)}{\partial Q_{n1}}\Big|_{C_{no}=0, c_{no}>0} &= \phi_n \Pr\{D_n > Q_{n1}\} + S_{n1} - C_{n1} + S_{n2} - C_{n2} \\
&\quad + \phi_{no}^{(2)} \Pr\{D_o > Q_{o2}, D_n < Q_{n1}, \\
&\qquad Q_{n1} - D_n < \min\{D_o, Q_{o1}\} - Q_{o2}\} \\
&> \phi_n \Pr\{D_n > Q_{n1}\} + S_{n1} - C_{n1} + S_{n2} - C_{n2} \\
&\quad + (p_o - S_{o1} - S_{n2}) \\
&\quad \times \Pr\{D_o > Q_{o2}, D_n < Q_{n1}, \\
&\qquad Q_{n1} - D_n < \min\{D_o, Q_{o1}\} - Q_{o2}\} \\
&= \frac{\partial \Pi(Q)}{\partial Q_{n1}}\Big|_{C_{no}=0, c_{no}=0}.
\end{aligned} \tag{27}$$

So, the second part also holds. Therefore, the proposition holds. □

Proposition 9. *The optimal order quantity of specific component of old generation product is larger for the case of considering mark-up value of component substitution than the case of not considering it.*

Proof. From (17), we have

$$\begin{aligned}
\frac{\partial \Pi(Q)}{\partial Q_{o1}}\Big|_{c_{no}>0} &= S_{o1} - C_{o1} + (p_o + c_{no} - S_{o1} - S_{n2}) \\
&\quad \times \Pr\{D_o > Q_{o1}, D_n \le Q_{n1}, Q_{n1} - D_n > Q_{o1} - Q_{o2}\} \\
&\ge S_{o1} - C_{o1} \\
&\quad + (p_o - S_{o1} - S_{n2}) \\
&\quad \times \Pr\{D_o > Q_{o1}, D_n \le Q_{n1}, Q_{n1} - D_n > Q_{o1} - Q_{o2}\} \\
&= \frac{\partial \Pi(Q)}{\partial Q_{o1}}\Big|_{c_{no}=0},
\end{aligned} \tag{28}$$

so $Q_{o1}^*(c_{no}) > Q_{o1}^*(0)$. The proposition holds. □

Proposition 10. *The optimal order quantity of specific component of old generation product is nondecreasing in mark-up value.*

Proof. The solution of $\Pr\{D_o \le Q_{o1}\} = (p_o + c_{no} - C_{o1} - S_{n2})/(p_o + c_{no} - S_{o1} - S_{n2})$ is increasing in mark-up value c_{no}. Therefore, the proposition is obvious. □

Proposition 11. *The optimal order quantity of substituted component of old generation product is less for the case of considering product substitution and component substitution simultaneously than the case of only considering component substitution.*

Proof. From (18), we have

$$\begin{aligned}
\frac{\partial \Pi(Q)}{\partial Q_{o2}}\Big|_{C_{no}>0, c_{no}>0} &= \phi_o \Pr\{D_o > Q_{o2}\} + S_{o1} - C_{o1} + S_{o2} - C_{o2} \\
&\quad - \phi_{no}^{(2)} \Pr\{D_o > Q_{o2}, D_o \le Q_{o1}, D_n \le Q_{n1}, \\
&\qquad Q_{n1} - D_n > D_o - Q_{o2}\} \\
&\quad - \theta(C_{no})\, \phi_{no}^{(1)} \Pr\{Q_{n1} > D_n, D_o > Q_{o2}, \\
&\qquad Q_{n1} - D_n > \theta(C_{no})(D_o - Q_{o2})\} \\
&< \phi_o \Pr\{D_o > Q_{o2}\} + S_{o1} - C_{o1} + S_{o2} - C_{o2} \\
&\quad - \phi_{no}^{(2)} \Pr\{D_o > Q_{o2}, D_o \le Q_{o1}, \\
&\qquad D_n \le Q_{n1}, Q_{n1} - D_n > D_o - Q_{o2}\} \\
&= \frac{\partial \Pi(Q)}{\partial Q_{o2}}\Big|_{C_{no}=0, c_{no}>0},
\end{aligned} \tag{29}$$

so $Q_{o2}^*(C_{no}) < Q_{o2}^*(0)$. The proposition holds. □

From the aforementioned proposition above, if the firm wants to decrease shortage by substitution, it must order more components than the case of no substitution behaviors. Moreover, when product substitution is also introduced, the order quantities for all types of component of new generation should be increased; but for old generation product, the order quantity of its substituted component should be decreased, and the order quantity of specific component should be increased.

The existence of mark-up value is a positive stimulation, to more effective cope with the emergent shortage, firm should store more specific components of old generation product, and it is also same for all type components of new generation product. For product substitution, the existence of substitution effort attracts partial customers of old generation product to buy new generation product, so the firm should order more components of new generation product in order to satisfy the demand of product substitution. However, increasing of the cost will decrease firm's activity of offering product substitution, so the order quantity should not be increasing as substitution effort increases.

From a more widely viewpoint of supply chain, introducing mark-up value and substitution effort are helpful for decreasing the shortage, and it is also an effective way of increasing the service level.

5. Conclusion

In this paper, we study an inventory decision problem with component substitution and product substitution, where a manufacturer produces two products with an updated relation, replenishes the component inventory in advance, and assembles the components into end products according to the customer's order. Since manufacturer makes the replenishment decisions of component inventories before the order arrivals, the shortage for component inventories is inevitable. Therefore, manufacturer may fill the shortage demand using an updated component. At the meanwhile, the manufacturer also can stimulate the customer to buy the other product himself by offering a discount price. We assume a proportion of shortage will purchase new products. To maximize firm's profit, a two-stage dynamic programming model was formulated. And decisions about assembled quantities of different types of products were given. By analyzing the expected profit function, we prove it to be concave in order quantities, and some bounds of decision variables are given. Finally, we investigate the management insights by theoretical method.

There are some possible extensions in the future research. Mark-up value and substitution effort are only regarded as system parameters, in fact, the firm also makes a decision on them. Therefore, the problem will be a joint inventory and a pricing problem, which is very interesting. Certainly, the extension also may result in a game problem between manufacturer and customers.

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Research Article

Parametric Analysis of Flexible Logic Control Model

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Based on deep analysis about the essential relation between two input variables of normal two-dimensional fuzzy controller, we used universal combinatorial operation model to describe the logic relationship and gave a flexible logic control method to realize the effective control for complex system. In practical control application, how to determine the general correlation coefficient of flexible logic control model is a problem for further studies. First, the conventional universal combinatorial operation model has been limited in the interval $[0,1]$. Consequently, this paper studies a kind of universal combinatorial operation model based on the interval $[a,b]$. And some important theorems are given and proved, which provide a foundation for the flexible logic control method. For dealing reasonably with the complex relations of every factor in complex system, a kind of universal combinatorial operation model with unequal weights is put forward. Then, this paper has carried out the parametric analysis of flexible logic control model. And some research results have been given, which have important directive to determine the values of the general correlation coefficients in practical control application.

1. Introduction

Fuzzy control has made the rapid development, and it has found a considerable number of successful industrial applications in recent years [1]. However, from the mathematical viewpoint, Professor Li revealed the interpolation mechanism of fuzzy control and proved that fuzzy controller is in essence an interpolator [2]. So, there are two problems in controlling some practical complex systems. One is that control rules will grow exponentially with the growing of inputs, and the other one is that the precision of control system is not high [3].

Compound controllers combine fuzzy control and other relatively mature control methods to obtain effective control effect, such as Fuzzy-PID controllers [4], fuzzy prediction control [5], and adaptive fuzzy H-infinity control [6]. To reduce dimensionality, hierarchical fuzzy logic controller separates the set of control rules into several sets based on different functions [7, 8]. The basic idea of adaptive fuzzy controllers based on variable universe is to keep the control rules unchanged and change the region bound of fuzzy variables with the values of input fuzzy variables in order to increase control rules indirectly [9]. Though a great deal of research has been done to improve the performance of fuzzy control, most of these methods are based on the basic idea that fuzzy controller is a piecewise approximation. However, to date, there has been relatively little research conducted on the internal relations among input variables of fuzzy controllers.

Universal Logic [10], proposed by He et al., is a kind of flexible logic. It considers the continuous change of not only the truth value of proposition, which is called truth value flexibility, but also the relation between propositions, which is called relational flexibility. Based on fuzzy logic, it puts up two important coefficients: *generalized correlation coefficient* "h" and *generalized self-correlation coefficient* "k." The flexible change of universal logic operations is based on "h" and "k." So, Universal Logic provides a new theoretical foundation to realize more accurate control for complex systems.

In our previous work [3], we focused on the basic physical meanings of fuzzy input variables of the normal two-dimensional fuzzy controller, such as E and EC. And we proved that the essential relation between them is just universal combinatorial relation in *Universal Logic* [10]. So, the simple universal combinatorial operation can be used instead of complex fuzzy reasoning process. As a result, a flexible logic control method was put forward.

Through the previous analysis, it is clear that flexible ability of flexible logic control model is resulted from the following aspect. We use universal combinatorial operation to reflect the essential relation between the deviation and the deviation change of control system, which considers the continuous change of the relation between things. And universal combinatorial operation model is not a single fixed operator, but a continuous cluster of combinatorial operators determined by the *general correlation coefficient h* between propositions. In practical control application, according to the general correlation between propositions, we can take the corresponding one from the cluster to realize effective control for complex system.

However, in practical control application, how to determine the *general correlation coefficient h* of flexible logic control model is a problem for further studies. First, the conventional universal combinatorial operation model has been limited in the interval $[0, 1]$. Consequently, this paper studies a kind of universal combinatorial operation model based on the interval $[a, b]$. And some important theorems are given and proved, which provide a foundation for the control application of universal combinatorial operation. Then, this paper carries out the parametric analysis of flexible logic control model. And some research results have been given, which have important directive to determine the values of the general correlation coefficients in practical control application.

The rest of the paper is organized as follows. Section 2 introduces necessary background on universal combinatorial operation model and flexible logic control method and gives and proves some important theorems of universal combinatorial operation model based on the interval $[a, b]$. Section 3 carries out the parametric analysis of flexible logic control model and gives some research results. Finally, concluding remarks are given in Section 4.

2. Universal Combinatorial Operation Model and Flexible Logic Control Method

2.1. Universal Combinatorial Operation Model. It is well known that there is complex relation between every factor of complex system, which may be conflictive or consistent. For dealing reasonably with the complex relations, various aggregation operators have been given which are mostly T-norm, S-norm, or *Mean* operators. Nevertheless, T-norm, S-norm, and *Mean* operators have the following properties:

$$\begin{gathered} 0 \leq T(x, y) \leq \min(x, y) \\ \max(x, y) \leq S(x, y) \leq 1 \\ \min(x, y) \leq M(x, y) \leq \max(x, y). \end{gathered} \tag{1}$$

As a result, T-norm or S-norm can only handle mutually conflictive relation. In contrast, *Mean* operators can only handle mutually consistent relation. Therefore, the operating regions of these aggregation operators are localized. Universal combinatorial operation model is the combinatorial connective of *Universal Logic*, whose operating region is the standard interval $[0, 1]$.

In the paper, we will only consider the *generalized correlation coefficient h*. So, zero-order universal combinatorial operation model is defined as follows.

Definition 1 (see [10]). Zero-order universal combinatorial operation model is the cluster

$$\begin{aligned} &C^e(x, y, h) \\ &\quad = \text{ite}\Big\{\Gamma^e\left[\left(x^m + y^m - e^m\right)^{1/m}\right] \mid x + y < 2e; \\ &\qquad 1 - \Big(\Gamma^{1-e}\left[(1-x)^m + (1-y)^m\right. \\ &\qquad\qquad \left.\left. -(1-e)^m\right]\Big)^{1/m} \mid x + y > 2e; e\right\}, \end{aligned} \tag{2}$$

where $m = (3 - 4h)/(4h(1 - h))$, $h \in [0, 1]$, $m \in R$, and $e \in [0, 1]$.

Remark 2. The conditional expression $\text{ite}\{\beta \mid \alpha; \gamma\}$ represents that if α is true, then the result is β, otherwise γ. Similarly, $\text{ite}\{\beta_1 \mid \alpha_1; \beta_2 \mid \alpha_2; \gamma\} = \text{ite}\{\beta_1 \mid \alpha_1; \text{ite}\{\beta_2 \mid \alpha_2; \gamma\}\}$. And $\Gamma^1[x] = \text{ite}\{1 \mid x > 1; 0 \mid x < 0$ or x is an imaginary number; $x\}$.

Universal combinatorial operation model is a continuous cluster of combinatorial operators, which can be continuously changeable with *generalized correlation coefficient h* between propositions. In practical application, according to the general correlation between propositions, we can take the corresponding one from the cluster. As *generalized correlation coefficient h* is equal to some special values, the corresponding combinatorial operators are given in Table 1.

2.2. Universal Combinatorial Operation Model on Any Interval $[a, b]$. In practical control application, fuzzy domain of fuzzy variables, E and EC, is mostly symmetrical, such as $[-6, 6]$. However, the conventional universal combinatorial operation model has been limited in the interval $[0, 1]$. So, Chen [11] put forward a kind of universal combinatorial operation model, which is on any interval $[a, b]$.

Definition 3 (see [11]). Normal universal *Not* operation model in any interval $[a, b]$ is defined as follows:

$$\text{GN}(x) = b + a - x. \tag{3}$$

For the previous definition, some common properties of normal universal *Not* operation model are given:

(1) closure

$$\text{GN}(x) \in [a, b], \tag{4}$$

(2) two polar law

$$\text{GN}(a) = b, \qquad \text{GN}(b) = a, \tag{5}$$

(3) symmetric involution

$$\text{GN}(\text{GN}(x)) = x. \tag{6}$$

Table 1: Some special combinatorial operators of universal combinatorial operation model.

Value of h	General correlation between propositions	Combinatorial operator	Name of combinatorial operator
1	Max-attracting	$C^e(x, y, 1)$ $= \text{ite}\{\min(x, y) \mid x + y < 2e;$ $\max(x, y) \mid x + y > 2e;$ $e\}$	Zadeh combinatorial operator C_3^e
0.75	Independent correlation	$C^e(x, y, 0.75)$ $= \text{ite}\{xy/e \mid x + y < 2e;$ $(x + y - xy - e)/(1 - e) \mid x + y > 2e;$ $e\}$	Probability combinatorial operator C_2^e
0.5	Max-rejecting	$C^e(x, y, 0.5) = \Gamma^1[x + y - e]$	Bounded combinatorial operator C_1^e
0	Max-restraining	$C^e(x, y, 0)$ $= \text{ite}\{0 \mid x, y < e;$ $1 \mid x, y > e;$ $e\}$	Drastic combinatorial operator C_0^e

Definition 4 (see [11]). Universal combinatorial operation model on any interval $[a, b]$ is the cluster

$$
\begin{aligned}
&\mathrm{GC}^e(x, y, h) \\
&= \text{ite}\Bigg\{\min\Bigg(e, (b-a) \times \left[\max\left(0, \frac{(x-a)^m + (y-a)^m - (e-a)^m}{(b-a)^m}\right)\right]^{1/m} + a\Bigg) \mid x + y < 2e; \\
&\quad b + a - \min\Bigg(e', (b-a) \times \left[\max\left(0, \frac{(b-x)^m + (b-y)^m - (b-e)^m}{(b-a)^m}\right)\right]^{1/m} + a\Bigg) \mid x + y > 2e; e\Bigg\},
\end{aligned}
\tag{7}
$$

where $m = (3 - 4h)/(4h(1 - h))$, $h \in [0, 1]$, $m \in R$, $e, e' \in [a, b]$, and $e' = \mathrm{GN}(e)$.

Without loss of generality, we assume that the fuzzy domain of fuzzy variables, E and EC, is the interval $[-1, 1]$. As a result, when a, b, and e are equal to -1, 1, and 0, respectively, the corresponding zero-order universal combinatorial operator is the cluster

$$
\begin{aligned}
&\mathrm{GC}^0(x, y, h) \\
&= \text{ite}\Bigg\{\min\Bigg(0, 2 \times \left[\max\left(0, \frac{(x+1)^m + (y+1)^m - 1}{2^m}\right)\right]^{1/m} - 1\Bigg) \mid x + y < 0; \\
&\quad -\min\Bigg(0, 2\left[\max\left(0, \frac{(1-x)^m + (1-y)^m - 1}{2^m}\right)\right]^{1/m} - 1\Bigg) \mid x + y > 0; 0\Bigg\}.
\end{aligned}
\tag{8}
$$

According to the definition of *Universal Combinatorial Operation Model* in any interval, the following characters [11] are attained:

(1) $\mathrm{GC}^e(x, y, h)$ conforms to the combination axiom:

(a) boundary condition GC1

when $x, y < e$, $\mathrm{GC}^e(x, y, h) \le \min(x, y)$;
when $x, y > e$, $\mathrm{GC}^e(x, y, h) \ge \max(x, y)$;
when $x + y = 2e$, $\mathrm{GC}^e(x, y, h) = e$;
otherwise, $\min(x, y) \le \mathrm{GC}^e(x, y, h) \le \max(x, y)$,

(b) monotonicity GC2

$\mathrm{GC}^e(x, y, h)$ increases monotonously along with x and y,

(c) continuity GC3

when $h \in (0, 1)$, $\mathrm{GC}^e(x, y, h)$ is continuous for all x and y,

(d) commutative law GC4

$$\mathrm{GC}^e(x, y, h) = \mathrm{GC}^e(y, x, h), \tag{9}$$

(e) law of identical element GC5

$$\mathrm{GC}^e(x, e, h) = x, \tag{10}$$

(2) closure

$$\mathrm{GC}^e(x, y, h) \in [a, b], \tag{11}$$

(3) inverse law

$$\mathrm{GC}^{e}(x, 2e-x, h) = e, \tag{12}$$

(4) renunciation law

$$\mathrm{GC}^{e}(e, e, h) = e. \tag{13}$$

Theorem 5. *Let* $GN(GC^{GN(e)}(GN(x), GN(y), h)) = GC^{e}(x, y, h)$, $x, y \in [a, b]$, $e \in [a, b]$.

Proof. $x, y \in [a, b]$, $e \in [a, b]$, according to the closure of central generalized negation operation and universal combinatorial operation:

$\mathrm{GN}(x), \mathrm{GN}(y) \in [a, b]$, $\mathrm{GN}(e) \in [a, b]$ and $\mathrm{GC}^{\mathrm{GN}(e)}(\mathrm{GN}(x), \mathrm{GN}(y), h) \in [a, b]$, and according to the definition of universal combinatorial operation:

(1) when $x + y < 2e$,

$$\begin{aligned} &\mathrm{GN}(x) + \mathrm{GN}(y) \\ &\quad = (b + a - x) + (b + a - y) \\ &\quad = 2(b + a) - (x + y) \\ &\quad > 2(b + a - e) = 2\mathrm{GN}(e). \end{aligned} \tag{14}$$

Then, according to the definition of $\mathrm{GC}^{e}(x, y, h)$,

$$\begin{aligned} &\mathrm{GC}^{\mathrm{GN}(e)}(\mathrm{GN}(x), \mathrm{GN}(y), h) \\ &\quad = b + a - \min\Bigg(b + a - \mathrm{GN}(e), (b-a) \\ &\qquad \times \Bigg[\max\Bigg(0, \frac{(b-\mathrm{GN}(x))^m}{(b-a)^m} \\ &\qquad + \frac{(b-\mathrm{GN}(y))^m}{(b-a)^m} \\ &\qquad - \frac{(b-\mathrm{GN}(e))^m}{(b-a)^m}\Bigg)\Bigg]^{1/m} + a\Bigg). \end{aligned} \tag{15}$$

According to the definition of central generalized negation operation,

$$\mathrm{GN}(x) = b + a - x \tag{16}$$

$$\mathrm{GN}(y) = b + a - y \tag{17}$$

$$\mathrm{GN}(e) = b + a - e. \tag{18}$$

Substituting (16), (17), and (18) separately into (15),

$$\begin{aligned} &\mathrm{GC}^{\mathrm{GN}(e)}(\mathrm{GN}(x), \mathrm{GN}(y), h) \\ &\quad = b + a - \min\Bigg(e, (b-a) \\ &\qquad \times \Bigg[\max\Bigg(0, \frac{(x-a)^m + (y-a)^m}{(b-a)^m} \\ &\qquad - \frac{(e-a)^m}{(b-a)^m}\Bigg)\Bigg]^{1/m} + a\Bigg), \end{aligned} \tag{19}$$

and then

$$\begin{aligned} &\mathrm{GN}\left(\mathrm{GC}^{\mathrm{GN}(e)}(\mathrm{GN}(x), \mathrm{GN}(y), h)\right) \\ &\quad = b + a - \Bigg(b + a \\ &\qquad - \min\Bigg(e, (b-a) \\ &\qquad \times \Bigg[\max\Bigg(0, \frac{(x-a)^m}{(b-a)^m} \\ &\qquad + \frac{(y-a)^m - (e-a)^m}{(b-a)^m}\Bigg)\Bigg]^{1/m} \\ &\qquad + a\Bigg)\Bigg) \\ &\quad = \min\Bigg(e, (b-a) \\ &\qquad \times \Bigg[\max\Bigg(0, \frac{(x-a)^m}{(b-a)^m} \\ &\qquad + \frac{(y-a)^m - (e-a)^m}{(b-a)^m}\Bigg)\Bigg]^{1/m} + a\Bigg). \\ &\quad = \mathrm{GC}^{e}(x, y, h). \end{aligned} \tag{20}$$

(2) When $x + y > 2e$,

$$\begin{aligned} &\mathrm{GN}(x) + \mathrm{GN}(y) \\ &\quad = (b + a - x) + (b + a - y) \\ &\quad = 2(b + a) - (x + y) \\ &\quad < 2(b + a - e) = 2\mathrm{GN}(e). \end{aligned} \tag{21}$$

So, according to the definition of $GC^e(x, y, h)$,

$$\begin{aligned}
&GC^{GN(e)}(GN(x), GN(y), h)\\
&= \min\Bigg(GN(e), (b-a)\\
&\quad\times\left[\max\left(0, \frac{(GN(x)-a)^m + (GN(y)-a)^m}{(b-a)^m}\right.\right.\\
&\quad\left.\left.-\frac{(GN(e)-a)^m}{(b-a)^m}\right)\right]^{1/m} + a\Bigg).
\end{aligned} \tag{22}$$

Substituting (16), (17), and (18) separately into (22),

$$\begin{aligned}
&GC^{GN(e)}(GN(x), GN(y), h)\\
&= \min\Bigg(b+a-e, (b-a)\\
&\quad\times\left[\max\left(0, \frac{(b-x)^m}{(b-a)^m}\right.\right.\\
&\quad\left.\left.+\frac{(b-y)^m-(b-e)^m}{(b-a)^m}\right)\right]^{1/m} + a\Bigg).
\end{aligned} \tag{23}$$

And then,

$$\begin{aligned}
&GN\left(GC^{GN(e)}(GN(x), GN(y), h)\right)\\
&= b+a-\min\Bigg(b+a-e, (b-a)\\
&\quad\times\left[\max\left(0, \frac{(b-x)^m}{(b-a)^m}\right.\right.\\
&\quad\left.\left.+\frac{(b-y)^m-(b-e)^m}{(b-a)^m}\right)\right]^{1/m} + a\Bigg)\\
&= GC^e(x, y, h).
\end{aligned} \tag{24}$$

(3) When $x + y = 2e$,

$$\begin{aligned}
&GN(x) + GN(y)\\
&= (b+a-x) + (b+a-y)\\
&= 2(b+a) - (x+y)\\
&= 2(b+a-e)\\
&= 2GN(e).
\end{aligned} \tag{25}$$

According to the definition of $GC^e(x, y, h)$,

$$GC^{GN(e)}(GN(x), GN(y), h) = GN(e). \tag{26}$$

Then,

$$GN\left(GC^{GN(e)}(GN(x), GN(y), h)\right) = GN(GN(e)) = e. \tag{27}$$

From the aforementioned, the theorem is true. ∎

Lemma 6. *Let $GC^{GN(e)}(GN(x), GN(y), h) = GN(GC^e(x, y, h))$.*

Proof. According to Theorem 5 and involution law of central generalized negation operator in interval $[a, b]$, the theorem can be proved simply. ∎

Lemma 7. *Let $C^e(x, y, h) = 1 - C^{1-e}(1-x, 1-y, h)$.*

Proof. Setting the interval $[a, b]$ of x, y as $[0, 1]$, the lemma can be proved simply. ∎

Lemma 8. *When interval $[a, b]$ relates to e symmetry, $GC^e(x^*, y^*, h) = (GC^e(x, y, h))^*$, wherein x^* represents the points of x relating to e symmetry, namely, $x^* = 2e - x$, y^*, e^*, $(GC^e(x, y, h))^*$ is similar, $e \in [a, b]$, $h \in [0, 1]$.*

Proof. Since interval $[a, b]$ relates to e symmetry, then $a + b = 2e$, and $GN(x) = a + b - x = 2e - x$. Thus,

$$GN(x) = x^*. \tag{28}$$

Similarly,

$$\begin{aligned}
GN(y) &= y^*,\\
GN(e) &= a + b - e = 2e - e = e.
\end{aligned} \tag{29}$$

From (28) and (29), the following could be obtained:

$$\begin{aligned}
GC^{GN(e)}(GN(x), GN(y), h) &= GC^e(x^*, y^*, h),\\
GN(GC^e(x, y, h)) &= (GC^e(x, y, h))^*.
\end{aligned} \tag{30}$$

And then from Lemma 6,

$$GC^e(x^*, y^*, h) = (GC^e(x, y, h))^*. \tag{31}$$

So, the theorem is true. ∎

Lemma 9. *Let $C^{0.5}(1-x, 1-y, h) = 1 - C^{0.5}(x, y, h)$, wherein $h \in [0, 1]$.*

Lemma 10. *When interval $[a, b]$ relates to symmetry of original point, $C^0(-x, -y, h) = -GC^0(x, y, h)$, $h \in [0, 1]$.*

This lemma indicates that when the interval $[a, b]$ is symmetrical about the origin point and identity element e

is 0, *Universal Combinatorial Operation* $\mathrm{GC}^e(x, y, h)$ is also symmetrical about the origin point.

As pointed out in the literature [3], the internal relation between fuzzy input variables, the deviation E, and the deviation change EC of normal two-dimensional fuzzy controller is the universal combination relation in universal logic. Consequently, the complex fuzzy rule inference process could be replaced by the simple universal combinatorial operation, and a flexible logic control model was presented accordingly. In fuzzy control, the domain of input variables and output variable is generally symmetrical about the original point, such as $[-5, 5]$. Obviously, it is the prerequisite of control model that the operation model relates to symmetry of original point. Therefore, Lemma 10 provides a basis for the *Universal Combinatorial Operation*'s application in control.

2.3. Universal Combinatorial Operation Model with Unequal Weights. In practical complex system, every factor is generally with unequal weight. But the existing universal combinatorial operation only discusses an ideal state that every factor is with equal weight.

2.3.1. Weighted Operator. For dealing reasonably with the complex relations of every factor in complex system, various properties which weighted operators should have are put forward. The weighted operator proposed by *Yager* is one of the famous ones.

The weighted operator proposed by Yager is defined as follows.

Definition 11 (see [12]). Assume that an operator $h(\alpha, x)$ is a mapping from $[0, 1]$ to $[0, 1]$.

$h(\alpha, x)$ is called a *Yager weighted operator* if it satisfies the following properties.

(I1) Monotonicity with respect to the value, x. In particular if $x > x'$, then we require

$$h(\alpha, x) \geq h(\alpha, x'). \tag{32}$$

(I2) We desire that elements with weight zero have no effect on the aggregation process; thus, if e is the fixed identity of the aggregation to be used on the resulting bag, we must have

$$h(0, x) = e. \tag{33}$$

(I3) A normalcy with respect to the weights

$$h(1, x) = x. \tag{34}$$

(I4) Finally, we desire that the transformation moves monotonically from its value for $\alpha = 0$ to $\alpha = 1$. That is, if $x \geq e$, $h(\alpha, x)$ increases monotonically with respect to the value α; if $x \leq e$, $h(\alpha, x)$ decreases monotonically with respect to the value α,

where α is the weight associated with an argument, and x is the argument. Both α and x are drawn from $[0, 1]$.

And due to the definition of *Yager weighted operator*, we can draw the following theorems.

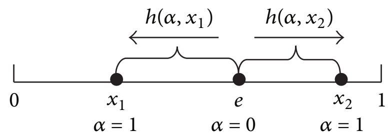

Figure 1: *Yager weighted operator* $h(\alpha, x)$ varies with the weight α.

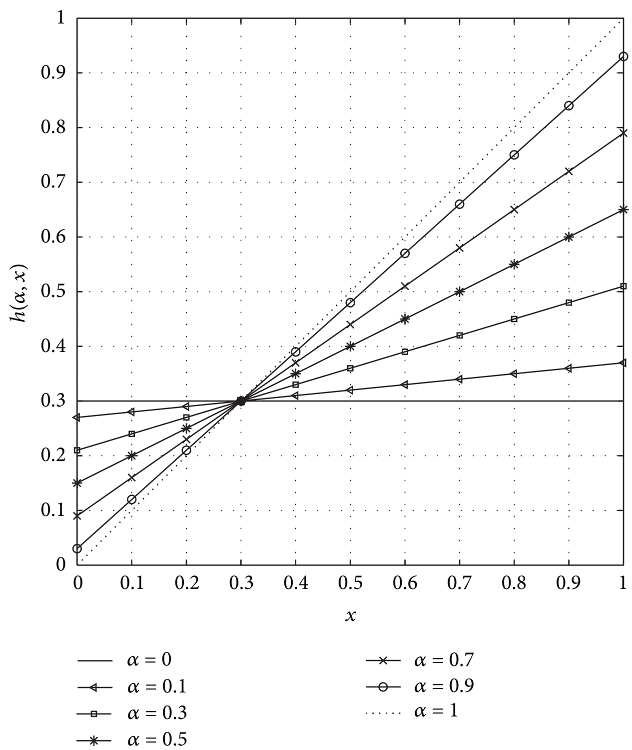

Figure 2: *Yager weighted operator* $h(\alpha, x) = \alpha x + (1 - \alpha)e$ with e is equal to 0.3.

Theorem 12. *Assume that* $h(\alpha, x)$ *is a Yager weighted operator. So, one has the following:*

$$\text{if } x \geq e, \text{ then } h(\alpha, x) \geq e, \text{ and if } x \leq e, \text{ then } h(\alpha, x) \leq e. \tag{35}$$

Proof. Due to the properties (I2) and (I4) of the definition, the theorem can be proved easily. ∎

Theorem 13. *Assume that* $h(\alpha, x)$ *is a Yager weighted operator. So, one has the following:*

$$\text{if } x \geq e, \text{ then } h(\alpha, x) \geq e, \text{ and if } x \leq e, \text{ then } h(\alpha, x) \leq e. \tag{36}$$

Proof. Due to the properties (I3) and (I4) of the definition, the theorem can be proved easily.

The chart of *Yager weighted operator* $h(\alpha, x)$ changing along with the weight α is given in Figure 1.

One formulation that satisfies these conditions is $h(\alpha, x) = \alpha x + (1 - \alpha)e$. It is described by Figure 2.

According to the previous analyses, we discover that *Yager weighted operator* has some shortcomings as follows.

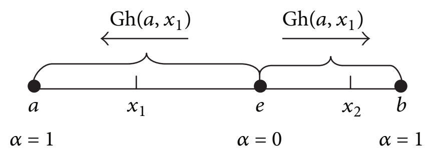

Figure 3: *General weighted operator* varies with the weight α.

(1) *Yager weighted operator* is likely to transform entire *True* (*False*) proposition into partial *True* (*False*) proposition. However, due to the view of logic, entire *True* (*False*) proposition should still be transformed into entire *True* (*False*) proposition by weighted operator.

(2) The weighted value changes in the interval $[e, x]$ or $[x, e]$ for any weight $\alpha \in [0, 1]$ as shown in Figure 1. However, the weighted value is desired to change in the total interval $[0, 1]$ in some practical applications.

(3) *Yager weighted operator* is limited in the interval $[0, 1]$. But weighted operators are desired to change in the general interval $[a, b]$ in some practical applications.

For solving the previous problems, the paper puts forward a kind of general weighted operators $\mathrm{Gh}(\alpha, x)$, which change in the general interval $[a, b]$. □

Definition 14. Assume that an operator $\mathrm{Gh}(\alpha, x)$ is a mapping from $[a, b]$ to $[a, b]$. $\mathrm{Gh}(\alpha, x)$ is called a *general weighted operator* if it satisfies the following properties.

(I1) Monotonicity with respect to the value, x. In particular, if $x > x^{'}$, then we require

$$\mathrm{Gh}(\alpha, x) \geq \mathrm{Gh}\left(\alpha, x^{'}\right). \tag{37}$$

(I2) $\mathrm{Gh}(0, x) = \mathrm{ite}\{a \mid x = a; b \mid x = b; e\}$.

(I3) $\mathrm{Gh}(\alpha, e) = e$.

(I4) $\mathrm{Gh}(\alpha, a) = a$ and $\mathrm{Gh}(\alpha, b) = b$.

(I5) Finally, we desire that the transformation moves monotonically for the weight $\alpha \in (0, 1)$. That is, if $x \geq e$, $\mathrm{Gh}(\alpha, x)$ increases monotonically with respect to the value α; if $x \leq e$, $\mathrm{Gh}(\alpha, x)$ decreases monotonically with respect to the value α.

(I6) $\mathrm{Gh}(1, x) = \mathrm{ite}\{e \mid x = e; a \mid x < e; b \mid x > e\}$.

α is the weight associated with an argument x, and e is the fixed identity of the aggregation operator GC^e. α is drawn from $[0, 1]$, and e is drawn from $[a, b]$.

According to the previous definition, we can know that the absolute value of the argument x decreases first and then increases with the weight α changing continuously from 0 to 1. The chart of *general weighted operator* changing along with the weight α is given in Figure 3.

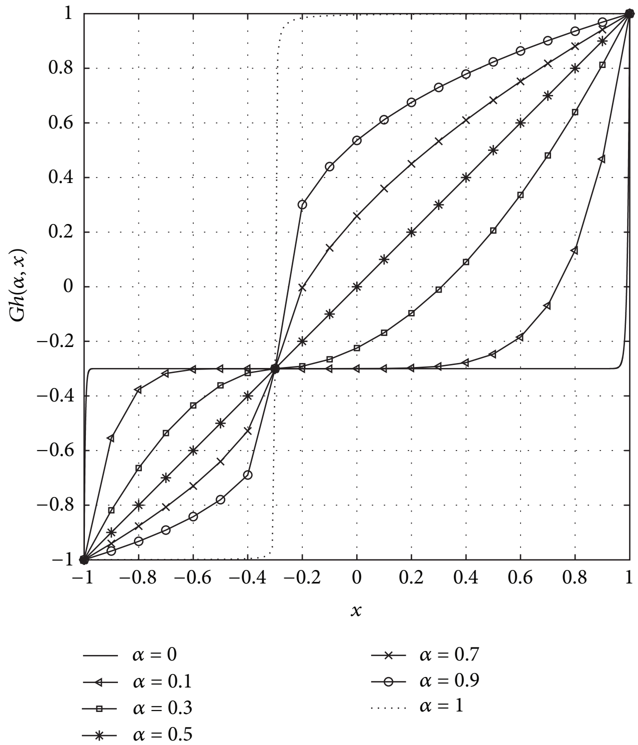

Figure 4: *General weighted operator of polynomial model* with $a = -1$, $b = 1$, and $e = -0.3$.

The common general weighted operation model is polynomial model:

$$\begin{aligned}
&\mathrm{Gh}(\alpha, x) \\
&\quad = \mathrm{ite}\left\{ \frac{(x-e)(b-e)(1+n)^{1/2}}{\left((b-x)+(1+n)^{1/2}(x-e)\right)} + e \mid x > e; \right. \\
&\qquad \left. e - \frac{(e-x)(e-a)(1+n)^{1/2}}{\left((x-a)+(1+n)^{1/2}(e-x)\right)} \mid x < e; e \right\}.
\end{aligned} \tag{38}$$

Assume that $n = (2\alpha - 1)/(1 - \alpha)^2$, $\alpha \in [0, 1]$, $n \geq -1$, and $\mathrm{Gh}(\alpha, x)$ is the limit as $\alpha = 0$ and $\alpha = 1$. So, (38) describes a *general weighted operator* as shown in Figure 4.

2.3.2. Universal Combinatorial Operation Model with Unequal Weights. According to the previous definition of general weighted operators, we can get the definition of universal combinatorial operation model with unequal weights as follows.

Definition 15. Assume that an operator $\mathrm{UGC}^e(x, y, \alpha_x, \alpha_y, h)$ is a mapping from $[a, b]$ to $[a, b]$. $\mathrm{UGC}^e(x, y, \alpha_x, \alpha_y, h)$ is called universal combinatorial operation model with unequal weights:

$$\mathrm{UGC}^e\left(x, y, \alpha_x, \alpha_y, h\right) = \mathrm{GC}^e\left(\mathrm{Gh}\left(\alpha_x, x\right), \mathrm{Gh}\left(\alpha_y, y\right), h\right), \tag{39}$$

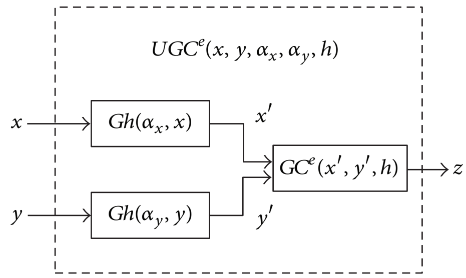

Figure 5: Universal combinatorial operation model with unequal weights.

Table 2: Fuzzy rules defining fuzzy composed variable GE_θ.

GE_θ		θ' NB	NM	NS	ZE	PS	PM	PB
θ	NB	NB	NB	NB	NM	NM	NS	ZE
	NM	NB	NB	NM	NM	NS	ZE	PS
	NS	NB	NM	NM	NS	ZE	PS	PM
	ZE	NM	NM	NS	ZE	PS	PM	PM
	PS	NM	NS	ZE	PS	PM	PM	PB
	PM	NS	ZE	PS	PM	PM	PB	PB
	PB	ZE	PS	PM	PM	PB	PB	PB

where $\mathrm{Gh}(\alpha, x)$ is general weighted operator, $\mathrm{GC}^e(x, y, h)$ is *universal combinatorial operator*, e is the fixed identity of $\mathrm{GC}^e(x, y, h)$, h is *the generalized correlation coefficient*, and α_x and α_y denote, respectively, the weights associated with the arguments x and y. x, y, and e are drawn from $[a, b]$, and h, α_x, and α_y are drawn from $[0, 1]$.

The universal combinatorial operation model with unequal weights is given in Figure 5.

2.4. Flexible Logic Control Method. In order to decrease effectively the number of fuzzy control rules for multivariable nonlinear system, Xiao et al. [13] gave a new concept of fuzzy composed variable. Its basic idea can be summarized as follows. According to characteristics of controlled system and internal relations between input variables, a fuzzy composed variable is constructed to reflect synthetically the deviation between reference and the process output with a fuzzy logic system.

There are four output variables in single inverted-pendulum, which are x, x', θ, and θ'. In the four input variables of control system, it is θ and θ' that describe the movement state of the rod. So, a fuzzy logic system can be designed, which is described by the fuzzy rules in Table 2, to define a fuzzy composed variable GE_θ with θ and θ'. The fuzzy composed variable GE_θ can describe synthetically the movement state of the rod. Similarly, a fuzzy composed variable GE_x can be defined with x and x' to describe synthetically the movement state of the cart. For multivariable system, one does not need to define, respectively, fuzzy logic system for every fuzzy composed variable. A uniform fuzzy rule table can be used, such as Table 2, but only select different quantification factors to obtain different fuzzy composed variables.

Remark 16. In the paper, the input variables of the fuzzy controllers discussed are e and ec, which denote, respectively, the deviation and the deviation change. The output variable u is the control signal. The variables, e, ec, and u are crisp values from the practical process. The fuzzy variables, E, EC, and U are the corresponding fuzzy ones, and the fuzzy domains are uniformed to be $[-1, 1]$ with fuzzy subsets, such as NB, NM, NS, ZE, PS, PM, and PB.

Apparently, the fuzzy rules in Table 2 describe the essential relation between all deviation and the deviation change. As shown Table 2 can be divided approximately into four parts, which describe, respectively, fuzzy rules used to define composed variable as E and EC are both negative, E is negative but EC is positive, E and EC are both positive, and E is positive but EC is negative. E and EC both describe the deviation between the reference and the process output; so, we can define a composed variable, denoted as E', based on the essential relation between them.

According to the physical meanings of fuzzy variables, E and EC, we can get the following conclusions.

(1) Suppose that both E and EC are positive. That is to say, the deviation is positive and it will increase continuously. So, the value of the composed variable E' should be positive and bigger than both of E and EC in this case. The combinatorial rules are shown by the right-bottom part of Table 2.

(2) Suppose that E is positive and EC is negative. That is to say, the deviation is positive but it will decrease. So, the value of the composed variable E' should be between E and EC in this case. The combinatorial rules are shown by the left-bottom part of Table 2.

(3) Similarly, we can obtain the value of E' in the two of other cases.

Based on the previous analysis, we get the conclusion that the essential relation among E, EC, and the composed variable E' is a kind of universal combinatorial one in *Universal Logic*. As a result, we have

$$E' = \mathrm{GC}^e(E, EC, h). \tag{40}$$

So, we can obtain the relation among E, EC, and the output variable U of control system as follows:

$$U = -\mathrm{GC}^e(E, EC, h), \tag{41}$$

where fuzzy variables, $E, EC, U \in [-1, 1]$, $e = 0$, $h \in [0, 1]$. Fu and He [3] named the method *Flexible Logic Control Method*.

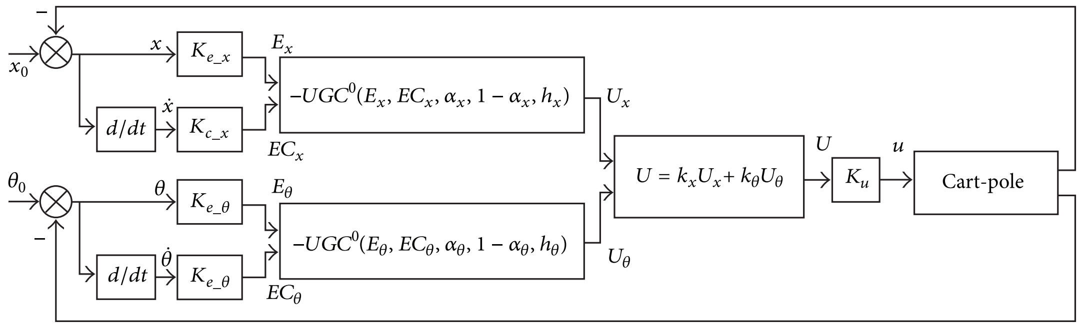

Figure 6: The flexible logic control model of single inverted-pendulum.

For effectively controlling different things, we can lead into a weighted factor $\alpha, \alpha \in [0,1]$. As the correlation coefficient h is equal to 0.5, we have

$$\begin{aligned} U &= -\mathrm{GC}^e(\alpha E, (1-\alpha) EC, 0.5) \\ &= \Gamma_{-1}^{1}[\alpha E + (1-\alpha) EC - e] \\ &= \alpha E + (1-\alpha) EC. \end{aligned} \tag{42}$$

However, (42) is just the fuzzy control method proposed by Long et al. [14]. They used a linear equation, such as (42), to describe fuzzy control rules. But the relation among E, EC, and U is not only linear. So, (41) is a cluster of operators determined by correlation coefficient h, and (42) is only a special operator in the cluster as h is equal to 0.5. As a result, flexible logic control method can realize the accurate control for complex system.

2.5. Flexible Logic Control Model of Single Inverted-Pendulum System. The objective is to maintain the pole in an upright position and the cart in an appointed position in the rail. There are four output variables and one input variable in single inverted-pendulum, which are x, x', θ, θ', and u. In the four input variables of control system, both θ and θ' describe the movement state of the pole, and both x and x' describe the movement state of the cart. So, we have designed two subcontrollers. One is to maintain the cart in an appointed position with two input variables E_x and EC_x. The other one is to maintain the pole in an upright position with two input variables E_θ and EC_θ.

We have designed the two subcontrollers with the flexible logic control method. And we led into weighted factors α_θ, α_x, $\alpha_x, \alpha_\theta \in [0,1]$. Two controllers are designed as follows:

$$U_\theta = -\mathrm{GC}^0(\mathrm{Gh}(\alpha_\theta, E_\theta), \mathrm{Gh}(1-\alpha_\theta, EC_\theta), h_\theta), \tag{43}$$

$$\alpha_\theta = (\alpha_{s\_\theta} - \alpha_{0\_\theta})|E_\theta| + \alpha_{0\_\theta}, \tag{44}$$

$$U_x = -\mathrm{GC}^0(\mathrm{Gh}(\alpha_x, E_x), \mathrm{Gh}(1-\alpha_x, EC_x), h_x), \tag{45}$$

$$\alpha_x = (\alpha_{s\_x} - \alpha_{0\_x})|E_x| + \alpha_{0\_x}. \tag{46}$$

When the control signal U is combined, we lead into two weighted factors $k_\theta, k_x, k_\theta, k_x \in [-1,1]$, for the two subcontrollers. According to (43) and (45), we can get the output of the controller as follows:

$$\begin{aligned} U &= k_x U_x + k_\theta U_\theta \\ &= k_x\left(-\mathrm{GC}^0(\mathrm{Gh}(\alpha_x, E_x), \mathrm{Gh}(1-\alpha_x, EC_x), h_x)\right) \\ &\quad + k_\theta\left(-\mathrm{GC}^0(\mathrm{Gh}(\alpha_\theta, E_\theta), \mathrm{Gh}(1-\alpha_\theta, EC_\theta), h_\theta)\right), \end{aligned} \tag{47}$$

where $E_x, EC_x, U_x, E_\theta, EC_\theta, U_\theta, U \in [-1, 1]$, $\mathrm{Gh}(\alpha, x)$ (38), $\mathrm{UGC}^0(x, y, \alpha_1, \alpha_2, h)$ is (39), h_x, h_θ are general correlation coefficients, $h_x, h_\theta \in [0,1]$, $\alpha_x, \alpha_\theta \in [0,1]$, $0 \le \alpha_{0\_x} \le \alpha_{s\_x} \le 1$, and $0 \le \alpha_{0\_\theta} \le \alpha_{s\_\theta} \le 1$.

By the previous analyses, we can obtain the control model of single inverted-pendulum as shown in Figure 6.

3. Parametric Analysis of Flexible Logic Control Model

Through the previous analysis, it is clear that flexible ability of flexible logic control model is resulted from the following aspect. Universal combinatorial operation model is not a single fixed operator, but a continuous cluster of combinatorial operators determined by the *general correlation coefficient* h between propositions. In practical control application, according to the general correlation between propositions, we can take the corresponding one from the cluster to realize effective control for complex system.

However, in practical control application, how to determine the *general correlation coefficient* h of flexible logic control model is a problem for further studies. In this section, we will analyze the general correlation coefficients, h_x, h_θ, of the flexible logic control model and give some research results.

3.1. Experimentation. We experiment the flexible logic control model in some single inverted-pendulum physical system. The physical parameters of the system are given in Table 3.

By simulating the inverted-pendulum and looking up the optimization with genetic algorithm, we can get the initial values of the control parameters. Then, by testing in real-time experimentations and repeatedly making some fine tuning,

Table 3: Physical parameters of the quadruple inverted-pendulum.

Symbol	Value	Meaning
m_0	0.924 kg	Mass of the cart
m_1	See Table 5	Mass of the pole
f_0	0.1 N·s/m	Dynamic friction coefficient between the cart and the track
f_1	0.007056 N·s/m	Dynamic friction coefficient for the pole
l_1	See Table 5	Distance from the position sensor to the center of gravity of the pole

Table 4: Control parameters of the control model.

Symbol	Value	Meaning
K_{e_x}	23.5294	Quantification factor for E_x
K_{c_x}	9.4118	Quantification factor for EC_x
α_{0_x}	0.1725	Minimum value of α_x
α_{s_x}	0.4683	Maximum value of α_x
K_{e_θ}	50.5882	Quantification factor for E_θ
K_{c_θ}	3.1765	Quantification factor for EC_θ
α_{0_θ}	0.2510	Minimum value of α_θ
α_{s_θ}	0.5953	Maximum value of α_θ
k_x	−0.2235	Weighted factor of the subcontroller for the cart
k_θ	0.4902	Weighted factor of the subcontroller for the pole
K_u	8.7843	Proportion factor for U
h_x	0.2118	General correlation coefficient between E_x and EC_x
h_θ	0.9490	General correlation coefficient between E_θ and EC_θ

Table 5: The variation of the length of the pole.

No.	m_1 (kg)	L_1 (m)
1	0.0149	0.1
2	0.0216	0.149
3	0.0284	0.199
4	0.0378	0.266
5	0.0493	0.354
6	0.0621	0.4436
7	0.0773	0.553
8	0.0966	0.691

we can get the control parameters shown in Table 4. And the variation of the length of the pole is shown in Table 5.

Genetic algorithm is used to optimize the general correlation coefficients, h_x, h_θ, and the fitness function is defined as follows:

$$\text{Dis} = \sum_{i=0}^{N} \frac{i^2 \times \left(x^2(i)/10 + x'^2(i)/20 + \theta^2(i) + \theta'^2(i)\right)}{N},$$
$$\text{fitness} = \frac{1}{(\text{Dis}/10)}. \tag{48}$$

Table 6: The optimization results of h_x, h_θ.

No.	m_1 (kg)	L_1 (m)	h_x	h_θ	Fitness
1	0.0149	0.1	0.985	0.845	0.4310
2	0.0216	0.149	0.975	0.845	0.3310
3	0.0284	0.199	0.96	0.845	0.2713
4	0.0378	0.266	0.955	0.87	0.2240
5	0.0493	0.354	0.63	0.88	0.1838
6	0.0621	0.4436	0.37	0.875	0.1531
7	0.0773	0.553	0.32	0.865	0.1260
8	0.0966	0.691	0.37	0.85	0.1029

Through simulated optimization, we can get the corresponding values of h_x, h_θ and the fitness shown in Table 6.

Figure 7 shows how the fitness varies with h_x, h_θ. And Figures 8 and 9 illustrate how the maximum fitness varies with h_θ taking different values when h_x, in turn, is equal to some value on standard interval $[0, 1]$. Figure 8 shows how the maximum fitness varies with h_x, and Figure 9 shows how h_θ varies with h_x when the fitness is the maximum.

Similarly, Figures 10 and 11 illustrate how the maximum fitness varies with h_x taking different values when h_θ, in turn, is equal to some value on standard interval $[0, 1]$. Figure 10 shows how the maximum fitness varies with h_θ, and Figure 11 shows how h_x varies with h_θ when the fitness is the maximum.

3.2. Parametric Analysis of Flexible Logic Control Model. Firstly, calculate the correlation coefficient between fitness and h_x when h_θ, in turn, is equal to some value on standard interval $[0, 1]$. The formula for calculating the correlation coefficient is defined as follows:

$$r = \frac{\sum_m \sum_n \left(A_{mn} - \overline{A}\right)\left(B_{mn} - \overline{B}\right)}{\sqrt{\left(\sum_m \sum_n \left(A_{mn} - \overline{A}\right)^2\right)\left(\sum_m \sum_n \left(B_{mn} - \overline{B}\right)^2\right)}}, \tag{49}$$

where A and B are $m * n$ matrices, $\overline{A}$ and $\overline{B}$ are the means of the values in A and B, respectively, and r is the correlation coefficient between A and B.

Figure 12 illustrates how the correlation coefficient between fitness and h_x varies with h_θ taking different values on standard interval $[0, 1]$.

Similarly, calculate the correlation coefficient between fitness and h_θ when h_x, in turn, is equal to some value on standard interval $[0, 1]$. And Figure 13 illustrates how the correlation coefficient between fitness and h_θ varies with h_x taking different values on standard interval $[0, 1]$.

After that, the correlation coefficients between the maximum fitness and h_x, h_θ are computed for poles of different lengths, as shown in Table 7.

By observing and analyzing the previous results, we can draw the following conclusions.

(1) The longer the pole, the worse the control effects. Figures 7, 8, and 10 all indicate that the longer the pole, the smaller the fitness with the optimum control parameters, and vice versa. Through experiment, we can know that the given single inverted-pendulum

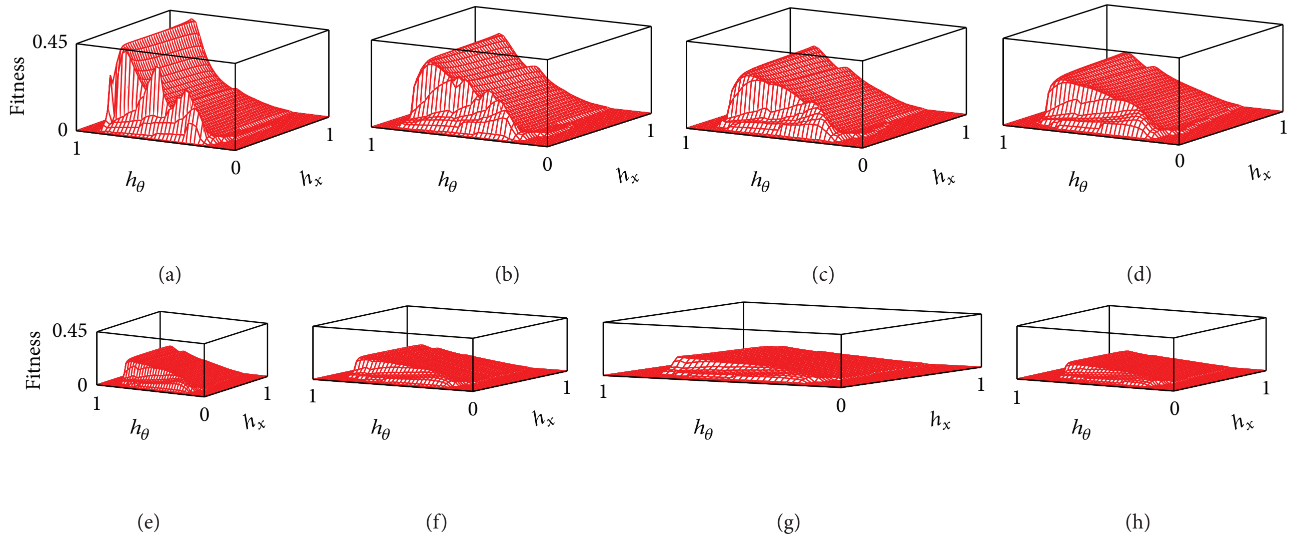

FIGURE 7: The fitness varies with h_x, h_θ.

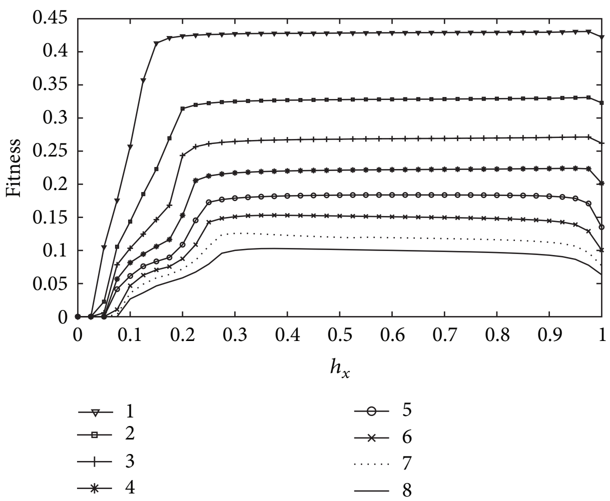

FIGURE 8: The maximum fitness varies with h_x.

physical system can be controlled effectively when the pole is in the range 0.1 m to 0.691 m. That is to say, when the pole is longer than 0.691 m or shorter than 0.1 m, we cannot realize the stable control for the given physical system.

(2) For some given physical system, the control effect is sensitive to the value of h_θ. This means that we can realize the effective control only when h_θ is in some very short interval. And the interval is relatively fixed. That is, the interval of h_θ does not change with the length of the pole. Figures 7, 9 and 10 all show that the fitness is relatively big only when h_θ is in the range 0.8 to 0.9, and the interval of h_θ remains unchanged for poles of different lengths.

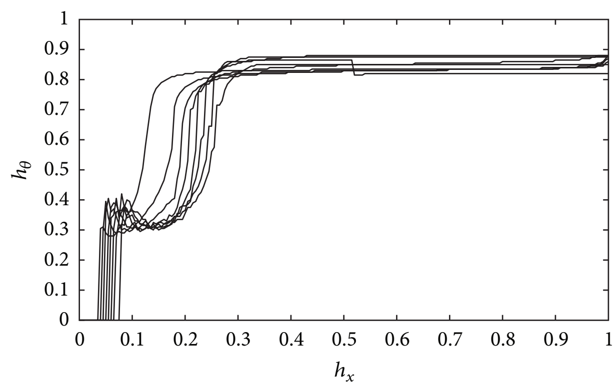

FIGURE 9: h_θ varies with h_x when the fitness is the maximum.

Figure 12 also shows that the correlation coefficient between fitness and h_x is relatively big only when h_θ is around 0.2 or in the range 0.8 to 0.9, and the interval of h_θ remains unchanged for poles of different lengths.

(3) For some given physical system, the control effect is not sensitive to the value of h_x. That is, we can realize the effective control when h_x is in some very long interval. Figure 10 shows that the control effect does not change much with h_x taking different values of the long interval.

Figure 13 also shows that the correlation coefficient between fitness and h_θ is relatively big when h_x is in the range 0.15 to 0.98.

However, the width of the interval varies with the length of pole. The longer the pole, the narrower the interval. The paper calls the interval of h_x as *h_x platform*.

(4) For poles of different lengths, there is much difference of h_x and little one of h_θ when we realize the most effective control. As shown in Table 7, the correlation

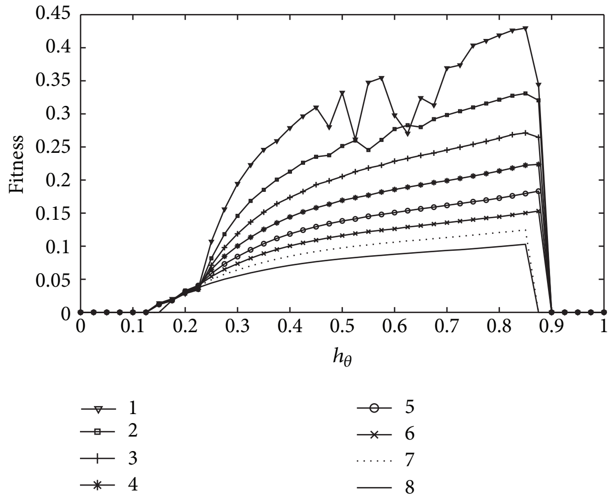

FIGURE 10: The maximum fitness varies with h_θ.

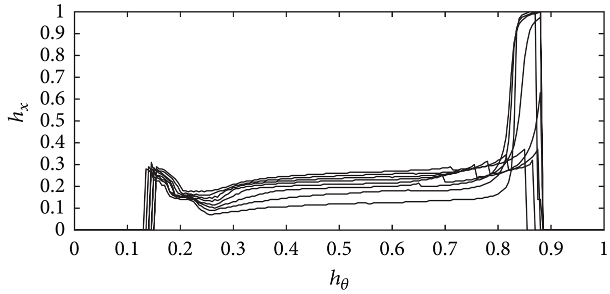

FIGURE 11: h_x varies with h_θ when the fitness is the maximum.

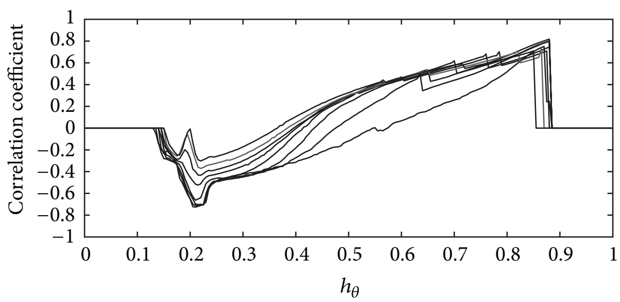

FIGURE 12: The correlation coefficient between fitness and h_x varies with h_θ.

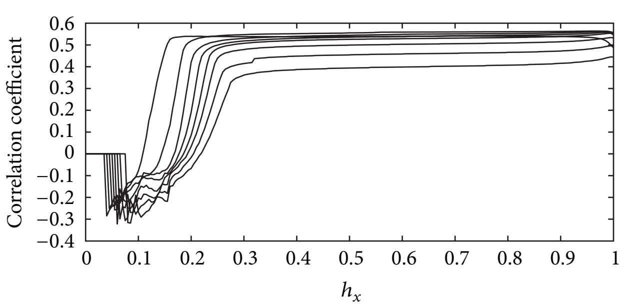

FIGURE 13: The correlation coefficient between fitness and h_θ varies with h_x.

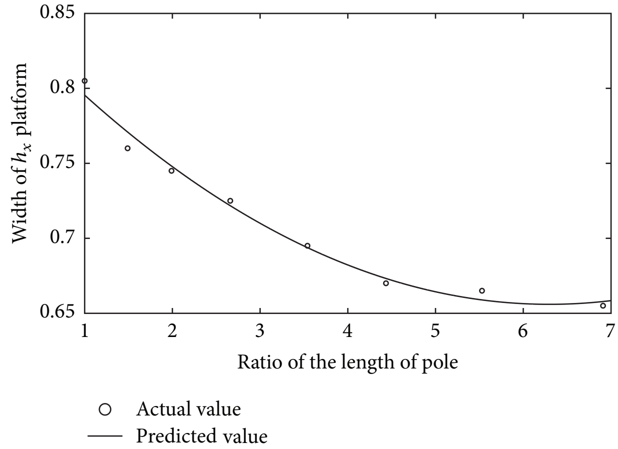

FIGURE 14: The width of h_x *platform* varies with the ratio of the length.

TABLE 7: The correlation coefficients between the maximum fitness and h_x, h_θ.

No.	m_1 (kg)	L_1 (m)	The correlation coefficient between the maximum fitness and h_x	The correlation coefficient between the maximum fitness and h_θ
1	0.0149	0.1	0.3801	0.9602
2	0.0216	0.149	0.3956	0.9503
3	0.0284	0.199	0.4104	0.9537
4	0.0378	0.266	0.4261	0.9604
5	0.0493	0.354	0.4285	0.9513
6	0.0621	0.4436	0.4294	0.9314
7	0.0773	0.553	0.4308	0.9125
8	0.0966	0.691	0.4327	0.8895

coefficient between the maximum fitness and h_x is small, but the one between the maximum fitness and h_θ is big. Table 6 shows that the longer the pole, the smaller the optimum h_x, and the optimum h_θ is in the range 0.8 to 0.9 with little difference.

From the third conclusion, we can know that the width of h_x *platform* varies with the length of pole. That is, the longer the pole, the smaller the width. For poles of different lengths, the corresponding h_x *platforms* are shown in Table 8. And the relation between the width of h_x *platform* and the length of pole and the one between the starting value of h_x *platform* and the length of pole can be obtained through fitting method according to the experimental results shown in Table 8.

Firstly, suppose that the smallest effective length of pole, that is, 0.1 m, is 1 unit. Then, the ratios of the other lengths to the smallest effective length of pole are shown in Table 8. The relation between the width of h_x*platform* and the ratio of the length of pole is depicted by means of the linear fit, which is as shown in (50) and Figure 14. Consider

$$y = 0.85314 - 0.062624x + 0.0049723x^2. \qquad (50)$$

Table 8: The corresponding h_x *platforms* for poles of different lengths.

No.	L_1 (m)	Ratio of the lengths of poles	The starting value of h_x *platform*	The finishing value of h_x *platform*	The width of h_x *platform*
1	0.1	1	0.175	0.98	0.805
2	0.149	1.49	0.22	0.98	0.76
3	0.199	1.99	0.235	0.98	0.745
4	0.266	2.66	0.25	0.975	0.725
5	0.354	3.54	0.26	0.955	0.695
6	0.4436	4.436	0.27	0.94	0.67
7	0.553	5.53	0.275	0.94	0.665
8	0.691	6.91	0.285	0.94	0.655

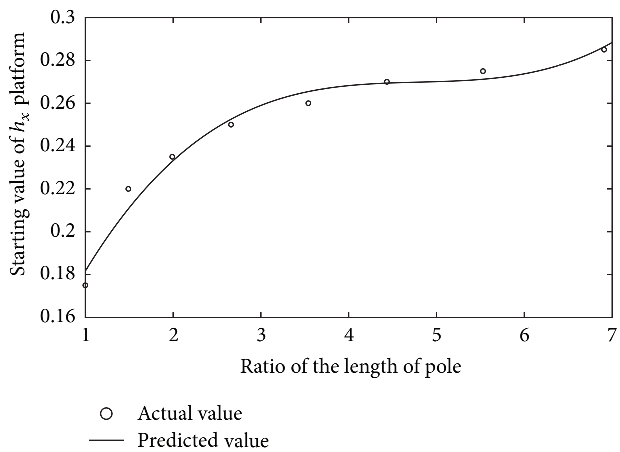

Figure 15: The starting value of h_x *platform* varies with the ratio of the length.

Similarly, the relation between the starting value of h_x *platform* and the ratio of the length of pole is depicted by means of the linear fit, which is as shown in (51) and Figure 15. Consider

$$y = 0.095132 + 0.10701x - 0.022054x^2 + 0.0015301x^3. \quad (51)$$

Hence, for the given physical system, when the length of pole takes different values in the previous controllable interval, we can calculate the corresponding interval of h_x *platform* by (50) and (51). Then, h_x can be any of the interval and h_θ should be any of the interval $[0.8, 0.9]$. Therefore, it does not need to optimize the control parameters again, and we can realize the effective control for the physical system with new length of pole.

4. Conclusion

Flexible logic method uses universal combinatorial operation model to describe the logic relation between E and EC, which are the fuzzy input variables of normal two-dimensional fuzzy controller. Universal combinatorial operation model is not a single fixed operator, but a continuous cluster of combinatorial operators determined by the *general correlation coefficient h* between propositions. In practical control application, according to the general correlation between propositions, we can take the corresponding one from the cluster to realize effective control for complex system.

However, in practical control application, how to determine the *general correlation coefficient h* of flexible logic control model is a problem for further studies. First, the conventional universal combinatorial operation model has been limited in the interval $[0, 1]$. Consequently, this paper studies a kind of universal combinatorial operation model based on any interval $[a, b]$. And some important theorems are given and proved, which provide a foundation for the flexible logic control method. For dealing reasonably with the complex relations of every factor in complex system, a kind of universal combinatorial operation model with unequal weights is put forward. Then, this paper has carried out the parametric analysis of flexible logic control model. And some research results have been given, which have important directive to determine the values of the general correlation coefficients in practical control application.

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Research Article

Optimal Investment Strategies for DC Pension with Stochastic Salary under the Affine Interest Rate Model

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We study the optimal investment strategies of DC pension, with the stochastic interest rate (including the CIR model and the Vasicek model) and stochastic salary. In our model, the plan member is allowed to invest in a risk-free asset, a zero-coupon bond, and a single risky asset. By applying the Hamilton-Jacobi-Bellman equation, Legendre transform, and dual theory, we find the explicit solutions for the CRRA and CARA utility functions, respectively.

1. Introduction

There are two radically different methods to design a pension fund: defined-benefit plan (hereinafter DB) and defined-contribution plan (hereinafter DC). In DB, the benefits are fixed in advance by the sponsor and the contributions are adjusted in order to maintain the fund in balance, where the associated financial risks are assumed by the sponsor agent; in DC, the contributions are fixed and the benefits depend on the returns on the assets of the fund, where the associated financial risks are borne by the beneficiary. Historically, DB is the more popular. However, in recent years, owing to the demographic evolution and the development of the equity markets, DC plays a crucial role in the social pension systems.

Our main objective in this paper is to find the optimal investment strategies for DC, which is a common model in the employment system. The paper extends the previous works of Cairns et al. [1] and Gao [2]. In particular, we consider the following framework: (i) the optimal investment strategies are derived with CARA and CRRA utility functions; (ii) the interest rate is affine (including the CIR model and the Vasicek model); (iii) the salary follows a general stochastic process.

Because the member of DC has some freedom in choosing the investment allocation of her pension fund in the accumulation phase, she has to solve an optimal investment strategies' problem. Traditionally, the usual method to deal with it has been the maximization of expected utility of final wealth. Consistently with the economics and financial literature, the most widely used utility function exhibits constant relative risk aversion (CRRA), that is, the power or logarithmic utility function (e.g., [1–5]). Some papers use the utility function that exhibits constant absolute risk aversion (CARA), that is, the exponential utility function (e.g., [6]). Some papers also adopt the CRRA and CARA utility functions simultaneously (e.g., [7, 8]). In this paper, we show the optimal investment strategies for DC pension with the CRRA and CARA utility functions.

The optimal portfolios for DC with stochastic interest rate have been widely discussed in the literatures. Some of them are by Boulier et al. [3], Battocchio and Menoncin [6], and Cairns et al. [1], where the interest rate is assumed to be of the Vasicek model. However, in the works of Deelstra et al. [4] and Gao [2], the interest rate has an affine structure, which includes the Cox-Ingersoll-Ross (CIR) model and the Vasicek model. In the Vasicek model, the volatility of interest rate is only a constant. It can generate a negative interest rate, which is not in accord with the facts. But in the CIR model, the volatility of interest rate is modified by the square of interest rate, which more tallies with practice. Obviously, the affine interest rate model does not only contain the Cox-Ingersoll-Ross (CIR) model and the Vasicek model, but also more accords with practice.

Meanwhile, Deelstra et al. [4] assumed that the stochastic interest rates followed the affine dynamics, described the contribution flow by a nonnegative, progressive measurable and square-integrable process, and then studied optimal investment strategies for different examples of guarantees and contributions. Battocchio and Menoncin [6] took into account two background risks (the salary risk and the inflation) in the Vasicek framework and analyzed in detail the behavior of the optimal portfolio with respect to salary and inflation. Cairns et al. [1] incorporated asset, salary (labor-income), and interest-rate risk (the Vasicek model), used the member's final salary as a numeraire, and then discussed various properties and characteristics of the optimal asset-allocation strategy both with and without the presence of nonhedgeable salary risk. However, except for them, the studies related with DC generally suppose that the salary is a constant, but the assumption is difficult to be accepted for the pension investment. In fact, the optimal investment for a pension fund involves quite a long period, generally from 20 to 40 years. The pension investment is considered to be a long-term investment problem. During the period, the salary switches violently; so it becomes crucial to take into account the salary risk. As a result, we consider the salary risk and use the member's final salary as a numeraire based on the work of Cairns et al. [1].

In addition, under the logarithmic utility function, Gao [2] just studied the portfolio problem of DC with the affine interest rate but did not consider the stochastic salary. The contribution of this paper: (i) extends the research of Gao [2] to the case of the power (CRRA) and exponential (CARA) utility functions under the stochastic salary; (ii) extends the research of Cairns et al. [1] to the case of the plan member with the CRRA and CARA utility functions under the affine interest rate model (including the CIR model and the Vasicek model). We consider that the financial market consists of three assets: a risk-less asset (i.e., cash), a zero-coupon bond, and a single risky asset (i.e., stock). Applying the maximum principle, we derive a nonlinear second-order partial differential equation (PDE) for the value function of the optimization problem. However, it is difficult to characterize the solution structure, especially under the framework of stochastic interest rates and stochastic salary. But the primary problem can be changed into a dual one by applying a Legendre transform. The transform methods can be found from the works of Xiao et al. [5] and Gao [2, 8].

The most novel feature of our research is the application of affine interest rate model and stochastic salary under the CRRA and CARA utility functions, which has not been reported in the existing literature. We assume that the term structure of the interest rates is affine, not a constant and the salary volatility is a hedgeable volatility whose risk source belongs to the set of the financial market risk sources. Consequently, a complicated nonlinear second-order partial differential equation is derived by using the methods of stochastic optimal control. However, we find that it is difficult to determine an explicit solution, and then we transform the primary problem into the dual one by applying a Legendre transform and derive a linear partial differential equation. Furthermore, we obtain the explicit solutions for the optimal strategies under the CRRA or CARA utility functions.

The rest of the paper is organized as follows. In Section 2, we introduce the mathematical model including the financial market, the stochastic salary, and the wealth process. In Section 3, we propose the optimization problems. In Section 4, we transform the nonlinear second partial differential equation into a linear partial differential equation by the Legendre transform and dual theory. In Section 5, we obtain the explicit solutions for the CRRA and CARA utility functions, respectively. In Section 6, we draw the conclusions.

2. Mathematical Model

In this section, we introduce the market structure and define the stochastic dynamics of the asset values and the salary.

We consider a complete and frictionless financial market which is continuously open over the fixed time interval $[0, T]$, where $T > 0$ denotes the retirement time of a representative shareholder.

2.1. The Financial Market. We suppose that the market is composed of three kinds of financial assets: a risk-free asset, a zero-coupon bond, and a single risky asset, and the investor can buy or sell continuously without incurring any restriction as short sales constraint or any trading cost. For the sake of simplicity, we will only consider a risky asset which can indeed represent the index of the stock market.

Let us begin with a complete probability space (Ω, F, P), where Ω is the real space, and P is the probability measure. $\{W_r(t), W_s(t) : t \geq 0\}$ is a standard, two-dimensional Brownian motion defined on a complete probability space (Ω, F, P). The filtration $F = \{F_t\}_{t\in[0,T]}$ is a right continuous filtration of sigma-algebras on this space and denotes the information structure generated by the Brownian motions.

We denote the price of the risk-free asset (i.e., cash) at time t by $S_0(t)$, which evolves according to the following equation:

$$dS_0(t) = r(t) S_0(t)\, dt, \quad S_0(0) = 1, \tag{1}$$

where the dynamics of the short interest rate process $r(t)$ are described by the following stochastic differential equation:

$$\begin{aligned} dr(t) &= (a - br(t))\, dt - \sigma_r dW_r(t), \\ \sigma_r &= \sqrt{k_1 r(t) + k_2}, \quad t \geq 0, \end{aligned} \tag{2}$$

with the coefficients $a, b, r(0),\ k_1$, and k_2 being positive real constants.

Notes that the dynamics recover, as a special case, the Vasicek [9] (resp., Cox et al. [10]) dynamics, when k_1 (resp., k_2) is equal to zero. So under these dynamics, the term structure of the interest rates is affine, which has been studied by Duffie and Kan [11], Deelstra et al. [4], and Gao [2].

We assume that the price of the risky asset is a continuous time stochastic process. We denote the price of the risky asset (i.e., stock) at time t by $S(t), t \geq 0$. The dynamics of $S(t)$ are given by

$$\frac{dS(t)}{S(t)} = r(t)\,dt + \sigma_s\left(dW_s(t) + \lambda_1 dt\right) + \eta_1\sigma_r\left(dW_r(t) + \lambda_2\sigma_r dt\right), \quad S(0) = S_0, \tag{3}$$

with λ_1, λ_2 (resp., σ_s, η_1) being constants (resp., positive constants) (see Deelstra et al. [4] and Gao [2]). Here, the two Brownian motions, $W_r(t)$ and $W_s(t)$, are supposed to be orthogonal.

The last asset is a zero-coupon bond with maturity T, whose price at time t is denoted by $B(t,T), t \geq 0$, which is described by the following stochastic differential equation (c.f. [2, 4]):

$$\frac{dB(t,T)}{B(t,T)} = r(t)\,dt + \sigma_B(T-t, r(t)) \times \left(dW_r(t) + \lambda_2\sigma_r dt\right), \quad B(T,T) = 1, \tag{4}$$

where $\sigma_B(T-t, r(t)) = f(T-t)\sigma_r$ with

$$f(t) = \frac{2\left(e^{mt}-1\right)}{m - (b - k_1\lambda_2) + e^{mt}(m + b - k_1\lambda_2)}, \qquad m = \sqrt{(b - k_1\lambda_2)^2 + 2k_1}. \tag{5}$$

2.2. The Stochastic Salary. Based on the works of Deelstra et al. [4], Battocchio and Menoncin [6], and Cairns et al. [1], we denote the salary at time t by $L(t)$ which is described by

$$\frac{dL(t)}{L(t)} = \mu_L(t, r(t))\,dt + \eta_2\sigma_r dW_r(t) + \eta_3\sigma_s dW_s(t), \quad L(0) = L_0, \tag{6}$$

where η_2, η_3 are real constants, which are two volatility scale factors measuring how the risk sources of interest rate and stock affect the salary. That is to say, the salary volatility is supposed to a hedgeable volatility whose risk source belongs to the set of the financial market risk sources. This assumption is in accordance with that of Deelstra et al. [4], but is different from those of Battocchio and Menoncin [6] and Cairns et al. [1] who also assumed that the salary was affected by nonhedgeable risk source (i.e., non-financial market). Moreover, we assume that the instantaneous mean of the salary is such that $\mu_L(t, r(t)) = r(t) + m_L$, where m_L is a real constant.

2.3. Pension Wealth Process. According to the viewpoint of Cairns et al. [1], we consider that the contributions are continuously into the pension fund at the rate of $kL(t)$. Let V_t denote the wealth of pension fund at time $t \in [0,T]$. $\pi_B(t)$ and $\pi_S(t)$ are denoted, respectively, by the proportion of the pension fund invested in the bond and the stock; so $\pi_0(t) = 1 - \pi_B(t) - \pi_S(t)$ is the proportion of the pension fund invested in the risk-free asset. The dynamics of the pension wealth are given by

$$\begin{aligned} dV(t) = {} & \left(1 - \pi_B - \pi_S\right) V(t) \frac{dS_0(t)}{S_0(t)} \\ & + \pi_B V(t) \frac{dB(t,T)}{B(t,T)} \\ & + \pi_s V(t) \frac{dS(t)}{S(t)} + kL(t)\,dt, \end{aligned} \tag{7}$$

where $V(0) = V_0$ stands for an initial wealth.

Taking into (1), (3), and (4), the evolution of pension wealth can be rewritten as

$$\begin{aligned} dV(t) = {} & V(t)\left(r(t) + \pi_B\lambda_2\sigma_r\sigma_B + \pi_S\left(\lambda_1\sigma_S + \lambda_2\eta_1\sigma_r^2\right)\right)dt \\ & + kL(t)\,dt + V(t)\left(\pi_B\sigma_B + \pi_S\eta_1\sigma_r\right)dW_r(t) \\ & + V(t)\,\pi_S\sigma_S dW_s(t). \end{aligned} \tag{8}$$

At the time of retirement, the plan member will be concerned about the preservation of his standard of living so he will be interested in his retirement income relative to his preretirement salary [1]. Considering the plan member's salary as a numeraire, we define a new state variable $X(t) = V(t)/L(t)$ (i.e., the relative wealth).

Taking into (6) and (8), by applying product law and Ito's formula, the stochastic differential equation for $X(t)$ is

$$\begin{aligned} dX(t) = {} & X(t)\Big[r(t) - \mu_L + \eta_2^2\sigma_r^2 + \eta_3^2\sigma_S^2 \\ & + \pi_B\sigma_r\sigma_B\left(\lambda_2 - \eta_2\right) \\ & + \pi_S\left(\lambda_1\sigma_S + \eta_3\sigma_S^2 + \lambda_2\eta_1\sigma_r^2 - \eta_1\eta_2\sigma_r^2\right)\Big]dt + kdt \\ & + X(t)\left(\pi_B\sigma_B + \pi_S\eta_1\sigma_r - \eta_2\sigma_r\right)dW_r(t) \\ & + X(t)\left(\pi_S - \eta_3\right)\sigma_S dW_s(t), \\ X(0) = {} & \frac{V(0)}{L(0)} = \frac{V_0}{L_0}. \end{aligned} \tag{9}$$

In the remainder, therefore, we will focus on $X(t)$ alone.

3. The Optimization Program

The plan member will retire at time T and is risk averse; so the utility function $U(x)$ is typically increasing and concave ($U''(x) < 0$). In this section, we are interested in maximizing the utility of the plan member's terminal relative wealth.

Let us denote a strategy π_t which is described by a dynamic process $(\pi_B(t), \pi_S(t))$. For a strategy π_t, we define the utility attained by the plan member from state x at time t as

$$H_{\pi_t}(t, r, x) = E_{\pi_t}\left[U(X(T)) \mid r(t) = r, X(t) = x\right]. \tag{10}$$

Our objective is to find the optimal value function:

$$H(t,r,x) = \sup_{\pi_t \in \pi} H_{\pi_t}(t,r,x), \tag{11}$$

and the optimal strategy is $\pi_t^* = (\pi_B^*(t), \pi_S^*(t))$ such that $H_{\pi_t^*}(t,r,x) = H(t,r,x)$.

The Hamilton-Jacobi-Bellman (HJB) equation associated with the optimization problem is

$$\begin{aligned} & H_t + (a-br) H_r + \frac{1}{2}\sigma_r^2 H_{rr} \\ & + \max_{\pi_t \in \pi} \{x(\alpha_1 + \pi_B \alpha_2 + \pi_S \alpha_3) H_x + kH_x \\ & + \frac{1}{2}x^2 (\pi_B \sigma_B + \pi_S \eta_1 \sigma_r - \eta_2 \sigma_r)^2 H_{xx} \\ & + \frac{1}{2}x^2 (\pi_S - \eta_3)^2 \sigma_S^2 H_{xx} \\ & - x\sigma_r (\pi_B \sigma_B + \pi_S \eta_1 \sigma_r - \eta_2 \sigma_r) H_{rx}\} = 0, \end{aligned} \tag{12}$$

with

$$\begin{aligned} \alpha_1 &= r - \mu_L + \eta_2^2 \sigma_r^2 + \eta_3^2 \sigma_S^2, \\ \alpha_2 &= \sigma_r \sigma_B (\lambda_2 - \eta_2), \\ \alpha_3 &= \lambda_1 \sigma_S + \eta_3 \sigma_S^2 + \lambda_2 \eta_1 \sigma_r^2 - \eta_1 \eta_2 \sigma_r^2, \\ H(T,r,x) &= U(x), \end{aligned} \tag{13}$$

where $H_t, H_r, H_x, H_{rx}, H_{rr}$, and H_{xx} denote partial derivatives of first and second orders with respect to time, short interest rate, and relative wealth.

The first-order maximizing conditions for the optimal strategies π_B^* and π_S^* are

$$\begin{aligned} & \alpha_2 H_x + x\sigma_B (\pi_B^* \sigma_B + \pi_S^* \eta_1 \sigma_r - \eta_2 \sigma_r) H_{xx} - \sigma_r \sigma_B H_{rx} = 0, \\ & \alpha_3 H_x + x\eta_1 \sigma_r (\pi_B^* \sigma_B + \pi_S^* \eta_1 \sigma_r - \eta_2 \sigma_r) H_{xx} \\ & \quad + x\sigma_S (\pi_S^* - \eta_3) H_{xx} - \eta_1 \sigma_r^2 H_{rx} = 0. \end{aligned} \tag{14}$$

We have

$$\begin{aligned} \pi_S^* &= \eta_3 - \frac{\lambda_1 + \eta_3 \sigma_S^2}{x\sigma_S} \frac{H_x}{H_{xx}}, \\ \pi_B^* &= \frac{\sigma_r (\eta_2 - \eta_1 \eta_3)}{\sigma_B} + \frac{\alpha_4 \sigma_r}{x\sigma_B} \frac{H_x}{H_{xx}} + \frac{\sigma_r}{x\sigma_B} \frac{H_{rx}}{H_{xx}}, \\ \alpha_4 &= \frac{(\eta_2 \sigma_S + \lambda_1 \eta_1 + \eta_1 \eta_3 \sigma_S^2 - \lambda_2 \sigma_S)}{\sigma_S}, \end{aligned} \tag{15}$$

Putting this in (12), we obtain a partial differential equation (PDE) for the value function H:

$$\begin{aligned} & H_t + (a-br) H_r + \frac{1}{2}\sigma_r^2 H_{rr} + (k + x\beta_0) H_x \\ & + \left(\beta_1 - \frac{1}{2}(\lambda_2 - \eta_2)^2 \sigma_r^2\right) \frac{H_x^2}{H_{xx}} \\ & + (\lambda_2 - \eta_2) \sigma_r^2 \frac{H_x H_{rx}}{H_{xx}} - \frac{1}{2}\sigma_r^2 \frac{H_{rx}^2}{H_{xx}} = 0, \\ & \beta_0 = \lambda_1 \eta_3 \sigma_S + \lambda_2 \eta_2 \sigma_r^2 + 2\eta_3^2 \sigma_S^2 - m_L, \\ & \beta_1 = \eta_3 \sigma_S \left(\frac{1}{2}\eta_3 \sigma_S^3 - \eta_3 \sigma_S^2 - \lambda_1\right) - \frac{1}{2}\lambda_1^2. \end{aligned} \tag{16}$$

with $H(T,r,x) = U(x)$.

Here, we notice that the stochastic control problem described in the previous section has been transformed into a PDE. The problem is now to solve (16) for the value function H and replace it in (15) in order to obtain the optimal investment strategies.

4. The Legendre Transform

In this section, we transform the non-linear second partial differential equation into a linear partial differential equation via the Legendre transform and dual theory.

Theorem 1. *Let $f : R^n \to R$ be a convex function, for $z > 0$, define the Legendre transform:*

$$L(z) = \max_x \{f(x) - zx\}. \tag{17}$$

The function $L(z)$ is called the Legendre dual of the function $f(x)$ (c.f. [12]).

If $f(x)$ is strictly convex, the maximum in the above equation will be attained at just one point, which we denote by x_0. It is attained at the unique solution to the first-order condition, namely, $df(x)/dx - z = 0$.

So, we may rewrite $L(z) = f(x_0) - zx_0$.

According to Theorem 1, we can take advantage of the assumed convexity of the value function $H(t,r,x)$ to define the Legendre transform:

$$\begin{aligned} & \widehat{H}(t,r,z) \\ & = \sup_{x>0} \{H(t,r,x) - zx \mid 0 < x < \infty\}, \quad 0 < t < T, \end{aligned} \tag{18}$$

where $z > 0$ denotes the dual variable to x, which is the same as those of Xiao et al. [5] and Gao [2, 8].

The value of x where this optimum is attained is denoted by $g(t,r,z)$, so that

$$\begin{aligned} & g(t,r,z) \\ & = \inf_{x>0} \left\{x \mid H(t,r,x) \geq zx + \widehat{H}(t,r,z)\right\}, \quad 0 < t < T. \end{aligned} \tag{19}$$

The two functions $g(t, r, z)$ and $\widehat{H}(t, r, z)$ are closely related, and we will refer to either one of them as the dual of H. In this paper, we will work mainly with the function g, as it is easier to compute numerically and suffices for the purpose of computing optimal investment strategies.

This leads to

$$\begin{aligned} \widehat{H}(t,r,z) &= H(t,r,g) - zg, \\ g(t,r,z) &= x, \qquad H_x = z. \end{aligned} \tag{20}$$

So the function $\widehat{H}$ is related to g by $g = -\widehat{H}_z$.

At the terminal time, we denote

$$\begin{aligned} \widehat{U}(z) &= \sup_{v>0} \{U(v) - zv \mid 0 < v < \infty\}, \\ G(z) &= \sup_{v>0} \left\{v \mid U(v) \geq zv + \widehat{U}(z)\right\}. \end{aligned} \tag{21}$$

As a result, $G(z) = (U')^{-1}(z)$.

Generally speaking, G is referred to as the inverse of marginal utility. Note that $H(T, r, x) = U(x)$, and then at the terminal time T, we can define

$$\begin{aligned} g(T,r,z) &= \inf_{x>0} \left\{x \mid U(x) \geq zx + \widehat{H}(T,r,z)\right\}, \\ \widehat{H}(T,r,z) &= \sup_{x>0} \{U(x) - zx\}. \end{aligned} \tag{22}$$

So $g(T, r, z) = (U')^{-1}(z)$.

By differentiating (20) with respect to t, r, and z, the transformation rules for the derivatives of the value function H and the dual function $\widehat{H}$ can be given by (e.g., [2, 5, 8, 12]):

$$\begin{aligned} &H_x = z, \qquad H_t = \widehat{H}_t, \\ &H_r = \widehat{H}_r, \qquad H_{rr} = \widehat{H}_{rr} - \frac{\widehat{H}_{rz}^2}{\widehat{H}_{zz}}, \\ &H_{rx} = -\frac{\widehat{H}_{rz}}{\widehat{H}_{zz}}, \qquad H_{xx} = -\frac{1}{\widehat{H}_{zz}}. \end{aligned} \tag{23}$$

Substituting the expression (23), we rewrite (16) and obtain the following partial differential equation:

$$\begin{aligned} &\widehat{H}_t + (a - br)\widehat{H}_r + \frac{1}{2}\sigma_r^2 \widehat{H}_{rr} + (k + x\beta_0) z \\ &\quad - \left(\beta_1 - \frac{1}{2}(\lambda_2 - \eta_2)^2 \sigma_r^2\right) z^2 \widehat{H}_{zz} \\ &\quad + (\lambda_2 - \eta_2)\sigma_r^2 z \widehat{H}_{rz} = 0, \\ &\beta_0 = \lambda_1 \eta_3 \sigma_S + \lambda_2 \eta_2 \sigma_r^2 + 2\eta_3^2 \sigma_S^2 - m_L, \\ &\beta_1 = \eta_3 \sigma_S \left(\frac{1}{2}\eta_3 \sigma_S^3 - \eta_3 \sigma_S^2 - \lambda_1\right) - \frac{1}{2}\lambda_1^2. \end{aligned} \tag{24}$$

Combining with $x = g = -\widehat{H}_z$ and differentiating the above equation for $\widehat{H}$ with respect to z, we derive

$$\begin{aligned} &g_t + (a - br) g_r + \frac{1}{2}\sigma_r^2 g_{rr} - k - \beta_0 g - \beta_0 z g_z \\ &\quad + (\lambda_2 - \eta_2)\sigma_r^2 g_r + (\lambda_2 - \eta_2)\sigma_r^2 z g_{rz} \\ &\quad - 2\left(\beta_1 - \frac{1}{2}(\lambda_2 - \eta_2)^2 \sigma_r^2\right) z g_z \\ &\quad - \left(\beta_1 - \frac{1}{2}(\lambda_2 - \eta_2)^2 \sigma_r^2\right) z^2 g_{zz} = 0, \\ &\beta_0 = \lambda_1 \eta_3 \sigma_S + \lambda_2 \eta_2 \sigma_r^2 + 2\eta_3^2 \sigma_S^2 - m_L, \\ &\beta_1 = \eta_3 \sigma_S \left(\frac{1}{2}\eta_3 \sigma_S^3 - \eta_3 \sigma_S^2 - \lambda_1\right) - \frac{1}{2}\lambda_1^2. \end{aligned} \tag{25}$$

Here, we notice that the non-linear second-order partial differential equation (16) has been transformed into a linear partial differential equation (25) by using the Legendre transform and dual theory. Under the given utility function, it is easy to find the solution of (25) by the classical variable decomposition approach.

Similarly, we can compute the optimal investment strategies as the feedback formulas in terms of derivatives of the value function. In terms of the dual function g, they are given by

$$\begin{aligned} \pi_0(t) &= 1 - \pi_B(t) - \pi_S(t), \\ \pi_S^* &= \eta_3 + \frac{\lambda_1 + \eta_3 \sigma_S^2}{x\sigma_S} z\widehat{H}_{zz} = \eta_3 - \frac{\lambda_1 + \eta_3 \sigma_S^2}{x\sigma_S} z g_z, \\ \pi_B^* &= \frac{(\eta_2 - \eta_1 \eta_3)}{f(T-t)} - \frac{\alpha_4 z \widehat{H}_{zz}}{x f(T-t)} + \frac{\widehat{H}_{rz}}{x f(T-t)} \\ &= \frac{(\eta_2 - \eta_1 \eta_3)}{f(T-t)} + \frac{\alpha_4}{x} z g_z - \frac{g_r}{x f(T-t)}, \\ \alpha_4 &= \frac{\left(\eta_2 \sigma_S + \lambda_1 \eta_1 + \eta_1 \eta_3 \sigma_S^2 - \lambda_2 \sigma_S\right)}{\sigma_S}, \\ f(t) &= \frac{2\left(e^{mt} - 1\right)}{m - (b - k_1 \lambda_2) + e^{mt}(m + b - k_1 \lambda_2)} \\ m &= \sqrt{(b - k_1 \lambda_2)^2 + 2k_1}. \end{aligned} \tag{26}$$

The problem is now to solve the linear partial differential equation (25) for g and to replace these solutions in (26) in order to obtain the optimal strategies.

5. Optimal Investment Strategies with Some Specific Utilities

This section provides the explicit solutions for the CRRA and CARA utility functions.

5.1. The Explicit Solution for The CRRA Utility Function. Assume that the plan member takes a power utility function

$$U(x)=\frac{x^p}{p},\quad (\text{with } p<1, p\neq 0). \tag{27}$$

The relative risk aversion of a decision maker with the utility described in (27) is constant, and (27) is a CRRA utility.

According to $g(T,r,z)=(U')^{-1}(z)$ and the CRRA utility function, we obtain

$$g(T,r,z)=z^{1/(p-1)}. \tag{28}$$

Therefore, we conjecture a solution to (25) with the following form:

$$g(t,r,z)=z^{1/(p-1)}h(t,r)+a(t), \tag{29}$$

with the boundary conditions given by $a(T)=0$, $h(T,r)=1$.

Then,

$$\begin{aligned}
&g_t=h_t z^{1/(p-1)}+a'(t), \qquad g_r=h_r z^{1/(p-1)},\\
&g_z=-\frac{h}{1-p}z^{(1/(p-1))-1}, \qquad g_{rr}=h_{rr}z^{1/(p-1)},\\
&g_{rz}=-\frac{h_r}{1-p}z^{(1/(p-1))-1},\\
&g_{zz}=\frac{(2-p)h}{(1-p)^2}z^{(1/(p-1))-2}.
\end{aligned} \tag{30}$$

Introducing these derivatives in (25), we derive

$$\begin{aligned}
&\left\{h_t+(a-br)h_r-\frac{(\lambda_2-\eta_2)p\sigma_r^2}{1-p}h_r+\frac{1}{2}\sigma_r^2h_{rr}\right.\\
&\quad\left.+\frac{\beta_0 p}{1-p}h-\frac{ph}{(1-p)^2}\left(\beta_1-\frac{1}{2}(\lambda_2-\eta_2)^2\sigma_r^2\right)\right\}z^{1/(p-1)}\\
&\quad+a'(t)-\beta_0a(t)-k=0,\\
&\beta_0=\lambda_1\eta_3\sigma_S+\lambda_2\eta_2\sigma_r^2+2\eta_3^2\sigma_S^2-m_L,\\
&\beta_1=\eta_3\sigma_S\left(\frac{1}{2}\eta_3\sigma_S^3-\eta_3\sigma_S^2-\lambda_1\right)-\frac{1}{2}\lambda_1^2.
\end{aligned} \tag{31}$$

We can split (31) into two equations in order to eliminate the dependence on $z^{1/(p-1)}$:

$$a'(t)-\beta_0a(t)-k=0, \tag{32}$$

$$\begin{aligned}
&h_t+(a-br)h_r-\frac{(\lambda_2-\eta_2)p\sigma_r^2}{1-p}h_r+\frac{1}{2}\sigma_r^2h_{rr}\\
&\quad+\frac{\beta_0 p}{1-p}h-\frac{ph}{(1-p)^2}\left(\beta_1-\frac{1}{2}(\lambda_2-\eta_2)^2\sigma_r^2\right)=0.
\end{aligned} \tag{33}$$

Taking into account the boundary condition $a(T)=0$, the solution to (32) is

$$\begin{aligned}
&a(t)=-k\left(\frac{1-e^{-\beta_0(T-t)}}{\beta_0}\right),\\
&\beta_0=\lambda_1\eta_3\sigma_S+\lambda_2\eta_2\sigma_r^2+2\eta_3^2\sigma_S^2-m_L,
\end{aligned} \tag{34}$$

where $\bar{a}_{\overline{T-t}|}=(1-e^{-\beta_0(T-t)})/\beta_0$ is a continuous annuity of duration $T-t$, and β_0 is the continuous technical rate.

Noting that (33) is a linear second-order PDE, we find the solution by the classical variable decomposition approach.

Let

$$h(t,r)=A(t)e^{B(t)r} \tag{35}$$

with the boundary conditions: $A(T)=1$, $B(T)=0$. Introducing this in (33), we obtain

$$\begin{aligned}
&\frac{A_t}{A}+\frac{a-((\lambda_2-\eta_2)k_1+a)p}{1-p}B+\frac{1}{2}k_2B^2\\
&\quad+\frac{p(\beta_0-\beta_1-p\beta_0)}{(1-p)^2}+\frac{(\lambda_2-\eta_2)^2pk_2}{2(1-p)^2}\\
&\quad+r\left(B_t-\frac{b+((\lambda_2-\eta_2)k_1-b)p}{1-p}B\right.\\
&\quad\left.+\frac{1}{2}k_1B^2+\frac{(\lambda_2-\eta_2)^2pk_1}{2(1-p)^2}\right)=0,\\
&\beta_0=\lambda_1\eta_3\sigma_S+\lambda_2\eta_2\sigma_r^2+2\eta_3^2\sigma_S^2-m_L,\\
&\beta_1=\eta_3\sigma_S\left(\frac{1}{2}\eta_3\sigma_S^3-\eta_3\sigma_S^2-\lambda_1\right)-\frac{1}{2}\lambda_1^2.
\end{aligned} \tag{36}$$

We can decompose (36) into two conditions in order to eliminate the dependence on r and t:

$$\begin{aligned}
&\frac{A_t}{A}+\frac{a-((\lambda_2-\eta_2)k_1+a)p}{1-p}B+\frac{1}{2}k_2B^2\\
&\quad+\frac{p(\beta_0-\beta_1-p\beta_0)}{(1-p)^2}+\frac{(\lambda_2-\eta_2)^2pk_2}{2(1-p)^2}=0,\\
&B_t-\frac{b+((\lambda_2-\eta_2)k_1-b)p}{1-p}B\\
&\quad+\frac{1}{2}k_1B^2+\frac{(\lambda_2-\eta_2)^2pk_1}{2(1-p)^2}=0.
\end{aligned} \tag{37}$$

Taking into account the boundary conditions, the solutions to (37) are

$$B(t) = \frac{m_1 - m_1 e^{(1/2)k_1(m_1-m_2)(T-t)}}{1-(m_1/m_2)\ e^{(1/2)k_1(m_1-m_2)(T-t)}},$$
$$A(t) = \exp\left\{\frac{((\lambda_2-\eta_2)k_1+a)p-a}{1-p}\int B(t)\,dt - \frac{1}{2}k_2\int B^2(t)\,dt - \frac{p(\beta_0-\beta_1-p\beta_0)}{(1-p)^2}t + C\right\}, \quad A(T)=1, \tag{38}$$

where

$$m_{1,2} = \left(b + ((\lambda_2-\eta_2)k_1 - b)p \pm\sqrt{(b+((\lambda_2-\eta_2)k_1-b)p)^2-(\lambda_2-\eta_2)^2k_1^2p}\right) \times ((1-p)k_1)^{-1}. \tag{39}$$

From the above calculation, we finally obtain the optimal investment strategies under the CRRA utility.

Proposition 2. *The optimal investment strategies are given by*

$$\pi_0(t) = 1-\pi_B(t)-\pi_S(t),$$
$$\pi_S^* = \eta_3 + \frac{\lambda_1+\eta_3\sigma_S^2}{(1-p)\sigma_S}I(t,r), \tag{40}$$
$$\pi_B^* = \frac{1}{f(T-t)}\left\{(\eta_2-\eta_1\eta_3) - \frac{\alpha_4}{1-p}I(t,r)J(t)\right\},$$

where

$$I(t,r) = 1+\frac{kl}{v}\bar{a}_{\overline{T-t}|}, \tag{41}$$
$$\bar{a}_{\overline{T-t}|} = \frac{1-e^{-\beta_0(T-t)}}{\beta_0},$$
$$J(t) = 1+\frac{(1-p)B(t)}{\alpha_4},$$
$$B(t) = \frac{m_1 - m_1 e^{(1/2)k_1(m_1-m_2)(T-t)}}{1-(m_1/m_2)\ e^{(1/2)k_1(m_1-m_2)(T-t)}},$$
$$f(t) = \frac{2(e^{mt}-1)}{m-(b-k_1\lambda_2)+e^{mt}(m+b-k_1\lambda_2)},$$
$$m = \sqrt{(b-k_1\lambda_2)^2+2k_1},$$
$$\beta_0 = \lambda_1\eta_3\sigma_S+\lambda_2\eta_2\sigma_r^2+2\eta_3^2\sigma_S^2-m_L,$$
$$\alpha_4 = \frac{(\eta_2\sigma_S+\lambda_1\eta_1+\eta_1\eta_3\sigma_S^2-\lambda_2\sigma_S)}{\sigma_S}, \tag{42}$$

$$m_{1,2} = \left(b + ((\lambda_2-\eta_2)k_1 - b)p \pm\sqrt{(b+((\lambda_2-\eta_2)k_1-b)p)^2-(\lambda_2-\eta_2)^2k_1^2p}\right) \times ((1-p)k_1)^{-1}. \tag{43}$$

Remark 3. Note that the power utility function (27) will degenerate into a logarithmic utility function $U(x)=\ln x$ as the limit $p\to 0$ (e.g., [7, 13, 14]). Meanwhile, in (6), if $\eta_2 = 0, \eta_3 = 0$, the salary is not stochastic; so the contributions are not stochastic. If we further assume that $l=1$, the model is the same as the model of Gao [2]. From Proposition 2, we find that as the limit $p\to 0$, the coefficients $m_{1,2}$ will reduce to $2b/k_1$ and zero, respectively. In this case, the coefficients $B(t)$ and $J(t)$ will, respectively, reduce to zero and one. As a result, the optimal investment strategies for a logarithmic utility function are

$$\pi_S^* = \frac{\lambda_1}{\sigma_S}\left(1+\frac{k}{v}\bar{a}_{\overline{T-t}|r}\right),$$
$$\pi_B^* = \frac{\sigma_r(\lambda_2\sigma_S-\lambda_1\eta_1)}{\sigma_B\sigma_S}\left(1+\frac{k}{v}\bar{a}_{\overline{T-t}|r}\right), \tag{44}$$

where π_S^* is the same as the result of Gao [2], but π_B^* is different from that result because Gao [2] made mistakes in calculation.

In this section, to make it easier for us to discuss the parameters' effect on the optimal investment strategies, we suppose that $\beta_0 > 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$, where the assumption is generally in line with reality.

Lemma 4. *Consider*

$$I(t,r) > 0, \qquad \frac{dI(t,r)}{dt} < 0, \qquad \frac{d\pi_S^*}{dt} < 0. \tag{45}$$

Proof. Since $p < 1$, $k > 0$, $\beta_0 > 0$, $\lambda_1 > 0$, $\eta_3 > 0$, and $\sigma_S > 0$, by differentiating $I(t,r)$ with the respect to t, we have

$$\bar{a}_{\overline{T-t}|} = \frac{1 - e^{-\beta_0 (T-t)}}{\beta_0} > 0, \quad I(t,r) > 0,$$
$$\frac{dI(t,r)}{dt} = \frac{kl}{v}\frac{d\bar{a}_{\overline{T-t}|}}{dt} = -\frac{kl}{v}e^{-\beta_0(T-t)} < 0, \tag{46}$$
$$\frac{d\pi_S^*}{dt} = \frac{\lambda_1 + \eta_3 \sigma_S^2}{(1-p)\sigma_S}\frac{dI(t,r)}{dt} < 0.$$

□

Lemma 5. *Consider*

$$\frac{dJ(t)}{dt} = \begin{cases} > 0, & (p < 0), \\ < 0, & (0 < p < 1), \end{cases}$$
$$J(t) = \begin{cases} \leq 1, & (p < 0), \\ \geq 1, & (0 < p < 1). \end{cases} \tag{47}$$

Proof. Since $p < 1$, we have

$$m_1 \times m_2 = \frac{(\lambda_2 - \eta_2)^2 p}{(1-p)^2} = \begin{cases} < 0, & (p < 0), \\ > 0, & (0 < p < 1). \end{cases} \tag{48}$$

Here, we just consider the condition of $\alpha_4 > 0$. Differentiating $B(t)$ with the respect to t, we have

$$\frac{dB(t)}{dt} = \frac{-(m_1 - m_2)^2 (k_1 m_1 / 2m_2) e^{(1/2)k_1(m_1 - m_2)(T-t)}}{\left(1 - (m_1/m_2)\ e^{(1/2)k_1(m_1-m_2)(T-t)}\right)^2}$$
$$= \begin{cases} > 0, & (p < 0) \\ < 0, & (0 < p < 1), \end{cases}$$
$$\frac{dJ(t)}{dt} = \frac{1-p}{\alpha_4}\frac{dB(t)}{dt} = \begin{cases} > 0, & (p < 0), \\ < 0, & (0 < p < 1). \end{cases} \tag{49}$$

In addition, noting that $B(T) = 0$ and $J(T) = 1$, we get

$$J(t) = \begin{cases} \leq 1, & (p < 0), \\ \geq 1, & (0 < p < 1). \end{cases} \tag{50}$$

□

Lemma 6. *Consider*

$$f(T-t) > 0, \qquad \frac{df(T-t)}{dt} < 0. \tag{51}$$

Proof. Since $T - t > 0$, and $k_1 > 0$, we have,

$$m = \sqrt{(b - k_1\lambda_2)^2 + 2k_1} > |b - k_1\lambda_2| > 0,$$
$$e^{m(T-t)} > 1,$$
$$f(T-t) = \frac{2\left(e^{m(T-t)} - 1\right)}{m - (b - k_1\lambda_2) + e^{m(T-t)}(m + b - k_1\lambda_2)} > 0,$$
$$\frac{df(T-t)}{dt} = -\frac{4m^2 e^{m(T-t)}}{\left(m - (b - k_1\lambda_2) + e^{m(T-t)}(m + b - k_1\lambda_2)\right)^2} < 0. \tag{52}$$

□

Lemma 7. *Whether $d\pi_B^*/dt$ is positive or negative or neither is not established, and it is affected by the coefficient of relative risk aversion p and the other parameters.*

Proof. By differentiating π_B^* with the respect to t, we have

$$\frac{d\pi_B^*}{dt} = \frac{-1}{f^2(T-t)}\frac{df(T-t)}{dt} \times \left\{(\eta_2 - \eta_1\eta_3) - \frac{\alpha_4}{1-p}I(t,r)J(t)\right\} - \frac{\alpha_4}{(1-p)f(T-t)}\left\{I(t,r)\frac{dJ(t)}{dt} + \frac{dI(t,r)}{dt}J(t)\right\}. \tag{53}$$

On the bases of Lemmas 4 and 6, we get

$$I(t,r) > 0, \qquad \frac{dI(t,r)}{dt} < 0,$$
$$f(T-t) > 0, \qquad \frac{df(T-t)}{dt} < 0. \tag{54}$$

Meanwhile, based on Lemma 5, we get

$$\frac{dJ(t)}{dt} = \begin{cases} > 0, & (p < 0), \\ < 0, & (0 < p < 1), \end{cases}$$
$$J(t) = \begin{cases} \leq 1, & (p < 0), \\ \geq 1, & (0 < p < 1). \end{cases} \tag{55}$$

Therefore, whether $d\pi_B^*/dt$ is positive or negative or neither is very complicated. □

Lemma 8. *Consider*

$$\frac{d\pi_S^*}{dl} > 0, \qquad \frac{d\pi_B^*}{dl} = \begin{cases} -, & (p < 0), \\ < 0, & (0 < p < 1). \end{cases} \tag{56}$$

Proof. Since $p < 1$, $k > 0$, $\beta_0 > 0$, $\lambda_1 > 0$, $\eta_3 > 0$, and $\sigma_S > 0$, therefore

$$\bar{a}_{\overline{T-t}|} > 0, \qquad \frac{dI(t,r)}{dl} = \frac{k}{v}\bar{a}_{\overline{T-t}|} > 0$$
$$\frac{d\pi_S^*}{dl} = \frac{\lambda_1 + \eta_3\sigma_S^2}{(1-p)\sigma_S}\frac{dI(t,r)}{dl} > 0. \tag{57}$$

According to Lemmas 5 and 6, we get

$$J(t) = \begin{cases} \leq 1, & (p < 0), \\ \geq 1, & (0 < p < 1), \end{cases} \qquad f(T-t) > 0. \tag{58}$$

Similarly, we just consider the condition of $\alpha_4 > 0$. So,

$$\frac{d\pi_B^*}{dl} = -\frac{\alpha_4 J(t)}{(1-p)f(T-t)}\frac{dI(t,r)}{dl} = \begin{cases} -, & (p < 0), \\ < 0, & (0 < p < 1). \end{cases} \tag{59}$$

□

Remark 9. The parameter p is the coefficient of the relative risk aversion. Hence, the plan member would like to avoid risk strongly if they get high p.

