# Applications of Methods of Numerical Linear Algebra in 

 EngineeringGuest Editors: Masoud Hajarian, Feng Ding, and Jein-Shan Chen


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## Editorial

# Applications of Methods of Numerical Linear Algebra in Engineering 

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Methods of numerical linear algebra are concerned with the theory and practical aspects of computing solutions of mathematical problems in engineering such as image and signal processing, telecommunication, data mining, computational finance, bioinformatics, optimization, and partial differential equations. In recent years, applications of methods of numerical linear algebra in engineering have received a lot of attention and a large number of papers have proposed several methods for solving engineering problems. This special issue is devoted to publishing the latest and significant methods of numerical linear algebra for computing solutions of engineering problems.

We received twenty-five papers in the interdisciplinary research fields. This special issue includes ten high quality peer-reviewed articles.

In the following, we briefly review each of the papers that are published.

In the paper entitled "A model based on cocitation for web information retrieval", Y. Xie and T.-Z. Huang propose a new hyperlink weighting scheme to describe the strength of the relevancy between any two webpages.

In the paper entitled "Optimal grasping manipulation for multifingered robots using semismooth Newton method," C.-H. Ko and J.-S. Chen perform the optimal grasping control to find both optimal motion path of the object and minimum grasping forces in the manipulation.

In the paper entitled "(Anti-)Hermitian generalized (anti-) Hamiltonian solution to a system of matrix equations,"
J. Yu et al. propose the necessary and sufficient conditions for the existence of and the expression for the (anti-)Hermitian generalized (anti-)Hamiltonian solutions to the system of matrix equations $A X=B, C X=D$.

In the paper entitled "Dynamic model of a wind turbine for the electric energy generation," J. de J. Rubio et al. introduce a novel dynamic model for the modeling of the wind turbine behavior.

In the paper entitled "The extrapolation-accelerated multilevel aggregation method in PageRank computation," B.Y. Pu et al. present an accelerated multilevel aggregation method for calculating the stationary probability vector of an irreducible stochastic matrix in PageRank computation.

In the paper entitled "A double-parameter GPMHSS method for a class of complex symmetric linear systems from Helmholtz equation," C.-X. Li and S.-L. Wu introduce a double-parameter GPMHSS (DGPMHSS) method for solving a class of complex symmetric linear systems from Helmholtz equation.

In the paper entitled "The explicit identities for spectral norms of circulant-type matrices involving binomial coefficients and harmonic numbers," J. Zhou et al. investigate the explicit formulae of spectral norms for circulant-type matrices.

In the paper entitled "Dynamic stability of Euler beams under axial unsteady wind force," Y.-Q. Huang et al. study the critical frequency equation for simply supported Euler beams with uniform section under arbitrary axial dynamic forces.

In the paper entitled "A modified conjugacy condition and related nonlinear conjugate gradient method," S. Yao et al. present a nonlinear conjugate gradient method that is globally convergent under the strong Wolfe Powell line search for general functions.

In the paper entitled "Spectral properties of the iteration matrix of the HSS method for saddle point problem," Q.-F. Cui et al. discuss spectral properties of the iteration matrix of the HSS method for saddle point problems.

## Acknowledgments

The editors of this special issue would like to express their gratitude to the authors who have submitted manuscripts for consideration. They also thank the many individuals who served as referees of the submitted manuscripts.

# A Double-Parameter GPMHSS Method for a Class of Complex Symmetric Linear Systems from Helmholtz Equation 

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#### Abstract

Based on the preconditioned MHSS (PMHSS) and generalized PMHSS (GPMHSS) methods, a double-parameter GPMHSS (DGPMHSS) method for solving a class of complex symmetric linear systems from Helmholtz equation is presented. A parameter region of the convergence for DGPMHSS method is provided. From practical point of view, we have analyzed and implemented inexact DGPMHSS (IDGPMHSS) iteration, which employs Krylov subspace methods as its inner processes. Numerical examples are reported to confirm the efficiency of the proposed methods.


## 1. Introduction

Let us consider the following form of the Helmholtz equation:

$$
\begin{equation*}
-\Delta u+\sigma_{1} u+i \sigma_{2} u=f \tag{1}
\end{equation*}
$$

where $\sigma_{1}, \sigma_{2}$ are real coefficient functions and $u$ satisfies Dirichlet boundary conditions in $D=[0,1] \times[0,1]$.

Using the finite difference method to discretize the Helmholtz equation (1) with both $\sigma_{1}$ and $\sigma_{2}$ strictly positive leads to the following system of linear equations:

$$
\begin{equation*}
A x=b, \quad A \in \mathbb{C}^{n \times n}, x, b \in \mathbb{C}^{n}, \tag{2}
\end{equation*}
$$

where $A=W+i T \in \mathbb{C}^{n \times n}$ and $i=\sqrt{-1}$ with real symmetric matrices $W, T \in \mathbb{R}^{n \times n}$ satisfying $-W \preceq T \prec W$ under certain conditions. Throughout the paper, for $\forall B, C \in \mathbb{R}^{n \times n}, B \prec C$ means that $B-C$ is symmetric negative definite and $B \preceq C$ means that $B-C$ is symmetric negative semidefinite.

Systems such as (2) are important and arise in a variety of scientific and engineering applications, including structural dynamics [1-3], diffuse optical tomography [4], FFT-based solution of certain time-dependent PDEs [5], lattice quantum chromodynamics [6], molecular dynamics and fluid dynamics [7], quantum chemistry, and eddy current problem [8, $9]$. One can see $[10,11]$ for more examples and additional references.

Based on the specific structure of the coefficient matrix $A$, one can verify that the Hermitian and skew-Hermitian parts of the complex symmetric matrix $A$, respectively, are

$$
\begin{equation*}
H=\frac{1}{2}\left(A+A^{*}\right)=W, \quad S=\frac{1}{2}\left(A-A^{*}\right)=i T . \tag{3}
\end{equation*}
$$

Obviously, the above Hermitian and skew-Hermitian splitting (HSS) of the coefficient matrix $A$ is in line with the real and imaginary parts splitting of the coefficient matrix $A$. Based on the HSS method [12], Bai et al. [2] skillfully designed the modified HSS (MHSS) method to solve the complex symmetric linear system (2). To generalize the concept of this method and accelerate its convergence rate, Bai et al. in [3, 13] designed the preconditioned MHSS (PMHSS) method. It is noted that MHSS and PMHSS methods can efficiently solve the linear system (2) with $W \succ 0$ and $T \succeq 0$.

Recently, Xu [14] proposed the following GPMHSS method for solving the complex symmetric linear systems (2) with $-W \preceq T \prec W$ and it is described in the following.

The GPMHSS Method. Let $x^{(0)} \in \mathbb{C}^{n}$ be an arbitrary initial guess. For $k=0,1,2, \ldots$ until the sequence of iterates $\left\{x^{(k)}\right\}_{k=0}^{\infty}$
converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$
\begin{gather*}
(\alpha V+W-T) x^{(k+1 / 2)}=(\alpha V-i(W+T)) x^{(k)}+(1+i) b \\
(\alpha V+W+T) x^{(k+1)}=(\alpha V+i(W-T)) x^{(k+1 / 2)}+(1-i) b \tag{4}
\end{gather*}
$$

where $\alpha$ is a given positive constant and $V \in \mathbb{R}^{n \times n}$ is a prescribed symmetric positive definite matrix.

Theoretical analysis in [14] shows that the GPMHSS method converges unconditionally to the unique solution of the complex symmetric linear system (2). Numerical experiments are given to show the effectiveness of the GMHSS method.

In this paper, based on the asymmetric HSS and generalized preconditioned HSS methods in [15, 16], a natural generalization for the GMHSS iteration scheme is that we can design a double-parameter GMHSS (DGPMHSS) iteration scheme for solving the complex symmetric linear systems (2) with $-W \preceq T \prec W$. That is to say, the DGPMHSS iterative scheme works as follows.

The DGPMHSS Method. Let $x^{(0)} \in \mathbb{C}^{n}$ be an arbitrary initial guess. For $k=0,1,2, \ldots$ until the sequence of iterates $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$
\begin{align*}
&(\alpha V+W-T) x^{(k+1 / 2)}=(\alpha V-i(W+T)) x^{(k)}+(1+i) b, \\
&(\beta V+W+T) x^{(k+1)}=(\beta V+i(W-T)) x^{(k+1 / 2)} \\
&+(1-i) b, \tag{5}
\end{align*}
$$

where $\alpha$ is a given nonnegative constant, $\beta$ is a given positive constant, and $V \in \mathbb{R}^{n \times n}$ is a prescribed symmetric positive definite matrix.

Just like the GPMHSS method (4), both matrices $\alpha V+$ $W-T$ and $\beta V+W+T$ are symmetric positive definite. Hence, the two linear subsystems in (5) can also be effectively solved either exactly by a sparse Cholesky factorization or inexactly by a preconditioned conjugate gradient scheme. Theoretical analysis shows that the iterative sequence produced by DGPMHSS iteration method converges to the unique solution of the complex symmetric linear systems (2) for a loose restriction on the choices of $\alpha$ and $\beta$. The contraction factor of the DGPMHSS iteration can be bounded by a function, which is dependent only on the choices of $\alpha$ and $\beta$, the smallest eigenvalues of matrices $V^{-1}(W-T)$ and $V^{-1}(W+$ T).

This paper is organized as follows. In Section 2, we study the convergence properties of the DGPMHSS method. In Section 3, we discuss the implementation of DGPMHSS method and the corresponding inexact DGPMHSS (IDPGMHSS) iteration method. Numerical examples are reported to confirm the efficiency of the proposed methods in Section 4. Finally, we end the paper with concluding remarks in Section 5.

## 2. Convergence Analysis for the DGPMHSS Method

In this section, the convergence of the DGPMHSS method is studied. The DGPMHSS iteration method can be generalized to the two-step splitting iteration framework. The following lemma is required to study the convergence rate of the DGPMHSS method.

Lemma 1 (see [13]). Let $A \in \mathbb{C}^{n \times n}, A=M_{i}-N_{i}(i=1,2)$ be two splittings of $A$, and let $x^{(0)} \in \mathbb{C}^{n}$ be a given initial vector. If $\left\{x^{(k)}\right\}$ is a two-step iteration sequence defined by

$$
\begin{gather*}
M_{1} x^{(k+1 / 2)}=N_{1} x^{(k)}+b, \\
M_{2} x^{(k+1)}=N_{2} x^{(k+1 / 2)}+b,  \tag{6}\\
k=0,1, \ldots, \text { then } \\
x^{(k+1)}=M_{2}^{-1} N_{2} M_{1}^{-1} N_{1} x^{(k)}+M_{2}^{-1}\left(I+N_{2} M_{1}^{-1}\right) b,  \tag{7}\\
\\
k=0,1, \ldots
\end{gather*}
$$

Moreover, if the spectral radius $\rho\left(M_{2}^{-1} N_{2} M_{1}^{-1} N_{1}\right)<1$, then the iterative sequence $\left\{x^{(k)}\right\}$ converges to the unique solution $x_{*} \in \mathbb{C}^{n}$ of system (1) for all initial vectors $x^{(0)} \in \mathbb{C}^{n}$.

Applying this lemma to the DGPMHSS method, we get convergence property in the following theorem.

Theorem 2. Let $\widehat{W}=R^{-T}(W-T) R^{-1}, \widehat{T}=R^{-T}(W+T) R^{-1}$, and $V=R^{T} R$, where $R \in \mathbb{R}^{n \times n}$ is a prescribed nonsingular matrix. Let $A=W+i T \in \mathbb{C}^{n \times n}$ be defined in (2), $\alpha$ a nonnegative constant, and $\beta$ a positive constant. Then the iteration matrix $M_{\alpha, \beta}$ of GPMHSS method is

$$
\begin{align*}
M_{\alpha, \beta}= & (\beta V+W+T)^{-1}(\beta V+i(W-T))  \tag{8}\\
& \times(\alpha V+W-T)^{-1}(\alpha V-i(W+T)),
\end{align*}
$$

which satisfies

$$
\begin{equation*}
\rho\left(M_{\alpha, \beta}\right) \leq \max _{\lambda_{i} \in \lambda(\bar{W})} \frac{\sqrt{\beta^{2}+\lambda_{i}^{2}}}{\alpha+\lambda_{i}} \max _{\mu_{i} \in \mu(\widehat{T})} \frac{\sqrt{\alpha^{2}+\mu_{i}^{2}}}{\beta+\mu_{i}}, \tag{9}
\end{equation*}
$$

where $\lambda(\widehat{W})$ and $\mu(\widehat{T})$, respectively, are the spectral sets of the matrices $\widehat{W}$ and $\widehat{T}$.

In addition, let $\mu_{\min }$ and $\lambda_{\min }$, respectively, be the smallest eigenvalues of the matrices $\widehat{W}$ and $\widehat{T}$. If

$$
\begin{equation*}
\sqrt{\alpha^{2}+\mu_{\min }^{2}}-\mu_{\min } \leq \beta<\sqrt{\alpha^{2}+2 \alpha \lambda_{\min }} \tag{10}
\end{equation*}
$$

then the DGPMHSS iteration (5) converges to the unique solution of the linear system (2).

Proof. Let

$$
\begin{array}{ll}
M_{1}=\alpha V+W-T, & N_{1}=\alpha V-i(W+T) \\
M_{2}=\beta V+W+T, & N_{2}=\beta V+i(W-T) \tag{11}
\end{array}
$$

Obviously, $\alpha V+W-T$ and $\beta V+W+T$ are nonsingular for any nonnegative constants $\alpha$ or positive constants $\beta$. So formula (8) is valid.

The iteration matrix $M_{\alpha, \beta}$ is similar to

$$
\begin{align*}
\widehat{M}_{\alpha, \beta}= & (\beta V+i(W-T))(\alpha V+W-T)^{-1} \\
& \times(\alpha V-i(W+T))(\beta V+W+T)^{-1} \tag{12}
\end{align*}
$$

That is,

$$
\begin{align*}
\widehat{M}_{\alpha, \beta}= & R^{T}\left(\beta I+i R^{-T}(W-T) R^{-1}\right) \\
& \times R R^{-1}\left(\alpha I+R^{-T}(W-T) R^{-1}\right)^{-1} R^{-T} \\
& \times R^{T}\left(\alpha I-i R^{-T}(W+T) R^{-1}\right) \\
& \times R R^{-1}\left(\beta I+R^{-T}(W+T) R^{-1}\right)^{-1} R^{-T} \\
= & R^{T}\left(\beta I+i R^{-T}(W-T) R^{-1}\right) \\
& \times\left(\alpha I+R^{-T}(W-T) R^{-1}\right)^{-1}  \tag{13}\\
& \times\left(\alpha I-i R^{-T}(W+T) R^{-1}\right) \\
& \times\left(\beta I+R^{-T}(W+T) R^{-1}\right)^{-1} R^{-T} \\
= & R^{T}(\beta I+i \widehat{W})(\alpha I+\widehat{W})^{-1} \\
& \times(\alpha I-i \widehat{T})(\beta I+\widehat{T})^{-1} R^{-T} .
\end{align*}
$$

Therefore, we have

$$
\begin{align*}
& \rho\left(M_{\alpha, \beta}\right) \\
&= \rho\left(\widehat{M}_{\alpha, \beta}\right) \\
&= \rho\left(R^{T}(\beta I+i \widehat{W})(\alpha I+\widehat{W})^{-1}\right. \\
&\left.\quad \times(\alpha I-i \widehat{T})(\beta I+\widehat{T})^{-1} R^{-T}\right)  \tag{14}\\
&= \rho\left((\beta I+i \widehat{W})(\alpha I+\widehat{W})^{-1}(\alpha I-i \widehat{T})(\beta I+\widehat{T})^{-1}\right) \\
& \leq\left\|(\beta I+i \widehat{W})(\alpha I+\widehat{W})^{-1}\right\|_{2} \\
& \quad \times\left\|(\alpha I-i \widehat{T})(\beta I+\widehat{T})^{-1}\right\|_{2} .
\end{align*}
$$

Since $W \in \mathbb{R}^{n \times n}$ and $T \in \mathbb{R}^{n \times n}$, respectively, are symmetric positive definite and symmetric positive semidefinite and $R$ is a nonsingular matrix, $\widehat{W}$ and $\widehat{T}$ are symmetric positive definite and symmetric positive semidefinite, respectively. Therefore, there exist orthogonal matrices $S_{1}, S_{2} \in \mathbb{R}^{n \times n}$ such that

$$
\begin{equation*}
S_{1}^{T} \widehat{W} S_{1}=\Lambda_{\widehat{W}}, \quad S_{2}^{T} \widehat{T} S_{2}=\Lambda_{\widehat{T}} \tag{15}
\end{equation*}
$$

where $\Lambda_{\widehat{W}}=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right)$ and $\Lambda_{\widehat{T}}=$ $\operatorname{diag}\left(\mu_{1}, \mu_{2}, \ldots, \mu_{n}\right)$ with $\lambda_{i}>0(1 \leq i \leq n)$ and
$\mu_{i} \geq 0(1 \leq i \leq n)$ being the eigenvalues of the matrices $\widehat{W}$ and $\widehat{T}$, respectively.

Through simple calculations, we can get that

$$
\begin{align*}
\rho\left(M_{\alpha, \beta}\right) & \leq \max _{\lambda_{i} \in \lambda(\widehat{W})}\left|\frac{\beta+i \lambda_{i}}{\alpha+\lambda_{i}}\right| \max _{\mu_{i} \in \mu(\hat{T})}\left|\frac{\alpha-i \mu_{i}}{\beta+\mu_{i}}\right| \\
& =\max _{\lambda_{i} \in \lambda(\bar{W})} \frac{\sqrt{\beta^{2}+\lambda_{i}^{2}}}{\alpha+\lambda_{i}} \max _{\mu_{i} \in \mu(\hat{T})} \frac{\sqrt{\alpha^{2}+\mu_{i}^{2}}}{\beta+\mu_{i}}, \tag{16}
\end{align*}
$$

which gives the upper bound for $\rho\left(M_{\alpha, \beta}\right)$ in (9). The next proof is similar to that of Theorem 1 in [17]. Here it is omitted.

The approach to minimize the upper bound is very important in theoretical viewpoint. However, it is not practical since the corresponding spectral radius of the iteration matrix $M_{\alpha, \beta}$ in (9) is not optimal. How to choose the suitable preconditioners and parameters for practical problem is still a great challenge.

## 3. The IDGPMHSS Iteration

In the DGPMHSS method, it is required to solve two systems of linear equations whose coefficient matrices are $\alpha V+W-T$ and $\beta V+W+T$, respectively. However, this may be very costly and impractical in actual implementation. To overcome this disadvantage and improve the computational efficiency of the DGPMHSS iteration method, we propose to solve the two subproblems iteratively $[12,18]$, which leads to IDGPMHSS iteration scheme. Its convergence can be shown in a similar way to that of the IHSS iteration method, using Theorem 3.1 of [12]. Since $\alpha V+W-T$ and $\beta V+W+T$ are symmetric positive definite, some Krylov subspace methods (such as CG) can be employed to gain its solution easily. Of course, if good preconditioners for matrices $\alpha V+W-T$ and $\beta V+W+T$ are available, we can use the preconditioned conjugate gradient (PCG) method instead of CG for the two inner systems that yields a better performance of IDGPMHSS method. If either $\alpha V+W-T$ or $\beta V+W+T$ (or both) is Toeplitz, we can use fast algorithms for solution of the corresponding subsystems [19].

## 4. Numerical Examples

In this section, we give some numerical examples to demonstrate the performance of the DGPMHSS and IDGPMHSS methods for solving the linear system (2). Numerical comparisons with the GPMHSS method are also presented to show the advantage of the DGPMHSS method.

In our implementation, the initial guess is chosen to be $x^{(0)}=0$ and the stopping criteria for outer iterations is

$$
\begin{equation*}
\frac{\left\|b-A x^{(k)}\right\|_{2}}{\|b\|_{2}} \leq 10^{-6} \tag{17}
\end{equation*}
$$

The preconditioner $V$ used in GPMHSS method is chosen to be $V=W-T$. For the sake of comparing, the corresponding

Table 1: The experimentally optimal parameters and the corresponding spectral radii for the iteration matrices of GPMHSS and DGPMHSS with $V=W-T$ and $m=8$.

|  | $\sigma_{2}$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | 10 | 50 | 80 | 100 |
| GPMHSS |  |  |  |  |
| $\alpha_{G}^{*}$ | 1.1 | 1.5 | 2.2 | 2 |
| $\rho\left(M_{\alpha_{G}^{*}}\right)$ | 0.5009 | 0.5222 | 0.5667 | 0.6274 |
| DGPMHSS |  |  |  |  |
| $\alpha_{D}^{*}$ | 1.1 | 1.5 | 2.2 | 2 |
| $\beta_{D}^{*}$ | 1 | 0.9 | 0.8 | 0.8 |
| $\rho\left(M_{\alpha_{D}^{*}, \beta_{D}^{*}}\right)$ | 0.5001 | 0.4986 | 0.5012 | 0.4901 |

preconditioner $V$ used in DGPMHSS method is chosen to be $V=W-T$. Since the numerical results in $[2,3]$ show that the PMHSS iteration method outperforms the MHSS and HSS iteration methods when they are employed as preconditioners for the GMRES method or its restarted variants [20], we just examine the efficiency of DGPMHSS iteration method as a solver for solving complex symmetric linear system (2) by comparing the iteration numbers (denoted as IT) and CPU times (in seconds, denoted as CPU(s)) of DGPMHSS method with GPMHSS method.

Example 3 (see [5, 21-23]). Consider the following form of the Helmholtz equation:

$$
\begin{equation*}
-\Delta u+\sigma_{1} u+i \sigma_{2} u=f \tag{18}
\end{equation*}
$$

where $\sigma_{1}, \sigma_{2}$ are real coefficient functions and $u$ satisfies Dirichlet boundary conditions in $D=[0,1] \times[0,1]$. The above equation describes the propagation of damped time harmonic waves. We take $H$ to be the five-point centered difference matrix approximating the negative Laplacian operator on a uniform mesh with mesh size $h=1 /(m+1)$. The matrix $H \in \mathbb{R}^{n \times n}$ possesses the tensor-product form $H=B_{m} \otimes I+I \otimes B_{m}$ with $B_{m}=h^{-2} \cdot \operatorname{tridiag}(-1,2,-1) \in \mathbb{R}^{m \times m}$. Hence, $H$ is an $n \times n$ block-tridiagonal matrix, with $n=m^{2}$. This leads to the complex symmetric linear system (2) of the form

$$
\begin{equation*}
\left[\left(H+\sigma_{1} I\right)+i \sigma_{2} I\right] x=b \tag{19}
\end{equation*}
$$

In addition, we set $\sigma_{1}=100$ and the right-hand side vector $b$ to be $b=(1+i) A \mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1 . As before, we normalize the system by multiplying both sides by $h^{2}$.

As is known, the spectral radius of the iteration matrix may be decisive for the convergence of the iteration method. The spectral radius corresponding to the iteration method is necessary to consider. The comparisons of the spectral radius of the two different iteration matrices derived by GPMHSS and DGPMHSS methods with different mesh size are performed in Tables 1, 2, 3, and 4. In Tables 1-4, we used the optimal values of the parameters $\alpha$ and $\beta$, denoted by $\alpha_{G}^{*}$

Table 2: The experimentally optimal parameters and the corresponding spectral radii for the iteration matrices of GPMHSS and DGPMHSS with $V=W-T$ and $m=16$.

|  | $\sigma_{2}$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 10 | 50 | 80 | 100 |  |  |
| GPMHSS |  |  |  |  |  |  |
| $\alpha_{G}^{*}$ | 1.1 | 1.5 | 2.2 | 1.8 |  |  |
| $\rho\left(M_{\alpha_{G}^{*}}\right)$ | 0.5010 | 0.5230 | 0.5697 | 0.6337 |  |  |
| DGPMHSS |  |  |  |  |  |  |
| $\alpha_{D}^{*}$ | 1.1 | 1.5 | 2.2 | 1.8 |  |  |
| $\beta_{D}^{*}$ | 1 | 1 | 0.9 | 0.9 |  |  |
| $\rho\left(M_{\alpha_{D}^{*}, \beta_{D}^{*}}\right)$ | 0.5004 | 0.5063 | 0.5235 | 0.5097 |  |  |

Table 3: The experimentally optimal parameters and the corresponding spectral radii for the iteration matrices of GPMHSS and DGPMHSS with $V=W-T$ and $m=24$.

|  | $\sigma_{2}$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | 10 | 50 | 80 | 100 |
| GPMHSS |  |  |  |  |
| $\alpha_{G}^{*}$ | 1.1 | 1.5 | 2.2 | 1.6 |
| $\rho\left(M_{\alpha_{G}^{*}}\right)$ | 0.5011 | 0.5234 | 0.5708 | 0.6360 |
| DGPMHSS |  |  |  |  |
| $\alpha_{D}^{*}$ | 1.1 | 1.5 | 2.2 | 1.8 |
| $\beta_{D}^{*}$ | 1 | 1 | 1 | 1 |
| $\rho\left(M_{\alpha_{D}^{*}, \beta_{D}^{*}}\right)$ | 0.5005 | 0.5081 | 0.5289 | 0.5091 |

Table 4: The experimentally optimal parameters and the corresponding spectral radii for the iteration matrices of GPMHSS and DGPMHSS with $V=W-T$ and $m=32$.

|  |  | $\sigma_{2}$ |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | 10 | 50 | 80 | 100 |
| GPMHSS |  |  |  |  |
| $\alpha_{G}^{*}$ | 1.1 | 1.5 | 2.2 | 1.7 |
| $\rho\left(M_{\alpha_{G}^{*}}\right)$ | 0.5011 | 0.5235 | 0.5714 | 0.6372 |
| DGPMHSS |  |  |  |  |
| $\alpha_{D}^{*}$ | 1.1 | 1.5 | 2.2 | 1.8 |
| $\beta_{D}^{*}$ | 1 | 1 | 1 | 1 |
| $\rho\left(M_{\alpha_{D}^{*}, \beta_{D}^{*}}^{*}\right)$ | 0.5005 | 0.5089 | 0.5310 | 0.5139 |

for GPMHSS method, and $\alpha_{D}^{*}, \beta_{D}^{*}$ for DGPMHSS method. These parameters are obtained experimentally with the least spectral radius for the iteration matrices of the two methods.

From Tables 1-4, one can see that with the mesh size creasing, the trend of the experimentally optimal parameter of the GPMHSS and DGPMHSS methods is relatively stable. In Tables 1-4, we observe that the optimal spectral radius of DGPMHSS method is still smaller than those of the GPMHSS method, which implies that the DGPMHSS method may outperform the GPMHSS method. To this end, we need

Table 5: RES, CPU(s), and IT for PMHSS and GPMHSS with $V=$ $W-T$ and $m=8$.

|  | $\sigma_{2}$ |  |  |  |
| :--- | :---: | :---: | :---: | :---: |
|  | 10 | 50 | 80 | 100 |
| GPMHSS |  |  |  |  |
| RES | $9.818 e-7$ | $9.8917 e-7$ | $9.1469 e-7$ | $6.306 e-7$ |
| CPU(s) | 0.015 | 0.016 | 0.016 | 0.016 |
| IT | 20 | 21 | 24 | 30 |
| DGPMHSS |  |  |  |  |
| RES | $9.0916 e-7$ | $7.0211 e-7$ | $4.9072 e-7$ | $6.5183 e-7$ |
| CPU(s) | 0.015 | 0.015 | 0.015 | 0.013 |
| IT | 20 | 19 | 18 | 17 |

Table 6: RES, CPU(s), and IT for PMHSS and GPMHSS with $V=$ $W-T$ and $m=16$.

|  |  | $\sigma_{2}$ |  |  |
| :--- | :---: | :---: | :---: | :---: |
|  | 10 | 50 | 80 | 100 |
| GPMHSS |  |  |  |  |
| RES | $9.8506 e-7$ | $9.4015 e-7$ | $8.9127 e-7$ | $9.4832 e-7$ |
| CPU(s) | 0.031 | 0.046 | 0.046 | 0.063 |
| IT | 20 | 21 | 24 | 29 |
| DGPMHSS |  |  |  |  |
| RES | $9.3425 e-7$ | $7.2018 e-7$ | $7.4847 e-7$ | $8.9905 e-7$ |
| CPU(s) | 0.031 | 0.032 | 0.031 | 0.031 |
| IT | 20 | 20 | 20 | 19 |

Table 7: RES, CPU(s), and IT for PMHSS and GPMHSS with $V=$ $W-T$ and $m=24$.

|  | $\sigma_{2}$ |  |  |  |
| :--- | :---: | :---: | :---: | :---: |
|  | 10 | 50 | 80 | 100 |
| GPMHSS |  |  |  |  |
| RES | $9.8907 e-7$ | $9.6059 e-7$ | $9.7633 e-7$ | $7.5427 e-7$ |
| CPU(s) | 0.094 | 0.094 | 0.11 | 0.125 |
| IT | 20 | 21 | 24 | 29 |
| DGPMHSS |  |  |  |  |
| RES | $9.5055 e-7$ | $9.5856 e-7$ | $7.8461 e-7$ | $7.6081 e-7$ |
| CPU(s) | 0.078 | 0.078 | 0.094 | 0.094 |
| IT | 20 | 20 | 21 | 20 |

to examine the efficiency of the GPMHSS and DGPMHSS methods for solving the systems of linear equations $A x=b$, where $A$ is described.

In Tables 5, 6, 7, and 8, we list the numbers of iteration steps and the computational times for GPMHSS and DGPMHSS iteration methods using the optimal parameters in Tables 1-4. In Tables 5-8, "RES" denotes the relative residual error.

From Tables 5-8, we see that the DGPMHSS method is the best among the two methods in terms of number of iteration steps and computational time. For the GPMHSS and DGPMHSS methods, the CPU's time grows with the problem size whereas the presented results in Tables 5-8 show that in all cases the DGPMHSS method is superior to the GPMHSS

Table 8: RES, CPU(s), and IT for PMHSS and GPMHSS with $V=$ $W-T$ and $m=32$.

|  |  | $\sigma_{2}$ |  |  |
| :--- | :---: | :---: | :---: | :---: |
|  | 10 | 50 | 80 | 100 |
| GPMHSS |  |  |  |  |
| RES | $9.9181 e-7$ | $9.8976 e-7$ | $6.0829 e-7$ | $9.1273 e-7$ |
| CPU(s) | 0.172 | 0.172 | 0.203 | 0.235 |
| IT | 20 | 21 | 25 | 28 |
| DGPMHSS |  |  |  |  |
| RES | $9.5956 e-7$ | $5.5697 e-7$ | $5.6167 e-7$ | $5.5736 e-7$ |
| CPU(s) | 0.156 | 0.156 | 0.187 | 0.172 |
| IT | 20 | 20 | 22 | 21 |

method. That is to say, under certain conditions, compared with the GPMHSS method, the DGPMHSS method may be given the priority for solving the complex symmetric linear system $(W+i T) x=b$ with $-W \preceq T \prec W$.

As already noted, in the two-half steps of the DGPMHSS iteration, there is a need to solve two systems of linear equations, whose coefficient matrices are $\alpha V+W-T$ and $\beta V+$ $W+T$, respectively. This can be very costly and impractical in actual implementation. We use the IDGPMHSS method to solve the systems of linear equations (2) in the actual implementation. That is, it is necessary to solve two systems of linear equations with $\alpha V+W-T$ and $\beta V+W+T$ by using the IDGPMHSS iteration. It is easy to know that $\alpha V+W-T$ and $\beta V+W+T$ are symmetric and positive definite. So, solving the above two subsystems, the CG method can be employed.

In our computations, the inner CG iteration is terminated if the current residual of the inner iterations satisfies

$$
\begin{equation*}
\frac{\left\|p^{(j)}\right\|_{2}}{\left\|r^{(k)}\right\|_{2}} \leq 0.1 \tau^{(k)}, \quad \frac{\left\|q^{(j)}\right\|_{2}}{\left\|r^{(k)}\right\|_{2}} \leq 0.1 \tau^{(k)} \tag{20}
\end{equation*}
$$

where $p^{(j)}$ and $q^{(j)}$ are, respectively, the residuals of the $j$ th inner CG for $\alpha V+W-T$ and $\beta V+W+T . r^{(k)}$ is the $k$ th outer IDGPMHSS iteration; $\tau$ is a tolerance.

Some results are listed in Tables 9 and 10 for $m=24$ and 32, which are the numbers of outer IDGPMHSS iteration (it.s), the average numbers (avg1) of inner CG iteration for $\alpha V+W-T$ and the average numbers (avg2) of CG iteration for $\beta V+W+T$.

In our numerical computations, it is easy to find the fact that the choice of $\tau$ is important to the convergence rate of the IDGPMHSS method. According to Tables 9 and 10, the iteration numbers of the IDGPMHSS method generally increase when $\tau$ decreases. Meanwhile, the iteration numbers of the IDGPMHSS method generally increase when $\sigma_{2}$ increases.

## 5. Conclusion

In this paper, we have generalized the GPMHSS method to the DGPMHSS method for a class of complex symmetric linear systems $(W+i T) x=b$ with real symmetric matrices $W, T \in \mathbb{R}^{n \times n}$ satisfying $-W \preceq T \prec W$ under certain conditions. Theoretical analysis shows that for any initial guess

Table 9: Convergence results for the IDGPMHSS iteration with $m=24$.

| $\left(\alpha_{G}^{*}, \beta_{G}^{*}\right)$ | $\sigma_{2}$ | it.s | $\tau=0.95$ |  |  |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| avg1 | avg2 | it.s | avg1 | avg2 | it.s | avg1 |  |  |  |  |
| $(1.1,1)$ | 10 | 51 | 20.9 | 18.2 | 68 | 26.1 | 23.7 | 84 | 30.4 |  |
| $(1.5,1)$ | 50 | 54 | 23.5 | 18.3 | 70 | 28.0 | 23.9 | 86 | 33.0 | 28.3 |
| $(2.2,1)$ | 80 | 57 | 25.0 | 20.1 | 72 | 30.0 | 25.3 | 88 | 34.2 | 30.4 |
| $(1.8,1)$ | 100 | 58 | 27.5 | 18.9 | 74 | 31.9 | 24.3 | 89 | 36.1 | 28.7 |

TABLE 10: Convergence results for the IDGPMHSS iteration with $m=32$.

| $\left(\alpha_{G}^{*}, \beta_{G}^{*}\right)\left(\alpha_{G}^{*}, \beta_{G}^{*}\right)$ | $\sigma_{2}$ | it.s | $\tau=0.95$ <br> avg1 | avg2 | it.s | $\tau=0.9$ <br> avg1 | avg2 | it.s | avg1 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(1.1,1)$ | 10 | 66 | 27.1 | 23.6 | 90 | 34.2 | 31.1 | 111 | 40.0 |
| $(1.5,1)$ | 50 | 70 | 30.6 | 23.7 | 93 | 37.5 | 31.3 | 114 | 43.4 |
| $(2.2,1)$ | 80 | 75 | 35.0 | 24.2 | 97 | 41.1 | 31.8 | 118 | 47.0 |
| $(1.8,1)$ | 100 | 77 | 36.1 | 25.3 | 98 | 42.2 | 32.4 | 119 | 48.0 |

the DGPMHSS method converges to the unique solution of the linear system for a wide range of the parameters. Then, an inexact version has been presented and implemented for saving the computational cost. Numerical experiments show that DGPMHSS method and IDGPMHSS method are efficient and competitive.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Spectral Properties of the Iteration Matrix of the HSS Method for Saddle Point Problem 

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We discuss spectral properties of the iteration matrix of the HSS method for saddle point problems and derive estimates for the region containing both the nonreal and real eigenvalues of the iteration matrix of the HSS method for saddle point problems.

## 1. Introduction

Consider the following saddle point problem:

$$
\mathscr{A} x \equiv\left[\begin{array}{cc}
A & B^{T}  \tag{1}\\
-B & 0
\end{array}\right]\left[\begin{array}{c}
u \\
v
\end{array}\right]=\left[\begin{array}{c}
f \\
-g
\end{array}\right] \equiv b
$$

with symmetric positive definite $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times n}$ with rank $(B)=m \leq n$. Without loss of generality, we assume that the coefficient matrix of (1) is nonsingular so that (1) has a unique solution. Systems of the form (1) arise in a variety of scientific and engineering applications, such as linear elasticity, fluid dynamics, electromagnetics, and constrained quadratic programming. One can see [1] for more applications and numerical solution techniques of (1).

Recently, based on the Hermitian and skew-Hermitian splitting of $\mathscr{A}: \mathscr{A}=H+S$, where

$$
H=\left[\begin{array}{cc}
A & 0  \tag{2}\\
0 & 0
\end{array}\right], \quad S=\left[\begin{array}{cc}
0 & B^{T} \\
-B & 0
\end{array}\right]
$$

the HSS method [2] has been extended by Benzi and Golub [3] to solve the saddle point problem (1) and it is as follows.

The HSS Method. Let $x^{(0)} \in \mathbb{C}^{n}$ be an arbitrary initial guess. For $k=0,1,2, \ldots$ until the sequence of iterates $\left\{x^{(k)}\right\}_{k=0}^{\infty}$
converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$
\begin{gather*}
(\alpha I+H) x^{(k+(1 / 2))}=(\alpha I-S) x^{(k)}+b \\
(\alpha I+S) x^{(k+1)}=(\alpha I-H) x^{(k+(1 / 2))}+b \tag{3}
\end{gather*}
$$

where $\alpha$ is a given positive constant.
By eliminating the intermediate vector $x^{(k+(1 / 2))}$, we obtain the following iteration in fixed point form as

$$
\begin{equation*}
x^{(k+1)}=M_{\alpha} x^{(k)}+N_{\alpha} b, \quad k=0,1,2, \ldots \tag{4}
\end{equation*}
$$

where $M_{\alpha}=(\alpha I+S)^{-1}(\alpha I-H)(\alpha I+H)^{-1}(\alpha I-S)$ and $N_{\alpha}=$ $2 \alpha(\alpha I+S)^{-1}(\alpha I+H)^{-1}$. Obviously, $M_{\alpha}$ is the iteration matrix of the HSS iteration method.

In addition, if we introduce matrices

$$
\begin{equation*}
B_{\alpha}=\frac{1}{2 \alpha}(\alpha I+H)(\alpha I+S), \quad C_{\alpha}=\frac{1}{2 \alpha}(\alpha I-H)(\alpha I-S) \tag{5}
\end{equation*}
$$

then $\mathscr{A}=B_{\alpha}-C_{\alpha}$ and $M_{\alpha}=B_{\alpha}^{-1} C_{\alpha}$. Therefore, one can readily verify that the HSS method is also induced by the matrix splitting $\mathscr{A}=B_{\alpha}-C_{\alpha}$.

The following theorem established in [3] describes the convergence property of the HSS method.

Theorem 1 (see [3]). Consider problem (1) and assume that $A$ is positive real and $B$ has full rank. Then the iteration (3) is unconditionally convergent; that is, $\rho\left(M_{\alpha}\right)<1$ for all $\alpha>0$.

In fact, one can see [4] for a comprehensive survey on the HSS method. As is known, the iteration method (3) converges to the unique solution of the linear system (1) if and only if the spectral radius $\rho\left(M_{\alpha}\right)$ of the iteration matrix $M_{\alpha}$ is less than 1. The spectral radius of the iteration matrix is decisive for the convergence and stability, and the smaller it is, the faster the iteration method converges when the spectral radius is less than 1. In this paper, we will discuss the spectral properties of the iteration matrix $M_{\alpha}$ of the HSS method for saddle point problems and derive estimates for the region containing both the nonreal and real eigenvalues of the iteration matrix $M_{\alpha}$ of the HSS method for saddle point problems.

Throughout the paper, $B^{T}$ denotes the transpose of a matrix $B$ and $u^{*}$ indicates its transposed conjugate. $\lambda_{n}, \lambda_{1} \geq 0$ are the smallest and largest eigenvalues of symmetric positive semidefinite $A$, respectively. We denote by $\sigma_{1}, \ldots, \sigma_{m}$ the decreasing ordered singular values of $B . \Re(\theta)$ and $\mathfrak{J}(\theta)$, respectively, denote the real part and imaginary part of $\theta \in \mathbb{C}$.

## 2. Main Results

In fact, the iteration matrix $M_{\alpha}$ can be written as

$$
\begin{equation*}
M_{\alpha}^{\prime}=(\alpha I+S)^{-1}(\alpha I+H)^{-1}(\alpha I-H)(\alpha I-S) . \tag{6}
\end{equation*}
$$

Therefore, we are just thinking about the spectral properties of matrix $M_{\alpha}^{\prime}$. That is, we consider the following eigenvalue problem:

$$
\begin{equation*}
(\alpha I-H)(\alpha I-S) x=\lambda(\alpha I+H)(\alpha I+S) x, \tag{7}
\end{equation*}
$$

where $(\lambda, x)$ is any eigenpair of $M_{\alpha}^{\prime}$. From (7), we have

$$
\begin{equation*}
(1-\lambda)\left(\alpha^{2} I+H S\right) x=(1+\lambda) \alpha \mathscr{A} x \tag{8}
\end{equation*}
$$

Note that $\rho\left(M_{\alpha}\right)<1$ for all $\alpha>0$. From (8), we have

$$
\begin{equation*}
\mathscr{A} x=\frac{(1-\lambda) \alpha}{(1+\lambda)}\left(I+\frac{1}{\alpha^{2}} H S\right) x \tag{9}
\end{equation*}
$$

Let

$$
\begin{equation*}
\theta=\frac{(1-\lambda) \alpha}{(1+\lambda)}, \quad \text { from which } \lambda=\frac{\alpha-\theta}{\alpha+\theta}=1-\frac{2 \theta}{\alpha+\theta} \tag{10}
\end{equation*}
$$

Obviously, $\theta \neq 0$. Therefore, (9) can be written as

$$
\begin{equation*}
(H+S) x=\theta\left(I+\frac{1}{\alpha^{2}} H S\right) x \tag{11}
\end{equation*}
$$

That is,

$$
\left[\begin{array}{cc}
A & B^{T}  \tag{12}\\
-B & 0
\end{array}\right] x=\theta\left[\begin{array}{cc}
I & \frac{1}{\alpha^{2}} A B^{T} \\
0 & I
\end{array}\right] x
$$

which is equal to

$$
\mathscr{B} x \equiv\left[\begin{array}{cc}
A+\frac{1}{\alpha^{2}} A B^{T} B & B^{T}  \tag{13}\\
-B & 0
\end{array}\right] x=\theta x .
$$

It is easy to see that the two eigenproblems (7) and (13) have the same eigenvectors, while the eigenvalues are related by (10). Obviously, if the spectrum of $\mathscr{B}$ can be obtained, then the spectrum of (7) can be also derived.

From [5, Lemma 2.1], we have the following result.
Lemma 2. Assume that $A$ is symmetric and positive definite and $K=I+\left(1 / \alpha^{2}\right) B^{T} B$. For each eigenpair $(\lambda, x)$ of $(7)$, all the eigenvalues $\lambda$ of the iteration matrix $M_{\alpha}$ are $\lambda=(\alpha-\theta) /(\alpha+\theta)$, where $\theta \neq 0$ satisfies the following.
(1) If $\mathfrak{J}(\theta) \neq 0$, then

$$
\begin{equation*}
\Re(\theta)=\frac{1}{2} \frac{u^{*} K A K u}{u^{*} K u}, \quad|\theta|^{2}=\frac{u^{*} K B^{T} B u}{u^{*} K u} . \tag{14}
\end{equation*}
$$

(2) If $\mathfrak{J}(\theta)=0$, then

$$
\begin{equation*}
\min \left\{\lambda_{n}, \frac{\alpha^{2} \sigma_{m}^{2}}{\lambda_{1}\left(\alpha^{2}+\sigma_{m}^{2}\right)}\right\} \leq \theta \leq \lambda_{1}\left(1+\frac{\sigma_{1}^{2}}{\alpha^{2}}\right) . \tag{15}
\end{equation*}
$$

From (14), it is easy to verify that $0 \leq|\theta|^{2} \leq \sigma_{1}^{2}$, and if $m=n$, then $\lambda_{n}\left(1+\left(\sigma_{m}^{2} / \alpha^{2}\right)\right) \leq 2 \mathfrak{R}(\theta) \leq \lambda_{1}\left(1+\left(\sigma_{1}^{2} / \alpha^{2}\right)\right)$, or if $m<n$, then $\lambda_{n} \leq 2 \mathfrak{R}(\theta) \leq \lambda_{1}\left(1+\left(\sigma_{1}^{2} / \alpha^{2}\right)\right)$.

In the sequel, we will present the main result, that is, Theorem 3.

Theorem 3. Under the hypotheses and notation of Lemma 2, all the eigenvalues $\lambda$ of the iteration matrix $M_{\alpha}$ are such that the following hold.
(1) If $\Im(\theta) \neq 0$, then

$$
\begin{equation*}
|\lambda|^{2} \leq 1-2 \frac{\lambda_{n}\left(1+\left(\sigma_{m}^{2} / \alpha^{2}\right)\right) \alpha}{\alpha^{2}+\lambda_{n}\left(1+\left(\sigma_{m}^{2} / \alpha^{2}\right)\right) \alpha+\sigma_{1}^{2}} . \tag{16}
\end{equation*}
$$

(2) If $\mathfrak{J}(\theta)=0$, then

$$
\begin{align*}
& \frac{\alpha-\lambda_{1}\left(1+\left(\sigma_{1}^{2} / \alpha^{2}\right)\right)}{\alpha+\lambda_{1}\left(1+\left(\sigma_{1}^{2} / \alpha^{2}\right)\right)} \\
& \quad \leq \lambda \leq \frac{\alpha-\min \left\{\lambda_{n}, \alpha^{2} \sigma_{m}^{2} / \lambda_{1}\left(\alpha^{2}+\sigma_{m}^{2}\right)\right\}}{\alpha+\min \left\{\lambda_{n}, \alpha^{2} \sigma_{m}^{2} / \lambda_{1}\left(\alpha^{2}+\sigma_{m}^{2}\right)\right\}} \tag{17}
\end{align*}
$$

Proof. Let $x=\left[u^{*}, v^{*}\right]^{*}$ be an eigenvector with respect to $\theta$. From (13), we get

$$
\begin{gather*}
A K u+B^{T} v=\theta u,  \tag{18}\\
-B u=\theta v . \tag{19}
\end{gather*}
$$

By (19), we get $v=-\theta^{-1} B u$. Substituting it into (18) yields

$$
\begin{equation*}
\theta A K u-B^{T} B u=\theta^{2} u . \tag{20}
\end{equation*}
$$

Multiplying (20) from the left by $u^{*} K$, we arrive at

$$
\begin{equation*}
\theta^{2} u^{*} K u-\theta u^{*} K A K u+u^{*} K B^{T} B u=0 . \tag{21}
\end{equation*}
$$

Let $\theta=\mathfrak{R}(\theta)+i \mathfrak{F}(\theta)$. For symmetric matrix $A$, the quadratic equation (21) has real coefficients so that its roots are given by

$$
\begin{equation*}
\theta_{ \pm}=\frac{1}{2} \frac{u^{*} K A K u}{u^{*} K u} \pm \sqrt{\frac{1}{4}\left(\frac{u^{*} K A K u}{u^{*} K u}\right)^{2}-\frac{u^{*} K B^{T} B u}{u^{*} K u}} . \tag{22}
\end{equation*}
$$

Eigenvalues with nonzero imaginary part arise if the discriminant is negative.

