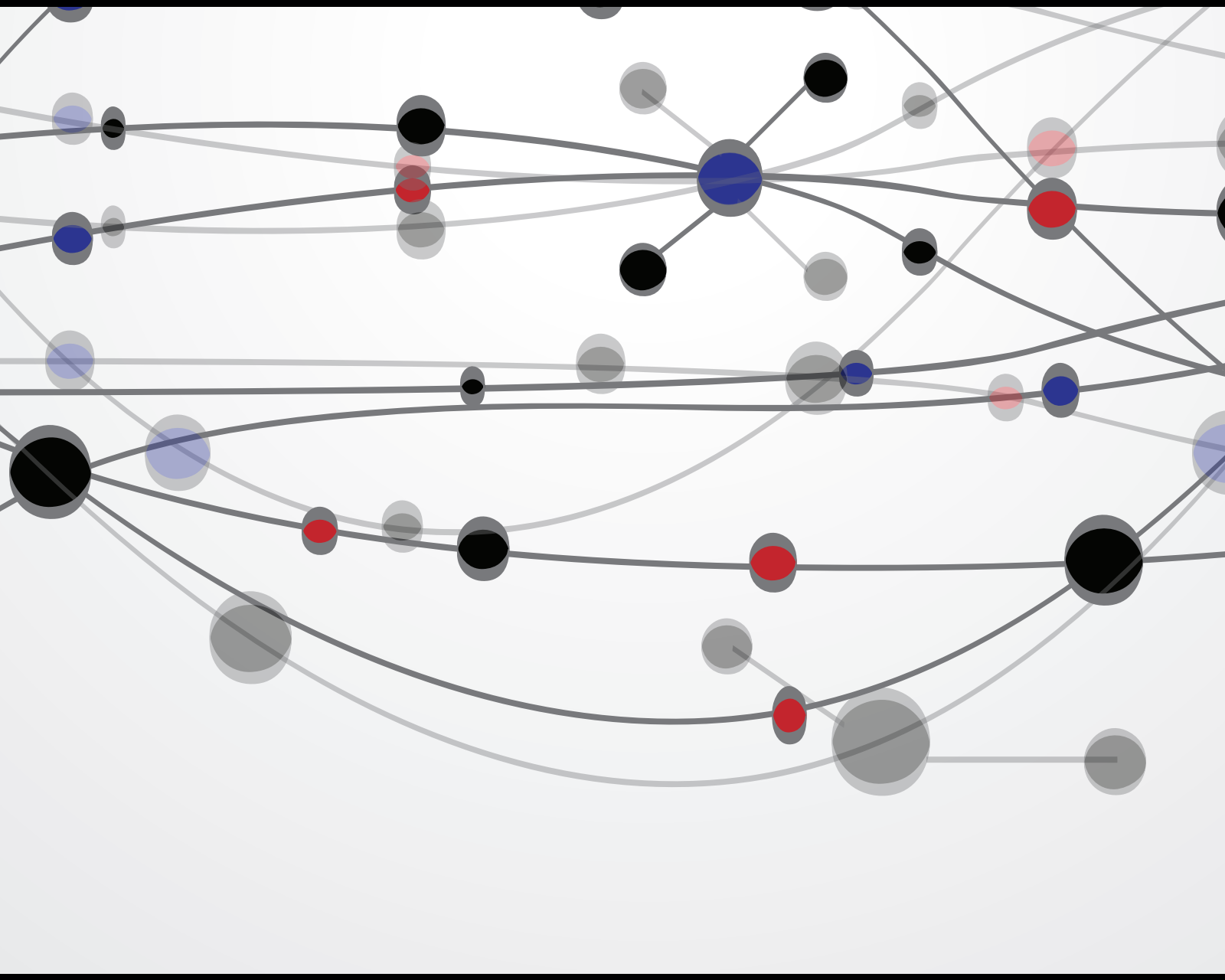
The Scientific World Journal

Recent Developments on Sequence Spaces and Compact Operators with Applications

Guest Editors: S. A. Mohiuddine, M. Mursaleen, Józef Banas, Suthep Suantai, and Abdullah Alotaibi



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Editorial

Recent Developments on Sequence Spaces and Compact Operators with Applications

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The aim of this special issue is to focus on recent developments and achievements in the theory of function spaces, sequences spaces and their geometry, and compact operators and their applications in various fields of applied mathematics, engineering, and other sciences. The theory of sequence spaces is powerful tool for obtaining positive results concerning Schauder basis and plays a fundamental role in creating the basis of several investigations conducted in nonlinear analysis. The compactness is very often used in fixed point theory and its applications to the theories of differential, functional differential, integral, and integrodifferential equations.

The research papers in this special issue cover various topics like geometry of Banach spaces, Schauder basis and dual spaces, sequence spaces and their topological and geometric properties, approximation of positive linear operators by matrix and nonmatrix methods, convergence and stability results for iterative process, applications of iterative methods to a nonlinear integral equation, Littlewood-Paley operators on Morrey spaces, matrix transformations between certain sequence spaces, measures of noncompactness and their applications in characterizing compact matrix operators, fixed point theorems and their applications, and so on.

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Research Article

A General Class of Derivative Free Optimal Root Finding Methods Based on Rational Interpolation

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We construct a new general class of derivative free n-point iterative methods of optimal order of convergence 2^{n-1} using rational interpolant. The special cases of this class are obtained. These methods do not need Newton's iterate in the first step of their iterative schemes. Numerical computations are presented to show that the new methods are efficient and can be seen as better alternates.

1. Introduction

The problem of root finding has been addressed extensively in the last few decades. In 1685, the first scheme to find the roots of nonlinear equations was published by John Wallis. Its simplified description was published in 1690 by Joseph Raphson and was called Newton-Raphson method. In 1740, Thomas Simpson was the first to introduce Newton's method as an iterative method for solving nonlinear equations. The method is quadratically convergent but it may not converge to real root if the initial guess does not lie in the vicinity of root and $f'(x)$ is zero in the neighborhood of the real root. This method is a without-memory method. Later, many derivative free methods were defined, for example, Secant's and Steffensen's methods. However, most of the derivative free iterative methods are with-memory methods; that is, they require old and new information to calculate the next approximation. Inspite of the drawbacks of Newton's method, many multipoint methods for finding simple root of nonlinear equations have been developed in the recent past using Newton's method as the first step. However, many higher order convergent derivative free iterative methods have also appeared most recently by taking Steffensen type methods at the first step. A large number of optimal higher order convergent iterative methods have been investigated recently up to order sixteen [1–3]. These methods used different interpolating techniques for approximating the first derivative.

In the era of 1960–1965, many authors used rational function approximation for finding the root of nonlinear equation, for example, Tornheim [4], Jarratt, and Nudds [5].

In 1967, Jarratt [6] effectively used rational interpolation of the form

$$y = \frac{x - a}{bx^2 + cx + d}, \tag{1}$$

to approximate $f(x)$ for constructing a with-memory scheme involving first derivative. The order of the scheme was 2.732 and its efficiency was 0.2182.

In 1987, Cuyt and Wuytack [7] described a with-memory iterative method involving first derivative based on rational interpolation and they also discussed two special cases of their scheme having order 1.84 with efficiency 0.1324 and order 2.41 with efficiency 0.1910.

In 1990, Field [8] used rational function to approximate the root of a nonlinear equation as follows:

$$x_{i+1} = x_i + d_i, \tag{2}$$

where d_i, the correction in each iterate, is the root of the numerator of the Pade approximant to the Taylor series:

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x_i)}{j!}(x - x_i)^j. \tag{3}$$

He proved that x_i converges to the root ξ with order $m + n + 1$, where m and n are degrees of the denominator and numerator of the Pade approximant.

In 1974, Kung and Traub [9] conjectured that a multipoint iterative scheme without memory for finding simple root of nonlinear equations requiring n functional evaluations for one complete cycle can have maximum order of convergence 2^{n-1} with maximal efficiency index $2^{(n-1)/n}$. Multipoint methods with this property are usually called optimal methods. Several researchers [1–3, 10] developed optimal multipoint iterative methods based on this hypothesis.

In 2011, Soleymani and Sharifi [10] developed a four-step without-memory fourteenth order convergent iterative method involving first derivative having an efficiency index of 1.6952. The first derivative at the fourth step is approximated using rational interpolation as follows:

$$w_n = \phi_8(x_n, y_n, z_n)$$

$$x_{n+1} = w_n - \frac{(1 + b_4(w_n - x_n))^2}{f'(x_n) + b_3(w_n - x_n)(2 + b_4(w_n - x_n))} f(w_n), \tag{4}$$

where ϕ_8 is an optimal eighth order convergent method and the rational interpolant is given as:

$$m(t) = \frac{b + b_2(t - x) + b_3(t - x)^2}{1 + b_4(t - x)}. \tag{5}$$

In 2012, Soleymani et al. [3] developed a three-step derivative free eighth order method using rational interpolation as follows:

$$z_n = \phi_4(x_n, y_n, w_n),$$

$$x_{n+1} = z_n - \frac{(1 + a_3(z_n - x_n))^2}{a_1 - a_0a_3 + 2a_2(z_n - x_n) + a_2a_3(z_n - x_n)^2} f(z_n), \tag{6}$$

where ϕ_4 is any two-step fourth order convergent derivative free iterative method. They used the same rational interpolant as given in (5). The constants are determined using the interpolating conditions.

In 2012, Soleymani et al. [2] added his contribution by developing a sixteenth order four-point scheme using Pade approximation. The scheme required four evaluations of functions and one evaluation of first derivative and achieved optimal order of sixteen and and an efficiency index 1.741. The scheme was of the form

$$w_n = \phi_8(x_n, y_n, z_n),$$

$$x_{n+1} = w_n - \left((1 + b_5(w_n - x_n))^2 f(w_n)\right) \times \left(f'(x_n) + 2b_3(w_n - x_n) + (3b_4 + b_3b_5)(w_n - x_n)^2 + 2b_4b_5(w_n - x_n)^3\right)^{-1}, \tag{7}$$

where ϕ_8 is eighth order optimal method. They used the rational interpolant of the following form:

$$p(t) = \frac{b_1 + b_2(t - x) + b_3(t - x)^2 + b_4(t - x)^3}{1 + b_5(t - x)}. \tag{8}$$

Recently in 2013, Sharma et al. [1] developed a three-step eighth order method and its extension to four-step sixteenth order method using rational interpolation. In the scheme, the first three steps are any arbitrary eight order convergent method. The fourth step is the root of the numerator of the method, which is given as

$$t_n = \phi_8(x_n, z_n, w_n),$$

$$x_{n+1} = x_n - \frac{P_1 f[z_n, w_n] + Q_1 f[x_n, w_n] + Rf[t_n, w_n]}{P_1L + Q_1M + RN} f(x_n), \tag{9}$$

where,

$$\begin{aligned} L &= \frac{f(w_n) f[x_n, z_n] - f(z_n) f[x_n, w_n]}{w_n - zn}, \\ M &= \frac{f(w_n) f'(x_n) - f(x_n) f[x_n, w_n]}{w_n - x_n}, \\ N &= \frac{f(w_n) f[x_n, t_n] - f(t_n) f[x_n, w_n]}{w_n - t_n}, \\ P_1 &= (x_n - t_n) f(x_n) f(t_n), \\ Q_1 &= (t_n - z_n) f(t_n) f(z_n), \\ R &= (z_n - x_n) f(z_n) f(x_n), \end{aligned} \tag{10}$$

and rational polynomial of the following form:

$$p_4(x) = \frac{(x - x_i) + \lambda}{\mu(x - x_i)^3 + \nu(x - x_i)^2 + \xi(x - x_i) + \eta}. \tag{11}$$

The efficiency index of the above sixteenth order method is 1.741. The method involves one derivative evaluation.

In this paper, we present a general class of derivative free n-point iterative method which satisfies Kung and Tarub's Hypothesis [9]. Proposed schemes require n functional evaluations to acquire the convergence order 2^{n-1} and efficiency index can have $2^{(n-1)/n}$. The contents of the paper are summarized as follows. In Section 2, we present a general class of n-point iterative scheme and its special cases with second, fourth, eighth, and sixteenth order convergence. Section 3 consists of the convergence analysis of the iterative methods discussed in Section 2. In the last section of the paper, we give concluding remarks and some numerical results to show the effectiveness of the proposed methods.

2. Higher Order Derivative Free Optimal Methods

In this section, we give a general class of n-point iterative method involving n functional evaluations having order of

convergence 2^{n-1}. Thus, the scheme is optimal in the sense of the conjecture of Kung and Traub [9].

Consider a rational polynomial of degree $n-1$ as follows:

$$r_{n-1}(t) = \frac{p_1(t)}{q_{n-2}(t)}, \tag{12}$$

where,

$$p_1(t) = a_0 + a_1(t-x),$$
$$q_{n-2}(t) = 1 + b_1(t-x) + \cdots + b_{n-2}(t-x)^{n-2}, \quad n \geq 2, \tag{13}$$
$$q_0 \equiv 1.$$

We approximate $f(x)$ by rational function given by (12) to construct a general class of n-point iterative scheme. Then, the root of nonlinear equation $f(x) = 0$ is the root of the numerator of the rational interpolant of degree $n-1$ for the n-point method. The unknowns $a_0, a_1, b_1, \ldots, b_{n-1}$ are determined by the following interpolating conditions:

$$r_{n-1}(x) = f(x),$$
$$r_{n-1}(w_k) = f(w_k), \qquad k = 1, \ldots, n-1, \ n \geq 2. \tag{14}$$

Then, the general n-point iterative method is given by

$$w_1 = x + \beta f(x),$$
$$\vdots \tag{15}$$
$$w_n = x - \frac{a_0}{a_1}, \quad n \geq 2.$$

Now, we are going to derive its special cases. For $n = 2$ in (14)-(15),

$$r_1(t) = a_0 + a_1(t-x). \tag{16}$$

We find a_0 and a_1 such that

$$r_1(x) = f(x), \qquad r_1(w_1) = f(w_1). \tag{17}$$

So, the two-point iterative scheme becomes

$$w_1 = x + \beta f(x),$$
$$w_2 = x - \frac{f(x)}{f[w_1, x]}. \tag{18}$$

The iterative scheme (18) is the same as given by Steffensen [11] for $\beta = 1$; thus, is a particular case of our scheme given by (14)-(15).

For $n = 3$, we have

$$r_2(t) = \frac{a_0 + a_1(t-x)}{1 + b_1(t-x)}. \tag{19}$$

We find a_0, a_1, and b_1 using the following conditions:

$$r_2(x) = f(x), \qquad r_2(w_1) = f(w_1),$$
$$r_2(w_2) = f(w_2). \tag{20}$$

By using conditions (20), we have

$$a_0 = f(x),$$
$$a_1 = f[w_2, x] + b_1 f(w_2), \tag{21}$$
$$b_1 = \frac{f[w_1, x] - f[w_2, x]}{f(w_2) - f(w_1)}.$$

Now, using (21), we have the following three-point iterative scheme:

$$w_1 = x + \beta f(x),$$
$$w_2 = x - \frac{f(x)}{f[w_1, x]}, \tag{22}$$
$$w_3 = x - \frac{f(x)(f(w_2) - f(w_1))}{f(w_2) f[w_1, x] - f(w_1) f[w_2, x]}.$$

For $n = 4$, we have the following rational interpolant:

$$r_3(t) = \frac{a_0 + a_1(t-x)}{1 + b_1(t-x) + b_2(t-x)^2}, \tag{23}$$

such that

$$r_3(x) = f(x), \qquad r_3(w_1) = f(w_1),$$
$$r_3(w_2) = f(w_2), \qquad r_3(w_3) = f(w_3). \tag{24}$$

The conditions (24) are used to determine the unknowns a_0, a_1, b_1, and b_2. Thus, we attain a four-point iterative method as follows:

$$w_1 = x + \beta f(x),$$
$$w_2 = x - \frac{f(x)}{f[w_1, x]},$$
$$w_3 = x - \frac{f(x)(f(w_2) - f(w_1))}{f(w_2) f[w_1, x] - f(w_1) f[w_2, x]}, \tag{25}$$
$$w_4 = x - \frac{f(x)(h_1 + h_2 + h_3)}{h_1 f[w_1, x] + h_2 f[w_2, x] + h_3 f[w_3, x]},$$

where,

$$h_1 = f(w_2) f(w_3)(w_3 - w_2),$$
$$h_2 = f(w_1) f(w_3)(w_1 - w_3), \tag{26}$$
$$h_3 = f(w_1) f(w_2)(w_2 - w_1).$$

For $n = 5$, we have the following five-point iterative scheme:

$$\begin{aligned}
w_1 &= x + \beta f(x), \\
w_2 &= x - \frac{f(x)}{f[w_1, x]}, \\
w_3 &= x - \frac{f(x)(f(w_2) - f(w_1))}{f(w_2) f[w_1, x] - f(w_1) f[w_2, x]}, \\
w_4 &= x - \frac{f(x)(h_1 + h_2 + h_3)}{h_1 f[w_1, x] + h_2 f[w_2, x] + h_3 f[w_3, x]}, \\
w_5 &= x - \frac{a_0}{a_1},
\end{aligned} \tag{27}$$

where,

$$r_4(t) = \frac{a_0 + a_1(t - x)}{1 + b_1(t - x) + b_2(t - x)^2 + b_3(t - x)^3}, \tag{28}$$

such that

$$\begin{aligned}
&r_4(x) = f(x), \qquad r_4(w_1) = f(w_1), \\
&r_4(w_2) = f(w_2), \qquad r_4(w_3) = f(w_3), \\
&r_4(w_4) = f(w_4).
\end{aligned} \tag{29}$$

The interpolating conditions (29) yield

$$\begin{aligned}
a_0 &= f(x), \\
a_1 &= (m_1 f[w_1, x] + m_2 f[w_2, x] + m_3 f[w_3, x] \\
&\quad + m_4 f[w_4, x])(m_1 + m_2 + m_3 + m_4)^{-1},
\end{aligned} \tag{30}$$

where,

$$\begin{aligned}
m_1 &= f(w_2) f(w_3) f(w_4) \\
&\quad \times \{-(w_3 - x)(w_4 - x)(w_4 - w_3) \\
&\quad + (w_2 - x)(w_4 - x)(w_4 - w_2)\} \\
&\quad - (w_2 - x)(w_3 - x) h_1 f(w_4) \\
m_2 &= f(w_1) f(w_3) f(w_4) \\
&\quad \times \{(w_3 - x)(w_4 - x)(w_4 - w_3) \\
&\quad - (w_1 - x)(w_4 - x)(w_4 - w_1)\} \\
&\quad - (w_1 - x)(w_3 - x) h_2 f(w_4) \\
m_3 &= f(w_1) f(w_2) f(w_4) \\
&\quad \times \{-(w_2 - x)(w_4 - x)(w_4 - w_2) \\
&\quad + (w_1 - x)(w_4 - x)(w_4 - w_1)\} \\
&\quad - (w_1 - x)(w_2 - x) h_3 f(w_4) \\
m_4 &= f(w_1) f(w_2) f(w_3) \\
&\quad \times \{(w_2 - x)(w_3 - x)(w_3 - w_2) \\
&\quad - (w_1 - x)(w_3 - x)(w_3 - w_1)\} \\
&\quad + (w_1 - x)(w_2 - x) h_3 f(w_3),
\end{aligned} \tag{31}$$

and h_1, h_2, h_3 are given as in (26). Hence, we obtain the following iterative method:

$$\begin{aligned}
w_1 &= x + \beta f(x), \\
w_2 &= x - \frac{f(x)}{f[w_1, x]}, \\
w_3 &= x - \frac{f(x)(f(w_2) - f(w_1))}{f(w_2) f[w_1, x] - f(w_1) f[w_2, x]}, \\
w_4 &= x - \frac{f(x)(h_1 + h_2 + h_3)}{h_1 f[w_1, x] + h_2 f[w_2, x] + h_3 f[w_3, x]}, \\
w_5 &= x - (f(x)(m_1 + m_2 + m_3 + m_4)) \\
&\quad \times (m_1 f[w_1, x] + m_2 f[w_2, x] \\
&\quad + m_3 f[w_3, x] + m_4 f[w_4, x])^{-1}.
\end{aligned} \tag{32}$$

We, now, give the convergence analysis of the proposed iterative methods (18), (22), (25), and (32).

3. Convergence Analysis

Theorem 1. *Let us consider $\omega \in I$ as the simple root of sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ in the neighborhood of the root for interval I. If x is sufficiently close to ω, then, for every $\beta \in \mathbb{R} \setminus \{-1\}$, the iterative methods defined by (18) and (22) are second and fourth order convergent, respectively, with the error equations given by*

$$\begin{aligned}
e_{n+1} &= c_2(1 + \beta) e_n^2 + O(e_n^3), \\
e_{n+1} &= (c_2^3 - c_2 c_3 + 2c_2^3\beta + c_2^3\beta^2 \\
&\quad - 2c_2 c_3 \beta - c_2 c_3 \beta^2) e_n^4 + O(e_n^5),
\end{aligned} \tag{33}$$

respectively, where,

$$c_k = \frac{1}{k!}\frac{f^{(k)}(\omega)}{f'(\omega)}, \qquad k = 2, 3, \ldots \tag{34}$$

Proof. Let $x = \omega + e_n$, where ω is the root of f and e_n is the error at nth step. Now, using Taylor expansion of $f(x)$ about the root ω, we have

$$f(x) = f'(\omega)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \cdots + c_8 e_n^8 + O\left(e_n^9\right)\right], \tag{35}$$

where c_k is defined by (34). Taylor's expansions for w_1 and $f(w_1)$ are

$$w_1 = e_n + \omega + \beta\left(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5\right) + O\left(e_n^6\right), \tag{36}$$

$$\begin{aligned} f(w_1) &= f'(\omega)\left[(1+\beta)e_n + \left(3\beta c_2 + c_2 + c_2\beta^2\right)e_n^2 \right. \\ &\quad + \left(2c_2^2\beta + 2c_2^2\beta^2 + c_3 + 4\beta c_3 + 3c_3\beta^2 + c_3\beta^3\right)e_n^3 \\ &\quad + \left(5c_2\beta c_3 + 8c_2\beta^2 c_3 + 3c_3\beta^3 + c_4 + 5\beta c_4 + 6\beta^2 c_4 + 4c_4\beta^3 + c_4\beta^4 + c_2^3\beta^2\right)e_n^4 \\ &\quad + \left(c_5 + 6\beta c_5 + 10c_5\beta^2 + 10c_5\beta^3 + 5c_5\beta^4 + c_5\beta^5 + 6c_2\beta c_4 + 14c_2c_4\beta^2 + 12c_4c_2\beta^3 \right. \\ &\quad \left. \left. + 4c_4c_2\beta^4 + 5c_2^2c_3\beta^2 + 3c_3^2\beta + 6c_3^2\beta^2 + 3c_3\beta^3c_2^2 + 3c_3^2\beta^3\right)e_n^5 + O\left(e_n^6\right)\right]. \end{aligned} \tag{37}$$

Using (35), (36), and (37), we have

$$w_2 = \omega + c_2(1+\beta)e_n^2 + \left(-2c_2^2 - 2c_2^2\beta + 2c_3 + 3\beta c_3 + c_3\beta^2 - c_2^2\beta^2\right)e_n^3 + \left(4c_2^3 + 5c_2^3\beta - 7c_2c_3 - 10c_2\beta c_3 - 7c_2\beta^2 c_3 + 3c_2^3\beta^2 - 2c_3\beta^3c_2 + 3c_4 + 6\beta c_4 + 4c_4\beta_2 + c_4\beta^3 + c_2^3\beta^3\right)e_n^4 + O\left(e_n^5\right), \tag{38}$$

which shows that the method (18) is quadratically convergent for all $\beta \in \mathbb{R} \setminus \{-1\}$. Again using Taylor expansion of $f(w_2)$, we have

$$\begin{aligned} f(w_2) = f'(\omega)\left[c_2(1+\beta)e_n^2\right. &+ \left(-2c_2^2 - 2c_2^2\beta + 2c_3 + 3\beta c_3 + c_3\beta^2 - c_2^2\beta^2\right)e_n^3 \\ &+ \left(5c_2^3 + 7c_2^3\beta - 7c_2c_3 - 10c_2\beta c_3 - 7c_2\beta^2c_3 + 4c_2^3\beta^2 - 2c_3\beta^3c_2 + 3c_4 + 6\beta c_4 \right. \\ &\left. \left. + 4c_4\beta^2 + c_4\beta^3 + c_2^3\beta^3\right)e_n^4 + O\left(e_n^5\right)\right]. \end{aligned} \tag{39}$$

Now, using (35)–(39), we see that the order of convergence of the method (22) is four and the error equation is given by

$$e_{n+1} = \left(c_2^3 + 2c_2^3\beta - c_2c_3 - 2c_2c_3\beta - c_2c_3\beta^2 + c_2^3\beta^2\right)e_n^4 + O\left(e_n^5\right). \tag{40}$$

□

Theorem 2. *Let us consider $\omega \in I$ as the simple root of sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ in the neighborhood of the root for interval I. If x is sufficiently close to ω, then for all $\beta \in \mathbb{R}\setminus\{-1\}$, the iterative methods defined by* (25) *and* (32) *are eighth and sixteenth order convergent, respectively, with the error equations given by*

$$\begin{aligned} e_{n+1} = c_2^2\left(c_2^5\beta^4 - 3c_3c_2^3\beta^4 + 4c_2^5\beta^3 + c_4c_2^2\beta^4 + 2c_3^2c_2\beta^4 - 12c_3c_2^3\beta^3 + 6c_2^5\beta^2 \right. \\ - c_4c_3\beta^4 + 4c_4c_2^2\beta^3 + 8c_3^2c_2\beta^3 - 18c_3c_2^3\beta^2 + 4c_2^5\beta - 4c_4c_3\beta^3 + 6c_4c_2^2\beta^2 \\ + 12c_3^2c_2\beta^2 - 12c_3c_2^3\beta + c_2^5 - 6c_4c_3\beta^2 + 4c_4c_2^2\beta + 8\beta c_2c_3^2 - 3c_2^3c_3 \\ \left. -4\beta c_3c_4 + c_2^2c_4 + 2c_2c_3^2 - c_3c_4\right)e_n^8 + O\left(e_n^9\right), \end{aligned}$$

$$\begin{aligned} e_{n+1} = \left(\left(4c_2^5c_3c_5 - c_3^2c_4c_5 + 2c_2c_3^2c_4^2 + 2c_2c_3^3c_5 - 4c_2^3c_3c_4^2 - 5c_2^3c_3^2c_5 - c_2^4c_4c_5 + 18c_2^4c_3^2c_4 \right. \right. \\ - 13c_2^6c_3c_4 - 9c_2^2c_3^3c_4 + 2c_2^5c_4^2 + c_2^4c_4 - 2c_2c_3^5 - c_2^7c_5 - 7c_2^9c_3 + 3c_2^8c_4 \end{aligned}$$

$$
\begin{aligned}
&+11c_2^3c_3^4-21c_2^5c_3^3+18c_2^7c_2^2\\
&+2c_2^2c_3c_4c_5+c_2^{11}\big)c_2^4\beta^8\\
&+\cdots+\big(4c_2^5c_3c_5-c_3^2c_4c_5+2c_2c_3^2c_4^2+2c_2c_3^3c_5\\
&\quad-4c_2^3c_3c_4^2-5c_2^3c_3^2c_5-c_2^4c_4c_5\\
&\quad+18c_2^4c_3^2c_4-13c_2^6c_3c_4-9c_2^2c_3^3c_4\\
&\quad+2c_2^5c_4^2+c_3^4c_4-2c_2c_3^5-c_2^7c_5-7c_2^9c_3\\
&\quad+3c_2^8c_4+11c_2^3c_3^4-21c_2^5c_3^3+18c_2^7c_3^2\\
&\quad+2c_2^2c_3c_4c_5+c_2^{11}\big)c_2^4\big)e_n^{16}+O\left(e_n^{17}\right),
\end{aligned}
\tag{41}
$$

where,

$$
c_k=\frac{1}{k!}\frac{f^{(k)}(\omega)}{f'(\omega)},\qquad k=2,3,\ldots \tag{42}
$$

Proof. Let $x=\omega+e_n$, where ω is the root of f and e_n is the error in the approximation at nth iteration. We will use (35), (37), (39), and (40) up to $O(e_n^{30})$ in this result and set

$$
\begin{aligned}
w_3&=\omega+\big(c_2^3+2c_2^3\beta-c_2c_3-2c_2c_3\beta\\
&\quad-c_2c_3\beta^2+c_2^3\beta^2\big)e_n^4\\
&\quad+\big(4c_3\beta^3c_2^2-2c_2^4\beta^3-c_3^2\beta^3-c_4\beta^3c_2\\
&\quad-4c_2\beta^2c_4+14c_2^2\beta^2c_3-6c_2^4\beta^2-4c_3^2\beta^2\\
&\quad+18c_2^2\beta c_3-5c_2\beta c_4-8c_2^4\beta-5c_3^2\beta\\
&\quad-4c_2^4-2c_2c_4+8c_2^2c_3-2c_3^2\big)e_n^5\\
&\quad+\cdots+O\left(e_n^{30}\right).
\end{aligned}
\tag{43}
$$

Again, using Taylor expansion of $f(w_3)$, we have

$$
\begin{aligned}
f(w_3)&=f'(\omega)\big[\big(c_2^3+2c_2^3\beta-c_2c_3-2c_2c_3\beta\\
&\quad-c_2c_3\beta^2+c_2^3\beta^2\big)e_n^4\\
&\quad+\big(4c_3\beta^3c_2^2-2c_2^4\beta^3-c_3^2\beta^3-c_4\beta^3c_2\\
&\quad-4c_2\beta^2c_4+14c_2^2\beta^2c_3-6c_2^4\beta^2-4c_3^2\beta^2\\
&\quad+18c_2^2\beta c_3-5c_2\beta c_4-8c_2^4\beta-5c_3^2\beta\\
&\quad-4c_2^4-2c_2c_4+8c_2^2c_3-2c_3^2\big)e_n^5\\
&\quad+\cdots+O\left(e_n^{30}\right)\big].
\end{aligned}
\tag{44}
$$

Now, using (35)–(39), (43), and (44), we see that the method (25) has eighth order convergence with the error equation

$$
\begin{aligned}
e_{n+1}&=c_2^2\big(c_2^5\beta^4-3c_3c_2^3\beta^4+4c_2^5\beta^3+c_4c_2^2\beta^4\\
&\quad+2c_3^2c_2\beta^4-12c_3c_2^3\beta^3+6c_2^5\beta^2\\
&\quad-c_4c_3\beta^4+4c_4c_2^2\beta^3+8c_3^2c_2\beta^3-18c_3c_2^3\beta^2\\
&\quad+4c_2^5\beta-4c_4c_3\beta^3+6c_4c_2^2\beta^2\\
&\quad+12c_3^2c_2\beta^2-12c_3c_2^3\beta+c_2^5-6c_4c_3\beta^2\\
&\quad+4c_4c_2^2\beta+8\beta c_2c_3^2-3c_2^3c_3\\
&\quad-4\beta c_3c_4+c_2^2c_4+2c_2c_3^2-c_3c_4\big)e_n^8+O\left(e_n^9\right).
\end{aligned}
\tag{45}
$$

To find the error equation of (32), we use (45) up to $O(e_n^{30})$ and set

$$
\begin{aligned}
w_4&=c_2^2\big(c_2^5\beta^4-3c_3c_2^3\beta^4+4c_2^5\beta^3+c_4c_2^2\beta^4\\
&\quad+2c_3^2c_2\beta^4-12c_3c_2^3\beta^3+6c_2^5\beta^2-c_4c_3\beta^4\\
&\quad+4c_4c_2^2\beta^3+8c_3^2c_2\beta^3-18c_3c_2^3\beta^2\\
&\quad+4c_2^5\beta-4c_4c_3\beta^3+6c_4c_2^2\beta^2\\
&\quad+12c_3^2c_2\beta^2-12c_3c_2^3\beta+c_2^5-6c_4c_3\beta^2\\
&\quad+4c_4c_2^2\beta+8\beta c_2c_3^2-3c_2^3c_3\\
&\quad-4\beta c_3c_4+c_2^2c_4+2c_2c_3^2-c_3c_4\big)e_n^8\\
&\quad+\cdots+O\left(e_n^{30}\right).
\end{aligned}
\tag{46}
$$

Taylor expansion of $f(w_4)$ is

$$
\begin{aligned}
f(w_4)&=f'(\omega)\big[\big(c_2^7\beta^4+c_2^4c_4+c_2^7+c_4\beta^4c_2^4\\
&\quad-c_2^2c_4c_3-c_4c_3\beta^4c_2^2+8c_2^3c_3^2\beta^3\\
&\quad+2\beta^4c_3^2c_2^3+8c_2^3\beta c_3^2+12c_2^3\beta^2c_3^2\\
&\quad-18c_2^5c_3\beta^2-12c_2^5c_3\beta-3\beta^4c_2^5c_3\\
&\quad+6c_2^4c_4\beta^2+4c_2^4c_4\beta^3+4c_2^4c_4\beta\\
&\quad-12c_2^5c_3\beta^3+6c_2^7\beta^2+4c_2^7\beta^3+4c_2^7\beta\\
&\quad-3c_2^5c_3+2c_2^3c_3^2-6c_3\beta^2c_4c_2^2\\
&\quad-4c_2^2c_4\beta c_3-4c_4c_2^2c_3\beta^3\big)e_n^8\\
&\quad+\cdots+O\left(e_n^{30}\right)\big].
\end{aligned}
\tag{47}
$$

Table 1: Test functions and exact roots.

Numerical example	Exact roots
$f_1(x) = \sqrt{x^4+8}\sin(\pi/(x^2+2)) + x^3/(x^4+1) - \sqrt{6} + 8/17$	$\omega_1 = -2$
$f_2(x) = \sin(x) - x/100$	$\omega_2 = 0$
$f_3(x) = (1/3)x^4 - x^2 - (1/3)x + 1$	$\omega_3 = 1$
$f_4(x) = e^{\sin(x)} - 1 - x/5$	$\omega_4 = 0$
$f_5(x) = xe^{x^2} - (\sin(x))^2 + 3\cos(x) + 5$	$\omega_5 \approx -1.207647827130919$
$f_6(x) = e^{-x} + \cos(x)$	$\omega_6 \approx 1.746139530408013$
$f_7(x) = 10xe^{-x^2} - 1$	$\omega_7 \approx 1.679630610428450$
$f_8(x) = x^3 + 4x^2 - 15$	$\omega_8 \approx 1.631980805566063$

Hence, using (35)–(39), (43), (44), (46), and (47), we see that iterative method (32) is sixteenth order convergent with the error equation given by

$$\begin{aligned} e_{n+1} = {} & \Big(\big(4c_2^5c_3c_5 - c_3^2c_4c_5 + 2c_2c_3^2c_4^2 + 2c_2c_3^3c_5 \\ & - 4c_2^3c_3c_4^2 - 5c_2^3c_3^2c_5 - c_2^4c_4c_5 \\ & + 18c_2^4c_3^2c_4 - 13c_2^6c_3c_4 - 9c_2^2c_3^3c_4 + 2c_2^5c_4^2 \\ & + c_3^4c_4 - 2c_2c_3^5 - c_2^7c_5 - 7c_2^9c_3 \\ & + 3c_2^8c_4 + 11c_2^3c_3^4 - 21c_2^5c_3^3 + 18c_2^7c_3^2 \\ & + 2c_2^2c_3c_4c_5 + c_2^{11}\big)c_2^4\beta^8 \\ & + \cdots + \big(4c_2^5c_3c_5 - c_3^2c_4c_5 + 2c_2c_3^2c_4^2 + 2c_2c_3^3c_5 \\ & - 4c_2^3c_3c_4^2 - 5c_2^3c_3^2c_5 - c_2^4c_4c_5 + 18c_2^4c_3^2c_4 \\ & - 13c_2^6c_3c_4 - 9c_2^2c_3^3c_4 + 2c_2^5c_4^2 + c_3^4c_4 - 2c_2c_3^5 \\ & - c_2^7c_5 - 7c_2^9c_3 + 3c_2^8c_4 + 11c_2^3c_3^4 \\ & - 21c_2^5c_3^3 + 18c_2^7c_3^2 + 2c_2^2c_3c_4c_5 \\ & + c_2^{11}\big)c_2^4\Big)e_n^{16} + O\left(e_n^{17}\right). \end{aligned} \tag{48}$$

□

Remark 3. From Theorems 1 and 2, it can be seen that the iterative schemes (18), (22), (25), and (32) are second, fourth, eighth, and sixteenth order convergent requiring two, three, four, and five functional evaluations, respectively. Hence, the proposed iterative schemes (18), (22), (25), and (32) are optimal in the sense of the hypothesis of Kung and Traub [9] with the efficiency indices 1.414, 1.587, 1.681, 1.741. Also, it is clear that (14)-(15) is a general n-point scheme with optimal order of convergence 2^{n-1}. The efficiency index of this scheme is $2^{(n-1)/n}$.

4. Numerical Results

In this section, we present some test functions to demonstrate the performance of the newly developed sixteenth order scheme (32) (FNMS-16). For the sake of comparison, we consider the existing higher order convergent methods based on rational interpolation. We consider the fourteenth order method of Soleymani and Sharifi (4) (SS-14), the sixteenth order method of Soleymani et al. (7) (SSS-16), and the sixteenth order method of Sharma et al. (9) (SGG-16). All the computations for the above-mentioned methods are performed using Maple 16 with 4000 decimal digits precision. The test functions given in Table 1 are taken from [1, 2, 10]. We used almost all types of nonlinear functions, polynomials, and transcendental functions to test the new methods. Table 2 shows that the newly developed sixteenth order methods are comparable with the existing methods of this domain in terms of significant digits and number of function evaluations per iteration. In many examples, the newly developed methods perform better than the existing methods. It can also be seen from the tables that, for the choice of initial guess, near to the exact root or far from the exact root, the performance of the new methods is better.

5. Attraction Basins

Let ω_i be the roots of the complex polynomial $p_n(x)$, $n \geq 1$, $x \in \mathbb{C}$, where $i = 1, 2, 3, \ldots, n$. We use two different techniques to generate basins of attraction on MATLAB software. We take a square box of $[-2, 2] \times [-2, 2] \in \mathbb{C}$ in the first technique. For every initial guess x_0, a specific color is assigned according to the exact root, and dark blue is assigned for the divergence of the method. We use $|f(x_k)| < 10^{-5}$ as the stopping criteria for convergence and the maximum number of iterations are 30. "Jet" is chosen as the colormap here. For the second technique, the same scale is taken but each initial guess is assigned a color depending upon the number of iterations for the method to converge to any of the roots of the given function. We use 25 as the maximum number of iterations; the stopping criteria are the same as above and colormap is selected as "hot." The method is considered divergent for that initial guess if it does not converge in the maximum number of iterations and this case is displayed by black color.

We take three test examples to obtain basins of attraction, which are given as $p_3(x) = x^3 - 1$, $p_4(x) = x^4 - 10x^2 + 9$, and $p_5(x) = x^5 - 1$. The roots of $p_3(x)$ are 1.0, $-0.5000 + 0.86605I$, and $-0.5000 - 0.86605I$, the roots of $p_4(x)$ are $-3, 3$,

Table 2: Comparison of various iterative methods.

$f(x), x_0$	(SS-14)	(SGG-16)	(SSS-16)	(FNMS-16)
$f_1, x_0 = -1.2$				
$\lvert f_1(x_1)\rvert$	$.3e-13$	$.1e-15$	$.1e-13$	$.1e-15$
$\lvert f_1(x_2)\rvert$	$.1e-181$	$.6e-248$	$.1e-211$	$.5e-245$
$\lvert f_1(x_3)\rvert$	$.6e-2538$	$.6e-3751$	$.7e-3379$	$.6e-3916$
$f_1, x_0 = -3$				
$\lvert f_1(x_1)\rvert$	$.1e-5$	$.3e-6$	$.3e-7$	$.2e-8$
$\lvert f_1(x_2)\rvert$	$.4e-75$	$.1e-96$	$.2e-113$	$.1e-132$
$\lvert f_1(x_3)\rvert$	$.1e-1050$	$.3e-1545$	$.2e-1812$	$.5e-2123$
$f_2, x_0 = 1.5$				
$\lvert f_2(x_1)\rvert$	.95	.95	.64	$.3e-4$
$\lvert f_2(x_2)\rvert$	$.1e-10$	$.1e-4$	$.3e-6$	$.5e-100$
$\lvert f_2(x_3)\rvert$	$.2e-88$	$.5e-90$	$.1e-111$	$.4e-1632$
$f_2, x_0 = 3$				
$\lvert f_2(x_1)\rvert$	$.2e-24$	$.1e-25$	$.3e-27$	$.2e-44$
$\lvert f_2(x_2)\rvert$	$.8e-358$	$.8e-425$	$.2e-452$	$.9e-743$
$\lvert f_2(x_3)\rvert$	0	0	0	0
$f_3, x_0 = 0.5$				
$\lvert f_3(x_1)\rvert$	$.5e-8$	$.6e-9$	$.3e-9$	$.9e-10$
$\lvert f_3(x_2)\rvert$	$.8e-114$	$.3e-144$	$.4e-149$	$.6e-238$
$\lvert f_3(x_3)\rvert$	$.2e-1595$	$.1e-2307$	$.1e-2386$	0
$f_3, x_0 = 1.5$				
$\lvert f_3(x_1)\rvert$	$.2e-11$	$.2e-13$	$.2e-11$	$.1e-11$
$\lvert f_3(x_2)\rvert$	$.2e-158$	$.6e-214$	$.1e-179$	$.6e-183$
$\lvert f_3(x_3)\rvert$	$.3e-2218$	$.8e-3423$	$.2e-2869$	$.6e-2935$
$f_4, x_0 = 5$				
$\lvert f_4(x_1)\rvert$	$.8e-1$	.25	$.8e-1$	$.1e-2$
$\lvert f_4(x_2)\rvert$	$.4e-15$	$.1e-6$	$.3e-16$	$.3e-56$
$\lvert f_4(x_3)\rvert$	$.1e-213$	$.7e-109$	$.9e-262$	$.1e-914$
$f_4, x_0 = 4$				
$\lvert f_4(x_1)\rvert$	.92	$.9e-1$	2.51	$.1e-17$
$\lvert f_4(x_2)\rvert$	$.4e-5$	$.1e-25$	$.6e-3$	$.3e-295$
$\lvert f_4(x_3)\rvert$	$.1e-73$	$.1e-423$	$.6e-49$	0
$f_5, x_0 = -1$				
$\lvert f_5(x_1)\rvert$	$.1e-3$	$.1e-7$	$.8e-4$	$.7e-5$
$\lvert f_5(x_2)\rvert$	$.2e-66$	$.5e-144$	$.9e-80$	$.4e-92$
$\lvert f_5(x_3)\rvert$	$.4e-946$	$.8e-2327$	$.1e-1294$	$.6e-1488$
$f_5, x_0 = -0.6$				
$\lvert f_5(x_1)\rvert$	$.2e44770$	$.4e-1$	$.2e44770$	.1
$\lvert f_5(x_2)\rvert$	$.4e44769$	$.4e-40$	$.7e44768$	$.3e-23$
$\lvert f_5(x_3)\rvert$	$.5e44767$	$.4e-664$	$.2e44767$	$.8e-386$
$f_6, x_0 = 0.5$				
$\lvert f_6(x_1)\rvert$	$.6e-7$	$.9e-9$	$.3e-8$	$.1e-14$
$\lvert f_6(x_2)\rvert$	$.8e-116$	$.1e-152$	$.7e-145$	$.1e-254$
$\lvert f_6(x_3)\rvert$	$.1e-1748$	$.7e-2456$	$.9e-2332$	$.1e-3999$
$f_6, x_0 = 3$				
$\lvert f_6(x_1)\rvert$	.7	.53	.79	$0.7e-8$
$\lvert f_6(x_2)\rvert$	$.9e-5$	$.4e-16$	$.2e-5$	$.2e-145$

Table 2: Continued.

$f(x), x_0$	(SS-14)	(SGG-16)	(SSS-16)	(FNMS-16)
$\lvert f_6(x_3)\rvert$	.7e − 102	.1e − 269	.5e − 125	0.3e − 2345
$f_7, x_0 = 0$				
$\lvert f_7(x_1)\rvert$	.1e − 6	.2e − 9	.1e − 5	.8e − 12
$\lvert f_7(x_2)\rvert$	.5e − 99	.3e − 161	.6e − 98	.1e − 199
$\lvert f_7(x_3)\rvert$	.3e − 1394	.6e − 2562	.5e − 1575	.3e − 3205
$f_7, x_0 = 2.2$				
$\lvert f_7(x_1)\rvert$	1	.9e − 2	1	.8e − 8
$\lvert f_7(x_2)\rvert$	1	.6e − 40	D	.9e − 136
$\lvert f_7(x_3)\rvert$	D	.5e − 651	D	.4e − 2183
$f_8, x_0 = 0.5$				
$\lvert f_8(x_1)\rvert$	.1e7	.97	.1e7	1.0
$\lvert f_8(x_2)\rvert$	16856.81	.2e − 25	16856.81	.8e − 15
$\lvert f_8(x_3)\rvert$	183.46	.3e − 435	183.46	.1e − 256
$f_8, x_0 = 1$				
$\lvert f_8(x_1)\rvert$	.2e − 2	.5e − 5	.2e − 2	.6e − 3
$\lvert f_8(x_2)\rvert$	.2e − 65	.1e − 109	.2e − 65	.7e − 67
$\lvert f_8(x_3)\rvert$	.6e − 1073	.1e − 1782	.6e − 1073	.1e − 1089

*D stands for divergence.

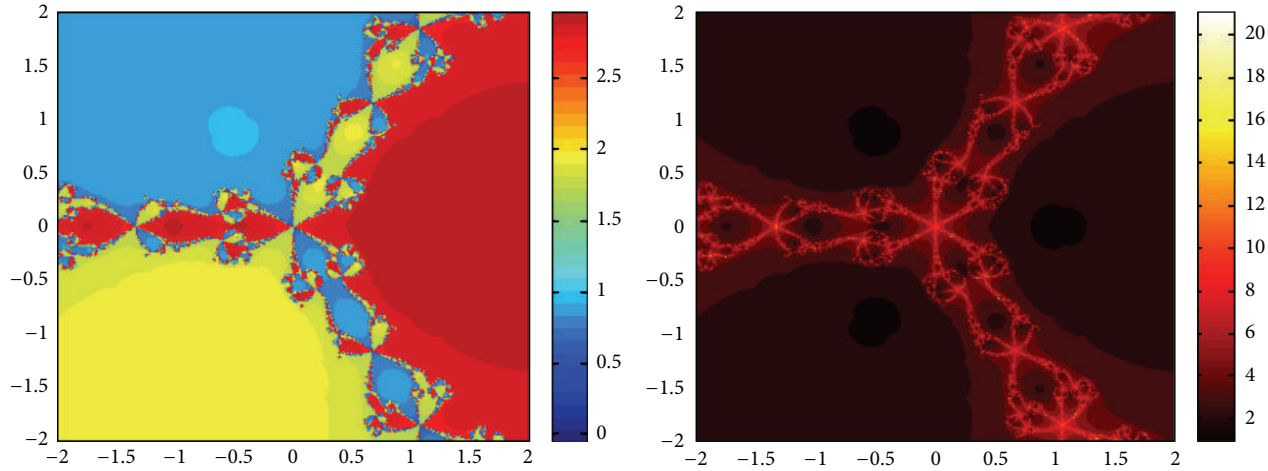

Figure 1: Basins of attraction of method (7) for $p_3(x)$.

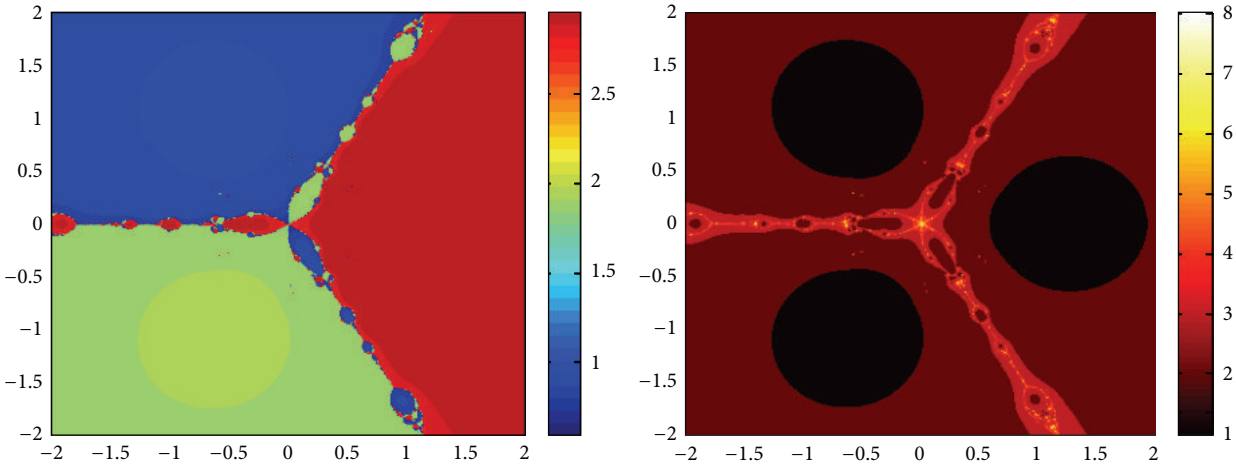

Figure 2: Basins of attraction of method (9) for $p_3(x)$.

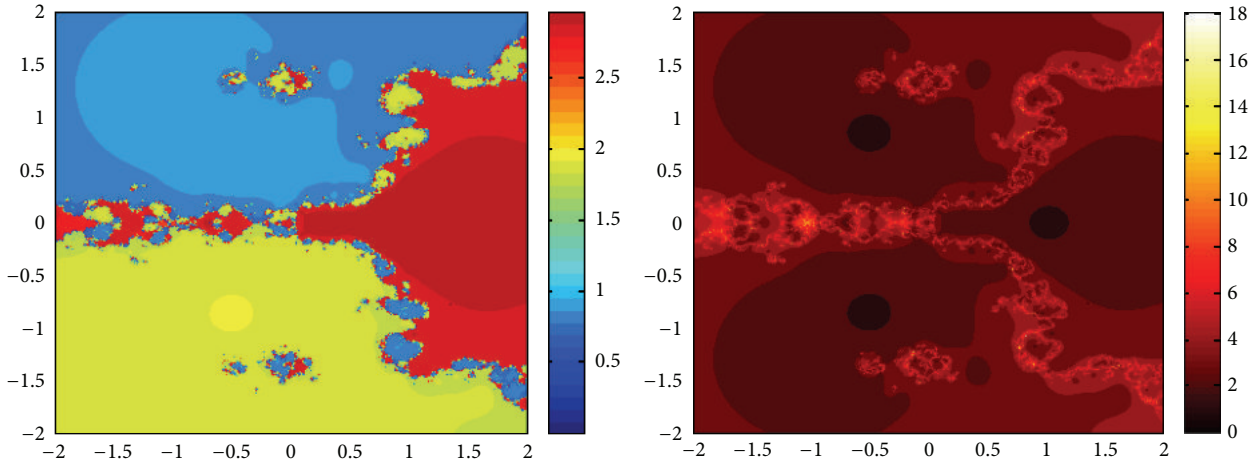

Figure 3: Basins of attraction of method (32) for $p_3(x)$.

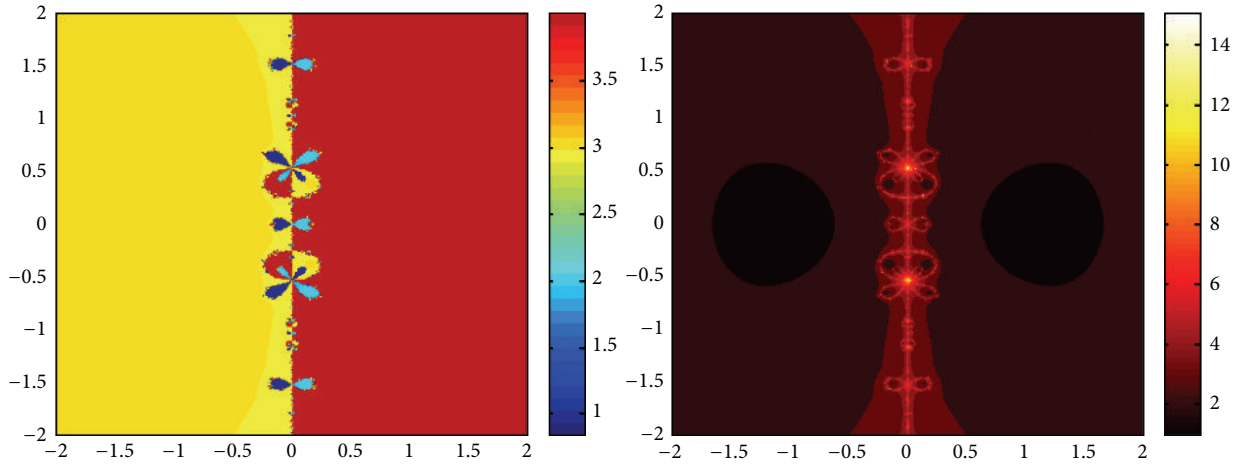

Figure 4: Basins of attraction of method (7) for $p_4(x)$.

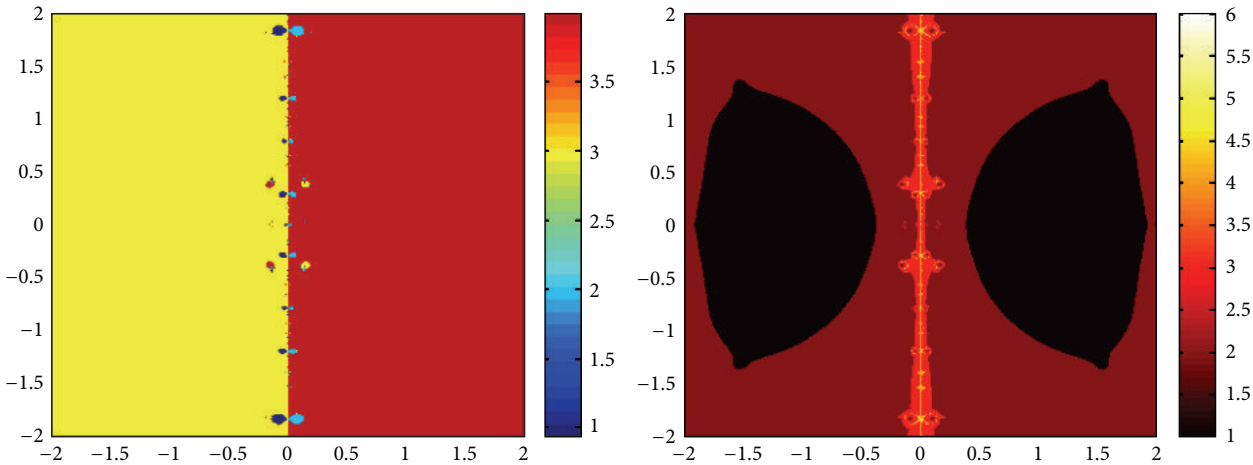

Figure 5: Basins of attraction of method (9) for $p_4(x)$.

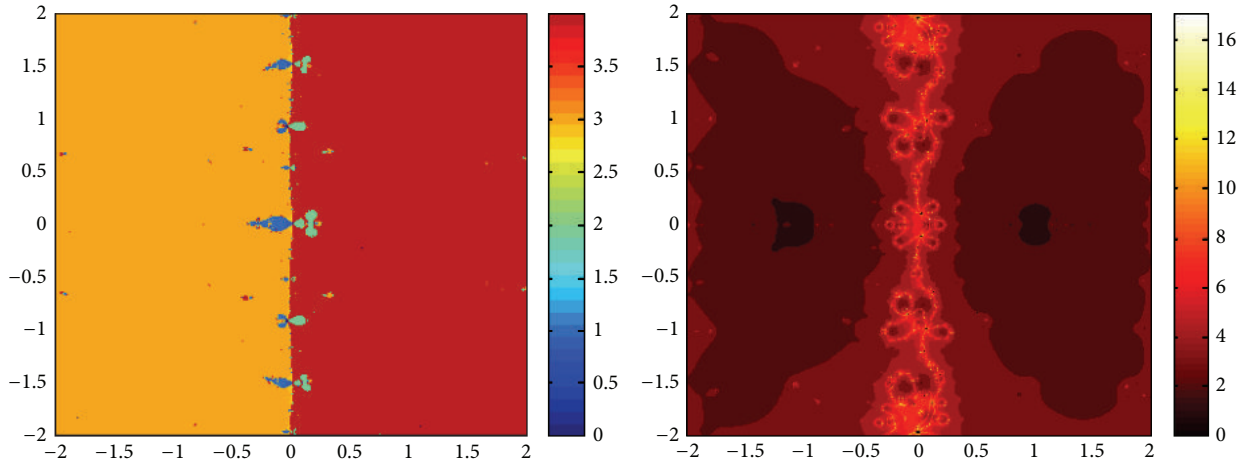

Figure 6: Basins of attraction of method (32) for $p_4(x)$.

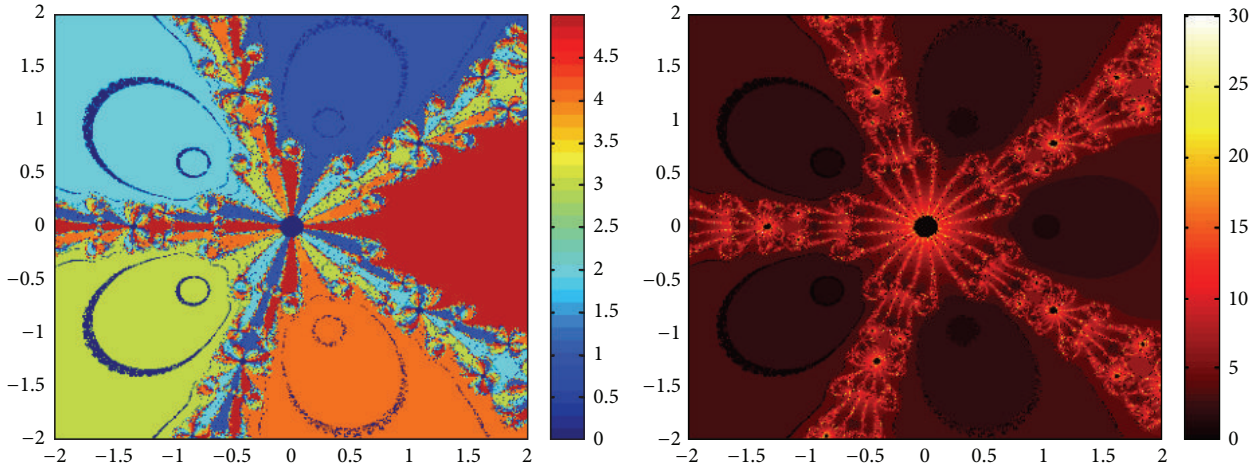

Figure 7: Basins of attraction of method (7) for $p_5(x)$.

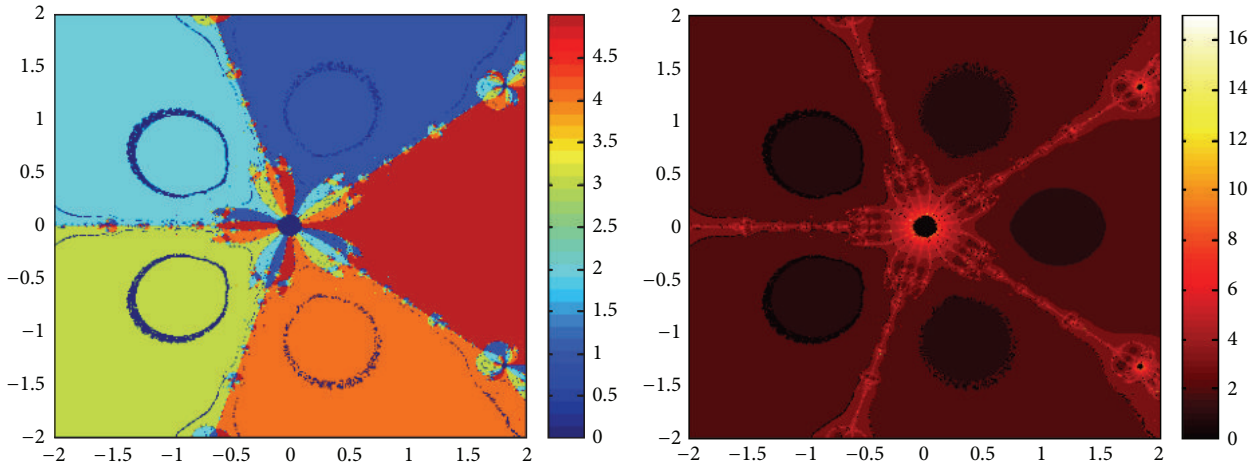

Figure 8: Basins of attraction of (9) for $p_5(x)$.

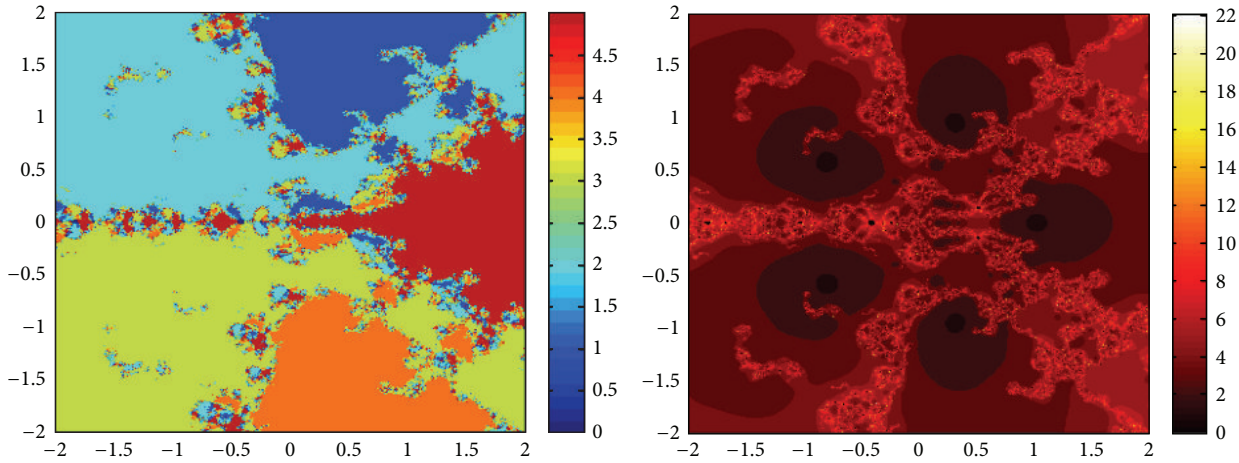

Figure 9: Basins of attraction of (32) for $p_5(x)$.

$-1, 1$, and for $p_5(x)$ roots are $1.0, 0.3090+0.95105I$, $-0.8090+0.58778I$, $-0.8090-0.58778I$, and $0.30902-0.95105I$.

We compare the results of our newly constructed method (32) with some existing methods (7) and (9), as given in Section 1. Figures 1, 2, 3, 4, 5, 6, 7, 8, and 9 show the dynamics of the methods (7), (9), and (32) for the polynomials x^3-1, x^4-1, and x^5-1. Two types of attraction basins are given in all figures. One can easily see that the appearance of darker region shows that the method consumes a fewer number of iterations. Color maps for both types are given with each figure which shows the root to which an initial guess converges and the number of iterations in which the convergence occurs.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Some New Sets of Sequences of Fuzzy Numbers with Respect to the Partial Metric

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In this paper, we essentially deal with Köthe-Toeplitz duals of fuzzy level sets defined using a partial metric. Since the utilization of Zadeh's extension principle is quite difficult in practice, we prefer the idea of level sets in order to construct some classical notions. In this paper, we present the sets of bounded, convergent, and null series and the set of sequences of bounded variation of fuzzy level sets, based on the partial metric. We examine the relationships between these sets and their classical forms and give some properties including definitions, propositions, and various kinds of partial metric spaces of fuzzy level sets. Furthermore, we study some of their properties like completeness and duality. Finally, we obtain the Köthe-Toeplitz duals of fuzzy level sets with respect to the partial metric based on a partial ordering.

1. Introduction

By $\omega(F)$, we denote the set of all sequences of fuzzy numbers. We define the classical sets $bs(H)$, $cs(H)$, and $cs_0(H)$ consisting of the sets of all bounded, convergent, and null series, respectively; that is

$$bs(H) := \left\{ u = (u_k) \in \omega(F) : \left(\sum_{k=0}^{n} u_k \right) \in \ell_\infty(H) \right\},$$

$$cs(H) := \left\{ u = (u_k) \in \omega(F) : \left(\sum_{k=0}^{n} u_k \right) \in c(H) \right\}, \quad (1)$$

$$cs_0(H) := \left\{ u = (u_k) \in \omega(F) : \left(\sum_{k=0}^{n} u_k \right) \in c_0(H) \right\}.$$

We can show that $bs(H)$, $cs(H)$, and $cs_0(H)$ are complete metric spaces with the partial metric H^s defined by

$$H_\infty^p(u, v) := \sup_{n \in \mathbb{N}} \left\{ \sum_{k=0}^{n} H^s(u_k, v_k) \right\}, \quad (2)$$

where $u = (u_k)$ and $v = (v_k)$ are the elements of the sets $bs(H)$, $cs(H)$, or $cs_0(H)$.

Secondly, we introduce the sets $bv(H)$, $bv_q(H)$, and $bv_\infty(H)$ consisting of sequences of q-bounded variation by using the partial metric H^s with respect to the partial ordering $\sqsubseteq_H$, as follows:

$$bv(H) := \left\{ u = (u_k) \in \omega(F) : \sum_{k=0}^{\infty} H^s\left[(\Delta u)_k, \overline{0} \right] < \infty \right\},$$

$$bv_q(H) := \left\{ u = (u_k) \in \omega(F) : \sum_{k=0}^{\infty} H^s\left[(\Delta u)_k, \overline{0} \right]^q < \infty \right\},$$

$$bv_\infty(H) := \left\{ u = (u_k) \in \omega(F) : \sup_{k \in \mathbb{N}} H^s\left[(\Delta u)_k, \overline{0} \right] < \infty \right\}, \quad (3)$$

where the distance function H^s denotes the partial metric of fuzzy level sets defined by

$$H(u, v) = \sup_{\lambda \in [0,1]} p\left([u]_\lambda, [v]_\lambda \right)$$

$$= \sup_{\lambda\in[0,1]} \left\{\max\left\{u_\lambda^+, v_\lambda^+\right\} - \min\left\{u_\lambda^-, v_\lambda^-\right\}\right\}$$

$$= \max\left\{u_0^+, v_0^+\right\} - \min\left\{u_0^-, v_0^-\right\},$$

$$H^s(u,v) = 2H(u,v) - H(u,u) - H(v,v), \tag{4}$$

for any $u, v \in E^1$ with the partial ordering $\sqsubseteq_H$. One can conclude that the sets $bv(H)$, $bv_q(H)$, and $bv_\infty(H)$ are complete metric spaces with the following partial metrics:

$$H^\Delta(u,v) := \sum_{k=0}^{\infty} H^s\left[(\Delta u)_k, (\Delta v)_k\right],$$

$$H_q^\Delta(u,v) := \left\{\sum_{k=0}^{\infty} H^s\left[(\Delta u)_k, (\Delta v)_k\right]^q\right\}^{1/q}, \tag{5}$$

$$H_\infty^\Delta(u,v) := \sup_{k\in\mathbb{N}} H^s\left[(\Delta u)_k, (\Delta v)_k\right],$$

respectively, where $u = (u_k)$ and $v = (v_k)$ are the elements of the sets $bv(H)$, $bv_q(H)$, or $bv_\infty(H)$ and $(\Delta u)_k = u_k - u_{k+1}$ for all $k \in \mathbb{N}$.

Many authors have extensively developed the theory of the different sets of sequences and its matrix transformations [1, 2]. Following Başar [3, page 347], we note that Mursaleen and Basarır [4] have recently introduced some new sets of sequences of fuzzy numbers generated by a nonnegative regular matrix A some of which reduced to the Maddox's spaces $\ell_\infty(p;F)$, $c(p;F)$, $c_0(p;F)$, and $\ell(p;F)$ of sequences of fuzzy numbers for the special cases of that matrix A. Quite recently, Talo and Başar [5] have extended the main results of Başar and Altay [6] to fuzzy numbers and defined the alpha-, beta-, and gamma-duals of a set of sequences of fuzzy numbers and gave the duals of the classical sets of sequences of fuzzy numbers together with the characterization of the classes of infinite matrices of fuzzy numbers transforming one of the classical set into another one. Also, Kadak and Başar [7–9] have recently studied fourier series of fuzzy valued functions and gave some properties of the level sets together with some inclusion relations, in [10]. Finally, Kadak and Ozluk [11–13] have introduced the sets $\ell_\infty(H)$, $c(H)$, $c_0(H)$, and $\ell_p(H)$ of classical sequences of fuzzy level sets and sufficient conditions for partial completeness of these are established by means of fuzzy level sets.

The rest of this paper is organized as follows. In Section 2, some required definitions and consequences related with the partial metric and fuzzy level sets, sequences, and convergence are given. Section 3 is devoted to the completeness of the sets of sequences $bs(H)$, $cs(H)$, $cs_0(H)$ and $bv(H)$, $bv_\infty(H)$, $bv_q(H)$ of fuzzy level sets and some related notions. In the final section of the paper, the Köthe-Toeplitz duals of some classical sets are determined and given some properties including solidness.

2. Preliminaries, Background and Notation

Motivated by experience from computer science, nonzero self-distance seen to be plausible for the subject of finite and infinite sequences.

Definition 1 (see [14]). Let X be a nonempty set and p be a function from $X \times X$ to the set $\mathbb{R}^+$ of nonnegative real numbers. Then the pair (X, p) is called a *partial metric space* and p is a *partial metric* for X, if the following partial metric axioms are satisfied for all $x, y, z \in X$:

(P1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$,

(P2) $0 \le p(x,x) \le p(x,y)$,

(P3) $p(x,y) = p(y,x)$,

(P4) $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$.

Proposition 2 (Nonzero self-distance [15]). *Let S^ω be the set of all infinite sequences $x = (x_0, x_1, x_2, \ldots)$ over a set S. For all such sequences x and y let $d_s(x,y) = 2^{-k}$, where k is the largest number (possibly ∞) such that $x_i = y_i$ for each $i < k$. Thus $d_s(x,y)$ is defined to be 1 over 2 to the power of the length of the longest initial sequence common to both x and y. It can be shown that (S^ω, d_s) is a metric space.*

Each partial metric space thus gives rise to a metric space with the additional notion of nonzero self-distance introduced. Also, a partial metric space is a generalization of a metric space; indeed, if an axiom $p(x,x) = 0$ is imposed, then the above axioms reduce to their metric counterparts. Thus, a metric space can be defined to be a partial metric space in which each self-distance is zero.

It is clear that $p(x,y) = 0$ implies $x = y$ from (P1) and (P2). But, $x = y$ does not imply $p(x,y) = 0$, in general. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$.

Remark 3 (see [16]). Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(\cdot,\cdot)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ imply $p(x_n, y_n) \to p(x,y)$. For example, if $X = [0, +\infty)$ and $p(x,y) = \max\{x,y\}$ for $x, y \in X$, then for $\{x_n\} = \{1\}$, $p(x_n, x) = x = p(x,x)$ for each $x \ge 1$ and so, for example, $x_n \to 2$ and $x_n \to 3$ when $n \to \infty$.