Lemma 4 shows that the optimal proportion invested in stock π_S^* depends on the time t and is a monotone decreasing function with respect to time t, but the trend is not affected by p. The stock is regarded as high risk, whose purpose is to satisfy the risk appetite of the plan member and hedge the risk. So as the retirement date approaches, the risk appetite begins to decrease so that the optimal proportion invested in stock is monotonically decreasing. It is concluded that as the retirement date approaches, there is a gradual switch from high-risk investment (i.e., stock) into low-risk investment (i.e., cash and bonds).

Thus it can be seen that, as the retirement date approaches, the plan member will think more about how to invest between cash and bonds. However, Lemma 7 indicates that the effect of the time t on π_B^* depends on the risk aversion coefficient p and the other parameters under the power utility. Consequently, as the retirement date approaches, how to invest between cash and bonds mainly depends on the risk aversion coefficient p and the other parameters.

In agreement with Cairns et al. [1], instead of switching from high-risk assets into low-risk assets, in the stochastic interest rate framework, the optimal investment strategies involve a switch between different types of low-risk assets (i.e., cash and bonds).

Lemma 8 reveals that the optimal proportion invested in stock π_S^* is a monotone increasing function with respect to the salary numeraire l, which means that the plan member will be more reluctant to invest in stock when the salary numeraire l becomes larger, but the trend is not affected by p. However, the effect of l on the optimal proportion invested in bonds π_B^* depends on the risk aversion coefficient p under the power utility. When $0 < p < 1$, π_B^* is a monotone decreasing function with respect to l. Because the plan members would like to avoid risk strongly if they get high p, they invest in cash more as l increases. But when the risk aversion coefficient $p < 0$, π_B^* depends on the risk aversion coefficient p and the other parameters.

5.2. The Explicit Solution for The CARA Utility Function. Assume that the plan member takes an exponential utility function:

$$U(x) = -\frac{1}{q}e^{-qx}, \quad (\text{with } q > 0). \tag{60}$$

The absolute risk aversion of a decision maker with the utility described in (60) is constant, and (60) is a CARA utility.

According to $g(T,r,z) = (U')^{-1}(z)$ and the CARA utility function, we obtain

$$g(T,r,z) = -\frac{1}{q}\ln z. \tag{61}$$

So, we conjecture a solution to (25) with the following form:

$$g(t,r,z) = -\frac{1}{q}[b(t)(\ln z + m(t,r))] + a(t), \tag{62}$$

with the boundary conditions given by $b(T) = 1$, $a(T) = 0$, $m(T,s) = 0$.

Therefore,

$$g_t = -\frac{1}{q}\left[b'(t)(\ln z + m(t,r)) + b(t)m_t\right] + a'(t),$$
$$g_r = -\frac{1}{q}b(t)m_r, \qquad g_z = -\frac{b(t)}{qz}, \tag{63}$$
$$g_{zz} = \frac{b(t)}{qz^2}, \qquad g_{rr} = -\frac{1}{q}b(t)m_{rr}, \qquad g_{rz} = 0.$$

Putting these derivatives into (25), we derive

$$\begin{aligned}&(\beta_0 b(t) - b'(t))\ln z + (a'(t) - \beta_0 a(t) - k)q \\ &\quad - \Big(m_t + \frac{1}{2}\sigma_r^2 m_{rr} - \beta_0 m + (a - br)m_r \\ &\quad + (\lambda_2 - \eta_2)\sigma_r^2 m_r - (\beta_0 + \beta_1) \\ &\quad + \frac{1}{2}(\lambda_2 - \eta_2)^2\sigma_r^2 + \frac{b'(t)}{b(t)}m\Big)b(t) = 0,\end{aligned} \tag{64}$$
$$\beta_0 = \lambda_1\eta_3\sigma_S + \lambda_2\eta_2\sigma_r^2 + 2\eta_3^2\sigma_S^2 - m_L,$$
$$\beta_1 = \eta_3\sigma_S\left(\frac{1}{2}\eta_3\sigma_S^3 - \eta_3\sigma_S^2 - \lambda_1\right) - \frac{1}{2}\lambda_1^2.$$

Again we can split this equation into three equations:

$$\beta_0 b(t) - b'(t) = 0, \tag{65}$$

$$a'(t) - \beta_0 a(t) - k = 0, \tag{66}$$

$$\begin{aligned} &m_t + \frac{1}{2}\sigma_r^2 m_{rr} - \beta_0 m + (a - br) m_r + (\lambda_2 - \eta_2)\sigma_r^2 m_r \\ &\quad - (\beta_0 + \beta_1) + \frac{1}{2}(\lambda_2 - \eta_2)^2 \sigma_r^2 + \frac{b'(t)}{b(t)} m = 0. \end{aligned} \tag{67}$$

Combining with the account boundary conditions: $b(T) = 1$ and $a(T) = 0$, the solutions to (65) and (66) are

$$\begin{aligned} b(t) &= e^{\beta_0(t-T)}, \\ a(t) &= -k\left(\frac{1 - e^{-\beta_0(T-t)}}{\beta_0}\right). \end{aligned} \tag{68}$$

We conjecture a solution of (67) with the following structure:

$$m(t, r) = A(t) + B(t) r, \tag{69}$$

with the boundary conditions: $A(T) = 0$ and $B(T) = 0$.

Putting this into (67), we obtain

$$\begin{aligned} &A_t + aB + (\lambda_2 - \eta_2) k_2 B + \frac{1}{2}(\lambda_2 - \eta_2)^2 k_2 - (\beta_0 + \beta_1) \\ &\quad + r\left\{B_t - bB + (\lambda_2 - \eta_2) k_1 B + \frac{1}{2}(\lambda_2 - \eta_2)^2 k_1\right\} = 0. \end{aligned} \tag{70}$$

By matching coefficients, we can decompose (70) into two conditions:

$$\begin{aligned} &B_t - bB + (\lambda_2 - \eta_2) k_1 B + \frac{1}{2}(\lambda_2 - \eta_2)^2 k_1 = 0, \\ &A_t + aB + (\lambda_2 - \eta_2) k_2 B + \frac{1}{2}(\lambda_2 - \eta_2)^2 k_2 - (\beta_0 + \beta_1) = 0. \end{aligned} \tag{71}$$

Taking into account the boundary conditions, the solutions to (71) are

$$\begin{aligned} B(t) &= \frac{\theta_3}{\theta_1}\left(1 - e^{\theta_1(t-T)}\right), \\ A(t) &= \left(\frac{\theta_2\theta_3}{\theta_1} + \theta_4 - \beta_0 - \beta_1\right)(T - t) + \frac{\theta_2\theta_3}{\theta_1^2}\left(e^{\theta_1(t-T)} - 1\right), \end{aligned} \tag{72}$$

where

$$\begin{aligned} &\theta_1 = b - (\lambda_2 - \eta_2) k_1, \qquad \theta_2 = a + (\lambda_2 - \eta_2) k_2 \\ &\theta_3 = \frac{1}{2}(\lambda_2 - \eta_2)^2 k_1, \qquad \theta_4 = \frac{1}{2}(\lambda_2 - \eta_2)^2 k_4, \\ &\beta_0 = \lambda_1\eta_3\sigma_S + \lambda_2\eta_2\sigma_r^2 + 2\eta_3^2\sigma_S^2 - m_L, \\ &\beta_1 = \eta_3\sigma_S\left(\frac{1}{2}\eta_3\sigma_S^3 - \eta_3\sigma_S^2 - \lambda_1\right) - \frac{1}{2}\lambda_1^2. \end{aligned} \tag{73}$$

From the above calculation, we finally obtain the optimal investment strategies under the CARA utility.

Proposition 10. *The optimal investment strategies are given by*

$$\begin{aligned} \pi_0(t) &= 1 - \pi_B(t) - \pi_S(t), \\ \pi_S^* &= \eta_3 + \frac{\left(\lambda_1 + \eta_3\sigma_S^2\right) l}{q\sigma_S v} b(t), \\ \pi_B^* &= \frac{1}{f(T-t)}\left\{(\eta_2 - \eta_1\eta_3) + \frac{l}{qv}(\alpha_4 + B(t)) b(t)\right\}, \end{aligned} \tag{74}$$

where

$$\begin{aligned} &b(t) = e^{\beta_0(t-T)}, \qquad B(t) = \frac{\theta_3}{\theta_1}\left(1 - e^{\theta_1(t-T)}\right), \\ &\beta_0 = \lambda_1\eta_3\sigma_S + \lambda_2\eta_2\sigma_r^2 + 2\eta_3^2\sigma_S^2 - m_L, \\ &\theta_1 = b - (\lambda_2 - \eta_2) k_1, \qquad \theta_3 = \frac{1}{2}(\lambda_2 - \eta_2)^2 k_1 \\ &\alpha_4 = \frac{\left(\eta_2\sigma_S + \lambda_1\eta_1 + \eta_1\eta_3\sigma_S^2 - \lambda_2\sigma_S\right)}{\sigma_S}, \\ &f(t) = \frac{2\left(e^{mt} - 1\right)}{m - (b - k_1\lambda_2) + e^{mt}(m + b - k_1\lambda_2)}, \\ &m = \sqrt{(b - k_1\lambda_2)^2 + 2k_1}. \end{aligned} \tag{75}$$

Lemma 11. *Consider*

$$\frac{db(t)}{dt} > 0, \qquad \frac{d\pi_S^*}{dt} > 0. \tag{76}$$

Proof. Since $\beta_0 > 0$, $q > 0$, $\lambda_1 > 0$, $\eta_3 > 0$, and $\sigma_S > 0$, by differentiating $b(t)$ with the respect to t, we have

$$\begin{aligned} \frac{db(t)}{dt} &= \beta_0 e^{\beta_0(t-T)} > 0, \\ \frac{d\pi_S^*}{dt} &= \frac{\left(\lambda_1 + \eta_3\sigma_S^2\right) l}{q\sigma_S v}\frac{db(t)}{dt} > 0. \end{aligned} \tag{77}$$

□

Lemma 12. *Consider*

$$\frac{d\pi_S^*}{dl} > 0. \tag{78}$$

Proof. Since $r > 0$, $\beta_0 > 0$, $q > 0$, $\lambda_1 > 0$, $\eta_3 > 0$, and $\sigma_S > 0$, we obtain

$$\begin{aligned} b(t) &= e^{\beta_0(t-T)} > 0, \\ \frac{d\pi_S^*}{dl} &= \frac{\left(\lambda_1 + \eta_3\sigma_S^2\right) b(t)}{q\sigma_S v} > 0. \end{aligned} \tag{79}$$

□

Remark 13. Lemma 11 shows that the effect of t on π_S^* is then different from the situation of the power utility. The optimal proportion invested in stock π_S^* depends on the time t and is a monotone increasing function with respect to time t. Meanwhile, we cannot find the monotone increasing or decreasing effect of t on π_B^*. So under the exponential utility, as the retirement date approaches, the plan member will distribute more assets to invest in stock or less asset to invest in low-risk assets (i.e., cash and bonds).

This can be explained by the risky tolerance, namely, $-U'(x)/U''(x) = q^{-1}$, which is only a constant. This indicates that for an exponential utility, due to the independence of a risk tolerance coefficient on wealth, the optimal proportion invested in stock π_S^* is independent of the profitability of risky assets and the wealth. As the wealth gives an insight into the accumulated profit gained from risky assets, the plan member will buy more risky assets as the wealth increases.

Lemma 12 reveals that the optimal proportion invested in stock π_S^* is a monotone increasing function with respect to the salary numeraire l, which is the same as the situation of the power utility. However, the regular change in the effect of l on π_B^* is not found.

Nevertheless, the change trend of t or l on π_S^* is not affected by the absolute risk aversion coefficient p, which is the same as the power utility.

6. Conclusions

We have analyzed an investment problem for a defined contribution pension plan with stochastic salary under the affine interest rate model. In view of the related literatures, we have adopted the CRRA and CARA utility functions. And then, the problem of the maximization of the terminal relative wealth's utility has been solved analytically by the Legendre transform and dual theory. As above mentioned, we have analyzed the effect of different parameters on the optimal investment strategies under the CRRA and CARA utility functions, respectively, and compared their differences. So, this paper extends the research of Gao [2] and Cairns et al. [1].

The further research on the stochastic optimal control of DC mainly spread our work under the more generalized situation: (i) assuming the salary to be affected by non-hedgeable risk source under the research framework; (ii) assuming the risky asset to follow a constant elasticity of variance (CEV) model, and so forth. It is noteworthy that the optimal solution with the extended framework is very difficult. Nevertheless, the above methodology cannot be applied to the extended framework, which will result in a more sophisticated nonlinear partial differential equation and cannot tackle it at present.

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Research Article

Modeling and Optimization of Inventory-Distribution Routing Problem for Agriculture Products Supply Chain

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Mathematical models of inventory-distribution routing problem for two-echelon agriculture products distribution network are established, which are based on two management modes, franchise chain and regular chain, one-to-many, interval periodic order, demand depending on inventory, deteriorating treatment cost of agriculture products, start-up costs of vehicles and so forth. Then, a heuristic adaptive genetic algorithm is presented for the model of franchise chain. For the regular chain model, a two-layer genetic algorithm based on oddment modification is proposed, in which the upper layer is to determine the distribution period and quantity and the lower layer is to seek the optimal order cycle, quantity, distribution routes, and the rational oddment modification number for the distributor. By simulation experiments, the validity of the algorithms is demonstrated, and the two management modes are compared.

1. Introduction

Agriculture products supply chain management is important for reducing circulation costs and improving supply chain efficiency. A key factor in agricultural products supply chain management is inventory-distribution routing problem. Federgruen and Zipkin [1] established an inventory-transportation integrated optimization model based on one-warehouse and multiretailer distribution network, and Bender's decomposition approach is used. Burns et al. [2] discussed the minimum cost problem on joint optimization of inventory and transportation based on one-dispatching point and multireceive point network under certain demand. Daganzo [3] researched the many-to-many inventory-transportation integrated optimization problem with bulk stations. Anily and Federgruen [4] studied the inventory and vehicle routing problem based on single product, certain demand, and continuous time. Trudeau and Dror [5] considered the situation about stochastic demand and the limited capacity of the vehicles. Ernst and Pyke [6] designed a heuristic algorithm to confirm the optimal delivery frequency, according to the inventory-transportation integrated optimization problem based on single distributor-single retailer under customer's random demand. Herer and Roundy [7], Viswanathan and Mathur [8], Bertazzi et al. [9], and Chen and Zheng [10] researched an inventory-transportation integrated optimization model of one-warehouse multiretailer distribution network based on heuristic algorithm to minimize the costs of inventory, ordering, and transportation. Bertazzi and Speranza [11] proposed an efficient heuristic algorithm for multilayer inventory-distribution system with certain demand, and single supplier, no shortage, single delivery vehicle with limited capacity. Wang et al. [12] designed a heuristic algorithm according to the two-echelon distribution system which takes cyclical replenishment strategy, and the terminal customer demand is uncertain. Zhao et al. [13] proposed a decomposition heuristic algorithm based on Markov decision processes and Modified *C-W* saving algorithm under random demand. Tang Jiafu researched delivery planning problem in distribution center with the strategy of vehicle outsourcing model and multiple direct transport; an integer programming model was presented, and a genetic algorithm is designed to solve this model.

So far, the researches about joint optimization of inventory and transportation focus on the industrial products, without considering the metamorphism of agricultural products. Furthermore, considering that chain operation has become the main way of distribution network operation, in which the distributor makes the unified supply for the retailer. At present, there are two models about chain operation: one is regular chain mode. The distributors and the detailers belong to the same enterprise, which makes the direct operation and centralized management. The other is the franchise chain mode, on which the distributors and the detailers do not belong to the same enterprise, and they conclude management relationship through contracts. In this paper, mathematical models are established about the routing problem based on the two modes, and corresponding algorithm is designed to give some analysis and comparison.

2. Assumptions and Notation

A two-echelon agriculture products supply chain system which consists of one distributor and multiple retailers is considered, in which the distributor has a single supplier and delivers agriculture products to the retailers. Both supplier and retailers adapt equal periodic order strategy and order at the beginning of period. For average profit maximization of supply chain, the optimal order time, quantity, and vehicle routing for distribution are to be determined.

The mathematical model is developed based on the following assumptions and notation.

Superscript and Subscript. Subscript 0 denotes the distributor. Subscripts i, j denote serial number of retailers. Superscripts r, m denote serial number of order interval period and delivery vehicles, respectively.

Constant and Set

V, M: denote, respectively, set of locations of distributor and retailers and set of delivery vehicles.

$N, \Delta T, n_{\max}$, and G: denote, respectively, total number of retailers, minimum order interval period, maximum order cycle length of the retailer, and maximum capacity of the delivery vehicles.

w: denotes wholesale price per unit of the retailers' order on the franchise chain mode.

θ, b: denote, respectively, the constant deterioration rate of agriculture products and deteriorating treatment cost per unit of deteriorating items.

F_0, F_j: denote, respectively, constant order cost of distributor and retailers for each time.

h_0, h: denote, respectively, inventory holding cost per unit per unit time of distributor and retailers.

s_j, v: denote, respectively, inventory shortage cost and salvage value of surplus item per unit.

B_j, α_j: denote, respectively, the customer's basic demand, demand, and inventory correlation coefficient for retailer j.

w_0, p: denote, respectively, wholesale price per unit of the distributor's order and retail price per unit of customers' order.

d_{ij}, c_{ij}: denote, respectively, the minimum distance and single vehicle freight between retailer i and retailer $j, i, j \in V$.

c, c_0, c': denote, respectively, single vehicle freight per distance unit between different retailers, start-up costs of single vehicle, and freight per unit of the distributor's order.

Variables and Symbols

n_j, Q_j: denote, respectively, order period's length and order quantity of retailer j.

H, Q: denote, respectively, order period's length and order quantity of the distributor.

Ω: distribution route, $\Omega = \{x_{ij}^{rm} \mid i, j \in V, m \in M, r = 1, 2, \ldots, L\}$, where x_{ij}^{rm} is the adjacency relationship of single vehicle m between retailer i and retailer j, $x_{ij}^{rm} \in \{0, 1\}$; L is the length of distribution cycle.

δ_j^r: order symbol of retailer j in the periodic interval $r, \delta_j^r \in \{0, 1\}$.

t_j, T_j: denote, respectively, the retailer j's time of the inventory level decreased to zero and retailer j's order cycle.

k, L: denote, respectively, the distributor's order cycle coefficient and the length of delivery cycle.

$D_j(t)$, $I_j(t)$: denote, respectively, the customers' demand and inventory level of retailer j at time t.

π, π_j: denote, respectively, the distributor's profit and retailer j's profit.

$\overline{\pi}$, $\overline{\pi}_j$: denote, respectively, the distributor's average profit and retailer j's average profit.

Π: average profit of supple chain system.

Assumption. (1) The retailers' demand is independent, and the demand rate is related to inventory level. Padmanabhan and Vrat's [14] demand model is adopted; at any time t, demand of the retailer j's customer, $D_j(t)$, is $D_j(t) = B_j + \alpha_j I_j(t)$, where α_j is demand and inventory correlation coefficient, $\alpha_j > 0$, B_j, is basic demand, $B_j \geq 0$. Without loss of generality, profit is minus when $B_j = 0$, that is, the retailer cannot gain profit without customers' basic demand.

(2) Considering the metamorphism of agricultural products, the longest length of retailers' order cycle is $n_{\max}$. At the ending of every order cycle, the surplus inventory is treatment at salvage value v, $v < w_0$.

(3) Agriculture products inventory of retailers deteriorates at constant deterioration rate θ, $0 < \theta < 1$. The

deteriorated items can be treated in time, and the treatment cost is b per deteriorated unit.

(4) Wholesale price per unit of distributor is w_0, and retailer price per unit is p. Inventory holding cost per unit per unit time of distributor and retailers is h_0 and h, $w_0+h_0+h<p$. For franchise chain mode, wholesale price per unit of retailer's order is w, $w_0+h_0<w$. Then, $w_0+h_0+h<w+h<p$.

(5) On franchise chain mode, distributor and retailers are risk neutral, sharing information based on wholesale contract. The distributor is leader, and the retailers act as the follower. For the distributor, shortage is not allowed and bear all freight. The average profit of distributor and retailer is concave functions of order cycle.

(6) Retailers' ordering lead time is zero. Both distributor and retailers adopt equal periodic order, and the order period is integral multiple of the minimum interval period ΔT. The distributor's order cycle H is integral multiple of the retailers' order cycle $n_j\Delta T$:

$$H = kL\Delta T, \tag{1}$$

where k is integer, $k \geq 1$, L is least common multiple of any retailer's order cycle:

$$L = \left[n_1, n_2, \ldots, n_N\right], \tag{2}$$

where $[\cdot,\cdot,\ldots,\cdot]$ denotes least common multiple

(7) Delivery vehicle capacity is G. Single vehicle transportation cost between retailer i and retailer j, c_{ij}, is in proportion to the distance. In view of vehicle start-up costs c_0,

$$c_{ij} = \begin{cases} cd_{ij}, & i \neq 0, \\ c_0 + cd_{ij}, & i = 0, \end{cases} \quad i \neq j,\ \forall i, j \in V. \tag{3}$$

From assumptions above, the retailer's maximum inventory level equal to order quantity Q_j. Define t_j as time of the inventory level decreased to zero; we have $dI_j(t)/dt = -\theta I_j(t) - D_j(t) = -(\theta+\alpha_j)I_j(t) - B_j$, $t \leq t_j$. With the boundary condition $I(t_j) = 0$, then we have

$$I_j(t) = \begin{cases} \dfrac{B_j}{\theta+\alpha_j}\left[e^{(\theta+\alpha_j)(t_j-t)} - 1\right], & t \leq t_j, \\ 0, & t > t_j. \end{cases} \tag{4}$$

Accordingly, the retailer's maximum inventory level and order quantity are

$$IH_j = Q_j = I_j(t)\big|_{t=0} = \frac{B_j}{\theta+\alpha_j}\left[e^{(\theta+\alpha_j)t_j} - 1\right]. \tag{5}$$

Define x_{ij}^{rm} as adjacent relationship in order interval period r and vehicle m:

$$x_{ij}^{rm} = \begin{cases} 1, & \text{if vehicle } m \text{ directly goes to retailer } j \text{ after serving retailer } i \text{ at the beginning of period } r, \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

$$i \neq j,\ \forall i, j \in V,\ m \in M,\ r = 1, 2, \ldots, kL.$$

Define Ω as distribution route:

$$\Omega = \left\{x_{ij}^{rm} \mid i, j \in V, m \in M, r = 1, 2, \ldots, L\right\}. \tag{7}$$

3. Model on Franchise Chain Mode

On franchise chain mode, supply chain is decentralized; the optimal order quantity and order cycle rely on wholesale price. The distributor delivery products according to the retailers' demand, and determine the distributor's optimal order quantity and order cycle.

3.1. Profit Analysis on Retailor

Proposition 1. *On franchise chain mode, as far as single order cycle $n_j\Delta T$ of retailer j is concerned, when profit of retailer j is maximumzied, time of the inventory level decreased to zero, t_j, satisfies*

$$t_j = n_j\Delta T. \tag{8}$$

From Proposition 1, profit is maximized when $t_j = T_j$, profit of retailer j, $\pi_j(T_j)$, is

$$\begin{aligned} \pi_j\left(T_j\right) = {} & p\int_0^{T_j}\left[B_j + \alpha_j I(t)\right]dt \\ & - w\int_0^{T_j}\left[B_j + \left(\alpha_j+\theta\right)I(t)\right]dt \\ & - b\int_0^{T_j}\theta I(t)\,dt - h_j\int_0^{T_j}I(t)\,dt - F_j. \end{aligned} \tag{9}$$

The partials of (9) are in sequence of sales revenue, purchase cost, deteriorating treatment cost, inventory holding cost, constant order cost. From (4), (5), and (9), we have the average profit of retailer j:

$$\begin{aligned} \overline{\pi}_j\left(T_j\right) &= \frac{\pi_j\left(T_j\right)}{T_j} \\ &= A\frac{e^{(\theta+\alpha_j)T_J} - 1}{T_j} - \frac{F_j}{T_j} + \frac{B_j}{\theta+\alpha_j}\left[\theta\left(p+b\right) + h_j\right], \end{aligned} \tag{10}$$

where

$$A = \frac{B_j}{\left(\theta+\alpha_j\right)^2}\cdot\left[\left(p-w\right)\alpha_j - \left(w+b\right)\theta - h_j\right]. \tag{11}$$

From assumption (6), Let $d\overline{\pi}_j(T_j)/dT_j = 0$; we have

$$\left[1 - \left(\theta+\alpha_j\right)T_j^*\right]e^{(\theta+\alpha_j)T_j^*} = 1 + \frac{F_j}{A}. \tag{12}$$

Then, when profit of retailer j is maximumzied, the optimal order cycle T_j^* satisfies (13). According to the constant order interval ΔT, we can find the optimal length of order cycle n_j^*, and $\overline{\pi}_j$ approach or reach the maximum value correspondingly:

$$n_j^* = \begin{cases} \left\lfloor \dfrac{T_j^*}{\Delta T} \right\rfloor, \\ \text{if } \overline{\pi}_j \left(T_j = \Delta T \left\lfloor \dfrac{T_j^*}{\Delta T} \right\rfloor \right) \geq \overline{\pi}_j \left(T_j = \Delta T \left\lceil \dfrac{T_j^*}{\Delta T} \right\rceil \right), \\ \left\lceil \dfrac{T_j^*}{\Delta T} \right\rceil, \\ \text{if } \overline{\pi}_j \left(T_j = \Delta T \left\lfloor \dfrac{T_j^*}{\Delta T} \right\rfloor \right) < \overline{\pi}_j \left(T_j = \Delta T \left\lceil \dfrac{T_j^*}{\Delta T} \right\rceil \right), \end{cases} \tag{13}$$

where $\lfloor \cdot \rfloor$and $\lceil \cdot \rceil$ denote, respectively, top rounding and round down.

Correspondingly, optimal profit of retailer j is

$$\overline{\pi}_j \left(n_j^* \Delta T \right) = A \frac{e^{(\theta+\alpha_j) n_j^* \Delta T} - 1}{n_j^* \Delta T} - \frac{F_j}{n_j^* \Delta T} + \frac{B_j}{\theta + \alpha_j} \left[\theta \left(p + b \right) + h_j \right]. \tag{14}$$

3.2. Profit Analysis on Distributor. From assumption (5) and Equation (1), the distributor's order quantity Q_0 equals his retailers' total demand in one order cycle:

$$Q_0 = \sum_{j=1}^{N} \sum_{r=1}^{k_0 L} \delta_j^r Q_j = \sum_{j=1}^{N} \sum_{l=1}^{k_0 L / n_j} Q_j = kL \sum_{j=1}^{N} \frac{Q_j}{n_j}. \tag{15}$$

Define δ_j^r as order symbol of retailer j in order interval period r:

$$\delta_j^r = \begin{cases} 1, & r = z n_j, \\ 0, & r \neq z n_j, \end{cases} \quad z \text{ is integer.} \tag{16}$$

At the beginning of order period r, the distributor's delivery quantity is Q_j^r, so

$$Q_j^r = \delta_j^r Q_j. \tag{17}$$

Profit of the distributor is

$$\pi (k, w, \Omega) = w Q_0 - w_0 Q_0 - h_0 \sum_{j=1}^{N} \sum_{l=1}^{kL/n_j - 1} n_j \Delta T Q_j l - c^l Q_0 - \sum_{r=1}^{kL} \sum_{m \in M} \sum_{i=0}^{N} \sum_{\substack{j=0 \\ j \neq i}}^{N} c_{ij} \delta_j^r \delta_i^r x_{ij}^{rm} - F_0. \tag{18}$$

The partials of (18) are in sequence of sales revenue, purchase cost, inventory holding cost, freight of purchase products, delivery cost, and constant order cost. From (17) and (18), we have the average profit of the distributor:

$$\begin{aligned} \overline{\pi} (k, w, \Omega) &= \frac{\pi (k, w, \Omega)}{kL\Delta T} \\ &= \frac{1}{\Delta T} \left[\left(w - w_0 - c^l - \frac{h_0}{2} kL\Delta T \right) \sum_{j=1}^{N} \frac{1}{n_j} Q_j \right] \\ &\quad + \frac{h_0}{2} \sum_{j=1}^{N} Q_j - \frac{1}{L\Delta T} \sum_{r=1}^{L} \sum_{m \in M} \sum_{i=0}^{N} \sum_{\substack{j=0 \\ j \neq i}}^{N} c_{ij} \delta_j^r \delta_i^r x_{ij}^{rm} \\ &\quad - \frac{1}{kL\Delta T} F_0. \end{aligned} \tag{19}$$

From (19), The average profit of distributor is concave functions of k. We can find the optimal k^*, $k^* \geq 1$, and $\overline{\pi}$ approach or reach the maximum value correspondingly. Let $d\overline{\pi}/dk = 0$, we have

$$k = \begin{cases} 1, & \lfloor \rho \rfloor \leq 1, \\ \lfloor \rho \rfloor, & \lfloor \rho \rfloor > 1, \ \overline{\pi}(k, w, \Omega)|_{k=\lfloor \rho \rfloor} > \overline{\pi}(k, w, \Omega)|_{k=\lceil \rho \rceil}, \\ \lceil \rho \rceil, & \lfloor \rho \rfloor > 1, \ \overline{\pi}(k, w, \Omega)|_{k=\lfloor \rho \rfloor} < \overline{\pi}(k, w, \Omega)|_{k=\lceil \rho \rceil}, \end{cases} \tag{20}$$

where

$$\rho = \frac{1}{L} \sqrt{\frac{2F_0}{h_0 \Delta T \sum_{j=1}^{N} \left(1/n_j \right) Q_j}}. \tag{21}$$

3.3. Mathematical Model. On franchise chain mode, the retailers determine their order interval $n_j \Delta T$ and order quantity according to wholesale price w, then the distributor determines optimal order interval $kL\Delta T$, order quantity Q, and distribute route Ω. The mathematical model of inventory-distribution routing problem is

$$\begin{aligned} &\max \overline{\pi} (k, w, \Omega) \\ &= \max_{k,w,\Omega} \left\{ \frac{1}{\Delta T} \left[\left(w - w_0 - c^l - \frac{h_0}{2} kL\Delta T \right) \sum_{j=1}^{N} \frac{1}{n_j} Q_j \right] \right. \\ &\quad + \frac{h_0}{2} \sum_{j=1}^{N} Q_j \\ &\quad \left. - \frac{1}{L\Delta T} \sum_{r=1}^{L} \sum_{i=0}^{N} \sum_{j=0}^{N} c_{ij} \delta_j^r \delta_i^r x_{ij}^{rm} - \frac{1}{kL\Delta T} F \right\} \end{aligned} \tag{22}$$

s.t.

$$Q_j = \frac{B_j}{\theta + \alpha_j}\left[e^{(\theta+\alpha_j)n_j\Delta T} - 1\right], \quad (23)$$

$$n_j = \begin{cases} \left\lfloor \dfrac{T_j^*}{\Delta T} \right\rfloor, \\ \text{if } \overline{\pi}_j\left(T_j = \Delta T \left\lfloor \dfrac{T_j^*}{\Delta T} \right\rfloor\right) \\ \geq \overline{\pi}_j\left(T_j = \Delta T \left\lceil \dfrac{T_j^*}{\Delta T} \right\rceil\right), \ \left\lfloor \dfrac{T_j^*}{\Delta T} \right\rfloor \leq n_{\max}, \\ \left\lceil \dfrac{T_j^*}{\Delta T} \right\rceil, \\ \text{if } \overline{\pi}_j\left(T_j = \Delta T \left\lfloor \dfrac{T_j^*}{\Delta T} \right\rfloor\right) \\ < \overline{\pi}_j\left(T_j = \Delta T \left\lceil \dfrac{T_j^*}{\Delta T} \right\rceil\right), \ \left\lceil \dfrac{T_j^*}{\Delta T} \right\rceil \leq n_{\max}, \\ n_{\max}, \\ \text{else}, \end{cases} \quad (24)$$

$$\left(\theta + \alpha_j\right) T_j^* = 1 - e^{-(\theta+\alpha_j)T_j^*}, \quad (25)$$

$$\begin{aligned} \overline{\pi}_j\left(T_j\right) = {} & \frac{B_j}{\left(\theta + \alpha_j\right)^2} \cdot \left[(p - w)\alpha_j - (w + b)\theta - h_j\right] \\ & \times \frac{e^{(\theta+\alpha_j)T_j} - 1}{T_j} - \frac{F_j}{T_j} + \frac{B_j}{\theta + \alpha_j}\left[\theta(p + b) + h_j\right] \\ & \geq 0, \end{aligned} \quad (26)$$

$$L = \left[n_1, n_2, \ldots, n_N\right], \quad (27)$$

$$k = \begin{cases} 1, & \lfloor\rho\rfloor \leq 1, \\ \lfloor\rho\rfloor, & \lfloor\rho\rfloor > 1,\ \overline{\pi}(k, w, \Omega)\big|_{k=\lfloor\rho\rfloor} > \overline{\pi}(k, w, \Omega)\big|_{k=\lceil\rho\rceil}, \\ \lceil\rho\rceil, & \lfloor\rho\rfloor > 1,\ \overline{\pi}(k, w, \Omega)\big|_{k=\lfloor\rho\rfloor} < \overline{\pi}(k, w, \Omega)\big|_{k=\lceil\rho\rceil}, \end{cases} \quad (28)$$

$$\rho = \frac{1}{L}\sqrt{\frac{2F}{h_0\Delta T \sum_{j=1}^{N}\left(1/n_j\right)Q_j}}, \quad (29)$$

$$\sum_{m\in M}\sum_{j=1}^{N}\sum_{i=1}^{N}\delta_i^r\delta_j^r Q_j x_{ij}^{rm} \leq G, \quad \forall r = 1, 2, \ldots, kL, \quad (30)$$

$$\sum_{m\in M}\sum_{j=1}^{N} x_{ij}^{rm} = 1, \quad i \neq j,\ \forall i = 1, 2, \ldots, N, \quad \forall r = 1, 2, \ldots, kL, \quad (31)$$

$$\sum_{j=1}^{N} x_{ij}^{rm} \leq 1, \quad i \neq j,\ \forall i = 1, 2, \ldots, N,\ \forall r = 1, 2, \ldots, kL, \quad \forall m \in M, \quad (32)$$

$$\sum_{j=1}^{N} x_{ij}^{rm} - \sum_{j=1}^{N} x_{ji}^{rm} = 0, \quad i \neq j,\ \forall i = 1, 2, \ldots, N, \quad \forall r = 1, 2, \ldots, kL,\ \forall m \in M, \quad (33)$$

$$\sum_{j=1}^{N} x_{0j}^{rm} = 1, \quad \forall r = 1, 2, \ldots, kL, \quad (34)$$

$$\sum_{j=1}^{N} x_{j0}^{rm} = 1, \quad \forall r = 1, 2, \ldots, kL, \quad (35)$$

$$x_{ij}^{rm} \in \{0, 1\}, \quad i \neq j,\ \forall i, j \in V,\ \forall m \in M, \quad (36)$$

$$\delta_j^r = \begin{cases} 1, & r = zn_j \\ 0, & r \neq zn_j \end{cases}, \quad z \text{ is integer}, \quad (37)$$

$$\Omega = \left\{x_{ij}^{rm} \mid i, j \in V, m \in M, r = 1, 2, \ldots, L\right\}. \quad (38)$$

Constraints (23) to (26) relate to optimal strategy of the retailer. Constraint (26) ensures that each retailer's profit is positive. Constraints (27) to (29) relate to optimal strategy of the distributor. Constraint (30) is capacity limit of delivery vehicle. Constraints (31) to (32) limit that one retailer is visited by exactly one vehicle. Constraint (33) ensures the equation of flow between retailers. Constraints (34) and (35) guarantee that the starting point and the terminal point of every distribute route are the distribute center. Finally, restrictions (34) to (38) define the nature of the variables.

4. Model on Regular Chain Mode

Proposition 2. *On regular chain mode, when the average profit of supply chain is maximumzied, there is no surplus products at terminal time of order cycle. That is, the relationship between any delivery cycle of retailer, $n_j\Delta T$, and retailer j time of the inventory level decreased to zero, t_j^r, satisfies*

$$t_j^r = n_j\Delta T - \varepsilon_j^r, \quad \varepsilon_j^r \geq 0,\ r = 1, 2, \ldots, \frac{L}{n_j},\ \forall j = 1, 2, \ldots, N. \quad (39)$$

Within one distribution cycle L, the sales revenue of retailer j minus inventory holding cost, shortage cost, and deteriorating treatment cost, we have

$$\begin{aligned} \pi_j^{/} = {} & p\sum_{r=1}^{L/n_j} Q_j^r - h_j\sum_{r=1}^{L/n_j}\int_0^{t_j^r} I_j^r(t)\,dt - s_jB_j\sum_{r=1}^{L/n_j}\left(n_j\Delta T - t_j^r\right) \\ & - b\sum_{r=1}^{L/n_j}\int_0^{t_j^r}\theta I(t)\,dt. \end{aligned} \quad (40)$$

Then, we have the average profit of retailer j:

$$\overline{\pi}_j^{/} = \frac{\pi_j^{/}}{L\Delta T} = \frac{p}{L\Delta T}\sum_{r=1}^{L/n_j} Q_j^r - \frac{s_j B_j}{L\Delta T}\sum_{r=1}^{L/n_j}\left(n_j\Delta T - t_j^r\right) - \frac{1}{L\Delta T}\frac{B_j\left(h_j + b\theta\right)}{\theta+\alpha_j}\sum_{r=1}^{L/n_j}\left[\frac{1}{B_j}Q_j^r - t_j^r\right]. \tag{41}$$

The distributor's order quantity, Q, equals the total delivery quantity in one order cycle $kL\Delta T$:

$$Q = \sum_{j=1}^{N}\sum_{r=1}^{kL}\delta_j^r Q_j^r = k\sum_{j=1}^{N}\sum_{r=1}^{L/n_j} Q_j^r. \tag{42}$$

The cost of distributor is

$$\pi^{/} = w_0 Q + c^{/}Q + c^{/}Q + \sum_{r=1}^{kL}\sum_{m\in M}\sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N} c_{ij}\delta_j^r\delta_i^r x_{ij}^{rm} + F. \tag{43}$$

The partials of (43) are in sequence of purchase cost, freight of purchase products, delivery cost, inventory holding cost, and constant order cost. Then, we have the average profit of the distributor:

$$\overline{\pi}^{/} = \frac{\pi^{/}}{kL\Delta T} = \frac{1}{L\Delta T}\left(w_0 + c^{/}\right)\sum_{j=1}^{N}\sum_{r=1}^{L/n_j} Q_j^r + \frac{1}{L}h_0\sum_{j=1}^{N}\sum_{r=1}^{L/n_j}\left[(r-1)n_j + \frac{k-1}{2}L\right]Q_j^r + \frac{1}{L\Delta T}\sum_{r=1}^{L}\sum_{m\in M}\sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N} c_{ij}\delta_j^r\delta_i^r x_{ij}^{rm} + \frac{1}{kL\Delta T}F. \tag{44}$$

From (41) and (44), we have the average profit of supply chain:

$$\begin{aligned}\overline{\Pi}^{/} &= \sum_{j=1}^{N}\overline{\pi}_j^{/} - \overline{\pi}^{/}\\ &= \frac{1}{L\Delta T}\sum_{j=1}^{N}\sum_{r=1}^{L/n_j}\left\{p - w_0 - c^{/} - \frac{h_j + b\theta}{\theta+\alpha_j} - h_0\Delta T \times\left[(r-1)n_j + \frac{k-1}{2}L\right]\right\}Q_j^r\\ &\quad + \frac{1}{L\Delta T}\sum_{j=1}^{N}\sum_{r=1}^{L/n_j} B_j\left(\frac{h_j + b\theta}{\theta+\alpha_j} + s_j\right)t_j^r\\ &\quad - \frac{1}{L\Delta T}\sum_{j=1}^{N}\sum_{r=1}^{L/n_j} s_j B_j n_j \Delta T\\ &\quad - \frac{1}{L\Delta T}\sum_{r=1}^{L}\sum_{m\in M}\sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N} c_{ij}\delta_j^r\delta_i^r x_{ij}^{rm}\\ &\quad - \frac{1}{kL\Delta T}F.\end{aligned} \tag{45}$$

From (45), the average profit of supply chain is concave functions of k. So, for constant order cycle of retailers, ΔTL, we can find the optimal k^*, $k^* \geq 1$, and $\overline{\Pi}^{/}$ approach or reach the maximum value correspondingly. Let $d\overline{\Pi}^{/}/dk = 0$; we have

$$k = \begin{cases} 1, & \lfloor\rho\rfloor \leq 1,\\ \lfloor\rho\rfloor, & \lfloor\rho\rfloor > 1,\ \overline{\Pi}^{/}\left(k^{/} = \lfloor\rho\rfloor\right) > \overline{\Pi}^{/}\left(k^{/} = \lceil\rho\rceil\right),\\ \lceil\rho\rceil, & \lfloor\rho\rfloor > 1,\ \overline{\Pi}^{/}\left(k^{/} = \lfloor\rho\rfloor\right) < \overline{\Pi}^{/}\left(k^{/} = \lceil\rho\rceil\right), \end{cases} \tag{46}$$

where

$$\rho = \frac{1}{L}\sqrt{\frac{2F}{h_0\Delta T\sum_{j=1}^{N}\sum_{r=1}^{L/n_j} Q_j^r}}. \tag{47}$$

On regular mode, the optimal problem is to determine the optimal order quantity Q and order cycle $kL\Delta T$, delivery quantity Q_j^r, $r = 1, 2, \ldots, L/n_j$, delivery cycle $n_j\Delta T$, and distribution route Ω, which maxmize the average profit of supply chain. The mathematical model of inventory-distribution routing problem is

$$\begin{aligned}\max\overline{\Pi}^{/}\left(k, n_j, Q_j^r, \Omega\right) = \max\Bigg\{&\frac{1}{L\Delta T}\sum_{j=1}^{N}\sum_{r=1}^{L/n_j}\left\{p - w_0 - c^{/} - \frac{h_j + b\theta}{\theta+\alpha_j} - h_0\Delta T\left[(r-1)n_j + \frac{k-1}{2}L\right]\right\}Q_j^r\\ &+ \frac{1}{L\Delta T}\sum_{j=1}^{N}\sum_{r=1}^{L/n_j} B_j\left(\frac{h_j + b\theta}{\theta+\alpha_j} + s_j\right)t_j^r\end{aligned}$$

$$-\frac{1}{L\Delta T}\sum_{j=1}^{N}\sum_{r=1}^{L/n_j} s_j B_j n_j \Delta T$$

$$\left.-\frac{1}{L\Delta T}\sum_{r=1}^{L}\sum_{m\in M}\sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N} c_{ij}\delta_j^r\delta_i^r x_{ij}^{rm} - \frac{1}{kL\Delta T}F\right\}, \tag{48}$$

s.t.

$$Q_j^r = \frac{B_j}{\theta+\alpha_j}\left[e^{(\theta+\alpha_j)t_j^r}-1\right], \quad \forall j = 1,2,\ldots,N, \tag{49}$$

$$n_j \text{ is integer},$$

$$L = [n_1, n_2, \ldots, n_N], \tag{50}$$

$$k = \begin{cases} 1, & \lfloor\rho\rfloor \le 1, \\ \lfloor\rho\rfloor, & \lfloor\rho\rfloor > 1,\ \overline{\Pi}^{l}\left(k^{l} = \lfloor\rho\rfloor\right) > \overline{\Pi}^{l}\left(k^{l} = \lceil\rho\rceil\right), \\ \lceil\rho\rceil, & \lfloor\rho\rfloor > 1,\ \overline{\Pi}^{l}\left(k^{l} = \lfloor\rho\rfloor\right) < \overline{\Pi}^{l}\left(k^{l} = \lceil\rho\rceil\right), \end{cases} \tag{51}$$

$$\rho = \frac{1}{L}\sqrt{\frac{2F}{h_0\Delta T\sum_{j=1}^{N}\sum_{r=1}^{L/n_j}Q_j^r}}, \tag{52}$$

$$\sum_{m\in M}\sum_{j=1}^{N}\sum_{i=1}^{N}\delta_i^r\delta_j^r Q_j x_{ij}^{rm} \le G, \quad \forall r = 1,2,\ldots,kL, \tag{53}$$

$$\sum_{m\in M}\sum_{j=1}^{N} x_{ij}^{rm} = 1, \quad i\neq j,\ \forall i = 1,2,\ldots,N, \quad \forall r = 1,2,\ldots,kL, \tag{54}$$

$$\sum_{j=1}^{N} x_{ij}^{rm} \le 1, \quad i\neq j,\ \forall i = 1,2,\ldots,N, \quad \forall r = 1,2,\ldots,kL,\ \forall m\in M, \tag{55}$$

$$\sum_{j=1}^{N} x_{ij}^{rm} - \sum_{j=1}^{N} x_{ji}^{rm} = 0, \quad i\neq j,\ \forall i = 1,2,\ldots,N, \quad \forall r = 1,2,\ldots,kL,\ \forall m\in M, \tag{56}$$

$$\sum_{j=1}^{N} x_{0j}^{rm} = 1, \quad \forall r = 1,2,\ldots,kL, \tag{57}$$

$$\sum_{j=1}^{N} x_{j0}^{rm} = 1, \quad \forall r = 1,2,\ldots,kL, \tag{58}$$

$$n_j \le n_{\max}, \tag{59}$$

$$x_{ij}^{rm} \in \{0,1\}, \quad i\neq j,\ \forall i,j\in V,\ \forall m\in M, \tag{60}$$

$$\delta_j^r = \begin{cases} 1, & r = zn_j \\ 0, & r\neq zn_j \end{cases}, \quad z \text{ is integer}, \tag{61}$$

$$\Omega = \left\{x_{ij}^{rm} \mid i,j\in V,\ m\in M,\ r = 1,2,\ldots,L\right\}. \tag{62}$$

Constraints (49) to (52) illustrate the relationship between n_j and Q_j, L, k. Constraint (53) is capacity limit of delivery vehicle. Constraints (54) and (55) limit that one retailer is visited by exactly one vehicle. Constraint (56) ensures the equation of flow between retailers. Constraints (57) and (58) guarantee that the starting point and the terminal point of every distribute route are the distribute center. Constraint (59) is length limit of retailer's order cycle. Finally, restrictions (60) to (62) define the nature of the variables.

5. Solution

In the following, improved genetic algorithms are developed to solve inventory-distribution routing problem of agriculture products supply chain.

5.1. HAGA for Franchise Chain Mode. On franchise chain mode, the optimal order quantity and order cycle rely on wholesale price, so the key to solve the optimal problem is distribution routing. A heuristic adaptive genetic algorithm (HAGA) is developed for it. Heuristic rules mean that adding some routes which obtained by nearest neighbor algorithm and saving algorithm to initial population. Adaptive rules are that crossover ratio and mutation ratio vary with the population's evolution state.

Natural number code is adopted, and the chromosome structure is

$$(0, a_{11}, a_{12}, \ldots, a_{1u}, 0, a_{21}, a_{22}, \ldots, a_{2v}, 0, \ldots, 0, a_{l1}, a_{l2}, \ldots, a_{lw}), \tag{63}$$

where a_{st} denotes serial number of task, 0 denotes distribution center, and gene fragment between two adjacent 0 is a delivery route.

To improve quality and stability of evolution, elitist model and elitist selection strategy is adopted.

PMX operator is used in crossover operation, and reversed operator is used in mutation operation. Crossover ratio P_c and mutation ratio P_m are expressed as follows:

$$P_c = \begin{cases} e_{c1}\dfrac{J_{\text{avg}} - J_{\min}}{J}, & J_{\text{avg}} - J_{\min} < J, \\ e_{c2}, & J_{\text{avg}} - J_{\min} \ge J, \end{cases} \tag{64}$$

where e_{c1}, e_{c2} are constant, $0 \le e_{c1} \le 1$, $0 \le e_{c2} \le 1$. J_{avg} is the average fitness value of the population. $J_{\min}$ is the fitness value of the best chromosome in the population. J is the better one in the cross couple

$$P_m = \begin{cases} e_{m1}\dfrac{J - J_{\min}}{J_{\text{avg}} - J_{\min}}, & J - J_{\min} < J_{\text{avg}} - J_{\min}, \\ e_{m2}, & J - J_{\min} \ge J_{\text{avg}} - J_{\min}, \end{cases} \tag{65}$$

where e_{m1}, e_{m2} are constant, $0 \le e_{m1} \le 1$, $0 \le e_{m2} \le 1$.

Take time limitation of calculating or generations-stagnancy as algorithm termination rule. The algorithm procedure is as follows.

Step 1. According to (23) to (25), determine the optimal order cycle and order quantity of the retailers. Then, determine the distribution cycle L and tasks in the order interval period. Furthermore, obtain the value of ρ via (29).

Step 2. If $\lfloor \rho \rfloor \leq 1$, let order cycle coefficient of the distributor $k = 1$ and obtain the optimal order cycle and order quantity of the distributor. Then, call subroutine of HAGA, seeking the optimal distribution route of every delivery interval period, go to Step 4.

Step 3. If $\lfloor \rho \rfloor > 1$, let $k = \lfloor \rho \rfloor$ and $k = \lceil \rho \rceil$ and calculate, respectively, the optimal order cycle and order quantity of the distributor. Then, call subroutine of HAGA, seeking the optimal distribution route of every delivery interval period; calculate the corresponding distributor's average profit. Furthermore, determine the optimal value of k.

The algorithm procedure of HAGA is as follows.

Step (1): yield initial population, evaluate every chromosome, and save the elitist chromosome.

Step (2): selection, crossover, and mutation operation, yield the new population. Evaluate the new population, find the elitist chromosome in the new population, and refresh the elitist chromosome saved.

Step (3): if the elitist model or the super iteration number reaches the upper limitation go to Step (5).

Step (4): go to Step (2).

Step (5): output the optimal solution.

Step 4. Output the optimal solution: include in the optimal order cycle, optimal order quantity of the distributor and retailers, and the optimal distribution routes.

5.2. Two-Layer Genetic Algorithm Based on Oddment Modification for Regular Chain Mode. On regular chain mode, the supply chain is concentrated, and it is needed to seek the optimal solution of all decision variables. On the other hand, owing to the capacity limitation and vehicle start-up costs, sometimes the delivery quantity of some retailer will be less than that have ordered to avoid the case of distribution route with little delivery quantity. Therefore, it is more difficult to solve the optimal problem on regular chain mode than that on franchise chain mode. A two-layer genetic algorithm based on oddment modification is developed to solve it. The algorithm procedure of the upper layer is as follows.

Step 1. In the upper layer, yield initial population and calculate the initial delivery quantity of every retailer according to (23).

Step 2. The upper layer transfers the initial delivery quantity of every retailer to the lower layer and obtains the modification schemes from the lower layer. Then, evaluate the modification schemes. If the current best solution is better than the maintaining optimum individual, refresh the optimum individual.

Step 3. If calculating time or generations-stagnancy reaches the upper limitation go to Step 5.

Step 4. Selection, crossover, and mutation operation yield the new population; go to Step 2.

Step 5. Output the optimal solution.

The algorithm procedure of the lower layer is as follows.

Step (1): according to the order cycle and order quantity of the retailers transferred by the upper layer, determine the distribution cycle L and delivery tasks in every order interval period by (49) to (52). Then, calculate ρ by (52).

Step (2): if $\lfloor \rho \rfloor \leq 1$, let order cycle coefficient of the distributor $k = 1$ and obtain the optimal order cycle and order quantity of the distributor. Then, call subroutine of HAGA, seeking the optimal distribution route of every delivery interval period; go to Step (4).

Step (3): if $\lfloor \rho \rfloor > 1$, let $k = \lfloor \rho \rfloor$ and $k = \lceil \rho \rceil$, calculate, respectively, the optimal order cycle and order quantity of the distributor. Then, call subroutine of HAGA, seeking the optimal distribution route of

Table 1: Parameter values of retailers.

	B_j	h_j	α_j
$j = 1$	40	0.22	0.2
$j = 2$	100	0.16	0.15
$j = 3$	80	0.1	0.16
$j = 4$	30	0.18	0.16
$j = 5$	150	0.2	0.1
$j = 6$	100	0.15	0.2
$j = 7$	80	0.1	0.18
$j = 9$	60	0.2	0.1
$j = 9$	120	0.2	0.2
$j = 10$	90	0.14	0.1
$j = 11$	80	0.25	0.18
$j = 12$	20	0.1	0.1
$j = 13$	130	0.2	0.16
$j = 14$	70	0.15	0.2
$j = 15$	40	0.25	0.15
$j = 16$	100	0.18	0.05
$j = 17$	50	0.25	0.15
$j = 18$	110	0.1	0.1
$j = 19$	30	0.2	0.2
$j = 20$	90	0.2	0.2

Table 2: Distance between distributor and retailers.

d_{ij}	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = 9$	$i = 10$	$i = 11$	$i = 12$	$i = 13$	$i = 14$	$i = 15$	$i = 16$	$i = 17$	$i = 18$	$i = 19$	$i = 20$
$j = 1$	4.0	0	8.0	8.0	12.0	18.0	21.7	13.7	14.5	23.7	21.7	29.7	30.1	9.7	15.7	21.7	10.0	26.1	6.0	22.0	9.7
$j = 2$	4.0	8.0	0	14.5	4.0	10.0	16.0	20.1	16.0	18.0	16.0	24.0	24.5	16.1	10.0	26.1	16.5	20.5	8.5	24.3	4.0
$j = 3$	10.5	8.0	14.5	0	18.5	22.0	24.0	8.0	12.9	26.0	20.9	28.9	29.4	4.0	18.0	16.0	8.5	26.9	6.0	22.0	12.0
$j = 4$	8.0	12.0	4.0	18.5	0	6.0	12.0	24.1	20.0	14.0	20.0	20.0	20.5	20.1	14.0	32.1	20.5	16.5	12.5	28.3	8.0
$j = 5$	14.0	18.0	10.0	22.0	6.0	0	6.0	27.7	22.0	8.0	22.0	14.0	20.0	23.7	16.0	35.7	24.0	10.5	12.5	30.3	10.0
$j = 6$	17.7	21.7	16.0	24.0	12.0	6.0	0	24.0	16.0	6.0	16.0	8.0	14.0	20.0	10.0	30.3	18.0	4.5	18.0	24.3	12.0
$j = 7$	16.1	13.7	20.1	8.0	24.1	27.7	24.0	0	8.0	30.0	16.0	24.0	24.5	4.0	14.0	8.0	6.0	22.0	11.7	14.0	17.7
$j = 8$	12.0	14.5	16.0	12.9	20.0	22.0	16.0	8.0	0	22.0	8.0	16.0	16.5	8.9	6.0	14.3	4.5	14.0	10.0	8.3	12.0
$j = 9$	19.7	23.7	18.0	26.0	14.0	8.0	6.0	30.0	22.0	0	20.5	12.5	18.5	26.0	16.0	36.3	24.0	8.0	20.0	30.3	14.0
$j = 10$	17.7	21.7	16.0	20.9	20.0	22.0	16.0	16.0	8.0	20.5	0	8.0	8.5	16.9	6.0	16.0	12.5	12.5	18.0	10.0	12.0
$j = 11$	25.7	29.7	24.0	28.9	20.0	14.0	8.0	24.0	16.0	12.5	8.0	0	6.0	24.9	14.0	24.0	20.5	4.5	26.0	18.0	20.0
$j = 12$	26.1	30.1	24.5	29.4	20.5	20.0	14.0	24.5	16.5	18.5	8.5	6.0	0	25.4	14.5	18.0	20.9	10.5	26.5	12.0	20.5
$j = 13$	12.1	9.7	16.1	4.0	20.1	23.7	20.0	4.0	8.9	26.0	16.9	24.9	25.4	0	14.0	12.0	4.5	22.9	7.7	16.5	13.7
$j = 14$	11.7	15.7	10.0	18.0	14.0	16.0	10.0	14.0	6.0	16.0	6.0	14.0	14.5	14.0	0	20.3	10.5	12.0	12.0	14.5	6.0
$j = 15$	24.1	21.7	26.1	16.0	32.1	35.7	30.3	8.0	14.3	36.3	16.0	24.0	18.0	12.0	20.3	0	14.0	28.5	19.7	6.0	25.7
$j = 16$	8.5	10.0	16.5	8.5	20.5	24.0	18.0	6.0	4.5	24.0	12.5	20.5	20.9	4.5	10.5	14.0	0	18.5	8.0	12.0	14.0
$j = 17$	22.1	26.1	20.5	26.9	16.5	10.5	4.5	22.0	14.0	8.0	12.5	4.5	10.5	22.9	12.0	28.5	18.5	0	22.5	22.5	16.5
$j = 18$	4.5	6.0	8.5	6.0	12.5	12.5	18.0	11.7	10.0	20.0	18.0	26.0	26.5	7.7	12.0	19.7	8.0	22.5	0	18.3	6.0
$j = 19$	20.9	22.0	24.3	22.0	28.3	30.3	24.3	14.0	8.3	30.3	10.0	18.0	12.0	16.5	14.5	6.0	12.0	22.5	18.3	0	20.3
$j = 20$	5.7	9.7	4.0	12.0	8.0	10.0	12.0	17.7	12.0	14.0	12.0	20.0	20.5	13.7	6.0	25.7	14.0	16.5	6.0	20.3	0

every delivery interval period, caculate the corresponding distributor's average profit. Furthermore, determine the optimal value of k.

Step (4): check the distribution scheme obtained by HAGA subroutine and divide the total delivery quantity in every distribution interval period by vehicle capacity G; if the remainders do not exceed the threshold ΔG, go to Step (6).

Step (5): relaxing the vehicle capacity limitation to $G+\Delta G$, call subroutine of HAGA and calculate again (put the current best distribution routes into initial population). Compare the new distribution routes with vehicle capacity and set the surplus delivery quantity as oddment. Then, by selecting retailers which have the lowest shortage cost and reducing their delivery quantity, obtain the modification delivery quantity. Furthermore, compare the optimal results after oddment modification with those of unmodified. If the new scheme is worse than the old, let the modification delivery quantity be zero.

Step (6): transfer the optimal result to the upper layer and include in the modification delivery quantity of retailers, the optimal order cycle and quantity of the distributor, and the optimal distribution routes.

6. Numerical Study

To illustrate the frontal model and algorithm, we consider a distribution network with one distributor and twenty retailers. Some parameter values are as follows: $\Delta T = 1$, $n_{\max} = 4$, $p = 10$, $\theta = 0.05$, $b = 0.1$, $s = 0.6$, $w = 7.5$, $w_0 = 3$,$F_j = 20$, $F_0 = 200$, $h_0 = 0.05$, $c = 0.5$, $c = 4$, $c_0 = 80$, and $G = 1500$. Other relevant parameter values are shown in Tables 1 and 2 in the appendix.

On franchise chain mode, the parameters of HAGA algorithm are as follows: the number of initial population is 30, $P_s = 0.1$, $e_{c1} = e_{c2} = 0.6$, $e_{m1} = e_{m2} = 0.2$, upper limit of the elitist model is 15, and super iteration number is 150.

In Table 3, the optimization results of order cycle length, order quantity, and average profit for every retailer are listed. Correspondingly, the optimal average profit of the retailers sums up to 3457.24. For the distributor, the optimal order cycle length $KL^* = 6$, the optimal order quantity $Q_0^* = 11533.17$, and the optimal average profit is 6880.468. On franchise chain mode, optimal average profit of the supply chain is 10337.71. An optimal distribution route schedule of the distributor, in which $KL = 6$, is shown in Table 4.

On regular chain mode, the parameters of the upper layer algorithm are as follows: the number of initial population is 40, $P_s = 0.1$, $P_c = 0.8$, $P_m = 0.2$, and upper limit of the elitist model is 20. The lower layer algorithm has the same parameters as those of algorithm on franchise chain mode and the oddment modification threshold $\Delta G = 100$.

In Table 5, the optimization results of order cycle length and order quantity for every retailer are listed. Correspondingly, for the distributor, the optimal order cycle length $KL^* = 2$ and the optimal order quantity $Q_0^* = 3543.11$. On regular chain mode, optimal average profit of the supply chain is 11290.3. An optimal distribution route schedule of the distributor, in which $KL = 2$, is shown in Table 6.

To illustrate the validity of oddment modification, we let the vehicle capacity to be 1000, then use the two-layer genetic algorithm based on oddment modification and without oddment modification, respectively, to calculate the problem of regular chain mode. The optimization results are shown in Table 7, in which oddment modification is done when $n = 1$; two routes are combined, and the corresponding modification delivery quantity is 85.

From Table 7, when the tasks of some routes are not adequate, by reducing the number of distribution routes, oddment modification may save more transportation cost at the cost of lesser shortage. It contributes to improvement of the supply chain's total profit.

Table 3: Optimal results of retailers on ranchise chain mode.

	n_j	Q_j	π_j
$j = 1$	3	178.72	85.50
$j = 2$	1	110.70	221.17
$j = 3$	2	198.84	182.60
$j = 4$	2	74.56	59.45
$j = 5$	1	161.83	328.96
$j = 6$	3	446.80	237.46
$j = 7$	3	345.64	188.74
$j = 9$	1	64.73	119.59
$j = 9$	2	311.38	278.58
$j = 10$	1	97.10	192.22
$j = 11$	2	203.16	173.11
$j = 12$	3	75.77	35.27
$j = 13$	1	144.66	292.43
$j = 14$	3	312.76	164.22
$j = 15$	2	98.36	78.29
$j = 16$	1	105.17	207.51
$j = 17$	2	122.96	100.37
$j = 18$	1	118.68	241.69
$j = 19$	3	134.04	63.63
$j = 20$	2	233.54	206.43

Table 4: Optimal distribution route of a distribution cycle on ranchise chain mode.

Serial number of distribution time	Ω
$n = 1$	0-2-5-10-8-16-13-18-0
$n = 2$	0-2-4-5-9-17-11-10-20-0, 0-18-16-8-15-13-3-0
$n = 3$	0-2-5-6-12-14-0, 0-1-18-13-7-16-8-19-10-0
$n = 4$	0-2-4-5-9-17-11-10-20-0, 0-18-16-8-15-13-3-0
$n = 5$	0-2-5-10-8-16-13-18-0
$n = 6$	0-2-4-5-9-20-0, 0-14-10-12-11-17-6-0, 0-1-3-13-7-15-19-8-16-18-0

Table 5: Optimal results of retailers on regular chain mode.

	n_j	Q_j
$j = 1$	2	103.79
$j = 2$	1	110.70
$j = 3$	2	198.84
$j = 4$	2	74.57
$j = 5$	1	161.83
$j = 6$	2	259.49
$j = 7$	2	203.16
$j = 9$	1	64.73
$j = 9$	1	136.33
$j = 10$	1	97.10
$j = 11$	1	89.95
$j = 12$	2	46.65
$j = 13$	1	144.66
$j = 14$	2	181.64
$j = 15$	2	98.36
$j = 16$	1	105.17
$j = 17$	1	55.35
$j = 18$	1	118.68
$j = 19$	2	77.85
$j = 20$	2	233.54

Table 6: Optimal distribution route of a distribution cycle on regular chain mode.

Serial number of distribution time	Ω
$n = 1$	0-2-5-9-17-11-10-8-16-13-18-0
$n = 2$	0-2-4-5-9-6-17-11-12-10-14-20-0, 0-1-3-13-7-15-19-8-16-18-0

Table 7: $G = 1000$, optimal routes of a distribution cycle on regular chain mode.

	Without oddment modification	With oddment modification
Ω		
$n = 1$	0-2-5-9-17-11-10-0, 0-8-16-13-18-0	0-2-5-9-17-11-10-8-16-13-18-0
$n = 2$	0-2-4-5-9-6-17-11-12-0, 0-1-3-13-7-15-19-16-18-0, 0-20-14-10-8-0	0-2-4-5-9-6-17-11-12-0, 0-1-3-13-7-15-19-16-18-0, 0-20-14-10-8-0
Average profit of supply chain	11081.3	11176.6

For supply chain, the character of centralized control is better than that of decentralized control. In the frontal case, the average profit of supply chain on regular chain mode, 11290.3, is higher than that on franchise chain mode, 10337.7. However, a fixed order cost, $F_0 = 20$, is considered for every retailer on franchise chain mode, while no order cost is considered on regular chain mode. Further, we remove the retailers' fixed order cost on franchise chain, that is, let $F_0 = 0$. Owing to no transportation cost, shorter order cycle means less cost, so the retailers' optimal length of order cycle is 1. Hence the distributor's optimal length of order cycle and distribution cycle will be 1. The corresponding optimal distribution routes are 0-2-4-5-9-6-17-11-12-10-14-20-0 and 0-1-3-13-7-15-19-8-16-18-0, the distributor's optimal average profit is 6587.65, and the optimal average profit of supply chain is 10361.03, which is higher than that with $F_0 = 20$.

Table 8: Profit of supply chain with different wholesale price, on ranchise chain mode.

Wholesale price	Average profit		
	Retailers	Distributor	Supply chain
$w = 7.48$	3495.89	6717.02	10212.91
$w = 7.49$	3476.48	6900.75	10377.23
$w = 7.50$	3457.24	6880.47	10337.71
$w = 7.54$	3380.40	6927.29	10307.69
$w = 7.60$	3266.69	6898.71	10165.40
$w = 8.00$	2534.11	7419.28	9953.39
$w = 8.99$	791.48	8794.95	9586.43

Table 8 shows the average profit with different wholesale price on franchise chain mode. From Table 8, while wholesale price contract cannot coordinate the supply chain on franchise chain mode, profit distribution can be regulated by changing the wholesale price. For retailers, higher wholesale price means less profit. For the distributor, the opposite tendency is true in most cases, but owing to transportation cost, the profit on $w = 7.50$ is less than that of $w = 7.49$. The average profit of supply chain reaches the maximum (10377.23) at $w = 7.49$, which is less than that on regular chain mode.

7. Conclusions

We have discussed the inventory-distribution routing problem of agriculture products supply chain with one distributor and multiple retailers, based on franchise chain mode and regular chain mode. The corresponding mathematical models are presented.

For the franchise chain model, the optimal order cycle and order quantity of the retailers are dependent on wholesale price; therefore the key to solve the optimal problem is to solve the static multiperiod distribution routing problem. Then, an heuristic adaptive genetic algorithm is proposed for the model of franchise chain. For the regular chain model, a two-layer genetic algorithm based on oddment modification is proposed, which searches for the optimal distribution routes, the optimal order cycle, and order quantity for both the retailers and the distributor. The upper layer provides the distribution cycle and delivery quantity for the retailers, and the lower layer seeks the optimal order cycle, order quantity, distribution routes, and the rational oddment modification number for the distributor. The validity of the algorithms is also demonstrated by numerical examples.

The supply chain on regular chain mode is centralized controlling, and its average profit is higher than that on franchise chain mode, which is decentralized controlling. On franchise chain mode, profit distribution between the retailers and the distributor can be regulated by changing the wholesale price.

Appendix

For more details see Table 2.

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Research Article

Stackelberg Game for Product Renewal in Supply Chain

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The paper studied the process of product renewal in a supply chain, which is composed of one manufacturer and one retailer. There are original product and renewal product in the supply chain. A market share shift model for renewal product was firstly built on a increment function and a shift function. Based on the model, the decision-making plane consisting of two variables was divided into four areas. Since the process of product renewal was divided into two stages, Stackelberg-Nash game model and Stackelberg-merger game model could be built to describe this process. The optimal solutions of product pricing strategy of two games were obtained. The relationships between renewal rate, cost, pricing strategy, and profits were got by numerical simulation. Some insights were obtained from this paper. Higher renewal rate will make participants' profits and total profit increase at the same margin cost. What is more important, the way of the optimal decision making of the SC was that RP comes onto the market with a great price differential between OP and RP.

1. Introduction

With the development of IT, the speed of product renewal process becomes faster and faster. This situation brings some new problems to traditional supply chain (SC), such as the dynamic nature of an SC and the relationship coordination between participants of an SC. The dynamic nature of an SC includes the dynamic variety of products; that is to say, the market requests SC to satisfy diverse needs of customer with the fastest speed, as well as the best quality, which calls for SC improving performance to adapt to the product variety. A valid way for SC enterprises to follow the variety of market is to renew their existing products, which can make full use of enterprise's existing resources and supply/distribution outlet of SC. Thus, SC will adapt to a variational market economically and quickly. Product renewal is an effective method that strengthens a SC enterprise's core competitiveness, and it will even impact on the survival of a enterprise. There are quite a few cases that enterprises collapse because of the mistakes of product renewal decision making, such as the failure of WANGAN Computer Corp.

Now, it is necessary to differentiate "renewal product (RP)" from "innovated product (IP)." IP's structure and principle are different from original product (OP), while RP's main structure and principle are the same as OP. Furthermore, RP has some appended components or upgraded functions. Thereby, RP is the renewal of existing product. At present, most researches are mainly focused on complete new product innovation, including management of product innovation process, optimization of product innovation investment, promotion of new product, and design of product innovation drive mechanism. Dereli et al. proposed a framework of the rapid response for innovative product development using reverse engineering approach [1]. Bourreau et al. studied the effects of modular design on firms' product innovation strategies and postinnovation competition in digital market [2]. Ju and Xiao built a model of product lifecycle evaluation based on context knowledge [3]. Wan et al. built a new product investment model [4]. Researches about product innovation based on SC frame have also emerged. Huo has studied the R&D strategy in SC using game theory [5].

At present, the researches concerning product renewal have been carried out. Bass P. I. and Bass F. M. had proposed a new product expansion model and studied the process of multigeneration renewal products coming onto market [6, 7]. Wu examined the reuse/redesign, quality, speed-to-market, and marketing decisions for two consecutive generations of a multicomponent modular product, utilizing stylized models [8]. Druehl et al. developed a model to gain insight into which factors drive the pace of product updates [9]. Huang and Ling studied the tracks of product update of ICT enterprises [10]. Koca et al. studied the product rollover strategy decision, where a firm decides whether to phase out an old generation product to be replaced by a new with either a dual or single roll [11]. Quan analyzed the relationship of the optimal launch time with parameters in the process of introducing a renewal product [12].

The other focus of product renewal is on the marketing process of renewal product, especially pricing decision. Luo and Tu studied a price decision problem for product renewal supply chain based on Nash game [13]. Wei studied the optimal decision of inventory and pricing in a supply chain, in which there are original products and renewal products [14]. But the work of product renewal is far less than enough.

The RP and OP are virtually differentiated products. When they are on sale at the same market, they can substitute for each other. Compared with common differentiated products, the substitution of RP for OP is unidirectional and partial. Therefore, the coming of RP onto market will cause some influences on OP market. At the same time, each entity in SC has independent profit. Therefore, the coordination of SC will become much more complex if there are RP and OP in the SC. A potent tool to coordinate SC is price. The fluctuation of price will make supply and demand tend to an equilibrium. On the other hand, price's fluctuation can make the profit of each entity in SC distributed reasonably, so as to achieve the aim of coordination of SC. Therefore pricing strategy is the main content of the paper.

The SC including OP and RP is called product renewal SC. The change of market share caused by product renewal in the SC is studied in this paper. When there is a RP in SC, the enterprises in SC will make strategies for both of OP and RP to achieve an equilibrium of enterprises' profits. Compared with other researches, this paper firstly studied Stackelberg game for product renewal, aiming at solving complex decision-making question in actual renewal SC, and obtained some novel insights.

The rest of this paper is organized as follows. Product renewal model is formulated in Section 2. In Section 3, we provide a Stackelberg-Nash Game based on product renewal model. Stackelberg-merger game is played in Section 4, and numerical examples are provided in Section 5. Finally, we conclude our paper in Section 6.

2. Model Description

2.1. Assumption. Consider a two-stage SC which is composed of one manufacturer and one retailer. First, suppose the manufacturer produces product A and sells it in market via the retailer. After a period of renewal process, the manufacturer develops product A's renewal product, B, and sells it with product A in market via the retailer.

In order to fully describe the model of the SC, we state the following hypotheses.

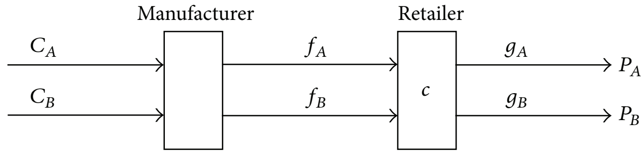

Figure 1: The structure of renewal products SC.

(i) The retailer is in a monopoly position; that is, only the retailer sells A and B in market.

(ii) Suppose that the consumer's repurchase is possible and each time one customer only buys one product. When there is an RP, some customers of OP may shift to buying RP in other words, there is repetition purchase for the products studied in this paper, such as toys, and small electric appliance equipment.

(iii) There is no stochastic demand in the market, so the demand is only decided by price.

(iv) Inventory is not considered.

(v) The manufacturer's productivity is sufficient enough to satisfy demand.

(vi) There is no fixed cost for unit product, and margin production cost is an invariant for manufacturer.

(vii) The product renewal generally causes its performance promotion.

The SC model of this paper is as follows: there is one manufacturer and one retailer in SC. Suppose that the manufacturer produces products A and B at marginal costs C_A and C_B, respectively, and then distributes them at the wholesale price of f_A and f_B to retailer. Based on the wholesale price, a price markup g_A and g_B is, respectively, added by retailer to product A and B to form the market retail price: $P_i = g_i + f_i$. Retailer's selling cost is c per unit. The product renewal SC including one manufacturer and one retailer that sells combination products is shown in Figure 1.

2.2. Market Model. Product renewal rate is used to measure the change of product performance caused by product renewal. Combined with price and performance price ratio ($E_{ij} = kP_j/P_i$), product renewal rate can be used to forecast the market share change. In fact, what is focused on is not performance value, but the ratio of OP's performance value to RP's [15], namely renewal rate k ($k > 1$).

The RP marketing process can be divided into two stages:

In Stage 1. there is only OP in market. Supposing the OP price of A is P_A, according to [15], the market demand for A is

$$T = a - bP_A. \tag{1}$$

In expression (1), a means market scale, the absolute value of b denotes price flexibility, and $b > 0$.

In Stage 2. RP comes onto the market and competes with OP. If RP comes onto the market, it will cause several changes in the market: (1) RP will stimulate new consumption demand because of its appended or upgraded functions, which makes the latent customer quantity of the total products market enlarge, and the increment is ΔT. ΔT is related to original market demand T and a decreasing function of RP price, while an increasing function of renewal rate k. (2) Customers have the tendency to purchase RP. In addition, the performance price ratio of RP is generally higher than that of OP. So some OP's customers will shift to buying RP. The number of shift customers is marked as T_s. (3) This will cause OP's price to be adjusted to market.

Definition 1. There exists an OP A and its RP B in the market. Suppose that the demand of A is T, and the product renewal rate is k. Then ΔT satisfies the following function:

$$\Delta T = w_1 T\left(k - w_2 P_B\right), \tag{2}$$

where w_1 is the influence coefficient of T and $0 < w_1 < 1$. w_2 is the influence coefficient of RP's price on its market increment, and $w_2 \ll 1$.

From $\Delta T = 0$ and $T = 0$, $P_{B1} = k/w_2$ and $P_{A1} = a/b$ can be got. If $P_A > P_{A1}$, OP's demand will be zero, which leads ΔT to be zero. If $P_B > P_{B1}$, no matter how great OP's demand is, demand increment ΔT will be equal to zero.

Theorem 2. *The market shift function T_s relates to OP price, RP price, and OP demand. T_s has the following features.*

(i) *If $P_B \le P_A$, $T_s = T$, that is, all of the OP demand will shift to RP. So OP will have to exit from the market, and RP will hold the market.*

(ii) *Existing $P_{B0} > PA$, if $P_B \ge P_{B0}$, $T_s = 0$. P_{B0} can be treated as the zero point of market shift.*

(iii) *If $P_A < P_B < P_{B0}$, T_s will be a monotone decreasing function of P_B, namely, $\partial T_s/\partial P_B < 0$.*

(iv) *If $P_A < P_B < P_{B0}$, T_s is to be a monotone increasing function of P_A, namely, $\partial T_s/\partial P_A > 0$.*

(v) *With the increase of k, P_{B0} increases correspondingly. T_s is an increasing function of k, namely, $\partial T_s/\partial k > 0$.*

(vi) *Let $\Delta T = 0$, and the corresponding price is P_{B1}, which is the upper limit of RP price.*

(vii) *Let $T_s = 0$, and there is still demand increment caused by RP; that is to say, on any condition, $P_{B0} < P_{B1}$.*

(viii) *T_s is a direct proportion increasing function with T.*

From Theorem 2, given P_A, the laws of changes of T_s and ΔT can be got with the variety of P_B, as shown in Figure 2. The specific shape of this figure is changing with the variety of P_A.

According to Figure 2, the shift function T_s in $[0, P_A]$ is

$$T_{s1} = T. \tag{3}$$

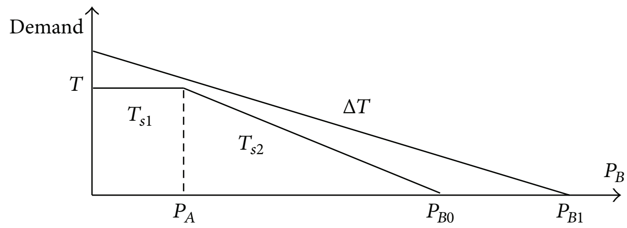

Figure 2: The changes of T_s and ΔT with the variety of P_B.

Definition 3. According to Figure 2, in $[P_A, P_{B0}]$, there exist an OP A and its RP B; suppose the prices of A, B are P_A, P_B, respectively, and the demand of A is T. Then, T_{s2} satisfies the following function:

$$T_{s2}\left(P_A, P_B\right) = w_3 T\left(k w_4 P_A - P_B\right), \tag{4}$$

where w_3 is the influence coefficient of P_A and P_B, $0 < w_1 < 1$, w_4 is the influence coefficient of P_A, and $w_4 > 1/k$.

The functions (3) and (4) meet eight features listed previewed, and can be used as the paper's model base. From the previous analysis, the shift function includes two portions, so it can be expressed as follows:

$$T_s = \begin{cases} T & P_B \le P_A, \\ \left[w_3 T\left(k w_4 P_A - P_B\right)\right]^+ & P_B > P_A. \end{cases} \tag{5}$$

Letting $T_s = 0$ and solve (5), the zero point of market shift function P_{B0} can be got:

$$P_{B0} = k w_4 P_A. \tag{6}$$

With the increase of k, P_{B0} obviously becomes higher. Therefore, when RP comes onto the market, the total demand will change. The market share of RP B is

$$T_B = \Delta T + T_s, \tag{7}$$

and OP A is

$$T_A = T - T_s. \tag{8}$$

In addition, product renewal will make marginal production cost change from C_A to C_B.

Figure 2 is a variety chart of shift demand T_s in the situation that P_A is given, so the shift demand T_s actually depends on P_A and P_B simultaneously. Accordingly, a three-dimensional coordinate can be built, P_A-P_B-T_s, as shown in Figure 3, and axis T_s is shift demand. Plane P_A-P_B is divided into the following four areas by six lines in Figure 3:

Area I: the area surrounded by axis P_A and lines $P_A = P_B$ and $P_A = a/b$,

Area II: the area surrounded by lines $P_A = P_B$, $P_A = a/b$ and $P_B = k w_4 P_A$,

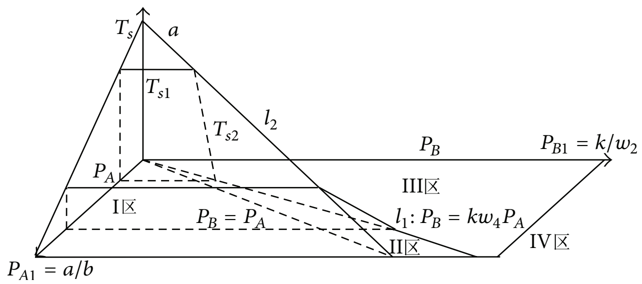

Figure 3: The change of T_s with the variety of P_B and P_A.

Table 1: The table of area's shift function and increment function.

	$T_s(P_A, P_B)$	$\Delta T(P_A, P_B)$
Area I	$T(P_A)$	$w_1T(P_A)(k - w_2P_B)$
Area II	$w_3T(P_A)(kw_4P_A - P_B)$	$w_1T(P_A)(k - w_2P_B)$
Area III	0	$w_1T(P_A)(k - w_2P_B)$
Area IV	0	0

Area III: the area surrounded by axis P_B and lines $P_B = kw_4P_A$, $P_A = a/b$ and $P_B = k/w_2$,

Area IV: the areas excepting area I, II, and III.

In area I, Letting $P_A = 0$, T_s's maximum value a can be obtained from (1) and (2). Given P_A, the line T_{s1} will be parallel to plane P_A-P_B, and it is clear that the slope of line T_{s1} is $k_{Ts1} = 0$. From (3), with the increasing of P_A, T will become small. Meanwhile, T_{s1} will become small. If $P_A = a/b$, $T = T_{s1} = 0$. Therefore, T_s will be a sloped plane in area I.

In area II, given P_A, $k_{Ts2} = -w_3T < 0$ can be obtained from (5). The line T_{s2} will become smoother and smoother with the increase of P_A and k_{Ts2}. If $P_A = a/b$, $k_{Ts2} = T_{s2} = 0$. Letting $T_{s2} = 0$ in (4), a line $l_1 : P_B = kw_4P_A$ in P_A-P_B plane can be got. The line linking maximal T_s point $(0, 0, a)$ to point $(a/b, a/b, 0)$ is a T_s line named l_2. The lines linking the points at which line l_1 or l_2 intersects with T_{s2} constitute T_s complex curving surface; in other words, T_s in area II is a complex curving surface.

In area III, $P_B > kw_4P_A$, $T_s = 0$ can be got from (5).

Considering T_s function's feature (vii), it is obvious that market increment $\Delta T > 0$ in area I, II, and III.

In area IV, $P_A > a/b$ or $P_B > k/w_2$, $\Delta T = T_s = 0$. OP and RP do not come onto the market in this area.

The shift functions and increment functions in different areas are shown in Table 1.

According to the feature (vii) of shift function, the zero point of increment function is bigger than that of shift function, namely, $P_{B1} > P_{B0} = kw_4P_A$. From expression (1), it is known that the maximum of P_A is a/b. So it requires $b > aw_2w_4$ to meet the seventh feature of shift function. Based on the previous analysis, the retailer's and manufacturer's profits including the profits of OP and RP are, respectively, as follows:

$$\begin{aligned}\pi_r &= (T_s + \Delta T)(g_B - c) + (T - T_s)(g_A - c),\\ \pi_m &= (\Delta T + T_s)(f_B - C_B) + (T - T_s)(f_A - C_A) - C_g.\end{aligned} \tag{9}$$

Where C_g is the fixed cost of the manufacturer. According to the price relationship between RP and OP in different areas, the profit functions can be shown as follows.

(i) In area I, $P_B \le P_A$, $\Delta T > 0$ and $T_s = T > 0$. The market share of RP B is $\Delta T + T_s$, and OP A is zero. The profits of retailer and manufacturer are

$$\begin{aligned}\pi_r(g_A, g_B) &= T(1 + w_1(k - w_2P_B))(g_B - c)\\ &= (a - b(f_A + g_A))\\ &\quad \times (1 + w_1(k - w_2(f_B + g_B)))(g_B - c),\\ \pi_m(f_A, f_B) &= T(1 + w_1(k - w_2P_B))(f_B - C_B) - C_g\\ &= (a - b(f_A + g_A))\\ &\quad \times (1 + w_1(k - w_2(f_B + g_B)))(f_B - C_B) - C_g.\end{aligned} \tag{10}$$

(ii) In area II, $P_A \le P_B \le P_{B0}$, $\Delta T > 0$ and $T_s > T > 0$. The market share of RP B is $\Delta T + T_s$, and OP A is $T - T_s$. The profits of retailer and manufacturer are

$$\begin{aligned}\pi_r(g_A, g_B) &= T\{(w_1(k - w_2P_B) + w_3(kw_4P_A - P_B))\\ &\quad \times (g_B - c) + (1 - w_3(kw_4P_A - P_B))\\ &\quad \times (g_A - c)\}\\ &= (a - b(f_A + g_A))\\ &\quad \times \{(w_1(k - w_2(f_B + g_B))\\ &\quad + w_3(kw_4(f_A + g_A)\\ &\quad - (f_B + g_B)))(g_B - c)\\ &\quad + (1 - w_3(kw_4(f_A + g_A) - (f_B + g_B)))\\ &\quad \times (g_A - c)\},\\ \pi_m(f_A, f_B) &= T\{(w_1(k - w_2P_B) + w_3(kw_4P_A - P_B))(f_B - C_B)\\ &\quad + (1 - w_3(kw_4P_A - P_B))(f_A - C_A)\} - C_g\\ &= (a - b(f_A + g_A))\\ &\quad \times \{(w_1(k - w_2(f_B + g_B))\\ &\quad + w_3(kw_4(f_A + g_A)\\ &\quad - (f_B + g_B)))(f_B - C_B)\\ &\quad + (1 - w_3(kw_4(f_A + g_A) - (f_B + g_B)))\\ &\quad \times (f_A - C_A)\} - C_g.\end{aligned} \tag{11}$$

(iii) In area III, $P_{B0} \le P_B \le P_{B1}$, $\Delta T > 0$ and $T_s = 0$. RP B's market share is ΔT, and OP A's market share is T. The profits of retailer and manufacturer are

$$\begin{aligned}
\pi_r(g_A, g_B) &= T\{w_1(k - w_2P_B)(g_B - c) + g_A - c\} \\
&= (a - b(f_A + g_A)) \\
&\quad \times \{w_1(k - w_2(f_B + g_B))(g_B - c) + g_A - c\}, \\
\pi_m(f_A, f_B) &= T\{w_1(k - w_2P_B)(f_B - C_B) + f_A - C_A\} - C_g \\
&= (a - b(f_A + g_A)) \\
&\quad \times \{w_1(k - w_2(f_B + g_B)) \\
&\quad \times (f_B - C_B) + f_A - C_A\} - C_g.
\end{aligned} \tag{12}$$

(iv) In area IV, $P_B > P_{B1}$ or $P_A > P_{A1}$, $\Delta T = T_s = 0$. The prices of OP and RP are so high that it could not come onto the market. It is not necessary to consider this situation.

3. Stackelberg-Nash Game

Product renewal process can be regarded as two stages. First, manufacturer and retailer make their optimal decisions for OP A, respectively, and synchronously. After OP A comes onto market, RP B is developed by the manufacturer and comes onto market, too. In this condition, both OP A and RP B are sold in the same market. Then the manufacturer and retailer make their optimal decisions for RP B, respectively, and synchronously. The optimal decision-making process is also divided into two stages. The two-stage games constitute a Stackelberg-Nash game, which is the real process of product renewal game.

In Stage 1: There is a decision for product A. The manufacturer and the retailer play a Nash game. The manufacturer makes a strategic decision of f_A, and the retailer makes a strategic decision of g_A.

In Stage 2: There is a decision for product B. The manufacturer and the retailer also play a Nash game. The manufacturer makes a strategic decision of f_B, and the retailer makes a strategic decision of g_B.

The game solving process is divided into three steps. Following three steps, the optimal solution of product renewal Stackelberg-Nash game can be derived.

In Step 1, given f_A and g_A, according to the Nash equilibrium of the manufacturer and retailer profit functions, the optimal reaction function of $f_B(f_A, g_A)$, $g_B(f_A, g_A)$ or the optimal value of g_B and f_B can be obtained from $\partial\pi_r/\partial g_B = 0$, $\partial\pi_m/\partial f_B = 0$.

In Step 2, put the optimal reaction function or the optimal value into the profit functions of Step 1, and the optimal solution $f_A{}^*$ and $g_A{}^*$ can be obtained from the Nash equilibrium of the manufacturer and retailer profit functions in Step 1.

In Step 3, letting $f_A{}^*$ and $g_A{}^*$ substitute f_A and g_A in the $f_B(f_A, g_A)$ and $g_B(f_A, g_A)$ of Step 2, the optimal value of $f_B{}^*$ and $g_B{}^*$ can be obtained.

Theorem 4. *In Stackelberg-Nash game, in areas I, II, and III, given f_A and g_A, π_r is a convex function of g_B, and π_m is convex functions of f_B.*

Proof. In area I

$$\begin{aligned}
\frac{\partial\pi_r(g_A, g_B)}{\partial g_B} &= (a - b(f_A + g_A)) \\
&\quad \times (-2w_1w_2g_B + w_1w_2(c - f_B) + w_1k + 1), \\
\frac{\partial\pi_m(f_A, f_B)}{\partial f_B} &= (a - b(f_A + g_A)) \\
&\quad \times (-2w_1w_2f_B + w_1w_2(C_B - g_B) + w_1k + 1), \\
\frac{\partial\pi_r^2(g_A, g_B)}{\partial g_B^2} &= -2w_1w_2(a - b(f_A + g_A)) < 0, \\
\frac{\partial\pi_m^2(f_A, f_B)}{\partial f_B^2} &= -2w_1w_2(a - b(f_A + g_A)) < 0.
\end{aligned} \tag{13}$$

Therefore, π_r is a convex function of g_B, and π_m is a convex function of f_B. The proofs in II and III are the same as area I. □

Theorem 5. *The Stackelberg-Nash decision-making process can be divided into the following three situations.*

(i) *In area I, there is no Stackelberg-Nash equilibrium.*

(ii) *In area II, there exists only one optimal pricing strategy for Stackelberg-Nash game.*

(iii) *In area III, there exists only one optimal pricing strategy for Stackelberg-Nash game.*

Proof. (i) In area I, according to (10), based on Theorem 4 and Step 1, the optimal value of g_B and f_B can be derived as follows:

$$\begin{aligned}
g_B{}^* &= \frac{(1 + w_1k + 3w_1w_2(2c - C_B))}{(3w_1w_2)}, \\
f_B{}^* &= \frac{(1 + w_1k + 3w_1w_2(2C_B - c))}{(3w_1w_2)}.
\end{aligned} \tag{14}$$

From the supply chain's structure, it is obvious that $g_B{}^* + f_B{}^* = P_B{}^*$. Because $P_B{}^*$ is greater than P_{A1} (from Theorem 6), considering the condition of area I, $P_B \le P_A$, the conclusion that $P_B{}^*$ is out of area I can be drawn. Accordingly, there is no Nash equilibrium in area I.

(ii) In area II, $P_A \le P_B \le P_{B0}$. According to (11), based on Theorem 4 and Step 1, the optimal reaction

$(f_B(f_A, g_A), g_B(f_A, g_A))$ of the manufacturer and retailer can be got as follows:

$$\begin{aligned} f_B(f_A, g_A) = {} & (w_3(kw_4+2) f_A + w_3(kw_4-1) g_A \\ & + w_1 k + 2w_3(C_B - C_A) \\ & + w_1 w_2(2C_B - c)) / (3w_3 + 3w_1 w_2), \\ g_B(f_A, g_A) = {} & (w_3(kw_4-1) f_A + w_3(kw_4+2) g_A \\ & + w_1 k + w_3(C_A - C_B) \\ & + w_1 w_2(2c - C_B)) / (3w_3 + 3w_1 w_2). \end{aligned} \tag{15}$$

From Step 2, two binary cubic equations can be got. Because the analytic solution of the equations cannot be found, the numerical method can be used to find the optimal approximate solution $(f_A{}^*, g_A{}^*)$. From Step 3, the optimal of $(f_B{}^*, g_B{}^*)$ can be got.

(iii) In area III, $P_{B0} \le P_B \le P_{B1}$. According to (12), based on Theorem 4 and Step 1, the optimal values of g_B and f_B of π_r and π_m are as follows:

$$\begin{aligned} g_B{}^* &= \frac{(k + w_2(2c - C_B))}{(3w_2)}, \\ f_B{}^* &= \frac{(k + w_2(2C_B - c))}{(3w_2)}. \end{aligned} \tag{16}$$

From Step 2, the optimal values $g_A{}^*$ and $f_A{}^*$ of π_r and π_m can be got as follows:

$$\begin{aligned} g_A{}^* = {} & -\left(w_1 b(k - w_2 C_B)^2 + 2w_1 w_2 bc(w_2 C_B - 1) + w_1 bc^2 w_2^2 \right. \\ & \left. + 9bw_2(C_A - 2c) - 9000w_2\right) / (27w_1 b), \\ f_A{}^* = {} & -\left(w_1 b(k - w_2 C_B)^2 + 2w_1 w_2 bc(w_2 C_B - 1) + w_1 bc^2 w_2^2 \right. \\ & \left. + 9bw_2(c - 2C_A) - 9000w_2\right) / (27w_1 b). \end{aligned} \tag{17}$$

□

Theorem 6. *The value of Nah equilibrium for area I, $P_B{}^*$, is out of area I.*

Proof. From the supply chain structure, it is obvious that: $g_B{}^* + f_B{}^* = P_B{}^*$:

$$P_B{}^* = g_B{}^* + f_B{}^* = \frac{(2 + 2w_1 k)}{(3w_1 w_2)} + C_B - c. \tag{18}$$

Because of $0 < w_1 < 1$ and $k > 1$,

$$P_B{}^* \ge \frac{(2+2k)}{(3w_2)} + C_B - c \ge \frac{4}{(3w_2)} + C_B - c, \tag{19}$$

whereas $w_2 \ll 1$,

$$P_B{}^* \ge P_{A1} = \frac{a}{b}. \tag{20}$$

It is clear that the value of Nah equilibrium, $P_B{}^*$, is greater than P_{A1}. Since P_{A1} is the maximum of P_A in area I, considering the condition of area I, $P_B \le P_A$, the value of Nah equilibrium, $P_B{}^*$, does not satisfy this condition. So $P_B{}^*$ is out of area I. □

4. Stackelberg-Merger Game

In the decision-making process of Stackelberg-Nash game mentioned previously, if the manufacturer and retailer constitute a manufacturing and sales league to maximize the SC profit, that is, there is no retailer, and manufacturer sells its product in market directly, the product renewal Stackelberg-Nash decision-making process will become a Stackelberg-merger decision-making process. This process can be divided into two stages, and this process will play a Stackelberg-merger game. In this condition, the league which consists of the manufacturer and the retailer can be regarded as a enterprise, because the league's decision variables are the retail price P_i of OP and RP (f_i and g_i can be regarded as the transfer price of the manufacturer and retailer profits distribution).

In Stage 1, the league sells OP A and makes a strategic decision of P_A.

In Stage 2, the league sells OP A and RP B at the same time, and the league makes strategic decisions of P_B and P_A.

The solving process of Stackelberg-merger game is divided into three steps, and by three steps the optimal solution of product renewal Stackelberg-merger game can be derived.

In Step 1, OP retail price P_A is given; the optimal reaction function $P_B(P_A)$ or the optimal value of P_B can be obtained from $\partial\pi/\partial P_B = 0$.

In Step 2, put the optimal reaction function $P_B(P_A)$ or the optimal value $P_B{}^*$ into the first stage; the optimal pricing $P_A{}^*$ can be obtained from $\partial\pi/\partial P_A = 0$.

In Step 3, put $P_A{}^*$ into the optimal reaction function $P_B(P_A)$; the optimal pricing $P_B{}^*$ can be derived.

Theorem 7. *In Stackelberg-merger game, in areas I, II, and III, given P_A, π is a convex function of P_B.*

Proof. In area I

$$\begin{aligned} \frac{\partial\pi(P_A, P_B)}{\partial P_B} = {} & (a - bP_A) \\ & \times(-2w_1 w_2 P_B + w_1 w_2(C_B + c) + w_1 k + 1), \end{aligned} \tag{21}$$

$$\frac{\partial\pi^2(P_A, P_B)}{\partial P_B^2} = -2w_1 w_2(a - bP_A) < 0. \tag{22}$$

Therefore, π is a convex function of P_B. The proofs in II and III are the same as those of area I. □

Theorem 8. *The process of the Stackelberg-merger game is divided into three situations according to areas I, II, III. There exists only one optimal pricing strategy for Stackelberg-merger game in each situation.*

Proof. (i) In area I, $P_B \leq P_A$. the total profit of the SC can be got from the sum of (10):

$$\pi(P_A, P_B) = (a - bP_A)(w_1(k - w_2 P_B) + 1) \times (P_B - C_B - c) - C_g. \tag{23}$$

Base on Theorem 7 and Step 1, the optimal value P_B of π can be obtained as follows:

$$P_B{}^* = \frac{(1 + w_1 k)}{(2w_1 w_2)} + \frac{(C_B + c)}{2}. \tag{24}$$

The value of w_2 is so small that the value of $P_B{}^*$ is greater than the maximum value of P_{A1}. Considering $P_B \leq P_A$ and Theorem 7, the maximum value of π can be obtained in the condition that $P_B = P_A$, and then (21) can be expressed as follows:

$$\pi(P_A) = (a - bP_A)(w_1(k - w_2 P_A) + 1)(P_A - C_B - c) - C_g. \tag{25}$$

From $\partial\pi/\partial P_A = 0$, the two values of P_A can be got. One solution is so small that it cannot match the situation, and it is cast out. The other is the optimal value of P_A. So there is only one solution for area I (from Theorem 9).

(ii) In area II, the total profit of SC can be got from the sum of (11):

$$\begin{aligned}\pi(P_A, P_B) = {} & (a - bP_A) \\ & \times \{(w_3(kw_4 P_A - P_B) + w_1(k - w_2 P_B)) \\ & \times (P_B - C_B - c) \\ & + (1 - w_3(kw_4 P_A - P_B))(P_A - C_A - c)\} - C_g.\end{aligned} \tag{26}$$

Based on Theorem 7 and Step 1, the optimal reaction $P_B(P_A)$ of π can be obtained as follows:

$$P_B(P_A) = \frac{(w_3(kw_4 + 1)P_A + w_1 w_2(C_B + c) + w_1 k + w_3 C_A)}{2(w_1 w_2 + w_3)}. \tag{27}$$

From Step 2, $P_A{}^*$ can be got, and the expression of $P_A{}^*$ is so long that it could not be written down here.

From Step 3, the optimal value of P_B for π can be obtained.

(iii) In area III, the total profit of SC can be derived from the sum of (12):

$$\pi(P_A, P_B) = (a - bP_A) \times (w_1(k - w_2 P_B)(P_B - C_B - c) + P_A - C_A - c). \tag{28}$$

Based on Theorem 7 and Step 1, the optimal value of P_B for π can be obtained as follows:

$$P_B{}^* = \frac{k}{2w_2} + \frac{(C_B + c)}{2}. \tag{29}$$

From Step 2, the optimal of P_A for π can be obtained as follows:

$$P_A{}^* = \frac{a}{2b} - \frac{(w_1(k - (C_B + c))(k - w_2(C_B + c)) - 4(C_A + c))}{8}. \tag{30}$$

□

Theorem 9. *In area I, the optimal value of P_A is the only one valid solution, which matches the real situation.*

Proof. In area I, from $\partial\pi(P_A)/\partial P_A = 0$,

$$\begin{aligned}&3bw_1 w_2 P_A^2 - (2w_1 w_2 a + b(C_B + c)(w_1 w_2)) \\ &\quad + (w_1 k + 1)(a - b(C_B + c)) + w_1 w_2 a(C_B + c) = 0.\end{aligned} \tag{31}$$

Two P_A can be got from (31):

$$P_A = \frac{((2w_1 w_2 a + b(C_B + c)(w_1 w_2 - 1)) \pm \sqrt{F})}{(6bw_1 w_2)}, \tag{32}$$

where

$$\begin{aligned}F = {} & (2w_1 w_2 a + b(C_B + c)(w_1 w_2))^2 \\ & - 12bw_1 w_2 \{(w_1 k + 1)(a - b(C_B + c)) + w_1 w_2 a(C_B + c)\}.\end{aligned} \tag{33}$$

In (32), P_A with the negative value of sqrt F is

$$\begin{aligned}P_A^1 &= \frac{((2w_1 w_2 a + b(C_B + c)(w_1 w_2 - 1)) - \sqrt{F})}{(6bw_1 w_2)} \\ &\approx \frac{a}{(3b)} + \frac{(C_B + c)}{6} - \frac{1}{(6w_1 w_2)} - \frac{\sqrt{F}}{(6bw_1 w_2)},\end{aligned} \tag{34}$$

where

$$\frac{\sqrt{F}}{(6bw_1 w_2)} \approx \frac{a}{(3b)} + \frac{(C_B + c)}{6} - \frac{1}{(6w_1 w_2)}. \tag{35}$$

Substituting (35) in (36),

$$P_A^1 \approx 0 < C_B + c. \tag{36}$$

It is obvious that the solution cannot match the real situation, and it was cast out. Therefore, the optimal value of P_A is the only one valid solution, which matches the real situation.

The key of the Stackelberg-merger game is how to distribute the total profit between manufacturer and retailer. Using Nash bargaining model can coordinate profit of each entity [16], and the study of this field needs to go further. □

Table 2: The effect of RP cost changing in Stackelberg-Nash game ($k = 1.2$).

Case result	$k=1.2$ $C_B=220$			$k=1.2$ $C_B=250$		
	$P_B \le P_A$	$P_A < P_B < P_{B0}$	$P_{B0} < P_B < P_{B1}$	$P_B \le P_A$	$P_A < P_B < P_{B0}$	$P_{B0} < P_B < P_{B1}$
f_A		284.3316	266.0267		286.2994	267.7067
g_A		104.3316	86.0267		106.2994	87.7067
f_B		371.5171	540.0000		393.8308	560.0000
g_B		171.5171	340.0000		163.8308	330.0000
P_A	365.9493	388.6632	352.0533	381.8052	392.5987	355.4133
P_B	365.9493	543.0343	880.0000	381.8052	557.6615	890.0000
T	268.1014	222.6737	295.8933	236.3896	214.8025	289.1733
ΔT	178.8881	117.0312	75.7487	154.7302	110.3807	71.7150
T_s	268.1014	29.6436	0	236.3896	13.1986	0
T_A	0	193.0301	295.8933	0	201.6040	289.1733
T_B	536.2027	252.3172	295.8933	472.7791	228.0011	289.1733
π_r		38502	43776		35173	41811
π_m		28502	33776		25173	31811
π	46298	67005	77553	33729	60346	73621

Table 3: The effect of RP cost changing in Stackelberg-Nash decision ($k = 1.4$).