If $\mathfrak{J}(\theta) \neq 0$, from (14) we have

$$
\begin{equation*}
2 \Re(\theta)=\frac{u^{*} K A K u}{u^{*} K u}>0, \quad|\theta|^{2}=\frac{u^{*} K B^{T} B u}{u^{*} K u} . \tag{23}
\end{equation*}
$$

In this case, from (23) we have

$$
\begin{align*}
|\lambda|^{2} & =\frac{\alpha-\theta}{\alpha+\theta} \cdot \frac{\overline{\alpha-\theta}}{\alpha+\theta} \\
& =\frac{\alpha-\theta}{\alpha+\theta} \cdot \frac{\alpha-\bar{\theta}}{\alpha+\bar{\theta}} \\
& =\frac{\alpha^{2}-2 \Re(\theta) \alpha+|\theta|^{2}}{\alpha^{2}+2 \Re(\theta) \alpha+|\theta|^{2}} \quad(<1)  \tag{<1}\\
& =1-2 \frac{2 \Re(\theta) \alpha}{\alpha^{2}+2 \Re(\theta) \alpha+|\theta|^{2}} \\
& =1-2 \frac{\left(u^{*} K A K u / u^{*} K u\right) \alpha}{\alpha^{2}+\left(u^{*} K A K u / u^{*} K u\right) \alpha+\left(u^{*} K B^{T} B u / u^{*} K u\right)} \\
& \leq 1-2 \frac{\lambda_{n}\left(1+\left(\sigma_{m}^{2} / \alpha^{2}\right)\right) \alpha}{\alpha^{2}+\lambda_{n}\left(1+\left(\sigma_{m}^{2} / \alpha^{2}\right)\right) \alpha+\sigma_{1}^{2}} . \tag{24}
\end{align*}
$$

If $\mathfrak{J}(\theta)=0$, then $\theta>0$ from (22). Combing (10) with (15), we have

$$
\begin{equation*}
\min \left\{\lambda_{n}, \frac{\alpha^{2} \sigma_{m}^{2}}{\lambda_{1}\left(\alpha^{2}+\sigma_{m}^{2}\right)}\right\} \leq \frac{(1-\lambda) \alpha}{(1+\lambda)} \leq \lambda_{1}\left(1+\frac{\sigma_{1}^{2}}{\alpha^{2}}\right) . \tag{25}
\end{equation*}
$$

That is,

$$
\begin{equation*}
\frac{1}{\alpha} \min \left\{\lambda_{n}, \frac{\alpha^{2} \sigma_{m}^{2}}{\lambda_{1}\left(\alpha^{2}+\sigma_{m}^{2}\right)}\right\} \leq \frac{1-\lambda}{1+\lambda} \leq \frac{1}{\alpha} \lambda_{1}\left(1+\frac{\sigma_{1}^{2}}{\alpha^{2}}\right) \tag{26}
\end{equation*}
$$

Further, we have

$$
\begin{equation*}
-\frac{1}{\alpha} \lambda_{1}\left(1+\frac{\sigma_{1}^{2}}{\alpha^{2}}\right) \leq \frac{\lambda-1}{\lambda+1} \leq-\frac{1}{\alpha} \min \left\{\lambda_{n}, \frac{\alpha^{2} \sigma_{m}^{2}}{\lambda_{1}\left(\alpha^{2}+\sigma_{m}^{2}\right)}\right\} \tag{27}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
-\frac{1}{\alpha} \lambda_{1}\left(1+\frac{\sigma_{1}^{2}}{\alpha^{2}}\right) & \leq 1-\frac{2}{\lambda+1}  \tag{28}\\
& \leq-\frac{1}{\alpha} \min \left\{\lambda_{n}, \frac{\alpha^{2} \sigma_{m}^{2}}{\lambda_{1}\left(\alpha^{2}+\sigma_{m}^{2}\right)}\right\}
\end{align*}
$$

So, we have

$$
\begin{equation*}
\frac{1}{\alpha} \min \left\{\lambda_{n}, \frac{\alpha^{2} \sigma_{m}^{2}}{\lambda_{1}\left(\alpha^{2}+\sigma_{m}^{2}\right)}\right\} \leq \frac{2}{\lambda+1}-1 \leq \frac{1}{\alpha} \lambda_{1}\left(1+\frac{\sigma_{1}^{2}}{\alpha^{2}}\right) \tag{29}
\end{equation*}
$$

That is,

$$
\begin{align*}
1+\frac{1}{\alpha} \min \left\{\lambda_{n}, \frac{\alpha^{2} \sigma_{m}^{2}}{\lambda_{1}\left(\alpha^{2}+\sigma_{m}^{2}\right)}\right\} & \leq \frac{2}{\lambda+1} \\
& \leq 1+\frac{1}{\alpha} \lambda_{1}\left(1+\frac{\sigma_{1}^{2}}{\alpha^{2}}\right) \tag{30}
\end{align*}
$$

By the simple computations, we have

$$
\begin{align*}
& \frac{1}{1+(1 / \alpha) \lambda_{1}\left(1+\left(\sigma_{1}^{2} / \alpha^{2}\right)\right)}  \tag{31}\\
& \quad \leq \frac{\lambda+1}{2} \leq \frac{1}{1+(1 / \alpha) \min \left\{\lambda_{n}, \alpha^{2} \sigma_{m}^{2} / \lambda_{1}\left(\alpha^{2}+\sigma_{m}^{2}\right)\right\}}
\end{align*}
$$

Obviously, we also have

$$
\begin{align*}
& \frac{2}{1+(1 / \alpha) \lambda_{1}\left(1+\left(\sigma_{1}^{2} / \alpha^{2}\right)\right)}  \tag{32}\\
& \quad \leq \lambda+1 \leq \frac{2}{1+(1 / \alpha) \min \left\{\lambda_{n}, \alpha^{2} \sigma_{m}^{2} / \lambda_{1}\left(\alpha^{2}+\sigma_{m}^{2}\right)\right\}}
\end{align*}
$$

That is to say,

$$
\begin{align*}
& \frac{1-(1 / \alpha) \lambda_{1}\left(1+\left(\sigma_{1}^{2} / \alpha^{2}\right)\right)}{1+(1 / \alpha) \lambda_{1}\left(1+\left(\sigma_{1}^{2} / \alpha^{2}\right)\right)} \\
& \quad \leq \lambda \leq \frac{1-(1 / \alpha) \min \left\{\lambda_{n}, \alpha^{2} \sigma_{m}^{2} / \lambda_{1}\left(\alpha^{2}+\sigma_{m}^{2}\right)\right\}}{1+(1 / \alpha) \min \left\{\lambda_{n}, \alpha^{2} \sigma_{m}^{2} / \lambda_{1}\left(\alpha^{2}+\sigma_{m}^{2}\right)\right\}} \tag{33}
\end{align*}
$$

## 3. Numerical Experiments

In this section, we consider the following two examples to illustrate the above result.

Example 1 (see [6-9]). Consider the following classic incompressible steady Stokes problem:

$$
\begin{gather*}
-\Delta u+\operatorname{grad} p=f, \quad \text { in } \Omega,  \tag{34}\\
-\operatorname{div} u=0, \quad \text { in } \Omega,
\end{gather*}
$$

with suitable boundary condition on $\partial \Omega$. That is to say, the boundary conditions are $u_{x}=u_{y}=0$ on the three fixed walls $(x=0, x=1$, and $y=0)$ and $u_{x}=1, u_{y}=0$ on the moving wall $(y=1)$. The test problem is a "leaky" twodimensional lid-driven cavity problem in square ( $0 \leq x \leq 1$, $0 \leq y \leq 1)$. Using the IFISS software [10] to discretize (34), the finite element subdivision is based on $8 \times 8$ and $16 \times 16$ uniform grids of square elements and the mixed finite

Table 1: The region for all the eigenvalues of $M_{\alpha}$ with $\mathfrak{\Im}(\theta)=0$ and $8 \times 8$.

| $\alpha$ | $E_{\min }$ | $L_{\min }$ | $\lambda_{\min }$ | $\lambda_{\max }$ | $U_{\max }$ | $E_{\max }$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1 | 0 | -0.9978 | -0.9978 | 0.9957 | 0.9971 | 0.0014 |
| 0.2 | 0 | -0.9846 | -0.9846 | 0.9977 | 0.9985 | 0.0008 |
| 0.3 | 0.0001 | -0.9568 | -0.9567 | 0.9985 | 0.9990 | 0.0005 |
| 0.4 | 0.0002 | -0.9176 | -0.9174 | 0.9988 | 0.9992 | 0.0004 |
| 0.5 | 0.0006 | -0.8723 | -0.8717 | 0.9990 | 0.9994 | 0.0004 |
| 0.6 | 0.001 | -0.8249 | -0.8239 | 0.9992 | 0.9995 | 0.0003 |
| 0.7 | 0.0015 | -0.7779 | -0.7764 | 0.9993 | 0.9996 | 0.0003 |
| 0.8 | 0.0021 | -0.7325 | -0.7304 | 0.9994 | 0.9996 | 0.0002 |
| 0.9 | 0.0026 | -0.6892 | -0.6866 | 0.9995 | 0.9997 | 0.0002 |
| 1 | 0.0031 | -0.6482 | -0.6451 | 0.9995 | 0.9997 | 0.0002 |

Table 2: The region for all the eigenvalues of $M_{\alpha}$ with $\mathfrak{\Im}(\theta)=0$ and $16 \times 16$.

| $\alpha$ | $E_{\min }$ | $L_{\min }$ | $\lambda_{\min }$ | $\lambda_{\max }$ | $U_{\max }$ | $E_{\max }$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1 | 0 | -0.9929 | -0.9929 | 0.9997 | 1 | 0.0003 |
| 0.2 | 0 | -0.9608 | -0.9608 | 0.9998 | 1 | 0.0002 |
| 0.3 | 0.0001 | -0.9135 | -0.9134 | 0.9999 | 1 | 0.0001 |
| 0.4 | 0.0002 | -0.8635 | -0.8633 | 0.9999 | 1 | 0.0001 |
| 0.5 | 0.0004 | -0.8154 | -0.8150 | 0.9999 | 1 | 0.0001 |
| 0.6 | 0.0006 | -0.7702 | -0.7696 | 0.99996 | 1 | 0.00004 |
| 0.7 | 0.0007 | -0.7278 | -0.7271 | 0.99998 | 1 | 0.00002 |
| 0.8 | 0.001 | -0.6880 | -0.6870 | 0.99997 | 1 | 0.00003 |
| 0.9 | 0.001 | -0.6504 | -0.6494 | 0.99998 | 1 | 0.00002 |
| 1 | 0.001 | -0.6147 | -0.6137 | 0.99998 | 1 | 0.00002 |

element used is the bilinear-constant velocity pressure: $Q_{1}-P_{0}$ pair with stabilization (the stabilization parameter is zero). The coefficient matrix generated by this package is singular because $B$ corresponding to the discrete divergence operator is rank deficient. The nonsingular matrix $\mathscr{A}$ is obtained by dropping the first two rows of $B$ and the first two rows and columns of matrix $C$. Note that matrix $C$ is a null matrix, which is the corresponding $(2,2)$ block of $(1)$. In this case, $n=162$ and $m=62$ correspond to $8 \times 8$, and $n=578$ and $m=254$ correspond to $16 \times 16$. For the Stokes problem, the $(1,1)$ block of the coefficient matrix corresponding to the discretization of the conservative term is symmetric positive definite.

By calculations, the values given in Tables 1 and 2 are obtained, which are to verify the results of Theorem 3. In Tables 1 and $2, L_{\text {min }}, U_{\text {max }}$, respectively, denote the lower and upper bounds of all the eigenvalues of $M_{\alpha} \cdot E_{\min }=\mid L_{\text {min }}$ $\lambda_{\text {min }} \mid$ and $E_{\text {max }}=\left|U_{\max }-\lambda_{\text {max }}\right|$.

From Tables 1 and 2, it is not difficult to find that the theoretical results are in line with the results of numerical experiments. Further, for $8 \times 8$, the average error in the lower bounds for 10 different values of $\alpha$ is 0.00112 and the average error in the upper bounds for 10 different values of $\alpha$ is 0.00047 . For $16 \times 16$, the average error in the lower bounds for 10 different values of $\alpha$ is 0.0005 and the average error in the upper bounds for 10 different values of $\alpha$ is 0.000091 .

Table 3: All the nonreal eigenvalues of $M_{\alpha}$ with $\mathfrak{J}(\theta) \neq 0$.

| $\alpha$ | $\max \|\lambda\|$ | Upper bound |
| :--- | :---: | :---: |
| 0.05 | 0.963349 | 0.999793 |
| 0.1 | 0.977199 | 0.999878 |
| 0.3 | 0.979200 | 0.999894 |
| 0.5 | 0.959713 | 0.999860 |
| 1 | 0.865509 | 0.999774 |

That is, Theorem 3 provides reasonably good bounds for the eigenvalue distribution of the iteration matrix $M_{\alpha}$ of the HSS method when the iteration parameter $\alpha$ is taken in different regions.

Example 2 (see [11]). The saddle point system is from the discretization of a groundwater flow problem using mixedhybrid finite elements [11]. In the example at hand, $n=270$ and $m=207$. By calculations, here we have $\lambda_{n}=0.0017$, $\lambda_{1}=0.010, \sigma_{1}=2.611$, and $\sigma_{m}=0.19743$.

In this case there are nonreal eigenvalues (except for very small $\alpha$ ). In Table 3 we list the upper bounds given in Theorem 3 when $\mathfrak{J}(\theta) \neq 0$. From Table 3, it is not difficult to find that the theoretical results are in line with the results of numerical experiments. That is, Theorem 3 provides reasonably good bounds for the eigenvalue distribution of the iteration matrix $M_{\alpha}$ with $\mathfrak{J}(\theta) \neq 0$ when the iteration parameter $\alpha$ is taken in different regions.

## Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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# (Anti-)Hermitian Generalized (Anti-)Hamiltonian Solution to a System of Matrix Equations 

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#### Abstract

We mainly solve three problems. Firstly, by the decomposition of the (anti-)Hermitian generalized (anti-)Hamiltonian matrices, the necessary and sufficient conditions for the existence of and the expression for the (anti-)Hermitian generalized (anti-)Hamiltonian solutions to the system of matrix equations $A X=B, X C=D$ are derived, respectively. Secondly, the optimal approximation solution $\min _{X \in K}\|\widehat{X}-X\|$ is obtained, where $K$ is the (anti-)Hermitian generalized (anti-)Hamiltonian solution set of the above system and $\widehat{X}$ is the given matrix. Thirdly, the least squares (anti-)Hermitian generalized (anti-)Hamiltonian solutions are considered. In addition, algorithms about computing the least squares (anti-)Hermitian generalized (anti-)Hamiltonian solution and the corresponding numerical examples are presented.


## 1. Introduction

Throughout this paper, the set of all $m \times n$ complex matrices, the set of all $n \times n$ Hermitian matrices, the set of all $n \times n$ anti-Hermitian matrices, the set of all $n \times n$ unitary matrices, and the set of all $n \times n$ antisymmetric orthogonal matrices are denoted, respectively, by $\mathbb{C}^{m \times n}, H \mathbb{C}^{n \times n}, A H \mathbb{C}^{n \times n}, U \mathbb{C}^{n \times n}$, and $A S O \mathbb{R}^{n \times n}$. The symbol $I_{n}$ represents an identity matrix of order $n$ and $r(A), A^{\dagger}$, and $A^{*}$, respectively, stand for the rank, the Moore-Penrose inverse, and the conjugate transpose of matrix $A$. For two matrices $A, B \in \mathbb{C}^{m \times n}$, the inner product is defined by $\langle A, B\rangle=\operatorname{tr}\left(B^{*} A\right)$. Obviously, $\mathbb{C}^{m \times n}$ is a complete inner product space. The norm $\|\cdot\|$, induced by the inner product, is called the Frobenius norm. $A * B$ stands for the Hadamard product of two matrices $A$ and $B$. For $A \in$ $\mathbb{C}^{m \times n}$, two matrices $L_{A}$ and $R_{A}$, respectively, represent two orthogonal projectors $L_{A}=I_{n}-A^{\dagger} A$ and $R_{A}=I_{m}-A A^{\dagger}$, both of which satisfy

$$
\begin{align*}
& L_{A}=\left(L_{A}\right)^{2}=\left(L_{A}\right)^{*}=\left(L_{A}\right)^{\dagger} \\
& R_{A}=\left(R_{A}\right)^{2}=\left(R_{A}\right)^{*}=\left(R_{A}\right)^{\dagger} \tag{1}
\end{align*}
$$

The Hamiltonian matrices defined as in [1] are very important in engineering (see [2] and the references therein). Moreover, using Hamiltonian matrices to solve algebraic matrix Riccati equation is a very effective method in optimal control theory [3-5]. As the extension of the Hamiltonian matrices, the following four definitions, which can also be found in $[1,6,7]$, are given. Without special statement, we in this paper always assume that $J \in A S O \mathbb{R}^{2 k \times 2 k}$ satisfies

$$
\begin{equation*}
J^{T}=-J, \quad J^{T} J=J J^{T}=I_{n} . \tag{2}
\end{equation*}
$$

Definition 1. A matrix $X \in H H \mathbb{C}^{2 k \times 2 k}$ is said to be a Hermitian generalized Hamiltonian matrix if $X=X^{*}$ and $J X J=$ $X^{*}$.

Definition 2. A matrix $X \in H A H \mathbb{C}^{2 k \times 2 k}$ is said to be a Hermitian generalized anti-Hamiltonian matrix if $X=X^{*}$ and $J X J=-X^{*}$.

Definition 3. A matrix $X \in A H A H \mathbb{C}^{2 k \times 2 k}$ is said to be an anti-Hermitian generalized anti-Hamiltonian matrix if $X=$ $-X^{*}$ and $J X J=-X^{*}$.

Definition 4. A matrix $X \in A H H \mathbb{C}^{2 k \times 2 k}$ is said to be an antiHermitian generalized Hamiltonian matrix if $X=-X^{*}$ and $J X J=X^{*}$ 。

The well-known system of matrix equations

$$
\begin{equation*}
A X=B, \quad X C=D \tag{3}
\end{equation*}
$$

with unknown matrix $X$, has attracted much attention and has been widely and deeply studied by many authors. For example, Khatri and Mitra [8] in 1976 established the Hermitian and nonnegative definite solution to the system (3). Mitra [9] in 1984 gave the system (3) the minimal rank solution over the complex field $\mathbb{C}$. Wang in [10] and Wang et al. [11], respectively, investigated the bisymmetric and centrosymmetric solutions over the quaternion algebra and obtained the bisymmetric nonnegative definite solutions with extremal ranks and inertias to the system (3). Xu in [12] studied the common Hermitian and positive solutions to the adjointable operator equations (3). Yuan in [13] presented the least squares solutions to the system (3). Some other results concerning the system (3) can be found in [14-23].

As special cases of the system (3), the classical matrix equations $A X=B$ and $X C=D$ have also been investigated (see, e.g., [1, 2, 5-7, 24-31]). For instance, Dai [24], by means of the singular value decomposition, derived the symmetric solution to equation $A X=B$. Guan and Jiang [6], using the decomposition of the anti-Hermitian generalized antiHamiltonian matrices, derived the least squares solution to equation $A X=B$. Zhang et al. in [29] and [1], respectively, obtained the general expression of the least squares Hermitian generalized Hamiltonian solutions to equation $X C=D$ and got the unite optimal approximation solution in the least squares solutions set and gave the solvable conditions and the general representation of the Hermitian generalized Hamiltonian solutions to equation $A X=B$, by using the singular value decomposition and the properties of Hermitian generalized Hamiltonian matrices.

As far as we know, there has been little information on studying the (anti-)Hermitian generalized (anti-) Hamiltonian solution to the system (3) over $\mathbb{C}^{2 k \times 2 k}$. So, motived by the work mentioned above, especially the work in $[6,7,26,29,30]$, we, in this paper, are mainly concerned with the following three problems.

Problem 5. Given $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$, find $X \in$ $H H \mathbb{C}^{2 k \times 2 k}\left(H A H \mathbb{C}^{2 k \times 2 k}, A H H \mathbb{C}^{2 k \times 2 k}\right.$, or $\left.A H A H \mathbb{C}^{2 k \times 2 k}\right)$ such that the system (3) holds.

Problem 6. Given $\widehat{X} \in \mathbb{C}^{2 k \times 2 k}$, find $\widetilde{X} \in K$ such that

$$
\begin{equation*}
\|\widehat{X}-\widetilde{X}\|=\min _{X \in K}\|\widehat{X}-X\|, \tag{4}
\end{equation*}
$$

where $K$ is the solution set of Problem 5.
Problem 7. Let $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$. Find $X \in$ $H H \mathbb{C}^{2 k \times 2 k}\left(H A H \mathbb{C}^{2 k \times 2 k}, A H H \mathbb{C}^{2 k \times 2 k}\right.$, or $\left.A H A H \mathbb{C}^{2 k \times 2 k}\right)$ such that

$$
\begin{equation*}
\min _{X}=\|A X-B\|^{2}+\|X C-D\|^{2} \tag{5}
\end{equation*}
$$

The remainder of this paper is arranged as follows. In Section 2, some lemmas will be introduced, which will be useful for us to obtain the solutions to Problems 5-7. In Section 3, by applying the decomposition of the (anti-)Hermitian generalized (anti-)Hamiltonian matrices, the solvability condition and the explicit expression of the solution to Problem 5 will be derived. In Section 4, the optimal approximation solution to Problem 6 will be established. In Section 5, the solution to Problem 7 will be investigated and meanwhile the minimum norm of the solution will be obtained. In Section 6, algorithms and numerical examples about computing the solution to Problem 7 will be provided. Finally, in Section 7, some conclusions will be made.

## 2. Preliminaries

In this section, we focus on introducing some lemmas, which will play key roles in solving Problems 5-7.

Taking into account Definitions 1-4 and the eigenvalue decomposition of the matrix $J \in A S O \mathbb{R}^{2 k \times 2 k}$, it is not difficult to conclude that the following decompositions of the (anti-) Hermitian generalized (anti-)Hamiltonian matrices hold, some of which can also be seen in $[6,26,29,30]$.

Lemma 8. Let the eigenvalue decomposition of matrix $J \in$ $A S O \mathbb{R}^{2 k \times 2 k} b e$

$$
J=P\left(\begin{array}{cc}
i I_{k} & 0  \tag{6}\\
0 & -i I_{k}
\end{array}\right) P^{*}
$$

where $P \in U \mathbb{C}^{2 k \times 2 k}$. Then $X \in H H \mathbb{C}^{2 k \times 2 k}$ if and only if $X$ can be expressed as

$$
X=P\left(\begin{array}{cc}
0 & X_{12}  \tag{7}\\
X_{12}^{*} & 0
\end{array}\right) P^{*}
$$

where $X_{12} \in \mathbb{C}^{k \times k}$ are arbitrary.
Lemma 9. Let the eigenvalue decomposition of matrix $J \in$ $A S O \mathbb{R}^{2 k \times 2 k}$ be (6). Then $X \in A H A H \mathbb{C}^{2 k \times 2 k}$ if and only if $X$ can be expressed as

$$
X=P\left(\begin{array}{cc}
0 & X_{12}  \tag{8}\\
-X_{12}^{*} & 0
\end{array}\right) P^{*}
$$

where $X_{12} \in \mathbb{C}^{k \times k}$ is arbitrary.
Lemma 10. Let the eigenvalue decomposition of matrix $J \in$ $A S O \mathbb{R}^{2 k \times 2 k}$ be (6). Then $X \in H A H \mathbb{C}^{2 k \times 2 k}$ if and only if $X$ can be expressed as

$$
X=P\left(\begin{array}{cc}
X_{11} & 0  \tag{9}\\
0 & X_{22}
\end{array}\right) P^{*}
$$

where $X_{11}, X_{22} \in H \mathbb{C}^{k \times k}$ are arbitrary.
Lemma 11. Let the eigenvalue decomposition of matrix $J \in$ $A S O \mathbb{R}^{2 k \times 2 k}$ be (6). Then $X \in A H H \mathbb{C}^{2 k \times 2 k}$ if and only if $X$ can be expressed as

$$
X=P\left(\begin{array}{cc}
X_{11} & 0  \tag{10}\\
0 & X_{22}
\end{array}\right) P^{*}
$$

where $X_{11}, X_{22} \in A H \mathbb{C}^{k \times k}$ are arbitrary.

Lemma 12 (see [20]). Given $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{p \times l}, C \in$ $\mathbb{C}^{m \times p}$, and $D \in \mathbb{C}^{n \times l}$, then the system of matrix equations

$$
\begin{equation*}
A X=C, \quad X B=D \tag{11}
\end{equation*}
$$

has a solution $X \in \mathbb{C}^{n \times p}$ if and only if

$$
\begin{equation*}
A A^{\dagger} C=C, \quad D B^{\dagger} B=D, \quad A D=C B \tag{12}
\end{equation*}
$$

in which case the general solutions can be expressed as

$$
\begin{equation*}
X=A^{\dagger} C+D B^{\dagger}-A^{\dagger} A D B^{\dagger}+\left(I-A^{\dagger} A\right) W\left(I-B B^{\dagger}\right) \tag{13}
\end{equation*}
$$

where $W \in \mathbb{C}^{n \times p}$ is arbitrary.
By applying the singular value decomposition, similar to the proof of Theorem 1 in [24], the following lemma can be shown.

Lemma 13. Assume $E, F \in \mathbb{C}^{m \times n}$. Let the singular value decomposition of $E$ be

$$
E=U\left(\begin{array}{ll}
\Sigma & 0  \tag{14}\\
0 & 0
\end{array}\right) V^{*}
$$

where

$$
\begin{array}{r}
U \in U \mathbb{C}^{m \times m}, \quad V \in U \mathbb{C}^{n \times n}, \\
\Sigma=\operatorname{diag}\left(\alpha_{1}, \ldots, \alpha_{r}\right), \quad \alpha_{i}>0,  \tag{15}\\
i=1, \ldots, r ; r=r(E) .
\end{array}
$$

## Partition

$$
\begin{align*}
V X V^{*} & =\left(\begin{array}{ll}
X_{11} & X_{12} \\
X_{12}^{*} & X_{22}
\end{array}\right), \\
U^{*} F V & =\left(\begin{array}{ll}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{array}\right), \tag{16}
\end{align*}
$$

where

$$
\begin{align*}
X_{11} \in H \mathbb{C}^{r \times r}, & X_{22} \in H \mathbb{C}^{(n-r) \times(n-r)} \\
F_{11} \in \mathbb{C}^{r \times r}, & F_{22} \in \mathbb{C}^{(m-r) \times(n-r)} \tag{17}
\end{align*}
$$

Then the matrix equation

$$
\begin{equation*}
E X=F \tag{18}
\end{equation*}
$$

has Hermitian solutions if and only if

$$
\begin{align*}
E E^{\dagger} F=F, & E F^{*}=F E^{*} \\
F_{21} & =0,  \tag{19}\\
F_{22} & =0
\end{align*}
$$

in which case the Hermitian solution can be expressed as

$$
X=V\left(\begin{array}{cc}
\Sigma^{-1} F_{11} & \Sigma^{-1} F_{12}  \tag{20}\\
F_{12}^{*} \Sigma^{-1} & X_{22}
\end{array}\right) V^{*}
$$

where $X_{22} \in H \mathbb{C}^{(n-r) \times(n-r)}$ is arbitrary.

By the similar way, the following lemma can also be verified.

Lemma 14. Assume $M, N \in \mathbb{C}^{m \times n}$. Let the singular value decomposition of $M$ be

$$
M=U\left(\begin{array}{cc}
\Pi & 0  \tag{21}\\
0 & 0
\end{array}\right) V^{*}
$$

where

$$
\begin{gather*}
U \in U \mathbb{C}^{m \times m}, \quad V \in U \mathbb{C}^{n \times n} \\
\Pi=\operatorname{diag}\left(\beta_{1}, \ldots, \beta_{s}\right), \quad \beta_{i}>0  \tag{22}\\
i=1, \ldots, s ; s=r(M)
\end{gather*}
$$

## Partition

$$
\begin{align*}
V X V^{*} & =\left(\begin{array}{cc}
X_{11} & X_{12} \\
-X_{12}^{*} & X_{22}
\end{array}\right)  \tag{23}\\
U^{*} N V & =\left(\begin{array}{ll}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{array}\right)
\end{align*}
$$

where

$$
\begin{align*}
X_{11} \in A H \mathbb{C}^{s \times s}, & X_{22} \in A H \mathbb{C}^{(n-s) \times(n-s)},  \tag{24}\\
N_{11} \in \mathbb{C}^{s \times s}, & N_{22} \in \mathbb{C}^{(m-s) \times(n-s)}
\end{align*}
$$

Then the matrix equation

$$
\begin{equation*}
M X=N \tag{25}
\end{equation*}
$$

has an anti-Hermitian solution if and only if

$$
\begin{gather*}
M M^{\dagger} N=N, \quad M N^{*}=-N M^{*}  \tag{26}\\
N_{21}=0, \quad N_{22}=0
\end{gather*}
$$

in which case the anti-Hermitian solution can be expressed as

$$
X=V\left(\begin{array}{cc}
\Pi^{-1} N_{11} & \Pi^{-1} N_{12}  \tag{27}\\
-N_{12}^{*} \Pi^{-1} & X_{22}
\end{array}\right) V^{*}
$$

where $X_{22} \in A H \mathbb{C}^{(n-s) \times(n-s)}$ is arbitrary.
Lemma 15 (see [31]). Given $A^{\prime}, B^{\prime} \in \mathbb{C}^{k \times(m+q)}, C^{\prime}, D^{\prime} \in$ $\mathbb{C}^{k \times(m+q)}$, suppose that the matrices $A^{\prime}$ and $C^{\prime}$, respectively, have the following singular value decompositions:

$$
A^{\prime}=P_{1}\left(\begin{array}{ll}
\Gamma & 0  \tag{28}\\
0 & 0
\end{array}\right) Q_{1}^{*}, \quad C^{\prime}=U_{1}\left(\begin{array}{cc}
\Lambda & 0 \\
0 & 0
\end{array}\right) V_{1}^{*}
$$

where

$$
\begin{gather*}
P_{1}=\left(\begin{array}{ll}
P_{11} & P_{12}
\end{array}\right) \in U \mathbb{C}^{k \times k}, \quad P_{11} \in \mathbb{C}^{k \times t_{1}} ; \\
Q_{1}=\left(\begin{array}{ll}
Q_{11} \quad Q_{12}
\end{array}\right) \in U \mathbb{C}^{(m+q) \times(m+q)}, \\
Q_{11} \in \mathbb{C}^{(m+q) \times t_{1}} ; \\
U_{1}=\left(\begin{array}{ll}
U_{11} & U_{12}
\end{array}\right) \in U \mathbb{C}^{k \times k}, \quad U_{11} \in \mathbb{C}^{k \times t_{2}} ; \\
V_{1}=\binom{V_{11}}{V_{12}} \in U \mathbb{C}^{(m+q) \times(m+q)}, \\
V_{11} \in \mathbb{C}^{(m+q) \times t_{2}} ;  \tag{29}\\
\Gamma=\operatorname{diag}\left(\delta_{1}, \delta_{2}, \ldots, \delta_{t_{1}}\right), \quad \delta_{i}>0, \\
1 \leq i \leq t_{1} ; t_{1}=r\left(A^{\prime}\right) ; \\
\Lambda=\operatorname{diag}\left(\gamma_{1}, \gamma_{2}, \ldots, \gamma_{t_{2}}\right), \quad \gamma_{i}>0, \\
1 \leq i \leq t_{2} ; t_{2}=r\left(C^{\prime}\right) .
\end{gather*}
$$

Then the solution set of the problem

$$
\begin{align*}
f\left(X_{12}\right) \stackrel{\Delta}{=} & \left\|\left(A^{\prime}\right)^{*} X_{12}-\left(B^{\prime}\right)^{*}\right\|^{2}  \tag{30}\\
& +\left\|X_{12} C^{\prime}-D^{\prime}\right\|^{2}=\min
\end{align*}
$$

consists of matrices $X_{12} \in \mathbb{C}^{k \times k}$ with the following form:
$X_{12}$

$$
=P_{1}\left(\begin{array}{cc}
\phi *\left(P_{11}^{*} D^{\prime} V_{11} \Lambda+\Gamma Q_{11}^{*}\left(B^{\prime}\right)^{*} U_{12}\right) & \Gamma^{-1} \mathrm{Q}_{11}^{*}\left(B^{\prime}\right)^{*} U_{12}  \tag{31}\\
P_{12}^{*} D^{\prime} V_{11} \Lambda^{-1} & X_{22}^{\prime}
\end{array}\right) U_{1}^{*},
$$

where

$$
\begin{gather*}
\phi=\left(\phi_{i j}\right), \quad \phi_{i j}=\frac{1}{\delta_{i}^{2}+\gamma_{j}^{2}}  \tag{32}\\
1 \leq i \leq t_{1}, \quad 1 \leq j \leq t_{2}
\end{gather*}
$$

and $X_{22}^{\prime} \in \mathbb{C}^{\left(k-t_{1}\right) \times\left(k-t_{2}\right)}$ is arbitrary.
Lemma 16. Given $E, F \in \mathbb{C}^{m \times n}$, let the singular value decomposition of $E$, the partitions of $V X V^{*}$ and $U^{*} F V$ be, respectively, as in (14)-(16). Then the least squares Hermitian solution to the matrix equation (18) can be expressed as

$$
X=V\left(\begin{array}{cc}
\Phi *\left(\Sigma F_{11}+F_{11}^{*} \Sigma\right) & \Sigma^{-1} F_{12}  \tag{33}\\
F_{12}^{*} \Sigma^{-1} & X_{22}
\end{array}\right) V^{*}
$$

where

$$
\begin{equation*}
\Phi=\left(\frac{1}{\alpha_{i}^{2}+\alpha_{j}^{2}}\right), \quad 1 \leq i, j \leq r \tag{34}
\end{equation*}
$$

and $X_{22} \in H \mathbb{C}^{(n-r) \times(n-r)}$ is arbitrary.

Proof. Combining (14)-(16) and the unitary invariance of the Frobenius norm, it is easy to obtain that

$$
\begin{align*}
\|E X-F\|^{2}= & \left\|\left(\begin{array}{ll}
\Sigma & 0 \\
0 & 0
\end{array}\right) V^{*} V\left(\begin{array}{ll}
X_{11} & X_{12} \\
X_{12}^{*} & X_{22}
\end{array}\right)-U^{*} F V\right\|^{2} \\
= & \left\|\left(\begin{array}{cc}
\Sigma X_{11} & \Sigma X_{12} \\
0 & 0
\end{array}\right)-\left(\begin{array}{ll}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{array}\right)\right\|^{2}  \tag{35}\\
= & \left\|\Sigma X_{11}-F_{11}\right\|^{2}+\left\|\Sigma X_{12}-F_{12}\right\|^{2} \\
& +\left\|F_{21}\right\|^{2}+\left\|F_{22}\right\|^{2} .
\end{align*}
$$

Then $\|E X-F\|^{2}$ reaches its minimum if and only if

$$
\begin{align*}
& \left\|\Sigma X_{11}-F_{11}\right\|^{2}  \tag{36}\\
& \left\|\Sigma X_{12}-F_{12}\right\|^{2} \tag{37}
\end{align*}
$$

reach their minimum. For $X_{11}=\left(x_{i j}\right) \in H \mathbb{C}^{r \times r}, F_{11}=\left(f_{i j}\right) \in$ $\mathbb{C}^{r \times r}$, since $x_{i j}=x_{i j}^{*}, 1 \leq i, j \leq r$, then

$$
\begin{align*}
\left\|\Sigma X_{11}-F_{11}\right\|^{2}= & \sum_{i=1}^{r} \sum_{j=1}^{r}\left(\alpha_{i} x_{i j}-f_{i j}\right)^{2} \\
= & \sum_{1 \leq i, j \leq r}^{r}\left[\left(\alpha_{i}^{2}+\alpha_{j}^{2}\right)\left|x_{i j}\right|^{2}\right. \\
& \left.+2\left(\alpha_{i} f_{i j}+\alpha_{j} f_{i j}^{*}\right) x_{i j}+2\left|f_{i j}\right|^{2}\right] \tag{38}
\end{align*}
$$

Hence, there exists a unique solution $X_{11}=\left(\widehat{x}_{i j}\right) \in H \mathbb{C}^{r \times r}$ for (36) such that

$$
\begin{equation*}
\widehat{x}_{i j}=\frac{\alpha_{i} f_{i j}+\alpha_{j} f_{i j}^{*}}{\alpha_{i}^{2}+\alpha_{j}^{2}}, \quad 1 \leq i, j \leq r \tag{39}
\end{equation*}
$$

That is,

$$
\begin{equation*}
X_{11}=\Phi *\left(\Sigma F_{11}+F_{11}^{*} \Sigma\right), \tag{40}
\end{equation*}
$$

where

$$
\begin{equation*}
\Phi=\left(\frac{1}{\alpha_{i}^{2}+\alpha_{j}^{2}}\right), \quad 1 \leq i, j \leq r . \tag{41}
\end{equation*}
$$

When $X_{12}$ can be expressed as

$$
\begin{equation*}
X_{12}=\Sigma^{-1} F_{12} \tag{42}
\end{equation*}
$$

(37) gets its minimum. Therefore, the least squares Hermitian solution to (18) can be described as (33).

By the similar way, the following result can be obtained.
Lemma 17. Given $M, N \in \mathbb{C}^{m \times n}$, let the singular value decomposition of $M$, the partitions of $V X V^{*}$, and $U^{*} N V$
be, respectively, as in (21)-(23). Then the least squares antiHermitian solution to the matrix equation $M X=N$ can be expressed as

$$
X=V\left(\begin{array}{cc}
\Psi *\left(\Pi N_{11}-N_{11}^{*} \Pi\right) & \Pi^{-1} N_{12}  \tag{43}\\
-N_{12}^{*} \Pi^{-1} & X_{22}
\end{array}\right) V^{*}
$$

where

$$
\begin{equation*}
\Psi=\left(\frac{1}{\beta_{i}^{2}+\beta_{j}^{2}}\right), \quad 1 \leq i, j \leq s \tag{44}
\end{equation*}
$$

and $X_{22} \in A H \mathbb{C}^{(n-s) \times(n-s)}$ is arbitrary.
Lemma 18 (see [20]). Given $F \in \mathbb{C}^{m \times n}, G \in \mathbb{C}^{p \times q}$, and $L \in$ $\mathbb{C}^{m \times q}$, then the matrix equation $F X G=L$ has a solution if and only if

$$
\begin{equation*}
F F^{\dagger} L G^{\dagger} G=L \tag{45}
\end{equation*}
$$

in which case the general solution is

$$
\begin{equation*}
X=F^{\dagger} L G^{\dagger}+Y-F^{\dagger} F Y G G^{\dagger} \tag{46}
\end{equation*}
$$

where $Y \in \mathbb{C}^{n \times p}$ is arbitrary.
The following lemma is due to [25, 32] or [29, Lemma 5].
Lemma 19. Let $M, N \in \mathbb{C}^{m \times n}$. Then there exists a unique matrix $W_{1} \in \mathbb{C}^{m \times n}$ such that

$$
\begin{align*}
& \left\|W_{1}-M\right\|^{2}+\left\|W_{1}-N\right\|^{2} \\
& \quad=\min _{W \in \mathbb{C}^{m \times n}}\left(\|W-M\|^{2}+\|W-N\|^{2}\right) \tag{47}
\end{align*}
$$

where

$$
\begin{equation*}
W_{1}=\frac{M+N}{2} . \tag{48}
\end{equation*}
$$

## 3. The Solvability Conditions and the Expression of the Solution to Problem 5

In this section, our purpose is to derive the necessary and sufficient conditions of and the explicit expression of the solution to Problem 5 by using the results introduced in Section 2.

Theorem 20. Given $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$, let the decomposition of $X \in H H \mathbb{C}^{2 k \times 2 k}$ be (7). Partition

$$
\begin{gather*}
A P=\left(\begin{array}{ll}
A_{1} & A_{2}
\end{array}\right), \quad A_{1} \in \mathbb{C}^{m \times k}, \quad A_{2} \in \mathbb{C}^{m \times k} ;  \tag{49}\\
B P=\left(\begin{array}{ll}
B_{1} & B_{2}
\end{array}\right), \quad B_{1} \in \mathbb{C}^{m \times k}, \quad B_{2} \in \mathbb{C}^{m \times k} ;  \tag{50}\\
P^{*} C=\binom{C_{1}}{C_{2}}, \quad C_{1} \in \mathbb{C}^{k \times q}, C_{2} \in \mathbb{C}^{k \times q} ;  \tag{51}\\
P^{*} D=\binom{D_{1}}{D_{2}}, \quad D_{1} \in \mathbb{C}^{k \times q}, D_{2} \in \mathbb{C}^{k \times q} ;  \tag{52}\\
A^{\prime}=\binom{A_{1}}{C_{1}^{*}}, \quad B^{\prime}=\binom{B_{2}}{D_{2}^{*}}  \tag{53}\\
C^{\prime}=\left(\begin{array}{ll}
A_{2}^{*} & C_{2}
\end{array}\right), \quad D^{\prime}=\left(\begin{array}{ll}
B_{1}^{*} & D_{1}
\end{array}\right) \tag{54}
\end{gather*}
$$

Then Problem 5 has a solution $X \in H H \mathbb{C}^{2 k \times 2 k}$ if and only if

$$
\begin{gather*}
A^{\prime}\left(A^{\prime}\right)^{\dagger} B^{\prime}=B^{\prime}, \quad D^{\prime}\left(C^{\prime}\right)^{\dagger} C^{\prime}=D^{\prime}  \tag{55}\\
A^{\prime} D^{\prime}=B^{\prime} C^{\prime}
\end{gather*}
$$

in which case the Hermitian generalized Hamiltonian solution to Problem 5 can be expressed as

$$
X=P\left(\begin{array}{cc}
0 & X_{12}  \tag{56}\\
X_{12}^{*} & 0
\end{array}\right) P^{*}
$$

where

$$
\begin{align*}
X_{12}= & \left(A^{\prime}\right)^{\dagger} B^{\prime}+D^{\prime}\left(C^{\prime}\right)^{\dagger}-\left(A^{\prime}\right)^{\dagger} A^{\prime} D^{\prime}\left(C^{\prime}\right)^{\dagger}  \tag{57}\\
& +L_{A^{\prime}} W R_{C^{\prime}}
\end{align*}
$$

and $W \in \mathbb{C}^{k \times k}$ is arbitrary.
Proof. It follows from (7) and (49)-(52) that the system (3) can be transformed into the following system of matrix equations:

$$
\begin{array}{ll}
A_{1} X_{12}=B_{2}, & X_{12} A_{2}^{*}=B_{1}^{*} \\
C_{1}^{*} X_{12}=D_{2}^{*}, & X_{12} C_{2}=D_{1} \tag{58}
\end{array}
$$

Then, combining (53) and (54) yields that

$$
\begin{equation*}
A^{\prime} X_{12}=B^{\prime}, \quad X_{12} C^{\prime}=D^{\prime} \tag{59}
\end{equation*}
$$

Thus, by Lemma 12, the system (59) has a solution $X_{12} \in \mathbb{C}^{k \times k}$ if and only if all equalities in (55) hold, in which case the solution can be written as (57). So the solution to system (3) can be expressed as (56).

Remark 21. Let $C$ and $D$ vanish in Theorem 20. Partition

$$
\begin{align*}
& A P=\left(\begin{array}{ll}
A_{1} & A_{2}
\end{array}\right), \quad A_{1} \in \mathbb{C}^{m \times k}, A_{2} \in \mathbb{C}^{m \times k} \\
& B P=\left(\begin{array}{ll}
B_{1} & B_{2}
\end{array}\right), \quad B_{1} \in \mathbb{C}^{m \times k}, B_{2} \in \mathbb{C}^{m \times k} \tag{60}
\end{align*}
$$

Then the matrix equation $A X=B$ has Hermitian generalized Hamiltonian solutions if and only if

$$
\begin{gather*}
A_{1} A_{1}^{\dagger} B_{2}=B_{2}, \quad A_{2} A_{2}^{\dagger} B_{1}=B_{1}  \tag{61}\\
A_{1} B_{1}^{*}=B_{2} A_{2}^{*}
\end{gather*}
$$

in which case its solution can be described as

$$
X=P\left(\begin{array}{cc}
0 & X_{12}  \tag{62}\\
X_{12}^{*} & 0
\end{array}\right) P^{*}
$$

where

$$
\begin{equation*}
X_{12}=A_{1}^{\dagger} B_{2}+B_{1}^{*}\left(A_{2}^{\dagger}\right)^{*}-A_{1}^{\dagger} A_{1} B_{1}^{*}\left(A_{2}^{\dagger}\right)^{*}+L_{A_{1}} W L_{A_{2}} \tag{63}
\end{equation*}
$$

and $W \in \mathbb{C}^{k \times k}$ is arbitrary. It is clear that this result is different from Theorem 3.1 given in [1].