Proposition 4 (see [17]). *Let $x, y \in X$ and define the partial distance function p by*

$$p : X \times X \longrightarrow \mathbb{R}^+$$

$$(x,y) \longmapsto p(x,y) = \max\{x,y\},$$

$$p : X \times X \longrightarrow \mathbb{R}^+ \tag{6}$$

$$(x,y) \longmapsto p(x,y) = -\min\{x,y\},$$

For $X = \mathbb{R}^+$ and $X = \mathbb{R}^-$, respectively. Then, $(\mathbb{R}^+, p)$ is complete partial metric space where the self-distance for any point $x \in \mathbb{R}^+$ is its value itself. The pair $(\mathbb{R}^-, p)$ is complete partial metric space for which p is called the usual partial metric on

$\mathbb{R}^-$, *and where the self-distance for any point* $x \in \mathbb{R}^-$ *is its absolute value.*

Proposition 5 (see [18]). *If* p *is a partial metric on* X*, then the function* p^s *defined by*

$$p^s : X \times X \longrightarrow \mathbb{R}^+$$
$$(x, y) \longmapsto p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y), \tag{7}$$

is a usual metric on X*. For example, in* $(\mathbb{R}^-, p)$*, where* p *is the usual partial metric on* $\mathbb{R}^-$*, we obtain the usual distance in* $\mathbb{R}^-$ *since for any* $x, y \in \mathbb{R}^-$, $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) = x + y - 2\min\{x, y\} = |x - y|$.

Definition 6 (see [15]). A partial order on X is a binary relation $\sqsubseteq$ on X such that

(i) $x \sqsubseteq x$ (reflexivity),

(ii) if $x \sqsubseteq y$ and $y \sqsubseteq x$ then $x = y$ (antisymmetry),

(iii) if $x \sqsubseteq y$ and $y \sqsubseteq z$ then $x \sqsubseteq z$ (transitivity).

A partially ordered set (or poset) is a pair $(X, \sqsubseteq)$ such that $\sqsubseteq$ is a partial order on X. For each partial metric space (X, p) let $\sqsubseteq_p$ be the binary relation over X such that $x \sqsubseteq_p y$ (to be read, x is part of y) if and only if $p(x, x) = p(x, y)$. Then it can be shown that $(X, \sqsubseteq_p)$ is a poset.

For the partial metric $\max\{a, b\}$ over the nonnegative reals, $\sqsubseteq_{\max}$ is the usual $\geq$ ordering. For intervals, $[a, b] \sqsubseteq_p [c, d]$ if and only if $[c, d]$ is a subset of $[a, b]$.

Definition 7 (cf. [17–20]). Let (X, p) be a partial metric space and (x_n) a sequence in (X, p). Then, we say the following:

(a) A sequence (x_n) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$.

(b) A sequence (x_n) is a *Cauchy sequence* if there exists (and is finite) $\lim_{m,n\to\infty} p(x_n, x_m)$.

(c) A partial metric space (X, p) is said to be *complete* if every Cauchy sequence (x_n) in X converges, with respect to the topology τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{m,n\to\infty} p(x_m, x_n)$. It is easy to see that every closed subset of a complete partial metric space is complete.

(d) A mapping $f : X \to X$ is called to be *continuous* at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0, \delta)) \subset B_p(f(x_0), \varepsilon)$.

(e) A sequence (x_n) in a partial metric space (X, p) converges to a point $x \in X$, for any $\epsilon > 0$ such that $x \in B_p(x, \epsilon)$, there exists $n_0 \geq 1$ so that for any $n \geq n_0$, $x_n \in B_p(x, \epsilon)$.

Lemma 8 (see [18]). *Let* (X, p) *be a partial metric space. Then,*

(i) (x_n) *is a Cauchy sequence in* (X, p) *if and only if it is a Cauchy sequence in the metric space* (X, p^s),

(ii) *a partial metric space* (X, p) *is complete if and only if the metric space* (X, p^s) *is complete. Furthermore,* $\lim_{n\to\infty} p^s(x_n, x) = 0$ *if and only if* $p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{m,n\to\infty} p(x_n, x_m)$.

In the partial metric space $(\mathbb{R}^-, p)$, the limit of the sequence $(-1/n)$ is 0 since one has $\lim_{n\to\infty} p^s(-1/n, 0)$, where p^s is the usual metric induced by p on $\mathbb{R}^-$.

2.1. The Level Sets of Fuzzy Numbers. A *fuzzy number* is a fuzzy set on the real axis, that is, a mapping $u : \mathbb{R} \to [0, 1]$ which satisfies the following four conditions.

(i) u is normal; that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(ii) u is fuzzy convex; that is, $u[\lambda x + (1 - \lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0, 1]$.

(iii) u is upper semicontinuous.

(iv) The set $[u]_0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact (cf. Zadeh [21]), where $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by E^1 and call it as *the space of fuzzy numbers*. *λ-level set* $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \lambda\}, & 0 < \lambda \leq 1, \\ \overline{\{x \in \mathbb{R} : u(x) > \lambda\}}, & \lambda = 0. \end{cases} \tag{8}$$

The set $[u]_\lambda$ is closed, bounded, and nonempty interval for each $\lambda \in [0, 1]$ which is defined by $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$. $\mathbb{R}$ can be embedded in E^1, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(x) = \begin{cases} 1, & x = r, \\ 0, & x \neq r. \end{cases} \tag{9}$$

Representation Theorem 1 (see [22]). *Let* $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ *for* $u \in E^1$ *and for each* $\lambda \in [0, 1]$*. Then the following statements hold.*

(i) u^- *is a bounded and nondecreasing left continuous function on* $]0, 1]$.

(ii) u^+ *is a bounded and nonincreasing left continuous function on* $]0, 1]$.

(iii) *The functions* u^- *and* u^+ *are right continuous at the point* $\lambda = 0$.

(iv) $u^-(1) \leq u^+(1)$.

Conversely, if the pair of functions u^- *and* u^+ *satisfies conditions (i)–(iv), then there exists a unique* $u \in E^1$ *such that* $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ *for each* $\lambda \in [0, 1]$*. The fuzzy number* u *corresponding to the pair of functions* u^- *and* u^+ *is defined by* $u : \mathbb{R} \to [0, 1]$, $u(x) = \sup\{\lambda : u^-(\lambda) \leq x \leq u^+(\lambda)\}$.

Now we give the definitions of triangular fuzzy numbers with the λ-level set.

Definition 9 (triangular fuzzy number, [23, Definition, page 137]). The membership function $\mu_{(u)}$ of a triangular fuzzy number u represented by (u_1, u_2, u_3) is interpreted, as follows:

$$\mu_{(u)}(x) = \begin{cases} \dfrac{x-u_1}{u_2-u_1}, & u_1 \le x \le u_2, \\ \dfrac{u_3-x}{u_3-u_2}, & u_2 \le x \le u_3, \\ 0, & x < u_1,\ x > u_3. \end{cases} \tag{10}$$

Then, the result $[u]_\lambda := [u^-(\lambda), u^+(\lambda)] = [(u_2-u_1)\lambda+u_1, (u_2-u_3)\lambda+u_3]$ holds for each $\lambda \in [0,1]$.

Let $u, v, w \in E^1$ and $\alpha \in \mathbb{R}$. Then the operations addition, scalar multiplication, and product defined on E^1 by $u+v=w \Leftrightarrow [w]_\lambda = [u]_\lambda + [v]_\lambda$ for all $\lambda \in [0,1]$ then $w^-(\lambda) = u^-(\lambda) + v^-(\lambda)$ and $w^+(\lambda) = u^+(\lambda) + v^+(\lambda)$ for all $\lambda \in [0,1]$.

Let W be the set of all closed bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$; that is, $A := [\underline{A}, \overline{A}]$. Define the relation d on W by $d(A,B) := \max\{|\underline{A}-\underline{B}|, |\overline{A}-\overline{B}|\}$. Then it can easily be observed that d is a metric on W (cf. Diamond and Kloeden [24]) and (W,d) is a complete metric space, (cf. Nanda [25]). Now, we can define the metric D on E^1 by means of the Hausdorff metric d as

$$\begin{aligned} D(u,v) &:= \sup_{\lambda\in[0,1]} d\left([u]_\lambda, [v]_\lambda\right) \\ &:= \sup_{\lambda\in[0,1]} \max\left\{\left|u^-(\lambda) - v^-(\lambda)\right|, \left|u^+(\lambda) - v^+(\lambda)\right|\right\}. \end{aligned} \tag{11}$$

Proposition 10 (see [26]). *Let $u, v, w, z \in E^1$ and $\alpha \in \mathbb{R}$. Then, the following statements hold.*

(i) *(E^1, D) is a complete metric space (cf. Puri and Ralescu [27]).*

(ii) $D(\alpha u, \alpha v) = |\alpha| D(u,v)$.

(iii) $D(u+v, w+v) = D(u,w)$.

(iv) $D(u+v, w+z) \le D(u,w) + D(v,z)$.

(v) $|D(u,\overline{0}) - D(v,\overline{0})| \le D(u,v) \le D(u,\overline{0}) + D(v,\overline{0})$.

Definition 11 (see [28]). The following statements hold.

(a) A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set $\mathbb{N}$ into the set E^1. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called as the general term of the sequence.

(b) A sequence $(u_n) \in \omega(F)$ is called convergent to $u \in E^1$, if and only if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $D(u_n, u) < \varepsilon$ for all $n \ge n_0$.

(c) A sequence $(u_n) \in \omega(F)$ is called bounded if and only if the set of its terms is a bounded set. That is to say that a sequence $(u_n) \in \omega(F)$ is said to be bounded if and only if there exist two fuzzy numbers m and M such that $m \preceq u_n \preceq M$ for all $n \in \mathbb{N}$. This means that $m^-(\lambda) \le u_n^-(\lambda) \le M^-(\lambda)$ and $m^+(\lambda) \le u_n^+(\lambda) \le M^+(\lambda)$ for all $\lambda \in [0,1]$.

The boundedness of the sequence $(u_n) \in \omega(F)$ is equivalent to the fact that

$$\sup_{n\in\mathbb{N}} D\left(u_n, \overline{0}\right) = \sup_{n\in\mathbb{N}} \sup_{\lambda\in[0,1]} \max\left\{\left|u_n^-(\lambda)\right|, \left|u_n^+(\lambda)\right|\right\} < \infty. \tag{12}$$

If the sequence $(u_k) \in \omega(F)$ is bounded then the sequences of functions $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are uniformly bounded in $[0,1]$.

3. Completeness of the Sets of Sequences with Respect to the Partial Metric

Following Kadak and Ozluk [11], we give the classical sets $\ell_\infty(H)$, $c(H)$, $c_0(H)$, and $\ell_p(H)$ consisting of the bounded, convergent, null, and p-summable sequences of fuzzy level sets with the partial metric H^s, as follows:

$$\begin{aligned} \ell_\infty(H) &:= \left\{u = (u_k) \in \omega(F) : \sup_{k\in\mathbb{N}} H^s\left(u_k, \overline{0}\right) < \infty, \right. \\ &\qquad \left. \overline{0} \in E^1\right\}, \\ c(H) &:= \left\{u = (u_k) \in \omega(F) : \lim_{k\to\infty} H^s(u_k, u) = 0 \right. \\ &\qquad \left. \text{for some } u \in E^1\right\}, \\ c_0(H) &:= \left\{u = (u_k) \in \omega(F) : \lim_{k\to\infty} H^s\left(u_k, \overline{0}\right) = 0\right\}, \\ \ell_p(H) &:= \left\{u = (u_k) \in \omega(F) : \sum_{k=0}^{\infty} H^s\left(u_k, \overline{0}\right)^p < \infty\right\}, \\ &\qquad \left(1 \le p < \infty\right). \end{aligned} \tag{13}$$

One can show that $\ell_\infty(H)$, $c(H)$, and $c_0(H)$ are complete metric spaces with the partial metric H_∞ defined by

$$H_\infty(u,v) := \sup_{k\in\mathbb{N}} \left\{H^s(u_k, v_k)\right\}, \tag{14}$$

where $u = (u_k)$ and $v = (v_k)$ are the elements of the sets $c(H)$, $c_0(H)$, or $\ell_\infty(H)$. Also, the space $\ell_p(H)$ is complete metric space with the partial metric H_p defined by

$$H_p(u,v) := \left\{\sum_{k=0}^{\infty} H^s(u_k, v_k)^p\right\}^{1/p}, \quad \left(1 \le p < \infty\right), \tag{15}$$

where $u = (u_k)$ and $v = (v_k)$ are the points of $\ell_p(H)$.

Theorem 12. *Let $\mu(H)$ denote any of the spaces $bs(H)$, $cs(H)$, and $cs_0(H)$, and $u = (u_k)$, $v = (v_k) \in \mu(H)$. Define the partial distance function H_∞^p on $\mu(H)$ by*

$$\begin{aligned} H_\infty^p &: \mu(H) \times \mu(H) \longrightarrow \mathbb{R}^+ \\ &(u,v) \longrightarrow H_\infty^p(u,v) := \sup_{n\in\mathbb{N}} \left\{\sum_{k=0}^{n} H^s(u_k, v_k)\right\}. \end{aligned} \tag{16}$$

Then, $(\mu(H), H_\infty^p)$ is a complete metric space.

Proof. Since the proof is similar for the spaces $cs(H)$ and $cs_0(H)$, we prove the theorem only for the space $bs(H)$. Let $u = (u_k)$, $v = (v_k)$, and $w = (w_k) \in bs(H)$. Then,

(i) by using the axiom (P1) in Definition 1, it is trivial that

$$\begin{aligned} u = v &\Longleftrightarrow H^p_\infty(u,v) = \sup_{n\in\mathbb{N}}\left\{\sum_{k=0}^{n} H^s(u_k,v_k)\right\} \\ &= \sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\{2H(u_k,v_k) - H(u_k,u_k) \\ &\qquad -H(v_k,v_k)\} = 0 \\ &\Longleftrightarrow \sup_n\sum_{k=0}^{n}\{H^s(u_k,u_k)\} = \sup_n\sum_{k=0}^{n}\{H^s(v_k,v_k)\} = 0 \\ &\Longleftrightarrow H^p_\infty(u,v) = H^p_\infty(u,u) = H^p_\infty(v,v). \end{aligned} \tag{17}$$

(ii) By using the axiom (P2) in Definition 1, it follows that

$$\begin{aligned} H^p_\infty(u,u) &= \sup_{n\in\mathbb{N}}\left\{\sum_{k=0}^{n} H^s(u_k,u_k)\right\} \\ &\le \sup_{n\in\mathbb{N}}\left\{\sum_{k=0}^{n} H^s(u_k,v_k)\right\} = H^p_\infty(u,v). \end{aligned} \tag{18}$$

(iii) By using the axiom (P3) in Definition 1, it is clear that

$$\begin{aligned} H^p_\infty(u,v) &= \sup_n\left\{\sum_{k=0}^{n} H^s(u_k,v_k)\right\} \\ &= \sup_n\left\{\sum_{k=0}^{n} H^s(v_k,u_k)\right\} = H^p_\infty(v,u). \end{aligned} \tag{19}$$

(iv) By using the axiom (P4) in Definition 1 with the inequalities $H(u_k,w_k) \le H(u_k,v_k) + H(v_k,w_k) - H(v_k,v_k)$ and $H^s(u_k,u_k) = 0$, we have

$$\begin{aligned} &H^p_\infty(u,w) \\ &= \sup_n\sum_{k=0}^{n}\{H^s(u_k,w_k) : k\in\mathbb{N}\} \\ &= \sup_n\sum_{k=0}^{n}\{2H(u_k,w_k) - H(u_k,u_k) - H(w_k,w_k)\} \\ &\le \sup_n\sum_{k=0}^{n}\{2[H(u_k,v_k) + H(v_k,w_k) - H(v_k,v_k)] \\ &\qquad -H(u_k,u_k) - H(w_k,w_k)\} \\ &= \sup_n\sum_{k=0}^{n}\{2H(u_k,v_k) - H(u_k,u_k) - H(v_k,v_k) \\ &\qquad +2H(v_k,w_k) - H(v_k,v_k) - H(w_k,w_k)\} \\ &\le \sup_n\sum_{k=0}^{n}\{2H(u_k,v_k) - H(u_k,u_k) - H(v_k,v_k)\} \\ &\quad + \sup_n\sum_{k=0}^{n}\{2H(v_k,w_k) - H(v_k,v_k) - H(w_k,w_k)\} \\ &= \sup_n\sum_{k=0}^{n}\{H^s(u_k,v_k)\} + \sup_n\sum_{k=0}^{n}\{H^s(v_k,w_k)\} \\ &\quad - \sup_n\sum_{k=0}^{n}\{H^s(v_k,v_k)\} \\ &= H^p_\infty(u,v) + H^p_\infty(v,w) - H^p_\infty(v,v). \end{aligned} \tag{20}$$

Therefore, one can conclude that $(bs(H), H^p_\infty)$ is a partial metric space on $bs(H)$. It remains to prove the completeness of the space $bs(H)$. Let (u_m) be any Cauchy sequence on $bs(H)$, where $u_m = \{u_1^{(m)}, u_2^{(m)}, \ldots\}$. Then, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ for all $m, r > N$ such that

$$H^p_\infty(u_m,u_r) = \sup_{n\in\mathbb{N}}\left\{\sum_{k=0}^{n} H^s\left(u_k^{(m)},u_k^{(r)}\right)\right\} < \epsilon. \tag{21}$$

A fortiori, for every fixed $k \in \mathbb{N}$ and for $m, r > N$

$$H^s\left(\sum_{k=0}^{n} u_k^{(m)}, \sum_{k=0}^{n} u_k^{(r)}\right) < \epsilon. \tag{22}$$

Hence for every fixed $k \in \mathbb{N}$, by using the completeness of (E^1, H^s) in Theorem 3.1 [11], we say the sequence $(u_k^{(m)}) = \{u_k^{(1)}, u_k^{(2)}, \ldots\}$ is a Cauchy sequence and is uniformly convergent. Now, we suppose that $\lim_{m\to\infty} u_k^{(m)} = u_k$ and $u = (u_1, u_2, \ldots)$. We must show that

$$\lim_{m\to\infty} H^p_\infty(u_m,u) = 0, \quad u \in bs(H). \tag{23}$$

The constant $N \in \mathbb{N}$ for all $m > N$, taking the limit for $r \to \infty$ in (22), we obtain

$$H^s\left(\sum_{k=0}^{n} u_k^{(m)}, \sum_{k=0}^{n} u_k\right) < \epsilon, \tag{24}$$

for all $k \in \mathbb{N}$. Since $(u_k^{(m)}) \in bs(H)$, there exists a number $M \succ \overline{0}$ such that $H^s(\sum_{k=0}^{n} u_k^{(m)}, \overline{0}) \le M$ for all $k \in \mathbb{N}$. Thus, (24) gives together with the triangle inequality of partial metric for $m > N$ that

$$\begin{aligned} &H^s\left(\sum_{k=0}^{n} u_k, \overline{0}\right) \\ &\le H^s\left(\sum_{k=0}^{n} u_k, \sum_{k=0}^{n} u_k^{(m)}\right) + H^s\left(\sum_{k=0}^{n} u_k^{(m)}, \sum_{k=0}^{n} \overline{0}\right) \\ &\quad - H^s\left(\sum_{k=0}^{n} u_k^{(m)}, \sum_{k=0}^{n} u_k^{(m)}\right) \le \epsilon + M. \end{aligned} \tag{25}$$

It is clear that (25) holds for every $k \in \mathbb{N}$ whose right-hand side does not involve k. This leads us to the consequence that $u = (u_1, u_2, \ldots)$ is bounded sequence of fuzzy numbers hence $u \in bs(H)$. Also, from (24) we obtain for $m > N$ that

$$H_\infty^p(u_m, u) = \sup_{k\in\mathbb{N}} \left\{ \sum_{k=0}^{n} H^s\left(u_k^{(m)}, u_k\right) \right\} \leq \epsilon. \tag{26}$$

This shows that (23) holds and $\lim_m H_\infty^p(u_m, u) = 0$. Since (u_m) is an arbitrary Cauchy sequence, $bs(H)$ is complete. □

Theorem 13. *Define the distance functions $H_\Delta(u, v)$, $H_q^\Delta(u, v)$, and $H_\infty^\Delta(u, v)$ by*

$$\begin{aligned} H^\Delta(u, v) &:= \sum_{k=0}^{\infty} \left\{ H^s\left[(\Delta u)_k, (\Delta v)_k\right] \right\}, \\ H_q^\Delta(u, v) &:= \left\{ \sum_{k=0}^{\infty} H^s\left[(\Delta u)_k, (\Delta v)_k\right]^q \right\}^{1/q}, \\ H_\infty^\Delta(u, v) &:= \sup_{k\in\mathbb{N}} \left\{ H^s\left[(\Delta u)_k, (\Delta v)_k\right] \right\}, \end{aligned} \tag{27}$$

where $u = (u_k)$, $v = (v_k)$ are the element of the spaces $bv(H)$, $bv_q(H)$, or $bv_\infty(H)$, respectively. Then, $(bv(H), H^\Delta)$, $(bv_q(H), H_q^\Delta)$, and $(bv_\infty(H), H_\infty^\Delta)$ are complete metric spaces.

Proof. Since the proof is similar for the spaces $bv(H)$ and $bv_\infty(H)$, we prove the theorem only for the space $bv_q(H)$. One can easily establish that H_q^Δ defines a metric on $bv_q(H)$. Let $x^i = \{x_0^{(i)}, x_1^{(i)}, \ldots\}$ be any Cauchy sequence on $bv_q(H)$. Then for every $\epsilon > 0$, there exists a positive integer $n_0(\epsilon) \in \mathbb{N}$ for all $i, j > n_0$, such that

$$H_q^\Delta\left(x^i, x^j\right) := \sum_{n=0}^{\infty} \left\{ H^s\left[(\Delta x)_n^i, (\Delta x)_n^j\right]^q \right\}^{1/q} < \epsilon, \tag{28}$$

where $(\Delta x)_n = x_n - x_{n-1}$ and $x_{-1} = \bar{0}$. We obtain for each fixed $n \in \mathbb{N}$ from (28) that

$$H^s\left[(\Delta x)_n^i, (\Delta x)_n^j\right] < \epsilon, \tag{29}$$

for all $i, j > n_0(\epsilon)$, which leads us to the fact that the sequence $\{(\Delta x)_n^i\}$ is a Cauchy sequence and is convergent. Now, we suppose that $(\Delta x)_n^i \to (\Delta x)_n$ as $n \to \infty$. We have from (29) for each $m \in \mathbb{N}$ and $i, j > n_0(\epsilon)$, that

$$\sum_{k=0}^{m} H^s\left[(\Delta x)_k^i, (\Delta x)_k^j\right]^q \leq H_q^\Delta\left(x^i, x^j\right)^q < \epsilon^q. \tag{30}$$

Take any $i > n_0(\epsilon)$. Let firstly $j \to \infty$ and nextly $m \to \infty$ in (30) to obtain $H_q^\Delta(x^i, x) \leq \epsilon$. Finally, by using Minkowski's inequality for each $m \in \mathbb{N}$

$$\begin{aligned} \left\{ \sum_{k=0}^{m} H^s\left(\Delta x\right)_k, \bar{0}\right)^q \Bigg\}^{1/q} &\leq H_q^\Delta\left(x^i, x\right) + H_q^\Delta\left(x^i, \bar{0}\right) \\ &\leq \epsilon + H_q^\Delta\left(x^i, \bar{0}\right) < \infty, \end{aligned} \tag{31}$$

which implies that $x \in bv_q(H)$. Since $H_q^\Delta(x^i, x) \leq \epsilon$ for all $i > n_0(\epsilon)$, it follows that $x^i \to x$ as $i \to \infty$. Since (x^i) is an arbitrary Cauchy sequence, the space $bv_q(H)$ is complete. This step concludes the proof. □

4. The Duals of the Sets of Sequence with the Partial Metric

The idea of dual sequence space, which plays an important role in the representation of linear functionals and the characterization of matrix transformations between sequence spaces, was introduced by Köthe and Toeplitz [29], whose main results concerned α-duals. An account of the duals of sequence spaces can be found in Köthe [30]. One can also find about different types of duals of sequence spaces in Maddox [31].

In this section, we focus on the alpha-, beta- and gamma-duals of the classical sets of sequences of fuzzy numbers with partial metric. For the sets $\lambda(H)$, $\mu(H)$, and $S(\lambda(H), \mu(H))$ of sequences defined by

$$\begin{aligned} S\left(\lambda(H), \mu(H)\right) := \{ &w = (w_k) \in \omega(F) : (w_k z_k) \in \mu(H) \\ &\forall z = (z_k) \in \lambda(H) \}, \end{aligned} \tag{32}$$

is called the multiplier sets of $\lambda(H)$ and $\mu(H)$ for all $k \in \mathbb{N}$. One can easily observe for a sequence set $\nu(H)$ of fuzzy level sets that the inclusions

$$\begin{aligned} S\left(\lambda(H), \mu(H)\right) &\subset S\left(\nu(H), \mu(H)\right) \quad \text{if } \nu(H) \subset \lambda(H), \\ S\left(\lambda(H), \mu(H)\right) &\subset S\left(\lambda(H), \nu(H)\right) \quad \text{if } \mu(H) \subset \nu(H), \end{aligned} \tag{33}$$

hold. The alpha-, beta- and gamma-duals $\{\lambda(H)\}^\alpha$, $\{\lambda(H)\}^\beta$, and $\{\lambda(H)\}^\gamma$ of a set $\lambda(H) \subset \omega(F)$ are, respectively, defined by

$$\begin{aligned} \{\lambda(H)\}^\alpha &:= \{ w = (w_k) \in \omega(F) : (w_k z_k) \in \ell_1(H) \\ &\qquad \forall z = (z_k) \in \lambda(H) \}, \\ \{\lambda(H)\}^\beta &:= \{ w = (w_k) \in \omega(F) : (w_k z_k) \in cs(H) \\ &\qquad \forall z = (z_k) \in \lambda(H) \}, \\ \{\lambda(H)\}^\gamma &:= \{ w = (w_k) \in \omega(F) : (w_k z_k) \in bs(H) \\ &\qquad \forall z = (z_k) \in \lambda(H) \}, \end{aligned} \tag{34}$$

where $(w_k z_k)$ the coordinatewise product of the sequences w and z of level sets for all $k \in \mathbb{N}$. Then $\{\lambda(H)\}^\beta$ is called β-dual of $\lambda(H)$ or the set of all factor sequences of $\lambda(H)$ are in $cs(H)$. Firstly, we give a remark concerning with the convergence factor sequences of fuzzy level sets with partial metric.

Remark 14. Let $\emptyset \neq \lambda(H) \subset \omega(F)$. Then the following statements are valid.

(a) $\{\lambda(H)\}^\beta$ is a set of sequence and $\varphi(F) < \{\lambda(H)\}^\beta < \omega(F)$ ("<" stands for "is a linear subset of") where

$$\varphi(F) := \left\{ u = (u_k) : \exists N \in \mathbb{N},\ \forall k \geq N,\ u_k = \bar{0} \right\}. \tag{35}$$

(b) If $\lambda(H) \subset \mu(H) \subset \omega(F)$ then $\{\mu(H)\}^\beta < \{\lambda(H)\}^\beta$.

(c) $\lambda(H) \subset \{\lambda(H)\}^{\beta\beta} := (\{\lambda(H)\}^\beta)^\beta$.

(d) $\{\varphi(F)\}^\beta = \omega(F)$ and $\{\omega(F)\}^\beta = \varphi(F)$.

Proof. Since the proof is trivial for conditions (b) and (c), we prove only (a) and (d). Let $m = (m_k)$ and $n = (n_k) \in \{\lambda(H)\}^\beta$.

(a) Let $l \in \lambda(H)$. Then we get $(m_k l_k) \in cs(H)$; $(n_k l_k) \in cs(H)$ and $(m_k + n_k)l_k = (m_k l_k) + (n_k l_k) \in cs(H)$. Since l is arbitrary, $m + n \in \{\lambda(H)\}^\beta$. For any $\alpha \in \mathbb{R}$ and $w = (w_k) \in \{\lambda(H)\}^\beta$ we have

$$(\alpha w_k) l_k = \alpha (w_k l_k) \in cs(H), \tag{36}$$

and we get $\alpha w \in \{\lambda(H)\}^\beta$. Therefore, $\{\lambda(H)\}^\beta$ is a linear subset of $\omega(F)$.

(d) Using (a) we need only show $\{\omega(F)\}^\beta \subset \varphi(F)$. Suppose that $w = (w_n) \in \{\omega(F)\}^\beta$ and $z = (z_n)$ be given with geometric division by $z_n := (1/w_n)$ if $w_n \neq \overline{0}$ and $z_n := \overline{0}$ otherwise. By taking into account the set $\varphi(F)$ from the case (a), then there exists an integer $N \in \mathbb{N}$ for all $n \geq N$ such that $w_n = \overline{0}$. Thus, we have

$$\sum_{n=0}^{\infty} w_n z_n = \sum_n 1 \quad \left(w_n \neq \overline{0}\right). \tag{37}$$

Further, $(w_n z_n) \in cs(H)$ implies that $w \in \varphi(F)$. The rest is an immediate consequence of this part; we omit the detail. □

Theorem 15. *The following statements hold.*

(a) $\{c_0(H)\}^\beta = \{c(H)\}^\beta = \{\ell_\infty(H)\}^\beta = \ell_1(H)$.

(b) $\{\ell_1(H)\}^\beta = \ell_\infty(H)$.

Proof. (a) Obviously $\{\ell_\infty(H)\}^\beta \subset \{c(H)\}^\beta \subset \{c_0(H)\}^\beta$ by Remark 14(b). Then we must show that $\ell_1(H) \subset \{\ell_\infty(H)\}^\beta$ and $\{c_0(H)\}^\beta \subset \ell_1(H)$. Now, consider $w = (w_k) \in \ell_1(H)$ and $z = (z_k) \in \ell_\infty(H)$ are given. Then

$$\sum_{k=0}^{n} H^s\left(w_k z_k, \overline{0}\right) \leq \sup_k H^s\left(z_k, \overline{0}\right) \sum_{k=0}^{n} H^s\left(w_k, \overline{0}\right) < \infty, \tag{38}$$

which implies that $wz \in cs(H)$. So the condition $\ell_1(H) \subset \{\ell_\infty(H)\}^\beta$ holds.

Conversely, for a given $y = (y_k) \in \omega(F) \setminus \ell_1(H)$ we prove the existence of an $x \in c_0(H)$ with $yx \notin cs(H)$. According to $y \notin \ell_1(H)$ we may take an index sequence (n_p) which is a strictly increasing real valued sequence with $n_0 = 0$ and $\sum_{k=n_{p-1}}^{n_p-1} H^s(y_k, \overline{0}) > p$ $(p \in \mathbb{N})$. If we define $x = (x_k) \in c_0(H)$ by $x_k := ((\operatorname{sgn} y_k)/p)$, where the real signum function defined by

$$\operatorname{sgn}(u) := \begin{cases} \dfrac{u}{|u|}, & u \neq \overline{0}, \\ 0, & u = \overline{0}, \end{cases} \tag{39}$$

for all $u = (u_k) \in E^1$, thus, we get

$$\sum_{k=n_{p-1}}^{n_p-1} (y_k x_k) = \frac{1}{p} \sum_{k=n_{p-1}}^{n_p-1} y_k (\operatorname{sgn} y_k) = \frac{1}{p} \sum_{k=n_{p-1}}^{n_p-1} H^s\left(y_k, \overline{0}\right) \geq 1, \tag{40}$$

for all $n_{p-1} \leq k < n_p$. Therefore $yx \notin cs(H)$ and thus $y \notin \{c_0(H)\}^\beta$. Hence $\{c_0(H)\}^\beta \subset \ell_1(H)$.

(b) From the condition (c) of Remark 14 we have $\ell_\infty(H) \subset (\{\ell_\infty(H)\}^\beta)^\beta = \{\ell_1(H)\}^\beta$ since $\{\ell_\infty(H)\}^\beta = \ell_1(H)$. Now we assume the existence of a $w = (w_n) \in \{\ell_1(H)\}^\beta \setminus \ell_\infty(H)$. Since w is an unbounded sequence there exists a subsequence (w_{n_k}) of (w_n) such that $H^s(w_{n_k}, \overline{0}) \geq (k+1)^2$ for all $k \in \mathbb{N}_1$. The sequence (x_n) is defined by $x_n := (\operatorname{sgn}(w_{n_k})/(k+1)^2)$ if $n = n_k$ and $\overline{0}$ otherwise. Then $x \in \ell_1(H)$. However

$$\sum_n w_n x_n = \sum_k \frac{H^s\left(w_{n_k}, \overline{0}\right)}{(k+1)^2} \geq \sum_k 1 = \infty. \tag{41}$$

Hence $w \notin \{\ell_1(H)\}^\beta$, which contradicts our assumption and $\{\ell_1(H)\}^\beta \subset \ell_\infty(H)$. This step completes the proof. □

Further to the statements in Remark 14 we make the following remarks which are immediate consequences of the definition of the ζ-duals ($\zeta \in \{\alpha, \beta, \gamma\}$).

Remark 16. Let $\emptyset \neq \lambda(H) \subset \omega(F)$. Then the following statements are valid.

(a) $\varphi(F) < \{\lambda(H)\}^\alpha < \{\lambda(H)\}^\beta < \{\lambda(H)\}^\gamma < \omega(F)$; in particular, $\{\lambda(H)\}^\zeta$ is a set of sequence.

(b) If $\lambda(H) < \mu(H) < \omega(F)$ then $\{\mu(H)\}^\zeta < \{\lambda(H)\}^\zeta$.

(c) If I is an index set, if $\lambda(H)_i$ are sets of sequences and if $\lambda(H) := \bigcup_{i \in I} \lambda(H)_i$, then

$$\langle \lambda(H) \rangle^\zeta = \bigcap_{i \in I} \{\lambda(H)_i\}^\zeta, \tag{42}$$

where the notation "$\langle\rangle$" stands for the span of linear subset in $\mathbb{R}$.

(d) If $\lambda(H) < \{\lambda(H)\}^{\zeta\zeta} := (\{\lambda(H)\}^\zeta)^\zeta$. $(\zeta \in \{\alpha, \beta, \gamma\})$

Proof. Condition (b) is obviously true, and (a) follows from $\ell_\infty(H) < cs(H) < bs(H)$. We only show conditions (c) and (d) taking $\zeta = \alpha$. Other parts can be obtained in a similar way.

(c) Now, as an immediate consequence $\lambda(H)_i \subset \langle \lambda(H) \rangle$ that the following conditions

$$\langle \lambda(H) \rangle^\alpha \subset \{\lambda(H)_i\}^\alpha, \quad \langle \lambda(H) \rangle^\alpha \subset \bigcap_{i \in I} \{\lambda(H)_i\}^\alpha, \tag{43}$$

hold by (b). On the other hand, if $y \in \bigcap_{i \in I} \{\lambda(H)_i\}^\alpha$, that is $y \in \{\lambda(H)_i\}^\alpha$, then $xy \in \ell_1(H)$ for all $x \in \lambda(H)$ and therefore $y \in \{\lambda(H)\}^\alpha \subset \langle \lambda(H) \rangle^\alpha$.

(d) We prove $\lambda(H) < \{\lambda(H)\}^{\alpha\alpha}$. Let $w \in \lambda(H)$; then, $wz \in \ell_1(H)$ for all $z \in \{\lambda(H)\}^{\alpha}$; thus, $w \in \{\lambda(H)\}^{\alpha\alpha}$ and $\lambda(H) < \{\lambda(H)\}^{\alpha\alpha}$ by (a). □

In general $\lambda(H) \neq \{\lambda(H)\}^{\zeta\zeta}$ as we get from Theorem 15(a) in the case of $\zeta = \beta$ and $\lambda(H) := c_0(H)$. We have $\{c_0(H)\}^{\beta\beta} = \ell_\infty(H) \neq c_0(H)$. This remark gives rise to the following definition.

Definition 17 (ζ-space, Köthe space). Let $\zeta \in \{\alpha, \beta, \gamma\}$, and let $\lambda(H)$ be a set of sequence. $\lambda(H)$ is called ζ-space if $\lambda(H) = \{\lambda(H)\}^{\zeta\zeta}$. Further, an α-space is also called a Köthe space or perfect sequence space.

From Remarks 4.3(d) and (b) we obtain immediately the following remark.

Remark 18. If $\lambda(H)$ is a set of sequence over real field and $(\zeta \in \{\alpha, \beta, \gamma\})$, then $\{\lambda(H)\}^{\zeta}$ is a ζ-space; that is, $\{\lambda(H)\}^{\zeta} = \{\lambda(H)\}^{\zeta\zeta\zeta}$.

Now we look for sufficient conditions for $\{\lambda(H)\}^{\alpha} = \{\lambda(H)\}^{\beta} = \{\lambda(H)\}^{\gamma}$. This gives rise to the notion of solidity.

Definition 19 (Solidness). Let $\lambda(H)$ be a set of sequence over the field $\mathbb{R}$. Then $\lambda(H)$ is called solid if

$$\left\{u = (u_k) \in \omega(F) : \exists (x_k) \in \lambda(H) \ \ \forall k \in \mathbb{N} : H^s(u_k, \overline{0}) \leq H^s(x_k, \overline{0})\right\} \subset \lambda(H). \tag{44}$$

Theorem 20. *Consider $\lambda(H) < \omega(F)$ is any set of sequence over the field $\mathbb{R}$; then, the following statements hold.*

(a) *If $\lambda(H)$ is a Köthe space, then $\lambda(H)$ is solid.*

(b) *If $\lambda(H)$ is solid, then $\{\lambda(H)\}^{\alpha} = \{\lambda(H)\}^{\beta} = \{\lambda(H)\}^{\gamma}$.*

(c) *If $\lambda(H)$ is a Köthe space, then $\lambda(H)$ is a ζ-space.*

Proof. Let $\lambda(H) < \omega(H)$ be a set of sequence over the field $\mathbb{R}$.

(a) If $\lambda(H)$ is a Köthe space and $u \in \omega(F)$, then $u \in \{\lambda(H)\}^{\alpha\alpha}$ if and only if the condition $uz \in \ell_1(H)$ holds for all $z \in \{\lambda(H)\}^{\alpha}$. Besides this we obtain $H^s(v_k, \overline{0}) \leq H^s(u_k, \overline{0})$ for $u = (u_k) \in \lambda(H)$ and $v = (v_k) \in \omega(F)$ and the statement

$$\sum_k H^s\left(v_k z_k, \overline{0}\right) \leq \sum_k H^s\left(u_k z_k, \overline{0}\right) < \infty, \tag{45}$$

holds for each $z \in \{\lambda(H)\}^{\alpha}$. Therefore $vz \in \ell_1(H)$. Hence $v \in \lambda(H)$ and $\lambda(H)$ is solid over the real field.

(b) Consider $\lambda(H)$ is solid. To show $\{\lambda(H)\}^{\alpha} = \{\lambda(H)\}^{\beta} = \{\lambda(H)\}^{\gamma}$, it suffices to verify $\{\lambda(H)\}^{\gamma} < \{\lambda(H)\}^{\alpha}$ as we have Remark 16 (a). So, let $v = (v_k) \in \{\lambda(H)\}^{\gamma}$; that is,

$$\sup_{n\in\mathbb{N}} \left\{\sum_{k=0}^{n} H^s\left(u_k v_k, \overline{0}\right)\right\} < \infty \quad \text{for every } u = (u_k) \in \lambda(H). \tag{46}$$

By taking into account solidness of $\lambda(H)$, for $z = (z_k) \in \lambda(H)$, where $z_k = u_k \operatorname{sgn}(u_k v_k)$ and the condition $H^s(z_k, \overline{0}) \leq H^s(u_k, \overline{0})$ holds; there exists a sequence $u = (u_k) \in \lambda(H)$ for all $k \in \mathbb{N}$. Therefore by combining this with the inclusion (46) we deduce that the condition

$$\sum_{k=0}^{N} H^s\left(u_k v_k, \overline{0}\right) = \sum_{k=0}^{N} v_k u_k \operatorname{sgn}\left(u_k v_k\right) = H^s\left(\sum_{k=0}^{N} v_k z_k, \overline{0}\right) \leq \sup_{n\in\mathbb{N}} \left\{H^s\left(\sum_{k=0}^{n} v_k z_k, \overline{0}\right)\right\} < \infty, \tag{47}$$

holds and $uv \in \ell_1(H)$. Hence $v \in \{\lambda(H)\}^{\alpha}$ and $\{\lambda(H)\}^{\gamma} < \{\lambda(H)\}^{\alpha}$.

(c) This is an obvious consequence of Remark 18 and conditions (a), (b) in Theorem 20. □

Theorem 21. *The following statements hold.*

(a) *The sets $\varphi(F)$, $\omega(F)$, $\ell_p(H)$, $c_0(H)$, and $\ell_\infty(H)$ of sequences are solid.*

(b) *The sets $c(H)$ and $bv(H)$ of sequences are not solid; therefore, none of them is a Köthe space.*

(c) *For each $\zeta \in \{\alpha, \beta, \gamma\}$, then*

(i) *$\{\ell_1(H)\}^{\zeta} = \ell_\infty(H)$ and $\{\ell_\infty(H)\}^{\zeta} = \ell_1(H)$,*

(ii) *$\{\omega(F)\}^{\zeta} = \varphi(F)$ and $\{\varphi(F)\}^{\zeta} = \omega(F)$.*

(d) *If $\zeta \in \{\alpha, \beta, \gamma\}$ and $c_0(H) < \mu(H) < \ell_\infty(H)$, then $\{\mu(H)\}^{\zeta} = \ell_1(H)$ and $\mu(H) \subset \{\mu(H)\}^{\zeta\zeta} = \ell_\infty(H)$. In particular $\{c_0(H)\}^{\zeta} = \{c(H)\}^{\zeta} = \ell_1(H)$, and each of $c(H), c_0(H)$ is not a ζ-space.*

Proof. Given specified sets are solid in (a) and (b) is an immediate consequence of their definition. Additionally, the parts (i) and (ii) of (c) can be obtained Theorem 15 and Remark 14(d). Since $c_0(H)$ and $\ell_\infty(H)$ are solid, we know that $\{c_0(H)\}^{\zeta} = \{\ell_\infty(H)\}^{\zeta} = \ell_1(H)$. So the statements in (d) obtain from Remark 16(b). □

Next, we determine the ζ-duals of the spaces $cs(H)$, $bs(H)$, $bv(H)$, and $bv_0(H)$. We will find that none of these sets is solid; in particular, none of them is a Köthe space.

Theorem 22. *The following statements hold.*

(a) $\{cs(H)\}^{\alpha} = \{bv(H)\}^{\alpha} = \{bv_0(H)\}^{\alpha} = \{bs(H)\}^{\alpha} = \ell_1(H)$.

(b) $\{cs(H)\}^{\beta} = bv(H)$, $\{bv(H)\}^{\beta} = cs(H)$, $\{bv_0(H)\}^{\beta} = bs(H)$, $\{bs(H)\}^{\beta} = bv_0(H)$.

(c) $\{cs(H)\}^{\gamma} = bv(H)$, $\{bv(H)\}^{\gamma} = bs(H)$, $\{bv_0(H)\}^{\gamma} = bs(H)$, $\{bs(H)\}^{\gamma} = bv(H)$.

In particular the sets $cs(H)$, $bs(H)$, $bv(H)$, and $bv_0(H)$ of sequences are β-spaces, but they are not Köthe spaces. Moreover, the sets $bs(H)$ and $bv(H)$ of sequences are γ-spaces, whereas both $cs(H)$ and $bv_0(H)$ are not γ-spaces. None of the spaces $cs(H)$, $bs(H)$, $bv(H)$, and $bv_0(H)$ is solid.

Proof. We prove the cases for the spaces $\{cs(H)\}^\zeta, \zeta \in \{\alpha, \beta, \gamma\}$ and the proofs of all other cases are quite similar.

(a) Let $x = (x_k) \in cs(H)$ and $y = (y_k) \in \ell_1(H)$. Then,

$$\sum_k H^s\left(y_k x_k, \overline{0}\right) \leq \sup_k H^s\left(x_k, \overline{0}\right) \sum_k H^s\left(y_k, \overline{0}\right) < \infty \quad \forall k \in \mathbb{N}. \tag{48}$$

Therefore, $y \in \{cs(H)\}^\alpha$ which gives that $\ell_1(H) \subset \{cs(H)\}^\alpha$.

Conversely, suppose that $y = (y_k) \in \{cs(H)\}^\alpha \setminus \ell_1(H)$. Then we can construct an index sequence (n_p) with $n_p < n_{p+1}$ and $\sum_{k=n_p+1}^{n_{p+1}} H^s(y_k, \overline{0}) > 4^p$. Define $x = (x_k)$ by

$$x_k := \begin{cases} (-1)^k 2^{-p}, & n_p < k \leq n_{p+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{49}$$

Then $x = (x_k) \in cs(H)$. According to the choice of n_p the inequalities

$$\sum_k H^s\left(y_k x_k, \overline{0}\right) \geq \sum_p 2^{-p} \sum_{k=n_p+1}^{n_{p+1}} H^s\left(y_k, \overline{0}\right) \geq \sum_p 2^p, \tag{50}$$

hold. Thus $xy \notin \ell_1(H)$, which implies $y \notin \{cs(H)\}^\alpha$. This contradicts that $y \in \{cs(H)\}^\alpha$. Therefore $\{cs(H)\}^\alpha \subset \ell_1(H)$.

As well if we take the sequence (x_k) by

$$x_k := \begin{cases} 2^{-p}, & n_p < k \leq n_{p+1}, \\ 0, & \text{otherwise,} \end{cases} \tag{51}$$

the condition $\{bv_0(H)\}^\alpha \subset \ell_1(H)$ holds.

(b) Let $u = (u_k) \in \{cs(H)\}^\beta$ and $w = (w_k) \in c_0(H)$. Define the sequence $v = (v_k) \in cs(H)$ by $v_k = (w_k - w_{k+1})$ for all $k \in \mathbb{N}$. Therefore, $\sum_k u_k v_k$ converges, but

$$\sum_{k=0}^{n} (w_k - w_{k+1}) u_k = \left[\sum_{k=1}^{n-1} w_k (u_k - u_{k-1})\right] - w_{n+1} u_n, \tag{52}$$

and the inclusion $\ell_1(H) \subset cs(H)$ yields that $(u_k) \in \{cs(H)\}^\beta \subset \{\ell_1(H)\}^\beta = \ell_\infty(H)$. Then we derive by passing to the limit in (52) as $n \to \infty$ which implies that

$$\sum_{k=0}^{\infty} (w_k - w_{k+1}) u_k = \sum_{k=0}^{\infty} w_k (u_k - u_{k-1}), \tag{53}$$

for every $k \in \mathbb{N}$. Hence $(u_k - u_{k-1}) \in \{c_0(H)\}^\beta = \{c_0(H)\}^\alpha = \ell_1(H)$; that is, $u \in bv(H)$. Therefore, $\{cs(H)\}^\beta \subseteq bv(H)$.

Conversely, suppose that $u = (u_k) \in bv(H)$. Then, $(u_k - u_{k-1}) \in \ell_1(H)$. Further, if $v = (v_k) \in cs(H)$, the sequence (w_n) defined by $w_n = \sum_{k=0}^{n} v_k$ for all $k \in \mathbb{N}$, is an element of the space $c(H)$. Since $\{c(H)\}^\alpha = \ell_1(H)$, the series $\sum_k w_k(u_k - u_{k+1})$ is convergent. Also, we have

$$\sum_{k=m}^{n} (w_k - w_{k-1}) u_k \leq \left[\sum_{k=m}^{n-1} w_k (u_k - u_{k+1})\right] + w_n u_n - w_{m-1} u_m. \tag{54}$$

Since $(w_n) \in c(H)$ and $(u_k) \in bv(H) \subset c(H)$, the right-hand side of inequality (54) converges to zero as $m, n \to \infty$. Hence, the series $\sum_{k=0}^{\infty}(w_k - w_{k-1})u_k$ or $\sum_{k=0}^{\infty} u_k v_k$ converges and $bv(H) \subseteq \{cs(H)\}^\beta$.

(c) By using (a), it is known that $bv(H) \subseteq \{cs(H)\}^\beta$ and since $\{cs(H)\}^\beta \subset \{cs(H)\}^\gamma$, so $bv(H) \subset \{cs(H)\}^\gamma$. We need to show that $\{cs(H)\}^\gamma \subset bv(H)$. Let $u = (u_n) \in \{cs(H)\}^\gamma$ and $v = (v_n) \in c_0(H)$. Then, for the sequence $(w_n) \in cs(H)$ defined by $w_n = (v_n - v_{n+1})$ for all $n \in \mathbb{N}$, we can find a number $K > 0$ such that $H^s(\sum_{k=0}^{n} u_k w_k, \overline{0}) \leq K$ for all $n \in \mathbb{N}$. Since $(v_n) \in c_0(H)$ and $(u_n) \in \{cs(H)\}^\gamma \subset \ell_\infty(H)$, there exists a real number $M > 0$ such that $H^s(u_n v_n, \overline{0}) \leq M$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} &H^s\left(\sum_{k=0}^{n} (u_k - u_{k-1}) v_k, \overline{0}\right) \\ &\leq H^s\left(\sum_{k=1}^{n+1} u_k (v_k - v_{k+1}), \overline{0}\right) \\ &\quad + H^s\left(v_{n+2} u_{n+1}, \overline{0}\right) \leq K + M. \end{aligned} \tag{55}$$

Hence $(u_k - u_{k-1}) \in \{c_0(H)\}^\gamma = \{c_0(H)\}^\alpha = \ell_1(H)$; that is, $(u_n) \in bv(H)$. Therefore, since the inclusion $\{cs(H)\}^\gamma \subset bv(H)$ holds, we conclude that $\{cs(H)\}^\gamma = bv(H)$, as desired. □

5. Conclusion

Partial metrics are more flexible than metrics; they generate partial orders and their topological properties are more general than the one for metrics, argued by the fact that the self-distance of each point need not be zero. They are useful in partially defined information for the study of domains and semantics in computer science.

The concept of level sets associated with a fuzzy set was originally introduced by Zadeh. With the aid of level sets we are able to provide a formulation for a fuzzy set in terms of crisp subsets via the representation theorem. The importance of having such a representation is that it can allow us to extend

operations defined on crisp sets to the case of fuzzy sets. Our focus here is on using the idea of level sets to construct the sets of sequences of fuzzy numbers within partial metric spaces.

This work presents the alpha-, beta-, and gamma-duals of the sets of bounded, convergent, and null series and the set of sequences of bounded variation of fuzzy level sets, based on the partial metric. The potential applications of the obtained results include the characterization of matrix transformations between these sets of sequences.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Existence of Tripled Fixed Points for a Class of Condensing Operators in Banach Spaces

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We give some results concerning the existence of tripled fixed points for a class of condensing operators in Banach spaces. Further, as an application, we study the existence of solutions for a general system of nonlinear integral equations.

1. Introduction and Preliminaries

Measures of noncompactness are very useful tools in functional analysis, for instance, in metric fixed point theory and in the theory of operator equations in Banach spaces. The first measure of noncompactness, denoted by μ, was defined and studied by Kuratowski [1] in 1930. In 1955, Banaś and Goebel [2] used the function μ to prove his fixed point theorem. Darbo's fixed point theorem [2] is a very important generalization of Schauder's fixed point theorem [3] and several authors had used this concept for the resolution of nonlinear equations, some of whom are Aghajani et al. [4, 5], Banaś [6], Banaś and Rzepka [7], Mursaleen and Mohiuddine [8], and many others. Recently in [9], Aghajani et al. give a generalization of Darbo's fixed point theorem. Moreover, they present some results on the existence of coupled fixed points for class of condensing operators. In this paper, we generalize these results to obtain the existence of tripled fixed points for the same class of operators.

Throughout this paper, X is assumed to be a Banach space and $\mathrm{BC}(\mathbb{R}^+)$ is the space of all real functions defined, bounded and continuous on $\mathbb{R}^+$. The family of bounded subset, closure, and closed convex hull of X are denoted by $\mathscr{B}_X$, $\overline{X}$, and $\mathrm{Conv}X$, respectively.

Definition 1 (see [10]). Let X be a Banach space and $\mathscr{B}_X$ the family of bounded subset of X. A map

$$\mu : \mathscr{B}_X \longrightarrow [0, \infty) \tag{1}$$

is called measure of noncompactness defined on X if it satisfies the following.

(1) $\mu(A) = 0 \Leftrightarrow A$ is a precompact set.

(2) $A \subset B \Rightarrow \mu(A) \leqslant \mu(B)$.

(3) $\mu(A) = \mu(\overline{A})$, $\forall A \in \mathscr{B}_X$.

(4) $\mu(\mathrm{Conv}A) = \mu(A)$.

(5) $\mu(\lambda A + (1-\lambda)B) \leqslant \lambda\mu(A) + (1-\lambda)\mu(B)$, for $\lambda \in [0,1]$.

(6) Let (A_n) be a sequence of closed sets from $\mathscr{B}_X$ such that $A_{n+1} \subseteq A_n$, $(n \geqslant 1)$, and $\lim_{n\to\infty}\mu(A_n) = 0$. Then, the intersection set $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty and A_∞ is precompact.

Theorem 2 (see [2]). *Let C be a nonempty closed, bounded, and convex subset of X. If $T : C \to C$ is a continuous mapping*

$$\mu(TA) \leqslant k\mu(A), \quad k \in [0,1), \tag{2}$$

then T has a fixed point.

Theorem 3 (see [9]). *Let C be a nonempty closed, bounded, and convex subset of X and $T : C \to C$ a continuous mapping such that for any subset A of C*

$$\mu(TA) \leqslant \beta(\mu(A))\mu(A), \tag{3}$$

where $\beta : \mathbb{R}_+ \to [0,1)$; that is, $\beta(t_n) \to 1$ implies $t_n \to 0$. Then, T has at least one fixed point.

The following result is a corollary of the previous theorem.

Corollary 4 (see [9]). *Let C be a nonempty closed, bounded, and convex subset of X and $T : C \to C$ a continuous mapping such that for any subset A of C*

$$\mu(TA) \leqslant \varphi(\mu(A)), \tag{4}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing and upper semicontinuous functions; that is, for every $t > 0$, $\varphi(t) < t$. Then, T has at least one fixed point.

Definition 5 (see [11]). A coupled fixed point of a mapping $G : X \times X \to X$ is an element $(x, y) \in X \times X$ such that $G(x, y) = x$ and $G(y, x) = y$.

Theorem 6 (see [12]). *Let $\mu_1, \mu_2, \ldots, \mu_n$ be measures of noncompactness in Banach spaces $E_1, E_2, \ldots, E_n$, (respectively).*

Then, the function

$$\widetilde{\mu}(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n)), \tag{5}$$

defines a measure of noncompactness in $E_1 \times E_2 \times \cdots \times E_n$, where X_i is the natural projection of X on E_i, for $i = 1, 2, \ldots, n$, and F is a convex function defined by

$$F : [0,\infty)^n \longrightarrow [0,\infty), \tag{6}$$

such that

$$F(x_1, x_2, \ldots, x_n) = 0 \Longleftrightarrow x_i = 0, \quad \text{for } i = 1, 2, \ldots, n. \tag{7}$$

Remark 7. Aghajani and Sabzali [13] illustrated the previous theorem by the following example. Let the mapping F be as follows:

$$F(x, y) = \max\{x, y\}, \quad \text{or} \quad F(x, y) = x + y, \quad \text{for any } (x, y) \in [0,\infty)^2. \tag{8}$$

They showed that

$$\widetilde{\mu}(X) = \max(\mu_1(X_1), \mu_2(X_2)), \tag{9}$$

or

$$\widetilde{\mu}(X) = \mu_1(X_1) + \mu_2(X_2) \tag{10}$$

defines a measure of noncompactness in the space $E_1 \times E_2$, where, for $i = 1, 2$, μ_i are measure of noncompactness in E_i and X_i, $i = 1, 2$ are the natural projections of X on E_i.

Theorem 8 (see [9]). *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. Assume that φ is a nondecreasing and upper semicontinuous function. Let $G : \Omega \times \Omega \to \Omega$ be a continuous operator satisfying*

$$\mu(G(X_1 \times X_2)) \leqslant \varphi\left(\frac{\mu(X_1) + \mu(X_2)}{2}\right), \quad X_1, X_2 \in \Omega, \tag{11}$$

for any measure of noncompactness μ. Then, G has at least a coupled fixed point.

2. Main Results

Definition 9 (see [14]). A tripled (x, y, z) of a mapping $G : X \times X \times X \to X$ is called a tripled fixed point if

$$G(x, y, z) = x, \quad G(y, x, z) = y, \quad G(z, y, x) = z. \tag{12}$$

Remark 10. We can notice that by taking

$$F(x, y, z) = \max\{x, y, z\}, \quad \text{for any } (x, y, z) \in [0,\infty)^3, \tag{13}$$

or

$$F(x, y, z) = x + y + z, \quad \text{for any } (x, y, z) \in [0,\infty)^3 \tag{14}$$

F satisfies the conditions of Theorem 6. Thus, for a measure of noncompactness μ_i $(i = 1, 2, 3)$, we have that

$$\widetilde{\mu}(X) = \max(\mu_1(X_1), \mu_2(X_2), \mu_3(X_3)), \tag{15}$$

or

$$\widetilde{\mu}(X) = \mu_1(X_1) + \mu_2(X_2) + \mu_3(X_3) \tag{16}$$

defines a measure of noncompactness in the space $E \times E \times E$ where X_i, $i = 1, 2, 3$ are the natural projections of X on E_i.

So, we obtain the following theorem.

Theorem 11. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing and upper semicontinuous function such that $\varphi(t) < t$ for all $t > 0$. Then, for any measure of noncompactness μ, the continuous operator $G : \Omega \times \Omega \times \Omega \to \Omega$ satisfying*

$$\mu(G(X_1 \times X_2 \times X_3)) \leqslant \varphi\left(\frac{\mu(X_1) + \mu(X_2) + \mu(X_3)}{3}\right), \quad X_1, X_2, X_3 \in \Omega \tag{17}$$

has at least a tripled fixed point.

Proof. To prove this theorem, let us define the measure of noncompactness $\widetilde{\mu}$ by

$$\widetilde{\mu}(X) = \mu_1(X_1) + \mu_2(X_2) + \mu_3(X_3) \tag{18}$$

and the mapping $\widetilde{G} : \Omega \times \Omega \times \Omega \to \Omega$

$$\widetilde{G}(x, y, z) = (G(x, y, z), G(y, x, z), G(z, y, x)). \tag{19}$$

Since

$$\begin{aligned}\widetilde{\mu}\left(\widetilde{G}(X)\right) &\leqslant \widetilde{\mu}\left(G\left(X_1\times X_2\times X_3\right)\times G\left(X_2\times X_1\times X_3\right)\right.\\ &\quad\left.\times G\left(X_3\times X_2\times X_1\right)\right)\\ &= \mu\left(G\left(X_1\times X_2\times X_3\right)\right)+\mu\left(G\left(X_2\times X_1\times X_3\right)\right)\\ &\quad+\mu\left(G\left(X_3\times X_2\times X_1\right)\right)\\ &\leqslant \varphi\left(\frac{\mu\left(X_1\right)+\mu\left(X_2\right)+\mu\left(X_3\right)}{3}\right)\\ &\quad+\varphi\left(\frac{\mu\left(X_1\right)+\mu\left(X_2\right)+\mu\left(X_3\right)}{3}\right)\\ &\quad+\varphi\left(\frac{\mu\left(X_1\right)+\mu\left(X_2\right)+\mu\left(X_3\right)}{3}\right)\\ &= 3\varphi\left(\frac{\mu\left(X_1\right)+\mu\left(X_2\right)+\mu\left(X_3\right)}{3}\right)\end{aligned} \tag{20}$$

and $\widetilde{\mu}' = (1/3)\,\widetilde{\mu}$ is a measure of noncompactness, we get

$$\widetilde{\mu}'\left(\widetilde{G}(X)\right) \leqslant \varphi\left(\widetilde{\mu}'(X)\right). \tag{21}$$

By Corollary 4, we obtain that G has at least a tripled fixed point. □

3. Applications

We can see an application of Theorem 11 in the study of existence of solutions for systems of integral equations defined on the Banach space $\mathrm{BC}(\mathbb{R}^+)$ endowed with the norm

$$\|x\| = \sup\{|x(t)| : t > 0\}. \tag{22}$$

The measure of noncompactness on $\mathrm{BC}(\mathbb{R}^+)$ for a positive fixed t on $\mathscr{B}_{\mathrm{BC}(\mathbb{R}^+)}$ is defined as follows:

$$\mu(X) = \omega_0(X) + \lim_{t\to\infty}\sup \operatorname{diam} X(t), \tag{23}$$

such that

$$\begin{aligned}\operatorname{diam} X(t) &= \sup\left\{\left|x(t)-y(t)\right| : x, y\in X\right\},\\ &\text{where } X(t) = \{x(t) : x\in X\}.\end{aligned} \tag{24}$$

Before defining $\omega_0(X)$, we need first to introduce the modulus of continuity.

Let $x \in X$ and $\epsilon > 0$;

$$\omega^T(x,\epsilon) = \sup\{|x(t)-x(s)| : t,s\in[0,T], |t-s|\leqslant\epsilon\}, \quad \text{for } T>0, \tag{25}$$

is the modulus of continuity of x on $[0,T]$ and let

$$\begin{aligned}\omega^T(X,\epsilon) &= \sup\left\{\omega^T(x,\epsilon) : x\in X\right\},\\ \omega_0^T(X) &= \lim_{\epsilon\to 0}\omega^T(X,\epsilon),\\ \omega_0(X) &= \lim_{T\to\infty}\omega_0^T(X).\end{aligned} \tag{26}$$

Assume that

(i) $\xi, \eta, q : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\xi(t) \to \infty$ as $t \to \infty$;

(ii) the function $\psi : \mathbb{R} \to \mathbb{R}$ is continuous and there exist positive δ, α such that

$$\left|\psi(t_1)-\psi(t_2)\right| \leqslant \delta\left|t_1-t_2\right|^{\alpha}, \tag{27}$$

for any $t_1, t_2 \in \mathbb{R}_+$;

(iii) $f : \mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and there exists a nondecareasing continuous function $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi(0) = 0$, so that

$$\begin{aligned}&\left|f(t,x_1,x_2,x_3,x_4)-f(t,y_1,y_2,y_3,y_4)\right|\\ &\leqslant \frac{1}{2}\left(\varphi\left(\left|x_1-y_1\right|+\left|x_2-y_2\right|+\left|x_3-y_3\right|\right)\right)\\ &\quad+\Phi\left(\left|x_4-y_4\right|\right);\end{aligned} \tag{28}$$

(iv) the function defined by $\left|f(t,0,0,0,0)\right|$ is bounded on $\mathbb{R}_+$; that is,

$$M_1 = \sup\{f(t,0,0,0,0) : t\in\mathbb{R}_+\} < \infty; \tag{29}$$

(v) $h : \mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function and there exists a positive solution r_0 of the inequality

$$\frac{1}{3}\varphi(3r) + M_1 + \Phi(\delta D) \leqslant r, \tag{30}$$

where D is positive constant defined by the equality

$$\begin{aligned}D = \sup\left\{\left|\int_0^{q(t)}\left(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\right)ds\right| :\right.\\ \left. t,s\in\mathbb{R}_+, x,y,z\in\mathrm{BC}\left(\mathbb{R}_+\right)\right\},\end{aligned}$$

$$\begin{aligned}\lim_{\epsilon\to\infty}\int_0^{q(t)}&\left[h\left(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\right)\right.\\ &\left.-h\left(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s))\right)\right]ds = 0,\end{aligned} \tag{31}$$

uniformly with respect to $x, y, z, u, v, w \in \mathrm{BC}(\mathbb{R}_+)$.

Theorem 12. *Suppose that (i)–(v) hold; then the system of integral equations*

$$\begin{aligned}
&x(t)\\
&= f\left(t, x(\xi(t)), y(\xi(t)), z(\xi(t)),\right.\\
&\qquad \left.\psi\left(\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\right)\right),\\
&y(t)\\
&= f\left(t, y(\xi(t)), x(\xi(t)), z(\xi(t)),\right.\\
&\qquad \left.\psi\left(\int_0^{q(t)} h(t,s,y(\eta(s)), x(\eta(s)), z(\eta(s)))\,ds\right)\right),\\
&z(t)\\
&= f\left(t, z(\xi(t)), y(\xi(t)), x(\xi(t)),\right.\\
&\qquad \left.\psi\left(\int_0^{q(t)} h(t,s,z(\eta(s)), y(\eta(s)), x(\eta(s)))\,ds\right)\right)
\end{aligned} \tag{32}$$

has at least one solution in the space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

Proof. Let $G : \mathrm{BC}(\mathbb{R}_+)\times\mathrm{BC}(\mathbb{R}_+)\times\mathrm{BC}(\mathbb{R}_+) \to \mathrm{BC}(\mathbb{R}_+)$ be an operator defined by

$$\begin{aligned}
&G(x,y,z)(t)\\
&= f\left(t, x(\xi(t)), y(\xi(t)), z(\xi(t)),\right.\\
&\qquad \left.\psi\left(\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\right)\right).
\end{aligned} \tag{33}$$

For $(x,y,z) \in \mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+)$, let

$$\begin{aligned}
&\|(x,y,z)\|_{\mathrm{BC}(\mathbb{R}_+)\times\mathrm{BC}(\mathbb{R}_+)\times\mathrm{BC}(\mathbb{R}_+)}\\
&\quad = \|x\|_\infty + \|y\|_\infty + \|z\|_\infty.
\end{aligned} \tag{34}$$

We can easily prove that the solution of (32) in $\mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+)$ is equivalent to the tripled fixed point of G.

Obviously, $G(x,y,z)(t)$ is continuous for any $(x,y,z) \in \mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+) \times \mathrm{BC}(\mathbb{R}_+)$. Hence, we have

$$\begin{aligned}
&\left|G(x,y,z)(t)\right|\\
&\leqslant \left|f\left(t, x(\xi(t)), y(\xi(t)), z(\xi(t)),\right.\right.\\
&\qquad \left.\left.\psi\left(\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\right)\right)\right.\\
&\qquad \left. - f(t,0,0,0)\right|\\
&\quad + \left|f(t,0,0,0,0)\right|\\
&\leqslant \frac{1}{2}\varphi\left(|x(\xi(t))| + |y(\xi(t))| + |y(\xi(t))|\right)\\
&\quad + \Phi\left(\left|\psi\left(\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\right)\right|\right)\\
&\quad + \left|f(t,0,0,0,0)\right|\\
&\leqslant \frac{1}{2}\varphi\left(|x(\xi(t))| + |y(\xi(t))| + |y(\xi(t))|\right)\\
&\quad + \Phi\left(\delta\left|\left(\int_0^{q(t)} h(t,s,x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\right)\right|^\alpha\right)\\
&\quad + \left|f(t,0,0,0,0)\right|.
\end{aligned} \tag{35}$$

Then, by (29) and (30), we get

$$\begin{aligned}
&\|G(x,y,z)\|_\infty\\
&\quad \leqslant \frac{1}{3}\varphi\left(\|x\|_\infty + \|y\|_\infty + \|z\|_\infty\right) + M_1 + \Phi(\delta D) \leqslant r_0.
\end{aligned} \tag{36}$$

So, we obtain

$$G\left(\overline{B}_{r_0} \times \overline{B}_{r_0} \times \overline{B}_{r_0}\right) \subset \overline{B}_{r_0}. \tag{37}$$

Now, we prove that $G : \overline{B}_{r_0} \times \overline{B}_{r_0} \times \overline{B}_{r_0} \to \overline{B}_{r_0}$ is continuous. Let $(x,y,z), (u,v,w) \in \overline{B}_{r_0} \times \overline{B}_{r_0} \times \overline{B}_{r_0}$ such that, for $\varepsilon > 0$,

$$\|(x,y,z) - (u,v,w)\|_{\overline{B}_{r_0}\times\overline{B}_{r_0}\times\overline{B}_{r_0}} < \epsilon. \tag{38}$$

Then,

$$\begin{aligned}
&\left|G(x,y,z)(t)-G(u,v,w)(t)\right| \\
&=\Bigg|f\Bigg(t,x(\xi(t)),y(\xi(t)),z(\xi(t)), \\
&\qquad \psi\left(\int_0^{q(t)}h(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\right)\Bigg) \\
&\quad -f\Bigg(t,x(\xi(t)),y(\xi(t)),z(\xi(t)), \\
&\qquad \psi\left(\int_0^{q(t)}h(t,s,u(\eta(s)),v(\eta(s)), \right. \\
&\qquad \left.w(\eta(s)))\,ds\right)\Bigg)\Bigg| \\
&\leqslant \frac{1}{2}\varphi\left(\left|x(\xi(t)-u(\xi(t)))\right|+\left|y(\xi(t))-v(\xi(t))\right|\right. \\
&\qquad \left.+\ \left|z(\xi(t))-w(\xi(t))\right|\right) \\
&\quad +\Phi\left(\psi\left(\int_0^{q(t)}h(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\right)\right. \\
&\qquad \left.-\psi\left(\int_0^{q(t)}h(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))\,ds\right)\right) \\
&\leqslant \frac{1}{2}\varphi\left(\left|x(\xi(t)-u(\xi(t)))\right|+\left|y(\xi(t))-v(\xi(t))\right|\right) \\
&\qquad +\ \left|z(\xi(t))-w(\xi(t))\right|\Big) \\
&\quad +\Phi\left(\delta\left|\int_0^{q(t)}h(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\right.\right. \\
&\qquad \left.\left.-\int_0^{q(t)}h(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))\,ds\right|^{\alpha}\right).
\end{aligned} \tag{39}$$

Using condition (iii) and (29), there exists $T>0$ such that if $t>T$, then

$$\begin{aligned}
&\Phi\left(\delta\left|\int_0^{q(t)}h(t,s,x(\eta(s)),y(\eta(s),z(\eta(s)))\,ds\right|^{\alpha}\right) \\
&\quad \leqslant \frac{1}{3}\epsilon,
\end{aligned} \tag{40}$$

for any $x,y,z\in \mathrm{BC}(\mathbb{R}_+)$. We notice two cases.

Case 1. If $t>T$, then from (39) and (40)

$$\left|G(x,y,z)(t)-G(u,v,w)(t)\right|\leqslant \frac{1}{3}\varphi(\epsilon)+\frac{1}{3}\epsilon. \tag{41}$$

Case 2. Similarly, for $t\in[0,T]$, we have

$$\begin{aligned}
&\left|G(x,y,z)(t)-G(u,v,w)(t)\right| \\
&\leqslant \frac{1}{2}\varphi\left(\left|x(\xi(t)-u(\xi(t)))\right|+\left|y(\xi(t))-v(\xi(t))\right|\right) \\
&\quad +\ \left|z(\xi(t))-w(\xi(t))\right|\Big) \\
&\quad +\Phi\left(\delta\left|\int_0^{q(t)}h(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\right.\right. \\
&\qquad \left.\left.-\int_0^{q(t)}h(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))\,ds\right|^{\alpha}\right) \\
&\leqslant \frac{1}{2}\varphi(\epsilon)+\Phi\left(\delta(q_T\beta(\epsilon))^{\alpha}\right) \\
&< \frac{1}{2}\epsilon+\Phi\left(\delta(q_T\beta(\epsilon))^{\alpha}\right),
\end{aligned} \tag{42}$$

where $q_T=\sup\{q(t):t\in[0,T]\}$, and

$$\begin{aligned}
\beta(\epsilon)=\ &\sup\{\left|h(t,s,x,y,z)-h(t,s,u,v,w)\right|: \\
&t\in[0,\mathrm{T}],s\in[0,q_T], \\
&x,y,z,u,v,w\in[-r_0,r_0], \\
&\left\|(x,y,z)-(u,v,w)\right\|<\epsilon\}.
\end{aligned} \tag{43}$$

Since β is continuous on $[0,T]\times[0,q_T]\times[-r_0,r_0]\times[-r_0,r_0]$, we have $\beta(\epsilon)\to 0$ and $\epsilon\to 0$. Thus, using (iii), we get

$$\Phi\left(\delta(q_T\beta(\epsilon))^{\alpha}\right)\longrightarrow 0,\quad \text{as } \epsilon\longrightarrow 0. \tag{44}$$

Finally, from (42) and (41), we conclude that G is a continuous function from $\overline{B}_{r_0}\times\overline{B}_{r_0}\times\overline{B}_{r_0}$ into $\overline{B}_{r_0}$.