Case result	$k=1.4$ $C_B=220$			$k=1.4$ $C_B=250$		
	$P_B \le P_A$	$P_A < P_B < P_{B0}$	$P_{B0} < P_B < P_{B1}$	$P_B \le P_A$	$P_A < P_B < P_{B0}$	$P_{B0} < P_B < P_{B1}$
f_A		279.2893	253.4637		281.5733	255.4993
g_A		99.2893	73.4637		101.5733	75.4993
f_B		399.1822	606.6667		422.1999	626.6667
g_B		199.1822	406.6667		192.1999	396.6667
P_A	366.3029	378.5785	326.9000	382.0862	383.1467	331.0000
P_B	366.3029	598.3644	1013.3000	382.0862	614.3999	1023.3000
T	267.3943	242.8429	346.1452	232.8276	233.7066	338.0030
ΔT	221.1238	155.7372	107.0742	192.0417	146.8800	101.8516
T_s	267.3943	73.1396	0	235.8276	54.7557	00
T_A	0	169.7034	346.1452	0	178.9509	338.0030
T_B	534.7886	315.9825	346.1452	471.6552	288.4624	338.0030
π_r		54466	59910		49319	57120
π_m		44466	49910		39319	47120
π	51701	98933	109820	37958	88639	104250

5. Numerical Simulation

5.1. Stackelberg-Nash Game Simulation. Look at mobile phone, for example. There are two types of mobile phone A and B in the same market, and B is the renewal product of A. The related parameters' values are as follows: $a = 1000, b = 2, w_1 = 0.8, w_2 = 0.001, w_3 = 0.008, w_4 = 1.2, C_A = 200$. Table 2 shows the effect of the change of RP cost on price, demand, and profits of both products in the condition $k = 1.2$. Table 3 shows the effect in the condition $k = 1.4$. From Tables 2 and 3, the effect of renewal rate k on price, demand, and profits can be concluded.

From numerical simulation in Tables 2 and 3, the following conclusions can be drawn.

(1) There is no Stackelberg-Nash equilibrium in area I, and so Stackelberg-merger is used to simulate this situation. The total profit in this situation is the optimal profit of area I. Obviously, the optimal total profit of area I is less than those of areas II and III. Therefore, area I ($P_B = P_A$) is not the optimal area of Stackelberg-Nash game. The rational manufacturer and retailer could not select area I.

(2) For all parameter conditions, the profits of manufacturer, retailer, and the total profit of SC in area III are greater than those of area II (in area II, there is a little difference between the price of RP and OP). Because the manufacturer and retailer are rational, the final decision-making result (P_A, P_B) of the whole SC should be in area III. In this condition, OP is set at a low price (break-even sales), while RP is priced at a high level (the price differential between RP and OP is huge); that is to say, RP comes onto the market at a high price, and OP comes onto the market at a low price, which leads to larger market

Table 4: The effect of RP cost changing in Stackelberg-merger model ($k = 1.2$).

Case result		$k = 1.2$ $C_B = 220$			$k = 1.2$ $C_B = 250$	
	$P_B \le P_A$	$P_A < P_B < P_{B0}$	$P_{B0} < P_B < P_{B1}$	$P_B \le P_A$	$P_A < P_B < P_{B0}$	$P_{B0} < P_B < P_{B1}$
P_A	365.9493	346.2141	267.8400	381.8052	346.2141	273.5100
P_B	365.9493	458.5284	720.0000	381.8052	473.5284	735.0000
T	268.1014	307.5718	464.3200	236.3896	307.5718	452.9800
ΔT	178.8881	182.4446	178.2989	154.7302	178.7538	168.5086
T_s	268.1014	98.4720	0	236.3896	61.5634	0
T_A	0	209.0998	464.3200	0	246.0084	452.9800
T_B	536.2027	406.0438	464.3200	472.7791	369.1352	452.9800
π	46298	87780	107800	33729	79981	102600

Table 5: The effect of RP cost changing in Stackelberg-merger model ($k = 1.4$).

Case result		$k = 1.4$ $C_B = 220$			$k = 1.4$ $C_B = 250$	
	$P_B \le P_A$	$P_A < P_B < P_{B0}$	$P_{B0} < P_B < P_{B1}$	$P_B \le P_A$	$P_A < P_B < P_{B0}$	$P_{B0} < P_B < P_{B1}$
P_A	366.3029	346.2141	225.4400	382.0862	346.2141	232.3100
P_B	366.3029	505.3881	820.0000	382.0862	520.3881	835.0000
T	267.3943	307.5718	549.1200	232.8276	307.5718	535.3800
ΔT	221.1238	220.1259	254.7917	192.0417	216.4351	241.9918
T_s	267.3943	187.6227	0	235.8276	150.7141	0
T_A	0	119.491	549.1200	0	156.8577	535.3800
T_B	534.7886	495.1945	549.1200	471.6552	458.2859	535.3800
π	51701	123350	150770	37958	111730	143320

scale and wider influence on product. In this case, even if RP is set at a high price, the shift demand and the increment demand of market are proportional to the size of OP scale, so high price has less influence on market demand. But the profit of unit renewal product rises substantially with its price increase. In this situation, RP comes onto the market with a great price differential, and so the profits of all products are the highest; that is, area III is the optimal pricing area of supply chain.

(3) The profit of retailer in areas II and III is greater than that of manufacturer, mainly because the retailer is a direct participator of market, and it reacts very rapidly when the demand of market changes. Therefore, the retailer can adjust the process of decision making in the fastest time to increase revenue and reduce losses. The manufacturer's reaction to the market relies on the retailer decision making information's transmission, and there is a delay in the transmission process. Accordingly, the manufacturer's profit is less than that of the retailer.

(4) With the cost of RP increasing, the profits of the manufacturer, retailer, and total will decrease.

(5) If the cost of RP is constant, with product renewal rate k increasing, the profits of the manufacturer, retailer and total will increase.

5.2. Stackelberg-Merger Game Simulation. The parameters are the same as Section 5.1. With the changing of product renewal rate or the cost of RP, the variety of price, demand, and profits are shown in Tables 4 and 5.

From numerical simulation in Tables 4 and 5, the following conclusions can be drawn:

(i) The profits in areas II and III of Stackelberg-merger game are greater than those in area I, so the Stackelberg-merger pricing strategy would not be in area I. The situation is consistent with Stackelberg-Nash game.

(ii) If OP adopts low-price strategy, and RP adopts high-price strategy, the total profit will be maximum value; in other words, area III is the optimal merger pricing area for the SC. The situation and its reason are the same as those of Stackelberg-Nash game.

(iii) If the cost of RP is constant, with the product renewal rate increasing, the total profit will increase obviously.

(iv) With the cost of RP increasing, the total profit will decrease.

From the analysis of Sections 5.1 and 5.2, the following can be drawn: (i) Both two kinds of decision-making processes are consistent with each other in profits, cost, and product renewal rate changing (with the cost of RP increasing, profits decrease, with product renewal rate increasing, profits increase). (ii) Both cases obtain the optimal profits in area III; in other words, the RP which comes onto the market has a great price differential with OP. (iii) Compared with Stackelberg-Nash game, the price in Stackelberg-merger game is lower while the profits are more, so Stackelberg-merger decision making is virtually a kind of win-win situation for enterprises and market.

6. Conclusion

This paper studied the process of product renewal. A market shift model of RP was built on incremental function and shift function. Based on the model, Stackelberg-Nash game model and Stackelberg-merger game model for RP in SC were built, and their theoretical analysis was carried out. The following conclusions can be drawn. (i) In both two models, the increase of RP cost will make participants' profits and total profit decrease, while higher renewal rate will make the profits increase at the same margin cost. (ii) Manufacturer and retailer obtain the optimal profits in area III; in other words, the way of the optimal decision making in SC is that RP comes onto the market with a great price differential with OP. (iii) Compared with Stackelberg-Nash game model, Stackelberg-merger game model's pricing is lower and the profits are higher, which is actually a kind of win-win situation for enterprises and market.

In this paper, a part of premise conditions of the model was built on the basis of some rational hypothesis, and the model parameters need to be confirmed by real statistics data. These problems need to be further studied.

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Research Article

A Three-Stage Optimization Algorithm for the Stochastic Parallel Machine Scheduling Problem with Adjustable Production Rates

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We consider a parallel machine scheduling problem with random processing/setup times and adjustable production rates. The objective functions to be minimized consist of two parts; the first part is related with the due date performance (i.e., the tardiness of the jobs), while the second part is related with the setting of machine speeds. Therefore, the decision variables include both the production schedule (sequences of jobs) and the production rate of each machine. The optimization process, however, is significantly complicated by the stochastic factors in the manufacturing system. To address the difficulty, a simulation-based three-stage optimization framework is presented in this paper for high-quality robust solutions to the integrated scheduling problem. The first stage (crude optimization) is featured by the ordinal optimization theory, the second stage (finer optimization) is implemented with a metaheuristic called differential evolution, and the third stage (fine-tuning) is characterized by a perturbation-based local search. Finally, computational experiments are conducted to verify the effectiveness of the proposed approach. Sensitivity analysis and practical implications are also discussed.

1. Introduction

The parallel machine scheduling problem has long been an important model for operations research because of its strong relevance with various industries, such as semiconductor manufacturing [1], automobile gear manufacturing [2], and train dispatching [3]. In the theoretical research, makespan (i.e., the completion time of the last job) is the most frequently adopted objective function [4]. However, this criterion alone does not provide a comprehensive characterization for the manufacturing costs in real-world situations. In this paper, we will focus on the minimization of operational cost and tardiness cost in a stochastic parallel machine production system. The former is mainly the running cost of the machines for daily operations, and the latter is brought about by the jobs that are finished later than the previously set due dates.

The operational cost, which is further categorized into fixed cost and variable cost, can be minimized by proper settings of the production rate (i.e., operating speed) of each machine. The production rate would produce opposite influences on the variable cost and the fixed cost. Setting the machine speed at a high level is beneficial for reducing the variable cost because the production time is shortened. However, achieving the high speed incurs considerable fixed costs at the same time (e.g., additional machine tools must be purchased). Therefore, an optimization approach is needed to determine the optimal values of the production rates for minimizing the operational cost.

The tardiness cost, which is usually represented by a penalty payment to the customer or a worsened service reputation, could be minimized by scheduling the jobs in a wise manner. Efficient schedules directly help to increase the on-time delivery ratio and reduce the waiting/queueing time in the production system [5]. In recent years, tardiness minimization has become a more important objective than makespan for many firms adopting the make-to-order strategy. Under the parallel machine environment, the objective functions that reflect tardiness cost include total tardiness

[6], total weighted tardiness [7], maximum lateness [8], and number of tardy jobs [9].

In the literature, the optimization of operational settings (including production rates) is usually performed separately of production scheduling. Actually, there exist strong interactions between the two decisions. For example, if the operating speed of a machine is set notably high, it is beneficial to allocate more jobs to the machine in order to fully utilize its processing capacity. On the other hand, if a large number of jobs have been assigned to a certain machine, it is necessary to maintain a high production rate for this machine in order to reduce the possibility of severe tardiness. Therefore, production scheduling and the selection of machine speeds should better be considered as an integrated optimization problem. A solution to this problem should include both the optimized machine speeds and the optimized production schedule that works well under this setting of machine speeds.

In real-world manufacturing systems, uncertainties are inevitable (due to, e.g., absent workers and material shortage). But the effect of random events has not been sufficiently considered in most existing research. For example, a common assumption in scheduling is that the processing time of each job is exactly known in advance, which, of course, is inconsistent with reality. If the random factors such as processing time variations are considered, we should rely on discrete-event simulation [10] to evaluate the performance of the manufacturing system. So, in this paper, we adopt a simulation-based optimization framework to find satisfactory settings of the production rates together with a satisfactory production schedule. The objective is to minimize the total manufacturing cost: sum of operational cost and tardiness cost.

The rest of the paper is organized as follows. Section 2 makes an introductory review on some topics closely related with our research. Section 3 depicts the production environment and formulates the integrated optimization problem. Section 4 describes the proposed approach for solving the stochastic optimization problem, which includes three stages with gradually increasing accuracy. Section 5 presents the main computational results and comparisons. Finally, Section 6 concludes the paper.

2. Related Research Background

2.1. Uniform Parallel Machine Scheduling. In the parallel machine scheduling problem, we consider n jobs that are waiting for processing. Each job consists of only a single operation which can be processed on any one of the m machines $M_1, \dots, M_m$. As a conventional constraint in scheduling models, each machine can process at most one job at a time, and each job may be processed by at most one machine at a time.

There are three types of parallel machines [11].

(i) *Identical Parallel Machines (P).* The processing time p_j^k of job j on machine k is identical for each machine; that is, $p_j^k = p_j$.

(ii) *Uniform Parallel Machines (Q).* The processing time p_j^k of job j on machine k is $p_j^k = p_j/q_k$, where q_k is the operating speed of machine k.

(iii) *Unrelated Parallel Machines (R).* The processing time p_j^k of job j on machine k is $p_j^k = p_j/q_{k,j}$, where $q_{k,j}$ is the job-dependent speed of machine k.

If preemption of operations is not allowed, scheduling uniform parallel machines with the makespan (i.e., maximum completion time) criterion (described as $Q\| C_{\max}$) is a strongly $\mathcal{NP}$-hard problem. According to the reduction relations between objective functions [11], scheduling uniform parallel machines under due date-based criteria (e.g., total tardiness) is also strongly $\mathcal{NP}$-hard. Therefore, metaheuristics have been widely used for solving these problems.

2.2. Simulation Optimization and Ordinal Optimization. The simulation optimization problem is generally defined as follows: find a solution ($\mathbf{x}$) which minimizes the given objective function $J(\mathbf{x})$, that is,

$$\min_{\mathbf{x}\in\mathcal{X}} J(\mathbf{x}), \tag{1}$$

where $\mathcal{X}$ represents the search space for the decision variable $\mathbf{x}$. The key assumption in simulation optimization is that $J(\mathbf{x})$ is not available as an analytical expression; so, simulation is the only way to obtain an evaluation of $\mathbf{x}$. Moreover, since applicable simulation algorithms must have a satisfactory time performance (which means that the simulation should be as fast as possible), some details must have been omitted when designing the simulation model, and thus simulation can only provide a noisy estimation of $J(\mathbf{x})$, which is usually denoted as $\widehat{J}(\mathbf{x})$.

However, two difficulties naturally exist in the implementation of simulation optimization: (1) the search space ($\mathcal{X}$) is often huge, containing zillions of choices for the decision variables; (2) simulation is subject to random errors, which means, a large number of simulation replications have to be adopted in order to guarantee a reliable evaluation for $\mathbf{x}$. These issues suggest that simulation optimization can be extremely costly in terms of computational burden.

For existing methods and applications of simulation optimization, interested readers may refer to [12–20]. Here, we will focus on the ordinal optimization (OO) methodology, which was first proposed by Ho et al. at Harvard [21].

OO attempts to settle the previous difficulties by emphasizing two important ideas: (1) order is much more robust against noise than value; (2) aiming at the single best solution is computationally expensive, and thus it is wiser to focus on the "good enough." The major contribution of OO is that it quantifies these ideas and thus provides accurate guidance for our optimization practice.

2.3. The Differential Evolution Algorithm. The differential evolution (DE) algorithm, which was first proposed in the mid-1990s [22], is a relatively new evolutionary optimizer. Characterized by a novel mutation operator, the algorithm has been found to be a powerful tool for continuous function optimization. Due to its easy implementation, quick convergence, and robustness, the DE algorithm is becoming increasingly popular in recent years.

Because of its continuous feature, the traditional DE algorithm cannot be directly applied to scheduling problems with inherent discrete nature. Indeed, in canonical DE, each solution is represented by a vector of floating-point numbers. But for scheduling problems, each solution is a permutation of integers. To address this issue, two kinds of approaches can be found in the literature. In the first category, a transformation scheme is established to convert permutations into real numbers and vice versa [23]. In the second category, the mutation and crossover operators in DE are modified to discrete versions which suit the permutation representation [24]. We would adopt the former strategy, where we only need to modify the encoding and decoding procedures without changing the implementation of DE itself. The clear advantage is that the search mechanism of DE is well preserved.

Despite the success on deterministic flow shop scheduling problems, the application of DE to simulation-based optimization has rarely been reported. To our knowledge, this is the first attempt in which DE is applied to an integrated operational optimization and production scheduling problem.

3. Problem Definition

3.1. System Configuration. In the production system, there are n jobs waiting to be processed by m uniform parallel machines. The basic processing time of job i is denoted by p_i ($i = 1, \dots, n$), and the basic setup time arising when job j is processed immediately after job i is denoted by s_{ij} (we assume that $s_{ij} > 0$ if $j \neq i$, and $s_{ij} = 0$ if $j = i$). To optimize the manufacturing performance, two decisions have to be made.

(i) *Production Rate Optimization*. For each machine k, the speed value (denoted by α_k) has to be determined. In other words, $\{\alpha_k : k = 1, \dots, m\}$ belong to the decision variables in the optimization problem. The relationship between these variables and the manufacturing cost will be introduced in the following.

(ii) *Production Scheduling*. The job assignment policy (i.e., which jobs are to be processed by each of the machines) should be determined. In addition, the processing order of the jobs assigned to each machine should also be specified.

The production rate is related with the controllable processing times and the controllable setup times. Indeed, if the speed of machine k is set as α_k, then the actual processing time of job i on machine k (denoted as p_i^k) has a mean value of p_i/α_k (i.e., $\mathrm{E}(p_i^k) = p_i/\alpha_k$), and the actual setup time between two consecutive jobs i and j on machine k (denoted as s_{ij}^k) has a mean of s_{ij}/α_k (i.e., $\mathrm{E}(s_{ij}^k) = s_{ij}/\alpha_k$). In most cases, the production process involves human participation (e.g., gathering materials and adjusting CNC machine status); so, the estimate of processing/setup lengths may not be completely precise. For this reason, we assume that the processing times and setup times are random variables following a certain distribution.

3.2. Cost Evaluation. In order to evaluate the cost corresponding to a given solution (i.e., $\mathbf{x} = \{\alpha_k, \mathrm{JPO}_k : k = 1, \dots, m\}$, where JPO_k denotes the processing order of the jobs assigned to machine k), simulation is used to obtain the necessary production information (e.g., the starting time and completion time of each job). Then, the realized total manufacturing cost can be calculated by adding the operational cost and the tardiness cost.

The tardiness cost is simply defined as

$$\mathrm{TC} = \sum_{i=1}^{n} w_i (C_i - d_i)^+, \tag{2}$$

where w_i is the unit tardiness cost for job i (which reflects the relative importance of the job), and C_i and d_i, respectively, denote the completion time and the due date of job i. $T_i = (C_i - d_i)^+ = \max\{C_i - d_i, 0\}$ defines the tardiness of job i.

Now, we focus on the operational cost, which is further divided into fixed cost and variable cost, as done in conventional financial research.

We will first discuss the fixed cost related with the settings of the production rates $\{\alpha_k\}$. The fixed cost of setting the speed at α_k for machine k is defined as

$$\mathrm{FOC}_k(\alpha_k) = a_k \cdot \alpha_k^2, \tag{3}$$

where a_k is a constant coefficient related with machine k. The square on α_k suggests that this type of fixed cost grows increasingly fast with α_k. In practice, when the machine speed is set notably higher above its normal mode, the energy consumption rises rapidly, and meanwhile, the expenses on status monitoring and preventative maintenance also add to the running cost. So, the previous equation form is defined to reflect such an actual situation.

Based on the previous description, the fixed operational cost can be evaluated as

$$\mathrm{FOC} = \sum_{k=1}^{m} \mathrm{FOC}_k(\alpha_k). \tag{4}$$

Once we have obtained the complete production information via simulation, we can calculate the variable operational cost, which is related with the operating time length of each machine. In particular, the variable operational cost can be categorized into two types according to the working mode: production cost (VOC_p) and setup cost (VOC_s). These costs are simply defined as follows:

$$\begin{aligned} \mathrm{VOC}_p &= \sum_{k=1}^{m} K_p \cdot \mathrm{TPT}_k, \\ \mathrm{VOC}_s &= \sum_{k=1}^{m} K_s \cdot \mathrm{TST}_k, \end{aligned} \tag{5}$$

where TPT_k and TST_k, respectively, represent the total production time and total setup time of machine k based on the actual performances; K_p and K_s are cost coefficients (positively correlated with α_k) known in advance. Thus, the variable operational cost is given by VOC = $\text{VOC}_p + \text{VOC}_s$.

Finally, the total manufacturing cost can be defined as

$$\text{TMC} = (\text{FOC} + \text{VOC}) + \text{TC}, \tag{6}$$

which is exactly the objective function to be minimized in our model.

It should be noted that, due to the systematic randomness, different simulation runs will yield distinct realizations of TMC. As a convention in the practice of simulation optimization, we take the average value of TMC obtained from a large number of simulation replications as an estimate for the expectation of TMC. Specifically, if $\text{RTMC}_u(\mathbf{x})$ denotes the realized manufacturing cost in the uth simulation for solution $\mathbf{x}$, the objective function can be stated as

$$\min \text{ E}(\text{TMC}(\mathbf{x})) \approx \frac{1}{U}\sum_{u=1}^{U}\text{RTMC}_u(\mathbf{x}), \tag{7}$$

where U is an appropriate number of simulation replications.

4. The Three-Stage Optimization Algorithm

Since the optimization of production rates and production scheduling are mutually interrelated, we develop a three-stage solution framework as follows.

(1) The first stage focuses on the production rates. The aim is to find a set of "good enough" values for the machine speeds. At this stage, it is unnecessary to overemphasize the accuracy of optimization; so, a fast and crude optimization algorithm will do the job.

(2) The second stage focuses on the production schedule. The aim is to find a schedule that works fine (achieves a low total cost) under the production rates set in the previous stage. Since the objective function is sensitive to job assignment and job sequencing, a finer optimization algorithm is required for this stage.

(3) The third stage focuses on the production rates again. Since the optimal machine speeds are also dependent on the production schedule, the aim of this stage is to fine-tune the machine speeds so as to achieve an ideal coordination between the two sets of decisions.

Based on the previous alternations, the entire optimization algorithm is expected to find high-quality solutions to the studied stochastic optimization problem. The details of the algorithm are given in the following subsections.

4.1. Stage 1: Coarse-Granular Optimization of α_k. In this stage, we try to find a set of satisfactory values for $\{\alpha_k : k = 1, \ldots, m\}$ using the ordinal optimization (OO) methodology.

Before going into the detailed algorithm description, we show a simple property of the optimal setting of α_k.

Theorem 1. *For any two machines k_1 and k_2, if the corresponding fixed cost coefficients satisfy $a_{k_1} > a_{k_2}$, then in the optimal solution we must have $\alpha_{k_1} \le \alpha_{k_2}$.*

Proof. The proof is by contradiction. Suppose that $a_{k_1} > a_{k_2}$ and a certain solution has indicated $\alpha_{k_1} > \alpha_{k_2}$; then, this solution can be improved by exchanging the production rates together with the processed job sequences of the two machines. After such an exchange is performed, the variable cost and the tardiness cost will remain the same because the production schedule is actually not changed. However, the fixed cost will be reduced since $a_{k_1}\alpha_{k_1}^2 + a_{k_2}\alpha_{k_2}^2 > a_{k_1}\alpha_{k_2}^2 + a_{k_2}\alpha_{k_1}^2$. □

4.1.1. Basics of Ordinal Optimization. We list the main procedure of OO as follows. Meanwhile, we would suggest interested readers to turn to [25] for more theories and proofs. Suppose that we want to find k solutions that belong to the top-g (normally $k < g$). Then, OO consists of the following steps.

Step 1. Uniformly and randomly select N solutions from $\mathcal{X}$ (this set of initial solutions is denoted by I).

Step 2. Use a crude and computationally fast model for the studied problem to estimate the performance of the N solutions in I.

Step 3. Pick the observed top s solutions of I (as estimated by the crude model) to form the selected subset S.

Step 4. Evaluate all the s solutions in S using the exact simulation model, and then output the top k ($1 \le k < s$) solutions.

As an example, let $g = 50$ and $k = 1$. If we take $N = 1000$ in Step 1 and the crude model in Step 2 has a moderate noise level, then OO theory ensures that the top solution in S (with $s \approx 30$) is among the actual top 50 of the N solutions with probability no less than 0.95. In practice, s is determined as a function of g and k; that is, $s = Z(g, k; N, \text{ noise level})$, where noise level reflects the degree of accuracy of the crude model. Since $J_{\text{crude\_model}} = J_{\text{exact\_simulation\_model}} + \text{noise}$, the noise level can be measured by the standard deviation of noise, that is, $\sqrt{\text{Var(noise)}}$. Intuitively, if the crude model is significantly inaccurate, then s should be set larger.

For our problem, the solution in this stage is represented by m real values $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. To facilitate the implementation of OO, these values are all discretized evenly into 10 levels between $[\alpha_{\min}, \alpha_{\max})$ (normally we have $\alpha_{\min} = 1$, while the speed limit $\alpha_{\max}$ is determined by specific machine conditions). If we assume that $\alpha_{\max} = 2$ in this paper, then $\alpha_k \in \{1.0, 1.1, 1.2, \ldots, 1.9\}$. Such a discretization reflects the coarse-granular nature of the optimization process in Stage 1. In addition, the conclusion of Theorem 1 helps to exclude a large number of nonoptimal solutions. So, the size of search space is at most 10^m.

In the implementation of OO, the crude model (used in Step 2) has to be designed specifically for a concrete problem. Although generic crude models (like the artificial neural network-based model presented in [26]) may be useful for some problems, No Free Lunch Theorems [27] suggest that incorporation of problem-specific information is the only way to promote the efficiency of optimization algorithms. So, here, we will devise a specialized crude model which can provide a quick and rough evaluation of the candidate solutions for the discussed parallel-machine problem.

4.1.2. The Crude Model Used by OO. Exact simulation is clearly very time consuming because a large number of replications (samplings of the stochastic processing/setup times) are needed to obtain a stable result. In order to apply OO, we devise a crude (quick-and-dirty) model for objective evaluation, which by definition is not so accurate as the exact simulation model but requires considerably shorter computational time.

The crude model presented here is deterministic rather than stochastic, which means that it needs to be run only once to obtain an approximate objective value. The crude model consists of 3 major steps, which will be detailed next.

Step 1. Schedule all the n jobs on an imaginary machine (with speed $\alpha = 1$). At each iteration, select the job that requires the shortest basic setup time to be the next job. This will lead to a production sequence including alternations of the n jobs and $(n-1)$ setups. The length of the entire sequence (i.e., summation of all the basic processing times and the basic setup times involved) is denoted as L.

Step 2. Split the production sequence into m subsequences such that the length of each subsequence is nearly equal to $L \times (\alpha_k / \sum_{k=1}^{m} \alpha_k)$ $(k = 1, \dots, m)$. The kth subsequence constitutes the production schedule for machine k.

Step 3. Approximate the tardiness cost, fixed cost, and variable cost. In this step, the total production time of machine k is calculated as $(1/\alpha_k) \times \sum_{j=1}^{n_k} p_{[j|k]}$, where n_k is the number of jobs assigned to machine k (i.e., the kth subsequence) and $p_{[j|k]}$ is the basic processing time of the jth job in this subsequence. The total setup time of machine k is calculated as $(1/\alpha_k) \times \sum_{j=1}^{n_k-1} s_{[j|k][j+1|k]}$, where $s_{[j|k][j+1|k]}$ is the basic setup time between the jth job and the $(j+1)$th job in the subsequence related with machine k.

When splitting the original sequence (Step 2), the order of each component (production period or setup period) should be kept unchanged. Meanwhile, since each output subsequence is actually thought of as the production schedule for a particular machine, "setup" should not appear at the end of any subsequence. In other words, some setups will be discarded during the splitting step. However, this will hardly influence the subsequent calculations because the total length of discarded setups is normally trivial compared with the complete length (L).

The desired length of each subsequence is deduced as follows. First, as this is a parallel machine manufacturing system, we hope the actual completion time of each machine is aligned (which is certainly the ideal situation) so that the makespan is minimized (no time resource is wasted). Then, if we use l_k to denote the length of the kth subsequence (i.e., the summation of all job lengths and setup lengths in this subsequence), the actual completion time of machine k (denoted by C_k) is l_k/α_k. The completion time alignment condition requires, for all $k \neq k'$, $C_k = C_{k'}$; that is, $l_k/\alpha_k = l_{k'}/\alpha_{k'}$. Solving the equation yields $l_k^* = \alpha_k L / \sum_{k=1}^{m} \alpha_k$.

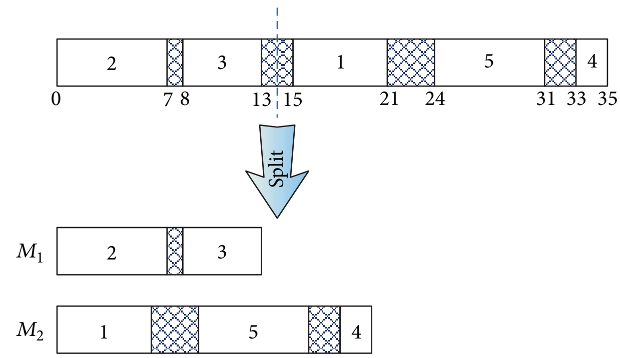


Figure 1: Splitting of the original job sequence in the cases $\alpha_1 = 1.0$ and $\alpha_2 = 1.5$.

A concrete example of the splitting process is shown in Figure 1, where the shaded areas represent setup periods. In this example, we assume that there are two machines with $\alpha_1 = 1.0$ and $\alpha_2 = 1.5$. Thus, the splitting point should be placed at $l_1^* = (1/(1+1.5)) \times 35 = 14$. By adopting the nearest feasible splitting policy, the five jobs are allocated to the two machines such that machine 1 should process jobs 2, 3 and machine 2 should process jobs 1, 5, 4.

4.2. Stage 2: Optimization of the Production Schedule. In this stage, we use a simulation-based differential evolution algorithm for finding a satisfactory production schedule. The production rates are fixed at the best values output by the first stage.

4.2.1. Basics of Differential Evolution. Like other evolutionary algorithms, DE is a population-based global optimizer. In DE, each individual in the population is represented by a D-dimensional real vector $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \dots, x_{i,D})$, $i = 1, \dots, \mathrm{SN}$, where SN is the population size. In each iteration, DE employs the mutation and crossover operators to generate new candidate solutions, and then it applies a one-to-one selection policy to determine whether the offspring or the parent can survive to the next generation. This process is repeated until a preset termination criterion is met. The DE algorithm can be described as follows.

Step 1 (Initialization). Randomly generate a population of SN solutions, $\{\mathbf{x}_1, \dots, \mathbf{x}_{\mathrm{SN}}\}$.

Step 2 (Mutation). For $i = 1, \dots, \mathrm{SN}$, generate a mutant solution $\mathbf{v}_i$ as follows:

$$\mathbf{v}_i = \mathbf{x}_{\text{best}} + F \times \left(\mathbf{x}_{r_1} - \mathbf{x}_{r_2}\right), \quad (8)$$

where $\mathbf{x}_{\text{best}}$ denotes the best solution in the current population; r_1 and r_2 are randomly selected from $\{1, \ldots, \text{SN}\}$ such that $r_1 \neq r_2 \neq i$; $F > 0$ is a weighting factor.

Step 3 (Crossover). For $i = 1, \ldots, \text{SN}$, generate a trial solution $\mathbf{u}_i$ as follows:

$$u_{i,j} = \begin{cases} v_{i,j}, & \text{if } \xi_j \leq \text{CR or } j = r_j, \\ x_{i,j}, & \text{otherwise}, \end{cases} \quad (j = 1, \ldots, D), \tag{9}$$

where r_j is an index randomly selected from $\{1, \ldots, D\}$ to guarantee that at least one dimension of the trial solution $\mathbf{u}_i$ differs from its parent $\mathbf{x}_i$; ξ_j is a random number generated from the uniform distribution $\mathcal{U}[0, 1]$; $\text{CR} \in [0, 1]$ is the crossover parameter.

Step 4 (Selection). If $\mathbf{u}_i$ is better than $\mathbf{x}_i$, let $\mathbf{x}_i = \mathbf{u}_i$.

Step 5. If the termination condition is not satisfied, go back to Step 2.

According to the algorithm description, DE has three important parameters, that is, SN, F, and CR. In order to ensure a good performance of DE, the setting of these parameters should be reasonably adjusted based on specific optimization problems.

4.2.2. Encoding and Decoding. The encoding of a solution in this stage is composed of n real numbers; so, a solution can be roughly expressed as $\mathbf{x} = [x_1, x_2, \ldots, x_n]$.

The encoding scheme is based on the random key representation and the smallest position value (SPV) rule. In the decoding process, the n real numbers $x_1, \ldots, x_n$ $(0 < x_i < m)$ will be transformed to a production schedule by the SPV rule. In particular, the integer part of x_i indicates the machine allocation for job i, while the decimal part of x_i determines the relative position of the job in the production sequence.

The decoding process is exemplified in Table 1 with a problem containing 9 jobs. The decoded information is shown in the last row, where (k, π) indicates that job i should be processed by machine k at the πth position. In fact, the index of the machine that should process job i is simply $k = \lceil x_i \rceil$. Hence, job 2 ($x_2 = 0.99$) and job 4 ($x_4 = 0.72$) are assigned to machine 1 according to this solution. When several jobs are assigned to the same machine, their relative orders are resolved by sorting the decimal parts. For instance, job 1 ($x_1 = 1.80$), job 5 ($x_5 = 1.45$), and job 9 ($x_9 = 1.90$) should all be processed by machine 2. Furthermore, because $x_5 < x_1 < x_9$, the processing order on machine 2 should be (5, 1, 9). Finally, the production schedule decoded from this solution can be expressed as $\sigma = (\text{M}_1 : 4, 2; \text{M}_2 : 5, 1, 9; \text{M}_3 : 3, 6, 7, 8)$.

4.2.3. Evaluation and Comparison of Solutions. Recall that the objective function is E(TMC), and in this stage, the FOC has been fixed with the setting of production rates. So, we only care about TC and VOC.

In order to evaluate a schedule σ, we need to implement σ under different realizations of the random processing/setup times. When σ has been evaluated for a sufficient number of times (U), its objective value can be approximated by (7), which is consistent with the idea of Monte Carlo simulation. However, this definitely increases the computational burden, especially when used in an optimization framework where frequent solution evaluations are needed. If we allow only one realization of the random processing/setup times, then a rational choice is to use the mean (expected) value of each processing/setup time. We can show the following property; that is, such an estimate is a lower bound of the true objective value.

Theorem 2. *Let σ denote a feasible schedule of the stochastic parallel machine scheduling problem. The following inequality must hold:*

$$E\left(TMC_\sigma\right) \geq \overline{TMC}_\sigma, \tag{10}$$

where TMC_σ (random variable) is the total manufacturing cost corresponding to the schedule, and $\overline{TMC}_\sigma$ (constant value) is the total manufacturing cost in the case where each random processing/setup time takes the value of its expectation.

Proof. In the proof, we will use "$\overline{X}$" to denote the realized value of the random variable X when all the processing/setup times are fixed at their mean values.

Under the given schedule σ, we have $\text{E}(\text{TPT}_k) = \text{E}(\sum_{j=1}^{n_k} p_{[j|k]}^k) = \sum_{j=1}^{n_k} \text{E}(p_{[j|k]}^k) = \overline{\text{TPT}}_k$ (where $p_{[j|k]}^k$ denotes the actual processing time of the jth job on machine k), and $\text{E}(\text{TST}_k) = \text{E}(\sum_{j=1}^{n_k-1} s_{[j|k][j+1|k]}^k) = \sum_{j=1}^{n_k-1} \text{E}(s_{[j|k][j+1|k]}^k) = \overline{\text{TST}}_k$ (where $s_{[j|k][j+1|k]}^k$ denotes the actual setup time between the jth and the $(j+1)$th job on machine k). Thus, it follows that $\text{E}(\text{VOC}_\sigma) = \overline{\text{VOC}}_\sigma$.

Under the given schedule σ, we denote the starting time of job i by t_i and the completion time of job i by C_i. As defined earlier, $\bar{t}_i$ (resp., $\overline{C}_i$) is used to denote the starting time (resp., completion time) of job i when each processing/setup time is replaced by its expected value. First, we will prove that, for any job i, $\text{E}(t_i) \geq \bar{t}_i$.

For the first job on each machine, we have $\text{E}(t_i) = \bar{t}_i$ (because $t_i = 0$ a.s.). Then, the proof procedure can be continued for the subsequent jobs on each machine. Suppose that we have already proved $\text{E}(t_i) \geq \bar{t}_i$ for each job before job j on machine k, and without loss of generality, we assume that job j immediately follows job i on machine k; then,

$$\begin{aligned} \text{E}\left(t_j\right) &= \text{E}\left(C_i + s_{ij}^k\right) \\ &= \text{E}\left(t_i + p_i^k + s_{ij}^k\right) \\ &\geq \bar{t}_i + \text{E}\left(p_i^k\right) + \text{E}\left(s_{ij}^k\right) \\ &= \bar{t}_j. \end{aligned} \tag{11}$$

Therefore, the reasoning applies to each job in the schedule.

Having proved $\text{E}(t_i) \geq \bar{t}_i$, we can now move to TC in the objective function. Recall that $T_i = (C_i - d_i)^+$ (with $x^+ = \max\{x, 0\}$) denotes the tardiness of job i and $\overline{T}_i$ denotes

Table 1: Illustration of the decoding process in DE.

i	1	2	3	4	5	6	7	8	9
x_i	1.80	0.99	2.01	0.72	1.45	2.25	2.30	2.80	1.90
(k,π)	(2, 2)	(1, 2)	(3, 1)	(1, 1)	(2, 1)	(3, 2)	(3, 3)	(3, 4)	(2, 3)

the tardiness in the case where each random processing/setup time takes its expected value. Meanwhile, suppose that k_i represents the machine which processes job i. Then,

$$\begin{aligned} \mathrm{E}\left(\mathrm{TC}_\sigma\right) &= \mathrm{E}\left(\sum_{i=1}^{n} w_i T_i\right) = \sum_{i=1}^{n} w_i \mathrm{E}\left[\left(C_i - d_i\right)^+\right] \\ &\geq \sum_{i=1}^{n} w_i \left[\mathrm{E}\left(C_i - d_i\right)\right]^+ = \sum_{i=1}^{n} w_i \left[\mathrm{E}\left(t_i + p_i^{k_i}\right) - d_i\right]^+ \\ &\geq \sum_{i=1}^{n} w_i \left[\bar{t}_i + \mathrm{E}\left(p_i^{k_i}\right) - d_i\right]^+ = \sum_{i=1}^{n} w_i \left(\overline{C}_i - d_i\right)^+ \\ &= \sum_{i=1}^{n} w_i \overline{T}_i = \overline{\mathrm{TC}}_\sigma. \end{aligned} \tag{12}$$

This completes the proof of $\mathrm{E}(\mathrm{TC}_\sigma) \geq \overline{\mathrm{TC}}_\sigma$.

Now that $\mathrm{E}(\mathrm{VOC}_\sigma) = \overline{\mathrm{VOC}}_\sigma$ and $\mathrm{E}(\mathrm{TC}_\sigma) \geq \overline{\mathrm{TC}}_\sigma$, we have shown that $\mathrm{E}(\mathrm{TMC}_\sigma) \geq \overline{\mathrm{TMC}}_\sigma$. □

The DE algorithm requires to compare two solutions in the Selection step. When facing a deterministic optimization problem, we can directly compare the exact objective values of two solutions to tell their quality difference. But in the stochastic case, the comparison of solutions may not be so straightforward because we can only obtain approximated (noisy) objective values from simulation. In this study, we will utilize the following two mechanisms for comparison purposes.

(A) *Prescreening*. Because $\overline{\mathrm{TMC}}_\sigma$ is a lower bound for $\mathrm{E}(\mathrm{TMC}_\sigma)$ (Theorem 2), we can arrive at the following conclusion which is useful for the prescreening of candidate solutions.

Corollary 3. *For two candidate solutions $\mathbf{x}_1$ (the equivalent schedule is denoted by σ_1) and $\mathbf{x}_2$ (the equivalent schedule is denoted by σ_2), if $\overline{TMC}_{\sigma_2} \geq E(TMC_{\sigma_1})$, then $\mathbf{x}_2$ must be inferior to $\mathbf{x}_1$ and thus can be discarded.*

When applying this property, the value of $\mathrm{E}(\mathrm{TMC}_{\sigma_1})$ is certainly not known exactly, and thus the Monte Carlo approximation based on U simulation replications is used instead.

(B) *Hypothesis Test*. If the candidate solutions have passed the pre-screening, then hypothesis test is used to compare the quality of two solutions.

Suppose that we have implemented U simulation replications for solution $\mathbf{x}_i$ whose true objective value is $f(\mathbf{x}_i) = \mathrm{E}(\mathrm{TMC}_{\sigma_i})$ $(i = 1, 2)$. Then, the sample mean and sample variance can be calculated by

$$\begin{aligned} \overline{f}_i &= \frac{1}{U}\sum_{j=1}^{U} f_i^{(j)}, \\ s_i^2 &= \frac{1}{U-1}\sum_{j=1}^{U}\left(f_i^{(j)} - \overline{f}_i\right)^2, \end{aligned} \tag{13}$$

where $f_i^{(j)}$ is the objective value obtained in the j-th simulation replication for solution $\mathbf{x}_i$.

Let the null hypothesis H_0 be "$f(\mathbf{x}_1) = f(\mathbf{x}_2)$", and thus the alternative hypothesis H_1 is "$f(\mathbf{x}_1) \neq f(\mathbf{x}_2)$". According to the statistical theory, the critical region of H_0 is

$$\left|\overline{f}_1 - \overline{f}_2\right| \geq Z = z_{\epsilon/2}\sqrt{\frac{\left(s_1^2 + s_2^2\right)}{U}}, \tag{14}$$

where $z_{\epsilon/2}$ is the value such that the area to its right under the standard normal curve is exactly $\epsilon/2$. Therefore, if $\overline{f}_1 - \overline{f}_2 \geq Z$, $\mathbf{x}_2$ is statistically better than $\mathbf{x}_1$; if $\overline{f}_1 - \overline{f}_2 \leq -Z$, $\mathbf{x}_1$ is statistically better than $\mathbf{x}_2$. Otherwise, if $|\overline{f}_1 - \overline{f}_2| < Z$ (i.e., the null hypothesis holds), it is concluded that there exists no statistical difference between $\mathbf{x}_1$ and $\mathbf{x}_2$ (in this case, DE may preserve either solution at random).

4.3. Stage 3: Fine-Tuning of α_k. Up till now, the production schedule (sequence of jobs on each machine) has been fixed by the second stage. It is found that fine-tuning the production rates $\{\alpha_k\}$ could further improve the solution quality to a noticeable extent (note that these variables are only roughly optimized on a grid basis in Stage 1). So, here, we propose a local search procedure based on systematic perturbations for fine-tuning $\{\alpha_k\}$. The directions of successive perturbations are not completely random but determined partly according to the knowledge gained from previous attempts.

Below are the detailed steps of the local search algorithm, which involves a parameter learning process similar to that of artificial neural networks for guiding the search direction. In the local search process, the optimal computing budget allocation (OCBA) technique [28] is used to identify the best solution among a set of randomly sampled solutions. Before applying OCBA, the allowed number of simulation replications is given. Then, OCBA can be used to allocate the limited computational resource to the solutions incrementally so that the probability of recognizing the truly best solution is maximized.

Step 1. Initialize the iteration index: $h = 1$. Let $\boldsymbol{\alpha}^{(h)} = \boldsymbol{\alpha}^*$ which is output by Stage 1 (now we express the production rates $\{\alpha_k : k = 1, \dots, m\}$ as a vector $\boldsymbol{\alpha}$).

Step 2. Randomly sample N_s solutions from the neighborhood of $\boldsymbol{\alpha}^{(h)}$. To produce the ith sample, first generate a random vector $\mathbf{r}$ (each component r_k is generated from the uniform distribution $\mathcal{U}[-1,1]$), and then let $\boldsymbol{\alpha}_i^{(h)} = \boldsymbol{\alpha}^{(h)} + \mathbf{r} \cdot \delta$.

Step 3. Use OCBA to allocate a total of U_s simulation replications to the set of temporary solutions $\{\boldsymbol{\alpha}_1^{(h)}, \boldsymbol{\alpha}_2^{(h)}, \ldots, \boldsymbol{\alpha}_{N_s}^{(h)}\}$ so that the best one among them can be identified and denoted as $\boldsymbol{\alpha}^{(h+1)}$.

Step 4. If $\boldsymbol{\alpha}^{(h+1)}$ has the best objective value (as reported by the OCBA) found so far, then set $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}^{(h+1)}$.

Step 5. If $h = h_{\max}$, go to Step 9. Otherwise, let $h \leftarrow h + 1$.

Step 6. If the best-so-far objective value has just been improved, then reinforcement is executed by letting $\boldsymbol{\alpha}^{(h)} \leftarrow \boldsymbol{\alpha}^{(h)} + \lambda \cdot (\boldsymbol{\alpha}^{(h)} - \boldsymbol{\alpha}^{(h-1)})$.

Step 7. If $\boldsymbol{\alpha}^*$ has not been updated during the most recent Q iterations, then backtracking is executed by letting $\boldsymbol{\alpha}^{(h)} = \boldsymbol{\alpha}^*$.

Step 8. Go back to Step 2.

Step 9. Output the optimization result, that is, $\boldsymbol{\alpha}^*$ and the corresponding objective value.

The parameters of the local search module include $h_{\max}$ (the total iteration number), λ (the reinforcement factor), Q (the allowed number of iterations without any improvement), and $\delta \in (0,1)$ (the amplitude of random perturbation). In addition, N_s controls the extensiveness of random sampling, and U_s controls the computational burden devoted to simulation (the detailed procedure of OCBA can be found in [28] and thus is omitted here). In the procedure, Step 6 applies a reinforcement strategy when the previous perturbation direction is beneficial for improving the estimated objective value. Step 7 is a backtracking policy which restores the solution to the best-so-far value when the latest Q perturbations do not result in any improvement. In Steps 2 and 6, the perturbed or reinforced new $\boldsymbol{\alpha}^{(h)}$ should be kept positive.

5. The Computational Experiments

To test the effectiveness of the proposed three-stage algorithm (abbreviated as TSA later), computational experiments are conducted on a number of randomly generated test instances. In each instance, the processing/setup times (bounded to be positive) are assumed to follow one of the three types of distributions: normal distribution, uniform distribution, and exponential distribution. In all cases, the basic processing times (p_i) are generated from the uniform distribution $\mathcal{U}(1,100)$, while the basic setup times (s_{ij}) are generated from the uniform distribution $\mathcal{U}(1,10)$. In the case of normal distributions, that is, $p_i^k \sim \mathcal{N}(p_i/\alpha_k, \sigma_i^2)$ and $s_{ij}^k \sim \mathcal{N}(s_{ij}/\alpha_k, \sigma_{ij}^2)$, the standard deviation is controlled by $\sigma_i = \theta \times p_i$ and $\sigma_{ij} = \theta \times s_{ij}$ ($\theta \in \{0.1, 0.2, 0.3\}$ describes the level of variability). In the case of uniform distributions, that is, $p_i^k \sim \mathcal{U}(p_i/\alpha_k - \omega_i, p_i/\alpha_k + \omega_i)$ and $s_{ij}^k \sim \mathcal{U}(s_{ij}/\alpha_k - \omega_{ij}, s_{ij}/\alpha_k + \omega_{ij})$, the width parameter is given by $\omega_i = \theta \times p_i$ and $\omega_{ij} = \theta \times s_{ij}$. In the case of exponential distributions, that is, $p_i^k \sim \mathrm{Exp}(\lambda_i^k)$ and $s_{ij}^k \sim \mathrm{Exp}(\lambda_{ij}^k)$, the only parameter is given by $\lambda_i^k = \alpha_k/p_i$ and $\lambda_{ij}^k = \alpha_k/s_{ij}$. The due dates are obtained by a series of simulations which apply dispatching rules (such as SPT and EDD [29]) to each machine with speed $\alpha_k = 1.5$, and the due date of each job is finally set as its average completion time. This method can generate reasonably tight due dates. Meanwhile, the weight of each job is an integer generated from the uniform distribution $\mathcal{U}(1,5)$. As for the machine-related parameters, the fixed cost coefficient a_k takes an integer value from the uniform distribution $\mathcal{U}(1,10)$, and the variable cost coefficients are directly given as $K_p = 0.01(5 + \alpha_k)$ and $K_s = 0.01(2 + \alpha_k)$. The following computational experiments are conducted with Visual C++ 2010 on an Intel Core i5-750/3GB RAM/Windows 7 desktop computer.

5.1. Parameter Settings. Since the three optimization stages are executed in a serial manner, the parameters for each stage can be studied independently of one another.

The parameters for Stage 1 include g, k, N, and s, which are required by the ordinal optimization procedure. For our problem, we empirically set $g = 20$ and $k = 1$ (which means that we want to find one solution that belongs to the top 20), $N = 1000$ (which means that 1000 solutions satisfying Theorem 1 will be randomly picked at first). On such a basis, the value for s can be estimated by the regression equation given in [25]: $s = 45$ (which means that we have to select the best 45 solutions from the 1000 according to the crude model, and subsequently each of them will undergo an exact evaluation). Finally, we define the average TMC obtained from 100 simulation replications as the "exact" evaluation for a considered solution; that is, we set $U = 100$.

When experimenting with the parameters of Stage 2 and Stage 3, we adopt an instance with 100 jobs and 10 machines under normally distributed processing/setup times and $\theta = 0.2$.

The parameters for Stage 2 include SN, F, and CR, which have full control on the searching behavior of DE. The termination criterion is an exogenously given computational time limit: 30 seconds (otherwise, the generation number and population size would be "the larger the better"). We apply a design of experiments (DOE) approach to determine satisfactory values for each parameter. In the full factorial design, we are considering 3 parameters, each with 3 value levels, thus leading to $3^3 = 27$ combinations. The DE is run 10 times, respectively, under each parameter combination, and the main effects plot based on mean objective values is shown as in Figure 2 (output by the Minitab software). From the results, we see that SN should take an intermediate value (either too large or too small will impair the searching performance). If SN is too large, much computational time will be consumed on the evaluation of solutions, which reduces the potential number of generations when the computational time is restricted. If SN is too small, the decreased solution

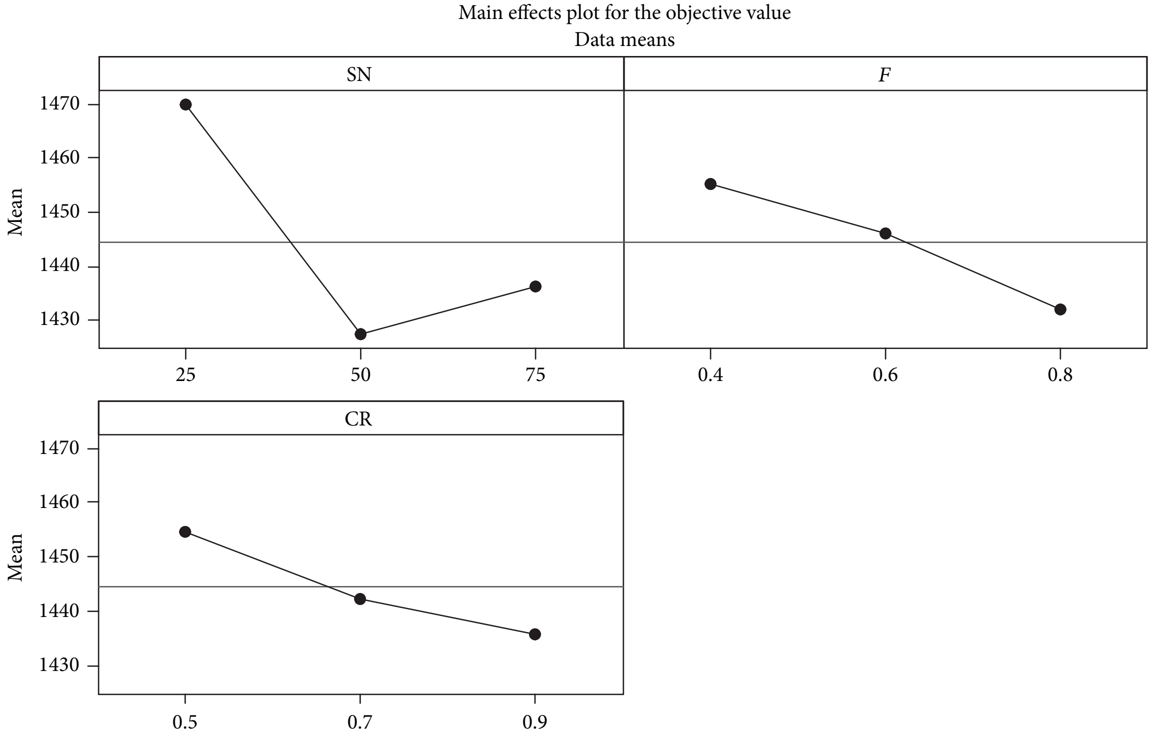


Figure 2: The impact of Stage 2 parameters.

diversity will limit the effectiveness of the mutation and crossover operation, which results in deterioration of solution quality in spite of more generations. Generally, setting SN = 50 is reasonable and recommended for most situations. With respect to F and CR, the results indicate that a larger value is more desirable under the given experimental condition. Since these two parameters control the intensity of mutation and crossover, assigning a reasonably large value is beneficial for enhancing the exploration ability of DE. Therefore, we set $F = 0.8$ and CR = 0.9 in the following experiments.

The parameters for Stage 3 include δ, λ, Q, $h_{\max}$, N_s, and U_s. If we still consider three possible levels for each parameter, full factorial design (including $3^6 = 729$ combinations) is almost unaffordable in this case. Therefore, we resort to the Taguchi design method [30] with the orthogonal array $L_{27}(3^6)$, which means that only 27 scenarios have to be tested. The local search procedure is run 10 times under each of the orthogonal combinations, and the main effects plot for means is shown in Figure 3 (output by the Minitab software). As the figure suggests, the most suitable values for these parameters are $\delta = 0.05$, $\lambda = 0.6$, $Q = 20$, $h_{\max} = 100$, $N_s = 15$, and $U_s = 100$. In particular, the desirable setting of δ (relative amplitude of local perturbations) tends to be small, because over-large perturbations will make the solution leap around the search range, and it is impossible to fine-tune the solution. The reinforcement step size (λ), however, would better be set relatively large, which suggests that reinforcement is a proper strategy for optimizing the production rates. The influence of Q (time to give up and start afresh) shows that it is unwise to backtrack too early or too late, and the searching process should have a moderate degree of tolerance for nonimproving trials. The impact of N_s indicates the importance of making a sufficient number of samplings around each visited solution. The best selection of U_s reflects the effectiveness of OCBA, which can reliably identify the promising solutions with a relatively small number of simulation replications and thus makes it possible to keep U_s low for saving computational time.

5.2. The Main Computational Results. Now, we will use the proposed three-stage algorithm (TSA) to solve different-sized problem instances. The results are compared with the hybrid meta-heuristic algorithm PSO-SA [31], which uses simulated annealing (SA) as a local optimizer for particle swarm optimization (PSO). PSO-SA also relies on hypothesis test to compare the quality of stochastic solutions, which makes it comparable to our approach. Although PSO-SA was initially proposed for stochastic flow shop scheduling, the algorithm does not explicitly utilize the information about machine environments. In fact, PSO-SA can be used for almost any stochastic combinatorial optimization problem. Therefore, PSO-SA can provide a baseline for comparison with our algorithm. The implemented PSO-SA for comparison optimizes the production rates and the production schedule at the same time (by adopting an integrated encoding scheme the first m digits express machine speeds and the last n digits express job sequences). The parameters of the PSO-SA have been calibrated for the discussed problem and finally set as follows: the swarm size $P_s = 40$, the inertia weight $\omega = 0.6$, the cognitive and social coefficients $c_1 = c_2 = 2$, the flying speed

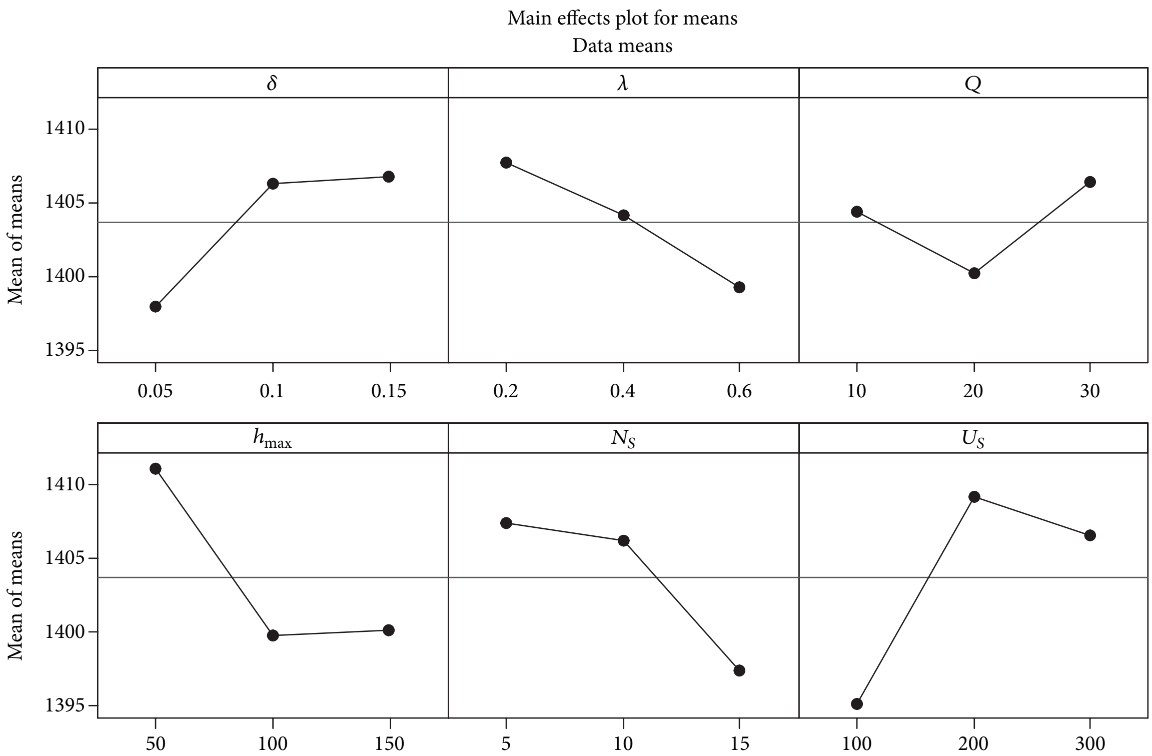


FIGURE 3: The impact of Stage 3 parameters.

limitations $v_{\min} = -1.0, v_{\max} = 1.0$, the initial temperature $T_0 = 3.0$, and the annealing rate $\eta = 0.95$.

In order to make the comparison meaningful, the computational time of PSO-SA is made equal to that of TSA. Specifically, in each trial, we run TSA first (DE is allowed to evolve 500 generations) and record its computational time as CT, and then we run PSO-SA under the time limit of CT (which controls the realized number of iterations for PSO-SA).

Tables 2, 3, and 4 display the optimization results for all the test instances, which involve 10 different sizes (denoted by (n, m)), 3 distribution patterns (normal, uniform, and exponential), and 3 variability levels ($\theta \in \{0.1, 0.2, 0.3\}$). Each algorithm is run for 10 independent times on each instance. In order to reduce random errors, 5 instances have been generated for each considered scenario, that is, each combination of size, distribution, and variability except for exponential distribution whose variance is not independently controllable. For each instance, the best, mean, and worst objective values (under "exact" evaluation) obtained by each algorithm from the 10 runs are, respectively, converted into relative numbers by taking the best objective value achieved by TSA as reference (the conversion is simply "the current value/the best objective value from TSA"). Finally, these values are averaged over the 5 instances of each scenario and listed in the tables.

Based on the presented results, we can conclude that TSA is more effective than the comparative method. The relative improvement of TSA over PSO-SA is greater when the variability level (θ) is higher. This suggests that a multistage optimization framework is more stable than a single-pass search method in the case of considerable uncertainty. The proposed algorithm implements the optimization process with a stepwise refinement policy (from crude optimization to systematic search and then to fine-tuning) so that the stochastic factors in the problem can be fully considered and gradually utilized to adjust the search direction. In addition, TSA outperforms PSO-SA to a greater extent when solving larger-scale instances. The potential search space grows exponentially fast with the increase of job and machine numbers. The advantage of TSA when faced with large solution space is that it utilizes the specific problem information (like the properties described by Theorem 1 and Corollary 3), which promotes the efficiency of evaluating and comparing solutions. By contrast, PSO-SA performs the search process in a quite generic way without special care about the structural property of the studied problem. Therefore, the superiority of TSA can also be explained from the perspective of the No Free Lunch Theorem (according to the No Free Lunch Theorem (NFLT) [27], all algorithms have identical performance when averaged across all possible problems; the NFLT implies that methods must incorporate problem-specific information to improve performance on a subset of problems).

To provide more information, we record the computational time consumed by TSA when solving the instances with normally distributed processing/setup times and $\theta = 0.2$. The time distribution among the three stages is also shown as percentage values in Figure 4. As the results show, the percentage of time consumed by Stage 1 decreases as

Table 2: The computational results under normal distributions.

Size (n, m)	TSA Best	TSA Mean	TSA Worst	PSO-SA Best	PSO-SA Mean	PSO-SA Worst
			$\theta = 0.1$			
100, 10	1.000	1.049	1.093	1.013	1.044	1.086
200, 10	1.000	1.022	1.089	1.010	1.034	1.089
300, 10	1.000	1.013	1.086	1.016	1.045	1.084
300, 15	1.000	1.012	1.039	1.008	1.014	1.045
400, 10	1.000	1.024	1.083	1.007	1.022	1.038
400, 20	1.000	1.024	1.054	1.010	1.032	1.069
500, 10	1.000	1.022	1.045	1.011	1.029	1.072
500, 20	1.000	1.026	1.079	1.012	1.048	1.098
600, 15	1.000	1.014	1.059	1.028	1.036	1.087
600, 20	1.000	1.023	1.037	1.021	1.039	1.078
Avg.	1.000	1.023	1.066	1.014	1.034	1.075
			$\theta = 0.2$			
100, 10	1.000	1.010	1.097	1.019	1.073	1.103
200, 10	1.000	1.015	1.043	1.017	1.040	1.112
300, 10	1.000	1.007	1.034	1.009	1.050	1.130
300, 15	1.000	1.005	1.010	1.017	1.035	1.066
400, 10	1.000	1.023	1.038	1.039	1.044	1.087
400, 20	1.000	1.034	1.050	1.034	1.051	1.072
500, 10	1.000	1.016	1.035	1.033	1.057	1.088
500, 20	1.000	1.014	1.065	1.021	1.049	1.116
600, 15	1.000	1.027	1.056	1.037	1.074	1.089
600, 20	1.000	1.031	1.084	1.038	1.055	1.111
Avg.	1.000	1.018	1.051	1.026	1.053	1.097
			$\theta = 0.3$			
100, 10	1.000	1.016	1.082	1.007	1.055	1.106
200, 10	1.000	1.026	1.053	1.017	1.066	1.102
300, 10	1.000	1.044	1.069	1.006	1.048	1.087
300, 15	1.000	1.009	1.038	1.035	1.057	1.098
400, 10	1.000	1.018	1.063	1.034	1.078	1.127
400, 20	1.000	1.017	1.022	1.024	1.048	1.086
500, 10	1.000	1.025	1.047	1.049	1.086	1.133
500, 20	1.000	1.016	1.069	1.011	1.078	1.091
600, 15	1.000	1.013	1.072	1.048	1.058	1.127
600, 20	1.000	1.031	1.072	1.015	1.068	1.122
Avg.	1.000	1.022	1.059	1.025	1.064	1.108

the problem size grows, which reflects the relative efficiency of the proposed crude model for OO. By contrast, Stage 2 and Stage 3 would require notably more computational time as the problem size increases.

5.3. Sensitivity Analysis for Cost Coefficients. The operational cost is directly affected by the following input parameters: the variable cost coefficients K_p and K_s (these reflect the operating cost to support the workings of the factory for a unit time, for example, the unit-time fuel cost, water and electricity fees, and the hourly wage rate), and the fixed cost coefficient related with each machine a_k(these are related with the cost to support the normal operation of machines for a period of time, for example, the investment on automatic status monitoring and early warning systems). These parameters are fixed at constant values in the short term (so, they have been treated as inputs for our problem), but they may be changed in the long run as the firm gradually increases the investment on the production equipment and manufacturing technology. For example, when new energy-saving technology is introduced into the production line, the variable cost coefficients K_p and K_s (which measure the cost incurred when a machine is working in production or setup mode for one hour) will be reduced to some extent. However, the introduction of new technology needs money; so, the

Table 3: The computational results under uniform distributions.

Size (n, m)	TSA			PSO-SA		
	Best	Mean	Worst	Best	Mean	Worst
			$\theta = 0.1$			
100, 10	1.000	1.012	1.086	1.014	1.052	1.120
200, 10	1.000	1.021	1.067	1.003	1.044	1.075
300, 10	1.000	1.012	1.087	1.010	1.033	1.069
300, 15	1.000	1.025	1.046	1.003	1.039	1.068
400, 10	1.000	1.059	1.089	0.994	1.043	1.073
400, 20	1.000	1.009	1.048	1.006	1.024	1.059
500, 10	1.000	1.034	1.069	1.000	1.038	1.106
500, 20	1.000	1.057	1.064	1.011	1.062	1.104
600, 15	1.000	1.039	1.063	1.018	1.038	1.054
600, 20	1.000	1.035	1.081	1.023	1.039	1.084
Avg.	1.000	1.030	1.070	1.008	1.041	1.081
			$\theta = 0.2$			
100, 10	1.000	1.040	1.147	1.064	1.096	1.145
200, 10	1.000	1.027	1.051	1.016	1.053	1.135
300, 10	1.000	1.037	1.102	1.037	1.082	1.129
300, 15	1.000	1.037	1.051	1.013	1.068	1.098
400, 10	1.000	1.044	1.109	1.051	1.082	1.118
400, 20	1.000	1.048	1.080	1.026	1.076	1.113
500, 10	1.000	1.063	1.092	1.033	1.081	1.139
500, 20	1.000	1.033	1.095	1.052	1.076	1.153
600, 15	1.000	1.057	1.105	1.039	1.100	1.129
600, 20	1.000	1.042	1.106	1.085	1.105	1.135
Avg.	1.000	1.043	1.094	1.042	1.082	1.129
			$\theta = 0.3$			
100, 10	1.000	1.030	1.095	1.029	1.050	1.137
200, 10	1.000	1.038	1.062	1.036	1.083	1.154
300, 10	1.000	1.068	1.105	1.017	1.073	1.135
300, 15	1.000	1.013	1.095	1.061	1.087	1.119
400, 10	1.000	1.068	1.103	1.037	1.092	1.147
400, 20	1.000	1.028	1.070	1.017	1.081	1.126
500, 10	1.000	1.022	1.070	1.046	1.077	1.174
500, 20	1.000	1.066	1.107	1.065	1.107	1.125
600, 15	1.000	1.067	1.093	1.078	1.097	1.156
600, 20	1.000	1.055	1.106	1.070	1.090	1.167
Avg.	1.000	1.046	1.091	1.046	1.084	1.144

question is how much investment is rational and economical for the firm to improve these long-term variables? Sensitivity analysis can provide an answer for such questions.

As an example of sensitivity analysis, we will focus on the impact of K_p on the setting of α_k. The (400, 10) instance under normally distributed processing/setup times and $\theta = 0.2$ is used in this experiment. The value of K_p varies from $0.01(1+\alpha_k)$ to $0.01(10+\alpha_k)$ (10 levels), and under each value of K_p, we run the proposed TSA for 10 independent times to get 10 optimized solutions (in the process of switching K_p, all the other input parameters are kept at their original values). For each solution i that is output by the i-th execution of TSA, we calculate the average value of α among all machines as $\overline{\alpha}_i = (1/m)\sum_{k=1}^{m}\alpha_k$, and we record the corresponding total cost as TMC_i. Finally, we calculate the averaged α in the 10 final solutions as $\alpha(K_p) = (1/10)\sum_{i=1}^{10}\overline{\alpha}_i$ and the averaged total cost as $\mathrm{TMC}(K_p) = (1/10)\sum_{i=1}^{10}\mathrm{TMC}_i$. The results are displayed in Figure 5.

Table 4: The computational results under exponential distributions.

Size (n, m)	TSA			PSO-SA		
	Best	Mean	Worst	Best	Mean	Worst
100, 10	1.000	1.051	1.093	1.001	1.071	1.119
200, 10	1.000	1.024	1.087	1.021	1.054	1.128
300, 10	1.000	1.037	1.076	0.995	1.054	1.088
300, 15	1.000	1.018	1.047	1.036	1.072	1.091
400, 10	1.000	1.050	1.078	1.016	1.051	1.076
400, 20	1.000	1.042	1.051	1.050	1.063	1.086
500, 10	1.000	1.042	1.056	1.061	1.078	1.089
500, 20	1.000	1.033	1.091	1.031	1.048	1.091
600, 15	1.000	1.048	1.095	1.017	1.069	1.085
600, 20	1.000	1.057	1.061	1.048	1.055	1.091
Avg.	1.000	1.040	1.074	1.028	1.062	1.094

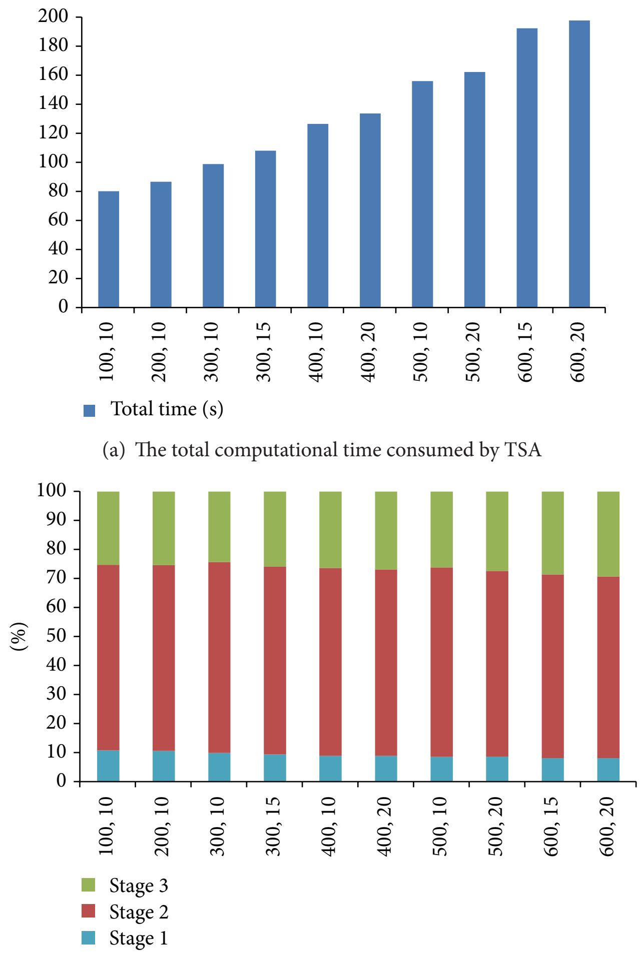
Table 5: The impact of the functional form of FOC_k.

Power on α_k	1	2	3	4
Optimized α	1.82	1.75	1.66	1.47

According to Figure 5(a), there is a clear rising trend in the total cost as the cost coefficient K_p increases. This is no surprise because K_p is an indicator of cost per unit time. Moreover, the slope of the regression line is 319.95, which suggests that reducing the value of K_p by one unit will result in a saving of 319.95 units (on average) in the total cost. Therefore, the firm should be willing to invest at most 319.95 for reducing the cost coefficient K_p by one (e.g., by promoting the energy efficiency of the production lines).

By observing the impact of K_p on the optimized production rate α (Figure 5(b)), we can obtain similar information. For example, if the value of K_p has been decreased by one, the optimal setting of α for each machine should be decreased by 0.1093 (on average). The underlying reason is that when the unit-time production cost decreases, the production pace does not need to be hurried to the original extent, and meanwhile, reducing α reasonably can help cutting down the fixed operational cost.

Finally, we examine the impact of the functional form used to describe the fixed operational cost. Recall that currently the fixed operational cost is defined as $\mathrm{FOC}_k(\alpha_k) = a_k \cdot \alpha_k^2$, and that the square on α_k is used to simulate the accelerated increase of the cost with the production rate. Now, we vary the power of α_k from 1 to 4 and run the optimization procedure in each case. The averaged α values (calculated as earlier) for the same instance are shown in Table 5. From the results, we see that as the power increases, the optimal settings for the production rates exhibit a downward trend. In practice, the detailed functional form for the fixed cost should be specified according to the historical production data.



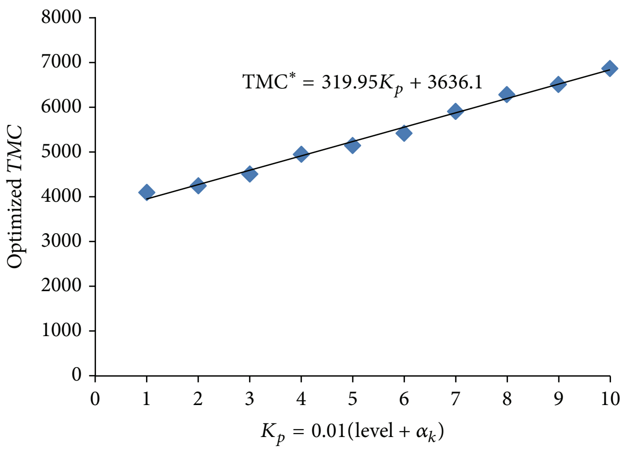
(a) The total computational time consumed by TSA

(b) The percentage of time consumed by each stage

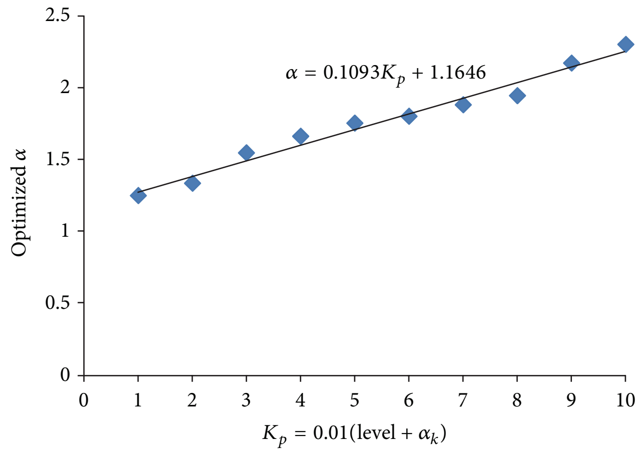
Figure 4: The computational time of TSA.

6. Conclusions

In this paper, we consider a stochastic uniform parallel machine production system with the aim of minimizing total manufacturing cost. The production rates of the machines are adjustable; so, they are treated as decision variables to be optimized together with the detailed production schedule. In accordance with the principle of simulation optimization,



(a) Impact of K_p on TMC*



(b) Impact of K_p on α

FIGURE 5: Sensitivity analysis based on K_p.

we propose a three-stage solution framework based on stepwise refinement for solving the stochastic optimization problem. The proposed algorithm is verified by its superior performance compared with another metaheuristic designed for stochastic optimization. Also, the procedure for sensitivity analysis is discussed. The future research may extend the main ideas presented here to more complicated and realistic production environments like flow shops and job shops.

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Research Article

Pricing and Remanufacturing Decisions of a Decentralized Fuzzy Supply Chain

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The optimal pricing and remanufacturing decisions problem of a fuzzy closed-loop supply chain is considered in this paper. Particularly, there is one manufacturer who has incorporated a remanufacturing process for used products into her original production system, so that she can manufacture a new product directly from raw materials or from collected used products. The manufacturer then sells the new product to two different competitive retailers, respectively, and the two competitive retailers are in charge of deciding the rates of the remanufactured products in their consumers' demand quantity. The fuzziness is associated with the customer's demands, the remanufacturing and manufacturing costs, and the collecting scaling parameters of the two retailers. The purpose of this paper is to explore how the manufacturer and the two retailers make their own decisions about wholesale price, retail prices, and the remanufacturing rates in the expected value model. Using game theory and fuzzy theory, we examine each firm's strategy and explore the role of the manufacturer and the two retailers over three different game scenarios. We get some insights into the economic behavior of firms, which can serve as the basis for empirical study in the future.

1. Introduction

In recent years, the management of closed-loop supply chains has gained growing attention from both business and academic research because of environmental consciousness, environmental concerns, and stringent environmental laws, for example, the legislation on producer responsibility, requiring companies to take back products from customers and to organize for proper recovery and disposal. This legislation is partially due to increased awareness of environmental issues. However, smart companies have also understood that used products often contain lots of value to be recovered. They manage closed-loop supply chains simply because it is a profitable business proposition. It is said that the costs derived from reverse-logistics activities in the USA exceed \$35 billion per year; remanufacturing is a \$53 billion industry in the USA [1].

Without a doubt, closed-loop supply chains has become a matter of strategic importance: an element that companies must consider in decision-making processes concerning the design and development of their supply chains [2]. A specific type of closed-loop supply chains is product manufacturing and remanufacturing supply chain. Product remanufacturing is the process that restores used products or product parts to an "as good as new" condition, after which they can be resold on the market of new products. The industrial operations involved with remanufacturing are of a very uncertain nature due to the uncertainty in timing, quantity, and quality of collected products. So one of the important management issues in product manufacturing and remanufacturing closed-loop supply chains is to effectively match demand, and supply by dealing with the uncertainty of the quality and quantities of the collected products and of the market demand.

In fact, in order to make effective closed-loop supply chain management, the uncertainties that happen in the real world cannot be ignored. Those uncertainties are usually associated with the product supply, used product collecting, the customer demand, and so on. Traditional probabilistic concepts have been used to model the various parameters among today's many studies published on the reverse logistics [3–5]. However, the probability-based approaches may not be sufficient enough to reflect all uncertainties that may arise in a real world manufacturing and remanufacturing closed-loop supply chains. Modelers may face some difficulties while trying to build a valid model of a manufacturing and remanufacturing closed-loop supply chains, in which the related costs cannot be determined precisely. For example, costs may be dependent on some foreign monetary unit, current interest rate, stock keeping unit's market price, and the quality of collected product, which may not be known precisely. Since some uncertainty within manufacturing and remanufacturing closed-loop supply chains cannot be considered appropriately using concepts of probability theory, the quantitative demand forecasts based on manager's judgements, intuitions, and experience seem to be more appropriate, and fuzzy theory rather than probability theory should be applied to model this kind of uncertainties [6]. Fuzzy theory provides a reasonable way to deal with the possibility and linguistic expressions. Zadeh [7] initialized the concept of a fuzzy set via membership function. From then on, many researchers such as Nahmias [8] and Kaufmann and Gupta [9] made great contributions to this field. Recently, Liu [10] B. Liu and Y. K. Liu [11] laid a new foundation for optimization problems in the fuzzy environment, in which the expected value was proposed to deal with optimization problems.

In recent supply chain studies, some researchers have already adopted fuzzy theory to depict uncertainties in supply chain models [12–16]. Li et al. [17] obtained the optimal order quantity for the fuzzy newsboy models through fuzzy ordering of fuzzy numbers with respect to their total integral values. Mukhopadhyay and Ma [18] addressed the issue of a hybrid system where both used and new parts can serve as inputs in the production process to satisfy an uncertain market demand. Kao and Hsu [19] proposed a newsboy model for cases of fuzzy demand. They obtained the optimal policy to minimize the total cost by adopting a method for ranking fuzzy numbers.

Although some researches on the forward supply chain have been given through considering the supply chain's fuzzy uncertainties, little researches on the reverse supply chain considering the fuzzy uncertainties has been established to our knowledge. So, in this paper, we consider a fuzzy manufacturing and remanufacturing closed-loop supply chain with one manufacturer and two competitive retailers; the fuzziness is associated with the consumer demand, the manufacturing and remanufacturing costs of new product, and the collecting cost of the used product. In the forward supply chain, the manufacturer has incorporated a remanufacturing process for used products into her original production system, so that she can manufacture a new product directly from raw materials, or remanufacture part or whole of a collected unit, and wholesales the new products to the two competitive retailers who then sell them to the end consumers. For the the reverse supply chain, the two competitive retailers are in charge of collecting the used products from the consumers, respectively. Using game theory and fuzzy theory, the optimal decisions for each supply chain participant are explored in the expected value model. Some management insights are given in this paper.

The rest of the paper is organized as follows. Section 2 gives the problem description and notations, and Section 3 details our key analytical results. Numerical studies are given in Section 4. Concluding remarks are presented in Section 5.

2. Problem Description

Consider a closed-loop supply chain in a fuzzy environment with one manufacturer and two competitive retailers, labeled retailer 1 and retailer 2. In the following discussion, "he" represents one of the two manufacturers, and "she" represents the retailer. In the forward supply chain, similar to Savaskan et al. [20], assume that the manufacturer has incorporated a remanufacturing process for used products into her original production system, so he can manufacture a new product directly from raw materials with unit manufacturing cost $\tilde{c}_m$, or from collected products with unit remanufacturing cost $\tilde{c}_r$. $\tilde{c}_m$ and $\tilde{c}_r$ are all fuzzy variables. (For the preliminaries of fuzzy theory used in this paper see the preliminaries in [16]). The manufacturer wholesales the new product to the two competitive retailers, respectively, with unit wholesale price w, then the two competitive retailers sell them to the consumers with unit retail price p_i, which is a decision variable of retailer i. We assume that the two retailers are equally powerful and compete in one common market, and all activities occur within a single period. The two competitive retailers face fuzzy linear consumer demands that are influenced by the retail prices of the new product made by the two retailers, respectively. The manufacturer and the two competitive retailers must make their pricing strategies in order to achieve optimal expected profits and behave as if they have perfect information of the demands and the cost structures of other channel members. In the reverse supply chain, the two competitive retailers are in charge of deciding the collecting rates of the remanufactured products in the consumers' demand quantity, denoted as τ_i, and taking back the used products from the end consumers with taking back cost $c(\tau_i)$ $(i = 1, 2)$, according to our survey results; assume that $c(\tau_i) = \tilde{k}_i \tau_i^2$, where $\tilde{k}_i$ is a scaling parameter, which is a fuzzy variable. The manufacturer will take back all the used products collected by the two competitive retailers with unit transfer cost $\tilde{c}_f$, which is a fuzzy variable.

We define the retailer i's price-dependent demand a

$$D_i\left(p_i, p_j\right) = \tilde{a} - p_i + \tilde{\beta} p_j, \quad i = 1, 2, \ \ j = 3 - i, \tag{1}$$

where $\tilde{a}$, $\tilde{\beta}$ are nonnegative fuzzy variables, $\tilde{a}$ denotes the primary demand of retailer i's product, $\tilde{\beta}$ denotes the measure of the responsiveness of each retailer's product's market demand to its competitor's price. We assume that the fuzzy linear demand (1) is symmetrical. This represents a situation in which two retailers have equal competing power in a duopolistic marketplace. We assume that $E[\tilde{\beta}] < 1$, which ensures that the response functions are negatively sloped which, in turn, ensures the existence of the equilibrium solutions. This seems reasonable since sales are relatively more sensitive to price at a retailer's own outlet(s) than at the competing retailer's outlets. In the past, similar demand function has been used widely in marketing research literature (see [21–24]) and in some economic literature (see [25–27]). Moreover, in this paper, assume that fuzzy variables $\tilde{c}_m$,

$\widetilde{c}_r, \widetilde{a}, \widetilde{\beta}, \widetilde{c}_f, \widetilde{k}_1, \widetilde{k}_2$ are all independently nonnegative, which is reasonable in the real world.

In our models, the manufacturer can influence the demand by setting the new product's wholesale price; the two competitive retailers can independently decide the retail price of the new product and the collecting rate of the used product. We do not assume any collusion or cooperation among firms; this assumption is typical in analytical model, although it overstates the information climate of the real world. The logistic cost components of the manufacturer and two retailers (e.g., carrying cost inventory cost, etc.) are without consideration for analytical convenience.

Assume each channel member has the same goal: to maximize his/her own expected profit. From the above descriptions, the two competitive retailers' objectives are to maximize their own expected profits (denoted as $E[\pi_{r_i}]$), which can be described as follows:

$$\begin{aligned}\operatorname*{Max}_{p_i,\tau_i} E\left[\pi_{r_i}\right] = \operatorname*{Max}_{p_i,\tau_i} E\big[&(p_i - w) D_i\left(p_i, p_j\right) \\ &-\widetilde{k}_i\tau_i^2 + \widetilde{c}_f\tau_i D_i\left(p_i, p_j\right)\big],\end{aligned} \tag{2}$$

where

$$\pi_{r_i} = \left(p_i - w\right) D_i\left(p_i, p_j\right) - \widetilde{k}_i\tau_i^2 + \widetilde{c}_f\tau_i D_i\left(p_i, p_j\right). \tag{3}$$

The manufacturer's objective is to maximize his own expected profit (denoted as $E[\pi_m]$), which can be described as follows:

$$\begin{aligned}&\operatorname*{Max}_{w} E\left[\pi_m\right] \\ &= \operatorname*{Max}_{w} E\left[\left(w - \left(\widetilde{c}_f - \widetilde{c}_m + \widetilde{c}_r\right)\tau_1 - \widetilde{c}_m\right) D_1\left(p_1, p_2\right)\right. \\ &\qquad \left. + \left(w - \left(\widetilde{c}_f - \widetilde{c}_m + \widetilde{c}_r\right)\tau_2 - \widetilde{c}_m\right) D_2\left(p_2, p_1\right)\right],\end{aligned} \tag{4}$$

where

$$\begin{aligned}\pi_m = &\left(w - \left(\widetilde{c}_f - \widetilde{c}_m + \widetilde{c}_r\right)\tau_1 - \widetilde{c}_m\right) D_1\left(p_1, p_2\right) \\ &+ \left(w - \left(\widetilde{c}_f - \widetilde{c}_m + \widetilde{c}_r\right)\tau_2 - \widetilde{c}_m\right) D_2\left(p_2, p_1\right).\end{aligned} \tag{5}$$

Note that so far we have not made any assumptions regarding the bargaining power possessed by each channel member. The assumption regarding bargaining power possessed by each firm can influence how the pricing game is solved in our model. Variation in bargaining power in a particular supply chain can create one of the following three scenarios: (1) Manufacturer Stackelberg: the manufacturer has more bargaining power than the two competitive retailers and thus is the Stackelberg leader. (2) Retailer Stackelberg: the two competitive retailers have more bargaining power than the manufacturer and are the Stackelberg leaders. (3) Vertical Nash: every firm in the system has equal bargaining power.

3. Model Analysis

To analyze our model, we follow a game theory approach. The leader in each scenario makes his decision to maximize his/her own expected profit, conditioned on the follower's response. The problem can be solved backwards. We begin by first solving for the decision of the follower of the game, given that he/she has observed the leader's decision. For example, in Manufacturer Stackelberg, the two competitive retailers' decisions are derived first, given that the two competitive retailers have observed the decision made by the manufacturer (on wholesale price). Then, the manufacturer solves his problem given that he knows how the two competitive retailers would react to his decision.

3.1. Manufacturer Stackelberg

3.1.1. Retailers' Decisions. In the Manufacturer Stackelberg game case, the manufacturer first announces his wholesale prices of the new product. The two competitive retailers observe the wholesale price and then simultaneously decide the retail prices they are going to charge for their own product and the collecting rates of the used products. Note that the two competitive retailers move simultaneously. Therefore, we need to calculate the Nash decisions between them first.

Proposition 1. *The two competitive retailers' optimal retail prices and optimal collecting rates of used products, given earlier decision w made by the manufacturer, are*

$$p_1^* = \frac{B_1}{A}w + \frac{B_2}{A}, \tag{6}$$

$$p_2^* = \frac{B_3}{A}w + \frac{B_4}{A}, \tag{7}$$

$$\tau_1^* = E_1 w + E_2, \tag{8}$$

$$\tau_2^* = E_3 w + E_4, \tag{9}$$

where

$$\begin{aligned}A = &\left(2E\left[\widetilde{k}_2\right]E\left[\widetilde{\beta}\right] - E\left[\widetilde{c}_f\widetilde{\beta}\right]E\left[\widetilde{c}_f\right]\right) \\ &\times\left(2E\left[\widetilde{k}_1\right]E\left[\widetilde{\beta}\right] - E\left[\widetilde{c}_f\widetilde{\beta}\right]E\left[\widetilde{c}_f\right]\right) \\ &-\left(E^2\left[\widetilde{c}_f\right] - 4E\left[\widetilde{k}_1\right]\right)\left(E^2\left[\widetilde{c}_f\right] - 4E\left[\widetilde{k}_2\right]\right), \\ B_1 = &\ 2E\left[\widetilde{k}_1\right]\left(E^2\left[\widetilde{c}_f\right] - 4E\left[\widetilde{k}_2\right]\right) \\ &- 2E\left[\widetilde{k}_2\right]\left(2E\left[\widetilde{k}_1\right]E\left[\widetilde{\beta}\right] - E\left[\widetilde{c}_f\widetilde{\beta}\right]E\left[\widetilde{c}_f\right]\right), \\ B_2 = &\left(2E\left[\widetilde{k}_1\right]E\left[\widetilde{a}\right] - E\left[\widetilde{c}_f\widetilde{a}\right]E\left[\widetilde{c}_f\right]\right) \\ &\times\left(E^2\left[\widetilde{c}_f\right] - 4E\left[\widetilde{k}_2\right]\right) \\ &-\left(2E\left[\widetilde{k}_2\right]E\left[\widetilde{a}\right] - E\left[\widetilde{c}_f\widetilde{a}\right]E\left[\widetilde{c}_f\right]\right) \\ &\times\left(2E\left[\widetilde{k}_1\right]E\left[\widetilde{\beta}\right] - E\left[\widetilde{c}_f\widetilde{\beta}\right]E\left[\widetilde{c}_f\right]\right),\end{aligned}$$

$$
\begin{aligned}
B_3 =\ & 2E\left[\widetilde{k}_2\right]\left(E^2\left[\widetilde{c}_f\right]-4E\left[\widetilde{k}_1\right]\right)\\
&-2E\left[\widetilde{k}_1\right]\left(2E\left[\widetilde{k}_2\right]E\left[\widetilde{\beta}\right]-E\left[\widetilde{c}_f\widetilde{\beta}\right]E\left[\widetilde{c}_f\right]\right),\\
B_4 =\ & \left(2E\left[\widetilde{k}_2\right]E\left[\widetilde{a}\right]-E\left[\widetilde{c}_f\widetilde{a}\right]E\left[\widetilde{c}_f\right]\right)\\
&\times\left(E^2\left[\widetilde{c}_f\right]-4E\left[\widetilde{k}_1\right]\right)\\
&-\left(2E\left[\widetilde{k}_1\right]E\left[\widetilde{a}\right]-E\left[\widetilde{c}_f\widetilde{a}\right]E\left[\widetilde{c}_f\right]\right)\\
&\times\left(2E\left[\widetilde{k}_2\right]E\left[\widetilde{\beta}\right]-E\left[\widetilde{c}_f\widetilde{\beta}\right]E\left[\widetilde{c}_f\right]\right),\\
E_1 =\ & \frac{1}{E\left[\widetilde{c}_f\right]}\left(1-\frac{2B_1}{A}+\frac{E\left[\widetilde{\beta}\right]B_3}{A}\right),\\
E_2 =\ & \frac{1}{E\left[\widetilde{c}_f\right]}\left(E\left[\widetilde{a}\right]-\frac{2B_2}{A}+\frac{E\left[\widetilde{\beta}\right]B_4}{A}\right),\\
E_3 =\ & \frac{1}{E\left[\widetilde{c}_f\right]}\left(1-\frac{2B_3}{A}+\frac{E\left[\widetilde{\beta}\right]B_1}{A}\right),\\
E_4 =\ & \frac{1}{E\left[\widetilde{c}_f\right]}\left(E\left[\widetilde{a}\right]-\frac{2B_4}{A}+\frac{E\left[\widetilde{\beta}\right]B_2}{A}\right).
\end{aligned}
\tag{10}
$$

Proof. Using (3), we can have the expected value of π_{r_i} as follows:

$$
\begin{aligned}
E\left[\pi_{r_i}\right] =\ & \left(p_i-w\right)\left(E\left[\widetilde{a}\right]-p_i+E\left[\widetilde{\beta}\right]p_j\right)-E\left[\widetilde{k}_i\right]\tau_i^2\\
&+E\left[\widetilde{c}_f\widetilde{a}\right]\tau_i-E\left[\widetilde{c}_f\right]\tau_ip_i+E\left[\widetilde{c}_f\widetilde{\beta}\right]\tau_ip_j.
\end{aligned}
\tag{11}
$$

From (11), the first order partial derivatives of $E[\pi_{r_1}]$ to p_1, τ_1 and $E[\pi_{r_2}]$ to p_2, τ_2 can be shown as

$$
\begin{aligned}
\frac{\partial E\left[\pi_{r_1}\right]}{\partial p_1} &= w-2p_1+E\left[\widetilde{a}\right]+E\left[\widetilde{\beta}\right]p_2-E\left[\widetilde{c}_f\right]\tau_1,\\
\frac{\partial E\left[\pi_{r_1}\right]}{\partial \tau_1} &= -2E\left[\widetilde{k}_1\right]\tau_1+E\left[\widetilde{c}_f\widetilde{a}\right]-E\left[\widetilde{c}_f\right]p_1+E\left[\widetilde{c}_f\widetilde{\beta}\right]p_2,\\
\frac{\partial E\left[\pi_{r_2}\right]}{\partial p_2} &= w-2p_2+E\left[\widetilde{a}\right]+E\left[\widetilde{\beta}\right]p_1-E\left[\widetilde{c}_f\right]\tau_2,\\
\frac{\partial E\left[\pi_{r_2}\right]}{\partial \tau_2} &= -2E\left[\widetilde{k}_2\right]\tau_2+E\left[\widetilde{c}_f\widetilde{a}\right]-E\left[\widetilde{c}_f\right]p_2+E\left[\widetilde{c}_f\widetilde{\beta}\right]p_1.
\end{aligned}
\tag{12}
$$

Then, we can have the first order conditions as follows:

$$
\begin{aligned}
& w-2p_1+E\left[\widetilde{a}\right]+E\left[\widetilde{\beta}\right]p_2-E\left[\widetilde{c}_f\right]\tau_1=0,\\
& -2E\left[\widetilde{k}_1\right]\tau_1+E\left[\widetilde{c}_f\widetilde{a}\right]-E\left[\widetilde{c}_f\right]p_1+E\left[\widetilde{c}_f\widetilde{\beta}\right]p_2=0,\\
& w-2p_2+E\left[\widetilde{a}\right]+E\left[\widetilde{\beta}\right]p_1-E\left[\widetilde{c}_f\right]\tau_2=0,\\
& -2E\left[\widetilde{k}_2\right]\tau_2+E\left[\widetilde{c}_f\widetilde{a}\right]-E\left[\widetilde{c}_f\right]p_2+E\left[\widetilde{c}_f\widetilde{\beta}\right]p_1=0.
\end{aligned}
\tag{13}
$$

Solving (13), simultaneously, we can easily have (6)–(9), so Proposition 1 is proven. □

3.1.2. Manufacturer's Decision. The manufacturer in this game is the Stackelberg leader. He announces his new product's wholesale price w. Using the retailers' decisions, we can derive the manufacturer's optimal wholesale price. This is carried out by maximizing the manufacturer's expected profit $E[\pi_m]$, given the two competitive retailers' decisions, which are given as in Proposition 1. The manufacturer chooses the wholesale price w to maximize his own individual expected profit $E[\pi_m]$, which can be given as follows:

$$
\begin{aligned}
&\operatorname*{Max}_{w} E\left[\pi_m\right]\\
&\quad= \operatorname*{Max}_{w} E\left[\left(w-\left(\widetilde{c}_f-\widetilde{c}_m+\widetilde{c}_r\right)\tau_1^*-\widetilde{c}_m\right)D_1\left(p_1^*,p_2^*\right)\right.\\
&\qquad\left.+\left(w-\left(\widetilde{c}_f-\widetilde{c}_m+\widetilde{c}_r\right)\tau_2^*-\widetilde{c}_m\right)D_2\left(p_2^*,p_1^*\right)\right],
\end{aligned}
\tag{14}
$$

where $p_1^*, p_2^*, \tau_1^*, \tau_2^*$ are defined as in (6)–(9), respectively.

Proposition 2. *In the Manufacturer Stackelberg game case, the manufacturer's optimal decision (denoted as w_m^*) is satisfied as follows:*

$$
\begin{aligned}
&2E\left[\widetilde{a}\right]+\frac{\left(B_2+B_4\right)\left(E\left[\widetilde{\beta}\right]-1\right)}{A}+2\left(E\left[\widetilde{\beta}\right]-1\right)\frac{B_1+B_3}{A}w_m^*\\
&\quad-\left(\frac{1}{2}\int_0^1\left(\widetilde{a}_\alpha^U\widetilde{c}_{f\alpha}^L+\widetilde{a}_\alpha^L\widetilde{c}_{f\alpha}^U\right)d\alpha-E\left[\widetilde{a}\widetilde{c}_m\right]\right.\\
&\qquad\left.+\frac{1}{2}\int_0^1\left(\widetilde{a}_\alpha^U\widetilde{c}_{r\alpha}^L+\widetilde{a}_\alpha^L\widetilde{c}_{r\alpha}^U\right)d\alpha\right)E_1\\
&\quad+E\left[\widetilde{c}_m\right]\frac{B_1}{A}-\frac{B_3}{2A}\int_0^1\left(\widetilde{\beta}_\alpha^U\widetilde{c}_{m\alpha}^L+\widetilde{\beta}_\alpha^L\widetilde{c}_{m\alpha}^U\right)d\alpha\\
&\quad-\left(\frac{1}{2}\int_0^1\left(\widetilde{a}_\alpha^U\widetilde{c}_{f\alpha}^L+\widetilde{a}_\alpha^L\widetilde{c}_{f\alpha}^U\right)d\alpha-E\left[\widetilde{a}\widetilde{c}_m\right]\right.\\
&\qquad\left.+\frac{1}{2}\int_0^1\left(\widetilde{a}_\alpha^U\widetilde{c}_{r\alpha}^L+\widetilde{a}_\alpha^L\widetilde{c}_{r\alpha}^U\right)d\alpha\right)E_3+E\left[\widetilde{c}_m\right]\frac{B_3}{A}-\frac{B_1}{2A}\\
&\quad\times\int_0^1\left(\widetilde{\beta}_\alpha^U\widetilde{c}_{m\alpha}^L+\widetilde{\beta}_\alpha^L\widetilde{c}_{m\alpha}^U\right)d\alpha+\left(E\left[\widetilde{c}_f\right]-E\left[\widetilde{c}_m\right]+E\left[\widetilde{c}_r\right]\right)\\
&\quad\times\left(\frac{2B_3E_3+2B_1E_1}{A}w_m^*+\frac{B_4E_3+B_3E_4+B_1E_2+B_2E_1}{A}\right)\\
&\quad-\left(\frac{1}{2}\int_0^1\left(\widetilde{\beta}_\alpha^U\widetilde{c}_{f\alpha}^L+\widetilde{\beta}_\alpha^L\widetilde{c}_{f\alpha}^U\right)d\alpha\right.\\
&\qquad\left.-E\left[\widetilde{\beta}\widetilde{c}_m\right]+\frac{1}{2}\int_0^1\left(\widetilde{\beta}_\alpha^U\widetilde{c}_{r\alpha}^L+\widetilde{\beta}_\alpha^L\widetilde{c}_{r\alpha}^U\right)d\alpha\right)\\
&\quad\times\left(\frac{2B_1E_3+2B_3E_1}{A}w_m^*\right.\\
&\qquad\left.+\frac{B_1E_4+B_2E_3+B_3E_2+B_4E_1}{A}\right)=0,
\end{aligned}
\tag{15}
$$

where A, B_1, B_2, B_3, B_4, E_1, E_2, E_3, E_4 are defined as in Proposition 1, respectively.

Proof. With some manipulations, the expected value $E[\pi_m]$ of π_m, defined in (5), can be rewritten as follows:

$$\begin{aligned}
E\left[\pi_m\right] = {} & \left(2E\left[\widetilde{a}\right] + \left(E\left[\widetilde{\beta}\right]-1\right)\left(p_1+p_2\right)\right)w \\
& - \left(\frac{1}{2}\int_0^1 \left(\widetilde{a}_\alpha^U \widetilde{c}_{f\alpha}^L + \widetilde{a}_\alpha^L \widetilde{c}_{f\alpha}^U\right) d\alpha - E\left[\widetilde{a}\,\widetilde{c}_m\right]\right. \\
& \left. + \frac{1}{2}\int_0^1 \left(\widetilde{a}_\alpha^U \widetilde{c}_{r\alpha}^L + \widetilde{a}_\alpha^L \widetilde{c}_{r\alpha}^U\right) d\alpha\right)\tau_1 \\
& + \left(E\left[\widetilde{c}_f\right] - E\left[\widetilde{c}_m\right] + E\left[\widetilde{c}_r\right]\right) p_1\tau_1 \\
& - \left(\frac{1}{2}\int_0^1 \left(\widetilde{\beta}_\alpha^U \widetilde{c}_{f\alpha}^L + \widetilde{\beta}_\alpha^L \widetilde{c}_{f\alpha}^U\right) d\alpha - E\left[\widetilde{\beta}\widetilde{c}_m\right]\right. \\
& \left. + \frac{1}{2}\int_0^1 \left(\widetilde{\beta}_\alpha^U \widetilde{c}_{r\alpha}^L + \widetilde{\beta}_\alpha^L \widetilde{c}_{r\alpha}^U\right) d\alpha\right) p_2\tau_1 \\
& - \frac{1}{2}\int_0^1 \left(\widetilde{a}_\alpha^U \widetilde{c}_{m\alpha}^L + \widetilde{a}_\alpha^L \widetilde{c}_{m\alpha}^U\right) d\alpha + E\left[\widetilde{c}_m\right] p_1 \\
& - \frac{p_2}{2}\int_0^1 \left(\widetilde{\beta}_\alpha^U \widetilde{c}_{m\alpha}^L + \widetilde{\beta}_\alpha^L \widetilde{c}_{m\alpha}^U\right) d\alpha \\
& - \left(\frac{1}{2}\int_0^1 \left(\widetilde{a}_\alpha^U \widetilde{c}_{f\alpha}^L + \widetilde{a}_\alpha^L \widetilde{c}_{f\alpha}^U\right) d\alpha - E\left[\widetilde{a}\,\widetilde{c}_m\right]\right. \\
& \left. + \frac{1}{2}\int_0^1 \left(\widetilde{a}_\alpha^U \widetilde{c}_{r\alpha}^L + \widetilde{a}_\alpha^L \widetilde{c}_{r\alpha}^U\right) d\alpha\right)\tau_2 \\
& + \left(E\left[\widetilde{c}_f\right] - E\left[\widetilde{c}_m\right] + E\left[\widetilde{c}_r\right]\right) p_2\tau_2 \\
& - \left(\frac{1}{2}\int_0^1 \left(\widetilde{\beta}_\alpha^U \widetilde{c}_{f\alpha}^L + \widetilde{\beta}_\alpha^L \widetilde{c}_{f\alpha}^U\right) d\alpha - E\left[\widetilde{\beta}\widetilde{c}_m\right]\right. \\
& \left. + \frac{1}{2}\int_0^1 \left(\widetilde{\beta}_\alpha^U \widetilde{c}_{r\alpha}^L + \widetilde{\beta}_\alpha^L \widetilde{c}_{r\alpha}^U\right) d\alpha\right) p_1\tau_2 \\
& - \frac{1}{2}\int_0^1 \left(\widetilde{a}_\alpha^U \widetilde{c}_{m\alpha}^L + \widetilde{a}_\alpha^L \widetilde{c}_{m\alpha}^U\right) d\alpha + E\left[\widetilde{c}_m\right] p_2 \\
& - \frac{p_1}{2}\int_0^1 \left(\widetilde{\beta}_\alpha^U \widetilde{c}_{m\alpha}^L + \widetilde{\beta}_\alpha^L \widetilde{c}_{m\alpha}^U\right) d\alpha.
\end{aligned} \tag{16}$$

□

With (6)–(9) and (16), the first order derivative of $E[\pi_m]$ to w can be shown as

$$\begin{aligned}
& \frac{\partial E\left[\pi_m\right]}{\partial w} \\
& = 2E\left[\widetilde{a}\right] + \left(E\left[\widetilde{\beta}\right]-1\right)\left(p_1^* + p_2^*\right) + w\left(E\left[\widetilde{\beta}\right]-1\right) \\
& \quad \times \left(\frac{\partial p_1^*}{\partial w} + \frac{\partial p_2^*}{\partial w}\right) - \left(\frac{1}{2}\int_0^1 \left(\widetilde{a}_\alpha^U \widetilde{c}_{f\alpha}^L + \widetilde{a}_\alpha^L \widetilde{c}_{f\alpha}^U\right) d\alpha - E\left[\widetilde{a}\widetilde{c}_m\right]\right. \\
& \quad \left. + \frac{1}{2}\int_0^1 \left(\widetilde{a}_\alpha^U \widetilde{c}_{r\alpha}^L + \widetilde{a}_\alpha^L \widetilde{c}_{r\alpha}^U\right) d\alpha\right) \frac{\partial \tau_1^*}{\partial w} \\
& \quad + \left(E\left[\widetilde{c}_f\right] - E\left[\widetilde{c}_m\right] + E\left[\widetilde{c}_r\right]\right)\left(\tau_1^* \frac{\partial p_1^*}{\partial w} + p_1^* \frac{\partial \tau_1^*}{\partial w}\right) \\
& \quad - \left(\frac{1}{2}\int_0^1 \left(\widetilde{\beta}_\alpha^U \widetilde{c}_{f\alpha}^L + \widetilde{\beta}_\alpha^L \widetilde{c}_{f\alpha}^U\right) d\alpha - E\left[\widetilde{\beta}\widetilde{c}_m\right]\right. \\
& \quad \left. + \frac{1}{2}\int_0^1 \left(\widetilde{\beta}_\alpha^U \widetilde{c}_{r\alpha}^L + \widetilde{\beta}_\alpha^L \widetilde{c}_{r\alpha}^U\right) d\alpha\right) \\
& \quad \times \left(\tau_1^* \frac{\partial p_2^*}{\partial w} + p_2^* \frac{\partial \tau_1^*}{\partial w}\right) + E\left[\widetilde{c}_m\right] \frac{\partial p_1^*}{\partial w} - \frac{\partial p_2^*}{\partial w} \\
& \quad \times \frac{1}{2}\int_0^1 \left(\widetilde{\beta}_\alpha^U \widetilde{c}_{m\alpha}^L + \widetilde{\beta}_\alpha^L \widetilde{c}_{m\alpha}^U\right) d\alpha \\
& \quad - \left(\frac{1}{2}\int_0^1 \left(\widetilde{a}_\alpha^U \widetilde{c}_{f\alpha}^L + \widetilde{a}_\alpha^L \widetilde{c}_{f\alpha}^U\right) d\alpha - E\left[\widetilde{a}\,\widetilde{c}_m\right]\right. \\
& \quad \left. + \frac{1}{2}\int_0^1 \left(\widetilde{a}_\alpha^U \widetilde{c}_{r\alpha}^L + \widetilde{a}_\alpha^L \widetilde{c}_{r\alpha}^U\right) d\alpha\right) \frac{\partial \tau_2^*}{\partial w} \\
& \quad + \left(E\left[\widetilde{c}_f\right] - E\left[\widetilde{c}_m\right] + E\left[\widetilde{c}_r\right]\right)\left(\tau_2^* \frac{\partial p_2^*}{\partial w} + p_2^* \frac{\partial \tau_2^*}{\partial w}\right) \\
& \quad - \left(\frac{1}{2}\int_0^1 \left(\widetilde{\beta}_\alpha^U \widetilde{c}_{f\alpha}^L + \widetilde{\beta}_\alpha^L \widetilde{c}_{f\alpha}^U\right) d\alpha - E\left[\widetilde{\beta}\widetilde{c}_m\right]\right. \\
& \quad \left. + \frac{1}{2}\int_0^1 \left(\widetilde{\beta}_\alpha^U \widetilde{c}_{r\alpha}^L + \widetilde{\beta}_\alpha^L \widetilde{c}_{r\alpha}^U\right) d\alpha\right)\left(\tau_2^* \frac{\partial p_1^*}{\partial w} + p_1^* \frac{\partial \tau_2^*}{\partial w}\right) \\
& \quad + E\left[\widetilde{c}_m\right] \frac{\partial p_2^*}{\partial w} - \frac{\partial p_1^*}{\partial w} \frac{1}{2}\int_0^1 \left(\widetilde{\beta}_\alpha^U \widetilde{c}_{m\alpha}^L + \widetilde{\beta}_\alpha^L \widetilde{c}_{m\alpha}^U\right) d\alpha.
\end{aligned} \tag{17}$$

Therefore, by setting (17) to zero, we can easily have (15).