Similarly, by Lemmas 9 and 12, we can get the antiHermitian generalized anti-Hamiltonian solution to system (3).

Theorem 22. Given $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$, let the decomposition of $X \in A H A H \mathbb{C}^{2 k \times 2 k}$ be (8). $A P, B P, P^{*} C$, and $P^{*} D$, respectively, have the partitions as in (49)-(52). Put

$$
\begin{array}{cc}
\widetilde{A}=\binom{A_{1}}{C_{1}^{*}}, \quad \widetilde{B}=\binom{B_{2}}{-D_{2}^{*}},  \tag{64}\\
\widetilde{C}=\left(\begin{array}{ll}
A_{2}^{*} & C_{2}
\end{array}\right), & \widetilde{D}=\left(\begin{array}{ll}
-B_{1}^{*} & D_{1}
\end{array}\right) .
\end{array}
$$

Then Problem 5 has a solution $X \in A H A H \mathbb{C}^{2 k \times 2 k}$ if and only if

$$
\begin{equation*}
\widetilde{A} \widetilde{A}^{\dagger} \widetilde{B}=\widetilde{B}, \quad \widetilde{D} \widetilde{C}^{\dagger} \widetilde{C}=\widetilde{D}, \quad \widetilde{A} \widetilde{D}=\widetilde{B} \widetilde{C} \tag{65}
\end{equation*}
$$

in which case the anti-Hermitian generalized anti-Hamiltonian solution to Problem 5 can be expressed as

$$
X=P\left(\begin{array}{cc}
0 & X_{12}  \tag{66}\\
-X_{12}^{*} & 0
\end{array}\right) P^{*}
$$

where

$$
\begin{equation*}
X_{12}=\widetilde{A}^{\dagger} \widetilde{B}+\widetilde{D} \widetilde{C}^{\dagger}-\widetilde{A}^{\dagger} \widetilde{A} \widetilde{D} \widetilde{C}^{\dagger}+L_{\widetilde{A}} Z R_{\widetilde{C}} \tag{67}
\end{equation*}
$$

and $Z \in \mathbb{C}^{k \times k}$ is arbitrary.
Now, we investigate the Hermitian generalized antiHamiltonian solution to the system (3).

Theorem 23. Given $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$, let the decomposition of $X \in H A H \mathbb{C}^{2 k \times 2 k}$ be (9). $A P, B P, P^{*} C$, and $P^{*} D$, respectively, have the partitions as in (49)-(52). Denote

$$
\begin{array}{ll}
\bar{A}=\binom{A_{1}}{C_{1}^{*}}, & \bar{B}=\binom{B_{1}}{D_{1}^{*}}, \\
\bar{C}=\binom{A_{2}}{C_{2}^{*}}, & \bar{D}=\binom{B_{2}}{D_{2}^{*}} . \tag{69}
\end{array}
$$

Let the singular value decompositions of $\bar{A}$ and $\bar{C}$ be, respectively,

$$
\begin{align*}
& \bar{A}=U\left(\begin{array}{ll}
\Sigma & 0 \\
0 & 0
\end{array}\right) V^{*},  \tag{70}\\
& \bar{C}=Q\left(\begin{array}{cc}
\Pi & 0 \\
0 & 0
\end{array}\right) R^{*}, \tag{71}
\end{align*}
$$

where

$$
\begin{array}{r}
U \in U \mathbb{C}^{(m+q) \times k}, \quad V \in U \mathbb{C}^{k \times k}, \\
\Sigma=\operatorname{diag}\left(\alpha_{1}, \ldots, \alpha_{r}\right), \quad \alpha_{i}>0, \\
i=1, \ldots, r ; r=r(\bar{A}), \\
Q \in U \mathbb{C}^{(m+q) \times k}, \quad R \in U \mathbb{C}^{k \times k}, \\
\Pi=\operatorname{diag}\left(\beta_{1}, \ldots, \beta_{s}\right), \quad \beta_{j}>0, \\
j=1, \ldots, s ; s=r(\bar{C}) .
\end{array}
$$

Set

$$
\begin{align*}
V X_{11} V^{*} & =\left(\begin{array}{ll}
\bar{X}_{11} & \bar{X}_{12} \\
\bar{X}_{12}^{*} & \bar{X}_{22}
\end{array}\right) ;  \tag{73}\\
U^{*} \bar{B} V & =\left(\begin{array}{ll}
\bar{B}_{11} & \bar{B}_{12} \\
\bar{B}_{21} & \bar{B}_{22}
\end{array}\right) ;  \tag{74}\\
R X_{22} R^{*} & =\left(\begin{array}{ll}
\widehat{X}_{11} & \widehat{X}_{12} \\
\widehat{X}_{12}^{*} & \widehat{X}_{22}
\end{array}\right) ;  \tag{75}\\
Q^{*} \bar{D} R & =\left(\begin{array}{ll}
\bar{D}_{11} & \bar{D}_{12} \\
\bar{D}_{21} & \bar{D}_{22}
\end{array}\right), \tag{76}
\end{align*}
$$

where

$$
\begin{align*}
\bar{X}_{11} \in H \mathbb{C}^{r \times r}, & \widehat{X}_{11} \in H \mathbb{C}^{s \times s}, \\
\bar{X}_{22} \in H \mathbb{C}^{(k-r) \times(k-r)}, & \widehat{X}_{22} \in H \mathbb{C}^{(k-s) \times(k-s)}, \\
\bar{B}_{11} \in \mathbb{C}^{r \times r}, & \bar{D}_{11} \in \mathbb{C}^{s \times s},  \tag{77}\\
\bar{B}_{22} \in \mathbb{C}^{(m+q-r) \times(k-r)}, & \bar{D}_{22} \in \mathbb{C}^{(m+q-s) \times(k-s)} .
\end{align*}
$$

Then Problem 5 has a solution $X \in H A H \mathbb{C}^{2 k \times 2 k}$ if and only if

$$
\begin{array}{cl}
\bar{A}(\bar{A})^{\dagger} \bar{B}=\bar{B}, & \bar{A}(\bar{B})^{*}=\bar{B}(\bar{A})^{*}, \\
\bar{B}_{21}=0, & \bar{B}_{22}=0, \\
\bar{C}(\bar{C})^{\dagger} \bar{D}=\bar{D}, & \bar{C}(\bar{D})^{*}=\bar{D}(\bar{C})^{*},  \tag{79}\\
\bar{D}_{21}=0, & \bar{D}_{22}=0,
\end{array}
$$

in which case the Hermitian generalized anti-Hamiltonian solution to Problem 5 can be described as

$$
X=P\left(\begin{array}{cc}
X_{11} & 0  \tag{80}\\
0 & X_{22}
\end{array}\right) P^{*}
$$

where

$$
\begin{align*}
& X_{11}=V\left(\begin{array}{cc}
\Sigma^{-1} \bar{B}_{11} & \Sigma^{-1} \bar{B}_{12} \\
\left(\bar{B}_{12}\right)^{*} \Sigma^{-1} & \bar{X}_{22}
\end{array}\right) V^{*},  \tag{81}\\
& X_{22}=R\left(\begin{array}{cc}
\Pi^{-1} \bar{D}_{11} & \Pi^{-1} \bar{D}_{12} \\
\left(\bar{D}_{12}\right)^{*} \Pi^{-1} & \widehat{X}_{22}
\end{array}\right) R^{*} \tag{82}
\end{align*}
$$

and $\bar{X}_{22} \in H \mathbb{C}^{(k-r) \times(k-r)}, \widehat{X}_{22} \in H \mathbb{C}^{(k-s) \times(k-s)}$ are arbitrary.
Proof. It can be derived from (9), (49)-(52), and (68)-(69) that the system (3) is consistent if and only if the following two equations:

$$
\begin{align*}
& \bar{A} X_{11}=\bar{B},  \tag{83}\\
& \bar{C} X_{22}=\bar{D}, \tag{84}
\end{align*}
$$

are solvable. By (70), (73), and (74), and then combining Lemma 13, we can obtain that there exists Hermitian solution
$X_{11}$ such that (83) holds if and only if all equalities in (78) hold, in which case the solution can be written as (81). By the similar way, there exists Hermitian solution $X_{22}$ such that (84) holds if and only if all equalities in (79) hold, in which case the solution can be described as (82). Therefore, the Hermitian generalized anti-Hamiltonian solution to Problem 5 can be expressed as (80).

From Lemmas 11 and 14, it is not difficult to obtain the anti-Hermitian generalized Hamiltonian solution to Problem 5 , which can be described as follows.

Theorem 24. Given $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$, let the decomposition of $X \in A H H \mathbb{C}^{2 k \times 2 k}$ be (10). $A P, B P, P^{*} C$, and $P^{*} D$, respectively, have the partitions as in (49)-(52). Denote

$$
\begin{array}{ll}
\widehat{A}=\binom{A_{1}}{C_{1}^{*}}, & \widehat{B}=\binom{B_{1}}{-D_{1}^{*}}, \\
\widehat{C}=\binom{A_{2}}{C_{2}^{*}}, & \widehat{D}=\binom{B_{2}}{-D_{2}^{*}} . \tag{85}
\end{array}
$$

Let the singular value decompositions of $\widehat{A}$ and $\widehat{C}$ be, respectively,

$$
\begin{align*}
& \widehat{A}=U\left(\begin{array}{ll}
\Sigma & 0 \\
0 & 0
\end{array}\right) V^{*} \\
& \widehat{C}=Q\left(\begin{array}{cc}
\Pi & 0 \\
0 & 0
\end{array}\right) R^{*} \tag{86}
\end{align*}
$$

where

$$
\begin{array}{r}
U \in U \mathbb{C}^{(m+q) \times k}, \quad V \in U \mathbb{C}^{k \times k} \\
\Sigma=\operatorname{diag}\left(\alpha_{1}, \ldots, \alpha_{r}\right), \quad \alpha_{i}>0 \\
i=1, \ldots, r ; r=r(\widehat{A}) \\
Q \in U \mathbb{C}^{(m+q) \times k}, \quad R \in U \mathbb{C}^{k \times k}  \tag{87}\\
\Pi=\operatorname{diag}\left(\beta_{1}, \ldots, \beta_{s}\right), \quad \beta_{j}>0 \\
j=1, \ldots, s ; s=r(\widehat{C})
\end{array}
$$

Set

$$
\begin{aligned}
V X_{11} V^{*} & =\left(\begin{array}{cc}
\bar{X}_{11} & \bar{X}_{12} \\
-\bar{X}_{12}^{*} & \bar{X}_{22}
\end{array}\right) ; \\
U^{*} \widehat{B} V & =\left(\begin{array}{cc}
\widehat{B}_{11} & \widehat{B}_{12} \\
\widehat{B}_{21} & \widehat{B}_{22}
\end{array}\right) ; \\
R X_{22} R^{*} & =\left(\begin{array}{cc}
\widehat{X}_{11} & \widehat{X}_{12} \\
-\widehat{X}_{12}^{*} & \widehat{X}_{22}
\end{array}\right) ; \\
Q^{*} \widehat{D} R & =\left(\begin{array}{cc}
\widehat{D}_{11} & \widehat{D}_{12} \\
\widehat{D}_{21} & \widehat{D}_{22}
\end{array}\right)
\end{aligned}
$$

where

$$
\begin{align*}
\bar{X}_{11} \in H \mathbb{C}^{r \times r}, & \widehat{X}_{11} \in H \mathbb{C}^{s \times s}, \\
\bar{X}_{22} \in H \mathbb{C}^{(k-r) \times(k-r)}, & \widehat{X}_{22} \in H \mathbb{C}^{(k-s) \times(k-s)}, \\
\widehat{B}_{11} \in \mathbb{C}^{r \times r}, & \widehat{D}_{11} \in \mathbb{C}^{s \times s}  \tag{89}\\
\widehat{B}_{22} \in \mathbb{C}^{(m+q-r) \times(k-r)}, & \widehat{D}_{22} \in \mathbb{C}^{(m+q-s) \times(k-s)}
\end{align*}
$$

Then Problem 5 has a solution $X \in A H H \mathbb{C}^{2 k \times 2 k}$ if and only if

$$
\begin{array}{rlrl}
\widehat{A} \widehat{A}^{\dagger} \widehat{B}=\widehat{B}, & \widehat{A} \widehat{B}^{*} & =-\widehat{B} \widehat{A}^{*}, \\
\widehat{B}_{21} & =0, & \widehat{B}_{22} & =0 \\
\widehat{C} \widehat{C}^{\dagger} \widehat{D}=\widehat{D}, & \widehat{C} \widehat{D}^{*} & =-\widehat{D} \widehat{C}^{*}  \tag{90}\\
\widehat{D}_{21} & =0, & \widehat{D}_{22} & =0
\end{array}
$$

in which case the anti-Hermitian generalized Hamiltonian solution to Problem 5 can be described as

$$
X=P\left(\begin{array}{cc}
X_{11} & 0  \tag{91}\\
0 & X_{22}
\end{array}\right) P^{*}
$$

where

$$
\begin{gathered}
X_{11}=V\left(\begin{array}{cc}
\Sigma^{-1} \widehat{B}_{11} & \Sigma^{-1} \widehat{B}_{12} \\
-\widehat{B}_{12}^{*} \Sigma^{-1} & \bar{X}_{22}
\end{array}\right) V^{*}, \\
X_{22}=R\left(\begin{array}{cc}
\Pi^{-1} \widehat{D}_{11} & \Pi^{-1} \widehat{D}_{12} \\
-\widehat{D}_{12}^{*} \Pi^{-1} & \widehat{X}_{22}
\end{array}\right) R^{*}, \\
\text { and } \bar{X}_{22} \in A H \mathbb{C}^{(k-r) \times(k-r)}, \widehat{X}_{22} \in A H \mathbb{C}^{(k-s) \times(k-s)} \text { are }
\end{gathered}
$$ arbitrary.

## 4. The Expression of the Unique Solution to Problem 6

In this section, our aim is to derive the optimal approximation solution to Problem 6.

Theorem 25. Given $\widehat{X} \in \mathbb{C}^{2 k \times 2 k}$, under the hypotheses of Theorem 20, let

$$
P^{*} \widehat{X} P=\left(\begin{array}{ll}
\widehat{X}_{11} & \widehat{X}_{12}  \tag{93}\\
\widehat{X}_{21} & \widehat{X}_{22}
\end{array}\right), \quad \widehat{X}_{11} \in \mathbb{C}^{k \times k}, \widehat{X}_{22} \in \mathbb{C}^{k \times k}
$$

If Problem 5 has Hermitian generalized Hamiltonian solutions, then Problem 6 has a unique solution $\widetilde{X} \in H H \mathbb{C}^{2 k \times 2 k}$ if and only if

$$
\begin{equation*}
L_{A^{\prime}}\left(\frac{\widehat{X}_{12}+\left(\widehat{X}_{21}\right)^{*}}{2}-X_{0}\right) R_{C^{\prime}}=\frac{\widehat{X}_{12}+\left(\widehat{X}_{21}\right)^{*}}{2}-X_{0} \tag{94}
\end{equation*}
$$

in which case the unique solution $\widetilde{X}$ can be expressed as

$$
\widetilde{X}=P\left(\begin{array}{cc}
0 & \overline{X_{0}}  \tag{95}\\
\left(\overline{X_{0}}\right)^{*} & 0
\end{array}\right) P^{*}
$$

where

$$
\begin{gather*}
\overline{X_{0}}=\frac{\widehat{X}_{12}+\left(\widehat{X}_{21}\right)^{*}}{2}  \tag{96}\\
X_{0}=\left(A^{\prime}\right)^{\dagger} B^{\prime}+D^{\prime}\left(C^{\prime}\right)^{\dagger}-\left(A^{\prime}\right)^{\dagger} A^{\prime} D^{\prime}\left(C^{\prime}\right)^{\dagger}
\end{gather*}
$$

Proof. When the Hermitian generalized Hamiltonian solution set $K$ of Problem 5 is nonempty, it is not difficult to verify that $K$ is a closed convex set. Then by [33], Problem 6 has a unique solution $\widetilde{X} \in H H \mathbb{C}^{2 k \times 2 k}$. From Theorem 20, for any $X \in K, X$ can be expressed as

$$
X=P\left(\begin{array}{cc}
0 & X_{0}  \tag{97}\\
X_{0}^{*} & 0
\end{array}\right) P^{*}+P\left(\begin{array}{cc}
0 & L_{A^{\prime}} W R_{C^{\prime}} \\
R_{C^{\prime}} W^{*} L_{A^{\prime}} & 0
\end{array}\right) P^{*}
$$

where

$$
\begin{equation*}
X_{0}=\left(A^{\prime}\right)^{\dagger} B^{\prime}+D^{\prime}\left(C^{\prime}\right)^{\dagger}-\left(A^{\prime}\right)^{\dagger} A^{\prime} D^{\prime}\left(C^{\prime}\right)^{\dagger} \tag{98}
\end{equation*}
$$

and $W \in \mathbb{C}^{k \times k}$ is arbitrary. Then it follows from the equalities in (93) and (97) and the unitary invariance of the Frobenius norm that

$$
\begin{align*}
\| \widehat{X} & -X \|^{2} \\
= & \left\|P^{*} \widehat{X} P-P^{*} X P\right\|^{2} \\
= & \left\|\left(\widehat{X}_{21}-X_{0}^{*}-\widehat{X}_{C^{\prime}} W^{*} L_{A^{\prime}} \quad \widehat{X}_{12}-X_{0}-L_{A^{\prime}} W R_{C^{\prime}}\right)\right\|_{22}^{2} \\
= & \left\|\widehat{X}_{11}\right\|^{2}+\left\|\widehat{X}_{22}\right\|^{2}+\left\|\widehat{X}_{12}-X_{0}-L_{A^{\prime}} W R_{C^{\prime}}\right\|^{2} \\
& +\left\|\widehat{X}_{21}-X_{0}^{*}-R_{C^{\prime}} W^{*} L_{A^{\prime}}\right\|^{2} \\
= & \left\|\widehat{X}_{11}\right\|^{2}+\left\|\widehat{X}_{22}\right\|^{2}+\left\|L_{A^{\prime}} W R_{C^{\prime}}-\left(-X_{0}+\widehat{X}_{12}\right)\right\|^{2} \\
& +\left\|L_{A^{\prime}} W R_{C^{\prime}}-\left(-X_{0}+\widehat{X}_{21}^{*}\right)\right\|^{2} . \tag{99}
\end{align*}
$$

Thus, Problem 6 has a unique solution $\widetilde{X} \in H H \mathbb{C}^{2 k \times 2 k}$ if and only if there exists $W$ such that

$$
\begin{align*}
& \left\|L_{A^{\prime}} W R_{C^{\prime}}-\left(-X_{0}+\widehat{X}_{12}\right)\right\|^{2} \\
& \quad+\left\|L_{A^{\prime}} W R_{C^{\prime}}-\left(-X_{0}+\left(\widehat{X}_{21}\right)^{*}\right)\right\|^{2} \tag{100}
\end{align*}
$$

reaches its minimum. Therefore, by Lemma 19, (100) arrives at its minimum if and only if there exists $W$ such that the matrix equation

$$
\begin{align*}
L_{A^{\prime}} W R_{C^{\prime}} & =\frac{-X_{0}+\widehat{X}_{12}-X_{0}+\left(\widehat{X}_{21}\right)^{*}}{2} \\
& =\frac{\widehat{X}_{12}+\left(\widehat{X}_{21}\right)^{*}}{2}-X_{0} \tag{101}
\end{align*}
$$

holds, which, by Lemma 18, has a solution if and only if (94) holds, in which case the solution can be expressed as

$$
\begin{equation*}
W=L_{A^{\prime}}\left(\frac{\widehat{X}_{12}+\left(\widehat{X}_{21}\right)^{*}}{2}-X_{0}\right) R_{C^{\prime}}+Z-L_{A^{\prime}} Z R_{C^{\prime}} \tag{102}
\end{equation*}
$$

where $Z \in \mathbb{C}^{k \times k}$ is arbitrary. Inserting (102) into (97), and then combining (94) yields (95).

Analogously, the following theorem can be shown.
Theorem 26. Given $\widehat{X} \in \mathbb{C}^{2 k \times 2 k}$, under the hypotheses of Theorem 22, let

$$
P^{*} \widehat{X} P=\left(\begin{array}{ll}
\widehat{X}_{11} & \widehat{X}_{12}  \tag{103}\\
\widehat{X}_{21} & \widehat{X}_{22}
\end{array}\right), \quad \widehat{X}_{11} \in \mathbb{C}^{k \times k}, \quad \widehat{X}_{22} \in \mathbb{C}^{k \times k} .
$$

If Problem 5 has anti-Hermitian generalized anti-Hamiltonian solutions, then Problem 6 has a unique solution $\widetilde{X} \in$ AHAH $\mathbb{C}^{2 k \times 2 k}$ if and only if

$$
\begin{equation*}
L_{\widetilde{A}}\left(\frac{\widehat{X}_{12}-\left(\widehat{X}_{21}\right)^{*}}{2}-X_{0}\right) R_{\tilde{C}}=\frac{\widehat{X}_{12}-\left(\widehat{X}_{21}\right)^{*}}{2}-X_{0} \tag{104}
\end{equation*}
$$

in which case the unique solution $\widetilde{X}$ can be expressed as

$$
\widetilde{X}=P\left(\begin{array}{cc}
0 & \overline{X_{0}}  \tag{105}\\
-\left(\overline{X_{0}}\right)^{*} & 0
\end{array}\right) P^{*}
$$

where

$$
\begin{gather*}
\overline{X_{0}}=\frac{\widehat{X}_{12}-\left(\widehat{X}_{21}\right)^{*}}{2},  \tag{106}\\
\left.X_{0}=(\widetilde{A})^{\dagger} \widetilde{B}+\widetilde{D}(\widetilde{C})^{\dagger}-(\widetilde{A})^{\dagger} \widetilde{A D} \widetilde{C}\right)^{\dagger}
\end{gather*}
$$

Now, we give the unique Hermitian generalized antiHamiltonian solution to Problem 6.

Theorem 27. Given $\widehat{X} \in \mathbb{C}^{2 k \times 2 k}$, under the hypotheses of Theorem 23, let

$$
\begin{gather*}
P^{*} \frac{\widehat{X}+\widehat{X}^{*}}{2} P=\left(\begin{array}{cc}
\widehat{X}_{11}^{\prime} & \widehat{X}_{12}^{\prime} \\
\left(\widehat{X}_{12}^{\prime}\right)^{*} & \widehat{X}_{22}^{\prime}
\end{array}\right),  \tag{107}\\
\widehat{X}_{11}^{\prime} \in H \mathbb{C}^{k \times k}, \widehat{X}_{22}^{\prime} \in H \mathbb{C}^{k \times k} ; \\
V^{*} \widehat{X}_{11}^{\prime} V=\left(\begin{array}{cc}
\widehat{X}_{11}^{\circ} & \widehat{X}_{12}^{\circ} \\
\left(\widehat{X}_{12}^{\circ}\right)^{*} & \widehat{X}_{22}^{\circ}
\end{array}\right),  \tag{108}\\
\widehat{X}_{11}^{\circ} \in H \mathbb{C}^{r \times r}, \widehat{X}_{22}^{\circ} \in H \mathbb{C}^{(k-r) \times(k-r)} ; \\
R^{*} \widehat{X}_{22}^{\prime} R=\left(\begin{array}{cc}
\widehat{X}_{11}^{\prime} & \widehat{X}_{12}^{\prime \prime} \\
\left(\widehat{X}_{12}^{\prime \prime}\right)^{*} & \widehat{X}_{22}^{\prime \prime}
\end{array}\right),  \tag{109}\\
\widehat{X}_{11}^{\prime \prime} \in H \mathbb{C}^{s \times s}, \widehat{X}_{22}^{\prime \prime} \in H \mathbb{C}^{(k-s) \times(k-s)}
\end{gather*}
$$

If Problem 5 has Hermitian generalized anti-Hamiltonian solutions, then the unique solution $\widetilde{X} \in H A H \mathbb{C}^{2 k \times 2 k}$ to Problem 6 can be expressed as

$$
\widetilde{X}=P\left(\begin{array}{cc}
X_{11}^{\circ} & 0  \tag{110}\\
0 & X_{22}^{\circ}
\end{array}\right) P^{*}
$$

where

$$
\begin{align*}
& X_{11}^{\circ}=V\left(\begin{array}{cc}
\Sigma^{-1} \bar{B}_{11} & \Sigma^{-1} \bar{B}_{12} \\
\left(\bar{B}_{12}\right)^{*} \Sigma^{-1} & \widehat{X}_{22}^{\circ}
\end{array}\right) V^{*}  \tag{111}\\
& X_{22}^{\circ}=R\left(\begin{array}{cc}
\Pi^{-1} \bar{D}_{11} & \Pi^{-1} \bar{D}_{12} \\
\left(\bar{D}_{12}\right)^{*} \Pi^{-1} & \widehat{X}_{22}^{\prime \prime}
\end{array}\right) R^{*} . \tag{112}
\end{align*}
$$

Proof. When the Hermitian generalized anti-Hamiltonian solution set $K$ of Problem 5 is nonempty, it is easy to prove that $K$ is a closed convex set. Then, Problem 6 has a unique solution $\widetilde{X} \in H A H \mathbb{C}^{2 k \times 2 k}$ by the aid of [33]. For any $X \in K$, due to Theorem 23, $X$ can be expressed as

$$
X=P\left(\begin{array}{cc}
X_{11} & 0  \tag{113}\\
0 & X_{22}
\end{array}\right) P^{*}
$$

where $X_{11}$ and $X_{22}$ have the expressions as in (81) and (82). Combining the equalities in (80)-(82) and (107) and the unitary invariance of the Frobenius norm yields that

$$
\begin{align*}
\|X-\widehat{X}\|^{2}= & \left\|X-\frac{\widehat{X}+\widehat{X}^{*}}{2}\right\|^{2}+\left\|\frac{\widehat{X}-\widehat{X}^{*}}{2}\right\|^{2} \\
= & \left\|P\left(\begin{array}{cc}
X_{11} & 0 \\
0 & X_{22}
\end{array}\right) P^{*}-\frac{\widehat{X}+\widehat{X}^{*}}{2}\right\|^{2} \\
& +\left\|\frac{\widehat{X}-\widehat{X}^{*}}{2}\right\|^{2} \\
= & \left\|\left(\begin{array}{cc}
X_{11} & 0 \\
0 & X_{22}
\end{array}\right)-\left(\begin{array}{c}
\widehat{X}_{11}^{\prime} \\
\left(\widehat{X}_{12}^{\prime}\right)^{\prime} \\
\widehat{X}_{22}^{\prime}
\end{array}\right)\right\|^{2}  \tag{114}\\
& +\left\|\frac{\widehat{X}-\widehat{X}^{*}}{2}\right\|^{2} \\
= & \left\|X_{11}-\widehat{X}_{11}^{\prime}\right\|^{2}+\left\|X_{22}-\widehat{X}_{22}^{\prime}\right\|^{2} \\
& +2\left\|\widehat{X}_{12}^{\prime}\right\|^{2}+\left\|\frac{\widehat{X}-\widehat{X}^{*}}{2}\right\|^{2}
\end{align*}
$$

So,

$$
\begin{align*}
& \min _{X \in H A H \mathbb{C}^{2 k \times 2 k}}\|X-\widehat{X}\|^{2} \text { holds } \\
& \Longleftrightarrow \min _{X_{11} \in H \mathbb{C}^{k \times k}}\left\|X_{11}-\widehat{X}_{11}^{\prime}\right\|^{2} \text { holds }  \tag{115}\\
& \quad \text { and } \min _{X_{22} \in H \mathbb{C}^{k \times k}}\left\|X_{22}-\widehat{X}_{22}^{\prime}\right\|^{2} \text { holds. }
\end{align*}
$$

By (81), (108), and the unitary invariance of the Frobenius norm, we obtain

$$
\begin{align*}
\| X_{11} & -\widehat{X}_{11}^{\prime} \|^{2} \\
= & \left\|\left(\begin{array}{cc}
\Sigma^{-1} \bar{B}_{11} & \Sigma^{-1} \bar{B}_{12} \\
\left(\bar{B}_{12}\right)^{*} \Sigma^{-1} & \bar{X}_{22}
\end{array}\right)-\left(\begin{array}{cc}
\widehat{X}_{11}^{\circ} & \widehat{X}_{12}^{\circ} \\
\left(\widehat{X}_{12}^{\circ}\right)^{*} & \widehat{X}_{22}^{\circ}
\end{array}\right)\right\|^{2}  \tag{116}\\
= & \left\|\Sigma^{-1} \bar{B}_{11}-\widehat{X}_{11}^{\circ}\right\|^{2}+\left\|\bar{X}_{22}-\widehat{X}_{22}^{\circ}\right\|^{2} \\
& +2\left\|\Sigma^{-1} \bar{B}_{12}-\widehat{X}_{12}^{\circ}\right\|^{2} .
\end{align*}
$$

Then

$$
\begin{align*}
& \min _{X_{11} \in H \mathbb{C}^{k \times k}}\left\|X_{11}-\widehat{X}_{11}^{\prime}\right\|^{2} \text { holds }  \tag{117}\\
& \Longleftrightarrow \min _{\bar{X}_{22} \in H \mathbb{C}^{(k-r) \times(k-r)}}\left\|\bar{X}_{22}-\widehat{X}_{22}^{\circ}\right\|^{2} \text { holds. }
\end{align*}
$$

Therefore, when $\bar{X}_{22}$ can be expressed as

$$
\begin{equation*}
\bar{X}_{22}=\widehat{X}_{22}^{\circ} \tag{118}
\end{equation*}
$$

$\min _{X_{11} \in H \mathbb{C}^{k \times k}}\left\|X_{11}-\widehat{X}_{11}^{\prime}\right\|^{2}$ holds. Then combining (81) yields (111). Similarly, we can derive the expression in (112) by (82) and (109). Thus, (110) is the unique solution to Problem 6.

By the method used in Theorem 27, the following theorem can also be shown.

Theorem 28. Given $\widehat{X} \in \mathbb{C}^{2 k \times 2 k}$, under the hypotheses of Theorem 24, let

$$
\begin{gather*}
P^{*} \frac{\widehat{X}-\widehat{X}^{*}}{2} P=\left(\begin{array}{cc}
\widehat{X}_{11}^{\prime} & \widehat{X}_{12}^{\prime} \\
-\left(\widehat{X}_{12}^{\prime}\right)^{*} & \widehat{X}_{22}^{\prime}
\end{array}\right), \\
\widehat{X}_{11}^{\prime} \in A H \mathbb{C}^{k \times k}, \widehat{X}_{22}^{\prime} \in A H \mathbb{C}^{k \times k} ; \\
V^{*} \widehat{X}_{11}^{\prime} V=\left(\begin{array}{cc}
\widehat{X}_{11}^{\circ} & \widehat{X}_{12}^{\circ} \\
-\left(\widehat{X}_{12}^{\circ}\right)^{*} & \widehat{X}_{22}^{\circ}
\end{array}\right),  \tag{119}\\
\widehat{X}_{11}^{\circ} \in A H \mathbb{C}^{r \times r}, \widehat{X}_{22}^{\circ} \in A H \mathbb{C}^{(k-r) \times(k-r)} ; \\
R^{*} \widehat{X}_{22}^{\prime} R=\left(\begin{array}{cc}
\widehat{X}_{11}^{\prime \prime} & \widehat{X}_{12}^{\prime \prime} \\
-\left(\widehat{X}_{12}^{\prime \prime}\right)^{*} & \widehat{X}_{22}^{\prime \prime}
\end{array}\right), \\
\widehat{X}_{11}^{\prime \prime} \in A H \mathbb{C}^{s \times s}, \widehat{X}_{22}^{\prime \prime} \in A H \mathbb{C}^{(k-s) \times(k-s)} .
\end{gather*}
$$

If Problem 5 has anti-Hermitian generalized Hamiltonian solutions, then the unique solution $\widetilde{X} \in A H H \mathbb{C}^{2 k \times 2 k}$ to Problem 6 can be expressed as

$$
\widetilde{X}=P\left(\begin{array}{cc}
X_{11}^{\circ} & 0  \tag{120}\\
0 & X_{22}^{\circ}
\end{array}\right) P^{*}
$$

where

$$
\begin{align*}
& X_{11}^{\circ}=V\left(\begin{array}{cc}
\Sigma^{-1} \widehat{B}_{11} & \Sigma^{-1} \widehat{B}_{12} \\
-\widehat{B}_{12}^{*} \Sigma^{-1} & \widehat{X}_{22}^{\circ}
\end{array}\right) V^{*} \\
& X_{22}^{\circ}=R\left(\begin{array}{cc}
\Pi^{-1} \widehat{D}_{11} & \Pi^{-1} \widehat{D}_{12} \\
-\widehat{D}_{12}^{*} \Pi^{-1} & \widehat{X}_{22}^{\prime \prime}
\end{array}\right) R^{*} \tag{121}
\end{align*}
$$

## 5. The Expression of the Solution to Problem 7

If the solvability conditions of linear matrix equations are not satisfied, the least squares solution is usually considered. So, in this section, the solution to Problem 7 is constructed.

Theorem 29. Given $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$, let the decomposition of $X \in H H \mathbb{C}^{2 k \times 2 k}$ be (7). $A P, B P, P^{*} C, P^{*} D, C^{\prime}$, and $D^{\prime}$, respectively, have the partitions as in (49)-(52) and (54). Denote

$$
A^{\prime}=\left(\begin{array}{ll}
A_{1}^{*} & C_{1}
\end{array}\right), \quad B^{\prime}=\left(\begin{array}{ll}
B_{2}^{*} & D_{2} \tag{122}
\end{array}\right) .
$$

Let the singular value decompositions of $A^{\prime}$ and $C^{\prime}$ be as given in (28). Then the least squares Hermitian generalized Hamiltonian solution to Problem 7 can be described as (7), where $X_{12}$ has the expression as in (31).

Proof. Combining (7), (49)-(52), (54), (122), and the unitary invariance of the Frobenius norm yields that

$$
\begin{align*}
& \|A X-B\|^{2}+\|X C-D\|^{2} \\
& \quad=\left\|\left(A^{\prime}\right)^{*} X_{12}-\left(B^{\prime}\right)^{*}\right\|^{2}+\left\|X_{12} C^{\prime}-D^{\prime}\right\|^{2} \tag{123}
\end{align*}
$$

Therefore, by Lemma 15, if $X_{12}$ has the expression as in (31), then (123) reaches its minimum. Then, substituting (31) into (7), we obtain the least squares Hermitian generalized Hamiltonian solution to Problem 7.

Corollary 30. Given $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$, under the conditions of Theorem 29, the least squares Hermitian generalized Hamiltonian solution with minimum norm to Problem 7 can be described as (7), where $X_{12}$ has the expression as in (31) with $X_{22}^{\prime}=0$.

By the same way, we can also derive the least squares antiHermitian generalized anti-Hamiltonian solution to Problem 7.

Theorem 31. Given $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$, let the decomposition of $X \in A H A H \mathbb{C}^{2 k \times 2 k}$ be (8). $A P, B P, P^{*} C$, and $P^{*} D$, respectively, have the partitions as in (49)-(52). Denote

$$
\begin{array}{ll}
A^{\prime}=\left(\begin{array}{ll}
A_{1}^{*} & C_{1}
\end{array}\right), & B^{\prime}=\left(\begin{array}{ll}
B_{2}^{*} & -D_{2}
\end{array}\right),  \tag{124}\\
C^{\prime}=\left(\begin{array}{ll}
A_{2}^{*} & C_{2}
\end{array}\right), & D^{\prime}=\left(\begin{array}{ll}
-B_{1}^{*} & D_{1}
\end{array}\right)
\end{array}
$$

Let the singular value decompositions of $A^{\prime}$ and $C^{\prime}$ be as in (28). Then the least squares anti-Hermitian generalized antiHamiltonian solution to Problem 7 can be described as (8), where $X_{12}$ has the expression as in (31).

Corollary 32. Given $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$, under the conditions of Theorem 31, the least squares anti-Hermitian generalized anti-Hamiltonian solution with minimum norm to Problem 7 can be described as (8), where $X_{12}$ has the expression as in (31) with $X_{22}^{\prime}=0$.

At present, we give the least squares Hermitian generalized anti-Hamiltonian solution to Problem 7.

Theorem 33. Assume $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$. Let the decomposition of $X \in H A H \mathbb{C}^{2 k \times 2 k}$ be (9). $A P, B P, P^{*} C, P^{*} D$, $\bar{A}, \bar{B}, \bar{C}$, and $\bar{D}$, respectively, have the partitions as in (49)(52), (68), and (69). Let the singular value decompositions of $\bar{A}$ and $\bar{C}$ be, respectively, (70) and (71), VX $11 V^{*}, U^{*} \bar{B} V$, $R X_{22} R^{*}$, and $Q^{*} \bar{D} R$ have the partitions as in (73)-(76). Then the least squares Hermitian generalized anti-Hamiltonian solution to Problem 7 can be expressed as (9) with

$$
\begin{gather*}
X_{11}=V\left(\begin{array}{cc}
\Phi_{1} *\left(\Sigma \bar{B}_{11}+\bar{B}_{11}^{*} \Sigma\right) & \Sigma^{-1} \bar{B}_{12} \\
\bar{B}_{12}^{*} \Sigma^{-1} & \bar{X}_{22}
\end{array}\right) V^{*},  \tag{125}\\
X_{22}=R\left(\begin{array}{cc}
\Phi_{2} *\left(\Pi \bar{D}_{11}+\bar{D}_{11}^{*} \Pi\right) & \Pi^{-1} \bar{D}_{12} \\
\bar{D}_{12}^{*} \Pi^{-1} & \widehat{X}_{22}
\end{array}\right) R^{*} \tag{126}
\end{gather*}
$$

where

$$
\begin{array}{ll}
\Phi_{1}=\left(\frac{1}{\alpha_{i}^{2}+\alpha_{j}^{2}}\right), & 1 \leq i, j \leq r \\
\Phi_{2}=\left(\frac{1}{\beta_{i}^{2}+\beta_{j}^{2}}\right), & 1 \leq i, j \leq s \tag{127}
\end{array}
$$

and $\bar{X}_{22} \in H \mathbb{C}^{(k-r) \times(k-r)}, \quad \widehat{X}_{22} \in H \mathbb{C}^{(k-s) \times(k-s)}$ are arbitrary.
Proof. It follows from (9), (49)-(52), (68), (69), and the unitary invariance of the Frobenius norm that

$$
\begin{align*}
& \|A X-B\|^{2}+\|X C-D\|^{2} \\
& \quad=\left\|\bar{A} X_{11}-\bar{B}\right\|^{2}+\left\|\bar{C} X_{22}-\bar{D}\right\|^{2} \tag{128}
\end{align*}
$$

Then

$$
\begin{equation*}
\|A X-B\|^{2}+\|X C-D\|^{2} \tag{129}
\end{equation*}
$$

gains its minimum value if and only if

$$
\begin{align*}
& \min =\left\|\bar{A} X_{11}-\bar{B}\right\|^{2} \text { holds }  \tag{130}\\
& \min =\left\|\bar{C} X_{22}-\bar{D}\right\|^{2} \text { holds } \tag{131}
\end{align*}
$$

So, by (68), (70), (73), and (74) and then combining Lemma 16, we get that if $X_{11}$ has the expression as in (125), then (130) holds. Similarly, if $X_{22}$ has the expression as in (126), then (131) holds. Thus, the least squares Hermitian generalized anti-Hamiltonian solution to Problem 7 can be expressed as (9), where $X_{11}$ and $X_{22}$ have the expressions as in (125) and (126).

Corollary 34. Given $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$, under the conditions of Theorem 33, the least squares Hermitian generalized anti-Hamiltonian solution with minimum norm to Problem 7 can be expressed as (9) with $X_{11}$ and $X_{22}$ having the expressions as in (125) and (126), where $\bar{X}_{22}=0, \widehat{X}_{22}=0$.

At last, on the basis of Lemma 17, we can obtain the least squares anti-Hermitian generalized Hamiltonian solution to Problem 7, the proof of which is analogous to the proof of Theorem 33.

Theorem 35. Given $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$, let the decomposition of $X \in A H H \mathbb{C}^{2 k \times 2 k}$ be (10). AP, $B P$, $P^{*} C, P^{*} D, \widehat{A}, \widehat{B}, \widehat{C}$, and $\widehat{D}$, respectively, have the partitions as in (49)-(52), (85). Assume that the singular value decompositions of $\widehat{A}$ and $\widehat{C}$ are, respectively, expressed as in (86) and $V X_{11} V^{*}, U^{*} \widehat{B} V, R X_{22} R^{*}, Q^{*} \widehat{D} R$ have the partitions as in (88). Then the least squares anti-Hermitian generalized Hamiltonian solution to Problem 7 can be expressed as (10) with

$$
\begin{align*}
& X_{11}=V\left(\begin{array}{cc}
\Psi_{1} *\left(\Sigma \widehat{B}_{11}-\widehat{B}_{11}^{*} \Sigma\right) & \Sigma^{-1} \widehat{B}_{12} \\
-\widehat{B}_{12}^{*} \Sigma^{-1} & \bar{X}_{22}
\end{array}\right) V^{*},  \tag{132}\\
& X_{22}=R\left(\begin{array}{cc}
\Psi_{2} *\left(\begin{array}{cc}
\left.\Pi \widehat{D}_{11}-\widehat{D}_{11}^{*} \Pi\right) & \Pi^{-1} \widehat{D}_{12} \\
-\widehat{D}_{12}^{*} \Pi^{-1} & \widehat{X}_{22}
\end{array}\right) R^{*}, ~
\end{array}\right.
\end{align*}
$$

where

$$
\begin{align*}
& \Psi_{1}=\left(\frac{1}{\alpha_{i}^{2}+\alpha_{j}^{2}}\right), \quad 1 \leq i, j \leq r \\
& \Psi_{2}=\left(\frac{1}{\beta_{i}^{2}+\beta_{j}^{2}}\right), \quad 1 \leq i, j \leq s, \tag{133}
\end{align*}
$$

and $\bar{X}_{22} \in A H \mathbb{C}^{(k-r) \times(k-r)}, \widehat{X}_{22} \in A H \mathbb{C}^{(k-s) \times(k-s)}$ are arbitrary.

Corollary 36. Given $A, B \in \mathbb{C}^{m \times 2 k}, C, D \in \mathbb{C}^{2 k \times q}$, under the conditions of Theorem 35, the least squares anti-Hermitian generalized Hamiltonian solution with minimum norm to Problem 7 can be expressed as (10) with $X_{11}$ and $X_{22}$ having the expressions as in (132), where $\bar{X}_{22}=0, \widehat{X}_{22}=0$.

## 6. Algorithms and Numerical Examples

In this section, algorithms are given to compute the solution to Problem 7, and meanwhile some numerical examples are presented to show that the algorithms provided are feasible. Note that all the tests are performed by MATLAB 7.6.

An algorithm is firstly presented to compute the least squares Hermitian generalized Hamiltonian solution to Problem 7.

Algorithm 37. Step 1. Input $A, B, C, D, J$.
Step 2. Compute the eigenvalue decomposition of $J$ according to (6).

Step 3. Compute $A P, B P, P^{*} C, P^{*} D$ according to (49)-(52).
Step 4. Compute $A^{\prime}, B^{\prime}, C^{\prime}, D^{\prime}$ according to (53) and (54). If the conditions in (55) hold, then compute the Hermitian generalized Hamiltonian solution to Problem 5 according to (56) and (57). Otherwise, turn to Step 5.

Step 5. Compute $A^{\prime}, B^{\prime}, C^{\prime}, D^{\prime}$ according to (53) and (122).
Step 6. Compute the singular value decompositions of $A^{\prime}$ and $C^{\prime}$ according to (28).

Step 7. Compute $X_{12}$ according to (31).
Step 8. Compute $X$ according to (7), and output $X$.
Example 38. Given

$$
\begin{gather*}
A=\left(\begin{array}{cccc}
3+6 i & 2+i & 7-2 i & 8+3 i \\
2-3 i & 5-4 i & 1+4 i & 9+3 i
\end{array}\right) \\
B=\left(\begin{array}{cccc}
2-4 i & 3+2 i & 5+i & 4+i \\
6+i & 2-5 i & 1+6 i & 5+3 i
\end{array}\right) \\
C=\left(\begin{array}{ccc}
4+7 i & 10+3 i & 7+i \\
8+7 i & 3+9 i & 1-6 i \\
2-5 i & 5+6 i & 2+7 i \\
2 & 3 i & 3+7 i
\end{array}\right),  \tag{134}\\
D=\left(\begin{array}{ccc}
7+3 i & 5 & 2 i \\
5+2 i & 2-3 i & 6-i \\
3+i & 9-i & 4 \\
4-2 i & 5+2 i & 1+4 i
\end{array}\right), \\
J=\left(\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{array}\right)
\end{gather*}
$$

it can be easily verified that the conditions in (55) are not satisfied. Then, according to Algorithm 37, the least squares Hermitian generalized Hamiltonian solution $X$ to Problem 7 can be expressed as

$$
X=\left(\begin{array}{cccc}
0.0865-i & 0.0141-0.0359 i & 0.0317-0.0312 i & 0.1174+0.0824 i  \tag{135}\\
0.0141+0.0359 i & -0.0148 & -0.0201-0.0682 i & 0.1189+0.0954 i \\
0.0317+0.0302 i & -0.0201+0.0682 i & -0.0724 & 0.1114+0.0359 i \\
0.1173-0.0824 i & 0.1189-0.0954 i & 0.1114-0.0359 i & 0.0006
\end{array}\right),
$$

Remark 39. (1) There exists a unique least squares Hermitian generalized Hamiltonian solution to Problem 7 if and only if both $A^{\prime}$ and $C^{\prime}$ in Theorem 29 have full row ranks. Example 38 just illustrates it.
(2) Similarly, the algorithm about computing the least squares anti-Hermitian generalized anti-Hamiltonian solution to Problem 7 can be shown. We omit it here.

Now, we provide another algorithm to compute the least squares Hermitian generalized anti-Hamiltonian solution to Problem 7.

Algorithm 40. Step 1. Input $A, B, C, D, J$.
Step 2. Compute the eigenvalue decomposition of J according to (6).