Now, we show that the map G satisfies all the conditions of Theorem 11. To do this, for an arbitrary fixed $T>0$ and $\epsilon>0$, assume that X_1, X_2, and X_3 are nonempty chosen subsets of $\overline{B}_{r_0}$ and t_1, $t_2\in[0,T]$, with $\left|t_2-t_1\right|\leqslant\epsilon$. Without loss of generality, let

$$q(t_1)<q(t_2). \tag{45}$$

For an arbitrary $(x,y,z)\in X_1\times X_2\times X_3$,

$$\begin{aligned}
&\left|G(x,y,z)(t_1)-G(x,y,z)(t_2)\right| \\
&\leqslant \Bigg|f\Bigg(t_1,x(\xi(t_1)),y(\xi(t_1)),z(\xi(t_1)), \\
&\qquad \psi\left(\int_0^{q(t_1)}h(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\right)\Bigg) \\
&\quad -f\Bigg(t_1,x(\xi(t_2)),y(\xi(t_2)),z(\xi(t_2)), \\
&\qquad \psi\left(\int_0^{q(t_1)}h(t_1,s,x(\eta(s)), \right. \\
&\qquad \left.y(\eta(s)),z(\eta(s)))\,ds\right)\Bigg)\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&+\left|f\left(t_1,x\left(\xi\left(t_2\right)\right),y\left(\xi\left(t_2\right)\right),z\left(\xi\left(t_2\right)\right),\right.\right.\\
&\qquad\left.\psi\left(\int_0^{q(t_1)}h\left(t_1,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)ds\right)\right)\\
&\quad-f\left(t_2,x\left(\xi\left(t_2\right)\right),y\left(\xi\left(t_2\right)\right),z\left(\xi\left(t_2\right)\right),\right.\\
&\qquad\psi\left(\int_0^{q(t_1)}h\left(t_1,s,x\left(\eta\left(s\right)\right),\right.\right.\\
&\qquad\left.\left.\left.y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)ds\right)\right)\Bigg|\\
&+\left|f\left(t_2,x\left(\xi\left(t_2\right)\right),y\left(\xi\left(t_2\right)\right),z\left(\xi\left(t_2\right)\right),\right.\right.\\
&\qquad\left.\psi\left(\int_0^{q(t_1)}h\left(t_1,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)ds\right)\right)\\
&\quad-f\left(t_2,x\left(\xi\left(t_2\right)\right),y\left(\xi\left(t_2\right)\right),z\left(\xi\left(t_2\right)\right),\right.\\
&\qquad\psi\left(\int_0^{q(t_1)}h\left(t_1,s,x\left(\eta\left(s\right)\right),\right.\right.\\
&\qquad\left.\left.\left.y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)ds\right)\right)\Bigg|\\
&+\left|f\left(t_2,x\left(\xi\left(t_2\right)\right),y\left(\xi\left(t_2\right)\right),z\left(\xi\left(t_2\right)\right),\right.\right.\\
&\qquad\left.\psi\left(\int_0^{q(t_1)}h(t_1,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right))ds\right)\right)\\
&\quad-f\left(t_2,x\left(\xi\left(t_2\right)\right),y\left(\xi\left(t_2\right)\right),z\left(\xi\left(t_2\right)\right),\right.\\
&\qquad\psi\left(\int_0^{q(t_2)}h\left(t_2,s,x\left(\eta\left(s\right)\right),\right.\right.\\
&\qquad\left.\left.\left.y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)ds\right)\right)\Bigg|\\
&\leqslant\frac{1}{3}\varphi\left(\left|x\left(\xi\left(t_1\right)-x\left(\xi\left(t_2\right)\right)\right)\right|+\left|y\left(\xi\left(t_1\right)-y\left(\xi\left(t_2\right)\right)\right)\right|\right.\\
&\qquad\left.+\left|z\left(\xi\left(t_1\right)-z\left(\xi\left(t_2\right)\right)\right)\right|+\omega_{r_0,D_1}^T\left(f,\epsilon\right)\right)\\
&\quad+\Phi\left(\delta\left|\int_0^{q(t_2)}\left[h\left(t_2,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)\right.\right.\right.\\
&\qquad\left.\left.\left.-h\left(t_1,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),\right.\right.\right.\\
&\qquad\left.\left.\left.z\left(\eta\left(s\right)\right)\right)\right]ds\right|^{\alpha}\right)\\
&\quad+\Phi\left(\delta\left|\int_{q(t_1)}^{q(t_2)}h\left(t_1,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)ds\right|^{\alpha}\right)\\
&\leqslant\frac{1}{3}\varphi\left(\omega^T\left(x,\omega^T\left(\xi,\epsilon\right)\right)+\omega^T\left(y,\omega^T\left(\xi,\epsilon\right)\right)\right.\\
&\qquad\left.+\omega^T\left(z,\omega^T\left(\xi,\epsilon\right)\right)+\omega_{r_0,D_1}^T\left(f,\epsilon\right)\right)\\
&\quad+\Phi\left(\delta\left(q_T\omega_{r_0}^T\left(h,\epsilon\right)\right)^{\alpha}\right)+\Phi\left(\delta\left(U_{r_0}^T\omega^T\left(q,\epsilon\right)\right)^{\alpha}\right),
\end{aligned}
\tag{46}
$$

where

$$
\begin{aligned}
\omega^T\left(\xi,\epsilon\right)&=\sup\left\{\left|\left(\xi\left(t_2\right)-\xi\left(t_1\right)\right)\right|:t_1,t_2\leqslant\epsilon,\left|t_2-t_1\right|\leqslant\epsilon\right\},\\
\omega^T\left(x,\omega^T\left(\xi,\epsilon\right)\right)&=\sup\left\{\left|\left(x\left(t_2\right)-x\left(t_1\right)\right)\right|:t_1,t_2\in\left[0,T\right],\right.\\
&\qquad\left.\left|t_2-t_1\right|\leqslant\omega^T\left(\xi,\epsilon\right)\right\},\\
D_1&=q_T\sup\left\{\left|h\left(t,s,x,y,z\right)\right|,t\in\left[0,T\right],s\in\left[0,q_T\right],\right.\\
&\qquad\left.x,y,z\in\left[-r_0,r_0\right]\right\}\\
\omega_{r_0,D_1}^T\left(f,\epsilon\right)&=\sup\left\{\left|f\left(t_2,x,y,z,d\right)-f\left(t_1,x,y,z,d\right)\right|,\right.\\
&\qquad t_1,t_2\in\left[0,T\right],\\
&\qquad\left|t_2-t_1\right|\leqslant\epsilon,x,y,z\in\left[-r_0,r_0\right],\\
&\qquad\left.d\in\left[-D_1,D_1\right]\right\},\\
\omega_{r_0}^T\left(f,\epsilon\right)&=\sup\left\{\left|h\left(t_2,s,x,y,z\right)-f\left(t_1,s,x,y,z\right)\right|,\right.\\
&\qquad t_1,t_2\in\left[0,T\right],\left|t_2-t_1\right|\leqslant\epsilon,\\
&\qquad\left.s\in\left[0,q_T\right],x,y,z\in\left[-r_0,r_0\right]\right\},\\
U_{r_0}^T&=\sup\left\{\left|h\left(t_1,s,x,y,z\right)\right|:t_1\in\left[0,T\right],\right.\\
&\qquad\left.s\in\left[0,q_T\right],x,y,z\in\left[-r_0,r_0\right]\right\},
\end{aligned}
\tag{47}
$$

we obtain

$$
\begin{aligned}
&\omega^T\left(G\left(X_1\times X_2\times X_3\right),\epsilon\right)\\
&\quad\leqslant\frac{1}{3}\varphi\left(\omega^T\left(X_1,\omega^T\left(\xi,\epsilon\right)\right)+\omega^T\left(X_2,\omega^T\left(\xi,\epsilon\right)\right)\right.\\
&\qquad\left.+\omega^T\left(X_3,\omega^T\left(\xi,\epsilon\right)\right)\right)+\omega_{r_0,D_1}^T\left(f,\epsilon\right)\\
&\qquad+\Phi\left(\delta\left(q_T\omega_{r_0}^T\left(h,\epsilon\right)\right)^{\alpha}\right)+\Phi\left(\delta\left(U_{r_0}^T\omega^T\left(q,\epsilon\right)\right)^{\alpha}\right).
\end{aligned}
\tag{48}
$$

Further, by the uniform continuity of f and h on the compact sets $[0,T]\times[-r_0,r_0]\times[-r_0,r_0]\times[-r_0,r_0]\times[-D_1,D_1]$ and $[0,T]\times[0,q_T]\times[-r_0,r_0]\times[-r_0,r_0]\times[-r_0,r_0]$, respectively, we get $\omega_{r_0,D_1}^T\left(f,\epsilon\right)\to 0$ and $\omega_{r_0}^T\left(h,\epsilon\right)\to 0$ as $\epsilon\to 0$.

Moreover, Φ is a nondecreasing continuous function with $\Phi(0) = 0$ and (iii), and we obtain

$$\Phi\left(\delta\left(q_T\omega_{r_0}^T(h,\epsilon)\right)^\alpha\right) + \Phi\left(\delta\left(U_{r_0}^T\omega^T(q,\epsilon)\right)^\alpha\right) \longrightarrow 0, \quad \epsilon \longrightarrow 0. \tag{49}$$

By (48), we get

$$\omega_0^T(G(X_1 \times X_2 \times X_3)) \leqslant \frac{1}{3}\varphi\left(\omega_0^T(X_1) + \omega_0^T(X_2) + \omega_0^T(X_3)\right). \tag{50}$$

Taking the limit $T \to \infty$ in (50), we obtain

$$\omega_0(G(X_1 \times X_2 \times X_3)) \leqslant \frac{1}{3}\varphi\left(\omega_0(X_1) + \omega_0(X_2) + \omega_0(X_3)\right). \tag{51}$$

Then, for arbitrary (x, y, z), $(u, v, w) \in X_1 \times X_2 \times X_3$, and $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\left|G(x,y,z)(t) - G(u,v,w)(t)\right| \\
&\leqslant \frac{1}{3}\varphi\left(\left|x(\xi(t)) - u(\xi(t))\right| + \left|y(\xi(t)) - v(\xi(t))\right|\right. \\
&\qquad \left. + \left|z(\xi(t)) - w(\xi(t))\right|\right) \\
&\quad + \Phi\left(\delta\left|\int_0^{q(t)} [h(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))) \right.\right. \\
&\qquad - h(t,s,u(\eta(s)), \\
&\qquad\quad \left.\left. v(\eta(s)),w(\eta(s)))]\,ds\right|^\alpha\right) \\
&\leqslant \frac{1}{2}\varphi\left(\operatorname{diam} X_1(\xi(t)) + \operatorname{diam} X_2(\xi(t)) + \operatorname{diam} X_3(\xi(t))\right) \\
&\quad + \Phi\left(\delta\left|\int_0^{q(t)} [h(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))) \right.\right. \\
&\qquad - h(t,s,u(\eta(s)), \\
&\qquad\quad \left.\left. v(\eta(s)),w(\eta(s)))]\,ds\right|^\alpha\right).
\end{aligned} \tag{52}$$

Since (x, y, z), (u, v, w), and t are arbitrary in (52),

$$\begin{aligned}
&\operatorname{diam} G(X_1 \times X_2 \times X_3)(t) \\
&\leqslant \frac{1}{3}\varphi\left(\operatorname{diam} X_1(\xi(t)) + \operatorname{diam} X_2(\xi(t)) + \operatorname{diam} X_3(\xi(t))\right) \\
&\quad + \Phi\left(\delta\left|\int_0^{q(t)} [h(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))) \right.\right. \\
&\qquad - h(t,s,u(\eta(s)),v(\eta(s)), \\
&\qquad\quad w(\eta(s)))(t,s,x(\eta(s)), \\
&\qquad\quad \left.\left. y(\eta(s)),z(\eta(s)))]\,ds\right|^\alpha\right).
\end{aligned} \tag{53}$$

Taking again $T \to \infty$ in (53), we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \operatorname{diam} G(X_1 \times X_2 \times X_3)(t) + \omega_0 \\
&\leqslant \frac{1}{3}\varphi\left(\limsup_{t\to\infty} \operatorname{diam} X_1(\xi(t)) + \limsup_{t\to\infty} \operatorname{diam} X_2(\xi(t))\right. \\
&\qquad \left. + \limsup_{t\to\infty} \operatorname{diam} X_3(\xi(t))\right).
\end{aligned} \tag{54}$$

We conclude from (51) and (54) that

$$\begin{aligned}
&\limsup_{t\to\infty} \omega_0(G(X_1 \times X_2 \times X_3))(t) + \omega_0(G(X_1 \times X_2 \times X_3)) \\
&\leqslant \frac{1}{3}\varphi\left(\limsup_{t\to\infty} \operatorname{diam} X_1(\xi(t)) + \limsup_{t\to\infty} \operatorname{diam} X_2(\xi(t))\right. \\
&\qquad \left. + \limsup_{t\to\infty} \operatorname{diam} X_3(\xi(t))\right) \\
&\quad + \frac{1}{3}\varphi\left(\omega_0(X_1) + \omega_0(X_2) + \omega_0(X_3)\right).
\end{aligned} \tag{55}$$

Since φ is a concave function, (55) implies

$$\begin{aligned}
&\limsup_{t\to\infty} \operatorname{diam} G(X_1 \times X_2 \times X_3)(t) \\
&\qquad + \omega_0(G(X_1 \times X_2 \times X_3)) \\
&\leqslant \varphi\left(\frac{1}{3}\left[\limsup_{t\to\infty} \operatorname{diam} X_1(\xi(t)) + \omega_0(X_1)\right]\right) \\
&\qquad + \varphi\left(\frac{1}{3}\left[\limsup_{t\to\infty} \operatorname{diam} X_2(\xi(t)) + \omega_0(X_2)\right]\right) \\
&\qquad + \varphi\left(\frac{1}{3}\left[\limsup_{t\to\infty} \operatorname{diam} X_3(\xi(t)) + \omega_0(X_3)\right]\right).
\end{aligned} \tag{56}$$

Finally, since μ is defined by

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \operatorname{diam} X(t), \tag{57}$$

we get

$$\mu\left(G\left(X_1 \times X_2 \times X_3\right)\right) \leqslant \varphi\left(\frac{\mu\left(X_1\right)+\mu\left(X_2\right)+\mu\left(X_3\right)}{3}\right). \tag{58}$$

Hence, by Theorem 11, T has at least a tripled fixed point in $\mathrm{BC}\left(\mathbb{R}_+\right) \times \mathrm{BC}\left(\mathbb{R}_+\right) \times \mathrm{BC}\left(\mathbb{R}_+\right)$. □

Example 1. We consider the following system of integral equations

$$\begin{aligned}
x(t) &= \frac{1}{3+t^2}x(t) + y(t) + z(t) \\
&\quad + \int_0^T \Big(x(s)\, s \left|\sin y(t)\right| |\cos z(t)| \\
&\qquad + e^s\left(1+x^2(s)\right)\left(1+\sin^2 y(s)\right) \\
&\qquad \cdot \left(1+\cos^2 z(s)\right)\Big) \\
&\quad \cdot \Big(e^t\left(1+x^2(s)\right)\left(1+\sin^2 y(s)\right) \\
&\qquad \cdot \left(1+\cos^2 z(s)\right)\Big)^{-1} ds, \\
y(t) &= \frac{1}{3+t^2}y(t) + x(t) + z(t) \\
&\quad + \int_0^T \Big(y(s)\, s \left|\sin x(t)\right| |\cos z(t)| \\
&\qquad + e^s\left(1+y^2(s)\right)\left(1+\sin^2 x(s)\right) \\
&\qquad \cdot \left(1+\cos^2 z(s)\right)\Big) \\
&\quad \cdot \Big(e^t\left(1+y^2(s)\right)\left(1+\sin^2 x(s)\right) \\
&\qquad \cdot \left(1+\cos^2 z(s)\right)\Big)^{-1} ds, \\
z(t) &= \frac{1}{3+t^2}z(t) + y(t) + x(t) \\
&\quad + \int_0^T \Big(z(s)\, s \left|\sin y(t)\right| |\cos x(t)| \\
&\qquad + e^s\left(1+z^2(s)\right)\left(1+\sin^2 y(s)\right) \\
&\qquad \cdot \left(1+\cos^2 x(s)\right)\Big) \\
&\quad \cdot \Big(e^t\left(1+z^2(s)\right)\left(1+\sin^2 y(s)\right) \\
&\qquad \cdot \left(1+\cos^2 x(s)\right)\Big)^{-1} ds.
\end{aligned} \tag{59}$$

We notice that by taking

$$\begin{aligned}
&f(t,x,y,z,p) = \frac{1}{3+t^2}x + \frac{1}{3}y + \frac{1}{3}z + p, \\
&h(t,s,x,y,z) \\
&= \frac{xs\left|\sin y\right| |\cos z| + e^s\left(1+x^2\right)\left(1+\sin^2 y\right)\left(1+\cos^2 z\right)}{e^t\left(1+x^2\right)\left(1+\sin^2 y\right)\left(1+\cos^2 z\right)}, \\
&\eta(t) = \xi(t) = q(t) = \Psi(t) = \Phi(t) = t, \\
&\varphi(t) = t-3,
\end{aligned} \tag{60}$$

we get the system integral equations (32).

To solve this system, we need to verify conditions (i)–(v).

Obviously, $\xi, \eta, q : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\xi \to \infty$ as $t \to \infty$. Further, the function $\psi : \mathbb{R} \to \mathbb{R}$ is continuous for $\delta = \alpha = 1$, and we have

$$\left|\psi(t_1) - \psi(t_2)\right| \leqslant \delta |t_1 - t_2|^{\alpha}, \tag{61}$$

for any $t_1, t_2 \in \mathbb{R}_+$. Conditions (i) and (ii) hold.

Now, let

$$\begin{aligned}
&\left|f(t,x,y,z,p) - f(t,u,v,w,\rho)\right| \\
&= \left|\frac{1}{3+t^2}x + \frac{1}{3}y + \frac{1}{3}z + p - \left(\frac{1}{3+t^2}u + \frac{1}{3}v + \frac{1}{3}w + \rho\right)\right| \\
&\leqslant \frac{1}{3}\left[|x-u| + |y-v| + |z-w|\right] + |p-\rho| \\
&= \frac{1}{3}\varphi\left(|x-u| + |y-v| + |z-w|\right) + \Phi\left(|p-\rho|\right).
\end{aligned} \tag{62}$$

Then, (iii) also holds.

Moreover,

$$M_1 = \sup\left|\left\{f(t,0,0,0,0) : t \in \mathbb{R}_+\right\}\right| = 0; \tag{63}$$

then, (iv) is valid.

Let us verify the last condition (v). First,

$$\begin{aligned}
&\left|h(t,s,x,y,z) - h(t,s,u,v,w)\right| \\
&= \left|\frac{xs\left|\sin y\right| |\cos z| + e^s\left(1+x^2\right)\left(1+\sin^2 y\right)\left(1+\cos^2 z\right)}{e^t\left(1+x^2\right)\left(1+\sin^2 y\right)\left(1+\cos^2 z\right)}\right. \\
&\quad \left. - \frac{us\left|\sin v\right| |\cos w| + e^s\left(1+u^2\right)\left(1+\sin^2 v\right)\left(1+\cos^2 w\right)}{e^t\left(1+u^2\right)\left(1+\sin^2 v\right)\left(1+\cos^2 w\right)}\right| \\
&= \left|\frac{xs\left|\sin y\right| |\cos z|}{e^t\left(1+x^2\right)\left(1+\sin^2 y\right)\left(1+\cos^2 z\right)}\right. \\
&\quad \left. - \frac{us\left|\sin v\right| |\cos w|}{e^t\left(1+u^2\right)\left(1+\sin^2 v\right)\left(1+\cos^2 w\right)}\right| \\
&\leqslant \left|\frac{x}{1+x^2}\frac{s}{e^t} - \frac{u}{1+u^2}\frac{s}{e^t}\right| \leqslant \frac{1}{2}\frac{s}{e^t} + \frac{1}{2}\frac{s}{e^t} \leqslant \frac{s}{e^t}.
\end{aligned} \tag{64}$$

Hence,

$$\lim_{t\to\infty}\int_0^t \left|h\left(t,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)\right.$$
$$\left.-h\left(t,s,u\left(\eta\left(s\right)\right),v\left(\eta\left(s\right)\right),w\left(\eta\left(s\right)\right)\right)\right| ds \quad (65)$$
$$\leqslant \lim_{t\to\infty}\int_0^t \frac{s}{e^t}ds = 0.$$

Furthermore, for any $x, y, z \in \mathrm{BC}\left(\mathbb{R}_+\right)\times\mathrm{BC}\left(\mathbb{R}_+\right)\times\mathrm{BC}\left(\mathbb{R}_+\right)$,

$$\begin{aligned}&\left|\int_0^t h\left(t,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)ds\right|\\ &\leqslant \int_0^t \left|h\left(t,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)\right| ds\\ &\leqslant \int_0^t \left(\frac{s}{2e^t}+\frac{e^s}{e^t}\right)ds = \frac{t^2}{4e^t}+1-\frac{1}{e^t}\\ &= \frac{t^2+4e^t-4}{4e^t}.\end{aligned} \quad (66)$$

Thus,

$$\begin{aligned}&\sup\left\{\left|\int_0^t h\left(t,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)ds\right|,\right.\\ &\qquad \left.t,s\in\mathbb{R}_+, x,y,z\in\mathrm{BC}\left(\mathbb{R}_+\right)\times\mathrm{BC}\left(\mathbb{R}_+\right)\times\mathrm{BC}\left(\mathbb{R}_+\right)\right\}\\ &= \sup\left\{\frac{t^2+4e^t-4}{4e^t}, t\in\mathbb{R}_+\right\} = 1.\end{aligned} \quad (67)$$

It is easy to see that, for any $r > 0$, we have that

$$\frac{1}{3}\varphi\left(3r\right)+M_1+\Phi\left(\delta D\right)\leqslant r \quad (68)$$

holds and condition (v) is valid.

Consequently, the system has at least one solution in $\mathrm{BC}\left(\mathbb{R}_+\right)\times\mathrm{BC}\left(\mathbb{R}_+\right)\times\mathrm{BC}\left(\mathbb{R}_+\right)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

On the Convergence and Stability Results for a New General Iterative Process

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We put forward a new general iterative process. We prove a convergence result as well as a stability result regarding this new iterative process for weak contraction operators.

1. Introduction and Preliminaries

Throughout this paper, by $\mathbb{N}$, we denote the set of all positive integers. In this paper, we obtain results on the stability and strong convergence for a new iteration process (3) in an arbitrary Banach space by using weak contraction operator in the sense of Berinde [1]. Also, we obtain that the iteration procedure (3) can be useful method for solution of delay differential equations. To obtain solution of delay differential equation by using fixed point theory, some authors have done different studies. One can find these works in [2, 3]. Many results of stability have been established by some authors using different contractive mappings. The first study on the stability of the Picard iteration under Banach contraction condition was done by Ostrowski [4]. Some other remarkable results on the concept of stability can be found in works of the following authors involving Harder and Hicks [5, 6], Rhoades [7, 8], Osilike [9], Osilike and Udomene [10], and Singh and Prasad [11]. In 1988, Harder and Hicks [5] established applications of stability results to first order differential equations. Osilike and Udomene [10] developed a short proof of stability results for various fixed point iteration processes. Afterward, in following studies, same technique given in [10] has been used, by Berinde [12], Olatinwo [13], Imoru and Olatinwo [14], Karakaya et al. [15], and some authors.

Let (E, d) be complete metric space and $T : E \to E$ a self-map on E; and the set of fixed points of T in E is defined by $F_T = \{p \in E : Tp = p\}$. Let $\{x_n\}_{n\in\mathbb{N}} \subset E$ be the sequence generated by an iteration involving T which is defined by

$$x_{n+1} = f(T, x_n) \quad n = 0, 1, \ldots, \tag{1}$$

where $x_0 \in E$ is the initial point and f is a proper function. Suppose that sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to a fixed point p of T. Let $\{y_n\}_{n\in\mathbb{N}} \subset E$ and set

$$\epsilon_n = d(y_{n+1}, f(T, y_n)) \quad n = 0, 1, \ldots. \tag{2}$$

Then, the iteration procedure (1) is said to be T stable or stable with respect to T if and only if $\lim_{n\to\infty}\epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

Now, let C be a convex subset of a normed space E and $T : C \to C$ a self-map on E. We introduce a new two-step iteration process which is a generalization of Ishikawa iteration process as follows:

$$\begin{aligned} x_0 &= x \in C, \\ f(T, x_n) &= (1 - \wp_n) x_n + \xi_n T x_n + (\wp_n - \xi_n) T y_n, \\ y_n &= (1 - \zeta_n) x_n + \zeta_n T x_n, \end{aligned} \tag{3}$$

for $n \geq 0$, where $\{\xi_n\}$, $\{\wp_n\}$, and $\{\zeta_n\}$ satisfy the following conditions

(C_1) $\wp_n \geq \xi_n$,
(C_2) $\{\wp_n - \xi_n\}_{n=0}^{\infty}, \{\wp_n\}_{n=0}^{\infty}, \{\zeta_n\}_{n=0}^{\infty}, \{\xi_n\}_{n=0}^{\infty} \in [0,1]$,
(C_3) $\sum_{n=0}^{\infty} \wp_n = \infty$.

In the following remark, we show that the new iteration process is more general than the Ishikawa and Mann iteration processes.

Remark 1.

(1) If $\xi_n = 0$, then (3) reduces to the Ishikawa iteration process in [16].

(2) If $\zeta_n = 0$, then (3) reduces to the Mann iteration process in [17].

Lemma 2 (see [18]). *If δ is a real number such that $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n\in\mathbb{N}}$ is a sequence of positive real numbers such that $\lim_{n\to\infty}\epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n\in\mathbb{N}}$ satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n \quad n = 0, 1, \ldots \tag{4}$$

one has

$$\lim_{n\to\infty} u_n = 0. \tag{5}$$

Lemma 3 (see [2]). *Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers including zero satisfying*

$$s_{n+1} \leq (1 - \mu_n)\, s_n. \tag{6}$$

If $\{\mu_n\} \subset (0,1)$ and $\sum_{n=0}^{\infty} \mu_n = \infty$, then $\lim_{n\to\infty} s_n = 0$.

A mapping $T : C \to E$ is said to be contraction if there is a fixed real number $a \in [0,1)$ such that

$$\|Tx - Ty\| \leq a \|x - y\| \tag{7}$$

for all $x, y \in C$.

This contraction condition has been generalized by many authors. For example, Kannan [19] shows that there exists $b \in [0, 1/2)$ such that, for all $x, y \in C$,

$$\|Tx - Ty\| \leq b \left[\|x - Tx\| + \|y - Ty\|\right]. \tag{8}$$

Chatterjea [20] shows that there exists $c \in [0, 1/2)$ such that, for all $x, y \in C$,

$$\|Tx - Ty\| \leq c \left[\|x - Ty\| + \|y - Tx\|\right]. \tag{9}$$

In 1972, Zamfirescu [21] obtained the following theorem.

Theorem 4 (see [21]). *Let (X, d) be a complete metric space and $T : X \to X$ a mapping for which there exist real numbers a, b, and c satisfying $a \in (0,1)$, $b, c \in (0, 1/2)$ such that, for each pair $x, y \in X$, at least one of the following conditions is performed:*

(i) $d(Tx, Ty) \leq ad(x, y)$,

(ii) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,

(iii) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n\in\mathbb{N}}$ defined by

$$x_{n+1} = Tx_n \quad n = 0, 1, 2, \ldots \tag{10}$$

converges to p for any arbitrary but fixed $x_0 \in X$.

In 2004, Berinde introduced the definition which is a generalization of the above operators.

Definition 5 (see [1]). A mapping T is said to be a weak contraction operator, if there exist $L \geq 0$ and $\delta \in (0,1)$ such that

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \|x - Tx\| \tag{11}$$

for all $x, y \in E$.

Theorem 6 (see [1]). *Let $(E, \|\cdot\|)$ be Banach space. Assume that $C \subseteq E$ is a nonempty closed convex subset and $T : C \to C$ is a mapping satisfying* (11). *Then $F(T) \neq \emptyset$.*

Definition 7 (see [22]). Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be two iteration processes and let both $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be converging to the same fixed point p of a self-mapping T. Assume that

$$\lim_{n\to\infty} \frac{\|u_n - p\|}{\|v_n - p\|} = 0. \tag{12}$$

Then, it is said that $\{u_n\}_{n\in\mathbb{N}}$ converges faster than $\{v_n\}_{n\in\mathbb{N}}$ to fixed point p of T.

The rate of convergence of the Picard and Mann iteration processes in terms of Zamfirescu operators in arbitrary Banach setting was compared by Berinde [22]. Using this class of operator, the Mann iteration method converges faster than the Ishikawa iteration method that was shown by Babu and Vara Prasad [23]. After a short time, Qing and Rhoades [24] showed that the claim of Babu and Vara Prasad [23] is false. There are many studies which have been made on the rate of convergence as given in [15, 25, 26] which are just a few of them.

2. Main Results

Theorem 8. *Let C be a nonempty closed convex subset of an arbitrary Banach space E and let $T : C \to C$ be a mapping satisfying* (11). *Let $\{x_n\}_{n\in\mathbb{N}}$ be defined through the new iteration* (3) *and $x_0 \in E$, where $\{\wp_n - \xi_n\}_{n=0}^{\infty}, \{\wp_n\}_{n=0}^{\infty}\{\xi_n\}, \{\zeta_n\} \in [0,1]$ with $\wp_n$ satisfying $\sum_{n=0}^{\infty} \wp_n = \infty$, $\wp_n \geq \xi_n$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to fixed point of T.*

Proof. From Theorems 4 and 6, it is clear that T has a unique fixed point in C and $F(T) \neq \emptyset$.

From (3), we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|(1 - \wp_n)\, x_n + (\wp_n - \xi_n)\, Ty_n + \xi_n Tx_n - p\| \\
&\leq (1 - \wp_n) \|x_n - p\| + (\wp_n - \xi_n) \|Ty_n - p\| + \xi_n \|Tx_n - p\|
\end{aligned}$$

$$\begin{aligned}
&\le (1-\wp_n)\|x_n-p\| + (\wp_n-\xi_n)\delta\|y_n-p\| \\
&\quad + (\wp_n-\xi_n)L\|p-Tp\| + \xi_n\delta\|x_n-p\| + \xi_nL\|p-Tp\| \\
&= [1-\wp_n+\xi_n\delta]\|x_n-p\| + (\wp_n-\xi_n)\delta\|y_n-p\| \\
&\quad + \wp_nL\|p-Tp\|.
\end{aligned} \tag{13}$$

In addition,

$$\begin{aligned}
\|y_n-p\| &= \|(1-\zeta_n)x_n + \zeta_nTx_n - p\| \\
&\le (1-\zeta_n)\|x_n-p\| + \zeta_n\|Tx_n-p\| \\
&\le (1-\zeta_n)\|x_n-p\| + \zeta_n\delta\|x_n-p\| + \zeta_nL\|p-Tp\| \\
&= (1-\zeta_n(1-\delta))\|x_n-p\| + \zeta_nL\|p-Tp\|.
\end{aligned} \tag{14}$$

Substituting (14) in (13), we have the following estimates:

$$\begin{aligned}
&\|x_{n+1}-p\| \\
&\le [1-\wp_n+\xi_n\delta]\|x_n-p\| \\
&\quad + (\wp_n-\xi_n)\delta[(1-\zeta_n(1-\delta))\|x_n-p\| + \zeta_nL\|p-Tp\|] \\
&\quad + \wp_nL\|p-Tp\| \\
&= [1-\wp_n+\xi_n\delta+(\wp_n-\xi_n)\delta(1-\zeta_n(1-\delta))]\|x_n-p\| \\
&\quad + [(\wp_n-\xi_n)\delta\zeta_n+\wp_n]L\|p-Tp\|.
\end{aligned} \tag{15}$$

Since $\|p-Tp\|=0$, we have

$$\begin{aligned}
\|x_{n+1}-p\| &\le [1-\wp_n(1-\delta)]\|x_n-p\| \\
\|x_{n+1}-p\| &\le (1-\wp_n(1-\delta))\|x_n-p\| \\
\|x_n-p\| &\le (1-\wp_{n-1}(1-\delta))\|x_{n-1}-p\| \\
\|x_{n-1}-p\| &\le (1-\wp_{n-2}(1-\delta))\|x_{n-2}-p\| \\
&\vdots \\
\|x_2-p\| &\le (1-\wp_1(1-\delta))\|x_1-p\| \\
\|x_1-p\| &\le (1-\wp_0(1-\delta))\|x_0-p\| \\
\|x_{n+1}-p\| &\le \prod_{i=0}^{n}[1-\wp_i(1-\delta)]\times\|x_0-p\| \\
&\le \|x_0-p\| \times e^{(\sum_{i=0}^{n}[-\wp_i(1-\delta)])} \\
&= \|x_0-p\| \times e^{(-(1-\delta)\sum_{i=0}^{n}\wp_i)}
\end{aligned} \tag{16}$$

for all $n\in\mathbb{N}$.

Since $0<\delta<1$, $\wp_n\in[0,1]$, and $\sum_{n=0}^{\infty}\wp_n=\infty$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\sup\|x_{n+1}-p\| \\
&\le \lim_{n\to\infty}\sup\|x_0-p\| \times e^{(-(1-\delta)\sum_{i=0}^{n}\wp_i)} \le 0.
\end{aligned} \tag{17}$$

So $\lim_{n\to\infty}\|x_n-p\|=0$ yields $x_n\to p\in F(T)$. This completes the proof of theorem. □

Theorem 9. *Let $(E,\|\cdot\|)$ be Banach space and $T: E\to E$ a self-mapping with fixed point p with respect to weak contraction condition in the sense of Berinde* (11). *Let $\{x_n\}_{n\in\mathbb{N}}$ be iteration process* (3) *converging to fixed point of T, where $\wp_n\ge\xi_n$ and $\{\wp_n-\xi_n\}_{n=0}^{\infty}, \{\wp_n\}_{n=0}^{\infty}, \{\xi_n\}_{n=0}^{\infty}, \{\zeta_n\}_{n=0}^{\infty}\in[0.1]$ such that $0<\wp\le\wp_n$ for all n. Then two-step iteration process is T stable.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be iteration process (3) converging to p. Assume that $\{y_n\}_{n\in\mathbb{N}}\subset E$ is an arbitrary sequence in E. Set

$$\epsilon_n = \|y_{n+1}-(1-\wp_n)y_n + (\wp_n-\xi_n)Tv_n + \xi_nTy_n\| \quad n=0,1,\ldots, \tag{18}$$

where $v_n=(1-\zeta_n)y_n+\zeta_nTy_n$. Suppose that $\lim_{n\to\infty}\epsilon_n=0$. Then, we shall prove that $\lim_{n\to\infty}y_n=p$. Using contraction condition (11), we have

$$\begin{aligned}
\|y_{n+1}-p\| &\le \|y_{n+1}-(1-\wp_n)y_n+(\wp_n-\xi_n)Tv_n+\xi_nTy_n\| \\
&\quad + \|(1-\wp_n)y_n+(\wp_n-\xi_n)Tv_n+\xi_nTy_n-p\| \\
&\le \epsilon_n + \|(1-\wp_n)y_n+(\wp_n-\xi_n)Tv_n+\xi_nTy_n-p\| \\
&\le \epsilon_n + (1-\wp_n)\|y_n-p\| + (\wp_n-\xi_n)\|Tv_n-p\| \\
&\quad + \xi_n\|Ty_n-p\| \\
&\le \epsilon_n + (1-\wp_n)\|y_n-p\| + (\wp_n-\xi_n)\delta\|v_n-p\| \\
&\quad + \xi_n\delta\|y_n-p\| + (\wp_n-\xi_n)L\|p-Tp\| \\
&\quad + \xi_nL\|p-Tp\| \\
&= \epsilon_n + (1-\wp_n+\xi_n\delta)\|y_n-p\| \\
&\quad + (\wp_n-\xi_n)\delta\|v_n-p\| \\
&\quad + (\wp_n-\xi_n)L\|p-Tp\| + \xi_nL\|p-Tp\|.
\end{aligned} \tag{19}$$

We estimate $\|v_n-p\|$ in (19) as follows:

$$\begin{aligned}
\|v_n-p\| &= \|(1-\zeta_n)y_n+\zeta_nTy_n-p\| \\
&\le (1-\zeta_n)\|y_n-p\| + \zeta_n\|Ty_n-p\| \\
&\le (1-\zeta_n)\|y_n-p\| + \zeta_n\delta\|y_n-p\| \\
&\quad + \zeta_n\varphi(\|p-Tp\|) \\
&= (1-\zeta_n(1-\delta))\|y_n-p\| + \zeta_n\varphi(\|p-Tp\|).
\end{aligned} \tag{20}$$

Substituting (20) in (19), we have

$$\begin{aligned}
&\|y_{n+1}-p\| \\
&\quad\le \varepsilon_n + (1-\wp_n+\xi_n\delta)\|y_n-p\| + \wp_n L\|p-Tp\| \\
&\qquad + (\wp_n-\xi_n)\delta\left[(1-\zeta_n(1-\delta))\|y_n-p\|\right. \\
&\qquad\qquad \left. + \zeta_n L\|p-Tp\|\right] \\
&\quad= \varepsilon_n + \left[1-\wp_n+\xi_n\delta+(\wp_n-\xi_n)\delta(1-\zeta_n(1-\delta))\right] \\
&\qquad \times\|y_n-p\| + \left[(\wp_n-\xi_n)\zeta_n\delta+\wp_n\right]L\|p-Tp\|.
\end{aligned} \tag{21}$$

Since $\|p-Tp\|=0$, we have

$$\begin{aligned}
&\|y_{n+1}-p\| \\
&\quad\le \varepsilon_n + \left[1-\wp_n+\xi_n\delta+(\wp_n-\xi_n)\delta(1-\zeta_n(1-\delta))\right] \\
&\qquad \times\|y_n-p\| \le \varepsilon_n + \left[1-\wp_n(1-\delta)\right]\|y_n-p\|.
\end{aligned} \tag{22}$$

Since $0<1-\wp_n(1-\delta)<1$ and using Lemma 2, we obtain $\lim_{n\to\infty}y_n=p$.

Conversely, letting $\lim_{n\to\infty}y_n = p$, we show that $\lim_{n\to\infty}\varepsilon_n=0$ as follows:

$$\begin{aligned}
\varepsilon_n &= \|y_{n+1}-(1-\wp_n)y_n-(\wp_n-\xi_n)Tv_n-\xi_nTy_n\| \\
&\le \|y_{n+1}-p\| + \|p-(1-\wp_n)y_n-(\wp_n-\xi_n)Tv_n-\xi_nTy_n\| \\
&\le \|y_{n+1}-p\| + (1-\wp_n)\|y_n-p\| + (\wp_n-\xi_n)\|Tv_n-p\| \\
&\quad + \xi_n\|Ty_n-p\| \\
&\le \|y_{n+1}-p\| + [1-\wp_n+\delta\xi_n]\|y_n-p\| \\
&\quad + (\wp_n-\xi_n)\delta\|v_n-p\| \\
&\le \|y_{n+1}-p\| + [1-\wp_n+\delta\xi_n] \\
&\quad \times\|y_n-p\| + (\wp_n-\xi_n)\delta(1-\zeta_n(1-\delta))\|y_n-p\| \\
&\le \|y_{n+1}-p\| + [1-\wp_n+\delta\xi_n \\
&\qquad\qquad + (\wp_n-\xi_n)\delta(1-\zeta_n(1-\delta))]\|y_n-p\| \\
&\le \|y_{n+1}-p\| + [1-\wp_n(1-\delta)]\|y_n-p\|.
\end{aligned} \tag{23}$$

Since $\lim_{n\to\infty}\|y_n-p\|=0$, it follows that $\lim_{n\to\infty}\varepsilon_n=0$. Therefore the iteration scheme is T stable. □

Example 10 (see [24]). Let $T:[0,1]\to[0,1]$, $Tx=x/2$, $\wp_n,\xi_n,\zeta_n,\vartheta_n=0$, $n=1,2,\ldots,15$, and $\xi_n=1/2-2/\sqrt{n}$, $\wp_n=1/2+2/\sqrt{n}$, $\zeta_n=\vartheta_n=4/\sqrt{n}$, for all $n\ge16$. It is easy to show that T is a weak contraction operator satisfying (11) with a unique fixed point 0. Furthermore, for all $n\ge16$, $4/\sqrt{n}, 1/2-2/\sqrt{n}, 1/2+2/\sqrt{n}\in[0,1]$, and $\sum_{n=0}^{\infty}(1/2+2/\sqrt{n})=\infty$. Then the new iterative process is faster than the Ishikawa iterative process. Assume that $u_0=w_0\neq0$ is initial point for the new and Ishikawa iterative processes, respectively. Firstly, we consider the new iterative process, and we have

$$\begin{aligned}
u_{n+1} &= (1-\wp_n)u_n + (\wp_n-\xi_n)T\left((1-\zeta_n)u_n+\zeta_nTu_n\right) \\
&\quad + \xi_nTu_n \\
&= \left(1-\left(\frac{1}{2}+\frac{2}{\sqrt{n}}\right)\right)u_n \\
&\quad + \left(\left(\frac{1}{2}+\frac{2}{\sqrt{n}}\right)-\left(\frac{1}{2}-\frac{2}{\sqrt{n}}\right)\right) \\
&\quad \times T\left(\left(1-\frac{4}{\sqrt{n}}\right)u_n+\frac{4}{\sqrt{n}}Tu_n\right) \\
&\quad + \left(\frac{1}{2}-\frac{2}{\sqrt{n}}\right)Tu_n \\
&= \left(1-\left(\frac{1}{2}+\frac{2}{\sqrt{n}}\right)\right)u_n \\
&\quad + \left(\left(\frac{1}{2}+\frac{2}{\sqrt{n}}\right)-\left(\frac{1}{2}-\frac{2}{\sqrt{n}}\right)\right) \\
&\quad \times\frac{1}{2}\left(\left(1-\frac{4}{\sqrt{n}}\right)u_n+\frac{4}{\sqrt{n}}\frac{1}{2}u_n\right) \\
&\quad + \left(\frac{1}{2}-\frac{2}{\sqrt{n}}\right)\frac{1}{2}u_n \\
&= \left(\frac{1}{2}-\frac{2}{\sqrt{n}}\right)u_n \\
&\quad + \frac{2}{\sqrt{n}}\left(1-\frac{2}{\sqrt{n}}\right)u_n+\left(\frac{1}{4}-\frac{1}{\sqrt{n}}\right)u_n \\
&= \left(\frac{1}{2}-\frac{2}{\sqrt{n}}+\frac{2}{\sqrt{n}}-\frac{4}{n}+\frac{1}{4}-\frac{1}{\sqrt{n}}\right)u_n \\
&= \left(1-\frac{1}{\sqrt{n}}-\frac{4}{n}-\frac{1}{4}\right)u_n = \prod_{i=16}^{n}\left(1-\frac{1}{\sqrt{i}}-\frac{4}{i}-\frac{1}{4}\right)u_0.
\end{aligned} \tag{24}$$

Secondly, we consider the Ishikawa iterative process, and we have

$$\begin{aligned}
w_{n+1} &= (1-\zeta_n)w_n + \zeta_nT\left((1-\vartheta_n)w_n+\vartheta_nTw_n\right) \\
&= \left(1-\frac{4}{\sqrt{n}}\right)w_n + \frac{4}{\sqrt{n}}T\left(\left(1-\frac{4}{\sqrt{n}}\right)w_n+\frac{4}{\sqrt{n}}Tw_n\right) \\
&= \left(1-\frac{4}{\sqrt{n}}\right)w_n + \frac{4}{\sqrt{n}}\frac{1}{2}\left(\left(1-\frac{4}{\sqrt{n}}\right)w_n+\frac{4}{\sqrt{n}}\frac{1}{2}w_n\right) \\
&= \left(1-\frac{4}{\sqrt{n}}\right)w_n + \frac{2}{\sqrt{n}}\left(\left(1-\frac{4}{\sqrt{n}}\right)w_n+\frac{2}{\sqrt{n}}w_n\right)
\end{aligned}$$

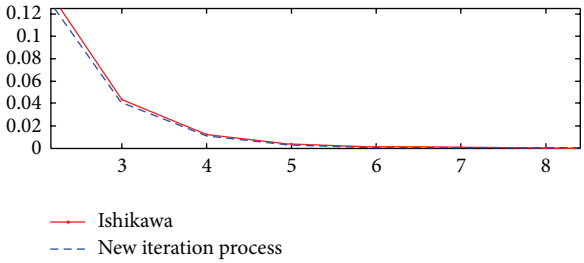

Figure 1: It shows the value functions found by successive steps of the Ishikawa and new iteration methods.

$$
\begin{aligned}
&= \left(1 - \frac{4}{\sqrt{n}}\right) w_n + \frac{2}{\sqrt{n}} \left(1 - \frac{2}{\sqrt{n}}\right) w_n \\
&= \left(1 - \frac{4}{\sqrt{n}} + \frac{2}{\sqrt{n}} - \frac{4}{n}\right) w_n \\
&= \left(1 - \frac{2}{\sqrt{n}} - \frac{4}{n}\right) w_n \\
&= \prod_{i=16}^{n} \left(1 - \frac{2}{\sqrt{i}} - \frac{4}{i}\right) w_0.
\end{aligned} \tag{25}
$$

Now, taking the above two equalities, we obtain

$$
\begin{aligned}
&\left|\frac{u_{n+1} - 0}{w_{n+1} - 0}\right| \\
&= \left|\frac{\prod_{i=16}^{n} \left(1 - 1/\sqrt{i} - 4/i - 1/4\right) u_0}{\prod_{i=16}^{n} \left(1 - 2/\sqrt{i} - 4/i\right) w_0}\right| \\
&= \left|\prod_{i=16}^{n} \frac{\left(1 - 1/\sqrt{i} - 4/i - 1/4\right)}{\left(1 - 2/\sqrt{i} - 4/i\right)}\right| \\
&= \left|\prod_{i=16}^{n} \left[1 - \frac{-1/\sqrt{i} + 1/4}{1 - 2/\sqrt{i} - 4/i}\right]\right| \\
&= \left|\prod_{i=16}^{n} \left[1 - \frac{i - 4\sqrt{i}}{4i - 8\sqrt{i} - 16}\right]\right|.
\end{aligned} \tag{26}
$$

It is clear that

$$
0 \leq \lim_{n \to \infty} \prod_{i=16}^{n} \left[1 - \frac{i - 4\sqrt{i}}{4i - 8\sqrt{i} - 16}\right] = 0. \tag{27}
$$

Therefore, the proof is completed.

Now, we can give Table 1 and Figures 1 and 2 to support and reinforce our claim in the Example 10.

Finally, we check that this iteration procedure can be applied to find the solution of delay differential equations.

2.1. An Application. Throughout the rest of this paper, the space $C[a, b]$ equipped with Chebyshev norm $\|x - y\|_\infty = \max_{t \in [a,b]} |x(t) - y(t)|$ denotes the space of all continuous functions. It is well known that $C[a, b]$ is a real Banach space with respect to $\|\cdot\|_\infty$ norm; more details can be found in [2, 27].

Table 1: Iterative values of the rate of convergence to zero of the Ishikawa and new iteration process.

x_n	Ishikawa	New iteration process
x_1	1,990000000000000	1,990000000000000
x_2	0,556472918237748	0,541618812060050
x_3	0,155609099865344	0,147412531445900
x_4	0,043513693420310	0,040121306615323
x_5	0,012167935658745	0,010919826345371
x_6	0,003402576213543	0,002972051946272
x_7	0,000951478148280	0,000808904142975
x_8	0,000266066242117	0,000220159648738
x_9	0,000074401335777	0,000059920908248
x_{10}	0,000020805190171	0,000016308689016
x_{11}	0,000005817851703	0,000004438740086
x_{12}	0,000001626872822	0,000001208093031
x_{13}	0,000000454929983	0,000000328806991
x_{14}	0,000000127214179	0,000000089491483
x_{15}	0,000000035573490	0,000000024356920
x_{16}	0,000000009947580	0,000000006629229
x_{17}	0,000000002781688	0,000000001804279
x_{18}	0,000000000777856	0,000000000491071
x_{19}	0,000000000217516	0,000000000133655
x_{20}	0,000000000060825	0,000000000036377
x_{21}	0,000000000017009	0,000000000009901
x_{22}	0,000000000004756	0,000000000002695
x_{23}	0,000000000001330	0,000000000000733
x_{24}	0,000000000000372	0,000000000000200
x_{25}	0,000000000000104	0,000000000000054
x_{26}	0,000000000000029	0,000000000000015
x_{27}	0,000000000000008	0,000000000000004
x_{28}	0,000000000000002	0,000000000000001
x_{29}	0,000000000000001	0,000000000000000

Now, we will consider a delay differential equation such that

$$
x'(t) = g(t, x(t), x(t - \varsigma)), \quad t \in [t_{0n}, b] \tag{28}
$$

and an assumed solution

$$
x(t) = \varphi(t), \quad t \in [t_0 - \varsigma, t_0]. \tag{29}
$$

Assume that the following conditions are satisfied:

(C_1) $t_0, b \in \mathbb{R}$, $\varsigma \geq 0$,

(C_2) $g \in C([t_0, b] \times \mathbb{R}^2, \mathbb{R})$,

(C_3) $\varphi \in C([t_0 - \varsigma, t_0], \mathbb{R})$,

(C_4) there exists the following inequality:

$$
\begin{aligned}
&|g(t, \gamma_1, \gamma_2) - g(t, \lambda_1, \lambda_2)| \\
&\quad \leq K_g \left[|\gamma_1 - \lambda_1| + |\gamma_2 - \lambda_2|\right] + L|\gamma_1 - T\gamma_1|,
\end{aligned} \tag{30}
$$

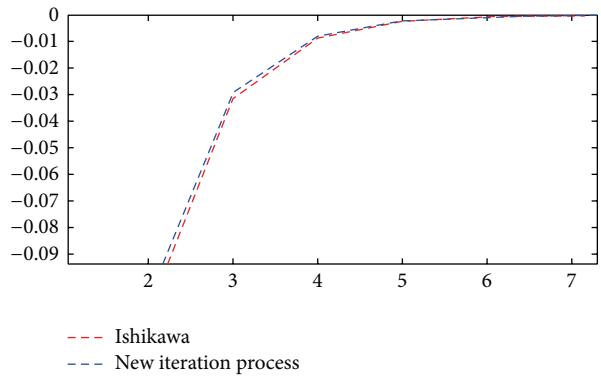

FIGURE 2: It shows the derivative functions of the Ishikawa and new iteration methods.

for all $\gamma_i, \lambda_i \in \mathbb{R}$ $(i = 1, 2)$ and $t \in [t_0, b]$ such that $K_g > 0$,

(C_5) $2K_g(b - t_0) < 1$, and according to a solution of problem (28)-(29) we infer the function $x \in C([t_0 - \varsigma, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$. The problem can be reconstituted as follows:

(C_6)

$$x(t) = \begin{cases} \varphi(t), & \text{if } t \in [t_0 - \varsigma, t_0], \\ \varphi(t_0) + \int_{t_0}^{t} g(t, x(s), x(s-\varsigma))\, ds, & \text{if } t \in [t_0, b]. \end{cases} \tag{31}$$

Also, the map $T : C([t_0 - \varsigma, b], \mathbb{R}) \to C([t_0 - \varsigma, b], \mathbb{R})$ is defined by the following form:

$$T(x)(t) = \begin{cases} \varphi(t), & \text{if } t \in [t_0 - \varsigma, t_0], \\ \varphi(t_0) + \int_{t_0}^{t} g(t, x(s), x(s-\varsigma))\, ds, & \text{if } t \in [t_0, b]. \end{cases} \tag{32}$$

Using weak-contraction mapping, we obtain the following.

Theorem 11. *We suppose that conditions* (C_1)–(C_5) *are performed. Then the problem* (28)-(29) *has a unique solution in* $C([t_0 - \varsigma, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$.

Proof. We consider iterative process (3) for the mapping T. The fixed point of T is shown via p such that $Tp = p$.

For the first part, that is, for $t \in [t_0 - \varsigma, t_0]$, it is clear that $\lim_{n\to\infty} x_n = p$. Therefore, letting $t \in [t_0, b]$, we obtain

$$\begin{aligned}
&\|x_{n+1} - p\|_\infty \\
&= \|(1 - \wp_n) x_n + \xi_n T x_n + (\wp_n - \xi_n) T y_n - p\|_\infty \\
&\le (1 - \wp_n) \|x_n - p\|_\infty + \xi_n \|T x_n - T p\|_\infty \\
&\quad + (\wp_n - \xi_n) \|T y_n - T p\|_\infty \\
&\le (1 - \wp_n) \|x_n - p\|_\infty \\
&\quad + \xi_n \|T x_n - T p\|_\infty + (\wp_n - \xi_n) \|T y_n - T p\|_\infty \\
&= (1 - \wp_n) \|x_n - p\|_\infty + \xi_n \max_{t\in[t_0-\varsigma,b]} |T x_n(t) - T p(t)| \\
&\quad + (\wp_n - \xi_n) \max_{t\in[t_0-\varsigma,b]} |T y_n(t) - T p(t)| \\
&= (1 - \wp_n) \|x_n - p\|_\infty \\
&\quad + \xi_n \max_{t\in[t_0-\varsigma,b]} \Big| \varphi(t_0) + \int_{t_0}^{t} g(t, x_n(s), x_n(s-\varsigma))\, ds \\
&\qquad - \varphi(t_0) - \int_{t_0}^{t} g(t, p(s), p(s-\varsigma))\, ds \Big| \\
&\quad + (\wp_n - \xi_n) \max_{t\in[t_0-\varsigma,b]} \Big| \varphi(t_0) \\
&\qquad + \int_{t_0}^{t} g(t, y_n(s), y_n(s-\varsigma))\, ds \\
&\qquad - \varphi(t_0) - \int_{t_0}^{t} g(t, p(s), \\
&\qquad p(s-\varsigma))\, ds \Big| \\
&= (1 - \wp_n) \|x_n - p\|_\infty \\
&\quad + \xi_n \max_{t\in[t_0-\varsigma,b]} \Big| \int_{t_0}^{t} g(t, x_n(s), x_n(s-\varsigma))\, ds \\
&\qquad - \int_{t_0}^{t} g(t, p(s), p(s-\varsigma))\, ds \Big| \\
&\quad + (\wp_n - \xi_n) \max_{t\in[t_0-\varsigma,b]} \Big| \int_{t_0}^{t} g(t, y_n(s), y_n(s-\varsigma))\, ds \\
&\qquad - \int_{t_0}^{t} g(t, p(s), p(s-\varsigma))\, ds \Big| \\
&\le (1 - \wp_n) \|x_n - p\|_\infty + \xi_n \\
&\quad \times \max_{t\in[t_0-\varsigma,b]} \int_{t_0}^{t} |g(t, x_n(s), x_n(s-\varsigma)) \\
&\qquad - g(t, p(s), p(s-\varsigma))|\, ds
\end{aligned}$$

$$
\begin{aligned}
&+ (\wp_n - \xi_n) \max_{t\in[t_0-\varsigma,b]} \int_{t_0}^{t} \left| g(t, y_n(s), y_n(s-\varsigma)) \right. \\
&\qquad \left. - g(t, p(s), p(s-\varsigma)) \right| ds \\
&\le (1-\wp_n) \left\| x_n - p \right\|_\infty \\
&\quad + \xi_n \max_{t\in[t_0-\varsigma,b]} \int_{t_0}^{t} \left[K_g \left(\left| x_n(s) - p(s) \right| \right. \right. \\
&\qquad \left. + \left| x_n(s-\varsigma) - p(s-\varsigma) \right| \right) \\
&\qquad + L \left| x_n(s) - Tx_n(s) \right| \Big] ds \\
&\quad + (\wp_n - \xi_n) \max_{t\in[t_0-\varsigma,b]} \int_{t_0}^{t} \left[K_g \left(\left| y_n(s) - p(s) \right| \right. \right. \\
&\qquad \left. + \left| y_n(s-\varsigma) - p(s-\varsigma) \right| \right) \\
&\qquad + L \left| y_n(s) - Ty_n(s) \right| \Big] ds.
\end{aligned} \tag{33}
$$

Hence, we obtain

$$
\begin{aligned}
\left\| x_{n+1} - p \right\|_\infty &\le (1-\wp_n) \left\| x_n - p \right\|_\infty \\
&\quad + \xi_n K_g (b - t_0) \max_{t\in[t_0-\varsigma,b]} \left\{ \left| x_n(t) - p(t) \right| \right. \\
&\qquad \left. + \left| x_n(t) - p(t) \right| \right\} \\
&\quad + \xi_n L (b - t_0) \max_{t\in[t_0-\varsigma,b]} \left| x_n(t) - Tx_n(t) \right| \\
&\quad + (\wp_n - \xi_n) K_g (b - t_0) \\
&\quad \times \max_{t\in[t_0-\varsigma,b]} \left\{ \left| y_n(t) - p(t) \right| + \left| y_n(t) - p(t) \right| \right\} \\
&\quad + (\wp_n - \xi_n) L (b - t_0) \\
&\quad \times \max_{t\in[t_0-\varsigma,b]} \left| y_n(t) - Ty_n(t) \right| \\
&= (1-\wp_n) \left\| x_n - p \right\|_\infty + 2\xi_n K_g (b - t_0) \\
&\quad \times \max_{t\in[t_0-\varsigma,b]} \left| x_n(t) - p(t) \right| \\
&\quad + 2 (\wp_n - \xi_n) K_g (b - t_0) \\
&\quad \times \max_{t\in[t_0-\varsigma,b]} \left| y_n(t) - p(t) \right| \\
&= (1-\wp_n) \left\| x_n - p \right\|_\infty \\
&\quad + 2\xi_n K_g (b - t_0) \left\| x_n - p \right\|_\infty \\
&\quad + 2 (\wp_n - \xi_n) K_g (b - t_0) \left\| y_n - p \right\|_\infty \\
&= \left(1 - \wp_n + 2\xi_n K_g (b - t_0) \right) \left\| x_n - p \right\|_\infty \\
&\quad + 2 (\wp_n - \xi_n) K_g (b - t_0) \left\| y_n - p \right\|_\infty.
\end{aligned} \tag{34}
$$

By continuing this way, we have

$$
\begin{aligned}
&\left\| y_n - p \right\|_\infty \\
&\quad = \left\| (1-\zeta_n) x_n + \zeta_n T x_n - p \right\|_\infty \\
&\quad \le (1-\zeta_n) \left\| x_n - p \right\|_\infty + \zeta_n \left\| T x_n - p \right\|_\infty \\
&\quad \le (1-\zeta_n) \left\| x_n - p \right\|_\infty \\
&\qquad + \zeta_n \max_{t\in[t_0-\varsigma,b]} \left| Tx_n(t) - Tp(t) \right| \\
&\quad = (1-\zeta_n) \left\| x_n - p \right\|_\infty \\
&\qquad + \zeta_n \max_{t\in[t_0-\varsigma,b]} \left| \varphi(t_0) + \int_{t_0}^{t} g(t, x_n(s), x_n(s-\varsigma)) ds \right. \\
&\qquad\qquad \left. - \varphi(t_0) - \int_{t_0}^{t} g(t, p(s), p(s-\varsigma)) ds \right| \\
&\quad = (1-\zeta_n) \left\| x_n - p \right\|_\infty \\
&\qquad + \zeta_n \max_{t\in[t_0-\varsigma,b]} \left| \int_{t_0}^{t} g(t, x_n(s), x_n(s-\varsigma)) ds \right. \\
&\qquad\qquad \left. - \int_{t_0}^{t} g(t, p(s), p(s-\varsigma)) ds \right| \\
&\quad \le (1-\zeta_n) \left\| x_n - p \right\|_\infty \\
&\qquad + \zeta_n \max_{t\in[t_0-\varsigma,b]} \int_{t_0}^{t} \left| g(t, x_n(s), x_n(s-\varsigma)) \right. \\
&\qquad\qquad \left. - g(t, p(s), p(s-\varsigma)) \right| ds \\
&\quad \le (1-\zeta_n) \left\| x_n - p \right\|_\infty \\
&\qquad + \zeta_n \int_{t_0}^{t} \left[K_g \left(\left| x_n(s) - p(s) \right| + \left| x_n(s-\varsigma) - p(s-\varsigma) \right| \right) \right. \\
&\qquad\qquad \left. + L \left| x_n(s) - Tx_n(s) \right| \right] ds.
\end{aligned} \tag{35}
$$

Hence, we obtain

$$
\begin{aligned}
&\left\| y_n - p \right\|_\infty \\
&\quad \le (1-\zeta_n) \left\| x_n - p \right\|_\infty \\
&\qquad + \zeta_n K_g (b - t_0) \max_{t\in[t_0-\varsigma,b]} \left\{ \left| x_n(t) - p(t) \right| \right. \\
&\qquad\qquad \left. + \left| x_n(t-\varsigma) - p(t-\varsigma) \right| \right\} \\
&\qquad + \zeta_n L (b - t_0) \max_{t\in[t_0-\varsigma,b]} \left| x_n(t) - Tx_n(t) \right| \\
&\quad \le (1-\zeta_n) \left\| x_n - p \right\|_\infty \\
&\qquad + \zeta_n 2K_g (b - t_0) \max_{t\in[t_0-\varsigma,b]} \left| x_n(t) - p(t) \right|
\end{aligned}
$$

$$= (1-\zeta_n)\|x_n - p\|_\infty + \zeta_n 2K_g(b-t_0)\|x_n - p\|_\infty$$
$$= \left(1-\zeta_n\left(1-2K_g(b-t_0)\right)\right)\|x_n - p\|_\infty. \quad (36)$$

Substituting (36) into (34), we obtain

$$\begin{aligned}\|x_{n+1} - p\|_\infty &\le \left(1-\wp_n+2\xi_n K_g(b-t_0)\right)\|x_n - p\|_\infty \\ &\quad + (\wp_n - \xi_n)2K_g(b-t_0) \\ &\quad \times \left(1-\zeta_n\left(1-2K_g(b-t_0)\right)\right)\|x_n - p\|_\infty \\ &= \left(1-\wp_n+2\xi_n K_g(b-t_0)\right. \\ &\quad + (\wp_n - \xi_n)2K_g(b-t_0) \\ &\quad \left.\times \left(1-\zeta_n\left(1-2K_g(b-t_0)\right)\right)\right)\|x_n - p\|_\infty. \quad (37)\end{aligned}$$

Since $(1-2K_g(b-t_0)) < 1$, we have

$$\|x_{n+1} - p\|_\infty \le \left(1-\zeta_n\left(1-2K_g(b-t_0)\right)\right)\|x_n - p\|_\infty. \quad (38)$$

We take $\zeta_n(1-2K_g(b-t_0)) = \mu_n < 1$ and $\|x_n - p\|_\infty = s_n$, and then the conditions of Lemma 3 immediately imply $\lim_{n\to\infty}\|x_n - p\|_\infty = 0$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A New Look at the Coefficients of a Reciprocal Generating Function

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We study a special property of free cumulants. We prove that coefficients of a reciprocal generating function correspond to "free cumulants with the first two elements in the same block."

1. Introduction

The original motivation for this paper comes from a desire to understand the results on free probability method in the book of Nica and Speicher [1]. Free probability is a mathematical theory that studies noncommutative random variables. The freeness or free independence property is the analogue of the classical notion of independence, and it is connected with free products. This theory was initiated by Dan Voiculescu around 1986 in order to attack the free group factors isomorphism problem, an important unsolved problem in the theory of operator algebras (see [2]).

It is natural to study relations between classical analysis and free probability. We will present a theorem which gives us a new relation between reciprocal generating function and free cumulants. The free cumulant (introduced by Speicher) [3] plays a major role in the free probability theory. It is related to the lattice of noncrossing partitions of the set $\{1, \ldots, n\}$ in the same way in which the classic cumulant functional is related to the lattice of all partitions of that set.

Since these beginnings of free probability theory have evolved into a theory with a lot of links in quite different fields, in particular, there exists a combinatorial facet: main aspects of free probability theory can be considered the combinatorics of noncrossing partitions. It is worthwhile to mention the work of [4], where authors introduced the so-called Bercovici-Pata bijection. This bijection, which we denote by Λ, is a correspondence between the probability measures on the real line that are infinitely divisible with respect to the classical convolution $*$ and the ones which are infinitely divisible with respect to the free convolution $\boxplus$. This mapping has several useful algebraic and topological properties and preserves the properties of cumulant, that is, determines the uniqueness between classical and free cumulant. Moreover, by suitable definition of the free cumulant transform, the connection between the free and classical Lévy-Khintchine representations of a probability law in $ID(*)$ and its counterpart Λ in $ID(\boxplus)$ is determined simply by μ and $\Lambda(\mu)$ having the same characteristic triplet (classical and free, resp.). The main aim of the paper is to show new property of free cumulants; that is, we will show that coefficients of a reciprocal generating function correspond to "cumulant with the first two elements in the same block." This topic has not been extensively studied in the past. The only result of which I am aware is Wright's [5]. Wright gave an asymptotic formula for the coefficient in some special case of power series. Recently, this topic has also been studied in [6], where the concept of composita of reciprocal generating function is used for obtaining a unique triangle, when only the generating function for the central coefficients of that triangle is known. In this paper, we present a new method using free cumulants to compute coefficients of a reciprocal generating function.

2. Free Cumulants and Indication

In this section, we provide a short and self-contained summary of the basic definitions and facts needed for our study.

In free probability, we assume that our probability space is a von Neumann algebra $\mathscr{A}$ with a normal faithful tracial state $\tau : \mathscr{A} \to \mathbb{C}$; that is, $\tau(\cdot)$ is linear and weak*-continuous and $\tau(\mathbb{X}\mathbb{Y}) = \tau(\mathbb{Y}\mathbb{X})$, $\tau(\mathbb{I}) = 1$, $\tau(\mathbb{X}\mathbb{X}^*) \geq 0$, and $\tau(\mathbb{X}\mathbb{X}^*) = 0$ imply $\mathbb{X} = 0$ for all $\mathbb{X}, \mathbb{Y} \in \mathscr{A}$. A (noncommutative) random variable $\mathbb{X}$ is a self-adjoint ($\mathbb{X} = \mathbb{X}^*$) element of $\mathscr{A}$. Below, we introduce the concept of noncrossing cumulants without using this abstract object.

Definition 1. Let $\pi = \{V_1, \ldots, V_p\}$ be a partition of the linear ordered set $1, \ldots, n$; that is, the $V_i \neq \emptyset$ are ordered and disjoint sets whose union is $\{1, \ldots, n\}$. Then π is called noncrossing if $a, c \in V_i$ and $b, d \in V_j$ with $a < b < c < d$ which implies $i = j$. The sets $V_i \in \pi$ are called blocks. We will denote the set of all noncrossing partitions of the set $\{1, \ldots, n\}$ by $\mathrm{NC}(n)$.

Now we can define the free cumulants by induction. An important technical tool is a formula, that is, (2), to factorize in a product according to the block structure of noncrossing partition. This formula is actually at the basis of many of our forthcoming results in this paper and allows elegant proofs of many statements.

Definition 2. Let $a_1, a_2, \ldots$ be a sequence of $\mathbb{C}$. The free (noncrossing) cumulants are the sequence $R_k := R_k(a_1, \ldots, a_k)$ defined by the recursive formula

$$a_n = \sum_{\nu \in \mathrm{NC}(n)} R_\nu (a_1, a_2, \ldots, a_n), \tag{1}$$

where

$$R_\nu (a_1, a_2, \ldots, a_n) := \Pi_{B \in \nu} R_{|B|} (a_i : i \in B) \tag{2}$$

and $\mathrm{NC}(n)$ is the set of all noncrossing partitions of $\{1, 2, \ldots, n\}$ (see [1, 7]).

Example 3. For $n = 3$, we have

$$\begin{aligned} \mathrm{NC}(3) &= \{\{(1,2),(3)\}, \{(1,3),(2)\}, \{(2,3),(1)\}, \\ &\qquad \{(1,2,3)\}, \{(1),(2),(3)\}\}, \\ a_3 &= \sum_{\nu \in \mathrm{NC}(3)} R_\nu (a_1, a_2, a_3) \\ &= R_2(a_1, a_2) R_1(a_3) + R_2(a_1, a_3) R_1(a_2) \\ &\quad + R_2(a_2, a_3) R_1(a_1) + R_3(a_1, a_2, a_3) \\ &\quad + R_1(a_1) R_1(a_2) R_1(a_3). \end{aligned} \tag{3}$$

Definition 4. The ordinary generating function of a sequence $(a_0, a_1, a_2, \ldots)$, where $a_i \in \mathbb{C}$, is

$$G(z) = \sum_{i=0}^{\infty} a_i z^i. \tag{4}$$

In this paper, we assume that $a_0 \neq 0$ above series is convergent for sufficiently small z.

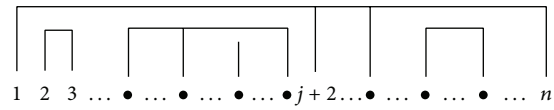

Figure 1: The main structure of noncrossing partitions of $\{1, 2, 3, \ldots, n\}$ with the first and $j + 2$ element in the same block.

The following lemma is a sequence version of Lemma 2.3 in [9] with $k = 1$ (the proof is also similar).

Lemma 5. *Let $a_1, a_2, \ldots$ be a sequence of $\mathbb{C}$. Then*

$$a_n = \sum_{i=0}^{n-2} a_i \sum_{\nu \in \mathrm{NC}''(n-i)} R_\nu (a_1, \ldots, a_{n-i}) + a_1 a_{n-1}, \tag{5}$$

where

(i) *n is positive integer greater than two;*

(ii) *$a_0 = 1$;*

(iii) *$\mathrm{NC}''(n)$ is the set of all noncrossing partitions of $\{1, 2, \ldots, n\}$ with the first two elements in the same block.*

Proof. At first, we will consider partitions with singleton 1; that is, $\pi \in \mathrm{NC}(n)$ and $\pi = \{V_1, \ldots, V_k\}$, where $V_1 = \{1\}$. It is clear that the sum over all noncrossing partitions of this form corresponds to the term $a_1 a_{n-1}$.