Proposition 3. *In the Manufacturer Stackelberg game case, the two competitive retailers' optimal retail prices (denoted as p_{m1}^* and p_{m2}^*, resp.) and the optimal collecting rates (denoted as τ_{m1}^* and τ_{m2}^*, resp.) are*

$$\begin{aligned}
p_{m1}^* &= \frac{B_1}{A} w_m^* + \frac{B_2}{A}, \\
p_{m2}^* &= \frac{B_3}{A} w_m^* + \frac{B_4}{A}, \\
\tau_{m1}^* &= E_1 w_m^* + E_2, \\
\tau_{m2}^* &= E_3 w_m^* + E_4,
\end{aligned} \tag{18}$$

where A, B_1, B_2, B_3, B_4, E_1, E_2, E_3, E_4 are defined as in Proposition 1, respectively. w_m^* is defined as in (15).

Proof. By Propositions 1 and 2, we can easily see that Proposition 3 holds. □

3.2. Retailer Stackelberg. The Retailer Stackelberg scenario arises in markets where the two competitive retailers' sizes are larger compared to their manufacturer. Because of their sizes, the two competitive retailers can maintain their margin on sales while squeezing profit from their suppliers. Similar game-theoretic framework as applied in the Manufacturer Stackelberg case is implemented to solve this problem. First, the manufacturer's problem is solved to derive the decision conditional on the retail prices and collecting rates chosen by the two competitive retailers. The two competitive retailers' problems are then solved given that the two competitive retailers know how the manufacturer would react to their retail prices and collecting rates.

Without loss of generality, let m_i be the margin of retailer i enjoyed by selling the new product, namely,

$$p_i = w + m_i, \quad i = 1, 2, \tag{19}$$

where $m_i > 0$.

3.2.1. Manufacturer's Decision. Since the two competitive retailers move first in this game, we need to calculate the manufacturer's decision. The manufacturer is trying to maximize his own expected profit $E[\pi_m]$, where π_m is defined as in (16).

Proposition 4. *In the Retailer Stackelberg game case, the manufacturer's optimal decision, given retail prices p_1 and p_2 and the collecting rates τ_1 and τ_2, is*

$$w^* = F_1 - \frac{1}{2}p_1 - \frac{1}{2}p_2 + F_2(\tau_1 + \tau_2), \tag{20}$$

where

$$\begin{aligned} F_1 &= \frac{E[\tilde{a}] + E[\tilde{c}_m] - (1/2)\int_0^1 \left(\tilde{\beta}^U_\alpha \tilde{c}^L_{m\alpha} + \tilde{\beta}^L_\alpha \tilde{c}^U_{m\alpha}\right) d\alpha}{1 - E[\tilde{\beta}]}, \\ F_2 &= \Bigg(E[\tilde{c}_f] - E[\tilde{c}_m] + E[\tilde{c}_r] - \frac{1}{2}\int_0^1 \left(\tilde{\beta}^U_\alpha \tilde{c}^L_{f\alpha} + \tilde{\beta}^L_\alpha \tilde{c}^U_{f\alpha}\right) d\alpha \\ &\quad - \frac{1}{2}\int_0^1 \left(\tilde{\beta}^U_\alpha \tilde{c}^L_{r\alpha} + \tilde{\beta}^L_\alpha \tilde{c}^U_{r\alpha}\right) d\alpha + E[\tilde{\beta}\tilde{c}_m]\Bigg) \\ &\quad \times \left(2\left(1 - E[\tilde{\beta}]\right)\right)^{-1}. \end{aligned} \tag{21}$$

Proof. Using (16), we have the first order derivative of $E[\pi_m]$ to w as follows:

$$\begin{aligned} \frac{\partial E[\pi_m]}{\partial w} &= 2E[\tilde{a}] + \left(E[\tilde{\beta}] - 1\right)(p_1 + p_2) + 2\left(E[\tilde{\beta}] - 1\right) w \\ &\quad + \left(E[\tilde{c}_f] - E[\tilde{c}_m] + E[\tilde{c}_r]\right)(\tau_1 + \tau_2) \\ &\quad - \Bigg(\frac{1}{2}\int_0^1 \left(\tilde{\beta}^U_\alpha \tilde{c}^L_{f\alpha} + \tilde{\beta}^L_\alpha \tilde{c}^U_{f\alpha}\right) d\alpha - E[\tilde{\beta}\tilde{c}_m] + 2E[\tilde{c}_m] \\ &\quad + \frac{1}{2}\int_0^1 \left(\tilde{\beta}^U_\alpha \tilde{c}^L_{r\alpha} + \tilde{\beta}^L_\alpha \tilde{c}^U_{r\alpha}\right) d\alpha\Bigg)(\tau_1 + \tau_2) \\ &\quad - \frac{1}{2}\int_0^1 \left(\tilde{a}^U_\alpha \tilde{c}^L_{m\alpha} + \tilde{a}^L_\alpha \tilde{c}^U_{m\alpha}\right) d\alpha \\ &\quad + E[\tilde{c}_m]\, p_2 - \frac{1}{2}\int_0^1 \left(\tilde{\beta}^U_\alpha \tilde{c}^L_{m\alpha} + \tilde{\beta}^L_\alpha \tilde{c}^U_{m\alpha}\right) d\alpha. \end{aligned} \tag{22}$$

We can easily see that Proposition 4 holds, by setting (22) to zero and solving it. □

3.2.2. Retailers' Decisions. Having the information about the decision of the manufacturer, each retailer would then use it to maximize her own expected profit $E[\pi_{r_i}]$, where π_{r_i} is defined as in (11).

Note that the two competitive retailers move simultaneously. Therefore, we need to calculate the Nash decisions between them first.

Proposition 5. *In the Retailer Stackelberg game case, the optimal retail price and collecting rate (denoted as p^*_{r1} and τ^*_{r1}, resp.) of retailer 1 and the optimal retail price and collecting rate (denoted as p^*_{r2} and τ^*_{r2}, resp.) of retailer 2 are given as follows:*

$$p^*_{r1} = \frac{G_1G_5 - G_3G_4}{G_3G_6 - G_2G_5}, \tag{23}$$

$$p^*_{r2} = \frac{G_1G_6 - G_2G_4}{G_2G_5 - G_3G_6}, \tag{24}$$

$$\begin{aligned} \tau^*_{r1} &= \frac{E[\tilde{c}_f\tilde{a}] - F_2E[\tilde{a}]}{2E[\tilde{k}_1]} + \frac{E[\tilde{c}_f\tilde{\beta}] - F_2E[\tilde{\beta}]}{2E[\tilde{k}_1]} p^*_{r2} \\ &\quad + \frac{F_2 - E[\tilde{c}_f]}{2E[\tilde{k}_1]} p^*_{r1}, \end{aligned} \tag{25}$$

$$\begin{aligned} \tau^*_{r2} &= \frac{E[\tilde{c}_f\tilde{a}] - F_2E[\tilde{a}]}{2E[\tilde{k}_2]} + \frac{E[\tilde{c}_f\tilde{\beta}] - F_2E[\tilde{\beta}]}{2E[\tilde{k}_2]} p^*_{r1} \\ &\quad + \frac{F_2 - E[\tilde{c}_f]}{2E[\tilde{k}_2]} p^*_{r2}, \end{aligned} \tag{26}$$

where

$$F_1 = \frac{E[\tilde{a}] + E[\tilde{c}_m] - (1/2)\int_0^1 \left(\tilde{\beta}^U_\alpha \tilde{c}^L_{m\alpha} + \tilde{\beta}^L_\alpha \tilde{c}^U_{m\alpha}\right) d\alpha}{1 - E[\tilde{\beta}]}, \tag{27}$$

$$\begin{aligned} F_2 &= \Bigg(E[\tilde{c}_f] - E[\tilde{c}_m] + E[\tilde{c}_r] - \frac{1}{2}\int_0^1 \left(\tilde{\beta}^U_\alpha \tilde{c}^L_{f\alpha} + \tilde{\beta}^L_\alpha \tilde{c}^U_{f\alpha}\right) d\alpha \\ &\quad - \frac{1}{2}\int_0^1 \left(\tilde{\beta}^U_\alpha \tilde{c}^L_{r\alpha} + \tilde{\beta}^L_\alpha \tilde{c}^U_{r\alpha}\right) d\alpha + E[\tilde{\beta}\tilde{c}_m]\Bigg) \\ &\quad \times \left(2\left(1 - E[\tilde{\beta}]\right)\right)^{-1}, \end{aligned}$$

$$G_1 = F_1 + \frac{3}{2}E[\tilde{a}] + \frac{F_2\left(E\left[\tilde{c}_f\tilde{a}\right] - F_2E[\tilde{a}]\right)}{2E\left[\tilde{k}_2\right]} + \frac{\left(F_2 - E\left[\tilde{c}_f\right]\right)\left(E\left[\tilde{c}_f\tilde{a}\right] - F_2E[\tilde{a}]\right)}{2E\left[\tilde{k}_1\right]},$$

$$G_2 = \frac{F_2\left(E\left[\tilde{c}_f\tilde{\beta}\right] - F_2E\left[\tilde{\beta}\right]\right)}{2E\left[\tilde{k}_2\right]} + \frac{\left(F_2 - E\left[\tilde{c}_f\right]\right)^2}{2E\left[\tilde{k}_1\right]} - 3, \tag{28}$$

$$G_3 = \frac{3E\left[\tilde{\beta}\right] - 1}{2} + \frac{\left(F_2 - E\left[\tilde{c}_f\right]\right)\left(E\left[\tilde{c}_f\tilde{\beta}\right] - F_2E\left[\tilde{\beta}\right]\right)}{2E\left[\tilde{k}_1\right]} + \frac{F_2\left(F_2 - E\left[\tilde{c}_f\right]\right)}{2E\left[\tilde{k}_2\right]},$$

$$G_4 = F_1 + \frac{3E[\tilde{a}]}{2} + \frac{\left(F_2 - E\left[\tilde{c}_f\right]\right)\left(E\left[\tilde{c}_f\tilde{a}\right] - F_2E[\tilde{a}]\right)}{2E\left[\tilde{k}_2\right]} + \frac{F_2\left(E\left[\tilde{c}_f\tilde{a}\right] - F_2E[\tilde{a}]\right)}{2E\left[\tilde{k}_1\right]},$$

$$G_5 = \frac{F_2\left(E\left[\tilde{c}_f\tilde{\beta}\right] - F_2E\left[\tilde{\beta}\right]\right)}{2E\left[\tilde{k}_1\right]} + \frac{\left(F_2 - E[\tilde{c}_f]\right)^2}{2E\left[\tilde{k}_2\right]} - 3,$$

$$G_6 = \frac{3E\left[\tilde{\beta}\right] - 1}{2} + \frac{\left(F_2 - E\left[\tilde{c}_f\right]\right)\left(E\left[\tilde{c}_f\tilde{\beta}\right] - F_2E\left[\tilde{\beta}\right]\right)}{2E\left[\tilde{k}_2\right]} + \frac{F_2\left(F_2 - E\left[\tilde{c}_f\right]\right)}{2E\left[\tilde{k}_1\right]}. \tag{29}$$

Proof. By (11) and (20), we have the first order partial derivatives of $E[\pi_{r_1}]$ to p_1 and τ_1 and the first order partial derivatives of $E[\pi_{r_2}]$ to p_2 and τ_2 as

$$\begin{aligned}
\frac{\partial E\left[\pi_{r_1}\right]}{\partial p_1} &= w^* - \frac{5}{2}p_1 + \frac{3}{2}\left(E[\tilde{a}] + E\left[\tilde{\beta}\right]p_2\right) - E\left[\tilde{c}_f\right]\tau_1,\\
\frac{\partial E\left[\pi_{r_1}\right]}{\partial \tau_1} &= \left(E\left[\tilde{c}_f\tilde{\beta}\right] - F_2E\left[\tilde{\beta}\right]\right)p_2 + \left(F_2 - E\left[\tilde{c}_f\right]\right)p_1 - 2E\left[\tilde{k}_1\right]\tau_1 + E\left[\tilde{c}_f\tilde{a}\right] - F_2E[\tilde{a}],\\
\frac{\partial E\left[\pi_{r_2}\right]}{\partial p_2} &= w^* - \frac{5}{2}p_2 + \frac{3}{2} \times \left(E[\tilde{a}] + E\left[\tilde{\beta}\right]p_1\right) - E\left[\tilde{c}_f\right]\tau_2,\\
\frac{\partial E\left[\pi_{r_2}\right]}{\partial \tau_2} &= \left(E\left[\tilde{c}_f\tilde{\beta}\right] - F_2E\left[\tilde{\beta}\right]\right)p_1 + \left(F_2 - E\left[\tilde{c}_f\right]\right)p_2 - 2E\left[\tilde{k}_2\right]\tau_2 + E\left[\tilde{c}_f\tilde{a}\right] - F_2E[\tilde{a}],
\end{aligned} \tag{30}$$

where w^* is defined in (20).

We can get the first order conditions as follows:

$$\begin{aligned}
&w^* - \frac{5}{2}p_1 + \frac{3}{2}\left(E[\tilde{a}] + E\left[\tilde{\beta}\right]p_2\right) - E\left[\tilde{c}_f\right]\tau_1 = 0,\\
&\left(E\left[\tilde{c}_f\tilde{\beta}\right] - F_2E\left[\tilde{\beta}\right]\right)p_2 + \left(F_2 - E\left[\tilde{c}_f\right]\right)p_1 - 2E\left[\tilde{k}_1\right]\tau_1 + E\left[\tilde{c}_f\tilde{a}\right] - F_2E[\tilde{a}] = 0,\\
&w^* - \frac{5}{2}p_2 + \frac{3}{2}\left(E[\tilde{a}] + E\left[\tilde{\beta}\right]p_1\right) - E\left[\tilde{c}_f\right]\tau_2 = 0,\\
&\left(E\left[\tilde{c}_f\tilde{\beta}\right] - F_2E\left[\tilde{\beta}\right]\right)p_1 + \left(F_2 - E\left[\tilde{c}_f\right]\right)p_2 - 2E\left[\tilde{k}_2\right]\tau_2 + E\left[\tilde{c}_f\tilde{a}\right] - F_2E[\tilde{a}] = 0.
\end{aligned} \tag{31}$$

Solving (31), simultaneously, we can easily see that Proposition 5 holds. □

Proposition 6. *In the Retailer Stackelberg game case, the manufacturer's optimal decision (denoted as w_r^*) is*

$$w_r^* = F_1 - \frac{1}{2}p_{r1}^* - \frac{1}{2}p_{r2}^* + F_2\left(\tau_{r1}^* + \tau_{r2}^*\right), \tag{32}$$

where p_{r1}^*, p_{r2}^*, τ_{r1}^*, τ_{r2}^* are defined as in (23)–(26), *respectively, and*

$$F_1 = \frac{E[\tilde{a}] + E[\tilde{c}_m] - (1/2)\int_0^1\left(\tilde{\beta}_\alpha^U\tilde{c}_{m\alpha}^L + \tilde{\beta}_\alpha^L\tilde{c}_{m\alpha}^U\right)d\alpha}{1 - E\left[\tilde{\beta}\right]}, \tag{33}$$

$$\begin{aligned}
F_2 = {}&\left(E\left[\tilde{c}_f\right] - E[\tilde{c}_m] + E[\tilde{c}_r] - \frac{1}{2}\int_0^1\left(\tilde{\beta}_\alpha^U\tilde{c}_{f\alpha}^L + \tilde{\beta}_\alpha^L\tilde{c}_{f\alpha}^U\right)d\alpha\right.\\
&\left.-\frac{1}{2}\int_0^1\left(\tilde{\beta}_\alpha^U\tilde{c}_{r\alpha}^L + \tilde{\beta}_\alpha^L\tilde{c}_{r\alpha}^U\right)d\alpha + E\left[\tilde{\beta}\tilde{c}_m\right]\right)\\
&\times\left(2\left(1 - E\left[\tilde{\beta}\right]\right)\right)^{-1}.
\end{aligned} \tag{34}$$

Proof. By Propositions 4 and 5, we can easily see that Proposition 6 holds. □

Table 1: Relation between linguistic expression and triangular fuzzy variable.

	Linguistic expression	Triangular fuzzy variable
	Low (about 7)	(6, 7, 9)
Remanufacturing cost $\tilde{c}_r$	Medium (about 11)	(9, 11, 14)
	High (about 16)	(14, 16, 19)
	Low (about 17)	(15, 17, 20)
Manufacturing cost $\tilde{c}_m$	Medium (about 23)	(20, 23, 25)
	High (about 29)	(25, 29, 35)
Market base $\tilde{a}$	Large (about 400)	(300, 400, 450)
	Small (about 200)	(150, 200, 280)
Price elasticity $\tilde{\beta}$	Very sensitive (about 0.8)	(0.6, 0.8, 0.9)
	Sensitive (about 0.5)	(0.3, 0.5, 0.6)
	Low (about 2)	(1, 2, 3)
Taking back transfer cost $\tilde{c}_f$	Medium (about 4)	(3, 4, 5)
	High (about 6)	(5, 6, 8)
	Low (about 500)	(450, 500, 650)
Scaling parameter $\tilde{k}_1$	Medium (about 800)	(700, 800, 1000)
	High (about 1100)	(1000, 1100, 1300)
	Low (about 550)	(400, 550, 650)
Scaling parameter $\tilde{k}_2$	Medium (about 850)	(650, 850, 1000)
	High (about 1200)	(1000, 1200, 1300)

3.3. Vertical Nash. In the Vertical Nash model, every firm has equal bargaining power and thus they make their decisions simultaneously. This scenario arises in a market in which there are relatively small- to medium-sized manufacturers and retailers. Since a manufacturer cannot dominate the market over the two competitive retailers, his price decision is conditioned on how the two competitive retailers price the new product. On the other hand, the two competitive retailers must also condition their own retail price and own collecting rate decisions on the wholesale price.

Consider that the decisions of the two competitive retailers and the manufacturer are already derived in the Manufacturer Stackelberg and Retailer Stackelberg game cases, respectively. From the Manufacturer Stackelberg game, the two competitive retailers' decisions for given wholesale price w are given in (6)–(9). From the Retailer Stackelberg game, the manufacturer's decision for given retail prices p_1 and p_2 and the collecting rates τ_1 and τ_2 is given in (20).

Solving (6)–(9) and (20) simultaneously yields the Nash decision solution. The optimal Nash decisions can be derived and be given Proposition 7.

Proposition 7. *In the Vertical Nash case, the optimal retail prices (denoted as p_{n1}^* and p_{n2}^*) chosen by retailer 1 and retailer 2, respectively, the optimal collecting rates (denoted as τ_{n1}^* and τ_{n2}^*) chosen by retailer 1 and retailer 2, respectively, and the optimal wholesale price (denoted as w_n^*) chosen by the manufacturer are*

$$w_n^* = \frac{AF_1 + F_2(B_2 + B_4) + AF_3(E_2 + E_4)}{A - F_2(B_1 + B_3) - AF_2(E_1 + E_3)},$$

$$p_1^* = \frac{B_1}{A} w_n^* + \frac{B_2}{A},$$

$$\begin{aligned} p_2^* &= \frac{B_3}{A} w_n^* + \frac{B_4}{A}, \\ \tau_1^* &= E_1 w_n^* + E_2, \\ \tau_2^* &= E_3 w_n^* + E_4, \end{aligned} \tag{35}$$

where A, B_1, B_2, B_3, B_4, E_1, E_2, E_3, E_4, F_1, F_2, F_3 are defined as in Propositions 1 and 4, respectively.

Proof. Solving (6)–(9) and (20), simultaneously, we can see that Proposition 7 holds. □

Table 2: Optimal expected profits of the manufacturer and the two retailers.

Game scenario	$E[\pi_m]$	$E[\pi_{r_1}]$	$E[\pi_{r_2}]$
Manufacturer Stackelberg	95194	15161	15159
Retailer Stackelberg	76463	26996	27519
Vertical Nash	91005	21868	21865

4. Numerical Studies

In this section, we compare the results obtained from the above three different decision scenarios using numerical approach and study the behavior of firms facing changing environment. By the results obtained from the above three different decision scenarios, we can easily see the expressions of the optimal wholesale price, retail prices, collected rates, and optimal expected profits under different decision scenarios.

Table 3: Optimal decisions of retail prices, wholesale price, and collecting rates.

Game scenario	p_1^*	p_2^*	w^*	τ_1^*	τ_2^*
Manufacturer Stackelberg	503.0028	503.0108	380.8939	0.3296	0.3247
Retailer Stackelberg	484.3593	481.2045	279.4503	0.6861	0.6988
Vertical Nash	455.7510	455.7604	309.0691	0.3883	0.3825

Here, assume that the relationship between linguistic expressions and triangular fuzzy variables for manufacturing cost, remanufacturing cost, market base, scaling parameter, collecting transfer cost, and price elasticity is determined by experts' experiences as shown in Table 1.

Consider the case that the remanufacturing and manufacturing costs $\tilde{c}_r$ and $\tilde{c}_m$ are high ($\tilde{c}_r$ is about 16, $\tilde{c}_m$ is about 29), the market base $\tilde{a}$ is large ($\tilde{a}$ is about 400), price elasticity $\tilde{\beta}$ is sensitive ($\tilde{\beta}$ is about 0.5), taking back transfer cost $\tilde{c}_f$ is medium ($\tilde{c}_f$ is about 4), and scaling parameters $\tilde{k}_1$ and $\tilde{k}_2$ are medium ($\tilde{k}_1$ is about 800, $\tilde{k}_2$ is about 850). Using Table 1, $\tilde{c}_r = (14, 16, 19)$, $\tilde{c}_m = (25, 29, 35)$, $\tilde{a} = (300, 400, 450)$, $\tilde{\beta} = (0.3, 0.5, 0.6)$, $\tilde{c}_f = (3, 4, 5)$, $\tilde{k}_1 = (700, 800, 1000)$, $\tilde{k}_2 = (650, 850, 1000)$. The expected values are $E[\tilde{c}_r] = (14+2\times16+19)/4 = 65/4$, $E[\tilde{c}_m] = (25 + 2 \times 29 + 35)/4 = 118/4$, $E[\tilde{a}] = (300+2\times400+450)/4 = 1550/4$, $E[\tilde{\beta}] = (0.3+2\times0.5+0.6)/4 = 1.9/4$, $E[\tilde{c}_f] = (3 + 2 \times 4 + 5)/4 = 4$, $E[\tilde{k}_1] = (700 + 2 \times 800 + 1000)/4 = 3300/4$, $E[\tilde{k}_2] = (650+2\times850+1000)/4 = 3350/4$. The α-optimistic value and α-pessimistic value of $\tilde{c}_r$, $\tilde{c}_m$, $\tilde{c}_f$, $\tilde{\beta}$ and $\tilde{a}$ are $\tilde{c}_{r\alpha}^L = 14 + 2\alpha$, $\tilde{c}_{r\alpha}^U = 19 - 3\alpha$, $\tilde{c}_{m\alpha}^L = 25 + 4\alpha$, $\tilde{c}_{m\alpha}^U = 35-6\alpha$, $\tilde{\beta}_\alpha^L = 0.3+0.2\alpha$, $\tilde{\beta}_\alpha^U = 0.6-0.1\alpha$, $\tilde{c}_{f\alpha}^L = 3+\alpha$, $\tilde{c}_{f\alpha}^U = 5-\alpha$, $\tilde{a}_\alpha^L = 300 + 100\alpha$, $\tilde{a}_\alpha^U = 450 - 50\alpha$, respectively. The results of expected profits and optimal decisions are shown as in Tables 2 and 3.

5. Observation

From Tables 2 and 3, we have the following results.

(1) For the three decentralized decision cases, the firm who is the leader in the supply chain has the advantage to get the higher profit; for example, the manufacturer's profit under Manufacturer Stackelberg game scenario is higher than that under Retailer Stackelberg game scenario and the Vertical Nash game case. For the two competitive retailers, they have their own minimal expected profits under Manufacturer Stackelberg game scenario.

(2) The new product's optimal retail prices charged by the two competitive retailers, respectively, under Vertical Nash decision case are lower than those under the Manufacturer Stackelberg and Retailer Stackelberg decision cases, and the optimal retail prices achieve the biggest value under Manufacturer Stackelberg game scenario.

(3) The new product achieves the highest wholesale price in the Manufacturer Stackelberg game, followed by the Vertical Nash game and then the Retailer Stackelberg game case.

(4) The optimal collecting rates of the used products charged by the two competitive retailers, respectively, achieve the highest wholesale price in the Retailer Stackelberg game, followed by the Vertical Nash game and then the Manufacturer Stackelberg game case.

6. Conclusions

Different from the conventional studies, this paper explores the roles of the two competitive retailers and the manufacturer and their bargaining powers by examining the supply chain in a fuzzy environment over three different game scenarios. We derive the expressions for optimal retail prices, wholesale price, and collecting rates with expected value model. By analyzing a numerical example, we further analyze the analytical solutions and give some managerial analysis.

Compared to the traditional approach used in the study of closed-loop supply chain, the proposed approach in this paper requires less data to model the fuzziness which is associated with the consumer demand, the manufacturing and remanufacturing costs of new product, and the collecting cost of the used product and can make use of the subjective estimation based on decision maker's judgment, experience, and intuitions. It is appropriate when the situation is ambiguous and lacks historical data.

However, we have made some assumptions that may be relaxed to improve the model in the future research. One assumption is that the demand function is linear; further work is desirable to test whether our conclusions extend to other forms of demand function. The other assumptions are that the closed-loop supply chains only with one period and competition only existing in retail process. Thus, the supply chain with competitive manufacturers and/or competitive retailers, and the model over multiple periods can be considered in the future.

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Research Article

Pricing Decisions of a Two-Echelon Supply Chain in Fuzzy Environment

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Pricing decisions of a two-echelon supply chain with one manufacturer and duopolistic retailers in fuzzy environment are considered in this paper. The manufacturer produces a product and sells it to the two retailers, who in turn retail it to end customers. The fuzziness is associated with the customers' demand and the manufacturing cost. The purpose of this paper is to analyze the effect of two retailers' different pricing strategies on the optimal pricing decisions of the manufacturer and the two retailers themselves in MS Game scenario. As a reference model, the centralized decision scenario is also considered. The closed-form optimal pricing decisions of the manufacturer and the two retailers are derived in the above decision scenarios. Some insights into how pricing decisions vary with decision scenarios and the two retailers' pricing strategies in fuzzy environment are also investigated, which can serve as the basis for empirical study in the future.

1. Introduction

There is abundant literature on the pricing/ordering policies for two-echelon supply chain management. Most of them focused on the two-echelon supply chain with one manufacturer/supplier and one retailer/buyer and adopted the following assumption on the channel structure: the manufacturer/supplier wholesales a product to the retailer/buyer who in turn retails it to end consumers. The retail market demand varies with the retail price according to a deterministic/stochastic demand function that is assumed to be known to both the manufacturer and the retailer, and the costs incurred in the manufacturing and inventory process are positive constant numbers. Moreover, a most common gaming assumption on the pricing/ordering decision process is that the manufacturer is a Stackelberg leader and the retailer is a Stackelberg follower (hereafter "MS Game") in the existing two-echelon supply chain literature. For example, A. H. L. Lau and H. S. Lau [1] studied the effects of different demand curves on the optimal solution of a two-echelon system in the manufacturer-Stackelberg process. They found out that under a downward-sloping price-versus-demand relationship the manufacturer's profit is the double of the retailer's. Subsequently, Lau et al. [2] considered a two-echelon system with one manufacturer and one retailer. They presented a procedure for the dominant manufacturer to design a profit maximizing volume-discount scheme with stochastic and asymmetric demand information by modeling this supply chain as a manufacturer-Stackelberg game. Zhao et al. [3] studied the pricing problem of substitutable products in a supply chain with one manufacturer and two competitive retailers.

In fact, in order to make effective supply chain management, uncertainties that happen in the real world cannot be ignored. Those uncertainties are usually associated with product supply, manufacturing cost, customer demand, and so on. The quantitative demand forecasts based on manager's judgements, intuitions, and experience seem to be more appropriate, and the fuzzy theory rather than probability theory should be applied to model this kind of uncertainties [4]. Zadeh [5] initialized the concept of a fuzzy set via membership function. From then on, many researchers such as Nahmias [6], Kaufmann and Gupta [7], Liu [8], and B. Liu and Y. K. Liu [9] made great contributions to this field. Fuzzy theory provides a reasonable way to deal with possibility and linguistic expressions (i.e., decision maker's judgements; e.g., manufacturing cost may be expressed as "low cost" or "high cost" to make rough estimates, and market base can be expressed as "large market base" or "small market base" to make rough estimates, etc.).

Many researchers have already adopted fuzzy theory to depict uncertainties in the supply chain model. Zhao et al. [3] considered the pricing problem of substitutable products in a fuzzy supply chain by using game theory in this paper. Xie et al. [10] developed a new two-level coordination strategy that aims to improve the overall supply chain performance through hierarchical inventories control and by introducing a coordination function. They supposed that the supply chain operates under uncertainty in customer demand, which is described by imprecise terms and modelled by fuzzy sets.

This paper extends the current model related to two-echelon supply chain pricing issue from two aspects: one is considering fuzziness associated with customer's demands and the manufacturing cost; the other is analyzing the effect of the two retailers' different pricing strategies (e.g., Bertrand, Cooperation and Stackelberg) on the optimal pricing decisions of the manufacturer and the duopolistic retailers in MS Game scenario. First, as a benchmark model, one centralized pricing model (namely, assume that the manufacturer and the duopolistic retailers behave as part of a unified system) is established. Second, based on the two retailers' different pricing strategies, three decentralized pricing models are constructed in fuzzy environment (e.g., the MSB model where the two retailers implement the Bertrand competition, the MSC model where the two retailers implement the cooperation strategy, and the MSS model where the two retailers implement the Stackelberg competition) and the effect of the two retailers' different pricing strategies on the pricing decisions of the manufacturer and the two retailers is considered. Third, the closed-form solutions for these models are provided. Finally, we provide numerical examples to show the difference among each firm's optimal pricing decisions, the difference among each firm's maximum expected profits, and the variation of each firm's optimal pricing strategy and maximum expected profit with the two retailers' pricing strategies and these decision scenarios in fuzzy environment.

The rest of the paper is organized as follows. Section 2 presents preliminaries of fuzzy theory for this paper. Section 3 gives the problem description and notations, and Section 4 details our key analytical results. Numerical studies are given in Section 5. Finally, some concluding comments are presented in Section 6.

2. Preliminaries

A possibility space is defined as a triplet $(\Theta, \mathscr{P}(\Theta), \text{Pos})$, where Θ is a nonempty set, $\mathscr{P}(\Theta)$ is the power set of Θ, and Pos is a possibility measure. Each element in $\mathscr{P}(\Theta)$ is called an event. For each event A, $\text{Pos}\{A\}$ indicates the possibility that A will occur. Nahmias [6] and Liu [11] gave the following four axioms.

Axiom 1. $\text{Pos}\{\Theta\} = 1$.

Axiom 2. $\text{Pos}\{\phi\} = 0$, where ϕ denotes the empty set.

Axiom 3. $\text{Pos}\{\bigcup_{i=1}^{m} A_i\} = \sup_{1\le i\le m} \text{Pos}\{A_i\}$ for any collection A_i in $\mathscr{P}(\Theta)$.

Axiom 4. Let Θ_i be a nonempty set, on which Pos_i is the possibility measure satisfying the above three axioms, $i = 1, 2, \ldots, n$, and $\Theta = \prod_{i=1}^{n} \Theta_i$, then

$$\text{Pos}(A) = \sup_{(\theta_1,\theta_2,\ldots,\theta_n)\in A} \text{Pos}_1(\theta_1) \wedge \text{Pos}_2(\theta_2) \wedge \cdots \wedge \text{Pos}_n(\theta_n), \tag{1}$$

for each $A \in \mathscr{P}(\Theta)$. In that case we write $\text{Pos} = \wedge_{i=1}^{n} \text{Pos}_i$.

Lemma 1 (see [12]). *Suppose that $(\Theta_i, \mathscr{P}(\Theta_i), \text{Pos}_i)$, $i = 1, 2, \cdots, n$ is a possibility space. By Axiom 4, $(\prod_{i=1}^{n}\Theta_i, \mathscr{P}(\prod_{i=1}^{n}\Theta_i), \wedge_{i=1}^{n} \text{Pos}_i)$ is also a possibility space, which is called the product possibility space.*

Definition 2 (see [6]). A fuzzy variable is defined as a function from the possibility space $(\Theta, \mathscr{P}(\Theta), \text{Pos})$ to the set of real numbers.

Definition 3 (see [12]). A fuzzy variable ξ is said to be nonnegative (or positive) if $\text{Pos}\{\xi < 0\} = 0$ (or $\text{Pos}\{\xi \le 0\} = 0$).

Definition 4 (see [12]). Let $f : R^n \to R$ be a function and let ξ_i be a fuzzy variable defined on the possibility space $(\Theta_i, \mathscr{P}(\Theta_i), \text{Pos}_i)$, $i = 1, 2, \ldots, n$, respectively. Then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is a fuzzy variable defined on the product possibility space $(\prod_{i=1}^{n}\Theta_i, \mathscr{P}(\prod_{i=1}^{n}\Theta_i), \wedge_{i=1}^{n} \text{Pos}_i)$ as $\xi(\theta_1, \theta_2, \ldots, \theta_n) = f(\xi_1(\theta_1), \xi_2(\theta_2), \ldots, \xi_n(\theta_n))$ for any $(\theta_1, \theta_2, \ldots, \theta_n) \in \prod_{i=1}^{n}\Theta_i$.

The independence of fuzzy variables was discussed by several researchers, such as Zadeh [13] and Nahmias [6].

Definition 5. The fuzzy variables $\xi_1, \xi_2, \ldots, \xi_n$ are independent if for any sets $\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_n$ of R,

$$\text{Pos}\{\xi_i \in \mathscr{B}_i, i = 1, 2, \ldots, n\} = \min_{1\le i\le n} \text{Pos}\{\xi_i \in \mathscr{B}_i\}. \tag{2}$$

Lemma 6 (see [11]). *Let ξ_i be independent fuzzy variable, and $f_i : R \to R$ function, $i = 1, 2, \ldots, m$, then $f_1(\xi_1), f_2(\xi_2), \ldots, f_m(\xi_m)$ are independent fuzzy variables.*

Definition 7 (see [12]). Let ξ be a fuzzy variable on the possibility space $(\Theta, \mathscr{P}(\Theta), \text{Pos})$, and $\alpha \in (0, 1]$, then

$$\xi_\alpha^L = \inf\{r \mid \text{Pos}\{\xi \le r\} \ge \alpha\}, \quad \xi_\alpha^U = \sup\{r \mid \text{Pos}\{\xi \ge r\} \ge \alpha\} \tag{3}$$

are called the α-pessimistic value and the α-optimistic value of ξ, respectively.

Example 8. The triangular fuzzy variable $\xi = (a_1, a_2, a_3)$ has its α-pessimistic value and α-optimistic value as

$$\xi_\alpha^L = a_2\alpha + a_1(1-\alpha), \qquad \xi_\alpha^U = a_2\alpha + a_3(1-\alpha). \tag{4}$$

Lemma 9 (see [14]). *Let ξ_i be an independent fuzzy variable defined on the possibility space $(\Theta_i, \mathscr{P}(\Theta_i), \mathrm{Pos}_i)$ with continuous membership function, $i = 1, 2, \ldots, n$, and $f : X \subset \mathscr{R}^n \to \mathscr{R}$ a measurable function. If $f(x_1, x_2, \ldots, x_n)$ is monotonic with respect to x_i, respectively, then*

(a) $f_\alpha^U(\xi) = f(\xi_{1\alpha}^V, \xi_{2\alpha}^V, \ldots, \xi_{n\alpha}^V)$, *where* $\xi_{i\alpha}^V = \xi_{i\alpha}^U$, *if* $f(x_1, x_2, \ldots, x_n)$ *is nondecreasing with respect to* x_i; $\xi_{i\alpha}^V = \xi_{i\alpha}^L$, *otherwise;*

(b) $f_\alpha^L(\xi) = f(\xi_{1\alpha}^{\overline{V}}, \xi_{2\alpha}^{\overline{V}}, \ldots, \xi_{n\alpha}^{\overline{V}})$, *where* $\xi_{i\alpha}^{\overline{V}} = \xi_{i\alpha}^L$, if $f(x_1, x_2, \ldots, x_n)$ *is nondecreasing with respect to* x_i; $\xi_{i\alpha}^{\overline{V}} = \xi_{i\alpha}^U$, *otherwise,*

where $f_\alpha^U(\xi)$ and $f_\alpha^L(\xi)$ denote the α-optimistic value and the α-pessimistic value of fuzzy variable $f(\xi)$, respectively.

Definition 10 (see [9]). Let $(\Theta, \mathscr{P}(\Theta), \mathrm{Pos})$ be a possibility space and A a set in $\mathscr{P}(\Theta)$. The credibility measure of A is defined as

$$\mathrm{Cr}\{A\} = \frac{1}{2}\left(1 + \mathrm{Pos}\{A\} - \mathrm{Pos}\{A^c\}\right), \tag{5}$$

where A^c denotes the complement of A.

Definition 11 (see [9]). Let ξ be a fuzzy variable; the expected value of ξ is defined as

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \ge x\}\,\mathrm{d}x - \int_{-\infty}^0 \mathrm{Cr}\{\xi \le x\}\,\mathrm{d}x, \tag{6}$$

provided that at least one of the two integrals is finite.

Example 12. The triangular fuzzy variable $\xi = (a_1, a_2, a_3)$ has an expected value

$$E[\xi] = \frac{a_1 + 2a_2 + a_3}{4}. \tag{7}$$

Definition 13 (see [9]). Let f be a function on $R \to R$ and let ξ be a fuzzy variable. Then the expected value $E[f(\xi)]$ is defined as

$$\begin{aligned} E[f(\xi)] = {} & \int_0^{+\infty} \mathrm{Cr}\{f(\xi) \ge x\}\,\mathrm{d}x \\ & - \int_{-\infty}^0 \mathrm{Cr}\{f(\xi) \le x\}\,\mathrm{d}x, \end{aligned} \tag{8}$$

provided that at least one of the two integrals is finite.

Lemma 14 (see [15]). *Let ξ be a fuzzy variable with finite expected value. Then*

$$E[\xi] = \frac{1}{2}\int_0^1 \left(\xi_\alpha^L + \xi_\alpha^U\right)\mathrm{d}\alpha. \tag{9}$$

Lemma 15 (see [15]). *Let ξ and η be independent fuzzy variables with finite expected values. Then for any numbers a and b, we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{10}$$

3. Problem Description

Consider a two-echelon supply chain with one monopolistic manufacturer and two duopolistic retailers (retailer 1 and retailer 2) in fuzzy environment. The monopolistic manufacturer manufactures products and sells them to the duopolistic retailers, who in turn retail them to end customers. The manufacturer produces products with unit manufacturing cost $\tilde{c}$, which is a fuzzy variable, and wholesales them to the retailers with unit wholesale price w, respectively. The retailer i then sells products to end consumers with unit retail price $p_i, i = 1, 2$. The manufacturer and retailers must make their pricing decisions in order to achieve the maximum expected profits and behave as if they have perfect information of the demand and the cost structures of other channel members.

Similar to McGuire and Staelin [16], we assume that the demand for each retailer's product is sensitive to the retail prices of the duopolistic retailers, which uses a set of basic characteristics of the type of demand of each product, for example, downward sloping in its own price, and increases with respect to the competitor's price. And we assume that all activities occur within a single period. Specifically, the demand faced by the retailer i can be expressed as

$$D_i\left(p_i, p_j\right) = \tilde{a}_i - \tilde{\beta} p_i + \tilde{\gamma} p_j, \quad i = 1, 2, \ j = 3 - i, \tag{11}$$

where $\tilde{a}_i$, $\tilde{\beta}$, and $\tilde{\gamma}$ are nonnegative fuzzy variables. $\tilde{a}_i$ denotes the primary demand faced by the retailer i $(i = 1, 2)$, $\tilde{\beta}$ denotes the measure of the responsiveness of each retailer's market demand to the price charged by herself, and $\tilde{\gamma}$ denotes the measure of the responsiveness of each retailer's market demand to her competitor's price. Here we assume that the fuzzy variables $\tilde{\beta}$ and $\tilde{\gamma}$ satisfy $E[\tilde{\beta}] > E[\tilde{\gamma}]$, which means that the expected demand for a retailer's product is more sensitive to the change in its own price than to the change in the price of the other competitor's product. This assumption is reasonable in reality.

In our model, the manufacturer can influence the market demand by setting his wholesale price, and the retailers can also influence the market demands by making their retail prices, respectively. We assume that the chain members are independent, risk neutral, and profit maximizing. The chain members choose their decisions sequentially in a manufacturer-Stackelberg game (namely, the manufacturer acts as the Stackelberg leader and the duopolistic retailers act as the followers), and they have complete information about the other members. Moreover, the logistic cost components of the chain members (i.e., carrying cost and inventory cost, etc.) are not considered in our paper for analytical convenience. As explained in A. H. L. Lau and H. S. Lau [1], by removing the confounding effect of the logistic cost components, their profit functions are more effective to reveal the effects of different game procedures.

From the above problem description, the manufacturer's objective is to maximize his expected profit $E[\pi_m(w)]$

(For convenience, it is $E[\pi_m]$ for short sometimes in this paper), which can be described as

$$\max_w E\left[\pi_m(w)\right] = \max_w E\left[\sum_{i=1}^{2}(w-\widetilde{c})\left(\widetilde{a}_i - \widetilde{\beta}p_i + \widetilde{\gamma}p_j\right)\right]. \tag{12}$$

The objectives of the retailers are to maximize their respective expected profits $E[\pi_{r1}(p_1, p_2)]$ and $E[\pi_{r2}(p_1, p_2)]$ (For convenience, abbreviated to $E[\pi_{r1}]$ and $E[\pi_{r2}]$ for short sometimes in this paper), which can be described as

$$\max_{p_1} E\left[\pi_{r1}(p_1,p_2)\right] = \max_{p_1} E\left[(p_1-w)\left(\widetilde{a}_1 - \widetilde{\beta}p_1 + \widetilde{\gamma}p_2\right)\right], \tag{13}$$

$$\max_{p_2} E\left[\pi_{r2}(p_1,p_2)\right] = \max_{p_2} E\left[(p_2-w)\left(\widetilde{a}_2 - \widetilde{\beta}p_2 + \widetilde{\gamma}p_1\right)\right]. \tag{14}$$

4. Analytical Results

4.1. Centralized Pricing Model (CD Model). As a benchmark to evaluate channel decisions under different decision cases, we first give the centralized pricing model; namely, there is one entity that aims to optimize the whole supply chain system performance, so both the duopolistic retailers' and the manufacturer's decisions are fully coordinated in the centralized decision case. The wholesale price charged by the manufacturer is seen as inner transfer price and thus will be neglected. The total profit is determined by the production cost and retail prices.

Let π_c be the total profit of the centralized supply chain; we have

$$\pi_c = \sum_{i=1}^{2}(p_i-\widetilde{c})\left(\widetilde{a}_i - \widetilde{\beta}p_i + \widetilde{\gamma}p_j\right), \quad j = 3-i. \tag{15}$$

To maximize the system expected profit $E[\pi_c(p_1, p_2)]$, the objective is

$$\max_{p_1,p_2} E\left[\pi_c\right] = \max_{p_1,p_2} E\left[\sum_{i=1}^{2}(p_i-\widetilde{c})\left(\widetilde{a}_i - \widetilde{\beta}p_i + \widetilde{\gamma}p_j\right)\right]. \tag{16}$$

Proposition 16. *In the CD model, the optimal retail prices p_{c1}^* and p_{c2}^* are given as*

$$\begin{aligned} p_{c1}^* &= \frac{A_1E[\widetilde{\beta}] + A_2E[\widetilde{\gamma}]}{2\left(E^2[\widetilde{\beta}] - E^2[\widetilde{\gamma}]\right)}, \\ p_{c2}^* &= \frac{A_2E[\widetilde{\beta}] + A_1E[\widetilde{\gamma}]}{2\left(E^2[\widetilde{\beta}] - E^2[\widetilde{\gamma}]\right)}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} A_1 &= E[\widetilde{a}_1] + E[\widetilde{c}\widetilde{\beta}] - \frac{1}{2}\int_0^1 \left(\widetilde{c}_\alpha^U\widetilde{\gamma}_\alpha^L + \widetilde{c}_\alpha^L\widetilde{\gamma}_\alpha^U\right)d\alpha, \\ A_2 &= E[\widetilde{a}_2] + E[\widetilde{c}\widetilde{\beta}] - \frac{1}{2}\int_0^1 \left(\widetilde{c}_\alpha^U\widetilde{\gamma}_\alpha^L + \widetilde{c}_\alpha^L\widetilde{\gamma}_\alpha^U\right)d\alpha. \end{aligned} \tag{18}$$

Proof. From (15), the expected profit $E[\pi_c]$ can be expressed as

$$\begin{aligned} E\left[\pi_c\right] = & -E[\widetilde{\beta}]\left(p_1^2+p_2^2\right) + 2E[\widetilde{\gamma}]p_1p_2 \\ & + \left(E[\widetilde{a}_1] + E[\widetilde{c}\widetilde{\beta}] - \frac{1}{2}\int_0^1 \left(\widetilde{c}_\alpha^U\widetilde{\gamma}_\alpha^L + \widetilde{c}_\alpha^L\widetilde{\gamma}_\alpha^U\right)d\alpha\right)p_1 \\ & + \left(E[\widetilde{a}_2] + E[\widetilde{c}\widetilde{\beta}] - \frac{1}{2}\int_0^1 \left(\widetilde{c}_\alpha^U\widetilde{\gamma}_\alpha^L + \widetilde{c}_\alpha^L\widetilde{\gamma}_\alpha^U\right)d\alpha\right)p_2 \\ & - \frac{1}{2}\int_0^1 \left(\widetilde{a}_{1\alpha}^U\widetilde{c}_\alpha^L + \widetilde{a}_{1\alpha}^L\widetilde{c}_\alpha^U\right)d\alpha \\ & - \frac{1}{2}\int_0^1 \left(\widetilde{a}_{2\alpha}^U\widetilde{c}_\alpha^L + \widetilde{a}_{2\alpha}^L\widetilde{c}_\alpha^U\right)d\alpha. \end{aligned} \tag{19}$$

Then, the first-order derivatives of $E[\pi_c]$ to p_1 and p_2 are

$$\begin{aligned} \frac{\partial E[\pi_c]}{\partial p_1} = & -2E[\widetilde{\beta}]p_1 + E[\widetilde{a}_1] + 2E[\widetilde{\gamma}]p_2 + E[\widetilde{c}\widetilde{\beta}] \\ & - \frac{1}{2}\int_0^1 \left(\widetilde{c}_\alpha^U\widetilde{\gamma}_\alpha^L + \widetilde{c}_\alpha^L\widetilde{\gamma}_\alpha^U\right)d\alpha, \\ \frac{\partial E[\pi_c]}{\partial p_2} = & -2E[\widetilde{\beta}]p_2 + E[\widetilde{a}_2] + 2E[\widetilde{\gamma}]p_1 \\ & + E[\widetilde{c}\widetilde{\beta}] - \frac{1}{2}\int_0^1 \left(\widetilde{c}_\alpha^U\widetilde{\gamma}_\alpha^L + \widetilde{c}_\alpha^L\widetilde{\gamma}_\alpha^U\right)d\alpha, \end{aligned} \tag{20}$$

and the second-order derivatives are

$$\begin{aligned} \frac{\partial^2 E[\pi_c]}{\partial p_1^2} &= \frac{\partial^2 E[\pi_c]}{\partial p_2^2} = -2E[\widetilde{\beta}] < 0, \\ \frac{\partial^2 E[\pi_c]}{\partial p_1 \partial p_2} &= \frac{\partial^2 E[\pi_c]}{\partial p_2 \partial p_1} = 2E[\widetilde{\gamma}]. \end{aligned} \tag{21}$$

By (21), together with the assumption $E[\widetilde{\beta}] > E[\widetilde{\gamma}]$, we can get a negative definite Hessian Matrix, so the expected profit $E[\pi_c]$ is jointly concave in p_1 and p_2. Let (20) be equal to zeros; we derive the first order conditions as

$$\begin{aligned} & -2E[\widetilde{\beta}]p_1 + E[\widetilde{a}_1] + 2E[\widetilde{\gamma}]p_2 + E[\widetilde{c}\widetilde{\beta}] \\ & \quad - \frac{1}{2}\int_0^1 \left(\widetilde{c}_\alpha^U\widetilde{\gamma}_\alpha^L + \widetilde{c}_\alpha^L\widetilde{\gamma}_\alpha^U\right)d\alpha = 0, \\ & -2E[\widetilde{\beta}]p_2 + E[\widetilde{a}_2] + 2E[\widetilde{\gamma}]p_1 + E[\widetilde{c}\widetilde{\beta}] \\ & \quad - \frac{1}{2}\int_0^1 \left(\widetilde{c}_\alpha^U\widetilde{\gamma}_\alpha^L + \widetilde{c}_\alpha^L\widetilde{\gamma}_\alpha^U\right)d\alpha = 0. \end{aligned} \tag{22}$$

So, solving (22) simultaneously, we get the solutions (17), and then the proposition is proved. □

4.2. Pricing Models in MS Game. In this section, we assume that the manufacturer acts as the Stackelberg leader and the duopolistic retailers act as the followers. The game-theoretical approach is used to analyze the models established in the following. For this case, the manufacturer chooses the wholesale prices of the product using the response functions of both the retailers. Then, given the wholesale prices made by the manufacturer, the duopolistic retailers determine their retail prices.

4.2.1. The MSB Model. When the two retailers pursue the Bertrand solution, the manufacturer first announces the wholesale price and the two retailers observe the wholesale price and then decide the retail prices simultaneously. Then the MSB model is formulated as

$$\begin{cases} \max_{w} E\left[\pi_m\left(w, p_1^*(w), p_2^*(w)\right)\right] \\ p_1^*(w), p_2^*(w) \text{ are derived from solving} \\ \text{the following problem} \\ \begin{cases} \max_{p_1} E\left[\pi_{r_1}\left(p_1, p_2^*(w)\right)\right] \\ \max_{p_2} E\left[\pi_{r_2}\left(p_2, p_1^*(w)\right)\right]. \end{cases} \end{cases} \tag{23}$$

We first derive the retailers' Bertrand decisions as follows.

Proposition 17. *When the duopolistic retailers pursue the Bertrand solution, the optimal retail prices (denoted by p_{msb1} and p_{msb2}, resp.), given earlier decision made by the manufacturer w, are*

$$\begin{aligned} p_{\text{msb1}} &= \frac{2E[\tilde{a}_1]E[\tilde{\beta}] + E[\tilde{a}_2]E[\tilde{\gamma}]}{4E^2[\beta] - E^2[\gamma]} + \frac{\left(E[\tilde{\beta}]E[\tilde{\gamma}] + 2E^2[\tilde{\beta}]\right)w}{4E^2[\beta] - E^2[\gamma]}, \\ p_{\text{msb2}} &= \frac{2E[\tilde{a}_2]E[\tilde{\beta}] + E[\tilde{a}_1]E[\tilde{\gamma}]}{4E^2[\beta] - E^2[\gamma]} + \frac{\left(E[\tilde{\beta}]E[\tilde{\gamma}] + 2E^2[\tilde{\beta}]\right)w}{4E^2[\beta] - E^2[\gamma]}. \end{aligned} \tag{24}$$

Proof. Using (13) and (14), we get the expected value of π_{r1} and π_{r2} as

$$E[\pi_{r1}] = (p_1 - w)\left(E[\tilde{a}_1] - E[\tilde{\beta}]p_1 + E[\tilde{\gamma}]p_2\right), \tag{25}$$

$$E[\pi_{r2}] = (p_2 - w)\left(E[\tilde{a}_2] - E[\tilde{\beta}]p_2 + E[\tilde{\gamma}]p_1\right). \tag{26}$$

By (25), the first order derivative of $E[\pi_{r1}]$ to p_1 is

$$\frac{\partial E[\pi_{r1}]}{\partial p_1} = E[\tilde{a}_1] - 2E[\tilde{\beta}]p_1 + E[\tilde{\gamma}]p_2 + E[\tilde{\beta}]w, \tag{27}$$

and the second order derivative is given below to check for the optimality:

$$\frac{\partial^2 E[\pi_{r1}]}{\partial p_1^2} = -2E[\tilde{\beta}] < 0. \tag{28}$$

From (28), the expected profit $E[\pi_{r1}]$ is concave in p_1. Let (27) be equal to zero; we get

$$E[\tilde{a}_1] - 2E[\tilde{\beta}]p_1 + E[\tilde{\gamma}]p_2 + E[\tilde{\beta}]w = 0. \tag{29}$$

Similarly, from (26), we get

$$E[\tilde{a}_2] - 2E[\tilde{\beta}]p_2 + E[\tilde{\gamma}]p_1 + E[\tilde{\beta}]w = 0. \tag{30}$$

Therefore, solving (29) and (30) simultaneously, we get the solutions (24), and thus Proposition 17 is proved. □

Having the information about the decisions of the two retailers, the manufacturer would then use them to maximize his expected profit $E[\pi_m]$. So, we get the following result.

Proposition 18. *When the duopolistic retailers pursue the Bertrand solution, the manufacturer's optimal wholesale price (denoted by w^*_{msb}) is*

$$w^*_{\text{msb}} = \frac{E[\tilde{a}_1] + E[\tilde{a}_2] + 2E[\tilde{c}\tilde{\beta}] - \int_0^1 \left(\tilde{c}_\alpha^U \tilde{\gamma}_\alpha^L + \tilde{c}_\alpha^L \tilde{\gamma}_\alpha^U\right) d\alpha}{4\left(E[\tilde{\beta}] - E[\tilde{\gamma}]\right)}. \tag{31}$$

Proof. By (13), together with some manipulations, we get

$$\begin{aligned} E[\pi_m] = {} & \left(E[\tilde{a}_1] + E[\tilde{a}_2] + \left(E[\tilde{\gamma}] - E[\tilde{\beta}]\right)(p_1 + p_2)\right)w \\ & + \left(E[\tilde{c}\tilde{\beta}] - \frac{1}{2}\int_0^1 \left(\tilde{c}_\alpha^U \tilde{\gamma}_\alpha^L + \tilde{c}_\alpha^L \tilde{\gamma}_\alpha^U\right) d\alpha\right)(p_1 + p_2) \\ & - \frac{1}{2}\int_0^1 \left(\tilde{a}_{1\alpha}^U \tilde{c}_\alpha^L + \tilde{a}_{1\alpha}^L \tilde{c}_\alpha^U\right) d\alpha \\ & - \frac{1}{2}\int_0^1 \left(\tilde{a}_{2\alpha}^U \tilde{c}_\alpha^L + \tilde{a}_{2\alpha}^L \tilde{c}_\alpha^U\right) d\alpha. \end{aligned} \tag{32}$$

Then, from (24) and (32), the first order derivative of $E[\pi_m]$ to w is

$$\begin{aligned} \frac{\partial E[\pi_m]}{\partial w} = {} & E[\tilde{a}_1] + E[\tilde{a}_2] \\ & + \left(E[\tilde{c}\tilde{\beta}] - \frac{1}{2}\int_0^1 \left(\tilde{c}_\alpha^U \tilde{\gamma}_\alpha^L + \tilde{c}_\alpha^L \tilde{\gamma}_\alpha^U\right) d\alpha\right) \\ & \times \frac{2E[\tilde{\beta}]}{2E[\tilde{\beta}] - E[\tilde{\gamma}]} \\ & + \frac{\left(E[\tilde{\gamma}] - E[\tilde{\beta}]\right)\left(E[\tilde{a}_1] + E[\tilde{a}_2]\right)}{2E[\tilde{\beta}] - E[\tilde{\gamma}]} \\ & + \frac{4E[\tilde{\beta}]\left(E[\tilde{\gamma}] - E[\tilde{\beta}]\right)}{2E[\tilde{\beta}] - E[\tilde{\gamma}]}w. \end{aligned} \tag{33}$$

Furthermore, its second order derivative satisfies

$$\frac{\partial^2 E[\pi_m]}{\partial w^2} = \frac{4E[\tilde{\beta}]\left(E[\tilde{\gamma}] - E[\tilde{\beta}]\right)}{2E[\tilde{\beta}] - E[\tilde{\gamma}]} < 0. \tag{34}$$

So, $E[\pi_m]$ is concave in w. Therefore, let (33) be equal to zero; the proposition is proved. □

Proposition 19. *When the duopolistic retailers pursue the Bertrand solution, their optimal retail prices (denoted by p^*_{msb1} and p^*_{msb2}, resp.) are given as*

$$\begin{aligned} p^*_{\mathrm{msb1}} &= \frac{2E[\tilde{a}_1]E[\tilde{\beta}] + E[\tilde{a}_2]E[\tilde{\gamma}]}{4E^2[\beta] - E^2[\gamma]} \\ &\quad + \frac{\left(E[\tilde{\beta}]E[\tilde{\gamma}] + 2E^2[\tilde{\beta}]\right)w^*_{\mathrm{msb}}}{4E^2[\beta] - E^2[\gamma]}, \\ p^*_{\mathrm{msb2}} &= \frac{2E[\tilde{a}_2]E[\tilde{\beta}] + E[\tilde{a}_1]E[\tilde{\gamma}]}{4E^2[\beta] - E^2[\gamma]} \\ &\quad + \frac{\left(E[\tilde{\beta}]E[\tilde{\gamma}] + 2E^2[\tilde{\beta}]\right)w^*_{\mathrm{msb}}}{4E^2[\beta] - E^2[\gamma]}, \end{aligned} \tag{35}$$

where w^*_{msb} is given in Proposition 18.

Proof. By Propositions 17 and 18, we can easily see that Proposition 19 holds. □

4.2.2. The MSC Model. In this decision case where the two retailers adopt the cooperation strategy, we assume that the retailers recognize their interdependence and agree to act in union in order to maximize the total expected profit of the downstream retail market. So, the manufacturer first announces the wholesale price and the retailers observe the wholesale price and then decide their retail prices with the objective to maximize the total expected profit of the downstream retail market. Thus, the MSC model is formulated as

$$\begin{cases} \max\limits_{w} E\left[\pi_m\left(w, p_1^*(w), p_2^*(w)\right)\right] \\ p_1^*(w), p_2^*(w) \text{ are derived from solving} \\ \text{the following problem} \\ \quad \max\limits_{p_1, p_2} E\left[\pi_{r_1}(p_1, p_2) + \pi_{r_2}(p_1, p_2)\right]. \end{cases} \tag{36}$$

We first derive the retailers' decisions.

Proposition 20. *When the two retailers adopt the cooperation strategy, their optimal retail prices p_{msc1} and p_{msc2}, given earlier decision made by the manufacturer w, are*

$$\begin{aligned} p_{\mathrm{msc1}} &= \frac{E[\tilde{a}_1]E[\tilde{\beta}] + E[\tilde{a}_2]E[\tilde{\gamma}] + \left(E^2[\tilde{\beta}] - E^2[\tilde{\gamma}]\right)w}{2\left(E^2[\tilde{\beta}] - E^2[\tilde{\gamma}]\right)}, \\ p_{\mathrm{msc2}} &= \frac{E[\tilde{a}_2]E[\tilde{\beta}] + E[\tilde{a}_1]E[\tilde{\gamma}] + \left(E^2[\tilde{\beta}] - E^2[\tilde{\gamma}]\right)w}{2\left(E^2[\tilde{\beta}] - E^2[\tilde{\gamma}]\right)}. \end{aligned} \tag{37}$$

Proof. By (13) and (14), we have

$$\begin{aligned} E[\pi_{r1} + \pi_{r2}] = {} & (p_1 - w)\left(E[\tilde{a}_1] - E[\tilde{\beta}]p_1 + E[\tilde{\gamma}]p_2\right) \\ & + (p_2 - w)\left(E[\tilde{a}_2] - E[\tilde{\beta}]p_2 + E[\tilde{\gamma}]p_1\right). \end{aligned} \tag{38}$$

Then

$$\begin{aligned} \frac{\partial E[\pi_{r1} + \pi_{r2}]}{\partial p_1} &= E[\tilde{a}_1] + \left(E[\tilde{\beta}] - E[\tilde{\gamma}]\right)w \\ &\quad - 2E[\tilde{\beta}]p_1 + 2E[\tilde{\gamma}]p_2, \\ \frac{\partial E[\pi_{r1} + \pi_{r2}]}{\partial p_2} &= E[\tilde{a}_2] + \left(E[\tilde{\beta}] - E[\tilde{\gamma}]\right)w \\ &\quad - 2E[\tilde{\beta}]p_2 + 2E[\tilde{\gamma}]p_1, \end{aligned} \tag{39}$$

$$\begin{aligned} \frac{\partial^2 E[\pi_{r1} + \pi_{r2}]}{\partial p_1^2} &= \frac{\partial^2 E[\pi_{r1} + \pi_{r2}]}{\partial p_2^2} = -2E[\tilde{\beta}], \\ \frac{\partial^2 E[\pi_{r1} + \pi_{r2}]}{\partial p_1 \partial p_2} &= \frac{\partial^2 E[\pi_{r1} + \pi_{r2}]}{\partial p_2 \partial p_1} = 2E[\tilde{\gamma}]. \end{aligned} \tag{40}$$

From (40) and the assumption $E[\tilde{\beta}] > E[\tilde{\gamma}]$, its Hessian Matrix is negative definite, so the expected profit $E[\pi_{r1} + \pi_{r2}]$ is jointly concave in p_1 and p_2. Let (39) be equal to 0, respectively; we get

$$\begin{aligned} E[\tilde{a}_1] + \left(E[\tilde{\beta}] - E[\tilde{\gamma}]\right)w - 2E[\tilde{\beta}]p_1 + 2E[\tilde{\gamma}]p_2 = 0, \\ E[\tilde{a}_2] + \left(E[\tilde{\beta}] - E[\tilde{\gamma}]\right)w - 2E[\tilde{\beta}]p_2 + 2E[\tilde{\gamma}]p_1 = 0. \end{aligned} \tag{41}$$

Thus, solving (41) simultaneously, we get (37), so the proposition is proved. □

Having the information about the decisions of the retailers, the manufacturer would then use them to maximize his expected profit $E[\pi_m]$. So, we get the following result.

Proposition 21. *When the two retailers adopt the cooperation strategy, the manufacturer's optimal wholesale price (denoted as w^*_{msc}) is*

$$w^*_{\mathrm{msc}} = \frac{E[\tilde{a}_1] + E[\tilde{a}_2] + 2E[\tilde{c}\tilde{\beta}] - \int_0^1 \left(\tilde{c}^U_\alpha \tilde{\gamma}^L_\alpha + \tilde{c}^L_\alpha \tilde{\gamma}^U_\alpha\right) d\alpha}{4\left(E[\tilde{\beta}] - E[\tilde{\gamma}]\right)}. \tag{42}$$

Proof. By (32) and (37), we get

$$\frac{\partial E[\pi_m]}{\partial w} = \frac{E[\tilde{a}_1]+E[\tilde{a}_2]}{2} + E[\tilde{c}\tilde{\beta}] - \frac{1}{2}\int_0^1 \left(\tilde{c}_\alpha^U \tilde{\gamma}_\alpha^L + \tilde{c}_\alpha^L \tilde{\gamma}_\alpha^U\right) d\alpha + 2\left(E[\tilde{\gamma}] - E[\tilde{\beta}]\right) w, \quad (43)$$

$$\frac{\partial^2 E[\pi_m]}{\partial w^2} = 2\left(E[\tilde{\gamma}] - E[\tilde{\beta}]\right) < 0. \quad (44)$$

By (44), $E[\pi_m]$ is concave in w. Then, let (43) be equal to zero; we can easily get the proposition. □

Proposition 22. *When the two retailers adopt the cooperation strategy, their optimal retail prices p^*_{msc1} and p^*_{msc2} are given as follows:*

$$p^*_{\text{msc1}} = \frac{E[\tilde{a}_1]E[\tilde{\beta}] + E[\tilde{a}_2]E[\tilde{\gamma}] + \left(E^2[\tilde{\beta}] - E^2[\tilde{\gamma}]\right) w^*_{\text{col}}}{2\left(E^2[\tilde{\beta}] - E^2[\tilde{\gamma}]\right)},$$

$$p^*_{\text{msc2}} = \frac{E[\tilde{a}_2]E[\tilde{\beta}] + E[\tilde{a}_1]E[\tilde{\gamma}] + \left(E^2[\tilde{\beta}] - E^2[\tilde{\gamma}]\right) w^*_{\text{col}}}{2\left(E^2[\tilde{\beta}] - E^2[\tilde{\gamma}]\right)}, \quad (45)$$

where w^*_{msc} is given in Proposition 20.

Proof. By Propositions 20 and 21, we can easily see that Proposition 22 holds. □

4.2.3. The MSS Model. In this decision case when the duopolistic retailers play Stackelberg Game, we assume that one of the duopolistic retailers (e.g., retailer 1) acts as a Stackelberg leader and the other (i.e., retailer 2) acts as a Stackelberg follower. The manufacturer first announces the wholesale price of the product, and retailer 1 then decides the retail price to maximize her expected profit and retailer 2 finally decides the retail price when knowing both the manufacturer and retailer 1 decisions. So, we first need to derive retailer 2's decision (as the Stackelberg game's follower). The MSS model is formulated as follows:

$$\begin{cases} \max\limits_{w} E\left[\pi_m\left(w, p_1^*(w), p_2^*(w, p_1^*(w))\right)\right] \\ p_1^*(w), p_2^*(w, p_1^*(w)) \text{ are derived from solving} \\ \text{the following problem} \\ \begin{cases} \max\limits_{p_1} E\left[\pi_{r_1}\left(p_1, p_2^*(w, p_1)\right)\right] \\ p_2^*(w, p_1) \text{ is derived from solving} \\ \text{the following problem} \\ \max\limits_{p_2} E\left[\pi_{r_2}(p_1, p_2)\right]. \end{cases} \end{cases} \quad (46)$$

We first derive retailer 2's decision as follows.

Proposition 23. *When the duopolistic retailers play Stackelberg Game, retailer 2's optimal decision (denoted as p_{mss2}), given earlier decisions made by the manufacturer and retailer 1 which are w and p_1, respectively, is*

$$p_{\text{mss2}} = \frac{E[\tilde{a}_2] + E[\tilde{\gamma}]p_1 + E[\tilde{\beta}]w}{2E[\tilde{\beta}]}. \quad (47)$$

Proof. Using (26), given earlier decisions made by the manufacturer and retailer 1 which are w and p_1, respectively, we can have the first-and second order derivatives of $E[\pi_{r2}]$ to p_2 as follows:

$$\frac{\partial E[\pi_{r2}]}{\partial p_2} = E[\tilde{a}_2] - 2E[\tilde{\beta}]p_2 + E[\tilde{\gamma}]p_1 + E[\tilde{\beta}]w, \quad (48)$$

$$\frac{\partial^2 E[\pi_{r2}]}{\partial p_2^2} = -2E[\tilde{\beta}] < 0. \quad (49)$$

By (49), we know that $E[\pi_{r2}]$ is concave in p_2 for given earlier decisions made by the manufacturer and retailer 1 which are w and p_1, respectively. Therefore, equating (48) to zero and solving it, we can easily have Proposition 23. □

Proposition 24. *When the duopolistic retailers play Stackelberg Game, retailer 1's optimal decision (denoted as p_{mss1}), given earlier decision made by the manufacturer which is w, is*

$$p_{\text{mss1}} = B_2 + B_1 w, \quad (50)$$

where

$$B_1 = \frac{E[\tilde{\beta}]E[\tilde{\gamma}] - E^2[\tilde{\gamma}] + E^2[\tilde{\beta}]}{3E^2[\tilde{\beta}] - 2E^2[\tilde{\gamma}]},$$

$$B_2 = \frac{2E[\tilde{a}_1]E[\tilde{\beta}] + E[\tilde{a}_2]E[\tilde{\gamma}]}{3E^2[\tilde{\beta}] - 2E^2[\tilde{\gamma}]}. \quad (51)$$

Proof. Using (25) and (47), given earlier decision made by the manufacturer which is w, we can have the first-and second order derivatives of $E[\pi_{r1}]$ to p_1 as follows:

$$\frac{\partial E[\pi_{r1}]}{\partial p_1} = E[\tilde{a}_1] + \frac{E[\tilde{a}_2]E[\tilde{\gamma}] + \left(E[\tilde{\beta}]E[\tilde{\gamma}] - E^2[\tilde{\gamma}] + E^2[\tilde{\beta}]\right)w}{2E[\tilde{\beta}]} + \frac{2E^2[\tilde{\gamma}] - 3E^2[\tilde{\beta}]}{2E[\tilde{\beta}]}p_1, \quad (52)$$

$$\frac{\partial^2 E[\pi_{r1}]}{\partial p_1^2} = \frac{2E^2[\tilde{\gamma}] - 3E^2[\tilde{\beta}]}{2E[\tilde{\beta}]} < 0. \quad (53)$$

By (53), we know that $E[\pi_{r1}]$ is concave in p_1 for given earlier decision made by the manufacturer which is w. Therefore, equating (52) to zero and solving it, we can easily have Proposition 24. □

Proposition 25. *When the duopolistic retailers play Stackelberg Game, retailer 2's optimal decision (denoted as p_{mss2}), given earlier decisions made by the manufacturer which is w, is*

$$\begin{aligned} p_{\text{mss2}} = {} & \frac{E[\tilde{a}_2]}{2E[\beta]} + \frac{w}{2} \\ & + E[\tilde{\gamma}]\left(2E[\tilde{a}_1]E[\tilde{\beta}] + E[\tilde{a}_2]E[\tilde{\gamma}] \right. \\ & \left. + \left(E[\tilde{\beta}]E[\tilde{\gamma}] - E^2[\tilde{\gamma}] + E^2[\tilde{\beta}]\right)w\right) \\ & \times \left(2E[\beta]\left(3E^2[\beta] - 2E^2[\tilde{\gamma}]\right)\right)^{-1}. \end{aligned} \tag{54}$$

Proof. Using Propositions 23 and 24, one can easily have Proposition 25. □

Proposition 26. *When the duopolistic retailers play Stackelberg Game, the manufacturer's optimal decision is (denoted by w^*_{mss}) given as follows:*

$$\begin{aligned} w^*_{\text{mss}} = {} & \frac{E[\tilde{a}_1] + E[\tilde{a}_2] + B_3(B_4 + B_1)}{2\left(E[\tilde{\beta}] - E[\tilde{\gamma}]\right)(B_4 + B_1)} \\ & + \frac{\left(E[\tilde{\gamma}] - E[\tilde{\beta}]\right)(B_2 + B_5)}{2\left(E[\tilde{\beta}] - E[\tilde{\gamma}]\right)(B_4 + B_1)}. \end{aligned} \tag{55}$$

Proof. Using (32), (50), and (54), one can have the first-and second order derivatives of $E[\pi_m]$ to w as follows:

$$\begin{aligned} \frac{\partial E[\pi_m]}{\partial w} = {} & E[\tilde{a}_1] + E[\tilde{a}_2] + B_3(B_4 + B_1) \\ & + \left(E[\tilde{\gamma}] - E[\tilde{\beta}]\right)(B_2 + B_5) \\ & + 2\left(E[\tilde{\gamma}] - E[\tilde{\beta}]\right)(B_4 + B_1)w, \end{aligned} \tag{56}$$

where

$$\begin{aligned} B_1 &= E[\tilde{c}\tilde{\beta}] - \frac{1}{2}\int_0^1 \left(\tilde{c}^U_\alpha \tilde{\gamma}^L_\alpha + \tilde{c}^L_\alpha \tilde{\gamma}^U_\alpha\right) \mathrm{d}\alpha, \\ B_4 &= \frac{1}{2} + \frac{E[\tilde{\gamma}]\left(E[\tilde{\beta}]E[\tilde{\gamma}] - E^2[\tilde{\gamma}] + E^2[\tilde{\beta}]\right)}{2E[\tilde{\beta}]\left(3E^2[\tilde{\beta}] - 2E^2[\tilde{\gamma}]\right)}, \\ B_5 &= \frac{E[\tilde{a}_2]}{2E[\tilde{\beta}]} + \frac{E[\tilde{\gamma}]\left(2E[\tilde{a}_1]E[\tilde{\beta}] + E[\tilde{a}_2]E[\tilde{\gamma}]\right)}{2E[\tilde{\beta}]\left(3E^2[\tilde{\beta}] - 2E^2[\tilde{\gamma}]\right)}, \\ \frac{\partial^2 E[\pi_m]}{\partial w^2} &= 2\left(E[\tilde{\gamma}] - E[\tilde{\beta}]\right)(B_4 + B_1)w. \end{aligned} \tag{57}$$

By (57), we know that $E[\pi_m]$ is concave in w. Therefore, equating (56) to zero and solving it, we can easily have Proposition 26. □

Proposition 27. *When the duopolistic retailers play Stackelberg Game, their optimal retail prices (denoted by p^*_{mss1} and p^*_{mss2}, resp.) are given as follows:*

$$\begin{aligned} p^*_{\text{mss1}} &= B_2 + B_1 w^*_{\text{msta}}, \\ p^*_{\text{mss2}} &= B_5 + B_4 w^*_{\text{msta}}, \end{aligned} \tag{58}$$

where w^*_{mss} is given as in Proposition 26.

Proof. By Propositions 24, 25, and 26, we can easily see that Proposition 27 holds. □

5. Numerical Studies

In this section, we compare analytical results obtained from the above different decision scenarios using numerical approach and study the behavior of firms facing changing fuzzy environment. Here, we assume that the fuzzy variables used in this paper are all triangular fuzzy variables which take values as follows: the manufacturing cost $\tilde{c}$ is high ($\tilde{c}$ is about 9), the market bases $\tilde{a}_1$ and $\tilde{a}_2$ are large ($\tilde{a}_1$ is about 1400 and $\tilde{a}_2$ is about 1200), and price elasticities $\tilde{\beta}$ and $\tilde{\gamma}$ are sensitive ($\tilde{\beta}$ is about 180 and $\tilde{\gamma}$ is about 130). More specifically, $\tilde{c} = (6, 9, 12)$, $\tilde{a}_1 = (1300, 1400, 1600)$, $\tilde{a}_2 = (1000, 1200, 1300)$, $\tilde{\beta} = (170, 180, 200)$, and $\tilde{\gamma} = (120, 130, 140)$. Similar to Example 8 in Preliminaries (see Section 2), the expected values of the above triangular fuzzy variables can be obtained as follows: $E[\tilde{a}_1] = (1300+2\times1400+1600)/4 = 5700/4$, $E[\tilde{a}_2] = (1000 + 2 \times 1200 + 1300)/4 = 4700/4$, and $E[\tilde{\beta}] = (170 + 2\times 180 + 200)/4 = 730/4$, $E[\tilde{\gamma}] = (120+2\times130+140)/4 = 520/4$, $E[\tilde{c}] = (6+2\times9+12)/4 = 9$. Similar to Example 12 in Preliminaries (see Section 2), the α-pessimistic values and α-optimistic values of triangular fuzzy variables $\tilde{c}$, $\tilde{a}_1, \tilde{a}_2, \tilde{\beta}$, and $\tilde{\gamma}$ are $\tilde{c}^L_\alpha = 6 + 3\alpha$, $\tilde{c}^U_\alpha = 12 - 3\alpha$, $\tilde{a}^L_{1\alpha} = 1300 + 100\alpha$, $\tilde{a}^U_{1\alpha} = 1600 - 200\alpha$, $\tilde{a}^L_{2\alpha} = 1000 + 200\alpha$, $\tilde{a}^U_{2\alpha} = 1300 - 100\alpha$, $\tilde{\beta}^L_\alpha = 170 + 10\alpha$, $\tilde{\beta}^U_\alpha = 200 - 20\alpha$, $\tilde{\gamma}^L_\alpha = 120 + 10\alpha$, $\tilde{\gamma}^U_\alpha = 140 - 10\alpha$, respectively. Using the analytical results obtained in this paper, we can easily have the following numerical results expressed in Tables 1 and 2 when the parameters take the values described above.

Remark 28. From Tables 1 and 2, we derive the following results when the two retailers have different market bases.

(1.1) The expected profit of the total supply chain in the centralized decision case is higher than that in all decentralized decision cases.

(1.2) One can observe directly from Table 1 that different pricing strategies of the two retailers affect the maximum expected profits of the manufacturer and the two retailers. The manufacturer achieves his largest expected profit in the MSB model, retailer 1 achieves her largest expected profit in MSC model, and retailer 2 achieves her largest expected profit in MSS model.

(1.3) From Table 1, we can see that the two retailers' Bertrand action benefits the manufacturer as well

Table 1: Maximum expect profit of total system and every firm under different pricing models.

Pricing model	$E[\pi_c]$	$E[\pi_m]$	$E[\pi_{r_1}]$	$E[\pi_{r_2}]$
CD model	7696.5			
MSB model	7388.7	6301.3	701.1	386.3
MSC model	6163.1	4604.8	1058.1	500.2
MSS model	6778.0	5492.1	546.5	739.4

Table 2: Optimal retail prices and wholesale price under different pricing models.

Pricing model	p_1^*	$p_1^* - w^*$	p_2^*	$p_2^* - w^*$	w^*
CD model	17.3190		16.9190		
MSB model	19.0790	1.9600	18.5740	1.4549	17.1190
MSC model	21.1405	4.0214	20.7405	3.6214	17.1190
MSS model	20.5076	3.4869	19.0337	2.0129	17.0208

as the total supply chain. On the other hand, the two retailers' cooperation action will always make the manufacturer and the total supply chain obtain the lowest expected profits, which implies that the manufacturer who acts as the leader does not always have the superiority of gaining expected profit in a two-echelon supply chain with two retailers. This is counterintuitive. Therefore, the manufacturer, as a Stackelberg leader, should find a way to induce the two retailers to implement Bertrand policy.

(1.4) From Table 1, we can also see that the cooperation action does not always benefit every retailer; for example, retailer 2's expected profit in the MSC model is lower than that in the MSS model. This insight is helpful to a retailer who is the follower, which tells the retailer that a suitable profit-split should be negotiated with his rival before agreeing to act in union.

(1.5) From Table 2 we find that the two retailers' cooperation behavior will result in the highest unit margins for themselves.

(1.6) From Table 2, we can see that the wholesale price in the MSB model is equal to that in the MSC model, which is consistent with the results expressed in both Propositions 18 and 21, and the wholesale price in the MSS model is the lowest one among the three decentralized decision models.

(1.7) We observe from Table 1 that the manufacturer's expected profit is bigger than the sum of the two retailers' expected profits in the above all Game cases. Moreover, retailer 2's expected profit is bigger than that of retailer 1 in the MSS model. However, retailer 1's expected profit is bigger than that of retailer 2 in other Game cases.

In order to see how the two retailers' different competitive behaviors affect the optimal pricing policy and the total expected profits of the manufacturer and the two retailers, we further assume that the retailers have the same market bases (here we set $\tilde{a}_1 = \tilde{a}_2 = (1000, 1200, 1300)$), which can be intuitively explained as the duopolistic retailers facing the similar market demand. Tables 3 and 4 present the optimal solutions when the two retailers have the same market bases.

Remark 29. From Tables 3 and 4, we can have the following results.

(2.1) The expected profit of the whole supply chain system in the centralized decision is higher than that in all decentralized decisions. This is consistent with the general case when the two retailers have different market bases.

(2.2) One can observe directly from Table 3 that the two retailers' different pricing strategies affect the total maximum expected profit of the manufacturer and the two retailers. First, retailer 1 achieves her highest expected profit in the RSC model while both the manufacturer and the whole supply chain achieve the lowest expected profits in this case. Secondly, the manufacturer achieves his highest expected profit in the MSB model while retailer 2 achieves her lowest expected profit in this case. This is consistent with the general case when the two retailers have different market bases. Thirdly, from Table 3, one can easily see that both retailer 1 and retailer 2 will achieve the same expected profit in the MSB model. Similar results occur in the MSC model. This is against to the general case when the two retailers have different market bases.

(2.3) From Table 3, we can see that the two retailers' Bertrand action benefits the manufacturer as well as the total supply chain, and the duopolistic retailers' cooperation action will always make the manufacturer and the total supply chain obtain the lowest expected profits. This is consistent with the general case when the two retailers have different market bases.

(2.4) From Table 3, we can also see that action in cooperation does not always benefit every duopolistic retailer; for example, retailer 2's expected profit in the MSC model is lower than that in the MSS model. This is also

Table 3: Maximum profit of total system and every firm under different pricing models.

Pricing model	$E[\pi_c]$	$E[\pi_m]$	$E[\pi_{r_1}]$	$E[\pi_{r_2}]$
CD model	5790.5			
MSB model	5572.3	4813.9	379.2	379.2
MSC model	4697.6	3604.8	546.4	546.4
MSS model	5212.3	4311.1	289.1	612.1

Table 4: Optimal retail prices and wholesale price under different pricing models.

Pricing model	p_1^*	$p_1^* - w^*$	p_2^*	$p_2^* - w^*$	w^*
CD model	15.9286		15.9286		
MSB model	17.3701	1.4415	17.3701	1.4415	15.9286
MSC model	19.1548	3.2262	19.1548	3.2262	15.9286
MSS model	18.4646	2.5361	17.7599	1.8313	15.9286

consistent with the general case when the two retailers have different market bases.

(2.5) One can observe directly from Table 4 that the duopolistic retailers' cooperation action will result in the highest unit margins for themselves. Moreover, we can have the following insights: firstly, the two retailers achieve the lowest unit margin in the MSB model, followed by the MSS model, then the MSC model; secondly, the two retailers will achieve equal unit margins in the MSB/MSC models. Thirdly, the retail prices charged by the two retailers achieve the highest value the retail prices charged by the two retailers achieve the highest value in the MSC model and achieve the lowest value in the CD model. Finally, the two retailers will charge the same retail price in the MSB and MSC models.

(2.6) From Table 4, we can see that the wholesale price charged by the manufacturer in three models does not vary with the two retailers' pricing strategies. This is against the general case when the two retailers have different market bases.

6. Conclusions

We have analyzed the duopolistic retailers' and the manufacturer's pricing decisions by considering the duopolistic retailers' three kinds of pricing strategies: Bertrand, Cooperation, and Stackelberg in fuzzy environment. As a benchmark to evaluate channel decision in different decision case, we first developed the pricing model in centralized decision case and derived the optimal retail prices. We then established the pricing models in decentralized decision cases by considering the duopolistic retailers' three kinds of pricing strategies and obtained the analytic equilibrium decisions. Finally, we provided comparison of the expected profits and optimal pricing decisions of the whole supply chain and every supply chain members in both the general case (namely, the two retailers have different market bases) and the special case (viz., the two retailers have the same market bases). The analytical and numerical results revealed some insights into the economic behavior of firms.

Our results, however, are based upon some assumptions about the two-echelon supply chain models. Thus, several extensions to the analysis in this paper are possible by considering the duopolistic retailers' three kinds of pricing strategies. First, as opposed to the risk neutral two-echelon supply chain members considered in this paper, one could study the case where the supply chain members with different attitudes toward risk and could also examine the influence of their attitudes toward risk on individual expected profits and the expected profit of the whole supply chain. This would add complications to the analysis of the two-echelon supply chain members' decisions. Second, we assumed that both the duopolistic retailers and the manufacturer have symmetric information about costs and demands. So, an extension would be to consider the two-echelon supply chain models in information asymmetry, such as asymmetry in cost information and demand information. Finally, we can also consider the coordination of the two-echelon supply chain under linear or isoelastic demand with symmetric and asymmetric information.

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Research Article

Coordination in a Single-Retailer Two-Supplier Supply Chain under Random Demand and Random Supply with Disruption

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This paper studies the coordination issue of a supply chain consisting of one retailer and two suppliers, a main supplier and a backup supplier. The main supplier's yield is subject to disruption and the retailer faces a random demand. We determine the retailer's optimal ordering policy and the main supplier's production quantity that maximize expected profit of the centralized supply chain. We also analyze the decentralized scenario, and a combination of overproduction risk sharing and buy-back contracts with a side payment from/to the backup supplier is provided to coordinate the supply chain. Numerical examples are given to gain some qualitative insights.

1. Introduction

This paper studies the coordination issue of a supply chain among one retailer and two suppliers under random demand and supply disruption. Different from the previous research that focuses on the supplier's random yield, we consider the supply disruption, during which the output of the production is zero. We also propose an overproduction risk sharing (OPRS) and buy-back contracts with a side payment for supply chain coordination. Since our work touches upon two areas of research: random yield with disruption and contracts in a supply chain with demand uncertainty, we confine ourselves to reviewing the following related works of these two topics.

In general, any production or logistics process is somewhat related to random yield; that is, with equal amount input, the output of the production usually varies. Due to damage that occurs during transferring, any transportation process can also be viewed as random yield process [1]; for example, in the liquid crystal display manufacturing industry, it is quite common to get a production yield of less than 50%. Then the actual response of the supplier to the retailer's order will be mandated by the supply contract between them; for example, if the cost of tracking partial orders is high or the transportation costs are high, the supply contract may specify delivery in a single shipment with the uncertainty in delivery time [2]; in other situations, the manufacturer may agree to accept partial shipment of the order quantity [3].

Gerchak and Grosfeld-Nir [4] analyzed the tradeoff between setup cost and production cost when making batch production decisions, where both the random yield and random demand follow a general discrete distribution. Granot and Yin [5] investigated the effect of sequential commitment in the decentralized newsvendor model with price-dependent demand, where the supplier and the retailer sequentially commit on prices and quantities. Mukhopadhyay and Ma [6] developed a single-period model to evaluate the optimal procurement and production decisions with uncertain demand and random yield of the used parts under three different cases. Wang [7] investigated the traditional and vendor-managed inventory arrangements in a supply chain and obtained the optimal production/inventory decisions under random yield and uncertain demand for both arrangements. He and Zhang [8] studied a seller/supplier commitment contract with minimum delivery commitment and analyzed the supply chain with the risk sharing contract under a constant secondary market price- and a yield-dependent secondary market price. He and Zhao [9] investigated the ordering policy of the retailer, raw material planning decision of the supplier, and the optimal contracts for a three-level supply chain with random yield and demand. Gurnani et al. [10] considered an assembly problem, where a firm faces random demand for a final product and a general yield distribution of two critical components from the individual suppliers. Using the modified cost function, they determined the combined component ordering quantity and assembly decisions for the firm. In the model of Güler and Bilgiç [11], they studied the coordination of an assembly system for arbitrary number of suppliers with random demand and random yield and established the concavity of expected supply chain profit and proposed two mixed type of contracts to coordinate the chain under forced compliance. Under the wholesale price contract, Keren [12] analyzed a two-tier supply chain, where a distributor facing a deterministic demand procures a product from a producer confronting a random production yield. And an analytical solution to the distributor's ordering decision is derived when the production yield follows the uniform distribution. Li et al. [13] extended and provided new results on the supply chain model with producer's random yield proposed by Keren [12]. They derived analytic solutions of the supply chain decisions under generalized yield distribution and pointed out that the distributor should order more than the demand if and only if his/her marginal profit from selling this product exceeds a threshold.

Supply disruption is not considered by the above-cited literature on random yield. However, in a practical reality, a supplier may be unable to satisfy the production order for a variety of reasons, such as equipment failures, damaged facilities, problems in procuring the necessary raw materials, and so forth. With more and more enterprises starting to realize that supply disruption severely affects their ability to successfully manage their own supply chains, supply disruption management has received increasing attention. Many researchers have devoted much effort to studying this issue. Based on the sample firms, Hendricks and Singhal [14] estimated the short-term effects of supply disruption such as production or shipment delays on shareholder value. They found that glitch announcements decrease shareholder value by 10.82%. Furthermore, Hendricks and Singhal [14] investigated the long-term negative effects of supply disruption on the financial performance of firms based. They found that the average abnormal stock returns of firms that experienced disruptions is nearly 40% and the firms cannot quickly recover from the negative effects of a disruption. Taking into account the disruption frequency and the loss of market share, Pochard [15] analyzed the value and the benefits of dual sourcing. Oke and Gopalakrishnan [16] investigated the types and management of risks in a retail supply chain. Yu et al. [17] studied the sourcing decision alternatives between single sourcing and dual sourcing in a two-stage supply chain, where the demand is price-sensitive and the market scale increases when a supply disruption occurs. Sarkar and Mohapatra [18] considered the risks of supply disruption due to occurrence of super, semisuper, and unique events and determined the optimal size of supply base. Tomlin [19] studied a single-product model in which a firm can source from a cheap but unreliable supplier and/or an expensive but reliable supplier. They established supplier characteristics and firm characteristics in determining the firm's optimal strategy for managing supply disruption risk. Meena et al. [20] developed an analytical model to determine the optimal number of suppliers under different failure probability, capacity, and compensation. Li et al. [21] investigated the sourcing strategy of a retailer and the pricing strategies of two suppliers in a supply chain with supply disruption. For more existing supply disruption models, we refer to the review in Snyder and Shen [22].

Since double marginalization [23] will directly lead to inefficient performance of the supply chain, coordination of activities among the different members in the supply chain is necessary for the whole supply chain's effective management. A great deal of effort has been devoted to researching the supply chain coordination issues. All kinds of popular contracts have been explored in the literature for the supply chain coordination, such as buy-back contracts or returns policies [24–26], revenue sharing contracts [27, 28], wholesale price contracts [27, 29], risk sharing contracts [30], quantity discount policies [31], quantity flexibility contracts [32], sales rebate contracts [33], and so on. Recently, Ding and Chen [34] studied how to fully coordinate a three-level supply chain with a flexible return policies by setting the rules of pricing while postponing the determination of the final contract prices. He et al. [35] studied supply chain contracts and coordination when the downstream retailer faces both effort- and price-dependent stochastic demand. In such a situation, since all traditional contracts fail to coordinate the retailer's action and distort the retailer's marginal incentive to exert system optimal effort and price, none of them can coordinate the supply chain. Hence they explored a variety of other combined contracts and found that only a properly designed returns policy with a sales rebate and a penalty contract is able to achieve channel coordination and lead to a Pareto improving win-win situation for supply chain members. Xiao et al. [36] studied coordination of a two-echelon supply chain with consumer return and random demand, where consumers also face uncertainty in their valuations for products. Granot and Yin [5] investigated the effect of sequential commitment in the decentralized newsvendor model with price-dependent demand, where the supplier and the retailer sequentially commit on prices and quantities. van der Rhee et al. [37] propose a new spanning revenue sharing contract mechanism that the most downstream entity initiates a single contract involving all upstream entities for multiechelon supply chains, and analyzed the new revenue sharing contract in the linear supply chain setting with stochastic demand. Lin et al. [38] proposed an insurance contract under which the supplier shares the risk of overstock and understock with the retailer, and showed that the insurance contract can coordinate the supply chain with a newsvendor-type product. Li et al. [39] explored a generalized supply chain model and showed the double marginalization effect in the supply chain under supply uncertainty. When the demand is deterministic, a wholesale price contract and a shortage penalty contract is developed to coordinate the supply chain. When the demand is random, two equivalent coordination contracts (called accept-all type contracts for the buyer) are proposed to induce the supplier into the systematically optimal production input level.

The works by He and Zhang [40] and Hou et al. [41] are similar to our study here. He and Zhang [40] studied the effects of random yield in a decentralized supply chain and proposed several risk sharing contracts. They also compared the results of proposed contracts and found that random yield might enhance the supply chain performance and decrease the double marginalization effect under certain conditions. But there is no alternative sources available and the yield disruption is not considered in their paper. Hou et al. [41] investigated a buy-back contract between a buyer and a backup supplier when the buyer's main supplier is subject to disruption. They derived the buyer's optimal order quantity and the backup supplier's optimal return price under the recurrent supply uncertainty and the demand uncertainty, respectively. But in their study, the coordination mechanism between the main supplier is not concerned, and they do not investigate the buyer's optimal decision when both recurrent supply uncertainty and the demand uncertainty exist.

The majority of the literature considers that the yield is a stochastic proportion of the planned quantity and does not consider the coordination and the yield disruption in the chain, and most of the existing literature that studies the coordination mechanism is confined to the supply chain consisting of one retailer and one supplier. In this paper, we integrate the works of He and Zhang [40] and Hou et al. [41] and study the retailer's optimal ordering policy and the main supplier's optimal planned production quantity under random demand and random supply with disruption. We investigate the coordination between one retailer and two suppliers—one is a main source and the other is a backup, and show that two contracts with three components: overproduction risk sharing, buy-back, and a payment from/to the backup supplier cannot only coordinate our supply chain but also divide the whole chain's expected profit at any proportion among the three members.

The rest of the paper is organized as follows. In Section 2, we provide a description of the model. In Section 3, we study optimal solution for the integrated supply chain. In Section 4, both the retailer's decision and the main supplier's decision are given in the decentralized scenario. In Section 5, we argue that simple wholesale and overproduction risk sharing contract cannot coordinate the chain. A combination of overproduction risk sharing and buy-back contracts with a side payment from the backup supplier to the main supplier is provided to coordinate the chain. Numerical examples to illustrate further insights are provided in Section 5. Finally, some conclusions and further research topics are given in Section 6.

2. Model Description

We consider a single-period supply chain consisting of one retailer and two supplier with the assumption that the information is symmetrical, where one supplier (called main supplier) is unreliable with cheaper wholesale price, and the other supplier (called backup supplier) is perfectly reliable with more expensive wholesale price. This supply phenomenon is often seen in the off-shoring situation [42].

We use L to denote the main supplier's decision on how many products to produce, and assume that the yields of the supplier will be YL, where Y is a random yield variable and positive support on $[0,1]$ which satisfies $P\{Y = 0\} = p$ and $P\{0 < Y \leqslant 1\} = \int_0^1 g(y)dy = 1 - p = q$, where $g(y)$ is a nonnegative function (note that the yield is subject to disruption with a probability p, and $g(y)$ is not a probability density function unless $p = 0$, i.e., there is no disruption). We assume that the backup supplier's production has a perfect yield, for example, the backup supplier can convert similar or better products from his inventory to satisfy the order. We also assume that the retailer faces a random demand, X, with cumulative distribution function, $F(x)$, probability density function, $f(x)$, and $E(X) = \mu_X$.

The main supplier's per unit production cost is c_m, and a marginal cost αc_m is incurred in the event of a disruption [21]. The unit understock cost to the retailer is c_u (this cost may include loss of reputation). In order to mitigate supply risks, the retailer orders Q units from the backup supplier at the beginning of the selling seasons. The main supplier's whole price is w_m per delivery unit, and the backup supplier charges the retailer a wholesale price w_b per delivery unit. The main supplier's production cost of each planned unit is c_m, and the unit production cost of the backup supplier is c_b. The retailer sells the products to the end customer at the unit price s. At the end of the selling season, all unsold products will be salvaged at a value of v per unit.

Based on the reliability of the backup supplier, it is reasonable to assume that $c_b > c_m > v$. In addition, we assume that $s > w_b > c_b$ and $s > w_m > c_m$. These inequalities make sure that each member has a positive profit.

3. Centralized Model

To establish a performance benchmark, we first analyze the optimal solution of an integrated supply chain. In the centralized model, the expected integrated supply chain profit is given by

$$\Pi_c(Q, L) = sE\min(X, YL + Q) + vE(YL + Q - X)^+ - c_u E(X - YL - Q)^+ - c_b Q - \delta c_m L, \tag{1}$$

where $\delta = \alpha p + q$. The first term in (1) is the expected revenue from sales, the second term is the salvaged value, the third term is the opportunity costs due to the lost sales, and the last two terms are the production costs.

From (1), the integrated supply chain's expected profit function can be rewritten as follows:

$$\begin{aligned}\Pi_c(Q,L) &= p\left\{\int_0^Q [sx+v(Q-x)]\,f(x)\,dx \right.\\ &\qquad \left. +\int_Q^\infty [sQ-c_u(x-Q)]\,f(x)\,dx\right\}\\ &\quad +\int_0^1\left\{\int_0^{yL+Q}[sx+v(yL+Q-x)]\,f(x)\,dx \right.\\ &\qquad \left. +\int_{yL+Q}^\infty [s(yL+Q)-c_u(x-yL-Q)]\,f(x)\,dx\right\}\\ &\quad \times g(y)\,dy - c_bQ-\delta c_mL.\end{aligned} \tag{2}$$

The following Proposition 1 states that the objective function in (2) is concave. Hence, the optimal production quantity and order quantity from the backup supplier can be determined easily. All proofs, if not provided in the paper, are in the appendix.

Proposition 1. *$\Pi_c(Q,L)$ is jointly concave in Q and L.*

From Proposition 1, we can characterize the optimal order quantity, Q^c, and the optimal planned production quantity, L^c, through the first-order conditions. In particular, we have the following Proposition 2.

Proposition 2. (1) *If $\delta c_m < c_b\mu_Y$ and $\int_0^1 F(yL_0)g(y)dy < (s-c_b+c_u)/(s-v+c_u)$, then (Q^c,L^c) is uniquely solved by*

$$pF(Q^c)+\int_0^1 F(yL^c+Q^c)\,g(y)\,dy = \frac{s-c_b+c_u}{s-v+c_u}, \tag{3}$$

$$\int_0^1 yF(yL^c+Q^c)\,g(y)\,dy = \frac{(s+c_u)\mu_Y-\delta c_m}{s-v+c_u}, \tag{4}$$

where L_0 satisfies $\int_0^1 yF(yL_0)g(y)dy = (\mu_Y(s+c_u)-\delta cm)/(s-v+c_u)$.

(2) *If $\delta c_m < \mu_Y c_b$ and $\int_0^1 F(yL_0)g(y)dy \geqslant (s-c_b+c_u)/(s-v+c_u)$, then $(Q^c,L^c)=(0,L_0)$.*

(3) *If $\delta c_m \geqslant \mu_Y c_b$, then $(Q^c,L^c) = (F^{-1}(1-(c_b-v)/(s-v+c_u)),0)$.*

From Proposition 2, we can obtain the maximum expected profit of the integrated supply chain as follows:

$$\Pi_c(Q^c,L^c) = (s-v+c_u)\,\pi(Q^c,L^c)-c_u\mu_X, \tag{5}$$

where $\pi(Q^c,L^c) = \int_0^1\int_0^{yL^c+Q^c} xf(x)g(y)dx\,dy + p\int_0^{Q^c} xf(x)dx$.

4. Decentralized Model under a Traditional Arrangement

Now we consider the decentralized scenario under the traditional arrangement; that is, the two suppliers and the retailer are independent agents and derive their individual expected profit, respectively. At the beginning of the selling season, the retailer places an order quantity R from the main supplier, and orders Q units of products from the backup supplier in order to mitigate supply risks. It is worth mentioning that the retailer does not charge the main supplier any penalty for the unsatisfied orders caused by the supply risks, but when its delivery quantity is higher than R, the retailer accepts only the amount equal to R.

4.1. Main Supplier's Production decision. For given order quantity R, the expected profit of the main supplier is

$$\Pi_m(L) = w_mE\min(YL,R)+vE(YL-R)^+-\delta c_mL. \tag{6}$$

Here, the first term is the supplier's revenue from the retailer's order, the second term is the salvaged value, and the last term is the production costs.

Proposition 3. *For given R, the main supplier's optimal planned production quantity $L^*(R)=\zeta R$, where ζ is independent of R and is determined by the following:*

$$(w_m-v)\int_0^{1/\zeta} yg(y)\,dy = \delta c_m - v\mu_Y. \tag{7}$$

Proof. Taking the first and second derivatives in (6), we obtain

$$\frac{d\Pi_m}{dL} = (w_m-v)\int_0^{R/L} yg(y)\,dy + v\mu_Y-\delta c_m, \tag{8}$$

$$\frac{d^2\Pi_m}{dL^2} = -(w_m-v)\frac{R^2}{L^3}g\left(\frac{R}{L}\right) < 0. \tag{9}$$

It follows from (9) that Π_m is concave. Hence, setting $d\Pi_m/dL=0$, we can derive (7). □

4.2. The Retailer's Ordering Policy. Taking into account the response from the main supplier, the retailer's expected profit, $\Pi_r(Q,R)$, is given by

$$\begin{aligned}\Pi_r(Q,R) &= sE\min(X,R+Q,Y\zeta R+Q)+vE(Y\zeta R+Q-X)^+\\ &\quad - c_uE[X-R\min(1,\zeta Y)-Q]^+\\ &\quad - w_mRE[\min(1,\zeta Y)]-w_bQ.\end{aligned} \tag{10}$$

The first term in (10) is the expected revenue from the sales, the second term is the earnings from salvaging the unsold products, the third term is the understock cost, and the last two terms are the cost of buying products from the two suppliers.

From (10), using the same argument as Proposition 1, we can obtain that $\Pi_r(Q,R)$ is jointly concave in Q and R, and the unique (Q^d, R^d) that maximizes $\Pi_r(Q, R)$ can be derived as follows.

Proposition 4. *(1) If $w_m < w_b$ and $F(R_0)\overline{G}(1/\zeta) + \int_0^{1/\zeta} F(y\zeta R_0)g(y)dy < (s-w_b+c_u)/(s-v+c_u)$, then (Q^d, R^d) is uniquely solved by*

$$pF\left(Q^d\right)+F\left(Q^d+R^d\right)\overline{G}\left(\frac{1}{\zeta}\right)+\int_0^{1/\zeta} F\left(Q^d+y\zeta R\right)g\left(y\right)dy = \frac{s-w_b+c_u}{s-v+c_u},$$

$$F\left(Q^d+R^d\right)\overline{G}\left(\frac{1}{\zeta}\right)+\int_0^{1/\zeta} F\left(Q^d+y\zeta R\right)\zeta y g\left(y\right)dy = \frac{s-w_m+c_u}{s-v+c_u}\left[\overline{G}\left(\frac{1}{\zeta}\right)+\frac{\zeta\left(\delta c_m - v\mu_Y\right)}{w_m - v}\right], \tag{11}$$

where R_0 satisfying $F(R_0)\overline{G}(1/\zeta) + \int_0^{1/\zeta} F(y\zeta R_0)\zeta y g(y)dy = ((s-w_m+c_u)/(s-v+c_u))[\overline{G}(1/\zeta)+(\zeta(\delta c_m - v\mu_Y)/(w_m-v))]$ and $\overline{G}(y) = \int_y^1 g(y)dy$.

(2) If $w_m < w_b$ and $F(R_0)\overline{G}(1/\zeta) + \int_0^{1/\zeta} \overline{F}(y\zeta R_0)g(y)dy \geqslant (s-w_b+c_u)/(s-v+c_u)$, then $(Q^d, R^d) = (0, R_0)$.

(3) If $w_m \geqslant w_b$, then $(Q^d, R^d) = (F^{-1}((s-w_b+c_u)/(s-v+c_u)), 0)$.

Proof. Similar to the proof of Proposition 2. □

Given the retailer's optimal order quantity Q^d and R^d, the retailer's expected profit is

$$\Pi_r^d = \left(s-v+c_u\right) \times\left[\pi\left(Q^d,\zeta R^d\right)-\int_{1/\zeta}^{1}\int_{R^d+Q^d}^{y\zeta R^d+Q^d} xf\left(x\right)g\left(y\right)dx\,dy\right] - c_u\mu_X, \tag{12}$$

and the backup supplier's expected profit and the main supplier's expected profit are $\Pi_b^d = (w_b - c_b)Q^d$ and $\Pi_m^d = (w_m - v)\overline{G}(1/\zeta)R^d$, respectively. Let $\Pi_d(Q^d, R^d)$ denote the total expected profit of the decentralized supply chain, then we have the following proposition.

Proposition 5. $\Pi_d(Q^d, R^d) < \Pi_c(Q^c, L^c)$.

Proposition 5 indicates that the decentralized supply chain's expected profit will be lower than that of an integrated supply chain. This phenomenon is well known as double marginalization [23]. It also shows the importance of supply chain coordination. From the main supplier's first-order condition; if $w_m \to \delta c_m/\mu_Y$, we have $\zeta \to 1$ and $L^d(R) \to R$. Furthermore, if $w_b \to c_b$, then we have $L^d = L^c$ and $Q^d = Q^c$ by the retailer's first-order conditions (11). In other words, the total expected profit of the decentralized supply chain is closer to the integrated system as the two supplier's wholesale price approaches their production cost, respectively.