Step 3. Compute $A P, B P, P^{*} C, P^{*} D$ according to (49)-(52).
Step 4. Compute $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ according to (68) and (69).

Step 5. Compute the singular value decompositions of $\bar{A}$ and $\bar{C}$ according to (70)-(71).

Step 6. Compute the partitions of $U^{*} \bar{B} V, Q^{*} \bar{D} R$ according to (74) and (76). If the conditions in (78) and (79) are all satisfied, then compute the Hermitian generalized anti-Hamiltonian solution to Problem 5 according to (80)-(82). Otherwise, turn to Step 7.

Step 7. Compute $X_{11}$ and $X_{22}$ according to (125) and (126).
Step 8. Compute $X$ according to (9), and output $X$.
Example 41. Let $A, B, C, D, J$ be as given in Example 38.
It is not difficult to prove that the conditions in (78) and (79) do not hold. So, according to Algorithm 40, the least squares Hermitian generalized anti-Hamiltonian solution to Problem 7 can be written as

$$
X=\left(\begin{array}{cccc}
0.3671 & 0.1579-0.1804 i & 0.1822+0.1400 i & 0.0179-0.0012 i  \tag{136}\\
0.1579+0.1804 i & 0.2324 & 0.0179+0.1149 i & -0.0662-0.1400 i \\
0.1822-0.1400 i & 0.0179-0.1149 i & 0.2143 & 0.1221-0.0849 i \\
0.0179+0.0012 i & -0.0662+0.1400 i & 0.1221+0.0849 i & 0.5764
\end{array}\right),
$$

Remark 42. (1) There exists a unique least squares Hermitian generalized anti-Hamiltonian solution to Problem 7 if and only if both $\bar{A}$ and $\bar{C}$ in Theorem 33 have full column ranks. Example 41 is just the case.
(2) Similarly, the algorithm about computing the least squares anti-Hermitian generalized Hamiltonian solution to Problem 7 can be obtained. We also omit it here.

## 7. Conclusions

In the previous sections, using the decomposition of the (anti-)Hermitian generalized (anti-)Hamiltonian matrices, the necessary and sufficient conditions for the existence of and the expression for the solution to Problem 5 have been firstly derived, respectively. Then the solutions to Problems 6 and 7 have been individually given. Finally, algorithms have been given to compute the least squares Hermitian generalized Hamiltonian solution and the least squares Hermitian generalized anti-Hamiltonian solution to Problem 7, and the corresponding examples have also been presented to show that the algorithms are reasonable.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Dynamic Stability of Euler Beams under Axial Unsteady Wind Force 

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#### Abstract

Dynamic instability of beams in complex structures caused by unsteady wind load has occurred more frequently. However, studies on the parametric resonance of beams are generally limited to harmonic loads, while arbitrary dynamic load is rarely involved. The critical frequency equation for simply supported Euler beams with uniform section under arbitrary axial dynamic forces is firstly derived in this paper based on the Mathieu-Hill equation. Dynamic instability regions with high precision are then calculated by a presented eigenvalue method. Further, the dynamically unstable state of beams under the wind force with any mean or fluctuating component is determined by load normalization, and the wind-induced parametric resonant response is computed by the RungeKutta approach. Finally, a measured wind load time-history is input into the dynamic system to indicate that the proposed methods are effective. This study presents a new method to determine the wind-induced dynamic stability of Euler beams. The beam would become dynamically unstable provided that the parametric point, denoting the relation between load properties and structural frequency, is located in the instability region, no matter whether the wind load component is large or not.


## 1. Introduction

The simply supported Euler beam, an important research object on the dynamic instability of elastic systems, is a common form of members in modern structures [1, 2]. Modern building structures are being constructed with greater height or span, resulting in smaller natural frequencies of beams in them, so parametric resonance of the beams may occur under unsteady strong wind forces. For example, many cases of roof collapse in windstorms caused by instability of significant beams have been reported $[3,4]$.

The dynamic instability of Euler beams has attracted wide attention of scholars in the early stage. Beliaev [5] analyzed the dynamic responses of simply supported Euler beams under axial load with the form of

$$
\begin{equation*}
P(t)=P_{0}+P_{t} \cos \theta t \tag{1}
\end{equation*}
$$

Main frequencies of parametric resonance were calculated, and the Mathieu-Hill equation for structural dynamic instability was derived. As a complement to Beliaev's analysis,

Krylov and Bogoliubov [6, 7] studied the influence of arbitrary boundary conditions on dynamic instability of Euler beams by the Galerkin method. To successfully solve the Mathieu-Hill equation for the problem of dynamic instability of Euler beams, Shtokalo [8] established the stability of linear differential equation with variable coefficients. McLachlan [9] discussed the theory and application of Mathieu equation. Based on these theories, Bolotin [10] built the critical frequency equation for structural dynamic instability and gave the approximate formula of the first three dynamic instability regions. With the development of finite element (FE) theories, Briseghella et al. [11] computed dynamic instability regions of Euler beams by FE method. He also compared the FE results with those obtained from the analytical method. Yang and Fu [12] discussed the dynamic instability of Euler beams with layered composite materials. Sochacki [13] concerned on simply supported Euler beams with additional elastic elements. Yan et al. [14] analyzed the effect of cracks on dynamic instability of Euler beams. However, in these previous studies for the engineering applications of the MathieuHill equation, axial loads are all assumed as harmonic loads,
whereas the wind load is arbitrary and cannot be simplified as a single harmonic wave in studying wind-induced parametric resonance. Therefore, it is necessary to establish an analytical method for computing the dynamic instability of Euler beams under arbitrary dynamic load.

In this research, the critical frequency equation of simply supported Euler beams with a uniform section under arbitrary dynamic load is firstly derived based on the MathieuHill equation. An eigenvalue algorithm is then developed to compute dynamic instability regions with high precision. Further, the approaches for judging dynamically unstable state under any unsteady wind force and calculating windinduced resonant responses are presented. Finally, a measured wind load time-history is adopted to describe the application of the proposed methods. At the same time, a discussion on the characteristics of wind-induced dynamic instability of Euler beams is also given.

## 2. Mathieu-Hill Equation for Arbitrary Dynamic Axial Load

The studied simply supported Euler beam with a uniform section is shown in Figure 1, whose length is $l$, Young's module is $E$, and the moment of inertia of the section is $I$. Under the action of axial unsteady wind force $f(t)$, the deflection of the beam can be denoted as [15]

$$
\begin{equation*}
v(x, t)=\sum_{i=1}^{\infty} q_{i}(t) \sin \frac{i \pi x}{l} \tag{2}
\end{equation*}
$$

where $v(x, t)$ is the time-varied deflection along the beam, $\sin (i \pi x / l)$ is the $i$ th modal shape of the beam, and $q_{i}(t)$ is the $i$ th modal coordinate.

The arbitrary force $f(t)$ with the duration of $t_{\max }$ can be transformed into Fourier series by even prolongation [16]:

$$
\begin{align*}
f(t) & =f_{0}+\sum_{k=1}^{\infty}\left(f_{k} \cos k \theta t\right), \quad t \in\left[0, t_{\max }\right]  \tag{3}\\
f_{0} & =\frac{1}{t_{\max }} \int_{0}^{t_{\max }} f(t) d t  \tag{4}\\
f_{k} & =\frac{2}{t_{\max }} \int_{0}^{t_{\max }} f(t) \cos k \theta t d t \tag{5}
\end{align*}
$$

where $f_{0}$ and $\sum_{k=1}^{\infty}\left(f_{k} \cos k \theta t\right)$ are the mean and fluctuating component of $f(t)$, respectively. $f_{k}$ and $k \theta$ are the amplitude and angular frequency of the $k$ th harmonic wave, respectively, and $\theta=\pi / t_{\text {max }}$.

If $f(t)$ is measured by wind force time-history composed of $N$ discrete data from wind tunnel lab or in field, the maximum $k$ is $N-1$.

Under the wind force, the Mathieu-Hill equation, corresponding to the $i$ th mode of the undamped Euler beam, can be written as [10]

$$
\begin{equation*}
\ddot{q}_{i}+\Omega_{i}^{2}\left[1-\sum_{k=1}^{\infty}\left(2 \mu_{i, k} \cos k \theta t\right)\right] q_{i}=0 \tag{6}
\end{equation*}
$$



Figure 1: The simply supported Euler beam under wind force.
where $\Omega_{i}$ is the $i$ th natural frequency of the beam loaded by $f_{0}$ and $\mu_{i, k}$ is the excitation parameter of the $k$ th harmonic wave for the $i$ th normal mode.
$\Omega_{i}$ and $\mu_{i, k}$ can be gained by

$$
\begin{align*}
\Omega_{i} & =\omega_{i} \sqrt{1-\frac{f_{0}}{f_{* i}}} \\
\mu_{i, k} & =\frac{f_{k}}{2\left(f_{* i}-f_{0}\right)}  \tag{7}\\
f_{* i} & =i^{2} f_{*}=\frac{i^{2} \pi^{2} E I}{l^{2}}
\end{align*}
$$

where $\omega_{i}$ is the $i$ th natural frequency of the unloaded beam and $f_{*}$ represents Euler's critical buckling axial load of the beam.

The Mathieu-Hill equations corresponding to normal modes of different orders have the same form as long as $\mu_{i, k}$ and $\Omega_{i}$ are calculated from the same order of mode, so the distributions of dynamic instability regions in different modes are identical. Consequently, the subscript $i$ in (6)(7) can be omitted, and all calculations later will use the data corresponding to $i=1$. The Mathieu-Hill equation is rewritten as

$$
\begin{equation*}
\ddot{q}+\Omega^{2}\left[1-\sum_{k=1}^{\infty}\left(2 \mu_{k} \cos k \theta t\right)\right] q=0 . \tag{8}
\end{equation*}
$$

As is well known, the dynamic instability regions are surrounded by two solutions of the Mathieu-Hill equation with the same period, while the dynamic stability regions are encompassed by two solutions with different periods [10]. Hence, provided the condition that there exist periodic solutions of the period $T$ or $2 T\left(T=2 t_{\max }\right)$ in (8) is found, the dynamic instability regions can be determined.

Periodic solutions of $2 T$ can be written in the form of Fourier series:

$$
\begin{equation*}
q(t)=\sum_{n=1,3,5}^{\infty}\left(a_{n} \sin \frac{n \theta t}{2}+b_{n} \cos \frac{n \theta t}{2}\right) \tag{9}
\end{equation*}
$$

where $a_{n}$ and $b_{n}$ are constant coefficients.
Substituting (9) into (8) and making the coefficients of terms of $\sin (n \theta t / 2)$ or $\cos (n \theta t / 2)$ be zero generate homogeneous system of linear equations with respect to $a_{n}$ and $b_{n}$ :

$$
\begin{aligned}
& \left(1+\mu_{1}-\frac{\theta^{2}}{4 \Omega^{2}}\right) a_{1}-\left(\mu_{1}-\mu_{2}\right) a_{3}-\left(\mu_{2}-\mu_{3}\right) a_{5}+\cdots=0 \\
& -\left(\mu_{1}-\mu_{2}\right) a_{1}+\left(1+\mu_{3}-\frac{9 \theta^{2}}{4 \Omega^{2}}\right) a_{3}-\left(\mu_{1}-\mu_{4}\right) a_{5}+\cdots=0
\end{aligned}
$$

$-\left(\mu_{2}-\mu_{3}\right) a_{1}-\left(\mu_{1}-\mu_{4}\right) a_{3}+\left(1+\mu_{5}-\frac{25 \theta^{2}}{4 \Omega^{2}}\right) a_{5}+\cdots=0, \quad-\left(\mu_{2}+\mu_{3}\right) b_{1}-\left(\mu_{1}+\mu_{4}\right) b_{3}+\left(1-\mu_{5}-\frac{25 \theta^{2}}{4 \Omega^{2}}\right) b_{5}+\cdots=0$,

$$
\begin{align*}
& \left(1-\mu_{1}-\frac{\theta^{2}}{4 \Omega^{2}}\right) b_{1}-\left(\mu_{1}+\mu_{2}\right) b_{3}-\left(\mu_{2}+\mu_{3}\right) b_{5}+\cdots=0  \tag{10}\\
& -\left(\mu_{1}+\mu_{2}\right) b_{1}+\left(1-\mu_{3}-\frac{9 \theta^{2}}{4 \Omega^{2}}\right) b_{3}-\left(\mu_{1}+\mu_{4}\right) b_{5}+\cdots=0
\end{align*}
$$

If and only if the coefficient determinant is equal to zero, (10) has nonzero solutions. Therefore, the condition that (8) has periodic solutions of $2 T$ is that the coefficient determinant of (10) is equal to zero. Considering both cases of $a_{n}$ and $b_{n}$, the critical frequency equation corresponding to $2 T$ periodic solutions is obtained:

$$
\left|\begin{array}{ccccc}
1 \pm \mu_{1}-\frac{\theta^{2}}{4 \Omega^{2}} & & &  \tag{11}\\
-\mu_{1} \pm \mu_{2} & 1 \pm \mu_{3}-\frac{9 \theta^{2}}{4 \Omega^{2}} & & & \text { symmetry } \\
-\mu_{2} \pm \mu_{3} & -\mu_{1} \pm \mu_{4} & 1 \pm \mu_{5}-\frac{25 \theta^{2}}{4 \Omega^{2}} & & \\
\ldots & \ldots & \ldots & \cdots & \cdots \\
-\mu_{i-1} \pm \mu_{i} & \ldots & -\mu_{i-j} \pm \mu_{i+j-1} & \cdots & 1 \pm \mu_{2 i-1}-\frac{(2 i-1)^{2} \theta^{2}}{4 \Omega^{2}} \\
\ldots & \ldots & \cdots & \cdots & \cdots \\
\ldots
\end{array}\right|=0
$$

The critical frequency equation contains the relationship between the frequency spectrum characteristics (amplitude and frequency characteristics) of wind load and structural natural frequency. By solving the equation, the boundary of dynamic instability region $\theta_{*} / 2 \Omega$, expressed by the ratio of critical load frequency and structural frequency, would be obtained. $\theta / 2 \Omega$ is called the frequency ratio, and $\theta_{*} / 2 \Omega$ is called the critical frequency ratio.

Similarly, in order to gain the periodic solution condition of the period $T$, the periodic solution of period $T$ is written in the following form:

$$
\begin{equation*}
q(t)=b_{0}+\sum_{n=2,4,6}^{\infty}\left(a_{n} \sin \frac{n \theta t}{2}+b_{n} \cos \frac{n \theta t}{2}\right) \tag{12}
\end{equation*}
$$

By substituting (12) into (8) and making the coefficients of terms of $\sin (n \theta t / 2)$ or $\cos (n \theta t / 2)$ be zero, the critical frequency equation corresponding to the periodic solutions of period $T$ is acquired:

$$
\begin{align*}
& \left\lvert\, \begin{array}{ccccc}
1+\mu_{2}-\frac{4 \theta^{2}}{4 \Omega^{2}} & & & & \\
-\mu_{1}+\mu_{3} & 1+\mu_{4}-\frac{16 \theta^{2}}{4 \Omega^{2}} & & \text { symmetry } & \\
-\mu_{2}+\mu_{4} & -\mu_{1}+\mu_{5} & 1+\mu_{6}-\frac{36 \theta^{2}}{4 \Omega^{2}} & & \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
-\mu_{i-1}+\mu_{i+1} & \ldots & -\mu_{i-j}+\mu_{i+j} & \ldots & 1+\mu_{2 i}-\frac{(2 i)^{2} \theta^{2}}{4 \Omega^{2}}
\end{array} \quad \ldots .\right.  \tag{13}\\
& \left|\begin{array}{cccccc}
1 & -\mu_{1} & -\mu_{2} & \ldots & -\mu_{i-1} & \ldots \\
-2 \mu_{1} & 1-\mu_{2}-\frac{4 \theta^{2}}{4 \Omega^{2}} & & & & \\
-2 \mu_{2} & -\mu_{1}-\mu_{3} & 1-\mu_{4}-\frac{16 \theta^{2}}{4 \Omega^{2}} & & \text { symmetry } & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
-2 \mu_{i-1} & \ldots & -\mu_{i-j}-\mu_{i+j-2} & \ldots & 1-\mu_{2 i-2}-\frac{(2 i-2)^{2} \theta^{2}}{4 \Omega^{2}} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}\right|=0 . \tag{14}
\end{align*}
$$

By solving (11) with $m$ order determinant, the first $m$ odd dynamic instability regions are obtained. Besides, by solving (13) with $m$ order determinant and (14) with $m+$ 1 order determinant, the first $m$ even dynamic instability regions are achieved. Thus, the first $2 m$ dynamic instability regions can be determined. Meanwhile, it is known from the above derivation that $2 m$ should not exceed the maximum harmonic number $N-1$.

## 3. Eigenvalue Algorithm for Computing Instability Boundary

Based on the critical frequency equation, the method of multiple scales or successive approximation is commonly used to determine the boundaries of dynamic instability regions [17-20]. However, the computation complexity or large time consuming of these methods generally increases rapidly with solution precision requirement. To ensure computation efficiency, less order of the determinant is usually selected, which results in less accurate instability boundaries. Here, a fast algorithm based on the eigenvalue problem is attempted to be established for acquiring instability boundaries with high precision.

Take the critical frequency equation (11); for example, matrices $\mathbf{A}$ and $\mathbf{B}$ can be defined as follows:

$$
\begin{align*}
& \mathbf{A}=\left[\begin{array}{cccccc}
1 \pm \mu_{1} & & & & & \\
-\mu_{1} \pm \mu_{2} & 1 \pm \mu_{3} & & & \\
-\mu_{2} \pm \mu_{3} & -\mu_{1} \pm \mu_{4} & 1 \pm \mu_{5} & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-\mu_{i-1} \pm \mu_{i} & \cdots & -\mu_{i-j} \pm \mu_{i+j-1} & \cdots & 1 \pm \mu_{2 i-1} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right], \\
& \mathbf{B}=\left[\begin{array}{llllll}
1 & & & & & \\
& 9 & & & & \\
& & 25 & & & \\
& & & \ddots & (2 i-1)^{2} & \\
& & & & & \\
& & & & &
\end{array}\right] . \tag{15}
\end{align*}
$$

Thus, (11) can be expressed as

$$
\begin{equation*}
\left|\mathbf{A}-\left(\frac{\theta}{2 \Omega}\right)^{2} \mathbf{B}\right|=0 \tag{16}
\end{equation*}
$$

It is obvious that (16) converts the problem of solving dynamic instability regions to the eigenvalue problem [21, 22]. Since the eigenvalue problem is easy to be solved by means of a computer and the order of determinant has few effects on calculated amount, much more order of the determinant can be selected to meet high precision requirement. For matrices $\mathbf{A}$ and $\mathbf{B}$ of $m$ order, there are $2 m$ eigenvalues $\theta_{*} / 2 \Omega$, which generate boundary points of $m$ dynamic instability regions. By altering the value of $\mu_{1}, \mu_{2}, \ldots, \mu_{2 m-1}$, the spatial instability region in $2 m$-dimensional coordinate system $\left(\theta / 2 \Omega, \mu_{1}, \mu_{2}, \ldots, \mu_{2 m-1}\right)$ can be constructed [23]. For the beam with known natural frequency, any unsteady wind force determines a point in the spatial coordinate system. If the point is located in an instability region, the beam would
become dynamically unstable under such wind force and dynamic instability wouldnot occur otherwise.

Analogously, the instability boundaries according to the periodic solution of $T$ can be determined by solving the critical frequency equations (13) and (14) through the eigenvalue algorithm. For (13), the matrices $\mathbf{A}$ and $\mathbf{B}$ are

$$
\begin{gather*}
\mathbf{A}=\left[\begin{array}{cccccc}
1+\mu_{2} & -\mu_{1}+\mu_{3} & -\mu_{2}+\mu_{4} & \ldots & -\mu_{i-1}+\mu_{i+1} & \cdots \\
-\mu_{1}+\mu_{3} & 1+\mu_{4} & -\mu_{1}+\mu_{5} & \cdots & \cdots & \cdots \\
-\mu_{2}+\mu_{4} & -\mu_{1}+\mu_{5} & 1+\mu_{6} & \cdots & -\mu_{i-j}+\mu_{i+j} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\mu_{i-1}+\mu_{i+1} & \cdots & -\mu_{i-j}+\mu_{i+j} & \cdots & 1+\mu_{2 i} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right], \\
\mathbf{B}=\left[\begin{array}{cccccc}
4 & & & & & \\
& 16 & & & & \\
& & 36 & & & \\
& & & \ddots & & \\
& & & (2 i)^{2} & \\
& & & & \ddots
\end{array}\right] . \tag{17}
\end{gather*}
$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ in (14) can be expressed as follows:

$$
\begin{gather*}
\mathbf{A}=\left[\begin{array}{cccccc}
1 & -\mu_{1} & -\mu_{2} & \cdots & -\mu_{i-1} & \cdots \\
-2 \mu_{1} & 1-\mu_{2} & -\mu_{1}-\mu_{3} & \cdots & \cdots & \cdots \\
-2 \mu_{2} & -\mu_{1}-\mu_{3} & 1-\mu_{4} & \cdots & -\mu_{i-j}-\mu_{i+j-2} & \cdots \\
-2 \mu_{i-1} & \cdots & \cdots & -\mu_{i-j}-\mu_{i+j-2} & \cdots & 1-\mu_{2 i-2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \\
\mathbf{B}=\left[\begin{array}{cccccc}
0 & & & & & \\
& 4 & & & & \\
& & 16 & & & \\
& & & \ddots & & \\
& & & & (2 i-2)^{2} & \\
& & & & \ddots
\end{array}\right],
\end{array}\right. \tag{18}
\end{gather*}
$$

## 4. Calculation of Wind-Induced Dynamic Instability Region and Resonant Response

The value of $m$ is usually fairly large to meet precision requirement, so it is difficult to graphically display the $2 m$ dimensional spatial instability region and prejudge the dynamic instability state. Here, a proportional load strategy is adopted to augment the mean or fluctuating component of wind force and set up the two- or three-dimensional dynamic instability regions [24, 25].

The wind force $f(t)$ as expressed in (3), called the initial load here, is firstly normalized as follows:

$$
\begin{align*}
\bar{f} & =\max _{k}\left|f_{k}\right|,  \tag{20}\\
f^{\prime \prime}(t) & =\frac{f(t)-f_{0}}{\bar{f}}=\sum_{k=1}^{\infty} \frac{f_{k}}{\bar{f}} \cos k \theta t \tag{21}
\end{align*}
$$

where $\bar{f}$ is the maximum harmonic amplitude of $f(t)$ and $f^{\prime \prime}(t)$ is the normalized wind force, called the basic load here.

According to (20) and (21), it is clear that the mean value and maximum harmonic amplitude of the basic load are

0 and 1 , respectively. Each harmonic amplitude of the basic load is $\bar{f}$ times less than that of the initial load $f(t)$.

Let $f_{k}^{\prime \prime}$ denote the harmonic amplitude of the basic load, and then the basic load can be written as

$$
\begin{equation*}
f^{\prime \prime}(t)=\sum_{k=1}^{\infty} f_{k}^{\prime \prime} \cos k \theta t \tag{22}
\end{equation*}
$$

Thus a new wind force $f^{\prime}(t)$ can be generated by defining two parameters $\alpha$ and $\beta$ to linearly increase the mean and fluctuating components of wind force based on the basic load:

$$
\begin{align*}
f^{\prime}(t) & =f_{0}^{\prime}+f_{k m}^{\prime} f^{\prime \prime}(t)=\alpha f_{*}+\beta f_{*} f^{\prime \prime}(t) \\
& =\alpha f_{*}+\sum_{k=1}^{\infty} \beta f_{*} f_{k}^{\prime \prime} \cos k \theta t=\alpha f_{*}+\sum_{k=1}^{\infty} \frac{\beta f_{*}}{\bar{f}} f_{k} \cos k \theta t \\
\alpha & =\frac{f_{0}^{\prime}}{f_{*}} \\
\beta & =\frac{f_{k m}^{\prime}}{f_{*}} \tag{23}
\end{align*}
$$

where $f^{\prime}(t)$ is the newly generated wind force; $f_{0}^{\prime}$ and $f_{k m}^{\prime}$ are the mean component and the maximum harmonic amplitude of $f^{\prime}(t)$, respectively; $\alpha$ and $\beta$ are the ratios of $f_{0}^{\prime}$ and $f_{k m}^{\prime}$ to Euler's critical load of the beam. The reason why Euler's critical load is adopted to denote the mean and fluctuating components is that the structural natural frequency and excitation parameter are both related to Euler's critical load as shown in (7).

For the new load, its mean component of is $\alpha f_{*}$ and its fluctuating component or harmonic amplitude is $\beta f_{*}$ times that of the basic load, while the harmonic frequency composition is identical to the initial load or the basic load.
$f^{\prime}(t)$ is further simplified as follows:

$$
\begin{equation*}
f^{\prime}(t)=f_{0}^{\prime}+\sum_{k=1}^{\infty} f_{k}^{\prime} \cos k \theta t \tag{24}
\end{equation*}
$$

where $f_{k}^{\prime}$ denotes the harmonic amplitude of $f^{\prime}(t)$.
Meanwhile, the natural frequency and excitation parameter corresponding to $f^{\prime}(t)$ can be computed by

$$
\begin{aligned}
\Omega^{\prime} & =\omega \sqrt{1-\frac{f_{0}^{\prime}}{f_{*}}}=\omega \sqrt{1-\frac{\alpha f_{*}}{f_{*}}}=\omega \sqrt{1-\alpha} \\
\mu_{k}^{\prime} & =\frac{f_{k}^{\prime}}{2\left(f_{*}-f_{0}^{\prime}\right)}=\frac{\left(\beta f_{*} / \bar{f}\right) f_{k}}{2\left(f_{*}-\alpha f_{*}\right)} \\
& =\frac{(\beta / \bar{f}) f_{k}}{2(1-\alpha)}=\frac{\beta}{2(1-\alpha)} f_{k}^{\prime \prime}
\end{aligned}
$$

The expression of excitation parameter can be further simplified by defining

$$
\begin{equation*}
\mu^{\prime}=\frac{\beta}{2(1-\alpha)} \tag{26}
\end{equation*}
$$

where $\mu^{\prime}$ represents the ratio of excitation parameter and harmonic amplitude of the basic load.

Thus, the excitation parameter of generated wind force can be calculated by harmonic amplitude of the basic load:

$$
\begin{equation*}
\mu_{k}^{\prime}=\mu^{\prime} f_{k}^{\prime \prime} \tag{27}
\end{equation*}
$$

Moreover, the Mathieu-Hill equation consistent with $f^{\prime}(t)$ is

$$
\begin{equation*}
\ddot{q}+\Omega^{2}\left[1-2 \mu^{\prime} \sum_{k=1}^{\infty}\left(f_{k}^{\prime \prime} \cos k \theta t\right)\right] q=0 . \tag{28}
\end{equation*}
$$

That is

$$
\begin{equation*}
\ddot{q}+\Omega^{2}\left[1-2 \mu^{\prime} f^{\prime \prime}(t)\right] q=0 . \tag{29}
\end{equation*}
$$

Therefore, various $\mu^{\prime}$ can be obtained only by adjusting $\alpha$ or $\beta$, and then the two-dimensional parametric plane $\left(\theta / 2 \Omega, \mu^{\prime}\right)$ or three-dimensional parametric space $(\theta / 2 \Omega, \alpha, \beta)$ for dynamic instability regions can be graphically established according to the theory in Section 3 of this paper. For different wind forces, various distributions of instability regions will be drawn. Because the mean component and the peak harmonic amplitude are generally less than Euler's critical load, there exist $\alpha \in[0,1]$ and $\beta \in$ [0, 1].

When the parametric point $\left(\theta / 2 \Omega, \mu^{\prime}\right)$ is located in the instability region, the beam is deemed to be dynamically unstable. In order to verify such a conclusion, modal response $q(t)$ of the beam can be calculated through the Mathieu-Hill equation (29). $q(t)$ should keep increasing with time when the beam is unstable.

The Mathieu-Hill equation shown in (29) can be converted to the following form:

$$
\begin{align*}
& \dot{x}=-\Omega^{2}\left[1-2 \mu^{\prime} f^{\prime \prime}(t)\right] q  \tag{30}\\
& \dot{q}=x .
\end{align*}
$$

Let

$$
\mathbf{y}=\left\{\begin{array}{l}
x  \tag{31}\\
q
\end{array}\right\}
$$

Equation (29) is expressed in the vector type

$$
\begin{equation*}
\dot{\mathbf{y}}=f(\mathbf{y}, t) \tag{32}
\end{equation*}
$$

For differential equations with the form of (32), the fourth-order Runge-Kutta method can be used to solve the


Figure 2: Rigid model in the wind tunnel tests.
unknowns [23, 26]. The response $q(t)$ at the $n$th iteration step can be obtained by the following iteration formula:

$$
\begin{align*}
& \mathbf{y}_{n}=\mathbf{y}_{n-1}+\frac{h}{6}\left(\mathbf{k}_{1}+2 \mathbf{k}_{2}+2 \mathbf{k}_{3}+\mathbf{k}_{4}\right), \\
& \mathbf{k}_{1}=f\left(t_{n-1}, \mathbf{y}_{n-1}\right) \\
& \mathbf{k}_{2}=f\left(t_{n-1}+\frac{h}{2}, \mathbf{y}_{n-1}+\frac{h}{2} \mathbf{k}_{1}\right),  \tag{33}\\
& \mathbf{k}_{3}=f\left(t_{n-1}+\frac{h}{2}, \mathbf{y}_{n-1}+\frac{h}{2} \mathbf{k}_{2}\right), \\
& \mathbf{k}_{4}=f\left(t_{n-1}+h, \mathbf{y}_{n-1}+h \mathbf{k}_{3}\right)
\end{align*}
$$

The time step length is set as $h=0.002 \mathrm{~s}$ and initial disturbance of response is $q(t=0)=0.001 \mathrm{~m}$ in the latter computation.

## 5. Case Study

In order to describe the application of the above approaches, a real wind force time-history is taken from a wind tunnel test. The rigid model tests of a cylindrical reticulated shell, with the span of 103 m , the longitudinal length of 140 m , and the height of 40 m , were carried in a TJ-2 wind tunnel of Tongji University, China, in order to obtain the time-histories of wind pressure on the shell (Figure 2). Detailed information on the test can be referred to Zhou et al. [27]. Here, wind pressure at one node of the shell is extracted and converted to full scale, and the area of $16 \mathrm{~m}^{2}$ is employed to construct wind force $f(t)$ on the beam, as shown in Figure 3.

The wind load data number is $N=6000$, and the sampling frequency is $F_{s}=9.58 \mathrm{~Hz}$, so the load duration is $t_{\max }=N / F_{s}=626.3 \mathrm{~s}$. The load is simulated by the periodic load $f(t)$ with period $2 t_{\text {max }}$ according to (3)-(5). Comparison of $f(t)$ and the measured data is shown in Figure 4(a). To appreciate the two sets of data more clearly, a comparison of shorter load duration is also given in Figure 4(b). It can be seen that the original data are well fitted by the Fourier series. Mean component of the wind force is $f_{0}=4.19 \mathrm{e} 3 \mathrm{~N}$ and the maximum harmonic amplitude is $\bar{f}=288.87 \mathrm{~N}$ by (4), (5), and (20). The relationship between harmonic amplitude $f_{k}$ and corresponding harmonic frequency $k /\left(2 t_{\max }\right)$ is shown in


Figure 3: Measured wind force acting on the beam.


Figure 4: Comparison of measured and fitted wind forces.

Figure 5, which indicates that larger amplitudes correspond to smaller frequencies. Moreover, the natural frequency of the beam is defined as $\omega=5 \mathrm{~Hz}$.

Normalizing the initial load through (20)-(22) produces the basic load $f^{\prime \prime}(t)$, whose magnitude and the relationship between harmonic amplitude and frequency are shown in Figure 6. Comparing with the initial load (Figures 4 and 5) shows that mean component and the maximum harmonic


Figure 5: Relationship between the harmonic amplitude and frequency in the fitted wind force.


Figure 6: Time-history and frequency feature of the basic load.
amplitude of the basic load are 0 and 1 m , respectively. Both the fluctuating components and harmonic amplitudes are reduced to $\bar{f}$ times, while the harmonic frequencies are invariant.

New wind load can be generated through (23)-(24) based on the basic load, and the dynamic instability regions can be established by substituting excitation parameters of the new load into (15)-(19). Figure 7 gives the left and right boundaries of the first dynamic instability region when $m$ is equal to 10,40 , and 80 , respectively. It is obvious that the boundary values vary with $m$ and larger $m$ results in close


Figure 7: Comparison of the first instability regions under different $m$ 's.


Figure 8: Comparison of the first instability regions by the approximate formula and eigenvalue method.
boundaries with higher precision. Since the results under $m=40$ and $m=80$ are close, $m$ is set for 40 in the following calculations. Meanwhile, for convenience, only the first 100 data of wind force are selected for later computations ( $N=$ 100).

Bolotin [10] has pointed out that, if diagonal elements of the critical frequency equation are ignored, the boundary of the $k$ th instability region under arbitrary dynamic load can be calculated by the following approximate formula:

$$
\begin{equation*}
\frac{\theta_{*}}{2 \Omega} \approx \frac{1}{k} \sqrt{1 \pm \mu_{k}} \tag{34}
\end{equation*}
$$

Equation (34) indicates that the $k$ th instability region only depends on the $k$ th harmonic wave. Figure 8 compares the


Figure 9: Modal response $q(t)$ of the parametric points A and B.
first instability regions gained by (34) and the eigenvalue method, respectively. It is seen that two methods produce different results, especially under larger excitation parameters. Comparing other regions brings similar conclusions. This is because the influence of the $i$ th harmonic wave on the width of the $k$ th instability region is the order of $\left(\mu_{k} \pm \mu_{i}\right)^{2}$ [10], while the value of $\left(\mu_{k} \pm \mu_{i}\right)^{2}$ under wind load could be fairly large. Therefore, the interaction between different harmonic waves cannot be ignored for wind-induced dynamic instability analysis.

In order to further verify the precision of the eigenvalue method, points $\mathrm{A}(0.65,0.8)$ and $\mathrm{B}(1.31,0.8)$ on the parametric plane of Figure 8 are selected. Point A is located in the unstable region of the approximate formula while in the stable region of the eigenvalue method. Point $B$ is located in the stable region of the approximate formula while in the unstable region of the eigenvalue method. Modal responses $q(t)$ corresponding to points A and point B are calculated through (33), as shown in Figure 9. The modal response of point $A$ is confined in a scope of about 0.01 m , while that of point $B$ increases rapidly to about 5.0 m in the same duration. Hence, the structural system corresponding to point $A$ is dynamically stable, but that corresponding to point $B$ is dynamically unstable, which is consistent with the conclusion of the eigenvalue algorithm. This indicates that the eigenvalue algorithm provides more accurate dynamic instability region for arbitrary dynamic load.

Distribution of the first 40 dynamic instability regions (the first 20 odd regions corresponding to period $2 T$ and the first 20 even regions corresponding to period $T$ ) is shown as the shaded area in Figure 10. When the parametric point $\left(\theta / 2 \Omega, \mu^{\prime}\right)$ corresponding to the specific structure and wind force falls in the shaded area, structural dynamic instability occurs. In addition, according to the instability regions, the scope of critical instability frequency can be obtained for


Figure 10: Distribution of the first 40 instability regions on the parametric plane $\left(\theta / 2 \Omega, \mu^{\prime}\right)$.
certain $\mu^{\prime}$ denoting the magnitude of wind force and the range of $\mu^{\prime}$ causing dynamic instability can also be evaluated when the frequency ratio $\theta / 2 \Omega$ is fixed.

When $\alpha=\beta=0.5$ is assumed, we have $\mu^{\prime}=0.5$ according to (26). By $t_{\max }=100 / 9.58=10.44 \mathrm{~s}$, we get $\theta=\pi / t_{\max }=0.3 \mathrm{~Hz}$ and $\theta / 2 \Omega=0.043$. Thus, the wind force and the beam correspond to a parametric point $\mathrm{C}(0.043$, $0.5)$, which is located in the shadow area of the parametric plane in Figure 11. Figure 11 is the partial enlarged view of Figure 10 in the scope of $\theta / 2 \Omega$ between 0.02 and 0.065 . The modal response $q(t)$ of point $C$ is also shown in Figure 12. It is shown that the deflection of the beam increases rapidly in a very short time. Therefore, obvious dynamic instability is caused by this wind force.

The above analysis indicates that the dynamic instability of simply supported Euler beam with uniform section under any wind load can be determined according to the values of $\alpha, \beta, t_{\text {max }}$, and $\omega$. For any wind load, its basic load can be constructed and the distribution of dynamic instability


Figure 11: Location of the parametric points on the parametric plane $\left(\theta / 2 \Omega, \mu^{\prime}\right)$.


Figure 12: Modal response $q(t)$ of the parametric point C.
regions similar to Figure 10 can be drawn to determine the state of dynamic stability.

In order to study the influence of mean or fluctuating component on the scope of critical load frequency, parametric planes $(\theta / 2 \Omega, \alpha)$ and $(\theta / 2 \Omega, \beta)$ can be built. Figure 13 provides the first 40 instability regions on the plane $(\theta / 2 \Omega$, $\beta$ ) when $\alpha=0,0.5$ and 0.99 . It is demonstrated that the increase of $\beta$ generally widens the scope of critical frequency ratio under the same $\alpha$. That is, when mean component of wind load is fixed, increase of fluctuating component may transform the beam from dynamically stable state to unstable state. Moreover, even though the mean component is zero, increasing fluctuating component could also cause dynamic instability. In addition, when the mean component is close
to Euler's critical load, the beam would become dynamically unstable regardless of the value of $\beta$ or $\theta$.

Figure 14 illustrates the first 40 instability regions on the plane $(\theta / 2 \Omega, \alpha)$ when $\beta=0.5$. It can be seen that the scope of critical frequency ratio is enlarged in general with the increase of $\alpha$ under constant fluctuating components. Therefore, mean component growth would make the beam easier to be dynamically unstable.

The impact of $\alpha$ or $\beta$ on the dynamic instability region is actually the impact of $\mu^{\prime}$ according to the solving process of dynamic instability regions. Increase (decrease) of $\alpha$ or $\beta$ will enhance (reduce) the value of $\mu^{\prime}$. The point $C_{1}$ ( 0.043 , 0.35 ) in the shaded area with smaller $\mu^{\prime}$ than the point $C$ is selected in Figure 11. Modal response of $\mathrm{C}_{1}$ is shown in Figure 15. It can be found that the wind load corresponding to $\mathrm{C}_{1}$ makes the beam become dynamically unstable, but the development trend of instability of $\mathrm{C}_{1}$ is not as obvious as that of C . Therefore, larger mean or fluctuating component of wind load would produce more evident trend of instability.

Another two points $C_{2}(0.043,0.13)$ and $C_{3}(0.043,0.09)$ are also selected in Figure 11. $\mathrm{C}_{2}$ is located in the unshaded area, while $\mathrm{C}_{3}$ is located in the shaded area. Modal responses corresponding to the two points are shown in Figure 16. It is clear that, even though the mean or fluctuating component corresponding to point $\mathrm{C}_{3}$ is less than that of $\mathrm{C}_{2}$, the beam is dynamically unstable under the wind load of $\mathrm{C}_{3}$, while it is stable under the load of $\mathrm{C}_{2}$. Consequently, when judging whether the structure is dynamically stable or not, the key does not lie in the magnitude of wind load but that the corresponding parametric point falls in dynamic instability regions or not. Because the shape of instability regions under complicated wind force is irregular, the beam might still become dynamically unstable even if the load component is very small. To determine the structural dynamic instability under any wind force, it is required to establish the parametric plane of instability regions and judge the position of the corresponding parametric point.

Furthermore, in order to express vividly the relationship between mean component, fluctuating component, and frequency ratio, three-dimensional parametric space $(\theta / 2 \Omega, \alpha$, $\beta$ ) can be built. Figure 17 shows the first 4 spatial instability regions. According to the spatial instability regions, dynamic instability of simply supported Euler beam can be determined by any load parameters $\alpha, \beta$ and the frequency relationship $\theta / 2 \Omega$ of wind force and the beam.

## 6. Conclusions

Through establishing the Mathieu-Hill equation under arbitrary wind force and the computation approach of dynamic instability regions based on the eigenvalue algorithm, a new method for analyzing the dynamic stability of simply supported Euler beam of uniform section under unsteady wind force is established. By introducing the basic load, two-dimensional or three-dimensional parametric coordinate system can be built to determine dynamic instability of the beam and the dynamic instability is verified by


Figure 13: The first 40 instability regions on the parametric plane $(\theta / 2 \Omega, \beta)$ under different $\alpha$ 's.


Figure 14: The first 40 instability regions on the parametric plane $(\theta / 2 \Omega, \alpha)$ when $\beta=0.5$.
calculating wind-induced resonant response. Analysis under the measured wind load shows that any unsteady wind load can be simulated by Fourier series for dynamic instability study and the eigenvalue method achieves sufficiently high precision in computing dynamic instability boundaries. When the parametric point corresponding to specific wind force and structure is located in the instability region, structural response augments divergently. Increase of mean or fluctuating component of wind force would generally


Figure 15: Modal response $q(t)$ of the parametric point $\mathrm{C}_{1}$.
widen the scope of instability region and produce more obvious development trend of dynamic instability. However, the magnitude of wind force is not the only factor deciding the dynamic instability state, and the key point is that the magnitude and frequency of wind load should meet a certain relationship with the natural frequency of the structure.


Figure 16: Modal response $q(t)$ of the parametric points $C_{2}$ and $C_{3}$.


Figure 17: Distribution of the first 4 instability regions on the parametric space $(\theta / 2 \Omega, \alpha, \beta)$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# A Modified Conjugacy Condition and Related Nonlinear Conjugate Gradient Method 

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#### Abstract

The conjugate gradient (CG) method has played a special role in solving large-scale nonlinear optimization problems due to the simplicity of their very low memory requirements. In this paper, we propose a new conjugacy condition which is similar to Dai-Liao (2001). Based on this condition, the related nonlinear conjugate gradient method is given. With some mild conditions, the given method is globally convergent under the strong Wolfe-Powell line search for general functions. The numerical experiments show that the proposed method is very robust and efficient.


## 1. Introduction

The conjugate gradient (CG) method has played a special role in solving large-scale nonlinear optimization problems due to the simplicity of their iterations and their very low memory requirements. In fact, the CG method is not among the fastest or most robust optimization methods for nonlinear problems available today, but it remains very popular for engineers and mathematicians who are interested in solving large-scale problems. The conjugate gradient method is designed to solve the following unconstrained optimization problem:

$$
\begin{equation*}
\min \left\{f(x) \mid x \in R^{n}\right\} \tag{1}
\end{equation*}
$$

where $f(x): R^{n} \rightarrow R$ is a smooth, nonlinear function whose gradient will be denoted by $g(x)$. The iterative formula of the conjugate gradient method is given by

$$
\begin{equation*}
x_{k+1}=x_{k}+s_{k}, \quad s_{k}=\alpha_{k} d_{k} \tag{2}
\end{equation*}
$$

where $\alpha_{k}$ is a step-length which is computed by carrying out a line search, and $d_{k}$ is the search direction defined by

$$
d_{k}= \begin{cases}-g_{k} & \text { if } k=1  \tag{3}\\ -g_{k}+\beta_{k} d_{k-1} & \text { if } k \geq 2\end{cases}
$$

where $\beta_{k}$ is a scalar and $g_{k}$ denotes the gradient $\nabla f\left(x_{k}\right)$. The different conjugate gradient methods correspond to different computing ways of $\beta_{k}$. If $f$ is a strictly convex quadratic function, namely,

$$
\begin{equation*}
f(x)=\frac{1}{2} x^{T} H x+b^{T} x \tag{4}
\end{equation*}
$$

where $H$ is a positive definite matrix, and if $\alpha_{k}$ is the exact one-dimensional minimizer along the direction $d_{k}$, then the method with (2) and (3) is called linear conjugate gradient method. Otherwise, it is called nonlinear conjugate gradient method. The most important feature of linear conjugate gradient method is that the search directions satisfy the following conjugacy condition:

$$
\begin{equation*}
d_{i}^{T} H d_{j}=0, \quad i \neq j \tag{5}
\end{equation*}
$$

For nonlinear conjugate gradient methods, (5) does not hold, since the Hessian $\nabla^{2} f(x)$ changes at different iterations.

Some well-known formulae for $\beta_{k}$ are the Fletcher-Reeves (FR), Polak-Ribière (PR), Hestense-Stiefel (HS), and DaiYuan (DY), which are given, respectively, by

$$
\begin{align*}
& \beta_{k}^{\mathrm{FR}}=\frac{\left\|g_{k}\right\|^{2}}{\left\|g_{k-1}\right\|^{2}} \\
& \beta_{k}^{\mathrm{PR}}=\frac{g_{k}^{T}\left(g_{k}-g_{k-1}\right)}{\left\|g_{k-1}\right\|^{2}} ;  \tag{6}\\
& \beta_{k}^{\mathrm{HS}}=\frac{g_{k}^{T}\left(g_{k}-g_{k-1}\right)}{\left(g_{k}-g_{k-1}\right)^{T} d_{k-1}} ; \\
& \beta_{k}^{\mathrm{DY}}=\frac{\left\|g_{k}\right\|^{2}}{\left(g_{k}-g_{k-1}\right)^{T} d_{k-1}}
\end{align*}
$$

where || || denotes the Euclidean norm. Their corresponding conjugate methods are abbreviated as FR, PR, HS, and DY methods. In the past two decades, the convergence properties of these methods have been intensively studied by many researchers (e.g., [1-9]). Although all these methods are equivalent in the linear case, namely, when $f$ is a strictly convex quadratic function and $\alpha_{k}$ are determined by exact line search, their behaviors for general objective functions may be far different.