On the other hand, for such partitions $\nu \in \mathrm{NC}(n)$, let $k = k(\nu) \in \{3, 4, \ldots, n\}$ denote the most-left element of the block containing 1. This decomposes $\mathrm{NC}(n)$ into the $n - 1$ classes $\mathrm{NC}''_j(n) = \{\nu \in \mathrm{NC}(n) : k(\nu) = j + 2\}$, $j = 0, 1, 2, \ldots, n - 2$. The set $\mathrm{NC}''_j(n)$ can be identified with the product $\mathrm{NC}(j) \times \mathrm{NC}''(n - j)$ for $j > 0$ and $\mathrm{NC}''_0(n) = \mathrm{NC}''(n)$. Indeed, the blocks of $\nu \in \mathrm{NC}''_j(n)$, which partition the elements $\{2, 3, 4, \ldots, j + 1\}$, can be identified with an appropriate partition in $\mathrm{NC}(j)$, and (under the additional constraint that the first two elements $1, j + 2$ are in the same block) the remaining blocks, which partition the set $\{1, j + 2, j + 3, \ldots, n\}$, can be uniquely identified with a partition in $\mathrm{NC}''(n - j)$. The above situation is illustrated in Figure 1. This gives the formula

$$\begin{aligned} a_n &= \sum_{\nu \in \mathrm{NC}(n)} R_\nu (a_1, \ldots, a_n) \\ &= \sum_{i=0}^{n-2} \sum_{\nu \in \mathrm{NC}(i)} R_\nu (a_1, \ldots, a_i) \\ &\quad \times \sum_{\nu \in \mathrm{NC}''(n-i)} R_\nu (a_1, \ldots, a_{n-i}) + a_1 a_{n-1} \\ &= \sum_{i=0}^{n-2} a_i \sum_{\nu \in \mathrm{NC}''(n-i)} R_\nu (a_1, \ldots, a_{n-i}) + a_1 a_{n-1}, \end{aligned} \tag{6}$$

which proves the lemma. □

Definition 6. Let $a_0, a_1, a_2, \ldots \in \mathbb{C}$ and $a_0 \neq 0$. We define

$$c_n^{(2)} := c_n^{(2)}(a_0, a_1, a_2, \ldots, a_n) = \sum_{\nu \in \mathrm{NC}''(n)} R_\nu\left(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \ldots, \frac{a_n}{a_0}\right) \tag{7}$$

for $n \geq 2$ and the functions (power series)

$$C^{(2)}(z) = \sum_{n=2}^{\infty} c_n^{(2)} z^n \tag{8}$$

for sufficiently small $|z| < \epsilon$ and $z \in \mathbb{C}$. This series is convergent because we consider only $(a_1, a_2, \ldots)$ such that $|\sum_{n=1}^{\infty} a_n z^n| < \infty$. This implies that $\sum_{n=2}^{\infty} c_n^{(2)} z^n$ is convergent and the proof of this fact follows from the main result.

3. The Main Result

The following is our main results of the paper.

Theorem 7. *Let $a_0, a_1, a_2, \ldots$ be a sequence of $\mathbb{C}$ $(a_0 \neq 0)$ with the generating function $G(z) = a_0 + \sum_{i=1}^{\infty} a_i z^i$, and then*

$$\frac{1}{G(z)} = \frac{1}{a_0}\left(1 - C^{(2)}(z) - \frac{a_1}{a_0} z\right). \tag{9}$$

In other words, sequence $(1/a_0, -a_1/a_0^2, -c_2^{(2)}/a_0, -c_3^{(2)}/a_0, \ldots)$ is the generating function of $1/G(z)$.

Proof. It is clear from Lemma 5 that we have

$$\begin{aligned}
&\frac{1}{a_0} G(z) - 1 \\
&= \frac{1}{a_0} \sum_{n=1}^{\infty} a_n z^n \\
&= \left(\frac{a_1}{a_0} z + \sum_{n=2}^{\infty} \frac{a_n}{a_0} z^n\right) \\
&= \frac{a_1}{a_0} z \\
&\quad + \sum_{n=2}^{\infty} \left[\sum_{i=0}^{n-2} \frac{a_i}{a_0} c_{n-i}^{(2)}(a_0, a_1, a_2, \ldots, a_{n-i}) + \frac{a_1}{a_0} \frac{a_{n-1}}{a_0}\right] z^n \\
&= \frac{a_1}{a_0} z + \sum_{n=2}^{\infty} \sum_{i=0}^{n-2} \frac{a_i}{a_0} z^i c_{n-i}^{(2)}(a_0, a_1, a_2, \ldots, a_{n-i}) z^{n-i} \\
&\quad + \frac{a_1}{a_0} z \sum_{n=2}^{\infty} \frac{a_{n-1}}{a_0} z^{n-1} \\
&= \sum_{n=2}^{\infty} \sum_{i=0}^{n-2} \frac{a_i}{a_0} z^i c_{n-i}^{(2)}(a_0, a_1, a_2, \ldots, a_{n-i}) z^{n-i} \\
&\quad + \frac{a_1}{a_0} z \left[1 + \sum_{n=2}^{\infty} \frac{a_{n-1}}{a_0} z^{n-1}\right] \\
&= \frac{1}{a_0} G(z) C^{(2)}(z) + \frac{a_1}{a_0^2} z G(z),
\end{aligned} \tag{10}$$

where $a_0 = 1$, and, in the last equality, we use the Cauchy product of two series. This proves the Theorem. □

Corollary 8. *If $a_0 = 1$, then one gets*

$$\frac{1}{G(z)} = 1 - C^{(2)}(z) - a_1 z. \tag{11}$$

Example 9. The nth Catalan number is given directly in terms of binomial coefficients by $(1/(n+1))\binom{2n}{n}$. Let $b_0 = 1$ and

$$b_n = \begin{cases} 0, & \text{for } n \text{ odd}, \\ \dfrac{1}{n/2+1}\binom{n}{\frac{n}{2}}, & \text{for } n \text{ even}. \end{cases} \tag{12}$$

Then the b_n is the nth moment of Wigner semicircle law of mean 0 and variance 1 (see [10]). The free cumulant of b_n is the number of noncrossing partitions of the set $\{1, \ldots 2n\}$ in which every block is of size 2 (see, e.g., [3, 8, 11–13]); that is, $b_n = \sum_{\nu \in \mathrm{NC}(n)} R_\nu(b_1, b_2, \ldots, b_n)$, where $R_\nu(b_1, a_2, \ldots, b_n) := \Pi_{B \in \nu} R_{|B|}(b_i : i \in B)$ and

$$R_n(b_1, b_2, \ldots, b_n) = \begin{cases} 1, & \text{for } n = 2, \\ 0, & \text{otherwise}. \end{cases} \tag{13}$$

Thus, we get

$$\begin{aligned}
c_n^{(2)} &= c_n^{(2)}(1, b_1, b_2, \ldots, b_n) \\
&= \begin{cases} 0, & \text{for } n \text{ odd}, \\ \dfrac{1}{(n-2)/2+1}\binom{n-2}{\frac{n-2}{2}}, & \text{for } n \text{ even}, \end{cases} \\
\frac{1}{G(z)} &= 1 - \sum_{i=2}^{\infty} z^n c_n^{(2)} = 1 - z^2 G(z).
\end{aligned} \tag{14}$$

Example 10. In random matrix theory, the Marčenko-Pastur distribution, or Marčenko-Pastur law, describes the asymptotic behavior of singular values of large rectangular random matrices. The theorem is named after Ukrainian mathematicians Marčenkoand Pastur who proved this result in 1967 (see [7]). Moments of the Marčenko-Pastur law (with rate 1 and jump size 1) are

$$d_n = \sum_{i=0}^{n-1} \frac{1}{i+1}\binom{n}{i}\binom{n-1}{i}. \tag{15}$$

Then, d_n have free cumulants, which are constant; that is, $R_n(d_1, d_2, \ldots, d_n) = 1$ (see [8, 10]). Thus, if $d_0 = 1$, then

$$c_n^{(2)} = c_n^{(2)}\left(1, d_1, d_2, \ldots, d_n\right) = d_{n-1}. \tag{16}$$

Indeed, the first block of $\nu \in \mathrm{NC}''(n)$ (which contains $\{1, 2\}$) can be identified with an appropriate block in $\mathrm{NC}(n-1)$ (i.e., which contains $\{1\}$) because R_n are constant. Thus, we get

$$\begin{aligned}\frac{1}{G(z)} &= 1 - z - \sum_{i=2}^{\infty} z^n c_n^{(2)} = 1 - z - z\left(G(z) - 1\right) \\ &= 1 - zG(z).\end{aligned} \tag{17}$$

Open Problem. It would be worth to show whether Theorem 7 is true for noncommutative c-free cumulants (for more details, see [14]).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Spectral Analysis of the Bounded Linear Operator in the Reproducing Kernel Space $W_2^m(D)$

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We first introduce some related definitions of the bounded linear operator L in the reproducing kernel space $W_2^m(D)$. Then we show spectral analysis of L and derive several property theorems.

1. Introduction

It is well known that spectral analysis of linear operators [1] is an important topic in functional analysis. For example, the matrix eigenvalue in linear algebra and eigenvalue problems for differential equation have been discussed emphatically. Two major reasons are as follows. Firstly, spectral analysis arises from vibration frequency problems and the stability theory of system. Secondly, spectral analysis comes from the need of discussing the structure of the operator and solving the corresponding equation by using the eigenvalue and spectral theorem. Also, spectral analysis can be used to study the structure of the solution for homogeneous or nonhomogeneous differential system and the normalized form of matrix which can be obtained clearly by matrix eigenvalues.

So far, the spectral decomposition method [2] has become a central topic in the theory of spectral analysis of linear operators. This method has been successfully applied in Hilbert space and perfect spectral decomposition theorem [3]. In recent years, the spectral decomposition method has been developed into spectral theorems of spectral operators and decomposable operators in Banach space [4, 5].

To our knowledge, reproducing kernel space has been applied in many fields, such as linear systems [6–8], nonlinear systems [9–11], operator equation, stochastic processes, wavelet transform, signal analysis, and pattern recognition [12–17]. Since the reproducing kernel space is a Hilbert space, this paper will apply the theory of spectral analysis for linear operator in the reproducing kernel space $W_2^m(D)$ and derive some useful conclusions.

The paper is organized as follows. In Section 2, we introduce some related definitions for the eigenvalue of the bounded linear operator L in the reproducing kernel space $W_2^m(D)$. In Section 3, the regular point and the spectral point of bounded linear operator L in $W_2^m(D)$ are given. In Section 4, we show spectral analysis of the bounded linear operator L and also establish several theorems. Section 5 ends this paper with a brief conclusion.

2. Related Definitions

Definition 1. Let D be an abstract set, $W_2^m(D)$ the reproducing kernel space, and $BL[W_2^m(D) \to W_2^n(D)]$ the bounded linear operator space. $\forall L \in BL[W_2^m(D) \to W_2^n(D)]$ with $m, n \in \mathbb{N}$, if there exists nonvanishing vector $u \in W_2^m(D)$, such that

$$Lu = \lambda u \quad \text{or} \quad (\lambda I - L)\, u = 0. \tag{1}$$

Then λ is called an eigenvalue of L and u is called the eigenvector of L according to λ, where I denotes the identity operator.

Definition 2. $\forall L \in BL[W_2^m(D) \to W_2^n(D)]$ and all eigenvectors and zero vector of L compose the eigenvector space which is denoted by E_λ.

Obviously, E_λ is a linear closed subspace of $W_2^m(D)$.

Definition 3. Denote the dimension of E_λ by $\dim E_\lambda$; it is called the multiplicity of eigenvalue λ. That is, $\dim E_\lambda$ is the number of vectors of maximum linear independence.

Example 4. Let $K(s,t)$ be a binary function on $D = \{(s,t) \mid a \le s \le b, a \le t \le b\}$, $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m,n \in \mathbb{N}$ with

$$Lu(s) = \int_a^b K(s,t)\,u(t)\,dt, \quad u \in W_2^m(D); \tag{2}$$

then λ is the eigenvalue of L if and only if the following integral equation has nonzero solution:

$$\lambda u(s) - \int_a^b K(s,t)\,u(t)\,dt = 0. \tag{3}$$

If $K(s,t) = \sum_{i=1}^n f_i(s)g_i(t)$, $\{f_i\}_{i=1}^n$ is a linear independence vector system, then (3) can be converted into the equivalent equation

$$\lambda u(s) - \sum_{i=1}^n f_i(s) \int_a^b g_i(t)\,u(t)\,dt = 0. \tag{4}$$

Thus we have the following results.

(a) If $\lambda = 0$, then (4) has nonzero solution if and only if $u \in W_2^m(D)$ and $\int_a^b g_i(t)u(t)dt = 0$, $i = 1,2,\ldots,n$. It follows that, for the eigenvalue $\lambda = 0$ of L, the eigenvector space is infinite-dimensional.

(b) If $\lambda \neq 0$, then the solution of (4) can be denoted by

$$u(s) = \sum_{i=1}^n C_i f_i(s), \tag{5}$$

where C_i $(i = 1,2,\ldots,n)$ are constants.

Combine (5) with (4) and, in view of the linear independence of $\{f_i\}_{i=1}^n$ in (5), C_i must satisfy the following linear equation system:

$$\sum_{j=1}^n C_j \int_a^b g_i(t)\,f_j(t)\,dt = \lambda C_i, \quad i = 1,2,\ldots,n. \tag{6}$$

Summing up the above results, we can see that eigenvalues of (3) and (5) are equivalent, where C_i $(i = 1,2,\ldots,n)$ are undetermined coefficients. In addition, in order to solve the eigenvector, we just need to solve C_i $(i = 1,2,\ldots,n)$ in (5).

3. Regular Point and Spectral Point

In Definitions 1–3, we introduce the eigenvalue, eigenvector, eigenvector space, and $\dim E_\lambda$ of L for the homogeneous equation (1). However, for many problems in mathematics and physics, we just need to solve the following nonhomogeneous equation:

$$(\lambda I - L)u = f, \tag{7}$$

where L is a given operator, f is a given vector, and u is an unknown vector. In order to discuss this problem, we need to introduce the following definitions and theorems.

Definition 5. Let $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m,n \in \mathbb{N}$, $\mathscr{D}(L) \subseteq W_2^m(D)$, and $\mathscr{R}(L) \subseteq W_2^n(D)$, where $\mathscr{D}(L)$ denotes the domain of L and $\mathscr{R}(L)$ denotes the range of values of L. If the inverse operator L^{-1} of L exists and is linearly bounded, then L is called a regular operator.

Let L be a linear operator; $\mathscr{D}(L) \subseteq W_2^m(D)$ and $\mathscr{R}(L) \subseteq W_2^n(D)$; if L^{-1} exists, then $L^{-1}L = I_{\mathscr{D}(L)}$ and $LL^{-1} = I_{\mathscr{R}(L)}$, where $I_{\mathscr{D}(L)}$ and $I_{\mathscr{R}(L)}$ are, respectively, identity operators of subspace $\mathscr{D}(L)$ and $\mathscr{R}(L)$. Inversely, if there exists a linear operator $C : W_2^n(D) \to W_2^m(D)$, such that $CL = I_{\mathscr{D}(L)}$ and $LC = I_{\mathscr{D}(C)}$, then L^{-1} exists and $I^{-1} = C$. In fact, $\forall u_1, u_2 \in \mathscr{D}(L)$; if $Lu_1 = Lu_2$, then $u_1 = CLu_1 = CLu_2 = u_2$. Hence, L is invertible. Since $LC = I_{\mathscr{D}(C)}$, then $\forall v \in \mathscr{D}(C)$; we have $u = Cv$ such that $Lu = v$. That is, $\mathscr{D}(C) \subseteq \mathscr{R}(L)$.

Summing up the above disscusion, $\mathscr{R}(L) = \mathscr{D}(C)$. Hence, we have $L^{-1} = (CL)L^{-1} = CLL^{-1} = C$. Particularly, when $\mathscr{D}(C) = W_2^n(D)$ and C is a bounded linear operator, we can derive the following results.

Theorem 6. *Let $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m,n \in \mathbb{N}$; then L is a regular operator if and only if $\exists C \in BL[W_2^n(D) \to W_2^m(D)]$, such that $CL = I_{\mathscr{D}(L)}$ and $LC = I_{\mathscr{D}(C)}$.*

Theorem 7. *Let $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m,n \in \mathbb{N}$; if L is a regular operator, then L^* is also a regular operator and $(L^*)^{-1} = (L^{-1})^*$.*

Proof. Since $L^{-1} \in BL[W_2^n(D) \to W_2^m(D)]$, $m,n \in \mathbb{N}$, and

$$LL^{-1} = I_{W_2^n(D)}, \qquad L^{-1}L = I_{W_2^m(D)}, \tag{8}$$

by taking conjugate on both sides of the above formulas, we obtain

$$L^*\left(L^{-1}\right)^* = I_{W_2^{m^*}(D)}, \qquad \left(L^{-1}\right)^* L^* = I_{W_2^{m^*}(D)}. \tag{9}$$

In view of Theorem 6, we can see that L^* is a regular operator and $(L^*)^{-1} = (L^{-1})^*$. The proof is complete. □

Definition 8. Let $\mathscr{D}(L) \subseteq W_2^m(D)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$; $\mathbb{C}$ denotes the complex number field.

(1) $\lambda I - L$ is a regular operator, that is, $\lambda I - L$ is a one-to-one linear operator from $\mathscr{D}(L)$ to $W_2^m(D)$. In addition, the inverse operator $(\lambda I - L)^{-1}$ is a linear bounded operator. Then λ is called a regular point of L. All regular points compose the regular set of L, which is denoted by $\rho(L)$.

(2) If λ is not a regular point, then λ is called a spectral point of L. All spectral points compose the spectral set of L, which is denoted by $\sigma(L)$.

In view of Definition 8, we have $\rho(L) \bigcup \sigma(L) = \mathbb{C}$. Then we have the following property results.

Lemma 9. *Let L be a bounded linear operator in reproducing kernel space $W_2^m(D)$, $m \in \mathbb{N}$; then λ is a regular point of L if and only if $\forall f \in W_2^m(D)$, there exists a solution g of $(\lambda I - L)g = f$, which satisfies $\|g\| \le m\|f\|$, where m is a positive constant.*

Proof. $\Rightarrow$ Since $\mathscr{R}(\lambda I - L) = W_2^m(D)$, then $\forall f \in W_2^m(D)$, $\exists g \in W_2^m(D)$ such that $(\lambda I - L)g = f$. In addition, in view of the boundedness of $(\lambda I - L)^{-1}$ and the Cauchy-Schwartz inequality, we have

$$\|g\| = \left\|(\lambda I - L)^{-1} f\right\| \le \left\|(\lambda I - L)^{-1}\right\| \|f\|. \tag{10}$$

Let $m = \|(\lambda I - L)^{-1}\| > 0$; then $\|g\| \le m\|f\|$.

$\Leftarrow$ Since $(\lambda I - L)g = f$, we have $\mathscr{R}(\lambda I - L) = W_2^m(D)$. Next, we will prove that $\lambda I - L$ is one-to-one. In fact, $\forall f \in W_2^m(D)$; if $(\lambda I - L)g_1 = f$, $(\lambda I - L)g_2 = f$, then

$$(\lambda I - L)(g_1 - g_2) = 0; \tag{11}$$

namely, the image of $g_1 - g_2$ is 0. Hence, $\|g_1 - g_2\| \le m\|0\|$; that is, $g_1 = g_2$. Therefore, $\lambda I - L$ is one-to-one and $(\lambda I - L)^{-1}$ exists. Furthermore, since $\|g\| \le m\|f\|$, we have

$$\left\|(\lambda I - L)^{-1} f\right\| \le m\|f\|; \tag{12}$$

that is,

$$\left\|(\lambda I - L)^{-1}\right\| \le m. \tag{13}$$

Hence, $(\lambda I - L)^{-1}$ exists and is a bounded linear operator. Summing up the above, $\lambda I - L$ is a regular operator, where λ is a regular point of L. □

Lemma 9 shows that $\forall f \in W_2^m(D)$; when L is a continuous linear operator and λ is the regular point of L, $(\lambda I - L)g = f$ has a unique solution g. Furthermore, the continuity of g depends on the right term. In other words, if $\{f_i\}_{i=1}^n$ are column vectors and $f_n \to f$, then $g_n \to g$.

Lemma 10. *Let L be a bounded linear operator in the reproducing kernel space $W_2^m(D)$, $m \in \mathbb{N}$. If λ is not the eigenvalue of L and $(\lambda I - L)g_1 = (\lambda I - L)g_2$, one has $L(g_1 - g_2) = \lambda(g_1 - g_2)$, $g_1 = g_2$. That is, $\lambda I - L$ is invertible.*

Proof. Otherwise, the invertible operator can convert the nonvanishing vector to nonvanishing vector. Hence, there exists $g \ne 0$ such that $(\lambda I - L)g = 0$. That is, λ is not the eigenvalue of L.

When $W_2^m(D)$ is a finite dimension space and λ is not the eigenvalue of L, we can derive that $C = \lambda I - L$ is an invertible mapping. Obviously, $\mathscr{R}(C) = W_2^m(D)$. In fact, let $\{e_i\}_{i=1}^n$ be the basis of $W_2^m(D)$; then $\{(\lambda I - L)e_i\}_{i=1}^n$ is a linear independent system in $W_2^n(D)$ and also a basis of $W_2^n(D)$. Therefore, $\mathscr{R}(L) = W_2^n(D)$. In view of the inverse operator Theorem, $(\lambda I - L)^{-1}$ is bounded. It follows that $\lambda \in \rho(L)$.

So, the proof of the theorem is complete. □

Lemma 10 shows that regular point and spectral point are absolutely opposite for finite dimension normed spaces. That is, spectral point of L can only be an eigenvalue in finite dimension normed space. This is entirely consistent with the conclusion of the theory of linear algebra. But if $W_2^m(D)$ is an infinite-dimensional space and λ is not the eigenvalue of L, then λ may not be a regular point of L, so far as $\lambda I - L$ is not a map from $W_2^m(D)$ to $W_2^n(D)$.

For example, let

$$Lu = \int_a^b u(t)\,dt, \quad u \in W_2^m(D); \tag{14}$$

$\forall \lambda \in \mathbb{C}$, $\int_a^b u(t)dt = \lambda u(t)$ has only zero solutions. Hence, L has not eigenvalue. That is, zero is not the eigenvalue. However, the range of values is all functions of the from $\int_a^b u(t)dt$ for $(0I - L)$. This shows that the spectral point is complex in infinite-dimensional space for the operator L.

Now, we will classify the spectral set by three situations.

(a) If $\lambda I - L$ is not one-to-one, then λ is called point spectral of L; the set of point spectral is denoted by $\sigma_p(L)$.

(b) If $\lambda I - L$ is one-to-one and $\mathscr{R}(\lambda I - L)$ is dense in $W_2^m(D)$, then λ is called continuous spectral of L; the set of continuous spectral is denoted by $\sigma_c(L)$.

(c) If $\lambda I - L$ is one-to-one and $\mathscr{R}(\lambda I - L)$ is not dense in $W_2^m(D)$, then λ is called residual spectral of L; the set of residual spectral is denoted by $\sigma_r(L)$.

Obviously, $\sigma_p(L)$, $\sigma_c(L)$, and $\sigma_r(L)$ are mutually disjoint sets and $\sigma(L) = \sigma_p(L) \cup \sigma_c(L) \cup \sigma_r(L)$.

4. Spectral Analysis

Let $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m, n \in \mathbb{N}$, $r = \lim_{n\to\infty} \sqrt[n]{\|L^n\|}$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}^*$, $\forall n > N$, such that $\sqrt[n]{\|L^n\|} < r + \varepsilon < 1$; that is, $\|L^n\| < (r + \varepsilon)^n$. In view of the completeness of $W_2^m(D)$, there exists $m > N$, such that

$$\begin{aligned}\left\|\sum_{n=m}^{\infty} L^n\right\| &\le \sum_{n=m}^{\infty} \left\|L^n\right\| \le \sum_{n=m}^{\infty} (r + \varepsilon)^n \\ &= (r + \varepsilon)^m (1 - r - \varepsilon)^{-1}.\end{aligned} \tag{15}$$

Hence, $\sum_{n=0}^{\infty} L^n$ converges in the sense of $\|\cdot\|$ and the limit is denoted by $C = \sum_{n=0}^{\infty} L^n$.

Let $C_m = \sum_{n=0}^{m} L^n$; then

$$C_m(I - L) = (I - L)C_m = I - L^{m+1}. \tag{16}$$

For $\|C_m - C\| \to 0$, $m \ge N$, we have

$$\left\|L^{m+1}\right\| \le (r + \varepsilon)^{m+1} \to 0. \tag{17}$$

If $m \to \infty$, then

$$C(I - L) = (I - L)C = I. \tag{18}$$

Namely, $1 \in \rho(L)$ and $(I - L)^{-1} = \sum_{n=0}^{\infty} L^n$.

For $\|L\| < 1$, one obtains

$$\left\|(I-L)^{-1}\right\| = \|C\| \le \sum_{n=0}^{\infty} \left\|L^n\right\| = \frac{1}{1-\|L\|}. \tag{19}$$

Summing up the above parts, we have the following theorems.

Theorem 11. *Let $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m, n \in \mathbb{N}$, then one has the following.*

(1) *Consider $1 \in \rho(L)$.*

(2) *Consider $(I-L)^{-1} = \sum_{n=0}^{\infty} L^n$.*

(3) *When $\|L\| < 1$, $\|(I-L)^{-1}\| \le 1/(1-\|L\|)$.*

Theorem 12. *Let $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m, n \in \mathbb{N}$; if $r = \lim_{n\to\infty} \sqrt[n]{\|L^n\|}$, then one has the following.*

(1) *$|\lambda| > r$ if and only if λ is a regular point of L.*

(2) *When $|\lambda| > r$, $(\lambda I - L)^{-1} = \sum_{n=0}^{\infty} (L^n/\lambda^{n+1})$.*

(3) *When $|\lambda| > \|L\|$, $\|(\lambda I - L)^{-1}\| \le (|\lambda| - \|L\|)^{-1}$.*

Proof. $\forall \lambda \ne 0$, since $(\lambda I - L) = \lambda(I - L/\lambda)$, $\lambda \in \rho(L)$ if and only if $1 \in \rho(L/\lambda)$. Replacing L by L/λ in Theorem 11, we have

$$\lim_{n\to\infty} \sqrt[n]{\left\|\frac{L^n}{\lambda^n}\right\|} = \frac{1}{|\lambda|} \lim_{n\to\infty} \sqrt[n]{\|L^n\|} < 1; \tag{20}$$

namely,

$$|\lambda| > \lim_{n\to\infty} \sqrt[n]{\|L^n\|} = r, \quad 1 \in \rho\left(\frac{L}{\lambda}\right). \tag{21}$$

Furthermore, we have

$$\left(I - \frac{L}{\lambda}\right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{L}{\lambda}\right)^n = \sum_{n=0}^{\infty} \frac{L^n}{\lambda^n}. \tag{22}$$

Then we obtain

$$(\lambda I - L)^{-1} = \frac{1}{\lambda}\left(I - \frac{L}{\lambda}\right)^{-1} = \sum_{n=0}^{\infty} \frac{L^n}{\lambda^{n+1}}. \tag{23}$$

It follows that λ is a regular point of L and $\|(\lambda I - L)^{-1}\| \le (|\lambda| - \|L\|)^{-1}$ with $|\lambda| > r$.

In addition, when $|\lambda| > \|L\|$, we have $\|L/\lambda\| < 1$. In view of (2) of Theorem 11, we have

$$\left\|(\lambda I - L)^{-1}\right\| < \frac{1}{|\lambda|}\left(1 - \left\|\frac{L}{\lambda}\right\|\right)^{-1} = (|\lambda| - \|L\|)^{-1}. \tag{24}$$

The proof is complete. □

Theorem 13. *Let $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m, n \in \mathbb{N}$; then one has the following.*

(1) *$\rho(L)$ is an open set.*

(2) *When $\rho(L)$ is nonempty, $\forall \lambda_0 \in \rho(L)$; if $r_{\lambda_0} = \lim_{n\to\infty} \sqrt[n]{\|(\lambda_0 I - L)^{-n}\|}$, then λ is a regular point of L and $(\lambda I - L)^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda_0 I - L)^{-(n+1)} (\lambda - \lambda_0)^n$, where $|\lambda - \lambda_0| < 1/r_{\lambda_0}$.*

(3) *$\sigma(L)$ is a closed set.*

(4) *Consider $\sup_{\lambda\in\sigma(L)} |\lambda| \le \lim_{n\to\infty} \sqrt[n]{\|L^n\|}$.*

Proof. (1) If $\rho(L) = \emptyset$, the conclusion is obvious. If $\rho(L) \ne \emptyset$, then

$$\begin{aligned} \lambda I - L &= (\lambda - \lambda_0) I + (\lambda_0 I - L) \\ &= \left[I + (\lambda - \lambda_0)(\lambda_0 I - L)^{-1}\right](\lambda_0 I - L), \end{aligned} \tag{25}$$

where $(\lambda_0 I - L)^{-1}$ is a bounded linear operator in the reproducing kernel space $W_2^m(D)$. We use $(\lambda - \lambda_0)(\lambda_0 I - L)^{-1}$ instead of L in Theorem 11, such that

$$\lim_{n\to\infty} \sqrt[n]{\left\|\left[-\lambda(\lambda - \lambda_0)(\lambda_0 I - L)^{-1}\right]^n\right\|} < 1. \tag{26}$$

That is, when $|\lambda - \lambda_0| < 1/r_{\lambda_0}$, $[I + (\lambda - \lambda_0)(\lambda_0 I - L)^{-1}]^{-1}$ exists and is bounded. Hence, when $|\lambda - \lambda_0| < 1/r_{\lambda_0}$, $\lambda \in \rho(L)$; that is, $\rho(L)$ is an open set.

(2) If $\rho(L)$ is nonempty, $\forall \lambda_0 \in \rho(L)$, let $r_{\lambda_0} = \lim_{n\to\infty} \sqrt[n]{\|(\lambda_0 I - L)^{-n}\|}$. In view of (2) of Theorem 11 and (1) of Theorem 13, we have

$$\begin{aligned} (\lambda I - L)^{-1} &= (\lambda_0 I - L)^{-1}\left[I + (\lambda - \lambda_0)(\lambda_0 I - L)^{-1}\right]^{-1} \\ &= \sum_{n=0}^{\infty} (-1)^n (\lambda_0 I - L)^{-(n+1)} (\lambda - \lambda_0)^n. \end{aligned} \tag{27}$$

(3) Since $\rho(L) \bigcup \sigma(L) = \mathbb{C}$ and (1), $\sigma(L)$ is a closed set.

(4) In view of (1) of Theorem 12 and $r = \lim_{n\to\infty} \sqrt[n]{\|L^n\|}$, we have $\sigma(L) \subseteq \{\lambda \mid |\lambda| \le r\}$, which means that $\sup_{\lambda\in\sigma(L)} |\lambda| \le \lim_{n\to\infty} \sqrt[n]{\|L^n\|}$.

The proof is complete. □

Definition 14. Let $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m, n \in \mathbb{N}$, $r(L) = \max_{\lambda\in\sigma(L)} |\lambda|$; $r(L)$ is called the spectral radius of L.

From the purpose of solving equations, spectral radius has the following meanings.

(1) For $|\lambda| > r(L)$, due to the fact that λ is a regular point of L, then for any $f \in W_2^m(D)$, $(\lambda I - L)g = f$ has a unique solution g.

(2) For $|\lambda| \le r(L)$, it cannot guarantee this equation has a solution for any $f \in W_2^m(D)$. In many practical problems, in order to calculate the spectral range, one needs to estimate the spectral radius. In terms of (4) of Theorem 13, we can get $r(L) \le \|L\|$. In practical terms, this estimate is convenient, but it is imprecise.

Theorem 15. *Let $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m, n \in \mathbb{N}$; then $r(L) = \sup_{\lambda\in\sigma(L)}|\lambda| = \lim_{n\to\infty}\sqrt[n]{\|L^n\|}$.*

Proof. In terms of (4) of Theorem 13, $r(L) \le \lim_{n\to\infty}\sqrt[n]{\|L^n\|}$. Hence, one only needs to prove that $r(L) \ge \lim_{n\to\infty}\sqrt[n]{\|L^n\|}$. For $|\lambda| > \|L\|$, one obtains

$$(\lambda I - L)^{-1} = \sum_{n=0}^{\infty}\frac{L^n}{\lambda^{n+1}}. \tag{28}$$

Consider $\forall f \in W_2^m(D)$, $f((\lambda I - L)^{-1}) = \sum_{n=0}^{\infty}(f(L^n)/\lambda^{n+1})$. If $|\lambda| > r(L)$, then λ is a regular point of L. In addition, since $\{\lambda \mid |\lambda| > r(L)\}$, then Laurent expansions of $f((\lambda I - L)^{-1})$ are established, where $|\lambda| > r(L)$.

Let $a = r(L)$, $\forall\varepsilon > 0$; we have

$$\sum_{n=0}^{\infty}\frac{f(L_n)}{(a+\varepsilon)^{n+1}} < \infty. \tag{29}$$

Let $B_n = L^n/(a+\varepsilon)^n$, $\forall f \in W_2^{m^*}(D)$; then

$$\sup_{n\ge1}|f(B_n)| < \infty. \tag{30}$$

In terms of the resonance Theorem, $\{B_n\}$ must be bounded. It follows that there exists a positive constant M, such that $\|B_n\| \le M$ and $\|L^n\| \le (a+\varepsilon)^n\|B^n\| \le (a+\varepsilon)^n M$. Namely,

$$\lim_{n\to\infty}\sqrt[n]{\|L^n\|} < a + \varepsilon. \tag{31}$$

Let $\varepsilon \to 0$; then

$$r(L) = a \ge \lim_{n\to\infty}\sqrt[n]{\|L^n\|}. \tag{32}$$

The proof is complete. □

Theorem 16. *Let $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m, n \in \mathbb{N}$; then $\sigma(L) \ne \emptyset$.*

Proof. If $\sigma(L) = \emptyset$, in view of the properties of the reproducing kernel space, $W_2^m(D) \ne \{0\}$; hence, $I \ne \{0\}$, where the unit element is denoted by I. In terms of the functional extension Theorem, $\exists f \in W_2^{m^*}(D)$, such that $f(I) \ne 0$. In addition, $\forall\lambda_0 \in \rho(L)$ and $\exists r_{\lambda_0} \in \mathbb{R}^+$; when $|\lambda - \lambda_0| < 1/r_{\lambda_0}$, we have

$$(\lambda I - L)^{-1} = \sum_{n=0}^{\infty}(-1)^n(\lambda_0 I - L)^{-n-1}(\lambda - \lambda_0)^n. \tag{33}$$

Note that

$$f\left((\lambda I - L)^{-1}\right) = \sum_{n=0}^{\infty}(-1)^n f\left((\lambda_0 I - L)^{-n-1}\right)(\lambda - \lambda_0)^n, \tag{34}$$

in terms of the assumption that $\sigma(L) = \emptyset$ and Theorem 15; when $|\lambda| > \|L\|$, one obtains

$$f\left((\lambda I - L)^{-1}\right) = \sum_{n=0}^{\infty}\frac{f(L^n)}{\lambda^{n+1}}. \tag{35}$$

Therefore, when $|\lambda| \ge \|L\| + 1$, we have

$$\left|f\left((\lambda I - L)^{-1}\right)\right| \le \sum_{n=0}^{\infty}\|f\|\frac{\|L^n\|}{\|\lambda\|^{n+1}} \le \|f\|\frac{1}{|\lambda| - \|L\|} \le \|f\|. \tag{36}$$

That is, $f((\lambda I - L)^{-1})$ is bounded. In terms of the Liouville Theorem, $f((\lambda I - L)^{-1})$ must be a constant, so we have $\sigma(L) \ne \emptyset$.

The proof is complete. □

Definition 17. If $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m, n \in \mathbb{N}$, $\lim_{n\to\infty}\sqrt[n]{\|L^n\|} = 0$, then L is called a generalized nilpotent operator.

Definition 17 is the finite-dimensional space concept nilpotent operator in the infinite-dimensional space to promote. In the spectral theory of operators, generalized nilpotent operator is a kind of important operator.

In terms of Theorem 16 and the spectral radius theorem, one can obtain that the generalized nilpotent operator has only a spectral point 0. For example, let $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m, n \in \mathbb{N}$, $[a, t] \subseteq D$,

$$(Lu)(t) = \int_a^t u(\mu)\,d\mu, \quad u \in W_2^m(D). \tag{37}$$

In terms of the property of L, one obtains

$$L^n u = \int_a^t\int_a^{t_1}\cdots\int_a^{t_{n-1}} u(\mu)\,d\mu dt_{n-1}\cdots dt_1. \tag{38}$$

Note that

$$\left|(L^n u)(t)\right| \le \|u\|\int_a^t\int_a^{t_1}\cdots\int_a^{t_{n-1}} u(\mu)\,d\mu dt_{n-1}\cdots dt_1; \tag{39}$$

we have

$$\left\|L^n u\right\| \le \frac{1}{n!}(b-a)^n\|u\|, \quad u \in W_2^m(D). \tag{40}$$

This shows that L is a generalized nilpotent operator; spectral point $\lambda = 0$ is not the eigenvalue of L.

Definition 18. Let $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m, n \in \mathbb{N}$, $\lambda \in \mathbb{C}$; if there exists $\{u_n\}_{n=1}^{\infty} \in W_2^m(D)$, such that $(\lambda I - L)u_n \to 0$, then λ is called an approximate spectral point. All approximate spectral points are denoted by $\sigma_a(L)$; the other spectral point is called remainder spectral point. All the remainder spectral points are denoted by $\sigma_r(L)$.

Theorem 19. *Let $L \in BL[W_2^m(D) \to W_2^n(D)]$, $m, n \in \mathbb{N}$; then*

(1) *$\sigma_p(L) \subseteq \sigma_a(L)$,*

(2) *$\sigma_a(L)\bigcap\sigma_r(L) = \emptyset$, and $\sigma_a(L)\bigcup\sigma_r(L) = \sigma(L)$,*

(3) *$\sigma_r(L)$ is an open set,*

(4) *$\partial\sigma(L) \subseteq \sigma_a(L)$, where $\partial\sigma(L)$ denotes the boundary of $\sigma(L)$,*

(5) *$\sigma_a(L)$ is a nonempty closed set.*

Proof. (1) If $\lambda \in \sigma_p(L)$, then there exists nonzero element u of $W_2^m(D)$, such that

$$(L - \lambda I)\, u = 0. \tag{41}$$

Without loss of the generality, let $\|u\| = 1$; we choose $u_n = u$, $n = 1, 2, \ldots$, and then $\|u_n\| = 1$ and $(L - \lambda I)u_n \to 0$; namely, $\lambda \in \sigma_a(L)$; this shows that $\sigma_p(L) \subseteq \sigma_a(L)$.

(2) In terms of Definition 18, one obtains $\sigma_a(L) \bigcap \sigma_r(L) = \emptyset$ and $\sigma_a(L) \bigcup \sigma_r(L) = \sigma(L)$.

(3) If $\lambda \in \sigma_r(L)$, then $\lambda \notin \sigma_a(L)$. Hence, $\exists \alpha \in \mathbb{N}^*$, such that

$$\|(L - \lambda I)\, u\| \geq \alpha \|u\|, \quad u \in W_2^m(D). \tag{42}$$

When $|\lambda' - \lambda| < \alpha/2$, $\forall u \in W_2^m(D)$, we have

$$\left\|\left(L - \lambda' I\right) u\right\| \geq \|(L - \lambda I)\, u\| - \left|\lambda - \lambda'\right| \cdot \|u\| \geq \frac{\alpha}{2} \|u\|. \tag{43}$$

It shows that for any λ' which satisfies $|\lambda' - \lambda| < \alpha/2$ it is impossible to be an approximate spectral point of L. Hence, if one can prove that when $|\lambda' - \lambda| < \alpha/2$, λ' is not a regular point of L, then $\lambda' \in \sigma_r(L)$. That is, λ is an inner point of $\sigma_r(L)$, so $\sigma_r(L)$ is an open set.

Now, we prove that when $|\lambda' - \lambda| < \alpha/2$, $\lambda' \notin \rho(L)$. But not vice versa, $\exists \lambda_0 \in \mathbb{C}$, $|\lambda_0 - \lambda| < \alpha/2$; then $\lambda_0 \in \rho(L)$. Note that $\|(L - \lambda' I)u\| \geq \|(L - \lambda I)u\| - |\lambda - \lambda'|\|u\| \geq (\alpha/2)\|u\|$; if $\lambda' = \lambda_0$, then

$$\left\|(L - \lambda_0 I)^{-1}\right\| < \frac{2}{\alpha}. \tag{44}$$

In view of (2) of Theorem 13, let μ be a regular point of L; if $|\mu - \lambda_0| < 1/r_{\lambda_0}$, then

$$r_{\lambda_0} = \lim_{n \to \infty} \sqrt[n]{\left\|(\lambda_0 I - L)^{-n}\right\|} \leq \left\|(\lambda_0 I - L)^{-1}\right\| = \frac{2}{\alpha}. \tag{45}$$

In a particular case, let $\mu = \lambda$; note that

$$|\mu - \lambda_0| = |\lambda - \lambda_0| < \frac{\alpha}{2} < \frac{1}{r_{\lambda_0}}; \tag{46}$$

then $\lambda \in \rho(L)$. This is a contradiction with $\lambda \in \sigma_r(L)$. It follows that $\sigma_r(L)$ is an open set.

(4) Since $\sigma(L)$ is a closed set, when $\lambda \in \partial\sigma(L)$, $\lambda \in \sigma(L)$. In addition, $\lambda \notin \sigma_r(L)$, $\partial\sigma(L) \nsubseteq \sigma_r(L)$, so we have $\partial\sigma(L) \subseteq \sigma_a(L)$.

(5) Since $\sigma_a(L) = \sigma(L) - \sigma_r(L)$, then $\sigma_a(L)$ is a closed set. Furthermore, since $\partial\sigma(L) \subseteq \sigma_a(L)$, $\sigma_a(L) \neq \emptyset$, this shows that $\partial\sigma(L)$ and $\sigma_a(L)$ are all nonempty sets.

The proof is complete. □

5. Conclusions

This paper first introduces the eigenvalue, eigenvector, eigenvector space, and $\dim E_\lambda$ of the bounded linear operator L in the reproducing kernel space $W_2^m(D)$. Then we show some definitions and properties of the regular operator. The regular set and spectral set of bounded linear operator are also introduced. From the solvability of the equation, we show the spectral classification and give three conditions. Finally, we introduce the spectral analysis of the bounded linear operator L. It includes the definitions of spectral radius, nilpotent operator, approximate spectral point, and remainder spectral point. We also establish some property theorems of the bounded linear operator in the reproducing kernel space $W_2^m(D)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

On $C_{\aleph}$-Fibrations in Bitopological Semigroups

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We extend the path lifting property in homotopy theory for topological spaces to bitopological semigroups and we show and prove its role in the $C_{\aleph}$-fibration property. We give and prove the relationship between the $C_{\aleph}$-fibration property and an approximate fibration property. Furthermore, we study the pullback maps for $C_{\aleph}$-fibrations.

1. Introduction

In homotopy theory for topological space (i.e., spaces), Hurewicz [1] introduced the concepts of fibrations and path lifting property of maps and showed its equivalence with the covering homotopy property. Coram and Duvall [2] introduced approximate fibrations as a generalization of cell-like maps [3] and showed that the uniform limit of a sequence of Hurewicz fibrations is an approximate fibration. In 1963, Kelly [4] introduced the notion of bitopological spaces. Such spaces were equipped with its two (arbitrary) topologies. The reader is suggested to refer to [4] for the detail definitions and notations. The concept of homotopy theory for topological semigroups has been introduced by Cerin in 2002 [5]. In this theory, he introduced $S_{\aleph}$-fibrations as extension of Hurewicz fibrations. In [6], we introduced the concepts of bitopological semigroups, c-bitopological semigroups, and $C_{\aleph}$-fibrations as extension of $S_{\aleph}$-fibrations.

This paper is organized as follows. It consists of five sections. After this Introduction, Section 2 is devoted to some preliminaries. In Section 3 we show the pullbacks of S-maps which have the $C_{\aleph}$-fibration property that will also have this property and the pullbacks of $C_{\aleph}$-fibrations are $C_{\aleph}$-fibrations under given conditions. In Section 4 we develop and extend path lifting property in homotopy theory for topological semigroups to theory for bitopological semigroups. Some results about Hurewicz fibrations carry over. In Section 5 we give and prove the relationship between the $C_{\mathcal{N}_\pi}$-fibration property and an approximate fibration property.

2. Preliminaries

Throughout this paper, by all X_τ we mean all topological spaces (X, τ) which will be assumed Hausdorff spaces. By all $X_{\tau_{12}}$ we mean all bitopological spaces (X, τ_1, τ_2). For two bitopological spaces $X_{\tau_{12}}$ and $Y_{\rho_{12}}$, a *p-map* $h : X_{\tau_{12}} \to Y_{\rho_{12}}$ is a function from X into Y that is continuous function (i.e., a map) from a space X_{τ_1} into a space Y_{ρ_1} and from X_{τ_2} into Y_{ρ_2} [4].

Recall [5] that a *topological semigroup* or an *S-space* is a pair $(X_\tau, *)$ consisting of a topological space X_τ and a map $* : X_\tau \times X_\tau \to X_\tau$ from the product space $X_\tau \times X_\tau$ into X_τ such that $*(x, *(y, z)) = *(*(x, y), z)$ for all $x, y, z \in X$. An S-space $(A, *')$ is called an *S-subspace* of $(X_\tau, *)$ if A is a subspace of X_τ and the map $*$ takes the product $A \times A$ into A and $*'(x, y) = *(x, y)$ for all $x, y \in A$. We denote the class of all S-spaces by $\aleph$. For every space X_τ, by $P(X_\tau)$, we mean the space of all paths from the unit closed interval $I = [0, 1]$ into X_τ with the compact-open topology. Recall [5] that, for every S-space $(X_\tau, *)$, $(P(X_\tau), p(*))$ is an S-space where $p(*) : P(X_\tau) \times P(X_\tau) \to P(X_\tau)$ is a map defined by $p(*)(\alpha, \beta)(t) = *(\alpha(t), \beta(t))$ for all $\alpha, \beta \in P(X_\tau)$, $t \in I$. The shorter notion for this S-space will be $P(X_\tau, *)$. For every space X_τ, the *natural S-space* is an S-space (X_τ, π_i), where

π_i is a continuous associative multiplication on X_τ given by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for all $x, y \in X$. We denote the class of all natural S-spaces (X_τ, π) by $\mathcal{N}_\pi$, where $\pi = \pi_1, \pi_2$.

Recall [5] that the function $f : (X_\tau, *) \to (O_\rho, \circ)$ is called an *S-map* if f is a map of a space X_τ into O_ρ and $f(*(x, y)) = \circ(f(x), f(y))$ for all $x, y \in X$. The function $f : X_\tau \to O_\rho$ of a natural S-space (X_τ, π) into (O_ρ, π) is an S-map if and only if it is continuous. The S-maps $f, g : (X_\tau, *) \to (O_\rho, \circ)$ are called *S-homotopic* and write $f \simeq_s g$ provided there is an S-map $H : (X_\tau, *) \to P(O_\rho, \circ)$ called an *S-homotopy* such that $H(x)(0) = f(x)$ and $H(x)(1) = g(x)$ for all $x \in X$.

A *bitopological semigroup* is a pair $(X_{\tau_{12}}, *)$ consisting of a bitopological space $X_{\tau_{12}}$ and the associative multiplication $*$ on X such that $*$ is an p-map from the product bitopological space $(X \times X, \tau_1 \times \tau_1, \tau_2 \times \tau_2)$ into $X_{\tau_{12}}$. For $B \subseteq X$, by $B|_{\tau_{12}}$ we mean the bitopological subspace $(B, \tau_1|_B, \tau_2|_B)$ of $X_{\tau_{12}}$. If the p-map $*$ takes the product $B \times B$ into B then the pair $(B|_{\tau_{12}}, *)$ will be a bitopological semigroup and will be called an *b-subspace* of $(X_{\tau_{12}}, *)$.

The function $h : (X_{\tau_{12}}, *) \to (Y_{\rho_{12}}, \circ)$ is called an *S_i-map* from $(X_{\tau_{12}}, *)$ into $(Y_{\rho_{12}}, \circ)$ provided h is an S-map from a function S-space $(X_{\tau_i}, *)$ into an S-space $(Y_{\rho_i}, \circ)$, where $i = 1, 2$. We say that h is an *Sp-map* if it is an S_1-map and S_2-map.

An *c-bitopological semigroup* is a triple $(X_{\tau_{12}}, *, \mathcal{X})$ consisting of bitopological semigroups $(X_{\tau_{12}}, *)$ and an S-map $\mathcal{X} : (X_{\tau_2}, *) \to (X_{\tau_1}, *)$ from an S-space $(X_{\tau_2}, *)$ into an S-space $(X_{\tau_1}, *)$. In our work, for any S-space, $(O_\rho, \circ)$ can be regarded as an c-bitopological semigroup $(O_{\rho\rho}, \circ, \mathrm{id})$ where id is the identity S-map on $(O_\rho, \circ)$. That is, $(O_\rho, \circ) := (O_{\rho\rho}, \circ, \mathrm{id})$.

An *c-map* from $(X_{\tau_{12}}, *, \mathcal{X})$ into $(O_\rho, \circ)$ is a pair $f_{12} = (f_1, f_2) : (X_{\tau_{12}}, *, \mathcal{X}) \to (O_\rho, \circ)$ of an S_1-map $f_1 : (X_{\tau_1}, *) \to (O_\rho, \circ)$ and S_2-map $f_2 : (X_{\tau_2}, *) \to (O_\rho, \circ)$ such that $f_1 \circ \mathcal{X} = f_2$.

Definition 1 (see [6]). Let $f : (X_\tau, *) \to (O_\rho, \circ)$ and $h : (X'_{\tau'}, *') \to (X_\tau, *)$ be two S-maps. An S-map f is said to have the $C_{\aleph}$-fibration property by an S-map h provided for every $(Y_\omega, \star) \in \aleph$ and, given two S-maps $g : (Y_\omega, \star) \to (X'_{\tau'}, *')$ and $G : (Y_\omega, \star) \to P(O_\rho, \circ)$ with $G_0 = f \circ (h \circ g)$, there exists an S-homotopy $H : (Y_\omega, \star) \to P(X_\tau, *)$ such that $H_0 = h \circ g$ and $f \circ H_t = G_t$ for all $t \in I$.

Definition 2 (see [6]). An c-map $f_{12} : (X_{\tau_{12}}, *, \mathcal{X}) \to (O_\rho, \circ)$ is called an $C_{\aleph}$-fibration if an S_1-map $f_1 : (X_{\tau_1}, *) \to (O_\rho, \circ)$ has the $C_{\aleph}$-fibration property by an S-map $\mathcal{X} : (X_{\tau_2}, *) \to (X_{\tau_1}, *)$. That is, for every $(Y_\omega, \star) \in \aleph$ and given two S-maps $g : (Y_\omega, \star) \to (X_{\tau_2}, *)$ and $G : (Y_\omega, \star) \to P(O_\rho, \circ)$ with $G_0 = f_2 \circ g$, there exists an S-homotopy $H : (Y_\omega, \star) \to P(X_{\tau_1}, *)$ such that $H_0 = \mathcal{X} \circ g$ and $f_1 \circ H_t = G_t$ for all $t \in I$.

Let $(X_{\tau_{12}}, *, \mathcal{B})$ be an c-bitopological semigroup and let $(B|_{\tau_{12}}, *)$ be an b-subspace of $(X_{\tau_{12}}, *)$. The c-bitopological semigroup $(B|_{\tau_{12}}, *, \mathcal{B})$ is called an *c-subspace* of $(X_{\tau_{12}}, *, \mathcal{X})$ provided $\mathcal{B}(b) = \mathcal{X}(b)$ for all $b \in B$.

Theorem 3 (see [6]). *Let $f_{12} : (X_{\tau_{12}}, *, \mathcal{X}) \to (O_\rho, \circ)$ be an c-map and $(B|_\rho, \circ)$ be an S-subspace of $(O_\rho, \circ)$ such that $f_1^{-1}(B) = f_2^{-1}(B)$. Then the triple $(B^-|_{\tau_{12}}, *, \mathcal{X}|_{B^-})$ is an c-subspace of $(X_{\tau_{12}}, *, \mathcal{X})$ and a pair $f_{12}|_{B^-} = (f_1|_{B^-}, f_2|_{B^-})$ is an c-map from an c-bitopological semigroup $(B^-|_{\tau_{12}}, *, \mathcal{X}|_{B^-})$ into $(B|_\rho, \circ)$, where $B^- = f_1^{-1}(B)$.*

Corollary 4 (see [6]). *Let $f_{12} : (X_{\tau_{12}}, *, \mathcal{X}) \to (O_\rho, \circ)$ be an $C_{\aleph}$-fibration and let $(B|_\rho, \circ)$ be an S-subspace of $(O_\rho, \circ)$ such that $f_1^{-1}(B) = f_2^{-1}(B)$. Then the restriction c-map*

$$f_{12}|_{B^-} : \left(B^-|_{\tau_{12}}, *, \mathcal{X}|_{B^-}\right) \longrightarrow \left(B|_\rho, \circ\right) \tag{1}$$

is an $C_{\aleph}$-fibration, where $B^- = f_1^{-1}(B)$.

3. The Pullback c-Maps

In this section, we show that the pullbacks of S-maps which have the $C_{\aleph}$-fibration property will also have this property and the pullbacks of $C_{\aleph}$-fibrations are $C_{\aleph}$-fibrations under given conditions.

Let $f_{12} : (X_{\tau_{12}}, *, \mathcal{X}) \to (O_\rho, \circ)$ be an c-map and let $h : (Q_\upsilon, \odot) \to (O_\rho, \circ)$ be an S-map. Let

$$X^{hi} = \{(x, b) \in X \times Q \mid f_i(x) = h(b)\}, \tag{2}$$

where $i = 1, 2$.

Lemma 5. *Let $f_{12} : (X_{\tau_{12}}, *, \mathcal{X}) \to (O_\rho, \circ)$ be an c-map and let $h : (Q_\upsilon, \odot) \to (O_\rho, \odot)$ be an S-map. Then the pair $(X^{hi}|_{\tau_{12}\times\upsilon}, *\times\odot)$ is an b-subspace of the bitopological semigroup $((X \times Q)_{\tau_{12}\times\upsilon}, * \times \odot)$, where $i = 1, 2$.*

Proof. It is clear that $X^{h1}|_{\tau_{12}\times\upsilon}$ and $X^{h2}|_{\tau_{12}\times\upsilon}$ are subspaces of a bitopological space $(X \times O)_{\tau_{12}\times\upsilon}$. Since h is an S-map and f_1 is an S_1-map, then, for all $(x, b), (x', b') \in X^{h1}$,

$$h\left(b \odot b'\right) = h(b) \circ h\left(b'\right) = f_1(x) \circ f_1\left(x'\right) = f_1\left(x * x'\right). \tag{3}$$

This implies

$$(x, b)(* \times \odot)\left(x', b'\right) = \left(x * x', b \odot b'\right) \in X^{h1} \tag{4}$$

for all $(x, b), (x', b') \in X^{h1}$. That is, $(X^{h1}|_{\tau_{12}\times\upsilon}, * \times \odot)$ is an b-subspace of the bitopological semigroup $((X \times O)_{\tau_{12}\times\upsilon}, * \times \odot)$. Similarly, $(X^{h2}|_{\tau_{12}\times\upsilon}, * \times \odot)$ is an b-subspace of the bitopological semigroup $((X \times O)_{\tau_{12}\times\upsilon}, * \times \odot)$. □

Henceforth, in this paper, by $\mathcal{J}_1$ and $\mathcal{J}_2$, we mean the usual first and the second projection S-maps (or maps), respectively.

Theorem 6. *Let $f_{12} : (X_{\tau_{12}}, *, \mathcal{X}) \to (O_\rho, \circ)$ be an $C_{\aleph}$-fibration and let $h : (Q_\upsilon, \odot) \to (O_\rho, \circ)$ be an S-map. Then the S-map $f^{h1} : (X^{h1}|_{\tau_1\times\upsilon}, * \times \odot) \to (Q_\upsilon, \odot)$ has the $C_{\aleph}$-fibration property by an S-map $\mathcal{X}^h = \mathcal{X} \times id|_{X^{h2}}$ such that $f^{h1}(x, b) = b$ for all $(x, b) \in X^{h1}$.*

Proof. Since f_{12} is an c-map then, for all $(x, b) \in X^{h2}$,

$$f_1(\mathcal{X}(x)) = f_2(x) = h(b). \tag{5}$$

That is, $(\mathscr{X}(x), b) \in X^{h1}$ for all $(x, b) \in X^{h2}$. Hence, by the last lemma, $\mathscr{X}^h$ is a well-defined S-map taking $(X^{h2}|_{\tau_2\times\upsilon}, * \times \odot)$ into $(X^{h1}|_{\tau_1\times\upsilon}, * \times \odot)$.

Now let $(Y_\omega, \star) \in \aleph$ and let $g : (Y_\omega, \star) \to (X^{h2}|_{\tau_2\times\upsilon}, * \times \odot)$ and $G : (Y_\omega, \star) \to P(Q_\upsilon, \odot)$ be two S-maps with $G_0 = f^{h1} \circ (\mathscr{X}^h \circ g)$.

Take an S-map $g' = \mathscr{J}_1 \circ g : (Y_\omega, \star) \to (X_{\tau_2}, *)$ and an S-homotopy

$$G' = h \circ G : (Y_\omega, \star) \longrightarrow P\left(O_\rho, \circ\right). \tag{6}$$

We observe that

$$\begin{aligned} G'(y)(0) &= h\left[G(y)(0)\right] = h\left[\left(f^{h1} \circ \mathscr{X}^h\right)(g(y))\right] \\ &= h\left\{f^{h1}\left[\mathscr{X}\left(\mathscr{J}_1(g(y))\right), \mathscr{J}_2(g(y))\right]\right\} \\ &= h\left[\mathscr{J}_2(g(y))\right] = f_2\left[\mathscr{J}_1(g(y))\right] = f_2\left(g'(y)\right) \end{aligned} \tag{7}$$

for all $y \in Y$. That is, $G'_0 = f_2 \circ g'$. Since f_{12} is an $C_\aleph$-fibration, then there is an S-homotopy $H' : (Y_\omega, \star) \to P(X_{\tau_1}, *)$ such that $H'_0 = \mathscr{X} \circ g'$ and $f_1 \circ H'_t = G'_t$ for all $t \in I$.

Define an S-homotopy $H : (Y_\omega, \star) \to P(X^{h1}|_{\tau_1\times\upsilon}, * \times \odot)$ by

$$H(y)(t) = \left[H'(y)(t), G(y)(t)\right] \tag{8}$$

for all $y \in Y, t \in I$. We observe that $f^{h1} \circ H_t = G_t$ for all $t \in I$ and

$$\begin{aligned} H(y)(0) &= \left[H'(y)(0), G(y)(0)\right] \\ &= \left[\mathscr{X}\left(g'(y)\right), \left(f^{h1} \circ \mathscr{X}^h\right)(g(y))\right] \\ &= \left[\mathscr{X}\left(\mathscr{J}_1(g(y))\right), \mathscr{J}_2(g(y))\right] \\ &= \mathscr{X}^h\left[\mathscr{J}_1(g(y)), \mathscr{J}_2(g(y))\right] \\ &= \mathscr{X}^h(g(y)) = \left(\mathscr{X}^h \circ g\right)(y) \end{aligned} \tag{9}$$

for all $y \in Y$. That is, $H_0 = \mathscr{X}^h \circ g$. Hence f^{h1} has the $C_\aleph$-fibration property by an S-map $\mathscr{X}^h$. □

In the last theorem, if $f_1 = f_2$ (i.e., f_{12} is an Sp-map), let $f = f_1 = f_2$; then

$$\mathscr{X}^h = \mathscr{X} \times id|_{X^h} : \left(X^h|_{\tau_2\times\upsilon}, * \times \odot\right) \longrightarrow \left(X^h|_{\tau_1\times\upsilon}, * \times \odot\right) \tag{10}$$

is a well-defined S-map taking $(X^h|_{\tau_2\times\upsilon}, * \times \odot)$ into $(X^h|_{\tau_1\times\upsilon}, * \times \odot)$, where $X^h = X^{h1} = X^{h2}$. That is, the triple $(X^h|_{\tau_{12}\times\upsilon}, * \times \odot, \mathscr{X}^h)$ is an c-bitopological semigroup, called a *pullback c-bitopological semigroup* of $(X_{\tau_{12}}, *, \mathscr{X})$ induced from f_{12} by h. The pair

$$f^h_{12} = \left(f^h, f^h\right) : (X^h, \tau_1 \times \upsilon|_{X^h}, \tau_2 \times \upsilon|_{X^h})_{\mathscr{X}^h} \longrightarrow (Q_\upsilon, \odot) \tag{11}$$

which is given by $f^h(x, b) = b$ for all $(x, b) \in X^h$ is an c-map, called a *pullback c-map* of f_{12} induced by h. We observe that

$$\left(f^h \circ \mathscr{X}^h\right)(x, b) = f^h(\mathscr{X}(x), b) = b = f^h(x, b) \tag{12}$$

for all $(x, b) \in X^h$.

Theorem 7. *Let $f_{12} = (f, f) : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ be an $C_\aleph$-fibration and let $h : (Q_\upsilon, \odot) \to (O_\rho, \circ)$ be an S-map such that $X^{h1} \cap X^{h2} \neq \phi$. Then the pullback c-map f^h_{12} of f_{12} induced by h is an $C_\aleph$-fibration.*

Proof. It is obvious by the last theorem and the second part in Definition 2. □

4. The c-Lifting Functions

In this section, we define the path lifting property for c-maps by giving the concept of an c-lifting property and we show its role in satisfying the $C_\aleph$-fibration property.

Recall [5] that for an S-map $f : (X_\tau, *) \to (O_\rho, \circ)$, the map: $\alpha \to f \circ \alpha$ for all $\alpha \in P(X_\tau)$ is an S-map from $P(X_\tau, *)$ into $P(O_\rho, \circ)$, denoted by $\widehat{f}$. Then for every c-bitopological semigroup $(X_{\tau_{12}}, *, \mathscr{X})$, $\widehat{\mathscr{X}}$ is an S-map from $P(X_{\tau_2}, *)$ into $P(X_{\tau_1}, *)$. That is, the triple $(P(X)_{\tau^c_{12}}, p(*), \widehat{\mathscr{X}})$ is an c-bitopological semigroup where τ^c_1 and τ^c_2 are compact-open topologies on $P(X)$ which are induced by τ_1 and τ_2, respectively. The shorter notion for this c-bitopological semigroup will be $P(X_{\tau_{12}}, *, \mathscr{X})$.

For a map $f : X_\tau \to O_\rho$, by $\Delta(f)$, we mean the set

$$\Delta(f) = \left\{(x, \alpha) \in X_\tau \times P\left(O_\rho\right) \mid \alpha(0) = f(x)\right\}. \tag{13}$$

Proposition 8. *Let $f : (X_\tau, *) \to (O_\rho, \circ)$ be an S-map. Then $(\Delta(f)|_{\tau\times\rho^c}, * \times p(\circ))$ is an S-subspace of an S-space $((X \times P(O))_{\tau\times\rho^c}, * \times p(\circ))$, where ρ^c is a compact-open topology on $P(O)$ which is induced by ρ.*

Proof. It is clear that $\Delta(f)|_{\tau\times\rho^c}$ is a subspace of a space $(X \times P(O))_{\tau\times\rho^c}$. We observe that, for all $(x, \alpha), (x', \alpha') \in \Delta(f)$,

$$\left(\alpha p(\circ)\alpha'\right)(0) = \alpha(0) \circ \alpha'(0) = f(x) \circ f\left(x'\right) = f\left(x * x'\right). \tag{14}$$

That is,

$$(x, \alpha) * \times p(\circ)\left(x', \alpha'\right) = \left(x * x', \alpha p(\circ)\alpha'\right) \in \Delta(f). \tag{15}$$

Hence $(\Delta(f)|_{\tau\times\rho^c}, * \times p(\circ))$ is an S-subspace of an S-space $((X \times P(O))_{\tau\times\rho^c}, * \times p(\circ))$. □

In the last theorem, the shorter notion for the S-space $(\Delta(f)|_{\tau\times\rho^c}, * \times p(\circ))$ will be $\Delta(f)|_{\tau\times\rho^c}$.

Definition 9. Let $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ be an c-map. An S-map

$$L : \Delta(f_2)|_{\tau_2\times\rho^c} \longrightarrow P\left(X_{\tau_1}, *\right) \tag{16}$$

from an S-space $\Delta(f_2)|_{\tau_2\times\rho^c}$ into $P(X_{\tau_1}, *)$ is called an *c-lifting function* for an c-map f_{12} provided L satisfies the following:

(1) $L(x,\alpha)(0) = \mathcal{X}(x)$ for all $(x,\alpha) \in \Delta(f_2)$;

(2) $f_1 \circ L(x,\alpha) = \alpha$ for all $(x,\alpha) \in \Delta(f_2)$.

And λ_f will be denoted to c-lifting function for an c-map f_{12}, if it exists.

Example 10. Let $(X_{\tau_{12}}, *, \mathcal{X})$ be an c-bitopological semigroup. For every S-space $(O_\rho, \circ)$, the Sp-map

$$f_{12} = (f,f) : \left((X \times O)_{\tau_{12}\times\rho}, * \times \circ, \mathcal{X} \times id\right) \longrightarrow \left(O_\rho, \circ\right) \tag{17}$$

is an c-map, where $f(x,y) = y$ for all $x \in X$, $y \in O$. Note that

$$\left[f \circ (\mathcal{X} \times \mathrm{id})\right](x,y) = f\left(\mathcal{X}(x), y\right) = y = f(x,y) \tag{18}$$

for all $x \in X$, $y \in O$. This c-map has an c-lifting function

$$\lambda_f : \Delta(f)|_{(\tau_2\times\rho)\times\rho^c} \longrightarrow P\left((X \times O)_{\tau_1\times\rho}, \tau_1 \times \circ\right) \tag{19}$$

which is given by

$$\begin{aligned} \lambda_f((x,b),\alpha)(t) &= (\mathcal{X}(x), \alpha(t)) \\ &\forall ((x,b),\alpha) \in \Delta(f),\ t \in I. \end{aligned} \tag{20}$$

Note that

$$\begin{aligned} \lambda_f((x,b),\alpha)(0) &= (\mathcal{X}(x), \alpha(0)) \\ &= \left(\mathcal{X}(x), f(x,b)\right) \\ &= (\mathcal{X}(x), b) \\ &= (\mathcal{X} \times \mathrm{id})(x,b), \end{aligned} \tag{21}$$

$$\left[f \circ \lambda_f((x,b),\alpha)\right](t) = f(\mathcal{X}(x), \alpha(t)) = \alpha(t)$$

for all $((x,b),\alpha) \in \Delta(f)$, $t \in I$.

The following theorem clarifies the existence property for c-lifting function in $C_{\aleph}$-fibration theory. That is, it clarifies that the existence of c-lifting function for any $C_{\aleph}$-fibration is necessary and sufficient condition.

Theorem 11. *An c-map $f_{12} : (X_{\tau_{12}}, *, \mathcal{X}) \to (O_\rho, \circ)$ is an $C_{\aleph}$-fibration if and only if there exists an c-lifting function for f_{12}.*

Proof. Suppose that f_{12} is an $C_{\aleph}$-fibration. Take $\Delta(f_2)|_{\tau_2\times\rho^c} \in \aleph$. Define two S-maps

$$\begin{aligned} g &: \Delta(f_2)\big|_{\tau_2\times\rho^c} \longrightarrow \left(X_{\tau_2}, *\right), \\ G &: \Delta(f_2)\big|_{\tau_2\times\rho^c} \longrightarrow P\left(O_\rho, \circ\right) \end{aligned} \tag{22}$$

by $g(x,\alpha) = x$ and $G(x,\alpha) = \alpha$ for all $(x,\alpha) \in \Delta(f_2)$, respectively. We observe that

$$G(x,\alpha)(0) = \alpha(0) = f_2(x) = \left(f_2 \circ g\right)(x,\alpha). \tag{23}$$

Since f_{12} is an $C_{\aleph}$-fibration, then there exists an S-homotopy $H : \Delta(f_2)|_{\tau_2\times\rho^c} \to P(X_{\tau_1}, *)$ such that $H_0 = \mathcal{X} \circ g$ and $f_1 \circ H_t = G_t$ for all $t \in I$. Define an S-map

$$\lambda_f : \Delta(f_2)\big|_{\tau_2\times\rho^c} \longrightarrow P\left(X_{\tau_1}, *\right) \tag{24}$$

by

$$\lambda_f(x,\alpha)(t) = H(x,\alpha)(t) \quad \forall (x,\alpha) \in \Delta\left(f_2\right). \tag{25}$$

We observe that, for all $(x,\alpha) \in \Delta(f_2)$,

$$\begin{aligned} \lambda_f(x,\alpha)(0) = H(x,\alpha)(0) &= \left(\mathcal{X} \circ g\right)(x,\alpha) = \mathcal{X}(x) \\ f_1 \circ \lambda_f(x,\alpha) = f_1 \circ H(x,\alpha) &= G(x,\alpha) = \alpha. \end{aligned} \tag{26}$$

That is, λ_f is an c-lifting function for f_{12}.

Conversely, suppose that there exists an c-lifting function λ_f for f_{12}. Let $(Y_\omega, \star) \in \aleph$ and let $g : (Y_\omega, \star) \to (X_{\tau_2}, *)$ and $G : (Y_\omega, \star) \to P(O_\rho, \circ)$ be two given S-maps with $G_0 = f_2 \circ g$. Define an S-homotopy $H : (Y_\omega, \star) \to P(X_{\tau_1}, *)$ by

$$H(y)(t) = \lambda_f\left[g(y), G(y)\right](t) \quad \forall y \in Y,\ t \in I. \tag{27}$$

We observe that

$$\begin{aligned} H(y)(0) &= \lambda_f\left[g(y), G(y)\right](0) = \left(\mathcal{X} \circ g\right)(y), \\ f_1\left[H(y)(t)\right] &= f_1\left[\lambda_f\left[g(y), G(y)\right](t)\right] = G(y)(t) \end{aligned} \tag{28}$$

for all $y \in Y$, $t \in I$. That is, $H_0 = \mathcal{X} \circ g$ and $f_1 \circ H_t = G_t$ for all $t \in I$. Hence f_{12} is an $C_{\aleph}$-fibration. □

Theorem 12. *Let $f_{12} : (X_{\tau_{12}}, *, \mathcal{X}) \to (O_\rho, \circ)$ be an $C_{\aleph}$-fibration. Then the c-map*

$$\widehat{f}_{12} = \left(\widehat{f}_1, \widehat{f}_2\right) : P\left(X_{\tau_{12}}, *, \mathcal{X}\right) \longrightarrow P\left(O_\rho, \circ\right) \tag{29}$$

is an $C_{\aleph}$-fibration.