5. Supply Chain Coordination

5.1. Overproduction Risk Sharing Contract. Risk sharing is one of the most common method of achieving supply chain coordination. This method aligns the objectives of the supply chain members and coordinates their activities to optimize system performance. The OPRS contract ensures the main supplier's risk of producing too many units (compared to the quantity ordered) being shared by the retailer [40]. Under the OPRS contract, the retailer commits to pay for all units produced by the main supplier. But he only pays the wholesale price w_m per unit for the order quantity R, and quantities that exceed this amount are compensated at a discount price w_d per unit. In order to guarantee that the main supplier is willing to sell the surplus products to the retailer and prevent the main supplier from producing an unlimited amount, we assume that $v < w_d < \delta c_m/\mu_Y$.

Under the OPRS contract, the main supplier's expected profit is

$$\Pi_m\left(L\right) = w_m E\min\left(YL, R\right) + w_d E(YL-R)^+ - \delta c_m L. \tag{13}$$

Proposition 6. *For given R, the main supplier's optimal planned production quantity $L^d(R) = \zeta_c R$, where ζ_c is independent of R and satisfies*

$$\left(w_m - w_d\right)\int_0^{1/\zeta_c} y g\left(y\right)dy = \delta c_m - w_d\mu_Y. \tag{14}$$

Proof. Similar to the proof of Proposition 3. □

Given the main supplier's best response function $L^d(R) = \zeta_c R$, the retailer's expected profit can be written as follows:

$$\begin{aligned}\Pi_r\left(Q,R\right) = {} & sE\min\left(X, YL+Q\right) + vE(YL+Q-X)^+ \\ & - c_u E[X-YL-Q]^+ \\ & - \left[w_m E\min\left(\zeta_c Y, 1\right) + w_d E(\zeta_c Y - 1)^+\right]R \\ & - w_b Q.\end{aligned} \tag{15}$$

Proposition 7. *The OPRS contract cannot coordinate the supply chain.*

Although the OPRS contract can reduce the main supplier's overproduction risk, it increases the retailer's overstock risk. Hence, the retailer will not choose the order quantity that is the same as the integrated supply chain. But if the suppliers are willing to pay the retailer back for unsold products, then the supply chain coordination may be achieved. In the next subsection, we will construct a combined contract with a side payment to coordinate the supply chain.

5.2. Overproduction Risk Sharing with Buy-Back and a Payment from/to the Backup Supplier. This contract is a mixture of overproduction risk sharing and buy-back contracts. For simplification, it is assumed that the main supplier deliver all of his products at the whole price w_m (i.e., $w_m = w_d$) whether his actual output of the product is more than the retailer's order quantity or not, and he will buy all of the retailer's unsold products back with a return price w_r per unit at the end of the selling season; for example, the main supplier can disassemble the unsold products and remanufacture at a lower production cost. We assume that $w_r - v > w_m - \delta c_m/\mu_Y$ to avoid the triviality of the main supplier producing an infinite amount (this inequality implies that the main supplier's compensation for a unsold product is more than the profit that he earns for the sale of it). Note that the main supplier may return the unsold quantities that exceed his output, for example, the unsold products will be $YL+Q-X > YL$ if $X < Q$ (in this case, the main supplier also returns some of the backup supplier's products). Hence, in order to compensate for the main supplier's product recycling risk, the backup supplier pays a side payment $T(>0)$ to him. If $T < 0$, it can be interpreted that the main supplier gives some reimbursement to the backup supplier for coordination, since he produce more products and earns more profit when the supply chain is coordinated. Hence the parameter T is adopted to split the expected profit of the coordinated system between two suppliers, and the parameter T can be interpreted as that the two suppliers share their expected profits.

Under the above combined contract, the expected profit for the retailer, the main supplier and the backup supplier are given by

$$\Pi_r = sE\left[\min(X, YL+Q)\right] + w_r E(YL+Q-X)^+ - w_m E(YL) - w_b Q, \tag{16}$$

$$\Pi_m = w_m E(YL) - (w_r - v) E(YL+Q-X)^+ - \delta c_m L + T, \tag{17}$$

$$\Pi_b = (w_b - c_b) Q - T, \tag{18}$$

respectively.

Proposition 8. *For given Q, if* $Q < F^{-1}((w_m - \delta c_m/\mu_Y)/(w_r - v))$*, then the main supplier's optimal planned production quantity* $L^d(Q)$ *is uniquely solved from*

$$(w_r - v)\int_0^1 F\left(yL^d(Q) + Q\right) yg(y)\,dy = w_m\mu_Y - \delta c_m, \tag{19}$$

otherwise if $Q \geqslant F^{-1}((w_m - \delta c_m/\mu_Y)/(w_r - v))$*, then* $L^d(Q) = 0$.

Proof. Taking the first and second derivatives of (15) with respect to L, we have

$$\frac{d\Pi_m}{dL} = w_m\mu_Y - \delta c_m - (w_r - v)\int_0^1 F(yL+Q)\, yg(y)\,dy, \tag{20}$$

$$\frac{d^2\Pi_m}{dL^2} = -(w_r - v)\int_0^1 f(yL+Q)\, y^2 g(y)\,dy < 0. \tag{21}$$

It follows from (20) that Π_m is concave on L for given Q. If $Q < F^{-1}((w_m - \delta c_m/\mu_Y)/(w_r - v))$, then $\lim_{L\to 0}(d\Pi_m/dL) = [w_m - (w_r - v)F(Q)]\mu_Y - \delta c_m > 0$ and $\lim_{L\to\infty}(d\Pi_m/dL) = (w_m - w_r + v)\mu_Y - \delta c_m < 0$, and hence there exists a unique $L^d(R)$ that satisfies the first-order condition in (18). Otherwise if $Q \geqslant F^{-1}((w_m - \delta c_m/\mu_Y)/(w_r - v))$, then $d\Pi_m/dL \leqslant 0$ for $L > 0$ and hence $L^d(Q) = 0$. □

Set

$$w_0 = s + c_u - (s - v + c_u) \times \left[\frac{\pi\left(Q^d, \zeta R^d\right)}{\pi\left(Q^c, L^c\right)} - \frac{1}{\pi\left(Q^c, L^c\right)}\int_{1/\zeta}^1 \int_{R^d+Q^d}^{y\zeta R^d+Q^d} xf(x)\, g(y)\,dx\,dy\right], \tag{22}$$

$$T_{\min} = (w_r - v)\left[\pi\left(Q^c, L^c\right) - \frac{(s - c_b + c_u)\, Q^c}{s - v + c_u}\right] - (w_m - v)\,\overline{G}\left(\frac{1}{\zeta}\right) R^d, \tag{23}$$

$$T_{\max} = (w_b - c_b)\left(Q^c - Q^d\right) \tag{24}$$

then we have the following proposition.

Proposition 9. *Under the combined contracts, if*

$$w_m = \frac{\delta c_m}{\mu_Y} + \frac{\left[(s + c_u)\mu_Y - \delta c_m\right](w_r - v)}{(s - v + c_u)\mu_Y}, \quad w_b = c_b + \frac{(s - c_b + c_u)(w_r - v)}{s - v + c_u}, \tag{25}$$

$w_r \in [v, w_0]$ *and* $T \in [T_{\min}, T_{\max}]$*, then the supply chain can be coordinated.*

Proposition 10. *Under the combined contract and contract parameters in* (25)*, an arbitrary allocation of the optimal supply chain profit among the three members can be achieved by varying* w_r *and* T.

Proposition 9 shows that the expected profit of the integrated supply chain can be shared with any specified ratios among three members. A simply effective but not exclusive way to set the profit allocated ratios is to let them equal to the profit ratios between the corresponding members before introducing the contracts, which can certainly ensure that each entity earns more as the contracts increase the total profit of the channel.

6. Numerical Examples

In this section, we provide some numerical examples to gain further insights. We assume that the retailer's demand follows

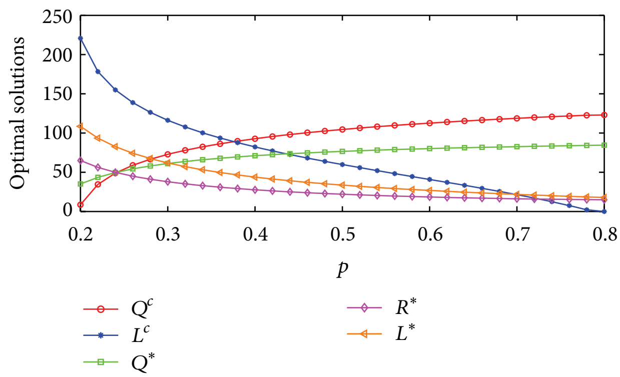

FIGURE 1: Effect of disruption probability on optimal solutions.

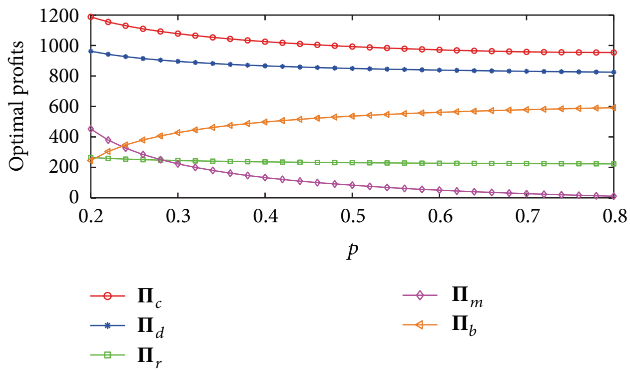

FIGURE 2: Effect of disruption probability on optimal profits.

a normal distribution with mean $\mu_X = 1000$ and standard deviation $\sigma_X = 200$. We also specify that $g(y) = q/(b-a)$, where $0 \leqslant a < b \leqslant 1$. Hence, the main supplier's yield has a mean of $\mu_Y = (a+b)q/2$ and a standard deviation of $\sigma_Y = \sqrt{[(4-3q)(a+b)^2 - 4ab]q/(2\sqrt{3})}$. The base values of the parameters are set as follows: $p = 0.25$; $c_m = 4$, $c_b = 11$, $c_u = 2$, $v = 3$, $s = 25$, $w_m = 16$, $w_b = 18$, $\alpha = 0.25$, $a = 0.4$, $b = 1$, $\mu_X = 100$, $\sigma_X = 50$.

6.1. Effect of the Disruption Probability. Under different values of p, we plot the optimal solutions in Figure 1 and plot the corresponding optimal profits in Figure 2, for the centralized and decentralized systems, respectively.

Figure 1 shows that the order quantity from the backup supplier increases and the planned production quantity of the main supplier decreases as p increases. Since larger disruption probability results in greater supply risk of the main supplier, the retailer will order more products from the backup supplier who has no production risk for avoiding risk increases, and the main supplier will decrease his production in order to reduce the production risk.

Figure 2 shows that an increase in the disruption probability p decreases the profits of both the retailer and the main supplier. But this is not the case for the backup supplier. Since the retailer's order quantity from the backup supplier increases with the disruption probability. For all values of p, the integrated supply chain's profit is significantly lager than the total expected profit of the three members in the decentralized supply chain.

6.2. Effect of the Random Uncertainty and Demand Uncertainty. Tables 1 and 2 present the profits of the supply chain with and without coordination under different yield and different demand, respectively, where $(Q^*, L^*, \zeta^*) = (Q^d, L^d, L^d/R^d)$ in the decentralized model and $(Q^*, L^*, \zeta^*) = (Q^c, L^c, 1)$ in the centralized model. Table 1 shows that the profits of three members decrease as the yield uncertainty increases, which gives the main supplier an incentive to decrease the planned production quantity for reducing yield uncertainty. Since the retailer also greatly benefits from the decrease in the yield uncertainty, he has an incentive to help the supplier to reduce the uncertainty through increasing the return price, and to place a larger order from the backup supplier for transferring the main supplier's supply risk. Table 2 indicates that lower demand uncertainty benefits the retailer and the backup supplier, but this is not the case for the main suppliers. Since the retailer will order smaller quantities as the demand variability decreases, it will directly reduce the main supplier's profit. We can also from Table 2 find that ζ^* is invariant. This is true for the reason that ζ^* is independent of the demand.

From Tables 1 and 2, we find that the retailer's expected profit significantly decreases and as the suppliers' whole price increase without coordination. Supply chain coordination greatly benefits the retailer and the whole supply chain, and this becomes more obvious when the demand uncertainty is more severe. We also see that the coordinated supply chain can achieve Pareto improvement by choosing the appropriate contract parameters w_r and T.

7. Conclusion

In this paper, we consider a supply chain including one retailer and two suppliers under random demand and random yield, where the yield is subject to disruption with a determinate probability. Both the centralized scenario and decentralized scenario are studied, and the retailer's optimal ordering policy and the main supplier's optimal planned production quantity are provided. In the decentralized setting, we propose a combined contract that is a combination of OPRS and buy-back contracts with an additional side payment to component regarding the worse performing supplier. With the combined contract, the appropriate whole price and return price can coordinate the supply and allocate the integrate supply chain's profit with any specified ratios between the retailer and the two suppliers, and the side payment can further split

Table 1: Impact of different yield variabilities on the supply chain.

Models	Contract parameters	(a, b)	Q^*	L^d	ζ^*	Π_r	Π_m	Π_b	Total
Decentralized model under traditional arrangement (w_m, w_b)	(17.96, 20.18)	(0.50, 0.90)	17.59	114.78	1.61	92.11	556.49	161.48	810.08
		(0.45, 0.95)	19.62	114.46	1.63	91.13	531.74	180.11	802.98
		(0.40, 1.00)	22.33	112.25	1.65	89.71	501.86	205.02	796.59
	(18.40, 20.52)	(0.50, 0.90)	16.49	112.83	1.61	63.84	563.02	156.96	783.82
		(0.45, 0.95)	18.51	112.46	1.63	62.90	537.68	176.29	776.87
		(0.40, 1.00)	21.22	110.21	1.65	61.56	507.10	202.06	770.71
	(18.84, 20.86)	(0.50, 0.90)	15.34	110.91	1.61	36.33	569.10	151.27	756.70
		(0.45, 0.95)	17.36	110.49	1.63	35.45	543.19	171.26	749.89
		(0.40, 1.00)	20.07	108.20	1.65	34.17	511.90	197.91	743.99
Coordination under the combined contract (w_m, w_b, w_r, T)	(17.9, 20.18, 16.5, 240)	(0.50, 0.90)	45.64	158.91	1.00	359.20	596.39	178.95	1134.54
		(0.45, 0.95)	49.85	152.91	1.00	355.88	553.78	217.65	1127.32
		(0.40, 1.00)	54.29	146.27	1.00	352.12	508.64	258.39	1119.15
	(18.40, 20.52, 17.0, 250)	(0.50, 0.90)	45.64	158.91	1.00	330.48	619.59	184.46	1134.54
		(0.45, 0.95)	49.85	152.91	1.00	327.31	575.41	224.60	1127.32
		(0.40, 1.00)	54.29	146.27	1.00	323.71	528.59	266.85	1119.15
	(18.84, 20.86, 17.5, 260)	(0.50, 0.90)	45.64	158.91	1.00	301.77	642.79	189.98	1134.54
		(0.45, 0.95)	49.85	152.91	1.00	298.73	597.03	231.55	1127.32
		(0.40, 1.00)	54.29	146.27	1.00	295.31	548.54	275.31	1119.15

Table 2: Impact of different demand variabilities on the supply chain.

Models	Contract parameters	(μ_X, σ_X)	Q^*	L^d	ζ^*	Π_r	Π_m	Π_b	Total
Decentralized model under traditional arrangement (w_m, w_b)	(18.40, 20.52)	(100, 40)	37.63	87.41	1.65	135.49	402.17	358.24	895.90
		(100, 45)	29.52	98.70	1.65	97.72	454.14	281.05	832.91
		(100, 50)	21.22	110.21	1.65	61.56	507.10	202.06	770.71
	(18.84, 20.86)	(100, 40)	36.73	85.76	1.65	106.70	405.76	362.17	874.62
		(100, 45)	28.50	96.86	1.65	69.62	458.29	281.03	808.94
		(100, 50)	20.07	108.20	1.65	34.17	511.90	197.91	743.99
	(19.28, 21.20)	(100, 40)	35.79	84.13	1.65	78.53	408.99	365.10	852.61
		(100, 45)	27.43	95.04	1.65	42.22	462.05	279.84	784.10
		(100, 50)	18.87	106.20	1.65	7.56	516.30	192.46	716.32
Coordination under the combined contract (w_m, w_b, w_r, T)	(18.40, 20.52, 17.0, 240)	(100, 40)	63.64	117.35	1.00	348.63	459.98	365.85	1174.45
		(100, 45)	58.99	131.86	1.00	336.06	488.69	321.63	1146.38
		(100, 50)	54.29	146.27	1.00	323.71	518.59	276.85	1119.15
	(18.84, 20.86, 17.5, 250)	(100, 40)	63.64	117.35	1.00	319.13	477.84	377.48	1174.45
		(100, 45)	58.99	131.86	1.00	307.12	507.57	331.69	1146.38
		(100, 50)	54.29	146.27	1.00	295.31	538.54	285.31	1119.15
	(19.28, 21.20, 18, 260)	(100, 40)	63.64	117.35	1.00	289.64	495.69	389.12	1174.45
		(100, 45)	58.99	131.86	1.00	278.18	526.45	341.75	1146.38
		(100, 50)	54.29	146.27	1.00	266.90	558.48	293.77	1119.15

the profit between the main supplier and the backup supplier. Numerical examples show that coordination can greatly benefit the retailer and the whole supply chain, and that the coordinated supply chain can achieve Pareto improvement by choosing the appropriate contract parameters.

One direct extension of this work is to study the proposed model under risk measures such as expected utility objective, mean-variance criterion, and conditional value-at-risk. Another extension is to consider our problems with asymmetric information on various factors such as the random yield and the random demand. We leave these problems for further research.

Appendix

Proof of Proposition 1. From (1), the first-order and second-order partial derivatives with respect to L and Q, respectively, are given by

$$\frac{\partial \Pi_c}{\partial Q} = -(s-v+c_u)\left[pF(Q)+\int_0^1 F(yL+Q)\,g(y)\,dy\right] + s - c_b + c_u, \quad (A.1)$$

$$\frac{\partial \Pi_c}{\partial L} = -(s-v+c_u)\int_0^1 yF(yL+Q)\,g(y)\,dy + (s+c_u)\mu_Y - \delta c_m, \quad (A.2)$$

$$\frac{\partial^2 \Pi_c}{\partial Q^2} = -(s-v+c_u)\left[pf(Q)+\int_0^1 f(yL+Q)\,g(y)\,dy\right] < 0, \quad (A.3)$$

$$\frac{\partial^2 \Pi_c}{\partial L^2} = -(s-v+c_u)\int_0^1 y^2 f(yL+Q)\,g(y)\,dy < 0, \quad (A.4)$$

$$\frac{\partial^2 \Pi_c}{\partial Q\partial L} = \frac{\partial^2 \Pi_c}{\partial L\partial Q} = -(s-v+c_u)\int_0^1 yf(yL+Q)\,g(y)\,dy. \quad (A.5)$$

Furthermore, it is easy to verify that

$$\begin{aligned}
&\begin{bmatrix} \frac{\partial^2 \Pi_c}{\partial Q^2} & \frac{\partial^2 \Pi_c}{\partial Q\partial L} \\ \frac{\partial^2 \Pi_c}{\partial L\partial Q} & \frac{\partial^2 \Pi_c}{\partial L^2} \end{bmatrix} \\
&= \frac{\partial^2 \Pi_c}{\partial Q^2}\frac{\partial^2 \Pi_c}{\partial L^2} - \left(\frac{\partial^2 \Pi_c}{\partial Q\partial L}\right)^2 \\
&= p(s-v+c_u)^2 f(Q)\int_0^1 f(yL+Q)\,g(y)\,dy \\
&\quad - (s-v+c_u)^2\left\{\int_0^1 y^2 f(yL+Q)\,g(y)\,dy \times \int_0^1 f(yL+Q)\,g(y)\,dy - \left(\int_0^1 yf(yL+Q)\,g(y)\,dy\right)^2\right\} \\
&\geqslant p(s-v+c_u)^2 f(Q)\int_0^1 f(yL+Q)\,g(y)\,dy > 0,
\end{aligned} \quad (A.6)$$

where the the first inequality follows from the Cauchy-Schwarz inequality. Therefore, the Hessian matrix of $\Pi(Q,L)$ is a negative definite matrix, which implies that $\Pi_c(Q,L)$ is jointly concave in (Q,L). □

Proof of Proposition 2. (i) From (A.4), it is obvious that $\partial\Pi_c/\partial L$ is a decreasing function of L for given Q. Let $Q_0 = F^{-1}(((s+c_u)\mu_Y-\delta c_m)/(s-v+c_u)\mu_Y)$, then for given $Q \in [0,Q_0]$, since $(\partial\Pi_c/\partial L)|_{L=0} > 0$ and $(\partial\Pi_c/\partial L)|_{L\to\infty} = v\mu_Y - \delta c_m < 0$, there exists a unique optimal $L^c(Q)$ that satisfies the first-order condition in (4). Substituting $L = L^c(Q)$ into (2), we have

$$\begin{aligned}
&\frac{\partial \Pi_c(Q,L^c(Q))}{\partial Q} \\
&= -(s-v+c_u) \times \left[pF(Q)+\int_0^1 F(yL_1^c(Q)+Q)\,g(y)\,dy\right] \\
&\quad + s - c_b + c_u.
\end{aligned} \quad (A.7)$$

Taking the first-order derivative with respect to Q in (A.1), we get

$$\int_0^1 y\left(y\frac{dL^c(Q)}{dQ}+1\right)f(yL^c(Q)+Q)\,g(y)\,dy = 0. \quad (A.8)$$

From (A.8), we have

$$\frac{dL^c(Q)}{dQ} = -\frac{\int_0^1 yf(yL^c(Q)+Q)\,g(y)\,dy}{\int_0^1 y^2 f(yL^c(Q)+Q)\,g(y)\,dy} < -1 \quad (A.9)$$

for all $Q \in [0,Q_0]$. By (A.8)-(A.9), we can conclude that there exists a unique constant $\lambda \in (0,1)$ such that

$$\lambda\frac{dL^c(Q)}{dQ} = \begin{cases} \leqslant -1, & y \leqslant \lambda, \\ > -1, & y > \lambda. \end{cases} \quad (A.10)$$

Since

$$\begin{aligned}
0 &= \int_0^{\lambda} y\left(y\frac{dL^c(Q)}{dQ}+1\right) f\left(yL^c(Q)+Q\right) g(y)\,dy \\
&\quad + \int_{\lambda}^{1} y\left(y\frac{dL^c(Q)}{dQ}+1\right) f\left(yL^c(Q)+Q\right) g(y)\,dy \\
&\leqslant \lambda\left[\int_0^{\lambda}\left(y\frac{dL^c(Q)}{dQ}+1\right) f\left(yL^c(Q)+Q\right) g(y)\,dy\right. \\
&\quad \left.+\int_{\lambda}^{1}\left(y\frac{dL^c(Q)}{dQ}+1\right) f\left(yL^c(Q)+Q\right) g(y)\,dy\right] \\
&\leqslant \int_0^{1}\left(y\frac{dL^c(Q)}{dQ}+1\right) f\left(yL^c(Q)+Q\right) g(y)\,dy,
\end{aligned} \tag{A.11}$$

hence, we have

$$\begin{aligned}
&\frac{d^2\Pi_c(Q, L^c(Q))}{dQ^2} \\
&\quad = -pf(Q) - \int_0^1\left(y\frac{dL^c(Q)}{dQ}+1\right) f\left(yL^c(Q)+Q\right) g(y)\,dy \\
&\quad < 0,
\end{aligned} \tag{A.12}$$

which implies that $\Pi_c(Q, L^c(Q))$ is concave in Q.

As a result, if $\delta c_m < c_b\mu_Y$ and $\int_0^1 F(yL_0)g(y)dy < (s - c_b + c_u)/(s - v + c_u)$, then $Q_0 > F^{-1}((s - c_b + c_u)/(s - v + c_u))$, and we have $(d\Pi_c(Q, L^c(Q))/dQ)|_{Q=Q_0} = \delta c_m/\mu_Y - c_b < 0$ and $(d\Pi_c(Q, L^c(Q))/dQ)|_{Q=0} = -(s - v + c_u)\int_0^1 F(yL_0)g(y)dy + s - c_b + c_u > 0$. Hence there exists a unique $Q^c \in (0, Q_0)$ such that $d\Pi_c(Q, L^c(Q))/dQ = 0$, and the optimal $L^c(Q)$ is solved from $\int_0^1 yF(yL^c(Q^c)+Q^c)g(y)dy = ((s+c_u)\mu_Y - \delta c_m)/(s - v + c_u)$. That is, (Q^c, L^c) can be uniquely solved by the first-order conditions in (3)-(4).

(ii) If $\delta c_m < c_b\mu_Y$ and $\int_0^1 F(yL_0)g(y)dy \geqslant (s - c_b + c_u)/(s - v + c_u)$, then $Q_0 > F^{-1}((s - c_b + c_u)/(s - v + c_u))$, $(d\Pi_c(Q, L^c(Q))/dQ)|_{Q=Q_0} < 0$ and $(d\Pi_c(Q, L^c(Q))/dQ)|_{Q=0} < 0$. Hence $\Pi_c(Q, L^c(Q))$ is decreasing on $Q \in [0, Q_0]$. Furthermore, it follows from (A.2) that $L^c(Q) = 0$ for all $Q > Q_0$. When $L^c(Q) = 0$, we can obtain that $\Pi_c(Q, 0)$ is decreasing on $Q \in (Q_0, \infty)$ since $d\Pi_c(Q, 0)/dQ < 0$ for all $Q > Q_0$. Hence, we have $Q^c = 0$ and $L^c = L_0$.

(iii) If $\delta c_m \geqslant c_b\mu_Y$, then $Q_0 \leqslant F^{-1}((s - c_b + c_u)/(s - v + c_u))$ and $(d\Pi_c(Q, L^c(Q))/dQ)|_{Q=Q_0} \geqslant 0$. Hence $\Pi_c(Q, L^c(Q))$ is increasing on $Q \in [0, Q_0]$. When $Q > Q_0$, it is easy to verify that $\Pi_c(Q, L^c(Q)) = \Pi_c(Q, 0)$ has the maximum value at the point $Q = F^{-1}((s - c_b + c_u)/(s - v + c_u))(> Q_0)$. Hence, we obtain that $\Pi_c(Q, L^c(Q))$ has the maximum value at $Q = F^{-1}((s - c_b + c_u)/(s - v + c_u))$ for all $Q \geqslant 0$, and we have $(Q^c, L^c) = (F^{-1}((s - c_b + c_u)/(s - v + c_u)), 0)$. □

Proof of Proposition 5. From Proposition 3, the total expected profit of the decentralized supply chain can be expressed as follows:

$$\begin{aligned}
&\Pi_d\left(Q^d, R^d\right) \\
&\quad = p\left\{\int_0^{Q^d}\left[(s-v)x + vQ^d\right] f(x)\,dx\right. \\
&\qquad \left.+\int_{Q^d}^{\infty}\left[(s+c_u)Q^d - c_u x\right] f(x)\,dx\right\} \\
&\qquad + \int_0^{1/\zeta}\left\{\int_0^{y\zeta R^d+Q^d}\left[(s-v)x + v\left(y\zeta R^d + Q^d\right)\right] f(x)\,dx\right. \\
&\qquad + \int_{y\zeta R^d+Q^d}^{\infty}\left[(s+c_u)\left(y\zeta R^d + Q^d\right) - c_u x\right] \\
&\qquad \left.\times f(x)\,dx\right\} g(y)\,dy \\
&\qquad + \int_{1/\zeta}^{1}\left\{\int_0^{R^d+Q^d}\left[(s-v)x + vQ^d\right] f(x)\,dx\right. \\
&\qquad + \int_{R^d+Q^d}^{\infty}\left[(s-v+c_u)\left(Q^d+R^d\right) + vQ^d\right. \\
&\qquad \left.\left. - c_u x\right] f(x)\,dx + vy\zeta R^d\right\} g(y)\,dy \\
&\qquad - c_b Q^d - \delta c_m \zeta R^d.
\end{aligned} \tag{A.13}$$

For given $Q \geqslant 0$, let

$$\begin{aligned}
&K(R \mid Q) \\
&\quad = \int_0^{R+Q}\left[(s-v)x + vQ\right] f(x)\,dx \\
&\qquad + \int_{R+Q}^{\infty}\left[(s-v+c_u)(Q+R) + vQ - c_u x\right] f(x)\,dx,
\end{aligned} \tag{A.14}$$

then

$$\frac{dK(R \mid Q)}{dR} = (s - v + c_u)\left[1 - F(Q+R)\right] > 0, \tag{A.15}$$

which implies that $K(R \mid Q)$ is an increasing function of R. Hence, we get

$$\begin{aligned}
&\int_{1/\zeta}^{1} \left[K\left(R^d \mid Q^d\right) + vy\zeta R^d\right] g(y)\, dy \\
&\quad < \int_{1/\zeta}^{1} \left[K\left(y\zeta R^d \mid Q^d\right) + vy\zeta R^d\right] g(y)\, dy \\
&\quad = \int_{1/\zeta}^{1} \left\{ \int_{0}^{Q^d + y\zeta R} \left[(s-v)x + v\left(y\zeta R^d + Q^d\right)\right] f(x)\, dx \right. \\
&\qquad \left. + \int_{Q^d + y\zeta R^d}^{\infty} \left[(s+c_u)\left(Q^d + y\zeta R^d\right) - c_u x\right] f(x)\, dx \right\} \\
&\qquad \times g(y)\, dy.
\end{aligned} \tag{A.16}$$

As a result, we have $\Pi_d(Q^d, R^d) < \Pi_c(Q^d, \zeta R^d) \leqslant \Pi_c(Q^c, L^c)$. □

Proof of Proposition 7. Take the first derivatives with Q and R in (15), we have

$$\begin{aligned}
\frac{\partial \Pi_r}{\partial Q} &= (s - w_b + c_u) - (s - v + c_u) \\
&\quad \times \left[pF(Q) + \int_0^1 F(y\eta R + Q)\, g(y)\, dy\right], \\
\frac{\partial \Pi_r}{\partial R} &= \left[(s + c_u)\mu_Y - \delta c_m \right. \\
&\quad \left. - (s - v + c_u) \int_0^1 F(y\eta R + Q)\, y g(y)\, dy\right] \eta \\
&\quad - (w_m - w_d)\, \overline{G}\left(\frac{1}{\eta}\right).
\end{aligned} \tag{A.17}$$

By comparing (A.17) with (A.1)-(A.2), we find that $Q^d = Q^c$ and $R^d = L^c/\eta$ if and only if $w_b = c_b$ and $w_m = w_d$. However, if $w_b = c_b$, then the backup supplier's expected profit is equal to zero, and $w_m = w_d$ will result in the main supplier producing an infinite amount. Hence, the OPRS contract cannot coordinate the supply chain. □

Proof of Proposition 9. From (15), we get

$$\begin{aligned}
\frac{d\Pi_r}{dQ} &= (s + c_u - w_r)\left[pF(Q) + \int_0^1 F(yL + Q)\, g(y)\, dy\right] \\
&\quad + s - w_b + c_u, \\
\frac{d\Pi_m}{dL} &= w_m \mu_Y - \delta c_m - (w_r - v) \int_0^1 F(yL + Q)\, y g(y)\, dy.
\end{aligned} \tag{A.18}$$

If conditions in (21)-(22) are fulfilled, then by comparing (A.18) with (A.1)-(A.2), we obtain $d\Pi_r/dQ = \partial\Pi_c/\partial L$ and $d\Pi_m/dL = \partial\Pi_c/\partial L$. Hence, we have $Q^d = Q^c$ and $L^d(Q^d) = L^c$.

If the supply chain coordination is achieved, then the expected profits for the retailer, the main supplier, and the backup supplier are $\Pi_m^c = (s - w_r + c_u)\pi(Q^c, L^c) - c_u\mu_X \Pi_m^c = (w_r - v)[\pi(Q^c, L^c) - (s - c_b + c_u)Q^c/(s - v + c_u)] + T$ and $\Pi_b^c = (w_b - c_b)Q^c - T$, respectively. The contract (w_r, T) will not be accepted unless the three member's expected profit is better than the reservation expected profit. Hence, by solving $\Pi_i^c \geqslant \Pi_i^d$, for $i = r, m, b$, we get $w_r \leqslant w_0$ and the feasible range of T, which is $[T_{\min}, T_{\max}]$. □

Proof of Proposition 10. For any given nonnegative numbers ϕ_r, ϕ_m, and ϕ_b that satisfy $\phi_r + \phi_m + \phi_b = 1$, if $w_r = \phi_r v + (\phi_m + \phi_b)(s + c_u - c_u\mu_X/\pi(Q^c, L^c))$ and $T = (w_r - v)[(s - c_b + c_u)Q^c/(s - v + c_u) - \phi_b \pi(Q^c, L^c)/(\phi_m + \phi_b)]$, then we have $\Pi_r^c : \Pi_m^c : \Pi_b^c = \phi_r : \phi_m : \phi_b$. □

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Research Article

A Dynamical Innovation Diffusion Model with Fuzzy Coefficient and Its Application to Local Telephone Diffusion in China

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This paper studies the innovation diffusion problem with the affection of urbanization, proposing a dynamical innovation diffusion model with fuzzy coefficient, and uses the shifting rate of people from rural areas stepping into urban areas to show the process of urbanization. The numerical simulation shows the diffusion process for telephones in China with Genetic Algorithms and this model is effective to show the process of innovation diffusion with the condition of urbanization process.

1. Introduction

Technological innovation promotes the evolution of industries by altering the competitive market structure and value chains of industries [1]. The market success of an innovation is determined not only by technological performance, but also by the interaction of numerous factors. Consequently, it is understandable that many studies choose to investigate the key factors that influence the acceptance or the rejection of an innovation during the process of diffusion. Based on these key factors including price and advertising, researchers have contributed to the development of the diffusion theory by suggesting analytical models for describing and forecasting the diffusion process of an innovation in a social system [2–16] such researchers include Fourt, Mansfield, and Mahajan. Fourt and Mansfield discussed the diffusion pattern in external influence such as mass advertisement or internal influence such as oral communication separately [8, 12]. The sales model for forecasting TV made by Bass in 1969 has settled the base for the following research on innovation diffusion with both external factors and internal factors taken into account [3]. However, the potential adopters in these models are static or fixed at the time an innovation is introduced and remains constant over the diffusion process. Obviously, such an assumption is not tenable with regard to either theory or practice.

In response to this shortcoming, Mahajan and Peterson proposed a dynamical diffusion model where the potential is permitted to vary over time [13]. There are many relevant factors affecting the potential, including socioeconomic conditions in the social system, increases or decreases in the population of the social system, government actions, and efforts to influence the diffusion process, such as advertising and pricing. Such a proposition was followed and developed by Lackman, Dodson, and several other researchers [17–24]. Kalish, for example, conjectured that the qualitative structure of an optimal pricing policy will remain the same upon the inclusion of additional marketing variables such as advertising, in the diffusion model. Jain and Rao show that price affects the diffusion rate via the coefficients of external and internal influence [20].

Because different adopters have different attitudes of risk or decisive patterns, there will be great difference in adoption possibility and rate. The consumption habit, the living standard, and the environment around also make a great difference in the adopting rate. For this reason, a diffusion model is presented in a different section in the paper [25]. The author believes that the characteristics of adopters in different patches make differences in the diffusion rate. Furthermore, there will be migration between patches, which would also have an effect on innovation diffusion. With great economic development and urbanization, we

Table 1: The explanation for conversion relation.

Conversion name/way of conversion	Explanation for conversion
(A) External information, oral communication	non-users become users in colony 1
(B) External information, oral communication	non-users become users in colony 2
(C) Give up using	Users become Non-users in colony 1
(D) Give up using	Users become Non-users in colony 2
(E) The members shifting between the colonies	Users in colony 1 continue using the innovation after shifting into colony 2
(F) The members shifting between the colonies	Users in colony 2 continue using the innovation after shifting into colony 1
(G) The members shifting between the colonies	Users in colony 2 give up using the innovation after shifting into colony 1
(H) The members shifting between the colonies	Users in colony 1 give up using the innovation after shifting into colony 2
(M) The members shifting between the colonies	Non-users in colony 2 still remain as non-users after shifting into colony 1
(N) The members shifting between the colonies	Non-users in colony 1 still remain as non-users after shifting into colony 2

should not ignore the potential adopters who have changed their economic conditions or status. They will have great consumption demands for goods which they would not have even considered before. For example, the data of telephone subscribers of China in 2002–2006 has got tremendous growth which cannot be explained by any diffusion model before. Furthermore, the rate of urbanization development is different in different stages. So it is more acceptable if we take the uncertain rate instead of the deterministic rate to simulate the process. We use the rate of people from rural areas stepping into urban to show the process of the urbanization development of this area. Consequently, this paper constructs a fuzzy innovation diffusion model and tries to find the principles of innovation diffusion with the effects of population increasing and urbanization. In this context, model uncertainty is portrayed through the fuzzy transition process from nonuser to the adopter of an innovation because of the uncertain rate of urbanization.

We organize this paper as follows. In Section 2, an innovation diffusion model with the population increasing and the conversion between two different colonies is constructed. In Section 3, a stability analysis is done on the population increasing and the diffusion rate of different groups. An empirical analysis is also done with the data of telephone subscribers of urban and rural areas in China compared with the Bass model in Section 4. Some concluding remarks are finally given in Section 5.

2. Modeling

There is a comparative independence and stability between colony 1 and colony 2. Each of the two colonies has its own main communication styles, broadcasting channels and ways, and consumption habits. However, there also exists migration between them for specific reasons such as improved living standards. The members moving into new community will be accepted into the new consumption system since they are affected by the new environment. In the interior of a community, for some related external factors and oral communication, nonusers adopt innovation and serve as users, or users give up the innovation for some reasons after experiencing it.

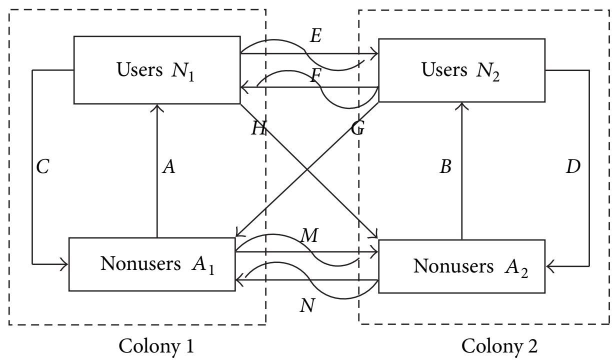

Figure 1: Customers flow diagram for innovation between two colonies.

2.1. Conceptional Model. Based on the foundation above, the concrete convert relations and relations between the inside and outside of the two communities are shown in Figure 1 and Table 1, respectively. The arrowheads denote the flow directions of the members.

2.2. Mathematical Model

Hypothesis 1. The members of each colony are divided into users and nonusers. The nonusers include those who adopted the innovation and have given up using it.

Hypothesis 2. The internal increasing rate of the two colonies obeys the logistic distribution functions.

Hypothesis 3. The nonusers of each colony are still nonusers when entering another colony, which can be shown in Figure 1 in the conversion way of M and N, and the users of each colony can either keep on or give up using the innovation for various reasons when entering another colony.

Hypothesis 4. The number of the members in one colony shifting to another colony is proportional to the number of the total members of the colony, and the coefficient is fuzzy for the uncertainty of urbanization.

Based on the conceptional model, the hypothesis, and the conversion relations above, we build the increase equation for variables Q_1, Q_2, where $Q_i(t)$ represents the number of members in colony i in time t, $i = 1, 2$. According to the empirical statistics, the birthrate and ratio to death are linear functions of the population of the colony. Suppose that $\beta_{i0} + (-d_{i0}Q_i)$ is the birthrate of colony i, β_{i0} is the birthrate of colony i without a resource limit, and $-d_{i0}Q_i$ is the birthrate of colony i with a resource limit, where β_{i0} and d_{i0} are both positive constants. The ratio of death of colony i is $\beta_{i1}+d_{i1}Q_i > 0$, where β_{i1} is the natural death ratio without a resource limit and $d_{i0}Q_i$ is the ratio of death increasing with a resource limit; that is to say, the birthrate will decrease with the increase of population and the death ratio will increase with the increase of population. Thus, the number of members increased in time t in colony 1 is $(\beta_{10} - \beta_{11})Q_1 - (d_{10} + d_{11})Q_1^2$ and $\beta_{10} - \beta_{11} > 0$ according to Hypothesis 2. From Hypothesis 4 and the conversion way of F and M, the number of increased members shifting from colony 2 in time t is $\theta_2 N_2 + \theta_2 A_2 = \theta_2 Q_2$, and the number of members shifting into colony 2 from conversion way of E and N is $\theta_1 N_1 + \theta_1 A_1 = \theta_1 Q_1$, as seen in Figure 1. Based on the analysis above, the increase equation of Q_1 is

$$\dot{Q}_1 = \frac{dQ_1}{dt} = (\beta_{10} - \beta_{11}) Q_1 - (d_{10} + d_{11}) Q_1^2 + \theta_2 Q_2 - \theta_1 Q_1. \tag{1}$$

Similarly, the increase equation of Q_2 is

$$\dot{Q}_2 = \frac{dQ_2}{dt} = (\beta_{20} - \beta_{21}) Q_2 - (d_{20} + d_{21}) Q_2^2 + \theta_1 Q_1 - \theta_2 Q_2. \tag{2}$$

Consequently the increasing model of colonies' members is governed by

$$\begin{aligned} \dot{Q}_1 &= (\beta_{10} - \beta_{11}) Q_1 - (d_{10} + d_{11}) Q_1^2 + \theta_2 Q_2 - \theta_1 Q_1, \\ \dot{Q}_2 &= (\beta_{20} - \beta_{21}) Q_2 - (d_{20} + d_{21}) Q_2^2 + \theta_1 Q_1 - \theta_2 Q_2, \\ Q_1(0) &= Q_{10} \geq 0, \qquad Q_2(0) = Q_{20} \geq 0. \end{aligned} \tag{3}$$

Suppose that $N_i(t)$ is the number of users in colony i at time t, and $A_i(t)$ is the number of nonusers in colony i at time t. Now we will build the equation of variable $N_1(t)$ in colony 1. From conversion way A in Figure 1, because of the internal oral communication and medium, the number of nonusers who have changed to become users is $a_1 A_1 + b_1 N_1 A_1$, in which a_1 is the probability for nonusers to become users caused by medium, and b_1 is the probability for nonusers to become users caused by oral communication between users and nonusers. Owing to natural deaths, the decreasing number of users is $(\beta_{11} + d_{11}Q_1)N_1$. From the conversion way C in Figure 1, there are $e_1 N_1$ users who have to give up using the innovation for some reason, because of movement of population, there are $\theta_1 N_1$ users shifting out of colony 1 and $k_2\theta_2 N_2$ users shifting into colony 1 from colony 2. Here θ_i is the probability for the members of colony i converting into another one and k_i is the probability for users to continue using the innovation in colony i after converting into another colony. Based on the analysis above, the increase equation of variable N_1 is

$$\begin{aligned} \dot{N}_1 = \frac{dN_1}{dt} &= a_1 A_1 + b_1 N_1 A_1 - (\beta_{11} + d_{11} Q_1) N_1 \\ &\quad - e_1 N_1 - \theta_1 N_1 + k_2 \theta_2 N_2. \end{aligned} \tag{4}$$

Similarly, the increase equation of variable N_2 is

$$\begin{aligned} \dot{N}_2 = \frac{dN_2}{dt} &= a_2 A_2 + b_2 N_2 A_2 - (\beta_{21} + d_{21} Q_2) N_2 \\ &\quad - e_2 N_2 - \theta_2 N_2 + k_1 \theta_1 N_1. \end{aligned} \tag{5}$$

Consequently, we can say that the dynamical equations of innovation diffusion based on members shifting between colony 1 and colony 2 are governed by

$$\begin{aligned} \dot{Q}_1 &= (\beta_{10} - \beta_{11}) Q_1 - (d_{10} + d_{11}) Q_1^2 + \theta_2 Q_2 - \theta_1 Q_1, \\ \dot{N}_1 &= a_1 A_1 + b_1 N_1 A_1 - (\beta_{11} + d_{11} Q_1) N_1 \\ &\quad - e_1 N_1 - \theta_1 N_1 + k_2 \theta_2 N_2, \\ \dot{Q}_2 &= (\beta_{20} - \beta_{21}) Q_2 - (d_{20} + d_{21}) Q_2^2 + \theta_1 Q_1 - \theta_2 Q_2, \\ \dot{N}_2 &= a_2 A_2 + b_2 N_2 A_2 - (\beta_{21} + d_{21} Q_2) N_2 \\ &\quad - e_2 N_2 - \theta_2 N_2 + k_1 \theta_1 N_1, \\ Q_1 &= A_1 + N_1, \\ Q_2 &= A_2 + N_2, \\ Q_1(0) &= Q_{10} \geq 0, \qquad Q_2(0) = Q_{20} \geq 0, \\ A_1(0) &= A_{10} \geq 0, \\ A_2(0) &= A_{20} \geq 0, \qquad N_1(0) = N_{10} \geq 0, \\ N_2(0) &= N_{20} \geq 0. \end{aligned} \tag{6}$$

This model focuses on the analysis of the affection on the customers by the process of urbanization. The urbanization process is expressed by the mutual conversion between various consumer groups. That is to say, with economic development and improved living standards, the low-consumption groups constantly improve their levels of consumption and step into consumer groups of higher levels. From the right expression of the following equation:

$$\dot{Q}_1 = (\beta_{10} - \beta_{11}) Q_1 - (d_{10} + d_{11}) Q_1^2 + \theta_2 Q_2 - \theta_1 Q_1, \tag{7}$$

$(\beta_{10} - \beta_{11})Q_1 - (d_{10} + d_{11})Q_1^2$ is fatally affected by natural population increase, and the policy of family planning, and little affected by economic development, while $\theta_2 Q_2 - \theta_1 Q_1$ is mainly affected by economic development. Although certain regularities of economic development can be followed, the effect of this on the customers is uncertain and will vary during the diffusion process of an innovation. However, the models mentioned in the introduction, including system (6), are crisp and obviously cannot meet the actual changes in the law of development. Since these models stressed some main influencing factors, the impact of which on the consumers might be exaggerated, and the combined effects of other factors were ignored, some serious disturbance might occur. Moreover, during the process of diffusion, the impact on the diffusion rate by these factors is of great difference in different stages. For this reason, we try to define $\theta_i,\ i = 1, 2$ as fuzzy numbers and use fuzzy theory to express the uncertainty of urbanization of China.

3. The Stability Analysis on the Model

Since the market is attracted to the variety of innovation users' size and the market potential, we only discuss the variables of members Q_1, Q_2 and the variables of users N_1, N_2. Before discussing the stability of the members increasing model, we will introduce some related concepts and then make the stability analysis on the innovation diffusion.

3.1. Related Concepts. We now recall some definitions needed through the paper.

Let D^n denote the set of upper semicontinuous normal fuzzy sets on R^n with compact support. That is, if $u \in D^n$, then $u : R^n \to [0, 1]$ is upper semicontinuous, $\operatorname{supp}(u) = x \in R^n : u(x) > 0$ is compact, and there exists at least one $\xi \in \operatorname{supp}(u)$ for which $u(\xi) = 1$. The α-set of u, $0 < \alpha \le 1$, is

$$[u]^\alpha = \{x \in R^n : u(x) > \alpha\}. \tag{8}$$

Definition 1. If for $\mu, \nu \in D^n$, there exists $\omega \in D^n$ such that $\mu = \nu + \omega$, then we say that the Hukuhara difference of μ and ν exists, call ω the H-difference of μ and ν, and denote $\mu - \nu = \omega$.

The approach of Hukuhara differentiation suffers a grave disadvantage in so far as the solution has the property that diam $(x(t))$ in nondecreasing in t. Consequently, this formulation cannot really reflect any of the reach behaviors of ordinary differential equations such as periodicity, stability, bifurcation, and the like and is ill suited for modeling. However, Hüllermeier suggested a different formulation of the FIVP based on a family of differential inclusions at each β level, $0 \le \beta \le 1$,

$$x'(t) \in [G(t, x(t))]^\alpha, \quad x(0) \in [x_0]^\alpha, \tag{9}$$

where now $[G(\cdot,\cdot)]^\alpha : R \times R^n \to K_C^n$, the space of nonempty convex compact subsets of R^n. The idea is that the set of all such solution $\Sigma_\alpha(X_0, t)$ would be the α level of a fuzzy set $\Sigma(X_0, t)$ in the sense that all attainable sets $A_\alpha(X_0, t)$, $0 < t \le T$, are levels of a fuzzy set $A(X_0, t)$ on R^n. Considering $\Sigma(X_0, T)$ to be the solution of the fuzzy DE $x' = G(t, x)$, $x(0) = X_0$, thus captures both uncertainty and the rich properties of differential inclusions in one and the same technique. It has been shown that the solution set and attainability set are fuzzy sets under fairly relaxed conditions on G [26].

Theorem 2 (Staking Theorem [27]). *Let $\{Y_\alpha \subset R^n \mid 0 \le \alpha \le 1\}$ be a family of compact subsets satisfying the following:*

(1) $Y_\alpha \in F(R)$ *for all* $0 \le \alpha \le 1$;

(2) $Y_\alpha \subseteq Y_\beta$ *for* $0 \le \beta \le \alpha \le 1$;

(3) $Y_\alpha = \cap_{i=1}^{\infty} Y_{\alpha_i}$ *for any nondecreasing sequence* $\alpha_i \to \alpha$ *in* $[0, 1]$.

Then there is a fuzzy set $u \in D^n$ such that $[u]^\alpha = Y_\alpha$. In particular, if the Y_α are also convex, then $u \in \varepsilon^n$. Conversely, the level sets $[u]^\beta$ of any $u \in D^n$ satisfy these conditions, while if $u \in \varepsilon^n$, then $[u]^\alpha$ are also convex.

Property 1 (Lyapunov Stability, [26]). Let $K \subset R^n$ be nonempty and suppose that $G : R^+ \times K \to K_C^n$ be such that initial value problems

$$x' \in G(t, x(t)), \quad t_0 \le t < \infty, \quad x(t_0) = x_0 \tag{10}$$

have solutions for every $t_0 \ge 0$ and $x_0 \in K$. So, the interval of existence of solutions is $J = [0, \infty)$.

A set M is stable for the inclusion (10) if for all $\varepsilon > 0$ and $t_0 \ge 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $x_0 \in M + \delta B^n$ implies that $x(t) \in M + \varepsilon B^n$ on $[t_0, \infty)$ for every solution $x(t)$ of (10). If $A(x_0, t)$ is the attainability set of (10), this may be rephrased as $x_0 \in M + \delta B^n$ implies that $\rho(A(x_0, t), M) \le \varepsilon$ on $[t_o, \infty)$. If $\delta = \delta(\varepsilon)$ is independent of t_0 and depends only on ε, M, (10) is said to be uniformly stable for the inclusion. If $\rho(A(x_0, t), M) \le \varepsilon \to 0$ as $t \to \infty$, and M is (uniformly) stable, the set M is said to be (uniformly) asymptotically stable.

Let $H : R \times R^n \to \varepsilon^n$, and consider the fuzzy differential equation(FDE):

$$x' = H(t, x), \qquad x(0) = X_0 \in \varepsilon^n, \tag{11}$$

interpreted as a family of differential inclusions. Set $[H(t, x)]^\beta = F(t, x; \beta)$ and identify the FDE with the family of differential inclusion:

$$x_\beta'(t) \in F\left(t, x_\beta(t); \beta\right), \quad x_\beta(0) = x_0 \in [X_0]_\beta, \quad 0 \le \beta \le 1, \tag{12}$$

where Ω is an open subset of R^{n+1} containing $(0, [X_0]^0)$, $\beta \in I := [0, 1]$ and $F : \Omega \times I \to K_C^n$.

Theorem 3 (see [26]). *Let $X_0 \in \varepsilon^n$, and let Ω be an open set in $R \times R^n$ containing $\{0\} \times \operatorname{supp}(X_0)$. Suppose that $H : \Omega \to \varepsilon^n$ is upper semicontinuous and write $F(t, x; \beta) = [H(t, x)]^\beta \in K_C^n$ for all $(t, x, \beta) \in R^{n+1} \times [0, 1]$. Let the boundedness assumption* *with constants b, M, T hold for all* $x_0 \in \operatorname{supp}(X_0)$ *and the inclusion:*

$$x'(t) \in F(t, x; 0), \quad x(0) \in \operatorname{supp}(X_0). \tag{13}$$

Then the attainable sets $A_\beta(X_\beta, T), \beta \in [0,1]$ *of the family of inclusions*

$$x'_\beta(t) \in F\left(t, x_\beta; \beta\right); \quad x_\beta(0) \in X_\beta := \left[X_0\right]^\beta, \quad \beta \in [0,1] \tag{14}$$

are the level sets of a fuzzy set $A(X_0, T) \subset D^n$*. The solution sets* $\Sigma_\beta(X_\beta, T)$ *of* (14) *are the level sets of a fuzzy set* $\Sigma(X_0, T)$ *defined on* $Z_T(R^n)$*, where* $Z_T(R^n) = \{x(\cdot) \in C([0,T]; R^n) : x'(\cdot) \in L^\infty([0,T]); R^n)\}$.

Property 2 (Lyapunov Stability of a Family of Differential Inclusion, [26]). If $U \in D^n$ is a fuzzy set and $\mathbf{U}, \mathbf{W} \subset D^n$ are closed subsets of D^n, define the distance from $\mathbf{W}$ and Hausdorff separation, respectively, by

$$\begin{aligned}\rho_*(U, \mathbf{W}) &= \inf\nolimits_{W \in \mathbf{W}} d_\infty(U, W), \\ \rho_D(\mathbf{U}, \mathbf{W}) &= \sup\nolimits_{U \in \mathbf{U}} \rho_*(U, \mathbf{W}).\end{aligned} \tag{15}$$

The significance of these definitions is that the metric space (D^n, d_∞) of fuzzy sets, $\rho_*(U, \mathbf{W})$, is the distance of $U \in D^n$ from $\mathbf{W} \subset D^n$ and is the analogue of $\rho(x, A)$ in R^n. Correspondingly, $\rho_D(\mathbf{U}, \mathbf{W})$ is the Hausdorff separation between $\mathbf{U}, \mathbf{W} \subset D^n$ with respect to the metric d_∞ and is the analogue of the Hausdorff separation $\rho(A, B)$ in K^n.

Let 0 be the fuzzy singleton $\chi_{\{0\}} \in D^n$, write $\|U\| = d_\infty(U, 0)$, and denote the open unit ball in D^n by $B^n = \{U \in D^n : \|U\| < 1\}$.

A set $\mathbf{U} \subset D^n$ is stable for the FDE (12) if for all $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta = \delta(\varepsilon, t_0)$ such that $X_0 \in \mathbf{U} + \delta B^n$ implies that $A(X_0, t) \in \mathbf{U} + \varepsilon B^n$ on $[t_0, \infty)$, where $A(X_0, t)$ is the fuzzy attainability set defined by the family (12); that is, $\rho_*(X_0, \mathbf{U}) < \delta$ implies that $\rho_D(A(X_0, t), \mathbf{U}) \leq \varepsilon$ on $[t_0, \infty)$. If $\delta = \delta(\varepsilon)$ is independent of t_0 and depends only on ε, $\mathbf{U}$ for the FDE (12) is said to be uniformly stable. If $\rho_D(A(X_0, t), \mathbf{U}) \to 0$ as $t \to \infty$ and $\mathbf{U}$ is (uniformly) stable, the set $\mathbf{U}$ is said to be (uniformly) asymptotically stable. Most frequently, $\mathbf{U}$ will consist of a single fuzzy set $U \in D^n$.

Theorem 4. *Let E be an open subset of* D^n *containing the origin, let* $F \in C^1(E)$*, and let* ϕ_t *be the flow of the nonlinear system* (12)*. Suppose that* $F(0) = 0$ *and that* $DF(0)$ *has k eigenvalues with negative real part and* $n - k$ *eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace* E^S *of the linear differential inclusions:*

$$x'_\beta(t) \in DF(0)\, x_\beta(t), \quad x_\beta(0) = x_0 \in \left[X_0\right]_\beta, \quad 0 \leq \beta \leq 1 \tag{16}$$

at 0 such that for all $t \geq 0$, $\phi_t(S) \subset S$ *and for all* $[X_0]_\beta \in S$,

$$\lim_{t \to \infty} \phi_t(x_0, \beta) = 0, \tag{17}$$

and there exists an $n - k$*-dimensional differentiable manifold U tangent to the unstable subspace* E^U *of* (16) *at 0 such that for all* $t \leq 0$, $\phi_t(U) \subset U$ *and for all* $x_0 \in U$,

$$\lim_{t \to -\infty} \phi_t(x_0, \beta) = 0. \tag{18}$$

Proof. If $F \in C^1(E)$ and $F(0) = 0$, system (12) can be written as

$$\begin{aligned}&x'_\beta(t) \in A_\beta x + f\left(t, x_\beta(t); \beta\right), \\ &x_\beta(0) = x_0 \in \left[X_0\right]_\beta, \quad 0 \leq \beta \leq 1,\end{aligned} \tag{19}$$

where $A_\beta = DF(0)$, $f(x) = F(x) - A_\beta x$, $f \in C^1(E)$, $f(0) = 0$, and $Df(0) = 0$. This in turn implies that for all $\varepsilon > 0$, there is a $\delta > 0$ such that $|x| \leq \delta$ and $|y| \leq \delta$ imply that

$$\left|f(x) - f(y)\right| \leq \varepsilon |x - y|. \tag{20}$$

Furthermore, for all matrix $A_0 \in A_\beta$ there is an $n \times n$ invertible matrix C_0 such that

$$B_0 = C_0^{-1} A_0 C_0 = \begin{bmatrix} P_0 & 0 \\ 0 & Q_0 \end{bmatrix} \tag{21}$$

in which the eigenvalues of $k \times k$ matrix P_0 have negative real part and the eigenvalues of the $(n-k) \times (n-k)$ matrix Q_0 have positive real part. We can choose $\alpha > 0$ sufficiently small that for $j = 1, \ldots, k$,

$$\operatorname{Re}\left(\lambda_j\right) < -\alpha < 0. \tag{22}$$

Letting $y = C^{-1}x$, $C = \{C_0 \mid B_0 = C_0^{-1} A_0 C_0, A_0 \in A_\beta\}$ the system (19) then has the form

$$\dot{y}_\beta \in B_\beta y_\beta + G\left(y_\beta\right), \tag{23}$$

where $B_\beta = \{B_0 = C_0^{-1} A_0 C_0 \mid A_0 \in A_\beta\}$, $G(y_\beta) = C^{-1} f(C y_\beta) \in C^1(\breve{E})$, where $\breve{E} = C^{-1}(E)$ and G satisfies the Lipschitz-type condition (20) above.

Consider the system (23). Let

$$U_\beta(t) = \begin{bmatrix} e^{P_\beta t} & 0 \\ 0 & 0 \end{bmatrix}, \qquad V_\beta(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^{Q_\beta t} \end{bmatrix}, \tag{24}$$

where $P_\beta = \{P_0 \mid B_0 = C_0^{-1} A_0 C_0, A_0 \in A_\beta\}$, $Q_\beta = \{Q_0 \mid B_0 = C_0^{-1} A_0 C_0, A_0 \in A_\beta\}$. Then, for $\forall U_0 \in U_\beta$ and $\forall V_0 \in V_\beta$, $\dot{U}_0 \in B_\beta U_\beta$, $\dot{V}_0 \in B_\beta V_\beta$, and

$$e^{B_\beta t} = U_\beta(t) + V_\beta(t). \tag{25}$$

It is not difficult to see that with $\alpha > 0$ chosen in (22), we can choose $K > 0$ sufficiently large and $\sigma > 0$ sufficiently small that

$$\begin{aligned}\max_{0 \leqslant \beta \leqslant 1} \left\|U_\beta(t)\right\| &\leq K e^{-(\alpha+\sigma)t} \quad t \geq 0, \\ \max_{0 \leqslant \beta \leqslant 1} \left\|V_\beta(t)\right\| &\leq K e^{\sigma t} \quad t \leq 0,\end{aligned} \tag{26}$$

Next consider the integral equation:

$$u(t,a,\beta) = U_\beta(t)a + \int_0^t U_\beta(t-s)G(u(t,a,\beta))\,ds - \int_t^\infty V_\beta(t-s)G(u(t,a,\beta))\,ds. \tag{27}$$

If $u(t,a,\beta)$ is a continuous solution of this integral equation, then it is a solution of the differential inclusion (23). We now solve this integral equation by the method of successive approximations. Let

$$u^{(0)}(t,a,\beta) = 0, \tag{28}$$

$$u^{(j+1)}(t,a,\beta) = U_\beta(t)a + \int_0^t U_\beta(t-s)G^{(j)}(u(s,a,\beta))\,ds - \int_t^\infty V_\beta(t-s)G^{(j)}(u(s,a,\beta))\,ds. \tag{29}$$

Assume that the induction hypothesis

$$\left|u^{(j)}(t,a,\beta) - u^{(j-1)}(t,a,\beta)\right| \le \frac{K|a|e^{-\alpha t}}{2^{j-1}} \tag{30}$$

holds for $j = 1,2,\dots,m$ and $t \ge 0$. It clearly holds for $j = 1$, provided $t \ge 0$. Then using the Lipschitz-type condition (20) satisfied by the function G and the above estimates on $\|U_\beta(t)\|$ and $\|V_\beta(t)\|$, it follows from the induction hypothesis that for $t \ge 0$, $0 \leqslant \beta \leqslant 1$,

$$\begin{aligned}
&\left|u^{(m+1)}(t,a,\beta) - u^{(m)(t,a,\beta)}\right| \\
&\quad\le \int_0^t \left\|U_\beta(t-s)\right\| \varepsilon \left|u^{(m)}(t,a,\beta) - u^{(m-1)}(t,a,\beta)\right| ds \\
&\qquad + \int_t^\infty \left\|V_\beta(t-s)\right\| \varepsilon \left|u^{(m)}(t,a,\beta) - u^{(m-1)}(t,a,\beta)\right| ds \\
&\quad\le \varepsilon \int_0^t K e^{-(a+\sigma)(t-s)} \frac{K|a|e^{-\alpha s}}{2^{m-1}} ds \\
&\qquad + \varepsilon \int_t^\infty K e^{\sigma(t-s)} \frac{K|a|e^{-\alpha s}}{2^{m-1}} ds \\
&\quad\le \frac{\varepsilon K^2|a|e^{-\alpha t}}{\sigma 2^{m-1}} + \frac{\varepsilon K^2|a|e^{-\alpha t}}{\sigma 2^{m-1}} \\
&\quad< \left(\frac{1}{4} + \frac{1}{4}\right)\frac{K|a|e^{-\alpha t}}{2^{m-1}} = \frac{K|a|e^{-\alpha t}}{2^m},
\end{aligned} \tag{31}$$

provided $\varepsilon K/\sigma < 1/4$, that is, provided we choose $\varepsilon < \sigma/4K$. In order that the condition (20) for the function G, it suffices to choose $K|a| < \sigma/2$.; that is, we choose $|a| < \sigma/2K$. It then follows by induction that (30) holds for all $j = 1,2,3,\dots$ and $t \ge 0$. Thus, for $n > m > N$ and $t \ge 0$,

$$\begin{aligned}
&\left|u^{(n)}(t,a,\beta) - u^{(m)}(t,a,\beta)\right| \\
&\quad\le \sum_{j=N}^{\infty} \left|u^{(j+1)}(t,a,\beta) - u^{(j)}(t,a,\beta)\right| \\
&\quad\le K|a| \sum_{j=N}^{\infty} \frac{1}{2^j} = \frac{K|a|}{2^{N-1}}.
\end{aligned} \tag{32}$$

This last quantity approaches zero as $N \to \infty$ and therefore $\{u^{(j)}(t,a,\beta)\}$ is a Cauchy sequence of continuous functions. Then we have

$$\lim_{j\to\infty} u^{(j)}(t,a,\beta) = u(t,a,\beta) \tag{33}$$

uniformly for all $t \ge 0$ and $|a| < \delta/2K$. Taking the limit of both sides of (29), it follows from the uniform convergence that the continuous function $u(t,a,\beta)$ satisfies the integral equation (27) and hence the differential equation (23). It follows by induction and the fact that $G \in C^1(\breve{E})$ that $u^{(j)}(t,a,\beta)$ is a differentiable function of a for $t \ge 0$ and $|a| < \delta/2K$. Thus, it follows from the uniform convergence that $u(t,a,\beta)$ is a differentiable function of a for $t \ge 0$ and $|a| < \delta/2K$. The estimate implies that

$$\left|u(t,a,\beta)\right| \le 2K|a|e^{-at} \tag{34}$$

for $t \le 0$ and $|a| < \delta/2K$.

It is clear from the integral equation (27) that the last $n-k$ components of vector a do not enter computation and hence they may be taken as 0. Thus, the components $u_j(t,a,\beta)$ of the solution $u(t,a,\beta)$ satisfy the initial conditions:

$$\begin{aligned}
u_j(0,a,\beta) &= a_j \quad j = 1,\dots,k, \\
u_j(0,a,\beta) &= -\int_0^\infty V_\beta(-s)G(u(s,a_1,\dots,a_k,0,\beta))\,ds \\
&\qquad j = k+1,\dots,n,
\end{aligned} \tag{35}$$

for $j = k+1,\dots,n$. We define the functions

$$\psi_j(a_1,\dots,a_k) = u_j(0,a_1,\dots,a_k,0,\dots,0,\beta). \tag{36}$$

Then the initial values $y_j = u_j(0,a_1,\dots,a_k,0,\dots,0,\beta)$ satisfy

$$y_j = \psi_j(y_1,\dots,y_k) \quad j = k+1,\dots,n. \tag{37}$$

according to the definition (36). These equations then define a differentiable manifold $\breve{S}$ for $\sqrt{y_1^2+\cdots+y_k^2} < \delta/2K$. Furthermore, if $y(t)$ is a solution of the differential inclusion (23) with $y(0) \in \breve{S}$, that is, with $y(0) = u(0,a,\beta)$, then

$$y(t) = u(t,a,\beta). \tag{38}$$

It follows that if $y(t)$ is a solution of (23) with $y(0) \in \check{S}$, then $y(t) \in \check{S}$ for all $t \geq 0$ and it follows from the estimate (34) that $y(t) \rightarrow 0$ as $t \rightarrow 0$. It can also be shown that if $y(t)$ is a solution of (23) with $y(0)$ not $\in \check{S}$, then $y(t) \nrightarrow 0$ as $t \rightarrow \infty$. From definition (36) we have

$$\frac{\partial \psi_j}{\partial y_i}(0) = 0 \tag{39}$$

for $i = 1, \ldots, k$ and $j = k+1, \ldots, n$; that is, the differentiable manifold $\check{S}$ is tangent to the stable subspace $E^S = \{y_1 = \cdots = y_k = 0\}$ of the linear system $\dot{y}_\beta \in B_\beta y_\beta$ at 0.

The existence of the unstable manifold $\check{U}$ of (23) is established in exactly the same way by considering the system (23) with $t \rightarrow -t$; that is,

$$\dot{y}_\beta \in -B_\beta y_\beta - G\left(y_\beta\right). \tag{40}$$

The stable manifold for this system will then be the unstable manifold $\check{U}$ for (23). Note that it is also necessary to replace the vector y by the vector $(y_{k+1}, \ldots, y_n, y_1, \ldots, y_k)$ in order to determine the $n-k$-dimensional manifold $\check{U}$ by the above process. This completes the proof. □

If the original point is replaced with the equilibrium point x, fussify it and the β level set of x is called *equilibrium core*, denoted by x_β. Now we prove that $\forall x^* \in x_\beta$ such as Theorem 4; that is, $F(x^*) = 0$ and $DF(x^*)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part, then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^S of the linear differential inclusions (16) at x^* such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $[x_0]_\beta \in S$,

$$\lim_{t \rightarrow \infty} \phi_t\left(x_0, \beta\right) = x^*, \tag{41}$$

then the equilibrium core x_β is stable, and there exists an $n-k$-dimensional differentiable manifold U tangent to unstable subspace E^U of (16) at x^* such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for all $x_0 \in U$,

$$\lim_{t \rightarrow -\infty} \phi_t\left(x_0, \beta\right) = x^*, \tag{42}$$

then the equilibrium core x_β is stable.

For all $x^* \in x_\beta$ corresponding to E^S,

$$\lim_{t \rightarrow \infty} \phi_t\left(x_0, \beta\right) = x^*. \tag{43}$$

That is, $\exists t_0$, if $t > t_0$,

$$\begin{gathered} \rho_*\left(x^*, \phi_t\left(x_0, \beta\right)\right) < \varepsilon, \\ \rho_D\left(x_\beta, \phi_t\left(x_0, \beta\right)\right) \leqslant \rho_*\left(x^*, \phi_t\left(x_0, \beta\right)\right) < \varepsilon. \end{gathered} \tag{44}$$

According to Property 2, set x_β to be stable. Similarly, the equilibrium core x_β corresponding to E^U is unstable and can be proved in the same way.

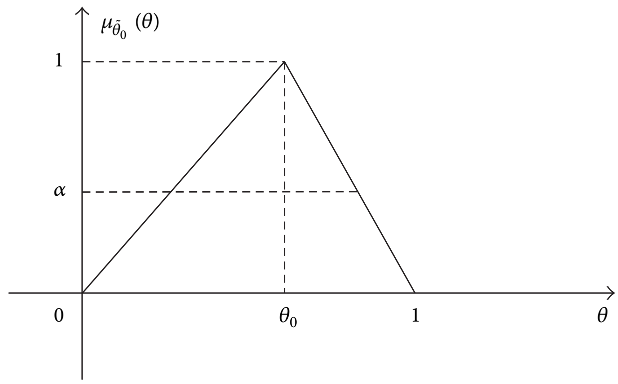

Figure 2: The definition of triangular fuzzy number.

3.2. Stability Analysis on Colonies' Members. Here we will discuss the stability of the colonies' members based on the the following different cases: (a) there is no shifting between members of different colonies; (b) one colony's members shift into another while another's members do not; (c) the two colonies' members are shifting into another, respectively.

In this section, we define θ as triangular fuzzy numbers, just as Figure 2. From Figure 2, $\theta_\alpha \in [\alpha\theta_0, 1 - ((1-\theta_0)/\theta_0)\alpha]$. And $\theta_i, i = 1, 2$ in system (6) follow this definition.

Theorem 5. *Suppose that* $[\theta_1]_\alpha = [\theta_2]_\alpha = 0$*, system* (3) *has one and only one positive equilibrium core, which is stable in* R_+^2.

Proof. If $[\theta_1]_\alpha = [\theta_2]_\alpha = 0$, system (3) has four equilibrium cores $(0, 0)$, $(M_{10}, 0)$, $(0, M_{20})$, (M_{10}, M_{20}), where $M_{10} = (\beta_{10} - \beta_{11})/(d_{10} + d_{11})$, $M_{20} = (\beta_{20} - \beta_{21})/(d_{20} + d_{21})$. It is easy to achieve that the equilibrium cores $(0, 0)$, $(M_{10}, 0)$, $(0, M_{20})$ are unstable and equilibrium core (M_{10}, M_{20}) is stable. So (M_{10}, M_{20}) is stable in R_+^2. This completes the proof. □

This indicates that if there is no shifting between the two colonies, the members of the two colonies will trend their maximum eventually and maintain stability.

Theorem 6. *Supposing* $[\theta_1]_\alpha = 0$, $[\theta_2]_\alpha > 0$*, if* $\beta_{20} - \beta_{21} - [\theta_2]_\alpha > 0$*, system* (3) *has one and only one positive equilibrium core, which is stable in* R_+^2.

Proof. If $[\theta_1]_\alpha = 0$, $[\theta_2]_\alpha > 0$, system (3) is reduced to

$$\begin{pmatrix} \dot{Q}_{1\alpha} \\ \dot{Q}_{2\alpha} \end{pmatrix} \in \begin{pmatrix} \left(\beta_{10} - \beta_{11}\right) Q_{1\alpha} - \left(d_{10} + d_{11}\right) Q_{1\alpha}^2 + [\theta_2]_\alpha Q_{2\alpha} \\ \left(\beta_{20} - \beta_{21}\right) Q_{2\alpha} - \left(d_{20} + d_{21}\right) Q_{2\alpha}^2 - [\theta_2]_\alpha Q_{2\alpha} \end{pmatrix}. \tag{45}$$

Obviously, when $\beta_{20} - \beta_{21} - [\theta_2]_\alpha > 0$, $M_{20} = (\beta_{20} - \beta_{21} - [\theta_2]_\alpha)/(d_{20} + d_{21})$ is stable for the second equation of system (45). Suppose that M_{10} is a positive solution of the following equation:

$$\left(\beta_{10} - \beta_{11}\right) Q_1 - \left(d_{10} + d_{11}\right) Q_1^2 + [\theta_2]_\alpha Q_{20} = 0 \tag{46}$$

then

$$M_{10} = \frac{\beta_{10} - \beta_{11} + \sqrt{\left(\beta_{10} - \beta_{11}\right)^2 + 4\left(d_{10} + d_{11}\right)[\theta_2]_\alpha M_{20}}}{2\left(d_{10} + d_{11}\right)} \tag{47}$$

and system (45) has unique positive equilibrium core (M_{10}, M_{20}).

Assuming $Q_{10} = Q_1 - M_{10}$, $Q_{20} = Q_2 - M_{20}$, and $\beta_i = \beta_{i0} - \beta_{i1}$, $d_i = d_{i0} + d_{i1}$, system (45) can be written as

$$\begin{pmatrix} \dot{Q}_{10\alpha} \\ \dot{Q}_{20\alpha} \end{pmatrix} \in \begin{pmatrix} \beta_1 (Q_{10\alpha}+M_{10}) - d_1 (Q_{10\alpha}+M_{10})^2 + [\theta_2]_\alpha (Q_{20\alpha}+M_{20}) \\ \beta_2 (Q_{20\alpha}+M_{20}) - d_2 (Q_{20\alpha}+M_{20})^2 - [\theta_2]_\alpha (Q_{20\alpha}+M_{20}) \end{pmatrix} \tag{48}$$

then the coefficient matrix of the linear system of system (48) is

$$\begin{bmatrix} \beta_1 - 2d_1 M_{10} & [\theta_2]_\alpha \\ 0 & \beta_2 - 2d_2 M_{20} - [\theta_2]_\alpha \end{bmatrix} = \begin{bmatrix} -\sqrt{\beta_1^2 + 4d_1 [\theta_2]_\alpha M_{20}} & [\theta_2]_\alpha \\ 0 & -\sqrt{\beta_2^2 + 4d_2 [\theta_2]_\alpha M_{20}} \end{bmatrix}. \tag{49}$$

The feature values of it are both less than 0, so the equilibrium core is stable.

Consequently, if $\beta_{20} - \beta_{21} - [\theta_2]_\alpha > 0$, the equilibrium core (M_{10}, M_{20}) is stable in R_+^2. This completes the proof. □

This indicates that if there is only one colony's member shifting into another while the other's members are not shifting, the members of the two colonies will trend their maximum eventually and maintain stability if satisfying some certain conditions.

Theorem 7. *Supposing $[\theta_1]_\alpha > 0$, $[\theta_2]_\alpha > 0$, system* (3) *has one and only one positive equilibrium core, which is stable in R_+^2.*

Proof. If $[\theta_1]_\alpha > 0$, $[\theta_2]_\alpha > 0$, we will show that system (3) has a unique positive equilibrium core. By setting the right-hand side of (3) to 0, we obtain

$$\begin{gathered} f_1 = Q_2 = \frac{(-\beta_1 + [\theta_1]_\alpha) Q_1 + d_1 Q_1^2}{[\theta_2]_\alpha}, \\ f_2 = Q_1 = \frac{(-\beta_2 + \theta_2) Q_2 + d_2 Q_2^2}{[\theta_1]_\alpha}, \\ \frac{df_1(Q_1)}{dQ_1} = \frac{(-\beta_1 + [\theta_1]_\alpha) + 2d_1 Q_1}{[\theta_2]_\alpha}, \\ \frac{df_2(Q_2)}{dQ_2} = \frac{(-\beta_2 + [\theta_2]_\alpha) + 2d_2 Q_2}{[\theta_1]_\alpha}, \\ \frac{df_1^2(Q_1)}{dQ_1^2} = \frac{2d_1}{[\theta_2]_\alpha}, \qquad \frac{df_2^2(Q_1)}{dQ_2^2} = \frac{2d_2}{[\theta_1]_\alpha}. \end{gathered} \tag{50}$$

By the equation $(-\beta_2 + [\theta_2]_\alpha) Q_2 + d_2 Q_2^2 - Q_1 [\theta_1]_\alpha = 0$, we have

$$\begin{aligned} h(Q_1) &= Q_2 \\ &= \frac{\left[(\beta_2 - [\theta_2]_\alpha) + \sqrt{(\beta_2 - [\theta_2]_\alpha)^2 + 4d_2 Q_1 [\theta_1]_\alpha}\right]}{2d_2}. \end{aligned} \tag{51}$$

It is easy to see that $h(Q_1) > 0$, $dh(Q_1)/dQ_1 > 0$, $dh^2(Q_1)/dQ_1^2 > 0$; therefore, $h(Q_1)$ is convex. Notice that $f_1(0) = 0$, $h(0) = 0$, $\lim_{Q_1 \to +\infty} [f_1(Q_1) - h(Q_1)] = +\infty$ and that f_1 is concave. The parabolic curves have one and only one positive equilibrium core $M(M_{10}, M_{20})$. Now we will prove that the equilibrium core is stable. Supposing $Q_{10} = Q_1 - M_{10}$, $Q_{20} = Q_2 - M_{20}$, and $\beta_i = \beta_{i0} - \beta_{i1} - \theta_i$, $d_i = d_{i0} + d_{i1}$, system (3) can be written as

$$\begin{pmatrix} \dot{Q}_{10\alpha} \\ \dot{Q}_{20\alpha} \end{pmatrix} \in \begin{pmatrix} \beta_1 (Q_{10\alpha}+M_{10}) - d_1 (Q_{10\alpha}+M_{10})^2 + [\theta_2]_\alpha (Q_{20\alpha}+M_{20}) \\ \beta_2 (Q_{20\alpha}+M_{20}) - d_2 (Q_{20\alpha}+M_{20})^2 + [\theta_1]_\alpha (Q_{10\alpha}+M_{10}) \end{pmatrix}. \tag{52}$$

Then the coefficient matrix of the linear system of system (52) is

$$\begin{bmatrix} \beta_1 - 2d_1 M_{10} & [\theta_2]_\alpha \\ [\theta_1]_\alpha & \beta_2 - 2d_2 M_{20} \end{bmatrix}. \tag{53}$$

Now let us check $\operatorname{Tr} A = \beta_1 - 2d_1 M_{10} + \beta_2 - 2d_2 M_{20} < 0$, $\det A = (\beta_1 - 2d_1 M_{10})(\beta_2 - 2d_2 M_{20}) - [\theta_1]_\alpha [\theta_2]_\alpha > 0$ and $(\operatorname{Tr} A)^2 - t \det A > 0$,

$$\begin{aligned} M_{10} M_{20} &\times \operatorname{Tr} A \\ &= (\beta_1 - 2d_1 M_{10} + \beta_2 - 2d_2 M_{20}) M_{10} M_{20} \\ &= M_{20} \left(\beta_1 M - 2d_1 M_{10}^2\right) + \left(\beta_2 M_{20} - 2d_2 M_{20}^2\right) M_{20} \end{aligned} \tag{54}$$

and since $M(M_{10}, M_{20})$ is the equilibrium core of system (3), M_{10}, M_{20} will satisfy the following equations, respectively:

$$\begin{gathered} \beta_1 Q_1 - d_1 Q_1^2 + [\theta_2]_\alpha Q_2 = 0, \\ \beta_2 Q_2 - d_2 Q_2^2 + [\theta_1]_\alpha Q_1 = 0. \end{gathered} \tag{55}$$

Consequently, $M_{10} M_{20} \times \operatorname{Tr} A = -([\theta_1]_\alpha M_{10}^2 + [\theta_2]_\alpha M_{20}^2)$, and then $\operatorname{Tr} A < 0$.

From (55), we can learn that

$$\begin{gathered} [\theta_1]_\alpha = -\frac{\left(\beta_2 M_{20} - d_2 M_{20}^2\right)}{M_{10}}, \\ [\theta_2]_\alpha = -\frac{\left(\beta_1 M_{10} - d_1 M_{10}^2\right)}{M_{20}}. \end{gathered} \tag{56}$$

Therefore,

$$\begin{aligned}\det A &= (\beta_1 - 2d_1M_{10})(\beta_2 - 2d_2M_{20}) - [\theta_1]_\alpha[\theta_2]_\alpha \\ &= (\beta_1 - 2d_1M_{10})(\beta_2 - 2d_2M_{20}) \\ &\quad - (\beta_1 - d_1M_{10})(\beta_2 - d_2M_{20}).\end{aligned} \tag{57}$$

From (56) we can conclude that $\beta_i - d_iM_{i0} < 0$ and $\det A > 0$. As far as $(\mathrm{Tr}\,A)^2 - 4$, $\det A > 0$ clearly comes into existence. Therefore, the equilibrium core $M(M_{10}, M_{20})$ is stable. Consequently, the equilibrium core $M(M_{10}, M_{20})$ is stable in R_+^2. This completes the proof. □

Accordingly, the members of the two colonies will trend their maximum eventually and maintain stability if satisfying certain conditions whether there are members shifting from one colony into another or not.

3.3. Stability Analysis on Innovation Diffusion. Since the equilibrium core in system (3) is stable and we are interested in the asymptotic behavior of system (6), we only discuss the stability of system (6) on the base of system (3) in it's equilibrium core. That is to say, we only discuss the stability of innovation diffusion when the members of the two colonies are in their stability. Suppose that the global equilibrium core of system (3) is (M_{10}, M_{20}), and $Q_{10} = M_{10}$, $Q_{20} = M_{20}$. At the same time suppose $\beta_1 = a_1 - b_1M_{10} + d_{11}M_{10} + \beta_{11} + e_1$, $\beta_2 = a_2 - b_2M_{20} + d_{21}M_{20} + \beta_{21} + e_2$, then we have the following theorem.

Theorem 8. *Systems*

$$\begin{pmatrix}\dot N_{1\alpha}\\ \dot N_{2\alpha}\end{pmatrix} \in \begin{pmatrix} a_1M_{10} - (\beta_1 + [\theta_1]_\alpha)N_{1\alpha} - b_1N_{1\alpha}^2 + k_2[\theta_2]_\alpha N_{2\alpha} \\ a_2M_{20} - (\beta_2 + [\theta_2]_\alpha)N_{2\alpha} - b_2N_{2\alpha}^2 + k_1[\theta_1]_\alpha N_{1\alpha}\end{pmatrix} \tag{58}$$

have one and only one positive equilibrium core, which is stable in R_+^2.

Proof. If $[\theta_1]_\alpha = [\theta_2]_\alpha = 0$, system (58) has a unique positive equilibrium core (C_1, C_2), where $C_i = (-\beta_i + \sqrt{\beta_i^2 + 4d_ia_iM_{i0}})/2d_i$. Obviously, this equilibrium core is stable in R_+^2; that is to say, if there is no shifting between two colonies, this innovation will occupy its potential eventually and remain stable.

If $[\theta_1]_\alpha = 0$, $[\theta_2]_\alpha > 0$, system (58) can be reduced as

$$\begin{pmatrix}\dot N_{1\alpha}\\ \dot N_{2\alpha}\end{pmatrix} \in \begin{pmatrix} a_1M_{10} - \beta_1N_{1\alpha} - b_1N_{1\alpha}^2 + k_2[\theta_2]_\alpha N_{2\alpha} \\ a_2M_{20} - (\beta_2 + [\theta_2]_\alpha)N_2 - b_2N_{2\alpha}^2\end{pmatrix}. \tag{59}$$

Take the second equation of system (6) into consideration. C_2 is the unique globe equilibrium core of this equation in R_+^2, where

$$C_2 = \frac{-(\beta_2 + [\theta_2]_\alpha) + \sqrt{(\beta_2 + [\theta_2]_\alpha)^2 + 4b_2a_2M_{20}}}{2b_2}. \tag{60}$$

Assume that C_1 is the positive solution of the following equation:

$$a_1M_{10} - \beta_1N_1 - b_1N_1^2 + k_2[\theta_2]_\alpha C_2 = 0, \tag{61}$$

then follow

$$C_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4b_1(a_1M_{10} + k_2[\theta_2]_\alpha C_2)}}{2b_1}. \tag{62}$$

Define $N_{i0} = N_i - C_i$, $(i = 1, 2)$, then system (59) can be rewritten as

$$\begin{pmatrix}\dot N_{10\alpha}\\ \dot N_{10\alpha}\end{pmatrix} \in \begin{pmatrix} a_1M_{10} - \beta_1(N_{10\alpha} + C_1) - b_1(N_{10\alpha} + C_1)^2 + k_2[\theta_2]_\alpha(N_{20\alpha} + C_2) \\ a_2M_{20} - (\beta_2 + [\theta_2]_\alpha)(N_{20\alpha} + C_2) - b_2(N_{20\alpha} + C_2)^2\end{pmatrix}. \tag{63}$$

The coefficient matrix of the linear system of system (63) is

$$A = \begin{bmatrix} -(\beta_1 + 2b_1(N_{10} + C_1)) & k_2\theta_2 \\ 0 & -(\beta_2 + \theta_2 + 2b_2(N_{20} + C_2))\end{bmatrix}. \tag{64}$$

Since the feature values of A are both less than 0, the equilibrium core (C_1, C_2) is stable. So this equilibrium core is stable in R_+^2.

If $[\theta_1]_\alpha > 0$, $[\theta_2]_\alpha > 0$, the demonstration process is similar to the process of Theorem 4. We can conclude that there exits a unique positive equilibrium core (C_1, C_2) and the equilibrium core is stable. Consequently, the innovation will reach its maximal potential and keep stable. This completes the proof. □

4. The Empirical Analysis

Owing to the household management system of planned economy in history, China has divided the population into urban population and rural population. Owing to the strict control, rare shifting, huge gaps between industry and agriculture, and the different environment and living level, there exists great difference between urban population and rural population in communication channels, information spreading, and consumption habit. However, with the reform and exposure to the outside world, the economic development converts China into urban society from village society, and the inhabitants begin to move. According to the related news, there is 1% rural population moving into town per year since the 1980s. The ratio between urban population and rural population may be 7 : 3 after 60 years. Thus, it is not enough to discuss their consumption systems, respectively, without taking the population moving between the cities and countries into consideration. Based on this, we will try to build the dynamical model for telephone diffusion under the condition of the existing large population moving between town and country.

To make it easier, we suppose that this model satisfies following hypotheses:

(1) since the standard of living in town is superior to that in the country of China, we suppose $\theta_2 = 0$, which indicates that there are no people in China shifting from town into country;

(2) since the communication by telephone is popular and necessary, we suppose $k_1 = 1$, $e_i = 0$ which indicates that the telephone users in country will continue using it after shifting into town and users in town will not give up using it;

(3) the population in China will reach 1.6 billion and maintain stable, which is obtained according to some related research on population in China [28].

4.1. Model Conversion. In this section, we first use the definition of subduction defined by Zadeh, which means $u - v = (u_1 - v_2, u_2 - v_1)$, in which $u = (u_1, u_2)$, $v = (v_1, v_2) \in D^n$. And then we use the H-difference to simulate the diffusion process of local telephone in China.

According to the hypothesis defined above, system (58) can be rewritten as

$$\begin{aligned} \dot{\widetilde{N}}_1 &= a_1\widetilde{A}_1 + b_1\widetilde{N}_1 - c_1\widetilde{N}_1^2 + d_1\widetilde{N}_2, \\ \dot{\widetilde{N}}_2 &= a_2\widetilde{A}_2 + b_2\widetilde{N}_2 - c_2\widetilde{N}_2^2. \end{aligned} \tag{65}$$

According to the definition of subduction defined by article [29], system (65) can be rewritten as

$$\begin{aligned} \underline{N}_1(t+1) - \underline{N}_1(t) &= a_1\underline{A}_1 + b_1\underline{N}_1(t) - c_1\overline{N}_1^2(t) + \underline{d}_1\underline{N}_2(t), \\ \overline{N}_1(t+1) - \overline{N}_1(t) &= a_1\overline{A}_1 + b_1\overline{N}_1(t) - c_1\underline{N}_1^2(t) + \overline{d}_1\overline{N}_2(t), \\ \underline{N}_2(t+1) - \underline{N}_2(t) &= a_2\underline{A}_2 + b_2\underline{N}_2(t) - c_2\overline{N}_2^2(t), \\ \overline{N}_2(t+1) - \overline{N}_2(t) &= a_2\overline{A}_2 + b_2\overline{N}_2(t) - c_2\underline{N}_2^2(t). \end{aligned} \tag{66}$$

That is,

$$\begin{aligned} \underline{N}_1(t+1) &= a_1\underline{A}_1 + (1+b_1)\underline{N}_1(t) - c_1\overline{N}_1^2(t) + \underline{d}_1\underline{N}_2(t), \\ \overline{N}_1(t+1) &= a_1\overline{A}_1 + (1+b_1)\overline{N}_1(t) - c_1\underline{N}_1^2(t) + \overline{d}_1\overline{N}_2(t), \\ \underline{N}_2(t+1) &= a_2\underline{A}_2 + (1+b_2)\underline{N}_2(t) - c_2\overline{N}_2^2(t), \\ \overline{N}_2(t+1) &= a_2\overline{A}_2 + (1+b_2)\overline{N}_2(t) - c_2\underline{N}_2^2(t). \end{aligned} \tag{67}$$

Table 2: The data of telephone subscribers in the year 1989–2011.

Year	Number of telephone subscribers in town	Number of telephone subscribers in country
1989	439.62	128.42
1990	538.45	146.58
1991	670.83	174.23
1992	920.57	226.34
1993	1407.37	325.79
1994	2246.78	482.75
1995	3263.56	807
1996	4277.82	1216.92
1997	5244.4	1786.63
1998	6259.81	2482.28
1999	7463.3	3408.4
2000	9311.6	5171.3
2001	11193.7	6843.1
2002	13579.1	7843.1
2003	17109.7	9165.0
2004	21025.1	10150.5
2005	23975.3	11069.2
2006	25132.9	11645.6
2007	24859.4	11685.5
2008	23151.5	10827.4
2009	20742.8	10432.8
2010	19671.0	9769.5
2011	18993.3	9477.3

Then we used Genetic Algorithms (GA) to model the coefficient of model (67), which were shown in Table 4, where error is the permitted maximum error ratio between the data predicted by model and history data.

If we take the coefficient value as shown in Table 2 to the minimum of the target values in Table 3, the plots of potential curve of telephone in urban area or in rural area governed by model (67) are shown in Figure 3 or Figure 4, respectively.

In Figure 3, the red line is made up of history data and the green solid line and the green dot line represent the upper limit and the lower limit of the telephone consumers in urban area, respectively.

In Figure 4, the red line is made up of history data and the blue solid line and the blue dot line represent the upper limit and the lower limit of the telephone consumers in rural area, respectively.

From Figures 3 and 4, the fuzzy simulation effect is not perfect, and the fuzzy intervals of the consumer numbers are too big to predict the diffusion process of innovation. We could only find the time of inflection point and the maximal potential market from Figures 3 and 4. So we try to use the definition of Hukuhara difference, which, $u - v = (\underline{u}(\alpha) - \underline{v}(\alpha), \overline{u}(\alpha) - \overline{v}(\alpha))$, comes into exist. Consequently, we have the following system:

Table 3: The error control target value of history data.

Error	Target value	a_1	b_1	c_1	$\underline{d}_1$	$\overline{d}_1$	a_2	b_2	c_2
40%	61768.5455	0.00096375	0.24507401	0.00469948	0.01684506	0.02466292	0.00001841	0.33879513	0.00426482
42%/45%	50022.8637	0.00095945	0.23580981	0.00458211	0.01059862	0.01564592	0.00010486	0.32689596	0.00722705
48%	53540.5485	0.00102646	0.26399854	0.00536816	0.00984631	0.01055605	0.00005475	0.34041871	0.00568559
50%	52452.7799	0.00110203	0.24089633	0.00428221	0.01080374	0.01947233	0.00015806	0.33330485	0.00639973
55%/60%	56275.6173	0.00138509	0.22832759	0.00437598	0.01320603	0.02571032	0.00010449	0.33776971	0.00441932
70%	67415.3809	0.00152391	0.2427015	0.00452018	0.01237617	0.01860317	0.00038169	0.30762658	0.01860317
80%	62990.3848	0.00182419	0.22243202	0.00546679	0.0168813	0.02976928	0.00026592	0.32409742	0.00742617
82%/100%	44348.1732	0.00098436	0.25674215	0.00472934	0.01607596	0.02030183	0.00065298	0.29850459	0.0078024
150%	68339.4199	0.00192974	0.22880764	0.00431324	0.01132408	0.01233833	0.00089076	0.28284555	0.00641127

Table 4: The error control target value of history data.

Error	F	a_1	b_1	$c_1 \times 10^3$	$\underline{d}_1$	$\widehat{d}_1$	a_2	b_2	$c_2 \times 10^3$
100%	43937.44	0.001795	0.226062	$5.54699E-06$	0.01164	0.027958	0.0003922	0.388549	$3.24809E-05$
80%	56921.23	0.001848	0.227387	$4.87714E-06$	0.018258	0.024537	0.00011641	0.416663	$4.76498E-05$
60%	63795.76	0.001114	0.252945	$4.65459E-06$	0.015721	0.023128	0.0001476	0.373296	$4.13123E-05$
55%	51876.72	0.001372	0.240317	$4.44283E-06$	0.018897	0.023776	0.00003187	0.39436	$3.84283E-05$
50%	54156.28	0.00115	0.268982	$5.77116E-06$	0.010228	0.029075	0.00005979	0.363491	$2.26276E-05$
45%	53254.21	0.000904	0.271314	$4.96505E-06$	0.014153	0.024267	0.00003765	0.357402	$2.08844E-05$

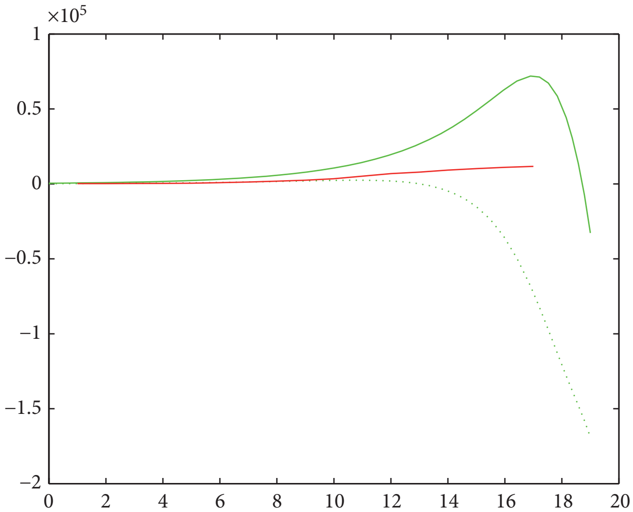

Figure 3: Plots of potential curve of telephone in urban area governed by model (67).

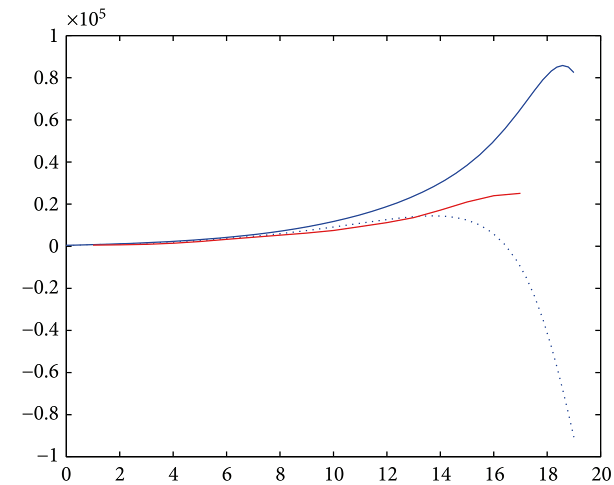

Figure 4: Plots of potential curve of telephone in rural area governed by model (67).

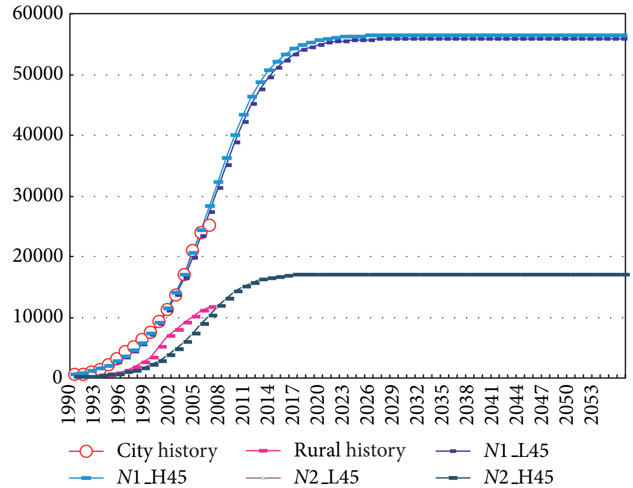

Figure 5: Plots of potential curve of telephone in urban area and rural area governed by model (68) (the control error rate is 45%).

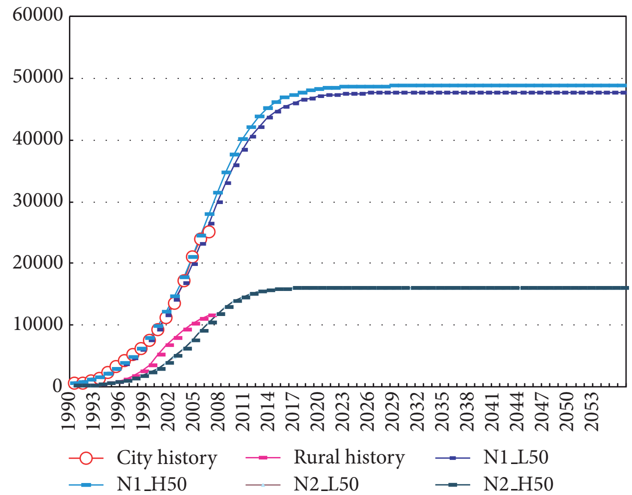

Figure 6: Plots of potential curve of telephone in urban area and rural area governed by model (68) (the control error rate is 50%).

$$\begin{aligned}
\underline{N}_1(t+1) &= a_1\underline{A}_1 + (1+b_1)\underline{N}_1(t) - c_1\underline{N}_1^2(t) + \underline{d}_1\underline{N}_2(t),\\
\overline{N}_1(t+1) &= a_1\overline{A}_1 + (1+b_1)\overline{N}_1(t) - c_1\overline{N}_1^2(t) + \overline{d}_1\overline{N}_2(t),\\
\underline{N}_2(t+1) &= a_2\underline{A}_2 + (1+b_2)\underline{N}_2(t) - c_2\underline{N}_2^2(t),\\
\overline{N}_2(t+1) &= a_2\overline{A}_2 + (1+b_2)\overline{N}_2(t) - c_2\overline{N}_2^2(t).
\end{aligned} \tag{68}$$

We give the predicted figures in Figures 5, 6, and 7, respectively, in which the error control ratios are 45%, 50%, and 100%, respectively.

Since the the effect of simulation under other control simulation error ratio is not good, we only give the potential market of local telephone under the control simulation error ratio which is 45%, 50%, and 100%, respectively. From Table 5, the number of local telephone consumers in China ranges from 54203.57 to 73615.08 in respect to different control error ratios.

4.2. Contrastive Analysis. In this section, firstly we consider the cases of system (58) without any fuzzy coefficients, then take the Bass model to simulate the process of local telephone diffusion in China.

We also suppose that this model satisfies the following hypotheses:

(1) since the standard of living in town is superior to that of country areas in China, we suppose $\theta_2 = 0$, $\theta_1 = 1\%$, which means that the population shifting from the country into the town is 1% of rural area inhabitants at that time while there are no people in China shifting from the town into the country;

Table 5: The predicted consumer numbers of China local telephone under different control simulation errors.

Error	The minimum value of urban area	The maximum value of urban area	The minimum value of rural area	The maximum value of rural area
100%	42194.89787	43026.2	12008.67	12012.68
50%	47665.11832	48739.22	16071.58	16072.23
45%	55876.63922	56496.48	17118.18	17118.6

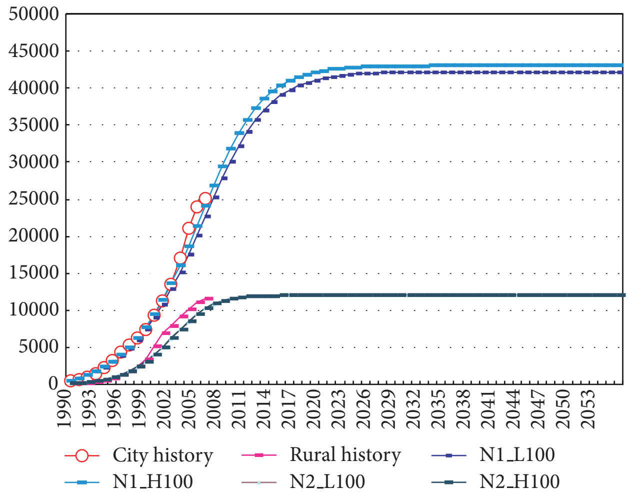

Figure 7: Plots of potential curve of telephone in urban area and rural area governed by model (68) (the control error rate is 100%).

(2) since communication by telephone is popular and necessary, we suppose $k_1 = 1$, which indicates that the telephone users in the country will continue using it after shifting into town and users in town will not give up using it;

(3) the population in China will reach 1.6 billion and maintain stability, according to data obtained from some related research on population in China.

Based on the hypotheses above, we can conclude that the population of the country will be stable in 0.48 billion and that of the town will be 1.12 billion, which is concluded by the nonlinear regression on system (3) without any fuzzy coefficients. According to the 14 groups of historical data, and the software *SPSS11.0*, we also make nonlinear regression on system (58) and obtain the following model:

$$\dot{N}_1 = 88.8608 + 0.21945N_1 - 5.9986 \times 10^{-6}N_1^2 + 0.01N_2,$$
$$\dot{N}_2 = 0.4800 + 0.39966N_2 - 3.0032 \times 10^{-5}N_2^2. \tag{69}$$

In which the coefficient of external influence in town $a_1 = 0.0007934$, the coefficient of the internal influence $b_1 = 0.22536$, and those in the country, respectively, are 0.00001 and 0.399696. With the calculation on the system (59), the market potential of telephone users in the town is 375684000 and that in country is 133090000. The R squared of the first equation of (69) is 0.95748 and that of the second is 0.91051.

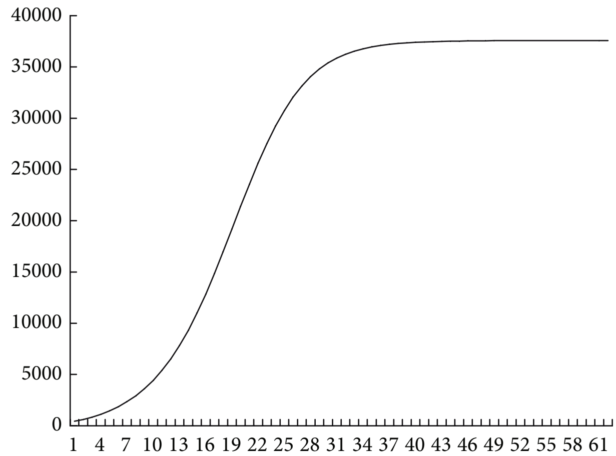

Figure 8: Plot of the diffusion of telephone users in urban area of China governed by model (69).

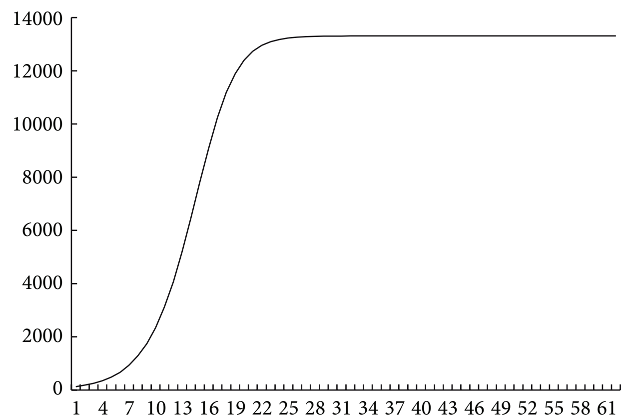

Figure 9: Plot of the diffusion of telephone users in rural area of China governed by model (69).

We use the software Mathematic 4 to forecast the numbers of telephone users in the town and the country, which are shown in Figures 8 and 9, respectively. And the origin point in these figures denotes the year 1989.

Consequently, we arrive at the following conclusions.

(1) When the population of the town and that of the country trend to stability, the users among them also trend to stability, the number of which is 13309000 and 375684000, respectively.

(2) Since the medium system in town is highly superior to that in the rural areas, the necessity and popularity of

communication in town are also better than those in the rural area and the coefficient of external affection in town (0.0007934) is obviously larger than that in rural area (0.00001).

(3) The coefficient of internal affection in town (0.22536) is much less than that in the rural area (0.3399696), which means that the inhabitants of the rural area are more reliable on oral communication than those in town. The main reason is that there are few communication channels in the rural area and the communication information mainly depends on oral communication. However, the network of communication in town is much developed and the necessity of oral communication is not so necessary.

(4) The population of telephone users in town will reach its maximal market potential in about 30 years, while in rural area it will be in about 15 years later. In fact, this dose not mean that the diffusion rate in rural area is faster than that in town. In fact, the diffusion rate in town is much faster than that in rural area. The main reason for this is that the market potential in town (375684000) is much higher than that in rural area (133090000).