For general functions, Zoutendijk [10] proved the global convergence of FR method with exact line search. (Here and throughout this paper, for global convergence, we mean that the sequence generated by the corresponding methods will either terminate after finite steps or contain a subsequence such that it converges to a stationary point of the objective function from a given initial point.) Although one would be satisfied with its global convergence properties, the FR method performs much worse than the PR and HS methods in real computations. Powell [11] analyzed a major numerical drawback of the FR method; namely, if a small step is generated away from the solution point, the subsequent steps may be also very short. On the other hand, in practical computation, the HS method resembles the PR method, and both methods are generally believed to be the most efficient conjugate gradient methods since these two methods essentially perform a restart if a bad direction occurs. However, Powell [12] constructed a counterexample and showed that the PR and HS methods without restarts can cycle infinitely without approaching the solution. This example suggests that these two methods have a drawback that they are not globally convergent for general functions. Therefore, over the past few years, much effort has been put to find out new formulae for conjugate methods such that they are not only globally convergent for general functions but also have robust and efficient numerical performance.

Recently, using a new conjugacy condition, Dai and Liao [13] proposed two new methods. Interestingly, one of their methods is not only globally convergent for general functions but also performs better than HS and PR methods. In this paper, similar to Dai and Liao's approach, we propose a new conjugacy condition. Based on the proposed condition, a new formula for computing $\beta_{k}$ is given. And then, we analyze
the convergence properties for the given method and also carry the numerical experiment which shows that the given method is robust and efficient.

The remainder of this paper is organized as follows. In Section 2, after a short description of Dai and Liao's conjugacy condition and related methods, the motivations of this paper are represented. According to the motivations, we propose a new conjugacy condition and related method at the end of Section 2. In Section 3, convergence analysis for the given method is presented. In the last Section we perform the numerical experiments by testing a set of largescale problems and do some numerical comparisons with some existing methods.

## 2. Motivations, New Conjugacy Condition, and Related Method

2.1. Dai-Liao's Methods. It is well-known that the linear conjugate gradient methods generate a sequence of search directions $d_{k}$ such that conjugacy condition (5) holds. Denote $y_{k-1}$ to be the gradient change, which means that

$$
\begin{equation*}
y_{k-1}=g_{k}-g_{k-1} . \tag{7}
\end{equation*}
$$

For a general nonlinear function $f$, we know by the mean value theorem that there exists some $t \in(0,1)$ such that

$$
\begin{equation*}
y_{k-1}^{T} d_{k}=\alpha_{k-1} d_{k}^{T} \nabla^{2} f\left(x_{k-1}+t \alpha_{k-1} d_{k-1}\right) d_{k-1} \tag{8}
\end{equation*}
$$

Therefore, it is reasonable to replace (5) with the following conjugacy condition:

$$
\begin{equation*}
y_{k-1}^{T} d_{k}=0 \tag{9}
\end{equation*}
$$

Recently, extension of (9) has been studied by Dai and Liao in [13]. Their approach is based on the quasi-Newton techniques. Recall that, in the quasi-Newton method, an approximation matrix $H_{k-1}$ of the Hessian $\nabla^{2} f\left(x_{k-1}\right)$ is updated such that the new matrix $H_{k}$ satisfies the following quasi-Newton equation:

$$
\begin{equation*}
H_{k} s_{k-1}=y_{k-1} \tag{10}
\end{equation*}
$$

The search direction $d_{k}$ in quasi-Newton method is calculated by

$$
\begin{equation*}
d_{k}=-H_{k}^{-1} g_{k} \tag{11}
\end{equation*}
$$

Combining these two equations, we obtain

$$
\begin{equation*}
d_{k}^{T} y_{k-1}=d_{k}^{T}\left(H_{k} s_{k-1}\right)=-g_{k}^{T} s_{k-1} \tag{12}
\end{equation*}
$$

The above relation implies that (9) holds if the line search is exact since in this case $g_{k}^{T} d_{k-1}=0$. However, practical numerical algorithms normally adopt inexact line searches instead of exact line search. For this reason, it seems more reasonable to replace conjugacy condition (9) with the condition

$$
\begin{equation*}
d_{k}^{T} y_{k-1}=-\operatorname{tg}_{k}^{T} s_{k-1} \quad t \geq 0 \tag{13}
\end{equation*}
$$

where $t \geq 0$ is a scalar.

To ensure that the search direction $d_{k}$ satisfies conjugacy condition (13), one only needs to multiply (3) with $y_{k-1}$ and use (13), yielding

$$
\begin{equation*}
\beta_{k}^{\mathrm{DL1}}=\frac{g_{k}^{T}\left(y_{k-1}-t s_{k-1}\right)}{d_{k-1}^{T} y_{k-1}} \tag{14}
\end{equation*}
$$

It is obvious that

$$
\begin{equation*}
\beta_{k}^{\mathrm{DL1}}=\beta_{k}^{\mathrm{HS}}-t \frac{g_{k}^{T} s_{k-1}}{d_{k-1}^{T} y_{k-1}} \tag{15}
\end{equation*}
$$

For simplicity, we call the method with (2), (3), and (14) as DL1 method. Dai and Liao also proved that the conjugate gradient method with DL1 is globally convergent for uniformly convex functions. For general functions, Powell [12] constructed an example showing that the PR method may cycle without approaching any solution point if the steplength $\alpha_{k}$ is chosen to be the first local minimizer along $d_{k}$. Since the DL1 method reduces to the PR method in the case that $g_{k}^{T} d_{k-1}=0$ holds, this implies that the method with (14) need not converge for general functions. To get the global convergence, Dai and Liao made a restriction on $\beta_{k}^{\text {DL1 }}$ as follows

$$
\begin{equation*}
\beta_{k}=\max \left\{\beta_{k}^{\mathrm{PR}}, 0\right\} \tag{16}
\end{equation*}
$$

Dai and Liao replaced (14) by

$$
\begin{align*}
\beta_{k}^{\mathrm{DL}} & =\max \left\{\frac{g_{k}^{T} y_{k-1}}{d_{k-1}^{T} y_{k-1}}, 0\right\}-t \frac{g_{k}^{T} s_{k-1}}{d_{k-1}^{T} y_{k-1}} \\
& =\max \left\{\beta_{k}^{\mathrm{HS}}, 0\right\}-t \frac{g_{k}^{T} s_{k-1}}{d_{k-1}^{T} y_{k-1}} \tag{17}
\end{align*}
$$

We also call the method with (2), (3), and (17) as DL method; Dai and Liao show that DL method is globally convergent for general functions under sufficient descent condition (31) and some suitable conditions. Besides, some numerical experiments in [13] indicate the efficiency of this method.

Similar to Dai and Liao's approach, Li et al. [14] proposed another conjugate condition and related conjugate gradient methods. And they also proved that the proposed methods are globally convergent under some assumptions.

Recently, based on a modified secant condition given by Zhang et al. [15], Yabe and Takano [16] derive an update parameter $\beta_{k}^{\mathrm{YT}}$ and show that the $\mathrm{YT}+$ scheme is globally convergent under some conditions:

$$
\begin{equation*}
\beta_{k}^{\mathrm{YT}}=\frac{g_{k+1}^{T}\left(z_{k}-t s_{k}\right)}{d_{k}^{T} z_{k}} \tag{18}
\end{equation*}
$$

where

$$
\begin{gather*}
z_{k}=y_{k}+\left(\frac{\rho \theta_{k}}{s_{k}^{T} u_{k}}\right) u_{k}  \tag{19}\\
\theta_{k}=6\left(f_{k}-f_{k+1}\right)+3\left(g_{k}+g_{k+1}\right)^{T} s_{k}
\end{gather*}
$$

$\rho \geq 0$ is a constant

$$
\begin{equation*}
\beta_{k}^{\mathrm{YT}+}=\max \left\{\frac{g_{k+1}^{T} z_{k}}{d_{k}^{T} z_{k}}, 0\right\}-t \frac{g_{k+1}^{T} s_{k}}{d_{k}^{T} z_{k}} . \tag{20}
\end{equation*}
$$

2.2. Motivations. From the above discussions, Dai and Liao's approach is effective; the main reason is that the search directions $d_{k}$ generated by DL1 method or DL method not only contain the gradient information but also contain some Hessian $\nabla^{2} f(x)$ information. From (15) and (17), $\beta_{k}^{\text {DL1 }}$ and $\beta_{k}^{\mathrm{DL}}$ are formed by two parts; the first part is $\beta_{k}^{\mathrm{HS}}$ and the second part is $-t\left(g_{k}^{T} s_{k-1} / d_{k-1}^{T} y_{k-1}\right)$. So we also consider DL1 and DL methods as the modified forms of the HS method by adding some information of Hessian $\nabla^{2} f(x)$ which is contained in the second part.

From the structure of (17), we know that the parameter $\beta_{k}^{D L}$ may be negative since the second part $-t\left(g_{k}^{T} s_{k-1} / d_{k-1}^{T} y_{k-1}\right)$ may be less than zero. In conjugate gradient methods, if the $\beta_{k}<0$ and $\left|\beta_{k}\right|$ is large, then the generated directions $d_{k}$ and $d_{k-1}$ may tend to be opposite. This type of methods is susceptible to jamming.

On the other hand, in conjugate gradient methods, the following strong Wolfe-Powell line search is often used to determine the step size $\alpha_{k}$ :

$$
\begin{gather*}
f\left(x_{k}+\alpha_{k} d_{k}\right)-f\left(x_{k}\right) \leq \delta \alpha_{k} g_{k}^{T} d_{k}  \tag{21}\\
\left|g\left(x_{k}+\alpha_{k} d_{k}\right)^{T} d_{k}\right| \leq \sigma\left|g_{k}^{T} d_{k}\right| \tag{22}
\end{gather*}
$$

where $0<\delta<\sigma<1$; a typical choice of $\sigma$ is $\sigma=0.1$. From the structure of (17), we know that $\beta_{k}^{\text {DL }}$ depends on the directional derivative $g_{k}^{T} d_{k-1}$ which is determined by the line search. For PRP+ algorithm with the strong Wolfe-Powell line search, in order to make sufficient descent condition (31) hold, people often used Lemarechal [17], Fletcher [18], or Moré and Thuente’s [19] strategy to make the directional derivative $\left|g_{k}^{T} d_{k-1}\right|$ sufficiently small. Under this strategy, the second part of $\beta_{k}^{\text {DL }}$ will tend to vanish. This means that the DL method is much line-search-dependent.

The above discussions motivate us to propose a modified conjugacy condition and the related conjugate gradient method, which should possess the following properties
(1) Nonnegative property $\beta_{k} \geq 0$.
(2) The new formula contains not only the gradient information but also some Hessian information.
(3) The formula should be less line-search-dependent.
2.3. The Modified Conjugacy Condition and Related Method. From the above discussion, it seems reasonable to replace conjugacy condition (13) with the following modified conjugacy condition:

$$
\begin{equation*}
d_{k}^{T} y_{k-1}=-\operatorname{tg}_{k-1}^{T} s_{k-1}, \quad t>0 \tag{23}
\end{equation*}
$$

To ensure that the search direction $d_{k}$ satisfies condition (23), one only needs to multiply (3) with $y_{k-1}$ and use (23), yielding

$$
\begin{equation*}
\beta_{k}^{\mathrm{MDL}}=\frac{g_{k}^{T} y_{k-1}-\operatorname{tg}_{k-1}^{T} s_{k-1}}{d_{k-1}^{T} y_{k-1}} \tag{24}
\end{equation*}
$$

It is obvious that

$$
\begin{equation*}
\beta_{k}^{\mathrm{MDL}}=\beta_{k}^{\mathrm{HS}}-t \frac{g_{k-1}^{T} s_{k-1}}{d_{k-1}^{T} y_{k-1}} \tag{25}
\end{equation*}
$$

For simplicity, we call the method with (2), (3), and (25) as MDL method. Similar to Gilbert and Nocedal's [4] approach, we propose the following restricted parameter $\beta_{k}^{\mathrm{MDL+}}$ :

$$
\begin{equation*}
\beta_{k}^{\mathrm{MDL}+}=\max \left\{\beta_{k}^{\mathrm{HS}}, 0\right\}-t \frac{g_{k-1}^{T} s_{k-1}}{d_{k-1}^{T} y_{k-1}} \tag{26}
\end{equation*}
$$

And we call the method with (2), (3), and (26) as MDL+ method and give the nonlinear conjugate gradient algorithm as below.

Algorithm 1 (MDL+). Step 1. Given $x_{1} \in R^{n}, \varepsilon \geq 0$, set $d_{1}=$ $-g_{1}, k=1$ if $\left\|g_{1}\right\| \leq \varepsilon$, then stop.

Step 2. Compute $\alpha_{k}$ by the strong Wolfe-Powell line search.
Step 3. Let $x_{k+1}=x_{k}+\alpha_{k} d_{k}, g_{k+1}=g\left(x_{k+1}\right)$; if $\left\|g_{k+1}\right\| \leq \varepsilon$, then stop.

Step 4. Compute $\beta_{k}$ by (26) and generate $d_{k+1}$ by (3).
Step 5. Set $k:=k+1$ and go to Step 2.

## 3. Convergence Analysis

In the convergence analysis of conjugate gradient methods, we often make the following basic assumptions on the objective functions.

Assumption A. (i) The level set $\Gamma=\left\{x \in R^{n}: f(x) \leq f\left(x_{1}\right)\right\}$ is bounded; namely, there exists a constant $B>0$ such that

$$
\begin{equation*}
\|x\| \leq B, \quad \forall x \in \Gamma \tag{27}
\end{equation*}
$$

(ii) In some neighborhood $N$ of $\Gamma, f$ is continuously differentiable, and its gradient is Lipschitz continuous; namely, there exists a constant $L>0$ such that

$$
\begin{equation*}
\|g(x)-g(y)\| \leq L\|x-y\|, \quad \forall x, y \in N \tag{28}
\end{equation*}
$$

Under the above assumptions of $f$, there exists a constant $\bar{\gamma} \geq$ 0 such that

$$
\begin{equation*}
\|\nabla f(x)\| \leq \bar{\gamma}, \quad \forall x \in \Gamma \tag{29}
\end{equation*}
$$

We say the descent condition holds if for each search direction $d_{k}$,

$$
\begin{equation*}
g_{k}^{T} d_{k}<0 \quad \forall k \geq 1 \tag{30}
\end{equation*}
$$

In addition, we say the sufficient descent condition holds if there exists a constant $c>0$ such that, for each search direction $d_{k}$, we have

$$
\begin{equation*}
g_{k}^{T} d_{k} \leq-c\left\|g_{k}\right\|^{2} \quad \forall k \geq 1 \tag{31}
\end{equation*}
$$

Under Assumption A, based on the Zoutendijk condition in [10], for any conjugate gradient method with the strong Wolfe-Powell line search, Dai et al. in [20] proved the following general result.

Lemma 2. Suppose that Assumption $A$ holds. Consider any conjugate gradient method in the form (2)-(3), where $d_{k}$ is a descent direction and $\alpha_{k}$ is obtained by the strong Wolfe-Powell line search. If

$$
\begin{equation*}
\sum_{k \geq 1} \frac{1}{\left\|d_{k}\right\|^{2}}=\infty \tag{32}
\end{equation*}
$$

we have that

$$
\begin{equation*}
\liminf _{k \rightarrow \infty}\left\|g_{k}\right\|=0 \tag{33}
\end{equation*}
$$

If the objective functions are uniformly convex, we can prove that the norm of $d_{k}$ generated by Algorithm 1 (MDL+) is bounded above. Thus by Lemma 2 we immediately have the following result.

Theorem 3. Suppose that Assumption A holds. Consider MDL+ method, where $d_{k}$ is a descent direction and $\alpha_{k}$ is obtained by the strong Wolfe-Powell line search. If the objective functions are uniformly convex, namely, there exists a constant $\mu>0$ such that

$$
\begin{array}{r}
(\nabla f(x)-\nabla f(y))^{T}(x-y) \geq \mu\|x-y\|^{2}  \tag{34}\\
\forall x, y \in \Gamma
\end{array}
$$

we have that

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|g_{k}\right\|=0 \tag{35}
\end{equation*}
$$

Proof. It follows from (34) that

$$
\begin{equation*}
d_{k-1}^{T} y_{k-1} \geq \mu \alpha_{k-1}\left\|d_{k-1}\right\|^{2} \tag{36}
\end{equation*}
$$

By (3), (24), (28), (29), and (36), we have that

$$
\begin{align*}
\left\|d_{k}\right\| & \leq\left\|g_{k}\right\|+\left|\beta_{k}^{\mathrm{MDL}}\right|\left\|d_{k-1}\right\| \\
& \leq\left\|g_{k}\right\|+\left|\frac{g_{k}^{T} y_{k-1}-t g_{k-1}^{T} s_{k-1}}{\mu \alpha_{k-1}\left\|d_{k-1}\right\|^{2}}\right|\left\|d_{k-1}\right\| \\
& \leq\left\|g_{k}\right\|+\frac{\left\|g_{k}\right\|\left\|g_{k}-g_{k-1}\right\|+t\left|g_{k-1}^{T} s_{k-1}\right|}{\mu \alpha_{k-1}\left\|d_{k-1}\right\|^{2}}\left\|d_{k-1}\right\|  \tag{37}\\
& \leq\left\|g_{k}\right\|+\frac{\left\|g_{k}\right\| L\left\|s_{k-1}\right\|+t\left\|g_{k-1}\right\|\left\|s_{k-1}\right\|}{\mu \alpha_{k-1}\left\|d_{k-1}\right\|^{2}}\left\|d_{k-1}\right\| \\
& \leq \bar{\gamma}+\frac{\bar{\gamma} L\left\|s_{k-1}\right\|+t \bar{\gamma}\left\|s_{k-1}\right\|}{\mu \alpha_{k-1}\left\|d_{k-1}\right\|^{2}}\left\|d_{k-1}\right\| \\
& =\bar{\gamma} \mu^{-1}(\mu+L+t)
\end{align*}
$$

which implies the truth of (32). Therefore, by Lemma 2, we have (33), which is equivalent to (35) for uniformly convex functions. The proof is completed.

For the method with $\beta_{k}^{\mathrm{MDL}+}$, if descent condition (30) holds and $\alpha_{k}$ satisfies Wolfe-Powell condition (22), then the parameter $\beta_{k}^{\mathrm{MDL}+}$ is nonnegative.

Lemma 4. In any conjugate gradient method, if the parameter $\beta_{k}$ is computed by (26) and $d_{k}$ is a descent direction, when $\alpha_{k}$ is determined by strong Wolfe-Powell line search conditions (21) and (22), then

$$
\begin{equation*}
\beta_{k}^{\mathrm{MDL}+} \geq 0 \tag{38}
\end{equation*}
$$

Proof. By SWP condition (22), we have $d_{k-1}^{T} y_{k-1} \geq \sigma g_{k-1}^{T}$ $d_{k-1}-g_{k-1}^{T} d_{k-1} \geq 0$ and $g_{k-1}^{T} s_{k-1}<0$. So we have

$$
\begin{align*}
\beta_{k}^{\mathrm{MDL}+} & =\max \left\{\beta_{k}^{\mathrm{HS}}, 0\right\}-t \frac{g_{k-1}^{T} s_{k-1}}{d_{k-1}^{T} y_{k-1}},  \tag{39}\\
& \geq 0
\end{align*}
$$

The proof is completed.
Theorem 5. Suppose that Assumption A holds. Consider MDL+ method, where $d_{k}$ is a sufficient descent direction and $\alpha_{k}$ is obtained by strong Wolfe-Powell line search. If there exists a constant $\gamma>0$ such that

$$
\begin{equation*}
\left\|g_{k}\right\| \geq \gamma, \quad \forall k \geq 1 \tag{40}
\end{equation*}
$$

then $d_{k} \neq 0$ and

$$
\begin{equation*}
\sum_{k \geq 2}\left\|u_{k}-u_{k-1}\right\|^{2}<\infty, \tag{41}
\end{equation*}
$$

where $u_{k}=d_{k} /\left\|d_{k}\right\|$.
Proof. First, note that $d_{k} \neq 0$; otherwise (31) is false. Therefore $u_{k}$ is well defined. In addition, by relation (40) and Lemma 2, we have that

$$
\begin{equation*}
\sum_{k \geq 1} \frac{1}{\left\|d_{k}\right\|^{2}}<\infty \tag{42}
\end{equation*}
$$

Now, we divide formula $\beta_{k}^{\mathrm{MDL}+}$ into two parts as follows:

$$
\begin{equation*}
\beta_{k}^{1}=\max \left\{\beta_{k}^{\mathrm{HS}}, 0\right\}, \quad \beta_{k}^{2}=-t \frac{g_{k-1}^{T} s_{k-1}}{d_{k-1}^{T} y_{k-1}} \tag{43}
\end{equation*}
$$

and define

$$
\begin{equation*}
r_{k}:=\frac{\vartheta_{k}}{\left\|d_{k}\right\|}, \quad \delta_{k}:=\beta_{k}^{1} \frac{\left\|d_{k-1}\right\|}{\left\|d_{k}\right\|} \tag{44}
\end{equation*}
$$

where $\vartheta_{k}=-g_{k}+\beta_{k}^{2} d_{k-1}$.
Then by (3) we have, for all $k \geq 2$,

$$
\begin{equation*}
u_{k}=r_{k}+\delta_{k} u_{k-1} . \tag{45}
\end{equation*}
$$

Using the identity $\left\|u_{k}\right\|=\left\|u_{k-1}\right\|=1$ and (45), we can obtain

$$
\begin{align*}
& \left\|u_{k}-\delta_{k} u_{k-1}\right\|^{2}=1+\delta_{k}^{2}-2 \delta_{k} u_{k}^{T} u_{k-1}  \tag{46}\\
& \left\|\delta_{k} u_{k}-u_{k-1}\right\|^{2}=1+\delta_{k}^{2}-2 \delta_{k} u_{k}^{T} u_{k-1}  \tag{47}\\
& \left\|r_{k}\right\|=\left\|u_{k}-\delta_{k} u_{k-1}\right\|=\left\|\delta_{k} u_{k}-u_{k-1}\right\| \tag{48}
\end{align*}
$$

Using the condition $\delta_{k}=\max \left\{\beta_{k}^{\mathrm{HS}}, 0\right\}\left(\left\|d_{k-1}\right\| /\left\|d_{k}\right\|\right) \geq 0$, the triangle inequality, and (48), we obtain

$$
\begin{align*}
\left\|u_{k}-u_{k-1}\right\| & \leq\left\|\left(1+\delta_{k}\right) u_{k}-\left(1+\delta_{k}\right) u_{k-1}\right\| \\
& \leq\left\|u_{k}-\delta_{k} u_{k-1}\right\|+\left\|\delta_{k} u_{k}-u_{k-1}\right\|  \tag{49}\\
& =2\left\|r_{k}\right\|
\end{align*}
$$

On the other hand, line search condition (22) gives

$$
\begin{equation*}
y_{k-1}^{T} d_{k-1} \geq(\sigma-1) g_{k-1}^{T} d_{k-1}>0 \tag{50}
\end{equation*}
$$

Equations (22), (31), and (50) imply that

$$
\begin{equation*}
\left|\frac{g_{k-1}^{T} d_{k-1}}{d_{k-1}^{T} y_{k-1}}\right| \leq \frac{1}{1-\sigma} \tag{51}
\end{equation*}
$$

It follows from the definition of $\vartheta_{k}$, (27), (29), and (51) that

$$
\begin{align*}
\left\|\vartheta_{k}\right\| & \leq\left\|g_{k}\right\|+t\left|\frac{g_{k-1}^{T} s_{k-1}}{d_{k-1}^{T} y_{k-1}}\right|\left\|d_{k-1}\right\| \\
& =\left\|g_{k}\right\|+t\left|\frac{g_{k-1}^{T} d_{k-1}}{d_{k-1}^{T} y_{k-1}}\right|\left\|s_{k-1}\right\|  \tag{52}\\
& \leq \bar{\gamma}+t \frac{1}{1-\sigma} 2 B .
\end{align*}
$$

So we have

$$
\begin{aligned}
\sum\left\|u_{k}-u_{k-1}\right\|^{2} & \leq 4 \sum\left\|r_{k}\right\|^{2} \leq 4 \sum \frac{\vartheta_{k}^{2}}{\left\|d_{k}\right\|^{2}} \\
& \leq 4\left(\bar{\gamma}+t \frac{1}{1-\sigma} 2 B\right)^{2} \sum \frac{1}{\left\|d_{k}\right\|^{2}} \\
& <\infty
\end{aligned}
$$

and the proof is completed.
Gilbert and Nocedal [4] introduced property ( ${ }^{*}$ ) which is very important for the convergence analysis of the conjugate gradient methods. In fact, with Assumption A, (40), and (50), if (31) holds with some constant $c>0$, the method with $\beta_{k}^{\text {MDL }+}$ possesses such property ( ${ }^{*}$ ).

Property $1\left(^{*}\right)$. Consider a method of forms (2) and (3). Suppose that

$$
\begin{equation*}
0<\gamma \leq\left\|g_{k}\right\| \leq \bar{\gamma} \quad \forall k \geq 1 \tag{54}
\end{equation*}
$$

We say that the method has property ( ${ }^{*}$ ), if, for all $k$, there exist constants $b>1, \lambda>0$ such that $\left|\beta_{k}\right| \leq b$, and if $\left\|s_{k-1}\right\| \leq$ $\lambda$, we have $\left|\beta_{k}\right| \leq 1 / 2 b$.

In fact, by (31), (40), and (50), we have

$$
\begin{align*}
d_{k-1}^{T} y_{k-1} & \geq(\sigma-1) g_{k-1}^{T} d_{k-1} \geq c(1-\sigma)\left\|g_{k-1}\right\|^{2}  \tag{55}\\
& \geq(1-\sigma) c \gamma^{2} .
\end{align*}
$$

Combining (55) with (27) and (28) and (29), we obtain

$$
\begin{equation*}
\left|\beta_{k}^{\mathrm{MDL}+}\right| \leq \frac{\bar{\gamma} L\left\|s_{k-1}\right\|+t \bar{\gamma}\left\|s_{k-1}\right\|}{(1-\sigma) c \gamma^{2}} \leq \frac{2 B(L+t) \bar{\gamma}}{(1-\sigma) c \gamma^{2}}=: b . \tag{56}
\end{equation*}
$$

Note that $b$ can be defined such that $b>1$. Therefore we can say that $b>1$. As a result, we define

$$
\begin{equation*}
\lambda:=\frac{(1-\sigma) c \gamma^{2}}{2 b(L+t) \bar{\gamma}} \tag{57}
\end{equation*}
$$

and we get from the first inequality in (56) that if $\left\|s_{k-1}\right\| \leq \lambda$, then

$$
\begin{align*}
\left|\beta_{k}^{\mathrm{MDL+}}\right| & \leq \frac{\bar{\gamma}(L+t) \lambda}{(1-\sigma) c \gamma^{2}} \\
& =\frac{\bar{\gamma}(L+t)}{(1-\sigma) c \gamma^{2}} \frac{(1-\sigma) c \gamma^{2}}{2 b(L+t) \bar{\gamma}}=\frac{1}{2 b} \tag{58}
\end{align*}
$$

Let $N^{*}$ denote the set of positive integers. For $\lambda>0$ and a positive integer $\Delta$, denote

$$
\begin{equation*}
K_{k, \Delta}^{\lambda}:=\left\{i \in N^{*}: k \leq i \leq k+\Delta-1,\left\|s_{k-1}\right\|>\lambda\right\} . \tag{59}
\end{equation*}
$$

Let $\left|K_{k, \Delta}^{\lambda}\right|$ denote the number of elements in $K_{k, \Delta}^{\lambda}$. From the above property $\left({ }^{*}\right)$, we can prove the following theorem.

Theorem 6. Suppose that Assumption A holds. Consider MDL+ method, where $d_{k}$ satisfies condition (31) with $c>0$, and $\alpha_{k}$ is obtained by the strong Wolfe-Powell line search. Then if (40) holds, there exists $\lambda>0$ such that, for any $\Delta \in N^{*}$ and any index $k_{0}$, there is an index $k \geq k_{0}$ such that

$$
\begin{equation*}
\left|K_{k, \Delta}^{\lambda}\right|>\frac{\Delta}{2} . \tag{60}
\end{equation*}
$$

The proof of this theorem is similar to the proof of Lemma 3.5 in [13]. So, we omit the proof.

According to the above lemmas and theorems, we can prove the following convergence result for the MDL+ method.

Theorem 7. Suppose that Assumption A holds. Consider MDL+ method, where $d_{k}$ satisfies condition (31) with $c>0$, and $\alpha_{k}$ is obtained by the strong Wolfe-Powell line search. Then we have $\lim \inf _{k \rightarrow \infty}\left\|g_{k}\right\|=0$.

Proof. We proceed by contradiction. If $\lim \inf _{k \rightarrow \infty}\left\|g_{k}\right\|>0$, then (40) must hold. Then the conditions of Theorem 6 hold. Defining $u_{i}=d_{i} /\left\|d_{i}\right\|$, we have, for any indices $l, k$, with $l \geq k$,

$$
\begin{align*}
x_{l}-x_{k-1} & =\sum_{i=k}^{l} x_{i}-x_{i-1} \\
& =\sum_{i=k}^{l} \alpha_{i-1} d_{i-1}=\sum_{i=k}^{l} u_{i-1}\left\|s_{i-1}\right\|  \tag{61}\\
& =\sum_{i=k}^{l}\left\|s_{i-1}\right\| u_{k-1}+\sum_{i=k}^{l}\left\|s_{i-1}\right\|\left(u_{i-1}-u_{k-1}\right) .
\end{align*}
$$

Consider $\left\|u_{i}\right\|=1$; (27) and (61) give that

$$
\begin{align*}
\sum_{i=k}^{l}\left\|s_{i-1}\right\| & \leq\left\|x_{l}-x_{k-1}\right\|+\sum_{i=k}^{l}\left\|s_{i-1}\right\|\left\|u_{i-1}-u_{k-1}\right\|  \tag{62}\\
& \leq 2 B+\sum_{i=k}^{l}\left\|s_{i-1}\right\|\left\|u_{i-1}-u_{k-1}\right\|
\end{align*}
$$

Let $\lambda>0$ be given by Theorem 6 and define $\Delta:=\lceil 8 B / \lambda\rceil$ to be the smallest integer not less than $8 B / \lambda$. By Theorem 6 , we can find an index $k_{0} \geq 1$ such that

$$
\begin{equation*}
\sum_{i \geq k_{0}}\left\|u_{i-1}-u_{k-1}\right\|^{2} \leq \frac{1}{4 \Delta} \tag{63}
\end{equation*}
$$

With this $\Delta$ and $k_{0}$, Theorem 6 gives an index $k \geq k_{0}$ such that

$$
\begin{equation*}
\left|K_{k, \Delta}^{\lambda}\right|>\frac{\Delta}{2} \tag{64}
\end{equation*}
$$

For any index $i \in[k, k+\Delta-1]$, by Cauchy-Schwarz, the geometric inequalities, and (63),

$$
\begin{align*}
\left\|u_{i}-u_{k-1}\right\| & \leq \sum_{j=k}^{i}\left\|u_{j}-u_{j-1}\right\| \\
& \leq(i-k+1)^{1 / 2}\left(\sum_{j=k}^{i}\left\|u_{j}-u_{j-1}\right\|^{2}\right)^{1 / 2}  \tag{65}\\
& \leq \Delta^{1 / 2}\left(\frac{1}{4 \Delta}\right)^{1 / 2}=\frac{1}{2}
\end{align*}
$$

From relations (64) and (65), by taking $l=k+\Delta-1$ in (62), we get

$$
\begin{equation*}
2 B \geq \frac{1}{2} \sum_{i=k}^{k+\Delta-1}\left\|s_{i-1}\right\|>\frac{\lambda}{2}\left|K_{k, \Delta}^{\lambda}\right|>\frac{\lambda \Delta}{4} \tag{66}
\end{equation*}
$$

Thus $\Delta<8 B / \lambda$, which contradicts the definition of $\Delta$. The proof is completed.

## 4. Numerical Results

In this section, we report the performance of Algorithm 1 (MDL+) on a set of test problems. The codes were written in Fortran 77 and in double precision arithmetic. All the tests were performed on the same PC (Intel Core i3 CPU M370 @ $2.4 \mathrm{GH}, 2 \mathrm{~GB}$ RAM). The experiments were performed on a set of 73 nonlinear unconstrained problems collected by Neculai Andrei. Some of the problems are from CUTE [21] library. For each test problem, we have performed 10 numerical experiments with a number of variables $n=1000$, 2000,..., 10000 .

In order to assess the reliability of the MDL+ algorithm, we also tested this method against the DL method and HS method using the same problems. All these algorithms terminate when $\left\|g_{k}\right\| \leq 10^{-5}$. We also force the routines to stop if the iterations exceed 1000 or the number of function evaluations reaches 2000. The parameters $\Delta$ and $\sigma$ in WolfePowell line search conditions (21) and (22) are set to be $10^{-4}$ and $10^{-1}$ respectively. For DL method, $t=0.1$, which is the same with [13]. We also test MDL+ algorithm with different parameters $t$ to see that $t=0.05$ is the best choice.

The comparing data contain the iterations, function, and gradient evaluations and CPU time. To approximatively assess the performance of MDL+, HS, and DL methods, we use the profile of Dolan and Moré [22] as an evaluated tool.

Dolan and Moré [22] gave a new tool to analyze the efficiency of algorithms. They introduced the notion of a performance profile as a means to evaluate and compare the performance of the set of solvers $S$ on a test set $P$. Assuming that there exist $n_{s}$ solvers and $n_{p}$ problems, for each problem $p$ and solver $s$, they defined $t_{p, s}=$ computing cost (iterations or function and gradient evaluations or CPU time) required to solve problem $p$ by solver $s$.

Requiring a baseline for comparisons, they compared the performance on problem $p$ by solver $s$ with the best performance by any solver on this problem; that is, using the performance ratio

$$
\begin{equation*}
r_{p, s}=\frac{t_{p, s}}{\min \left\{t_{p, s}: s \in S\right\}} \tag{67}
\end{equation*}
$$

Suppose that a parameter $M \geq r_{p, s}$ for all $p, s$. Set $r_{p, s}=M$ if and only if solver $s$ does not solve problem $p$. Then they defined

$$
\begin{equation*}
\rho_{s}(\tau)=\frac{1}{n_{p}} \operatorname{size}\left\{p \in P: r_{p, s} \leq \tau\right\} . \tag{68}
\end{equation*}
$$

Thus $\rho_{s}(\tau)$ is the probability for solver $s$ that a performance ratio $r_{p, s}$ is within factor $\tau \geq 1$ of the best possible ratio. Then function $\rho_{s}$ is the distribution function for the performance ratio. The performance profile $\rho_{s}$ is a nondecreasing, piecewise constant function. That is, for subset of the methods being analyzed, we plot the fraction $P$ of the problems for which any given method is within a factor $\tau$ of the best.

For the testing problems, if all three methods can not terminate successfully, then we got rid of it. In case one method fails, but there is another method that terminates successfully, then the performance ratio of the failed method


Figure 1: Iterations.


Figure 2: Function and gradient evaluations.
is set to be $M$ ( $M$ is the maxima of the performance ratios). The performance profiles based on iterations, function and gradient evaluations, and CPU time of the three methods are plotted in Figures 1, 2, and 3, respectively.

From Figure 1, which plots the performance profile based on iterations, when $\tau=1$, the HS method performs better than MDL+ and DL methods. With the increasing of $\tau$, when $\tau \geq 1.3$, the profile of MDL+ method outperforms HS and DL


Figure 3: CPU time.
methods. This means that, from the iteration points of view, for a subset of problems, HS method is better than MDL+ and DL methods. But, for all the testing problems, DML+ method is much robuster than HS and DL methods.

From Figure 2, which plots the performance profile based on function and gradient evaluations, it is easy to see that, for all $\tau \geq 1$, MDL+ method performs much better than HS and DL methods. It is an interesting phenomenon, since, when $\tau \leq 1.3$, the profiles of HS based on iterations outperform DML+ method. This means that, during process of iteration, the required function and gradient evaluations of MDL+ method are much less than HS and DL methods. Form this point of view, the CPU time consumed by MDL+ method should be much less than HS and DL methods, since the CPU time is mainly dependent on function and gradient evaluations. Figure 3 validates that the CPU time consumed by MDL+ method is much less than HS and DL methods.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# A Model Based on Cocitation for Web Information Retrieval 

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#### Abstract

According to the relationship between authority and cocitation in HITS, we propose a new hyperlink weighting scheme to describe the strength of the relevancy between any two webpages. Then we combine hyperlink weight normalization and random surfing schemes as used in PageRank to justify the new model. In the new model based on cocitation (MBCC), the pages with stronger relevancy are assigned higher values, not just depending on the outlinks. This model combines both features of HITS and PageRank. Finally, we present the results of some numerical experiments, showing that the MBCC ranking agrees with the HITS ranking, especially in top 10 . Meanwhile, MBCC keeps the superiority of PageRank, that is, existence and uniqueness of ranking vectors.


## 1. Introduction

In the past, search engines ranked pages by using word frequency or similar measures. However, the relevancy of webpages returned by this traditional web information retrieval is still lacking, because the webpages are created with varying qualities. Recently, some new algorithms have been created that greatly improve rankings. One of the popular ideas is to use hyperlinks to determine the value of different webpages. This hyperlink graph contains useful information: if webpage $i$ has a link pointing to webpage $j$, it usually indicates that the creator of $i$ considers $j$ to contain relevant information for $i$. Such useful opinions and knowledge are therefore registered in the form of adjacency matrix which is denoted by $L . L_{i j}=1$ if there is a link from $i$ to $j$, or 0 , otherwise.

Two most popular ranking algorithms based on hyperlink analysis are the PageRank algorithm [1,2] and the HITS (Hyper-text Induced Topic Selection) algorithm [3]. Generally, PageRank considers the hyperlink weight normalization and the equilibrium distribution of random surfers as the citation score. For more information about the calculation methods of PageRank refer to [4-6]. HITS makes the distinction between hubs and authorities and then computes them in a mutually reinforcing way. For each of these two algorithms, the ranking vector is the dominant eigenvector of some matrix describing the network. How this matrix is defined differs in each method. There are other works which
have recognized that the hyperlink structure can be very valuable for locating information $[3,7,8]$.

This paper is organized as follows. In Section 2, we introduce the PageRank and HITS algorithms and briefly discuss the limitations in HITS. Then in Section 3, we emphasize the role of cocitation (Figure 1) and provide a hyperlink weighting scheme to describe the strength of the relevancy between any two webpages. In order to ensure the existence of solutions and uniqueness of solutions in the new model (MBCC), we also combine ideas from PageRank. In Section 4, some experiments are presented. The result shows that the MBCC ranking is close well to the HITS ranking. Conclusions are given in Section 5.

## 2. PageRank and HITS

We treat the web as a directed graph $D=(V, E)$ : the nodes in $V$ correspond to the pages, and a directed edge $(i, j) \in E$ indicates the existence of a link from $i$ to $j$. We say that the out-degree of a node $i$ denoted by $d_{\text {out }}(i)$ is the number of nodes it has links point to, and the in-degree of $i$ denoted by $d_{\mathrm{in}}(i)$ is the number of nodes that have links point to it. We also denote that

$$
\begin{align*}
& d_{\mathrm{out}}=\left(d_{\mathrm{out}}(1), \ldots, d_{\mathrm{out}}(n)\right)^{T},  \tag{1}\\
& d_{\mathrm{in}}=\left(d_{\mathrm{in}}(1), \ldots, d_{\mathrm{in}}(n)\right)^{T} .
\end{align*}
$$

2.1. Review of PageRank. PageRank [1, 2] uses a web surfing model based on a random walk process. Suppose there is a link from page $i$ to page $j$; that is, $(i, j) \in E$. Consider a random surfer visiting page $i$ at time $t$. Then at the next time $t+1$, the surfer lands at page $j$ with probability $1 / d_{\text {out }}(i)$. Once the above is done, the PageRank algorithm assigns a rank value $x_{i}$ for the page $i$ as a function of the rank of the pages that point to it:

$$
\begin{equation*}
x_{i}=\sum_{(j, i) \in E} \frac{r_{i}}{d_{\text {out }}(j)} . \tag{2}
\end{equation*}
$$

If the page $i$ has no outlink, that is, $d_{\text {out }}(i)=0$, then, at time $t+1$, the surfer chooses any page with probability $1 / n$. Thus, we replace $d_{\text {out }}(i)=0$ with $d_{\text {out }}(i)=n$. Then the stationary distribution $x$ is determined by the following matrix form:

$$
\begin{equation*}
x=P^{T} x, \quad P^{T}=\left(L+d e^{T}\right)^{T} D_{\text {out }}^{-1} . \tag{3}
\end{equation*}
$$

Here $x=\left(x_{1}, \ldots, x_{n}\right)^{T}, L$ is the adjacency matrix of the directed web graph, $D_{\text {out }}=\operatorname{diag}\left(d_{\text {out }}\right)$, and $e=(1, \ldots, 1)^{T}$. In the vector $d$, the element $d_{i}=1$ if the $i$ th row of $L$ corresponds to a dangling node $\left(d_{\text {out }}(i)=0\right)$, or 0 , otherwise.

In order to calculate the above recursive equation and get a unique stationary probability distribution, it is important to guarantee that (3) is convergent. This problem can be solved if the directed graph $D$ is strongly connected, which is generally not the case for the directed graph. In the context of computing PageRank, the standard way of ensuring this property is to add a new set of complete outgoing transitions, with small transition probabilities (in this work, we set each of them as $1 / n$ ), to all nodes in $D$. Then the modified transition probability called Google matrix is

$$
\begin{equation*}
G^{T}=\alpha\left(L+d e^{T}\right)^{T} D_{\text {out }}^{-1}+(1-\alpha)\left(\frac{1}{n}\right) e e^{T}, \tag{4}
\end{equation*}
$$

where $\alpha=0.8 \sim 0.9$. Here $e=(1, \ldots, 1)^{T}$; thus $e e^{T}$ is a matrix of all 1's. The PageRank algorithm is to solve the eigenvector of the Google matrix $G^{T}$

$$
\begin{equation*}
x=G^{T} x, \quad \sum_{i=1}^{n} x_{i}=1, \tag{5}
\end{equation*}
$$

where $G^{T}$ is stochastic and irreducible.
PageRank models two types of random jumps on the Internet. With probability $1-\alpha$ a surfer randomly chooses a new page. Otherwise, the surfer follows one of directed edges from the present node.
2.2. Review of HITS. In the HITS algorithm [3], each webpage $i$ has both a hub score $y_{i}$ (based on the links going from the page) and an authority score $x_{i}$ (based on the links going to the page). Let $x=\left(x_{1}, \ldots, x_{n}\right)^{T}$ denote the vector of all authority weights, let $y=\left(y_{1}, \ldots, y_{n}\right)^{T}$ denote the vector of all hub weights, and let $L$ be the adjacency matrix of the directed web graph. In HITS, there are two operations
at each iteration. One is defined as operation $\mathscr{J}$ which sets the authority vector to $x=L^{T} y$. It indicates that a good authority is pointed by many good hubs. Another is defined as operation $\mathcal{O}$ which sets the hub vector to $y=L x$. It indicates that a good hub points to many good authorities. This mutually reinforcing relationship can be written in the following matrix representations:

$$
\begin{array}{ll}
x=\frac{1}{\lambda^{*}} L^{T} L x, & \sum_{i=1}^{n} x_{i}^{2}=1, \\
y=\frac{1}{\lambda^{*}} L L^{T} y, & \sum_{i=1}^{n} y_{i}^{2}=1 . \tag{6}
\end{array}
$$

The final authority and hub scores are the principal eigenvectors of $L^{T} L$ and $L L^{T}$ which are corresponding to the dominant eigenvalue $\lambda^{*}$. Since $L^{T} L$ and $L L^{T}$ determine the authority ranking and hub ranking, we call $L^{T} L$ the authority matrix and $L L^{T}$ the hub matrix.

In the fields of citation analysis and bibliometrics, it has shown that the authority matrix has interesting connections to cocitation [3]. Here cocitation is defined as the number of webpages that cocite $i, j$ [9]. In the authority matrix, $\left(L^{T} L\right)_{i i}=$ $\sum_{k=1}^{n} L_{k i} L_{k i}=\sum_{k=1}^{n} L_{k i}$ is the in-degree of page $i$; that is,

$$
\begin{equation*}
\left(L^{T} L\right)_{i i}=d_{\mathrm{in}}(i) . \tag{7}
\end{equation*}
$$

This implies that

$$
\begin{equation*}
\operatorname{diag}\left(L^{T} L\right)=D_{\mathrm{in}} . \tag{8}
\end{equation*}
$$

For $i \neq j,\left(L^{T} L\right)_{i j}=\sum_{k=1}^{n} L_{k i} L_{k j}$ is the number of webpages that cocite $i, j$ that is denoted by $C_{i j}$. Therefore the authority matrix $L^{T} L$ is the sum of in-degree and cocitation $[10,11]$

$$
\begin{equation*}
L^{T} L=D_{\mathrm{in}}+C \tag{9}
\end{equation*}
$$

The self cocitation $C_{i i}$ in $C$ is not defined and is usually set to 0 .
2.3. Existence and Uniqueness of Ranking Vectors. In this section, we present the existence and uniqueness of ranking vectors in the above two algorithms.

Since the Google matrix $G^{T}$ in (4) is stochastic and irreducible, for the PageRank algorithm, the PageRank ranking vector exists, and it is unique and positive. See the equivalent theorem in [12, Theorem 3.8]. For the HITS algorithm, it has been proved that the hub and authority ranking vectors exist but may not be unique. In [12], they show that the HITS algorithm badly behaved on certain networks, meaning that (i) it can return ranking vectors that are not unique but depend on the initial seed vector or (ii) it can return ranking vectors that inappropriately assign zero weights to parts of the network.

There are also other limitations for HITS; see [12, 13]. Thus, to address these limitations, a modification for HITS is needed, for example, exponentiated input method in [12]. In the next section, we combine both features of HITS and PageRank. The ranking produced by the new model is expected to be unique and close to the HITS ranking.


Figure 1: Webpages $i, j$ are cocited by webpage $k$.