Proof. Since f_{12} is an $C_{\aleph}$-fibration, then there exists c-lifting function

$$\lambda_f : \Delta(f_2)\big|_{\tau_2\times\rho^c} \longrightarrow P\left(X_{\tau_1}, *\right) \tag{30}$$

for f_{12} such that

$$\begin{aligned} \lambda_f(x,\alpha)(0) &= \mathcal{X}(x), \\ f_1 \circ \lambda_f(x,\alpha) &= \alpha \end{aligned} \tag{31}$$

for all $(x,\alpha) \in \Delta(f_2)$. Let $(Y_\omega, \star) \in \aleph$ and let $g : (Y_\omega, \star) \to P(X_{\tau_2}, *)$ and $G : (Y_\omega, \star) \to P[P(O)_{\rho^c}, p(\circ)]$ be two given S-maps with

$$\left[G(y)(s)\right](0) = \left(\widehat{f}_2 \circ g\right)(y)(s) \tag{32}$$

for all $y \in Y$, $s \in I$, where ρ^c is a compact-open topology on $P(O)$ which is induced by ρ. Define an S-homotopy $H : (Y_\omega, \star) \to P[P(X)_{\tau_1^c}, p(*)]$ by

$$\begin{aligned} \left[H(y)(s)\right](t) &= \lambda_f\left[g(y)(s), G(y)(s)\right](t) \\ &\forall y \in Y,\ s,t \in I. \end{aligned} \tag{33}$$

We observe that

$$\begin{aligned}[H(y)(s)](0) &= \lambda_f[g(y)(s), G(y)(s)](0)\\ &= \mathcal{X}[g(y)(s)] = (\widehat{\mathcal{X}}\circ g)(y)(s),\\ (\widehat{f}_1\circ H_r)(y)(t) &= (f_1\circ\lambda_f[g(y)(s), G(y)(s)])(t)\\ &= [G(y)(s)](t),\end{aligned} \tag{34}$$

for all $y \in Y, s, t \in I$. That is, $H_0 = \widehat{\mathcal{X}}\circ g$ and $\widehat{f}_1\circ H_s = G_s$ for all $s \in I$. Hence $\widehat{f}_{12}$ is an $C_{\aleph}$-fibration. □

An c-lifting function λ_f is called *regular* if for every $x \in X_{\tau_2}$, $\lambda_f(x, f_2\circ\widetilde{x}) = \widetilde{\mathcal{X}(x)}$, where $\widetilde{x}$ is the constant path in X_{τ_2} (i.e., $\widetilde{x}(t) = x$), similar for $\widetilde{\mathcal{X}(x)}$. An $C_{\aleph}$-fibration f_{12} is called *regular* if it has regular c-lifting function.

Example 13. In Example 10, the c-lifting function λ_f which is given by

$$\begin{aligned}\lambda_f((x,b),\alpha)(t) &= (\mathcal{X}(x), \alpha(t))\\ \forall((x,b),\alpha) &\in \Delta(f),\ t \in I\end{aligned} \tag{35}$$

is regular. Note that, for every $(x, b) \in (X\times O)_{\tau_2}$,

$$\begin{aligned}\lambda_f\left((x,b), f_2\circ\widetilde{(x,b)}\right)(t) &= \left(\mathcal{X}(x), \left(f_2\circ\widetilde{(x,b)}\right)(t)\right)\\ &= (\mathcal{X}(x), f_2(x,b))\\ &= (\mathcal{X}(x), b) = (\mathcal{X}\times\mathrm{id})(x,b)\\ &= (\mathcal{X}\times\widetilde{\mathrm{id})(x,b)}(t)\end{aligned} \tag{36}$$

for all $t \in I$.

The following theorem is an analogue of results of Fadell in Hurewicz fibration theory [7].

Theorem 14. *Let $f_{12} : (X_{\tau_{12}}, *, \mathcal{X}) \to (O_\rho, \circ)$ be a regular $C_{\aleph}$-fibration and let*

$$M_i : P\left(X_{\tau_i}, *\right) \longrightarrow \Delta(f_i)\big|_{\tau_i\times\rho^c} \tag{37}$$

be an S-map defined by $M_i(\alpha) = (\alpha(0), f_i\circ\alpha)$ for all $\alpha \in P(X_{\tau_i})$ where $i = 1, 2$. Then

(1) $M_1\circ\lambda_f = \mathcal{X}\times\mathrm{id}|_{\Delta(f_2)}$;

(2) $\lambda_f\circ M_2 \simeq_s \widehat{\mathcal{X}}$ *preserving projection. That is, there is an S-homotopy*

$$H : P\left(X_{\tau_2}, *\right) \longrightarrow P\left[P(X)_{\tau_1^c}, p(*)\right] \tag{38}$$

between two S-maps $\lambda_f\circ M_2$ and $\widehat{\mathcal{X}}$ such that $f_1[(H(\alpha)(s))(t)] = f_2(\alpha(t))$ for all $t, s \in I, \alpha \in P(X_{\tau_2})$.

Proof. For the first part, we observe that, for every $(x, \alpha) \in \Delta(f_2)$,

$$\begin{aligned}\left(M_1\circ\lambda_f\right)(x,\alpha) &= M_1\left[\lambda_f(x,\alpha)\right]\\ &= \left(\lambda_f(x,\alpha)(0), f_1\circ\lambda_f(x,\alpha)\right)\\ &= (\mathcal{X}(x),\alpha) = (\mathcal{X}\times\mathrm{id})(x,\alpha).\end{aligned} \tag{39}$$

That is, $M_1\circ\lambda_f = \mathcal{X}\times\mathrm{id}|_{\Delta(f_2)}$.

For the second part, for $\alpha \in P(X_{\tau_2})$ and $s \in I$, define a path $\beta^{1-s} \in P(O_\rho)$ by

$$\beta^{1-s}(t) = \begin{cases} f_2(\alpha(s+t)), & \text{for } 0 \le t \le 1-s,\\ f_2(\alpha(1)), & \text{for } 1-s \le t \le 1.\end{cases} \tag{40}$$

By the regularity of λ_f, we can define an S-homotopy $H : P(X_{\tau_2}, *) \to P[P(X)_{\tau_1^c}, p(*)]$ by

$$[H(\alpha)(s)](t) = \begin{cases} \mathcal{X}[\alpha(t)], & \text{for } 0 \le t \le s,\\ \lambda_f\left(\alpha(s), \beta^{1-s}\right)(t-s), & \text{for } s \le t \le 1,\end{cases} \tag{41}$$

for all $s \in I, \alpha \in P(X_{\tau_2})$. Then

$$\begin{aligned}[H(\alpha)(0)](t) &= \lambda_f\left(\alpha(0), \beta^1\right)(t)\\ &= \lambda_f(\alpha(0), \beta)(t)\\ &= \lambda_f(\alpha(0), f_2\circ\alpha)(t)\\ &= \left(\lambda_f\circ M_2\right)(\alpha)(t),\\ [H(\alpha)(1)](t) &= \mathcal{X}[\alpha(t)] = \widehat{\mathcal{X}}(\alpha)(t)\end{aligned} \tag{42}$$

for all $\alpha \in P(X_{\tau_2})$, $t \in I$. That is, $\lambda_f\circ M_2 \simeq_s \widehat{\mathcal{X}}$. Also we get that

$$\begin{aligned}&[f_1\circ(H(\alpha)(s))](t)\\ &= \begin{cases} f_1(\mathcal{X}[\alpha(t)]), & \text{for } 0 \le t \le s,\\ f_1\left[\lambda_f\left(\alpha(s), \beta^{1-s}\right)(t-s)\right], & \text{for } s \le t \le 1;\end{cases}\\ &= \begin{cases} f_2(\alpha(t)), & \text{for } 0 \le t \le s,\\ \beta^{1-s}(t-s), & \text{for } s \le t \le 1;\end{cases}\\ &= \begin{cases} f_2(\alpha(t)), & \text{for } 0 \le t \le s,\\ f_2(\alpha(s+t-s)), & \text{for } s \le t \le 1;\end{cases}\\ &= \begin{cases} f_2(\alpha(t)), & \text{for } 0 \le t \le s,\\ f_2(\alpha(t)), & \text{for } s \le t \le 1;\end{cases}\\ &= f_2(\alpha(t)),\end{aligned} \tag{43}$$

for all $s, t \in I, \alpha \in P(X_{\tau_2})$. Hence $\lambda_f\circ M_2 \simeq_s \widehat{\mathcal{X}}$ preserving projection. □

5. Approximate Fibrations

Coram and Duvall [2] introduced approximate fibrations as a generalization of cell-like maps [3] and showed that the uniform limit of a sequence of Hurewicz fibrations is an approximate fibration. A map $f : X_\tau \to O_\rho$ of compact metrizable spaces X_τ and O_ρ is called an *approximate fibration* if, for every space Y_ω and for given $\epsilon > 0$, there exists $\delta > 0$ such that whenever $g : Y_\omega \to X_\tau$ and $H : Y_\omega \times I \to O_\rho$ are maps with $d[H(y,0),(f\circ g)(y)] < \delta$, then there is homotopy $G : Y_\omega \times I \to X_\tau$ such that $G_0 = g$ and

$$d\left[H(y,t),(f\circ G)(y,t)\right] < \epsilon \quad \forall y \in Y,\ t \in I. \tag{44}$$

One notable exception is that the pullback of approximate fibration need not be an approximate fibration.

The following theorem shows the role of the $C_{\mathcal{N}_\pi}$-fibration property in inducing an approximate fibration property.

For an S-map $f : (X_\tau,\pi) \to (O_\rho,\pi)$ with metrizable spaces X_τ and O_ρ, by d_τ and d_ρ, we mean the metric functions on X and O, respectively; by $X \times O$ we mean the product metrizable space of X_τ and O_ρ with a metric function

$$d\left((x,b),\left(x',b'\right)\right) = \max\left\{d_\tau\left(x,x'\right),d_\rho\left(b,b'\right)\right\}; \tag{45}$$

by $\mathcal{G}^f$ we mean the graph of f (i.e., $\mathcal{G}^f = \{(x,f(x)) : x \in X\}$) which is an S-subspace of $((X\times O)_{\tau\times\rho},\pi)$; for a positive integer $n > 0$, by $\mathcal{G}^n(f)$, we mean the $(1/n)$-neighborhood of $\mathcal{G}^f$ in a metrizable space $X \times O$ which is also S-subspace of $((X\times O)_{\tau\times\rho},\pi)$.

Theorem 15. *Let $f : X_\tau \to O_\rho$ be a map with compact metrizable spaces X_τ and O_ρ. Then f is an approximate fibration if and only if, for every positive integer $n > 0$, there exists a positive integer $m \geq n$ such that the S-map $f_n : (\mathcal{G}^n(f),\pi) \to (O_\rho,\pi)$ has the $C_{\mathcal{N}_\pi}$-fibration property by the inclusion S-map $\mathcal{I}^n_m : (\mathcal{G}^m(f),\pi) \to (\mathcal{G}^n(f),\pi)$, where $f_n(x,b) = b$ for all $(x,b) \in \mathcal{G}^n(f)$.*

Proof. Let n be any positive integer. For $\epsilon = 1/n > 0$, let δ be given in the definition of approximate fibration. Since $\delta/2 > 0$ and f is a continuous function, then let δ' be chosen such that if $x, x' \in X$ and $d_\tau(x,x') < \delta'$, then $d_\rho(f(x), f(x')) < \epsilon/2$. Choose a positive integer $m \geq n$, such that $1/m \leq \delta', \delta/2$.

Now let $(Y_\omega,\pi) \in \mathcal{N}_\pi$ and let $g : (Y_\omega,\pi) \to (\mathcal{G}^m(f),\pi)$ and $G : (Y_\omega,\pi) \to P(O_\rho,\pi)$ be two given S-maps with $G_0 = f_n \circ (\mathcal{I}^n_m \circ g)$. Define a map $g' : Y_\omega \to X_\tau$ by $g'(y) = \mathcal{J}_1[g(y)]$ and a homotopy $G' : Y_\omega \times I \to O_\rho$ by $G'(y,t) = G(y)(t)$ for all $y \in Y$ and $t \in I$. We get that $g(y) = (g'(y), G'(y,0))$ for all $y \in Y$. Since $g(y) \in \mathcal{G}^m(f)$, then there exists $x \in X$ such that

$$d\left[(x,f(x)),g(y)\right] < \frac{1}{m}. \tag{46}$$

Then

$$\begin{aligned} &d_\tau\left(x,g'(y)\right) < \frac{1}{m} \leqslant \delta', \\ &d_\rho\left(f(x),G'(y,0)\right) < \frac{1}{m} \leqslant \frac{\delta}{2}, \\ &d_\rho\left(f(x),f\left(g'(y)\right)\right) < \frac{1}{m} \leqslant \frac{\delta}{2} \end{aligned} \tag{47}$$

for all $y \in Y$. This implies

$$\begin{aligned} &d_\rho\left(f\left(g'(y)\right),G'(y,0)\right) \\ &\quad\leqslant d_\rho\left(f\left(g'(y)\right),f(x)\right) + d_\rho\left(f(x),G'(y,0)\right) \\ &\quad< \delta \end{aligned} \tag{48}$$

for all $y \in Y$. Hence, since f is an approximate fibration, there exists a homotopy $H' : Y_\omega \times I \to X_\tau$ such that $H'_0 = g'$ and

$$d_\tau\left(G'(y,t),\left(f\circ H'_t\right)(y)\right) < \epsilon \tag{49}$$

for all $y \in Y$, $t \in I$. Define an S-homotopy $H : (Y_\omega,\pi) \to P(\mathcal{G}^n(f),\pi)$ by

$$H(y)(t) = \left(H'(y,t),G(y)(t)\right) \quad \forall y \in Y,\ t \in I. \tag{50}$$

Then we get that

$$\begin{aligned} H(y)(0) &= \left(H'(y,0),G(y)(0)\right) = \left(g'(y),G(y)(0)\right) \\ &= g(y) = \left(\mathcal{I}^n_m \circ g\right)(y) \end{aligned} \tag{51}$$

for all $y \in Y$ and $f_n \circ H_t = G_t$ for all $t \in I$. Hence f_n has the $C_{\mathcal{N}_\pi}$-fibration property by $\mathcal{I}^n_m$.

Conversely, let $\epsilon > 0$ be given. Since f is a continuous function, then let δ' be chosen such that if $x, x' \in X$ and $d_\tau(x,x') < \delta'$, then $d_\rho(f(x), f(x')) < \epsilon/2$. Choose a positive integer $n > 0$ such that $1/n \leq \delta', \epsilon/2$. By hypothesis, there exists a positive integer $m \geq n$ such that f_n has the $C_{\mathcal{N}_\pi}$-fibration property by $\mathcal{I}^n_m$.

Take $\delta = 1/m$. Let Y_ω be any space and let $g : Y_\omega \to X_\tau$ and $G : Y_\omega \times I \to O_\rho$ be two given maps with

$$d_\rho\left[G(y,0),(f\circ g)(y)\right] < \delta \tag{52}$$

for all $y \in Y$. Define an S-map $g' : (Y_\omega,\pi) \to (\mathcal{G}^m(f),\pi)$ by $g'(y) = (g(y), G(y,0))$ and an S-homotopy $G' : (Y_\omega,\pi) \to P(O_\rho,\iota)$ by $G'(y)(t) = G(y,t)$ for all $y \in Y$ and $t \in I$. Since $G'_0 = f_n \circ (\mathcal{I}^n_m \circ g')$, then there exists an S-homotopy $F : (Y_\omega,\pi) \to P(\mathcal{G}^n(f),\pi)$ such that $F_0 = \mathcal{I}^n_m \circ g'$ and $f_n \circ F_t = G'_t$ for all $t \in I$. By the last part, we can define a homotopy $H : Y_\omega \times I \to X_\tau$ by

$$H(y,t) = \mathcal{J}_1\left[F(y)(t)\right] \quad \forall y \in Y,\ t \in I. \tag{53}$$

We get that $F(y)(t) = (H(y,t), G(y,t))$. Since $F(y)(t) \in \mathcal{G}^n(f)$, then there exists $x \in X$ such that

$$d\left[(x,f(x)),F(y)(t)\right] < \frac{1}{n}. \tag{54}$$

Then

$$d_\tau(x, H(y,t)) < \frac{1}{n} \leqslant \delta', \quad d_\rho(f(x), G(y,t)) < \frac{1}{n} \leqslant \frac{\epsilon}{2},$$
$$d_\rho[f(x), f(H(y,t))] < \frac{1}{n} \leqslant \frac{\epsilon}{2}. \tag{55}$$

This implies

$$d_\rho[G(y,t), f(H(y,t))] \leqslant d_\rho[f(H(y,t)), f(x)] + d_\rho(f(x), G(y,t)) < \epsilon \tag{56}$$

for all $y \in Y, t \in I$. Hence f is an approximate fibration. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Optimal Sixteenth Order Convergent Method Based on Quasi-Hermite Interpolation for Computing Roots

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We have given a four-step, multipoint iterative method without memory for solving nonlinear equations. The method is constructed by using quasi-Hermite interpolation and has order of convergence sixteen. As this method requires four function evaluations and one derivative evaluation at each step, it is optimal in the sense of the Kung and Traub conjecture. The comparisons are given with some other newly developed sixteenth-order methods. Interval Newton's method is also used for finding the enough accurate initial approximations. Some figures show the enclosure of finitely many zeroes of nonlinear equations in an interval. Basins of attractions show the effectiveness of the method.

1. Introduction

Let us consider the problem of approximating the simple root x^* of the nonlinear equation involving a nonlinear univariate function f:

$$f(x) = 0. \tag{1}$$

Newton's method and its variants have always remained as widely used one-point without memory and one-step methods for solving (1). However, the usage of single point and one-step methods puts limit on the order of convergence and computational efficiency is given as

$$E = p^{1/\theta}, \tag{2}$$

where p is the order of convergence of the iterative method and θ is the cost of evaluating f and its derivatives.

To overcome the drawbacks of one-point, one-step methods, many multipoint multistep higher order convergent methods have been introduced in the recent past by using inverse, Hermite, and rational interpolation [1, 2]. In developing these methods, so far, the conjecture of Kung and Traub has remained the focus of attention. It states the following.

Conjecture 1. *An optimal iterative method without memory based on n evaluations would achieve an optimal convergence order of 2^{n-1}, hence, a computational efficiency of $2^{(n-1)/n}$.*

In [3, 4], Petković presented a general optimal n-point iterative scheme without memory defined by

$$x_{k+1} = \phi_n(x_k) = N_{n-1}\left(N_{n-2}\left(\cdots\left(N_2\left(\psi_f(x_k)\right)\right)\cdots\right)\right), \quad k = 0, 1, 2, \ldots, \tag{3}$$

where x_k is the approximation of the root x^* at the kth iteration and $\psi_f(x_k) = \phi_2(x_k)$ is an arbitrary fourth-order, two-point method requiring three function evaluations:

$$\phi_{m+1}(x_k) = N_m(\phi_m) = \phi_m - \frac{f(\phi_m)}{f'(\phi_m)}, \quad m = 2, \ldots, n-1 \tag{4}$$

is Newton's method. The derivative at $m+1$-step is approximated through quasi-Hermite interpolatory polynomial of degree $m+1$, denoted by $h'_{m+1}(\phi_m)$.

Using this approach, Sargolzaei and Soleymani [5] presented a three-step optimal eighth-order iterative method.

However, since the authors approximated the derivative at the fourth step by using Hermite interpolatory polynomials of degree three, therefore the fourth-step method given by Sargolzaei and Soleymani has order of convergence fourteen including five function evaluations, which is not optimal in the sense of Kung and Traub.

In this paper, we present an optimal four-step four-point sixteenth-order convergent method by using quasi-Hermite interpolation from the general class of Petković [3, 4]. The interpolation is done by using the Newtonian formulation given by Traub [6]. The numerical comparisons are given in Section 4 with recent optimal sixteenth-order convergent methods based on rational interpolants. Since, the first step of our method is Newton's method, thus to overcome the drawbacks of Newton's method we have calculated, in Section 5, accurate initial guess required for the convergence of this method for some oscillatory functions.

2. Construction of Method

We define the following:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= \psi_f(x_n, y_n), \\ t_n &= \varphi_f(x_n, y_n, z_n), \\ x_{n+1} &= t_n - \frac{f(t_n)}{f'(t_n)}, \end{aligned} \tag{5}$$

where $\psi_f(x_n, y_n)$ and $\varphi_f(x_n, y_n, z_n)$ are any arbitrary fourth- and eighth-order, multipoint methods. We, now, approximate $f'(t_n)$ with a quasi-Hermite interpolatory polynomial of degree four satisfying

$$\begin{aligned} f(x_n) &= h_4(x_n), \\ f'(x_n) &= h_4'(x_n), \\ f(y_n) &= h_4(y_n), \\ f(z_n) &= h_4(z_n), \\ f(t_n) &= h_4(t_n). \end{aligned} \tag{6}$$

To construct the interpolatory polynomial $h_4(t)$, satisfying the above conditions, we apply the Newtonian representation of the interpolatory polynomial satisfying the conditions

$$\begin{aligned} P^{(k_j)}\left(x_{i-j}\right) &= f^{(k_j)}\left(x_{i-j}\right), \quad k_j = 0, 1, \ldots, \gamma_j - 1, \\ \gamma_j &\geq 1, \quad j = 0, 1, 2, 3, \ldots. \end{aligned} \tag{7}$$

Traub [6, p. 243] have given this as follows:

$$\begin{aligned} h_4(t) &\equiv P_{3,\gamma}(t) \\ &= \sum_{j=0}^{3} \sum_{l=0}^{\gamma_j - 1} C_{l,j}^{\gamma_j}(t)\, f\left[x_i, \gamma_0; x_{i-1}, \gamma; \ldots; x_{i-j}, l+1\right], \end{aligned} \tag{8}$$

$$C_{l,j}^{\gamma_j}(t) = \left(t - x_{i-j}\right)^l \prod_{k=0}^{j-1} (t - x_{i-k})^{\gamma_k}, \qquad \prod_{0}^{-1} = 1. \tag{9}$$

The confluent divided differences involved here are defined as

$$\begin{aligned} &f\left[x_i, \gamma_0; \ldots; x_{i-q}, \gamma_q; \ldots; x_{i-r}, \gamma_r; \ldots; x_{i-n}, \gamma_n\right] \\ &= \frac{1}{\left(x_{i-q} - x_{i-r}\right)} \\ &\quad \times \Big(f\left[x_i, \gamma_0; \ldots; x_{i-q}, \gamma_q; \ldots; x_{i-r}, \gamma_r - 1; \ldots; x_{i-n}, \gamma_n\right] \\ &\qquad - f\left[x_i, \gamma_0; \ldots; x_{i-q}, \gamma_q - 1; \right. \\ &\qquad\qquad \left. \ldots; x_{i-r}, \gamma_r; \ldots; x_{i-n}, \gamma_n\right] \Big), \\ &\qquad f\left[x_{i-j}; l+1\right] = \frac{f^{(l)}\left(x_{i-j}\right)}{l!}. \end{aligned} \tag{10}$$

In particular, $f[x_{i-j}, 1] = f[x_{i-j}]$ is the usual divided difference. Here, we take $x_i = t_n$, $x_{i-1} = z_n$, $x_{i-2} = y_n$, $x_{i-3} = x_n$ and hence, $\gamma_0 = 1$, $\gamma_1 = 1$, $\gamma_2 = 1$, and $\gamma_3 = 2$. Expanding (8), we get

$$\begin{aligned} h_4(t) &= f(t_n) + (t - t_n) f[t_n, z_n] \\ &\quad + (t - t_n)(t - z_n) f[t_n, z_n, y_n] \\ &\quad + (t - t_n)(t - z_n)(t - y_n) f[t_n, z_n, y_n, x_n] \\ &\quad + (t - t_n)(t - z_n)(t - y_n)(t - x_n) f[t_n, z_n, y_n, x_n, 2]. \end{aligned} \tag{11}$$

Differentiating (11) with respect to "t" and substituting $t = t_n$ in the above equation, we obtain

$$\begin{aligned} h_4'(t_n) &= f[t_n, z_n] + (t_n - z_n) f[t_n, z_n, y_n] \\ &\quad + (t_n - z_n)(t_n - y_n) f[t_n, z_n, y_n, x_n] \\ &\quad + (t_n - z_n)(t_n - y_n)(t_n - x_n) f[t_n, z_n, y_n, x_n, 2], \end{aligned} \tag{12}$$

where

$$\begin{aligned} f[t_n, z_n, y_n] &= \frac{1}{(t_n - y_n)} \left(f[t_n, z_n] - f[t_n, y_n]\right), \\ f[t_n, z_n, y_n, x_n] &= \frac{1}{(t_n - x_n)(t_n - y_n)} \left(f[t_n, z_n] - f[z_n, y_n]\right) \\ &\quad - \frac{1}{(t_n - x_n)(z_n - x_n)} \left(f[z_n, y_n] - f[y_n, x_n]\right), \end{aligned}$$

$$
\begin{aligned}
&f\left[t_n, z_n, y_n, x_n, 2\right] \\
&= \frac{1}{\left(t_n - x_n\right)^2 \left(t_n - y_n\right)} \left(f\left[t_n, z_n\right] - f\left[z_n, y_n\right]\right) \\
&\quad - \frac{1}{\left(t_n - x_n\right)^2 \left(z_n - x_n\right)} \left(f\left[z_n, y_n\right] - f\left[y_n, x_n\right]\right) \\
&\quad - \frac{1}{\left(z_n - x_n\right)^2 \left(t_n - x_n\right)} \left(f\left[z_n, y_n\right] - f\left[y_n, x_n\right]\right) \\
&\quad + \frac{1}{\left(t_n - x_n\right)\left(z_n - x_n\right)\left(y_n - x_n\right)} \\
&\quad \times \left(f\left[y_n, x_n\right] - f'\left(x_n\right)\right).
\end{aligned} \tag{13}
$$

Using representation (12) of $h_4'(t_n)$ in place of $f'(t_n)$ at the fourth step, the new four-step iterative method is obtained as

$$
\begin{aligned}
z_n &= \psi_f\left(x_n, y_n\right), \\
t_n &= \varphi_f\left(x_n, y_n, z_n\right), \\
x_{n+1} &= t_n - \frac{f\left(t_n\right)}{h_4'\left(t_n\right)},
\end{aligned} \tag{14}
$$

where $\psi_f(x_n, y_n)$ and $\varphi_f(x_n, y_n, z_n)$ are any fourth- and eighth-order convergent methods, respectively, and

$$
\begin{aligned}
h_4'\left(t_n\right) &= f\left[t_n, z_n\right] + \left(t_n - z_n\right) f\left[t_n, z_n, y_n\right] \\
&\quad + \left(t_n - z_n\right)\left(t_n - y_n\right) f\left[t_n, z_n, y_n, x_n\right] \\
&\quad + \left(t_n - z_n\right)\left(t_n - y_n\right)\left(t_n - x_n\right) f\left[t_n, z_n, y_n, x_n, 2\right].
\end{aligned} \tag{15}
$$

Theorem 2. *Let one consider x^* as a root of nonlinear equation* (1) *in the domain D and assume that $f(x)$ is sufficiently differentiable in the neighbourhood of the root. Then the iterative method defined by* (14) *is of optimal order sixteen and has the following error equation:*

$$
e_{n+1} = x_{n+1} - x^* = -c_2 b_1 \left(a_1 c_5 - b_1\right) e_n^{16} + O\left(e_n^{17}\right), \tag{16}
$$

where c_i, for $i \geq 2$, are defined by

$$
c_i = \frac{1}{i!}\left(\frac{f^{(i)}\left(x^*\right)}{f'\left(x^*\right)}\right), \quad i = 0, 1, 2, 3, \ldots. \tag{17}
$$

Proof. We write the Taylor series expansion of the function f about the simple root x^* in nth iteration. Let $e_n = x_n - x^*$. Therefore, we have

$$
\begin{aligned}
f\left(x_n\right) = f'\left(x^*\right)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5\right. \\
\left. + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O\left(e_n^9\right)\right].
\end{aligned} \tag{18}
$$

Also, we obtain

$$
\begin{aligned}
f'\left(x_n\right) = f'\left(x^*\right)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4\right. \\
\left. + 6c_6 e_n^5 + 7c_7 e_n^6 + 8c_8 e_n^7 + O\left(e_n^8\right)\right].
\end{aligned} \tag{19}
$$

Now, we find the Taylor expansion of y_n, the first step, by using the above two expressions (18) and (19). Hence, we have

$$
\begin{aligned}
y_n &= c_2 e_n^2 + \left(2c_3 - 2c_2^2\right) e_n^3 + \left(3c_4 - 7c_2 c_3 + 4c_2^3\right) e_n^4 \\
&\quad + \left(4c_5 - 10c_2 c_4 - 6c_3^2 + 20 c_3 c_2^2 - 8c_2^4\right) e_n^5 + O\left(e_n^6\right).
\end{aligned} \tag{20}
$$

Also, we need the Taylor expansion of $f(y_n)$; that is

$$
\begin{aligned}
&f\left(y_n\right) \\
&= f'\left(x^*\right) \\
&\quad \times \left[c_2 e_n^2 - 2\left(c_2^2 - c_3\right) e_n^3 + \left(3c_4 - 7c_2 c_3 + 5c_2^3\right) e_n^4\right. \\
&\qquad \left. - 2\left(-2c_5 + 5c_2 c_4 + 3c_3^3 - 12 c_3 c_2^2 + 6c_2^4\right) e_n^5 + O\left(e_n^6\right)\right].
\end{aligned} \tag{21}
$$

In second step, we take a general fourth-order convergent method as

$$
\begin{aligned}
z_n &= a_1 e_n^4 + a_2 e_n^5 + a_3 e_n^6 + a_4 e_n^7 + a_5 e_n^8 + O\left(e_n^9\right), \\
f\left(z_n\right) &= f'\left(x^*\right)\left[a_1 e_n^4 + a_2 e_n^5 + a_3 e_n^6\right. \\
&\qquad \left. + a_4 e_n^7 + \left(c_2 a_1^2 + a_5\right) e_n^8 + O\left(e_n^9\right)\right].
\end{aligned} \tag{22}
$$

Now, we find the Taylor expansion of each divided difference used at the third step. We thus obtain

$$
\begin{aligned}
&f\left[x_n, y_n\right] \\
&= f'\left(x^*\right)\left[1 + c_2 e_n + \left(c_3 + c_2^2\right) e_n^2\right. \\
&\qquad \left. + \left(c_4 + 3c_2 c_3 - 2c_2^3\right) e_n^3 + \cdots + O\left(e_n^9\right)\right], \\
&f\left[x_n, z_n\right] \\
&= f'\left(x^*\right)\left[1 + c_2 e_n + c_3 e_n^2 + c_4 e_n^3\right. \\
&\qquad \left. + \left(c_5 + c_2 a_1\right) e_n^4 + \cdots + O\left(e_n^9\right)\right], \\
&f\left[y_n, z_n\right] \\
&= f'\left(x^*\right)\left[1 + c_2^2 e_n^2 - 2c_2\left(-c_3 + c_2^2\right) e_n^3\right. \\
&\qquad \left. + c_2\left(3c_4 - 6c_2 c_3 + 4c_2^3 + a_1\right) e_n^4 + \cdots + O\left(e_n^9\right)\right], \\
&f\left[y_n, x_n, x_n\right] \\
&= f'\left(x^*\right)\left[c_2 + 2c_3 e_n + \left(3c_4 + c_2 c_3\right)^2 e_n^2\right. \\
&\qquad \left. + \left(2c_2 c_4 - 2c_3 c_2^2 + 4c_5 + 2c_3^2\right) e_n^3 + \cdots + O\left(e_n^9\right)\right].
\end{aligned} \tag{23}
$$

In the third step, we take a general eighth-order convergent method as follows:

$$
t_n = b_1 e_n^8 + b_2 e_n^9 + b_3 e_n^{10} + b_4 e_n^{11} + b_5 e_n^{12} + \cdots + O\left(e_n^{17}\right), \tag{24}
$$

and the Taylor expansion for $f(t_n)$ is

$$\begin{aligned} & f(t_n) \\ & = f'(x^*) \\ & \quad \times \left[b_1 e_n^8 + b_2 e_n^9 + b_3 e_n^{10} + b_4 e_n^{11} + b_5 e_n^{12} + \cdots + O\left(e_n^{17}\right)\right]. \end{aligned} \tag{25}$$

Now, we find the Taylor expansion of divided differences used at the last step. We, thus, obtain

$$\begin{aligned} & f[t_n, z_n, y_n] \\ & = f'(x^*)\left[c_2 + c_2 c_3 e_n^2 - 2c_3\left(-c_3 + c_2^2\right)e_n^3 + \cdots + O\left(e_n^{17}\right)\right], \\ & f[t_n, z_n, y_n, x_n] \\ & = f'(x^*)\left[c_3 + c_4 e_n + \left(c_2 c_4 + c_5\right)e_n^2 \right. \\ & \qquad \left. + \left(2c_3 c_4 - c_2^2 c_4 + c_6 + c_2 c_5\right)e_n^3 + \cdots + O\left(e_n^{17}\right)\right], \\ & f[t_n, z_n, y_n, x_n, 2] \\ & = f'(x^*)\left[c_4 + 2c_5 e_n + \left(3c_6 + c_2 c_5\right)e_n^2 \right. \\ & \qquad \left. + \left(2c_2 c_6 + 4c_7 + 2c_3 c_5 - 2c_2^2 c_5\right)e_n^3 + \cdots + O\left(e_n^{17}\right)\right]. \end{aligned} \tag{26}$$

Hence, our fourth step defined in (14) becomes

$$x_{n+1} = -c_2 b_1 \left(a_1 c_5 - b_1\right) e_n^{16} + O\left(e_n^{17}\right), \tag{27}$$

which manifests that (14) is a four-step iterative method of optimal order of convergence of sixteen consuming four function evaluations and one derivative evaluation. □

Remark 3. It is concluded from Theorem 2 that the new sixteenth-order convergent iterative method (14) for solving nonlinear equations satisfies the conjecture of Kung and Traub that a multipoint method without memory with four evaluations of functions and a derivative evaluation can achieve an optimal sixteenth order of convergence ($2^4 = 16$) and an efficiency index of $2^{4/5} = 1.741$.

3. Some Particular Methods

In this section, we consider some particular methods from the newly developed family of the sixteenth-order convergent iterative methods.

3.1. Iterative Method M1. Here, we take $\psi_f(x_n, y_n)$ as two-step fourth-order convergent method defined by Geum and Kim [7] and the third-step $\varphi_f(x_n, y_n, z_n)$ is replaced by the third step of eighth-order convergent method given by [5] using Hermite interpolation. Hence, our four-step method becomes

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \left(1 + \frac{f(y_n)}{f(x_n)}\right)^2 \frac{f(y_n)}{f'(x_n)}, \\ t_n &= z_n - f(z_n) \\ &\quad \times \left(2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n] \right. \\ &\quad \left. + (y_n - z_n) f[y_n, x_n, x_n]\right)^{-1}, \\ x_{n+1} &= t_n - \frac{f(t_n)}{h_4'(t_n)}, \end{aligned} \tag{28}$$

where $h_4'(t_n)$ is given by (15).

3.2. Iterative Method M2. Here, we define $\psi_f(x_n, y_n)$ as King's two-step fourth-order convergent method [8] with $\beta = 0$, as

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n)}{f(x_n) - 2f(y_n)}. \end{aligned} \tag{29}$$

Hence, our four-step iterative method becomes

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n)}{f(x_n) - 2f(y_n)}, \\ t_n &= z_n - f(z_n) \\ &\quad \times \left(2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n] \right. \\ &\quad \left. + (y_n - z_n) f[y_n, x_n, x_n]\right)^{-1}, \\ x_{n+1} &= t_n - \frac{f(t_n)}{h_4'(t_n)}, \end{aligned} \tag{30}$$

where $h_4'(t_n)$ is given by (15).

4. Numerical Results and Computational Cost

In this section, we compare our newly constructed family of iterative methods of optimal sixteenth-order M1 and M2 defined in (28) and (30), respectively, with some famous equation solvers. For the sake of comparison, we consider the fourteenth-order convergent method (PF) given by Sargolzaei and Soleymani [5] and the optimal sixteenth-order convergent methods (JRP) and (FSH) given by Sharma et al. [1] and Soleymani et al. [2], respectively. All the computations

Table 1: Numerical examples.

Numerical example	Exact zero
$f_1(x) = \sin(x) - \frac{x}{100}$	$x^* = 0.0000000000000000$
$f_2(x) = e^{\sin(x)} - 1 - \frac{x}{5}$	$x^* = 0.0000000000000000$
$f_3(x) = e^{-x} + \cos(x)$	$x^* = 1.7461395304080130$
$f_4(x) = x^3 + 4x^2 - 15$	$x^* = 1.6319808055660635$
$f_5(x) = 10xe^{-x^2} - 1$	$x^* = 1.6796306104284499$
$f_6(x) = \sqrt{x^2+2x+5} - 2\sin(x) - x^2 + 3$	$x^* = 2.3319676558839640$

Table 2: Comparison table for $f_1(x)$.

$f_1(x)$, $x_0 = -0.9$	n	$\lvert f_1(x_1)\rvert$	$\lvert f_1(x_2)\rvert$	$\lvert f_1(x_3)\rvert$
PF	3	$0.602\,10^{-5}$	$0.244\,10^{-105}$	$0.907\,10^{-2013}$
JRP	3	$0.234\,10^{-5}$	$0.205\,10^{-124}$	$0.133\,10^{-2624}$
FSH	3	$0.149\,10^{-5}$	$0.286\,10^{-129}$	$0.248\,10^{-2727}$
M1	3	$0.138\,10^{-5}$	$0.561\,10^{-130}$	$0.349\,10^{-2742}$
M2	3	$0.419\,10^{-7}$	$0.737\,10^{-162}$	$0.107\,10^{-3411}$

Table 3: Comparison table for $f_2(x)$.

$f_2(x)$, $x_0 = 1.0$	n	$\lvert f_2(x_1)\rvert$	$\lvert f_2(x_2)\rvert$	$\lvert f_2(x_3)\rvert$
PF	3	$0.178\,10^{-6}$	$0.455\,10^{-94}$	$0.227\,10^{-1320}$
JRP	3	$0.815\,10^{-9}$	$0.143\,10^{-146}$	$0.124\,10^{-2350}$
FSH	3	$0.375\,10^{-8}$	$0.299\,10^{-126}$	$0.139\,10^{-1899}$
M1	3	$0.239\,10^{-8}$	$0.610\,10^{-137}$	$0.192\,10^{-2194}$
M2	3	$0.202\,10^{-8}$	$0.631\,10^{-141}$	$0.510\,10^{-2261}$

Table 4: Comparison table for $f_3(x)$.

$f_3(x)$, $x_0 = 0.5$	n	$\lvert f_3(x_1)\rvert$	$\lvert f_3(x_2)\rvert$	$\lvert f_3(x_3)\rvert$
PF	3	$0.628\,10^{-7}$	$0.899\,10^{-116}$	$0.197\,10^{-1748}$
JRP	3	$0.992\,10^{-9}$	$0.111\,10^{-152}$	$0.734\,10^{-2456}$
FSH	3	$0.335\,10^{-8}$	$0.701\,10^{-145}$	$0.942\,10^{-2332}$
M1	3	$0.143\,10^{-8}$	$0.850\,10^{-151}$	$0.204\,10^{-2426}$
M2	3	$0.103\,10^{-8}$	$0.197\,10^{-153}$	$0.655\,10^{-2469}$

are done using software Maple 13 with tolerance $\varepsilon = 10^{-1000}$ and 4000 digits precision. The stopping criterion is

$$\left|f\left(x_n\right)\right| < \varepsilon. \tag{31}$$

Here, x^* is the exact zero of the function and x_0 is the initial guess. In Tables 1–9, columns show the number of iterations n, in which the method converges to x^*, the absolute value of function $|f(x_n)|$ at nth step, for $n = 1, 2, 3$. The numerical examples are taken from [1, 2].

We now give the numerical results of our new schemes in comparison with Newton's method for three oscillatory nonlinear functions, $f_7(x) = -\cos(2 - x^2) + \log(x/7) + (1/10)$ in the domain $[1, 15]$ having 69 zeroes, $f_8(x) = (x^2 - 4)\sin(100x)$ on the interval $[0, 10]$ having 320 zeroes, and $f_9(x) = e^{\sin(\log(x)\cos(20x))} - 2$ in the domain $[2, 10]$ having 51 zeroes using the same precision, stopping criterion, and tolerance as given above. The first two functions $f_7(x)$ and $f_8(x)$

Table 5: Comparison table for $f_4(x)$.

$f_4(x)$, $x_0 = 3.0$	n	$\lvert f_4(x_1)\rvert$	$\lvert f_4(x_2)\rvert$	$\lvert f_4(x_3)\rvert$
PF	3	0.00023181	$0.471\,10^{-81}$	$0.410\,10^{-1324}$
JRP	3	0.00001307	$0.455\,10^{-103}$	$0.214\,10^{-1678}$
FSH	3	0.00023181	$0.471\,10^{-81}$	$0.410\,10^{-1324}$
M1	3	0.00023181	$0.471\,10^{-81}$	$0.410\,10^{-1324}$
M2	3	$0.890\,10^{-5}$	$0.159\,10^{-106}$	$0.180\,10^{-1734}$

Table 6: Comparison table for $f_5(x)$.

$f_5(x)$, $x_0 = 0.0$	n	$\lvert f_5(x_1)\rvert$	$\lvert f_5(x_2)\rvert$	$\lvert f_5(x_3)\rvert$
PF	3	$0.322\,10^{-17}$	$0.430\,10^{-260}$	$0.247\,10^{-3660}$
JRP	3	$0.186\,10^{-19}$	$0.128\,10^{-332}$	0
FSH	3	$0.101\,10^{-19}$	$0.230\,10^{-337}$	0
M1	3	$0.955\,10^{-20}$	$0.884\,10^{-338}$	0
M2	3	$0.860\,10^{-20}$	$0.511\,10^{-339}$	0

Table 7: Comparison table for $f_6(x)$.

$f_6(x)$, $x_0 = 2.0$	n	$\lvert f_6(x_1)\rvert$	$\lvert f_6(x_2)\rvert$	$\lvert f_6(x_3)\rvert$
PF	3	$0.125\,10^{-15}$	$0.336\,10^{-236}$	$0.354\,10^{-3324}$
JRP	3	$0.263\,10^{-18}$	$0.273\,10^{-313}$	0
FSH	3	$0.346\,10^{-18}$	$0.549\,10^{-311}$	$0.2\,10^{-3998}$
M1	3	$0.138\,10^{-18}$	$0.119\,10^{-317}$	0
M2	3	$0.156\,10^{-18}$	$0.165\,10^{-316}$	0

Table 8: Comparison table for $f_7(x)$, $f_8(x)$, and $f_9(x)$.

$f(x)$		NM	M1	M2
$f_7(x)$	n	10	3	3
$x_0 = 3.2$	$\lvert f_7(x_n)\rvert$	$0.118\,10^{-1013}$	$0.135\,10^{-3415}$	$0.243\,10^{-3666}$
	x^*	3.253180973	3.253180973	3.253180973
$f_7(x)$	n	13	4	4
$x_0 = 3.0$	$\lvert f_7(x_n)\rvert$	$0.126\,10^{-1768}$	$0.10\,10^{-3998}$	$0.10\,10^{-3998}$
	x^*	3.253180973	3.253180973	3.253180973
$f_7(x)$	n	—	—	—
$x_0 = 0.9$	$\lvert f_7(x_n)\rvert$	—	—	—
	x^*	D	D	D
$f_8(x)$	n	10	3	3
$x_0 = 2.8$	$\lvert f_8(x_n)\rvert$	$0.126\,10^{-1765}$	$0.115\,10^{-3996}$	$0.115\,10^{-3996}$
	x^*	2.796017462	2.796017462	2.796017462
$f_8(x)$	n	11	4	3
$x_0 = 2.4$	$\lvert f_8(x_n)\rvert$	$0.221\,10^{-1921}$	$0.726\,10^{-3997}$	$0.706\,10^{-1061}$
	x^*	2.481858196	2.481858196	2.387610416
$f_8(x)$	n	12	4	3
$x_0 = 14.0$	$\lvert f_8(x_n)\rvert$	$0.587\,10^{-1226}$	$0.594\,10^{-3994}$	$0.580\,10^{-1410}$
	x^*	14.01150324	14.01150324	14.01150324
$f_9(x)$	n	11	3	3
$x_0 = 2.18$	$\lvert f_9(x_n)\rvert$	$0.170\,10^{-1209}$	$0.175\,10^{-2003}$	$0.156\,10^{-2402}$
	x^*	2.188557091	2.188557091	2.188557091
$f_9(x)$	n	12	4	4
$x_0 = 2.15$	$\lvert f_9(x_n)\rvert$	$0.120\,10^{-1044}$	$0.10\,10^{-3998}$	$0.10\,10^{-3998}$
	x^*	2.188557091	2.188557091	2.188557091
$f_9(x)$	n	—	—	—
$x_0 = -1.5$	$\lvert f_9(x_n)\rvert$	—	—	—
	x^*	D	D	D

*D stands for divergence.

Table 9: Comparison of computational costs.

	Order	Addition/subtraction	Multiplication/division	Total
PF	14	27	21	48
JR	16	30	41	71
FS	16	69	66	135
M1	16	32	35	67
M2	16	31	34	65

are taken from [9] and $f_9(x)$ is taken from [2]. Table 8 shows the importance of accurate initial guesses for the convergence of Newton's method (NM) for these types of highly fluctuating functions. The results include the number of iterations n, the absolute value of each function at the nth iterate $|f(x_n)|$, and the root x^* to which the methods converge.

Table 9 shows the cost of executing each method for solving a nonlinear equation. The table clearly depicts that except that of the fourteenth-order convergent method given by Sargolzaei and Soleymani (PF) [5] all other methods of respective domain require more computational effort compared to our methods M1 and M2.

5. Newton's Method and Zeroes of Functions

The new sixteenth-order iterative method developed in this paper includes Newton's method as the first step. Although Newton's method is one of the most widely used methods, still it has many drawbacks; that is, proper initial guess plays a crucial role in the convergence of this method; an initial guess, which is not close enough to the root of the function, may lead to divergence as shown in Table 8. Moreover, another drawback is the involvement of derivative which may not exist at some points of the domain. To overcome these two main drawbacks of Newton's method, Moore et al. in 1966 ([10], Chapter 9) gave a method called interval Newton's method which can generate the safe initial guesses to ensure the convergence of Newton's method in vicinity of the root. However, interval Newton's method for handling nonlinear equations has a restriction that if the interval extension of initial guess $X^{(0)}$ contains a zero of the function $f'(x)$, then every kth iteration $X^{(k)}$ contains the zero of $f'(x)$ for all $k = 0, 1, 2, 3, \ldots$, which thus leads to failure of this method. Thus, $X^{(k)}$ forms a nested sequence converging to x only if $0 \notin F'(X^{(0)})$. To remove this restriction and to allow the range of values of the derivative $f'(x)$ to contain zero, Moore et al. ([10], Chapter 5) gave an extension of this method by splitting the quotient $f(x)/f'(X)$ occurring in interval Newton's method into two subintervals, where each subinterval though contains a zero of the function but excludes the zero of the derivative of $f(x)$. This method is known as extended interval Newton's method. We, herein, find the intervals enclosing all the zeroes of the function by using extended interval Newton's method defined in [10]. The endpoints of these subintervals are approximated up to 10 decimal places which may serve as initial guesses, good enough to show convergence for all the zeroes of oscillatory nonlinear functions.

By using Maple, we find the subintervals for $f_7(x)$, $f_8(x)$, and $f_9(x)$ defined above in Section 4. For $f_7(x)$, 69 subintervals are calculated as follows:

[14.91409469, 14.91907799] , [14.87618033, 14.87882406] , [14.70253913, 14.70832763] ,

[14.66422750, 14.66485977] , [14.48793933, 14.49176427] , [14.44786835, 14.44826560] ,

[14.26995241, 14.27656721] , [14.22604037, 14.23066461] , [14.04552680, 14.05497489] ,

[14.00282091, 14.00616288] , [13.81848674, 13.82996307] , [13.77545098, 13.77899594] ,

[13.59080017, 13.60161684] , [13.54513440, 13.54762734] , [13.36096809, 13.36958249] ,

[13.31111271, 13.31195699] , [13.12910597, 13.13311931] , [13.06857536, 13.07318658] ,

[12.88867541, 12.89225623] , [12.82502646, 12.82959150] , [12.64414769, 12.64629550] ,

[12.57794309, 12.58168834] , [12.39221606, 12.39604347] , [12.32427994, 12.33155620] ,

[12.13933578, 12.14152125] , [12.06589114, 12.06949432] , [11.87681567, 11.88836316] ,

[11.80257913, 11.80785022] , [11.61017387, 11.61417336] , [11.52580093, 11.53582338] ,

[11.33855318, 11.34565854] , [11.25371239, 11.25707075] , [11.05782638, 11.06375050] ,

[10.97106243, 10.97617210] , [10.77402365, 10.77570682] , [10.67229899, 10.68264897] ,

[10.48006360, 10.48312670] , [10.37979979, 10.38123836] , [10.17800639, 10.17943511] ,

[10.07090553, 10.07387957] , [9.866768415, 9.868206263] , [9.752713256, 9.754776613] ,

[9.542656144, 9.546467439] , [9.423111972, 9.425145592] , [9.209671918, 9.212909453] ,

[9.081311827, 9.085205858] , [8.866105341, 8.868806136] , [8.723516543, 8.727512993] ,
[8.507376755, 8.509833917] , [8.355789454, 8.356573732] , [8.133209509, 8.133391926] ,
[7.967541337, 7.968419812] , [7.740477001, 7.741246284] , [7.560115543, 7.560177140] ,
[7.327235639, 7.327826668] , [7.128324205, 7.128337654] , [6.889745558, 6.889772255] ,
[6.667942072, 6.668081225] , [6.422938778, 6.423097661] , [6.172706697, 6.173230855] ,
[5.917397368, 5.920842381] , [5.632674465, 5.632904614] , [5.372683988, 5.375017239] ,
[5.031577748, 5.033245710] , [4.765041785, 4.765502794] , [4.346558913, 4.346610460] ,
[4.073130785, 4.073647556] , [3.516479084, 3.516796449] , [3.253019201, 3.253316980] . (32)

Likewise, for the nonlinear function $f_8(x)$, the interval $[0, 10]$ is subdivided into 320 subintervals given as

[9.990184044, 10] , [9.955946724, 9.959695063] , [9.921726460, 9.927528356] ,
[9.895991269, 9.901813845] , [9.860932807, 9.867395738] , [9.833013483, 9.842299773] ,
[9.798356795, 9.802371500] , [9.764625542, 9.770416585] , [9.738902076, 9.744714230] ,
[9.703232575, 9.709925323] , [9.675902793, 9.685025935] , [9.641179727, 9.645245491] ,
[9.607542832, 9.613332772] , [9.581821638, 9.587633073] , [9.546128339, 9.552831077] ,
[9.518599577, 9.527156111] , [9.483395827, 9.487943109] , [9.450441211, 9.456225830] ,
[9.424726523, 9.430533040] , [9.388258511, 9.395312094] , [9.361797477, 9.362112523] ,
[9.327254150, 9.331188503] , [9.293391113, 9.299187867] , [9.267665764, 9.273484771] ,
[9.232219640, 9.238827421] , [9.204564556, 9.213401510] , [9.169580781, 9.173885974] ,
[9.136290459, 9.142079783] , [9.110573681, 9.116385739] , [9.074453010, 9.081345324] ,
[9.047440909, 9.056141780] , [9.012398012, 9.016770408] , [8.979207819, 8.984996872] ,
[8.953492707, 8.959304932] , [8.917330060, 8.924242464] , [8.890031492, 8.898238370] ,
[8.854549244, 8.859511211] , [8.822106558, 8.827894687] , [8.796393961, 8.802205670] ,
[8.759426649, 8.766743202] , [8.733620068, 8.737367181] , [8.700194918, 8.703778580] ,
[8.665140340, 8.670997704] , [8.639361807, 8.645242888] , [8.605076793, 8.611338461] ,
[8.576524492, 8.576819799] , [8.542723242, 8.546366963] , [8.508031342, 8.513867105] ,
[8.482275207, 8.488135298] , [8.447441739, 8.453855727] , [8.419442565, 8.419733454] ,
[8.385642791, 8.389273297] , [8.350952363, 8.356789480] , [8.325196573, 8.331058521] ,
[8.290472928, 8.296852876] , [8.262306529, 8.262599657] , [8.227975566, 8.231819589] ,
[8.193839383, 8.199655788] , [8.168105345, 8.173947253] , [8.132631531, 8.139267983] ,
[8.105301657, 8.108754352] , [8.071887350, 8.075485577] , [8.036821970, 8.042687844] ,
[8.011041930, 8.016933569] , [7.976738061, 7.983014238] , [7.948206312, 7.948506710] ,

[7.914414622, 7.918069963] , [7.879712050, 7.885556070] , [7.853954900, 7.859825409] ,
[7.819100301, 7.825530412] , [7.791127469, 7.791426462] , [7.757383380, 7.761016594] ,
[7.722636327, 7.728484673] , [7.696876988, 7.702752410] , [7.662184799, 7.668566032] ,
[7.633999479, 7.634293225] , [7.599741530, 7.603562482] , [7.565522523, 7.571349577] ,
[7.539785779, 7.545640712] , [7.504367753, 7.510993040] , [7.473598085, 7.477903415] ,
[7.445535699, 7.445741224] , [7.409526765, 7.415709324] , [7.376633149, 7.382748116] ,
[7.350628015, 7.356769948] , [7.316205621, 7.320727946] , [7.288433978, 7.288634901] ,
[7.252187667, 7.258500302] , [7.219382270, 7.225665666] , [7.193227785, 7.199537852] ,
[7.159100616, 7.163628382] , [7.131354615, 7.131554070] , [7.095214411, 7.101441457] ,
[7.062406422, 7.068587545] , [7.036345992, 7.042555094] , [7.001343483, 7.006374052] ,
[6.974208505, 6.974430846] , [6.937739087, 6.944121686] , [6.911502119, 6.917912067] ,
[6.878629934, 6.885058820] , [6.845557212, 6.849712217] , [6.817229629, 6.817372349] ,
[6.781328958, 6.787520738] , [6.748326672, 6.754431355] , [6.722341155, 6.728475853] ,
[6.688217195, 6.692543011] , [6.660135374, 6.660349224] , [6.624020054, 6.630322401] ,
[6.591095561, 6.597348936] , [6.564978649, 6.571261929] , [6.531350276, 6.535539635] ,
[6.503067732, 6.503242835] , [6.467150640, 6.473357071] , [6.434150065, 6.440272246] ,
[6.408154615, 6.414308468] , [6.373988633, 6.378364826] , [6.345971330, 6.346185408] ,
[6.314591592, 6.314801811] , [6.283130684, 6.283189482] , [6.251439225, 6.251792397] ,
[6.219732001, 6.223771957] , [6.188797959, 6.189152266] , [6.157480535, 6.157523164] ,
[6.120271452, 6.126143816] , [6.094516232, 6.100424165] , [6.062632401, 6.066666282] ,
[6.031713707, 6.032062120] , [6.000404602, 6.000443419] , [5.963159907, 5.969063625] ,
[5.937378306, 5.937652790] , [5.905762388, 5.910128802] , [5.874676316, 5.875173796] ,
[5.843267432, 5.843366441] , [5.811903897, 5.811997045] , [5.780439491, 5.780578109] ,
[5.748846678, 5.753534709] , [5.717642569, 5.718446131] , [5.686026171, 5.686297214] ,
[5.649589507, 5.654965745] , [5.623410417, 5.623560314] , [5.591995021, 5.592085462] ,
[5.560617933, 5.560647031] , [5.529050201, 5.529379409] , [5.494655569, 5.498548325] ,
[5.466333793, 5.466594000] , [5.434917226, 5.435007765] , [5.403538504, 5.403560842] ,
[5.371986243, 5.372310224] , [5.337627395, 5.341510491] , [5.309259066, 5.309710904] ,
[5.277842856, 5.277931374] , [5.246459480, 5.246465096] , [5.214949689, 5.215260016] ,
[5.180745879, 5.184567812] , [5.152175433, 5.152483750] , [5.120757973, 5.120850212] ,
[5.089379473, 5.089393976] , [5.057853016, 5.058169633] , [5.023599984, 5.027444216] ,
[4.995089639, 4.995134322] , [4.963669800, 4.963761292] , [4.931962692, 4.932342959] ,
[4.898577516, 4.902594687] , [4.869459724, 4.871816598] , [4.838049762, 4.838119448] ,
[4.806585813, 4.806656799] , [4.774807982, 4.775256312] , [4.740143422, 4.744730700] ,
[4.712320646, 4.712554905] , [4.680971224, 4.681013199] , [4.649505766, 4.649581859] ,

[4.617729602, 4.618176851] , [4.586717450, 4.586839669] , [4.555222666, 4.555463069] ,
[4.523886327, 4.524034761] , [4.492422625, 4.492489967] , [4.460687549, 4.461092742] ,
[4.429626928, 4.429896951] , [4.398061805, 4.398325942] , [4.366778548, 4.366815736] ,
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[0.1256635747, 0.1256707687] , [0.09424771546, 0.09424778634] ,
[0.06283185165, 0.06283185330] , [0.03141592644, 0.03141592674] , [0., 0.00003862611601] . (33)

Similarly, we find out that the function $f_9(x)$ has 51 zeroes in subintervals

[9.989560628, 9.992137465] , [9.800306886, 9.800592868] , [9.676233039, 9.681699494] ,
[9.479678599, 9.487508694] , [9.361916677, 9.366649388] , [9.167915476, 9.172598808] ,
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[7.596487817, 7.599819788] , [7.478101387, 7.482505409] , [7.280935917, 7.286251169] ,
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[5.708719363, 5.714342230] , [5.596962948, 5.601063643] , [5.394445916, 5.397139080] ,
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[2.864997815, 2.865471627] , [2.789473642, 2.791666167] , [2.543047934, 2.545369212] ,
[2.484628981, 2.484810607] , [2.212216544, 2.212720982] , [2.188542314, 2.188577325] . (34)

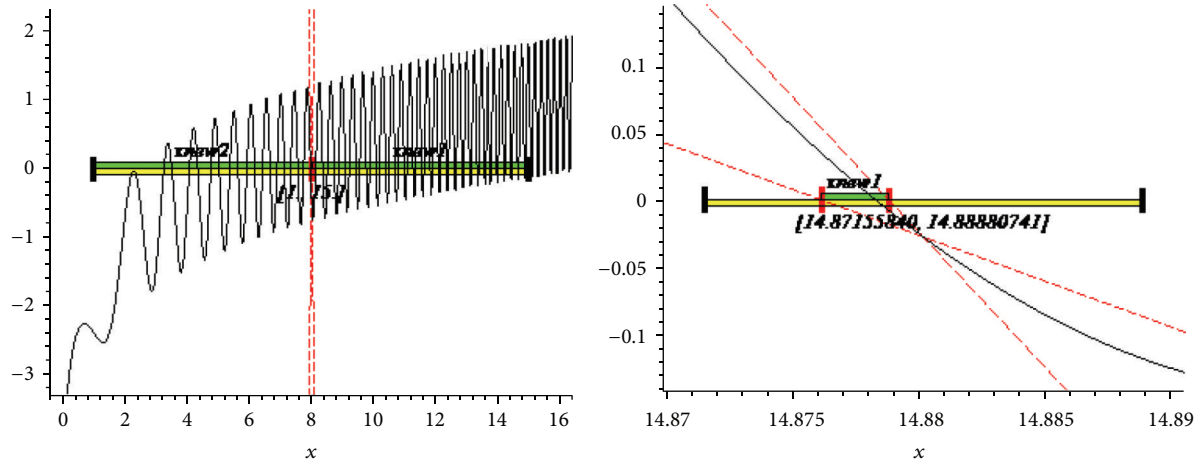

FIGURE 1: Graph of first and tenth iteration of extended interval Newton's method for $f_7(x)$.

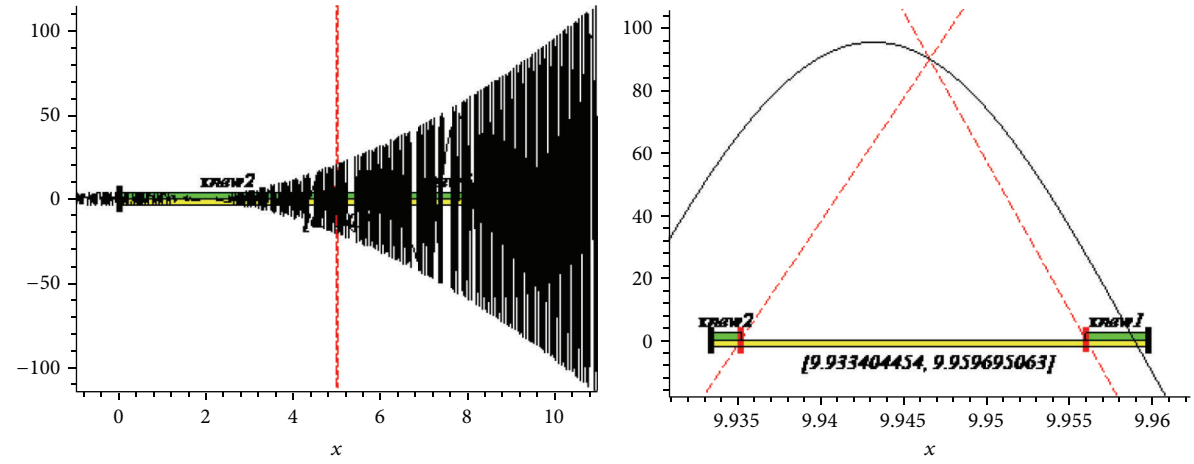

FIGURE 2: Graph of first and tenth iteration of extended interval Newton's method for $f_8(x)$.

The graphs of the first and tenth iteration of extended interval Newton's method for each function obtained by using Maple are shown in Figures 1, 2, and 3, representing the enclosure of exact zeroes at each interval.

6. Basins of Attraction

We consider complex polynomials $p_n(x)$, $n \geq 1$, $x \in \mathbb{C}$. To generate basins of attraction, we use two different techniques. In the first technique, we take a square box of $[-2, 2] \times [-2, 2] \in \mathbb{C}$. Now for every initial guess x_0, we assign a colour according to the root to which an iterative method converges. For divergence, we assign the colour dark blue. The stopping criteria for convergence are $|f(x_k)| < 10^{-5}$, and the maximum number of iterations is 30. In the second technique, we take the same scale, but we assign a colour for each initial guess depending upon the number of iterations in which the iterative method converges to any of the roots of the given function. The maximum number of iterations taken here is 25; stopping criterion is same as given earlier. If an iterative method does not converge in the maximum number of iterations, we consider the method as divergent for that initial guess and the method thus is represented by black colour.

To obtain basins of attraction, we take four test examples which are given as $p_3(x) = x^3 - 1$, $p_4(x) = x^4 - 10x^2 + 9$, $p_5(x) = x^5 - 1$, and $p_7(x) = x^7 - 1$. The roots of $p_3(x)$ are 1.0, $-0.5000 + 0.86605I$, $-0.5000 - 0.86605I$; roots of $p_4(x)$ are $-3, 3, -1, 1$, and for $p_5(x)$ roots are 1.0, $0.3090 + 0.95105I$, $-0.8090 + 0.58778I$, $-0.8090 - 0.58778I$, $0.30902 - 0.95105I$. And roots of $p_7(x)$ are 1.0, $0.6234 + 0.78183I$, $-0.2225 + 0.97492I$, $-0.9009 + 0.43388I$, $-0.9009 - 0.43388I$, $-0.2225 - 0.974927I$, $0.6234 - 0.781831I$.

We compare the results of our newly constructed method M1 with those of well-known sixteenth-order convergent methods PF [5], JR [1], and FS [2] (see Figures 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, and 19).

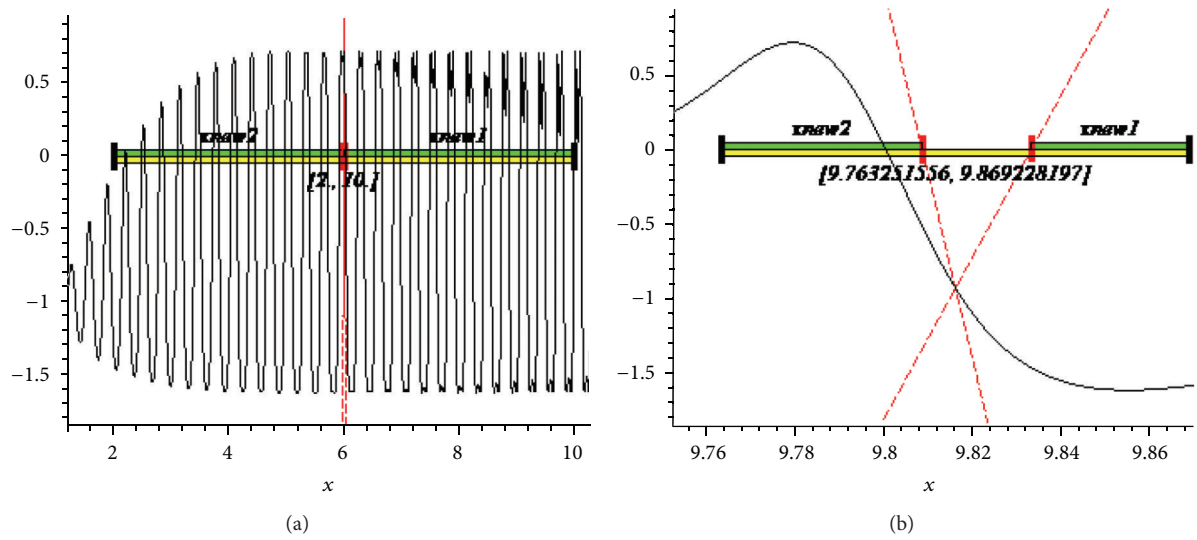

Figure 3: Graph of first and tenth iteration of extended interval Newton's method for $f_9(x)$.

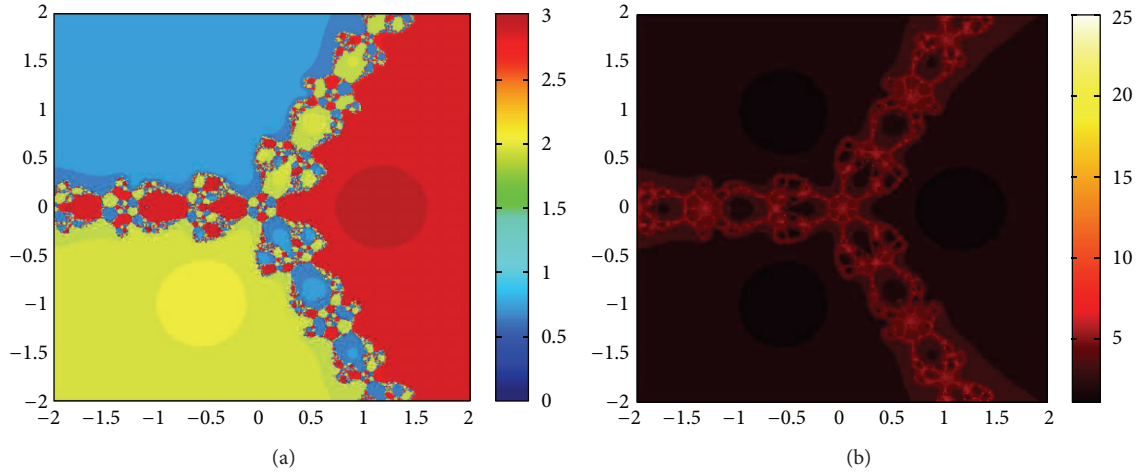

Figure 4: Basins of attraction of method (28) for $p_3(x)$.

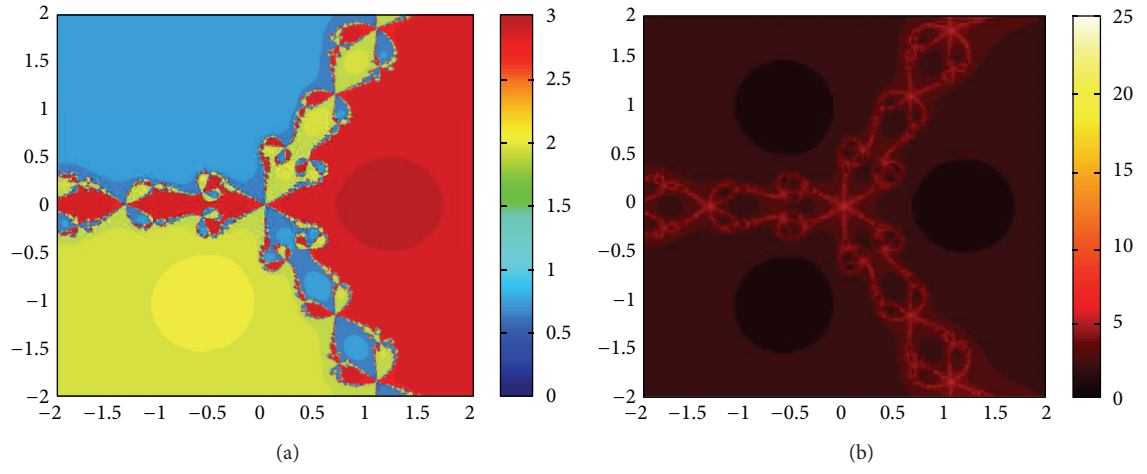

Figure 5: Basins of attraction of method PF for $p_3(x)$.

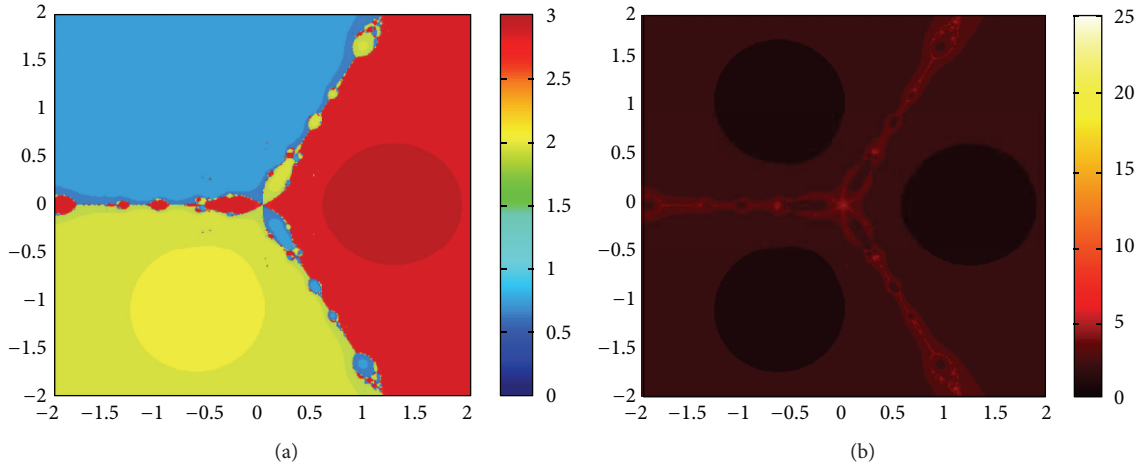

Figure 6: Basins of attraction of method JRP for $p_3(x)$.