We also use Bass model to describe and forecast the numbers of the telephone users in the town or in rural areas, respectively, without taking the people shifting between them into account. With the nonlinear regression, another forecasting model is governed by

$$\begin{aligned}\dot{N}_1 &= \left(0.000313699 + 3.61914 \times 10^{-6} N_1\right) \\ &\quad \times \left(64443.178967 - N_1\right), \\ \dot{N}_2 &= 3.1750 \times 10^{-5} N_2 \left(13446.495087 - N_2\right)\end{aligned} \tag{70}$$

in which the coefficient of external affection in town $a_1 = 0.000313699$, the coefficient of the internal affection $b_1 = 0.233073$, and those in rural area, respectively, are 0 and 0.42672. The market potential of telephone users in town is 644431789 and that in rural area is 134464950. The R squared of the first equation of (70) is 0.94766 and that of the second is 0.9047.

Furthermore, we also use the Bass model to describe and forecast the number of the telephone users without considering where the users are from. The model is as follows:

$$\begin{aligned}\dot{N} &= \left(0.000412046 + 3.70996 \times 10^{-6} N\right) \\ &\quad \times \left(71216.171268 - N\right).\end{aligned} \tag{71}$$

The market potential of telephone users in China is 712161700 and the prediction curve of this model is shown in Figure 10. The origin points in these figures denote the year 1989.

Comparing model (71) with model (70) we can see that if we distinguish where people settle down, the total market potential of telephone users in China will be 778896700 or, if not considering the place of setting, it will be 712161700. That is to say, there will be another 66 million in the market potential of China if distinguishing where people settle down.

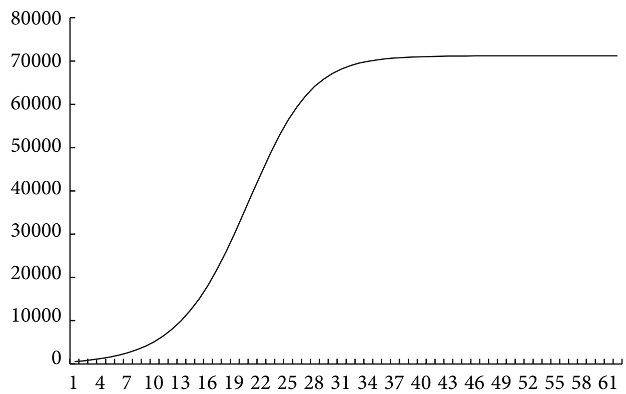

Figure 10: Plot of the diffusion curve of telephone in China governed by model (71).

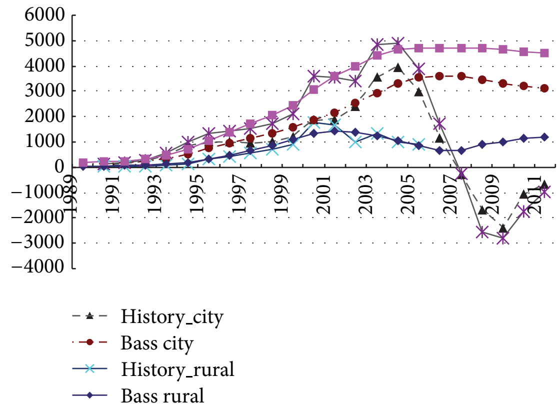

Figure 11: Plots of the diffusion rate of telephone by Bass model.

In order to distinguish which prediction model is more accurate, we have drawn the plots of the prediction curves of different models and compared them with those of the historical data, which are shown in Figures 11, 12, and 13, respectively. y_u, y_c, y_a denote the diffusion rate of telephones in town, in the rural area, or the total in China, respectively, and p_yu, p_yc, y_ya denote the prediction numbers of those, respectively, by Bass model. p_c_me, p_u_me denote the prediction numbers of the diffusion rate of telephones in town or in the rural area by model (69). From Figure 11, there is no difference in accuracy between model (70) and model (71). However, the prediction numbers of model (69) are more accurate than those of model (70), as deduced by Figures 12 and 13.

We also drew the prediction curves of telephone users in town and in rural areas governed by model (69) or model (70), which are shown in Figures 15 and 14, respectively. The curves are almost identical in Figure 15, which means that the potential of telephone usage in rural areas is virtually the same whether governed by model (69) or model (70) at any given point time t, while there is a wide difference between those in town, as shown in Figure 14. With the course of China's stepping into an urban society from a rural society,

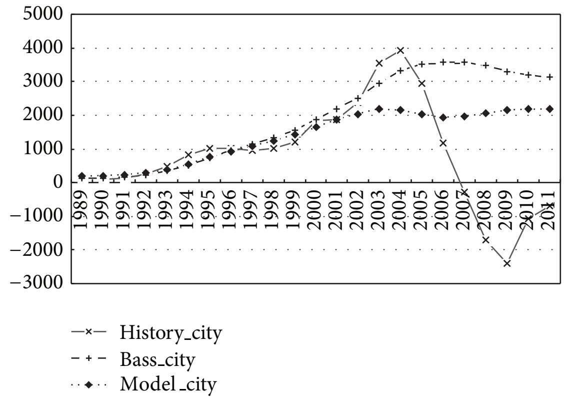

Figure 12: Plots of the diffusion rate of telephone in urban area by model (69) or model (70).

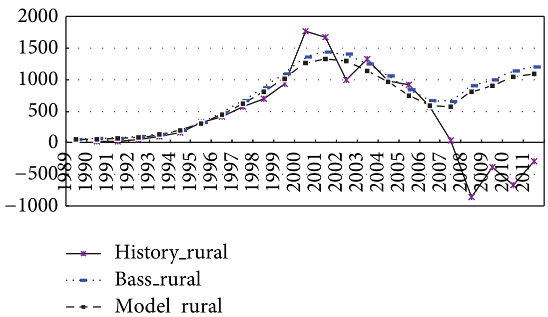

Figure 13: Plots of the diffusion rate of telephone in rural area by model (69) or model (70).

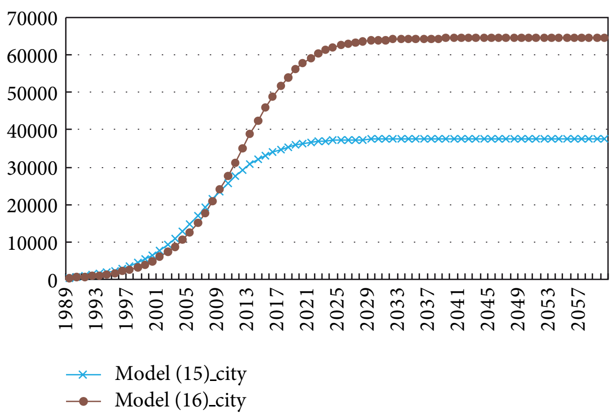

Figure 14: Plots of potential curve of telephone in urban area governed by model (69) or model (70).

more people are moving into town from rural areas and have a stronger ability and demand for communication. So the potential predicted by model (69) should be bigger than that predicted by model (70). However, on the contrary, the potential predicted by model (69) is much smaller than that predicted by model (70) as shown in Figure 14.

The cause may be that the Bass model uses historical data to predict the market potential, and the historical data of

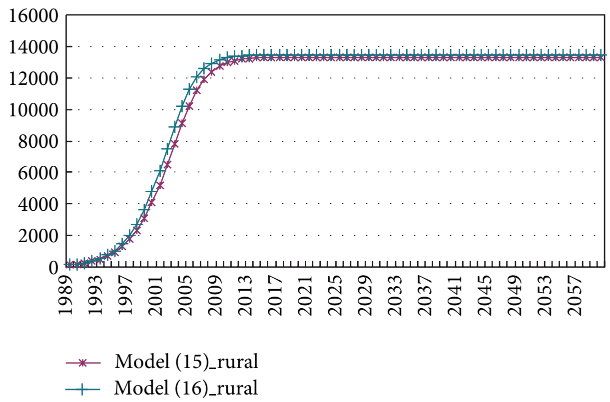

Figure 15: Plots of potential curve of telephone in rural area governed by model (69) or model (70).

towns have involved those who have shifted from rural areas. So the Bass model magnifies the potential of innovation in town.

5. Conclusion

In this paper, we constructed an innovation diffusion model with migration between two different colonies. With the analysis of the innovation diffusion model based on the members of the two colonies shifting between each other, and on the empirical analysis on the telephone users of China, we conclude that the subscribers will trend to their maximal market potential whether or not there are members shifting between town and rural areas. During the process of urbanization and the relaxation of household management in China, such phenomena should not be ignored since its influence on the economic and social system may be vital and significant. Since obtaining data is very difficult, the population data is directly obtained from related information and empirical estimation. Moreover, the proposed model is acceptable without competing technologies taken into consideration. The simulation of data is not accurate enough since we have not taken the economic environment, competing technologies, and the effect of decreasing costs in communication into consideration, all of which will be included in future research.

Acknowledgments

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Research Article

The M/M/N Repairable Queueing System with Variable Breakdown Rates

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This paper considers the M/M/N repairable queuing system. The customers' arrival is a Poisson process. The servers are subject to breakdown according to Poisson processes with different rates in idle time and busy time, respectively. The breakdown servers are repaired by repairmen, and the repair time is an exponential distribution. Using probability generating function and transform method, we obtain the steady-state probabilities of the system states, the steady-state availability of the servers, and the mean queueing length of the model.

1. Introduction

In queuing researches, many researchers have studied the queuing system with repairable servers. Most of the works of the repairable queuing system deal with the single-server models [1–9]. The works about multiserver repairable systems is not sufficient. Mitrany and Avi-Itzhak [10] analyzed the model with N units of servers and the same amount of repairmen, they obtained the steady-state mean queuing length of customers. Neuts and Lucantoni [11] studied the model with N units of repairable servers and c ($0 \le c \le N$) repairmen by matrix analysis method and obtained the steady-state properties of the model.

In recent years, many flexible policies have been introduced to the repairable systems. Gray et al. [5] studied the model with a single server which may take a vacation in idle times and may breakdown in busy times; they obtained the mean queue length. Altman and Yechiali [12] presented a comprehensive analysis of the M/M/1 and M/G/1 queues, as well as of the M/M/c queue with server vacations; they obtained various closed-form results for the probability generating function (PGF) of the number of the customers. Zhang and Hou [6] analyzed an M/G/1 queue with working vacations and vacation interruptions; they obtained the queue length distribution and steady-state service status probability. Yang et al. [7] analyzed an M/G/1 queuing system with second optional service, server breakdowns, and general startup times under (N, p)-policy, they obtained the explicit closed-form expression of the joint optimum threshold values of (N, p) at the minimum cost. Chang et al. [8] studied the optimal management problem of a finite capacity M/H2/1 queuing system, where the unreliable server operates F-policy, a cost model is developed to determine the optimal capacity K, the optimal threshold F, the optimal setup rate, and the optimal repair rate at a minimum cost. Wang [9] used a quasi-birth-and-death (QBD) modeling approach to model queuing-inventory systems with a single removable server, performance measures are obtained by using both hybrid and standard procedures; an optimal control policy is proposed and verified through numerical studies.

The most works of repairable queuing system assumed that the server breakdown rate is constant, but the breakdown rate of a server may be variable in a real system. It is well known that many kinds of machine are easy to breakdown at their busy times, and some equipments may be easy to fail after a long idle period. For example, the tires of the truck prefer to breakdown when the truck is running on the road. On the other hand, the storage battery in an automobile may not work if the automobile is idle for long period. For the actual demands of the above cases, we study a multiserver repairable queuing system in this paper, and assume that the unreliable servers have different breakdown rates in their busy times and idle times, respectively.

The rest of this paper is organized as follows. Section 2 describes the model and gives the balance equations. Section 3 presents the equations of PGF. The steady-state availability of the system is derived in Section 4. The steady-state probabilities of the system states and mean queuing length are obtained in Section 5. Case analysis is given in Section 6. Section 7 presents the conclusions.

2. Model Description

The model characteristics are as follows.

(1) There are N units of identical servers in the system. The servers are subject to breakdown according to Poisson processes with different rates which are ξ_1 in idle times and ξ_2 in busy times, respectively.

(2) Customers arrive according to a Poisson process with rate λ. The service discipline is first come first served (FCFS). The service which is interrupted by a server breakdown will become the first one of the queue of customers. The service time distribution is an exponential distribution with parameter μ.

(3) There are c $(1 \leq c \leq N)$ reliable repairmen to maintain the unreliable servers. The repair discipline is first come first repaired (FCFR). The repair time distribution is an exponential distributions with parameter η. A server is as good as a new one after repair.

We define

$X(t) \equiv$ the number of available servers in the system at the moment t $(0 \leqslant X(t) \leqslant N)$,

$Y(t) \equiv$ the number of customers in the system at the moment t $(0 \leqslant Y(t))$.

The stochastic process $\{X(t), Y(t), t \geq 0\}$ is a two-dimensional Markov process which is called quasi-birth-and-death (QBD) process [11] with state space $\{(i, j),\ 0 \leq i \leq N,\ j \geq 0\}$.

Let $P_{i,j}(t)$ denote the probability that the system is in a state of (i, j) at the moment t, and $P_{i,j}$ denote the steady-state probability of $P_{i,j}(t)$, then we have

$$P_{i,j} = \begin{cases} \lim\limits_{t\to\infty} P_{i,j}(t), & i = 0, 1, 2, \ldots, N,\quad j = 0, 1, 2, \ldots, \\ 0, & \text{other}. \end{cases} \tag{1}$$

Assuming that the system is positive recurrent, the balance equations are as follows:

$$\begin{aligned}
&(\lambda + c\eta) P_{0,0} = \xi_1 P_{1,0},\\
&(\lambda + c\eta) P_{0,j} = \lambda P_{0,j-1} + \xi_2 P_{1,j}, \quad j > 0,\\
&(\lambda + c\eta + i\xi_1) P_{i,0} = c\eta P_{i-1,0} + \mu P_{i,1} + (i+1)\xi_1 P_{i+1,0},\\
&\qquad 0 < i \leq N - c,\ \ j = 0,\\
&P_{i,j}\left[\lambda + c\eta + j\mu + (i-j)\xi_1 + j\xi_2\right] = \lambda P_{i,j-1} + c\eta P_{i-1,j}\\
&\qquad + (j+1)\mu P_{i,j+1} + \left[(i+1-j)\xi_1 + j\xi_2\right] P_{i+1,j},\\
&\qquad 0 < i \leq N - c,\ 0 < j < i,\\
&P_{i,j}\left[\lambda + c\eta + i\mu + i\xi_2\right]\\
&\quad = \lambda P_{i,j-1} + c\eta P_{i-1,j} + i\mu P_{i,j+1} + (\xi_1 + j\xi_2) P_{i+1,j},\\
&\qquad 0 < i \leq N - c,\ \ j = i,\\
&P_{i,j}\left[\lambda + c\eta + i\mu + i\xi_2\right]\\
&\quad = \lambda P_{i,j-1} + c\eta P_{i-1,j} + i\mu P_{i,j+1} + (i+1)\xi_2 P_{i+1,j},\\
&\qquad 0 < i \leq N - c,\ \ j > i,\\
&P_{i,0}\left[\lambda + (N-i)\eta + i\xi_1\right]\\
&\quad = (N-i+1)\eta P_{i-1,0} + \mu P_{i,1} + (i+1)\xi_1 P_{i+1,0},\\
&\qquad N - c < i < N,\ \ j = 0,\\
&P_{i,j}\left[\lambda + (N-i)\eta + j\mu + (i-j)\xi_1 + j\xi_2\right]\\
&\quad = (N-i+1)\eta P_{i-1,j} + \lambda P_{i,j-1} + (j+1)\mu P_{i,j+1}\\
&\qquad + \left[(i+1-j)\xi_1 + j\xi_2\right] P_{i+1,j},\\
&\qquad N - c < i < N,\ 0 < j < i,\\
&P_{i,j}\left[\lambda + (N-i)\eta + i\mu + i\xi_2\right]\\
&\quad = \lambda P_{i,j-1} + (N-i+1)\eta P_{i-1,j} + i\mu P_{i,j+1}\\
&\qquad + (\xi_1 + j\xi_2) P_{i+1,j}, \quad N - c < i < N,\ \ j = i,\\
&P_{i,j}\left[\lambda + (N-i)\eta + i\mu + i\xi_2\right]\\
&\quad = \lambda P_{i,j-1} + (N-i+1)\eta P_{i-1,j} + i\mu P_{i,j+1}\\
&\qquad + (i+1)\xi_2 P_{i+1,j}, \quad N - c < i < N,\ \ j > i,\\
&P_{N,0}(\lambda + N\xi_1) = \eta P_{N-1,0} + \mu P_{N,1}, \quad i = N,\ \ j = 0,\\
&P_{N,j}\left[\lambda + j\mu + (N-j)\xi_1 + j\xi_2\right]\\
&\quad = \lambda P_{N,j-1} + \eta P_{N-1,j} + (j+1)\mu P_{N,j+1},\\
&\qquad i = N,\ 0 < j < N,\\
&P_{N,j}(\lambda + N\mu + N\xi_2) = \lambda P_{N,j-1} + \eta P_{N-1,j} + N\mu P_{N,j+1},\\
&\qquad i = N,\ \ j \geq N.
\end{aligned} \tag{2}$$

Here, we give the derivation of the second equation of (2). Since the process $\{X(t), Y(t), t \geq 0\}$ is a vector Markov process of continuous time, we write the equations of the state of $(0, j)$ by considering the transitions occurring between the moments t and $t + \Delta t$ $(\Delta t > 0)$ as follows:

$$\begin{aligned} P_{0,j}(t+\Delta t) = {} & P_{0,j-1}(t)\lambda\Delta t + P_{1,j}(t)\xi_2\Delta t \\ & + P_{0,j}(t)\left[1-(\lambda+c\eta)\Delta t\right] + o(\Delta t), \end{aligned} \tag{3}$$

then we have

$$\begin{aligned} P_{0,j}(t+\Delta t) - P_{0,j}(t) &= P_{0,j-1}(t)\lambda\Delta t + P_{1,j}(t)\xi_2\Delta t \\ &\quad - P_{0,j}(t)(\lambda+c\eta)\Delta t + o(\Delta t), \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{P_{0,j}(t+\Delta t) - P_{0,j}(t)}{\Delta t} &= P_{0,j-1}(t)\lambda + P_{1,j}(t)\xi_2 \\ &\quad - P_{0,j}(t)(\lambda+c\eta) + \frac{o(\Delta t)}{\Delta t}. \end{aligned} \tag{5}$$

Letting $\Delta t \to 0$ in (5), we have

$$P_{0,j}(t)' = P_{0,j-1}(t)\lambda + P_{1,j}(t)\xi_2 - P_{0,j}(t)(\lambda+c\eta). \tag{6}$$

If the system is positive recurrent, we have the formulas $\lim_{t\to\infty}P_{0,j}(t)' = 0$ [13]. Letting $t \to 0$ in (6), we obtain the second equation of (2). The derivations of other formulas in (2) are similar.

3. Equations of Probability Generating Functions

The PGFs of the number of customers are defined as follows:

$$G_i(z) \equiv \sum_{j=0}^{\infty} z^j P_{i,j}, \qquad G(z) \equiv \sum_{i=0}^{N} G_i(z), \qquad 0 \leq i \leq N,\ |z| \leq 1. \tag{7}$$

Then

$$G_i(1) = \sum_{j=0}^{\infty} P_{i,j} \quad (i = 0, 1, 2, \ldots, N), \tag{8}$$

where $G_i(1)$ is the steady-state probability that the number of the available servers of the system is i. Hence,

$$\sum_{i=0}^{N} G_i(1) = 1. \tag{9}$$

Multiplying the two sides of every equation of (2) by z^{j+1}, and summing over j $(j = 0, 1, 2, \ldots)$ for every i, we obtain

$$\begin{aligned}
& z(\lambda - \lambda z + c\eta)G_0(z) - z\xi_2 G_1(z) = z(\xi_1 - \xi_2)P_{1,0}, \\
& -c\eta z G_{i-1}(z) + \left[z(i\mu + i\xi_2 + \lambda + c\eta) - \lambda z^2 - i\mu\right]G_i(z) \\
& \quad - (i+1)z\xi_2 G_{i+1}(z) \\
& \quad = \sum_{m=0}^{i-1}(\xi_2 - \xi_1)(i-m)P_{i,m}z^{m+1} \\
& \qquad + \sum_{m=0}^{i}(\xi_1 - \xi_2)(i+1-m)P_{i+1,m}z^{m+1} \\
& \qquad + (z-1)\sum_{m=0}^{i-1}\mu(i-m)P_{i,m}z^m, \quad 0 < i \leq N - c, \\
& -(N-i+1)\eta z G_{i-1}(z) \\
& + \left\{z\left[i\mu + i\xi_2 + \lambda + (N-i)\eta\right] - \lambda z^2 - i\mu\right\}G_i(z) \\
& - (i+1)z\xi_2 G_{i+1}(z) \\
& \quad = \sum_{m=0}^{i-1}(\xi_2 - \xi_1)(i-m)P_{i,m}z^{m+1} \\
& \qquad + \sum_{m=0}^{i}(\xi_1 - \xi_2)(i+1-m)P_{i+1,m}z^{m+1} \\
& \qquad + (z-1)\sum_{m=0}^{i-1}\mu(i-m)P_{i,m}z^m, \quad N - c < i < N, \\
& -\eta z G_{N-1}(z) + \left[z(N\mu + N\xi_2 + \lambda) - \lambda z^2 - N\mu\right]G_N(z) \\
& \quad = \sum_{m=0}^{N-1}(\xi_2 - \xi_1)(N-m)P_{N,m}z^{m+1} \\
& \qquad + (z-1)\sum_{m=0}^{N-1}\mu(N-m)P_{N,m}z^m.
\end{aligned} \tag{10}$$

We give some explanations of (10), the first equation of (2) multiplied by z, we get

$$(\lambda + c\eta)P_{0,0}z = \xi_1 P_{1,0}z. \tag{11}$$

The second equation of (2) multiplied by z^{j+1}, we get

$$(\lambda + c\eta)P_{0,j}z^{j+1} = \lambda P_{0,j-1}z^{j+1} + \xi_2 P_{1,j}z^{j+1}, \quad j > 0. \tag{12}$$

Summing (11) and (12) over j and using (7), we obtain the first equation of (10). The other equations of (10) are obtained in the same way.

In order to simplify (10), we defined the following notations:

$$f_i(z) \equiv (i\mu + i\xi_2 + \lambda + c\eta) z - \lambda z^2 - i\mu, \quad i = 0, 1, 2, \dots, N-c,$$

$$f_i(z) \equiv [i\mu + i\xi_2 + \lambda + (N-i)\eta] z - \lambda z^2 - i\mu, \quad i = N-c+1,\ N-c+2, \dots, N, \tag{13}$$

$$A(z) \equiv \begin{bmatrix} f_0(z) & -\xi_2 z & & & & & & & \\ -c\eta z & f_1(z) & -2\xi_2 z & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & -c\eta z & f_{N-c+1}(z) & -(N-c+2)\xi_2 z & & & \\ & & & & (c-1)\eta z & f_{N-c+2}(z) & -(N-c+3)\xi_2 z & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & \ddots & \ddots & \ddots \\ & & & & & & -2\eta z & f_{N-1}(z) & -N\xi_2 z \\ & & & & & & & -\eta z & f_N(z) \end{bmatrix}, \tag{14}$$

$$\begin{aligned} b_0(z) &\equiv z(\xi_1 - \xi_2) P_{1,0}, \\ b_i(z) &\equiv \sum_{m=0}^{i-1} (\xi_2 - \xi_1)(i-m) P_{i,m} z^{m+1} \\ &\quad + \sum_{m=0}^{i} (\xi_1 - \xi_2)(i+1-m) P_{i+1,m} z^{m+1} \\ &\quad + (z-1) \sum_{m=0}^{i-1} \mu (i-m) P_{i,m} z^m, \quad 0 < i < N, \\ b_N(z) &\equiv \sum_{m=0}^{N-1} (\xi_2 - \xi_1)(N-m) P_{N,m} z^{m+1} \\ &\quad + (z-1) \sum_{m=0}^{N-1} \mu (N-m) P_{N,m} z^m, \\ \overline{b}(z) &\equiv \begin{bmatrix} b_0(z) \\ b_1(z) \\ \vdots \\ b_N(z) \end{bmatrix}, \qquad \overline{g}(z) \equiv \begin{bmatrix} G_0(z) \\ G_1(z) \\ \vdots \\ G_N(z) \end{bmatrix}. \end{aligned} \tag{15}$$

Using the above notations, (10) is rewritten as follows:

$$A(z)\,\overline{g}(z) = \overline{b}(z). \tag{16}$$

Using Cramer's rule we obtain

$$|A(z)|\, G_i(z) = |A_i(z)|, \quad i = 0, 1, 2, \dots, N, \tag{17}$$

where $|A(z)|$ denotes the determinant of $A(z)$, and $A_i(z)$ is a matrix obtained by replacing the $(i+1)$th column of $A(z)$ with $\overline{b}(z)$. In (17), the functions of z are continuous and bounded in the interval $[0, 1]$, so the equations in (17) are valid in the interval $[0, 1]$ no matter $|A(z)| = 0$ or not.

4. Steady-State Availability

In this section, we discuss the steady-state availabilities $G_i(1)$ $(i = 0, 1, 2, \dots, N)$.

Letting $z = 1$ in (10) we obtain

$$\begin{aligned} & c\eta G_0(1) - \xi_2 G_1(1) = (\xi_1 - \xi_2) P_{1,0}, \\ & c\eta G_{i-1}(1) - i\xi_2 G_i(1) - [c\eta G_i(1) - (i+1)\xi_2 G_{i+1}(1)] \\ &\qquad = (\xi_1 - \xi_2) \sum_{m=1}^{i} m P_{i,i-m} - (\xi_1 - \xi_2) \sum_{m=1}^{i+1} m P_{i+1,i+1-m}, \\ &\qquad\qquad 0 < i \le N-c, \\ & [N-(i-1)]\eta G_{i-1}(1) - i\xi_2 G_i(1) \\ &\quad - [(N-i)\eta G_i(1) - (i+1)\xi_2 G_{i+1}(1)] \\ &\qquad = (\xi_1 - \xi_2) \sum_{m=1}^{i} m P_{i,i-m} - (\xi_1 - \xi_2) \sum_{m=1}^{i+1} m P_{i+1,i+1-m}, \\ &\qquad\qquad N-c < i < N, \\ & \eta G_{N-1}(1) - N\xi_2 G_N(1) = (\xi_1 - \xi_2) \sum_{m=1}^{N} m P_{N,N-m}. \end{aligned} \tag{18}$$

The $(N+1)$ equations of (18) are simplified to N independent equations, joined with (9), we have

$$\begin{aligned} & c\eta G_i(1) - (i+1)\xi_2 G_{i+1}(1) \\ &\qquad = (\xi_1 - \xi_2) \sum_{m=1}^{i+1} m P_{i+1,i+1-m}, \quad 0 \le i \le N-c, \end{aligned}$$

$$\begin{gathered}(N-i)\,\eta G_i(1)-(i+1)\,\xi_2 G_{i+1}(1) \\ =(\xi_1-\xi_2)\sum_{m=1}^{i+1} mP_{i+1,i+1-m}, \quad N-c\le i\le N-1, \\ \sum_{i=0}^{N} G_i(1)=1.\end{gathered} \tag{19}$$

Using (2), $P_{i,j}$ $(0 \le j \le i-1, 1 \le i \le N)$ are reduced to $P_{i,0}$ $(1 \le i \le N)$ which will be solved in Section 5. Given $P_{i,0}$ $(1 \le i \le N)$, we obtain $G_i(1)$ $(i = 0, 1, 2, \ldots, N)$ by solving (19), then the steady-state availability of the system is as follows:

$$A = 1 - G_0(1). \tag{20}$$

5. Steady-State Probabilities of the System States and Mean Queuing Length

5.1. The Roots of $|A(z)|$ *in the Interval* $(0,1)$. In order to get the steady-state probabilities $P_{i,0}$ $(1 \le i \le N)$, we need all N independent linear equations. Further, for getting the linear equations of $P_{i,0}$ $(1 \le i \le N)$ we need the roots of $|A(z)|$ in the interval $(0,1)$, so we discuss the roots of $|A(z)|$ as follows.

We define the following notations:

$$Q_0(z) \equiv 1, \qquad Q_1(z) \equiv f_N(z), \tag{21}$$

$$(z) \equiv \begin{vmatrix} f_{N-1}(z) & -N\xi_2 z \\ -\eta z & f_N(z)\end{vmatrix},$$
$$\vdots \qquad \vdots$$

$$Q_N(z) \equiv \begin{vmatrix} f_1(z) & -2\xi_2 z & & & & & & \\ -c\eta z & f_2(z) & -3\xi_2 z & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & -c\eta z & f_{N-c+1}(z) & -(N-c+2)\,\xi_2 z & & & \\ & & & (c-1)\,\eta z & f_{N-c+2}(z) & -(N-c+3)\,\xi_2 z & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & -2\eta z & f_{N-1}(z) & -N\xi_2 z \\ & & & & & & -\eta z & f_N(z)\end{vmatrix}, \tag{22}$$

$$Q_{N+1}(z) \equiv |A(z)|, \quad (0 \le z \le \infty), \tag{23}$$

where $Q_i(z)$ $(i = 1, 2, \ldots, N)$ is formed by the last i rows and last i columns of $|A(z)|$. According to (14) we have

$$\begin{gathered} Q_{k+1}(z) = f_{N-k}(z)\,Q_k(z) - (N-k+1)\,\xi_2 k\eta z^2 Q_{k-1}(z), \\ 1 \le k \le c-1, \\ Q_{k+1}(z) = f_{N-k}(z)\,Q_k(z) - (N-k+1)\,\xi_2 c\eta z^2 Q_{k-1}(z), \\ c \le k \le N. \end{gathered} \tag{24}$$

The properties of $Q_i(z)$ $(i = 1, 2, \ldots, N)$ and the necessary proofs are as follows.

(a) $Q_0(z)$ has no roots.

(b) $Q_k(z)$ and $Q_{k+1}(z)$ have no common roots in the interval $(0, \infty)$ $(k = 1, 2, \ldots, N)$.

Proof. Suppose that (b) is not true and z_0 (>0) is a common roots of $Q_k(z)$ and $Q_{k+1}(z)$, then $Q_{k-1}(z_0) = 0$ due to (24). Similarly, $Q_{k-2}(z_0) = 0$, so we get $Q_0(z_0) = 0$ which contradicts the statement (a). □

(c) If z_0 is a positive root of $Q_k(z)$ $(k = 1, 2, \ldots, N)$, then $Q_{k-1}(z_0)$ and $Q_{k+1}(z_0)$ are opposite in sign. This property readily follows from (24).

(d) $Q_k(1) > 0$, $k = 0, 1, 2, \ldots, N$.

Proof. Substituting $z = 1$ into $Q_k(z)$, we get $Q_k(1) = (N-k+1)(N-k+2)\cdots(N-1)N\xi_2^k > 0$, $k = 1, 2, \ldots, N$, and $Q_0(1) = 1$. □

(e) $Q_{N+1}(1) = |A(1)| = 0$, and $Q_{N+1}(0) = |A(0)| = 0$.

Proof. The first row of $|A(z)|$ has a common factor z, so $Q_{N+1}(0) = |A(0)| = 0$. The sum of every column of $|A(z)|$ has a common factor $(z-1)$. Replacing every element of the last row of $|A(z)|$ with the sum of the corresponding column and extracting the common factor $(z-1)$, $|A(z)|$ is written as follows:

$$|A(z)| = z(z-1) \times \begin{bmatrix} \lambda + c\eta - \lambda z & -\xi_2 & 0 & \cdots & 0 & 0 \\ -c\eta z & f_1(z) & -2\xi_2 z & \cdots & 0 & 0 \\ 0 & -c\eta & f_2(z) & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & f_{N-1}(z) & -N\xi_2 z \\ -\lambda z & -\lambda z + \mu & -\lambda z + 2\mu & \cdots & -\lambda z + (N-1)\mu & -\lambda z + N\mu \end{bmatrix}. \quad (25)$$

If we define

$$D(z) \equiv \begin{vmatrix} \lambda + c\eta - \lambda z & -\xi_2 & 0 & \cdots & 0 & 0 \\ -c\eta z & f_1(z) & -2\xi_2 z & \cdots & 0 & 0 \\ 0 & -c\eta & f_2(z) & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & f_{N-1}(z) & -N\xi_2 z \\ -\lambda z & -\lambda z + \mu & -\lambda z + 2\mu & \cdots & -\lambda z + (N-1)\mu & -\lambda z + N\mu \end{vmatrix}, \quad (26)$$

then

$$|A(z)| = z(z-1)D(z). \quad (27)$$

□

(f) $\mathrm{Sign}[Q_k(0)] = (-1)^k$ $(k = 0, 1, 2, \ldots, N)$.

Proof. From the definitions of $f_i(z)$ $(i = 0, 1, 2, \ldots, N)$, we got $f_0(0) = 0$ and $f_k(0) < 0$ $(k = 1, 2, \ldots, N)$, so we get this property from (24). □

(g) $\mathrm{Sign}[Q_k(+\infty)] = (-1)^k$ $(k = 0, 1, 2, \ldots, N + 1)$.

Proof. It is since the highest power term of $Q_k(z)$ is $(-\lambda z^2)^k$ $(k = 0, 1, 2, \ldots, N + 1)$ and the sign of $Q_k(+\infty)$ is determined by its highest power term. □

Theorem 1. *If $D(1) > 0$, the polynomial $|A(z)|$ has exactly $(N-1)$ distinct roots in the interval $(0, 1)$.*

Proof. Since $Q_1(z) = f_N(z) = [N(\mu + \xi_2) + \lambda]z - \lambda z^2 - N\mu$, $Q_1(z)$ is a 2-power polynomial of z. Further, we find that $Q_1(1) = \xi_2 > 0$ and $Q_1(0) = -\mu < 0$, so $Q_1(z)$ has two distinct roots which are denoted by $z_{1,1}$ $(0 < z_{1,1} < 1)$ and $z_{1,2}$ (> 1).

With the fact that $z_{1,1}$ and $z_{1,2}$ are roots of $Q_1(z)$, and $Q_0(z) = 1 > 0$, according to the property (c) or (24), we get $Q_2(z_{1,1}) < 0$ and $Q_2(z_{1,2}) < 0$.

$Q_2(z)$ is a 4-power polynomial of z. From the properties (c), (d), (f), and (g), we find that $Q_2(z)$ has one and only one root in each interval of $(0, z_{1,1})$, $(z_{1,1}, 1)$, $(1, z_{1,2})$, and $(z_{1,2}, +\infty)$.

So on, we find that $Q_N(z)$ is a $2N$-power polynomial of z, it has N distinct roots in the interval $(0, 1)$ and N distinct roots in the interval $(1, \infty)$. We denote the $2N$ roots of $Q_N(z)$ by $z_{N,i}$ $(i = 1, 2, \ldots, 2N)$ orderly.

From the properties (c), (d), (e), and (f), we find that $|A(z)|$ has one and only one root in each interval $(z_{N,i}, z_{N,i+1})$ $(i = 1, 2, \ldots, N-1, N+1, \ldots, 2N-1)$, all of them are $2(N-1)$ distinct roots of $|A(z)|$.

Since $|A(1)| = 0$ and $D(1) = (|A(z)|)'_{z=1} > 0$, it has a real number ε (>0) satisfies $1 + \varepsilon < z_{N,N+1}$ and $|A(1+\varepsilon)| > 0$. On the other hand, from (c) and (d) we get

$$\mathrm{Sign}[Q_{N+1}(z_{N,i})] = (-1)^{N+i}, \quad i = N+1, N+2, \ldots, 2N, \quad (28)$$

then $\mathrm{Sign}[Q_{N+1}(z_{N,N+1})] = (-1)^{N+N+1}$ or $|A(z_{N,N+1})| = Q_{N+1}(z_{N,N+1}) < 0$, so $|A(z)|$ has at least one root in the interval $(1, z_{N,N+1})$.

From (28), we get $\mathrm{Sign}[|A(z_{N,2N})|] = \mathrm{Sign}[Q_{N+1}(z_{N,2N})] = (-1)^{N+2N} = (-1)^{3N} = (-1)^N$. From the property (g), we get $\mathrm{Sign}[|A(+\infty)|] = \mathrm{Sign}[Q_{N+1}(+\infty)] = (-1)^{N+1}$. So we know that $|A(z)|$ has at least one root in the interval $(z_{N,2N}, \infty)$.

From the properties (e) and (f), we know that 0 and 1 are roots of $|A(z)|$.

In conclusion, $|A(z)|$ is a $2(N+1)$-power polynomial of z, it has $2(N+1)$ distinct roots at most. Now we make certain all roots of $|A(z)|$ and find that it has $N-1$ distinct roots in the interval $(0, 1)$. □

From the proof, we find that the $(N-1)$ distinct roots in the interval $(0, 1)$ of $|A(z)|$ are also the roots of $D(z)$.

5.2. Steady-State Probabilities. Assuming that the system parameters meet $D(1) > 0$. Letting z_k $(k = 1, 2, \ldots, N-1)$ denote the roots of $|A(z)|$ in the interval $(0, 1)$. Substituting z_1 in (17), we obtain a set of linear equations about the steady-state probabilities of $P_{i,0}$ $(i = 1, 2, \ldots, N)$, but these equations are similar to each other. However, the equations belong to different z_k $(k = 1, 2, \ldots, N-1)$ are independent mutually, so we obtain $(N-1)$ independent equations by the $(N-1)$ different roots of z_k, respectively.

In the following, we discuss about the Nth-independent linear equation of $P_{i,0}$ $(i = 1, 2, \ldots, N)$. Similar to (27), $|A_i(z)|$ is written as follows:

$$|A_i(z)| = z(z-1)D_i(z), \quad i = 0, 1, 2, \ldots, N, \quad (29)$$

where

$$D_i(z) = \left| \begin{array}{ccccccc} c\eta + \lambda(1-z) & -\xi_2 & \cdots & (\xi_1 - \xi_2) P_{1,0} & \cdots & 0 \\ -c\eta z & f_1(z) & \cdots & b_1(z) & \cdots & 0 \\ 0 & -c\eta z & \cdots & b_2(z) & \cdots & 0 \\ & & \vdots & & \vdots & \\ 0 & 0 & \cdots & b_{N-1}(z) & \cdots & -N\xi_2 z \\ -\lambda z & -\lambda z + \mu & \cdots & \sum_{i=1}^{N} \sum_{m=1}^{i} m\mu p_{i,i-m} z^{i-m} & \cdots & -\lambda z + N\mu \end{array} \right|. \quad (30)$$

where the $\sum$ column is the $(i+1)$th column.

Substituting (27) and (29) into (17), we obtain

$$D(z) G_i(z) = D_i(z), \quad i = 0, 1, 2, \ldots, N. \quad (31)$$

Substituting $z = 1$ in (31), we obtain

$$D(1) G_i(1) = D_i(1), \quad i = 0, 1, 2, \ldots, N. \quad (32)$$

From (19), we know that $G_i(1)$ $(i = 0, 1, 2, \ldots, N)$ can be expressed by $P_{i,0}$ $(i = 1, 2, \ldots, N)$, so (32) are linear equations of $P_{i,0}$ $(i = 1, 2, \ldots, N)$, but they are similar to each other. However, every equation of (32) is independent with the $(N-1)$ linear equations obtained by the roots of $|A_i(z)|$ in the interval $(0, 1)$. Then all independent linear equations of $P_{i,0}$ $(i = 1, 2, \ldots, N)$ are as follows:

$$\begin{gathered} |A_0(z_k)| = 0, \quad k = 1, 2, \ldots, N-1, \\ D(1) G_0(1) = D_0(1). \end{gathered} \quad (33)$$

Further, from (29), (33) is equivalent to the follows:

$$\begin{gathered} D_0(z_k) = 0, \quad k = 1, 2, \ldots, N-1, \\ D(1) G_0(1) = D_0(1). \end{gathered} \quad (34)$$

The steady-state probabilities of $P_{i,0}$ $(i = 0, 1, 2, \ldots, N)$ are obtained by solving (34). Using $P_{i,0}$ $(i = 0, 1, 2, \ldots, N)$ and (2), we obtain the other steady-state probabilities of $P_{i,j}$ $(i = 0, 1, 2, \ldots, N, j = 1, 2, \ldots)$.

5.3. Mean Queuing Length. After getting the probabilities $P_{i,j}$ $(0 \le j \le i-1, 1 \le i \le N)$ in (19) we obtain $G_i(1)$ $(i = 0, 1, 2, \ldots, N)$ by solving (19) and obtain the steady-state availability (A) of the model by (20).

From (31), we obtain

$$G_i(z) = \frac{D_i(z)}{D(z)}, \quad D(z) \neq 0, \ |z| \le 1, \ i = 0, 1, 2, \ldots, N. \quad (35)$$

The PGF of $G(z)$ is obtained by (7). Using the property of PGF [13] we obtain the steady-state mean queuing length is as follows:

$$L = \left. \frac{dG(z)}{dz} \right|_{z=1}. \quad (36)$$

6. Case Analysis

We analyze the case of $N = 2$ and $c = 1$ in this section. According to the above discussion, the determinant $|A(z)|$ (or $D(z)$) of this case has only one root z_1 in the interval $(0, 1)$. The notations of this case are as follows:

$$\begin{aligned} f_0(z) &= (\lambda + \eta) z - \lambda z^2, \\ f_1(z) &= (\mu + \xi_2 + \lambda + \eta) z - \lambda z^2 - \mu, \\ f_2(z) &= (2\mu + 2\xi_2 + \lambda) z - \lambda z^2 - 2\mu, \\ b_0(z) &= (\xi_1 - \xi_2) P_{1,0} z, \\ b_1(z) &= (\mu + \xi_2 - \xi_1) P_{1,0} z \\ &\quad + \sum_{m=1}^{2} m (\xi_1 - \xi_2) P_{2,2-m} z^{3-m} - \mu P_{1,0} \\ &= \mu P_{1,0} (z-1) + z (\xi_1 - \xi_2)(P_{2,1} z + 2P_{2,0} - P_{1,0}), \\ b_2(z) &= \sum_{m=1}^{2} m (\mu + \xi_2 - \xi_1) P_{2,2-m} z^{3-m} \\ &\quad - \sum_{m=1}^{2} m\mu P_{2,2-m} z^{2-m}, \\ A(z) &= \begin{bmatrix} f_0(z) & -\xi_2 z & 0 \\ -\eta z & f_1(z) & -2\xi_2 z \\ 0 & -\eta z & f_2(z) \end{bmatrix}, \\ |A(z)| &= z(z-1) \\ &\quad \times \begin{vmatrix} \lambda + \eta - \lambda z & -\xi_2 & 0 \\ -\eta z & (\mu + \xi_2 + \lambda + \eta) z - \lambda z^2 - \mu & -2\xi_2 z \\ -\lambda z & -\lambda z + \mu & -\lambda z + 2\mu \end{vmatrix}, \\ D(z) &= \begin{vmatrix} \lambda + \eta - \lambda z & -\xi_2 & 0 \\ -\eta z & (\mu + \xi_2 + \lambda + \eta) z - \lambda z^2 - \mu & -2\xi_2 z \\ -\lambda z & -\lambda z + \mu & -\lambda z + 2\mu \end{vmatrix} \\ &= -(\eta + \lambda - z\lambda)(z\lambda - 2\mu)[-\mu + z(\eta + \lambda - z\lambda + \mu)] \\ &\quad + z\xi_2 [z\lambda(-2\eta - 3\lambda + 3z\lambda) \\ &\qquad + 2(\eta - 2z\lambda + 2\lambda)\mu - 2z\lambda\xi_2], \end{aligned} \quad (37)$$

then

$$D(1)=\begin{vmatrix}\eta & -\xi_2 & 0\\ -\eta & \eta+\xi_2 & -2\xi_2\\ -\lambda & -\lambda+\mu & -\lambda+2\mu\end{vmatrix} = 2\mu\left(\xi_2\eta+\eta^2\right)-\lambda\left(2\xi_2^2+2\xi_2\eta+\eta^2\right), \tag{38}$$

and $D(1)>0$ is equivalent to

$$2\mu\left(\xi_2\eta+\eta^2\right)-\lambda\left(2\xi_2^2+2\xi_2\eta+\eta^2\right)>0, \tag{39}$$

or

$$\frac{\lambda}{\mu}<\frac{2\xi_2\eta+2\eta^2}{2\xi_2^2+2\xi_2\eta+\eta^2}=\frac{2\eta/\left(\xi_2+\eta\right)}{1+\left(\xi_2/\left(\xi_2+\eta\right)\right)^2}. \tag{40}$$

The left of (40) is the mean service quantities that all customers need per unit time. The right of (40) is the mean service quantities that the two servers provide per unit time. So (40) is the necessary and sufficient condition of recurrence of the system.

Equations (19) in this case are as follows:

$$\begin{gathered}\eta G_0(1)-\xi_2 G_1(1)=\left(\xi_1-\xi_2\right)P_{1,0},\\ \eta G_1(1)-2\xi_2 G_2(1)=\sum_{m=1}^{2} m\left(\xi_1-\xi_2\right)P_{2,2-m},\\ \sum_{i=0}^{2} G_i(1)=1.\end{gathered} \tag{41}$$

Equations (2) in this case are as follows:

$$\begin{gathered}\left(\lambda+\eta\right)P_{0,0}=\xi_1 P_{1,0},\quad i=0,\ j=0,\\ \left(\lambda+\eta\right)P_{0,j}=\lambda P_{0,j-1}+\xi_2 P_{1,j},\quad i=0,\ j>0,\\ \left(\lambda+\eta+\xi_1\right)P_{1,0}=\eta P_{0,0}+\mu P_{1,1}+2\xi_1 P_{2,0},\\ i=1,\ j=0,\\ \left(\lambda+\eta+\xi_2+\mu\right)P_{1,1}\\ =\lambda P_{1,0}+\eta P_{0,1}+\mu P_{1,2}+\left(\xi_1+\xi_2\right)P_{2,1},\quad i=1,\ j=1,\\ \left(\lambda+\eta+\xi_2+\mu\right)P_{1,j}\\ =\lambda P_{1,j-1}+\eta P_{0,j}+\mu P_{1,j+1}+2\xi_2 P_{2,j},\quad i=1,\ j>1,\\ \left(\lambda+2\xi_1\right)P_{2,0}=\eta P_{1,0}+\mu P_{2,1},\quad i=2,\ j=0,\\ \left(\lambda+\xi_1+\xi_2+\mu\right)P_{2,1}=\lambda P_{2,0}+\eta P_{1,1}+2\mu P_{2,2},\\ i=2,\ j=1,\\ \left(\lambda+2\mu+2\xi_2\right)P_{2,j}=\lambda P_{2,j-1}+\eta P_{1,j}+2\mu P_{2,j+1},\\ i=2,\ j\geq 2.\end{gathered} \tag{42}$$

Using (42), we obtain

$$P_{2,1}=\frac{\left(2\xi_1+\lambda\right)P_{2,0}-\eta P_{1,0}}{\mu}. \tag{43}$$

Using (41) and (43), $G_0(1)$, $G_1(1)$, and $G_2(1)$ are expressed in an algebraic expressions of $P_{1,0}$ and $P_{2,0}$.

If (40) is satisfied, we obtain z_1 by solving

$$D(z)=0. \tag{44}$$

Equations (34) in this case are as follows:

$$\begin{gathered}D_0\left(z_1\right)=0,\\ D(1)\,G_0(1)=D_0(1).\end{gathered} \tag{45}$$

Solving (45), we obtain $P_{1,0}$ and $P_{2,0}$.

Using (42), we obtain $P_{i,j}$ $(i=0,1,2,\ j=1,2,\ldots)$.

Using (20) and (41), we obtain the steady-state availability A.

For the mean queuing lengths, we have

$$\begin{gathered}D_0(z)=\begin{vmatrix}\left(\xi_1-\xi_2\right)P_{1,0} & -\xi_2 & 0\\ b_1(z) & \left(\mu+\xi_2+\lambda+\eta\right)z-\lambda z^2-\mu & -2\xi_2\\ \mu\left(P_{2,1}z+2P_{2,0}+P_{1,0}\right) & -\lambda z+\mu & \lambda z+2\mu\end{vmatrix},\\ D_1(z)=\begin{vmatrix}\lambda+\eta-\lambda z & \left(\xi_1-\xi_2\right)P_{1,0} & 0\\ -\eta z & b_1(z) & -2\xi_2\\ -\lambda z & \mu\left(P_{2,1}z+2P_{2,0}+P_{1,0}\right) & \lambda z+2\mu\end{vmatrix},\\ D_2(z)=\begin{vmatrix}\lambda+\eta-\lambda z & -\xi_2 & \left(\xi_1-\xi_2\right)P_{1,0}\\ -\eta z & \left(\mu+\xi_2+\lambda+\eta\right)z-\lambda z^2-\mu & b_1(z)\\ -\lambda z & -\lambda z+\mu & \mu\left(P_{2,1}z+2P_{2,0}+P_{1,0}\right)\end{vmatrix},\end{gathered} \tag{46}$$

$$G(z)=\frac{D_0(z)+D_1(z)+D_2(z)}{D(z)}. \tag{47}$$

Using (36), we obtain the mean queuing lengths L is as follows:

Table 1: The availability A and mean queuing length L ($N = 2$, $c = 1$, and $\lambda = 1$).

μ	$\xi_1 = 0$, $\xi_2 = 0.5$, and $\eta = 1.2$		$\xi_1 = 0.3$, $\xi_2 = 0.5$, and $\eta = 1$		$\xi_1 = 0.5$, $\xi_2 = 0.5$, and $\eta = 1$	
	L	A	L	A	L	A
1.1	3.0797	0.8951	4.6956	0.8209	4.8505	0.8
1.2	2.3810	0.9058	3.5018	0.8266	3.6549	0.8
1.3	1.9425	0.9147	2.8136	0.8315	2.9650	0.8
1.4	1.6411	0.9224	2.3641	0.8354	2.5138	0.8
1.5	1.4207	0.9290	2.0464	0.8383	2.1947	0.8
1.6	1.2524	0.9348	1.8095	0.8429	1.9563	0.8
1.7	1.1195	0.9399	1.6256	0.8458	1.7710	0.8
1.8	1.0119	0.9443	1.4784	0.8484	1.6225	0.8
μ	$\xi_1 = 0.5$, $\xi_2 = 0.5$, and $\eta = 0.8$		$\xi_1 = 0.5$, $\xi_2 = 1$, and $\eta = 1.5$		$\xi_1 = 1$, $\xi_2 = 0.8$, and $\eta = 1.5$	
	L	A	L	A	L	A
1.1	9.4410	0.7423	9.9094	0.7412	5.0503	0.7734
1.2	6.1455	0.7423	5.7928	0.7509	3.7017	0.7700
1.3	4.6345	0.7423	4.1374	0.7608	2.9584	0.7671
1.4	3.7636	0.7423	3.2405	0.7687	2.4853	0.7646
1.5	3.1952	0.7423	2.6761	0.7756	2.1565	0.7623
1.6	2.7937	0.7423	2.2873	0.7817	1.9140	0.7603
1.7	2.4944	0.7423	2.0025	0.7872	1.7273	0.7585
1.8	2.2389	0.7423	1.7846	0.7920	1.5788	0.7569

$$
\begin{aligned}
L &= \left.\frac{dG(z)}{dz}\right|_{z=1} \\
&= \frac{1}{\left[\lambda\eta^2 - 2\mu\eta^2 + 2\xi_2\left(\eta\lambda - \eta\mu + \lambda\xi_2\right)\right]^2} \\
&\quad \times \Big\{-\Big[\eta\left(-2\eta\lambda + 2\lambda^2 + 2\eta\mu - 5\lambda\mu + 2\mu^2\right) \\
&\qquad\qquad + \xi_2\left(-4\eta\lambda + 3\lambda^2 + 2\eta\mu - 4\lambda\mu - 4\lambda\xi_2\right)\Big] \\
&\qquad \times \Big[\mu\left(\eta^2 + 2\xi_1\eta + 2\xi_1\xi_2\right)P_{1,0} \\
&\qquad\qquad + \mu\left(\eta^2 + \eta\xi_2 + \eta\xi_1 + 2\xi_1\xi_2\right)\left(2P_{2,0} + P_{2,1}\right)\Big] \\
&\qquad + \mu\left[-\eta^2\lambda + 2\mu\eta^2 - 2\xi_2\left(\eta\lambda - \eta\mu + \lambda\xi_2\right)\right] \\
&\qquad \times \Big[\left(\eta^2 - 2\eta\lambda + 2\eta\mu - 2\lambda\xi_2 + 2\xi_1\eta\right. \\
&\qquad\qquad \left. - \lambda\xi_1 + 2\mu\xi_1 + 2\xi_1\xi_2\right)P_{1,0} \\
&\qquad\qquad + \left(\eta^2 + \eta\xi_2 + \eta\xi_1 + 2\xi_1\xi_2\right)P_{2,1} \\
&\qquad\qquad + \left(\eta^2 - 2\eta\lambda + \eta\mu + \eta\xi_2 - 2\lambda\xi_2 + \eta\xi_1\right. \\
&\qquad\qquad \left. - \lambda\xi_1 + 2\xi_1\xi_2\right)\left(2P_{2,0} + P_{2,1}\right)\Big]\Big\}.
\end{aligned} \tag{48}
$$

Numerical Example. Letting $N = 2$, $c = 1$, $\lambda = 1$, $\xi_1 = 0.5$, $\xi_2 = 1$, $\eta = 1$, and $\mu = 2$, we have

$$
\frac{\lambda}{\mu} = \frac{1}{2} < \frac{4}{5} = \frac{2\eta/\left(\xi_2 + \eta\right)}{1 + \left(\xi_2/\left(\xi_2 + \eta\right)\right)^2}. \tag{49}
$$

The roots of $D(z) = 0$ are as follows:

$$
\begin{aligned}
z_1 &= 0.349123, & z_2 &= 1.84513, \\
z_3 &= 4.46896, & z_4 &= 8.33679,
\end{aligned} \tag{50}
$$

and only z_1 in the interval $(0, 1)$.

Solving (45), we obtain

$$
P_{1,0} = 0.151375, \qquad P_{2,0} = 0.297974. \tag{51}
$$

Using (42) we obtain

$$
P_{0,0} = 0.037844, \qquad P_{2,1} = 0.146599. \tag{52}
$$

Solving (41), we obtain

$$
\begin{aligned}
&G_0(1) = 0.280333, \qquad G_1(1) = 0.35602, \\
&G_2(1) = 0.363647.
\end{aligned} \tag{53}
$$

Finally, the availability and mean queuing lengths of this example are as follows:

$$
A = 0.719667, \qquad L = 6.04039. \tag{54}
$$

The other numerical results are shown in Table 1. All the system parameters in Table 1 satisfy (40).

We find that the mean queuing length (L) decreases with the increasing of the parameter μ in Table 1, it is because of the greater service rate the less customers in the system. Furthermore, we find that the availability (A) increases with the increasing of the parameter μ, where $\xi_1 < \xi_2$ (the cases:

$\xi_1 = 0, \xi_2 = 0.5$; $\xi_1 = 0.3, \xi_2 = 0.5$; $\xi_1 = 0.5, \xi_2 = 1$); on the contrary, the availability decreases with the increasing of the parameter μ, where $\xi_1 > \xi_2$ (the case: $\xi_1 = 1, \xi_2 = 0.8$); otherwise, the availability is constant, where $\xi_1 = \xi_2$ (the case: $\xi_1 = 0.5, \xi_2 = 0.5$).

7. Conclusions

In Section 5.1, the inequality $D(1) > 0$ of Theorem 1 is the necessary and sufficient condition for the system to be positive recurrent, and a probability explanation of this condition is given by (40).

We find that the idle time breakdown rate ξ_1 does not appear in (40). This is because the busy time breakdown rate ξ_2 is at work when the number of the customers is greater than or equal to the number of the available servers, and the criteria of positive recurrence depends on the busy time breakdown rate.

A case analysis is given to illustrate the analysis of this paper, and the numerical results indicate that the variation of breakdown rates has a significant effect on the steady-state availability and steady-state queue length of the system.

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Research Article

Combined Sales Effort and Inventory Control under Demand Uncertainty

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We study the joint inventory and sales effort management problems of a retailer in a broad context and investigate the optimal policies for a single item, periodic-review system. In each period, the demand is uncertain depending on the sales effort level exerted by the retailer, which incurs an associated cost. The retailer's objective is to find a joint optimal inventory replenishment and sales effort policy to maximize the discounted profit over a finite horizon. We first consider a basic setting with zero setup cost and no batch ordering, under which the base stock list sales effort policy is optimal. Two extensions are then investigated: (1) the case with nonzero setup cost, under which we show that (s, S, e) policy is optimal; and (2) the case with batch ordering, under which we prove the optimality of the (r, Nq, e) policy. Finally, we conduct numerical studies to provide additional managerial insights.

1. Introduction

Inventory and demand management has received much attention due to the increasingly competitive environment. Traditional inventory problems often consider an exogenously determined uncertain demand and thus the key decision facing firms is to decide on an appropriate inventory policy to mitigate the mismatch cost of demand and supply. However, demand could be endogenously influenced by many factors such as price, freshness level of the product, and the sales effort exerted by firms. Within these factors, changing sales effort level is an important driver to match supply and demand and achieve business goals. In this paper, we focus on the optimal decision on the sales effort which includes, to list a few, the incentives to the sales people and the promotions with short-term effects such as in-store displays, product-service bundling, and so forth.

Specifically, we study the joint inventory and sales effort management of a retailer in a broad context and investigate his/her optimal policies under a single item, periodic-review system. In each period, the retailer decides on the replenishment quantity and the sales effort to be exerted jointly. For the basic setting without considering the setup cost and batch ordering, we assume that the procurement cost is linearly increasing in the order quantity and the associated cost for exerting sales effort (e.g., rentals for in-store displays and advertisement expenditure in media like TV and newspaper) is increasing and convex in the sales effort level. All replenishment arrives immediately and all unmet demand is fully backlogged. The inventory holding and shortage cost are charged based on the inventory leftover at the end of each period. The retailer's objective is to find a joint optimal inventory replenishment and sales effort policy to maximize the discounted profit over a finite horizon. In addition to the basic setting mentioned above, we will go further to investigate the retailer's optimal polices when each ordering incurs a non-zero setup cost and when the order quantity for each procurement must be in batch size respectively.

The model studied in this paper is closely related to those dealing with the coordination between marketing and operations management in general and those studying the jointly pricing and inventory control problem under a multi period framework in particular. Here we briefly review the most relevant work. As to the literature on joint pricing and inventory control, reader of interest can refer to Yano and Gilbert [1] for detailed reviews. Karakul [2], Serel [3], and Webster and Weng [4] study these problems with a single period setting. In a multi period setting, when there is no ordering setup cost, Federgruen and Heching [5] show the optimality of the so-called base stock list price policy. When there is a setup cost for each ordering, several authors have identified the conditions under which an (s, S, p) policy is optimal. In a finite horizon periodic-review setting, Chen and Simchi-Levi [6] show the optimality of the (s, S, p) policy or a variation of such a policy. Chen and Simchi-Levi [7] extend the optimality of a stationary (s, S, p) policy to an infinite horizon setting. The sales effort considered in our paper is different from the pricing decisions, as it will not influence the selling price of the product. Such sales effort is rather reasonable as the retailers tend to avoid frequent price changes, which on one hand may bring down the customer's perception on the product, and on the other hand often incur additional cost (see, e.g., Chen et al. [8]).

Our research is also closely related to literature investigating the impact of various sales efforts on operational inventory decisions. Balcer [9] consider a joint inventory and advertising strategy problem. Cheng and Sethi [10] study the joint inventory-promotion problem with Markov-dependent demand state. Porteus and Whang [11] and Chen [12] analyze the impact of incentive schemes of sales force compensation on manufacturing decisions. Ernst and Kouvelis [13] study the joint decision of goods bundling and inventory control in a newsvendor setting. In a continuous-review setting, Chen et al. [14] show that the (s, S) type policy is optimal for product inventory control, and an inventory level-based service package composite is optimal for service offerings. Zhang et al. [15] discuss the joint optimization of inventory and pricing, and promotion, and ignore the fixed ordering cost in their setting. Wei and Chen [16] study the computation method for the joint inventory and sales effort model in an infinite horizon framework, when the optimal policy has an (s, S, e) type. However, to the best of our knowledge, few researchers have been devoted to study the optimal joint inventory and sales effort policy under a periodic-review setting.

Our current work shares some similarities with Wei and Chen [16]. However, the model is different from theirs in the following ways. First, our model bases on the finite horizon setting and the system parameters can be nonstationary, while their analysis focuses on the infinite horizon with stationary policy. Second, we derive the optimal policy, while they only optimize the policy parameters given the policy form. Third, we study the case when the demand has a multiplicative form with the sales effort level. Moreover, we also extend the analysis to the case when the order quantity is in batch size. Through our analysis, we find that the optimal policy is a base stock list sales effort policy when the setup cost is negligible and is a (s, S, e) policy when the setup cost is considered. When the order size in each period is in batch, we show the optimality of a (r, Nq, e) policy when there is no setup cost. Numerical study shows that both the base stock level and the optimal sales effort level are decreasing with the unit cost and the diseconomy of scale of the sales effort cost. However, although the base stock level is increasing in the demand uncertainty, the corresponding sales effort level is decreasing, which implies that the cost caused by additional demand variance is larger than the profits made by additional expected demand when the demand has a multiplicative form in the sales effort level.

The reminder of this paper is organized as follows. In Section 2, we first setup and analyze the basic model in which there is no setup cost for each ordering. Then, in Section 3, we extend the analysis to the case when there is a setup cost and the case when the order size is in batch, respectively. The optimal policies for both cases are identified. In Section 4, we employ a numerical study to gain more insights on the joint inventory and sales effort problem. Section 5 concludes the paper with possible future directions.

2. Model Formulation

Consider a multi period inventory planning problem faced by a retailer with sales effort sensitive demand. There are T periods. At the beginning of each period t, $t = 1, 2, \ldots, T$, depending on his/her on hand inventory level x_t, the retailer should decide how many to order from the supplier at a linear unit cost c_t (actually, we will extend our analysis to the case when there is a setup cost for each procurement) and how much sales effort, e_t, to exert to stimulate the demand. The replenishment becomes available instantaneously; that is, the lead time is zero. Let y_t be the inventory level after orders arrive. The cost to exert effort e_t is $v(e_t)$, which is increasing convex in e_t with $v(1) = 0$. That is, the marginal cost to exert the effort is increasing in the effort level. The new demand after exerting sales effort e_t is given by $e_t\xi$, in which ξ is a random variable when there is no sales effort exerts (i.e., $e = 1$). Thus, the sales effort, although will increase the expected sales, will also bring up the variance of the demand; however, the coefficient of variant of the demand keeps constant. Let $F(\cdot)$ and $f(\cdot)$ be the cumulative distribution function and the probability density function of ξ, respectively. Without loss of generality, the demands in consecutive periods are assumed to be independent and nonnegative.

The unit selling price of the product, p, is exogenously given. Such exogenously determined selling price reflects the retailer's limited pricing power when facing intense competition in the market. The demand realizes. If demand exceeds the available inventory, unsatisfied demand is backlogged and incurs shortage cost; otherwise, excess inventory incurs holding cost and will be carried to the next period. Let $H_t(I)$ be the holding or backlogging cost incurred in period t with ending inventory level I. Let $L_t(y, e) = E[H_t(y - e\xi_t)]$ denote the one-period expected inventory and backlogging costs for period t, when the inventory level after order arrives is y and the exerted sales effort is e. We impose the following assumption on the function $L_t(y, e)$ and $H_t(I)$.

Assumption 1. (i) $\lim_{y\to\infty} L_t(y, e) = \lim_{y\to-\infty}[c_t y + L_t(y, e)] = \lim_{y\to\infty}[(c_t - \alpha c_{t+1})y + L_t(y, e)] = \infty$ for all $e \in [e_{\min}, e_{\max}]$; (ii) $H_t(I)$ is convex in I.

The first part of this assumption holds whenever the inventory (and backlogging) cost function, L_t, tends to infinity as the inventory level (or backlog size) increases to infinity; the latter applies to any reasonable inventory cost structure in which the loss associated with a stockout exceeds the unit's purchase price. Such assumption is to ensure the existence of finite order-up-to level for each level of sales effort. The second part is a technical assumption to ensures the existence of the optimal policy. Actually, linear inventory holding and backlogging cost setting, that is, $H_t(I) = h_t I^+ + b_t I^-$ (h_t is the linear holding cost and b_t is the linear backlogging cost), which is widely adopted in literature, satisfies this assumption. From this assumption, it is a direct result that $L_t(y, e)$ is jointly convex in (y, e).

Now, we begin to develop the multi period dynamic programming model for this finite horizon planning problem. Let $\pi_t^*(x_t)$ denote maximum expected discounted profit when starting period t with state x, then, the optimality equation of this finite horizon problem is

$$\pi_t^*(x_t) = c_t x_t + \max_{y_t \ge x_t, e_t \in [e_{\min}, e_{\max}]} J_t'(y_t, e_t), \tag{1}$$

where

$$\begin{aligned} J_t'(y_t, e_t) = {} & p_t E[e_t \xi_t] - c_t y_t - L_t(y_t, e_t) \\ & - v_t(e_t) + \alpha E[\pi_{t+1}^*(y_t - e_t \xi_t)] \end{aligned} \tag{2}$$

and $\alpha \le 1$ is the discount factor for future periods. In the above formula, $p_t E[e_t \xi_t]$ is the revenue collected in this period. Thus, we implicitly assume that we still collect money today for the backlog demand. Without loss of generality, we assume $\pi_{T+1}^*(x_{T+1}) = c_{T+1} x_{T+1}$; that is, all demand at the end of the planning horizon will be satisfied at the unit cost c_{T+1} and all leftover inventory is salvaged at this unit cost.

To ease analysis, let $\Pi_t^*(x_t) = \pi_t^*(x_t) - c_t x_t$. Then, $\Pi_{T+1}^*(x_{T+1}) = 0$, and the original dynamic programming problem can be reformulated as follows:

$$\Pi_t^*(x_t) = \max_{y_t \ge x_t, e_t \in [e_{\min}, e_{\max}]} J_t(y_t, e_t), \tag{3}$$

where

$$\begin{aligned} J_t(y_t, e_t) &= p_t E[e_t \xi_t] - c_t y_t - L_t(y_t, e_t) - v_t(e_t) \\ &\quad + \alpha c_{t+1} E[y_t - e_t \xi_t] + \alpha E[\Pi_{t+1}^*(y_t - e_t \xi_t)] \\ &= (p_t - \alpha c_{t+1}) e_t \mu_t - (c_t - \alpha c_{t+1}) y_t - L_t(y_t, e_t) \\ &\quad - v_t(e_t) + \alpha E[\Pi_{t+1}^*(y_t - e_t \xi_t)]. \end{aligned} \tag{4}$$

Thus, the inner part of the objective function, that is, $J_t(y_t, e_t)$, is called the profit-to-go function and is independent of the initial inventory level x_t. The following analysis will be based on the value function $\Pi_t^*(x)$ instead of $\pi_t^*(x)$.

Before further analyzing the optimal policy, we first define the so-called (s, S, e) policy as follows.

Definition 2. An (s, S, e) policy is characterized by two stock levels (s_t^*, S_t^*) and a list sales effort e_t^*. If the inventory level is below the stock level s_t^*, it is increased to the stock level S_t^* and the sales effort e_t^* is exerted. If the inventory level is above the stock level s_t^*, then, nothing is ordered but only sales effort is exerted. In addition, the sales effort level is nondecreasing in the initial inventory level.

When $s_t^* = S_t^* (\equiv y_t^*)$, this policy is reduced the so-called base stock list sales effort policy, which is similar to the base stock list price policy proposed by Federgruen and Heching [5]. As we will show in the following analysis, when there is no setup cost for procurement, that is, the model (1), a base stock list sales effort policy is optimal, while when there is a fixed setup cost for each procurement, the (s, S, e) policy is optimal.

2.1. Analysis of the Finite Horizon Model. Before we derive the optimal policy, we first present the following result on the properties of the value function and the profit-to-go function.

Theorem 3. *For any $t = 1, 2, \ldots, T$, the function $J_t(y_t, e_t)$ is jointly concave in y_t and e_t. Moreover, the function $\Pi_t^*(x_t)$ is concave and nondecreasing in x_t.*

Proof. The proof will be done by induction. It is easy to verify $J_T(\cdot, \cdot)$ is jointly concave. Now, by assuming $J_{t+1}(y_{t+1}, e_{t+1})$ is jointly concave in y_{t+1} and e_{t+1}, we will show that the property holds even for $J_t(y_t, e_t)$, which is given by

$$\begin{aligned} J_t(y_t, e_t) = {} & (p_t - \alpha c_{t+1}) e_t \mu_t - (c_t - \alpha c_{t+1}) y_t - L_t(y_t, e_t) \\ & - v_t(e_t) + \alpha E[\Pi_{t+1}^*(y_t - e_t \xi_t)]. \end{aligned} \tag{5}$$

It is straightforward to verify that $(p_t - \alpha c_{t+1}) e_t \mu_t$ is jointly concave in p_t and e_t. For $E[\Pi_{t+1}^*(y_t - e_t \xi_t)]$, let $\lambda \in [0, 1]$ and consider two pairs $(y_t^{(1)}, e_t^{(1)})$ and $(y_t^{(2)}, e_t^{(2)})$, then, we get

$$\begin{aligned} & \Pi_{t+1}^* \left[\lambda y_t^{(1)} + (1 - \lambda) y^{(2)} - \left(\lambda e_t^{(1)} + (1 - \lambda) e_t^{(2)} \right) \xi_t \right] \\ & \quad = \Pi_{t+1}^* \left[\lambda \left(y^{(1)} - e_t^{(1)} \xi_t \right) + (1 - \lambda) \left(y_t^{(2)} - e_t^{(2)} \xi_t \right) \right] \\ & \quad \ge \lambda \Pi_{t+1}^* \left[y^{(1)} - e_t^{(1)} \xi_t \right] + (1 - \lambda) \Pi_{t+1}^* \left(y_t^{(2)} - e_t^{(2)} \xi_t \right). \end{aligned} \tag{6}$$

The above inequality holds since $\Pi_{t+1}^*(\cdot)$ is concave. This implies that $E[\Pi_{t+1}^*(y_t - e_t \xi_t)]$ is jointly concave in y_t and e_t. As the other terms of $J_t(y_t, e_t)$ are jointly concave in y_t and e_t, then $J_t(y_t, e_t)$ is jointly concave in y_t and e_t. The concavity and monotonicity of $\Pi_t^*(\cdot)$ is immediate. □

This theorem indicates the concavity of the value function and the existence of the optimal solution to the maximization problem. Thus, we can define the values (y_t^*, e_t^*) and $e^*(x)$ as follows:

$$\begin{aligned} (y_t^*, e_t^*) &= \arg\max_{(y_t, e_t)} \{J_t(y_t, e_t)\}, \\ e^*(x) &= \arg\max_{e_t} \{J_t(x, e_t)\}. \end{aligned} \tag{7}$$

Then, the above theorem shows that whenever the inventory level is less than y_t^*, the retailer should order to bring the inventory up to y_t^* and exert sales effort e_t^*. However, to show that the optimal policy is a base stock list sales effort policy, we still need to show that the optimal sales effort is increasing in the initial inventory level. This will be stated by the following theorem.

Theorem 4. *For any $t = 1, 2, \ldots, T$,*

(1) *the optimal sales effort $e_t^*(x)$ is increasing in x with $e_t^*(x) \geq e_t^*$;*

(2) *a base stock list sales effort with base stock y_t^* and e_t^* is optimal.*

Proof. (1) We first show that $J_t(y_t, e_t)$ is supermodular. Since the sum of supermodular function is supermodular, it suffices to establish supermodularity for each term of $J_t(y_t, e_t)$.

Because $\partial^2(p_t - \alpha c_{t+1})e_t\mu_t/\partial y_t \partial e_t \geq 0$, then, $p_t E[\min(y_t, e_t\xi_t)]$ is supermodular. Consider $-L_t(y_t, e_t) = -E[H_t(y_t - e_t\xi_t)]$,

$$-\frac{\partial^2 H_t\left(y_t - e_t\xi_t\right)}{\partial y_t \partial e_t} = H''\xi_t \geq 0. \tag{8}$$

So, $(\partial^2 - L_t(y_t - e_t\xi_t))/\partial y_t \partial e_t \geq 0$; thus, $L_t(\cdot, \cdot)$ is supermodular.

To show $E[\Pi_{t+1}^*(y_t - e_t\xi_t)]$ is supermodular, fix ξ_t and consider an arbitrary pair of inventory level $(y_t^{(1)}, y_t^{(2)})$ and any pair of sales effort levels $(e_t^{(1)}, e_t^{(2)})$ with $y_t^{(1)} \geq y_t^{(2)}$ and $e_t^{(1)} \geq e_t^{(2)}$. By the concavity and monotonicity of $\Pi_t^*(\cdot)$,

$$\begin{aligned}\Pi_t^*\left(y^{(1)} - e^{(1)}\xi_t\right) - \Pi_t^*\left(y^{(2)} - e^{(1)}\xi_t\right) \\ \geq \Pi_t^*\left(y^{(1)} - e^{(2)}\xi_t\right) - \Pi_t^*\left(y^{(2)} - e^{(2)}\xi_t\right),\end{aligned} \tag{9}$$

that is, $\Pi_t^*(y^{(1)} - e^{(1)}\xi_t) + \Pi_t^*(y^{(2)} - e^{(2)}\xi_t) \geq \Pi_t^*(y^{(2)} - e^{(1)}\xi_t) + \Pi_t^*(y^{(1)} - e^{(2)}\xi_t)$, because concave function has decreasing difference. Then, $E[\Pi_{t+1}^*(y_t - e_t\xi_t)]$ is supermodular in y_t and e_t.

The other terms of $J_t(y_t, e_t)$ are supermodular obviously. Thus, we obtain that $J_t(y_t, e_t)$ is supermodular. Then, by Theorem 2.3 in Vives [17], $e_t^*(x)$ is increasing in x.

(2) Immediate from part (1) and Theorem 3. □

Federgruen and Heching [5] show that the optimal selling price $p^*(x)$ is nonincreasing in x; that is, when the beginning inventory is too high, the retailer will choose a lower price to achieve a higher demand. Our result has the similar principle: when the beginning inventory is too high, the retailer will exert more sales effort in order for high demand.

Remark 5. We can see from the proofs that the structure of the optimal policy does not depend on the terminal value $\pi_{T+1}^*(x) = c_{T+1}x$. It only requires that $\Pi_{T+1}^*(x) = \pi_{T+1}^*(x) - c_{T+1}x$ is concave and decreasing in x. For example, when all unsatisfied demand is lost and all leftover inventory is salvaged at zero, that is, the terminal value $\pi_{T+1}^*(x) = \gamma \min\{x, 0\}$ in which $\gamma > \max_t p_t$ is the shortage cost, we have $c_{T+1} = \gamma$ and $\Pi_{T+1}^*(x) = \gamma \min\{-x, 0\}$, which is also concave and nonincreasing in x. Thus, the results still hold. Moreover, in the case of $\pi_{T+1}^*(x) = c_{T+1}x$, we can prove that the myopic policy is optimal when the system parameters are stationary.

3. Extension

So far, we have established analytical results associated with the sales efforts demand, in which the ordering cost is proportional to the order quantity and the order size is continuous in the sense that the retailer can order any quantity he/she wishes. However, in practice, the placement of an order always incurs additional transaction cost other than the cost of the product, for example, the cost of paper work, the cost of transhipment, and so forth. These transaction costs, or setup cost, are not neglectable in most times. Moreover, in some cases, there may be some constraints on the order amount the retailer should order for each transaction; for example, each order should be multiple of certain fixed amount. In this section, we extend our analysis to the combination of sales effort and inventory management with fixed order cost and batch orders.

3.1. Combining Sales Effort and Inventory Control with Setup Cost. Now, we first extend the analysis in the last section to the case where there is a setup cost for each order; that is, when we replenish the inventory, we need to pay the fixed order cost besides variable order cost. Keeping other notations unchanged, let k_t be fixed order cost at the beginning of period t. Then, when the order-up-to level $y_t > x_t$, a cost k_t occurs in addition to the variable cost, and when $y_t \leq x_t$, no procurement cost occurs. To make the problem tractable, we need the following assumption on the relationship between k_t, $t = 1, 2, \ldots, T$.

Assumption 6. For each period $0 \leq t \leq T$, $k_t \geq \alpha k_{t+1}$.

This assumption means that the discounted setup cost in future periods is less than the current setup cost. The case when the setup costs are constant among periods satisfies this assumption.

Then, we can formulate this problem as follows:

$$\pi_t^*\left(x_t\right) = c_t x_t + \max_{y_t \geq x_t, e_t \in [e_{\min}, e_{\max}]} - k_t \delta\left(y_t - x_t\right) + J_t\left(y_t, e_t\right), \tag{10}$$

where

$$\begin{aligned}J_t\left(y_t, e_t\right) = p_t E\left[e_t\xi_t\right] - c_t y_t - L_t\left(y_t, e_t\right) \\ - v_t\left(e_t\right) + \alpha E\left[\pi_{t+1}^*\left(y_t - e_t\xi_t\right)\right].\end{aligned} \tag{11}$$

With the definition of K-convex widely studied in traditional inventory literature (see, e.g., Scarf [18] and Veinott [19]), we can get the following results associating with the properties of the value function and the resulting optimal policy.

Theorem 7. *(a) For $t = 1, 2, \ldots, T$, $J_t(y_t, e_t)$ is continuous in (y_t, e_t) and $\lim_{|y|\to\infty} J_t(y_t, e_t) = -\infty$ for any $e_t \in [e_{\min}, e_{\max}]$. Hence, for any fixed y_t, there is a finite best sales effort $e_t^*(y_t)$.*

(b) For $t = 1, 2, \ldots, T$, both $J_t(y_t, e_t^*(y_t))$ and $\pi_t^*(x_t)$ are k_t-concave.

(c) For $t = 1, 2, \ldots, T$, there exist s_t and S_t with $s_t \le S_t$ such that it is optimal to order $S_t - x_t$ and exert sales effort $e_t^*(S_t)$ when $x_t < s_t$; otherwise, do not order and exert sales effort $e_t^*(x_t)$.

Proof. We will prove this theorem by induction. Assume parts (a), (b), and (c) hold for $t + 1$. Then, as it is easy to check that $J_t(y_t, e_t)$ is continuous, there is an optimal sales effort $e_t^*(y_t)$ for any fixed y_t.

We now focus on part (b). We will show that both $J_t(y_t.e^*(y_t))$ and $\pi^*(x_t)$ are k_t-concave.

For any $y \le y'$ and $\lambda \in [0, 1]$, we have π_{t+1}^* is k_{t+1}-concave. Then,

$$\begin{aligned} &\pi_{t+1}^*\left(\lambda\left(y - e_t^*(y)\xi_t\right) + (1-\lambda)\left(y' - e_t^*\left(y'\right)\xi_t\right)\right) \\ &\quad \ge \lambda \pi_{t+1}^*\left(y - e_t^*(y)\xi_t\right) \\ &\qquad + (1-\lambda)\pi_{t+1}^*\left(y' - e_t^*\left(y'\right)\xi_t\right) - (1-\lambda)k_{t+1}. \end{aligned} \tag{12}$$

Because H_t is convex, and $p_t E[e_t\xi_t] - L_t(y_t, e_t) = p_t e_t \mu_t - EH(y_t - e_t\xi_t) := L^+(y_t, e_t)$, it is easy to verify that $L^+(\cdot,\cdot)$ is jointly concave. Then,

$$\begin{aligned} &L^+\left(\lambda y + (1-\lambda)y', \lambda e^*(y) + (1-\lambda)e^*\left(y'\right)\right) \\ &\quad \ge \lambda L^+\left(y, e^*(y)\right) + (1-\lambda)L^+\left(y', e^*\left(y'\right)\right). \end{aligned} \tag{13}$$

Because $v(\cdot)$ is convex, then

$$\begin{aligned} &v\left(\lambda e_t^*(y) + (1-\lambda)e^*\left(y'\right)\right) \\ &\quad \ge \lambda v\left(e^*(y)\right) + (1-\lambda)v\left(e^*\left(y'\right)\right). \end{aligned} \tag{14}$$

Thus, we have proved that

$$\begin{aligned} &J_t\left(\lambda y + (1-\lambda)y', \lambda e_t^*(y) + (1-\lambda)e_t^*\left(y'\right)\right) \\ &\quad \ge \lambda J_t\left(y, e_t^*(y)\right) + (1-\lambda)J_t\left(y', e_t^*\left(y'\right)\right) - (1-\lambda)\alpha k_{t+1}. \end{aligned} \tag{15}$$

Because $e^*(\lambda y + (1-\lambda)y')$ is the best sales effort for inventory level $\lambda y + (1-\lambda)y'$, then

$$\begin{aligned} &J_t\left(\lambda y + (1-\lambda)y', e_t^*\left(\lambda y + (1-\lambda)\left(y'\right)\right)\right) \\ &\quad \ge J_t\left(\lambda y + (1-\lambda)y', \lambda e_t^*(y) + (1-\lambda)e_t^*\left(y'\right)\right). \end{aligned} \tag{16}$$

Therefore, $J_t(y_t, e_t^*(y_t))$ is αk_{t+1}-concave, so it is also k_t-concave, because of Assumption 6.

Thus, there exist $s_t < S_t$, such that S_t maximizes $J_t(y_t, e^*(y_t))$ and s_t is the smallest value of y, such that $J_t(S_t, e_t^*(S_t)) = J_t(y, e_t^*(y)) - k_t$, and

$$\pi_t^*(x_t) = \begin{cases} -k + J_t\left(S_t, e_t^*\left(S_t\right)\right) + c_t x & \text{if } x \le s_t, \\ J_t\left(x_t, e_t^*\left(x_t\right)\right) + c_t x & \text{if } x \ge s_t. \end{cases} \tag{17}$$

The k_t-concavity of $\pi_t^*(x_t)$ can be checked directly from k_t-concavity of $J_t(y_t, e_t^*(y_t))$; see Zipkin [20] for a proof. □

3.2. Combining Sales Effort and Batch Ordering. In this section, we will extend the analysis of the last section to the case where each order is in batch size. That is, orders must be placed in multiples of some standard batch size q, for example, a case, a barrel, or a truck load. To make the problem tractable, we assume the cost parameters are stationary, because even for a general inventory problem, a (r, Nq) policy is not optimal when the parameters are nonstationary. Let $c_1 = c_2 = \cdots = c_{T+1} = c$, $p_1 = p_2 = \cdots = p_{T+1} = p$, and $L_1(\cdot,\cdot) = L_2(\cdot,\cdot) = \cdots = L_{T+1}(\cdot,\cdot) = L(\cdot,\cdot)$. Therefore, for any policy Θ, the total discounted expected profit for this finite horizon dynamic programming problem is

$$\begin{aligned} \pi^{(T)}\left(x_0 \mid \Theta\right) &= \sum_{t=0}^{t=T} \alpha^t \left[-c\left(y_t - x_t\right) + pE\left[e_t\xi_t\right]\right. \\ &\qquad \left. - L\left(y_t, e_t\right) - v\left(e_t\right)\right] + \alpha^{T+1} c x_{T+1} \\ &= c x_0 + \sum_{t=1}^{t=T} \alpha^t \left[-(1-\alpha)c y_t + (p - \alpha c)e_t\mu_t\right. \\ &\qquad \left. - L\left(y_t, e_t\right) - v\left(e_t\right)\right] \\ &= c x_0 + \sum_{t=1}^{t=T} C^+\left(y_t, e_t\right), \end{aligned} \tag{18}$$

where x_0 is initial inventory and

$$C^+(y, e) = -(1-\alpha)cy + (p - \alpha c)e\mu - L(y, e) - v(e). \tag{19}$$

Then, we can get the following result for the joint optimal inventory and sales effort policy when each order is in batch size.

Theorem 8. *(1) $C^+(y, e)$ is jointly concave in y and e.*

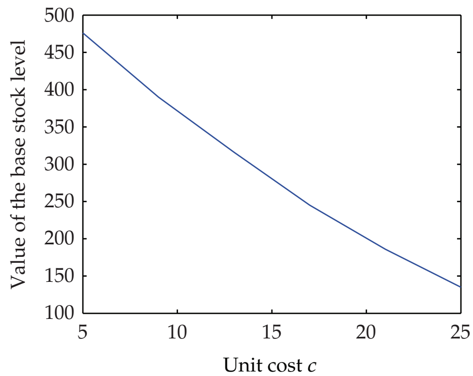
(2) $C^+(y, e)$ is continuous in y and e and $\lim_{|y|\to\infty}(C^+(y, e)) = -\infty$ for any fixed $e \in [e_{\min}, e_{\max}]$. Hence, for any fixed y, $C^+(y, e)$ has a finite maximizer, denoted by $e^*(y)$, which is increasing in y, and $C^+(y, e^*(y))$ is concave in y.

(3) The optimal policy is in a (r, Nq, e) type; that is, if the inventory at the beginning of each period is lower than r, then order minimum multiple of q, such that the inventory level y after ordering is greater than r, and exert sales effort $e^*(y)$; otherwise, do not order and exert sales effort $e^*(x)$.

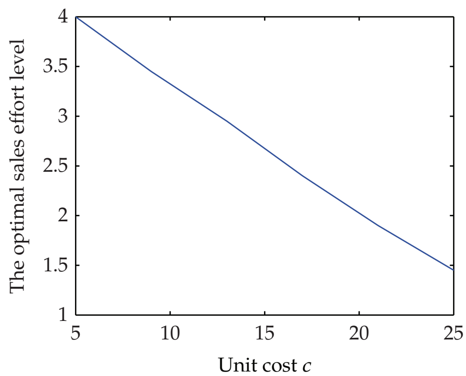
Proof. (1) Recalling Theorem 3, $C^+(y, e)$ is jointly concave in y and e.

Table 1: Basic parameter settings.

Demand Dist.	c	b	h	a_1	a_2	p	α
$\mu = 100, \sigma = 20$	10	10	5	400	0	30	0.8



(a) Base stock level



(b) Optimal sales effort level

Figure 1: An illustration of the optimal policies with respect to unit cost.

(2) From part (a), for any fixed y, there is an optimal sales effort $e^*(y)$. For the monotony of $e^*(y)$, consider the cross-derivative of $C^+(y,e)$,

$$\begin{aligned}\frac{\partial^2 C^+(y,e)}{\partial y \partial e} &= \frac{\partial^2 (p-\alpha c) e\mu}{\partial y \partial e} - \frac{\partial^2 L(y,e)}{\partial y \partial e} \\ &= \mu + E\left[H''(y-e\xi)\,\xi\right] \geq 0.\end{aligned} \tag{20}$$

Thus, $C^+(y,e)$ is supermodular in y and e. So, $e^*(y)$ is increasing in y.

For the concavity of $C^+(y, e^*(y))$, we consider

$$\begin{aligned}&C^+(\alpha y_1 + (1-\alpha) y_2, e^*(\alpha y_1 + (1-\alpha) y_2)) \\ &\quad\geq C^+(\alpha y_1 + (1-\alpha) y_2, \alpha e^*(y_1) + (1-\alpha) e^*(y_2)) \\ &\quad\geq \alpha C^+(y_1, e^*(y_1)) + (1-\alpha) C^+(y_2, e^*(y_2)).\end{aligned} \tag{21}$$

The first inequality holds, because $e^*(\alpha y_1 + (1-\alpha)y_2)$ is the best sales effort for inventory level $\alpha y_1 + (1-\alpha)y_2$; the second inequality holds, because of $C^+(y,e)$'s joint concavity.

(3) Let $y^* = \arg\max C^+(y, e^*(y))$, and choose r, such that $C^+(r, e^*(r)) = C^+(r+q, e^*(r+q))$ and $r \leq y^* \leq r+q$. Denote the (r, Nq, e) policy by Θ^*. For any other policy Θ, under the same demand sample path, we have $y_t^* = y_t$ or $|y_t^* - y_t| \geq q$ for any $1 \leq t \leq T$. Then, $C^+(y_t^*, p^*(y_t^*)) \geq C^+(y_t, p^*(y_t))$ for any $1 \leq t \leq T$. Thus, $\pi^{(T)}(x_0 \mid \Theta^*) \geq \pi^{(T)}(x_0 \mid \Theta)$. Finally, Θ^* is an optimal policy. □

4. Numerical Study

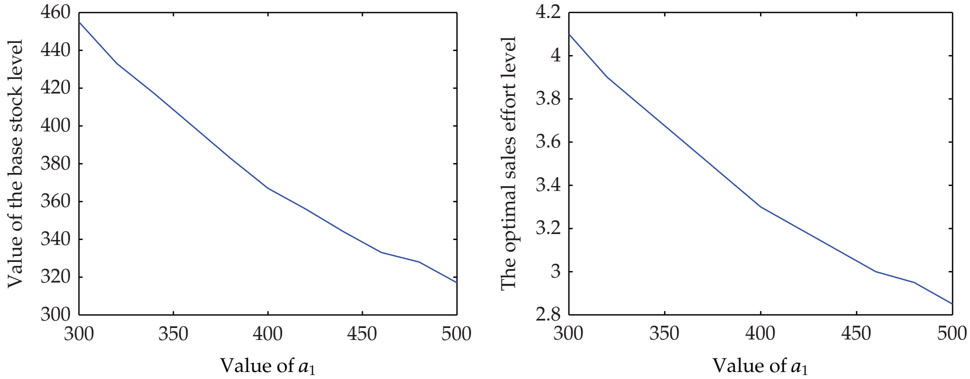
In this section, basing on model (1), we report a numerical study conducted to attain qualitative insights into the structure of optimal policies and their sensitivity with respect to several parameters. Among the major questions investigated, we focus in particular on how the optimal decisions vary against system parameters, that is, the demand uncertainty, the unit cost, and the sales effort cost.

Before conducting the numerical studies, we first setup the configuration of the basic model. The single period inventory holding and backlogging cost setting is $H_t(I) = h_t I^+ + b_t I^-$. The system is assumed to run in a stationary setting under a planning horizon with 5 periods, that is, $T = 5$, $h_1 = \cdots h_T = h$, $p_1 = \cdots p_T = p$, $c_1 = \cdots c_T = c$, and $b_1 = \cdots b_T = b$. The sales effort cost function $v(e) = a_1(e-1)^2 + a_2(e-1)$ for $6 \geq e \geq 1$, in which a_1 measures the diseconomy of scale of the sales effort level. This sales effort function is sufficient for us to study the impact of cost parameters in our problem as we can approximate other forms of functions by a quadratic function as a direct result of Taylor expansion. This setting appears in several literature references, for example, Taylor [21]. The random part of the demand, ξ, is assumed to follow a normal distribution, which is a typical setting both in practice and academy. The values of the system parameters are provided in Table 1.

Figures 1(a) and 1(b) illustrate that both the optimal base stock level and the sales effort level decrease as the unit cost increases. This is rather intuitive as the high unit cost reduces the retailer's motivation to order more, which results in less sales effort exerted.

Figures 2(a) and 2(b) illustrate a similar pattern for the sales effort cost parameters a_1. However, the undermining reason is different. In this case, the high sales effort cost reduces the incentives to stimulate demand, which results in the low base stock level.

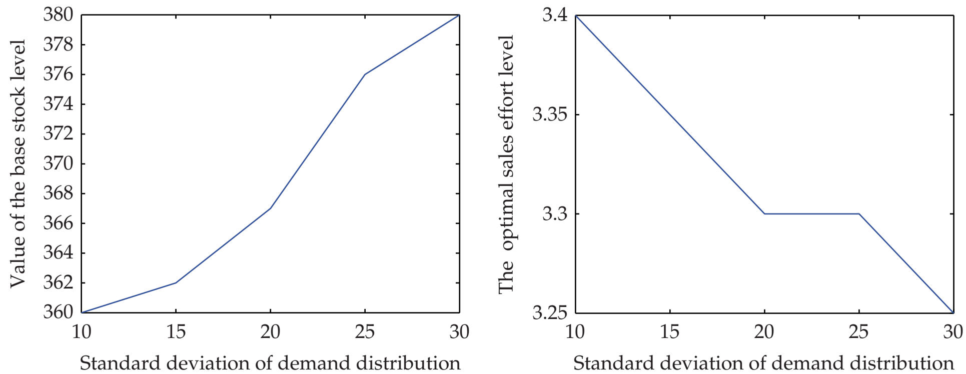
Figures 3(a) and 3(b) illustrate the effect of demand uncertainty on the optimal policies. Interestingly, it shows that when the demand uncertainty becomes large, the optimal order-up-to level is large too. The reason is that when the shortage penalty is high, given the inventory level, large



(a) Base stock level

(b) Optimal sales effort leve

Figure 2: An illustration of the optimal policies with respect to sales effort cost.



(a) Base stock level

(b) Optimal sales effort level

Figure 3: An illustration of the optimal policies with respect to demand uncertainty.

demand uncertainty will result in high penalty cost, which induce the retailer to order more to avoid such shortage. However, the numerical results show that the optimal sales effort level is decreasing in the demand uncertainty. This is because, when the demand has a multiplicative form with the sales effort level, high sales effort level also induces high variance of the demand distribution, although it increases the expected demand; that is, the cost caused by additional demand variance is larger than the profits made by additional expected demand.

5. Conclusion

In this paper, we study joint inventory and sales effort management of a retailer in a broad context and investigate optimal policies for a single item, periodic-review system. The retailer decides on the order replenishment and sales effort level jointly in each period, in which the demand has a multiplicative form with the sales effort level. His objective is to find a joint optimal inventory replenishment and sales effort policy to maximize the discounted profit over a finite horizon. We consider three cases depending on the cost structure and the constrains on order quantity: linear ordering cost without setup cost, linear ordering cost with setup cost, and batch size for each ordering. Through our analysis, we find that a base stock list sales effort policy, an (s, S, e) policy, and a (r, Nq, e) policy are optimal for these three cases, respectively. Numerical studies show that the base stock level and the optimal sales effort level are decreasing with the unit cost and the diseconomy of scale for the sales effort. However, although the base stock level is increasing in the demand uncertainty, the corresponding sales effort level is decreasing, which implies that the cost caused by additional demand variance is larger than the profits made by additional expected demand.

Several extensions based on our model can be analyzed in future research. First, models should be developed to address the joint consideration on the price, sales effort, and inventory decisions. In our paper, we assume that the selling price is exogenously determined and provide some reasonable arguments. However, in practices, both the price change and the sales effort are exerted, especially in the B2C e-commence industry. In this case, why does the retailer take both actions? what is the joint effect of pricing and sales effort decisions on the optimal inventory decisions? Second, the financial state of the retailer and its effect on the optimal decisions should be considered. In our analysis, we implicitly assume the retailer has enough cash on hand to make procurement and sales effort decisions. However, in practice, the working capital is always limited. In this case, how should the retailer allocate these limited resources between marketing and inventory? The effect of financial states on the optimal policy is worth investigation.

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Research Article

Several Types of Convergence Rates of the M/G/1 Queueing System

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We study the workload process of the M/G/1 queueing system. Firstly, we give the explicit criteria for the geometric rate of convergence and the geometric decay of stationary tail. And the parameters ε_0 and s_0 for the geometric rate of convergence and the geometric decay of the stationary tail are obtained, respectively. Then, we give the explicit criteria for the rate of convergence and decay of stationary tail for three specific types of subgeometric cases. And we give the parameters ε_1 and s_1 of the rate of convergence and the decay of the stationary tail, respectively, for the subgeometric rate $r(n) = \exp(sn^{1/(1+\alpha)})$, $s > 0$, $\alpha > 0$.

1. Introduction

We consider several types of convergence rates of the M/G/1 queueing system by using drift conditions. The M/G/1 queueing system discussed here is that the arrivals form a Poisson process with parameter λ. The service times $v_1, v_2, \ldots$ for the customers are independently identically distributed random variables with a common distribution function $B(x)$. Let

$$\frac{1}{\mu} \equiv \int_0^\infty x dB(x), \qquad \rho \equiv \frac{\lambda}{\mu}, \tag{1}$$

where $\mu > 0$ is a constant, and ρ is called the service intensity. Denote the workload process of the M/G/1 queueing system by $W(t)$; then, $\{W(t), t \geq 0\}$ is a Markov process.

Ergodicity, specially ordinary ergodicity, has been well studied for Markov processes. There are a large volume of references devoted to the geometric case (or exponential case) and the subgeometric case (e.g., see [1–3]). Hou and Liu [4, 5] discussed ergodicity of embedded M/G/1 and GI/M/n queues, polynomial and geometric ergodicity for M/G/1-type Markov chain, and processes by generating function of the first return probability. Hou and Li [6, 7] obtained the explicit necessary and sufficient conditions for polynomial ergodicity and geometric ergodicity for the class of quasi-birth-and-death processes by using matrix geometric solutions.

There is much work on decay of the tail in the stationary distribution. Li and Zhao [8, 9] studied heavy-tailed asymptotic and light-tailed asymptotic of stationary probability vectors of Markov chains of GI/G/1 type. Jarner and Roberts [10] discussed Foster-Lyapounov-type drift conditions for Markov chains which imply polynomial rate convergence to stationarity in appropriate V-norms. Jarner and Tweedie [11] proved that the geometric decay of the tail in the stationary distribution is a necessary condition for the geometric-ergodicity for random walk-type Markov chains. We will discuss several types of ergodicity and the tail asymptotic behavior of the stationary distribution by Foster-Lyapounov-drift conditions. We give the relationship of ergodicity and the decay of the tail in the stationary distribution for h-skeleton chain in M/G/1 queueing system, which is different from the former; ergodicity and the decay of the tail are discussed, respectively. We shall give the bounded interval in which geometric and subexponential parameter s lies and prove that it is determined by the tail of the service distribution. The parameters ε_0 and s_0 for geometric rate of convergence and the geometric decay of the stationary tail are obtained, respectively. We shall also give explicit criteria for the rate of convergence and decay of stationary tail for three specific types of subgeometric cases (Case 1: the rate function $r(n) = \exp(sn^{1/(1+\alpha)})$, $\alpha > 0$, $s > 0$; Case 2: polynomial rate function

$r(n) = n^{\alpha}, \alpha > 0$; Case 3: logarithmic rate function $r(n) = \log^{\alpha} n, \alpha > 0$). And we give the parameters ε_1 and s_1 of the rate of convergence and the decay of the stationary tail, respectively, for the subgeometric rate in Case 1.

We organize the paper as follows. In Section 2, we shall introduce basic definitions and theorems, including the main result, Theorem 6. In Section 3, we shall prove the geometric rates of convergence in Theorem 6. In Section 4, we shall prove the rates of convergence for the subgeometric Cases 1–3 in Theorem 6.

2. Basic Definitions and the Main Results

Let $\{X_n, n \geq 0\}$ be a discrete time Markov chain on the state space $(E, \mathscr{E})$ with transition kernel P. Assume that it is ψ-irreducible, aperiodic, and positive recurrent. Now, we discuss the convergence in f-norm of the iterates P^n of the kernel to the stationary distribution π at rate $r := (r(n), n \geq 0)$; that is, for all $A \in \mathscr{E}$,

$$\lim_{n \to \infty} r(n) \left\| P^{(n)}(x, A) - \pi(A) \right\|_f = 0, \quad \pi\text{-a.e.}, \tag{2}$$

where $f : E \to [1, +\infty)$ satisfies $\pi(f) < +\infty$, and for all signed measures σ, the f-norm $\|\sigma\|_f$ is defined as $\sup_{|g| \leq f} |\sigma(g)|$.

Geometric Rate Function. That is, the function r satisfies

$$0 < \liminf \frac{\log r(n)}{n} \leq \limsup \frac{\log r(n)}{n} < +\infty. \tag{3}$$

Subgeometric Rate Function. That is, the function r satisfies

$$\lim_{n \to \infty} \frac{\log r(n)}{n} = 0. \tag{4}$$

The class of subgeometric rates function includes polynomial rates functions; that is, $r(n) = n^{\alpha}, \alpha > 0$, and rate functions which increase faster than the polynomial ones; $r(n) = \exp(s n^{1/(1+\alpha)}), \alpha > 0, s > 0$.

We shall discuss geometric rates of convergence $r(n) = \exp(sn), s > 0$, subgeometric rate of convergence $r(n) = \exp(s n^{1/(1+\alpha)}), \alpha > 0, s > 0$, polynomial rate of convergence $r(n) = n^{\alpha}, \alpha > 0$, and logarithmic rate of convergence $r(n) = \log^{\alpha} n, \alpha > 0$.

Condition $D(\phi, V, C)$. There exist a function $V : E \to [1, \infty)$, a concave monotone nondecreasing differentiable function $\phi : [1, \infty] \to (0, \infty]$, a measurable set C, and a finite constant b such that

$$\Delta V(x) = PV(x) - V(x) \leq \phi \circ V + b I_C, \tag{5}$$

where I_C is the indicator function of the set C.

Now we shall give Theorems 1 and 2 which we will use in this paper.

Theorem 1 (Theorem 14.0.1 in [1]). *If $D(\phi, V, C)$ holds for some petite set C and there exists $x_0 \in E$ such that $V(x_0) < \infty$, then there exists a unique invariant distribution π, $\pi(\phi \circ V) < \infty$ and*

$$\lim_{n \to \infty} \left\| P^{(n)}(x, A) - \pi(A) \right\|_{\phi \circ V} = 0, \quad \pi\text{-a.e.}, \tag{6}$$

where $\phi \circ V(x) \geq 1, x \in E$.

Theorem 2 (Proposition 2.5 in Douc et al. [12]). *Let P be a ψ-irreducible and aperiodic kernel. Assume that $D(\phi, V, C)$ holds for function ϕ with $\lim_{t \to +\infty} \phi'(t) = 0$, a petite set C, and a function V with $\{V < +\infty\} \neq \emptyset$. Then, there exists an invariant probability measure π, and for all x in the full and absorbing set $\{V < \infty\}$,*

$$\lim_{n \to \infty} r_{\phi}(n) \left\| P^{(n)}(x, A) - \pi(A) \right\| = 0, \tag{7}$$

where $(r_{\phi}(n)) = \phi \circ H_{\phi}^{-1}(n)$, $H_{\phi}(v) := \int_1^v (1/\phi(x)) dx$.

Since ϕ is a concave monotone nondecreasing differentiable function, ϕ' is nonincreasing. Then, there exists $c \in [0, 1)$, such that $\lim_{t \to +\infty} \phi'(t) = c$. In Theorem 2, for the case $c \in (0, 1)$, condition $D(\phi, V, C)$ implies that the chain is geometric ergodic, but the rate in the geometric convergence property cannot be achieved under the condition that $\lim_{t \to +\infty} \phi'(t) = c > 0$.

The workload process $\{W(t), t \geq 0\}$ of the M/G/1 queueing system is a Markov process on the state space $\{R_+, \mathscr{B}(R_+)\}$. $\{W(nh)\}$ is an h-skeleton of $\{W(t), t \geq 0\}$. We choose $h = 1$, and denote $\{W(nh)\}_{h=1}$ by $\{X_n\}$. Suppose that the workload can be decreased by $\min\{1, X_n\}$ during the time interval $[n, n+1]$. And suppose that the transition kernel of $\{X_n\}$ is $P(x, \cdot)$. For convenience, let $\sigma_k = \nu_1 + \nu_2 + \cdots + \nu_k$. Then,

$$X_{n+1} = \sum_{k=1}^{+\infty} I_{\{\xi = k\}} \sigma_k + \min\{X_n - 1, 0\}, \quad n = 1, 2, \ldots, \tag{8}$$

where ξ is the number of arrivals in a time interval of unit length.

Lemma 3. *$\{X_n\}$ is irreducible and aperiodic.*

Proof. Let φ be a measure on R_+ with $\varphi(\{0\}) = 1, \varphi(\{0\}^c) = 0$. For all $x \in R_+$, there exists a k satisfying $k - 1 < t \leq k$, such that

$$P^k(x, \{0\}) \geq \exp(-\lambda k) > 0. \tag{9}$$

Hence, $\{X_n\}$ is irreducible. From

$$P^n(x, \{0\}) > 0, \quad n \geq 1, \tag{10}$$

we know that $\{X_n\}$ is also aperiodic. □

Lemma 4. $C = [0, c]$ *is petite set, where c ($c \geq 0$) is a real number.*

Proof. Let $[c]$ be the maximum integer no more than c. Since

$$P^{[c]+1}(x, \{0\}) \geq \exp\{-\lambda([c] + 1)\} > 0, \quad \forall x \in C, \tag{11}$$

and C is a closed set, we know that $\min_{x\in C}P^{[c]+1}(x,\{0\}) > 0$. Let ν_2 be a measure on R_+ satisfying, for all $B \in \mathscr{B}(R_+)$,

$$\nu_2(B) = \begin{cases} 0, & \{0\} \notin B, \\ \min_{x\in C} P^{[c]+1}(x,\{0\}), & \{0\} \in B. \end{cases} \tag{12}$$

Obviously, for all $x \in C$,

$$P^{[c]+1}(x,B) \geq \nu_2(B), \quad \forall \mathscr{B}(R_+). \tag{13}$$

Thus, we get that C is a petite set. □

Lemma 5. *The Markov chain $\{X_n\}$ is stochastically monotonic.*

Proof. For every fixed y, from

$$\begin{aligned} P\{X_{n+1} \leq y \mid X_n = x\} &= P\left\{\sum_{k=1}^{+\infty} I_{\{\xi=k\}}\sigma_k \leq y\right\}, \quad \forall x \in [0,1], \\ P\{X_{n+1} \leq y \mid X_n = x\} &= P\left\{\sum_{k=1}^{+\infty} I_{\{\xi=k\}}\sigma_k + x - 1 \leq y\right\}, \\ &= P\left\{\sum_{k=1}^{+\infty} I_{\{\xi=k\}}\sigma_k - y - 1 \leq -x\right\}, \quad \forall x \in [1,+\infty), \end{aligned} \tag{14}$$

we obtain that $P\{X_{n+1} \leq y \mid X_n = x\}$ is nonincreasing in x. That is, $\{X_n\}$ is stochastically monotonic. □

For two sequences u_n and v_n, we write $u_n \asymp v_n$, if there exist positive constants c_1 and c_2 such that, for large n, $c_1 u_n \leq v_n \leq c_2 u_n$.