## 3. A Model Based on Cocitation (MBCC)

In HITS, according to (9), the authority ranking value $x_{i}$ can be expressed as

$$
\begin{equation*}
x_{i}=\frac{1}{\lambda^{*}} d_{\mathrm{in}}(i) x_{i}+\frac{1}{\lambda^{*}} \sum_{j=1}^{n} C_{i j} x_{j}, \tag{10}
\end{equation*}
$$

revealing the close relationship between authorities and cocitations. It also implies that, if two distinct webpages $i, j$ are cocited by many other webpages $k$ as shown in Figure 1, then $i, j$ are likely to be related in some sense. In this paper, we present a property for HITS corresponding to (10).

Property 1 (relationship between authority value and cocitation). If the number of webpages that cocite webpages $i$ and $j$, that is, $C_{i j}$, is larger, the page $i$ could receive more authority value from the page $j$, even though there are no links between $i$ and $j$.

The fact that the webpages cocite two distinct webpages $i$ and $j$ indicates that $i, j$ have certain commonality. Therefore, we say that the number of cocitations represents the relevancy among the pages. Then, in the following, we focus on the use of cocitation for analyzing the relevancy among the pages.

Note that, in Section 2.1, the rank of a page in PageRank is divided among its forward links evenly; see (2); that is, a web surfer could chose the forward outlinks randomly. However, this process of dividing the rank equally may seem unrealistic; that is, a web surfer may have a priori idea of the value of pages, favoring pages from the relevant sites. Since it shows that the number of cocitations could represent the relevancy among the pages, we say that the number of cocitations between two pages can impact the behavior of web surfers. Therefore, we define a new hyperlink weighting scheme based on cocitation as follows:

Definition 2 (hyperlink weighting scheme based on cocitation). Let $Q_{i j}$ be the number of webpages that cocite two webpages $i, j$. Specially, $Q_{i i}=d_{\mathrm{in}}(i)$, and $d_{\mathrm{in}}(i)$ is the in-degree

Table 1: The data of Example 1.

|  | $\left(j_{1}, j_{1}\right)$ | $\left(j_{1}, j_{2}\right)$ | $\left(j_{1}, j_{3}\right)$ |
| :--- | :---: | :---: | :---: |
| $Q$ | 3 | 3 | 1 |
| $W$ | $3 / 7$ | $3 / 7$ | $1 / 7$ |

of webpage $i$. Then we define the following function as the value of $j$ which will receive form $i$ :

$$
\begin{equation*}
W_{i j}=\frac{Q_{i j}}{\sum_{k \in V} Q_{i k}}=\frac{Q_{i j}}{Q_{i}} \tag{11}
\end{equation*}
$$

where $Q_{i}=\sum_{k \in V} Q_{i k}$.
Under this assignment method, the rank value for the page $j$ is determined by

$$
\begin{equation*}
x_{j}=\sum_{m \in V} W_{m j} x_{m} . \tag{12}
\end{equation*}
$$

The matrix form of above equation is $x=W x$, where $x=\left(x_{1}, \ldots, x_{n}\right)^{T}$. The problem is that, if at least one page has zero in-degree, that is, no in-links and $Q_{i}=0$, then the matrix $W$ is absorbing and its dominant eigenvector does not exist. In order to resolve this, similarly to PageRank, we assume that, if the page $i$ has no link that points to it, then at time $t+1$, the page $i$ divides its value equally to any other page with probability $1 / n$. The modified matrix $W$ is given by

$$
\begin{equation*}
W=\left(L^{\prime} L+v e^{T}\right)^{T} D_{\mathrm{Q}}^{-1}=\left(L^{\prime} L+e v^{T}\right) D_{\mathrm{Q}}^{-1} \tag{13}
\end{equation*}
$$

where we replace $Q_{i}=0$ with $Q_{i}=n, D_{\mathrm{Q}}=\operatorname{diag}(\widetilde{Q})$ and $\widetilde{Q}=\left(Q_{1}, \ldots, Q_{n}\right)^{T} . \widetilde{Q}$ can be computed as

$$
\begin{equation*}
\widetilde{Q}=\left(L^{\prime} L+v e^{T}\right) e . \tag{14}
\end{equation*}
$$

In the vector $v$, the element $v_{i}=1$ if the $i$-th row of $L$ corresponds to a page with no in-degree, or 0 , otherwise. Therefore, the modified matrix $W$ becomes a stochastic matrix, that is, each column in $W$ sum to 1 .

In order to get a unique stationary probability distribution, it is important to guarantee that $W$ is strongly connected. Similarly to PageRank, we add a new set of complete outgoing transitions. The final transition probability matrix based on using cocitation as a hyperlink weighting scheme is

$$
\begin{equation*}
W=\beta\left(L^{\prime} L+e v^{T}\right) D_{\mathrm{Q}}^{-1}+(1-\beta)\left(\frac{1}{n}\right) e e^{T}, \tag{15}
\end{equation*}
$$

where $0<\beta<1$ and $e=(1, \ldots, 1)^{T}$. The model based on cocitation (MBCC) is to solve the following function:

$$
\begin{gather*}
x^{*}=\left\{\beta\left(L^{\prime} L+e v^{T}\right) D_{\mathrm{Q}}^{-1}+(1-\beta)\left(\frac{1}{n}\right) e e^{T}\right\} x^{*} \\
\sum_{i=1}^{n} x_{i}=1 \tag{16}
\end{gather*}
$$



Figure 2: (a) Link structure of PageRank. (b) Resigned link structure of MBCC.

TABLE 2: HITS authority ranking, MBCC authority ranking, and PageRank ranking in top 20.

| HITS | MBCC | PageRank | URL |
| :---: | :---: | :---: | :---: |
| 1 | 1 | 9 | http://www.ics.uci.edu/~eppstein/geom.html |
| 2 | 2 | 10 | http://www.geom.uiuc.edu/software/cglist/ |
| 3 | 3 | 6 | http://www.cs.uu.nl/CGAL |
| 4 | 4 | 18 | http://www.ics.uci.edu/~eppstein/junkyard |
| 5 | 6 | 14 | http://www.scs.carleton.ca/ ccsgs/resources/cg.html |
| 6 | 7 | 20 | http://www.geom.uiuc.edu/ |
| 7 | 5 | >20 | http://www.ics.uci.edu/~eppstein |
| 8 | 8 | $>20$ | http://www.mpi-sb.mpg.de/LEDA/leda.html |
| 9 | 10 | $>20$ | http://www.inria.fr/prisme/personnel/bronnimann/cgt |
| 10 | 9 | >20 | http://www.cs.sunysb.edu/~algorith/ |
| 11 | 13 | $>20$ | http://graphics.lcs.mit.edu/~seth/pubs/taskforce/techrep.html |
| 12 | 11 | $>20$ | http://www.cs.smith.edu/~orourke |
| 13 | 12 | $>20$ | http://www.cs.brown.edu/people/rt |
| 14 | 18 | $>20$ | http://www.geom.uiuc.edu/~nina/ |
| 15 | >20 | $>20$ | http://compgeom.cs.uiuc.edu/~jeffe/compgeom |
| 16 | 14 | $>20$ | http://www.cs.princeton.edu/~chazelle |
| 17 | >20 | >20 | http://www.dcc.unicamp.br/~guialbu/geompages.html |
| 18 | 16 | 8 | http://www.yahoo.com |
| 19 | $>20$ | >20 | http://www.ics.uci.edu/~eppstein/gina/authors.html |
| 20 | $>20$ | $>20$ | http://www.cs.brown.edu/people/rt/sdcr/report.html |

We assume that the solution of (16) denoted by $x^{*}=$ $\left(x_{1}^{*}, \ldots, x_{n}^{*}\right)$ is the MBCC authority ranking vector, and $y=$ $L L^{T} x^{*}$ is the MBCC hub ranking vector. Since the matrix $W$ in (15) is stochastic and irreducible, just like the Google matrix $G$ in PageRank, the solution of (16) exists, and it is unique and positive.

## 4. Numerical Experiments

First, we present an example to describe the assignment process in Definition 2.

Example 1. Suppose that there are six webpages $V=\left(j_{1}, j_{2}\right.$, $\left.j_{3}, a, b, c\right)$, and the directed graph is shown in Figure 2. The conclusion can be found from Table 1 and Figure 2. In Table 1, $Q(i, j)$ is the number of webpages that cocite webpages $i$ and
$j$; $W(i, j)$ is obtained by (11). In Figure 2, the left one is the original link structure of PageRank where the value of the page $j_{1}$ is divided equally to the pages that it points to, and the right one divides the value of $j_{1}$ based on cocitation.

Then, we compare the MBCC model with HITS and PageRank, experimenting with dataset from http://www.cs .toronto.edu/~tsap/experiments/datasets/. The dataset is about the topic computational geometry which contains a total of 1100 webpages. We set $\beta=0.9$. Meanwhile, we use $\tau=10^{-10}$ as the convergence tolerance and measure the convergence rates of the three algorithms using the L1 norm of the residual vector. Table 2 shows the list of the top 20 authorities with HITS, MBCC, and PageRank. Table 3 shows the list of the top 20 hubs with HITS and MBCC. It shows that MBCC authority ranking is closer to HITS

TAbLE 3: HITS hub ranking and MBCC hub ranking in top 20.

| HITS | MBCC | URL |
| :---: | :---: | :---: |
| 1 | 2 | http://mother.lub.lu.se/ae/bytitle/043501-043550.html |
| 2 | 1 | http://compgeom.cs.uiuc.edu/~jeffe/compgeom/preprehistory.html |
| 3 | 3 | http://www.dcc.unicamp.br/~guialbu/geompages.html |
| 4 | 4 | http://www.softlab.ece.ntua.gr/~cfrag |
| 5 | 5 | http://corelab.cs.ntua.gr/courses/compgeom |
| 6 | 6 | http://compgeom.cs.uiuc.edu/~jeffe/compgeom/direct.html |
| 7 | 7 | http://mother.lub.lu.se/ae/bydomain/010101-010150.html |
| 8 | 9 | http://www-sop.inria.fr/prisme/personnel/bronnimann/cgt/WWW.html |
| 9 | 10 | http://members.tripod.com/~GeomWiz/develop.html |
| 10 | 11 | http://geomwiz.tripod.com/develop.html |
| 11 | 8 | http://www.scs.carleton.ca/~csgs/resources/cg.html |
| 12 | 13 | http://www.ams.sunysb.edu/~jsbm/hotlist.html |
| 13 | 14 | http://www.graphics.lcs.mit.edu/~fredo/Book/geoAlgo.html |
| 14 | 16 | http://www.cs.uwaterloo.ca/~yganjali/r_links.html |
| 15 | 12 | http://cs.smith.edu/~streinu/bookmarks.html |
| 16 | 15 | http://www.cs.umn.edu/scg98 |
| 17 | >20 | http://forum.swarthmore.edu/library/topics/comp_geom |
| 18 | 19 | http://www.geom.umn.edu/~mucke/GeomDir/people.html |
| 19 | >20 | http://cis.poly.edu/~aronov |
| 20 | 20 | http://intra.cmkos.cz/~honza/osvr/odkazy.html |

Table 4: Comparison between MBCC and HITS ranking vectors, for example, top 10 represents a ranking vector agreeing with another ranking vector in top 10 .

|  | Top 10 | Top 20 | Top 30 | Top 40 | Top 50 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Authority (HITS MBCC) | $100 \%$ | $80 \%$ | $96.7 \%$ | $85 \%$ | $90 \%$ |
| Hub (HITS MBCC) | $90 \%$ | $90 \%$ | $90 \%$ | $92.5 \%$ | $96 \%$ |

authority ranking than PageRank ranking which is close to HITS authority ranking. The comparison between MBCC and HITS ranking vectors in Table 4 indicates that MBCC ranking agrees well with HITS ranking, especially in top 10.

## 5. Conclusion

In this work, we emphasize the role of cocitation in defining authorities. First, we observe that, in the HITS algorithm, if two distinct webpages $i, j$ are cocited by many other webpages $k$, then $i, j$ are likely to be related in some sense or have certain commonality. According to this close relationship, we come to the conclusion that the higher the number of webpages that cocite webpages $i$ and $j$, the stronger the relevancy between the two pages. The page $j$ with stronger relevancy should obtain more values from page $i$. Therefore, we develop a hyperlink weighting scheme for extracting information from the link structure. Then we combine hyperlink weight normalization and random surfing schemes as used in PageRank to justify the model.

The experimental results show that the MBCC authority (hub) ranking is close well to the HITS authority (hub) ranking in top 20, and in general a surfer seldomly browses
beyond these webpages in top 20 [11]. Moreover, MBCC keeps the superiority of PageRank: the authority vector of MBCC in (16) exists, and it is unique and positive, while the authority and hub vectors of HITS may not be unique. Therefore, we can use the authority (hub) ranking vector of MBCC as the authority (hub) ranking vector of HITS.

## Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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## Research Article

# Dynamic Model of a Wind Turbine for the Electric Energy Generation 

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A novel dynamic model is introduced for the modeling of the wind turbine behavior. The objective of the wind turbine is the electric energy generation. The analytic model has the characteristic that considers a rotatory tower. Experiments show the validity of the proposed method.

## 1. Introduction

Researchers are often trying to improve the total power of a wind turbine. The dynamic model of a wind turbine plays an important role in some applications as the control, classification, or prediction.

Some authors have proposed the equations to model the dynamic behavior of the wind turbine as shown in [1-5].

This paper presents a novel dynamic model of a wind turbine; the first dynamic model is for the wind turbine, the second is for the tower, and both are related. Because the rotatory tower can turn, it may help the wind turbine to increase the air intake. Some companies propose wind turbines that consider rotatory blades; however, in this study, a rotatory tower is proposed instead of the rotatory blades because the first dynamic model is easier to obtain than the second.

The proposed model is linear in the states as the works analyzed by [6-10] show. The technique provides an acceptable approximation of the wind turbine behavior.

The paper is structured as follows. In Section 2, the dynamic model of a windward wind turbine of three blades with a rotatory tower is introduced. In Section 3, the simulations of the dynamic model are compared with the real
data obtained by a real wind turbine prototype. Finally, in Section 4, the conclusion and future research are detailed.

## 2. Dynamic Model of the Wind Turbine with a Rotatory Tower

This section is divided into four parts: the first is the description of the mechanic model, the second is the description of the aerodynamic model, the third is the description of the electric model, and, finally, the fourth is the combination of the aforementioned models to obtain the final dynamic model.

The dynamic model of the wind turbine is, first, the equations that represent the change between the wind energy and mechanic energy and, second, the equations that represent the change between the mechanic energy and electric energy.
2.1. The Mechanic Model. A windward wind turbine of three blades with a rotatory tower is considered. First, the Euler Lagrangian method [11, 12] is used to obtain the model that represents the change from the wind energy to mechanic energy for the wind turbine and the change from the electric energy to mechanic energy for the tower. The masses are


Figure 1: Lateral view of the wind turbine.


Figure 2: Upper view of the wind turbine.
concentrated at the center of mass. Consider the lateral view of Figure 1 and upper view of Figure 2.

From Figures 1 and 2, it can be seen that

$$
\begin{gather*}
x_{2}=-l_{c 2} S_{2}\left(-C_{1}\right)=l_{c 2} S_{2} C_{1}, \\
y_{2}=-l_{c 2} S_{2}\left(-S_{1}\right)=l_{c 2} S_{2} S_{1}, \tag{1}
\end{gather*}
$$

where $S_{1}=\sin \left(\theta_{1}\right), S_{2}=\sin \left(\theta_{2}\right), C_{1}=\cos \left(\theta_{1}\right), C_{2}=\cos \left(\theta_{2}\right)$, $\theta_{1}$ is the angular position of the tower motor in rad, $\theta_{2}$ is the angular position of a wind turbine blade in rad, and $l_{c 2}$ is the length to the center of the wind turbine blade in $m$.

Consequently, the kinetic energy $K_{1}, K_{2}$ and potential energy $P_{1}, P_{2}$ are given as

$$
\begin{gather*}
K_{1}=0, \\
K_{2}=\frac{1}{2} m_{2} l_{c 2}^{2} S_{2}^{2} \dot{\theta}_{1}^{2}+\frac{1}{2} m_{2} l_{c 2}^{2} \dot{\theta}_{2}^{2},  \tag{2}\\
P_{1}=m_{1} g l_{c 1}, \\
P_{2}=l_{1}+m_{2} g l_{c 2} C_{2},
\end{gather*}
$$

where $m_{1}$ is the tower mass in $\mathrm{kg}, m_{2}$ is the blade mass in $\mathrm{kg}, g$ is the acceleration gravity in $\mathrm{m} / \mathrm{s}^{2}, l_{1}$ is the constant length of the tower in m , and $l_{c 1}$ is the length to the center of the tower in m . The torques $\tau_{T 1 a}$ and $\tau_{T 2 a}$ are given as

$$
\begin{align*}
& \tau_{T 1 a}=\tau_{1 a}-k_{b 1} \theta_{1}-b_{b 1} \dot{\theta}_{1},  \tag{3}\\
& \tau_{T 2 a}=\tau_{2 a}-k_{b 2} \theta_{2}-b_{b 2} \dot{\theta}_{2},
\end{align*}
$$

where $\tau_{2 a}$ is the torque of the generator moved by the blade in $\mathrm{kg} \mathrm{m}{ }^{2} \mathrm{rad} / \mathrm{s}^{2}, \tau_{1 a}$ is the torque of the motor used to move the tower in $\mathrm{kg} \mathrm{m}{ }^{2} \mathrm{rad} / \mathrm{s}^{2}, k_{b 1}$ and $k_{b 2}$ are the spring effects presented when the blade is near to a stop in $\mathrm{kg} \mathrm{m}^{2} / \mathrm{s}^{2}$, and $b_{b 1}$ and $b_{b 2}$ are the shock absorbers in $\mathrm{kg} \mathrm{m}^{2} \mathrm{rad} / \mathrm{s}$. Using the Euler Lagrange method in [11, 12] gives the following equations:

$$
\begin{gather*}
m_{2} l_{c 2}^{2} S_{2}^{2} \ddot{\theta}_{1}+2 m_{2} l_{c 2}^{2} S_{2} C_{2} \dot{\theta}_{1} \dot{\theta}_{2}+b_{b 1} \dot{\theta}_{1}+k_{b 1} \theta_{1}=\tau_{1 a}  \tag{4}\\
m_{2} l_{c 2}^{2} \ddot{\theta}_{2}+m_{2} g l_{c 2} S_{2}+b_{b 2} \dot{\theta}_{2}+k_{b 2} \theta_{2}=\tau_{2 a}
\end{gather*}
$$

The angular position of the blade 1 is related to the angular position of the blades 2 and 3 as follows:

$$
\begin{align*}
& \theta_{3}=\theta_{2}+\frac{2}{3} \pi \mathrm{rad} \\
& \theta_{4}=\theta_{2}+\frac{4}{3} \pi \mathrm{rad} \tag{5}
\end{align*}
$$

where $\theta_{3}$ and $\theta_{4}$ are the angular positions for the blades 2 and 3 , respectively. Then, using (4), it yields the equations for blades 2 and 3 as a function of $\theta_{2}$ as follows:

$$
\begin{gather*}
m_{2} l_{c 2}^{2} S_{2+(2 / 3) \pi}^{2} \ddot{\theta}_{1}+2 m_{2} l_{c 2}^{2} S_{2+(2 / 3) \pi} C_{2+(2 / 3) \pi} \dot{\theta}_{1} \dot{\theta}_{2} \\
+b_{b 1} \dot{\theta}_{1}+k_{b 1} \theta_{1}=\tau_{1 b}, \\
m_{2} l_{c 2}^{2} \ddot{\theta}_{2}+m_{2} g l_{c 2} S_{2+(2 / 3) \pi}+b_{b 2} \dot{\theta}_{2}+k_{b 2}\left(\theta_{2}+\frac{2}{3} \pi\right)=\tau_{2 b},  \tag{6}\\
m_{2} l_{c 2}^{2} S_{2+(4 / 3) \pi}^{2} \ddot{\theta}_{1}+2 m_{2} l_{c 2}^{2} S_{2+(4 / 3) \pi} C_{2+(4 / 3) \pi} \dot{\theta}_{1} \dot{\theta}_{2} \\
+b_{b 1} \dot{\theta}_{1}+k_{b 1} \theta_{1}=\tau_{1 c}, \\
m_{2} l_{c 2}^{2} \ddot{\theta}_{2}+m_{2} g l_{c 2} S_{2+(4 / 3) \pi}+b_{b 2} \dot{\theta}_{2}+k_{b 2}\left(\theta_{2}+\frac{4}{3} \pi\right)=\tau_{2 c}, \tag{7}
\end{gather*}
$$

where $\tau_{1 b}, \tau_{2 b}, \tau_{1 c}$, and $\tau_{2 c}$ are the torques applied to move the blades 2 and 3, respectively. Adding $S_{2}^{2}+S_{2+(2 / 3) \pi}^{2}+S_{2+(4 / 3) \pi}^{2}$,
$S_{2} C_{2}+S_{2+(2 / 3) \pi} C_{2+(2 / 3) \pi}+S_{2+(4 / 3) \pi} C_{2+(4 / 3) \pi}$, and $S_{2}+S_{2+(2 / 3) \pi}+$ $S_{2+(4 / 3) \pi}$, respectively, it gives

$$
\begin{gather*}
S_{2}^{2}+S_{2+(2 / 3) \pi}^{2}+S_{2+(4 / 3) \pi}^{2}=1.5  \tag{8}\\
S_{2} C_{2}+S_{2+(2 / 3) \pi} C_{2+(2 / 3) \pi}+S_{2+(4 / 3) \pi} C_{2+(4 / 3) \pi}=0  \tag{9}\\
S_{2}+S_{2+(2 / 3) \pi}+S_{2+(4 / 3) \pi}=0 \tag{10}
\end{gather*}
$$

Now, adding the three equations of (4), (6), and (7) and using (8), (9), and (10), it gives

$$
\begin{gather*}
4.5 m_{2} l_{c 2}^{2} \ddot{\theta}_{1}+3 b_{b 1} \dot{\theta}_{1}+3 k_{b 1} \theta_{1}=\tau_{1}  \tag{11}\\
3 m_{2} l_{c 2}^{2} \ddot{\theta}_{2}+3 b_{b 2} \dot{\theta}_{2}+3 k_{b 2} \theta_{2}+2 \pi k_{b 2}=\tau_{2}
\end{gather*}
$$

where $\tau_{1}=\tau_{1 a}+\tau_{1 b}+\tau_{1 c}$ and $\tau_{2}=\tau_{2 a}+\tau_{2 b}+\tau_{2 c}$. From Figures 1 and $2, \tau_{2}$ of (11) is defined as follows:

$$
\begin{gather*}
\tau_{2}=C_{1} F_{2}  \tag{12}\\
F_{2}=F_{2 a}+F_{2 b}+F_{2 c},
\end{gather*}
$$

where $C_{1}=\cos \left(\theta_{1}\right) F_{2 a}$ and $F_{2 b}$ and $F_{2 c}$ are the force of the air received by the three blades. Equation (12) describes the assumption that the air goes in one direction; if $\theta_{1}=0$, then the maximum air intake moves the blades of the wind turbine, but if the tower turns to the left or to the right and $\theta_{1}$ changes, then the wind turbine turns and the air intake decreases. From Figures 1 and 2 and [12], $\tau_{1}$ of (11) is defined as follows:

$$
\begin{equation*}
\tau_{1}=k_{m} i_{1} \tag{13}
\end{equation*}
$$

where $k_{m}$ is a motor magnetic flux constant of the tower in Wb and $i_{1}$ is the motor armature current of the tower in A . Equations (11), (12), and (13) are the main equations of the mechanic model that represents the wind turbine and tower.
2.2. Aerodynamic Model. The aerodynamic model is the dynamic model of the torque applied to the blades. The mechanic power captured by the wind turbine $P_{a}$ is given by [2-5]

$$
\begin{equation*}
P_{a}=F_{2 a} \dot{\theta}_{2}=\frac{1}{2} \rho A C_{p}(\lambda, \beta) V_{\omega}^{3} \tag{14}
\end{equation*}
$$

where $\rho$ is the air density in $\mathrm{Kg} / \mathrm{m}^{3}, A=\pi R^{2}$ is the area swept by the rotor blades in $\mathrm{m}^{2}$ with radius $R$ in $\mathrm{m}, V_{\omega}$ is the wind speed in $\mathrm{m} / \mathrm{s}$, and $C_{p}(\lambda, \beta)$ is the performance coefficient of the wind turbine, whose value is a function of the tip speed ratio $\lambda$, defined as [2-5]

$$
\begin{equation*}
\lambda=\frac{\dot{\theta}_{2} R}{V_{\omega}} \tag{15}
\end{equation*}
$$

For the purpose of simulation, the following model of $C_{p}(\lambda, \beta)$ is presented [2-5]:

$$
\begin{equation*}
C_{p}(\lambda, \beta)=c_{1}\left(\frac{c_{2}}{\lambda_{i}}-c_{3} \beta-c_{4}\right) e^{-c_{5} / \lambda_{i}}+c_{6} \lambda \tag{16}
\end{equation*}
$$



Figure 3: The wind turbine generator.


Figure 4: The tower motor.
where

$$
\begin{equation*}
\frac{1}{\lambda_{i}}=\frac{1}{\lambda+0.08 \beta}-\frac{0.035}{\beta^{3}+1} \tag{17}
\end{equation*}
$$

and the coefficients are $c_{1}=0.5176, c_{2}=116, c_{3}=0.4, c_{4}=5$, $c_{5}=21, c_{6}=0.0068$, and $\beta$ is the blade pitch angle in rad. Using (12), (14), (15), (16), and (17) gives the dynamic model of the torque applied to the wind turbine blades as follows:

$$
\begin{align*}
& \tau_{2}=C_{1}\left(F_{2 a}+F_{2 b}+F_{2 c}\right), \\
& F_{2 a}=\frac{1}{2 \dot{\theta}_{2}} \rho A C_{p}(\lambda, \beta) V_{\omega}^{3}, \\
& F_{2 b}=\frac{1}{2 \dot{\theta}_{2}} \rho A C_{p}(\lambda, \beta) V_{\omega}^{3},  \tag{18}\\
& F_{2 c}=\frac{1}{2 \dot{\theta}_{2}} \rho A C_{p}(\lambda, \beta) V_{\omega}^{3},
\end{align*}
$$

where $C_{1}$ is defined in (12), $\lambda$ is defined in (15), $C_{p}(\lambda, \beta)$ is defined in (16), and $\lambda_{i}$ is defined in (17).
2.3. The Electric Model. Now, analyze the change from the mechanic to electric energy for the wind turbine generator and the electric energy to mechanic energy for the tower motor. Figure 3 shows the wind turbine generator and Figure 4 shows the tower motor.

In [12, 13], they presented the model of a motor, and similarly, from Figure 3, a model for the generator is obtained
by using the Kirchhoff voltage law. Therefore, the dynamic models of the motor and generator are as follows:

$$
\begin{align*}
& V_{1}=R_{1} i_{1}+L_{1} \dot{i}_{1}+k_{1} \dot{\theta}_{1} \\
& k_{2} \dot{\theta}_{2}=R_{2} i_{2}+L_{2} \dot{i}_{2}+V_{2} \tag{19}
\end{align*}
$$

where $k_{1}$ is the motor back emf constant in V s/rad, $k_{2}$ is the generator back emf constant in $\mathrm{V} \mathrm{s} / \mathrm{rad}, R_{1}$ is the motor armature resistance in $\Omega, R_{2}$ is the generator armature resistance in $\Omega, L_{1}$ is the motor armature inductance in H , $L_{2}$ is the generator armature inductance in $\mathrm{H}, V_{1}$ is the motor armature voltage in $\mathrm{V}, V_{2}$ is the generator armature voltage in $\mathrm{V}, i_{1}$ is the motor armature current in A , and $i_{2}$ is the generator armature current in A . For the generator of this paper, $V_{2}=R_{e} i_{2}$. Thus, (19) becomes

$$
\begin{gather*}
V_{1}=R_{1} i_{1}+L_{1} \dot{i}_{1}+k_{1} \dot{\theta}_{1} \\
k_{2} \dot{\theta}_{2}=\left(R_{2}+R_{e}\right) i_{2}+L_{2} \dot{i}_{2}  \tag{20}\\
V_{2}=R_{e} i_{2} .
\end{gather*}
$$

2.4. The Final Dynamic Model. Thus, (11), (13), and (18) that represent the change between the wind energy and mechanic energy and (20) that represents the change between the mechanic energy and electric energy are considered as the wind turbine dynamic model with a rotatory tower.

Define the state variables as $x_{1}=i_{2}, x_{2}=\theta_{2}, x_{3}=\dot{\theta}_{2}$, $x_{4}=i_{1}, x_{5}=\theta_{1}, x_{6}=\dot{\theta}_{1}$, the inputs as $u_{1}=F_{2}$ and $u_{2}=V_{1}$, and the output as $y=V_{2}$. Consequently, the dynamic model of (11), (13), (15), (18), and (20) becomes

$$
\begin{gather*}
\dot{x}_{1}=-\frac{\left(R_{2}+R_{e}\right)}{L_{2}} x_{1}+\frac{k_{2}}{L_{2}} x_{3}, \\
\dot{x}_{2}=x_{3} \\
\dot{x}_{3}=-\frac{k_{b 2}}{m_{2} l_{c 2}^{2}} x_{2}-\frac{b_{b 2}}{m_{2} l_{c 2}^{2}} x_{3}-\frac{2 \pi k_{b 2}}{3 m_{2} l_{c 2}^{2}}+\frac{\cos \left(x_{5}\right)}{3 m_{2} l_{c 2}^{2}} u_{1}, \\
\dot{x}_{4}=-\frac{R_{1}}{L_{1}} x_{4}-\frac{k_{1}}{L_{1}} x_{6}+\frac{1}{L_{1}} u_{2} \\
\dot{x}_{5}=x_{6} \\
\dot{x}_{6}=-\frac{3 k_{b 1}}{4.5 m_{2} l_{c 2}^{2}} x_{5}-\frac{3 b_{b 1}}{4.5 m_{1} l_{c 2}^{2}} x_{6}+\frac{k_{m}}{4.5 m_{2} l_{c 2}^{2}} x_{4}  \tag{21}\\
y=R_{e} x_{1}, \\
u_{1}=F_{2 a}+F_{2 b}+F_{2 c} \\
F_{2 a}=\frac{1}{2 x_{3}} \rho A C_{p}(\lambda, \beta) V_{\omega}^{3} \\
F_{2 b}=\frac{1}{2 x_{3}} \rho A C_{p}(\lambda, \beta) V_{\omega}^{3} \\
F_{2 c}=\frac{1}{2 x_{3}} \rho A C_{p}(\lambda, \beta) V_{\omega}^{3} \\
\lambda=\frac{x_{3} R}{V_{\omega}}
\end{gather*}
$$

where $C_{p}(\lambda, \beta)$ is defined in (16) and $\lambda_{i}$ is defined in (17).


Figure 5: Prototype of a wind turbine with rotatory tower.

TAbLE 1: Parameters of the prototype.

| Parameter | Value |
| :--- | :---: |
| $l_{c 2}$ | 0.5 m |
| $m_{2}$ | 0.5 kg |
| $k_{b 2}$ | $1 \times 10^{-6} \mathrm{kgm}^{2} / \mathrm{s}^{2}$ |
| $b_{b 2}$ | $1 \times 10^{-1} \mathrm{kgm}^{2} \mathrm{rad} / \mathrm{s}$ |
| $k_{2}$ | $0.45 \mathrm{Vs} / \mathrm{rad}$ |
| $R_{2}$ | $6.96 \Omega$ |
| $L_{2}$ | $6.031 \times 10^{-1} \mathrm{H}$ |
| $R$ | $l_{c 2} \mathrm{~m}$ |
| $\rho$ | $1.225 \mathrm{~kg} / \mathrm{m}^{3}$ |
| $g$ | $9.81 \mathrm{~m} / \mathrm{s}^{2}$ |
| $R_{e}$ | $30 \Omega$ |
| $k_{m}$ | 0.09 Wb |
| $k_{b 1}$ | $1 \times 10^{-6} \mathrm{kgm}^{2} / \mathrm{s}^{2}$ |
| $b_{b 1}$ | $1 \times 10^{-1} \mathrm{kgm}{ }^{2} \mathrm{rad} / \mathrm{s}$ |
| $k_{1}$ | $0.0045 \mathrm{Vs} / \mathrm{rad}^{2}$ |
| $R_{1}$ | $18 \Omega$ |
| $L_{1}$ | $6.031 \times 10^{-1} \mathrm{H}$ |
| $V_{\omega}$ | $5 \mathrm{~m} / \mathrm{s}$ |
| $\beta$ | 0.5 rad |

Remark 1. The model of this research considers a direct current motor and a direct current generator. This wind turbine is proposed to be used on the house roof for the daily usage to feed a light as a renewable source. If more than one wind turbine is used, more electricity is generated. The proposed model could be extended to other kinds of machines by changing (20).

Remark 2. Note that the dynamic model states of the wind turbine with a rotatory tower (21) is a linear system [6-10].

## 3. Experiments to Validate the Dynamic Model

Figure 5 shows the prototype of a wind turbine with a rotatory tower which is considered for the simulations of the dynamic model. This prototype has three blades with a rotatory tower which does not use a gear box. Table 1 shows the parameters of the prototype. The parameters $m_{2}$ and $l_{c 2}$ are obtained from


Figure 6: Inputs and output of the wind turbine of Example 1.
the wind turbine blades. The parameters $R_{1}, L_{1}$, and $k_{1}$ are obtained from the tower motor. The parameters $k_{2}, R_{2}, R_{e}$, and $L_{2}$ are obtained from the wind turbine generator. The parameters $R, \rho, V_{\omega}$, and $\beta$ are obtained from [2-5].

The dynamic model of the wind turbine with a rotatory tower is given by (21) with the parameters of Table $1.1 \times 10^{-5}$ are considered as the initial conditions for the plant states $x_{1}=i_{2}, x_{2}=\theta_{2}, x_{3}=\dot{\theta}_{2}, x_{4}=i_{1}, x_{5}=\theta_{1}$, and $x_{6}=\dot{\theta}_{1}$. The root mean square error (RMSE) is used, and it is given as [13-15]

$$
\begin{equation*}
\operatorname{RMSE}=\left(\frac{1}{T} \int_{0}^{T} e^{2} d \tau\right)^{1 / 2} \tag{22}
\end{equation*}
$$

where $e^{2}=e_{u j}^{2}=\left(u_{j r}-u_{j}\right)^{2}, e^{2}=e_{x i}^{2}=\left(x_{i r}-x_{i}\right)^{2}$, or $e^{2}=$ $e_{y}^{2}=\left(y_{r}-y\right)^{2}$ and $u_{j r}, x_{i r}$, and $y_{r}$ are the real data of $u_{j}, x_{i}$, and $y$, respectively, $i=1,2, \ldots, 6, j=1,2$.
3.1. Example 1: The First Behavior. Figures 6 and 7 show the wind turbine inputs, outputs, and states. Table 2 shows the RMSE for the errors.

From Figures 6 and 7, it can be seen that the dynamic model has good behavior described as follows: (1) from 0 s to 2 s , both inputs are fed; consequently, the tower moves far of the maximum air intake, the generator current is decreased, and the wind turbine blades stop moving; (2) from 2 s to 4 s , both inputs are not fed; consequently, current is not generated and both the tower and wind turbine blades do not move; (3) from 4 s to 6 s , both inputs are fed, but the air intake is positive and tower voltage is negative; consequently, the tower returns to the maximum air intake, the generator current is increased, and the wind turbine blades move; (4) from 6 s to 8 s , both

Table 2: The RMSE for Example 1.

|  | RMSE |
| :--- | :---: |
| $e_{u 1}^{2}$ | 0.0204 |
| $e_{u 2}^{2}$ | 1.1835 |
| $e_{y}^{2}$ | 0.0171 |
| $e_{x 1}^{2}$ | $5.7158 \times 10^{-4}$ |
| $e_{x 2}^{2}$ | 0.9843 |
| $e_{x 3}^{2}$ | 0.0470 |
| $e_{x 4}^{2}$ | 0.0658 |
| $e_{x 5}^{2}$ | 0.0872 |
| $e_{x 6}^{2}$ | 0.0163 |

inputs are not fed; consequently, current is not generated and the tower and wind turbine blades do not move. The dynamic model is a good approximation of the wind turbine behavior because the signals of the first are near to the signals of the second. From Table 2, it is shown that the dynamic model is a good approximation of the real process because the RMSE is near to zero.
3.2. Example 2: The Second Behavior. Figures 8 and 9 show the wind turbine inputs, outputs, and states. Table 3 shows the RMSE for the errors.

From Figures 8 and 9, it can be seen that the dynamic model has good behavior described as follows: (1) from 0 s to 2 s , the input air is fed and the tower input is not fed; consequently, the tower remains in the maximum air intake, the generator current is maximum, and the wind turbine blades have motion; (2) from 2 s to 4 s , the air is not fed and


Figure 7: States of the wind turbine of Example 1.


Figure 8: Inputs and output of the wind turbine of Example 2.


Figure 9: States of the wind turbine of Example 2.

Table 3: The RMSE for Example 2.

|  | RMSE |
| :--- | :---: |
| $e_{u 1}^{2}$ | 0.0288 |
| $e_{u 2}^{2}$ | 0.9423 |
| $e_{y}^{2}$ | 0.0280 |
| $e_{x 1}^{2}$ | $9.3212 \times 10^{-4}$ |
| $e_{x 2}^{2}$ | 1.7209 |
| $e_{x 3}^{2}$ | 0.0766 |
| $e_{x 4}^{2}$ | 0.0523 |
| $e_{x 5}^{2}$ | 0.1324 |
| $e_{x 6}^{2}$ | 0.0162 |

the tower input is fed; consequently, current is not generated, the tower moves far of the maximum air intake, and the wind turbine blades do not have motion; (3) from 4 s to 6 s , the air is fed and the tower input is not fed; consequently, the tower does not move, the generator current is minimum, and the wind turbine blades almost do not move; (4) from 6 s to 8 s , the air is not fed and the tower input is fed with a negative voltage; consequently, current is not generated, the tower returns to the maximum air intake, and the wind turbine blades do not have motion. The dynamic model is a good approximation of the wind turbine behavior because
the signals of the first are near to the signals of the second. From Table 2, it is shown that the dynamic model is a good approximation of the real process because the RMSE is near to zero.

## 4. Conclusion

In this paper, a dynamic model of a wind turbine with a rotatory tower was introduced; the simulations using the parameters of a prototype showed that the proposed dynamic model has an acceptable approximation of the wind turbine behavior. The proposed dynamic model could be used on control, prediction, or classification. As a future research, the modeling will be improved using other interesting methods as the least squares in [16-26], neural networks in [14, 27-32], or fuzzy systems in [15, 33-35].

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# The Explicit Identities for Spectral Norms of Circulant-Type Matrices Involving Binomial Coefficients and Harmonic Numbers 

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#### Abstract

The explicit formulae of spectral norms for circulant-type matrices are investigated; the matrices are circulant matrix, skew-circulant matrix, and $g$-circulant matrix, respectively. The entries are products of binomial coefficients with harmonic numbers. Explicit identities for these spectral norms are obtained. Employing these approaches, some numerical tests are listed to verify the results.


## 1. Introduction

The classical hypergeometric summation theorems are exploited to derive several striking identities on harmonic numbers [1]. In numerical analysis, circulant matrices (named "premultipliers" in numerical methods) are important because they are diagonalized by a discrete Fourier transform, and hence linear equations that contain them may be quickly solved using a fast Fourier transform. Furthermore, circulant, skew-circulant, and $g$-circulant matrices play important roles in various applications, such as image processing, coding, and engineering model. For more details, please refer to [2-13] and the references therein. The skew-circulant matrices were collected to construct preconditioners for LMF-based ODE codes; Hermitian and skew-Hermitian Toeplitz systems were considered in [1417]; Lyness employed a skew-circulant matrix to construct $s$ dimensional lattice rules in [18]. Recently, there are lots of research on the spectral distribution and norms of circulanttype matrices. In [19], the authors pointed out the processes based on the eigenvalue of circulant-type matrices and the convergence to a Poisson random measure in vague topology. There were discussions about the convergence in probability and distribution of the spectral norm of circulanttype matrices in [20]. The authors in [21] listed the limiting spectral distribution for a class of circulant-type matrices with heavy tailed input sequence. Ngondiep et al. showed that
the singular values of $g$-circulants in [22]. Solak established the lower and upper bounds for the spectral norms of circulant matrices with classical Fibonacci and Lucas numbers entries in [23]. İpek investigated an improved estimation for spectral norms in [24].

In this paper, we derive some explicit identities of spectral norms for some circulant-type matrices with product of binomial coefficients with harmonic numbers.

The outline of the paper is as follows. In Section 2, the definitions and preliminary results are listed. In Section 3, the spectral norms of some circulant matrices are studied. In Section 4, the formulae of spectral norms for skew-circulant matrices are established. Section 5 is devoted to investigate the explicit formulae for $g$-circulant matrices. The numerical tests are given in Section 6.

## 2. Preliminaries

The binomial coefficients are defined by $\binom{n}{k}$ for all natural numbers $k$ at once by

$$
\begin{equation*}
(1+X)^{n}=\sum_{k \geq 0}\binom{n}{k} X^{k} . \tag{1}
\end{equation*}
$$

Note that $\binom{n}{k}$ is the $k$ th binomial coefficient of $n$. It is clear that $\binom{n}{0}=1,\binom{n}{n}=1$, and $\binom{n}{k}=0$, for $k>n$.

The generalized harmonic numbers are defined to be partial sums of the harmonic series [1]:

$$
\begin{equation*}
H_{0}(x)=0, \quad H_{i}(x)=\sum_{k=0}^{i} \frac{1}{x+k} \quad(i=1,2, \ldots) \tag{2}
\end{equation*}
$$

For $x=0$ in particular, they reduce to classical harmonic numbers:

$$
\begin{equation*}
H_{0}=0, \quad H_{i}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{i} \quad(i=1,2, \ldots) . \tag{3}
\end{equation*}
$$

We recall the following harmonic number identities [1]:

$$
\begin{gather*}
\sum_{i=0}^{n}\binom{n}{i}^{2}\binom{2 n+i}{i}\left(H_{2 n+i}-H_{i}\right) \\
=2\binom{2 n}{n}^{2}\left(H_{2 n}-H_{n}\right) \\
\sum_{i=0}^{n}\binom{n}{i}\binom{2 n}{i}\binom{3 n+i}{i}\left(H_{3 n+i}-H_{i}\right)  \tag{4}\\
=\binom{3 n}{n}^{2}\left(2 H_{3 n}-H_{2 n}-H_{n}\right)
\end{gather*}
$$

Definition 1 (see $[6,8]$ ). A circulant matrix is an $n \times n$ complex matrix with the following form:

$$
A_{c}=\left(\begin{array}{cccc}
a_{0} & a_{1} & \cdots & a_{n-1}  \tag{5}\\
a_{n-1} & a_{0} & \cdots & a_{n-2} \\
a_{n-2} & a_{n-1} & \cdots & a_{n-3} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1} & a_{2} & \cdots & a_{0}
\end{array}\right)_{n \times n} .
$$

The first row of $A_{c}$ is $\left(a_{0}, a_{1}, \ldots, a_{n-1}\right)$; its $(j+1)$ th row is obtained by giving its $j$ th row a right circular shift by one position.

Equivalently, a circulant matrix can be described with polynomial as

$$
\begin{equation*}
A_{c}=f\left(\eta_{c}\right)=\sum_{i=0}^{n-1} a_{i} \eta_{c}^{i} \tag{6}
\end{equation*}
$$

where

$$
\eta_{c}=\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0  \tag{7}\\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{array}\right)_{n \times n}
$$

Obviously, $\eta_{c}^{n}=I_{n}$.
Now, we discuss the eigenvalues of $A_{c}$. We declare that the eigenvalues of $\eta_{c}$ are the corresponding eigenvalues of $A_{c}$ with the function $f$ in (6), which is

$$
\begin{equation*}
\lambda\left(A_{c}\right)=f\left(\lambda\left(\eta_{c}\right)\right)=\sum_{i=0}^{n-1} a_{i} \lambda\left(\eta_{c}\right)^{i} \tag{8}
\end{equation*}
$$

Whereas $\lambda_{j}\left(\eta_{c}\right)=\omega^{j},(j=0,1, \ldots, n-1)$, then $\lambda_{j}\left(A_{c}\right)$ can be calculated by

$$
\begin{equation*}
\lambda_{j}\left(A_{c}\right)=\sum_{i=0}^{n-1} a_{i}\left(\omega^{j}\right)^{i} \tag{9}
\end{equation*}
$$

where $\omega=\cos (2 \pi / n)+i \sin (2 \pi / n)$.
Similarly, we recall a skew-circulant matrix.
Definition 2 (see $[6,8]$ ). A skew-circulant matrix is an $n \times n$ complex matrix with the following form:

$$
A_{\mathrm{sc}}=\left(\begin{array}{cccc}
a_{0} & a_{1} & \cdots & a_{n-1}  \tag{10}\\
-a_{n-1} & a_{0} & \cdots & a_{n-2} \\
-a_{n-2} & -a_{n-1} & \cdots & a_{n-3} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{1} & -a_{2} & \cdots & a_{0}
\end{array}\right)_{n \times n} .
$$

Moreover, a skew-circulant matrix can be described with polynomial as

$$
\begin{equation*}
A_{\mathrm{sc}}=f\left(\eta_{\mathrm{sc}}\right)=\sum_{i=0}^{n-1} a_{i} \eta_{\mathrm{sc}}^{i}, \tag{11}
\end{equation*}
$$

where

$$
\eta_{\mathrm{sc}}=\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0  \tag{12}\\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \\
0 & 0 & 0 & \cdots & 1 \\
-1 & 0 & 0 & \cdots & 0
\end{array}\right)_{n \times n}
$$

Obviously, $\eta_{\mathrm{sc}}^{n}=-I_{n}$.
Thus we have to calculate the eigenvalues of $A_{\mathrm{sc}}$. For the same reason, we obtain that

$$
\begin{equation*}
\lambda\left(A_{\mathrm{sc}}\right)=f\left(\lambda\left(\eta_{\mathrm{sc}}\right)\right)=\sum_{i=0}^{n-1} a_{i} \lambda^{i}\left(\eta_{\mathrm{sc}}\right) \tag{13}
\end{equation*}
$$

Whereas $\lambda_{j}\left(\eta_{\mathrm{sc}}\right)=\omega^{j} \alpha,(j=0,1, \ldots, n-1) \lambda_{j}\left(A_{\mathrm{sc}}\right)$ can be computed by

$$
\begin{equation*}
\lambda_{j}\left(A_{\mathrm{sc}}\right)=\sum_{i=0}^{n-1} a_{i}\left(\omega^{j} \alpha\right)^{i} \tag{14}
\end{equation*}
$$

where $\omega=\cos (2 \pi / n)+i \sin (2 \pi / n), \alpha=\cos (\pi / n)+i \sin (\pi / n)$.
Definition 3 (see [21, 25]). A $g$-circulant matrix is an $n \times n$ complex matrix with the following form:

$$
A_{g}=\left(\begin{array}{cccc}
a_{0} & a_{1} & \ldots & a_{n-1}  \tag{15}\\
a_{n-g} & a_{n-g+1} & \ldots & a_{n-g-1} \\
a_{n-2 g} & a_{n-2 g+1} & \ldots & a_{n-2 g-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{g} & a_{g+1} & \cdots & a_{g-1}
\end{array}\right)_{n \times n}
$$

where $g$ is a nonnegative integer and each of the subscripts is understood to be reduced modulo $n$.