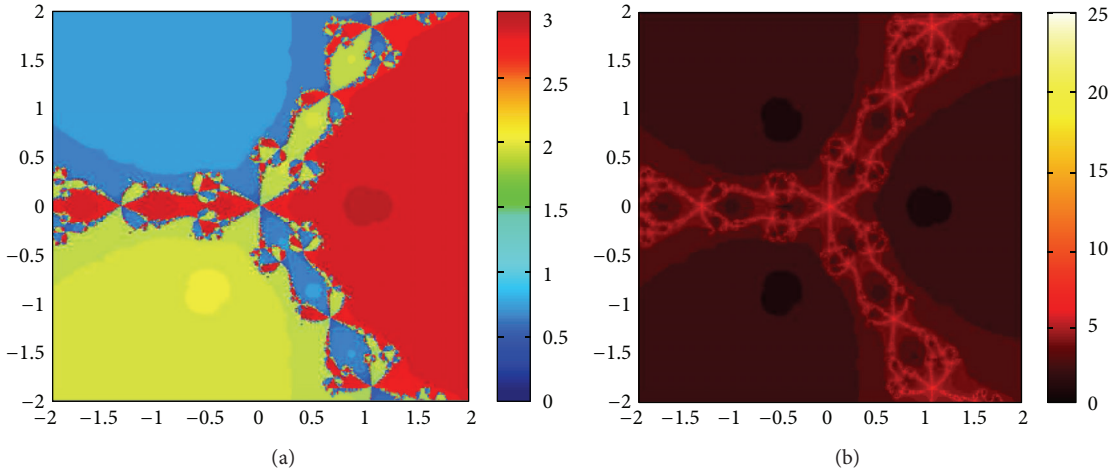

Figure 7: Basins of attraction of method FSH for $p_3(x)$.

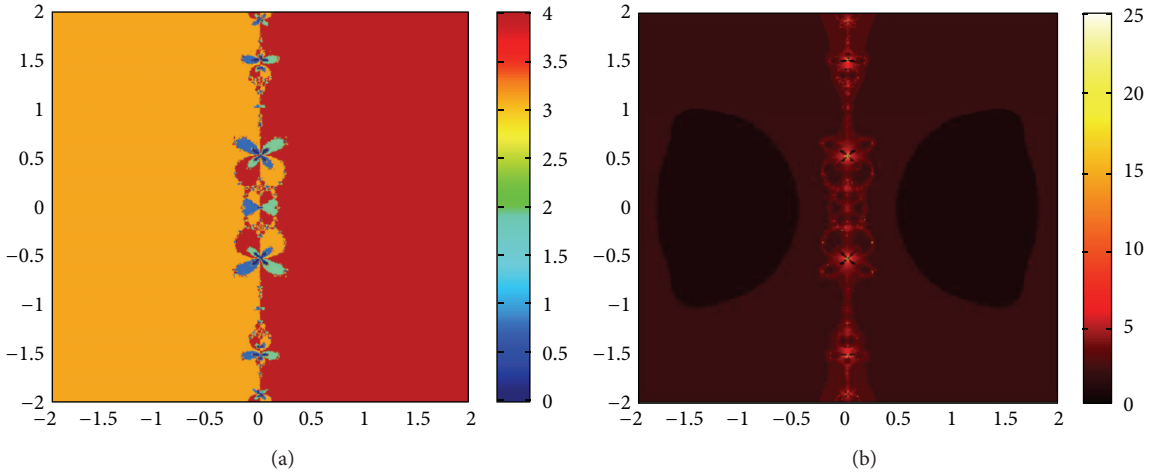

Figure 8: Basins of attraction of method (28) for $p_4(x)$.

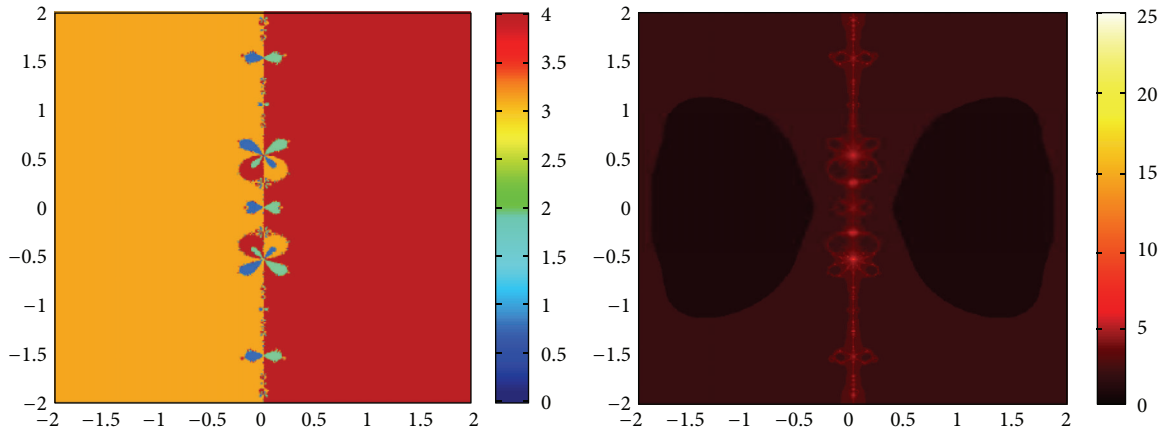

Figure 9: Basins of attraction of method PF for $p_4(x)$.

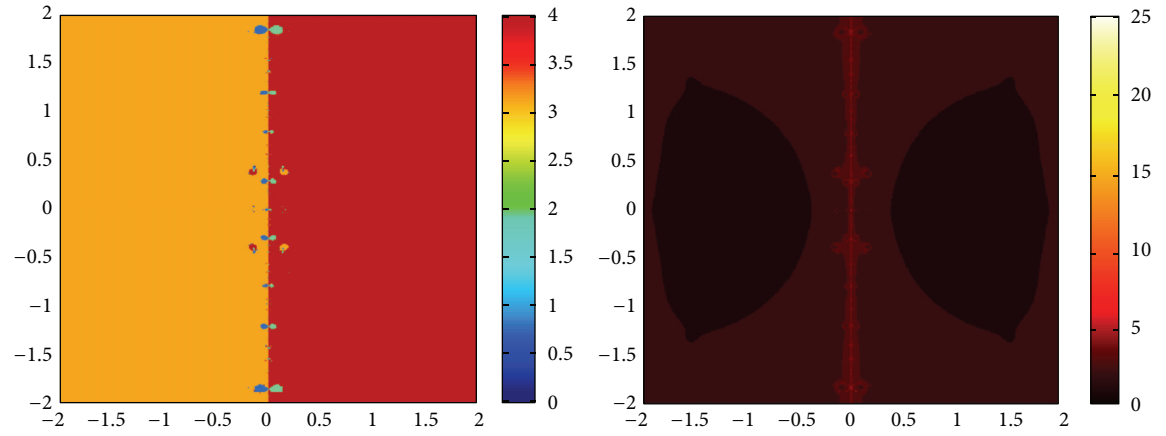

Figure 10: Basins of attraction of method JRP for $p_4(x)$.

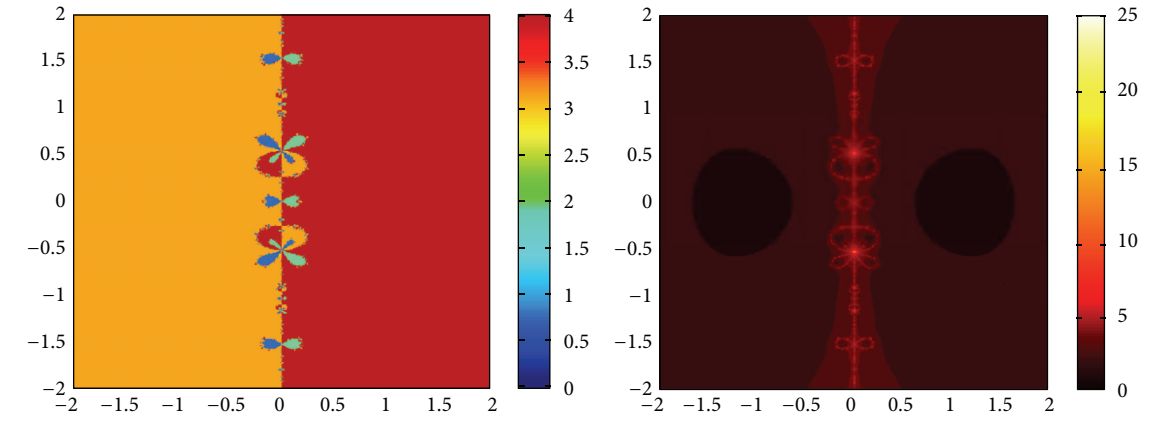

Figure 11: Basins of attraction of method FSH for $p_4(x)$.

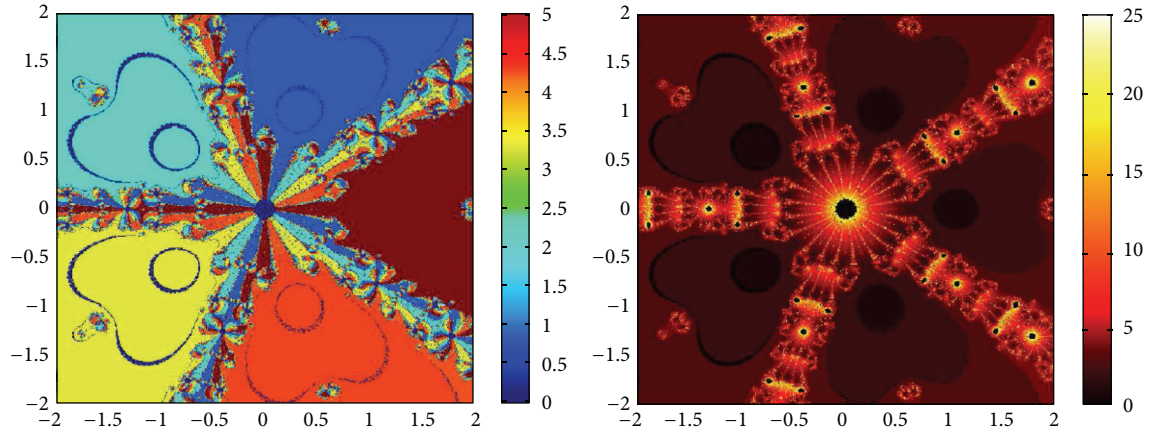

Figure 12: Basins of attraction of method (28) for $p_5(x)$.

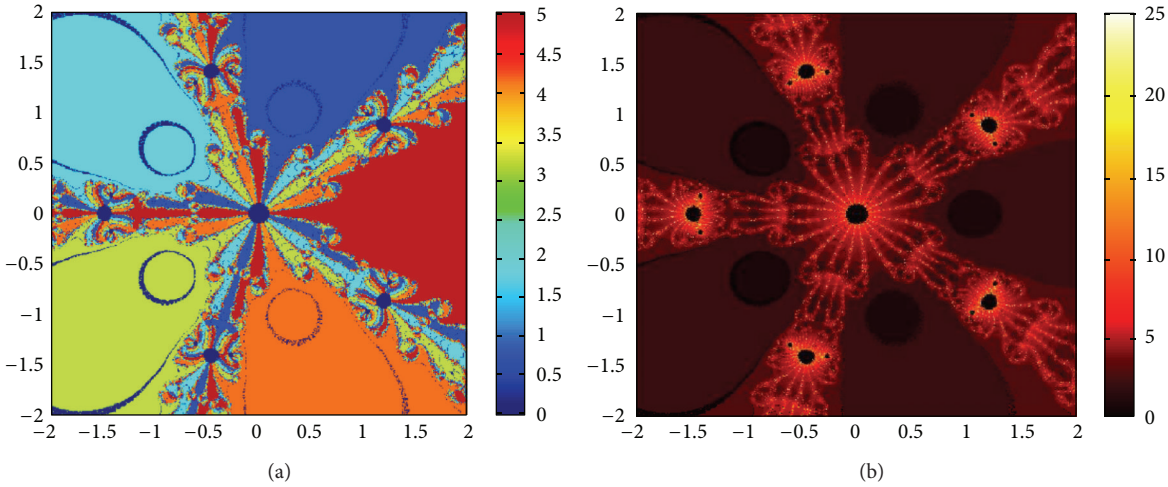

Figure 13: Basins of attraction of method PF for $p_5(x)$.

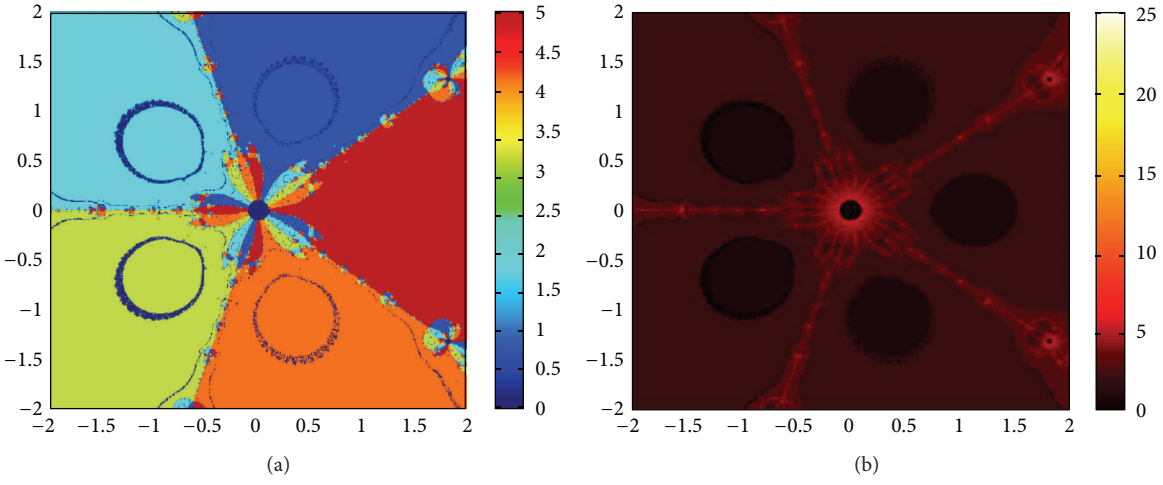

Figure 14: Basins of attraction of method JRP for $p_5(x)$.

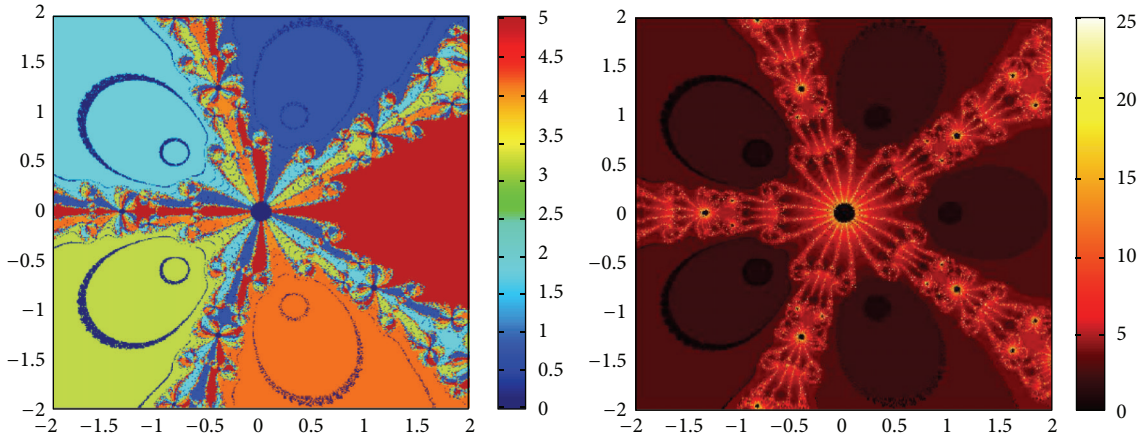

Figure 15: Basins of attraction of method FSH for $p_5(x)$.

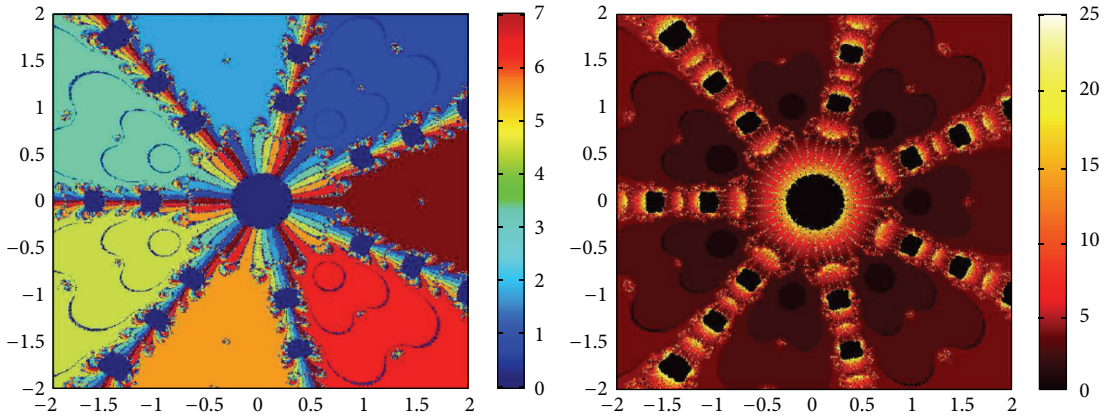

Figure 16: Basins of attraction of method (28) for $p_7(x)$.

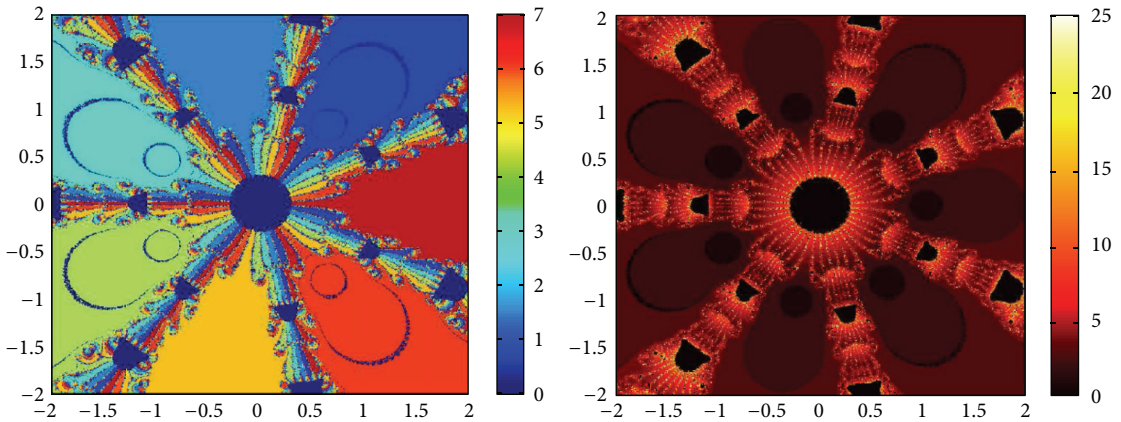

Figure 17: Basins of attraction of method PF for $p_7(x)$.

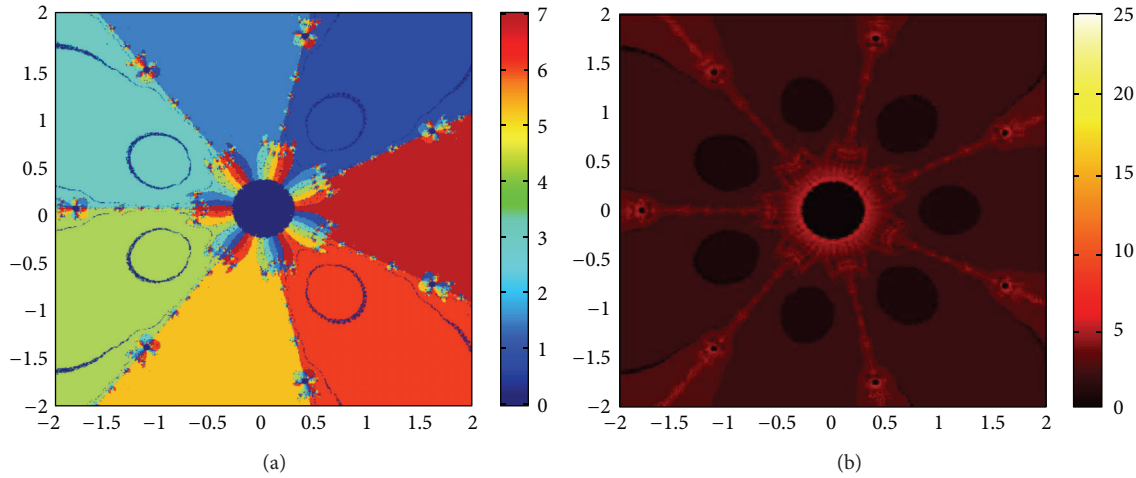

Figure 18: Basins of attraction of method JRP for $p_7(x)$.

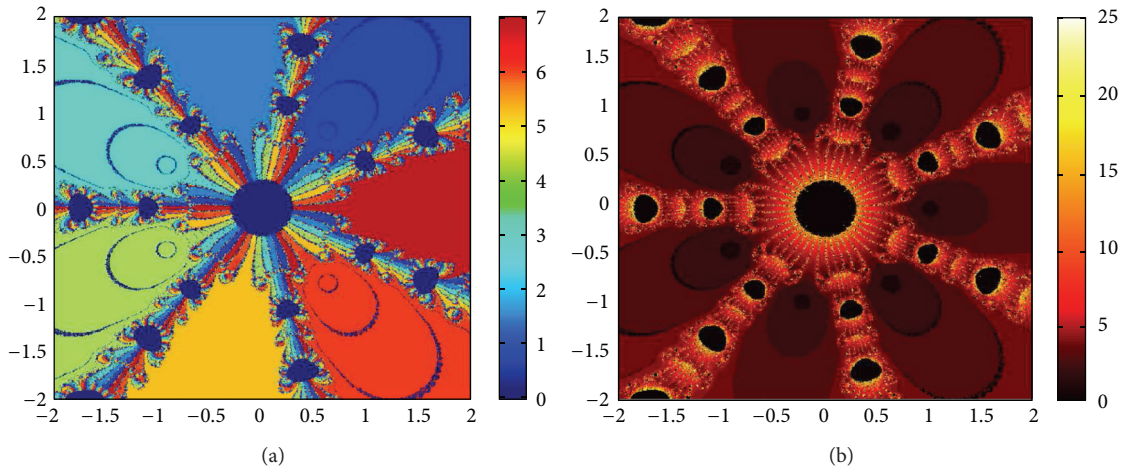

Figure 19: Basins of attraction of method FSH for $p_7(x)$.

7. Conclusions

A general four-step four-point iterative method without memory has been given for solving nonlinear equations. This iterative method has been obtained by approximating the first derivative of the function at the fourth step by using quasi-Hermite interpolation. An analytic proof for the order of convergence of this method was given which demonstrates that the method has an optimal order of sixteen. For this method the number of function evaluations is five per full step, so the efficiency index of the method is 1.741. Numerical comparisons in the form of Tables 2, 3, 4, 5, 6, and 7 with the methods based on rational interpolation, that is, methods with comparably more arithmetic cost as shown in Table 9, reveal the robust performance of this method compared to existing methods of this domain. Moreover, extended interval Newton's method is also introduced which is very effective in finding enough accurate initial guesses for solving nonlinear functions having finitely many zeroes in an interval. The basins of attraction show that our new method requires less number of iterations to converge to a root compared to the methods of [2, 5].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Applications of Normal S-Iterative Method to a Nonlinear Integral Equation

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It has been shown that a normal S-iterative method converges to the solution of a mixed type Volterra-Fredholm functional nonlinear integral equation. Furthermore, a data dependence result for the solution of this integral equation has been proven.

1. Introduction

The scientists working in almost every field of science are faced with nonlinear problems, because nature itself is intrinsically nonlinear. Such problems can be modelled as nonlinear mathematical equations. Solving nonlinear equations is, of course, considered to be a matter of the uttermost importance in mathematics and its manifold applications. There are numerous systematic approaches which are classified as direct and iterative methods to solve such equations in the existing literature. Indeed, by using direct methods, finding solutions to a complicated nonlinear equation can be an almost insurmountable challenge. In this context, iterative methods have become very important mathematical tools for finding solutions to a nonlinear equation. For a comprehensive review and references to the extensive literature on the iterative methods, the interested reader may refer to some recent works [1–8].

Recently, Sahu [9] and Khan [10], who was probably unaware of Sahu's work, introduced the following iterative process which has been called normal S-iterative method and Picard-Mann hybrid iterative process by Sahu and Khan, respectively, and hereinafter referred to as the "normal S-iterative method."

Definition 1. Let X be an ambient space and let T be a self-map of X. A normal S-iterative method is defined by

$$\begin{aligned} x_0 &\in X, \\ x_{n+1} &= Ty_n, \\ y_n &= (1-\xi_n)\, x_n + \xi_n T x_n, \quad n \in \mathbb{N}, \end{aligned} \tag{1}$$

where $\{\xi_n\}_{n=0}^{\infty}$ is a real sequence in $[0,1]$ satisfying certain control condition(s).

It has been shown both analytically and numerically in [9, 10] that iterative method (1) converges at a rate faster than all Picard [11], Mann [12], and Ishikawa [13] iterative processes in the sense of Berinde [14] for the class of contraction mappings.

This iterative method, due to its simplicity and fastness, has attracted the attention of many researchers and has been examined in various aspects; see [15–20].

In this paper, inspired by the performance and achievements of normal S-iterative method (1), we will give some of its applications. We will show that normal S-iterative method (1) converges strongly to the solution of the following mixed type Volterra-Fredholm functional nonlinear integral equation which was considered in [21]:

$$\begin{aligned} x(t) = F\Bigg(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t,s,x(s))\, ds, \\ \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t,s,x(s))\, ds\Bigg), \end{aligned} \tag{2}$$

where $[a_1;b_1]\times\cdots\times[a_m;b_m]$ is an interval in $\mathbb{R}^m$, K, H : $[a_1;b_1]\times\cdots\times[a_m;b_m]\times[a_1;b_1]\times\cdots\times[a_m;b_m]\times\mathbb{R}\to\mathbb{R}$ continuous functions, and $F:[a_1;b_1]\times\cdots\times[a_m;b_m]\times\mathbb{R}^3\to\mathbb{R}$.

Also we give a data dependence result for the solution of integral equation (2) with the help of normal S-iterative method (1).

We end this section with some known results which will be useful in proving our main results.

Theorem 2 (see [21]). *We suppose that the following conditions are satisfied:*

(A_1) K, $H\in C([a_1;b_1]\times\cdots\times[a_m;b_m]\times[a_1;b_1]\times\cdots\times[a_m;b_m]\times\mathbb{R})$;

(A_2) $F\in C([a_1;b_1]\times\cdots\times[a_m;b_m]\times\mathbb{R}^3)$;

(A_3) *there exist nonnegative constants* α, β, *and* γ *such that*

$$\begin{aligned}&\left|F(t,u_1,v_1,w_1)-F(t,u_2,v_2,w_2)\right|\\&\quad\le\alpha\left|u_1-u_2\right|+\beta\left|v_1-v_2\right|+\gamma\left|w_1-w_2\right|,\end{aligned}\tag{3}$$

for all $t\in[a_1;b_1]\times\cdots\times[a_m;b_m]$, u_i, v_i, $w_i\in\mathbb{R}$, $i=1,2$;

(A_4) *there exist nonnegative constants* L_K *and* L_H *such that*

$$\begin{aligned}&\left|K(t,s,u)-K(t,s,v)\right|\le L_K\left|u-v\right|,\\&\left|H(t,s,u)-H(t,s,v)\right|\le L_H\left|u-v\right|,\end{aligned}\tag{4}$$

for all t, $s\in[a_1;b_1]\times\cdots\times[a_m;b_m]$, u, $v\in\mathbb{R}$;

(A_5) $\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)<1$.

Then (2) *has a unique solution* $x^*\in C([a_1;b_1]\times\cdots\times[a_m;b_m])$.

Lemma 3 (see [22]). *Let* $\{\beta_n\}_{n=0}^{\infty}$ *be a nonnegative sequence for which one assumes there exists* $n_0\in\mathbb{N}$, *such that for all* $n\ge n_0$ *one has satisfied the inequality*

$$\beta_{n+1}\le(1-\mu_n)\beta_n+\mu_n\gamma_n,\tag{5}$$

where $\mu_n\in(0,1)$, *for all* $n\in\mathbb{N}$, $\sum_{n=0}^{\infty}\mu_n=\infty$, *and* $\gamma_n\ge0$, *for all* $n\in\mathbb{N}$. *Then the following inequality holds:*

$$0\le\limsup_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\gamma_n.\tag{6}$$

2. Main Results

Theorem 4. *One opines that all conditions* (A_1)–(A_5) *in Theorem 2 are performed. Let* $\{\xi_n\}_{n=0}^{\infty}\subset[0,1]$ *be a real sequence satisfying* $\sum_{n=0}^{\infty}\xi_n=\infty$. *Then* (2) *has a unique solution, say* x^*, *in* $C([a_1;b_1]\times\cdots\times[a_m;b_m])$ *and normal S-iterative method* (1) *converges to* x^*.

Proof. We consider the Banach space $B=C([a_1;b_1]\times\cdots\times[a_m;b_m],\|\cdot\|_C)$, where $\|\cdot\|_C$ is Chebyshev's norm. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by normal S-iterative method (1) for the operator $A:B\to B$ defined by

$$\begin{aligned}A(x)(t)=F\Bigg(&t,x(t),\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}K(t,s,x(s))\,ds,\\&\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}H(t,s,x(s))\,ds\Bigg).\end{aligned}\tag{7}$$

We will show that $x_n\to x^*$ as $n\to\infty$.

From (1), (2), and assumptions (A_1)–(A_4), we have that

$$\begin{aligned}&\left\|x_{n+1}-x^*\right\|\\&=\left\|Ay_n-x^*\right\|=\left|A(y_n)(t)-A(x^*)(t)\right|\\&=\Bigg|F\Bigg(t,y_n(t),\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}K(t,s,y_n(s))\,ds,\\&\qquad\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}H(t,s,y_n(s))\,ds\Bigg)\\&\quad-F\Bigg(t,x^*(t),\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}K(t,s,x^*(s))\,ds,\\&\qquad\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}H(t,s,x^*(s))\,ds\Bigg)\Bigg|\\&\le\alpha\left|y_n(t)-x^*(t)\right|\\&\quad+\beta\left|\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}K(t,s,y_n(s))\,ds\right.\\&\qquad\left.-\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}K(t,s,x^*(s))\,ds\right|\\&\quad+\gamma\left|\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}H(t,s,y_n(s))\,ds\right.\\&\qquad\left.-\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}H(t,s,x^*(s))\,ds\right|\\&\le\alpha\left|y_n(t)-x^*(t)\right|\\&\quad+\beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\left|K(t,s,y_n(s))-K(t,s,x^*(s))\right|ds\\&\quad+\gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left|H(t,s,y_n(s))-H(t,s,x^*(s))\right|ds\end{aligned}$$

$$\begin{aligned}
&\le \alpha \left|y_n(t) - x^*(t)\right| + \beta \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} L_K \left|y_n(s) - x^*(s)\right| ds \\
&\quad + \gamma \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} L_H \left|y_n(s) - x^*(s)\right| ds \\
&\le \left[\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (b_i - a_i)\right] \left\|y_n - x^*\right\|,
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\left\|y_n - x^*\right\| \\
&\le (1 - \xi_n) \left|x_n(t) - x^*(t)\right| \\
&\quad + \xi_n \left|A(x_n)(t) - A(x^*)(t)\right| \\
&= (1 - \xi_n) \left|x_n(t) - x^*(t)\right| \\
&\quad + \xi_n \left| F\left(t, x_n(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t, s, x_n(s))\, ds, \right.\right. \\
&\qquad \left. \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t, s, x_n(s))\, ds \right) \\
&\qquad - F\left(t, x^*(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t, s, x^*(s))\, ds, \right. \\
&\qquad \left.\left. \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t, s, x^*(s))\, ds \right) \right| \\
&\le (1 - \xi_n) \left|x_n(t) - x^*(t)\right| + \xi_n \alpha \left|x_n(t) - x^*(t)\right| \\
&\quad + \xi_n \beta \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} L_K \left|x_n(s) - x^*(s)\right| ds \\
&\quad + \xi_n \gamma \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} L_H \left|x_n(s) - x^*(s)\right| ds \\
&\le \left\{1 - \xi_n \left(1 - \left[\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (b_i - a_i)\right]\right)\right\} \\
&\quad \times \left\|x_n - x^*\right\|.
\end{aligned} \tag{9}$$

Combining (8) with (9), we obtain

$$\begin{aligned}
&\left\|x_{n+1} - x^*\right\| \\
&\le \left[\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (b_i - a_i)\right] \\
&\quad \times \left\{1 - \xi_n \left(1 - \left[\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (b_i - a_i)\right]\right)\right\} \\
&\quad \times \left\|x_n - x^*\right\|,
\end{aligned} \tag{10}$$

or, from assumption (A_5),

$$\begin{aligned}
&\left\|x_{n+1} - x^*\right\| \\
&\le \left\{1 - \xi_n \left(1 - \left[\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (b_i - a_i)\right]\right)\right\} \\
&\quad \times \left\|x_n - x^*\right\|.
\end{aligned} \tag{11}$$

Thus, by induction, we get

$$\begin{aligned}
\left\|x_{n+1} - x\right\| &\le \left\|x_0 - x^*\right\| \\
&\quad \times \prod_{k=0}^{n} \left\{1 - \xi_k \left(1 - \left[\alpha + (\beta L_K + \gamma L_H) \right.\right.\right. \\
&\qquad \left.\left.\left. \times \prod_{i=1}^{m} (b_i - a_i)\right]\right)\right\}.
\end{aligned} \tag{12}$$

Since $\xi_k \in [0, 1]$ for all $k \in \mathbb{N}$, assumption (A_5) yields

$$1 - \xi_k \left(1 - \left[\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (b_i - a_i)\right]\right) < 1. \tag{13}$$

Having regard to the fact that $e^x \ge 1 - x$ for all $x \in [0, 1]$, we can rewrite (12) as

$$\left\|x_{n+1} - x^*\right\| \le \left\|x_0 - x^*\right\| e^{-(1 - [\alpha + (\beta L_K + \gamma L_H) \prod_{i=1}^{m} (b_i - a_i)]) \sum_{k=0}^{n} \xi_k}, \tag{14}$$

which yields $\lim_{n \to \infty} \|x_n - x^*\| = 0$. □

We now prove the data dependence of the solution for integral equation (2) with the help of the normal S-iterative method (1).

Let B be as in the proof of Theorem 4 and $T, \widetilde{T} : B \to B$ two operators defined by

$$\begin{aligned}
T(x)(t) = F\left(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t, s, x(s))\, ds, \right. \\
\left. \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t, s, x(s))\, ds \right),
\end{aligned} \tag{15}$$

$$\begin{aligned}
\widetilde{T}(x)(t) = F\left(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} \widetilde{K}(t, s, x(s))\, ds, \right. \\
\left. \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \widetilde{H}(t, s, x(s))\, ds \right),
\end{aligned} \tag{16}$$

where $K, \widetilde{K}, H, \widetilde{H} \in C([a_1; b_1] \times \cdots \times [a_m; b_m] \times [a_1; b_1] \times \cdots \times [a_m; b_m] \times \mathbb{R})$.

Theorem 5. *Let F, K, and H be defined as in Theorem 2 and let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence defined by normal*

S-iterative method (1) *associated with* T. *Let* $\{\widetilde{x}_n\}_{n=0}^{\infty}$ *be an iterative sequence generated by*

$$\begin{aligned} \widetilde{x}_0 &\in B, \\ \widetilde{x}_{n+1} &= \widetilde{T}\widetilde{y}_n, \\ \widetilde{y}_n &= (1-\xi_n)\widetilde{x}_n + \xi_n \widetilde{T}\widetilde{x}_n, \quad n \in \mathbb{N}, \end{aligned} \tag{17}$$

where B *is defined as in the proof of Theorem 4 and* $\{\xi_n\}_{n=0}^{\infty}$ *is a real sequence in* $[0,1]$ *satisfying (i)* $1/2 \le \xi_n$, *for all* $n \in \mathbb{N}$, *and (ii)* $\sum_{n=0}^{\infty} \xi_n = \infty$. *One supposes further that (iii) there exist nonnegative constants* ε_1 *and* ε_2 *such that* $|K(t,s,u) - \widetilde{K}(t,s,u)| \le \varepsilon_1$ *and* $|H(t,s,u) - \widetilde{H}(t,s,u)| \le \varepsilon_2$, *for all* $u \in \mathbb{R}$ *and for all* $t, s \in [a_1; b_1] \times \cdots \times [a_m; b_m]$.

If x^* *and* $\widetilde{x}^*$ *are solutions of corresponding equations* (15) *and* (16), *respectively, then one has that*

$$\left\|x^* - \widetilde{x}^*\right\| \le \frac{3(\beta\varepsilon_1 + \gamma\varepsilon_2)\prod_{i=1}^{m}(b_i - a_i)}{1 - [\alpha + (\beta L_K + \gamma L_H)\prod_{i=1}^{m}(b_i - a_i)]}. \tag{18}$$

Proof. Using (1), (15), (16), (17), and assumptions (A_1)–(A_4) and (iii), we obtain

$$\begin{aligned} &\left\|x_{n+1} - \widetilde{x}_{n+1}\right\| \\ &= \left\|Ty_n - \widetilde{T}\widetilde{y}_n\right\| \\ &= \left| F\left(t, y_n(t), \int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K(t,s,y_n(s))\,ds, \right.\right. \\ &\qquad \left.\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H(t,s,y_n(s))\,ds\right) \\ &\quad - F\left(t, \widetilde{y}_n(t), \int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} \widetilde{K}(t,s,\widetilde{y}_n(s))\,ds, \right. \\ &\qquad \left.\left.\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} \widetilde{H}(t,s,\widetilde{y}_n(s))\,ds\right)\right| \\ &\le \alpha\left|y_n(t) - \widetilde{y}_n(t)\right| \\ &\quad + \beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\left|K(t,s,y_n(s)) - \widetilde{K}(t,s,\widetilde{y}_n(s))\right| ds \\ &\quad + \gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left|H(t,s,y_n(s)) - \widetilde{H}(t,s,\widetilde{y}_n(s))\right| ds \\ &\le \alpha\left|y_n(t) - \widetilde{y}_n(t)\right| \\ &\quad + \beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\left(\left|K(t,s,y_n(s)) - K(t,s,\widetilde{y}_n(s))\right|\right. \\ &\qquad \left. + \left|K(t,s,\widetilde{y}_n(s)) - \widetilde{K}(t,s,\widetilde{y}_n(s))\right|\right) ds \\ &\quad + \gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left(\left|H(t,s,y_n(s)) - H(t,s,\widetilde{y}_n(s))\right|\right. \\ &\qquad \left. + \left|H(t,s,\widetilde{y}_n(s)) - \widetilde{H}(t,s,\widetilde{y}_n(s))\right|\right) ds \\ &\le \alpha\left|y_n(t) - \widetilde{y}_n(t)\right| \\ &\quad + \beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\left(L_K\left|y_n(s) - \widetilde{y}_n(s)\right| + \varepsilon_1\right) ds \\ &\quad + \gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left(L_H\left|y_n(s) - \widetilde{y}_n(s)\right| + \varepsilon_2\right) ds \\ &\le \alpha\left\|y_n - \widetilde{y}_n\right\| + \beta\left(L_K\left\|y_n - \widetilde{y}_n\right\| + \varepsilon_1\right)\prod_{i=1}^{m}(b_i - a_i) \\ &\quad + \gamma\left(L_H\left\|y_n - \widetilde{y}_n\right\| + \varepsilon_2\right)\prod_{i=1}^{m}(b_i - a_i) \\ &\le \left[\alpha + (\beta L_K + \gamma L_H)\prod_{i=1}^{m}(b_i - a_i)\right]\left\|y_n - \widetilde{y}_n\right\| \\ &\quad + (\beta\varepsilon_1 + \gamma\varepsilon_2)\prod_{i=1}^{m}(b_i - a_i), \end{aligned} \tag{19}$$

$$\begin{aligned} &\left\|y_n - \widetilde{y}_n\right\| \\ &\le (1-\xi_n)\left|x_n(t) - \widetilde{x}_n(t)\right| \\ &\quad + \xi_n\left|T(x_n)(t) - \widetilde{T}(\widetilde{x}_n)(t)\right| \\ &\le (1-\xi_n)\left|x_n(t) - \widetilde{x}_n(t)\right| \\ &\quad + \xi_n\left\{\alpha\left|x_n(t) - \widetilde{x}_n(t)\right|\right. \\ &\qquad + \beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\left\{L_K\left|x_n(s) - \widetilde{x}_n(s)\right| + \varepsilon_1\right\} ds \\ &\qquad \left. + \gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left\{L_H\left|x_n(s) - \widetilde{x}_n(s)\right| + \varepsilon_2\right\} ds\right\} \\ &\le \left\{1 - \xi_n\left(1 - \left[\alpha + (\beta L_K + \gamma L_H)\prod_{i=1}^{m}(b_i - a_i)\right]\right)\right\} \\ &\quad \times \left\|x_n - \widetilde{x}_n\right\| + \xi_n(\beta\varepsilon_1 + \gamma\varepsilon_2)\prod_{i=1}^{m}(b_i - a_i). \end{aligned} \tag{20}$$

Combining (19) with (20) and using assumptions (A_5) and $1/2 \le \xi_n$ in the resulting inequality, we get

$$\begin{aligned} &\left\|x_{n+1} - \widetilde{x}_{n+1}\right\| \\ &\le \left\{1 - \xi_n\left(1 - \left[\alpha + (\beta L_K + \gamma L_H)\prod_{i=1}^{m}(b_i - a_i)\right]\right)\right\} \\ &\quad \times \left\|x_n - \widetilde{x}_n\right\| \end{aligned}$$

$$+\xi_n\left(1-\left[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)\right]\right)\times\frac{3(\beta\varepsilon_1+\gamma\varepsilon_2)\prod_{i=1}^{m}(b_i-a_i)}{1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)]}. \quad (21)$$

Denote that

$$\beta_n=\|x_n-\widetilde{x}_n\|,$$
$$\mu_n=\xi_n\left(1-\left[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)\right]\right)\in(0,1),$$
$$\gamma_n=\frac{3(\beta\varepsilon_1+\gamma\varepsilon_2)\prod_{i=1}^{m}(b_i-a_i)}{1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)]}\geq 0. \quad (22)$$

It is clear that inequality (21) satisfies all conditions in Lemma 3, and hence it follows that

$$\|x^*-\widetilde{x}^*\|\leq\frac{3(\beta\varepsilon_1+\gamma\varepsilon_2)\prod_{i=1}^{m}(b_i-a_i)}{1-[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)]}. \quad (23)$$

□

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Convergence Analysis for a Modified SP Iterative Method

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We consider a new iterative method due to Kadioglu and Yildirim (2014) for further investigation. We study convergence analysis of this iterative method when applied to class of contraction mappings. Furthermore, we give a data dependence result for fixed point of contraction mappings with the help of the new iteration method.

1. Introduction

Recent progress in nonlinear science reveals that iterative methods are most powerful tools which are used to approximate solutions of nonlinear problems whose solutions are inaccessible analytically. Therefore, in recent years, an intensive interest has been devoted to developing faster and more effective iterative methods for solving nonlinear problems arising from diverse branches in science and engineering.

Very recently the following iterative methods are introduced in [1] and [2], respectively:

$$\begin{aligned} x_0 &\in D, \\ x_{n+1} &= Ty_n, \\ y_n &= (1-\alpha_n)\, z_n + \alpha_n T z_n, \\ z_n &= (1-\beta_n)\, x_n + \beta_n T x_n, \quad n \in \mathbb{N}, \end{aligned} \tag{1}$$

$$\begin{aligned} u_0 &\in D, \\ u_{n+1} &= Tv_n, \\ v_n &= (1-\alpha_n)\, Tu_n + \alpha_n T w_n, \\ w_n &= (1-\beta_n)\, u_n + \beta_n T u_n, \quad n \in \mathbb{N}, \end{aligned} \tag{2}$$

where D is a nonempty convex subset of a Banach space B, T is a self map of D, and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$.

While the iterative method (1) fails to be named in [1], the iterative method (2) is called Picard-S iteration method in [2]. Since iterative method (1) is a special case of SP iterative method of Phuengrattana and Suantai [3], we will call it here Modified SP iterative method.

It was shown in [1] that Modified SP iterative method (1) is faster than all Picard [4], Mann [5], Ishikawa [6], and S [7] iterative methods in the sense of Definitions 1 and 2 given below for the class of contraction mappings satisfying

$$\|Tx - Ty\| \leq \delta \|x - y\|, \quad \delta \in (0, 1), \ \forall x, y \in B. \tag{3}$$

Using the same class of contraction mappings (3), Gürsoy and Karakaya [2] showed that Picard-S iteration method (2) is also faster than all Picard [4], Mann [5], Ishikawa [6], S [7], and some other iterative methods in the existing literature.

In this paper, we show that Modified SP iterative method converges to the fixed point of contraction mappings (3). Also, we establish an equivalence between convergence of iterative methods (1) and (2). For the sake of completness, we give a comparison result between the rate of convergences of iterative methods (1) and (2), and it thus will be shown that Picard-S iteration method is still the fastest method. Finally, a data dependence result for the fixed point of the contraction mappings (3) is proven.

The following definitions and lemmas will be needed in order to obtain the main results of this paper.

Definition 1 (see [8]). Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers with limits a and b, respectively. Suppose that

$$\lim_{n\to\infty}\frac{|a_n-a|}{|b_n-b|}=l \tag{4}$$

exists.

(i) If $l=0$, then we say that $\{a_n\}_{n=0}^{\infty}$ converges faster to a than $\{b_n\}_{n=0}^{\infty}$ to b.

(ii) If $0<l<\infty$, then we say that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Definition 2 (see [8]). Assume that for two fixed point iteration processes $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ both converging to the same fixed point p, the following error estimates,

$$\begin{aligned}\|u_n-p\|&\le a_n\quad\forall n\in\mathbb{N},\\ \|v_n-p\|&\le b_n\quad\forall n\in\mathbb{N},\end{aligned} \tag{5}$$

are available where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero). If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ to p.

Definition 3 (see [9]). Let $T,\widetilde{T}:B\to B$ be two operators. We say that $\widetilde{T}$ is an approximate operator of T if for all $x\in B$ and for a fixed $\varepsilon>0$ we have

$$\|Tx-\widetilde{T}x\|\le\varepsilon. \tag{6}$$

Lemma 4 (see [10]). *Let $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\rho_n\}_{n=0}^{\infty}$ be nonnegative real sequences and suppose that for all $n\ge n_0$, $\tau_n\in(0,1)$, $\sum_{n=1}^{\infty}\tau_n=\infty$, and $\rho_n/\tau_n\to 0$ as $n\to\infty$*

$$\sigma_{n+1}\le(1-\tau_n)\sigma_n+\rho_n \tag{7}$$

holds. Then $\lim_{n\to\infty}\sigma_n=0$.

Lemma 5 (see [11]). *Let $\{\sigma_n\}_{n=0}^{\infty}$ be a nonnegative sequence such that there exists $n_0\in\mathbb{N}$, for all $n\ge n_0$; the following inequality holds. Consider*

$$\sigma_{n+1}\le(1-\tau_n)\sigma_n+\tau_n\mu_n, \tag{8}$$

where $\tau_n\in(0,1)$, for all $n\in\mathbb{N}$, $\sum_{n=0}^{\infty}\tau_n=\infty$ and $\eta_n\ge 0$, $\forall n\in\mathbb{N}$. Then

$$0\le\limsup_{n\to\infty}\sigma_n\le\limsup_{n\to\infty}\mu_n. \tag{9}$$

2. Main Results

Theorem 6. *Let D be a nonempty closed convex subset of a Banach space B and $T:D\to D$ a contraction map satisfying condition* (3). *Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by* (1) *with real sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying $\sum_{k=0}^{\infty}\alpha_k=\infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to a unique fixed point of T, say x_*.*

Proof. The well-known Picard-Banach theorem guarantees the existence and uniqueness of x_*. We will show that $x_n\to x_*$ as $n\to\infty$. From (3) and (1) we have

$$\begin{aligned}\|x_{n+1}-x_*\|&=\|Ty_n-Tx_*\|\\ &\le\delta\|y_n-x_*\|\\ &\le\delta\{(1-\alpha_n)\|z_n-x_*\|+\alpha_n\delta\|z_n-x_*\|\}\\ &\le\delta[1-\alpha_n(1-\delta)]\|z_n-x_*\|\\ &\le\delta[1-\alpha_n(1-\delta)]\\ &\quad\times\{(1-\beta_n)\|x_n-x_*\|+\beta_n\delta\|x_n-x_*\|\}\\ &\le\delta[1-\alpha_n(1-\delta)][1-\beta_n(1-\delta)]\|x_n-x_*\|\\ &\le\delta[1-\alpha_n(1-\delta)]\|x_n-x_*\|.\end{aligned} \tag{10}$$

By induction on the inequality (10), we derive

$$\begin{aligned}\|x_{n+1}-x_*\|&\le\|x_0-x_*\|\delta^{n+1}\prod_{k=0}^{n}[1-\alpha_k(1-\delta)]\\ &\le\|x_0-x_*\|\delta^{n+1}e^{-(1-\delta)\sum_{k=0}^{n}\alpha_k}.\end{aligned} \tag{11}$$

Since $\sum_{k=0}^{\infty}\alpha_k=\infty$, taking the limit of both sides of inequality (11) yields $\lim_{n\to\infty}\|x_n-x_*\|=0$; that is, $x_n\to x_*$ as $n\to\infty$. □

Theorem 7. *Let D, B, and T with fixed point x_* be as in Theorem 6. Let $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ be two iterative sequences defined by* (1) *and* (2), *respectively, with real sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying $\sum_{k=0}^{\infty}\alpha_k\beta_k=\infty$. Then the following are equivalent:*

(i) *$\{x_n\}_{n=0}^{\infty}$ converges to x_*;*

(ii) *$\{u_n\}_{n=0}^{\infty}$ converges to x_*.*

Proof. We will prove (i)⇒(ii). Now by using (1), (2), and condition (3), we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\|&=\|Ty_n-Tv_n\|\\ &\le\delta\|y_n-v_n\|\\ &=\delta\|(1-\alpha_n)z_n+\alpha_nTz_n-(1-\alpha_n)Tu_n\\ &\qquad-\alpha_nTw_n\|\\ &\le\delta\{(1-\alpha_n)\|z_n-Tz_n\|+(1-\alpha_n)\delta\|z_n-u_n\|\\ &\qquad+\alpha_n\delta\|z_n-w_n\|\}\end{aligned}$$

$$\begin{aligned}
&\le (1-\alpha_n)\,\delta\,\|x_n-u_n\| \\
&\quad + (1-\alpha_n)\,\delta\beta_n\,\|Tx_n-x_n\| \\
&\quad + \alpha_n\delta\,\|z_n-w_n\| + (1-\alpha_n)\,\|z_n-Tz_n\| \\
&\le (1-\alpha_n)\,\delta\,\|x_n-u_n\| + (1-\alpha_n)\,\delta\beta_n\,\|Tx_n-x_n\| \\
&\quad + \alpha_n\delta\,[1-\beta_n(1-\delta)]\,\|x_n-u_n\| \\
&\quad + (1-\alpha_n)\,\|z_n-Tz_n\| \\
&\le [1-\alpha_n(1-\delta)]\,\|x_n-u_n\| \\
&\quad + (1-\alpha_n)\,\delta\beta_n\,\|Tx_n-x_n\| + (1-\alpha_n)\,\|z_n-Tz_n\|.
\end{aligned} \tag{12}$$

Define

$$\begin{aligned}
\sigma_n &:= \|x_n-u_n\|, \\
\tau_n &:= \alpha_n(1-\delta) \in (0,1), \\
\rho_n &:= (1-\alpha_n)\,\delta\beta_n\,\|x_n-Tx_n\| + (1-\alpha_n)\,\|z_n-Tz_n\|.
\end{aligned} \tag{13}$$

Since $\lim_{n\to\infty}\|x_n-x_*\| = 0$ and $Tx_* = x_*$, $\lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|z_n - Tz_n\| = 0$ which implies $\rho_n/\tau_n \to 0$ as $n \to \infty$. Since also $\alpha_n, \beta_n \in [0,1]$ for all $n \in \mathbb{N}$

$$\alpha_n\beta_n < \alpha_n; \tag{14}$$

hence the assumption $\sum_{k=0}^{\infty}\alpha_k\beta_k = \infty$ leads to

$$\sum_{k=0}^{\infty}\alpha_k = \infty. \tag{15}$$

Thus all conditions of Lemma 4 are fulfilled by (12), and so $\lim_{n\to\infty}\|x_n-u_n\| = 0$. Since

$$\begin{aligned}
\|u_n-x_*\| &\le \|x_n-u_n\| + \|x_n-x_*\|, \\
\lim_{n\to\infty}\|u_n-x_*\| &= 0.
\end{aligned} \tag{16}$$

Using the same argument as above one can easily show the implication (ii)⇒(i); thus it is omitted here. □

Theorem 8. *Let D, B, and T with fixed point x_* be as in Theorem 6. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ be real sequences in $(0,1)$ satisfying*

(i) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$.

For given $x_0 = u_0 \in D$, consider iterative sequences $\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ defined by (1) and (2), respectively. Then $\{u_n\}_{n=0}^{\infty}$ converges to x_* faster than $\{x_n\}_{n=0}^{\infty}$ does.

Proof. The following inequality comes from inequality (10) of Theorem 6:

$$\begin{aligned}
\|x_{n+1}-x_*\| &\le \|x_0-x_*\|\,\delta^{n+1} \\
&\quad \times \prod_{k=0}^{n}[1-\alpha_k(1-\delta)]\,[1-\beta_k(1-\delta)].
\end{aligned} \tag{17}$$

The following inequality is due to ([2], inequality (2.5) of Theorem 1):

$$\begin{aligned}
\|u_{n+1}-x_*\| &\le \|u_0-x_*\|\,\delta^{2(n+1)} \\
&\quad \times \prod_{k=0}^{n}[1-\alpha_k\beta_k(1-\delta)].
\end{aligned} \tag{18}$$

Define

$$\begin{aligned}
a_n &:= \|u_0-x_*\|\,\delta^{2(n+1)}\prod_{k=0}^{n}[1-\alpha_k\beta_k(1-\delta)], \\
b_n &:= \|x_0-x_*\|\,\delta^{n+1}\prod_{k=0}^{n}[1-\alpha_k(1-\delta)]\,[1-\beta_k(1-\delta)].
\end{aligned} \tag{19}$$

Since $x_0 = u_0$

$$\theta_n := \frac{a_n}{b_n} = \frac{\delta^{n+1}\prod_{k=0}^{n}[1-\alpha_k\beta_k(1-\delta)]}{\prod_{k=0}^{n}[1-\alpha_k(1-\delta)]\,[1-\beta_k(1-\delta)]}. \tag{20}$$

Therefore, taking into account assumption (i), we obtain

$$\begin{aligned}
\lim_{n\to\infty}\frac{\theta_{n+1}}{\theta_n} &= \lim_{n\to\infty}\frac{\delta\,[1-\alpha_{n+1}\beta_{n+1}(1-\delta)]}{[1-\alpha_{n+1}(1-\delta)]\,[1-\beta_{n+1}(1-\delta)]} \\
&= \delta < 1.
\end{aligned} \tag{21}$$

It thus follows from well-known ratio test that $\sum_{n=0}^{\infty}\theta_n < \infty$. Hence, we have $\lim_{n\to\infty}\theta_n = 0$ which implies that $\{u_n\}_{n=0}^{\infty}$ is faster than $\{x_n\}_{n=0}^{\infty}$. □

In order to support analytical proof of Theorem 8 and to illustrate the efficiency of Picard-S iteration method (2), we will use a numerical example provided by Sahu [12] for the sake of consistent comparison.

Example 9. Let $B = \mathbb{R}$ and $D = [0,\infty)$. Let $T : D \to D$ be a mapping and for all $x \in D$, $Tx = \sqrt[3]{3x+18}$. T is a contraction with contractivity factor $\delta = 1/\sqrt[3]{18}$ and $x_* = 3$; see [12]. Take $\alpha_n = \beta_n = \gamma_n = 1/(n+1)$ with initial value $x_0 = 1000$. Tables 1, 2, and 3 show that Picard-S iteration method (2) converges faster than all SP [3], Picard [4], Mann [5], Ishikawa [6], S [7], CR [13], S^* [14], Noor [15], and Normal-S [16] iteration methods including a new three-step iteration method due to Abbas and Nazir [17].

We are now able to establish the following data dependence result.

Theorem 10. *Let $\widetilde{T}$ be an approximate operator of T satisfying condition (3). Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (1) for T and define an iterative sequence $\{\widetilde{x}_n\}_{n=0}^{\infty}$ as follows:*

$$\begin{aligned}
\widetilde{x}_0 &\in D, \\
\widetilde{x}_{n+1} &= \widetilde{T}\widetilde{y}_n, \\
\widetilde{y}_n &= (1-\alpha_n)\,\widetilde{z}_n + \alpha_n\widetilde{T}\widetilde{z}_n, \\
\widetilde{z}_n &= (1-\beta_n)\,\widetilde{x}_n + \beta_n\widetilde{T}\widetilde{x}_n, \quad n \in \mathbb{N},
\end{aligned} \tag{22}$$

Table 1: Comparison speed of convergence among various iteration methods.

Number of iterations	Picard-S	Abbas and Nazir	Modified SP	S^*
1	3.101431265	3.944094141	3.101431265	3.101431265
2	3.000970459	3.032885422	3.003472396	3.006099262
3	3.000010797	3.001099931	3.000191044	3.000474311
4	3.000000126	3.000033381	3.000012841	3.000040908
5	3.000000001	3.000000928	3.000000964	3.000003733
6	3.000000000	3.000000024	3.000000078	3.000000354
7	3.000000000	3.000000000	3.000000007	3.000000034
8	3.000000000	3.000000000	3.000000001	3.000000004
9	3.000000000	3.000000000	3.000000000	3.000000000
⋮	⋮	⋮	⋮	⋮

Table 2: Comparison speed of convergence among various iteration methods.

Number of iterations	CR	Normal S	S	Picard
1	3.101431265	3.944094141	3.944094141	14.45128320
2	3.004853706	3.056995075	3.079213170	3.944094141
3	3.000341967	3.004449310	3.007910488	3.101431265
4	3.000027911	3.000384457	3.000829879	3.011228065
5	3.000002459	3.000035123	3.000088928	3.001247045
6	3.000000227	3.000003324	3.000009637	3.000138554
7	3.000000022	3.000000323	3.000001051	3.000015395
8	3.000000003	3.000000032	3.000000115	3.000001710
9	3.000000000	3.000000003	3.000000013	3.000000190
10	3.000000000	3.000000000	3.000000001	3.000000021
11	3.000000000	3.000000000	3.000000000	3.000000002
12	3.000000000	3.000000000	3.000000000	3.000000000
⋮	⋮	⋮	⋮	⋮

Table 3: Comparison speed of convergence among various iteration methods.

Number of iterations	SP	Noor	Ishikawa	Mann
1	3.101431265	3.101431265	3.944094141	14.45128320
2	3.017380074	3.053700718	3.500544608	9.197688670
3	3.006056041	3.037176288	3.346563527	7.322609407
4	3.002849358	3.028678163	3.267333303	6.346715746
5	3.001583841	3.023463545	3.218710750	5.744057924
6	3.000979045	3.019921623	3.185684999	5.333095485
7	3.000651430	3.017350921	3.161716071	5.034023149
8	3.000457519	3.015395770	3.143487338	4.806124994
9	3.000334906	3.013856108	3.129133091	4.626397579
10	3.000253300	3.012610550	3.117521325	4.480838008
11	3.000196722	3.011581068	3.107924338	4.360421594
12	3.000156165	3.010715155	3.099852480	4.259063398
13	3.000126272	3.009976148	3.092963854	4.172507477
⋮	⋮	⋮	⋮	⋮

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ *are real sequences in* $[0,1]$ *satisfying (i)* $1/2 \le \alpha_n$, *(ii)* $\beta_n \le \alpha_n$ *for all* $n \in \mathbb{N}$, *and (iii)* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *If* $Tx_* = x_*$ *and* $\widetilde{T}\widetilde{x}_* = \widetilde{x}_*$ *such that* $\widetilde{x}_n \to \widetilde{x}_*$ *as* $n \to \infty$, *then we have*

$$\|x_* - \widetilde{x}_*\| \le \frac{4\varepsilon}{1-\delta}, \tag{23}$$

where $\varepsilon > 0$ *is a fixed number and* $\delta \in (0,1)$.

Proof. It follows from (1), (3), (22), and assumption (ii) that

$$\begin{aligned}
\|x_{n+1} - \widetilde{x}_{n+1}\| &= \|Ty_n - T\widetilde{y}_n + T\widetilde{y}_n - \widetilde{T}\widetilde{y}_n\| \\
&\le \|Ty_n - T\widetilde{y}_n\| + \|T\widetilde{y}_n - \widetilde{T}\widetilde{y}_n\| \\
&\le \delta \|y_n - \widetilde{y}_n\| + \varepsilon \\
&\le \delta (1-\alpha_n) \|z_n - \widetilde{z}_n\| \\
&\quad + \delta\alpha_n \|Tz_n - T\widetilde{z}_n\| + \delta\alpha_n \|T\widetilde{z}_n - \widetilde{T}\widetilde{z}_n\| + \varepsilon \\
&\le \delta [1 - \alpha_n (1-\delta)] \|z_n - \widetilde{z}_n\| + \delta\alpha_n\varepsilon + \varepsilon \\
&\le \delta [1 - \alpha_n (1-\delta)] [1 - \beta_n (1-\delta)] \|x_n - \widetilde{x}_n\|
\end{aligned}$$

$$+\delta\left[1-\alpha_n(1-\delta)\right]\beta_n\varepsilon+\delta\alpha_n\varepsilon+\varepsilon$$
$$\leq\left[1-\alpha_n(1-\delta)\right]\left\|x_n-\widetilde{x}_n\right\|+2\alpha_n\varepsilon+\varepsilon. \tag{24}$$

From assumption (i) we have

$$1\leq 2\alpha_n, \tag{25}$$

and thus, inequality (24) becomes

$$\begin{aligned}\left\|x_{n+1}-\widetilde{x}_{n+1}\right\| &\leq\left[1-\alpha_n(1-\delta)\right]\left\|x_n-\widetilde{x}_n\right\|+4\alpha_n\varepsilon\\ &\leq\left[1-\alpha_n(1-\delta)\right]\left\|x_n-\widetilde{x}_n\right\|\\ &\quad+\alpha_n(1-\delta)\frac{4\varepsilon}{1-\delta}.\end{aligned} \tag{26}$$

Denote that

$$\sigma_n:=\left\|x_n-\widetilde{x}_n\right\|,\qquad \tau_n:=\alpha_n(1-\delta)\in(0,1),$$
$$\mu_n:=\frac{4\varepsilon}{1-\delta}. \tag{27}$$

It follows from Lemma 5 that

$$0\leq\limsup_{n\to\infty}\left\|x_n-\widetilde{x}_n\right\|\leq\limsup_{n\to\infty}\frac{4\varepsilon}{1-\delta}. \tag{28}$$

From Theorem 6 we know that $\lim_{n\to\infty}x_n=x_*$. Thus, using this fact together with the assumption $\lim_{n\to\infty}\widetilde{x}_n=\widetilde{x}_*$ we obtain

$$\left\|x_*-\widetilde{x}_*\right\|\leq\frac{4\varepsilon}{1-\delta}. \tag{29}$$

□

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Littlewood-Paley Operators on Morrey Spaces with Variable Exponent

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By applying the vector-valued inequalities for the Littlewood-Paley operators and their commutators on Lebesgue spaces with variable exponent, the boundedness of the Littlewood-Paley operators, including the Lusin area integrals, the Littlewood-Paley g-functions and g_μ^*-functions, and their commutators generated by BMO functions, is obtained on the Morrey spaces with variable exponent.

1. Introduction and Main Results

Let $\psi \in L^1(\mathbb{R}^n)$ and satisfy the following:

(i) $\int_{\mathbb{R}^n} \psi(x)dx = 0$;

(ii) $|\psi(x)| \leqslant C(1+|x|)^{-n-\varepsilon}$;

(iii) $|\psi(x+y)-\psi(x)| \leqslant C|y|^{\gamma}(1+|x|)^{-n-\gamma-\varepsilon}$, $|x| \geqslant 2|y|$,

where C, ε, γ are all positive constants. Denote $\psi_t(x) = t^{-n}\psi(x/t)$ with $t > 0$ and $x \in \mathbb{R}^n$. Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Lusin area integral of f is defined by

$$S_{\psi,a}(f)(x) = \left(\int_{\Gamma_a(x)} |\psi_t * f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2}, \quad (1)$$

where $\Gamma_a(x)$ denote the usual cone of aperture one

$$\Gamma_a(x) = \left\{(t,y) \in \mathbb{R}^{n+1}_+ : |y-x| < at,\ a \geq 0\right\}. \quad (2)$$

As $a = 1$, we denote $S_{\psi,a}(f)$ as $S_\psi(f)$.

Now let us turn to the introduction of the other two Littlewood-Paley operators. It is well known that the Littlewood-Paley operators include also the Littlewood-Paley g-functions and the Littlewood-Paley g_μ^*-functions besides the Lusin area integrals. The Littlewood-Paley g-functions, which can be viewed as a "zero-aperture" version of S_ψ, and g_μ^*-functions, which can be viewed as an "infinite-aperture" version of S_ψ, are, respectively, defined by

$$g(f)(x) = \left(\int_0^\infty |\psi_t * f(y)|^2 \frac{dt}{t}\right)^{1/2}, \quad (3)$$

$$g_\mu^*(f)(x) = \left(\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\mu n} |\psi_t * f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2}, \quad \mu > 0. \quad (4)$$

If we take ψ to be the Poisson kernel, then the functions defined above are the classical Littlewood-Paley operators.

Letting $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $m \geq 1$, the corresponding m-order commutators of Littlewood-Paley operators above generated by a function b are defined by

$$\left[b^m, g_\psi\right](f)(x) = \left(\int_0^\infty \left|\int_{\mathbb{R}^n} [b(x)-b(y)]^m \psi_t(x-y) f(y)\,dy\right|^2 \frac{dt}{t}\right)^{1/2}, \quad (5)$$

$$\left[b^m, S_{\psi,a}\right](f)(x) = \left(\int_{\Gamma_a(x)}\left|\int_{\mathbb{R}^n}[b(x)-b(z)]^m\psi_t(y-z)f(z)\,dz\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2}, \tag{6}$$

$$\left[b^m, g_\mu^*\right](f)(x) = \left(\int_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\mu n} \times\left|\int_{\mathbb{R}^n}[b(x)-b(z)]^m\psi_t(y-z)f(z)\,dz\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2}, \tag{7}$$

where $\mu > 0$.

The Littlewood-Paley operators are a class of important integral operators. Due to the fact that they play very important roles in harmonic analysis, PDE, and the other fields (see [1–3]), people pay much more attention to this class of operators. In 1995, Lu and Yang investigated the behavior of Littlewood-Paley operators in the space $\mathrm{CBMO}_p(\mathbb{R}^n)$ in [4]. In 2005, Zhang and Liu proved the commutator $[b, g_\psi]$ is bounded on $L^p(\omega)$ in [5]. In 2009, Xue and Ding gave the weighted estimate for Littlewood-Paley operators and their commutators (see [6]). There are some other results about Littlewood-Paley operators in [7–9] and so forth.

On the other hand, Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ become one class of important research subject in analysis filed due to the fundamental paper [10] by Kováčik and Rákosník. In the past twenty years, the theory of these spaces has made progress rapidly, and the study of which has many applications in fluid dynamics, elasticity, calculus of variations, and differential equations with nonstandard growth conditions (see [11–15]). In [16], Cruz-Uribe et al. stated that the extrapolation theorem leads the boundedness of some classical operators including the commutator on $L^{p(\cdot)}(\mathbb{R}^n)$. Karlovich and Lerner also independently obtained the boundedness of the singular integrals commutator on Lebesgue spaces with variable exponent in [17]. In 2009 and 2010, Izuki considered the boundedness of vector-valued sublinear operators and fractional integrals on Herz-Morrey spaces with variable exponent in [18, 19], respectively. In 2013, Ho in [20] introduced a class of Morrey spaces with variable exponent $\mathscr{M}_{p(\cdot),u}$ and studied the boundedness of the fractional integral operators on $\mathscr{M}_{p(\cdot),u}$.

Inspired by the results mentioned previously, in this paper we will consider the vector-valued inequalities of the Littlewood-Paley operators and their m-order commutators on Morrey spaces with variable exponent. Before stating our main results, we need to recall some relevant definitions and notations.

Let E be a Lebesgue measurable set in $\mathbb{R}^n$ with measure $|E| > 0$.

Definition 1 (see [10]). Let $p(\cdot): E \to [1,\infty)$ be a measurable function.

The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E) = \left\{f \text{ is measurable: } \int_E\left(\frac{|f(x)|}{\eta}\right)^{p(x)}dx < \infty \text{ for some constant } \eta > 0\right\}. \tag{8}$$

The space $L^{p(\cdot)}_{\mathrm{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(E) = \left\{f \text{ is measurable: } f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset E\right\}. \tag{9}$$

The Lebesgue space $L^{p(\cdot)}(E)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\eta > 0 : \int_E\left(\frac{|f(x)|}{\eta}\right)^{p(x)}dx \le 1\right\}. \tag{10}$$

Remark 2. (1) Note that if the function $p(x) = p_0$ is a constant function, then $L^{p(\cdot)}(\mathbb{R}^n)$ equals $L^{p_0}(\mathbb{R}^n)$. This implies that the Lebesgue spaces with variable exponent generalize the usual Lebesgue spaces. And they have many properties in common with the usual Lebesgue spaces.

(2) Denote $p_- := \operatorname{ess\,inf}\{p(x) : x \in E\}$, $p_+ := \operatorname{ess\,sup}\{p(x) : x \in E\}$. Then $\mathscr{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

(3) The Hardy-Littlewood maximal operator M is defined by

$$M(f)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|\,dy. \tag{11}$$

Denote $\mathscr{B}(E)$ to be the set of all functions $p(\cdot) \in \mathscr{P}(E)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(E)$.

(4) Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Denote $\kappa_{p(\cdot)} = \sup\{q > 1 : p(\cdot)/q \in \mathscr{B}(\mathbb{R}^n)\}$ and $e_{p(\cdot)}$ is the conjugate exponent of $\kappa_{p'(\cdot)}$ (see [20]).

Definition 3 (see [20]). Let $p(x) \in L^\infty(\mathbb{R}^n)$, $1 < p(x) < \infty$. If there exists a constant $C > 0$ such that, for any $x \in \mathbb{R}^n$ and $r > 0$, Lebesgue measurable function $u(x,r): \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ satisfying

$$\sum_{j=0}^{\infty}\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}u\left(x,2^{j+1}r\right) \le Cu(x,r), \tag{12}$$

then one says u is a Morrey weight function for $L^{p(\cdot)}(\mathbb{R}^n)$. One denotes the class of Morrey weight functions by $\mathbb{W}_{p(\cdot)}$.

Definition 4 (see [20]). Let $p(x) \in \mathscr{B}(\mathbb{R}^n)$, $u(x,r) \in \mathbb{W}_{p(\cdot)}$. Then the Morrey spaces with variable exponent $\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)$ are defined by

$$\mathscr{M}_{p(\cdot),u}\left(\mathbb{R}^n\right) = \left\{f \text{ is measurable: } \|f\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty\right\}, \quad (13)$$

where

$$\|f\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, R>0} \frac{1}{u(x,R)} \|f\chi_{B(x,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \quad (14)$$

Remark 5. (1) If $u(x,r) \equiv 1$, then $\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)$ is the Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$.