Let us say that the distribution function G of a random variable ξ is in $\mathscr{G}^+(r)$ if

$$Ee^{s\xi} = \int_0^{+\infty} e^{sx} G(dx) < +\infty, \quad 0 < s \leq r; \tag{15}$$

the distribution function G of a random variable ξ is in $\mathscr{S}^+(r,\alpha)$ if

$$Ee^{s\xi^{1/(1+\alpha)}} = \int_0^{+\infty} e^{sx^{1/(1+\alpha)}} G(dx) < +\infty, \quad 0 < s \leq r, \tag{16}$$

where $r > 0$, and $\alpha > 0$.

Now, we give the main result.

Theorem 6. *Suppose that $\rho < 1$ and π is the stationary distribution of $\{X_n\}$.*

(1) If $B \in \mathscr{G}^+(r)$, then one has

$$\int_1^{+\infty} \exp(sx)\,\pi(dx) < +\infty, \quad \forall 0 < s < s_0, \tag{17}$$

where s_0 is the minimum positive root of the equation $Ee^{s\nu_1} = 1 + (s/\lambda)$. Moreover, $\{X_n\}$ is geometrically ergodic,

$$\lim_{n\to\infty} e^{\varepsilon_0 n} \left\| P^{(n)}(x,A) - \pi(A) \right\| = 0, \tag{18}$$

where $\varepsilon_0 = \lambda + \bar{s} - \lambda E e^{\bar{s}\nu_1}$, and $\bar{s} \in (0,s_0)$ is a root of the equation $E\nu_1 e^{s\nu_1} = 1/\lambda$.

(2) If $B \in \mathscr{S}^+(r,\alpha)$, then one has

$$\int_1^{+\infty} x^{-\alpha/(1+\alpha)} \exp\left(sx^{1/(1+\alpha)}\right) \pi(dx) < \infty, \quad 0 < s < s_1, \tag{19}$$

where s_1 is the minimal positive solution of $\beta(s) = 0$ ($\beta(s) = x^{\alpha/(1+\alpha)}\{1 - \sum_{k=0}^{\infty}(\lambda^k e^{-\lambda}/k!)E\exp\{s(\sigma_k - 1 + x)^{1/(1+\alpha)} - sx^{1/(1+\alpha)}\}\}$). And

$$\lim_{n\to\infty} n^{(-\alpha/(1+\alpha))} \exp\left(\varepsilon_1 n^{1/(1+\alpha)}\right) \left\| P^{(n)}(x,\cdot) - \pi(\cdot) \right\| = 0, \tag{20}$$

where $\varepsilon_1 = \max_{s\in(0,s_1)}[(\alpha+1)\beta(s)]^{1/(\alpha+1)}$.

(3) If there exists a constant $\alpha > 1$, such that $E\nu_1^\alpha = \int_0^{+\infty} x^\alpha B(dx) < +\infty$, then

$$\begin{aligned} &\lim_{n\to\infty} n^{\alpha-1} \left\| P^{(n)}(x,\cdot) - \pi(\cdot) \right\| = 0, \\ &\int_1^{+\infty} x^{\alpha-1} \pi(dx) < +\infty. \end{aligned} \tag{21}$$

(4) If there exists an integer number $\alpha > 0$, such that $E\nu_1 \log^\alpha(\nu_1 + 1) = \int_0^{+\infty} x\log^\alpha(x+1)B(dx) < +\infty$, then

$$\begin{aligned} &\lim_{n\to\infty} \log^\alpha n \left\| P^{(n)}(x,\cdot) - \pi(\cdot) \right\| = 0, \\ &\int_1^{+\infty} \log^\alpha x\,\pi(dx) < +\infty. \end{aligned} \tag{22}$$

We shall prove Theorem 6 in Sections 3 and 4.

3. Geometric Rate of Convergence

The Markov chain $\{X_n\}$ is geometrically ergodic if (2) holds with $r(n) = e^{sn}$ for some $s > 0$. By Theorem 15.0.1 in [1], an equivalent condition of geometric ergodicity is that there exist a petite set C, constants $\beta > 0$ and $b < \infty$, and a function $V \geq 1$ finite for at least one $x_0 \in E$ satisfying

$$\Delta V(x) < -\beta V(x) + bI_C, \quad x \in E. \tag{23}$$

By using the drift previous condition, we usually obtain the geometric ergodicity, but we could not get the parameters for the geometric rate of convergence. Now, we will study the geometric decay of the stationary tail and geometric rate of convergence to the stationary distribution.

Let $V_s(x) = \exp(sx)$, $0 < s \leq r$, $x \in R_+$. Taking the petite set $C = [0,1]$, for all $x \in [0,1]$,

$$\begin{aligned} \Delta V_s(x) &= PV_s(x) - V_s(x) \\ &< \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\exp\{s(\nu_1 + \cdots + \nu_k + x)\} \\ &\leq \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(Ee^{s\nu_1}\right)^k e^x = \exp\left(\lambda E e^{s\nu_1}\right) e^x. \end{aligned} \tag{24}$$

Since $B \in \mathcal{G}^+(r)$(i.e., $Ee^{s\nu_1} < \infty, 0 < s \le r$), we know that

$$\Delta V_s(x) < \exp\left(\lambda E e^{s\nu_1}\right) e^x < \infty, \quad \forall x \in [0,1]. \quad (25)$$

For all $x > 1$ (i.e., $x \in C^C$),

$$\begin{aligned} \Delta V_s(x) &= PV_s(x) - V_s(x) \\ &= -\exp(sx) + \exp(sx)\sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\exp\{s(\sigma_k - 1)\} \\ &= -\exp(sx)\left\{1 - \exp(-\lambda - s)\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\left(Ee^{s\nu_1}\right)^k\right\} \\ &= -\exp(sx)\left\{1 - \exp\left(-\lambda - s + \lambda E e^{s\nu_1}\right)\right\}. \end{aligned} \quad (26)$$

Let $\beta(s) = 1 - \exp(-\lambda - s + \lambda E e^{s\nu_1})$. Now, we prove that there exists an $s' > 0$ such that $\beta(s') > 0$. By the stated condition $B \in \mathcal{G}^+(r)$, we know that $\beta(s)$ is a finite differentiable function for $s \in [0, r)$. Furthermore,

$$\begin{aligned} \beta(0) &= \left\{1 - \exp\left(-\lambda - s + \lambda E e^{s\nu_1}\right)\right\}_{s=0} = 0, \\ \beta'(s)\big|_{s=0} &= \left\{-\left(-1 + \lambda E \nu_1 e^{s\nu_1}\right)\exp\left(-\lambda - s + \lambda E e^{s\nu_1}\right)\right\}_{s=0} \\ &= 1 - \rho > 0. \end{aligned} \quad (27)$$

Proposition 7. *Suppose that $\rho < 1$ and π is the stationary distribution of $\{X_n\}$. If $B \in \mathcal{G}^+(r)$; then,*

$$\int_1^{+\infty} \exp(sx)\,\pi(dx) < +\infty, \quad \forall 0 < s < s_0, \quad (28)$$

where s_0 is the minimum positive root of the equation $Ee^{s\nu_1} = 1 + (s/\lambda)$.

Proof. By (27), we know that $\beta(0) = 0$ and $\beta'(0) > 0$. So, there exists an $s' \in (0, r)$ such that $\beta(s') > 0$. The function $\beta(s)$ is continuous in the interval $[s', r]$, and it is easy to see that $\beta(r) < 0$, $\beta(s') > 0$. By the zero theorem, we know that there exists at least one root of the equation $\beta(s) = 0$ (i.e., $Ee^{s\nu_1} = 1 + (s/\lambda)$). Let s_0 be the minimum positive root; then, $\beta(s) > 0$, for all $0 < s < s_0$.

Let $b = \sup_{s\in(0,s_0)} \exp(\lambda E e^{s\nu_1}) < \infty$; then, we have

$$\begin{aligned} \Delta V_s(x) &\le -\beta(s)V_s(x) + bI_{[0,1]} \\ &= -\phi \circ V_s(x) + bI_{[0,1]}, \quad 0 < s < s_0, \ x \in R_+, \end{aligned} \quad (29)$$

where $\phi(x) = \beta(s)x$ (i.e., condition $D(\phi, V_s, C)$ holds). By Theorem 1, we know that $\pi(\phi \circ V_s) < \infty$; that is,

$$\int_1^{+\infty} \exp(sx)\,\pi(dx) < \infty, \quad 0 < s < s_0. \quad (30)$$

□

Proposition 8. *Suppose that $\rho < 1$ and π is the stationary distribution of $\{X_n\}$. If $B \in \mathcal{G}^+(r)$; then,*

$$\lim_{n\to\infty} e^{\varepsilon_0 n}\left\|P^{(n)}(x, A) - \pi(A)\right\| = 0, \quad (31)$$

where $\varepsilon_0 = \alpha(\bar{s})$, $\alpha(s) = \lambda + s - \lambda E e^{s\nu_1}$, and $\bar{s} \in (0, s_0)$ is the root of the equation $1 - \lambda E \nu_1 e^{s\nu_1} = 0$.

Proof. From (29),

$$\Delta V_s(x) \le -\beta(s)V_s(x) + bI_{[0,1]}, \quad 0 < s < s_0, \ x \in R, \quad (32)$$

where $\beta(s) = 1 - \exp(-\lambda - s + \lambda E e^{s\nu_1})$, and s_0 is the minimum positive root of the equation $Ee^{s\nu_1} = 1 + (s/\lambda)$. We have

$$\begin{aligned} PV_s(x) &\le \exp\left(-\lambda - s + \lambda E e^{s\nu_1}\right)V_s(x) + bI_{[0,1]}, \\ &\qquad 0 < s < s_0, \ x \in R_+. \end{aligned} \quad (33)$$

From Lemma 5, we know that $\{X_n\}$ is a stochastically monotonic Markov chain. By using Theorem 1.1 in [13], we have

$$\begin{aligned} \lim_{n\to\infty} \exp\left\{\left(\lambda + s - \lambda E e^{s\nu_1}\right)n\right\}\left\|P^{(n)}(x,\cdot) - \pi(\cdot)\right\| = 0, \\ 0 < s < s_0. \end{aligned} \quad (34)$$

Let $\alpha(s) = \lambda + s - \lambda E e^{s\nu_1}$. From $\alpha''(s) = -\lambda E \nu_1^2 e^{s\nu_1} < 0$, we know that $\alpha(x)$ is a concave function. Together with $\alpha(0) = 0$, $\alpha(s_0) = 0$, there exists a unique point $\bar{s} \in (0, s_0)$, such that $\alpha'(\bar{s}) = 1 - \lambda E \nu_1 e^{\bar{s}\nu_1} = 0$, and $\alpha(s)$ has a maximum $\alpha(\bar{s})$ at the point $\bar{s}$ in the interval $(0, s_0)$. So,

$$\lim_{n\to\infty} e^{\varepsilon_0 n}\left\|P^{(n)}(x, A) - \pi(A)\right\| = 0, \quad (35)$$

where $\varepsilon_0 = \alpha(\bar{s})$. The proof is completed. □

4. Subgeometric Rates of Convergence for Cases 1–3

Case 1 (The Rate Function $r(n) = \exp(sn^{1/(1+\alpha)})$). The rate function $r(n) = \exp(sn^{1/(1+\alpha)})$, which increases to infinity faster than the polynomial one, and slower than the geometrical one, has been discussed only recently in the literature.

Proposition 9. *Suppose that $\rho < 1$ and π is the stationary distribution of $\{X_n\}$. If $B \in \mathcal{S}^+(r, \alpha)$, then one has*

$$\int_1^{+\infty} x^{-\alpha/(1+\alpha)} \exp\left(sx^{1/(1+\alpha)}\right)\pi(dx) < \infty, \quad 0 < s < s_1, \quad (36)$$

where s_1 is the minimal positive solution of $\beta(s) = 0$ ($\beta(s) = x^{\alpha/(1+\alpha)}\{1 - \sum_{k=0}^{\infty}(\lambda^k e^{-\lambda}/k!)E\exp\{s(\sigma_k - 1 + x)^{1/(1+\alpha)} - sx^{1/(1+\alpha)}\}\}$). And

$$\lim_{n\to\infty} n^{-\alpha/(1+\alpha)} \exp\left(\varepsilon_1 n^{1/(1+\alpha)}\right)\left\|P^{(n)}(x,\cdot) - \pi(\cdot)\right\| = 0, \quad (37)$$

where $\varepsilon_1 = \max_{s\in(0,s_1)}[(\alpha + 1)\beta(s)]^{(1/(1+\alpha))}$.

Proof. Let $V_s(x) = \exp(sx^{1/(1+\alpha)})$, $0 < s \le r, x \in R_+$. For all $x \in C$,

$$\begin{aligned}
&\Delta V_s(x) \\
&= PV_s(x) - V_s(x) \le -\exp\left(sx^{1/(1+\alpha)}\right) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E \exp\left\{s(\sigma_k + x)^{1/(1+\alpha)}\right\} \\
&\le -\exp\left(sx^{1/(1+\alpha)}\right) + \exp\left(sx^{1/(1+\alpha)}\right) \\
&\quad \cdot \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E \exp\left\{s\left(v_1^{1/(1+\alpha)} + v_2^{1/(1+\alpha)} + \cdots + v_k^{1/(1+\alpha)}\right)\right\} \\
&= -\exp\left(sx^{1/(1+\alpha)}\right) + \exp\left(sx^{1/(1+\alpha)}\right) \\
&\quad \cdot \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \left[E \exp\left(sv_1^{1/(1+\alpha)}\right)\right]^k = -\exp\left(sx^{1/(1+\alpha)}\right) \\
&\quad + \exp\left(sx^{1/(1+\alpha)}\right) \exp\left\{\lambda E \exp\left(sv_1^{1/(1+\alpha)}\right)\right\},
\end{aligned} \tag{38}$$

where the second inequality holds by using the condition that $f(x) = x^{1/(1+\alpha)}$ is concave. Let $\alpha(s) = -\exp(sx^{1/(1+\alpha)}) + \exp(sx^{1/(1+\alpha)}) \exp\{\lambda E \exp(sv_1^{1/(1+\alpha)})\}$. Since $E\exp(sv_1^{1/(1+\alpha)}) < \infty$, we know that

$$\Delta V_s(x) \le \alpha(s) < \infty, \quad x \in C. \tag{39}$$

For all $x \in C^C$,

$$\begin{aligned}
\Delta V_s(x) &= PV_s(x) - V_s(x) \\
&= -\exp\left(sx^{1/(1+\alpha)}\right) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E \exp\left\{s(\sigma_k - 1 + x)^{1/(1+\alpha)}\right\} \\
&= -\exp\left(sx^{1/(1+\alpha)}\right) \\
&\quad \cdot \left\{1 - \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E \exp\left\{s(\sigma_k - 1 + x)^{1/(1+\alpha)} - sx^{1/(1+\alpha)}\right\}\right\} \\
&= -\frac{\exp\left(sx^{1/(1+\alpha)}\right)}{x^{1/(1+\alpha)}} x^{\alpha/(1+\alpha)} \\
&\quad \cdot \left\{1 - \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E \exp\left\{s(\sigma_k - 1 + x)^{1/(1+\alpha)} - sx^{1/(1+\alpha)}\right\}\right\}.
\end{aligned} \tag{40}$$

Let

$$\begin{aligned}
&\beta(s) \\
&= x^{\alpha/(1+\alpha)} \\
&\quad \cdot \left\{1 - \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E \exp\left\{s(\sigma_k - 1 + x)^{1/(1+\alpha)} - sx^{1/(1+\alpha)}\right\}\right\}.
\end{aligned} \tag{41}$$

Now, we prove that there exists an $s_1 > 0$ such that $\beta(s) > 0$ for all $s \in (0, s_1)$. Similar to the proof of the case $x \in C$, we know that $\beta(s)$ is a finite function for $s \in [0, r)$. Furthermore,

$$\begin{aligned}
\beta(0) &= x^{\alpha/(1+\alpha)} \left(1 - \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}\right) = 0, \\
\beta'(s)|_{s=0} &= -x^{\alpha/(1+\alpha)} \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{(\sigma_k - 1 + x)^{1/(1+\alpha)} - x^{1/(1+\alpha)}\right\} \\
&= x\left[1 - \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left(1 + \frac{\sigma_k - 1}{x}\right)^{1/(1+\alpha)}\right] \\
&\ge x\left[1 - \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \left(1 + \frac{(k/\mu) - 1}{x}\right)^{1/(1+\alpha)}\right] \\
&\ge x\left[1 - \left(1 + \frac{\rho - 1}{x}\right)^{1/(1+\alpha)}\right] > 0.
\end{aligned} \tag{42}$$

Let s_1 be the minimum positive root of the equation $\beta(s) = 0$; then, we have $\beta(s) > 0$ for all $0 < s < s_1$.

Let $V_s(x) = \exp(sx^{1/(1+\alpha)})$, for all $s \in (0, s_1)$, and let $b = \max_{s \in (0,s_1)}\{\alpha(s)\} < \infty$; then, we have

$$\begin{aligned}
\Delta V_s(x) &\le -\beta(s) \frac{\exp\left(sx^{1/(1+\alpha)}\right)}{x^{\alpha/(1+\alpha)}} + bI_C, \\
&= -\phi \circ V_s(x) + bI_C, \quad x \in R_+,
\end{aligned} \tag{43}$$

where $\phi(x) = \beta(s)(x/\log^\alpha(x))$, (i.e., Condition $D(\phi, V_s, C)$ holds). By Theorem 1, we know that there exists a unique invariant distribution π, $\pi(\phi \circ V) < \infty$, that is

$$\int_1^{+\infty} x^{-\alpha/(1+\alpha)} \exp\left(sx^{1/(1+\alpha)}\right) \pi(dx) < \infty. \tag{44}$$

From

$$H_\phi(x) = \int_1^x \frac{dx}{\phi(x)} = \int_1^x \frac{\log^\alpha(x)\, dx}{\beta(s)\, x} = \frac{\log^{1+\alpha}(x)}{(\alpha+1)\beta(s)}, \tag{45}$$

we have $H_\phi^{-1}(x) = \exp[((\alpha+1)\beta(s)x)^{1/(1+\alpha)}]$. So,

$$\begin{aligned} r_\phi(n) &= \phi \circ H_\phi^{-1}(n) \\ &= \beta(s) \frac{\exp\left[((\alpha+1)\beta(s)n)^{1/(1+\alpha)}\right]}{((\alpha+1)\beta(s)n)^{\alpha/(1+\alpha)}} \\ &= \beta(s)^{1/(1+\alpha)}[(\alpha+1)n]^{-\alpha/(1+\alpha)} \\ &\quad \cdot \exp\left[((\alpha+1)\beta(s)n)^{1/(1+\alpha)}\right] \\ &\asymp n^{-\alpha/(1+\alpha)} \exp\left[((\alpha+1)\beta(s)n)^{1/(1+\alpha)}\right]. \end{aligned} \tag{46}$$

Let $\varepsilon_1 = \max_{s\in(0,s_1)}\{[(\alpha+1)\beta(s)]^{1/(1+\alpha)}\}$; then we have,

$$\lim_{n\to\infty} n^{-\alpha/(1+\alpha)} \exp\left(\varepsilon_1 n^{1/(1+\alpha)}\right) \left\| P^{(n)}(x,\cdot) - \pi(\cdot) \right\| = 0. \tag{47}$$

The proof is completed. □

Case 2 (Polynomial Rate of Convergence). Consider the following.

Proposition 10. *If $\rho < 1$ and there exists a constant $\alpha > 1$ such that*

$$E\nu_1^\alpha = \int_0^{+\infty} x^\alpha B(dx) < +\infty, \tag{48}$$

then

$$\begin{gathered} \lim_{n\to\infty} n^{\alpha-1} \left\| P^{(n)}(x,\cdot) - \pi(\cdot) \right\| = 0, \\ \int_1^{+\infty} x^{\alpha-1} \pi(dx) < +\infty. \end{gathered} \tag{49}$$

Proof. Let $V(x) = (x+1)^\alpha \geq 1$, $x \in R_+$, and let m_α be the αth moment of the poisson distribution with parameter λ. From $(a+b)^\alpha \leq 2^\alpha(a^\alpha + b^\alpha)$, where $a > 0, b > 0$, we have, for all $x \in C$ (where C is the petite $[0,c]$),

$$\begin{aligned} \Delta V(x) &= PV(x) - V(x) \leq -(x+1)^\alpha \\ &\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E(\nu_1 + \nu_2 + \cdots + \nu_k + x + 1)^\alpha \\ &\leq -(x+1)^\alpha + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} 2^\alpha E\left((\sigma_k)^\alpha + (x+1)^\alpha\right) \\ &\leq -(x+1)^\alpha + 2^\alpha (x+1)^\alpha + 2^\alpha \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} k^\alpha E\nu_1^\alpha \\ &= -(x+1)^\alpha + 2^\alpha(x+1)^\alpha + 2^\alpha m_\alpha E\nu_1^\alpha. \end{aligned} \tag{50}$$

Let $b = -(c+1)^\alpha + 2^\alpha(c+1)^\alpha + 2^\alpha m_\alpha E\nu_1^\alpha$. Since $E\nu_1^\alpha < \infty$, we know that

$$\Delta V(x) \leq b < \infty, \quad x \in C. \tag{51}$$

Let C_n^i denote the binomial coefficient, and let $n = \lfloor \alpha \rfloor$, $\theta = \alpha - n$; then; $\alpha = n + \theta$. For all $x \in C^C$,

$$\begin{aligned} \Delta V(x) &= PV(x) - V(x) = -(x+1)^\alpha \\ &\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E(\sigma_k - 1 + x + 1)^\alpha = -(x+1)^\alpha \\ &\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E(\sigma_k - 1 + x + 1)^{n+\theta} = -(x+1)^\alpha \\ &\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \\ &\qquad \cdot E\left\{\left[(x+1)^n + C_n^1 (x+1)^{n-1}(\sigma_k - 1)\right.\right. \\ &\qquad \left.\left. + \sum_{i=2}^{n} C_n^i (x+1)^{n-i} (\sigma_k - 1)^i \right] (\sigma_k + x)^\theta \right\} \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{(x+1)^n (\sigma_k + x)^\theta - (x+1)^\alpha\right\} \\ &\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{C_n^1 (x+1)^{n-1}(\sigma_k - 1)(\sigma_k + x)^\theta\right\} \\ &\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \\ &\qquad \cdot E\left\{\sum_{i=2}^{n} \left[C_n^i (x+1)^{n-i} (\sigma_k - 1)^i\right] (\sigma_k + x)^\theta \right\}. \end{aligned} \tag{52}$$

Since $f_1(\xi) = \xi^\theta$ is a concave function, we know that

$$E(\xi + x)^\theta \leq (E\xi + x)^\theta. \tag{53}$$

Thus, the first part of (52) is

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{(x+1)^n (\sigma_k + x)^\theta - (x+1)^\alpha\right\} \\ &\quad \leq (x+1)^n (\rho + x)^\theta - (x+1)^\alpha < 0. \end{aligned} \tag{54}$$

If α is integer (i.e., $\theta = 0$), then the second part of (52) is

$$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{C_n^1 (x+1)^{n-1}(\sigma_k - 1)\right\} = C_n^1 (\rho - 1)(x+1)^{\alpha-1}. \tag{55}$$

If α is not integer (i.e., $\theta \neq 0$), then the second part of (52) is

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{C_n^1 (x+1)^{n-1} (\sigma_k - 1)(\sigma_k + x)^{\theta}\right\} \\
&\quad = C_n^1 (x+1)^{n-1} \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left[(\sigma_k - 1)(x+1)^{\theta}\right] \\
&\qquad + C_n^1 (x+1)^{n-1} \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \\
&\qquad \cdot E\left\{(\sigma_k - 1)\left[(\sigma_k + x)^{\theta} - (x+1)^{\theta}\right]\right\} \\
&\quad \leq C_n^1 (\rho - 1)(x+1)^{\alpha-1} \\
&\qquad + C_n^1 (x+1)^{n-1} \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{\sigma_k > 1\}} \sigma_k^{\theta+1}\right\} \\
&\qquad + E\left\{1_{\{\sigma_k < 1\}} \left((x+1)^{\theta} - x^{\theta}\right)\right\} \\
&\quad \leq C_n^1 (\rho - 1)(x+1)^{\alpha-1} \\
&\qquad + C_n^1 (x+1)^{n-1} m_{\theta+1} \left(E\nu_1^{\theta+1} + 1\right) \\
&\quad = C_n^1 (\rho - 1)(x+1)^{\alpha-1} + a_1 (x+1)^{n-1},
\end{aligned} \tag{56}$$

where $a_1 = C_n^1 m_{\theta+1}(E\nu_1^{\theta+1}+1)$. From (55) and (56), we obtain that the second part of (52) is

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{C_n^1 (x+1)^{n-1} (\sigma_k - 1)(\sigma_k + x)^{\theta}\right\} \\
&\quad = C_n^1 (\rho - 1)(x+1)^{\alpha-1} + a_1 (x+1)^{n-1},
\end{aligned} \tag{57}$$

where $a_1 = 0$ if α is integer. The third part of (52) is

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \sum_{i=2}^{n} C_n^i (x+1)^{n-i} E\left[(\sigma_k - 1)^i (\sigma_k + x)^{\theta}\right] \\
&\leq \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \sum_{i=2}^{n} C_n^i (x+1)^{n-i} \\
&\qquad \cdot E\left\{\left(\left((\sigma_k)^i + 1\right)(\sigma_k)^{\theta} + (x+1)^{\theta}\right)\right\} \\
&\leq \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \sum_{i=2}^{n} C_n^i (x+1)^{n-i} \\
&\qquad \cdot E\left\{(\sigma_k)^{i+\theta} + (\sigma_k)^i (x+1)^{\theta} + (\sigma_k)^{\theta} + (x+1)^{\theta}\right\} \\
&= \sum_{i=2}^{n} C_n^i (x+1)^{n-i} \\
&\qquad \cdot \left\{m_{i+\theta} E\nu_1^{i+\theta} + m_i E\nu_1^i (x+1)^{\theta} + m_{\theta} E\nu_1^{\theta} + (x+1)^{\theta}\right\} \\
&\leq \sum_{i=2}^{n} C_n^i \left(m_{i+\theta} E\nu_1^{i+\theta} + m_i E\nu_1^i + m_{\theta} E\nu_1^{\theta} + 1\right)(x+1)^{n-i+\theta} \\
&= \sum_{i=2}^{n} a_i (x+1)^{\alpha-i},
\end{aligned} \tag{58}$$

where $a_i = C_n^i(m_{i+\theta}E\nu_1^{i+\theta}+m_iE\nu_1^i+m_\theta E\nu_1^\theta+1) < \infty$, $1 \leq i \leq n$ (by $E\nu_1^\alpha < \infty$). Combining (52), (54), (57), and (58), we have

$$\begin{aligned}
\Delta V(x) &\leq C_n^1 (\rho - 1)(x+1)^{\alpha-1} \\
&\quad + a_1 (x+1)^{n-1} + \sum_{i=2}^{n} a_i (x+1)^{\alpha-i},
\end{aligned} \tag{59}$$

where $a_1 = 0$ if α is integer. Choose c large enough such that, for all $x > c$ (i.e., $x \in C^c, C = [0, c]$),

$$a_1 (x+1)^{n-1} + \sum_{i=2}^{n} a_i (x+1)^{\alpha-i} < -\frac{1}{2} C_n^1 (\rho - 1)(x+1)^{\alpha-1}. \tag{60}$$

Thus

$$\Delta V(x) \leq \frac{1}{2} C_n^1 (\rho - 1)(x+1)^{\alpha-1}, \quad \forall x \in C^c. \tag{61}$$

Together with (51), we have

$$\begin{aligned}
\Delta V(x) &\leq -\beta (x+1)^{\alpha-1} + bI_C, \\
&= -\phi \circ V(x+1) + bI_C, \quad \forall x \in R_+,
\end{aligned} \tag{62}$$

where $\beta = (1/2)C_n^1(1-\rho) > 0$, $\phi(z) = \beta z^{(\alpha-1)/\alpha}$ (i.e., condition $D(\phi, V, C)$ holds). By Theorem 1, we know that there exists a unique invariant distribution π, $\pi(\phi \circ V) < \infty$ (i.e., $\int_1^{+\infty} x^{\alpha-1}\pi(dx) < \infty$) and

$$\lim_{n \to \infty} \left\| P^{(n)}(x, A) - \pi(A) \right\|_{\phi \circ V} = 0, \quad \pi\text{-a.e.}. \tag{63}$$

From

$$H_{\phi}(x) = \int_1^x \frac{dz}{\phi(z)} = \frac{1}{\beta} \int_1^x z^{(1-\alpha)/\alpha} dz = \frac{\alpha}{\beta}\left(x^{1/\alpha} - 1\right), \tag{64}$$

we have $H_\phi^{-1}(x) = ((\beta/\alpha)x+1)^\alpha$. So,

$$\begin{aligned}
r_{\phi}(n) &= \phi \circ H_{\phi}^{-1}(n) \\
&= \beta\left[\left(\frac{\beta}{\alpha} n + 1\right)^{\alpha}\right]^{(\alpha-1)/\alpha} \\
&= \beta\left(\frac{\beta}{\alpha} n + 1\right)^{\alpha-1} \asymp n^{\alpha-1}.
\end{aligned} \tag{65}$$

That is,

$$\lim_{n \to \infty} n^{\alpha-1} \left\| P^{(n)}(x, \cdot) - \pi(\cdot) \right\| = 0. \tag{66}$$

□

Case 3 (Logarithmic Rate of Convergence). Now, we consider the logarithmic case which is slower than that for any polynomial.

Proposition 11. *If $\rho < 1$ and there exists a positive integer α such that*

$$E\left(v_1 \log^\alpha (v_1 + 1)\right) = \int_0^{+\infty} x \log^\alpha (x+1) B(dx) < +\infty, \tag{67}$$

then

$$\lim_{n\to\infty} \log^\alpha n \left\| P^{(n)}(x,\cdot) - \pi(\cdot)\right\| = 0, \tag{68}$$

$$\int_1^{+\infty} \log^\alpha x \pi(dx) < \infty. \tag{69}$$

Proof. For all $x \in R_+$, let $V(x) = (x+e)\log^\alpha(x+e)$; then, we have $V(x) > e, x \in R_+$. Let $c > e^\alpha$, and choose $C = [0,c]$. For all $x \in E$,

$$\begin{aligned}
&\Delta V(x) \\
&= PV(x) - V(x) \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left(\sigma_k - 1 + x + e\right)\log^\alpha\left(\sigma_k - 1 + x + e\right) \\
&\quad - (x+e)\log^\alpha(x+e) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left(\sigma_k - 1\right)\log^\alpha(x+e) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left(\sigma_k - 1 + x + e\right) \\
&\qquad \cdot \left[\log^\alpha\left(\sigma_k - 1 + x + e\right) - \log^\alpha(x+e)\right] = (\rho - 1)\log^\alpha(x+e) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \\
&\qquad \cdot E\left\{\left(\sigma_k - 1 + x + e\right)\right. \\
&\qquad \left. \cdot \left[\log^\alpha\left((x+e)\left(1 + \frac{\sigma_k - 1}{x+e}\right)\right) - \log^\alpha(x+e)\right]\right\} \\
&\le (\rho - 1)\log^\alpha(x+e) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{\sigma_k > 1\}}\left(\sigma_k - 1 + x + e\right)\right. \\
&\qquad \left. \cdot \sum_{i=1}^{\alpha} C_\alpha^i \log^{\alpha-i}(x+e)\log^i\left(1 + \frac{\sigma_k - 1}{x+e}\right)\right\} \\
&\le (\rho - 1)\log^\alpha(x+e) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{1 < \sigma_k \le x+e\}} 2(x+e)\right. \\
&\qquad \left. \cdot \sum_{i=1}^{\alpha} C_\alpha^i \log^{\alpha-i}(x+e)\log^i\left(1 + \frac{\sigma_k - 1}{x+e}\right)\right\} \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{\sigma_k > x+e\}} 2\sigma_k \sum_{i=1}^{\alpha} C_\alpha^i \log^{\alpha-i}\right. \\
&\qquad \left. \cdot (x+e)\log^i\left(1 + \frac{\sigma_k - 1}{x+e}\right)\right\} \\
&\le (\rho - 1)\log^\alpha(x+e) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{1 \le \sigma_k \le x+e\}} 2\left(\sigma_k - 1\right) \sum_{i=1}^{\alpha} C_\alpha^i \log^{\alpha-i}(x+e)\right\} \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{\sigma_k > x+e\}} 2\sigma_k\right. \\
&\qquad \left. \cdot \sum_{i=1}^{\alpha} C_\alpha^i \log^{\alpha-i}(x+e)\log^i\sigma_k\right\} \\
&\le (\rho - 1)\log^\alpha(x+e) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{1 \le \sigma_k \le x+e\}} 2^{\alpha+1}\sigma_k \log^{\alpha-1}(x+e)\right\} \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{\sigma_k > x+e\}} 2^{\alpha+1}\sigma_k\right. \\
&\qquad \left. \cdot \log^{\alpha-1}(x+e)\log^\alpha\sigma_k\right\} \\
&\le (\rho - 1)\log^\alpha(x+e) \\
&\quad + 2^{\alpha+1}\log^{\alpha-1}(x+e) \\
&\quad \cdot \left[\rho + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{\sigma_k > x+e\}}\sigma_k \log^\alpha\sigma_k\right\}\right] \\
&\le (\rho - 1)\log^\alpha(x+e) + 2^{\alpha+1}\log^{\alpha-1}(x+e) \\
&\quad \cdot \left[\rho + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{\sigma_k > x+e\}}\sigma_k 2^\alpha\right.\right. \\
&\qquad \left.\left. \cdot \left(\log^\alpha k + \log^\alpha\left(v_1 + 1\right)\right)\right\}\right] \\
&\le (\rho - 1)\log^\alpha(x+e) + 2^{\alpha+1}\log^{\alpha-1}(x+e) \\
&\quad \cdot \left[\rho + 2^\alpha\left(m_2 E v_1 + m_1 E\left(v_1 \log^\alpha\left(v_1 + 1\right)\right)\right)\right] \\
&= (\rho - 1)\log^\alpha(x+e) + a_0 \log^{\alpha-1}(x+e),
\end{aligned} \tag{70}$$

where $a_0 = 2^{\alpha+1}[\rho + 2^{\alpha}(m_2 E\nu_1 + m_1 E(\nu_1 \log^{\alpha}(\nu_1 + 1)))]$. Since $E(\nu_1 \log^{\alpha}(\nu_1 + 1)) < \infty$, we get that $a_0 < +\infty$. Let $b = a_0 \log^{\alpha-1}(c+e)$; then, we have

$$\Delta V(x) \le b < \infty, \quad x \in C. \tag{71}$$

Choose c large enough such that, if $x > c$ (i.e., $x \in [0,c]^c$),

$$a_0 \log^{\alpha-1}(x+e) < \frac{1}{2}(1-\rho)\log^{\alpha}(x+e). \tag{72}$$

Thus,

$$\Delta V(x) \le -\frac{1}{2}(1-\rho)\log^{\alpha}(x+e), \quad \forall x \in C^c. \tag{73}$$

Together with (71), we have

$$\begin{aligned} \Delta V(x) \le & -\beta \log^{\alpha}(x+e) + bI_C, \\ = & -\phi \circ V(x) + bI_C, \quad \forall x \in R_+, \end{aligned} \tag{74}$$

where $\beta = (1/2)(1-\rho) > 0$, $\phi(z) = \beta \log^{\alpha} z$ (i.e., condition $D(\phi, V, C)$ holds). A straightforward calculation shows that

$$r_{\phi}(n) \asymp \log^{\alpha} n. \tag{75}$$

That is,

$$\lim_{n\to\infty} \log^{\alpha} n \left\| P^{(n)}(x,\cdot) - \pi(\cdot) \right\| = 0. \tag{76}$$

By Theorem 1, we know that there exists a unique invariant distribution π, $\pi(\phi \circ V) < \infty$ (i.e., $\int_1^{+\infty} \log^{\alpha} x\pi(dx) < \infty$) and

$$\lim_{n\to\infty} \left\| P^{(n)}(x,A) - \pi(A) \right\|_{\phi\circ V} = 0, \quad \pi\text{-}a.e.. \tag{77}$$

□

4.1. Conclusion and Future Research. We studied the M/G/1 queueing system, and the waiting time process of the queueing system is a Markov process. For the workload process of the M/G/1 queueing system, we got an h-skeleton process and discussed its properties of the irreducible and aperiodic and the property of stochastic monotone. Then, we got the parameters ε_0 and s_0 for geometric rate of convergence and the geometric decay of the stationary tail, respectively. For three specific types of subgeometric cases: Case 1: the rate function $r(n) = \exp(sn^{1/(1+\alpha)})$, $\alpha > 0$, $s > 0$; Case 2: polynomial rate function $r(n) = n^{\alpha}$, $\alpha > 0$; Case 3: logarithmic rate function $r(n) = \log^{\alpha} n$, $\alpha > 0$, we gave explicit criteria for the rate of convergence and decay of stationary tail. We gave the parameters ε_1 and s_1 of the rate of convergence and the decay of the stationary tail, respectively, for the subgeometric rate $r(n) = \exp(sn^{1/(1+\alpha)})$, $\alpha > 0$, $s > 0$. These results are important in the study of the stability of M/G/1 queueing system.

For future research, much could be done. Our work could be used to the convergence analysis of Markov chain Monte Carlo (MCMC) theory. It could also be used to further discuss queue length, congestion, and so forth. Using similar techniques, these results may be extended to storage models, nonlinear autoregressive model, stochastic unit root models, multidimensional random walk, and other queueing systems.

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Research Article

Dynamic Proportional Reinsurance and Approximations for Ruin Probabilities in the Two-Dimensional Compound Poisson Risk Model

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We consider the dynamic proportional reinsurance in a two-dimensional compound Poisson risk model. The optimization in the sense of minimizing the ruin probability which is defined by the sum of subportfolio is being ruined. Via the Hamilton-Jacobi-Bellman approach we find a candidate for the optimal value function and prove the verification theorem. In addition, we obtain the Lundberg bounds and the Cramér-Lundberg approximation for the ruin probability and show that as the capital tends to infinity, the optimal strategies converge to the asymptotically optimal constant strategies. The asymptotic value can be found by maximizing the adjustment coefficient.

1. Introduction

In an insurance business, a reinsurance arrangement is an agreement between an insurer and a reinsurer under which claims are split between them in an agreed manner. Thus, the insurer (cedent company) is insuring part of a risk with a reinsurer and pays premium to the reinsurer for this cover. Reinsurance can reduce the probability of suffering losses and diminish the impact of the large claims of the company. Proportional reinsurance is one of the reinsurance arrangement, which means the insurer pays a proportion, say a, when the claim occurs and the remaining proportion, $1 - a$, is paid by the reinsurer. If the proportion a can be changed according to the risk position of the insurance company, this is the dynamic proportional reinsurance. Researches dealing with this problem in the one-dimensional risk model have been done by many authors. See for instance, Højgaard and Taksar [1, 2], Schmidli [3] considered the optimal proportional reinsurance policies for diffusion risk model and for compound Poisson risk model, respectively. Works combining proportional and other type of reinsurance polices for the diffusion model were presented in Zhang et al. [4]. If investment or dividend can be involved, this problem was discussed by Schmidli [5] and Azcue and Muler [6], respectively. References about dynamic reinsurance of large claim are Taksar and Markussen [7], Schmidli [8], and the references therein.

Although literatures on the optimal control are increasing rapidly, seemly that none of them consider this problem in the multidimensional risk model so far. This kind of model depicts that an unexpected claim event usually triggers several types of claims in an umbrella insurance policy, which means that a single event influences the risks of the entire portfolio. Such risk model has become more important for the insurance companies due to the fact that it is useful when the insurance companies handle dependent class of business. The previous work relating to multidimensional model without dynamic control mainly focuses on the ruin probability. See for example, Chan et al. [9] obtained the simple bounds for the ruin probabilities in two-dimensional case, and a partial integral-differential equation satisfied by the corresponding ruin probability. Yuen et al. [10] researched the finite-time survival probability of a two-dimensional compound Poisson model by the approximation of the so-called bivariate compound binomial model. Li et al. [11] studied the ruin probabilities of a two-dimensional perturbed insurance risk model and obtained a Lundberg-type upper bound for the infinite-time ruin probability. Dang et al. [12] obtained explicit expressions for recursively calculating the survival probability of the two-dimensional risk model by applying the partial integral-differential equation when claims are exponentially distributed. More literatures can be found in the references within the above papers.

In this paper, we will discuss the dynamic proportional reinsurance in a two-dimensional compound Poisson risk model. From the insurers point of view, we want to minimize the ruin probability or equivalently to maximize the survival probability.

We start with a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. $\mathcal{F}_t$ represents the information available at time t, and any decision is made upon it. Suppose that an insurance portfolio consists of two subportfolios $\{X_t^a\}$ and $\{Y_t^b\}$. $\{(U_n, V_n)\}$ is a sequence of *i.i.d* random vectors which denote the claim size for (X_t^a, Y_t^b). Let $G(u,v)$ denote their joint distribution function, and suppose $G(u,v)$ is continuous. At any time t the cedent may choose proportional reinsurance strategy (a_t, b_t). This implies that at time t the cedent company pays $(a_t U, b_t V)$. The reinsurance company pays the amount $((1-a_t)U, (1-b_t)V)$. $a = \{a_t\}$ and $b = \{b_t\}$ are admissible if they are adapted processes with value in $[0,1]$. By $\mathcal{U}$ we denote the set of all admissible strategies. The model can be stated as

$$\begin{pmatrix} X_t^a \\ Y_t^b \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \int_0^t c_1(a_s)\mathrm{d}s \\ \int_0^t c_2(b_s)\mathrm{d}s \end{pmatrix} - \sum_{n=1}^{N_t} \begin{pmatrix} a_{\sigma_n-} U_n \\ b_{\sigma_n-} V_n \end{pmatrix} \tag{1.1}$$

u_1, u_2 are the initial capital of $\{X_t^a\}$ and $\{Y_t^b\}$, respectively. $c_1(a_t)$ and $c_2(b_t)$ denote the premium rates received by the insurance (cedent) company for the subportfolio $\{X_t^a\}$ and $\{Y_t^b\}$ at time t. Suppose $c_1(a)$ is continuous about a and $c_2(b)$ is continuous about b. Note that if full reinsurance, that is, $a = b = 0$ is chosen the premium rates, $c_1(0)$ and $c_2(0)$ are strictly negative. Otherwise, the insurer would reinsure the whole portfolio, then ruin would never occur for it. Let c_1, c_2 denote the premium if no reinsurance is chosen. Then $c_1(a_t) \leq c_1$, $c_2(b_t) \leq c_2$. For (U_n, V_n), their common arrival times constitute a counting process $\{N_t\}$, which is a Poisson process with rate λ and independent of (U_n, V_n). The net profit conditions are $c_1 > \lambda \mathrm{E} U_n$ and $c_2 > \lambda \mathrm{E} V_n$. $a_{\sigma_n -} U_n$ and $b_{\sigma_n -} V_n$ are the claim size that the cedent company pays at σ_n (time of the nth claim arrivals). This reinsurance form chosen prior to the claim prevents the insurer change the strategies to full reinsurance when the claim occurs and avoid the insurer owning all the premium while the reinsurer pays all the claims.

In realities, if the insurance company deals with multidimensional risk model, they may adjust the capital among every subportfolio. If the adjustment is reasonable, the company may run smoothly. So the actuaries care more about how the aggregate loss for the whole book of business effects the insurance company. Hence, in our problem we focus on the aggregate surplus:

$$R_t^{a,b} = X_t^a + Y_t^b = u + \int_0^t (c_1(a_s) + c_2(b_s))\mathrm{d}s - \sum_{n=1}^{N_t} (a_{\sigma_n -} U_n + b_{\sigma_n -} V_n), \tag{1.2}$$

where $u = u_1 + u_2$. Ruin time is defined by

$$\tau_{a,b} = \inf\left\{ t \geq 0; R_t^{a,b} < 0 \right\}, \tag{1.3}$$

which denotes the first time that the total of X_t^a and Y_t^b is negative. The ruin probability is

$$\psi_{a,b}(u) = \mathrm{P}\left(\tau_{a,b} < \infty \mid R_0^{a,b} = u \right). \tag{1.4}$$

The corresponding survival probability is

$$\delta_{a,b}(u) = \mathrm{P}\left(\tau_{a,b} = \infty \mid R_0^{a,b} = u \right). \tag{1.5}$$

Our optimization criterion is maximization of survival probability from the insurer (cedent company) point of view. So the objective is to find the optimal value function $\delta(u)$ which is defined by

$$\delta(u) = \sup_{(a,b) \in \mathcal{U}} \delta_{a,b}(u). \tag{1.6}$$

If the optimal strategy (a^*, b^*) exists, we try to determine it. Let $\{R_t\}$ denote the process under the optimal strategy (a^*, b^*) and τ^* the corresponding ruin time.

The paper is organized as follows. After the brief introduction of our model, in Section 2, we proof some useful properties of $\delta(u)$. The HJB equation satisfied by the optimal value function is presented in Section 3. Furthermore, we show that there exists a unique solution with certain boundary condition and give a proof of the verification theorem. Taking advantage of a very important technique of changing of measure, the Lundberg bounds for the controlled process are obtained in Section 4. In Section 5, we get the Cramér-Lundberg approximation for $\psi(u)$. The convergence of the optimal strategy is proved in Section 6. In the last section, we give a numerical example to illustrate how to get the upper bound of $\psi(u)$.

2. Some Properties of $\delta(u)$

We first give some useful properties of $\delta(u)$.

Lemma 2.1. *For any strategy (a,b), with probability 1, either ruin occurs or $R_t^{a,b} \to \infty$ as $t \to \infty$.*

Proof. Let (a,b) be a strategy. If the full reinsurance of each subportfolio is chosen, we denote $c_1^0 < 0$, $c_2^0 < 0$ be the premium left to the cedent insurance company. Let $B = \{(a,b) : c_1(a) + c_2(b) \geq (c_1^0 + c_2^0)/2\}$, let and $\overline{B}$ be its complementary set. Choose $\varepsilon < -(c_1^0 + c_2^0)/2$ and $\kappa = (-c_1^0 - c_2^0 - 2\varepsilon)/(2(c_1+c_2) - c_1^0 - c_2^0)$. First, if $\int_t^{t+1} 1_{(a_s,b_s)\in B} \mathrm{d}s \leq \kappa$, then

$$\begin{aligned}
R_{t+1}^{a,b} &= R_t^{a,b} + \int_t^{t+1} (c_1(a) + c_2(b))\mathrm{d}s - \sum_{i=N(t)+1}^{N(t+1)} (a_{\sigma_{i-}} U_i + b_{\sigma_{i-}} V_i) \\
&\leq R_t^{a,b} + \int_t^{t+1} (c_1(a) + c_2(b))\mathrm{d}s \\
&= R_t^{a,b} + \int_t^{t+1} (c_1(a) + c_2(b)) 1_{(a,b)\in B}\mathrm{d}s + \int_t^{t+1} (c_1(a) + c_2(b)) 1_{(a,b)\in \overline{B}}\mathrm{d}s \\
&\leq R_t^{a,b} + (c_1 + c_2)\int_t^{t+1} 1_{(a,b)\in B}\mathrm{d}s + \frac{c_1^0 + c_2^0}{2}\int_t^{t+1} 1_{(a,b)\in \overline{B}}\mathrm{d}s \\
&\leq R_t^{a,b} + (c_1 + c_2)\kappa + (1-\kappa)\frac{c_1^0 + c_2^0}{2} \\
&= R_t^{a,b} - \varepsilon.
\end{aligned} \tag{2.1}$$

Otherwise, if $\int_t^{t+1} 1_{b_s\in B}\mathrm{d}s > \kappa$. Because $c_1(a)$, $c_2(b)$, au, and bv are continuous, we assume that ε is small enough such that

$$\mathrm{P}\left[\inf_{(a,b)\in B} aU + bV > \varepsilon\right] > 0. \tag{2.2}$$

Also

$$\mathrm{P}\left[\int_t^{t+1} 1_{(a_s,b_s)\in B}\mathrm{d}N_s \geq 1 + \frac{c_1 + c_2}{\varepsilon}\right] \geq \mathrm{P}\left[N_\kappa \geq 1 + \frac{c_1 + c_2}{\varepsilon}\right] > 0. \tag{2.3}$$

While

$$\begin{aligned}
&\sum_{i=N_t+1}^{N_{t+1}} a_{\sigma_i-} U_i + b_{\sigma_i-} V_i \\
&\quad = \sum_{i=N_t+1}^{N_{t+1}} (a_{\sigma_i-} U_i + b_{\sigma_i-} V_i) 1_{(a,b)\in B} + \sum_{i=N_t+1}^{N_{t+1}} (a_{\sigma_i-} U_i + b_{\sigma_i-} V_i) 1_{(a,b)\in \overline{B}} \\
&\quad \geq \sum_{i=N_t+1}^{N_{t+1}} (a_{\sigma_i-} U_i + b_{\sigma_i-} V_i) 1_{(a,b)\in B}.
\end{aligned} \tag{2.4}$$

Because $\mathrm{P}[\sum_{i=N_t+1}^{N_{t+1}}(a_{\sigma_i-}U_i + b_{\sigma_i-}V_i)1_{(a,b)\in B} \geq (1 + ((c_1 + c_2)/\varepsilon))\varepsilon = (c_1 + c_2) + \varepsilon] > 0$, then

$$\mathrm{P}\left[\sum_{i=N_t+1}^{N_{t+1}}(a_{\sigma_i-}U_i + b_{\sigma_i-}V_i) \geq (c_1 + c_2) + \varepsilon\right] > 0. \tag{2.5}$$

We denote a lower bound by $\delta > 0$. Choose $M > 0$. Let $t_0 = 0$ and $t_{k+1} = \inf\{t \geq t_k + 1; R_t^{a,b} \leq M\}$. Here we define $t_{k+1} = \infty$ if $t_k = \infty$ or if $R_t^{a,b} > M$ for all $t \geq t_k + 1$. Because

$$\begin{aligned} M - R_{t_k+1}^{a,b} &\geq R_{t_k}^{a,b} - R_{t_k+1}^{a,b} \\ &= \sum_{i=N_{t_k}+1}^{N_{t_k+1}}(a_{\sigma_i-}U_i + b_{\sigma_i-}V_i) - \int_{t_k}^{t_k+1}(c_1(a) + c_2(b))\mathrm{d}s \\ &\geq \sum_{i=N_{t_k}+1}^{N_{t_k+1}}(a_{\sigma_i-}U_i + b_{\sigma_i-}V_i) - (c_1 + c_2). \end{aligned} \tag{2.6}$$

Then

$$\mathrm{P}\left[M - R_{t_k+1}^{a,b} \geq \varepsilon \mid \mathcal{F}_{t_k}\right] \geq \delta, \tag{2.7}$$

which can also be expressed by

$$\mathrm{P}\left[R_{t_k+1}^{a,b} \leq M - \varepsilon \mid \mathcal{F}_{t_k}\right] \geq \delta. \tag{2.8}$$

Let $W_k = 1_{t_k<\infty, R_{t_k+1}^{a,b}<M-\varepsilon}$, $Z_k = \delta 1_{t_k<\infty}$ and $S_n = \sum_{k=1}^{n}(W_k - Z_k)$. Because

$$\begin{aligned} \mathrm{E}|S_n| &= \mathrm{E}\left(\left|\sum_{k=1}^{n}1_{t_k<\infty, R_{t_k+1}^{a,b}<M-\varepsilon} - \sum_{k=1}^{n}\delta 1_{t_k<\infty}\right|\right) \leq 2n < \infty, \\ \mathrm{E}[S_{n+1} \mid \mathcal{F}_n] &= \mathrm{E}\left[\sum_{k=1}^{n+1}(W_k - Z_k) \mid \mathcal{F}_n\right] \\ &= \mathrm{E}\left[\sum_{k=1}^{n}(W_k - Z_k) + W_{n+1} - Z_{n+1} \mid \mathcal{F}_n\right] \\ &= S_n + \mathrm{E}[W_{n+1} - Z_{n+1} \mid \mathcal{F}_n] \\ &= S_n + \mathrm{E}\left[1_{t_{n+1}<\infty, R_{t_{n+1}+1}^{a,b}<M-\varepsilon} - \delta 1_{t_{n+1}<\infty} \mid \mathcal{F}_n\right] \\ &= S_n + \left(\mathrm{P}\left[R_{t_{n+1}+1}^{a,b} < M - \varepsilon \mid \mathcal{F}_{t_n}\right] - \delta\right)\mathrm{P}[t_{n+1} < \infty] \\ &\geq S_n. \end{aligned} \tag{2.9}$$

From above, we know that $\{S_n\}$ is a submartingale and $\{S_n\}$ satisfied the conditions of Lemma 1.15 in Schmidli [13]. So

$$P\left[\sum_{k=1}^{\infty} 1_{t_k<\infty, R^{a,b}_{t_k+1}<M-\varepsilon} < \infty, \sum_{k=1}^{\infty} \delta 1_{t_k<\infty} = \infty\right] = 0. \tag{2.10}$$

Thus $R^{a,b}_{t_k+1} < M-\varepsilon$ infinitely often. If $\liminf R^{a,b}_t \leq N$, then for $M = N+\varepsilon/2$, $t_n < \infty$ for all n. Then $R^{a,b}_{t_n+1} \leq N-\varepsilon/2$ infinitely often. In particular, $\liminf R^{a,b}_t \leq N-\varepsilon/2$. We can conclude that $\liminf R^{a,b}_t < \infty$ implies $\liminf R^{a,b}_t < -\varepsilon/2$. Therefore ruin occurs. While $\liminf R^{a,b}_t = \infty$ implies $R^{a,b}_t \to \infty$ as $t \to \infty$. □

Lemma 2.2. *The function $\delta(u)$ is strictly increasing.*

Proof. If $u < z$, we can use the same strategy (a,b) for initial capital u and z. Then we can conclude that $\delta_{a,b}(u) < \delta_{a,b}(z)$, so $\delta(u) = \sup_{(a,b)\in\mathcal{U}}\delta_{a,b}(u) \leq \sup_{(a,b)\in\mathcal{U}}\delta_{a,b}(z) = \delta(z)$. Suppose that $\delta(u) = \delta(z)$.

(a) From Lemma 2.1, we know that if $c_1(a) + c_2(b) \leq (c_1^0 + c_2^0)/2$ on the interval $[0,T_1)$, where $T_1 = [2(u+\kappa(c_1+c_2))]/(-c_1^0-c_2^0)+\kappa$ for all t except a set with measure κ, then

$$R_{T_1} \leq u+\kappa(c_1+c_2) + \frac{2(u+\kappa(c_1+c_2))}{-c_1^0-c_2^0}\frac{c_1^0+c_2^0}{2} \leq 0. \tag{2.11}$$

Then ruin occurs.

(b) Otherwise, let $T_2 = \inf\{t : \int_0^t 1_{(c_1(a)+c_2(b)\leq(c_1^0+c_2^0)/2)}ds > \kappa\}$. Similar to Lemma 2.1, we have

$$\begin{gathered} P\left[\inf_{(a,b)\in B}(aU+bV) > \varepsilon\right] > 0, \\ P\left[\int_0^{T_2} 1_{(a_s,b_s)\in B}dN_s \geq \frac{u+\kappa(c_1+c_2)}{\varepsilon}\right] \geq P\left[N_\kappa \geq \frac{u+\kappa(c_1+c_2)}{\varepsilon}\right] > 0. \end{gathered} \tag{2.12}$$

Thus

$$P\left[\sum_{i=1}^{N_{T_2}}(a_{\sigma_i-}U_i + b_{\sigma_i-}V_i) \geq u+\kappa(c_1+c_2)\right] > 0. \tag{2.13}$$

This implies that ruin occurs with strictly positive probability.

From (a) and (b) above, we conclude that $\delta(u) < 1$.

The process $\{\delta_{a,b}(R^{a,b}_{\tau_{a,b}\wedge t})\}$ is a martingale, if we stop the the process starting in u at the first time T_z where $R^{a,b}_t = z$. Define $\overline{R}^{a,b}_t = R^{a,b}_t + z - u$ for $t \leq T_z$, and choose arbitrary strategy

$(\overline{a}, \overline{b})$ after time T_z. To the process $\{\overline{R}_t^{a,b}\}$, we define its corresponding characteristics by a bar sign. Then

$$\begin{aligned}\overline{\delta}_{a,b}(z) &= \mathrm{E}\left[\overline{\delta}_{a,b}\left(\overline{R}_{T_z \wedge \overline{\tau}_{a,b}}\right)\right] \\ &= \overline{\delta}_{a,b}(2z-u)\mathrm{P}[T_z < \overline{\tau}_{a,b}] \geq \overline{\delta}_{a,b}(2z-u)\mathrm{P}[T_z < \tau_{a,b}].\end{aligned} \tag{2.14}$$

There exists a strategy such that $\mathrm{P}[T_z < \tau_{a,b}]$ is arbitrarily close to 1 due to $\delta_{a,b}(u) = \delta_{a,b}(z)\mathrm{P}[T_z < \tau_{a,b}]$. From the arbitrary property of $(\overline{a}, \overline{b})$, we have $\delta(2z-u) = \delta(z) = \delta(u)$. Thus, $\delta(z)$ would be a constant for all $z \geq u$. While $\delta(z) \to 1$ as $z \to \infty$, this is only possible if $\delta(u) = 1$. Then this is contract with $\delta(u) < 1$. From all above, we conclude that $\delta(u)$ is strictly increasing. □

3. HJB Equation and Verification of Optimality

In this section, we establish the Hamilton-Jacobi-Bellman (HJB for short) equation associated with our problem and give a proof of verification theorem.

We first derive the HJB equation. Let $(a,b) \in [0,1]$ be two arbitrary constants and $\varepsilon > 0$. If the initial capital $u = 0$, we assume that $c_1(a) + c_2(b) \geq 0$ in order to avoid immediate ruin. If $u > 0$, assume that $h > 0$ is small enough such that $u + (c_1(a) + c_2(b))h > 0$. Define

$$\left(u_t^1, u_t^2\right) = \begin{cases} (a,b), & \text{for } t \leq \sigma_1 \wedge h, \\ \left(a^{\varepsilon}_{t-(\sigma_1 \wedge h)}, b^{\varepsilon}_{t-(\sigma_1 \wedge h)}\right), & \text{for } t > \sigma_1 \wedge h, \end{cases} \tag{3.1}$$

where $(a_t^{\varepsilon}, b_t^{\varepsilon})$ are strategies satisfying $\delta_{a_t^{\varepsilon}, b_t^{\varepsilon}}(x) > \delta(x) - \varepsilon$. The first claim happens with density $\lambda e^{-\lambda t}$ and $\mathrm{P}(\sigma_1 > h) = e^{-\lambda h}$. This yields by conditioning on $\mathcal{F}_{\sigma_1 \wedge h}$

$$\begin{aligned}\delta(u) \geq \delta_{u^1,u^2}(u) &= e^{-\lambda h}\delta_{a^{\varepsilon},b^{\varepsilon}}(u + (c_1(a) + c_2(b))h) \\ &\quad + \int_0^h \int_0^{(u+(c_1(a)+c_2(b))t)/a} \int_0^{(u+(c_1(a)+c_2(b))t-ax)/b} \delta_{a^{\varepsilon},b^{\varepsilon}}(u + (c_1(a) + c_2(b))t - au - bv) \\ &\qquad \times \mathrm{d}G(u,v)\lambda e^{-\lambda t}\mathrm{d}t \\ &\geq e^{-\lambda h}\delta(u + (c_1(a) + c_2(b))h) \\ &\quad + \int_0^h \int_0^{(u+(c_1(a)+c_2(b))t)/a} \int_0^{(u+(c_1(a)+c_2(b))t-ax)/b} \delta(u + (c_1(a) + c_2(b))t - au - bv) \\ &\qquad \times \mathrm{d}G(u,v)\lambda e^{-\lambda t}\mathrm{d}t - \varepsilon.\end{aligned} \tag{3.2}$$

Because ε is arbitrary, let $\varepsilon = 0$. The above expression can be expressed as

$$\begin{aligned}&\frac{\delta(u+(c_1(a)+c_2(b))h)-\delta(u)}{h}-\frac{1-e^{-\lambda h}}{h}\delta(u+(c_1(a)+c_2(b))h)\\&\quad+\frac{1}{h}\int_0^h\int_0^{u/a}\int_0^{(u-ax)/b}\delta(u+(c_1(a)+c_2(b))t-au-bv)\mathrm{d}G(u,v)\lambda e^{-\lambda t}\mathrm{d}t\le 0.\end{aligned}\tag{3.3}$$

If we assume that $\delta(u)$ is differentiable and $h\to 0$, yields

$$[c_1(a)+c_2(b)]\delta'(u)+\lambda\int_0^{u/a}\int_0^{(u-ax)/b}\delta(u-ax-by)\mathrm{d}G(x,y)-\lambda\delta(u)\le 0.\tag{3.4}$$

For all $(a,b)\in\mathcal{U}$, (3.4) is true. We first consider such a HJB equation

$$\sup_{(a,b)\in[0,1]\times[0,1]}[c_1(a)+c_2(b)]f'(u)+\lambda\int_0^{\infty}\int_0^{\infty}f(u-ax-by)\mathrm{d}G(x,y)-\lambda f(u)=0.\tag{3.5}$$

For the moment, we are not sure whether $\delta(u)$ fulfills the HJB equation and just conjecture that $\delta(u)$ is one of the solutions, so we replace $\delta(u)$ by $f(u)$. Because $\delta(u)$ is a survival function, we are interested in a function $f(x)$ which is strictly increasing, $f(x)=0$ for $x<0$ and $f(0)>0$. Because the function for which the supremum is taken is continuous in a, b, and $[0,1]\times[0,1]$ is compact, for $u\ge 0$, there are values $a(u)$, $b(u)$ for which the supremum is attained. In (3.5), we also need $c_1(a)+c_2(b)\ge 0$. Otherwise, (3.5) will never be true. Furthermore, $\mathrm{P}(aU_n+bV_n>0)>0$, so $c_1(a)+c_2(b)>0$. We rewrite (3.5) by

$$\sup_{(a,b)\in\tilde{U}}[c_1(a)+c_2(b)]f'(u)+\lambda\int_0^{u/a}\int_0^{(u-ax)/b}f(u-ax-by)\mathrm{d}G(x,y)-\lambda f(u)=0,\tag{3.6}$$

where $\tilde{U}=\{(a,b)\in[0,1]\times[0,1]:c_1(a)+c_2(b)>0\}$ and $u\ge 0$. Define that $u/0=\infty$.

From (3.6), we have

$$f'(u)\le\frac{\lambda}{c_1(a)+c_2(b)}\left[f(u)-\int_0^{u/a}\int_0^{(u-ax)/b}f(u-ax-by)\mathrm{d}G(x,y)\right].\tag{3.7}$$

When $(a,b)=(a^*,b^*)$, equality holds. Then $f(u)$ also satisfies the following equivalent equation:

$$f'(u)=\inf_{(a,b)\in\tilde{U}}\frac{\lambda}{c_1(a)+c_2(b)}\left[f(u)-\int_0^{u/a}\int_0^{(u-ax)/b}f(u-ax-by)\mathrm{d}G(x,y)\right].\tag{3.8}$$

Equations (3.4) and (3.8) are equivalent for strictly increasing functions. Solutions solved from (3.8) are only up to a constant, and we can choose $f(0)=1$.

In the next theorem we prove the existence of a solution of HJB equation and also give the properties of the solution.

Theorem 3.1. *There is a unique solution to the HJB equation* (3.8) *with* $f(0) = 1$*. The solution is bounded, strictly increasing, and continuously differentiable.*

Proof. Reformulate the expression by integrating by part,

$$\begin{aligned}
f(u) &- \int_0^{u/a}\int_0^{(u-ax)/b} f(u-ax-by)\mathrm{d}G(x,y) \\
&= f(u) - \int_0^u f(u-x)\mathrm{d}G_{aU+bV}(x) \\
&= f(u) - \int_0^u\left(\int_0^{u-x} f'(y)\mathrm{d}y - 1\right)\mathrm{d}G_{aU+bV}(x) \\
&= \int_0^u f'(y)(1-G_{aU+bV}(u-y))\mathrm{d}y + 1 - G_{aU+bV}(u).
\end{aligned} \tag{3.9}$$

Let $\mho$ be an operator, and let g be a positive function, define

$$\mho g(u) = \inf_{(a,b)\in\overline{U}} \frac{\lambda}{c_1(a)+c_2(b)}\left[\int_0^u g(y)(1-G_{aU+bV}(u-y))\mathrm{d}y + 1 - G_{aU+bV}(u)\right]. \tag{3.10}$$

First we will show the existence of a solution. If no reinsurance is taken to every subportfolio, the survival probability $\delta_1(u)$ satisfied the equation (See Rolski et al. [14]) as follows:

$$\begin{aligned}
\delta_1'(u) &= \frac{\lambda}{c_1+c_2}\left[\delta_1(u) - \int_0^u \delta_1(u-x)\mathrm{d}G_{U+V}(x)\right] \\
&= \frac{\lambda}{c_1+c_2}\left[\int_0^u \delta_1'(y)(1-G_{U+V}(u-y))\mathrm{d}y + 1 - G_{U+V}(u)\right].
\end{aligned} \tag{3.11}$$

Let

$$g_0(u) = \frac{(c_1+c_2)\delta_1'(u)}{\lambda(\mathrm{E}(U+V))} = \frac{\delta_1'(u)}{\delta_1(0)}, \tag{3.12}$$

where $\delta_1(0) = (\lambda\mathrm{E}(U+V))/(c_1+c_2)$ (this result can be found in Schmidli [13] Appendix D.1.) Next we define recursively $g_n(u) = \mho g_{n-1}(u)$. Because

$$\begin{aligned}
g_1(u) &= \mho g_0(u) \\
&= \inf_{(a,b)\in U} \frac{\lambda}{c_1(a)+c_2(b)}\left[\int_0^u g_0(y)(1-G_{aU+bV}(u-y))\mathrm{d}y + 1 - G_{aU+bV}(u)\right] \\
&\le \frac{\lambda}{c_1+c_2}\left[\int_0^u g_0(y)(1-G_{U+V}(u-y))\mathrm{d}y + 1 - G_{U+V}(u)\right] \\
&= \frac{\lambda}{c_1+c_2}\left[\int_0^u \frac{\delta_1'(y)}{\delta_1(0)}(1-G_{U+V}(u-y))\mathrm{d}y + 1 - G_{U+V}(u)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\delta_1(0)} \frac{\lambda}{c_1 + c_2} \left[\int_0^u \delta_1'(y)(1 - G_{U+V}(u-y))\mathrm{d}y + (1 - G_{U+V}(u))\delta_1(0) \right] \\
&\le \frac{1}{\delta_1(0)} \frac{\lambda}{c_1 + c_2} \left[\int_0^u \delta_1'(y)(1 - G_{U+V}(u-y))\mathrm{d}y + (1 - G_{U+V}(u)) \right] \\
&= \frac{\delta_1'(u)}{\delta_1(0)} = g_0(u).
\end{aligned}
\tag{3.13}
$$

Then $g_1(u) \le g_0(u)$. We conclude that $g_n(u)$ is decreasing in n. Indeed, suppose that $g_{n-1}(u) \ge g_n(u)$. Let (a_n, b_n) be the points where $\mho g_{n-1}(u)$ attains the minimum. Such a pair of points exist because the right side of (3.8) is continuous in both a and b, the set $\{(a,b) : c_1(a)+c_2(b) \ge 0\}$ is compact, and the right side of (3.8) converges to infinity as (a,b) approach the point (a_0, b_0) where $c_1(a_0) = 0$, $c_2(b_0) = 0$. Then

$$
\begin{aligned}
g_n(u) - g_{n+1}(u) &= \mho g_{n-1}(u) - \mho g_n(u) \\
&\ge \frac{\lambda}{c_1(a_n) + c_2(b_n)} \left[\int_0^u (g_{n-1}(y) - g_n(y))(1 - G_{a_nU+b_nV}(u-y))\mathrm{d}y \right] \ge 0.
\end{aligned}
\tag{3.14}
$$

So $g_n(u) \ge g_{n+1}(u) > 0$, and we have $g(u) = \lim_{n\to\infty} g_n(u)$ exists point wise. By the bounded convergence, for each u, a, and b

$$
\lim_{n\to\infty} \int_0^u g_n(y)(1 - G_{aU+bV}(u-y))\mathrm{d}y = \int_0^u g(y)(1 - G_{aU+bV}(u-y))\mathrm{d}y. \tag{3.15}
$$

Let a, b be points which $\mho g(u)$ attains its minimum. For

$$
\begin{aligned}
g_n(u) &= \frac{\lambda}{c_1(a_n) + c_2(b_n)} \left[1 - G_{a_nU+b_nV}(u) + \int_0^u g_{n-1}(y)(1 - G_{a_nU+b_nV}(u-y))\mathrm{d}y \right] \\
&\le \frac{\lambda}{c_1(a) + c_2(b)} \left[1 - G_{aU+bV}(u) + \int_0^u g_{n-1}(y)(1 - G_{aU+bV}(u-y))\mathrm{d}y \right].
\end{aligned}
\tag{3.16}
$$

So $g(u) \le \mho g(u)$ by letting $n \to \infty$. On the other hand, $g_n(z)$ is decreasing, then

$$
\begin{aligned}
g_n(u) &= \frac{\lambda}{c_1(a_n) + c_2(b_n)} \left[1 - G_{a_nU+b_nV}(u) + \int_0^u g_{n-1}(y)(1 - G_{a_nU+b_nV}(u-y))\mathrm{d}y \right] \\
&\ge \frac{\lambda}{c_1(a_n) + c_2(b_n)} \left[1 - G_{a_nU+b_nV}(u) + \int_0^u g(y)(1 - G_{a_nU+b_nV}(u-y))\mathrm{d}y \right] \\
&\ge \frac{\lambda}{c_1(a) + c_2(b)} \left[1 - G_{aU+bV}(u) + \int_0^u g(y)(1 - G_{aU+bV}(u-y))\mathrm{d}y \right].
\end{aligned}
\tag{3.17}
$$

So $g(u) = \mho g(u)$. Define $f(u) = 1 + \int_0^u g(x)\mathrm{d}x$. By the bounded convergence, $f(u)$ fulfills (3.8). Then $f(u)$ is increasing, continuously differentiable and bounded by $(c_1 + c_2)/(\lambda \mathrm{E}(U + V))$. From (3.8), $f'(0) > 0$. Let $x_0 = \inf\{z : f'(z) = 0\}$. Because $f(u)$ is strictly increasing in $[0, x_0]$, we must have $G_{aU+bV}(x_0) = 1$ and $ax + by = 0$ for all points of increase of $G_{aU+bV}(z)$. But this would be $a = b = 0$, which is impossible. Thus $f(u)$ is strictly increasing.

Next we want to show the uniqueness of the solution. Suppose that $f_1(u)$ and $f_2(u)$ are the solutions to (3.8) with $f_1(0) = f_2(0) = 1$. Define $g_i(u) = f_i'(u)$, and (a_i, b_i) is the value which minimize (3.8). To a constant $\overline{x} > 0$, because the right hand of (3.8) is continuous both in a and b and tends to infinity as $c_1(a) + c_2(b)$ approach 0, the $c_1(a) + c_2(b)$ is bounded away from 0 on $(0, \overline{x}]$. Let $x_1 = \inf\{\min_i c_1(a_i(x)) + c_2(b_i(x)) : 0 \le u \le \overline{x}\}/(2\lambda)$ and $x_n = nx_1 \wedge \overline{x}$. Suppose we have proved that $f_1(u) = f_2(u)$ on $[0, x_n]$. For $n = 0$, it is obviously true. Then for $u \in [x_n, x_{n+1}]$, with $m = \sup_{x_n \le u \le x_{n+1}} |g_1(u) - g_2(u)|$

$$
\begin{aligned}
g_1(u) - g_2(u) &= \mho g_1(u) - \mho g_2(u) \\
&\le \frac{\lambda}{c_1(a_2) + c_2(b_2)} \left[\int_0^u (g_1(y) - g_2(y))(1 - G_{a_2U+b_2V}(u - y))\mathrm{d}y \right] \\
&= \frac{\lambda}{c_1(a_2) + c_2(b_2)} \left[\int_{x_n}^u (g_1(y) - g_2(y))(1 - G_{a_2U+b_2V}(u - y))\mathrm{d}y \right] \\
&\le \frac{\lambda}{c_1(a_2) + c_2(b_2)} m(u - x_n) \\
&\le \frac{\lambda}{c_1(a_2) + c_2(b_2)} m(x_{n+1} - x_n) \\
&\le \frac{\lambda}{c_1(a_2) + c_2(b_2)} m x_1 \\
&\le \frac{\lambda}{c_1(a_2) + c_2(b_2)} m \frac{c_1(a_2) + c_2(b_2)}{2\lambda} = \frac{m}{2}.
\end{aligned}
\tag{3.18}
$$

Once revers the role of $g_1(u)$ and $g_2(u)$, then $|g_1(u) - g_2(u)| \le m/2$. This is impossible for all $u \in [x_n, x_{n+1}]$ if $m \neq 0$. This shows that $f_1(u) = f_2(u)$ on $[0, x_{n+1}]$. So $f_1(u) = f_2(u)$ on $[0, \overline{x}]$. The uniqueness is true from the arbitrary of $\overline{x}$. □

Denoted by $a^*(u), b^*(u)$ the value of a and b maximize (3.6).

From the next theorem, so-called verification theorem, we conclude that a solution to the HJB equation which satisfies some conditions really is the desired value function.

Theorem 3.2. *Let $f(u)$ be the unique solution to the HJB equation* (3.8) *with $f(0) = 1$. Then $f(u) = \delta(u)/\delta(0)$. An optimal strategy is given by (a_t^*, b_t^*), which minimize* (3.8)*, and $\{R_t\}$ is the process under the optimal strategy.*

Proof. Let (a, b) be an arbitrary strategy with the risk processes $\{R_t^{a,b}\}$. Since $f(u)$ is bounded, then for each $t \ge 0$,

$$
\mathrm{E}\left(\sum_{n:\sigma_n \le t} \left| f\left(R^{a,b}(\sigma_n)\right) - f\left(R^{a,b}(\sigma_n -)\right) \right| \right) < \infty. \tag{3.19}
$$

Let $\mathscr{A}$ denotes the generator of $\{R_t^{a,b}\}$. From Theorem 11.2.2 in Rolski et al. [14], we know that $f \in \mathfrak{D}(\mathscr{A})$, where $\mathfrak{D}(\mathscr{A})$ is the domain of $\mathscr{A}$. Then

$$f\left(R^{a,b}_{\tau_{a,b}\wedge t}\right) - \int_0^{\tau_{a,b}\wedge t}\Bigg[(c_1(a)+c_2(b))f'\left(R_s^{a,b}\right) + \lambda\left(\int_0^{R_t^{a,b}/a}\int_0^{(R_t^{a,b}-ax)/b} f\left(R_s^{a,b}-ax-by\right)\mathrm{d}G(x,y) - f\left(R_s^{a,b}\right)\right)\Bigg]\mathrm{d}s \tag{3.20}$$

is a martingale. From (3.6) we know that $\{f(R_t^{a,b})1_{\tau_{a,b}>t}\}$ is a supermartingale, then

$$\mathrm{E}\left(f\left(R_t^{a,b}\right)1_{\tau_{a,b}>t}\right) = \mathrm{E}\left(f\left(R^{a,b}_{\tau_{a,b}\wedge t}\right)\right) \le f(u). \tag{3.21}$$

If $(a,b)=(a^*,b^*)$, then $\{f(R_{\tau^*\wedge t})\}$ is a martingale. So $\mathrm{E}(f(R_t)1_{\tau^*>t}) = f(u)$. Let $t\to\infty$, from the bounded property of $f(u)$, we have

$$f(\infty)\delta_{a,b}(u) = f(\infty)\mathrm{P}[\tau_{a,b}=\infty] \le f(u) = f(\infty)\delta_{a^*,b^*}(u) = \delta(u)f(\infty). \tag{3.22}$$

For $u=0$, we obtain that $f(\infty)=1/\delta(0)$. Then $\delta(u)=f(u)/f(\infty)=f(u)\delta(0)$. Furthermore, the associated policy with (a^*,b^*) is indeed an optimal strategy. □

4. Lundberg Bounds and the Change of Measure Formula

In Section 3, we have seen when considering the dynamic reinsurance police the explicit expression of ruin probability is not easy to derive. Therefore the asymptotic optimal strategies are very important. In the classical risk theory, we have Lundberg bounds and Cramér-Lundberg approximation for the ruin probability. The former gives the upper and lower bounds for ruin probability, and the latter gives the asymptotic behavior of ruin probability as the capital tends to infinity. They both provide the useful information in getting the nature of underlying risks. In researching the two-dimensional risk model controlled by reinsurance strategy, we can also discuss the analogous problems. References are Schmidli [15, 16], Hipp and Schmidli [17], and so forth. The key in researching the asymptotic behavior is adjustment coefficient. Next we will discuss it in detail.

Assume that $\mathrm{E}e^{r(U+V)}<\infty$ for $r>0$. To the *fixed* (a,b), let $R(a,b)$ be adjustment coefficient satisfied

$$\theta(r;a,b) := \lambda\left(\mathrm{E}e^{r(aU+bV)}-1\right) - r(c_1(a)+c_2(b)) = 0. \tag{4.1}$$

We focus on $R = \sup_{(a,b)\in[0,1]\times[0,1]} R(a,b)$, which is the adjustment coefficient for our problem. By the assumption that $c_1(a)$ and $c_2(b)$ are continuous, then $\theta(r;a,b)$ is continuous both in a and b. Moreover

$$\frac{\partial^2\theta(r;a,b)}{\partial r^2} = \lambda \mathrm{E}(aU+bV)^2 e^{r(aU+bV)} > 0,$$
$$\theta(0;a,b) = 0, \qquad \theta(R(a,b);a,b) = 0. \tag{4.2}$$

We can get that $\theta(r;a,b)$ is strictly convex in r and $\theta(R;a,b) > 0$. If $r < R$, then there are a and b such that $R(a,b) > r$ and $\theta(r;a,b) < 0$. Because $\theta(R;a,b)$ is continuous in a and b, also $[0,1]\times[0,1]$ is compact, there exist $\tilde{a}$ and $\tilde{b}$ for which $\theta(R;\tilde{a},\tilde{b}) = 0$.

Lemma 4.1. *Suppose that $M(r,a,b)$, $c_1(a)$, and $c_2(b)$ are all twice differentiable (with respect to r, a, and b). Moreover that*

$$c_1''(a) \le 0, \qquad c_2''(b) \le 0, \tag{4.3}$$

then there is a unique maximum of $R(a,b)$.

Proof. $R(a,b)$ satisfies (4.1):

$$\lambda\left(\mathrm{E}e^{R(a,b)(aU+bV)} - 1\right) - (c_1(a) + c_2(b))R(a,b) = 0. \tag{4.4}$$

Let $M(r,a,b) = \mathrm{E}e^{r(aU+bV)}$, and $M_r(r,a,b)$, $M_a(r,a,b)$, $M_b(r,a,b)$, R_a, and R_b denote the partial derivatives.

Taking partial derivative of (4.4) with respect to a,

$$\lambda M_r R_a(a,b) + \lambda M_a - c_1'(a)R(a,b) - (c_1(a)+c_2(b))R_a(a,b) = 0. \tag{4.5}$$

Because the left-side hand of (4.4) is a convex function in r, we have $\lambda M_r - (c_1(a)+c_2(b)) > 0$. So

$$R_a(a,b) = -\frac{\lambda M_a - c_1'(a)R(a,b)}{\lambda M_r - (c_1(a)+c_2(b))}. \tag{4.6}$$

Similarly

$$R_b(a,b) = -\frac{\lambda M_b - c_2'(b)R(a,b)}{\lambda M_r - (c_1(a)+c_2(b))}. \tag{4.7}$$

Let $(\tilde{a},\tilde{b})$ be the point such that $R_a(\tilde{a},\tilde{b}) = R_b(\tilde{a},\tilde{b}) = 0$. Then

$$\begin{aligned}
R_{a,a}\left(\tilde{a},\tilde{b}\right) &= -\frac{\lambda M_{a,a}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - c_1''(a)R\left(\tilde{a},\tilde{b}\right)}{\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)} \\
&= -\frac{\lambda \mathrm{E}\left(R\left(\tilde{a},\tilde{b}\right)U\right)^2 e^{R(\tilde{a},\tilde{b})(\tilde{a}U+\tilde{b}V)} - c_1''(\tilde{a})R\left(\tilde{a},\tilde{b}\right)}{\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)} < 0, \\
R_{b,b}\left(\tilde{a},\tilde{b}\right) &= -\frac{\lambda M_{b,b}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - c_2''(b)R\left(\tilde{a},\tilde{b}\right)}{\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)} \\
&= -\frac{\lambda \mathrm{E}\left(R\left(\tilde{a},\tilde{b}\right)V\right)^2 e^{R(\tilde{a},\tilde{b})(\tilde{a}U+\tilde{b}V)} - c_2''\left(\tilde{b}\right)R\left(\tilde{a},\tilde{b}\right)}{\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)} < 0, \\
R_{a,b}\left(\tilde{a},\tilde{b}\right) &= -\frac{\lambda M_{a,b}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right)}{\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)}.
\end{aligned} \tag{4.8}$$

While

$$\begin{aligned}
&R_{a,a}\left(\tilde{a},\tilde{b}\right)R_{b,b}\left(\tilde{a},\tilde{b}\right) - R_{a,b}^2\left(\tilde{a},\tilde{b}\right) \\
&\quad= \frac{\left[\lambda M_{a,a}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - c_1''(a)R\left(\tilde{a},\tilde{b}\right)\right]\left[\lambda M_{b,b}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - c_2''(b)R\left(\tilde{a},\tilde{b}\right)\right]}{\left[\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)\right]^2} \\
&\qquad - \frac{\lambda^2 M_{a,b}^2\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right)}{\left[\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)\right]^2} \\
&\quad= \frac{\left[M_{a,a}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right)M_{b,b}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - M_{a,b}^2\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right)\right]\lambda^2}{\left[\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)\right]^2} \\
&\qquad + \frac{c_1''(a)c_2''(b)R^2 + \left[M_{b,b}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right)c_1''(a) + M_{a,a}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right)c_2''(b)R\left(\tilde{a},\tilde{b}\right)\right]\lambda R}{\left[\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)\right]^2}.
\end{aligned} \tag{4.9}$$

From Hölder inequality, we have that the first term of above expression is positive. Owning to the conditions given by the lemma, we also find that the second term of above is positive. Therefore, $R(\tilde{a},\tilde{b})$ is a maximum value. □

We now let $\psi(u)$ be the ruin probability under the optimal strategy. First we give a Lundberg upper bound of $\psi(u)$.

Theorem 4.2. *The minimal ruin probability $\psi(u)$ is bounded by e^{-Ru}, that is, $\psi(u) < e^{-Ru}$.*

Proof. To the fixed proportional reinsurance $(\tilde{a},\tilde{b})$, $\psi_{a,b}(u)$ can be calculated by the result on ruin probability of the classical risk model. We have the following expression of $\psi_{\tilde{a},\tilde{b}}(u)$:

$$\begin{aligned}\psi_{\tilde{a},\tilde{b}}(u) &= \mathrm{P}\left(\tau_{\tilde{a},\tilde{b}} < \infty\right) \\ &= \mathrm{E}^{(R)}\left[\exp\left\{RR^{\tilde{a},\tilde{b}}_{\tau_{\tilde{a},\tilde{b}}}\right\}\right]e^{-Ru} \\ &< e^{-Ru}.\end{aligned} \tag{4.10}$$

So the minimal ruin probability is bounded by $\psi(u) \le \psi_{\tilde{a},\tilde{b}}(u) < e^{-Ru}$. □

From Theorem 4.2, the adjustment coefficient R can be looked upon as a risk measure to estimate the optimal ruin probability.

For the considerations below we define the strategy: if $u < 0$, we let $a^*(u) = b^*(u) = 1$. In order to obtain the lower bound, we start by defining a process M_t as follows:

$$\begin{aligned}M_t &= \exp\left\{-R(R_t - u) - \int_0^t \theta(R; a^*(R_s), b^*(R_s))\mathrm{d}s\right\} \\ &= \exp\left\{\sum_{n=1}^{N_t} R(a^*(R_{\sigma_n-})U_n + b^*(R_{\sigma_n-})V_n) - \int_0^t \lambda\left(\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)} - 1\right)\mathrm{d}s\right\}.\end{aligned} \tag{4.11}$$

Lemma 4.3. *The process M_t is a strictly positive martingale with mean value 1.*

Proof. First we will show that $\{M_{\sigma_n \wedge t}\}$ is a martingale. Indeed, $\mathrm{E}M_{\sigma_0\wedge t} = \mathrm{E}M_{\sigma_0} = 1$, and we suppose that $\mathrm{E}M_{\sigma_{n-1}\wedge t} = 1$. Given $\mathcal{F}_{\sigma_{n-1}}$, the progress $\{(X_t, Y_t)\}$ is deterministic on $[\sigma_{n-1}, \sigma_n)$. We split into the event $\{\sigma_n > t\}$ and $\{\sigma_n \le t\}$. From the Markov property of M_t and for $\sigma_{n-1} < t$, we have

$$\begin{aligned}\mathrm{E}M_{\sigma_n\wedge t} &= \mathrm{E}\{\mathrm{E}[M_{\sigma_n\wedge t} \mid \mathcal{F}_{\sigma_{n-1}}]\} \\ &= \mathrm{E}\{\mathrm{E}[M_{\sigma_n\wedge t} \mid M_{\sigma_{n-1}}]\} \\ &= \mathrm{E}\{\mathrm{E}[1_{\sigma_n > t}M_t \mid M_{\sigma_{n-1}}]\} + \mathrm{E}\{\mathrm{E}[1_{\sigma_n \le t}M_{\sigma_n} \mid M_{\sigma_{n-1}}]\}.\end{aligned} \tag{4.12}$$

For convenience, let $A = \mathrm{E}\{\mathrm{E}[1_{\sigma_n > t}M_t \mid M_{\sigma_{n-1}}]\}$ and $B = \mathrm{E}\{\mathrm{E}[1_{\sigma_n \le t}M_{\sigma_n} \mid M_{\sigma_{n-1}}]\}$. Next we calculate A and B, respectively,

$$\begin{aligned}A &= \mathrm{E}\{\mathrm{E}[1_{\sigma_n > t}M_t \mid M_{\sigma_{n-1}}]\} \\ &= \mathrm{E}\left\{1_{\sigma_n > t > \sigma_{n-1}}\mathrm{E}\left[M_{\sigma_n-1}\exp\left\{-\lambda\int_{\sigma_{n-1}}^t \left(\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)} - 1\right)\mathrm{d}s\right\} \mid M_{\sigma_{n-1}}\right]\right\}\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}M_{\sigma_{n-1}}\mathrm{P}(\sigma_{n-1} < t < \sigma_n)\exp\left\{-\lambda\int_{\sigma_{n-1}}^{t}\left(\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)}-1\right)\mathrm{d}s\right\}\\
&= e^{-\lambda(t-\sigma_{n-1})}\exp\left\{-\lambda\int_{\sigma_{n-1}}^{t}\left(\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)}-1\right)\mathrm{d}s\right\}\\
&= \exp\left\{-\lambda\int_{\sigma_{n-1}}^{t}\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)}\mathrm{d}s\right\},
\end{aligned}
$$

$$
\begin{aligned}
B &= \mathrm{E}\{\mathrm{E}[1_{\sigma_n\le t}M_n \mid M_{\sigma_{n-1}}]\}\\
&= \mathrm{E}\left\{1_{\sigma_n\le t}\mathrm{E}\left[M_{\sigma_n-1}\exp\left\{R(a^*(R_{\sigma_n-})U+b^*(R_{\sigma_n-})V)\right.\right.\right.\\
&\qquad\left.\left.\left.-\lambda\int_{\sigma_{n-1}}^{\sigma_n}\left(\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)}-1\right)\mathrm{d}s\right\} \mid M_{\sigma_{n-1}}\right]\right\}\\
&= \int_{\sigma_{n-1}}^{t}\mathrm{E}\,\exp\left\{R(a^*(R_{s-})U+b^*(R_{s-})V)-\lambda\int_{\sigma_{n-1}}^{s}\left(\mathrm{E}e^{R(a^*(R_w)U+b^*(R_w)V)}-1\right)\mathrm{d}w\right\}\\
&\quad\times\lambda e^{-\lambda(s-\sigma_{n-1})}\mathrm{d}s\\
&= \int_{\sigma_{n-1}}^{t}\lambda\mathrm{E}e^{R(a^*(R_{s-})U+b^*(R_{s-})V)}\exp\left\{-\lambda\int_{\sigma_{n-1}}^{s}\mathrm{E}e^{R(a^*(R_w)U+b^*(R_w)V)}\mathrm{d}w\right\}\mathrm{d}s.
\end{aligned}
\tag{4.13}
$$

Let $f(s)=\lambda\mathrm{E}e^{R(a^*(R_{s-})U+b^*(R_{s-})V)}$, then

$$
\mathrm{E}M_{\sigma_n\wedge t} = e^{-\int_{\sigma_{n-1}}^{t}f(s)\mathrm{d}s}+\int_{\sigma_{n-1}}^{t}f(s)e^{-\int_{\sigma_{n-1}}^{s}f(w)\mathrm{d}w}\mathrm{d}s. \tag{4.14}
$$

Because $(e^{-\int_{\sigma_{n-1}}^{t}f(s)\mathrm{d}s})' = f(t)e^{-\int_{\sigma_{n-1}}^{t}f(w)\mathrm{d}w}$, using the integration by part, we have $\mathrm{E}M_{\sigma_n\wedge t}=1$. From above we know that $\mathrm{E}[M_{\sigma_n\wedge t}\mid \mathcal{F}_{\sigma_{n-1}}]=M_{\sigma_{n-1}\wedge t}$. Furthermore, following the assumption that $\mathrm{E}e^{R(U+V)}<\infty$, then

$$
\exp\left\{\sum_{n=1}^{N_t}R(a^*(R_{\sigma_n-})U_n+b^*(R_{\sigma_n-})V_n)\right\}\le\exp\left\{\sum_{n=1}^{N_t}R(U_n+V_n)\right\}<\infty. \tag{4.15}
$$

So for each t, $\{M_{\sigma_n\wedge t}\}$ is uniform integrable. This finishes the proof of Lemma 4.3. □

Based on the martingale $\{M(t), t\ge 0\}$ given above, we consider a family of new measure $\mathrm{P}_t^*[A]=\mathrm{E}[M_t;A]$, $A\in\mathcal{F}_t$. From the Kolmogorov's extension theorem, there exists a probability measure P^* such that the restriction of P^* to $\mathcal{F}_t$ is P_t^*. Moreover, if T is an $\mathcal{F}_t$-stopping time and $A\subset\{T<\infty\}$ such that $A\in\mathcal{F}_T$, then $\mathrm{P}^*[A]=\mathrm{E}[M_T;A]$. The change of measure technique is a powerful tool in investigating ruin probability. The following theorem gives us the feature of R_t under the new measure.

Theorem 4.4. *Under the new measure P^*, the process $\{R_t\}$ is a piecewise deterministic Markov process (PDMP for short) with jump intensity $\lambda^*(x) = \lambda E e^{R(a^*(x)U+b^*(x)V)}$ and claim size distribution*

$$G_x^*(u,v) = \frac{1}{Ee^{R(a^*(x)U+b^*(x)V)}} \int_0^u \int_0^v e^{R(a^*(x)r+b^*(x)s)} dG(r,s). \tag{4.16}$$

The premium rates for each subportfolios are $c_1(a^*(x))$ and $c_2(b^*(x))$, respectively.

Proof. Let B be a Borel set. Refer to Lemma C.1 in Schmidli [13], we have

$$\begin{aligned} \mathrm{P}^*[R_{t+s} \in B \mid \mathcal{F}_t] &= \mathrm{E}\Big[M_t^{-1} M_{t+s}; R_{t+s} \in B \mid \mathcal{F}_t\Big] \\ &= \mathrm{E}\Big[M_t^{-1} M_{t+s}; R_{t+s} \in B \mid \mathrm{X}_t\Big]. \end{aligned} \tag{4.17}$$

This means that under the new measure P^*, $\{R_t\}$ is still a Markov process. On the other hand, the path between jumps is deterministic. So $\{R_t\}$ is a PDMP under P^*. Next we will calculate the distribution of σ_1 (the time of the first claim happens), U, and V. Let r_s denote the deterministic path on $[0,\sigma_1)$. The distribution of σ_1 can be obtained by

$$\begin{aligned} \mathrm{P}^*[\sigma_1 > t] &= \mathrm{E}[M_t; \sigma_1 > t] \\ &= e^{-\lambda t} \exp\left\{ -\int_0^t \lambda\left(\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)} - 1\right) \mathrm{d}s \right\} \\ &= \exp\left\{ -\int_0^t \lambda \mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)} \mathrm{d}s \right\}. \end{aligned} \tag{4.18}$$

So $\lambda^*(x) = \lambda \mathrm{E}e^{R(a^*(x)U+b^*(x)V)}$.

Next we consider the first claim size (U_1, V_1). Let B_1, B_2 be two Borel sets.

$$\begin{aligned} &\mathrm{P}^*[\sigma_1 \le t,\ U_1 \in B_1, V_1 \in B_2] \\ &\quad = \mathrm{E}[M_t; \sigma_1 \le t,\ U_1 \in B_1, V_1 \in B_2] \\ &\quad = \mathrm{E}[\mathrm{E}[M_t; \sigma_1 \le t,\ U_1 \in B_1, V_1 \in B_2 \mid \mathcal{F}_{\sigma_1}]] \\ &\quad = \mathrm{E}[M_{\sigma_1}; \sigma_1 \le t,\ U_1 \in B_1, V_1 \in B_2] \\ &\quad = \int_0^t \int_0^\infty \int_0^\infty \exp\Big\{ R(a^*(r_s)u + b^*(r_s)v) \\ &\qquad\qquad -\lambda \int_0^s \Big(\mathrm{E}e^{R(a^*(r_w)U_1+b^*(r_w)V_1)} - 1\Big) \mathrm{d}w \Big\} 1_{B_1\times B_2}(u,v) \mathrm{d}G(u,v) \lambda e^{-\lambda s} \mathrm{d}s \\ &\quad = \int_0^t \int_0^\infty \int_0^\infty 1_{B_1\times B_2}(u,v) \mathrm{d}G^*(u,v) \lambda^*(r_s) e^{-\int_0^s \lambda^*(r_w)\mathrm{d}w} \mathrm{d}s. \end{aligned} \tag{4.19}$$

At last, since the set of trajectories of R_t is same under P and P*, it is clear that the deterministic premium rates remain $c_1(a^*)$ and $c_2(b^*)$. □

If we consider the drift of R_t under the new measure P*, then

$$\begin{aligned}
&c_1(a^*(x)) + c_2(b^*(x)) - \lambda^*(x)\int_0^\infty\int_0^\infty (a^*(x)u + b^*(x)v)\mathrm{d}G_x^*(u,v)\\
&\quad = c_1(a^*(x)) + c_2(b^*(x)) - \lambda \mathrm{E}e^{R(a^*(x)U+b^*(x)V)}\\
&\qquad \times \int_0^\infty\int_0^\infty (a^*(x)u + b^*(x)v)\frac{1}{\mathrm{E}e^{R(a^*(x)U+b^*(x)V)}}e^{R(a^*(x)u+b^*(x)v)}\mathrm{d}G(u,v)\\
&\quad = c_1(a^*(x)) + c_2(b^*(x)) - \lambda \mathrm{E}R(a^*(x)U + b^*(x)V)e^{R(a^*(x)U+b^*(x)V)}\\
&\quad = -\frac{\partial\theta}{\partial R}(R; a^*(x), b^*(x)).
\end{aligned} \tag{4.20}$$

From the convexity property of $\theta(r; a^*(x), b^*(x))$ about r, we know that $\theta'_r(R; a^*(x), b^*(x)) > 0$. This implies that $\mathrm{P}^*[\tau^* < \infty] = 1$, and

$$\psi(u) = \mathrm{E}^*\left[e^{RR_{\tau^*}+\int_0^{\tau^*}\theta(R;a^*(R_s),b^*(R_s))\mathrm{d}s}\right]e^{-Ru}. \tag{4.21}$$

The following theorem gives a lower bound for $\psi(u)$.

Theorem 4.5. *Let*

$$C_- = \inf_z \frac{1}{E\left[e^{R(U+V-z)} \mid U+V>z\right]}, \tag{4.22}$$

where z is taken over the set $\{z : P[U+V>z] > 0\}$. Then $\psi(u) \geq C_- e^{-Ru}$.

Proof. Suppose that $R_{\tau^*-} = z$, then

$$\begin{aligned}
&\mathrm{E}^*\left[\exp\{RR_{\tau^*}\} \mid R_{\tau^*-} = z\right]\\
&\quad = \mathrm{E}^*\left[\exp\{R(z - a^*(z)U - b^*(z)V)\} \mid a^*(z)U + b^*(z)V > z\right]\\
&\quad = \frac{1}{\mathrm{E}\left[\exp\{R((a^*(z)U + b^*(z)V) - z)\} \mid a^*(z)U + b^*(z)V > z\right]}\\
&\quad = \frac{1}{\mathrm{E}\left[\exp\left\{Ra^*(z)b^*(z)\left[\dfrac{U}{b^*(z)} + \dfrac{V}{a^*(z)} - \dfrac{z}{a^*(z)b^*(z)}\right]\right\} \mid \dfrac{U}{b^*(z)} + \dfrac{V}{a^*(z)} > \dfrac{z}{a^*(z)b^*(z)}\right]}\\
&\quad \geq \inf_{a,b}\frac{1}{\mathrm{E}\left[\exp\left\{Rab\left[\dfrac{U}{b^*(z)} + \dfrac{V}{a^*(z)} - \dfrac{z}{a^*(z)b^*(z)}\right]\right\} \mid \dfrac{U}{b^*(z)} + \dfrac{V}{a^*(z)} > \dfrac{z}{a^*(z)b^*(z)}\right]}.
\end{aligned} \tag{4.23}$$

Thus,

$$
\begin{aligned}
&\mathrm{E}^{*}\left[\exp\{RR_{\tau^{*}}\}\right] \\
&\quad\geq \inf_{a,b,z}\frac{1}{\mathrm{E}\left[\exp\left\{R\left[\dfrac{ab}{b^{*}(z)}U+\dfrac{ab}{a^{*}(z)}V-\dfrac{ab}{a^{*}(z)b^{*}(z)}z\right]\right\}\mid\dfrac{ab}{b^{*}(z)}U+\dfrac{ab}{a^{*}(z)}V>\dfrac{ab}{a^{*}(z)b^{*}(z)}z\right]} \\
&\quad\geq \inf_{a,b,z}\frac{1}{\mathrm{E}\left[\exp\{R[abU+abV-abz]\}\mid abU+abV>abz\right]} \\
&\quad\geq \inf_{z}\frac{1}{\mathrm{E}\left[\exp\{R[U+V-z]\}\mid U+V>z\right]}=C_{-}.
\end{aligned}
\tag{4.24}
$$

Then $\psi(u)\geq\mathrm{E}^{*}[\exp\{RR_{\tau^{*}}\}]e^{-Ru}\geq C_{-}e^{-Ru}$. □

5. The Cramér-Lundberg Approximation

In this section we will consider the asymptotic behavior of $\psi(x)e^{Rx}$, called Cramér-Lundberg approximation. First from the Fubini's theorem, we transform the expression below:

$$
\begin{aligned}
&\int_{0}^{x/a^{*}(x)}\int_{0}^{(x-a^{*}(x)u)/b^{*}(x)}\psi(x-a^{*}(x)u-b^{*}(x)v)\mathrm{d}G(u,v) \\
&\quad=\int_{0}^{x/a^{*}(x)}\int_{0}^{(x-a^{*}(x)u)/b^{*}(x)}\left(\int_{0}^{x-a^{*}(x)u-b^{*}(x)v}\psi'(z)\mathrm{d}z+\psi(0)\right)\mathrm{d}G(u,v) \\
&\quad=\psi(0)\mathrm{P}[a^{*}(x)U+b^{*}(x)V<x]+\int_{0}^{x}\int_{0}^{x-r}\psi'(z)\mathrm{d}z\mathrm{d}G_{a^{*}(x)U+b^{*}(x)V}(r) \\
&\quad=\psi(0)\mathrm{P}[a^{*}(x)U+b^{*}(x)V<x]+\int_{0}^{x}\int_{0}^{x-z}\mathrm{d}G_{a^{*}(x)U+b^{*}(x)V}(r)\psi'(z)\mathrm{d}z \\
&\quad=\psi(0)\mathrm{P}[a^{*}(x)U+b^{*}(x)V<x]+\int_{0}^{x}\mathrm{P}(a^{*}(x)U+b^{*}(x)V<x-z)\psi'(z)\mathrm{d}z \\
&\quad=\psi(0)\int_{0}^{x/a^{*}(x)}\int_{0}^{(x-a^{*}(x)u)/b^{*}(x)}\mathrm{d}G(u,v) \\
&\qquad+\int_{0}^{x}\left(\int_{0}^{(x-z)/a^{*}(x)}\int_{0}^{(x-a^{*}(x)u)/b^{*}(x)}\mathrm{d}G(u,v)\right)\psi'(z)\mathrm{d}z \\
&\quad=\psi(0)\int_{0}^{x/a^{*}(x)}\int_{0}^{(x-a^{*}(x)u)/b^{*}(x)}\mathrm{d}G(u,v)+\int_{0}^{x}\psi'(z)G_{a^{*}(x)U+b^{*}(x)V}(x-z)\mathrm{d}z.
\end{aligned}
\tag{5.1}
$$

Because $\psi(x) = 1 - \delta(x)$, then the HJB equation can be changed into

$$\begin{aligned}
&(c_1(a^*(x)) + c_2(b^*(x)))\psi'(x) \\
&\quad + \lambda\Big[1 - G_{a^*(x)U+b^*(x)V}(x) - \psi(0)\big(1 - G_{a^*(x)U+b^*(x)V}(x)\big) \\
&\quad\quad - \int_0^x \psi'(z)\big(1 - G_{a^*(x)U+b^*(x)V}(x-z)\big)\mathrm{d}z\Big] = 0.
\end{aligned} \tag{5.2}$$

Let $f(x) = \psi(x)e^{Rx}$, then $\psi'(x)e^{Rx} = f'(x) - Rf(x)$, and

$$\begin{aligned}
&[c_1(a^*(x)) + c_2(b^*(x))]\big(f'(x) - Rf(x)\big) \\
&\quad + \lambda\Big[\delta(0)\big(1 - G_{a^*(x)U+b^*(x)V}(x)\big)e^{Rx} \\
&\quad\quad + \int_0^x \big(Rf(z) - f'(z)\big)\big(1 - G_{a^*(x)U+b^*(x)V}(x-z)\big)e^{R(x-z)}\mathrm{d}z\Big] = 0.
\end{aligned} \tag{5.3}$$

Because $\psi(x)$ is strictly decreasing, then $\psi'(x)e^{Rx} < 0$. So $f'(x) < Rf(x)$. Thus $f'(x)$ is bounded from above. Let $g(x) = Rf(x) - f'(x)$, we get

$$\begin{aligned}
&\lambda\Big[\delta(0)\big(1 - G_{a^*(x)U+b^*(x)V}(x)\big)e^{Rx} + \int_0^x g(z)\big(1 - G_{a^*(x)U+b^*(x)V}(x-z)\big)e^{R(x-z)}\mathrm{d}z\Big] \\
&\quad - g(x)[c_1(a^*(x)) + c_2(b^*(x))] = 0.
\end{aligned} \tag{5.4}$$

Changing the order of the integral, we have

$$\begin{aligned}
&\lambda\Big[\delta(0)\big(1 - G_{a^*(x)U+b^*(x)V}(x)\big)e^{Rx} + \int_0^x g(x-y)\big(1 - G_{a^*(x)U+b^*(x)V}(y)\big)e^{Ry}\mathrm{d}y\Big] \\
&\quad - g(x)[c_1(a^*(x)) + c_2(b^*(x))] = 0.
\end{aligned} \tag{5.5}$$

If we replace $a^*(x)$, $b^*(x)$ by $\tilde{a}$, $\tilde{b}$, we will obtain the inequality

$$\begin{aligned}
&\lambda\Big[\delta(0)\big(1 - G_{\tilde{a}U+\tilde{b}V}(x)\big)e^{Rx} + \int_0^x g(x-y)\big(1 - G_{\tilde{a}U+\tilde{b}V}(y)\big)e^{Ry}\mathrm{d}y\Big] \\
&\quad - g(x)\Big[c_1(\tilde{a}) + c_2\big(\tilde{b}\big)\Big] \geq 0.
\end{aligned} \tag{5.6}$$

Note that

$$\mathrm{E}e^{R(\tilde{a}U+\tilde{b}V)} = \int_0^\infty \int_0^\infty e^{R(\tilde{a}u+\tilde{b}v)} \mathrm{d}G(u,v)$$
$$= \int_0^\infty e^{Rx} \mathrm{d}G_{\tilde{a}U+\tilde{b}V}(x) \tag{5.7}$$
$$= 1 + R\int_0^\infty e^{Ry}(1 - G_{\tilde{a}U+\tilde{b}V}(y))\mathrm{d}y.$$

From the definition of $\tilde{a}$ and $\tilde{b}$,

$$\lambda\left[\mathrm{E}e^{R(\tilde{a}U+\tilde{b}V)} - 1\right] = \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)R. \tag{5.8}$$

Thus $c_1(\tilde{a}) + c_2(\tilde{b}) = \lambda\int_0^\infty e^{Ry}(1 - G_{\tilde{a}U+\tilde{b}V}(y))\mathrm{d}y$. Take the expression of $c_1(\tilde{a}) + c_2(\tilde{b})$ into the above inequality, and obtain

$$\lambda\left[\delta(0)(1 - G_{\tilde{a}U+\tilde{b}V}(x))e^{Rx} + \int_0^x g(x-y)(1 - G_{\tilde{a}U+\tilde{b}V}(y))e^{Ry}\mathrm{d}y\right]$$
$$- g(x)\lambda\int_0^\infty e^{Ry}(1 - G_{\tilde{a}U+\tilde{b}V}(y))\mathrm{d}y \geq 0. \tag{5.9}$$

After transforming

$$\int_0^x [g(x-y) - g(x)]\left(1 - G_{\tilde{a}U+\tilde{b}V}(y)\right)e^{Ry}\mathrm{d}y$$
$$\geq \int_x^\infty (1 - G_{\tilde{a}U+\tilde{b}V}(y))e^{Ry}\mathrm{d}y \cdot g(x) - \delta(0)(1 - G_{\tilde{a}U+\tilde{b}V}(x))e^{Rx}. \tag{5.10}$$

From Lemma A.12 in Schmidli [13], we know $\lim_{x\to\infty}(1 - G_{\tilde{a}U+\tilde{b}V}(x))e^{Rx} = 0$. First we consider two functions $f(x) = \psi(x)e^{Rx}$ and $g(x) = Rf(x) - f'(x)$, which are important in investigating the Cramér-Lundberg approximation. Repeating the proof of Lemma 4.10 in Schmidli [13] (note (5.10) will be uesd in the proof) gives the analogous results.