The first row of $A_{g}$ is $\left(a_{0}, a_{1}, \ldots, a_{n-1}\right)$; its $(j+1)$ th row is obtained by giving its $j$ th row a right circular shift by $g$ positions (equivalently, $g \bmod n$ positions). Note that $g=1$ or $g=n+1$ yields the standard circulant matrix. If $g=n-1$, then we obtain the so-called reverse circulant matrix [21].

Definition 4 (see [26]). The spectral norm $\|\cdot\|_{2}$ of a matrix $A$ with complex entries is the square root of the largest eigenvalue of the positive semidefinite matrix $A^{*} A$ :

$$
\begin{equation*}
\|A\|_{2}=\sqrt{\lambda_{\max }\left(A^{*} A\right)} \tag{16}
\end{equation*}
$$

where $A^{*}$ denotes the conjugate transpose of $A$. Therefore if $A$ is an $n \times n$ real symmetric matrix or $A$ is a normal matrix, then

$$
\begin{equation*}
\|A\|_{2}=\max _{1 \leq i \leq n}\left|\lambda_{i}\right| \tag{17}
\end{equation*}
$$

where $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}$ are the eigenvalues of $A$.

## 3. Spectral Norms of Some Circulant Matrices

Now, we will analyse spectral norms of some given circulant matrices, whose entries are binomial coefficients combined with harmonic numbers.

Our main results for those matrices are stated as follows.
Theorem 5. Let $(n+1) \times(n+1)$-circulant matrix $B_{1}$ is as in (5), and the first row of $B_{1}$ is

$$
\begin{align*}
& \left(\binom{n}{0}^{2}\binom{2 n}{0}\left(H_{2 n}-H_{0}\right)\right. \\
& \binom{n}{1}^{2}\binom{2 n+1}{1}\left(H_{2 n+1}-H_{1}\right), \ldots  \tag{18}\\
& \left.\binom{n}{n}^{2}\binom{2 n+n}{n}\left(H_{2 n+n}-H_{n}\right)\right)
\end{align*}
$$

where $a_{i}=\binom{n}{i}^{2}\binom{2 n+i}{i}\left(H_{2 n+i}-H_{i}\right)$. Then one has

$$
\begin{equation*}
\left\|B_{1}\right\|_{2}=2\binom{2 n}{n}^{2}\left(H_{2 n}-H_{n}\right) \tag{19}
\end{equation*}
$$

Proof. Since circulant matrix $B_{1}$ is normal, employing Definition 4, we claim that the spectral norm of $B_{1}$ is equal to its spectral radius. Furthermore, applying the irreducible and entrywise nonnegative properties, we claim that $\left\|B_{1}\right\|_{2}$ (i.e., its spectral norm) is equal to its Perron value. We select an $(n+1)$-dimensional column vector $v=(1,1, \ldots, 1)^{T}$; then

$$
\begin{equation*}
B_{1} v=\left(\sum_{i=0}^{n}\binom{n}{i}^{2}\binom{2 n+i}{i}\left(H_{2 n+i}-H_{i}\right)\right) v . \tag{20}
\end{equation*}
$$

Obviously, $\sum_{i=0}^{n}\binom{n}{i}^{2}\binom{2 n+i}{i}\left(H_{2 n+i}-H_{i}\right)$ is an eigenvalue of $B_{1}$ associated with $v$, which is necessarily the Perron value of $B_{1}$. Employing (4), we obtain

$$
\begin{equation*}
\left\|B_{1}\right\|_{2}=2\binom{2 n}{n}^{2}\left(H_{2 n}-H_{n}\right) \tag{21}
\end{equation*}
$$

This completes the proof.

Hence, employing the same approaches, we get the following corollary.

Corollary 6. Let $(n+1) \times(n+1)$-circulant matrix $B_{2}$ be as in (5), and the first row of $B_{2}$ is

$$
\begin{align*}
& \left(\binom{n}{0}\binom{2 n}{0}\binom{3 n}{0}\left(H_{3 n}-H_{0}\right),\right. \\
& \binom{n}{1}\binom{2 n}{1}\binom{3 n+1}{1}\left(H_{3 n+1}-H_{1}\right), \ldots  \tag{22}\\
& \left.\binom{n}{n}\binom{2 n}{n}\binom{4 n}{n}\left(H_{4 n}-H_{n}\right)\right),
\end{align*}
$$

where $a_{i}=\binom{n}{i}\binom{2 n}{i}\binom{3 n+i}{i}\left(H_{3 n+i}-H_{i}\right)$. Then

$$
\begin{equation*}
\left\|B_{2}\right\|_{2}=\binom{3 n}{n}^{2}\left(2 H_{3 n}-H_{2 n}-H_{n}\right) \tag{23}
\end{equation*}
$$

Now, we investigate some even-order alternative as follows, where $m$ is odd (i.e., $m+1$ is even).

Theorem 7. Let $(m+1) \times(m+1)$-circulant matrix $B_{3}$ be as in (5), and the first row of $B_{3}$ is

$$
\begin{align*}
& \left(\binom{m}{0}^{2}\binom{2 m}{0}\left(H_{2 m}-H_{0}\right)\right. \\
& -\binom{m}{1}^{2}\binom{2 m+1}{1}\left(H_{2 m+1}-H_{1}\right), \ldots  \tag{24}\\
& \left.\quad-\binom{m}{m}^{2}\binom{2 m+m}{m}\left(H_{2 m+m}-H_{m}\right)\right)
\end{align*}
$$

where $a_{i}=(-1)^{i}\binom{m}{i}^{2}\binom{2 m+i}{i}\left(H_{2 m+i}-H_{i}\right)$. Then there holds the following identity:

$$
\begin{equation*}
\left\|B_{3}\right\|_{2}=2\binom{2 m}{m}^{2}\left(H_{2 m}-H_{m}\right) \tag{25}
\end{equation*}
$$

Proof. Noticing (9) and (17), it is clear that the spectral norm of $B_{3}$ can be calculated by

$$
\begin{align*}
\left\|B_{3}\right\|_{2} & =\max _{0 \leq t \leq m}\left|\lambda_{t}\left(B_{3}\right)\right|=\max _{0 \leq t \leq m}\left|\sum_{i=0}^{m} a_{i}\left(\omega^{t}\right)^{i}\right| \\
& \leq \max _{0 \leq t \leq m}\left\{\sum_{i=0}^{m}\left|a_{i}\right| \cdot\left|\left(\omega^{t}\right)^{i}\right|\right\}=\sum_{i=0}^{m}\left|a_{i}\right|, \tag{26}
\end{align*}
$$

where $a_{i}=(-1)^{i}\binom{m}{i}^{2}\binom{2 m+i}{i}\left(H_{2 m+i}-H_{i}\right)$, and we employed that all circulant matrices are normal.

Note that, if $m$ is odd, then $m+1$ is even, and $\lambda_{t_{0}}\left(\eta_{c}\right)=$ $\omega^{t_{0}}=-1$ is an eigenvalue of $\eta_{c}$, so

$$
\begin{equation*}
\left\|B_{3}\right\|_{2}=\sum_{i=0}^{m}\left|a_{i}\right| \tag{27}
\end{equation*}
$$

Combining (4) and (27) yields

$$
\begin{equation*}
\left\|B_{3}\right\|_{2}=2\binom{2 m}{m}^{2}\left(H_{2 m}-H_{m}\right) \tag{28}
\end{equation*}
$$

This completes the proof.

Employing the same approaches, we get the following corollary.

Corollary 8. Let $(m+1) \times(m+1)$-circulant matrix $B_{4}$ be as in (5), and the first row of $B_{4}$ is

$$
\begin{align*}
& \left(\binom{m}{0}\binom{2 m}{0}\binom{3 m}{0}\left(H_{3 m}-H_{0}\right)\right. \\
& -\binom{m}{1}\binom{2 m}{1}\binom{3 m+1}{1}\left(H_{3 m+1}-H_{1}\right), \ldots  \tag{29}\\
& \left.\binom{m}{m}\binom{2 m}{m}\binom{4 m}{m}\left(H_{4 m}-H_{m}\right)\right)
\end{align*}
$$

where $a_{i}=(-1)^{i}\binom{m}{i}\binom{2 m}{i}\binom{3 m+i}{i}\left(H_{3 m+i}-H_{i}\right)$. Then one has the following identity:

$$
\begin{equation*}
\left\|B_{4}\right\|_{2}=\binom{3 m}{m}^{2}\left(2 H_{3 m}-H_{2 m}-H_{m}\right) . \tag{30}
\end{equation*}
$$

Similarly, we set $\widetilde{B}_{3}=-B_{3}, \widetilde{B}_{4}=-B_{4}$.
Corollary 9. Let $\widetilde{B}_{3}, \widetilde{B}_{4}$ be as above, respectively, and $m$ is odd. Then

$$
\begin{gather*}
\left\|\widetilde{B}_{3}\right\|_{2}=2\binom{2 m}{m}^{2}\left(H_{2 m}-H_{m}\right),  \tag{31}\\
\left\|\widetilde{B}_{4}\right\|_{2}=\binom{3 m}{m}^{2}\left(2 H_{3 m}-H_{2 m}-H_{m}\right) .
\end{gather*}
$$

## 4. Spectral Norms of Skew-Circulant Matrices

An odd-order alternative skew-circulant matrix is defined as follows, where $s$ is even.

Theorem 10. Let $(s+1) \times(s+1)$-circulant matrix $B_{5}$ be as in (10), and the first row of $B_{5}$ is

$$
\begin{align*}
& \left(\binom{s}{0}^{2}\binom{2 s}{0}\left(H_{2 s}-H_{0}\right)\right. \\
& \quad-\binom{s}{1}^{2}\binom{2 s+1}{1}\left(H_{2 s+1}-H_{1}\right), \ldots  \tag{32}\\
& \left.\quad\binom{s}{s}^{2}\binom{2 s+s}{s}\left(H_{2 s+s}-H_{s}\right)\right)
\end{align*}
$$

where $a_{i}=(-1)^{i}\binom{s}{i}^{2}\binom{2 s+i}{i}\left(H_{2 s+i}-H_{i}\right)$. Then one obtains

$$
\begin{equation*}
\left\|B_{5}\right\|_{2}=2\binom{2 s}{s}^{2}\left(H_{2 s}-H_{s}\right) \tag{33}
\end{equation*}
$$

Proof. We employ (14) and (17) to calculate the spectral norm of $B_{5}$ as follows, for all $t=0,1, \ldots, s$ :

$$
\begin{align*}
\left|\lambda_{t}\left(B_{5}\right)\right| & =\left|\sum_{i=0}^{s} a_{i}\left(\omega^{t} \alpha\right)^{i}\right| \leq \sum_{i=0}^{s}\left|a_{i}\right| \cdot\left|\left(\omega^{t} \alpha\right)^{i}\right|  \tag{34}\\
& =\sum_{i=0}^{s}\left|a_{i}\right|=\sum_{i=0}^{s}\binom{s}{i}^{2}\binom{2 s+i}{i}\left(H_{2 s+i}-H_{i}\right)
\end{align*}
$$

where $a_{i}=(-1)^{i}\binom{s}{i}^{2}\binom{2 s+i}{i}\left(H_{2 s+i}-H_{i}\right)$.
Since the skew-circulant matrix is normal, we deduce that

$$
\begin{equation*}
\left\|B_{5}\right\|_{2}=\max _{0 \leq t \leq s}\left|\lambda_{t}\left(B_{5}\right)\right| \tag{35}
\end{equation*}
$$

If $s$ is even, then $s+1$ is odd. We declare that $\lambda_{\mathrm{sc}}=-1$ is an eigenvalue of $\eta_{\mathrm{sc}}$; then we calculate the corresponding eigenvalue of $B_{5}$ as follows:

$$
\begin{align*}
\lambda_{\bar{t}}\left(B_{5}\right) & =\sum_{i=0}^{s} a_{i} \lambda_{\mathrm{sc}}^{i}=\sum_{i=0}^{s} a_{i}(-1)^{i} \\
& =\sum_{i=0}^{s}\binom{s}{i}^{2}\binom{2 s+i}{i}\left(H_{2 s+i}-H_{i}\right), \tag{36}
\end{align*}
$$

where we had employed (14).
Noticing (34), we claim that $\lambda_{\hat{t}}\left(B_{5}\right)$ is the maximum of $\left|\lambda_{t}\left(B_{5}\right)\right|$, which means

$$
\begin{equation*}
\left\|B_{5}\right\|_{2}=\sum_{i=0}^{s}\binom{s}{i}^{2}\binom{2 s+i}{i}\left(H_{2 s+i}-H_{i}\right) . \tag{37}
\end{equation*}
$$

Thus, from (4) we obtain

$$
\begin{equation*}
\left\|B_{5}\right\|_{2}=2\binom{2 s}{s}^{2}\left(H_{2 s}-H_{s}\right) \tag{38}
\end{equation*}
$$

This completes the proof.
Similarly, we can calculate the identity for $B_{6}$.
Corollary 11. Let $(s+1) \times(s+1)$-circulant matrix $B_{6}$ be as in (10), and the first row of $B_{6}$ is

$$
\begin{align*}
& \left(\binom{s}{0}\binom{2 s}{0}\binom{3 s}{0}\left(H_{3 s}-H_{0}\right),\right. \\
& -\binom{s}{1}\binom{2 s}{1}\binom{3 s+1}{1}\left(H_{3 s+1}-H_{1}\right), \ldots,  \tag{39}\\
& \left.\binom{s}{s}\binom{2 s}{s}\binom{3 s+s}{s}\left(H_{3 s+s}-H_{s}\right)\right),
\end{align*}
$$

where $a_{i}=(-1)^{i}\binom{s}{i}\binom{2 s}{i}\binom{3 s+i}{i}\left(H_{3 s+i}-H_{i}\right)$. Then there holds

$$
\begin{equation*}
\left\|B_{6}\right\|_{2}=\binom{3 s}{s}^{2}\left(2 H_{3 s}-H_{2 s}-H_{s}\right) . \tag{40}
\end{equation*}
$$

Corollary 12. Let $\widetilde{B}_{5}=-B_{5}$ and $\widetilde{B}_{6}=-B_{6}$, and sis even. Then one has the identities for spectral norm

$$
\begin{gather*}
\left\|\widetilde{B}_{5}\right\|_{2}=2\binom{2 s}{s}^{2}\left(2 H_{2 s}-H_{s}\right), \\
\left\|\widetilde{B}_{6}\right\|_{2}=\binom{3 s}{s}^{2}\left(2 H_{3 s}-H_{2 s}-H_{s}\right) . \tag{41}
\end{gather*}
$$

## 5. Spectral Norms of $\boldsymbol{g}$-Circulant Matrices

Inspired by the above propositions, we analyse spectral norms of some given $g$-circulant matrices in this section.

Lemma 13 (see [25]). The $(n+1) \times(n+1)$ matrix $Q_{g}$ is unitary if and only if

$$
\begin{equation*}
(n+1, g)=1 \tag{42}
\end{equation*}
$$

where $Q_{g}$ is a g-circulant matrix with the first row $e^{*}=[1$, $0, \ldots, 0]$.

Lemma 14 (see [25]). $A$ is a $g$-circulant matrix with the first row $\left[a_{0}, a_{1}, \ldots, a_{n}\right]$ if and only if

$$
\begin{equation*}
A=Q_{g} C \tag{43}
\end{equation*}
$$

where

$$
\begin{equation*}
C=\operatorname{circ}\left(a_{0}, a_{1}, \ldots, a_{n}\right) \tag{44}
\end{equation*}
$$

In the following part, we set $(n+1, g)=1$.
Theorem 15. Let $(n+1) \times(n+1)$-circulant matrix $B_{7}$ be as in (15), and the first row of $B_{7}$ is

$$
\begin{align*}
& \left(\binom{n}{0}^{2}\binom{2 n}{0}\left(H_{2 n}-H_{0}\right)\right. \\
& \binom{n}{1}^{2}\binom{2 n+1}{1}\left(H_{2 n+1}-H_{1}\right), \ldots  \tag{45}\\
& \left.\binom{n}{n}^{2}\binom{2 n+n}{n}\left(H_{2 n+n}-H_{n}\right)\right)
\end{align*}
$$

where $a_{i}=\binom{n}{i}^{2}\binom{2 n+i}{i}\left(H_{2 n+i}-H_{i}\right)$. Then

$$
\begin{equation*}
\left\|B_{7}\right\|_{2}=2\binom{2 n}{n}^{2}\left(H_{2 n}-H_{n}\right) \tag{46}
\end{equation*}
$$

Proof. With the help of Lemmas 13 and 14, we know that the $g$-circulant matrix $B_{7}$ is normal; then we claim that the spectral norm of $B_{7}$ is equal to its spectral radius. Furthermore, applying the irreducible and entrywise nonnegative properties, we claim that $\left\|B_{7}\right\|_{2}$ (i.e., its spectral norm) is equal to its Perron value. We select a $(n+1)$-dimensional column vector $v=(1,1, \ldots, 1)^{T}$; then

$$
\begin{equation*}
B_{7} v=\left(\sum_{i=0}^{n}\binom{n}{i}^{2}\binom{2 n+i}{i}\left(H_{2 n+i}-H_{i}\right)\right) v . \tag{47}
\end{equation*}
$$

Obviously, $\sum_{i=0}^{n}\binom{n}{i}^{2}\binom{2 n+i}{i}\left(H_{2 n+i}-H_{i}\right)$ is an eigenvalue of $B_{7}$ associated with $v$, which is necessarily the Perron value of $B_{7}$. Employing (4), we obtain

$$
\begin{equation*}
\left\|B_{7}\right\|_{2}=2\binom{2 n}{n}^{2}\left(H_{2 n}-H_{n}\right) . \tag{48}
\end{equation*}
$$

This completes the proof.
Corollary 16. Let $(n+1) \times(n+1)$-circulant matrix $B_{8}$ be as in (15), and the first row of $B_{8}$ is

$$
\begin{align*}
& \left(\binom{n}{0}\binom{2 n}{0}\binom{3 n}{0}\left(H_{3 n}-H_{0}\right),\right. \\
& \binom{n}{1}\binom{2 n}{1}\binom{3 n+1}{1}\left(H_{3 n+1}-H_{1}\right), \ldots,  \tag{49}\\
& \left.\binom{n}{n}\binom{2 n}{n}\binom{4 n}{n}\left(H_{4 n}-H_{n}\right)\right),
\end{align*}
$$

where $a_{i}=\binom{n}{i}\binom{2 n}{i}\binom{3 n+i}{i}\left(H_{3 n+i}-H_{i}\right)$. Then one obtains

$$
\begin{equation*}
\left\|B_{8}\right\|_{2}=\binom{3 n}{n}^{2}\left(2 H_{3 n}-H_{2 n}-H_{n}\right) \tag{50}
\end{equation*}
$$

## 6. Numerical Examples

Example 1. In this example, we give the numerical results for $B_{1}$ and $B_{2}$.

Comparing the data in Table 1, we declare that the identities of spectral norms for $B_{i}(i=1,2)$ hold.

Example 2. In this example, we list the numerical results for $B_{i}, \widetilde{B}_{i}(i=3,4)$.

With the help of data in Table 2, it is clear that the identities of spectral norms for $B_{i}, \widetilde{B}_{i}(i=3,4)$ hold.

Example 3. In this example, we reveal the numerical results for alternative skew-circulant matrices $B_{i}, \widetilde{B}_{i}(i=5,6)$.

Combining the data in Table 3, we deduce that the identities of spectral norms for $B_{i}, \widetilde{B}_{i}(i=5,6)$ hold.

Example 4. In this example, we show numerical results for $B_{7}$ and $B_{8}$.

Considering the data in Table 4, we deduce that the identities of spectral norms for $B_{i}(i=7,8)$ hold.

The above results demonstrate that the identities of spectral norms for the given matrices hold.

## 7. Conclusion

This paper had discussed the explicit formulae for identical estimations of spectral norms for circulant, skew-circulant and $g$-circulant matrices, whose entries are binomial coefficients combined with harmonic numbers. Furthermore, it is easy to take other entries to obtain more interesting identities, and the same approaches can be used to verify those identities. Furthermore, explicit formulas for both

TABLE 1: Spectral norms of $B_{i}(i=1,2), C_{1, n}=2\binom{2 n}{n}^{2}\left(H_{2 n}-H_{n}\right)$, and $C_{2, n}=\binom{3 n}{n}^{2}\left(2 H_{3 n}-H_{2 n}-H_{n}\right)$.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\left\\|B_{1}\right\\|_{2}$ | 0 | 4 | 42 | $4.93 e+2$ | $6.22 e+3$ | $8.20 e+4$ | $1.12 e+6$ |
| $\left\\|B_{2}\right\\|_{2}$ | 0 | $1.05 e+1$ | $2.96 e+2$ | $9.69 e+3$ | $3.44 e+5$ | $1.28 e+7$ | $4.95 e+8$ |
| $C_{1, n}$ | 0 | 4 | 42 | $4.93 e+2$ | $6.22 e+3$ | $8.20 e+4$ | $1.12 e+6$ |
| $C_{2, n}$ | 0 | $1.05 e+1$ | $2.96 e+2$ | $9.69 e+3$ | $3.44 e+5$ | $1.28 e+7$ | $4.95 e+8$ |

TABLE 2: Spectral norms of $B_{i}, \widetilde{B}_{i}(i=3,4), C_{1, m}=2\binom{2 m}{m}^{2}\left(H_{2 m}-H_{m}\right)$, and $C_{2, m}=\binom{3 m}{m}^{2}\left(2 H_{3 m}-H_{2 m}-H_{m}\right)$.

| $m$ | 1 | 3 | 5 | 7 | 9 | 11 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $\left\\|B_{3}\right\\|_{2}$ | 4 | $4.93 e+2$ | $8.20 e+4$ | $1.55 e+7$ | $3.15 e+9$ | $6.68 e+11$ |
| $\left\\|B_{4}\right\\|_{2}$ | 10.5 | $9.69 e+3$ | $1.28 e+7$ | $1.96 e+10$ | $3.20 e+13$ | $5.49 e+16$ |
| $\left\\|\widetilde{B}_{3}\right\\|_{2}$ | 4 | $4.93 e+2$ | $8.20 e+4$ | $1.55 e+7$ | $3.15 e+9$ | $6.68 e+11$ |
| $\left\\|\widetilde{B}_{4}\right\\|_{2}$ | 10.5 | $9.69 e+3$ | $1.28 e+7$ | $1.96 e+10$ | $3.20 e+13$ | $5.49 e+16$ |
| $C_{1, m}$ | 4 | $4.93 e+2$ | $8.20 e+4$ | $1.55 e+7$ | $3.15 e+9$ | $6.68 e+11$ |
| $C_{2, m}$ | 10.5 | $9.69 e+3$ | $1.28 e+7$ | $1.96 e+10$ | $3.20 e+13$ | $5.49 e+16$ |

TABLE 3: Spectral norms of $B_{i}, \widetilde{B}_{i}(i=5,6), C_{1, s}=2\binom{2 s}{s}^{2}\left(H_{2 s}-H_{s}\right)$, and $C_{2, s}=\binom{3 s}{s}^{2}\left(2 H_{3 s}-H_{2 s}-H_{s}\right)$.

| $s$ | 0 | 2 | 4 | 6 | 8 | 10 |
| :--- | :--- | :---: | :---: | :---: | :---: | :---: |
| $\left\\|B_{5}\right\\|_{2}$ | 0 | 42 | $6.22 e+3$ | $1.12 e+6$ | $2.20 e+8$ | $4.57 e+10$ |
| $\left\\|B_{6}\right\\|_{2}$ | 0 | $2.96 e+2$ | $3.44 e+5$ | $4.95 e+5$ | $7.86 e+11$ | $1.32 e+15$ |
| $\left\\|\widetilde{B}_{5}\right\\|_{2}$ | 0 | 42 | $6.22 e+3$ | $1.12 e+6$ | $2.20 e+8$ | $4.57 e+10$ |
| $\left\\|\widetilde{B}_{6}\right\\|_{2}$ | 0 | $2.96 e+2$ | $3.44 e+5$ | $4.95 e+5$ | $7.86 e+11$ | $1.32 e+15$ |
| $C_{1, s}$ | 0 | 42 | $6.22 e+3$ | $1.12 e+6$ | $2.20 e+8$ | $4.57 e+10$ |
| $C_{2, s}$ | 0 | $2.96 e+2$ | $3.44 e+5$ | $4.95 e+5$ | $7.86 e+11$ | $1.32 e+15$ |

Table 4: Spectral norms of $B_{i}(i=7,8), C_{1, n}=2\binom{2 n}{n}^{2}\left(H_{2 n}-H_{n}\right)$, and $C_{2, n}=\binom{3 n}{n}^{2}\left(2 H_{3 n}-H_{2 n}-H_{n}\right)$.

| $n+1$ |  | 5 |  | 6 |  | 7 |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $g$ | 2 | 3 | 4 | 5 | 2 | 3 | 4 | 5 |
| $\left\\|B_{7}\right\\|_{2}$ | $6.22 e+3$ | $6.22 e+3$ | $6.22 e+3$ | $8.20 e+4$ | $1.12 e+6$ | $1.12 e+6$ | $1.12 e+6$ | $1.12 e+6$ |
| $\left\\|B_{8}\right\\|_{2}$ | $3.44 e+5$ | $3.44 e+5$ | $3.44 e+5$ | $1.28 e+7$ | $4.95 e+8$ | $4.95 e+8$ | $4.95 e+8$ | $4.95 e+8$ |
| $C_{1, n}$ | $6.22 e+3$ | $6.22 e+3$ | $6.22 e+3$ | $8.20 e+4$ | $1.12 e+6$ | $1.12 e+6$ | $1.12 e+6$ | $1.12 e+6$ |
| $C_{2, n}$ | $3.44 e+5$ | $3.44 e+5$ | $3.44 e+5$ | $1.28 e+7$ | $4.95 e+8$ | $4.95 e+8$ | $4.95 e+8$ | $1.12 e+6$ |

norms $\|A\|$ and $\left\|A^{-1}\right\|$ help us to estimate the so-called condition number. It is an interesting problem to investigate the properties of $B_{i}(i=1,2, \ldots, 8)$, such as the explicit formulations for determinants and inverses, by just using the entries in the first row.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Optimal Grasping Manipulation for Multifingered Robots Using Semismooth Newton Method 

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#### Abstract

Multifingered robots play an important role in manipulation applications. They can grasp various shaped objects to perform point-to-point movement. It is important to plan the motion path of the object and appropriately control the grasping forces for multifingered robot manipulation. In this paper, we perform the optimal grasping control to find both optimal motion path of the object and minimum grasping forces in the manipulation. The rigid body dynamics of the object and the grasping forces subjected to the second-order cone (SOC) constraints are considered in optimal control problem. The minimum principle is applied to obtain the system equalities and the SOC complementarity problems. The SOC complementarity problems are further recast as the equations with the Fischer-Burmeister (FB) function. Since the FB function is semismooth, the semismooth Newton method with the generalized Jacobian of FB function is used to solve the nonlinear equations. The 2 D and 3 D simulations of grasping manipulation are performed to demonstrate the effectiveness of the proposed approach.


## 1. Introduction

Multifingered robots have attracted much attention in robotics manipulation applications. They can grasp various shaped objects and dexterously perform point-to-point manipulations. Many researches [1-6] have been proposed for grasping and manipulating objects with multifingered robots. Miller and Allen [2] proposed a user interface with grasp quality evaluation for the robot hand design. Yokokohji et al. [6] proposed a measure of dynamic manipulability of multifingered grasping for the systems consisting of a multifingered hand and a grasped object. Xu and Li [5] proposed a modeling method for the manipulation involving finger gaits. Kawamura et al. [1] used soft finger tips for stable grasping. Takahashi et al. [4] proposed robust force and position control with the information of tactile sensor. It is important to appropriately control the grasping forces for multifingered robot manipulation.

Since the grasping manipulation utilizes the contact and friction forces to hold and move an object, the grasping forces should satisfy the point-contact friction constraint and be equal to the dynamic wrench of the grasped object. It is required to find the minimum forces for moving the grasped object in the manipulation. Boyd and Wegbreit [7] used the semidefinite programming and second-order cone programming to efficiently find the grasping forces. Helmke et al. [8] proposed quadratically convergent algorithms for optimal dexterous grasping. Han et al. [9] used the convex optimization involving linear matrix inequalities for grasping forces computation. Liu et al. [10] presented a unified geometric framework for efficient grasping force optimization. Zheng et al. [11] developed an algorithm to determine the minimum required friction coefficient and the corresponding reliable minimum contact forces in practice. Ko et al. [12] proposed a neural network to calculate the optimal grasping forces. Because the external wrench of the object varies with
the manipulation path and orientation of the object, it is important to plan a manipulation trajectory [13] for achieving the minimum grasping forces.

In this paper, we perform the optimal grasping control to find both optimal manipulation path of the object and minimum grasping forces. The rigid body dynamics of the object and the grasping forces subjected to the second-order cone (SOC) constraints are considered in the grasping control problem. The minimum principle [14] is applied to obtain the system equalities and the SOC complementarity problems. The SOC complementarity problems can be recast as the equations with the Fischer-Burmeister (FB) function. The semismooth Newton method with the generalized Jacobian of FB function is then used to solve the equations. Finally, simulations of optimal grasping manipulation are performed to demonstrate the effectiveness of the proposed approach.

The remainder of this paper is organized as follows: Section 2 describes the optimal grasping control problem. In Section 3, the semismooth Newton method with the generalized Jacobian of Fischer-Burmeister function is addressed. Section 4 presents the simulation results of 2D and 3D grasping manipulations. Finally, concluding remarks are given in Section 5.

## 2. Optimal Grasping Control

Figure 1 shows the multifingered robot grasping manipulation. The multifingered robot grasps and moves the object from the initial position to the final position. The dynamic equation of the object can be expressed with Newton-Euler equations $[15,16]$ as

$$
\begin{gather*}
\dot{y}=v \\
\dot{v}=\frac{1}{m} R G_{1} u+\left[\begin{array}{lll}
0 & 0 & -g
\end{array}\right]^{T},  \tag{1}\\
\dot{q}=Q \omega \\
\dot{\omega}=I^{-1}\left(R G_{2} u-\omega \times(I \omega)\right),
\end{gather*}
$$

where $y$ is the position, $v$ is the velocity, $q=\left[\begin{array}{lll}q_{1} & q_{2} & q_{3}\end{array}\right]^{T}$ is the quaternion, $\omega$ is the angular velocity, $m$ is the object mass, $I$ is the matrix of moment of inertia, $g$ is the gravity constant, $u$ means the grasping forces which is represented by a matrix, [ $G_{1} G_{2}$ ] is the contact matrix, $R$ is the rotation matrix of the object, and $Q$ can be expressed as

$$
Q=\frac{1}{2}\left[\begin{array}{ccc}
q_{0} & q_{3} & -q_{2}  \tag{2}\\
-q_{3} & q_{0} & q_{1} \\
q_{2} & -q_{1} & q_{0}
\end{array}\right] \quad \text { with } q_{0}=\sqrt{q_{1}^{2}+q_{2}^{2}+q_{3}^{2}}
$$

Moreover, the grasping forces are subject to the contact friction constraint, expressed as

$$
\begin{equation*}
\left\|\left(u_{i 2}, u_{i 3}\right)\right\| \leq \mu u_{i 1} \tag{3}
\end{equation*}
$$

where $u_{i 1}$ is the normal force of the $i$ th finger, $u_{i 2}$ and $u_{i 3}$ are the friction forces of the $i$ th finger, $\|\cdot\|$ is the 2 -norm, and $\mu$ is the friction coefficient.


Figure 1: Multifingered robot manipulation.

To find the path that can be achieved with the minimum grasping forces, the optimal control problem can be recast as

$$
\begin{align*}
& \min \int_{0}^{T} L d t \\
& \text { s.t. } \quad \dot{x}=f(x, u), \\
& \quad x(0)=x_{0}  \tag{4}\\
& x(T)=x_{T} \\
& \quad D u \in \mathscr{K}^{d} \times \mathscr{K}^{d} \times \cdots \times \mathscr{K}^{d}
\end{align*}
$$

where

$$
L=\frac{u^{T} u}{2}, \quad x=\left[\begin{array}{c}
y  \tag{5}\\
v \\
q \\
\omega
\end{array}\right]
$$

In addition, $f$ represents the right hand side of system (1), $T$ is the control duration, $x_{0}$ and $x_{T}$ are the initial and final states, respectively, $D$ is the diagonal matrix with the friction coefficient, and $\mathscr{K}$ denotes the second-order cone which is given by

$$
\mathscr{K}^{d}:=\left\{\left.\left[\begin{array}{l}
z_{1}  \tag{6}\\
z_{2}
\end{array}\right] \in \mathbb{R} \times \mathbb{R}^{d-1} \right\rvert\,\left\|z_{2}\right\| \leq z_{1}\right\}
$$

The optimal control problem (4) can be solved by using the Pontryagin's minimum principle, see [14, 17, 18]. In optimization language, it is to write out the KKT conditions for problem (4) which consist of two parts. The first part involves a few equalities about Lagrange multipliers, while the other part is related to complementarity conditions. More specifically, with the Hamiltonian function, the first part can be reformulated as follows:

$$
\begin{gather*}
\dot{x}-H_{\lambda}=\dot{x}-f(x, u)=0, \\
\dot{\lambda}+H_{x}=\dot{\lambda}+\lambda^{T} f_{x}=0, \\
H_{u}=L_{u}+\lambda^{T} f_{u}+\eta^{T} D=0, \\
\phi(x(0), x(T))=0,  \tag{7}\\
\lambda(0)+\phi_{x(0)}^{T} \sigma=0, \\
\lambda(T)+\phi_{x(T)}^{T} \sigma=0,
\end{gather*}
$$



Figure 2: The manipulation path of 90 degrees rotation with $T=1 \mathrm{~s}$ and $\mu=0.3$.
where $\lambda, \eta$, and $\sigma$ are the Lagrange multipliers, and $\phi(x(0), x(T))=\left[\begin{array}{l}x(0)-x_{0} \\ x(T)-x_{T}\end{array}\right]$. The second part forms a secondorder cone complementarity problem (SOCCP) as follows:

$$
\begin{equation*}
-\eta \in \mathscr{K}, \quad D u \in \mathscr{K}, \quad \eta^{T} D u=0 \tag{8}
\end{equation*}
$$

where $\mathscr{K}=\mathscr{K}^{d} \times \mathscr{K}^{d} \times \cdots \times \mathscr{K}^{d}$.
From [19, 20], we see that the previous SOCCP (8) can be further recast as a system of equations:

$$
\begin{equation*}
\phi_{\mathrm{FB}}(D u,-\eta)=0 \tag{9}
\end{equation*}
$$

by employing the so-called complementarity function $\phi_{\mathrm{FB}}$ which is a vector-valued function defined as

$$
\begin{equation*}
\phi_{\mathrm{FB}}(\mathbf{a}, \mathbf{b}):=\left(\mathbf{a}^{2}+\mathbf{b}^{2}\right)^{1 / 2}-(\mathbf{a}+\mathbf{b}) \tag{10}
\end{equation*}
$$

for $\mathbf{a}=\left[\begin{array}{l}a_{1} \\ a_{2}\end{array}\right] \in \mathbb{R} \times \mathbb{R}^{d-1}, \mathbf{b}=\left[\begin{array}{l}b_{1} \\ b_{2}\end{array}\right] \in \mathbb{R} \times \mathbb{R}^{d-1}$. We point out that the square term and square-root term in (10) are calculated via Jordan product

$$
\mathbf{a} \circ \mathbf{b}=\left[\begin{array}{c}
\mathbf{a}^{\mathrm{T}} \mathbf{b}  \tag{11}\\
a_{1} b_{2}+b_{1} a_{2}
\end{array}\right]
$$

In particular, the expressions for $\mathbf{a}^{2}$ and $\mathbf{a}^{1 / 2}$ are given by

$$
\begin{gather*}
\mathbf{a}^{2}=\left[\begin{array}{c}
\|\mathbf{a}\|^{2} \\
2 a_{1} a_{2}
\end{array}\right]  \tag{12}\\
\mathbf{a}^{1 / 2}=\left[\begin{array}{c}
s \\
\frac{a_{2}}{2 s}
\end{array}\right] \quad \text { with } \boldsymbol{s}=\sqrt{\frac{1}{2}\left(\mathbf{a}_{1}+\sqrt{\mathbf{a}_{1}^{2}-\left\|\mathbf{a}_{2}\right\|^{2}}\right)} \tag{13}
\end{gather*}
$$

respectively.
In summary, the optimal grasping forces can be obtained by solving (7) and (9). Then, the semismooth Newton method is used to solve these equations, which will be described in next section.


Figure 3: The trajectories of the variables $x, y, \theta, v_{x}, v_{y}$, and $\omega$ in 90 degrees rotation simulation.

## 3. Semismooth Newton Method with Generalized Jacobian of FB Function

In order to apply the semismooth Newton method [21,22] to (7) and (9), we need the following three linear equations:

$$
\begin{aligned}
{\left[\begin{array}{c}
\Delta \dot{x} \\
\Delta \dot{\lambda}
\end{array}\right] } & -\left[\begin{array}{cc}
f_{x}^{(k)} & 0 \\
-H_{x x}^{(k)} & -\left(f_{x}^{(k)}\right)^{T}
\end{array}\right]\left[\begin{array}{c}
\Delta x \\
\Delta \lambda
\end{array}\right]-\left[\begin{array}{cc}
f_{u}^{(k)} & 0 \\
-H_{x u}^{(k)} & 0
\end{array}\right]\left[\begin{array}{l}
\Delta u \\
\Delta \eta
\end{array}\right] \\
& =-\left[\begin{array}{c}
\dot{x}^{(k)}-f^{(k)} \\
\dot{\lambda}^{(k)}+H_{x}^{(k)}
\end{array}\right]
\end{aligned}
$$

$$
\begin{align*}
& {\left[\begin{array}{ccc}
\phi_{x(0)}^{(k)} & 0 & 0 \\
0 & I & \left(\phi_{x(0)}^{(k)}\right)^{T} \\
0 & 0 & -\left(\phi_{x(0)}^{(k)}\right)^{T}
\end{array}\right]\left[\begin{array}{c}
\Delta x(0) \\
\Delta \lambda(0) \\
\Delta \sigma(0)
\end{array}\right]+\left[\begin{array}{ccc}
\phi_{x(T)}^{(k)} & 0 & 0 \\
0 & 0 & 0 \\
0 & I & 0
\end{array}\right]\left[\begin{array}{c}
\Delta x(T) \\
\Delta \lambda(T) \\
\Delta \sigma(T)
\end{array}\right]}  \tag{14}\\
& \quad=-\left[\begin{array}{c}
\phi^{(k)} \\
\lambda^{(k)}(0)+\left(\phi_{x(0)}^{(k)}\right)^{T} \sigma^{(k)} \\
\lambda^{(k)}(T)+\left(\phi_{x(T)}^{(k)}\right)^{T} \sigma^{(k)}
\end{array}\right], \tag{15}
\end{align*}
$$



Figure 4: The trajectories of the grasping forces in 90 degrees rotation simulation.


Figure 5: The manipulation path of 90 degrees rotation with $T=1 \mathrm{~s}$ and $\mu=0.1$.


Figure 6: The manipulation path of 180 degrees rotation with $T=$ 1 s and $\mu=0.3$.


Figure 7: The manipulation path of 180 degrees rotation with $T=$ 2 s and $\mu=0.3$.

$$
\begin{align*}
& {\left[\begin{array}{cc}
I & D^{T} \\
V_{\mathbf{a}}^{(k)} D & -V_{\mathbf{b}}^{(k)}
\end{array}\right]\left[\begin{array}{c}
\Delta u \\
\Delta \eta
\end{array}\right]+\left[\begin{array}{cc}
\lambda^{T} f_{x u}^{(k)} & f_{u}^{(k)} \\
0 & 0
\end{array}\right]\left[\begin{array}{l}
\Delta x \\
\Delta \lambda
\end{array}\right]} \\
& \quad=-\left[\begin{array}{c}
H_{u}^{(k)} \\
\phi_{\mathrm{FB}}\left(D u^{(k)},-\eta^{(k)}\right)
\end{array}\right] \tag{16}
\end{align*}
$$

Most information in linear equations (14)-(16) is known except the generalized Jacobian $V_{\mathrm{a}}, V_{\mathrm{b}}$ in (16). What do they represent? We provide a brief introduction here. First, we recall the concept of the $B$-subdifferential. Given a mapping $\Psi: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$, if $\Psi$ is locally Lipschitz continuous, then the set

$$
\begin{align*}
\partial_{B} H & (z) \\
& :=\left\{V \in \mathbb{R}^{m \times n} \mid \exists\left\{z^{k}\right\} \subseteq D_{\Psi}: z^{k} \longrightarrow z, \Psi^{\prime}\left(z^{k}\right) \longrightarrow V\right\} \tag{17}
\end{align*}
$$



Figure 8: The 3D manipulation path of the five-fingered robot.




Figure 9: The trajectories of the variables $x, y, z, q_{0}, q 1, q_{2}, q_{3}$ in fivefingered robot simulation.


Figure 10: The trajectories of the variables $v_{x}, v_{y}, v_{z}, \omega_{x}, \omega_{y}, \omega_{z}$ in five-fingered robot simulation.
is nonempty and is called the $B$-subdifferential of $\Psi$ at $z$, where $D_{\Psi} \subseteq \mathbb{R}^{n}$ denotes the set of points at which $\Psi$ is differentiable. The convex hull

$$
\begin{equation*}
\partial \Psi(z):=\operatorname{conv} \partial_{B} \Psi(z) \tag{18}
\end{equation*}
$$

is the generalized Jacobian of Clarke [23]. From this definition, we see that the generalized Jacobian of $\phi_{\mathrm{FB}}$ can be obtained by computing $\partial_{B} \phi_{\mathrm{FB}}$. From [24, Proposition 3.1], the $B$-subdifferential of $\phi_{\mathrm{FB}}$ in (16) is exactly expressed as

$$
\partial_{B} \phi_{\mathrm{FB}}(\mathbf{a}, \mathbf{b})=\left[\begin{array}{c}
V_{\mathbf{a}}^{T}  \tag{19}\\
V_{\mathbf{b}}^{T}
\end{array}\right] .
$$

Moreover, by denoting $\mathbf{c}:=\left(\mathbf{a}^{2}+\mathbf{b}^{2}\right)^{1 / 2}$, we have
(a) If $\mathbf{a}^{2}+\mathbf{b}^{2} \in \operatorname{int}\left(\mathscr{K}^{\mathbf{d}}\right)$, then $V_{\mathbf{a}}=L_{\mathbf{c}}^{-1} L_{\mathbf{a}}$ and $V_{\mathbf{b}}=L_{\mathbf{c}}^{-1} L_{\mathbf{b}}$.
(b) If $\mathbf{a}^{2}+\mathbf{b}^{2} \in \operatorname{bd}\left(\mathscr{K}^{\mathbf{d}}\right)$ and $(\mathbf{a}, \mathbf{b}) \neq(\mathbf{0}, \mathbf{0})$, then

$$
\begin{align*}
& V_{\mathbf{a}} \in\left\{\frac{1}{2 \sqrt{2 w_{1}}}\left(\begin{array}{cc}
1 & \bar{w}_{2}^{T} \\
\bar{w}_{2} & 4 I-3 \bar{w}_{2} \bar{w}_{2}^{T}
\end{array}\right) L_{\mathbf{a}}+\frac{1}{2}\binom{1}{-\bar{w}_{2}} \alpha^{T}\right\}, \\
& V_{\mathbf{b}} \in\left\{\frac{1}{2 \sqrt{2 w_{1}}}\left(\begin{array}{cc}
1 & \bar{w}_{2}^{T} \\
\bar{w}_{2} & 4 I-3 \bar{w}_{2} \bar{w}_{2}^{T}
\end{array}\right) L_{\mathbf{b}}+\frac{1}{2}\binom{1}{-\bar{w}_{2}} \beta^{T}\right\} \tag{20}
\end{align*}
$$







$$
\begin{array}{lll}
- & u_{51} \\
\ldots \ldots & u_{52}
\end{array} \quad---u_{53}
$$

Figure 11: The trajectories of the grasping forces in five-fingered robot simulation.
for some $\alpha=\left[\begin{array}{l}\alpha_{1} \\ \alpha_{2}\end{array}\right] \in \mathbb{R} \times \mathbb{R}^{d-1} \beta=\left[\begin{array}{c}\beta_{1} \\ \beta_{2}\end{array}\right] \in \mathbb{R} \times \mathbb{R}^{d-1}$ satisfying $\left|\alpha_{1}\right| \leq\left\|\alpha_{2}\right\| \leq 1$ and $\left|\beta_{1}\right| \leq\left\|\beta_{2}\right\| \leq 1$, where $\bar{w}_{2}=w_{2} /\left\|w_{2}\right\|$.
(c) If $(\mathbf{a}, \mathbf{b})=(\mathbf{0}, \mathbf{0})$, then $V_{\mathbf{a}} \in\left\{L_{\widehat{\mathbf{a}}}\right\}, V_{\mathbf{b}} \in\left\{L_{\widehat{\mathbf{b}}}\right\}$ for some $\widehat{\mathbf{a}}, \widehat{\mathbf{b}}$ with $\|\widehat{\mathbf{a}}\|^{2}+\|\widehat{\mathbf{b}}\|^{2}=1$, or
$V_{\mathbf{a}} \in\left\{\frac{1}{2}\left(\frac{1}{\bar{w}_{2}}\right) \xi^{T}+\frac{1}{2}\binom{1}{-\bar{w}_{2}} \alpha^{T}\right.$ $\left.+2\left(\begin{array}{cc}0 & 0 \\ \left(I-\bar{w}_{2} \bar{w}_{2}^{T}\right) \delta_{2} & \left(I-\bar{w}_{2} \bar{w}_{2}^{T}\right) \delta_{1}\end{array}\right)\right\}$,


Figure 12: The 3D manipulation path of the four-fingered robot.