(2) Notice that if $p(x) \equiv p$, $1 < p < \infty$, is a constant function, then formula (12) can be rewritten as an integral in form. To be precise, formula (12) can be rewritten in the following form (see [20]):

$$\int_r^\infty \frac{u(x,t)}{t^{n/p+1}}dt \le C\frac{u(x,r)}{r^{n/p}}, \quad r>0, \ \forall x \in \mathbb{R}^n. \quad (15)$$

Let $0 < \alpha < n$. By the the conditions of Morrey weight functions mentioned in [21]

$$\int_r^\infty \frac{u^p(x,t)}{t^{n-\alpha/p+1}}dt \le C\frac{u^p(x,r)}{r^{n-\alpha/p}}, \quad r>0, \ \forall x \in \mathbb{R}^n \quad (16)$$

and Hölder's inequality, via simple calculation, we have

$$\begin{aligned}\int_r^\infty &\frac{u(x,t)}{t^{n/p+1}}dt \\ &\le C\left\{\int_r^\infty \frac{u^p(x,t)}{t^{n-\alpha/p+1}}dt\right\}^{1/p}\left\{\int_r^\infty \frac{1}{t^{\alpha p'+1}}dt\right\}^{1/p'} \\ &\le C\frac{u(x,r)}{r^{n/p}}.\end{aligned} \quad (17)$$

From this, it follows that if $p(x) \equiv p, 1 < p < \infty$, is a constant function, then condition (12) is weaker than condition (16). Thus, the class of the Morrey spaces introduced in Definition 4 is more wide than that satisfying condition (1.8) in [21]. More studies of common Morrey spaces can be seen in [22, 23] and so forth.

(3) If $u(x,r) = |B(x,r)|^{1/p(x)-1/q(x)}$, $p(x) \le q(x)$, then the space mentioned in Definition 4 is the Morrey space with variable exponent introduced in [24]. And when $1 < s < \kappa_{p'(x)}$, $1/p(x) - 1/q(x) < 1 - 1/s$, it is easy to see $u(x,r)$ satisfying condition (12). That is because it follows from $p(x) \in \mathscr{B}(\mathbb{R}^n)$, $1 < s < \kappa_{p'(x)}$, that (see [20])

$$\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C2^{jn(1/s-1)}, \quad \forall x \in \mathbb{R}^n, \ r>0, \ j \in \mathbb{N}. \quad (18)$$

For Littlewood-Paley operators $S_{\psi,a}, g_\psi$, and g_μ^*, in this paper, we have the following results.

Theorem 6. *Suppose that function $\psi \in L^1(\mathbb{R}^n)$ satisfies (i)–(iii) and $S_{\psi,a}$ is defined by* (1). *If $u \in \mathbb{W}_{p(\cdot)}$, $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $1 < r < \infty$, then there exists a constant $C > 0$ independent of f such that, for any function sequences $\{f_h\}_{h=1}^\infty$ with $\|\{\sum_h |f_h|^r\}^{1/r}\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty$, the following inequality holds:*

$$\left\|\left\{\sum_h \left|S_{\psi,a}(f_h)\right|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \le C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}. \quad (19)$$

Theorem 7. *Suppose that g_ψ is defined by* (3). *Then under the same condition as the one in Theorem 6, there exists a constant $C > 0$ independent of f such that, for any function sequences $\{f_h\}_{h=1}^\infty$ with $\|\{\sum_h |f_h|^r\}^{1/r}\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty$, the following inequality holds:*

$$\left\|\left\{\sum_h \left|g_\psi(f_h)\right|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \le C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}. \quad (20)$$

Theorem 8. *Suppose that g_μ^* is defined by* (4) *and $\mu > 3 + 2(\varepsilon+\gamma)/n$, $0 < \gamma < \varepsilon$. Then under the same condition as the one in Theorem 6, there exists a constant $C > 0$ independent of f such that, for any function sequences $\{f_h\}_{h=1}^\infty$ with $\|\{\sum_h |f_h|^r\}^{1/r}\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty$, the following inequality holds:*

$$\left\|\left\{\sum_h \left|g_\mu^*(f_h)\right|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \le C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}. \quad (21)$$

For commutators $[b^m, S_{\psi,a}]$, $[b^m, g_\psi]$, and $[b^m, g_\mu^*]$, we have the following results.

Theorem 9. *Suppose that function $\psi \in L^1(\mathbb{R}^n)$ satisfies (i)–(iii) and $[b^m, S_{\psi,a}]$ is defined by* (5). *Let $b \in BMO(\mathbb{R}^n)$, $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $m \ge 1$, $1 < r < \infty$. If, for any $x \in \mathbb{R}^n$ and $r_0 > 0$, function u satisfies*

$$\sum_{j=0}^\infty \frac{\|\chi_{B(x,r_0)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r_0)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} j^m u\left(x, 2^{j+1}r_0\right) \le Cu(x,r_0), \quad (22)$$

then there exists a constant $C > 0$ independent of f such that, for any function sequences $\{f_h\}_{h=1}^\infty$ with $\|\{\sum_h |f_h|^r\}^{1/r}\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty$, the following inequality holds:

$$\begin{aligned}&\left\|\left\{\sum_h \left|\left[b^m, S_{\psi,a}\right](f_h)\right|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \\ &\le C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}.\end{aligned} \quad (23)$$

Theorem 10. *Suppose that $[b^m, g_\psi]$ is defined by* (6). *Then under the same condition as the one in Theorem 9, there exists*

a constant $C > 0$ independent of f such that, for any function sequences $\{f_h\}_{h=1}^{\infty}$ with $\|\{\sum_h |f_h|^r\}^{1/r}\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty$, the following inequality holds:

$$\left\|\left\{\sum_h \left|\left[b^m, g_\psi\right](f_h)\right|^r\right\}^{1/r}\right\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)} \le C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)}. \tag{24}$$

Theorem 11. *Suppose that $[b^m, g_\mu^*]$ is defined by (7), $\mu > 3 + 2(\varepsilon + \gamma)/n$, and $0 < \gamma < \varepsilon$. Then under the same condition as the one in Theorem 9, there exists a constant $C > 0$ independent of f such that, for any function sequences $\{f_h\}_{h=1}^{\infty}$ with $\|\{\sum_h |f_h|^r\}^{1/r}\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty$, the following inequality holds:*

$$\left\|\left\{\sum_h \left|\left[b^m, g_\mu^*\right](f_h)\right|^r\right\}^{1/r}\right\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)} \le C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)}. \tag{25}$$

Remark 12. (1) It is easy to see that condition (22) in Theorem 9 is stronger than condition (12) in Definition 3. Therefore, if a function u satisfies condition (22), then $u \in \mathbb{W}_{p(\cdot)}$.

(2) The function u which satisfies (22) exists. In fact, if we take $\tau : 0 \le \tau < 1/e_{p(\cdot)}$ such that function u satisfies

$$\frac{u(x,2r)}{u(x,r)} \le 2^{n\tau}, \quad \forall x \in \mathbb{R}^n,\ r > 0, \tag{26}$$

then $u \in \mathbb{W}_{p(\cdot)}$. That is because, for any $\tau < 1/e_{p(\cdot)}$, there exists $s < \kappa_{p'(\cdot)}$ such that $\tau < 1 - 1/s < 1 - 1/\kappa_{p'(\cdot)} = 1/e_{p(\cdot)}$. Hence, it follows from (18) that

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} j^m u\left(x, 2^{j+1}r\right) \le C\sum_{j=0}^{\infty} j^m 2^{-jn(1-1/s)} 2^{jn\tau} u(x,r) \le Cu(x,r). \tag{27}$$

We end this section by introducing some conventional notations which will be used later. Throughout this paper, given a function f, we denote the mean value of f on E by $f_E =: (1/|E|)\int_E f(x)dx$. $p'(\cdot)$ means the conjugate exponent of $p(\cdot)$; namely, $1/p(x) + 1/p'(x) = 1$ holds. C always means a positive constant independent of the main parameters and may change from one occurrence to another.

2. Preliminary Lemmas

In this section, we introduce some conclusions which will be used in the proofs of our main results.

Lemma 13 (see [10] (generalized Hölder's inequality)). *Let $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.*

(1) *For any $f \in L^{p(\cdot)}(\mathbb{R}^n), g \in L^{p'(\cdot)}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x)\,g(x)|\,dx \le C_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \tag{28}$$

where $C_p = 1 + 1/p_- - 1/p_+$.

(2) *For any $f \in L^{p_1(\cdot)}(\mathbb{R}^n), g \in L^{p_2(\cdot)}(\mathbb{R}^n)$, when $1/p(x) = 1/p_1(x) + 1/p_2(x)$, one has*

$$\|f(x)\,g(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_{p_1,p_2} \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}, \tag{29}$$

where $C_{p_1,p_2} = (1 + 1/p_{1_-} - 1/p_{1_+})^{1/p_-}$.

Lemma 14 (see [17]). *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exist constants $\delta_1, \delta_2, C > 0$, such that, for all balls $B \subset \mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2}. \tag{30}$$

Remark 15. From formula (12), it follows that

$$\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u\left(x, 2^{j+1}r\right) \le Cu(x,r), \quad \forall j \in \mathbb{N}. \tag{31}$$

Thus, by Lemma 14, we have, $\forall j \in \mathbb{N}$,

$$\frac{u\left(x, 2^j r\right)}{u(x,r)} \le C\frac{\|\chi_{B(x,2^j r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{\left|B\left(x, 2^j r\right)\right|}{|B(x,r)|} \le C2^{nj}. \tag{32}$$

Lemma 16 (see [18]). *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exists constant $C > 0$, such that, for all balls $B \subset \mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C. \tag{33}$$

Lemma 17 (see [25]). *Let $b \in BMO(\mathbb{R}^n)$; m is a positive integer. There exist constants $C > 0$, such that, for any $k, j \in \mathbb{Z}$ with $k > j$,*

(1) $C^{-1}\|b\|_*^m \le \sup_B (1/\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)})\|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*^m$;

(2) $\|(b - b_{B_j})^m \chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(k-j)^m \|b\|_*^m \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Lemma 18 (see [26]). *Let $\psi \in L^1(\mathbb{R}^n)$ satisfy (i)–(iii). If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $1 < q < \infty$, then for all bounded compactly support functions f_j such that $\{f_j\}_{j=1}^{\infty} \in L^{p(\cdot)}(l^q)$, that is, $\|(\sum_j (|f_j|)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$, the following vector-valued inequalities hold:*

(1) $\|(\sum_j |S_{\psi,a}(f_j)|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|(\sum_j |f_j|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

(2) $\|(\sum_j |g_\psi(f_j)|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|(\sum_j |f_j|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

(3) $\|(\sum_j |g_\mu^*(f_j)|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|(\sum_j |f_j|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

where $\mu > 2$, $0 < \gamma < \min\{(\mu-2)n/2, \varepsilon\}$.

Lemma 19 (see [26]). *Let* $b \in BMO, \psi \in L^1(\mathbb{R}^n)$ *satisfy (i)–(iii),* $m \in \mathbb{N} \setminus \{0\}$. *If* $p(\cdot) \in \mathscr{B}(\mathbb{R}^n), 1 < q < \infty$, *then for all bounded compactly support functions* f_j *such that* $\{f_j\}_{j=1}^{\infty} \in L^{p(\cdot)}(l^q)$, *that is,* $\|(\sum_j (|f_j|)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$, *the following vector-valued inequalities hold:*

(1) $\|(\sum_j |[b^m, S_{\psi,a}](f_j)|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|(\sum_j |f_j|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

(2) $\|(\sum_j |[b^m, g_\psi](f_j)|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|(\sum_j |f_j|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

(3) $\|(\sum_j |[b^m, g_\mu^*](f_j)|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|(\sum_j |f_j|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

where $\mu > 2$, $0 < \gamma < \min\{(\mu-2)n/2, \varepsilon\}$.

3. Proofs of Main Results

Next, let us show the proofs of Theorems 6–11, respectively.

Proof of Theorem 6. Let $\|\{f_h\}_h\|_{l^r} \in \mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)$; for any $x_0 \in \mathbb{R}^n$, $r_0 > 0$, denote

$$f_h(x) \triangleq f_h^0(x) + \sum_{j=1}^{\infty} f_h^j(x), \tag{34}$$

where $f_h^0 = f_h \chi_{B(x_0,2r_0)}$, $f_h^j = f_h \chi_{B(x_0,2^{j+1}r_0)\setminus B(x_0,2^j r_0)}$, $j \in \mathbb{N} \setminus \{0\}$.

Noting that, in order to prove Theorem 6, it is enough to show that the following inequality holds:

$$\frac{1}{u(x_0,r_0)} \left\| \left\{ \sum_h \left| S_{\psi,a}(f_h) \right|^r \right\}^{1/r} \chi_{B(x_0,r_0)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \left\| \left\{ \sum_h |f_h|^r \right\}^{1/r} \right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}. \tag{35}$$

Thus,

$$\begin{aligned} &\frac{1}{u(x_0,r_0)} \left\| \left\{ \sum_h \left| S_{\psi,a}(f_h) \right|^r \right\}^{1/r} \chi_{B(x_0,r_0)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le \frac{C}{u(x_0,r_0)} \left\| \left\{ \sum_h \left| S_{\psi,a}(f_h^0) \right|^r \right\}^{1/r} \chi_{B(x_0,r_0)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\quad + \frac{C}{u(x_0,r_0)} \sum_{j=1}^{\infty} \left\| \left\{ \sum_h \left| S_{\psi,a}(f_h^j) \right|^r \right\}^{1/r} \chi_{B(x_0,r_0)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\triangleq D_1 + D_2. \end{aligned} \tag{36}$$

For the term D_1, notice that $\operatorname{supp} f_h^0 \subset B(x_0, 2r_0)$; using Lemma 18 and (32), it is easy to see that

$$\begin{aligned} D_1 &\le \frac{C}{u(x_0,r_0)} \left\| \left\{ \sum_h |f_h^0|^r \right\}^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le C \frac{u(x_0,2r_0)}{u(x_0,r_0)} \frac{1}{u(x_0,2r_0)} \\ &\quad \times \left\| \left\{ \sum_h |f_h|^r \right\}^{1/r} \chi_{B(x_0,2r_0)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le C 2^n \sup_{y \in \mathbb{R}^n, R>0} \frac{1}{u(y,R)} \\ &\quad \times \left\| \left\{ \sum_h |f_h|^r \right\}^{1/r} \chi_{B(y,R)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le C \left\| \left\{ \sum_h |f_h|^r \right\}^{1/r} \right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}. \end{aligned} \tag{37}$$

We now turn to estimate D_2. To do this, we need to consider $S_{\psi,a}$ first. Without loss of generality, we may assume that $a \ge 1$. Let

$$\begin{aligned} \Gamma' &= \left\{ y \in \mathbb{R}^n : |y-x| < at,\ y \le 2^{j+1} r_0 \right\}, \\ \Gamma'' &= \left\{ y \in \mathbb{R}^n : |y-x| < at,\ y > 2^{j+1} r_0 \right\}. \end{aligned} \tag{38}$$

Then, by Minkowski's inequality, we have

$$\begin{aligned}
&S_{\psi,a}\left(f_h^j\right)(x)\\
&\quad=\left(\int_{\Gamma_a(x)}\left|\int_{\mathbb{R}^n}t^{-n}\psi\left(\frac{y-z}{t}\right)f_h^j(z)\,dz\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2}\\
&\quad\le\int_{\mathbb{R}^n}\left|f_h^j(z)\right|\left(\int_{\Gamma_a(x)}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\right)^{1/2}dz\\
&\quad\le\int_{\mathbb{R}^n}\left|f_h^j(z)\right|\left(\int_0^\infty\int_{\Gamma'}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\right)^{1/2}dz\\
&\qquad+\int_{\mathbb{R}^n}\left|f_h^j(z)\right|\left(\int_0^\infty\int_{\Gamma''}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\right)^{1/2}dz.
\end{aligned}\tag{39}$$

Observe that if $x\in B(x_0,r_0)$, $z\in B(x_0,2^{j+1}r_0)\setminus B(x_0,2^jr_0)$, $j\ge1$, then

$$\begin{aligned}
t+|y-z|&\ge\frac{|x-y|}{a}+|y-z|\\
&\ge\frac{|x-y|+|y-z|}{a}\ge\frac{|z|-|x|}{a}\ge\frac{|z|}{2a}.
\end{aligned}\tag{40}$$

Therefore, it follows from condition (ii) that

$$\begin{aligned}
&\int_0^\infty\int_{\Gamma'}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\\
&\quad\le\int_0^\infty\int_{\Gamma'}t^{-3n-1}\left(1+\frac{|y-z|}{t}\right)^{-2(n+\varepsilon)}dy\,dt\\
&\quad\le\int_0^{2^{j+1}r_0}\int_{\Gamma'}\frac{t^{2\varepsilon-n-1}}{\left(t+|y-z|\right)^{2(n+\varepsilon)}}dy\,dt\\
&\qquad+\int_{2^{j+1}r_0}^\infty\int_{\Gamma'}\frac{t^{2\varepsilon-n-1}}{\left(t+|y-z|\right)^{2(n+\varepsilon)}}dy\,dt\\
&\quad\le C\frac{a^{2(n+\varepsilon)}}{|z|^{2(n+\varepsilon)}}\int_0^{2^{j+1}r_0}\int_{|x-y|<at}t^{2\varepsilon-n-1}dy\,dt\\
&\qquad+C\frac{a^{2n}}{|z|^{2n}}\int_{2^{j+1}r_0}^\infty\int_{|x-y|<at}t^{-3n-1}dy\,dt\\
&\quad\le C\frac{a^{2(n+\varepsilon)}}{|z|^{2(n+\varepsilon)}}\int_0^{2^{j+1}r_0}a^nt^{2\varepsilon-1}dy\,dt\\
&\qquad+C\int_{2^{j+1}r_0}^\infty a^nt^{-2n-1}dy\,dt\\
&\quad\le Ca^{3n+2\varepsilon}\left(2^jr_0\right)^{-2n}.
\end{aligned}\tag{41}$$

On the other hand, we denote

$$\begin{aligned}
&\int_0^\infty\int_{\Gamma''}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\\
&\quad\le C\int_0^\infty\int_{\Gamma''}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)-\psi\left(\frac{y}{t}\right)\right|^2dy\,dt\\
&\qquad+C\int_0^\infty\int_{\Gamma''}t^{-3n-1}\left|\psi\left(\frac{y}{t}\right)\right|^2dy\,dt\\
&\quad\triangleq I+II.
\end{aligned}\tag{42}$$

Note that if $y>2^{j+1}r_0$, then $t>|x-y|/a\ge(|x|-|y|)/a>(2^{j+1}r_0-r_0)/a\ge2^jr_0/a$. Thus, by condition (iii), we get

$$\begin{aligned}
I&\le C\int_{2^jr_0/a}^\infty\int_{\Gamma''}t^{-3n-1}\left(\frac{|z|}{t}\right)^{2\gamma}\left(1+\frac{|y|}{t}\right)^{-2(n+\gamma+\varepsilon)}dy\,dt\\
&\le C\left(2^jr_0\right)^{2\gamma}\int_{2^jr_0/a}^\infty\int_{|x-y|<at}t^{-3n-2\gamma-1}dy\,dt\\
&\le C\left(2^jr_0\right)^{2\gamma}a^n\int_{2^jr_0/a}^\infty t^{-2n-2\gamma-1}dy\,dt\\
&\le Ca^{3n+2\gamma}\left(2^jr_0\right)^{-2n}.
\end{aligned}\tag{43}$$

And by condition (ii), similar to the estimate of I, we obtain

$$\begin{aligned}
II&\le C\int_{2^jr_0/a}^\infty\int_{\Gamma''}t^{-3n-1}\left(\frac{|z|}{t}\right)^{2\gamma}\left(1+\frac{|y|}{t}\right)^{-2(n+\varepsilon)}dy\,dt\\
&\le C\int_{2^jr_0/a}^\infty\int_{\Gamma''}\frac{t^{2\varepsilon-n-1}}{\left(t+|y|\right)^{2(n+\varepsilon)}}dy\,dt\\
&\le Ca^{3n}\left(2^jr_0\right)^{-2n}.
\end{aligned}\tag{44}$$

Hence, from the estimates above, it follows that

$$S_{\psi,a}\left(f_h^j\right)(x)\le Ca^{3n/2+\gamma+\varepsilon}\left(2^jr_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}.\tag{45}$$

Thus,

$$\begin{aligned}
D_2&\le\frac{C}{u(x_0,r_0)}\sum_{j=1}^\infty\left\|\left\{\sum_h\left|a^{3n/2+\gamma+\varepsilon}\left(2^jr_0\right)^{-n}\right.\right.\right.\\
&\qquad\left.\left.\left.\times\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le a^{3n/2+\gamma+\varepsilon}\frac{C}{u(x_0,r_0)}\sum_{j=1}^\infty\left(2^jr_0\right)^{-n}\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\qquad\times\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^1(\mathbb{R}^n)}.
\end{aligned}\tag{46}$$

Therefore, applying the generalized Hölder's inequality, Lemma 16, and (12), we have

$$\begin{aligned}
D_2 &\le a^{3n/2+\gamma+\varepsilon}\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^j r_0\right)^{-n}\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\le\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^j r_0\right)^{-n}\left|B\left(x_0,2^{j+1}r_0\right)\right|\\
&\quad\times\frac{\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\times\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\frac{\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}u\left(x_0,2^{j+1}r_0\right)\\
&\quad\times\frac{1}{u(x_0,2^{j+1}r_0)}\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\sup_{y\in\mathbb{R}^n,R>0}\frac{1}{u(y,R)}\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\chi_{B(y,R)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}.
\end{aligned}\tag{47}$$

Adding up the estimates of D_1, D_2, we obtain

$$\begin{aligned}
&\frac{1}{u(x_0,r_0)}\left\|\left\{\sum_h\left|S_{\psi,a}(f_h)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\le C\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{\mathrm{M}_{p(\cdot),u}(\mathbb{R}^n)}.
\end{aligned}\tag{48}$$

This completes the proof of Theorem 6. □

Now let us prove Theorems 7 and 8 in brief.

Proof of Theorem 7. For g_ψ, similar to the estimate of $S_{\psi,a}$, via a simple calculation, we get that (see [26]) if $x \in B(x_0,r_0)$, supp $f_h^j \subset B(x_0,2^{j+1}r_0)\setminus B(x_0,2^j r_0)$, $j\ge 1$, then

$$g_\psi\left(f_h^j\right)(x)\le C\left(2^j r_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}.\tag{49}$$

Hence, similar to the proof of Theorem 6, it follows from inequality (2) in Lemma 18 that

$$\left\|\left\{\sum_h\left|g_\psi(f_h)\right|^r\right\}^{1/r}\right\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)}\le C\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)}.\tag{50}$$

This accomplishes the proof of Theorem 7. □

Proof of Theorem 8. For g_μ^*, by the definitions of $S_{\psi,a}$ and g_μ^*, we have

$$\begin{aligned}
&g_\mu^*(f)(x)\\
&=\left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\mu n}|\psi_t*f(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2}\\
&\le\left(\int_0^\infty\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{\mu n}|\psi_t*f(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2}\\
&\quad+\sum_{l=1}^{\infty}\left(\int_0^\infty\int_{2^{l-1}t\le|x-y|<2^l t}\left(\frac{t}{t+|x-y|}\right)^{\mu n}\right.\\
&\qquad\left.\times|\psi_t*f(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2}\\
&\le S_{\psi,a}\left(f_h^j\right)(x)+\sum_{l=1}^{\infty}\left(1+2^{l-1}\right)^{-\mu n/2}S_{\psi,2^l a}\left(f_h^j\right)(x).
\end{aligned}\tag{51}$$

According to the estimate of $S_{\psi,a}$ in the proof of Theorem 6, we know that if $x\in B(x_0,r_0)$, supp $f_h^j\subset B(x_0,2^{j+1}r_0)\setminus B(x_0,2^j r_0)$, $j\ge 1$, then

$$S_{\psi,a}\left(f_h^j\right)(x)\le Ca^{3n/2+\varepsilon+\gamma}\left(2^j r_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}.\tag{52}$$

Thus, as $\mu>3+2(\varepsilon+\gamma)/n$, we obtain

$$\begin{aligned}
g_\mu^*\left(f_h^j\right)(x)&\le Ca^{3n/2+\varepsilon+\gamma}\left(1+\sum_{l=1}^{\infty}2^{(3n/2+\varepsilon+\gamma-\mu n/2)}\right)\\
&\quad\times\left(2^j r_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}\\
&\le Ca^{3n/2+\varepsilon+\gamma}\left(2^j r_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}.
\end{aligned}\tag{53}$$

Hence, also similar to the proof of Theorem 6, and from inequality (3) in Lemma 18, it follows that

$$\left\|\left\{\sum_h\left|g_\mu^*(f_h)\right|^r\right\}^{1/r}\right\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)}\le C\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)}.\tag{54}$$

This finishes the proof of Theorem 8. □

Proof of Theorem 9. Let $b\in \mathrm{BMO}$, $\|\{f_h\}_h\|_{l^r}\in\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)$. For any $x_0\in\mathbb{R}^n$, $r_0>0$, denote

$$f_h(x)\triangleq f_h^0(x)+\sum_{j=1}^{\infty}f_h^j(x),\tag{55}$$

where $f_h^0=f_h\chi_{B(x_0,2r_0)}$ and $f_h^j=f_h\chi_{B(x_0,2^{j+1}r_0)\setminus B(x_0,2^j r_0)}$, $j\in\mathbb{N}\setminus\{0\}$.

To finish the proof of Theorem 9, we only need to prove

$$\frac{1}{u(x_0,r_0)}\left\|\left\{\sum_h\left|\left[b^m,S_{\psi,a}\right](f_h)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}. \tag{56}$$

Thus,

$$\begin{aligned}&\frac{1}{u(x_0,r_0)}\left\|\left\{\sum_h\left|\left[b^m,S_{\psi,a}\right](f_h)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le \frac{C}{u(x_0,r_0)}\left\|\left\{\sum_h\left|\left[b^m,S_{\psi,a}\right](f_h^0)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\quad+\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left\|\left\{\sum_h\left|\left[b^m,S_{\psi,a}\right](f_h^j)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\triangleq E_1+E_2.\end{aligned} \tag{57}$$

For the term E_1, notice that $\operatorname{supp} f_h^0 \subset B(x_0,2r_0)$; by Lemma 19 and inequality (32), we have

$$\begin{aligned}E_1&\le\frac{C}{u(x_0,r_0)}\left\|\left\{\sum_h|f_h^0|^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\frac{u(x_0,2r_0)}{u(x_0,r_0)}\frac{1}{u(x_0,2r_0)}\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\chi_{B(x_0,2r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C2^n\sup_{y\in\mathbb{R}^n,R>0}\frac{1}{u(y,R)}\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\chi_{B(y,R)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}.\end{aligned} \tag{58}$$

Now we turn to estimate E_2. According to the estimate of $S_{\psi,a}$ in the proof of Theorem 6, we see that if $x\in B(x_0,r_0)$, $z\in B(x_0,2^{j+1}r_0)\setminus B(x_0,2^jr_0)$, $j\ge1$, then

$$S_{\psi,a}\left(f_h^j\right)(x)\le Ca^{3n/2+\varepsilon+\gamma}\left(2^jr_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}. \tag{59}$$

Therefore,

$$\begin{aligned}&\left|\left[b^m,S_{\psi,a}\right]\left(f_h^j\right)(x)\right|\\ &=\left|S_{\psi,a}\left[(b(x)-b)^mf\right](x)\right|\\ &\le Ca^{3n/2+\varepsilon+\gamma}\left(2^jr_0\right)^{-n}\left\|(b(\cdot)-b)^mf_h^j\right\|_{L^1(\mathbb{R}^n)}.\end{aligned} \tag{60}$$

Thus,

$$\begin{aligned}E_2&\le\frac{C}{u(x_0,r_0)}\times\sum_{j=1}^{\infty}\left\|a^{3n/2+\gamma+\varepsilon}\left(2^jr_0\right)^{-n}\left\{\sum_h\left\|(b(\cdot)-b)^mf_h^j\right\|_{L^1(\mathbb{R}^n)}^r\right\}^{1/r}\times\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^jr_0\right)^{-n}\times\left\|\left\{\sum_h\left|(b(\cdot)-b)^mf_h^j\right|^r\right\}^{1/r}\right\|_{L^1(\mathbb{R}^n)}\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^jr_0\right)^{-n}\left\|\left(b-b_{B(x_0,r_0)}\right)^m\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\times\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^1(\mathbb{R}^n)}\\ &\quad+\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^jr_0\right)^{-n}\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\times\left\|\left\{\sum_h\left|\left(b-b_{B(x_0,r_0)}\right)^mf_h^j\right|^r\right\}^{1/r}\right\|_{L^1(\mathbb{R}^n)}.\end{aligned} \tag{61}$$

Using Hölder's inequality and Lemma 17, we get

$$\begin{aligned}E_2&\le\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^jr_0\right)^{-n}\|b\|_*^m\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\times\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\ &\quad+\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^jr_0\right)^{-n}\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\times\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|\left(b-b_{B(x_0,r_0)}\right)^m\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\end{aligned}$$

$$\leq \frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^j r_0\right)^{-n}\left[j^m+1\right]\|b\|_*^m \times \left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \times \left\|\left\{\sum_h \left|f_h^j\right|^r\right\}^{\frac{1}{r}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \tag{62}$$

And then, it follows from Lemma 16 and (22) that

$$\begin{aligned} E_2 &\leq C\|b\|_*^m \frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^j r_0\right)^{-n} j^m \left|B\left(x_0,2^{j+1}r_0\right)\right| \\ &\quad\times \frac{\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \times u\left(x_0,2^{j+1}r_0\right)\frac{1}{u\left(x_0,2^{j+1}r_0\right)} \\ &\quad\times \left\|\left\{\sum_h \left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq C\|b\|_*^m \frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty} j^m \frac{\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \\ &\quad\times u\left(x_0,2^{j+1}r_0\right) \\ &\quad\times \sup_{y\in\mathbb{R}^n,R>0}\frac{1}{u(y,R)}\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\chi_{B(y,R)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}. \end{aligned} \tag{63}$$

Hence, Adding up the results of E_1, E_2, we have

$$\begin{aligned} &\frac{1}{u(x_0,r_0)}\left\|\left\{\sum_h \left|\left[b^m,S_{\psi,a}\right](f_h)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\quad\leq C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}. \end{aligned} \tag{64}$$

The proof of Theorem 9 is accomplished. □

Proof of Theorem 10. For $[b^m, g_\psi]$, according to the estimate of g_ψ in the proof of Theorem 7, we see that if $x \in B(x_0,r_0)$, supp $f_h^j \subset B(x_0,2^{j+1}r_0)\setminus B(x_0,2^j r_0)$, $j \geq 1$, then

$$g_\psi\left(f_h^j\right)(x) \leq C\left(2^j r_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}. \tag{65}$$

Thus,

$$\begin{aligned} \left[b^m,g_\psi\right] &= \left|g_\psi\left[(b(x)-b)^m f\right](x)\right| \\ &\leq C\left(2^j r_0\right)^{-n}\left\|(b(\cdot)-b)^m f_h^j\right\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{66}$$

Hence, similar to the proof of Theorem 9, and from inequality (2) in Lemma 19, it follows that

$$\left\|\left\{\sum_h \left|g_\psi(f_h)\right|^r\right\}^{1/r}\right\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)} \leq C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)}. \tag{67}$$

The proof of Theorem 10 is completed. □

Proof of Theorem 11. For $[b^m, g_\mu^*]$, according to the estimate of $[b^m, g_\mu^*]$ in the proof of Theorem 8, we get that if $x \in B(x_0,r_0)$, supp $f_h^j \subset B(x_0,2^{j+1}r_0)\setminus B(x_0,2^j r_0)$, $j \geq 1$, then

$$g_\mu^*\left(f_h^j\right)(x) \leq Ca^{3n/2+\varepsilon+\gamma}\left(2^j r_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}. \tag{68}$$

Thus,

$$\begin{aligned} \left[b^m,g_\mu^*\right] &= \left|g_\mu^*\left[(b(x)-b)^m f\right](x)\right| \\ &\leq Ca^{3n/2+\varepsilon+\gamma}\left(2^j r_0\right)^{-n}\left\|(b(\cdot)-b)^m f_h^j\right\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{69}$$

Hence, also similar to the proof of Theorem 9, it follows from inequality (3) in Lemma 19 that

$$\left\|\left\{\sum_h \left|\left[b^m,g_\mu^*\right](f_h)\right|^r\right\}^{1/r}\right\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)} \leq C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)}. \tag{70}$$

The proof of Theorem 11 is accomplished. □

Conflict of Interests

The authors declare that they have no conflict of interests.

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Research Article

Generalized Equilibrium Problem with Mixed Relaxed Monotonicity

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We extend the concept of relaxed α-monotonicity to mixed relaxed α-β-monotonicity. The concept of mixed relaxed α-β-monotonicity is more general than many existing concepts of monotonicities. Finally, we apply this concept and well known KKM-theory to obtain the solution of generalized equilibrium problem.

1. Introduction

Generalized monotonicities provide a way of finding parameter moves that yield monotonicity of model solutions and allow studying the monotonicity of functions or subset of variables. In recent past, many researchers have proposed many important generalizations of monotonicity such as pseudomonotonicity, relaxed monotonicity, relaxed α-β-monotonicity, quasimonotonicity, and semimonotonicity; see [1–3]. Karamardian and Schaible [4] introduced various kinds of monotone mappings which in the case of gradient mappings are related to generalized convex functions. For more details, we refer to [5–7].

Many problems of practical interest in optimization, economics, and engineering involve equilibrium in their description. The techniques involved in the study of equilibrium problems are applicable to a variety of diverse areas and proved to be productive and innovative. Blum and Oettli [8] and Noor and Oettli [9] have shown that the mathematical programming problem can be viewed as special realization of abstract equilibrium problems.

Inspired and motivated by the recent development of equilibrium problems and their solutions methods, in this paper, we extend the concept of relaxed α-monotonicity to mixed relaxed α-β-monotonicity. Finally, this concept is applied with KKM-theory to solve a generalized equilibrium problem. The results of this paper can be viewed as generalization of many known results; see [10–13].

2. Preliminaries

Let K be a nonempty subset of real Banach space X. Let $\phi : K \times K \to R$ be a real-valued function and let $f : K \times K \to R$ be an equilibrium function; that is, $f(x, x) = 0$, for all $x \in K$. We consider the following generalized equilibrium problem: find $\overline{x} \in K$ such that

$$f(\overline{x}, y) + \phi(\overline{x}, y) - \phi(\overline{x}, \overline{x}) \geq 0, \quad \forall y \in K. \tag{1}$$

Problem (1) has been studied by many authors in different settings; see, for instance, [14].

If $\phi \equiv 0$, then the problem (1) reduces to the classical equilibrium problem, that is, to find $\overline{x} \in K$ such that

$$f(\overline{x}, y) \geq 0, \quad \text{with } f(x, x) = 0, \quad \forall y \in K. \tag{2}$$

Problem (2) was introduced and studied by Blum and Oettli [8].

We need the following definition and results in the sequel.

Definition 1. A real-valued function defined on a convex subset K of X is said to be hemicontinuous if

$$\lim_{t \to 0^+} f(tx + (1-t)y) = f(y), \quad \text{for each } x, y \in K. \tag{3}$$

Definition 2. Let $f : K \to 2^X$ be a multivalued mapping. The f is said to be a KKM-mapping if, for any finite subset $\{y_1, y_2, \ldots, y_n\}$ of K, $\mathrm{co}\{y_1, y_2, \ldots, y_n\} \subset \bigcup_{i=1}^{n} f(y_i)$, where co denotes the convex hull.

Lemma 3 (see [15]). *Let K be a nonempty subset of a topological vector space X and let $f : K \to 2^X$ be a KKM-mapping. If $f(y)$ is closed in X for all $y \in K$ and compact for at least one $y \in K$, then $\bigcap_{y \in K} f(y) \neq \phi$.*

Definition 4. Let X be a Banach space. A mapping $f : X \to R$ is said to be lower semicontinuous at $x_0 \in X$, if

$$f(x_0) \leq \lim_n \inf f(x_n), \tag{4}$$

for any sequence $\{x_n\}$ of X such that $x_n \to x_0$.

Definition 5. Let X be a Banach space. A mapping $f : X \to R$ is said to be weakly upper semicontinuous at $x_0 \in X$, if

$$f(x_0) \geq \lim_x \sup f(x_n), \tag{5}$$

for any sequence $\{x_n\}$ of X such that $x_n \to x_0$.

Now, we extend the definition of relaxed α-monotonicity [11] to mixed relaxed α-β-monotonicity.

Definition 6. A mapping $f : K \times K \to R$ is said to be mixed relaxed α-β-monotone, if there exist mappings $\alpha : K \to R$ with $\alpha(tx) = t^p \alpha(x)$, for all $t > 0$ and $\beta : K \times K \to R$, such that

$$f(x, y) + f(y, x) \leq \alpha(y - x) + \beta(x, y), \quad \forall x, y \in K, \tag{6}$$

where

$$\lim_{t \to 0} \left[\frac{t^p \alpha(y - x)}{t} + \frac{\beta(x, ty + (1 - t)x)}{t} \right] = 0, \tag{7}$$

and $p > 1$ is a constant.

If $\beta = 0$, then Definition 6 reduces to the definition of generalized relaxed α-monotone; that is,

$$f(x, y) + f(y, x) \leq \alpha(y - x), \quad \forall x, y \in K, \tag{8}$$

where

$$\lim_{t \to 0} \left[\frac{t^p \alpha(y - x)}{t} \right] = 0, \quad p > 1 \text{ is a constant.} \tag{9}$$

If $\alpha = 0$, then Definition 6 reduces to the definition of generalized relaxed β-monotone; that is,

$$f(x, y) + f(y, x) \leq \beta(x, y), \quad \forall x, y \in K, \tag{10}$$

where

$$\lim_{t \to 0} \frac{\beta(x, ty + (1 - t)x)}{t} = 0. \tag{11}$$

If both $\alpha = 0 = \beta$, then Definition 6 coincides with the definition of monotonicity; that is,

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in K. \tag{12}$$

Definition 7. A mapping $\phi : K \times K \to R \cup \{\pm 0\}$ is said to be o-diagonally convex if, for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of K and $\lambda_i \geq 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $\overline{x} = \sum_{i=1}^{n} \lambda_i x_i$, one has

$$\sum_{i=1}^{n} \lambda_i \phi(\overline{x}, x_i) \geq 0. \tag{13}$$

3. Existence of Solution for Generalized Equilibrium Problem

We establish this section with the discussion of existence of solution for generalized equilibrium problem by using mixed relaxed α-β-monotonicity.

Theorem 8. *Suppose $f : K \times K \to R$ is mixed relaxed α-β-monotone, hemicontinuous in the first argument and convex in the second argument with $f(x, x) = 0$, for all $x \in K$. Let $\phi : K \times K \to R$ be convex in the second argument. Then, generalized equilibrium problem* (1) *is equivalent to the following problem.*

Find $\overline{x} \in K$ such that

$$f(y, \overline{x}) + \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, y) \leq \alpha(y - \overline{x}) + \beta(\overline{x}, y), \quad \forall y \in K, \tag{14}$$

where $\alpha(tx) = t^p \alpha(x)$ and $p > 1$ is a constant.

Proof. Suppose that the generalized equilibrium problem (1) admits a solution; that is, there exists $\overline{x} \in K$ such that

$$f(\overline{x}, y) + \phi(\overline{x}, y) - \phi(\overline{x}, \overline{x}) \geq 0, \quad \forall y \in K. \tag{15}$$

Since f is mixed relaxed α-β-monotone, we have

$$f(\overline{x}, y) + f(y, \overline{x}) \leq \alpha(y - \overline{x}) + \beta(\overline{x}, y), \quad \forall y \in K, \tag{16}$$

$$f(y, \overline{x}) \leq \alpha(y - \overline{x}) + \beta(\overline{x}, y) - f(\overline{x}, y), \quad \forall y \in K. \tag{17}$$

Adding $\phi(\overline{x}, \overline{x}) - \phi(\overline{x}, x)$ on both sides of (17), we have

$$\begin{aligned} &f(y, \overline{x}) + \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, y) \\ &\quad \leq \alpha(y - \overline{x}) + \beta(\overline{x}, y) - [f(\overline{x}, y) + \phi(\overline{x}, y) - \phi(\overline{x}, x)] \\ &\quad \leq \alpha(y - \overline{x}) + \beta(\overline{x}, y), \quad \forall y \in K. \end{aligned} \tag{18}$$

Hence, $\overline{x} \in K$ is a solution of problem (14).

Conversely, suppose that $\overline{x} \in K$ is a solution of problem (14); that is,

$$f(y, \overline{x}) + \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, y) \leq \alpha(y - \overline{x}) + \beta(\overline{x}, y), \quad \forall y \in K. \tag{19}$$

Let $x_t = ty + (1-t)\overline{x}, t \in [0,1]$, and $y \in K$; then clearly $x_t \in K$ as K is convex. Thus from (17), we have

$$f(x_t, \overline{x}) + \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, x_t) \leq \alpha(x_t - \overline{x}) + \beta(\overline{x}, x_t). \tag{20}$$

Since f is convex in the second argument, we have

$$0 = f(x_t, x_t) \leq tf(x_t, y) + (1-t) f(x_t, \overline{x}) \tag{21}$$

which implies that

$$t[f(x_t, \overline{x}) - f(x_t, y)] \leq f(x_t, \overline{x}). \tag{22}$$

Also as ϕ is convex in the second argument, we have

$$\begin{aligned} \phi(\overline{x}, x_t) &\leq t\phi(\overline{x}, y) + (1-t)\phi(\overline{x}, \overline{x}), \\ -t\phi(\overline{x}, y) &\leq -\phi(\overline{x}, x_t) + (1-t)\phi(\overline{x}, \overline{x}), \\ t\phi(\overline{x}, \overline{x}) - t\phi(\overline{x}, y) &\leq \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, x_t), \end{aligned} \tag{23}$$

$$t[\phi(\overline{x}, \overline{x}) - \phi(\overline{x}, y)] \leq \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, x_t). \tag{24}$$

Adding (22) and (24), we have

$$\begin{aligned} &t[f(x_t, \overline{x}) - f(x_t, y) + \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, y)] \\ &\quad \leq f(x_t, \overline{x}) + \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, x_t) \\ &\quad \leq \alpha(x_t - \overline{x}) + \beta(\overline{x}, x_t). \end{aligned} \tag{25}$$

It follows that

$$\begin{aligned} &f(x_t, \overline{x}) - f(x_t, y) + \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, y) \\ &\quad \leq \frac{\alpha(x_t - \overline{x})}{t} + \frac{\beta(\overline{x}, x_t)}{t} \\ &\quad \leq \frac{t^p \alpha(y - \overline{x})}{t} + \frac{\beta(\overline{x}, x_t)}{t}, \quad p > 1. \end{aligned} \tag{26}$$

Since f is hemicontinuous in the first argument, taking $t \to 0$, we have

$$f(\overline{x}, \overline{x}) - f(\overline{x}, y) + \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, y) \leq 0; \tag{27}$$

that is, we have

$$f(\overline{x}, y) + \phi(\overline{x}, y) - \phi(\overline{x}, \overline{x}) \geq 0, \quad \forall y \in K. \tag{28}$$

Hence $\overline{x} \in K$ is a solution of generalized equilibrium problem (1). □

Theorem 9. *Let K be a nonempty bounded closed convex subset of a real Banach space X. Let $f : K \times K \to R$ be a mixed relaxed α-β-monotone, hemicontinuous in the first argument, convex in the second argument with $f(x,x) = 0$, o-diagonally convex, and lower semicontinuous. Let $\phi : K \times K \to R$ be convex in the second argument, o-diagonally convex, and lower semicontinuous; $\alpha : K \to R$ is weakly upper semicontinuous and $\beta : K \times K \to R$ is weakly upper semicontinuous in the second argument. Then the mixed equilibrium problem* (1) *admits a solution.*

Proof. Consider a multivalued mapping $F : K \to 2^X$ such that

$$F(y) = \{\overline{x} \in K : f(\overline{x}, y) + \phi(\overline{x}, y) - \phi(\overline{x}, \overline{x}) \geq 0\}, \quad \forall y \in K. \tag{29}$$

We show that $\bigcap_{y \in K} F(y) = \phi$; that is, $\overline{x} \in K$ is a solution of generalized equilibrium problem (1).

Our claim is that F is a KKM-mapping. Suppose to contrary that is F is not a KKM-mapping; then there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of K and $\lambda_i \geq 0$ $(i = 1, 2, \ldots n)$ with $\sum_{i=1}^n \lambda_i = 1$ such that

$$x_0 = \sum_{i=1}^n \lambda_i x_i \notin \bigcup_{i=1}^n F(y_i). \tag{30}$$

It follows that

$$f(x_0, x_i) + \phi(x_0, x_i) - \phi(x_0, x_0) < 0, \quad \text{for } i = 1, 2, \ldots, n. \tag{31}$$

Also we have

$$\sum_{i=1}^n \lambda_i [f(x_0, x_i) + \phi(x_0, x_i) - \phi(x_0, x_0)] < 0, \quad \text{for } i = 1, 2, \ldots, n, \tag{32}$$

which contradicts the o-diagonal convexity of f and ϕ. Hence F is a KKM-mapping.

Now consider another multivalued mapping $G : K \to 2^X$ such that

$$G(y) = \{\overline{x} \in K : f(y, \overline{x}) + \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, y) \leq \alpha(y - \overline{x}) + \beta(\overline{x}, y)\}, \quad \forall y \in K. \tag{33}$$

We will show that $F(y) \subset G(y), \forall y \in K$. For any given $y \in K$, let $\overline{x} \in F(y)$; then

$$f(\overline{x}, y) + \phi(\overline{x}, y) - \phi(\overline{x}, \overline{x}) \geq 0. \tag{34}$$

It follows from the mixed relaxed α-β-monotonicity of f that

$$\begin{aligned} &f(y, \overline{x}) + \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, y) \\ &\quad \leq \alpha(y - \overline{x}) + \beta(\overline{x}, y) - [f(\overline{x}, y) + \phi(\overline{x}, y) - \phi(\overline{x}, \overline{x})] \\ &\quad \leq \alpha(y - \overline{x}) + \beta(\overline{x}, y); \end{aligned} \tag{35}$$

that is, $\overline{x} \in G(y)$. Thus $F(y) \subset G(y)$ and consequently G is also KKM-mapping.

Since f and ϕ both are convex in the second argument and lower semicontinuous, thus they both are weakly lower semicontinuous. From weakly upper semicontinuity of α, weakly upper semicontinuity of β in the second argument, and the construction of G, it is accessible to see that $G(y)$ is weakly closed for all $y \in K$. Since K is closed, bounded, and convex, it is weakly compact and consequently $G(y)$ is weakly

compact in K for all $y \in K$. Therefore, from Lemma 3 and Theorem 8, we have

$$\bigcap_{y\in K} F(y) = \bigcap_{y\in K} G(y) \neq \phi; \tag{36}$$

that is, there exists $\overline{x} \in K$ such that

$$f(\overline{x}, y) + \phi(\overline{x}, y) - \phi(\overline{x}, \overline{x}) \geq 0, \quad \forall y \in K. \tag{37}$$

Thus, the generalized equilibrium problem (1) admits a solution. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Weighted A-Statistical Convergence for Sequences of Positive Linear Operators

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We introduce the notion of weighted A-statistical convergence of a sequence, where A represents the nonnegative regular matrix. We also prove the Korovkin approximation theorem by using the notion of weighted A-statistical convergence. Further, we give a rate of weighted A-statistical convergence and apply the classical Bernstein polynomial to construct an illustrative example in support of our result.

1. Background, Notations, and Preliminaries

We begin this paper by recalling the definition of natural (or asymptotic) density as follows. Suppose that $E \subseteq \mathbb{N} := \{1, 2, \ldots\}$ and $E_n = \{k \leq n : k \in E\}$. Then

$$\delta(E) = \lim_n \frac{1}{n} \left|E_n\right| \tag{1}$$

is called the *natural density* of E provided that the limit exists, where $|\cdot|$ represents the number of elements in the enclosed set.

The term "statistical convergence" was first presented by Fast [1] which is a generalization of the concept of ordinary convergence. Actually, a root of the notion of statistical convergence can be detected by Zygmund [2] (also, see [3]), where he used the term "almost convergence" which turned out to be equivalent to the concept of statistical convergence. The notion of Fast was further investigated by Schoenberg [4], Šalát [5], Fridy [6], and Conner [7].

The following notion is due to Fast [1]. A sequence $x = (x_k)$ is said to be *statistically convergent* to L if $\delta(K_\epsilon) = 0$ for every $\epsilon > 0$, where

$$K_\epsilon := \{k \in \mathbb{N} : \left|x_k - L\right| \geq \epsilon\}. \tag{2}$$

Equivalently,

$$\lim_n n^{-1} \left|\{k \leq n : \left|x_k - L\right| \geq \epsilon\}\right| = 0. \tag{3}$$

In symbol, we will write $S\text{-}\lim x = L$. We remark that every convergent sequence is statistically convergent but not conversely.

Let X and Y be two sequence spaces and let $A = (a_{n,k})$ be an infinite matrix. If for each $x = (x_k)$ in X the series

$$A_n x = \sum_k a_{n,k} x_k = \sum_{k=1}^{\infty} a_{n,k} x_k \tag{4}$$

converges for each $n \in \mathbb{N}$ and the sequence $Ax = (A_n x)$ belongs to Y, then we say that matrix A maps X into Y. By the symbol (X, Y) we denote the set of all matrices which map X into Y.

A matrix A (or a matrix map A) is called regular if $A \in (c, c)$, where the symbol c denotes the spaces of all convergent sequences and

$$\lim_n A_n x = \lim_k x_k \tag{5}$$

for all $x \in c$. The well-known Silverman-Toeplitz theorem (see [8]) asserts that $A = (a_{n,k})$ is regular if and only if

(i) $\lim_n a_{n,k} = 0$ for each k;

(ii) $\lim_n \sum_k a_{n,k} = 1$;

(iii) $\sup_n \sum_k |a_{n,k}| < \infty$.

Kolk [9] extended the definition of statistical convergence with the help of nonnegative regular matrix $A = (a_{n,k})$ calling it A-statistical convergence. The definition of A-statistical convergence is given by Kolk as follows. For any nonnegative regular matrix A, we say that a sequence is A-statistically convergent to L provided that for every $\epsilon > 0$ we have

$$\lim_n \sum_{k:|x_n - L| \geq \epsilon} a_{n,k} = L. \tag{6}$$

In 2009, the concept of weighted statistical convergence was defined and studied by Karakaya and Chishti [10] and further modified by Mursaleen et al. [11] in 2012. In 2013, Belen and Mohiuddine [12] presented a generalization of this notion through de la Vallée-Poussin mean. Quite recently, Esi [13] defined and studied the notion statistical summability through de la Vallée-Poussin mean in probabilistic normed spaces.

Let $p = (p_k)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and

$$P_n = \sum_{k=0}^{n} p_k \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{7}$$

Let

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k x_k, \quad n = 0, 1, 2 \ldots. \tag{8}$$

We say that the sequence $x = (x_k)$ is $(\overline{N}, p_n)$-*summable* to L if $\lim_{n \to \infty} t_n = L$.

The lower and upper weighted densities of $E \subseteq \mathbb{N}$ are defined by

$$\underline{\delta}_{\overline{N}}(E) = \liminf_n \frac{1}{P_n} \left|\{k \leq P_n : k \in E\}\right|, \tag{9}$$

$$\overline{\delta}_{\overline{N}}(E) = \limsup_n \frac{1}{P_n} \left|\{k \leq P_n : k \in E\}\right|, \tag{10}$$

respectively. We say that E has *weighted density*, denoted by $\delta_{\overline{N}}(E)$, if the limits of both of the above densities exist and are equal; that is, one writes

$$\delta_{\overline{N}}(E) = \lim_n \frac{1}{P_n} \left|\{k \leq P_n : k \in E\}\right|. \tag{11}$$

The sequence $x = (x_k)$ is said to be *weighted statistically convergent* (or $S_{\overline{N}}$-*convergent*) to L if, for every $\epsilon > 0$, the set $\{k \in \mathbb{N} : p_k |x_k - L| \geq \epsilon\}$ has weighted density zero; that is,

$$\lim_n \frac{1}{P_n} \left|\{k \leq P_n : p_k \left|x_k - L\right| \geq \epsilon\}\right| = 0. \tag{12}$$

In this case we write $L = S_{\overline{N}}\text{-}\lim x$.

Remark 1. If $p_k = 1$ for all k, then $(\overline{N}, p_n)$-summable is reduced to $(C, 1)$-summable (or Cesàro summable) and weighted statistical convergence is reduced to statistical convergence.

On the other hand, let us recall that $\mathscr{C}[a, b]$ is the space of all functions f continuous on $[a, b]$. We know that $\mathscr{C}[a, b]$ is a Banach space with norm

$$\|f\|_\infty := \sup_{x \in [a,b]} |f(x)|, \quad f \in \mathscr{C}[a, b]. \tag{13}$$

Suppose that L is a linear operator from $\mathscr{C}[a, b]$ into $\mathscr{C}[a, b]$. It is clear that if $f \geq 0$ implies $Lf \geq 0$, then the linear operator L is positive on $\mathscr{C}[a, b]$. We denote the value of Lf at a point $x \in [a, b]$ by $L(f; x)$. The classical Korovkin approximation theorem states the following [14].

Theorem 2. *Let (T_n) be a sequence of positive linear operators from $\mathscr{C}[a, b]$ into $\mathscr{C}[a, b]$. Then,*

$$\lim_n \|T_n(f, x) - f(x)\|_\infty = 0, \tag{14}$$

for all $f \in \mathscr{C}[a, b]$ if and only if

$$\lim_n \|T_n(f_i, x) - f_i(x)\|_\infty = 0, \tag{15}$$

where $f_i(x) = x^i$ and $i = 0, 1, 2$.

Many mathematicians extended the Korovkin-type approximation theorems by using various test functions in several setups, including Banach spaces, abstract Banach lattices, function spaces, and Banach algebras. Firstly, Gadjiev and Orhan [15] established classical Korovkin theorem through statistical convergence and display an interesting example in support of our result. Recently, Korovkin-type theorems have been obtained by Mohiuddine [16] for almost convergence. Korovkin-type theorems were also obtained in [17] for λ-statistical convergence. The authors of [18] established these types of approximation theorem in weighted L_p spaces, where $1 \leq p < \infty$, through A-summability which is stronger than ordinary convergence. For these types of approximation theorems and related concepts, one can be referred to [19–27] and references therein.

2. Korovkin-Type Theorems by Weighted A-Statistical Convergence

Kolk [9] introduced the notion of A-statistical convergence by considering nonnegative regular matrix A instead of Cesáro matrix in the definition of statistical convergence due to Fast. Inspired from the work of Kolk, we introduce the notion of weighted A-statistical convergence of a sequence and then we establish some Korovkin-type theorems by using this notion.

Definition 3. Let $A = (a_{n,k})$ be a nonnegative regular matrix. A sequence $x = (x_k)$ of real or complex numbers is said to be *weighted A-statistically convergent*, denoted by $S_A^{\overline{N}}$-convergent, to L if for every $\epsilon > 0$

$$\lim_n \sum_{k \in E(p,\epsilon)} a_{n,k} = 0, \tag{16}$$

where

$$E(p,\epsilon) = \{k \in \mathbb{N} : p_k |x_k - L| \geq \epsilon\}. \quad (17)$$

In symbol, we will write $S_A^{\overline{N}}\text{-lim}\, x = L$.

Remark 4. One has the following.

(i) If we take $A = I$, where I denotes the identity matrix, then weighted A-statistical convergence of a sequence is reduced to ordinary convergence.

(ii) If we take $A = (C,1)$, where $(C,1)$ denotes the Cesáro matrix of order one, then weighted A-statistical convergence of a sequence reduces to weighted statistical convergence.

(iii) If we take $A = (C,1)$ and $p_k = 1$ for all k, then weighted A-statistical convergence of a sequence reduces to statistical convergence.

Note that convergent sequence implies weighted A-statistically convergent to the same value but the converse is not true in general. For example, take $A = (C,1)$ and $p_k = 1$ for all k and define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1, & \text{if } k = n^2, \\ 0, & \text{otherwise,} \end{cases} \quad (18)$$

where $n \in \mathbb{N}$. Then this sequence is statistically convergent to 0 but not convergent; in this case, weighted A-statistical convergence of a sequence coincides with statistical convergence.

Theorem 5. *Let $A = (a_{n,k})$ be a nonnegative regular matrix. Consider a sequence of positive linear operators (M_k) from $\mathscr{C}[a,b]$ into itself. Then, for all $f \in \mathscr{C}[a,b]$ bounded on whole real line,*

$$S_A^{\overline{N}}\text{-}\lim_k \|M_k(f,x) - f(x)\|_\infty = 0 \quad (19)$$

if and only if

$$\begin{aligned} S_A^{\overline{N}}\text{-}\lim_k \|M_k(1,x) - 1\|_\infty &= 0, \\ S_A^{\overline{N}}\text{-}\lim_k \|M_k(v,x) - x\|_\infty &= 0, \\ S_A^{\overline{N}}\text{-}\lim_k \|M_k(v^2,x) - x^2\|_\infty &= 0. \end{aligned} \quad (20)$$

Proof. Equation (20) directly follows from (19) because each of $1, x, x^2$ belongs to $\mathscr{C}[a,b]$. Consider a function $f \in \mathscr{C}[a,b]$. Then there is a constant $C > 0$ such that $|f(x)| \leq C$ for all $x \in (-\infty,\infty)$. Therefore,

$$|f(v) - f(x)| \leq 2C, \quad -\infty < v, x < \infty. \quad (21)$$

Let $\epsilon > 0$ be given. By hypothesis there is a $\delta := \delta(\varepsilon) > 0$ such that

$$|f(v) - f(x)| < \epsilon \quad \forall |v - x| < \delta. \quad (22)$$

Solving (21) and (22) and then substituting $\Omega(v) = (v-x)^2$, one obtains

$$|f(v) - f(x)| < \epsilon + \frac{2C}{\delta^2}\Omega, \quad \forall |v - x| < \delta. \quad (23)$$

Equation (23) can also be written as

$$-\epsilon - \frac{2C}{\delta^2}\Omega < f(v) - f(x) < \epsilon + \frac{2M}{\delta^2}\Omega. \quad (24)$$

Operating $M_k(1,x)$ to (24) since $M_k(f,x)$ is linear and monotone, one obtains

$$\begin{aligned} M_k(1,x)\left(-\epsilon - \frac{2C}{\delta^2}\Omega\right) &< M_k(1,x)(f(v) - f(x)) \\ &< M_k(1,x)\left(\epsilon + \frac{2C}{\delta^2}\Omega\right). \end{aligned} \quad (25)$$

Note that x is fixed, so $f(x)$ is constant number. Thus, we obtain from (25) that

$$\begin{aligned} -\epsilon M_k(1,x) - \frac{2C}{\delta^2}M_k(\Omega,x) &< M_k(f,x) - f(x)M_k(1,x) \\ &< \epsilon M_k(1,x) + \frac{2C}{\delta^2}M_k(\Omega,x). \end{aligned} \quad (26)$$

The term "$M_k(f,x) - f(x)M_k(1,x)$" in (26) can also be written as

$$\begin{aligned} M_k(f,x) - f(x)M_k(1,x) &= M_k(f,x) - f(x) \\ &\quad - f(x)[M_k(1,x) - 1]. \end{aligned} \quad (27)$$

Now substituting the value of $M_k(f,x) - f(x)M_k(1,x)$ in (26), we get that

$$\begin{aligned} M_k(f,x) - f(x) &< \epsilon M_k(1,x) + \frac{2C}{\delta^2}M_k(\Omega,x) \\ &\quad + f(x)(M_k(1,x) - 1). \end{aligned} \quad (28)$$

We can rewrite the term "$M_k(\Omega,x)$" in (28) as follows:

$$\begin{aligned} M_k(\Omega,x) &= M_k((v-x)^2,x) \\ &= M_k(v^2,x) + 2xM_k(v,x) + x^2M_k(1,x) \\ &= [M_k(v^2,x) - x^2] - 2x[M_k(v,x) - x] \\ &\quad + x^2[M_k(1,x) - 1]. \end{aligned} \quad (29)$$

Equation (28) with the above value of $M_k(\Omega, x)$ becomes

$$\begin{aligned} M_k(f,x) - f(x) &< \epsilon M_k(1,x) \\ &\quad + \frac{2C}{\delta^2}\left\{\left[M_k\left(v^2,x\right) - x^2\right] + 2x\left[M_k(v,x) - x\right]\right. \\ &\qquad \left. + x^2\left[M_k(1,x) - 1\right]\right\} + f(x)\left(M_k(1,x) - 1\right) \\ &= \epsilon\left[M_k(1,x) - 1\right] + \epsilon \\ &\quad + \frac{2C}{\delta^2}\left\{\left[M_k\left(v^2,x\right) - x^2\right] + 2x\left[M_k(v,x) - x\right]\right. \\ &\qquad \left. + x^2\left[M_k(1,x) - 1\right]\right\} + f(x)\left[M_k(1,x) - 1\right]. \end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned} \left|M_k(f,x) - f(x)\right| \le \left(\epsilon + \frac{2Cb^2}{\delta^2} + C\right)\left|M_k(1,x) - 1\right| \\ + \frac{2C}{\delta^2}\left|M_k\left(v^2,x\right) - x^2\right| \\ + \frac{4Cb}{\delta^2}\left|M_k(v,x) - x\right|, \end{aligned} \tag{31}$$

where $b = \max|x|$. Taking supremum over $x \in [a,b]$, one obtains

$$\begin{aligned} \left\|M_k(f,x) - f(x)\right\|_\infty \le \left(\epsilon + \frac{2Cb^2}{\delta^2} + C\right)\left\|M_k(1,x) - 1\right\|_\infty \\ + \frac{2C}{\delta^2}\left\|M_k\left(v^2,x\right) - x^2\right\|_\infty \\ + \frac{4Cb}{\delta^2}\left\|M_k(v,x) - x\right\|_\infty, \end{aligned} \tag{32}$$

or

$$\begin{aligned} &\left\|M_k(f,x) - f(x)\right\|_\infty \\ &\quad \le T\left\{\left\|M_k(1,x) - 1\right\|_\infty + \left\|M_k\left(v^2,x\right) - x^2\right\|_\infty\right. \\ &\qquad \left. + \left\|M_k(v,x) - x\right\|_\infty\right\}, \end{aligned} \tag{33}$$

where

$$T := \max\left\{\epsilon + \frac{2Cb^2}{\delta^2} + C, \frac{2C}{\delta^2}, \frac{4Cb}{\delta^2}\right\}. \tag{34}$$

Hence,

$$\begin{aligned} &p_k\left\|M_k(f,x) - f(x)\right\|_\infty \\ &\quad \le T\left\{p_k\left\|M_k(1,x) - 1\right\|_\infty + p_k\left\|M_k\left(v^2,x\right) - x^2\right\|_\infty\right. \\ &\qquad \left. + p_k\left\|M_k(v,x) - x\right\|_\infty\right\}. \end{aligned} \tag{35}$$

For a given $\alpha > 0$, choose $\epsilon > 0$ such that $\epsilon < \alpha$, and we will define the following sets:

$$\begin{aligned} E &= \left\{k \in \mathbb{N} : p_k\left\|M_k(f,x) - f(x)\right\|_\infty \ge \alpha\right\}, \\ E_1 &= \left\{k \in \mathbb{N} : p_k\left\|M_k(1,x) - 1\right\|_\infty \ge \frac{\alpha - \epsilon}{3T}\right\}, \\ E_2 &= \left\{k \in \mathbb{N} : p_k\left\|M_k(v,x) - x\right\|_\infty \ge \frac{\alpha - \epsilon}{3T}\right\}, \\ E_3 &= \left\{k \in \mathbb{N} : p_k\left\|M_k\left(v^2,x\right) - x^2\right\|_\infty \ge \frac{\alpha - \epsilon}{3T}\right\}. \end{aligned} \tag{36}$$

It is easy to see that

$$E \subset E_1 \cup E_2 \cup E_3. \tag{37}$$

Thus, for each $n \in \mathbb{N}$, we obtain from (35) that

$$\sum_{k\in E} a_{n,k} \le \sum_{k\in E_1} a_{n,k} + \sum_{k\in E_2} a_{n,k} + \sum_{k\in E_3} a_{n,k}. \tag{38}$$

Taking limit $n \to \infty$ in (38) and also (20) gives that

$$\lim_n \sum_{k\in E} a_{n,k} = 0. \tag{39}$$

This yields that

$$S_A^{\overline{N}}\text{-}\lim_k \left\|M_k(f,x) - f(x)\right\|_\infty = 0, \tag{40}$$

for all $f \in \mathscr{C}[a,b]$. □

We also obtain the following Korovkin-type theorem for weighted statistical convergence by writing Cesáro matrix $(C,1)$ instead of nonnegative regular matrix A in Theorem 5.

Theorem 6. *Consider a sequence of positive linear operators (M_k) from $\mathscr{C}[a,b]$ into itself. Then, for all $f \in \mathscr{C}[a,b]$*

$$S_{\overline{N}}\text{-}\lim_k \left\|M_k(f,x) - f(x)\right\|_\infty = 0 \tag{41}$$

if and only if

$$S_{\overline{N}}\text{-}\lim_k \left\|M_k(1,x) - 1\right\|_\infty = 0, \tag{42}$$

$$S_{\overline{N}}\text{-}\lim_k \left\|M_k(v,x) - x\right\|_\infty = 0, \tag{43}$$

$$S_{\overline{N}}\text{-}\lim_k \left\|M_k\left(v^2,x\right) - x^2\right\|_\infty = 0. \tag{44}$$

Proof. Following the proof of Theorem 5, one obtains

$$E \subset E_1 \cup E_2 \cup E_3 \tag{45}$$

and so

$$\delta_{\overline{N}}(E) \subset \delta_{\overline{N}}(E_1) + \delta_{\overline{N}}(E_2) + \delta_{\overline{N}}(E_3). \tag{46}$$

Equations (42)–(44) give that

$$S_{\overline{N}}\text{-}\lim_k \left\|M_k(f,x) - f(x)\right\|_\infty = 0. \tag{47}$$

□

Remark 7. If we replace nonnegative regular matrix A by Cesáro matrix and choose $p_k = 1$ for all k, in Theorem 5, then we obtain Theorem 1 due to Gadjiev and Orhan [15].

Remark 8. By Theorem 2 of [10], we have that if a sequence $x = (x_k)$ is weighted statistically convergent to L, then it is strongly $(\overline{N}, p_n)$-summable to L provided that $p_k|x_k - L|$ is bounded; that is, there exists a constant C such that $p_k|x_k - L| \leq C$ for all $k \in \mathbb{N}$. We write

$$\left|\overline{N}, p_n\right| = \left\{x = (x_k) : \lim_n \frac{1}{P_n}\sum_{k=0}^{n} p_k \left|x_k - L\right| = 0 \text{ for some } L\right\} \tag{48}$$

for the set of all sequences $x = (x_k)$ which are strongly $(\overline{N}, p_n)$-summable to L.

Theorem 9. *Let $M_k : \mathscr{C}[a,b] \to \mathscr{C}[a,b]$ be a sequence of positive linear operators which satisfies* (43)-(44) *of Theorem 6 and the following condition holds:*

$$\lim_k \left\|M_k(1,x) - 1\right\|_\infty = 0. \tag{49}$$

Then,

$$\lim_n \frac{1}{P_n}\sum_{k=0}^{n} p_k \left\|M_k(f,x) - f(x)\right\|_\infty = 0, \tag{50}$$

for any $f \in \mathscr{C}[a,b]$.

Proof. It follows from (49) that $\|M_k(1,x)\|_\infty \leq C'$, for some constant $C' > 0$ and for all $k \in \mathbb{N}$. Hence, for $f \in \mathscr{C}[a,b]$, one obtains

$$\begin{aligned} p_k\left\|M_k(f,x) - f(x)\right\|_\infty &\leq p_k\left(\|f\|_\infty \|M_k(1,x)\|_\infty + \|f\|_\infty\right) \\ &\leq p_k C\left(C' + 1\right). \end{aligned} \tag{51}$$

Right hand side of (51) is constant, so $p_k\|M_k(f,x) - f(x)\|_\infty$ is bounded. Since (49) implies (42), by Theorem 6 we get that

$$S_{\overline{N}}\text{-}\lim \left\|M_k(f,x) - f(x)\right\|_\infty = 0. \tag{52}$$

By Remark 8, (51) and (52) together give the desired result. □

3. Rate of Weighted A-Statistical Convergence

First we define the rate of weighted A-statistically convergent sequence as follows.

Definition 10. Let $A = (a_{n,k})$ be a nonnegative regular matrix and let (a_k) be a positive nonincreasing sequence. Then, a sequence $x = (x_k)$ is weighted A-statistically convergent to L with the rate of $o(a_k)$ if for each $\epsilon > 0$

$$\lim_n \frac{1}{a_n}\sum_{k \in E(p,\epsilon)} a_{n,k} = 0, \tag{53}$$

where

$$E(p,\epsilon) = \left\{k \in \mathbb{N} : p_k\left|x_k - L\right| \geq \epsilon\right\}. \tag{54}$$

In symbol, we will write

$$x_k - L = S_A^{\overline{N}}\text{-}o(a_k) \quad \text{as } k \longrightarrow \infty. \tag{55}$$

We will prove the following auxiliary result by using the above definition.

Lemma 11. *Let $A = (a_{n,k})$ be a nonnegative regular matrix. Suppose that (a_k) and (b_k) are two positive nonincreasing sequences. Let $x = (x_k)$ and $y = (y_k)$ be two sequences such that $x_k - L_1 = S_A^{\overline{N}}\text{-}o(a_k)$ and $y_k - L_2 = S_A^{\overline{N}}\text{-}o(b_k)$. Then,*

(i) $(x_k - L_1) \pm (y_k - L_2) = S_A^{\overline{N}}\text{-}o(c_k)$,

(ii) $(x_k - L_1)(y_k - L_2) = S_A^{\overline{N}}\text{-}o(c_k)$,

(iii) $\alpha(x_k - L_1) = S_A^{\overline{N}}\text{-}o(a_k)$, *for any scalar α,*

where $c_k = \max\{a_k, b_k\}$.