Lemma 5.1.

(a) $g(x)$ is bounded. In particular, $f'(x)$ is bounded.

(b) Let $\xi = \limsup_{x\to\infty} g(x)/R$, then $\limsup_{x\to\infty} f(x) = \xi$. In particular, $\xi > 0$ if $C_- > 0$.

(c) For any $\beta > 0$, $x_0 > 0$, and $\varepsilon > 0$, there is an $x \geq x_0$ such that $f(y) > \xi - \varepsilon$ for $y \in [x-\beta, x]$.

The main result of this section is as follows.

Theorem 5.2. *Suppose that $C_- > 0$. Then $\lim_{u\to\infty}\psi(u)e^{Ru} = \xi > 0$, where ξ is defined in Lemma 5.1.*

Proof. Choose $\beta > 0$, $\varepsilon > 0$. There exists $x_0 \geq \beta$ such that $f(x) > \xi - \varepsilon$ for $x \in [x_0 - \beta, x_0]$. If $x \geq 2x_0$ and define $T = \inf\{t > 0, R_t < x_0\}$, then

$$\begin{aligned} f(x) &= \mathrm{E}^*\left[e^{RR_{\tau^*}+\int_0^{\tau^*}\theta(R;a^*(R_s),b^*(R_s))\mathrm{d}s}\right] \\ &= \mathrm{E}^*\left[\mathrm{E}^*\left[e^{RR_{\tau^*}+\int_0^{\tau^*}\theta(R;a^*(R_s),b^*(R_s))\mathrm{d}s} \mid \mathcal{F}_T\right]\right] \\ &= \mathrm{E}^*\left[\mathrm{E}^*\left[e^{RR_{\tau^*}+\int_T^{\tau^*}\theta(R;a^*(R_s),b^*(R_s))\mathrm{d}s} \mid R_T\right]e^{\int_0^T\theta(R;a^*(R_s),b^*(R_s))\mathrm{d}s}\right] \\ &\geq \mathrm{E}^*\left[f(R(T))\right] \geq \mathrm{E}^*\left[f(R(T))1_{x_0-\beta\leq R_T\leq x_0}\right] \\ &> (\xi-\varepsilon)\mathrm{P}^*\left[x_0 - R_T \leq \beta\right]. \end{aligned} \tag{5.11}$$

By choosing β appropriately, we can get $\mathrm{P}^*[x_0 - R_T \leq \beta] > 1 - \varepsilon$. Then $f(x) > (\xi - \varepsilon)(1 - \varepsilon)$. Thus

$$\liminf_{x\to\infty} f(x) \geq \xi = \limsup_{x\to\infty} f(x). \tag{5.12}$$

By Lemma 5.1, this theorem can be proved. □

6. Convergence of the Strategies

After discussing the asymptotic behavior of $\psi(u)e^{Ru}$, in this section we will study the behavior of the optimal strategies (a^*, b^*) when the capital is large enough. If the optimal strategies converge, then using the convergent limit value we can obtain the asymptotic behavior of the optimal ruin probability. The following theorem indicates the convergence of (a^*, b^*).

Theorem 6.1. *Suppose that $C_- > 0$. Then $\lim_{x\to\infty} f'(x) = 0$. Moreover if $(\tilde{a}, \tilde{b})$ is unique, then $\lim_{x\to\infty} a^*(x) = \tilde{a}$, $\lim_{x\to\infty} b^*(x) = \tilde{b}$.*

Proof. First we replace $\psi(x)$ by $f(x)e^{-Rx}$ in the HJB equation to get

$$\begin{aligned} &[c_1(a^*(x)) + c_2(b^*(x))]\left[f'(x) - Rf(x)\right] \\ &\quad - \lambda\left[G_{a^*(x)U+b^*(x)V}(x)e^{R(x)} - \int_0^x f(x-y)e^{Ry}\mathrm{d}G_{a^*(x)U+b^*(x)V}(y) - e^{Rx} + f(x)\right] = 0. \end{aligned} \tag{6.1}$$

That is

$$\begin{aligned} &\lambda\left[\int_0^x f(x-y)e^{Ry}\mathrm{d}G_{a^*(x)U+b^*(x)V}(y) - f(x)\right] + \lambda e^{Rx}\left(1 - G_{a^*(x)U+b^*(x)V}(x)\right) \\ &\quad + [c_1(a^*(x)) + c_2(b^*(x))]f'(x) - [c_1(a^*(x)) + c_2(b^*(x))]Rf(x) = 0. \end{aligned} \tag{6.2}$$

Then

$$\begin{aligned}
&[c_1(a^*(x)) + c_2(b^*(x))]f'(x) \\
&\quad = -\lambda\left[\int_0^x f(x-y)e^{Ry}\mathrm{d}G_{a^*(x)U+b^*(x)V}(y) - f(x) - \xi\left(\mathrm{E}e^{R(a^*(x)U+b^*(x)V)} - 1\right)\right] \\
&\qquad - \lambda\xi\left(\mathrm{E}e^{R(a^*(x)U+b^*(x)V)} - 1\right) + [c_1(a^*(x)) + c_2(b^*(x))]Rf(x) \\
&\qquad - [c_1(a^*(x)) + c_2(b^*(x))]R\xi + [c_1(a^*(x))c_2(b^*(x))]R\xi - \lambda e^{Rx}\left(1 - G_{a^*(x)U+b^*(x)V}(x)\right) \\
&\quad = -\lambda\left[\int_0^{x/a^*(x)}\int_0^{(x-a^*(x)u)/b^*(x)} f(x-a^*(x)u-b^*(x)v)e^{-R(a^*(x)u+b^*(x)v)}\mathrm{d}G(u,v) - f(x)\right. \\
&\qquad\qquad \left. -\xi\left(\mathrm{E}e^{R(a^*(x)U+b^*(x)V)} - 1\right)\right] - \xi\theta(R;a^*(x),b^*(x)) \\
&\qquad + [c_1(a^*(x)) + c_2(b^*(x))]R\left(f(x)-\xi\right) - \lambda e^{Rx}\left(1 - G_{a^*(x)U+b^*(x)V}(x)\right) \\
&\quad < \lambda\left|\left[\int_0^{x/a^*(x)}\int_0^{(x-a^*(x)u)/b^*(x)} f(x-a^*(x)u-b^*(x)v)e^{-R(a^*(x)u+b^*(x)v)}\mathrm{d}G(u,v) - f(x)\right.\right. \\
&\qquad\qquad \left.\left. -\xi\left(\mathrm{E}e^{R(a^*(x)U+b^*(x)V)} - 1\right)\right]\right| \\
&\qquad + \left|[c_1(a^*(x)) + c_2(b^*(x))]R\left(f(x)-\xi\right)\right| - \xi\theta(R;a^*(x),b^*(x)).
\end{aligned} \tag{6.3}$$

Note that when $x \to \infty$

$$\begin{gathered}
\left|\left[\int_0^{x/a^*(x)}\int_0^{(x-a^*(x)u)/b^*(x)} f(x-a^*(x)u-b^*(x)v)e^{-R(a^*(x)u+b^*(x)v)}\mathrm{d}G(u,v) - f(x)\right.\right. \\
\left.\left. -\xi\left(\mathrm{E}e^{R(a^*(x)U+b^*(x)V)} - 1\right)\right]\right| < \frac{\varepsilon}{4}, \\
\left|[c_1(a^*(x)) + c_2(b^*(x))]R\left(f(x)-\xi\right)\right| < \frac{\varepsilon}{4}.
\end{gathered} \tag{6.4}$$

Then

$$[c_1(a^*(x)) + c_2(b^*(x))]f'(x) < -\xi\theta(R;a^*(x),b^*(x)) + \frac{\varepsilon}{2}. \tag{6.5}$$

If for each $\varepsilon > 0$, there exists x_0 such that $c_1(a^*(x_0)) + c_2(b^*(x_0)) < \varepsilon$. Because $\theta(R;a^*(x_0), b^*(x_0)) > 0$, and $f'(x)$ is bounded, under this case we cannot get (6.5). So $c_1(a^*(x)) + c_2(b^*(x))$ cannot be arbitrary small. That means $c_1(a^*(x)) + c_2(b^*(x))$ are bounded away from 0. Therefore $\limsup_{x\to\infty} f'(x) \le 0$. Clearly, we have $\liminf_{x\to\infty} f'(x) = R\xi - \limsup_{x\to\infty} g(x) = 0$.

Thus $\lim_{x\to\infty} f'(x) = 0$. If x is large enough, we get $\lambda e^{Rx}(1 - \int_0^{x/a^*(x)} \int_0^{(x-a^*(x)u)/b^*(x)} \mathrm{d}G(u,v)) < \varepsilon/4$ and $[c_1(a^*(x)) + c_2(b^*(x))]|f'(x)| < \varepsilon/4$. Thus we have

$$-\frac{\varepsilon}{4} < [c_1(a^*(x)) + c_2(b^*(x))]f'(x) < -\xi\theta(R; a^*(x), b^*(x)) + \frac{3\varepsilon}{4}, \tag{6.6}$$

which is equal to

$$0 \le \xi\theta(R; a^*(x), b^*(x)) < \varepsilon. \tag{6.7}$$

This proves that $\lim_{x\to\infty}\theta(R; a^*(x), b^*(x)) = 0$. If $(\tilde{a}, \tilde{b})$ is unique, this is only possible if $\lim_{x\to\infty} a^*(x) = \tilde{a}$, $\lim_{x\to\infty} b^*(x) = \tilde{b}$. □

7. Example

To the multidimensional risk model, it seems impossible to get a closed form solution for the optimal ruin probability $\psi(u)$. In this section, from a numerical example, we will give an explicit procedure to obtain an exponential upper bound of $\psi(u)$ and the asymptotic optimal reinsurance strategies.

Example 7.1. Suppose that U_n and V_n are independent. The distribution function of them are given by $F_U = 1 - e^{-2x}$ and $F_V = 1 - e^{-x}$, respectively. So the joint distribution function of (U_n, V_n) is $G(x,y) = (1 - e^{-2x})(1 - e^{-y})$, and the joint density function is $p(x,y) = 2e^{-2x-y}$. Then $\mu_1 = \mathrm{E}U_n = 1/2$ and $\mu_2 = \mathrm{E}V_n = 1$. Let $\lambda = 1$. The expected value principle is used for our premium. Suppose that the relative safety loading for each subportfolios from the insurer point of view $\theta_1 = \theta_2 = 0.5$, and from the reinsurer $\eta_1 = \eta_2 = 0.7$. So $c_1(a) = (1.7a - 0.2)/2$, $c_2(b) = 1.7b - 0.2$.

Theorem 4.2 shows us that e^{-Ru} is an exponential type upper bound for $\psi(u)$. R can be get from $R = \sup_{(a,b)\in U} R(a,b)$, where $R(a,b)$ satisfied (4.1), that is, $\lambda(\mathrm{E}e^{R(a,b)(aU+bV)} - 1) = (c_1(a) + c_2(b))R(a,b)$. We can easily get when $\tilde{a} = 0.77$ and $\tilde{b} = 0.38$, $R(a,b)$ solved from previous equation reaches the maximum $R = 0.4194$. So $\psi(u) \le e^{-0.4194u}$. Moreover, $\tilde{a}$, $\tilde{b}$ work well as the optimal reinsurance constant strategies for "large" capital according to Theorem 6.1. So the asymptotic optimal constant strategies are $(\tilde{a}, \tilde{b}) = (0.77, 0.38)$.

Remark 7.2. When considering the two-dimensional risk model without dynamic control, the problem of the sum of two subportfolio indeed can be convert back to the one-dimensional case (e.g., Yuen et al. [18]). If we consider the dynamic proportional reinsurance in the two-dimensional compound Poisson risk model again from the this point, then the aggregate process $\tilde{R}_t$ is as follows:

$$\tilde{R}_t = X_t + Y_t = u + (c_1 + c_2)t - \sum_{n=1}^{N_t}(U_n + V_n). \tag{7.1}$$

We consider the dynamic proportional reinsurance strategy $\{\alpha_t\}$ on the one-dimensional risk model $\widetilde{R}_t$:

$$\widetilde{R}_t^{\alpha} = u + \int_0^t (c_1(\alpha_s) + c_2(\alpha_s))\mathrm{d}s - \sum_{n=1}^{N_t} \alpha_{\sigma_n -}(U_n + V_n). \tag{7.2}$$

The optimal reinsurance in one-dimensional case had been discussed in Schmidli [5]. So from the equation $\lambda(\mathrm{E}e^{R_1(\alpha)\alpha(U+V)} - 1) = (c_1(\alpha) + c_2(\alpha))R_1(\alpha)$ for some fixed α, we can calculate the maximal adjustment coefficient $R_1 = 0.40412$.

Obviously, $R_1 < R$. From the point of comparing the upper bound of $\psi(u)$, this tells us that from the two-dimensional point of view to consider the dynamic proportional reinsurance strategies for each subportfolio is better than considering the strategy just for the aggregate portfolio.

Remark 7.3. Another approach based on one-dimensional risk model to deal with our problem is that we may view each subportfolio as a one-dimensional case and discuss them, respectively. So we can handle the example as follows. First, we consider the subportfolio $\{X_t^a\}$. Similar to Schmidli [5], from $\lambda(\mathrm{E}e^{R(a)aU} - 1) = c_1(a)R(a)$ we derive the asymptotic optimal constant reinsurance strategy $\widetilde{a} = 0.504847$ for $\{X_t^a\}$. Meanwhile using the same way to $\{Y_t^b\}$, we can get the asymptotic optimal reinsurance strategy $\widetilde{b} = 0.504847$ for $\{Y_t^b\}$. Till now, we have get a constant strategy $(\widetilde{a}, \widetilde{b}) = (0.504847, 0.504847)$. Next we think over the sum of two subportfolio, that is, $R_t^{\widetilde{a},\widetilde{b}} = X_t^{\widetilde{a}} + Y_t^{\widetilde{b}}$. From $\lambda(\mathrm{E}e^{R_2(\widetilde{a},\widetilde{b})(\widetilde{a}U+\widetilde{b}V)} - 1) = (c_1(\widetilde{a}) + c_2(\widetilde{b}))R_2(\widetilde{a}, \widetilde{b})$, the adjustment coefficient $R_2 = 0.40408$ for $\{R_t^{\widetilde{a},\widetilde{b}}\}$ can be obtained.

We find that $R_2 < R$. This implies that even though $\widetilde{a}$ and $\widetilde{b}$ are the asymptotic optimal constant strategy for $\{X_t^a\}$ and $\{Y_t^b\}$, respectively, under $(\widetilde{a}, \widetilde{b})$ the upper bound ruin probability of $\{R_t^{\widetilde{a},\widetilde{b}}\}$ is not optimized at the same time.

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Research Article

How to Implement Return Policies in a Two-Echelon Supply Chain?

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We integrate a retailer's return policy and a supplier's buyback policy within a modeling framework. In this setting, consumers decide whether to buy and then whether to return the product, the retailer sets the retail price, quantity, and refund price, and the supplier chooses the wholesale price and buyback price. Both the demand uncertainty and consumers' valuation uncertainty are considered; consumers realize their valuations only after purchase. We discuss four scenarios for each party in the supply chain that may offer or not offer return policy. We characterize each party's optimal decisions for all scenarios and we show that the supplier's best choice is to provide buyback policy and the retailer's optimal response is to set refund price to be the same as supplier's buyback price.

1. Introduction

There has been a growing trend towards the consumer returns in recent years. A large portion of customer returns may be nondefective because the consumer does not like the product as much as anticipated, or the returned product does not fit the customer's need or expectation. Sciarrotta [1] reports that the nondefective returns rate was very high. Lawton [2] points out that only about 5% of customer returns were truly defective. Accepting customer returns is a common practice in the retail industry, where it has the objective of keeping customers' loyalty and maintaining customer satisfaction. In practice, most retailers implement various customer returns policies, such as full refund, partial refund, or store credit. "Customer returns policy" refers here to the money back policy offered by a retailer to her customers. It reduces the customer's risk of having to keep an unwanted product.

Suppliers whose products are subject to uncertain demand face a problem of inducing retailers to stock those products. A supplier may attempt to compensate his retailers by accepting returns of unsold goods for full or partial refunds of their purchase price. The practice of returns policy has been reported widely in both research literature and business, see Bose and Anand [3]. The main objective of the return policy is to mitigate the risk of overstocking, caused by uncertain demand that retailers face. Buyback policies are commonly used in many industries, especially for products with short life cycles such as books, CDs, holiday gifts, and computers.

Although the approaches to dealing with customer returns and the supplier's buyback policies have been well studied, a very few research integrates customer returns policy with a buyback policy. In reality, supply chain is a network of suppliers, retailers, and customers. Some electronic firms (e.g., HP, Lenovo, Nokia) procure modules from their suppliers and sell various fashion electronic products to their customers. To keep the customer's loyalty, the firms offer return policies to their customers; on the other hand, the upstream supplier might also offer return policies to encourage the firms to order more modules to improve the service level for their customers. So consideration of return policies should not be limited within only two levels.

We consider a two-echelon supply chain with a single supplier who supplies a product to a retailer, and the retailer sells the product to the uncertain market. They both offer a return policy to their adjacent downstream firm. The consumers return unsatisfactory product back to the retailer, then the retailer will return unsold items, together with the consumer returns, back to the supplier. In order to differentiate supplier-retailer refund policy from retailer-customer return policy, we use the term "return policy" to refer to the retailer-customer agreement, and "buyback policy" to refer specifically to arrangements between the supplier and his retailers. Most research on buyback agreements only considers unsold inventory resulting from demand uncertainty being returned to the supplier. We also consider a buyback agreement that includes customer returns.

In this paper, we develop an integrated model that investigates the supplier's buyback policy and the retailer's return policy. Based on firms' operational policies, we study both the demand uncertainty and the customers' valuation uncertainty. Specially, we model the above decisions as a Stackelberg game.

Stage 1. The supplier, acting as the leader, offers the retailer a take-it-or-leave-it contract, specifying the wholesale price and buyback price for the returned items.

Stage 2. Given the supplier's decisions, the retailer chooses the order quantity, retail price, and refund price for the customers' returns under both demand uncertainty and valuation uncertainty.

Stage 3. Market demand uncertainty is realized. Based on the retailer's decisions, consumers make two sequential decisions. Initially, facing uncertainty in their own valuations, they decide whether to purchase the product or not. If they buy the product, after realizing their own valuation, they go on to decide whether to keep or return the product.

Stage 4. Unsatisfied consumers return products back to retailer for a refund, the supplier buys back all leftovers from the retailer. These leftover units include units that were unsold and units that were sold but returned.

The purpose of this paper is to shed light on the supplier's and retailer's optimal policy in the presence of customer valuation uncertainty and demand uncertainty. Considering supplier and retailer may offer or not offer return policy, we examine four combinations.

For each scenario, we derive each party's optimal decisions. Finally, we figure out that it is optimal for the supplier to present buyback policy and the retailer's ideal response refund price should be as same as supplier's buyback price.

In summary, this paper contributes to the literature by: (1) constructing a game model of two-echelon return policies that involves both demand uncertainty and valuation uncertainty; (2) analyzing the equilibrium solution of the model; (3) providing new insights for the return policies in a two echelon supply chain.

The remainder of this paper is organized as follows. Section 2 provides literature review. Section 3 introduces the basic model. In Section 4, we discuss both parties' optimal solutions for all scenarios. We conduct performance comparison under uniform distribution in Section 5. Finally, we provide concluding remarks and offer some directions for future work in Section 6.

2. Literature Review

The model setting we consider in this paper is a combination of three distinctive features: (1) a buyback policy offered by the upstream supplier to the downstream retailer; (2) a return policy offered by the retailer to the end consumer; (3) consumers are uncertain over the valuation. In the following, we provide a brief review of papers that relate to these model features.

A buyback contract is an arrangement where an upstream supplier agrees to provide a retailer credit for unsold product (see [4, 5]). Pasternack [6] first considers buyback policy for a seasonal product with stochastic demand under the newsvendor framework, with underage and overage costs as a vehicle to share the risks resulting from demand uncertainty. Most research on buyback agreements only considers unsold inventory resulting from demand uncertainty being returned to the supplier [6–8]. We consider a buyback agreement that also includes customer returns to retailers as that in Chen and Bell [9].

There is a stream of literatures on consumer returns. Some papers investigate how to prevent inappropriate returns from consumers with no intention of keeping their purchase (Hess et al. [10–12]). Most of the literatures focus on full refunds, including Marvel and Peck [13], Xiao et al. [14], Chen and Bell [9, 15, 16]. Su [17] discusses partial refund and shows that it is optimal for a retailer to offer the partial refund policy. We do not restrict our analysis on full refund policy.

Fit risk is an important component of purchase uncertainty. For example, when purchasing clothes, buyers are not completely sure whether the new clothes fit into their daily life and the rest of their wardrobe; children may not like the musical instrument that their parents bought for them [18]. Dana and Spier [19], Xie and Shugan [20] consider firms may wish to offer advance purchase discounts to compensate them for bearing risk when consumers face valuation uncertainty. Alexandrov and Lariviere [21] study the value of offering reservations to consumers who face valuation uncertainty. Su [17], Xiao et al. [14] study the role of consumer returns policy as a risk-sharing mechanism when valuations of consumers are uncertain. Along the line, we characterize the consumers' purchase and return behavior depend on the retailer's decision variables (such as retail price and refund).

Only a few researchers integrate the buyback policy and consumer return policy. The supplier's buyback policy is exogenously given in Su [17], he studies possible approaches to achieve the channel coordination. In Xiao et al. [14]'s model, the retailer only makes quantity decision, the supplier finds an appropriate buyback policy to coordinate the whole supply chain. In Chen and Bell [16], the retailer implements a full refund policy to consumers and retail price is determined exogenously. Chen and Bell [9] investigate full refund policy and customer returns are a fixed proportion of quantity sold. Different to these papers, both the buyback policy and refund policy are decisions in our model.

Overall, our model incorporates consumers' valuation uncertainty as well as demand uncertainty. We also consider consumer returns that depend on the retailer's decision variables. This work integrates the supplier's buyback contract, retailer's pricing, refund policy, and consumer's purchase and return behavior within a unified framework.

3. The Model

Consider a supply chain consisting of one supplier (he), one retailer (she), and end consumers. The upstream supplier initiates the process by offering a wholesale price w, for which he will sell to the retailer prior to the selling season, and a refund price r^S, for which he will buy items back from the retailer at the end of selling season. The supplier's production cost is c per unit. In response to the offered wholesale and buyback prices, the retailer determines a quantity Q for the supplier to deliver, a retail price p during the season, and a refund price r^R to the consumer at the end of the season. $r^R = 0$ means that no return policy is provided and $r^R = p$ represents a full refund policy, that is, 100% money-back-guarantee is offered to ensure consumer satisfaction [22, 23]. We refer to return policy with $0 < r^R < p$ as partial refund policy. Unsold goods and returned items have no scrap value to either the retailer or the supplier. To make our results more transparent, we let the other modeling elements be as simple as possible.

Similar to Che [24], we assume that each consumer purchases at most a unit product and consumers do not fully know their preferences for the products until they obtain some experiences with the products. Consumers make two sequential decisions. First, they decide whether to purchase the product or not. If they buy the product, they then decide whether to keep it or not after privately observing their own expected valuation.

As in Su [17], we assume the customer valuations V are identically and independently drawn from the distribution with CDF $G(\cdot)$, PDF $g(\cdot)$. The consumers seek to maximize individual expected surplus. If a consumer chooses to buy, the consumer will keep the product if his valuation is at least as high as the refund ($V \geq r^R$), otherwise, he will return this product for refund r^R. So in the first stage, should the consumer decides to buy, his expected utility will be $E \max(V, r^R)$, this is the customer's reservation price. In other words, when retail price p exceeds $E \max(V, r^R)$, consumer leaves the market without making a purchase; however, when p is less than or equal to $E \max(V, r^R)$, every consumer will buy the product. The market demand X is a random factor with CDF $F(\cdot)$ and PDF $f(\cdot)$. Let $h(x) = f(x)/[1 - F(x)]$ denote the failure rate function of the demand function. From the retailer's perspective, demand is

$$D = \begin{cases} X, & \text{if } E \max\left(V, r^R\right) \geq p, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Then the retailer makes three decisions: retail price p, order quantity Q, and refund price r^R.

Her profit function is given as:

$$\begin{aligned}
\prod^{R}\left(p,Q,r^{R}\right) &= pE\min(D,Q) - r^{R}G\left(r^{R}\right)E\min(D,Q) \\
&\quad + r^{S}\left[Q - E\min(D,Q) + G\left(r^{R}\right)E\min(D,Q)\right] - wQ \\
&= \underbrace{p\overline{G}\left(r^{R}\right)E\min(D,Q)}_{\text{net sale}} + \underbrace{\left(p - r^{R} + r^{S}\right)G\left(r^{R}\right)E\min(D,Q)}_{\text{returned by customer}} \\
&\quad + \underbrace{r^{S}(Q - E\min(D,Q))}_{\text{not sold}} - wQ.
\end{aligned} \tag{3.2}$$

The first term is the retailer's revenue from net sale (total sales minus total returns), each returned unit yields $p-r^R$ from the consumer and r^S from the supplier, and each unsold item yields only the buyback value r^S. The last term is the retailer's order cost.

Similarly, we can give the supplier's profit function as:

$$\prod^{S}\left(w,r^{S}\right) = \underbrace{w\overline{G}\left(r^{R}\right)E\min(D,Q)}_{\text{net sale}} + \underbrace{\left(w - r^{S}\right)\left[Q - \overline{G}\left(r^{R}\right)E\min(D,Q)\right]}_{\text{returned by retailer}} - cQ. \tag{3.3}$$

In (3.3), we see that the supplier gains w from each product that is sold and kept by the consumer, each returned unit yields $w - r^S$ from the retailer (unsold items and returned items).

4. Model Analysis

Considering each party in the supply chain system may offer or not offer return policy, we discuss four combinations in this section. There are neither the supplier nor the retailer offers a return policy (i.e., $r^S = 0,\ r^R = 0$); the retailer offers a return policy but the supplier does not (i.e., $r^S = 0,\ r^R \geq 0$); the retailer offers no return policy but the supplier does (i.e., $r^S \geq 0,\ r^R = 0$) and both the supplier and the retailer offer a return policy (i.e., $r^S \geq 0,\ r^R \geq 0$).

We introduce a dummy variable that takes the values 0 or 1 to indicate the absence or presence of return policy. We will use subscripts and superscripts to facilitate the expression of the model variables. The superscripts R and S will denote the retailer and the supplier. The first subscript refers to the supplier's decision and the second subscript characterizes the retailer's decision. For example, $\Pi^S_{0,1}$ is the supplier's profit of the setting, in which the supplier does not offer while the retailer provides a return policy, notations introduced later indicate the similar meanings.

4.1. Scenario 1: Supplier and Retailer Do Not Offer Return Policy

In this scenario, the returns are not accepted in both the supplier and retailer, which means $r^S = 0$ and $r^R = 0$. Now the retailer decides her retail price and order quantity which maximizes the expected profit:

$$\prod_{0,0}^{R}(p,Q) = pE\ \min(D,Q) - wQ. \tag{4.1}$$

Observe that when consumer returns are not accepted, the highest price consumers are willing to pay is EV, their expected valuation for the product. In this case, the retailer faces a standard newsvendor problem with the selling price $p^*_{0,0} = EV$ and stocking quantity satisfy $\overline{F}(Q^*_{0,0}) = w/EV$.

Anticipating this, the upstream supplier chooses his optimal wholesale price to maximize

$$\prod_{0,0}^{S}(w) = (w-c)Q^*_{0,0}. \tag{4.2}$$

The following proposition characterizes the supplier's optimal decision.

Proposition 4.1. *If $dh(x)/dx > 0$, the optimal $w^*_{0,0}$ is determined uniquely by:*

$$wQ^*_{0,0}h\left(Q^*_{0,0}\right) - w + c = 0. \tag{4.3}$$

All the proofs are provided in the appendix.

4.2. Scenario 2: Supplier Does Not Offer Return Policy, Retailer Offer Return Policy

In this case, the returns are only accepted in the retailer, which means $r^S = 0$ and $r^R \geq 0$. Now the retailer decides her retail price, refund price and order quantity which maximizes the expected profit:

$$\prod_{0,1}^{R}\left(p,Q,r^R\right) = \left[p\overline{G}\left(r^R\right) + \left(p-r^R\right)G\left(r^R\right)\right]E\ \min(D,Q) - wQ. \tag{4.4}$$

Proposition 4.2. *The retailer's optimal price $p^*_{0,1}$, quantity $Q^*_{0,1}$, and refund $r^{R*}_{0,1}$ are given by*

$$p^*_{0,1} = EV, \tag{4.5}$$

$$\overline{F}\left(Q^*_{0,1}\right) = \frac{w}{p^*_{0,1}} = \frac{w}{EV}, \tag{4.6}$$

$$r^{R*}_{0,1} = 0. \tag{4.7}$$

If the supplier does not offer buyback policy, he just find optimal wholesale price to maximize his profit:

$$\prod_{0,1}^{S}(w) = (w - c)Q_{0,1}^{*}. \tag{4.8}$$

Proposition 4.3. *If $dh(x)/dx > 0$, the optimal $w_{0,1}^{*}$ is determined uniquely by:*

$$wQ_{0,1}^{*}h\left(Q_{0,1}^{*}\right) - w + c = 0. \tag{4.9}$$

Remember that the retailer's optimal refund price in this scenario is zero, so when the upstream supplier does not buyback returns, the retailer's optimal response is to choose not to provide return policy to customer either. Notice the other decisions are all the same as the correspondence in scenario 1, so scenario 2 is degenerate into the simply scenario 1.

From the analysis of scenario 1 and scenario 2, we find that if the upstream supplier do not provide buyback policy to the retailer, then the retailer's optimal response is do not allow return behavior of the downstream customer either.

4.3. Scenario 3: Supplier Offers Return Policy, Retailer Does Not Offer Return Policy

In this case, the returns are only allowed in the supplier, which means $r^S \geq 0$ and $r^R = 0$. Now the retailer decides retail price and order quantity to maximize her expected profit:

$$\begin{aligned}\prod_{1,0}^{R}(p, Q) &= pE\ \min(D, Q) + r^S[Q - E\ \min(D, Q)] - wQ \\ &= \left(p - r^S\right)E\ \min(D, Q) - \left(w - r^S\right)Q.\end{aligned} \tag{4.10}$$

If the retailer does not provide return policy to the consumers, the highest price consumers are willing to pay is EV. The retailer then faces a standard newsvendor problem with the selling price $p_{1,0}^{*}$ and stocking quantity $Q_{1,0}^{*}$ satisfying:

$$p_{1,0}^{*} = EV, \tag{4.11}$$

$$\overline{F}\left(Q_{1,0}^{*}\right) = \frac{w - r^S}{p_{1,0}^{*} - r^S} = \frac{w - r^S}{EV - r^S}. \tag{4.12}$$

Observing the retailer's optimal response will be (4.11) and (4.12), the supplier determines (w, r^S) to maximize his profit:

$$\begin{aligned}\prod_{1,0}^{S}\left(w, r^S\right) &= (w-c)Q_{1,0}^{*} - r^S\left[Q_{1,0}^{*} - E\ \min\left(D, Q_{1,0}^{*}\right)\right] \\ &= r^S E\ \min\left(D, Q_{1,0}^{*}\right) + \left(w - c - r^S\right)Q_{1,0}^{*}.\end{aligned} \tag{4.13}$$

We assume $r^S \le w - c$, so the supplier can gain profit from producing products. The term $Q_{1,0}^{*}$ is shown in (4.12), which is a function of w and r^S. We reduce (4.13) to an optimization problem over the single variable r^S by first solving for the optimal value of w as a function of r^S, and then substituting the result back into $\Pi_{1,0}^{S}(w, r^S)$ to search optimal r^S.

Proposition 4.4. *For any fixed r^S, if $dh(x)/dx > 0$, the optimal $w_{1,0}^{*}$ is determined uniquely as a function of r^S :*

$$\left(w_{1,0}^{*} - r^S\right)Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right) - \left[r^S\overline{F}\left(Q_{1,0}^{*}\right) + w_{1,0}^{*} - r^S - c\right] = 0, \tag{4.14}$$

where $Q_{1,0}^{*}$ is given by (4.12) *replacing w with $w_{1,0}^{*}$.*

Combine (4.12) and (4.14), we can present $Q_{1,0}^{*}$ as a function of only r^S :

$$\left[r^S Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right) + EV\left(1 - Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right)\right)\right]\overline{F}\left(Q_{1,0}^{*}\right) = c. \tag{4.15}$$

We can clearly describe how $Q_{1,0}^{*}$ is affected by r^S in the following proposition.

Proposition 4.5. *$Q_{1,0}^{*}$ is increasing in r^S.*

Now we plug $w_{1,0}^{*}$ into (4.13) to find optimal r^S, the following proposition characterizes our results.

Proposition 4.6. *$\Pi_{1,0}^{S}(w_{1,0}^{*}, r^S)$ is increasing in r^S and reaches its maximum at $r_{1,0}^{S*} = w_{1,0}^{*} - c$.*

4.4. Scenario 4: Supplier and Retailer Offer Return Policy

In this case, a supplier sells the product at unit price w and will buyback each returned items at r^S from the retailer. The retailer then chooses her response order quantity Q, retail price p and refund price r^R to the customer. We derive the overall channel solutions by backward-induction procedure. We begin our analysis by focusing on the retailer's decisions.

For given wholesale price w and buyback price r^S offered by the supplier, and knowing the demand is characterized by (3.1), the retailer's problem is to choose the optimal order quantity Q, the price p sell to the market, and refund price r^R to the unsatisfied customers, to maximize her own profit which is shown in (3.2).

The optimization problem has a structure similar to that of the centralized problem in Proposition 4.2. The following proposition characterizes the decentralized decisions $(p^*_{1,1}, Q^*_{1,1}, r^{R*}_{1,1})$.

Proposition 4.7. *The retailer's optimal price $p^*_{1,1}$, quantity $Q^*_{1,1}$, and refund $r^{R*}_{1,1}$ are given by*

$$p^*_{1,1} = E\ \max\left(V, r^S\right), \tag{4.16}$$

$$\overline{F}\left(Q^*_{1,1}\right) = \frac{w - r^S}{p^*_{1,1} - r^S} = \frac{w - r^S}{E\ \max(V, r^S) - r^S}, \tag{4.17}$$

$$r^{R*}_{1,1} = r^S. \tag{4.18}$$

Knowing that the retailer chooses $(p^*_{1,1}, Q^*_{1,1}, r^{R*}_{1,1})$ according to (4.16) to (4.18) in response to given w and r^S. The supplier sets on (w, r^S) to maximize his own profit, which is given by (3.3).

Plugging $(p^*_{1,1}, Q^*_{1,1}, r^{R*}_{1,1})$ into the supplier's profit function, then (3.3) can be rearranged as:

$$\prod_{1,1}^{S}\left(w, r^S\right) = r^S\overline{G}\left(r^S\right)E\ \min\left(X, Q^*_{1,1}\right) + \left(w - c - r^S\right)Q^*_{1,1}. \tag{4.19}$$

The term $Q^*_{1,1}$ is shown in (4.17), which is a function of w and r^S. We reduce (4.19) to an optimization problem over the single variable r^S by first solving for the optimal value of w as a function of r^S, and then substituting the result back into $\Pi^S_{1,1}(w, r^S)$. Go along the same line as in scenario 3, we have.

Proposition 4.8. *For any fixed r^S, if $dh(x)/dx > 0$, the optimal $w^*_{1,1}$ is determined uniquely as a function of r^S:*

$$\left(w^*_{1,1} - r^S\right)Q^*_{1,1}h\left(Q^*_{1,1}\right) - \left[r^S\overline{G}\left(r^S\right)\overline{F}\left(Q^*_{1,1}\right) + w^*_{1,1} - r^S - c\right] = 0, \tag{4.20}$$

where $Q^*_{1,1}$ is given by (4.17) *replacing w with $w^*_{1,1}$.*

After substituting $w^*_{1,1}$ in (4.19), we find that the solution depends on F and G distribution, and it is difficult to track analytical solution of r^S even if the demand distribution function follows a uniform distribution. If the inverse distribution functions (such as normal distribution) are not available, the analytical solution is unsolvable.

5. Performance Comparisons under Uniform Distribution

In Section 4, we completely characterize the optimal decisions of both the supplier and retailer for all scenarios except scenario 4. Actually, the equilibrium solutions are not in closed form, so we turn to a specific distribution to gain insights into the performance comparisons. In some papers on supply chain management with uncertainty demand, researchers ([25, 26]) use a uniform distribution in theoretical problems because of the complexity of the supply chain management. This assumption is in real existence for example in the fashion clothing and apparels market, new products market, and so forth [27]. In our paper, we suppose that the random market demand X follows a uniform distribution on $[0, b]$. Then, we have $f(x) = 1/b$, $\overline{F}(x) = (b - x)/b$, and $h(x) = 1/(b - x)$. The uniform distribution is an IFR distribution that ensures the existence of the unique optimal wholesale price.

Recall the analysis of scenario 1 in Section 4, from (4.5) to (4.7), we have

$$\begin{cases} p_{0,0}^{*} = EV, \\ \overline{F}\left(Q_{0,0}^{*}\right) = \dfrac{w}{EV}, \\ wQ_{0,0}^{*}h\left(Q_{0,0}^{*}\right) - w + c = 0, \end{cases} \Longrightarrow \begin{cases} p_{0,0}^{*} = EV, \\ Q_{0,0}^{*} = \dfrac{b(EV - c)}{2EV}, \\ w_{0,0}^{*} = \dfrac{EV + c}{2}. \end{cases} \tag{5.1}$$

Substituting all the optimal decisions into the retailer's and supplier's profit functions:

$$\begin{aligned} \prod_{0,0}^{R*} &= p_{0,0}^{*}E\min\left(D, Q_{0,0}^{*}\right) - w_{0,0}^{*}Q_{0,0}^{*} \\ &= w_{0,0}^{*}\left[\frac{E\min\left(D, Q_{0,0}^{*}\right)}{\overline{F}\left(Q_{0,0}^{*}\right)} - Q_{0,0}^{*}\right] \\ &= w_{0,0}^{*}\frac{\left(Q_{0,0}^{*}\right)^{2}}{2\left(b - Q_{0,0}^{*}\right)} \\ &= \frac{b}{8EV}(EV - c)^{2}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} \prod_{0,0}^{S*} &= \left(w_{0,0}^{*} - c\right)Q_{0,0}^{*} \\ &= \frac{EV - c}{2} \cdot \frac{b(EV - c)}{2EV}. \end{aligned} \tag{5.3}$$

From (4.11), (4.12), (4.14), and Proposition 4.6 in scenario 3, we have

$$\begin{cases} p_{1,0}^{*} = EV, \\ \overline{F}\left(Q_{1,0}^{*}\right) = \dfrac{w - r^{S}}{EV - r^{S}}, \\ r_{1,0}^{S*} = w_{1,0}^{*} - c, \\ \left(w_{1,0}^{*} - r^{S}\right)Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right) - \left[r^{S}\overline{F}\left(Q_{1,0}^{*}\right) + w_{1,0}^{*} - r^{S} - c\right] = 0, \end{cases}$$
$$\Longrightarrow \begin{cases} p_{1,0}^{*} = EV, \\ Q_{1,0}^{*} = b\left(1 - \sqrt{\dfrac{c}{EV}}\right), \\ r_{1,0}^{S*} = EV - \sqrt{c \cdot EV}, \\ w_{1,0}^{*} = EV + c - \sqrt{c \cdot EV}. \end{cases} \tag{5.4}$$

Substituting all the optimal decisions into the retailer's and supplier's profit functions:

$$\begin{aligned} \prod_{1,0}^{R*} &= \left(p_{1,0}^{*} - r_{1,0}^{S*}\right)E\min\left(D, Q_{1,0}^{*}\right) - \left(w_{1,0}^{*} - r_{1,0}^{S*}\right)Q_{1,0}^{*} \\ &= \frac{c}{\overline{F}\left(Q_{1,0}^{*}\right)}E\min\left(D, Q_{1,0}^{*}\right) - cQ_{1,0}^{*} \\ &= c\left[\frac{E\min\left(D, Q_{1,0}^{*}\right)}{\overline{F}\left(Q_{1,0}^{*}\right)} - Q_{1,0}^{*}\right] \\ &= c\frac{\left(Q_{1,0}^{*}\right)^{2}}{2\left(b - Q_{1,0}^{*}\right)} \\ &= \frac{b}{2}\sqrt{c \cdot EV}\left(1 - \sqrt{\frac{c}{EV}}\right)^{2}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \prod_{1,0}^{S*} &= r_{1,0}^{S*}E\min\left(D, Q_{1,0}^{*}\right) + \left(w_{1,0}^{*} - c - r_{1,0}^{S*}\right)Q_{1,0}^{*} \\ &= r_{1,0}^{S*}E\min\left(D, Q_{1,0}^{*}\right) \\ &= \left(EV - \sqrt{c \cdot EV}\right) \cdot \frac{b(EV - c)}{2EV}. \end{aligned} \tag{5.6}$$

Make comparison between (5.2) and (5.5), (5.3) and (5.6), we can draw the following conclusion.

Proposition 5.1. $\Pi_{0,0}^{R*} \geq \Pi_{1,0}^{R*}$ *and* $\Pi_{0,0}^{S*} \leq \Pi_{1,0}^{S*}$.

From Proposition 5.1, we can get the conclusion that if the retailer does not intend to provide return policy to the consumer, the supplier's best choice is to offer the most generous buyback policy to the retailer. Actually, the supplier can gain more while the retailer earns less compared to that of scenario 1.

From Propositions 4.2 and 4.3 in scenario 2, we can get

$$\begin{cases} p_{0,1}^{*} = EV, \\ \overline{F}\left(Q_{0,1}^{*}\right) = \dfrac{w}{EV}, \\ r_{0,1}^{R*} = 0, \\ wQ_{0,1}^{*}h\left(Q_{0,1}^{*}\right) - w + c = 0, \end{cases} \Longrightarrow \begin{cases} p_{0,1}^{*} = EV, \\ Q_{0,1}^{*} = \dfrac{b(EV - c)}{2EV}, \\ r_{0,1}^{R*} = 0, \\ w_{0,1}^{*} = \dfrac{EV + c}{2}. \end{cases} \tag{5.7}$$

Substituting all the optimal decisions into the retailer's profit function (4.4) and supplier's profit function (4.8), we have:

$$\begin{gathered} \prod_{0,1}^{R*} = \left[p_{0,1}^{*}\overline{G}\left(r_{0,1}^{R*}\right) + \left(p - r_{0,1}^{R*}\right)G\left(r_{0,1}^{R*}\right)\right]E\,\min\left(D, Q_{0,1}^{*}\right) - w_{0,1}^{*}Q_{0,1}^{*} = \frac{b}{8EV}(EV - c)^{2}, \\ \prod_{0,1}^{S*} = \left(w_{0,1}^{*} - c\right)Q_{0,1}^{*} = \frac{EV - c}{2} \cdot \frac{b(EV - c)}{2EV}. \end{gathered} \tag{5.8}$$

From Proposition 4.7 in scenario 4, we find that if the supplier presents buyback policy to the retailer, it is better for the retailer to present return policy too. Specially, the retailer's optimal return price is the same as the supplier's buyback price. The derivation of supplier's buyback price r^S depends on the demand uncertainty distribution F and valuation uncertainty distribution G. Comparing the results in scenario 2 and scenario 4, we can derive the following proposition.

Proposition 5.2. $\Pi_{0,1}^{S*} \leq \Pi_{1,1}^{S*}$.

However, we cannot analytically make a comparison between $\Pi_{1,1}^{R*}$ and $\Pi_{1,0}^{R*}$. When we do numerical simulations, we find the result relies critically on the parameters and distribution G.

From the above analysis, we can draw the following conclusions: whether the retailer offers return policy to consumers or not, the upstream supplier's optimal policy is always offer buyback policy to the retailer. This is because the supplier has an incentive to enlarge the market by offering buyback policy. In addition, if the retailer does not present return policy, the supplier can profit from the most generous buyback price; if the retailer presents return policy, the supplier can gain profit from the buyback policy, the specific buyback price depends on the system parameters. For the retailer, supplier's buyback policy may hurt her profit. As she is the follower in the supply chain, her optimal response to supplier's buyback price is to implement the same return price.

We provide a guideline for the supplier offering a buyback policy for unsold inventory and customer returns as how to contract a buyback price with the retailer and also guide the retailer how to decide the return policy for the end consumers.

6. Conclusion

In this paper, we examine return policies in a two-echelon supply chain that comprises an upstream supplier, a downstream retailer, and end consumers. In this environment, the upstream supplier decides his wholesale price and buyback price for returned items; the downstream retailer then chooses her order quantity, retail price, and refund price for customers' returns. The end consumers face uncertainty in their valuation for products. With returns polices, the consumer can then decide whether to keep or return the product. Using this model, we put forth the following results.

(1) If the upstream supplier does not provide buyback policy, the retailer's optimal response is not to provide return policy either; otherwise, the retailer's optimal refund price would be the same as that of the buyback price.

(2) The supplier will adopt buyback policy, as he can always gain more profit than absence of buyback price. Retailer would expect the supplier not to adopt buyback policy, because she may gain a lower profit by facing the buyback policy.

(3) As supplier is the leader of this Stackelberg game, so in the two-echelon system, the supplier will present buyback policy, then the retailer will offer the same amount return policy. The analysis of the specific amount relies on all environmental parameters.

We believe this research will provide new insights for the return policies in a two echelon supply chain. This research can be enriched in several directions. In practice, many retailers permit consumer returns up to a certain time limit, to understand how the duration of returns policy take effect may be necessary. It would be interesting to investigate a related but different context, in which consumers return the product directly to the supplier rather than to the retailer. It is difficult to generate closed-form solutions for general demand distributions; this needs to be investigated in further research.

Appendix

Mathematical Proofs

Proof of Proposition 4.1. From (4.2), taking the first derivative with w, we have

$$\begin{aligned}\frac{d\Pi_{0,0}^{S}(w)}{dw} &= \frac{\partial\Pi_{0,0}^{S}(w)}{\partial w} + \frac{\partial\Pi_{0,0}^{S}(w)}{\partial Q}\frac{dQ_{0,0}^{*}}{dw} \\ &= Q_{0,0}^{*} + (w-c)\frac{dQ_{0,0}^{*}}{dw} \\ &= Q_{0,0}^{*} - \frac{(w-c)}{w}\frac{1}{h\left(Q_{0,0}^{*}\right)}.\end{aligned} \tag{A.1}$$

The last equation is derived by applying the implicit function rule on $\overline{F}(Q_{0,0}^{*}) = w/EV$:

$$\frac{dQ_{0,0}^{*}}{dw} = -\frac{1}{f\left(Q_{0,0}^{*}\right)}\frac{1}{EV} = -\frac{1}{f\left(Q_{0,0}^{*}\right)}\frac{\overline{F}\left(Q_{0,0}^{*}\right)}{w} = -\frac{1}{w}\frac{1}{h\left(Q_{0,0}^{*}\right)}. \tag{A.2}$$

Then we have

$$\frac{d^2\Pi_{0,0}^{S}(w)}{dw^2} = \frac{dQ_{0,0}^{*}}{dw} - \frac{c}{w^2}\frac{1}{h\left(Q_{0,0}^{*}\right)} - \frac{w-c}{w}\frac{-h'\left(Q_{0,0}^{*}\right)}{h^2\left(Q_{0,0}^{*}\right)}\frac{dQ_{0,0}^{*}}{dw}. \tag{A.3}$$

So if $dh(x)/dx > 0$, $\Pi_{0,0}^{S}(w)$ is concave in w, which reaches its maximum at the first order condition. □

Proof of Proposition 4.2. The first step in maximizing (4.4) is to find optimal $p_{0,1}^{*}$, $r_{0,1}^{R*}$ that maximizes the expression $[p\overline{G}(r^R) + (p - r^R)G(r^R)]$, subject to the constraint $p \le E\ \max(V, r^R)$ so that consumers are willing to buy in the first place. Obviously, the optimal price is $p = E\ \max(V, r^R)$, so we have an expression in terms of only r^R :

$$p\overline{G}\left(r^R\right) + \left(p - r^R\right)G\left(r^R\right) = p - r^R G\left(r^R\right) = E\ \max\left(V, r^R\right) - r^R G\left(r^R\right) = \int_{r^R}^{\infty} vg(v)dv. \tag{A.4}$$

This term is decreasing in r^R and maximized when $r^R = 0$. So we have $r_{0,1}^{R*} = 0$ and $p_{0,1}^{*} = EV$. Then we solve the resulting newsvendor problem in Q to obtain (4.6). □

Proof of Proposition 4.3. The proof here is totally the same as that of Proposition 4.1. □

Proof of Proposition 4.4. From (4.12), by the implicit function rule, we can derive

$$\frac{\partial Q_{1,0}^{*}}{\partial w} = -\frac{1}{f\left(Q_{1,0}^{*}\right)}\frac{1}{EV - r^S} = -\frac{1}{w - r^S}\frac{1}{h\left(Q_{1,0}^{*}\right)}. \tag{A.5}$$

Then taking the partial derivative of (4.13) w.r.t. w,

$$\begin{aligned}
\frac{\partial\Pi_{1,0}^{S}(w, r^S)}{\partial w} &= Q_{1,0}^{*} + \left[\left(w - r^S - c\right) + r^S\overline{F}\left(Q_{1,0}^{*}\right)\right]\frac{\partial Q_{1,0}^{*}}{\partial w} \\
&= Q_{1,0}^{*} - \left[\frac{(w - r^S - c) + r^S\overline{F}\left(Q_{1,0}^{*}\right)}{w - r^S}\right]\frac{1}{h\left(Q_{1,0}^{*}\right)} \\
&= Q_{1,0}^{*} - \left[\frac{EV(w - r^S) - c(EV - r^S)}{(w - r^S)(EV - r^S)}\right]\frac{1}{h\left(Q_{1,0}^{*}\right)}.
\end{aligned} \tag{A.6}$$

The last equation is obtained by substituting (4.12) into the second equation, we continue to have

$$\frac{\partial^2 \Pi_{1,0}^S(w,r^S)}{\partial w^2} = \frac{\partial Q_{1,0}^*}{\partial w} - \frac{c}{(w-r^S)^2}\frac{1}{h\left(Q_{1,0}^*\right)} - \left[\frac{EV(w-r^S)-c(EV-r^S)}{(w-r^S)(EV-r^S)}\right]\frac{-h'\left(Q_{1,0}^*\right)}{h^2\left(Q_{1,0}^*\right)}\frac{\partial Q_{1,0}^*}{\partial w}. \tag{A.7}$$

Therefore, if $h'(x) > 0$, $\Pi_{1,0}^S(w,r^S)$ is concave in w, there exists a unique optimal $w_{1,0}^*$. Replacing w with $w_{1,0}^*$ in (4.12) and $\partial\Pi_{1,0}^S(w,r^S)/\partial w$, setting $\partial\Pi_{1,0}^S(w,r^S)/\partial w = 0$, we have (4.14). □

Proof of Proposition 4.5. We define

$$\nabla = \left[r^S Q_{1,0}^* h\left(Q_{1,0}^*\right) + EV\left(1 - Q_{1,0}^* h\left(Q_{1,0}^*\right)\right)\right]\overline{F}\left(Q_{1,0}^*\right) - c, \tag{A.8}$$

by the implicit function rule, we have

$$\begin{aligned}
&\frac{dQ_{1,0}^*}{dr^S}\\
&= -\frac{\partial\nabla/\partial r^S}{\partial\nabla/\partial Q_{1,0}^*}\\
&= \frac{-Q_{1,0}^* h\left(Q_{1,0}^*\right)\overline{F}\left(Q_{1,0}^*\right)}{-f\left(Q_{1,0}^*\right)\left[r^S Q_{1,0}^* h\left(Q_{1,0}^*\right) + EV\left(1-Q_{1,0}^* h\left(Q_{1,0}^*\right)\right)\right] - (EV-r^S)\overline{F}\left(Q_{1,0}^*\right)\left[h\left(Q_{1,0}^*\right) + Q_{1,0}^* h'\left(Q_{1,0}^*\right)\right]}\\
&= \frac{-Q_{1,0}^* f\left(Q_{1,0}^*\right)}{-ch\left(Q_{1,0}^*\right) - (EV-r^S)\overline{F}\left(Q_{1,0}^*\right)\left[h\left(Q_{1,0}^*\right) + Q_{1,0}^* h'\left(Q_{1,0}^*\right)\right]} > 0.
\end{aligned} \tag{A.9}$$

□

Proof of Proposition 4.6. We can rewrite (4.15) as

$$\left[r^S Q_{1,0}^* h\left(Q_{1,0}^*\right) + EV\left(1 - Q_{1,0}^* h\left(Q_{1,0}^*\right)\right)\right]\left(w_{1,0}^* - r^S\right) = c\left(EV - r^S\right), \tag{A.10}$$

then

$$w_{1,0}^* - r^S - c = \frac{\left[(EV-r^S)Q_{1,0}^* h\left(Q_{1,0}^*\right) - r^S\right]c}{r^S Q_{1,0}^* h\left(Q_{1,0}^*\right) + EV\left(1 - Q_{1,0}^* h\left(Q_{1,0}^*\right)\right)}. \tag{A.11}$$

So

$$\prod_{1,0}^{S}\left(w_{1,0}^{*}, r^{S}\right) = r^{S}E\ \min\left(D, Q_{1,0}^{*}\right) + \left(w_{1,0}^{*} - c - r^{S}\right)Q_{1,0}^{*}$$
$$= r^{S}E\ \min\left(D, Q_{1,0}^{*}\right) + \frac{\left[(EV - r^{S})Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right) - r^{S}\right]cQ_{1,0}^{*}}{r^{S}Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right) + EV\left(1 - Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right)\right)}. \tag{A.12}$$

Taking the derivative with r^{S}, we have

$$\frac{d\Pi_{1,0}^{S}\left(w_{1,0}^{*}, r^{S}\right)}{dr^{S}} = \frac{\partial\Pi_{1,0}^{S}\left(w_{1,0}^{*}, r^{S}\right)}{\partial r^{S}} + \frac{\partial\Pi_{1,0}^{S}\left(w_{1,0}^{*}, r^{S}\right)}{\partial Q}\frac{dQ_{1,0}^{*}}{dr^{S}}$$
$$= E\ \min\left(D, Q_{1,0}^{*}\right) + \frac{-EVcQ_{1,0}^{*}}{\left[r^{S}Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right) + EV\left(1 - Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right)\right)\right]^{2}}$$
$$+ \left[r^{S}\overline{F}\left(Q_{1,0}^{*}\right) + \frac{\left[(EV - r^{S})Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right) - r^{S}\right]c}{r^{S}Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right) + EV\left(1 - Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right)\right)}\right.$$
$$\left.+\frac{(EV - r^{S})^{2}\left[h\left(Q_{1,0}^{*}\right) + Q_{1,0}^{*}h'\left(Q_{1,0}^{*}\right)\right]cQ_{1,0}^{*}}{\left[r^{S}Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right) + EV\left(1 - Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right)\right)\right]^{2}}\right]\frac{dQ_{1,0}^{*}}{dr^{S}}$$
$$= E\ \min\left(D, Q_{1,0}^{*}\right) + \frac{-EVQ_{1,0}^{*}\overline{F}^{2}\left(Q_{1,0}^{*}\right)}{c}$$
$$+ \left[r^{S}\overline{F}\left(Q_{1,0}^{*}\right) + \left[\left(EV - r^{S}\right)Q_{1,0}^{*}h\left(Q_{1,0}^{*}\right) - r^{S}\right]\overline{F}\left(Q_{1,0}^{*}\right)\right.$$
$$\left.+\left(EV - r^{S}\right)^{2}\left[h\left(Q_{1,0}^{*}\right) + Q_{1,0}^{*}h'\left(Q_{1,0}^{*}\right)\right]cQ_{1,0}^{*}\frac{\overline{F}^{2}\left(Q_{1,0}^{*}\right)}{c^{2}}\right]\frac{dQ_{1,0}^{*}}{dr^{S}}$$
$$= E\ \min\left(D, Q_{1,0}^{*}\right) - \frac{EVQ_{1,0}^{*}\overline{F}^{2}\left(Q_{1,0}^{*}\right)}{c}$$

$$+\left[\left(EV-r^S\right)Q_{1,0}^* h\left(Q_{1,0}^*\right)\overline{F}\left(Q_{1,0}^*\right)\right.$$
$$\left.+\left(EV-r^S\right)^2\left[h\left(Q_{1,0}^*\right)+Q_{1,0}^* h'\left(Q_{1,0}^*\right)\right]Q_{1,0}^*\frac{\overline{F}^2\left(Q_{1,0}^*\right)}{c}\right]\frac{dQ_{1,0}^*}{dr^S}. \tag{A.13}$$

The third equation is derived from (4.15), from Proposition 4.5, we know $dQ_{1,0}^*/dr^S$ is positive, so is the last term. To the end, it suffices to show that the sum of the first two terms is nonnegative. In fact, we get

$$\begin{aligned}
&\frac{d}{dQ_{1,0}^*}\left[E\min\left(D,Q_{1,0}^*\right)-\frac{EVQ_{1,0}^*\overline{F}^2\left(Q_{1,0}^*\right)}{c}\right]\\
&=\overline{F}\left(Q_{1,0}^*\right)-\frac{EV}{c}\left[\overline{F}^2\left(Q_{1,0}^*\right)-2Q_{1,0}^* f\left(Q_{1,0}^*\right)\overline{F}\left(Q_{1,0}^*\right)\right]\\
&=\frac{c\overline{F}\left(Q_{1,0}^*\right)-EV\overline{F}^2\left(Q_{1,0}^*\right)+2EVQ_{1,0}^* f\left(Q_{1,0}^*\right)\overline{F}\left(Q_{1,0}^*\right)}{c}\\
&=\frac{\overline{F}^2\left(Q_{1,0}^*\right)\left[r^S Q_{1,0}^* h\left(Q_{1,0}^*\right)+EV\left(1-Q_{1,0}^* h\left(Q_{1,0}^*\right)\right)\right]-EV\overline{F}^2\left(Q_{1,0}^*\right)+2EVQ_{1,0}^* f\left(Q_{1,0}^*\right)\overline{F}\left(Q_{1,0}^*\right)}{c}\\
&=\frac{(EV+r^S)Q_{1,0}^* f\left(Q_{1,0}^*\right)\overline{F}\left(Q_{1,0}^*\right)}{c}\geq 0.
\end{aligned} \tag{A.14}$$

The third equation is derived from (4.15), and $E\min(D,Q_{1,0}^*)-EVQ_{1,0}^*\overline{F}^2(Q_{1,0}^*)/c$ equals zero at $Q_{1,0}^*=0$, then this term turns out to be nonnegative for any nonnegative $Q_{1,0}^*$.

Above all, $\Pi_{1,0}^S(w_{1,0}^*, r^S)$ is increasing in r^S and reaches its maximum at the upper bound $w_{1,0}^*-c$. □

Proof of Proposition 4.7. We can rewrite (3.2) as:

$$\prod_{1,1}^{R}\left(p,Q,r^R\right)=\left[\left(p-r^S\right)\overline{G}\left(r^R\right)+\left(p-r^R\right)G\left(r^R\right)\right]E\min(D,Q)-\left(w-r^S\right)Q. \tag{A.15}$$

Firstly, we aim to find optimal $p_{1,1}^*$ and $r_{1,1}^{R*}$ that maximizes $[(p - r^S)\overline{G}(r^R) + (p - r^R)G(r^R)]$ under the constraint $p \le E\max(V, r^R)$. Clearly, the retailer will set the maximum possible price $p = E\max(V, r^R)$, then we have

$$\begin{aligned}\left(p - r^S\right)\overline{G}\left(r^R\right) + \left(p - r^R\right)G\left(r^R\right) &= p - r^S\overline{G}\left(r^R\right) - r^R G\left(r^R\right) \\ &= E\max\left(V, r^R\right) - r^S\overline{G}\left(r^R\right) - r^R G\left(r^R\right) \\ &= \int_{r^R}^{\infty}\left(v - r^S\right)g(v)dv.\end{aligned} \tag{A.16}$$

This is maximized when $r^R = r^S$, so we have $r_{1,1}^{R*} = r^S$ and $p_{1,1}^* = E\max(V, r^S)$. We can easily get $Q_{1,1}^*$ by solving a resulting newsvendor problem. □

Proof of Proposition 4.8. The analysis here is strikingly similar to that of Proposition 4.4, so we omit the details here. □

Proof of Proposition 5.1. To show $\Pi_{0,0}^{R*} \ge \Pi_{1,0}^{R*}$ is equivalent to show

$$\begin{gathered}4EV\sqrt{c\cdot EV}\left(1 - \sqrt{\frac{c}{EV}}\right)^2 \le (EV - c)^2, \\ \Longleftarrow 4(EV + c)\sqrt{c\cdot EV} \le (EV + c)^2 + 4c\cdot EV, \\ \Longleftarrow 0 \le \left[(EV + c) - 2\sqrt{c\cdot EV}\right]^2.\end{gathered} \tag{A.17}$$

To show $\Pi_{0,0}^{S*} \le \Pi_{1,0}^{S*}$ is equivalent to show $EV + c \ge 2\sqrt{c\cdot EV}$. □

Proof of Proposition 5.2. From (4.16), (4.17), (4.18), and Proposition 4.8, we have

$$\begin{gathered}p_{1,1}^* = E\max\left(V, r^S\right), \\ \overline{F}\left(Q_{1,1}^*\right) = \frac{w - r^S}{p_{1,1}^* - r^S} = \frac{w - r^S}{E\max(V, r^S) - r^S}, \\ r_{1,1}^{R*} = r^S, \\ \left(w_{1,1}^* - r^S\right)Q_{1,1}^* h\left(Q_{1,1}^*\right) - \left[\left(w_{1,1}^* - r^S - c\right) + r^S\overline{G}\left(r^S\right)\overline{F}\left(Q_{1,1}^*\right)\right] = 0.\end{gathered} \tag{A.18}$$

All of decisions are decided by r^S, and we find that it is difficult to find optimal r^S even for F that follows this uniform distribution.

If we fix $r^S = 0$, we can easily find that the decisions in scenario 4 are the same as those in scenario 2, then $\Pi_{1,1}^S|_{(r^S=0)} = \Pi_{0,1}^{S*}$. As $\Pi_{1,1}^{S*}$ can be achieved for an optimal r^S, so $\Pi_{1,1}^{S*} \ge \Pi_{1,1}^S|_{(r^S=0)}$, then $\Pi_{1,1}^{S*} \ge \Pi_{1,0}^{S*}$. □

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Research Article

A Time Scheduling Model of Logistics Service Supply Chain with Mass Customized Logistics Service

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With the increasing demand for customized logistics services in the manufacturing industry, the key factor in realizing the competitiveness of a logistics service supply chain (LSSC) is whether it can meet specific requirements with the cost of mass service. In this case, in-depth research on the time-scheduling of LSSC is required. Setting the total cost, completion time, and the satisfaction of functional logistics service providers (FLSPs) as optimal targets, this paper establishes a time scheduling model of LSSC, which is constrained by the service order time requirement. Numerical analysis is conducted by using Matlab 7.0 software. The effects of the relationship cost coefficient and the time delay coefficient on the comprehensive performance of LSSC are discussed. The results demonstrate that with the time scheduling model in mass-customized logistics services (MCLSs) environment, the logistics service integrator (LSI) can complete the order earlier or later than scheduled. With the increase of the relationship cost coefficient and the time delay coefficient, the comprehensive performance of LSSC also increases and tends towards stability. In addition, the time delay coefficient has a better effect in increasing the LSSC's comprehensive performance than the relationship cost coefficient does.

1. Introduction

In the face of growing demand for customized logistics services, numerous logistics enterprises are not only providing customers with mass services, but are also beginning to meet the demand for customized services and are considering a change in their logistics service modes. Specifically, they try to provide mass customized logistics services (MCLS) instead of mass logistics services [1]. MCLS represents a significant development in logistics services, and a logistics company's capability to offer MCLS is crucial to enhance its market competitiveness. In the MCLS environment, to meet customized service demand and offer large-scale services, several logistics enterprises form a logistics service supply chain (LSSC) via union and integration [2, 3]. LSSC is a new supply chain of which logistics service integrator (LSI) is the core enterprise. The basic structure of LSSC is functional logistics service provider (FLSP) $\rightarrow$ LSI $\rightarrow$ customer. FLSP is integrated by LSI when LSI builds the integrated logistics to customer. The main purpose of LSSC is to provide the flexible logistics service for manufacturing supply chain [3]. As the core enterprise of a LSSC, the LSI integrates the advantages of the FLSPs, such as various logistics processes and logistics service functions and then provides flexible logistics services to customers.

In such an environment, the key to improve the LSSC's competitiveness is its capability to offer customized logistics services with mass logistics costs. An in-depth research on the time scheduling of the LSSC is necessary to meet customers' requirements. However, the existing research on supply chain scheduling model has three deficiencies, which are discussed as follows.

First, the existing research on supply chain scheduling does not consider the feature that the operation time of suppliers may be delayed or compressed. It also ignores the fact that the customer's order is flexible in many cases, and that the order completion time may be delayed or compressed within an acceptable range. Thus, questions arise in expressing features of time delay or compression in model building and effects of time delay or compression on the comprehensive performance of LSSC.

Second, in most studies on supply chain scheduling, cost control is regarded as the primary scheduling goal. However, this is not always the case for an LSSC in the MCLS environment. The flexibility of the order completion time and the satisfaction of FLSPs are also significant. Expressing these objective functions in an optimization model and searching for a more reasonable method of solving this model are pertinent questions.

Third, the conclusions drawn from traditional manufacturing supply chain scheduling models may not be fully applicable to LSSC scheduling problems. Therefore, we will determine how LSI can use the time scheduling model proposed in this paper to better manage its LSSC.

This paper is organized as follows. Section 2 is the literature review in which the existing supply chain optimization scheduling models and methods are systematically summarized. Section 3 focuses on model building, in which a time scheduling optimization model of LSSC in the MCLS environment is established. Section 4 describes the model solution: a method of solving the multiobjective programming model. Section 5 presents numerical analysis, including a discussion of the influence of relevant parameters on LSSC's scheduling performance. Section 6 provides management insights and recommendations to improve actual operation of LSSC. The last section presents the conclusions and future research directions in this field.

2. Literature Review

Most studies on supply chain scheduling have focused on the manufacturing industry and have achieved significant results. Many earlier studies discussed job shop scheduling within a single enterprise (see, e.g., [4, 5]). They are primarily interested in the arrangement of processing procedures and the order operation sequence. Kreipl and Pinedo [6] introduced what is planning and what is scheduling and gave an overview of the various planning and scheduling models in general. Several scholars wrote about the coordination of the assembly system in manufacturing enterprise (see, e.g., [7]). But in early studies, many researches emphases were focused within one manufacturing enterprise and did not pay much attention to the roles outside the enterprise. As is known to all, customer is the most important role in the competitive market. Philipoom [8] transferred his attention to the most important part, which is the customers' requirements and their feelings. The dilemma for the manufacturing manager when planning the lead-times was discussed. He took delivery reliability and time as the vital factors in scheduling. The idea of thinking highly of customer satisfactory is worthy of our research. A systematic study of the supply chain scheduling model had been established by Hall and Potts [9] and from then on planning and scheduling problems began getting more attention.

However, studies on the supply chain scheduling with the mass-customization production mode are a relatively new development (see, e.g., [10, 11]). The contents of these studies included the application of postponement strategy, the positioning of customer order decoupling point (CODP), and the time scheduling problem. In terms of the supply chain time scheduling, several scholars examined the dominant contradictions analysis and its optimization solution of supply chain scheduling in mass customization (see, e.g., [12, 13]). Yao and Pu [13] introduced a dynamic and multiobjective optimization model to balance the contradiction between scale production effect and customized demand. They considered the delay coefficient but did not give out the time delay punishment in mathematical expression. Other scholars studied the differences between the ideal timetables of the different supply chain members and that of customer demand, and then explored ways to solve this discrepancy (see, e.g., [14]). From the perspective of integrated supply chain production planning and scheduling process, Mishra et al. [15] designed a mixed integer programming model. Moreover, in the scheduling of manufacturing supply chain, the main objective of scheduling optimization is supply chain cost (see, e.g., [16–18]). Most studies assumed that the order completion time or the delivery time of suppliers was fixed. However, in a number of cases, as an index reflecting supply chain agility, the order time requirements may be changed (see, e.g., [19–21]). Thus, considering the influence of time compression or delay on the scheduling results is necessary.

Similar to studies on manufacturing supply chain, research on the service supply chain mainly focuses on service process scheduling (see, e.g., [22, 23]), order assignment scheduling (see, e.g., [3, 24]), and so forth. Time scheduling, although is a significant part of LSSC scheduling, has not received sufficient attention. Time scheduling is a key aspect of the service supply chain operation. It is directly related to whether the logistics service can be completed successfully based on customer requirements. Therefore, examining time scheduling of LSSC is necessary.

This paper aims to address these issues and to evaluate the time scheduling model of LSSC in the MCLS environment. The main contributions of this study are listed as follows.

(1) The effects of time delay and compression of the customer order on LSSC's comprehensive performance are analyzed. Time delay in the order requirements contributes to an increase in the satisfaction of FLSPs. However, their satisfaction level remains stable after it reaches a certain value. Moreover, the extent of time compression is also limited.

(2) The effect of the relationship cost coefficient on the comprehensive performance of LSSC is discussed. With the increase of LSI's relationship cost coefficient, the comprehensive performance of LSSC also increases. This comprehensive performance cannot be infinitely enhanced, but remains stable upon reaching a certain value.

(3) LSI can improve the comprehensive performance of LSSC by both the relationship cost coefficient and time delay. However, compared with the relationship cost coefficient, using time delay is a superior way to improve the comprehensive performance.

3. Time Scheduling Model of LSSC

3.1. Model Assumptions and Variables

A two-echelon LSSC is assumed to consist of one LSI and several FLSPs. The logistics service comprises multiple service processes. Every service process is completed by many cooperative FLSPs. Each FLSP's service capability is different from that of the others; different FLSPs have varying normal completion times for the same work. Table 1 presents the notations of the model.

The other assumptions of the model are as follows.

(1) The $(i+1)$th service process cannot begin until the ith service process is finished.

(2) In each service process, the quantity of service orders assigned by LSI to FLSP is assumed given. FLSP can only complete these orders based on the order quantity assigned by LSI. The service cost of FLSP is equal to the service cost per unit time multiplied by service time.

(3) The normal service time of FLSP refers to the typical length of time needed to complete a task using its capability without considering any time delay or compression. While working within the normal time, the satisfaction of FLSP is at its highest, and the FLSP does not have to pay any additional cost.

(4) In each service process, time delay or compression to complete an order produces additional costs. The costs of time delay or compression per unit time are the same.

(5) All the service processes are outsourced to FLSPs by LSI and are completed by the former. The model does not consider the operation cost of LSI.

(6) The assignment of the service orders is finished. And the CODP has been decided. The model does not consider the location problem of the CODP.

(7) T_{ij} means the normal service time of the jth FLSP in the ith service process, which is a parameter decided by the technology of the corresponding FLSP and differs from each other. In the section of numerical analysis (see Section 5), it is a definitive parameter. In practice, the normal service time of finishing a specific order is usually definitive, but not stochastic.

3.2. Model Building

Considering multiple goals in LSSC time scheduling is necessary. To minimize the total cost incurred in LSSC, to minimize the difference between the expected and the actual time of completing the service order, and to maximize the satisfaction of FLSPs, this paper establishes a time-scheduling model that is constrained by the service order time requirement. The modeling process is as follows.

Table 1: Notations for the model.

Notations	Description
C_{ij}	The normal service cost per unit time of the jth FLSP in the ith service process, which is $i = 1, 2, 3, \ldots, I_0$, $j = 1, 2, 3, \ldots, J_0$, the same as below
C_{ij}^{ext}	The additional service cost per unit time of the jth FLSP in the ith service process
C_i^{ext}	When the ith service process is delayed, the delay cost per unit time incurred in the $(i+1)$th service process
p_i^{ext}	When the time of the ith service process is compressed, the time compression cost per unit time incurred in the $(i+1)$th service process
T_{ij}	The normal service time of the jth FLSP in the ith service process
T_{ij}^{ext}	The additional service time of the jth FLSP in the ith service process
T_i^{exp}	The expected operation time for the ith service process set by LSI
U_{ij}	The satisfaction of the jth FLSP in the ith service process
ϕ_{ij}	Weights of the jth FLSP's satisfaction in the ith service process
β_i	Time delay coefficient for the ith service process set by LSI
T_i^*	The upper limit of time delay or compression incurred in the $(i-1)$th service process, which can be endured by the ith service process
U_{ij}^0	The lower limit of the satisfaction of the jth FLSP in the ith service process
R	The adjustment coefficient of the time delay or compression, in which $R > 0$ means that the operation time is delayed. $R < 0$ indicates that the operation time is compressed
Z_1	The total cost of LSSC
Z_2	The total delivery time of all processes in LSSC
Z_3	The total satisfaction of FLSPs in LSSC
$Z_1^{\min}$	The minimum of the objective function Z_1 when the objective functions Z_2 and Z_3 are not considered
$Z_2^{\min}$	The minimum of the objective function Z_2 when the objective functions Z_1 and Z_3 are not considered
$Z_3^{\min}$	The minimum of the objective function Z_3 when the objective functions Z_1 and Z_2 are not considered
$Z_2^{\max}$	The maximum of the objective function Z_2 when the objective functions Z_1 and Z_3 are not considered
$Z_3^{\max}$	The maximum of the objective function Z_3 when the objective functions Z_1 and Z_2 are not considered
C	The relationship cost coefficient of the LSI, $C > 1$
Z	The objective function synthesized by Z_2 and Z_3, which is also called the comprehensive performance of LSSC
K_1	Weight coefficients of the objective function Z_2 in the comprehensive performance of LSSC
K_2	Weight coefficients of the objective function Z_1 in the comprehensive performance of LSSC
F_{ij}	The jth FLSP in the ith service process

We set that $[f(x)]^+ = \max\{0, f(x)\}$.

Then

$$\begin{aligned} \min Z_1 = {} & \sum_{i=1}^{I_0}\sum_{j=1}^{J_0}\left(C_{ij}T_{ij} + C_{ij}^{\text{ext}}\left|T_{ij}^{\text{ext}}\right|\right) + \sum_{i=1}^{I_0}\left(\max_{j=1}^{J_0}\left[T_{ij} + T_{ij}^{\text{ext}} - T_i^{\exp}\right]^+\right)C_i^{\text{ext}} \\ & + \sum_{i=1}^{I_0}\left(\min_{j=1}^{J_0}\left[T_i^{\exp} - T_{ij} - T_{ij}^{\text{ext}}\right]^+\right)p_i^{\text{ext}}, \end{aligned} \tag{3.1}$$

$$\min Z_2 = \sum_{i=1}^{I_0}\left(\min_{j=1}^{J_0}\left[T_i^{\exp} - T_{ij} - T_{ij}^{\text{ext}}\right]^+\right), \tag{3.2}$$

$$\max Z_3 = \sum_{i=1}^{I_0}\sum_{j=1}^{J_0}\left(1 - \frac{\left|T_{ij} - T_i^{\exp}\right|}{T_{ij}}\right)\frac{T_{ij}C_{ij}}{\left(T_{ij}C_{ij} + \left|T_{ij}^{\text{ext}}\right|C_{ij}^{\text{ext}}\right)}\phi_{ij}, \tag{3.3}$$

$$\begin{aligned} \text{subject to}\quad & T_{ij} + T_{ij}^{\text{ext}} \le (1 + \beta_i R)T_i^{\exp}, \\ & \left|T_i^{\exp} - T_{ij} - T_{ij}^{\text{ext}}\right| \le T_{i+1}^{*}, \\ & \left(1 - \frac{\left|T_{ij} - T_i^{\exp}\right|}{T_{ij}}\right)\frac{T_{ij}C_{ij}}{\left(T_{ij}C_{ij} + \left|T_{ij}^{\text{ext}}\right|C_{ij}^{\text{ext}}\right)} \ge U_{ij}^{0}. \end{aligned} \tag{3.4}$$

In (3.1), the objective function Z_1 is made to minimize LSSC total operation cost. The first part of (3.1) represents the total operation cost of all processes in the LSSC. The second part of (3.1) represents the time delay cost incurred in the next process, which is caused by the previous process. The third part of (3.1) represents the cost of time compression incurred in the next process, which is caused by the previous process too.

In (3.2), the objective function Z_2 makes the service order completed on time as much as possible.

In (3.3), the objective function Z_3 is made to maximize the weighted satisfaction of all FLSPs. This indicator consists of two parts: one is the satisfaction with operation time and the other is related to the cost. In Z_3, the first part $(1 - |T_{ij} - T_i^{\exp}|/T_{ij})$ represents the proximity degree between the normal operation time and the order expected time of the jth FLSPs in the ith service process, which represents the satisfaction in the time aspect. The second part $T_{ij}C_{ij}/(T_{ij}C_{ij} + |T_{ij}^{\text{ext}}|C_{ij}^{\text{ext}})$ means the proportion of the normal operation cost in the total cost of the jth FLSP in the ith service process, which represents the satisfaction in the cost aspect.

In (3.4), the first constraint represents the requirements of FLSP's order completion time proposed by the LSI. Specifically, it requires that the actual completion time must be less than a certain multiple of the expected time. This multiple is determined by the product of β_i and R, which represents the limits that LSI can endure the time delay or compression. The second constraint means that the difference between the actual order completion time and the expected time cannot exceed the upper limit of the time variation (time compression or delay) incurred in the $(i-1)$th service process, which can be endured by the ith service process. This is a strong constraint that must be observed because during the entire service provision process, the continued link of operation time exists between the upstream and downstream service processes. The starting time of each process should meet certain time requirements. The third constraint means that the satisfaction of each FLSP must be more than the lower limit that they can accept.

4. Model Solution

4.1. Simplifying the Multiobjective Programming Model

The model described has three objectives and three constraints. It is a typical multiobjective programming problem. Multiobjective programming problems have numerous mature solutions, such as the evaluation function method (including linear weighting method, reference target method, maximin method), goal programming method, delaminating sequence method, interactive programming method, and subordinate function method. For the specific issues, it is necessary to choose an appropriate solution method to solve the practical problems. This paper is based on the MCLS environment, so we should consider not only the cost target but also the customized target. In time scheduling, LSI tends to conform to the customized time requirements and set corresponding requirements related to order completion time for their FLSPs. Each FLSP's satisfaction will directly determine the quality and the possibility of order completion. Thus, order completion time and FLSPs' satisfaction are two critical goals. Operation cost is often not the primary consideration. Completing logistics service orders with the minimum cost is not required; the total cost of LSSC must be maintained within a certain range. Besides, to build and maintain a good relationship with FLSPs, LSI is usually willing to make a certain amount of cost concessions. Considering these actual situations, we introduce the following parameter called the relationship cost coefficient C $(C > 1)$ into our model and use it to represent the cost augmentation limits (4.1).

$$Z_1 < Z_1^{\min} \times C. \tag{4.1}$$

Thus, (4.1) can be regarded as a new constraint and be combined with (3.4) to form the new constraints. The original model then becomes a twin-goal programming problem. Given that a certain degree of conflict and incommensurability occurs among the targets in multiobjective decision-making problems, determining an absolute optimal solution is difficult. In this paper, we choose the commonly used linear weighting method to solve our model, and attempt to transform this multiobjective programming model into a single-objective programming model.

The objective function Z_2 and the objective function Z_3 mean to make the service order completed on time as much as possible and to maximize the satisfaction of FLSPs, respectively. Therefore, each of them has to be normalized. Z_2 and Z_3 are thus divided by the possible maximum value $Z_2^{\max}$ and $Z_3^{\max}$, respectively, and their corresponding results are added up. The synthesized objective function is as follows:

$$\max Z = K_2 \times \frac{Z_3}{Z_3^{\max}} - K_1 \times \frac{Z_2}{Z_2^{\max}}. \tag{4.2}$$

In (4.2), K_1 and K_2 represent the weights of Z_2 and Z_3, respectively, which are determined by the linear weighting method. Consider the following:

$$\begin{aligned}
\max \quad & Z = K_2 \times \frac{Z_3}{Z_3^{\max}} - K_1 \times \frac{Z_2}{Z_2^{\max}}, \\
\text{subject to} \quad & T_{ij} + T_{ij}^{\text{ext}} \leq \left(1 + \beta_i R\right) T_i^{\exp}, \\
& \left| T_i^{\exp} - T_{ij} - T_{ij}^{\text{ext}} \right| \leq T_{i+1}^{*}, \\
& \left(1 - \frac{\left| T_{ij} - T_i^{\exp} \right|}{T_{ij}}\right) \frac{T_{ij} C_{ij}}{\left(T_{ij} C_{ij} + \left| T_{ij}^{\text{ext}} \right| C_{ij}^{\text{ext}}\right)} \geq U_{ij}^{0}, \\
& Z_1 < Z_1^{\min} C.
\end{aligned} \tag{4.3}$$

Here,

$$\begin{aligned}
Z_1 &= \sum_{i=1}^{I_0} \sum_{j=1}^{J_0} \left(C_{ij} T_{ij} + C_{ij}^{\text{ext}} \left| T_{ij}^{\text{ext}} \right| \right) + \sum_{i=1}^{I_0} \left(\max_{j=1}^{J_0} \left[T_{ij} + T_{ij}^{\text{ext}} - T_i^{\exp} \right]^{+} \right) C_i^{\text{ext}} \\
&\quad + \sum_{i=1}^{I_0} \left(\min_{j=1}^{J_0} \left[T_i^{\exp} - T_{ij} - T_{ij}^{\text{ext}} \right]^{+} \right) p_i^{\text{ext}}, \\
Z_2 &= \sum_{i=1}^{I_0} \left(\min_{j=1}^{J_0} \left[T_i^{\exp} - T_{ij} - T_{ij}^{\text{ext}} \right]^{+} \right), \\
Z_3 &= \sum_{i=1}^{I_0} \sum_{j=1}^{J_0} \left(1 - \frac{\left| T_{ij} - T_i^{\exp} \right|}{T_{ij}} \right) \frac{T_{ij} C_{ij}}{\left(T_{ij} C_{ij} + \left| T_{ij}^{\text{ext}} \right| C_{ij}^{\text{ext}} \right)} \phi_{ij}.
\end{aligned} \tag{4.4}$$

4.2. Using the Genetic Algorithm to Solve the Multiobjective Programming Problem

The genetic algorithm (GA) is an effective method of searching for the optimal solution by simulation of the natural selection process. It uses multiple starting points to begin the search; it has global search capability. Similar to the natural evolutionary process, the computational process of genetic algorithms is iterative, which means that it includes selection, crossover, and mutation. For reference, GA is described in [25].

5. Numerical Analysis

This section first illustrates the validity of the model via a numerical analysis, and then explores the influence of relevant parameters on time scheduling results. Additionally, some effective recommendations are given for use in the actual operation-based on numerical analysis.

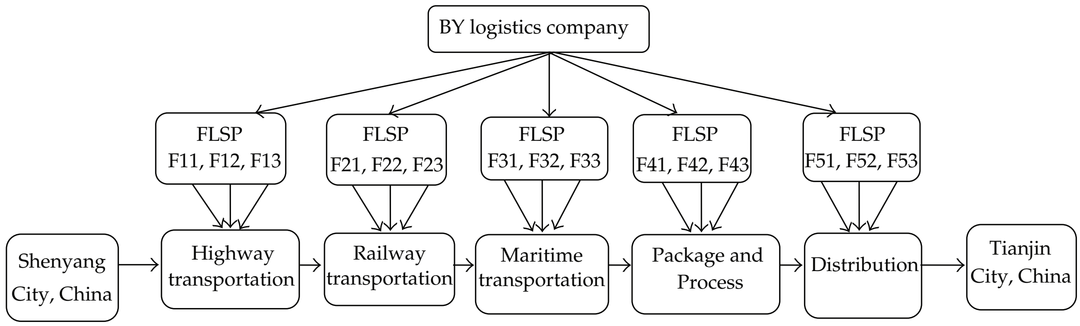

Figure 1: The logistics service process of one order of BY Company.

5.1. Numerical Data Description and Basic Results

BY is a logistics company based in northern China. It is a logistics integrator that has integrated several third-party logistics firms. This company has received an order requiring it to deliver goods from Shenyang City to Tianjin City. During delivery, five service processes are needed: road transportation, rail transportation, maritime transportation, packaging and processing, and distribution. The CODP has been decided and it will be at the 4th service process which is packaging process. Obviously, the processes before the CODP are provided with mass operations and the processes after the CODP are provided with customized operations. Each service process is to be completed by three FLSPs (see Figure 1).

The service capability of each FLSP is different from that of the others because the normal operation times T_{ij} vary when different FLSPs complete the same order. BY company, the LSI, has set an expected operation time $T_i^{\exp}$ for each order in every process. The parameters of each process and of each FLSP are different (Tables 2 and 3). We assume that i be the service process, $i = 1,2,3,4,5$, j represents the number of FLSPs in each process, so that $j = 1,2,3$. C_{ij} represents the normal service cost per unit time of the jth FLSP in the ith process.

In the model solving, genetic algorithms and Matlab 7.0 software are used. Assuming the genetic population to be 200, the hereditary algebra to be 400, the adjustment coefficient $R = 1$, and the relationship cost coefficient $C = 1.2$, then the calculation result of comprehensive performance Z is equal to 0.6802. The time parameters of each FLSP in every process are shown in Table 4.

We can find that among the 15 FLSPs, seven completed the order with time compression ($T_{ij}^{\exp} < 0$), and eight completed the order with time delay ($T_{ij}^{\exp} > 0$).

5.2. The Effect of Time Delay Coefficient on LSSC Scheduling Results

In practice, responding rapidly to customized demand is one of the main features of the mass-customized LSSC. It is often the case that the LSI requires its FLSPs to compress their service time frequently to respond to the customer's demand. Therefore, the order is completed ahead of the expected time, which is reflected in the model as $\beta_i R < 0$. In some cases, if the time requirement of the order is not that urgent, LSI may permit its FLSPs to delay their work. That is, the completion time is later than the expected time, which is reflected in the model as $\beta_i R > 0$. The permitted order completion time is directly related to the difficulty faced by FLSPs in completing the order. It has a direct impact on the cost of these FLSPs. To describe

Table 2: Basic data (3.1).

C_{ij}	$j = 1$	$j = 2$	$j = 3$
$i = 1$	5	6	5.5
$i = 2$	7	8	8.5
$i = 3$	11	12	10.5
$i = 4$	15	18	14.5
$i = 5$	6	5	7.5
C_{ij}^{ext}	$j = 1$	$j = 2$	$j = 3$
$i = 1$	5.5	6.5	6
$i = 2$	8	9	9.5
$i = 3$	12	14	12
$i = 4$	16	20	16
$i = 5$	7	6	8
T_{ij}	$j = 1$	$j = 2$	$j = 3$
$i = 1$	8	7	6
$i = 2$	7	6	5
$i = 3$	10	9	9.5
$i = 4$	12	10	11
$i = 5$	8	9	10
U_{ij}^{0}	$j = 1$	$j = 2$	$j = 3$
$i = 1$	0.4	0.5	0.3
$i = 2$	0.5	0.6	0.55
$i = 3$	0.4	0.5	0.55
$i = 4$	0.3	0.3	0.3
$i = 5$	0.3	0.25	0.3
ϕ_{ij}	$j = 1$	$j = 2$	$j = 3$
$i = 1$	0.4	0.3	0.4
$i = 2$	0.3	0.4	0.3
$i = 3$	0.5	0.3	0.2
$i = 4$	0.3	0.2	0.5
$i = 5$	0.4	0.3	0.3

Table 3: Basic data (3.2).

C_i^{ext}	C_1^{ext} 2	C_2^{ext} 4	C_3^{ext} 6	C_4^{ext} 8	C_5^{ext} 3
p_i^{ext}	p_1^{ext} 3	p_2^{ext} 5	p_3^{ext} 7	p_4^{ext} 9	p_5^{ext} 4
T_i^{exp}	T_1^{exp} 7	T_2^{exp} 6	T_3^{exp} 8	T_2^{exp} 14	T_5^{exp} 8
β_i	β_1 0.1	β_2 0.2	β_3 0.2	β_4 0.15	β_5 0.15
T_i^*	T_2^* 1	T_3^* 2	T_4^* 3	T_5^* 3	T_6^* 2

the two situations, we discuss the influence of the time delay (or compression) coefficient on the satisfaction of LSSC and on the order completion time. We use the synthesized objective function Z to denote the comprehensive performance of LSSC. Keeping the model parameters unchanged and changing only the value of adjustment coefficient R, we obtain the calculation results of Z. The results are presented in Table 5.

When the calculation results in Table 5 are plotted, a curve is shown in Figure 2.

Table 4: Calculation results.

Process i	F_{ij}	$T_{ij}^{\exp}$	The total operation time of each FLSP
$i = 1$	F_{11}	−0.4181	7.5819
	F_{12}	0.0223	7.0223
	F_{13}	0.0204	6.0204
$i = 2$	F_{21}	0.0287	7.0287
	F_{22}	−0.0039	5.9961
	F_{23}	0.0001	5.0001
$i = 3$	F_{31}	−0.4028	9.5972
	F_{32}	−0.0014	8.9986
	F_{33}	−0.0297	9.4703
$i = 4$	F_{41}	0.0087	12.0087
	F_{42}	4.0001	14.0001
	F_{43}	0.0363	11.0363
$i = 5$	F_{51}	−0.0151	7.9849
	F_{52}	0.0110	9.0110
	F_{53}	−0.8012	9.1988

Table 5: The effect of time delay (compression) coefficient on the comprehensive performance of LSSC.

R	Z
2.4	0.7019
2	0.7019
1.6	0.6948
1.2	0.6827
1	0.6802
0.8	0.6671
0.4	0.6615
0	0.6272
−0.2	0.5999
−0.4	0.5738
−0.6	0.4429
−0.8	0.3941
−0.85	0.1175

5.2.1. Analysis of the Overall Change Trend

From Figure 2, we can observe that along with the increase of R (from negative to positive values), the synthesized objective function Z rises gradually, which indicates that the satisfaction of LSSC increases and remains stable after reaching a certain value. The slope in the delay part is smaller; using the time delay strategy, the margin of increase in the LSSC's comprehensive performance is slow. However, the slope of the curve in the time compression part is bigger than that in the time delay part. Hence, the influence on the comprehensive performance of LSSC is greater. It is due to that along with the increase of R, these FLSPs can get time relaxation, but they are required to pay the delay cost at the same time. In addition, their additional time is occupied by the order, thus they could not use it to finish other service orders. And the more time relaxation is offered, the effect described above is larger. Therefore, Z value is shown marginal decrease with the increasing of adjustment

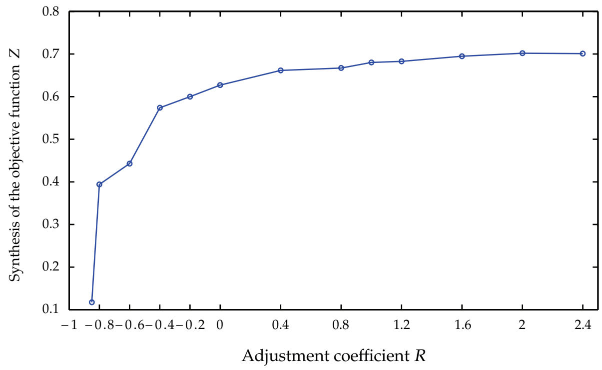

Figure 2: Curve of Z changed with R.

coefficient R. Based on these facts, it can be inferred that in practice, before the LSI requires its FLSPs to operate with time compression (to shorten their operation time in finishing the same order), it is necessary to carefully weigh the advantages and disadvantages. Furthermore, following certain measures (such as offering cost compensation) to minimize the decline in the FLSPs' satisfaction is necessary.

To further analyze the influence on the LSSC performance caused by the time delay (or time compression) coefficient, Section 5.2.2 and Section 5.2.3 cover the relevant conclusions drawn from the different parts of Figure 2.

5.2.2. *Analysis of the Time Compression Part*

Figure 3 shows the curve of Z changed with R when operation time is compressed. When $R < 0$, the LSSC is in the operation status of time compression. Figure 3 indicates that with the increase of the absolute value of R, Z decreases, which means that the performance of LSSC is declining. In the part of R decreasing from 0 to −0.9, from the slope of the curve (Figure 3), the Z value decreases by a much larger margin (from 0.6272 to 0.1175). Thus, if FLSPs are required to operate in the time compression state, their total satisfaction will decline sharply. Moreover, a greater increase in time compression indicates a greater decline in the satisfaction of FLSPs. In our numerical example, if $R < -0.9$, no solution in the model occurs, which means LSSC has collapsed and can no longer operate. This also shows that in practice, the order completion time of FLSP cannot be compressed infinitely. Upon reaching a certain point, the FLSP's operation time can no longer be compressed. When working within this limitation time, the comprehensive performance of LSSC is at its lowest.

5.2.3. *Analysis of the Time Delay Part*

Figure 4 shows the curve of Z changed with R when operation time is delayed. When $R > 0$, LSSC is in the operation status of time delay. Figure 4 indicates that along with the increase of R, Z also increases. That is, the comprehensive performance of LSSC continues to increase. However, when $R > 0$, Z only increases slightly. The range of increase of Z is far less than the range of its decline when operation time is delayed or compressed at the same extent. This occurs because in the case of time delay, FLSPs must pay the penalty cost. In addition, the same with that in Figure 2, the Z value is shown that marginal decrease with the increasing of adjustment coefficient R, the more time relaxation is offered, the more the service capacities

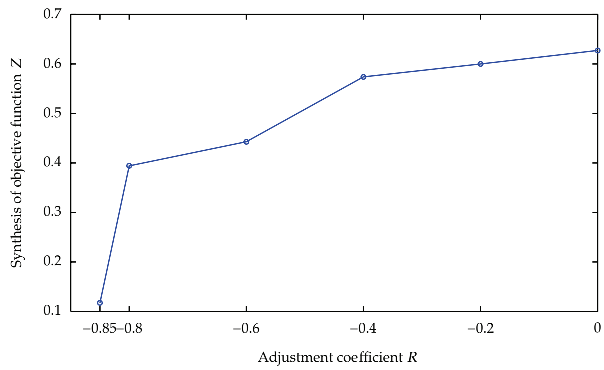

Figure 3: Curve of Z changed with R when operation time is compressed.

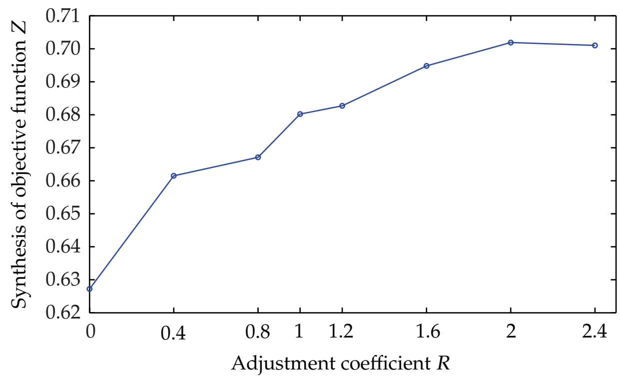

Figure 4: Curve of Z changed with R when operation time is delayed.

of FLSPs are occupied and cannot be used to complete other service orders. For the reasons mentioned, FLSPs are generally unwilling to delay their completion time.

Figure 4 illustrates that Z remains unchanged after $R > 2$, which shows that the time delay has reached the upper limit. Increasing the time delay coefficient is of no use to improve the Z value anymore. In other words, Z has reached the maximum at $R = 2$. It also explains that in practice, the service time cannot be delayed infinitely.

In summary, allowing the FLSPs to delay the order completion time to a certain extent will contribute to LSCC performance. However, constrained by the time connections between the upstream and downstream service processes, the LSSC's performance improvement faces an upper limit.

5.3. The Effect of Relationship Cost Coefficient of LSI on LSSC Scheduling Results

The relationship cost coefficient is introduced into the model solving approach, and the cost of LSI is allowed to increase appropriately. In this part, we explore the effect of the LSI's relationship cost coefficient on LSSC scheduling results to provide a theoretical basis for the LSI's decision-making.

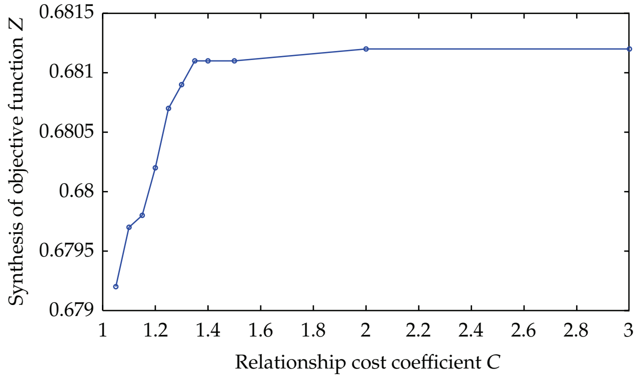

Figure 5: Curve of Z changed with C.

Table 6: The effect of C on Z.

C	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.5	1	2
Z	0.6792	0.6797	0.6798	0.6802	0.6807	0.6809	0.6811	0.6811	0.6811	0.6812	0.6812

The relationship cost coefficient is made by LSI. Its size will have a direct influence on the FLSP's satisfaction and on the overall performance of LSSC. To explore the relationship between the synthesized objective function and the cost relationship coefficient, we assign different values to C, and then obtain the corresponding Z (Table 6 and Figure 5).

Figure 5 shows the curve of Z changed with C. It indicates that along with the increase of C, Z also increases and ultimately tends to stabilize at 0.6812. Along with the increase of the LSI's relationship cost coefficient, the comprehensive performance of LSSC increases as well. However, this improvement is not unlimited. Instead, it remains stable after reaching a certain value due to the mandatory requirement on service time that FLSPs must meet. As mentioned, the service time cannot be delayed or compressed without limit. Thus, when the cost augmentation increases to a certain level, continuing to increase the cost does not contribute to the improvement of the LSSC's comprehensive performance. In the actual time scheduling, LSI may consider using the relationship cost coefficient strategy to improve the comprehensive performance of LSSC, but this improvement is limited.

5.4. Comparison of the Effects of Different Parameters on LSSC Scheduling Results

Following the preceding analysis, LSI can use both the relationship cost coefficient and the time delay coefficient to improve the comprehensive performance of LSSC. Indeed, these two strategies are often used by LSIs in actual time scheduling. To compare the effects of these two strategies, we respectively set $C = 1.2$ and $R = 1.2$ as benchmarks and figure out the proportion of the variations of Z with R and C. The results are shown in Table 7.

Plotting the data in Table 7 into a line chart, we obtain Figure 6. Figure 6 shows the comparison of the performance changes caused by C and R. It clearly shows the following.

(1) The slope of the curve of Z% varied with R% is bigger than that of Z% varied with C%. Therefore, compared with the cost augmentation, time delay is better in improving the comprehensive performance of LSSC.

Table 7: Influence on the comprehensive performance of LSSC caused by time delay and cost increase.

R	Z	$R\%$	$Z\%$
0.8	0.6671	−33.33%	−2.29%
1	0.6802	−16.67%	−0.37%
1.2	**0.6827**	**0.00%**	**0.00%**
1.4	0.6938	16.67%	1.63%
1.6	0.6948	33.33%	1.77%
1.8	0.6972	50.00%	2.12%
2	0.7019	66.67%	2.81%
C	Z	$C\%$	$Z\%$
1.05	0.6792	−12.50%	−0.15%
1.1	0.6797	−8.33%	−0.07%
1.15	0.6798	−4.17%	−0.06%
1.2	**0.6802**	**0.00%**	**0.00%**
1.25	0.6807	4.17%	0.07%
1.3	0.6809	8.33%	0.10%
1.35	0.6811	12.50%	0.13%
1.4	0.6811	16.67%	0.13%
1.5	0.6811	25.00%	0.13%
2	0.6812	66.67%	0.15%

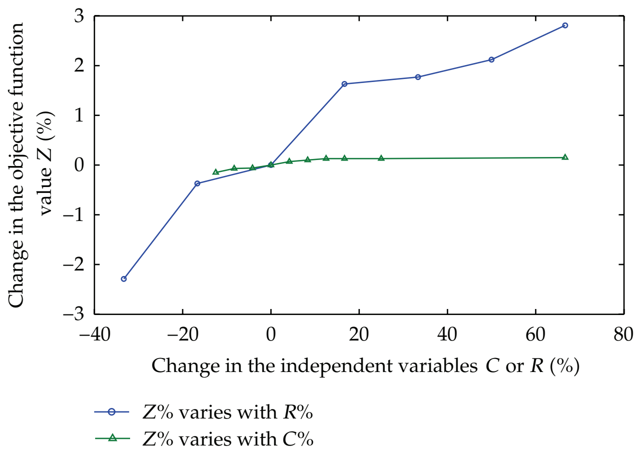

Figure 6: Comparison of the performance changes caused by C and R.

(2) In terms of the curve of $Z\%$ varied with $R\%$, when $R\%$ is less than 0, the slope of the curve is bigger. Thus, if FLSPs operate in the time-compressed state and then the LSI allows some time delay, the improvement in the comprehensive performance of LSSC is much more significant. By contrast, if FLSPs operate in the time delay state, and then the LSI allows more time delay, the improvement in the comprehensive performance of LSSC is not substantial.

6. Main Conclusions and Management Insights

Based on the preceding analysis, the following conclusions can be made.

(1) Along with the increase of R (from negative to positive value), the synthesized objective function Z rises gradually, which shows that the overall satisfaction of

LSSC increases and remains stable upon reaching a certain value with the time delay. Furthermore, time compression decreases the satisfaction level of FLSPs, whereas delaying the completion time increases their satisfaction.

(2) Along with the increase in the LSI's relationship cost coefficient, the comprehensive performance of LSSC increases as well. The effect on improving the comprehensive performance is relatively slow and improvement is limited and remains unchanged upon reaching a certain value.

(3) Both time delay and the LSI's relationship cost can improve the LSSC's comprehensive performance to a certain degree. However, compared with the cost augmentation, time delay is better in improving the comprehensive performance of the LSSC. Based on this result, LSI should address the time scheduling problem reasonably and try to reduce unnecessary time compression requirements to prevent the sharp decline of the LSSC's comprehensive performance. If the comprehensive performance of LSSC has to be improved, LSI should prioritize the use of the time delay strategy.

(4) The results of numerical analysis indicate that the time scheduling model tends to reduce the operation time difference among various FLSPs that are in the same service process and always tries to bring the actual service time close to the expected service time. Thus, it is strongly recommended that the LSI could try its best to choose FLSPs with normal completion times that are close to one another's to minimise the difference in the time of order completion. What the best case scenario is, the normal completion times of those FLSPs are the same as the expected working time. Furthermore, the time scheduling model indicates that with flexible scheduling, we could reach the goals of making the operation time to be compressed or to be delayed in accordance with customer's needs. Time flexibility is a quite important characteristic of flexible supply chain. All the points mentioned previously contribute to make the 3rd-party logistics more reliable to customers, especially the manufacturers. It is helpful to release outsourcing pressure of the manufacturing industry.

(5) Through the model established in this paper, it indicates that it is possible to get an optimized scheduling plan of the FLSPs' operation time if LSI could get the parameters used in the time scheduling model. This model is suitable for the situation which is under the MC background and the CODP has been decided in advance. LSI could use this model to choose a better time delay coefficient and relationship cost coefficient, thus better manage his FLSPs.

7. Research Limitations and Future Work

After reviewing the literature on the LSSC scheduling model to minimize the total cost incurred in the LSSC, minimize the difference between the expected and actual times of completing the service order, and maximize the satisfaction of FLSPs, this paper has established a time scheduling model of LSSC that is constrained by the service order time requirement. Numerical analysis has been conducted by using the Matlab 7.0 software. The effects of the relationship cost coefficient and the time delay coefficient on the LSSC's comprehensive performance have been discussed. Finally, the main conclusions and management insights have been presented.

However, certain limitations remain in this paper. For example, the model solving and analysis are only in accordance with a real numerical example, which cannot represent all actual situations. Besides, we consider the normal service time for a specific order (T_{ij}) is certain, but in some cases, this parameter is stochastic. In the future, we could continue this research considering the stochastic factors. Furthermore, in this paper, it is assumed that the upstream service capability matches the downstream service capability without considering the unmatched cases. In the future, we can build a time scheduling model in which the capability is matched and the time flexibility is considered. This paper also assumes that mutual trust and collaboration have already been established among the participants in LSSC. However, actual scheduling problems indicate asymmetric information, which should be examined in future research.

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