Figure 13: The 3D manipulation path of the six-fingered robot.

$$
\begin{align*}
V_{\mathbf{b}} \in\{ & \frac{1}{2}\left(\frac{1}{w_{2}}\right) \tau^{T}+\frac{1}{2}\binom{1}{-\bar{w}_{2}} \beta^{T} \\
& \left.+2\left(\left(I-\bar{w}_{2} \bar{w}_{2}^{T}\right) \gamma_{2}\left(I-\bar{w}_{2} \bar{w}_{2}^{T}\right) \gamma_{1}\right)\right\} \tag{21}
\end{align*}
$$

for some $\alpha=\left[\begin{array}{c}\alpha_{1} \\ \alpha_{2}\end{array}\right], \beta=\left[\begin{array}{l}\beta_{1} \\ \beta_{2}\end{array}\right], \xi=\left[\begin{array}{l}\xi_{1} \\ \xi_{2}\end{array}\right], \tau=\left[\begin{array}{l}\tau_{1} \\ \tau_{2}\end{array}\right] \in \mathbb{R} \times \mathbb{R}^{l-1}$ such that

$$
\begin{array}{ll}
\left|\alpha_{1}\right| \leq\left\|\alpha_{2}\right\| \leq 1, & \left|\beta_{1}\right| \leq\left\|\beta_{2}\right\| \leq 1 \\
\left|\xi_{1}\right| \leq\left\|\xi_{2}\right\| \leq 1, & \left|\tau_{1}\right| \leq\left\|\tau_{2}\right\| \leq 1 \tag{22}
\end{array}
$$

$\bar{w}_{2} \in \mathbb{R}^{l-1}$ satisfying $\left\|\bar{w}_{2}\right\|=1$, and $\delta=\left[\begin{array}{l}\delta_{1} \\ \delta_{2}\end{array}\right], \gamma=\left[\begin{array}{l}\gamma_{1} \\ \gamma_{2}\end{array}\right] \in$ $\mathbb{R} \times \mathbb{R}^{d-1}$ satisfying $\|\delta\|^{2}+\|\gamma\|^{2} \leq 1 / 2$.

Note that the calculations of $L_{\mathbf{a}}$ and $L_{\mathbf{a}}^{-1}$ are given by

$$
\begin{gather*}
L_{\mathbf{a}}=\left[\begin{array}{cc}
a_{1} & a_{2}^{T} \\
a_{2} & a_{1} I
\end{array}\right], \\
L_{\mathbf{a}}^{-1}=\frac{1}{a_{1}^{2}-\left\|a_{2}\right\|^{2}}\left[\begin{array}{cc}
a_{1} & -a_{2}^{T} \\
-a_{2} & \frac{a_{1}^{2}-\left\|a_{2}\right\|^{2}}{a_{1}} I+\frac{1}{a_{1}} a_{2} a_{2}^{T}
\end{array}\right] . \tag{23}
\end{gather*}
$$

For more details, please refer to [24]. Now, we write down the iterative scheme of semismooth Newton method for solving the optimal grasping control problem.

## Algorithm

Step 1. Choose $x^{0}, u^{0}, \lambda^{0}, \eta^{0}, \sigma^{0}$ and set $k=0$.
Step 2. If convergence criterion is satisfied, stop.
Step 3. Compute the direction $\Delta x^{k}, \Delta u^{k}, \Delta \lambda^{k}, \Delta \eta^{k}, \Delta \sigma^{k}$ from the linear equations (14)-(16).

Step 4. Set

$$
\left[\begin{array}{l}
x^{k+1}  \tag{24}\\
u^{k+1} \\
\lambda^{k+1} \\
\eta^{k+1} \\
\sigma^{k+1}
\end{array}\right]=\left[\begin{array}{c}
x^{k} \\
u^{k} \\
\lambda^{k} \\
\eta^{k} \\
\sigma^{k}
\end{array}\right]+\left[\begin{array}{c}
\Delta x^{k} \\
\Delta u^{k} \\
\Delta \lambda^{k} \\
\Delta \eta^{k} \\
\Delta \sigma^{k}
\end{array}\right], \quad k=k+1
$$

and go to Step 2.
A few words about the implementations. From (14) and (16), the parameters $\left[\begin{array}{c}\Delta u \\ \Delta \eta\end{array}\right]$ can be eliminated and the differential equations regarding $\left[\begin{array}{c}\Delta x \\ \Delta \lambda\end{array}\right]$ are obtained. With the boundary conditions (15), the solutions of [ $\left.\begin{array}{c}\Delta x \\ \Delta \lambda\end{array}\right]$ can be achieved. Finally, the solutions of $\left[\begin{array}{c}\Delta u \\ \Delta \eta\end{array}\right]$ can be obtained by (16). Once all linear equations are solved by the above procedures, the iterative scheme for the calculation of optimal grasping force is kept going.

## 4. Simulations

To evaluate the performance of the proposed approach, we do simulations for 2D and 3D multifingered robots for grasping manipulations. The 2D grasping simulations are performed with a plane three-fingered robot. The parameter values of the
object are $m=1 \mathrm{~kg}, I=0.01 \mathrm{kgm}^{2}, \mu=0.6$, and the grasping matrices are

$$
\begin{gather*}
G_{1}=\left[\begin{array}{cccccc}
0 & 1 & 0 & -1 & 0 & -1 \\
-1 & 0 & 1 & 0 & 1 & 0
\end{array}\right]  \tag{25}\\
G_{2}=\left[\begin{array}{llllll}
0 & -0.03 & -0.03 & 0.03 & 0.03 & 0.03
\end{array}\right] .
\end{gather*}
$$

The first 2D simulation is the manipulation of 90 degrees rotation of the object. The start and end points $(x, y, \theta)$ are set to be ( $0 \mathrm{~m}, 0 \mathrm{~m}, 0 \mathrm{rad}$ ) and ( $1 \mathrm{~m}, 1 \mathrm{~m},-(\pi / 2) \mathrm{rad})$, respectively. Figure 2 shows the manipulation path of 90 degrees rotation with the time $T=1 \mathrm{~s}$ and the friction coefficient $\mu=0.3$. The simulation result indicates that the proposed scheme grasps the object to the end point smoothly and accurately. Figure 3 depicts the trajectories of the variables $x, y, \theta, v_{x}, v_{y}$, and $\omega$. We observe that the rotation angle $\theta$ varies around 0 which results in a small grasping force. Moreover, the translation speeds are kept within $1.8 \mathrm{~m} / \mathrm{s}$ and the turning speed within $5.5 \mathrm{rad} / \mathrm{s}$. The trajectories of the grasping forces are shown in Figure 4. The simulation results show that the normal forces are all nonnegative and the tangent forces satisfy the friction constraint. To evaluate the effect of the friction, simulation with a different value of friction coefficient is also conducted for 90 degrees rotation simulation. Figure 5 shows the manipulation path with the friction coefficient $\mu=0.1$. We observe that the manipulation path length becomes longer as the friction coefficient decreases. Meanwhile, the value of the objective function $J$ is computed to be 3.07 when $\mu$ was 0.3 and it becomes 3.39 when $\mu$ reduces to 0.1 , leading to the increase in grasping force.

The second 2D simulation is the manipulation of 180 degrees rotation of the object. The start and end points $(x, y, \theta)$ are set to be $(0 \mathrm{~m}, 0 \mathrm{~m},-(\pi / 2) \mathrm{rad})$ and ( $1 \mathrm{~m}, 1 \mathrm{~m},(\pi / 2) \mathrm{rad})$, respectively. The friction coefficient is set as $\mu=0.3$. Figures 6-7 depict the manipulation paths of 180 degrees rotation with the time $T=1 \mathrm{~s}$ and $T=2 \mathrm{~s}$, respectively. We observe that the object moves down initially and reaches to the end point accurately. Moreover, the mean of rotation angle decreases as $T$ increases.

The 3D grasping simulation is performed with a fivefingered robot which has not been implemented in the literature. The object is considered as a block and its parameter values were set as

$$
\begin{gather*}
m=1 \mathrm{~kg}, \quad I_{11}=8.33 e-3 \mathrm{kgm}^{2}  \tag{26}\\
I_{22}=4.17 e-3 \mathrm{kgm}^{2}, \quad I_{33}=1.08 e-2 \mathrm{kgm}^{2}
\end{gather*}
$$

The two fingers of the robot grasp the top of the object, while the other three grasp the bottom of the object. The matrices $G_{1}$ and $G_{2}$ are

$$
\begin{align*}
G_{1} & =\left[\begin{array}{ccccccccccccccc}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
-1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}\right], \\
G_{2} & =0.01 \cdot\left[\begin{array}{cccccccccccccccc}
0 & 0 & 5 & 0 & 0 & 5 & 6 & 0 & 5 & -3 & 0 & 5 & -3 & 0 & 5 \\
3 & 5 & 0 & -3 & 5 & 0 & 0 & -5 & 0 & -3 \sqrt{3} & -5 & 0 & 3 \sqrt{3} & -5 & 0 \\
0 & 0 & -3 & 0 & 0 & 3 & 0 & -6 & 0 & 0 & 3 & 3 \sqrt{3} & 0 & 3 & -5 \sqrt{3}
\end{array}\right] . \tag{27}
\end{align*}
$$

The start and end points $\left(x, y, z, q_{0}, q_{1}, q_{2}, q_{3}\right)$ are set to be $(0 \mathrm{~m}, 0 \mathrm{~m}, 0 \mathrm{~m}, 1,0,0,0)$ and ( $2 \mathrm{~m}, 2 \mathrm{~m}, 2 \mathrm{~m}, 0.5,-0.5,0.5$, -0.5 ), respectively. Figure 8 shows the 3D manipulation path with the time $T=2 \mathrm{~s}$ and the friction coefficient $\mu=0.3$. As we can see, the object moves and rotates to the end point smoothly and accurately. Figures 9-10 depict the trajectories of the variables $x, y, z, q_{0}, q_{1}, q_{2}, q_{3}$ and $v_{x}, v_{y}, v_{z}, \omega_{x}$, $\omega_{y}, \omega_{z}$, respectively. We observe that the trajectories are smooth. The translation speeds are kept within $1.67 \mathrm{~m} / \mathrm{s}$ and the turning speed within $3.77 \mathrm{rad} / \mathrm{s}$. The trajectories of the grasping forces are shown in Figure 11. The simulation results indicate that the grasping forces satisfy the friction constraint. The 3D grasping simulations are also performed with a fourfingered robot and a six-fingered robot, respectively. The four-fingered and six-fingered robots place one finger and three fingers on the top center of the object, respectively, while their other three fingers grasping the bottom. Figures 12-13 show the 3D manipulation paths with the four-fingered and six-fingered robots, respectively. From Figures 8, 12, and 13 , we observe that increasing the number of the robot fingers can reduce manipulation path length and enhance the maneuverability. Consequently, the simulation results show that the proposed scheme can achieve the accurate manipulation for multifingered robots.

## 5. Conclusion

In this paper, we have proposed an effective method for multifingered robot path planning and grasping forces computation. The optimal grasping control problem was formulated with the rigid body dynamics of the object and the second-order cone constraints of grasping forces. The SOC complementarity problem was recast as the equations with the Fischer-Burmeister (FB) function, and the semismooth Newton method with the generalized Jacobian of FB function was used to solve the system equations. The simulation results show that the optimal grasping forces can accurately move the object to a goal, demonstrating the effectiveness of the proposed method.

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# The Extrapolation-Accelerated Multilevel Aggregation Method in PageRank Computation 

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#### Abstract

An accelerated multilevel aggregation method is presented for calculating the stationary probability vector of an irreducible stochastic matrix in PageRank computation, where the vector extrapolation method is its accelerator. We show how to periodically combine the extrapolation method together with the multilevel aggregation method on the finest level for speeding up the PageRank computation. Detailed numerical results are given to illustrate the behavior of this method, and comparisons with the typical methods are also made.


## 1. Introduction

The PageRank algorithm for assigning a rank of importance to web pages has been the key technique in web search [1]. The core of the PageRank computation consists in the principal eigenvector of a stochastic matrix representing the hyperlink structure of the web. Due to the large size of the web graph (over eight billion nodes [2]), computing PageRank is faced with the big challenge of computational resources, both in terms of the space of CPU and RAM required and in terms of the speed of updating in time; that is, as a new crawl is completed, it can be soon available for searching. Among all the numerical methods to compute PageRank, the Power method is one of the standard ways for its stable and reliable performances [1], whereas the low rate of convergence is its fatal flaw. Many accelerating techniques have been proposed to speed up the convergence of the Power method, including aggregation/disaggregation [3-5], vector extrapolation [6$10]$, multilevel [11-14], lumping [15, 16], Arnoldi-type [10, 17], and adaptive methods [4, 18]. To some extent, our work follows in this vein, that is, seeking a method to accelerate the convergence of PageRank computation. In this research, we introduce a multilevel aggregation method to accelerate the computation of PageRank, and the acceleration is performed
by vector extrapolation, which aims at combining the old iteration sequences.

The remainder of this paper is organized as follows. Section 2 provides preliminary and related works on PageRank and some acceleration techniques. Section 3 describes our main acceleration algorithm. Section 4 covers our numerical results and comparisons among the main PageRank algorithms. Section 5 contains conclusion and points out directions for future research.

## 2. Preliminaries and Related Works

2.1. Background on PageRank. PageRank [19] is the heart of software, as Google claims on its webpage, and continues to provide the basis for all the web search tools. Link-based PageRank views the web as a directed graph, where each node represents a page, and each edge from node $i$ to node $j$ represents the existence of a link from page $i$ to page $j$, and at this time, one can view page $j$ as "important." Page and Brin [1] made another basic assumption: the amount of importance distributed to $j$ by $i$ is proportional to the importance of $i$ and inversely proportional to the number of pages $i$ is pointing to. This definition suggests an iterative
fixed-point computation for determining the importance for every page on the web. Each page has a fixed rank score $\pi_{i}$, forming the rank vector $\pi$. Page and Brin convert the computation of PageRank to the computation of stationary distribution vector of the linear system $\pi^{T}=\pi^{T} P$, where $P$ is the adjacency matrix of the web graph $G$ normalized so that each row sums to 1 . For $P$ to be a valid transition probability matrix, every node must have at least 1 outlink; that is, $P$ should have no rows consisting of all zeros (pages without outlinks are called dangling nodes). Meanwhile, to guarantee the uniqueness of a unitary norm eigenvector corresponding to the eigenvalue 1, by the ergodic theorem for Markov chains [20], $P$ should be aperiodic and irreducible. To solve these problems, Page and Brin have introduced a parameter $\alpha$ and transferred matrix $P$ to a new one $P^{\prime}: P^{\prime}=\alpha\left(P+d V^{T}\right)+$ $(1-\alpha) E V^{T}$, where $E$ is the vector with all entries equal to $1, V$ is the column vector representing a uniform probability distribution over all nodes, $V=[1 / n]_{n \times 1}, d=1$, while page $i$ is dangling and $d=0$ otherwise, assuming the total page number is $n$. If viewing a surfer as a random walker on the web, the constant $\alpha$ can be explained below: a web surfer visiting a page will jump to any other page on the web with probability $(1-\alpha)$, rather than following an outlink. Matrix $P^{\prime}$ is usually called the Google matrix. Then, PageRank vector $\pi$ is the stationary distribution vector of $P^{\prime}$; that is,

$$
\begin{equation*}
\pi^{T}=\pi^{T} P^{\prime} \tag{1}
\end{equation*}
$$

obviously, (1) can be rewritten as

$$
\begin{equation*}
\pi^{T} A=0 \tag{2}
\end{equation*}
$$

where $A=I-P^{\prime}, I$ is an identity matrix, and $A$ is a singular $M$-matrix with the diagonal elements are negative column sums of its off-diagonal elements. So what remains to do is to solve the linear system (1) or its homogeneous one (2), corresponding to an irreducible Markov chain.
2.2. The Vector Extrapolation-Acceleration of PageRank. The Power method is the standard algorithm for PageRank, that is, giving the initial uniform distribution $x^{(0)}$ of the system (1), to compute successive iterates $x^{(k)}=x^{(k-1)} P^{\prime}$, until convergence, that is, when $\lim _{k \rightarrow \infty} x^{(k)}$ exists, which is exactly the PageRank vector. As mentioned in Section 1, although the Power method is a stable and reliable [1] or even a fast iteration algorithm [21], accelerating the computation is still important, since every search engine crawls a huge number of pages, and each matrix multiplication in iteration is so expensive, requiring considerable resources both in terms of CPU and RAM, and hence, the rate of convergence deteriorates as the number of pages grows larger. Now there are many acceleration algorithms for PageRank computation, and among all the algorithms the vector extrapolation method is a very popular and efficient one. A detailed review of the vector extrapolation method can be found in the work of Kamvar et al., Smith et al., and Sidi [9, 22, 23]. To our knowledge, there are several kinds of extrapolation methods, such as quadratic extrapolation [9], two polynomial-type
methods including the minimal polynomial extrapolation method (MPE) of Cabay and Jackson [24] and the reduced rank extrapolation (RRE) of Eddy [25], and the three epsilon vector extrapolation methods of Wynn [26]. In recent years, many papers have discussed the application of vector extrapolation method to compute the stationary probability vector of Markov chains and web ranking problems, see [6, 8, $9,22,23,27]$ for details. Numerical experiments in these papers suggest that the polynomial-type methods are in general more economical than the epsilon vector extrapolation methods in the sense of computation time and storage requirements. For this reason, our concern in the context is to consider a polynomial-type vector extrapolation, that is, the generalization of quadratic extrapolation method (GQE) proposed in [23]. Now let us discuss simply the strategy of GQE. As it is known, the starting point of the extrapolation method is to accelerate the convergence of the sequences $\left\{x_{j}\right\}$ generated from a fixed-point iterative method in Markov chain of the form

$$
\begin{equation*}
x_{j+1}=F\left(x_{j}\right), \quad j=0,1, \ldots ; F: \mathbb{R}^{n} \longrightarrow \mathbb{R}^{n} \tag{3}
\end{equation*}
$$

where $x_{0}$ is an initial guess. Supposve that we have produced a sequence of iterates $\left\{x_{i}\right\}_{i=1}^{\infty}$, where $x_{i} \geq 0$. Then, at the $k$ th outer iteration, let

$$
\begin{equation*}
X=\left[x_{k}, x_{k-1}, \ldots, x_{k-m+1}\right] \in \mathbb{R}^{n \times m} \tag{4}
\end{equation*}
$$

be a matrix consisting of the last $m$ iterates with $x_{k}$ being the newest, where $m$ is called the window size as usual. It is evident that $X$ has the following properties:

$$
\begin{equation*}
x_{i} \geq 0, \quad\left\|x_{i}\right\|_{1}=1, \quad i=1,2, \ldots \tag{5}
\end{equation*}
$$

The problem to be solved is transformed into obtaining a vector $z$ satisfying $\sum_{i=1}^{m} z_{i}=1$, and, thus we have an updated probability vector

$$
\begin{equation*}
\widehat{x}_{k}=X z=z_{1} x_{k}+z_{2} x_{k-1}+\cdots+z_{m} x_{k-m+1} \tag{6}
\end{equation*}
$$

a linear combination of the last $m$ iterates. That is to say, the current iterate can be expressed as a linear combination of some of the first eigenvectors, combined with the Power method, up to the converge of the principal eigenvector. GQE is derived in this light and can be given as follows [23].

Algorithm 1 (the generalization of quadratic extrapolation (GQE)). (1) Input the vectors $x_{0}, x_{1}, \ldots, x_{k+1}$.
(2) Compute $u_{i}=x_{i+1}-x_{0}, i=0,1, \ldots, k$, set $U_{k}=$ [ $u_{0}, u_{1}, \ldots, u_{k}$ ]. Compute the QR-factorization of $U_{k}$, namely, $U_{k}=Q_{k} R_{k}$. Obtain $R_{k-1}:=R_{k}(1: k, 1: k), Q_{k-1}=Q_{k}(1:$ $k, 1: k)$.
(3) Solve the linear system $R_{k-1} d=-Q_{k-1}^{T} u_{k}, d=$ $\left[d_{0}, d_{1}, \ldots, d_{k-1}\right]^{T}$.
(4) Set $d_{k}=1$ and compute $c=\left[c_{0}, c_{1}, \ldots, c_{k}\right]^{T}$ by $c_{i}=$ $\sum_{j=i}^{k} d_{j}, i=0,1, \ldots, k$.
(5) Compute $\widehat{x}_{k+1}=\left(\sum_{i=0}^{k} c_{i}\right) x_{0}+Q_{k}\left(R_{k} c\right)$.

## 3. Accelerated Aggregation Strategy in PageRank Computation

The core of PageRank computation, as discussed earlier, is to solve the large sparse Markov chain problem (2). Thus, it is evocative of using a especially useful method-multilevel method based on aggregation of Markov states, which has attracted considerable attention [11-14, 28, 29]. Isensee and Horton considered a kind of multilevel methods for the steady state solution of continuous-time and discrete-time Markov chains in [13, 14], respectively. De Sterck et al. proposed a multilevel adaptive aggregation method for calculating the stationary probability vector of Markov matrices in [11], as shown in their context, which is a special case of the adaptive smoothed aggregation [30] and adaptive algebraic multigrid methods [31] for sparse linear systems.

The central idea of multilevel aggregation method is to convert a large linear system to a smaller one by some aggregation strategies, and thus, the stationary state solution can be obtained in an efficient way.

However, aggregation/disaggregation cannot always dissolve the algorithmic scalability due to poor approximation of $A$ in problem (2) by unsmoothed intergrid operators, so a careful choice of aggregation strategy is crucial for the efficiency of the multilevel aggregation hierarchies. Now there are many aggregation methods, such as graph aggregation [32], neighborhood-based aggregation, and (double) pairwise aggregation $[4,13,14]$. In our study, we consider the neighborhood aggregation method as described in [1214], since it is able to result in well-balanced aggregates of approximately equal size and provide a more regular coarsening throughout the automatic coarsening process [12, $29,33]$. Our aggregation strategies are based on the problem matrix by the current iterate $\bar{A}=A \times \operatorname{diag}\left(x_{i}\right)$, rather than the original coefficient matrix $A$. Let $x_{k}$ denote the current iterate; then, we say node $i$ is strongly connected to node $j$ in the graph of $\bar{A}$ if

$$
\begin{equation*}
-\bar{a}_{i j} \geq \theta_{k \neq i}\left\{-\bar{a}_{i k}\right\} \quad \text { or } \quad-\bar{a}_{j i} \geq \theta_{k \neq j}\left\{-\bar{a}_{j k}\right\}, \tag{7}
\end{equation*}
$$

where $\theta$ is a strength of connection parameter and $\theta=0.25$ is used there. Suppose that $\mathcal{N}_{i}$ is the set of all points which are strongly connected to $i$ in the graph of $\bar{A}$ including node $i$ itself. Then, we have the neighborhood-based algorithm as follows (see [12, 29] for more details).

Algorithm 2 (neighborhood-based aggregation, $\left\{Q_{j}\right\}_{j=1}^{J} \leftarrow$ $\operatorname{NBA}(\bar{A}, \theta))$. (1) Preparing: set $R=\{1, \ldots, n\}$ and $J=0$.
(2) Assign entire neighborhoods to aggregates: for $i \in$ $\{1, \ldots, n\}$, build strong neighborhoods $\mathscr{N}_{i}$ based on (*); if $\mathcal{N}_{i} \subset R$, then $J \leftarrow J+1, \mathrm{Q}_{J} \leftarrow \mathcal{N}_{i}, \widetilde{\mathrm{Q}}_{J} \leftarrow \mathcal{N}_{i}, R \leftarrow R \backslash \mathcal{N}_{i}$.
(3) Put the remaining points in the most connected aggregates: while $R \neq \emptyset$, pick $i \in R$ and set $j=$ $\operatorname{argmax}_{k=1, \ldots, J} \operatorname{card}\left(N_{i} \cap \widetilde{Q}_{k}\right)$, set $Q_{j} \leftarrow Q_{j} \cup\{i\}$ and $R \leftarrow$ $R \backslash\{i\}$.
(4) Obtain the aggregation matrix $Q$ : if $i \in Q_{j}, j=$ $1, \ldots, J$, then $Q_{i j}=1$, otherwise $Q_{i j}=0$.

Note that the aggregation matrix $Q$ in Algorithm 2 has the following form

$$
Q=\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & \ldots  \tag{8}\\
1 & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & \ldots \\
0 & 0 & 1 & 0 & \ldots \\
0 & 0 & 0 & 1 & \ldots \\
0 & 0 & 0 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]_{n \times J}
$$

From (8), we find that there exists only one element $Q_{i j}=$ 1 in each row, but each column may have several elements $Q_{i j}=1$, and the sum of the elements in the $j$ th column denotes the number of the nodes combined into the $j$ th aggregation.

The multilevel aggregation method, based on the neighborhood-based aggregation strategy, introduced by De Sterck et al. in [11, 12, 28, 29] can be expressed as follows.

Algorithm 3 (multilevel aggregation method, $\left.x \leftarrow \operatorname{MA}\left(A, x, v_{1}, v_{2}, \alpha\right)\right)$. (1) Presmoothing: do $v_{1}$ times $x \leftarrow N(\operatorname{Relax}(A, x))$.
(2) Build $Q$ according to the automatic coarsening process described below. Obtain $R \leftarrow Q^{T}$ and $P \leftarrow \operatorname{diag}(x) Q$.
(3) Form the coarse-level matrix $A_{c} \leftarrow$ RAP and compute $x_{c} \leftarrow Q^{T} x$.
(4) If on the coarsest level, solve $A_{c} x_{c}=0$ by a direct method. Otherwise, apply $\alpha$ iterations of this algorithm $x_{c} \leftarrow$ $\operatorname{MA}\left(A_{c} \operatorname{diag}\left(Q^{T} x\right)^{-1}, x_{c}, \nu_{1}, v_{2}, \alpha\right)$.
(5) Coarse-level correction: $x \leftarrow P\left(\operatorname{diag}\left(Q^{T} x\right)^{-1}\right) x_{c}$.
(6) Postsmoothing: do $v_{2}$ times $x \leftarrow N(\operatorname{Relax}(A, x))$.

In Algorithm 3, $A=A_{1}$ is given in system (2), $x$ is an initial guess vector, $A_{c}$ denotes the coarse-level matrix, and $x_{c}$ is the corresponding coarse-level vector, where $c=$ $1,2, \ldots, L$ is the number of levels, the finest level is 1 , and the coarsest level is $L . Q$ is the aggregation matrix generated by the aggregation method-Algorithm 2 , and $P_{l}$ and $R_{l}, l=$ $1,2, \ldots, L-1$ are the prolongation and restriction operators, respectively, which are created by an automatic coarsening process in step 2, and then, we get the coarse-level matrices by the equation $A_{l}=R_{l} \times A_{l} \times P_{l}$. Setting the weighted Jacobi method with weight $\omega$, a variant of Jacobi method, as the pre- and postsmoothing approaches and letting $\nu_{1}$ and $\nu_{2}$ be their iteration times, respectively, the matrix $A$ in system (2) is splitted into the following form

$$
\begin{equation*}
A=D-L-U \tag{9}
\end{equation*}
$$

where $D$ is the diagonal part of the matrix $A$ with $d_{i i}>0$ for all $i$ and $L$ and $U$ are the negated strictly lower- and uppertriangular parts of $A$, respectively. Then, the weighted Jacobi relaxation method can be written as

$$
\begin{equation*}
x \longleftarrow N\left((1-\omega) x+\omega D^{-1}(L+U) x\right) \tag{10}
\end{equation*}
$$

with weight $\omega \in(0,1)$. Here, we use $N(\cdot)$ to denote the normalization operator defined by $N(x):=x /\|x\|_{1}$ for all
$x \neq 0$. Note that we have carried out the normalization after each relaxation process in Algorithm 3 to ensure that the finelevel iterates $x_{i}$ can be interpreted as approximations to the stationary probability vector.

It should be noted that the direct method on the coarsest level, used at step 4 of Algorithm 3, is based on the following theorem.

Theorem 4 (see [34, Theorem 4.16]). If $A$ is an irreducible singular M-matrix, then each of its principal submatrices other than A itself is a nonsingular M-matrix.

If $A_{L}$ is the coarsest-level operator, then we use the direct method presented in the coarsest-level algorithm below to solve the coarsest-level equation $A_{L} x_{L}=0$.

## Coarsest-Level Algorithm

Step 1. Compute $N_{L}:=\operatorname{size}\left(A_{L}, 1\right) ; \%$ the size of coarsest-level operator $A_{L}$.
Step 2. Obtain $A_{L p}:=A_{L}\left(1: N_{L}-1,1: N_{L}-1\right) ; \%$ the $N_{L}-1$ order principal submatrix of $A_{L}$.
Step 3. Obtain $b_{L p}:=-A_{L}\left(1: N_{L}-1, N_{L}\right) ; \%$ the right-hand vector corresponding to $A_{L p}$.
Step 4. Compute $x_{L p}:=A_{L p} \backslash b_{L p} ;$ let $x_{L p}\left(N_{L}\right)=1$; and obtain the coarsest-level solution $x_{L}=x_{L p} /\left\|x_{L p}\right\|_{1}$.

Now, let us introduce the main idea of this paper and the implementation details of the main algorithm. In fact, our idea is derived from the excellent papers [12, 28, 29]. In the literature, De Sterck et al. studied several strategies to accelerate the convergence of the multilevel aggregation method, including the application of a smoothing technique to the interpolation and restriction operators in [28] and the analysis of a recursively accelerated multilevel aggregation method by computing quadratic programming problems with inequality constraints in [29]. Of particular note, in [12], they introduced a top-level acceleration of adaptive algebraic multilevel method by selecting a linear combination of old iterates to minimize a function over a subset of the set of probability vectors. Their acceleration strategy was involved mainly in cases when the window size $m=2$, and for larger window size $m$, such as $m=3$ or 4 , they used the active set method from Matlab's quadprog function [35]. With this in mind, enough interest is aroused to look for a general acceleration method for any case of window size. In view of the effectiveness of vector extrapolation in improving the convergence of the iterate sequence, now consider a new accelerated multilevel aggregation method, combining the multilevel aggregation method and the vector extrapolation method. Giving the finest level iterate sequence $\left\{x_{n}\right\}$ produced by Algorithm 3 in multilevel aggregation, the updated iterate $\widehat{x}_{k}$ is generated as their linear combination by Algorithm 1. And so we can present our accelerated multilevel aggregation as follows.

Algorithm 5 (extrapolation-accelerated multilevel aggregation methods, $\left.x \leftarrow \operatorname{EAMA}\left(A, \widehat{x}_{0}, m, \epsilon\right)\right)$. (1) Set $k=1$, if no initial guess is given, choose $\widehat{x}_{0}$.
(2) Run the multilevel aggregation method, $x_{k} \leftarrow$ $\operatorname{MA}\left(A, \widehat{x}_{k-1}, \nu_{1}, v_{2}, \alpha\right)$.
(3) Set $m \leftarrow \min \{M, k\}$. $\%$ set the window size.
(4) Set $X \leftarrow\left[x_{k}, x_{k-1}, \ldots, x_{k-m-1}\right] . \%$ the last $m+2$ iterates.
(5) Apply GQE algorithms to obtain $\widehat{x}_{k}$ from $X$, respectively.
(6) If $\left\|A \widehat{x}_{k}\right\|_{1}>\left\|A x_{k}\right\|_{1}$, then $\widehat{x}_{k} \leftarrow x_{k}$.
(7) Check convergence, if $\left\|A \widehat{x}_{k}\right\|_{1} /\left\|A \widehat{x}_{0}\right\|_{1}<\epsilon$, then $\widehat{x}_{k} \leftarrow$ $\widehat{x}_{k} /\left\|\widehat{x}_{k}\right\|_{1}$, otherwise set $k \leftarrow k+1$ and go to step 2 .

Note that $m$ in Algorithm 5 is the window size and $\epsilon$ is a prescribed tolerance. In particular, there exists a difference in constructing the matrix $X$ between our Algorithm 5 and the algorithm of [36]. From Algorithm 1, the main reason is that the GQE needs $k+2$ vectors $x_{0}, x_{1}, \ldots, x_{k+1}$ as their input. That is to say, when the window size is $m=2$, four approximate probability vectors given in the matrix $X=$ $\left[x_{0}, x_{1}, x_{2}, x_{3}\right] \in \mathbb{R}^{n \times 4}$ are required as its input of GQE algorithms. In particular, the window size $m=2$ corresponds to the quadratic vector extrapolation method presented in [9]. Hence, the matrix $X$ is given as the form of step 4 in Algorithm 5. The efficiency of our accelerated multilevel aggregation algorithms will be shown in the next section, where the relative advantage of our method over Matlab's quadprog function in the Markov chain and over other methods in PageRank computation will be demonstrated.

## 4. Numerical Results and Analysis

In this section, we report some numerical experiments to show the numerical behavior of extrapolation-accelerated multilevel aggregation methods. All the numerical results are obtained with a Matlab 7.0.1 implementation on a 2.93 GHz processor with 2 GB main memory.

For the sake of justice, the initial guess vector is selected as $x_{0}=e /\|e\|_{1}$ for all the algorithms. All the iterations are terminated when the residual 1-norm

$$
\begin{equation*}
\frac{\left\|A x_{k}\right\|_{1}}{\left\|A x_{0}\right\|_{1}} \leq \text { tol } \tag{11}
\end{equation*}
$$

where $x_{k}$ is the current approximate solution and tol is a user described tolerance. For convenience, in all the previous tables we have abbreviated the Power method, the generation of quadratic extrapolation method, and the extrapolationaccelerated multilevel aggregation method as Power, GQE, and EAMA, respectively. We use "res" to denote the 1-norm residual, "lev" to the number of levels in the multilevel aggregation, "ite" to the number of iterations, and "CPU" to the CPU time used in seconds.

Example 1. As described in Section 3, the active set method from Matlab's quadprog function [12,35] is usually used for the acceleration strategy in Markov chains and other problems. And in this paper, we have suggested the extrapolationacceleration strategy instead of the Matlab's one. This example aims to compare the two methods and to test the efficiency of our new method. The test example is a birth-death chain

TAbLE 1: Numerical comparison results for EAMA and Q-matlab for Example 1.

| Method |  | Window size |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | 2 |  |  | 3 |  |  | 4 |  |  |
| $n$ | lev | $C_{o p}$ | res | it | $\mathrm{C}_{\text {op }}$ | res | it | $\mathrm{C}_{\text {op }}$ | res | it |
| 27 | 2 | 2.0127 | 7.4095 - 6 | 10 | 2.0127 | $1.9435 e-6$ | 10 | 2.0127 | $3.9440 e-6$ | 9 |
| 81 | 3 | 3.0166 | $7.0117 e-6$ | 14 | 3.0166 | 6.1956 - 6 | 15 | 3.0166 | $9.0883 e-6$ | 13 |
| 243 | 4 | 4.5433 | $5.4196 e-6$ | 22 | 4.0481 | $7.4238 e-6$ | 20 | 4.0481 | $7.3145 e-6$ | 16 |
| 1024 | 5 | 4.9831 | $1.2408 e-6$ | 48 | 5.5179 | $6.2320 e-6$ | 40 | 4.9831 | $8.4259 e-6$ | 31 |
| Q-matlab |  | 2 |  |  | 3 |  |  | 4 |  |  |
| $n$ | lev | $\mathrm{C}_{\text {op }}$ | res | it | $\mathrm{C}_{\text {op }}$ | res | it | $\mathrm{C}_{\text {op }}$ | res | it |
| 27 | 2 | 2.0127 | $5.3065 e-6$ | 10 | 2.0127 | $3.8436 e-6$ | 8 | 2.0127 | $6.3351 e-6$ | 7 |
| 81 | 3 | 3.0166 | $7.4583 e-6$ | 15 | 3.0166 | $8.0007 e$ - 6 | 14 | 3.0166 | $6.4303 e-6$ | 13 |
| 243 | 4 | 4.0481 | $7.5300 e-6$ | 22 | 4.0481 | $9.1973 e-6$ | 22 | 4.0481 | 8.9225 - 6 | 18 |
| 1024 | 5 | 5.8769 | $3.9990 e-6$ | 77 | 4.9831 | $9.4107 e-6$ | 50 | 4.9987 | $8.6577 e-6$ | 48 |



Figure 1: Graph of an M/PH/1 queue.
with invariant birth and death rates as shown in Figure 1 [3739], which is usually used in queuing theory, demographics, performance engineering, or in biology [32, 40].

Setting $\mu=0.96$, relaxation parameter $\omega=0.7$, and the strength parameter of connection $\theta=0.25$ in our experiment, we use $Q$-matlab to denote the Matlab's quadprog function acceleration method that improves the $W$-cycles on the finest level and " $n$ " to denote the length of Markov chains. " $C_{o p}$ " denotes the operator complexity of the last cycle, which is defined as the sum of the number of nonzero entries in all operators on all levels $A_{l}, l=1, \ldots, L$, divided by the number of nonzero entries in the finest-level operator $A_{1}$. That is, $C_{o p}=\sum_{l=1}^{L} \mathrm{nnz}\left(A_{l}\right) / \mathrm{nnz}\left(A_{1}\right)$ where $\mathrm{nnz}(A)$ is the number of nonzero entries in the matrix $A$. Numerical results for EAMA and Q-matlab methods have been reported in Table 1.

From Table 1, it is evident that our accelerated multilevel aggregation method EAMA has better performance than Matlab's function Q-matlab for the testing problem from both an operator complexity and the iteration count perspective, and moreover, our method is more efficient with the length of Markov chain increasing. For instance, when $n=1024$ and the window size $m$ : $m=2,3,4$, the EAMA method cuts the number of iterations by up to $37.7 \%, 20 \%$, and $35.4 \%$, respectively, compared to the $Q$-matlab method.

Example 2. In this example, we consider the performance of EAMA in the PageRank computation. We take a typical website of "Hollions" as our numerical example, which has been listed in Chapter 14 of [41], which contains 6012 web pages and 23875 hyperlinks. We compare the Power method, the generalization of quadratic extrapolation method, and the extrapolation-accelerated multilevel aggregation method for the PageRank problem.

In Algorithm 5, we consider that accelerators, $W$-cycles $(r=2)$, and $V$-cycles can be treated in a similar way.

Table 2: Comparisons of the residual vectors for the Power, GQE, and EAMA methods when given the number of iterations.

| Method | ite $=5$ | ite $=10$ | ite $=20$ | ite $=30$ |
| :--- | :---: | :---: | :---: | :---: |
| Power | 0.148848 | 0.02414 | 0.002329 | $3.6687 e-004$ |
| GQE | 0.029646 | 0.00210 | $4.0479 e-005$ | $1.2882 e-007$ |
| EAMA | 0.003400 | $1.4285 e-004$ | $2.0306 e-007$ | $4.7062 e-010$ |

Some specific sets of parameters are used. We use the weighted Jacobi as the pre- and postsmoothing approaches in Algorithm 2, with relaxation parameter $\omega=0.7$. Direct coarsest-level solvers are performed by the coarsest-level algorithm, and the strength parameter of connection is chosen as $\theta=0.25$. There are some numerical results reported in Tables 2 and 3 in the following parts.

The residual analysis in Table 2 indicates that EAMA does much better in PageRank computation than the other two methods, namely, the classic Power method and GQE, and EAMA has a more obvious advantage over the other two methds, with the iteration count increasing.

From Table 3, when an error tolerance is provided and no matter what value the window size is, the accelerated multilevel aggregation method outperforms the Power and GQE, in terms of both iteration count and CPU time. For instance, when the window size is 3 , the number of iterations by EAMA is about half of the one by GQE and less than one third of the one by Power.

Example 3. The test matrix is the widely used CS-Stanford Web matrix, which is available from http://www.cise.ufl.edu/ research/sparse/matrices/index.html. It contains 9914 nodes and 35,555 links. In this example, we run the Power method, the generalized quadratic extrapolation method, and our extrapolation-accelerated multilevel aggregation method. The convergence tolerance is tol $=10^{-5}$, and the window size is selected as $m=3$. Table 4 lists the results.

It is seen from Table 4 that the numerical behavior of the three algorithms strongly relies on the choice of the damping factor $\alpha$ in PageRank computation. When $\alpha$ is high, say 0.95 and 0.99 , the computing time and iteration count are sizable. However, when $\alpha$ is moderate, say 0.85 , the computing time

Table 3: Comparisons of the iteration counts and CPU for the Power, GQE, and EAMA methods when the residual vector is $10^{-5}$.

| Window size | Method |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Power | GQE |  |  | EAMA |  |  |
|  |  | 2 | 3 | 4 | 2 | 3 | 4 |
| ite | 50 | 26 | 24 | 26 | 18 | 14 | 15 |
| CPU | 34.25 | 28.16 | 28.67 | 27.57 | 23.13 | 22.63 | 22.75 |

Table 4: Numerical results of the three algorithms (with various $\alpha$ ) on the CS-Stanford Web matrix.

|  | Power | Method <br> GQE | EAMA |
| :---: | :---: | :---: | :---: |
| $\alpha=0.85$ |  |  |  |
| ite | 273 | 147 | 98 |
| CPU | 39.78 | 36.61 | 25.31 |
| res | $9.9972 e-006$ | $6.1911 e-006$ | $3.2462 e-006$ |
| $\alpha=0.90$ |  |  |  |
| ite | 379 | 214 | 126 |
| CPU | 54.13 | 45.31 | 38.42 |
| res | $9.9718 e-006$ | $6.3426 e-006$ | $6.5672 e-006$ |
| $\alpha=0.95$ |  |  |  |
| ite | 629 | 425 | 253 |
| CPU | 90.08 | 68.33 | 52.41 |
| res | $9.9834 e-006$ | $8.32671 e-006$ | $7.4938 e-006$ |
| $\alpha=0.99$ |  |  |  |
| ite | 1920 | 734 | 527 |
| CPU | 263.54 | 198.32 | 139.42 |
| res | $9.9940 e-006$ | $7.1028 e-006$ | $7.3542 e-006$ |

Table 5: Numerical comparison results of the three algorithms (with various convergence tolerances) for Example 4.

| Method | Power |  | GQE |  | EAMA |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| tol | ite | CPU | ite | CPU | ite | CPU |
| $10^{-5}$ | 284 | 68.76 | 256 | 57.33 | 227 | 48.32 |
| $10^{-6}$ | 316 | 97.55 | 283 | 86.42 | 248 | 80.21 |
| $10^{-8}$ | 392 | 145.72 | 334 | 121.75 | 291 | 106.43 |
| $10^{-10}$ | 514 | 234.61 | 438 | 178.92 | 387 | 152.49 |

and iteration count are greatly reduced, like, about oneseventh of the case $\alpha=0.99$. This just confirms Page and Brin's point [1]: $\alpha=0.85$ is the most common choice.

Besides, just as expected, it is obvious to see that the GQE method is superior to the Power method, while the new multilevel aggregation method (EAMA) performs the best, in both computing time and iteration count terms. For instance, when $\alpha$ is 0.85 , comparing with the results by Power and GQE, the CPU time for EAMA has been reduced by $36.4 \%$ and $30.9 \%$, respectively, and the iteration count has been reduced by $64.1 \%$ and $33.3 \%$, respectively.

Example 4. This example aims to examine the influence of the choice of the convergence tolerance on the algorithms. The test matrix is the Stanford-Berkeley Web matrix (available from http://www.stanford.edu/~sdkamvar/research.html). It
contains 683,446 pages and 7.6 million links. We run the Power method, the generalized quadratic extrapolation method, and our extrapolation-accelerated multilevel aggregation method on this problem and choose tol to be $10^{-5}$, $10^{-6}, 10^{-8}$, and $10^{-10}$, respectively. The window size for the extrapolation procedure is selected as $m=3$, and $\alpha$ in PageRank is set as 0.85 . Table 5 reports the results obtained.

It is easy to see from Table 5 that our new algorithm, EAMA, performs the best in most cases, regardless of the convergence tolerance and both in terms of computing time and iteration numbers. For instance, when tol varies from $10^{-6}$ to $10^{-8}$, the iteration count needed for the three algorithms (power, GQE, and EAMA) increases $24.1 \%, 18.0 \%$, and $17.3 \%$, respectively, but nonetheless, the count for EAMA is still less than the corresponding one for GQE and far less than that for the Power, only about $74.2 \%$ of the latter. It can be seen that about the same goes for CPU time.

## 5. Conclusions

This paper illustrates the accelerated multilevel aggregation method for calculating the stationary probability vector of an irreducible stochastic matrix, Google matrix, in PageRank computation. It is conducted to combine the vector extrapolation method and the multilevel aggregation method, where the neighborhood-based aggregation strategy $[11,12,28,29]$ is used, and the vector extrapolation acceleration is carried out on the finest level. Our approach is inspired by De Sterck et al. in [12] where the active set method from Matlab's quadprog function was used therein for window size $m$, which is greater than two. Thus, it is natural to seek for a general acceleration method for any case of window size, and thus, our EAMA is produced.

Although our method is effective, however, there are still many places worth exploring. The Google matrix, for example, can be reordered according to dangling nodes and nondangling nodes of the matrix [16, 42-44], which reduces the computation of PageRank to that of solving a much smaller problem. And for the special linear system (2) in Markov chains, there maybe other even better options than the neighborhood-based aggregation strategy in multilevel method. With some improvement, our algorithm will be worth watching and will be a part of future work.

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