Proof. (i) Suppose that

$$x_k - L_1 = S_A^{\overline{N}}\text{-}o\left(a_k\right), \qquad y_k - L_2 = S_A^{\overline{N}}\text{-}o\left(b_k\right). \tag{56}$$

Given $\epsilon > 0$, define

$$\begin{aligned} E' &= \left\{k \in \mathbb{N} : p_k\left|\left(x_k - L_1\right) \pm \left(y_k - L_2\right)\right| \geq \epsilon\right\}, \\ E'' &= \left\{k \in \mathbb{N} : p_k\left|x_k - L_1\right| \geq \frac{\epsilon}{2}\right\}, \\ E''' &= \left\{k \in \mathbb{N} : p_k\left|y_k - L_2\right| \geq \frac{\epsilon}{2}\right\}. \end{aligned} \tag{57}$$

It is easy to see that

$$E' \subset E'' \cup E'''. \tag{58}$$

This yields that

$$\frac{1}{c_n}\sum_{k \in E'} a_{n,k} \leq \frac{1}{c_n}\sum_{k \in E''} a_{n,k} + \frac{1}{c_n}\sum_{k \in E'''} a_{n,k} \tag{59}$$

holds for all $n \in \mathbb{N}$. Since $c_k = \max\{a_k, b_k\}$, (59) gives that

$$\frac{1}{c_n}\sum_{k \in E'} a_{n,k} \leq \frac{1}{a_n}\sum_{k \in E''} a_{n,k} + \frac{1}{b_n}\sum_{k \in E'''} a_{n,k}. \tag{60}$$

Taking limit $n \to \infty$ in (60) together with (56), we obtain

$$\lim_n \frac{1}{c_n}\sum_{k \in E'} a_{n,k} = 0. \tag{61}$$

Thus,

$$\left(x_k - L_1\right) \pm \left(y_k - L_2\right) = S_A^{\overline{N}}\text{-}o\left(c_k\right). \tag{62}$$

Similarly, we can prove (ii) and (iii). □

Recall that the modulus of continuity of f in $\mathscr{C}[a,b]$ is defined by

$$\omega(f,\delta)=\sup\{|f(x)-f(y)|:x,y\in[a,b],|x-y|<\delta\}. \tag{63}$$

It is well known that

$$|f(x)-f(y)|\le\omega(f,\delta)\left(\frac{|x-y|}{\delta}+1\right). \tag{64}$$

Theorem 12. *Let $A=(a_{n,k})$ be a nonnegative regular matrix. If the sequence of positive linear operators $M_k:\mathscr{C}[a,b]\to\mathscr{C}[a,b]$ satisfies the conditions*

(i) $\|M_k(1;x)-1\|_\infty=S_A^{\overline{N}}\text{-}o(a_k)$,

(ii) $\omega(f,\lambda_k)=S_A^{\overline{N}}\text{-}o(b_k)$ *with* $\lambda_k=\sqrt{M_k(\varphi_x;x)}$ *and* $\varphi_x(y)=(y-x)^2$,

where (a_k) and (b_k) are two positive nonincreasing sequences, then

$$\|M_k(f;x)-f(x)\|_\infty=S_A^{\overline{N}}\text{-}o(c_k), \tag{65}$$

for all $f\in\mathscr{C}[a,b]$, where $c_k=\max\{a_k,b_k\}$.

Proof. Equation (27) can be reformed into the following form:

$$\begin{aligned}
|M_k(f;x)-f(x)|&\le M_k(|f(y)-f(x)|;x)\\
&\quad+|f(x)|\,|M_k(1;x)-1|\\
&\le M_k\left(1+\frac{|y-x|}{\delta};x\right)\omega(f,\delta)\\
&\quad+|f(x)|\,|M_k(1;x)-1|\\
&\le M_k\left(1+\frac{(y-x)^2}{\delta^2};x\right)\omega(f,\delta)\\
&\quad+|f(x)|\,|M_k(1;x)-1|\\
&\le\left(M_k(1;x)+\frac{1}{\delta^2}M_k(\varphi_x;x)\right)\omega(f,\delta)\\
&\quad+|f(x)|\,|M_k(1;x)-1|\\
&\le|M_k(1;x)-1|\,\omega(f,\delta)\\
&\quad+|f(x)|\,|M_k(1;x)-1|+\omega(f,\delta)\\
&\quad+\frac{1}{\delta^2}M_k(\varphi_x;x)\,\omega(f,\delta).
\end{aligned} \tag{66}$$

Choosing $\delta=\lambda_k=\sqrt{M_k(\varphi_x;x)}$, one obtains

$$\begin{aligned}
\|M_k(f;x)-f(x)\|_\infty&\le T\|M_k(1;x)-1\|_\infty+2\omega(f,\lambda_k)\\
&\quad+\|M_k(1;x)-1\|_\infty\omega(f,\lambda_k),
\end{aligned} \tag{67}$$

where $T=\|f\|_\infty$. For a given $\epsilon>0$, we will define the following sets:

$$\begin{aligned}
E_1'&=\{k\in\mathbb{N}:p_k\|M_k(f,x)-f(x)\|_\infty\ge\epsilon\},\\
E_2'&=\left\{k\in\mathbb{N}:p_k\|M_k(1,x)-1\|_\infty\ge\frac{\epsilon}{3T}\right\},\\
E_3'&=\left\{k\in\mathbb{N}:p_k\omega(f,\lambda_k)\ge\frac{\epsilon}{6}\right\},\\
E_4'&=\left\{k\in\mathbb{N}:p_k\omega(f,\lambda_k)\|M_k(1,x)-1\|_\infty\ge\frac{\epsilon}{3}\right\}.
\end{aligned} \tag{68}$$

It follows from (67) that

$$\frac{1}{c_n}\sum_{k\in E_1'}a_{n,k}\le\frac{1}{c_n}\sum_{k\in E_2'}a_{n,k}+\frac{1}{c_n}\sum_{k\in E_3'}a_{n,k}+\frac{1}{c_n}\sum_{k\in E_4'}a_{n,k} \tag{69}$$

holds for $n\in\mathbb{N}$. Since $c_k=\max\{a_k,b_k\}$, we obtain from (69) that

$$\frac{1}{c_n}\sum_{k\in E_1'}a_{n,k}\le\frac{1}{a_n}\sum_{k\in E_2'}a_{n,k}+\frac{1}{b_n}\sum_{k\in E_3'}a_{n,k}+\frac{1}{c_n}\sum_{k\in E_4'}a_{n,k}. \tag{70}$$

Taking limit $n\to\infty$ in (70) together with Lemma 11 and our hypotheses (i) and (ii), one obtains

$$\lim_n\frac{1}{c_n}\sum_{k\in E_1'}a_{n,k}=0. \tag{71}$$

This yields

$$\|M_k(f;x)-f(x)\|_\infty=S_A^{\overline{N}}\text{-}o(c_k). \tag{72}$$

□

4. Example and the Concluding Remark

The operators $B_n:\mathscr{C}[0;1]\to\mathscr{C}[0;1]$ given by

$$B_n(f,x):=\sum_{k=0}^{n}p_{n,k}(x)\,f\left(\frac{k}{n}\right), \tag{73}$$

where $p_{n,k}(x)$ are the fundamental Bernstein polynomials defined by

$$p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k} \tag{74}$$

for any $x\in[0,1]$, any $k\in\{0,1,\dots,n\}$, and any $n\in\mathbb{N}$, are called Bernstein operators and were first introduced in [28]. Let the sequence (A_n) be defined by $A_n:\mathscr{C}[0,1]\to\mathscr{C}[0,1]$ with $A_n(f,x)=(1+x_n)B_n(f,x)$, where $x=(x_k)$ is a sequence defined by

$$x=(x_k)=\begin{cases}\sqrt{k}, & \text{if } k=n^2,\ n\in\mathbb{N},\\ 0, & \text{otherwise.}\end{cases} \tag{75}$$

That is, $(x_k) = (1,0,0,2,0,0,0,0,3,0,\ldots,0,4,0,0,\ldots)$. Let $p_k = k$ $(k = 1,2,\ldots)$ and consider a nonnegative regular matrix $A = (C,1)$. Then,

$$p_k x_k = (1,0,0,8,0,0,0,0,27,0,\ldots,0,64,0,0,\ldots),$$
$$P_n = \sum_{k=1}^{n} p_k = \frac{n(n+1)}{2}. \tag{76}$$

Since

$$\lim_n \frac{1}{P_n} \left|\{k \le P_n : p_k \left|x_k - 0\right| \ge \epsilon\}\right| \le \lim_n \frac{1}{P_n}\sqrt{P_n} = \frac{1}{\sqrt{n(n+1)/2}} = 0, \tag{77}$$

(x_k) is weighted statistically convergent to 0 but not convergent. It is not difficult to see that

$$B_n(1,x) = 1, \qquad B_n(t,x) = x,$$
$$B_n\left(t^2,x\right) = x^2 + \frac{x - x^2}{n} \tag{78}$$

and the sequence (A_n) satisfies conditions (20). This yields that

$$S^{\overline{N}}_{(C,1)}\text{-}\lim \left\|A_n(f,x) - f(x)\right\|_\infty = 0. \tag{79}$$

On the other hand, one obtains $A_n(f,0) = (1 + x_n)f(0)$, since $B_n(f,0) = f(0)$, and hence

$$\left\|A_n(f,x) - f(x)\right\|_\infty \ge \left|A_n(f,0) - f(0)\right| = x_n \left|f(0)\right|. \tag{80}$$

It follows that (A_n) does not satisfy the Korovkin theorem, since (x_n) and hence (A_n) is not convergent. Finally, we conclude that Theorem 5 is stronger than Theorem 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Matrix Transformations between Certain Sequence Spaces over the Non-Newtonian Complex Field

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In some cases, the most general linear operator between two sequence spaces is given by an infinite matrix. So the theory of matrix transformations has always been of great interest in the study of sequence spaces. In the present paper, we introduce the matrix transformations in sequence spaces over the field $\mathbb{C}^*$ and characterize some classes of infinite matrices with respect to the non-Newtonian calculus. Also we give the necessary and sufficient conditions on an infinite matrix transforming one of the classical sets over $\mathbb{C}^*$ to another one. Furthermore, the concept for sequence-to-sequence and series-to-series methods of summability is given with some illustrated examples.

1. Introduction

The theory of sequence spaces is the fundamental of summability. Summability is a wide field of mathematics, mainly in analysis and functional analysis, and has many applications, for instance, in numerical analysis to speed up the rate of convergence, in operator theory, the theory of orthogonal series, and approximation theory. This subsection serves as a motivation of what follows. The classical summability theory deals with the generalization of the convergence of sequences or series of real or complex numbers. The idea is to assign a limit of some sort to divergent sequences or series by considering a transform of a sequence or series rather than the original sequence or series. One can ask why we employ the special transformations represented by infinite matrices instead of general linear operators. The answer to this question is that, in many cases, the most general linear operators between two sequence spaces are given by an infinite matrix. Many authors have extensively developed the theory of the matrix transformations between some sequence spaces we refer the reader to [1–13].

As an alternative to the classical calculus, Grossman and Katz [14–16] introduced the non-Newtonian calculus consisting of the branches of geometric, quadratic, and harmonic calculus, and so forth. All these calculi can be described simultaneously within the framework of a general theory. We decided to use the adjective *non-Newtonian* to indicate any of calculi other than the classical calculus. Every property in classical calculus has an analogue in non-Newtonian calculus which is a methodology that allows one to have a different look at problems which can be investigated via calculus. In some cases, for example, for wage-rate (in dollars, euro, etc.) related problems, the use of bigeometric calculus which is a kind of non-Newtonian calculus is advocated instead of a traditional Newtonian one.

Bashirov et al. [17, 18] have recently concentrated on the non-Newtonian calculus and gave the results with applications corresponding to the well-known properties of derivatives and integrals in the classical calculus. Also, Uzer [19] has extended the non-Newtonian calculus to the complex valued functions and was interested in the statements of some fundamental theorems and concepts of multiplicative complex calculus and demonstrated some analogies between the multiplicative complex calculus and classical calculus by theoretical and numerical examples. Further, Misirli and Gurefe have introduced multiplicative Adams Bashforth-Moulton methods for differential equations in [20]. Quite recently, Kadak [21, 22] have determinated Köthe-Toeplitz dual between classical sets of sequences over

the non-Newtonian complex field and have constructed Hilbert spaces over the non-Newtonian field.

The main purpose of the present paper is to characterize some matrix classes between certain sequence spaces over the non-Newtonian complex field.

2. β-Arithmetics and Some Related Applications

A *generator* is a one-to-one function whose domain is $\mathbb{R}$ and whose range is a subset of $B \subseteq \mathbb{R}$, the set of real numbers. Each generator generates exactly one arithmetic, and, conversely, each arithmetic is generated by exactly one generator. For example, the identity function generates classical arithmetic and exponential function generates geometric arithmetic. As a generator, we choose the function β such that its basic algebraic basic algebraic operations are defined as follows:

$$\begin{aligned}
&\beta\text{-addition } x \ddot{+} y = \beta\left\{\beta^{-1}(x) + \beta^{-1}(y)\right\};\\
&\beta\text{-subtraction } x \ddot{-} y = \beta\left\{\beta^{-1}(x) - \beta^{-1}(y)\right\},\\
&\beta\text{-multiplication } x \ddot{\times} y = \beta\left\{\beta^{-1}(x) \times \beta^{-1}(y)\right\};\\
&\beta\text{-division } x \ddot{/} y, \frac{x}{y} := \beta\left\{\beta^{-1}(x) \div \beta^{-1}(y)\right\}\\
&\qquad\qquad \left(\beta^{-1}(y) \neq 0\right),\\
&\beta\text{-order } x \ddot{<} y \Longleftrightarrow \beta^{-1}(x) < \beta^{-1}(y),
\end{aligned} \tag{1}$$

for all $x, y \in \mathbb{R}_\beta$ where the non-Newtonian real field $\mathbb{R}_\beta := \{\beta\{x\} : x \in \mathbb{R}\}$ as in [23].

The β-positive real numbers, denoted by $\mathbb{R}_\beta^+$, are the numbers x in $\mathbb{R}_\beta$ such that $\ddot{0} \ddot{<} x$; the β-negative real numbers, denoted by $\mathbb{R}_\beta^-$, are those for which $x \ddot{<} \ddot{0}$. The *beta*-zero, $\ddot{0}$, and the *beta*-one, $\ddot{1}$, turn out to be $\beta(0)$ and $\beta(1)$. Further, $\beta(-p) = \beta\{\beta^{-1}(\ddot{p})\} = \ddot{-}\ddot{p}$ holds for all $p \in \mathbb{Z}^+$. Thus the set of all β-integers turns out to be the following:

$$\begin{aligned}
\mathbb{Z}_\beta &= \{\ldots, \beta(-2), \beta(-1), \beta(0), \beta(1), \beta(2), \ldots\}\\
&= \{\ldots, \ddot{-}\ddot{2}, \ddot{-}\ddot{1}, \ddot{0}, \ddot{1}, \ddot{2}, \ldots\}.
\end{aligned} \tag{2}$$

Definition 1 (see [21]). Let X be a nonempty set and let $d^* : X \times X \to \mathbb{R}_\beta$ be a function such that for all $x, y, z \in X$, then the following axioms hold:

(NM1) $d^*(x, y) = \ddot{0}$ if and only if $x = y$,

(NM2) $d^*(x, y) = d^*(y, x)$,

(NM3) $d^*(x, y) \ddot{\leq} d^*(x, z) \ddot{+} d^*(z, y)$.

Then, the pair (X, d^*) and d^* are called a non-Newtonian metric space and a non-Newtonian metric on X, respectively.

Definition 2 (see [23]). Let $X = (X, d^*)$ be a non-Newtonian metric space. Then the basic notions can be defined as follows.

(a) A sequence $x = (x_k)$ is a function from the set $\mathbb{N}$ into the set $\mathbb{R}_\beta$. The β-real number x_k denotes the value of the function at $k \in \mathbb{N}$ and is called the kth term of the sequence.

(b) A sequence (x_n) in a metric space $X = (X, d^*)$ is said to be $*$-convergent if for every given $\varepsilon \ddot{>} \ddot{0}$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $x \in X$ such that $d^*(x_n, x) \ddot{<} \varepsilon$ for all $n > n_0$ and is denoted by ${}^*\lim_{n\to\infty} x_n = x$ or $x_n \overset{*}{\to} x$, as $n \to \infty$.

(c) A sequence (x_n) in $X = (X, d^*)$ is said to be non-Newtonian Cauchy ($*$-Cauchy) if for every $\varepsilon \ddot{>} \ddot{0}$ there is an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $d^*(x_n, x_m) \ddot{<} \varepsilon$ for all $m, n > n_0$.

Remark 3 (see [21]). Let $\ddot{b} \in B \subseteq \mathbb{R}$. Then the number $\ddot{b} \ddot{\times} \ddot{b}$ is called the β-square and is denoted by $\ddot{b}^{\ddot{2}}$. Let $\ddot{b} \in \mathbb{R}_\beta^+$. Then $\beta\{\sqrt{\beta^{-1}(\ddot{b})}\}$ is called the β-square root of $\ddot{b}$ and is denoted by $\sqrt[\ddot{2}]{\ddot{b}}$. Further, for each $\ddot{b} \in B$ we can write $\ddot{b}^{\ddot{p}} = \beta\{[\beta^{-1}(\ddot{b})]^p\} = \beta\{b^p\}$ for all $p \in \mathbb{N}$.

The β-absolute value of a number x in $B \subset \mathbb{R}(N)$ is defined as $\beta(|\beta^{-1}(x)|)$ and is denoted by $|x|_\beta$. For each number x in $A \subset \mathbb{R}(N)$, $\sqrt[\ddot{2}]{\ddot{x}^{\ddot{2}}} = |x|_\beta = \beta(|\beta^{-1}(x)|)$. Then we say

$$|x|_\beta = \begin{cases} x, & x \ddot{>} \ddot{0}\\ \ddot{0}, & x = \ddot{0} = \beta\left\{\left|\beta^{-1}(x)\right|\right\}\\ \ddot{0} \ddot{-} x, & x \ddot{<} \ddot{0}. \end{cases} \tag{3}$$

The non-Newtonian distance between two numbers x_1 and x_2 is defined by $|x_1 \ddot{-} x_2|_\beta = \beta\{|\beta^{-1}(x_1) - \beta^{-1}(x_2)|\} = \beta\{|\beta^{-1}(x_2) - \beta^{-1}(x_1)|\} = |x_2 \ddot{-} x_1|_\beta$. Similarly, by taking into account the definition for *alpha*-generator in (3), one can conclude that the equality $|x_1 \dot{-} x_2|_\alpha = |x_2 \dot{-} x_1|_\alpha$ holds for all $x_1, x_2 \in \mathbb{R}$.

Now, we give a new type calculus for non-Newtonian complex terms, denoted by $*$-calculus, which is a branch of non-Newtonian calculus. From now on we will use $*$-calculus type with respect to two arbitrarily selected generator functions.

2.1. $*$-Arithmetic. Suppose that α and β are two arbitrarily selected generators and ("star-") also is the ordered pair of arithmetics (β-arithmetic, α-arithmetic). The sets $(B, \ddot{+}, \ddot{-}, \ddot{\times}, \ddot{/})$ and $(A, \dot{+}, \dot{-}, \dot{\times}, \dot{/})$ are complete ordered fields and *beta*(*alpha*)-generator generates *beta*(*alpha*)-arithmetics, respectively. Definitions given for β-arithmetic are also valid for α-arithmetic.

The important point to note here is that α-arithmetic is used for arguments and β-arithmetic is used for values; in particular, changes in arguments and values are measured by α-differences and β-differences, respectively. The operators of this calculus type are applied only to functions with arguments in A and values in B. The $*$-limit of a function f at an element a in A is, if it exists, the unique number b in B such that for every sequence (a_n) of arguments of f distinct

from a, if (a_n) is α-convergent to a, then $\{f(a_n)\}$ β-converges to b and is denoted by ${}^*\lim_{x\to a} f(x) = b$. That is,

$$\begin{aligned} {}^*\lim_{x\to a} f(x) = b \Longleftrightarrow \forall \varepsilon \dot{>} \ddot{0},\ \exists \delta \dot{>} \dot{0} \ni \left| f(x) \ddot{-} b \right|_\beta \ddot{<} \varepsilon \\ \forall x \in A, \quad \left| x \dot{-} a \right|_\alpha \dot{<} \delta. \end{aligned} \tag{4}$$

A function f is $*$-continuous at a point a in A if and only if a is an argument of f and ${}^*\lim_{x\to a} f(x) = f(a)$. When α and β are the identity function I, the concepts of $*$-limit and $*$-continuity are identical with those of classical limit and classical continuity.

The isomorphism from α-arithmetic to β-arithmetic is the unique function ι (iota) that possesses the following three properties.

(i) ι is one to one.

(ii) ι is from A onto B.

(iii) For any numbers u and v in A,

$$\begin{aligned} &\iota(u \dot{+} v) = \iota(u) \ddot{+} \iota(v), \qquad \iota(u \dot{-} v) = \iota(u) \ddot{-} \iota(v), \\ &\iota(u \dot{\times} v) = \iota(u) \ddot{\times} \iota(v), \qquad \iota\left(u \dot{/} v\right) = \iota(u) \ddot{/} \iota(v); \\ &v \neq \dot{0}, \quad u \leq v \Longleftrightarrow \iota(u) \ddot{\leq} \iota(v). \end{aligned} \tag{5}$$

It turns out that $\iota(x) = \beta\{\alpha^{-1}(x)\}$ for every x in A and that $\iota(\dot{n}) = \ddot{n}$ for every integer n. Since, for example, $u \dot{+} v = \iota^{-1}\{\iota(u) \ddot{+} \iota(v)\}$, it should be clear that any statement in α-arithmetic can readily be transformed into a statement in β-arithmetic.

2.2. Non-Newtonian Complex Field and Some Inequalities. Let $\alpha(a) = \dot{a} \in (A, \dot{+}, \dot{-}, \dot{\times}, \dot{/})$ and $\beta(b) = \ddot{b} \in (B, \ddot{+}, \ddot{-}, \ddot{\times}, \ddot{/})$ be arbitrarily chosen elements from corresponding arithmetics. Then the ordered pair $(\dot{a}, \ddot{b})$ is called a $*$-point. The set of all $*$-points is called the set of $*$-complex numbers and is denoted by $\mathbb{C}^*$; that is,

$$\mathbb{C}^* := \left\{ z^* = \left(\dot{a}, \ddot{b}\right) \mid \dot{a} \in A \subseteq \mathbb{R}, \ddot{b} \in B \subseteq \mathbb{R} \right\}. \tag{6}$$

Define the binary operations addition ($\oplus$) and multiplication ($\odot$) of $*$-complex numbers $z_1^* = (\dot{a}_1, \ddot{b}_1)$ and $z_2^* = (\dot{a}_2, \ddot{b}_2)$ as follows:

$$\begin{aligned} \oplus:\ &\mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^* \\ &\left(z_1^*, z_2^*\right) \longmapsto {z_1}^* \oplus z_2^* \\ &\qquad = \left(\alpha\{a_1 + a_2\}, \beta\{b_1 + b_2\}\right) \\ &\qquad = \left(\dot{a}_1 \dot{+} \dot{a}_2, \ddot{b}_1 \ddot{+} \ddot{b}_2\right) \\ \odot:\ &\mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^* \\ &\left(z_1^*, z_2^*\right) \longmapsto z_1^* \odot z_2^* \\ &\qquad = \left(\alpha\{a_1 a_2 - b_1 b_2\}, \right. \\ &\qquad\quad \left. \beta\{a_1 b_2 + b_1 a_2\}\right), \end{aligned} \tag{7}$$

where $\dot{a}_1, \dot{a}_2 \in A$ and $\ddot{b}_1, \ddot{b}_2 \in B$.

Theorem 4 (see [24]). *$(\mathbb{C}^*, \oplus, \odot)$ is a field.*

Following Grossman and Katz [15], we can give the definition of $*$-distance and some applications with respect to the $*$-calculus which is a kind of calculi of non-Newtonian calculus.

The $*$-distance d^* between two arbitrarily elements $z_1^* = (\dot{a}_1, \ddot{b}_1)$ and $z_2^* = (\dot{a}_2, \ddot{b}_2)$ of the set $\mathbb{C}^*$ is defined by

$$\begin{aligned} d^*:\ &\mathbb{C}^* \times \mathbb{C}^* \longrightarrow [\ddot{0}, \infty) = B' \subset B \\ &\left(z_1^*, z_2^*\right) \longmapsto d^*\left(z_1^*, z_2^*\right) \\ &\qquad = \sqrt[\ddot{2}]{\left[\iota\left(\dot{a}_1 \dot{-} \dot{a}_2\right)\right]^{\ddot{2}} \ddot{+} \left(\ddot{b}_1 \ddot{-} \ddot{b}_2\right)^{\ddot{2}}} \\ &\qquad = \beta\left\{\sqrt{\left(a_1 - a_2\right)^2 + \left(b_1 - b_2\right)^2}\right\}. \end{aligned} \tag{8}$$

Up to now, we know that $\mathbb{C}^*$ is a field and the distance between two points in $\mathbb{C}^*$ is computed by the function d^*, defined by (8).

Definition 5 (see [21]). Given a sequence (z_k^*) of $*$-complex numbers, the formal notation

$$\sum_{k=0}^{\infty}{}^* z_k^* = z_0^* \oplus z_1^* \oplus z_2^* \oplus \cdots \oplus z_k^* \oplus \cdots, \quad \forall k \in \mathbb{N}, \tag{9}$$

is called an infinite series with $*$-complex terms, or simply complex N-series. Also, for integers $n \in \mathbb{N}$, the finite $*$-sums $s_n^* = {}_*\sum_{k=0}^{n} z_k^*$ are called the partial sums of complex N-series. If the sequence $*$-converges to a complex number s^* then we say that the series $*$-converges and write $s^* = {}_*\sum_{n=0}^{\infty} z_n^*$. The number s^* is then called the $*$-sum of this series. If (s_n) $*$-diverges, we say that the series $*$-diverges, or that it is $*$-divergent.

Remark 6. Given a sequence (x_k) of β-real numbers $\mathbb{R}_\beta$, the formal notation

$$\sum_{k=0}^{\infty}{}_\beta x_k = \beta\left\{\sum_{k=0}^{\infty} \beta^{-1}\{x_k\}\right\} = x_1 \ddot{+} x_2 \ddot{+} x_3 \ddot{+} \cdots \ddot{+} x_k \ddot{+} \cdots \tag{10}$$

is called an infinite non-Newtonian series with β real terms. Also, for integers $n \in \mathbb{N}$, the finite sums $s_n = {}_\beta\sum_{k=0}^{n} x_k$ are called the partial sums of the N-series. If the sequence β-converges to a real number s then we say that the series β-converges and write $s = {}_\beta\sum_{n=0}^{\infty} x_k$. The number s is then called the sum of this series. If (s_n) β-diverges, we say that the N-series is β-divergent.

Proposition 7 (see [24]). *For any $z_1^*, z_2^* \in \mathbb{C}^*$. Then the following statements hold.*

(i) $\ddot{\|} z_1^* \oplus z_2^* \ddot{\|} \ddot{\leq} \ddot{\|} z_1^* \ddot{\|} \ddot{+} \ddot{\|} z_2^* \ddot{\|}$. *($*$-triangle inequality)*

(ii) $\ddot{\|} z_1^* \odot z_2^* \ddot{\|} = \ddot{\|} z_1^* \ddot{\|} \ddot{\times} \ddot{\|} z_2^* \ddot{\|}$.

(iii) *Let* $p \ddot{>} \ddot{1}$ *and* $z_k^*, t_k^* \in \mathbb{C}^*$ *for* $k \in \{0, 1, 2, 3, \ldots, n\}$. *Then,*

$$\left(\sum_{k=0}^{n} {}_{*} \ddot{\|} z_k^* \oplus t_k^* \ddot{\|}^p \right)^{1 \ddot{/} p} \ddot{\leq} \left(\sum_{k=0}^{n} {}_{*} \ddot{\|} z_k^* \ddot{\|}^p \right)^{1 \ddot{/} p} \ddot{+} \left(\sum_{k=0}^{n} {}_{*} \ddot{\|} t_k^* \ddot{\|}^p \right)^{1 \ddot{/} p} \quad (11)$$

$$(\textit{Minkowski's inequality}).$$

Folllowing Tekin and Başar [24], we can give the $*$-norm and next derive some required inequalities in the sense of non-Newtonian complex calculus.

Let $z^* \in \mathbb{C}^*$ be an arbitrary element. The distance function $d^*(z^*, \theta^*)$ is called $*$-norm of z^* and is denoted by $\ddot{\|} \cdot \ddot{\|}$. In other words,

$$\begin{aligned} \ddot{\|} z^* \ddot{\|} &= d^* (z^*, \theta^*) \\ &= \ddot{\sqrt{}} \left[\iota \left(\dot{a} \dot{-} \dot{0} \right) \right]^{\ddot{2}} \ddot{+} \left(\ddot{b} \ddot{-} \ddot{0} \right)^{\ddot{2}} \\ &= \beta \left\{ \sqrt{a^2 + b^2} \right\}, \end{aligned} \quad (12)$$

where $z^* = (\dot{a}, \ddot{b})$ and $\theta^* = (\dot{0}, \ddot{0})$. Moreover, since for all $z_1^*, z_2^* \in \mathbb{C}^*$ we have $d^*(z_1^*, z_2^*) = \ddot{\|} z_1^* \ominus z_2^* \ddot{\|}$ which d^* is the induced metric from $\ddot{\|} \cdot \ddot{\|}$ norm.

Definition 8 (see [21], complex conjugate). Let $z^* = (\dot{a}, \ddot{b}) \in \mathbb{C}^*$. We define the $*$-complex conjugate $\overline{z}^*$ of z^* by $\overline{z}^* = (\alpha\{a\}, \beta\{-\beta^{-1}(\ddot{b})\}) = (\dot{a}, \ddot{-}\ddot{b})$. Conjugation changes the sign of the imaginary part of z^* but leaves the real part the same. Thus,

$$\begin{aligned} \mathscr{R}e\left(\overline{z}^*\right) &= \mathscr{R}e\left(z^*\right) = \left(z^* \oplus \overline{z}^*\right) \dot{/} \dot{2} = \dot{a}, \\ \mathscr{I}m\left(\overline{z}^*\right) &= \ddot{-} \mathscr{I}m\left(z^*\right) = \left(z^* \ominus \overline{z}^*\right) \ddot{/} \ddot{2} = \ddot{b}. \end{aligned} \quad (13)$$

Theorem 9 (see [24]). $(\mathbb{C}^*, d^*)$ *is a complete metric space, where* d^* *is defined by* (8).

Corollary 10 (see [24]). $\mathbb{C}^*$ *is a Banach space with the* $*$*-norm* $\ddot{\|} \cdot \ddot{\|}$ *defined by* $\ddot{\|} z^* \ddot{\|} = \ddot{\sqrt{}} [\iota(\dot{a})]^{\ddot{2}} + (\ddot{b})^{\ddot{2}}$; $z^* = (\dot{a}, \ddot{b}) \in \mathbb{C}^*$.

3. Non-Newtonian Infinite Matrices

A non-Newtonian infinite matrix $A = (a_{ij}^*)$ of non-Newtonian complex numbers is a double sequence of complex numbers defined by a function A from the set $\mathbb{N} \times \mathbb{N}$ into the complex field C^*, where $\mathbb{N}$ denotes the set of natural numbers, that is, $\mathbb{N} = \{0, 1, 2, \ldots\}$. The complex number a_{ij}^* denotes the value of the function at $(i, j) \in \mathbb{N} \times \mathbb{N}$ and is called the entry of the matrix in the ith row and jth column.

The addition $(\oplus)$ and scalar multiplication $(\odot)$ of the infinite matrices $A = (a_{ij}^*) = (\dot{\epsilon}_{ij}, \ddot{\delta}_{ij})$ and $B = (b_{ij}^*) = (\dot{\mu}_{ij}, \ddot{\eta}_{ij})$ are defined by

$$\begin{aligned} A \oplus B &= \left(a_{ij}^* \oplus b_{ij}^*\right) = \left(\dot{\epsilon}_{ij}, \ddot{\delta}_{ij}\right) \oplus \left(\dot{\mu}_{ij}, \ddot{\eta}_{ij}\right) \\ &= \left(\dot{\epsilon}_{ij} \dot{+} \dot{\mu}_{ij}, \ddot{\delta}_{ij} \ddot{+} \ddot{\eta}_{ij}\right) \\ &= \left(\alpha\left\{\epsilon_{ij} + \mu_{ij}\right\}, \beta\left\{\delta_{ij} + \eta_{ij}\right\}\right), \\ \lambda^* \odot A &= \left(\lambda^* \odot a_{ij}^*\right) = \left(\dot{\lambda}, \ddot{\lambda}\right) \odot \left(\dot{\epsilon}_{ij}, \ddot{\delta}_{ij}\right) \\ &= \left(\alpha\left\{\lambda\epsilon_{ij} - \lambda\delta_{ij}\right\}, \beta\left\{\lambda\epsilon_{ij} + \lambda\delta_{ij}\right\}\right), \end{aligned} \quad (14)$$

where the elements $\varepsilon, \delta, \mu, \eta$ are in $\mathbb{R}$ and $\lambda^* = (\dot{\lambda}, \ddot{\lambda})$ is a non-Newtonian scalar in C^*. The product $A \odot B$ of the infinite matrices $A = (a_{ij}^*)$ and $B = (b_{ij}^*)$ is defined by

$$\begin{aligned} (A \odot B)_{ij} &= \sum_{k=0}^{\infty} {}_{*} a_{ik}^* \odot b_{kj}^* \\ &= \left(\alpha \left\{ \sum_{k=0}^{\infty} \left(\varepsilon_{ik}\mu_{kj} - \delta_{ik}\eta_{kj} \right) \right\}, \right. \\ &\qquad \left. \beta \left\{ \sum_{k=0}^{\infty} \left(\varepsilon_{ik}\eta_{kj} + \delta_{ik}\mu_{kj} \right) \right\} \right), \quad \forall i, j \in \mathbb{N}, \end{aligned} \quad (15)$$

provided that the series on the right hand side of (15) $*$-converge for all $i, j \in \mathbb{N}$, where $(A \odot B)_{ij}$ denotes the entry of the matrix $A \odot B$ in the ith row and jth column. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. On the other hand, the series on the right hand side of (15) $*$-converges if and only if

$$\sum_k \left(\varepsilon_{ik}\mu_{kj} - \delta_{ik}\eta_{kj} \right), \qquad \sum_k \left(\varepsilon_{ik}\eta_{kj} + \delta_{ik}\mu_{kj} \right) \quad (16)$$

are convergent classically for all $k, n \in \mathbb{N}$. However the series (15) may $*$-diverge for some, or all, values of i, j; the product $A \odot B$ of the infinite matrices may not exist.

Definition 11 (see [25]). Consider the following system of an infinite number of linear equations in infinitely many unknown $x_0^*, x_1^*, x_2^*, \ldots$ elements by ${}_{*}\sum_k a_{ik}^* \odot x_k^* = y_i$ for all $i \in \mathbb{N}$. If we construct a non-Newtonian infinite matrix $A = (a_{ik}^*)$ with the coefficients a_{ik}^* of the unknowns x_k^* and denote the $*$-vectors of unknowns and constants by X and Y, then the above sum can be expressed in matrix form as $A \odot X = Y$. Also $I_* \odot A = A \odot I_* = A$, where $I_* = (\delta_{ij}^*)$ is called $*$-unit matrix and is defined by

$$\delta_{ij}^* = \begin{cases} 1^* = \left(\dot{1}, \ddot{1}\right), & i = j, \\ \theta^* = \left(\dot{0}, \ddot{0}\right), & i \neq j. \end{cases} \quad (17)$$

A very important application of infinite matrices is used in the theory of summability of divergent sequences and series which is considered based on non-Newtonian mean in next chapter. A simple example of the non-Newtonian Cesaro mean, denoted by $*$-Cesaro mean of order one, which is the analog of the well-known method of summability given below.

Example 12 (Cesàro mean). Define the matrix $C_1^* = (c_{nk}^*)$ by

$$c_{nk}^* = \begin{cases} \left(\frac{1}{n+1}\right)^*, & 0 \le k \le n, \\ \theta^*, & k > n. \end{cases} \tag{18}$$

If we choose the generator functions as $\alpha = \exp$ and $\beta = \exp$ the calculus is bigeometric calculus [14, 15], then we obtain an infinite matrix with complex terms as follows:

$$\begin{pmatrix} (e,e) & (1,1) & \cdots & \cdots & \cdots & \cdots \\ \left(\sqrt{e},\sqrt{e}\right) & \left(\sqrt{e},\sqrt{e}\right) & (1,1) & \cdots & \cdots & \cdots \\ \left(\sqrt[3]{e},\sqrt[3]{e}\right) & \left(\sqrt[3]{e},\sqrt[3]{e}\right) & \left(\sqrt[3]{e},\sqrt[3]{e}\right) & (1,1) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \left(\sqrt[n+1]{e},\sqrt[n+1]{e}\right) & \left(\sqrt[n+1]{e},\sqrt[n+1]{e}\right) & \left(\sqrt[n+1]{e},\sqrt[n+1]{e}\right) & \cdots & \left(\sqrt[n+1]{e},\sqrt[n+1]{e}\right) & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}, \tag{19}$$

where $\theta^* = (\alpha(0), \beta(0)) = (1,1)$ and e is a logarithmic number. The important point to note here is that the infinite matrix can be obtained in a similar way by using different generator functions above mentioned.

The $*$-zero matrix θ^* is the matrix whose entries are all equal to θ^*. Thus, it is obvious that $A \odot \theta^* = \theta^* \odot A = \theta^*$. But, as classical, $A \odot B = \theta^*$ does not imply $A = \theta^*$ or $B = \theta^*$. Further, the conjugate $\overline{A}$ of a complex matrix $A = (a_{ij}^*)$ is the matrix $\overline{A} = (\overline{a}_{ij}^*)$ where $\overline{a}_{ij}^*$ is the conjugate of the complex number a_{ij}^* in Definition 8.

3.1. Non-Newtonian Matrix Transformations. Let μ_1^*, $\mu_2^* \subset w^*$ and $A = (a_{nk}^*) = (\dot{\varepsilon}_{nk}, \ddot{\delta}_{nk})$ be an infinite matrix of non-Newtonian complex numbers for all $\dot{\varepsilon}_k \in \mathbb{R}_\alpha$ and $\ddot{\delta}_k \in \mathbb{R}_\beta$. Then, we say that A defines a matrix mapping from μ_1^* into μ_2^* and denote it by writing $A : \mu_1^* \to \mu_2^*$, if for every sequence $z = (z_k^*) \in \mu_1^*$ the sequence $A \odot z = \{(Az)_n\}$, the A-transform of z, exists and is in μ_2^*, where

$$A \odot z = \begin{pmatrix} a_{00}^* & a_{01}^* & \cdots & a_{0k}^* & \cdots \\ a_{10}^* & a_{11}^* & \cdots & a_{1k}^* & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ a_{n0}^* & a_{n1}^* & \cdots & a_{nk}^* & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix} \odot \begin{pmatrix} z_0^* \\ z_1^* \\ \vdots \\ z_k^* \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} a_{00}^* \odot z_0^* \otimes a_{01}^* \odot z_1^* \oplus \cdots \\ a_{10}^* \odot z_0^* \otimes a_{11}^* \odot z_1^* \oplus \cdots \\ \vdots \\ a_{n0}^* \odot z_0^* \otimes a_{n1}^* \odot z_1^* \oplus \cdots \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} {}_*\sum_k a_{0k}^* \odot z_k^* \\ {}_*\sum_k a_{1k}^* \odot z_k^* \\ \vdots \\ {}_*\sum_k a_{nk}^* \odot z_k^* \\ \vdots \end{pmatrix} = \begin{pmatrix} (Az)_0 \\ (Az)_1 \\ \vdots \\ (Az)_n \\ \vdots \end{pmatrix}, \tag{20}$$

and, in this way, we transform the sequence $z = (z_k^*) = (\dot{\mu}_k, \ddot{\eta}_k)$, with $\dot{\mu}_k \in \mathbb{R}_\alpha$ and $\ddot{\eta}_k \in \mathbb{R}_\beta$, into the sequence $\{(Az)_n\}$ by

$$(Az)_n := {}_*\sum_k a_{nk}^* \odot z_k^* = \left(\alpha \left\{ \sum_k \left(\varepsilon_{nk}\mu_k - \delta_{nk}\eta_k\right) \right\}, \beta \left\{ \sum_k \left(\varepsilon_{nk}\eta_k + \delta_{nk}\mu_k\right) \right\} \right) \tag{21}$$

for all $n \in \mathbb{N}_0$ and $\varepsilon, \delta, \mu, \eta \in \mathbb{R}$. Thus, $A \in (\mu_1^* : \mu_2^*)$ if and only if the series on the right side of (21) $*$-converges for each $n \in \mathbb{N}$ and every $z \in \mu_1^*$, and we have $A \odot z = \{(Az)_n\}_{n\in\mathbb{N}} \in \mu_2^*$ for all $z \in \mu_1^*$. On the other hand, we say $A \in (\mu_1^* : \mu_2^*)$ if and only if the series $\sum_k(\varepsilon_{nk}\mu_k - \delta_{nk}\eta_k)$ and $\sum_k(\varepsilon_{nk}\eta_k + \delta_{nk}\mu_k)$ are convergent classically for all $k, n \in \mathbb{N}_0$. A sequence z is said to be A-summable to γ if $A \odot z$ $*$-converges to $\gamma \in \mathbb{C}^*$ which is called as the A-*lim of z. We denote the nth row of a matrix $A = (a_{nk}^*)$ by A_n^* for all $n \in \mathbb{N}$; that is, $A_n^* := \{a_{nk}^*\}_{k=0}^\infty$ for all $n \in \mathbb{N}$. Following Başar [25], we give some lines about ordinary and absolute summability of non-Newtonian complex numbers.

Let $A = (a_{nk}^*)$ be an infinite matrix of non-Newtonian complex numbers throughout. We define two kinds of

summability: ordinary and absolute summability, as shortly mentioned, below.

(a) Ordinary Summability. A sequence $z = (z_k) \in w^*$ is said to be summable A to a $\gamma \in \mathbb{C}^*$ if the A-$^*\lim$ of z is $\gamma = (\dot{\gamma}_1, \ddot{\gamma}_2)$ for all $\gamma_1, \gamma_2 \in \mathbb{R}$; that is, $^*\lim_{n\to\infty} d^*((Az)_n, \gamma) = \theta^*$ which implies that

$$\sum_k (\varepsilon_{nk}\mu_k - \delta_{nk}\eta_k) \longrightarrow \gamma_1,$$
$$\sum_k (\varepsilon_{nk}\eta_k + \delta_{nk}\mu_k) \longrightarrow \gamma_2 \tag{22}$$

in classical mean for each $k, n \in \mathbb{N}_0$. The matrix A defines a summability method A or a matrix transformation by (21).

(b) Absolute Summability. A sequence $z = (z_k) \in w^*$ is said to be absolutely summable with index p to a number $\zeta \in \mathbb{C}^*$ if the series on the right hand side of (21) $*$-converge for each $n \in \mathbb{N}$ and

$$\sum_{n=0}^{\infty} d^*((Az)_n, \theta^*)^{\ddot{p}} = \zeta \quad (1 < p < \infty). \tag{23}$$

The Cesàro transform of a sequence $z = (z_k^*) \in w^*$ is given by $C_1^* \odot z = \{(C_1^* z)_n\}_{n\in\mathbb{N}}$, where the Cesàro method C_1^* of one order is given by Example 12. Now, following Example 12, we may state the Cesàro summability with respect to the non-Newtonian calculus which is analogous to the classical Cesàro summable.

Example 13. Suppose that $z = (z_k^*)$ is an infinite sequence defined by

$$z_k^* = \begin{cases} 1^*, & k \text{ even}, \\ \ominus 1^*, & k \text{ odd}, \end{cases} \tag{24}$$

where $\ominus 1^* = (\dot{-}\dot{1}, \ddot{-}\ddot{1}) \in C^*$. One can easily conclude that $z_k^* \in \ell_\infty^* \setminus c^*$. Then, since $\theta^* \leq \ddot{\|}(C_1^* z)_n\ddot{\|} \leq (1/(n+1))^*$ for all $n \in \mathbb{N}$, $^*\lim_{n\to\infty}(C_1^* z)_n = \theta^*$. This means that the $*$-divergent sequence (z_k^*) is C_1^*-summable to θ^*.

Tekin and Başar [24] have introduced the sets ℓ_∞^*, c^*, c_0^* and ℓ_p^* of all bounded, convergent, null, and absolutely p-summable sequences over the complex field $\mathbb{C}^*$ which correspond to the sets ℓ_∞, c, c_0 and ℓ_p over the complex field $\mathbb{C}$, respectively. That is to say that

$$\ell_\infty^* = \left\{ z^* = (z_k^*) \in \omega^* : \sup_{k\in\mathbb{N}} \ddot{\|} z_k^* \ddot{\|} < \infty \right\},$$
$$c^* = \left\{ z^* = (z_k^*) \in \omega^* : \exists l \in \mathbb{C}^* \ni {}^*\lim_{k\to\infty} z_k^* = l \right\},$$
$$c_0^* = \left\{ z^* = (z_k^*) \in \omega^* : {}^*\lim_{k\to\infty} z_k^* = \theta^* \right\},$$
$$\ell_p^* = \left\{ z^* = (z_k^*) \in \omega^* : {}_*\sum_{k=1}^{\infty} \ddot{\|} z_k^* \ddot{\|}^{\ddot{p}} < \infty \right\}, \quad (1 \leq p < \infty). \tag{25}$$

It is not hard to show that the sets ℓ_∞^*, c^*, c_0^*, and ℓ_p^* are the subspaces of the space ω^*. This means that ℓ_∞^*, c^*, c_0^*, and ℓ_p^* are classical sequence spaces over the field $\mathbb{C}^*$ and complete metric spaces with corresponding metrics.

Quite recently, Kadak [21] have introduced the sets bs^*, cs^*, and cs_0^* consisting of the sets of all bounded, convergent, and null series based on the non-Newtonian calculus, as follows:

$$bs^* := \left\{ x = (x_k) \in \omega^* : \|x\|_{bs}^* = \sup_{n\in\mathbb{N}} \ddot{\|} {}_*\sum_{k=0}^{n} x_k \ddot{\|} < \infty \right\},$$
$$cs^* := \left\{ x = (x_k) \in \omega^* : \left({}_*\sum_{k=0}^{n} x_k \right) \in c^* \right\},$$
$$cs_0^* := \left\{ x = (x_k) \in \omega^* : \left({}_*\sum_{k=0}^{n} x_k \right) \in c_0^* \right\}, \tag{26}$$
$$\omega^* = \{ x = (x_k) : x_k \in \mathbb{C}^* \ \forall k \in \mathbb{N} \}.$$

Theorem 14 (see [24]). *The following statements hold.*

(a) *The sets ℓ_∞^*, c^*, c_0^*, and ℓ_p^*; $p \geq 1$ are sequence spaces.*

(b) *Let λ^* denote any of the spaces ℓ_∞^*, c^*, and c_0^* and $z = (z_k^*)$, $t = (t_k^*) \in \lambda^*$. Define d_∞^* on the space λ^* by $d_\infty^*(z,t) = \sup_{k\in\mathbb{N}} \ddot{\|} z_k^* \ominus t_k^* \ddot{\|}$. Then, (λ^*, d_∞^*) is a complete metric space.*

(c) *The spaces ℓ_∞^*, c^*, and c_0^* are Banach spaces with the norm $\|z\|_\infty^*$ defined by*

$$\|z\|_\infty^* := \sup_{k\in\mathbb{N}} \ddot{\|} z_k^* \ddot{\|}; \quad z = (z_k^*) \in \lambda^*, \quad \lambda \in \{\ell_\infty, c, c_0\}. \tag{27}$$

(d) *The space ℓ_p^* is Banach spaces with the norm $\|z\|_p^*$ defined by*

$$\|z\|_p^* := \left({}_*\sum_{k=0}^{\infty} \ddot{\|} z_k^* \ddot{\|}^{\ddot{p}} \right)^{\ddot{1}/\ddot{p}}; \quad z = (z_k^*) \in \ell_p^*. \tag{28}$$

Theorem 15 (see [21]). *Let μ^* denote any of the spaces bs^*, cs^*, and cs_0^*, and $z = (z_k^*)$, $t = (t_k^*) \in \mu^*$. Define d_∞^N on the space μ^* by*

$$d_\infty^N : \mu^* \times \mu^* \longrightarrow \mathbb{R}_\beta$$
$$(z,t) \longrightarrow d_\infty^N(z,t) := \sup_{n\in\mathbb{N}} \left\{ d^* \left({}_*\sum_{k=0}^{n} z_k^*, {}_*\sum_{k=0}^{n} t_k^* \right) \right\} \tag{29}$$

for arbitrarily chosen α, β operators and corresponding function $\iota = \beta\alpha^{-1}$. Then, (μ^*, d_∞^N) is a complete metric space.

Corollary 16 (see [21]). *The spaces bs^*, cs^*, and cs_0^* are Banach spaces with the norm $\|x\|_{bs}^*$ defined by*

$$\|x\|_{bs}^* = \|x\|_{cs}^* = \sup_{n\in\mathbb{N}} \ddot{\|} \sum_{k=1}^{n} x_k \ddot{\|}; \quad x = (x_n) \in \lambda^*, \quad \lambda \in \{bs, cs, cs_0\}. \tag{30}$$

Theorem 17 (see [21]). *Let d_Δ be defined on the space bv^* by*

$$d_\Delta : bv^* \times bv^* \longrightarrow \mathbb{R}_\beta, \quad (z,t) \longrightarrow d_\Delta(z,t) := \sum_{k=0}^{\infty} \left\{ d^* \left[(\Delta z)_k', (\Delta t)_k' \right] \right\}, \tag{31}$$

where $z = (z_k)$, $t = (t_k) \in bv^*$, and $(\Delta z)_k' = z_k \ominus z_{k+1}$. Then, (bv^*, d_Δ) is a complete metric space.

Firstly, we give the alpha-, beta-, and gamma-duals of a set $\lambda^* \subset \omega^*$ which are, respectively, denoted by $\{\lambda^*\}^\alpha$, $\{\lambda^*\}^\beta$, and $\{\lambda^*\}^\gamma$, as follows:

$$\begin{aligned} \{\lambda^*\}^\alpha &:= \{w = (w_k^*) \in \omega^* : w \odot z = (w_k^* \odot z_k^*) \in \ell_1^* \ \forall z = (z_k^*) \in \lambda^*\}, \\ \{\lambda^*\}^\beta &:= \{w = (w_k^*) \in \omega^* : w \odot z = (w_k^* \odot z_k^*) \in cs^* \ \forall z = (z_k^*) \in \lambda^*\}, \\ \{\lambda^*\}^\gamma &:= \{w = (w_k^*) \in \omega^* : w \odot z = (w_k^* \odot z_k^*) \in bs^* \ \forall z = (z_k^*) \in \lambda^*\}, \end{aligned} \tag{32}$$

where $(w_k^* \odot z_k^*)$ is the coordinatewise product of $*$-complex numbers w and z for all $k \in \mathbb{N}$. Then $\{\lambda^*\}^\beta$ is called beta-dual of λ^* or the set of all convergence factor sequences of λ^* in cs^*. Firstly, we give a remark concerning with the $*$-convergence factor sequences.

Theorem 18 (see [21]). *The following statements hold.*

(a) $\{c_0^*\}^\beta = \{c^*\}^\beta = \{\ell_\infty^*\}^\beta = \ell_1^*$.

(b) $\{\ell_1^*\}^\beta = \ell_\infty^*$.

Theorem 19 (see [21]). *The following statements hold.*

(a) $\{cs^*\}^\alpha = \{bv^*\}^\alpha = \{bv_0^*\}^\alpha = \ell_1^*$.

(b) $\{cs^*\}^\beta = bv^*$, $\{bv^*\}^\beta = cs^*$, $\{bv_0^*\}^\beta = bs^*$, $\{bs^*\}^\beta = bv_0^*$.

(c) $\{cs^*\}^\gamma = bv^*$, $\{bv^*\}^\gamma = bs^*$, $\{bv_0^*\}^\gamma = bs^*$, $\{bs^*\}^\gamma = bv^*$.

Now, we give the characterizations of some matrix classes and state the necessary and sufficient condition on non-Newtonian matrix transformations by using the results given on Köthe-Toeplitz duals in [21].

Theorem 20. *The following statements hold:*

(i) $A = (a_{nk}) \in (\ell_\infty^* : \ell_\infty^*)$ *if and only if*

$$M = \sup_{n\in\mathbb{N}} \sum_{\beta k} \ddot{\|} a_{nk}^* \ddot{\|} \ddot{<} \infty. \tag{33}$$

(ii) $A = (a_{nk}) \in (c^* : \ell_\infty^*)$ *if and only if* (33) *holds.*

(iii) $A = (a_{nk}) \in (c_0^* : \ell_\infty^*)$ *if and only if* (33) *holds.*

(iv) $A = (a_{nk}) \in (\ell_p^* : \ell_\infty^*)$ *if and only if*

$$C = \sup_{n\in\mathbb{N}} \sum_{\beta k} \ddot{\|} a_{nk}^* \ddot{\|}^{\ddot{p}} \ddot{<} \infty, \quad (\ddot{0} \ddot{\leq} \ddot{p} < \infty). \tag{34}$$

Proof. Since the proof can also be obtained in the similar way for other cases, to avoid the repetition of the similar statements, we prove only case (i).

Suppose that condition (33) holds and $x = (x_k) \in \ell_\infty^*$. In this situation, since $(a_{nk}^*)_{k\in\mathbb{N}} \in \{\ell_\infty^*\}^\beta = \ell_1^*$ for every fixed $n \in \mathbb{N}$, the $*A$-transform of x exists. Taking into account the hypothesis, one can easily observe that

$$\begin{aligned} \sup_{n\in\mathbb{N}} d^*((Ax)_n, \theta^*) &= \sup_{n\in\mathbb{N}} d^* \left(\sum_{*k} a_{nk}^* \odot x_k, \theta^* \right) \\ &\ddot{\leq} \|x\|_\infty^* \ddot{\times} \sup_{n\in\mathbb{N}} \sum_{\beta k} \ddot{\|} a_{nk}^* \ddot{\|} < \infty, \end{aligned} \tag{35}$$

which leads us to the fact that $A \odot x \in \ell_\infty^*$, as desired.

Conversely, suppose that $A \in (\ell_\infty^* : \ell_\infty^*)$. Put $A \odot x = \{(Ax)_n\}_{n\in\mathbb{N}}$ and observe that $(Ax)_n$ is a sequence of bounded linear operators on ℓ_∞^* such that $\sup_n d^*((Ax)_n, \theta^*) < \infty$. Hence the results are obtained similarly from an application of Banach-Steinhaus theorem in classical mean. □

Example 21. Let $(x_k^*) = (\dot{\varepsilon}_k, \ddot{\delta}_k) \in \ell_\infty^*$ and define the matrix $A = (a_{nk}^*)$ by

$$a_{nk}^* := \begin{cases} x_k^*, & k = n, \\ \theta^*, & k \neq n, \end{cases} \tag{36}$$

for all $k, n \in \mathbb{N}$. Then $\ddot{\|} a_{nk}^* \ddot{\|} = \beta\{\sqrt{\varepsilon_k^2 + \delta_k^2}\}$ holds for $k = n$ otherwise θ^*. By taking into account $(x_k^*) \in \ell_\infty^*$, we obtain

$$\sup_{n\in\mathbb{N}} \sum_{\beta k} \ddot{\|} a_{nk}^* \ddot{\|} = \sup_{n\in\mathbb{N}} \left\{ \beta\sqrt{\varepsilon_0^2 + \delta_0^2}, \beta\sqrt{\varepsilon_1^2 + \delta_1^2}, \ldots, \beta\sqrt{\varepsilon_n^2 + \delta_n^2}, \ldots \right\} < \infty \tag{37}$$

for all $\varepsilon_k, \delta_k \in \mathbb{R}$. This shows, by (i) of Theorem 20, that $A = (a_{nk}^*) \in (\ell_\infty^* : \ell_\infty^*)$.

We state and prove the Kojima-Schur theorem which gives the necessary and sufficient conditions on an infinite matrix with respect to the non-Newtonian calculus, that maps the space c^* into itself. A matrix satisfying the conditions of the Kojima-Schur theorem is called a conservative matrix or convergence preserving matrix.

Theorem 22 (Kojima-Schur). *$A = (a_{nk}^*) \in (c^* : c^*)$ if and only if* (33) *holds, and there exist $\alpha_k, l \in \mathbb{C}^*$ such that*

$$ {}^*\lim_{n\to\infty} a_{nk}^* = \alpha_k \quad \text{for each } k \in \mathbb{N}, \tag{38} $$

$$ {}^*\lim_{n\to\infty} {}_*\sum_k a_{nk}^* = l. \tag{39} $$

Proof. Suppose that the conditions (33), (38), and (39) hold and $x = (x_k^*) \in c^*$ with $x_k^* \to s \in \mathbb{C}^*$ as $k \to \infty$. Then, since $(a_{nk})_{k\in\mathbb{N}} \in \{c^*\}^\beta = \ell_1^*$ for each $n \in \mathbb{N}$, the $*A$-transform of x exists. In this situation, the equality

$$ {}_*\sum_k a_{nk}^* \odot x_k^* = \left\{ {}_*\sum_k a_{nk}^* \odot \left(x_k^* \ominus s\right) \right\} \oplus \left\{ s \odot {}_*\sum_k a_{nk}^* \right\} \tag{40} $$

holds for each $n \in \mathbb{N}$. In (40), since the terms on the right hand side tend to ${}_*\sum_k \alpha_k \odot (x_k^* \ominus s)$ by (38) and the second term on the right hand side tends to $l \odot s$ by (39) as $n \to \infty$, in the sense of $*$-limit, we have

$$ {}^*\lim_{n\to\infty} {}_*\sum_k a_{nk}^* \odot x_k^* = \left\{ {}_*\sum_k \alpha_k \odot \left(x_k^* \ominus s\right) \right\} \oplus l \odot s. \tag{41} $$

Hence, $Ax \in c^*$; that is the conditions are sufficient.

Conversely, suppose that $A = (a_{nk}^*) \in (c^* : c^*)$. Then $A \odot x$ exists for every $x \in c^*$. By e and $e^{(n)}$, we denote the sequences such that $e_k = 1^*$ for $k = 0, 1, \ldots$, and $e_n^{(n)} = 1^*$ and $e_k^{(n)} = \theta^*$ $(k \neq n)$. The necessity of the conditions (38) and (39) is immediate by taking $x = e^{(k)}$ and $x = e$, respectively. Since $c^* \subset \ell_\infty^*$, the necessity of the condition (33) is obtained from Theorem 20(i). □

Theorem 23. *$A = (a_{nk}^*) \in (c_0^* : c^*)$ if and only if* (33) *holds and there exists $(\alpha_k^*) \in w^*$ such that*

$$ {}^*\lim_{n\to\infty} d^*\left(a_{nk}^*, \alpha_k^*\right) = \theta^* \tag{42} $$

for each $k \in \mathbb{N}$. If $A = (a_{nk}^*) \in (c_0^* : c^*)$, then $(\alpha_k^*) \in \ell_1^*$ and ${}^*\lim_{n\to\infty} {}_*\sum_k a_{nk}^* \odot z_k^* = {}_*\sum_k \alpha_k^* \odot z_k^*$.

Proof. Suppose that (33) and (42) hold. Then there exists an $n_K \in \mathbb{N}$ for $K \in \mathbb{N}$ and $\ddot{\varepsilon} \ddot{>} 0$ such that

$$ {}_\beta\sum_{k=0}^{k} d^*\left(a_{nk}^*, \alpha_k^*\right) < \ddot{\varepsilon}, \tag{43} $$

for all $n \geq n_K$. Since

$$ \begin{aligned} {}_\beta\sum_{k=0}^{k} d^*\left(\alpha_k^*, \theta^*\right) &\ddot{\leq} {}_\beta\sum_{k=0}^{k} d^*\left(a_{nk}^*, \alpha_k^*\right) \ddot{+} {}_\beta\sum_{k=0}^{k} d^*\left(a_{nk}^*, \theta^*\right) \\ &\ddot{\leq} \ddot{\varepsilon} \ddot{+} M \end{aligned} \tag{44} $$

for $n \geq n_K$, by (42), one can see that $(\alpha_k^*) \in \ell_1^*$ and ${}_\beta\sum_k d^*(\alpha_k^*, \theta^*) \leq M_0$. Let $z = (z_k^*) \in c_0^*$. Then, one can choose a $k_0 \in \mathbb{N}$ for $\ddot{\varepsilon}_1 \ddot{>} 0$ such that $d^*(z_k^*, \theta^*) \ddot{<} \ddot{\varepsilon}_1$ for each fixed $k \geq k_0$. Additionally, since $a_{nk}^* \overset{*}{\to} \alpha_k^*$, as $n \to \infty$ by (42), we have $a_{nk}^* \odot z_k^* \overset{*}{\to} \alpha_k^* \odot z_k^*$, as $n \to \infty$ for each fixed $k \in \mathbb{N}$. That is to say that ${}^*\lim_{n\to\infty} d^*(a_{nk}^* \odot z_k^*, \alpha_k^* \odot z_k^*) = \theta^*$. Hence, there exists an $N = N(k_0) \in \mathbb{N}$ such that ${}_\beta\sum_{k=0}^{k_0} d^*(a_{nk}^* \odot z_k^*, \alpha_k^* \odot z_k^*) \ddot{<} \ddot{\varepsilon}_2$ for all $n \geq N$. Thus, since

$$ \begin{aligned} & d^*\left({}_*\sum_k a_{nk}^* \odot z_k^*, {}_*\sum_k \alpha_k^* \odot z_k^* \right) \\ & \ddot{\leq} {}_\beta\sum_k d^*\left(a_{nk}^* \odot z_k^*, \alpha_k^* \odot z_k^*\right) \\ & = {}_\beta\sum_{k=0}^{k_0} d^*\left(a_{nk}^* \odot z_k^*, \alpha_k^* \odot z_k^*\right) \\ & \quad \ddot{+} {}_\beta\sum_{k=k_0+1}^{\infty} d^*\left(a_{nk}^* z_k^*, \alpha_k^* z_k^*\right) \\ & \ddot{\leq} \ddot{\varepsilon}_2 \ddot{+} {}_\beta\sum_{k=k_0+1}^{\infty} \left[d^*\left(a_{nk}^* \odot z_k^*, \theta^*\right) \ddot{+} d^*\left(\alpha_k^* \odot z_k^*, \theta^*\right) \right] \\ & \ddot{\leq} \ddot{\varepsilon}_2 \ddot{+} {}_\beta\sum_{k=k_0+1}^{\infty} d^*\left(a_{nk}^*, \theta^*\right) d^*\left(z_k^*, \theta^*\right) \\ & \quad \ddot{+} {}_\beta\sum_{k=k_0+1}^{\infty} d^*\left(\alpha_k^*, \theta^*\right) d^*\left(z_k^*, \theta^*\right) \\ & \ddot{\leq} \ddot{\varepsilon}_2 \ddot{+} \left\{ \varepsilon_1 \ddot{\times} \left({}_\beta\sum_{k=k_0+1}^{\infty} d^*\left(a_{nk}^*, \theta^*\right) \right.\right. \\ & \qquad \left.\left. \ddot{+} {}_\beta\sum_{k=k_0+1}^{\infty} d^*\left(\alpha_k^*, \theta^*\right) \right) \right\} \\ & \ddot{\leq} \ddot{\varepsilon}_2 \ddot{+} \left[\varepsilon_1 \ddot{\times} \left(M \ddot{+} M_0\right) \right] \end{aligned} \tag{45} $$

for all $n \geq N$, the series ${}_*\sum_k a_{nk}^* \odot z_k^*$ are $*$-convergent for each $n \in \mathbb{N}$ and ${}_*\sum_k a_{nk}^* \odot z_k^* \overset{*}{\to} {}_*\sum \alpha_k^* \odot z_k^*$, as $n \to \infty$. This means that $A \odot z \in c^*$.

Conversely, let $A = (a_{nk}^*) \in (c_0^* : c^*)$ and let $z = (z_k^*) \in c_0^*$. Then, since $A \odot z \in c^*$ exists and the inclusion $(c_0^* : c^*) \subset (c_0^* : \ell_\infty^*)$ holds, the necessity of (33) is trivial by (iii) of Theorem 20. Now, if we take the sequence $z^{(n)} = \{z_k^{(n)}\} \in c_0^*$, then $A \odot z^{(n)} = \{a_{nk}^*\}_{n=0}^{\infty} \in c^*$ holds for each fixed $k \in \mathbb{N}$;

that is, condition (42) is also necessary. Thus, the proof is completed. □

As an easy consequence of Theorem 23, we have the following corollary.

Corollary 24. *$A = (a_{nk}^*) \in (c_0^* : c_0^*)$ if and only if* (33) *holds and* (42) *also holds with $\alpha_k^* = \theta^*$ for all $k \in \mathbb{N}$.*

Example 25. Let $k, n, r \in \mathbb{N}$ and $r \geq 0$. The Cesaro means of order r is defined by the matrix $C_r^* = (c_{nk}^{*(r)})$ as

$$\left(c_{nk}^{*(r)}\right) = \begin{cases} \left(\dfrac{\binom{n-k+r-1}{n-k}}{\binom{n+r}{n}}\right)^*; & \text{if } k \leq n \\ 0^*; & \text{otherwise.} \end{cases} \tag{46}$$

Taking $r = 2$ we obtain an infinite matrix as follows:

$$C_2^* = \begin{pmatrix} 1^* & \theta^* & \theta^* & \theta^* & \theta^* & \cdots \\ \left(\frac{2}{3}\right)^* & \left(\frac{1}{3}\right)^* & \theta^* & \theta^* & \theta^* & \cdots \\ \left(\frac{3}{6}\right)^* & \left(\frac{2}{6}\right)^* & \left(\frac{1}{6}\right)^* & \theta^* & \theta^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(\frac{2}{n+2}\right)^* & \left(\frac{2n}{(n+1)(n+2)}\right)^* & \cdots & \left(\frac{2(n-k+1)}{(n+1)(n+2)}\right)^* & \theta^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{47}$$

One can easily conclude that $\sup_{n\in\mathbb{N}} {}_\beta\sum_k \|c_{nk}^{*(2)}\| < \infty$ for all $k \in \mathbb{N}$ and (33) holds. On the other hand,

$$\lim_{n\to\infty}{}^* \|c_{nk}^{*(2)}\| = \lim_{n\to\infty}{}^* \left\| \left(\frac{2(n-k+1)}{(n+1)(n+2)} \right)^* \right\| = \theta^* \tag{48}$$

so (42) also holds with $\alpha_k^* = \theta^*$ for all $k \in \mathbb{N}$. Therefore $C_2^* \in (c_0^* : c_0^*)$.

A matrix satisfying the conditions of the Silverman-Toeplitz theorem is called a Toeplitz matrix or regular matrix. By $(c^* : c^*; p)$, we denote the class of Toeplitz matrices. Now, we may give the corollaries characterizing the classes of $(c^* : c^*; p)$.

Corollary 26 (Silverman-Toeplitz Theorem). *$A = (a_{nk}^*) \in (c^* : c^*; p)$ if and only if* (33) *holds and* (38) *and* (39) *also hold with $\alpha_k^* = \theta^*$ for all $k \in \mathbb{N}$ and $l = 1^*$, respectively.*

Example 27. Example 25 can be given as an example of Silverman-Toeplitz theorem. Because the conditions (33) and (38) hold with $\alpha_k^* = \theta^*$. Furthermore we have ${}^*\lim_{n\to\infty} {}_\beta\sum_k \|c_{nk}^{*(2)}\| = 1^*$ and (39) also holds.

Theorem 28. *$A = (a_{nk}^*) \in (\ell_\infty^* : c_0^*)$ if and only if*

$$\lim_{n\to\infty}{}^* {}_*\sum_k d^*\left(a_{nk}^*, \theta^*\right) = \theta^*. \tag{49}$$

Proof. Let $A = (a_{nk}^*) \in (\ell_\infty^* : c_0^*)$ and $u = (u_k^*) \in \ell_\infty^*$. Then, the series ${}_*\sum_k a_{nk}^* \odot u_k^*$ $*$-converges to θ^* for each fixed $n \in \mathbb{N}$, since $A \odot u$ exists. Hence, $A_n^* := \{a_{nk}^*\}_{k=0}^\infty \in \{\ell_\infty^*\}^\beta$ for all $n \in \mathbb{N}$. Define the sequence $u = (u_k^*) \in \ell_\infty^*$ by $u_k^* := (1^*, 1^*, 1^*, \ldots,)$ for all $k \in \mathbb{N}$. Then, $A \odot u \in c_0^*$ which yields for all $n \in \mathbb{N}$ that

$$\begin{aligned} \lim_{n\to\infty}{}^* {}_*\sum_k a_{nk}^* \odot u_k^* &= \lim_{n\to\infty}{}^* {}_*\sum_k a_{nk}^* \odot 1^* \\ &= \lim_{n\to\infty}{}^* {}_*\sum_k a_{nk}^* = \theta^*. \end{aligned} \tag{50}$$

Furthermore we obtain

$$\begin{aligned} \lim_{n\to\infty}{}^* {}_*\sum_k d^*\left(a_{nk}^*, \theta^*\right) &= \lim_{n\to\infty}{}^* {}_\beta\sum_k \|a_{nk}^*\| \\ &\leq \left\| \lim_{n\to\infty}{}^* {}_*\sum_k a_{nk}^* \right\| = \theta^*. \end{aligned} \tag{51}$$

Conversely, suppose that (49) holds and $u = (u_k^*) \in \ell_\infty^*$. Then, since $A_n^* \in \{\ell_\infty^*\}^\beta = \ell_1^*$ for each $n \in \mathbb{N}$, $A \odot u$ exists. Therefore, one can observe, by using condition (49), that

$$\begin{aligned} \lim_{n\to\infty}{}^* d^*\left((Au)_n, \theta^*\right) &= \lim_{n\to\infty}{}^* d^*\left({}_*\sum_k a_{nk}^* \odot u_k^*, \theta^* \right) \\ &\leq \lim_{n\to\infty}{}^* {}_*\sum_k d^*\left(a_{nk} \odot u_k, \theta^*\right) \\ &\leq \lim_{n\to\infty}{}^* {}_*\sum_k d^*\left(a_{nk}, \theta^*\right) \mathbin{\ddot{\times}} d^*\left(u_k, \theta^*\right) \\ &\leq \sup_{k\in\mathbb{N}} d^*\left(u_k, \theta^*\right) \\ &\mathbin{\ddot{\times}} \lim_{n\to\infty}{}^* {}_*\sum_k d^*\left(a_{nk}, \theta^*\right) = \theta^* \end{aligned} \tag{52}$$

which means that $A \odot u \in c_0^*$, as desired. □

Theorem 29. *$A = (a_{nk}^*) \in (cs^* : c^*)$ if and only if* (38) *holds, and*

$$\sup_{n\in\mathbb{N}} {}_*\sum_k d^*\left(\Delta a_{nk}^*, \theta^*\right) \ddot{<} \infty \quad \text{where } \Delta a_{nk}^* = a_{nk}^* \ominus a_{n,k+1}^* \quad \forall n,k \in \mathbb{N}. \tag{53}$$

Proof. Let $x = (x_k^*) \in cs^*$ with ${}_*\sum_k x_k^* = s$ and $y_k^* = {}_*\sum_{j=0}^k x_j^*$ for all $k \in \mathbb{N}$. Define the infinite matrix $B = (b_{nk}^*)$ by $B = (b_{nk}^*) = (\Delta a_{nk}^*)$ for all $k,n \in \mathbb{N}$. Suppose that $A \in (cs^* : c^*)$. Then, $A \odot x$ exists for every $x = (x_k^*) \in cs^*$ and is in c^*. Since this also holds for $x = e^{(k)} \in cs^*$ for each fixed $k \in \mathbb{N}$, the necessity of (38) is clear. Consider the following relation obtained from mth-partial sums of the series ${}_*\sum_k a_{nk}^* \odot x_k^*$ by applying Abel's partial summation. In this situation, the equalities

$$\begin{aligned}
{}_*\sum_{k=0}^m a_{nk}^* \odot x_k^* &= \left\{ {}_*\sum_{k=0}^{m-1} \left(\Delta a_{nk}^*\right) \odot y_k^* \right\} \oplus a_{nm}^* \odot y_m^* \\
&= \left\{ {}_*\sum_{k=0}^{m-1} \left(\Delta a_{nk}^*\right) \odot \left(y_k^* \ominus s\right) \right\} \\
&\quad \oplus \left\{ s \odot {}_*\sum_{k=0}^{m-1} \Delta a_{nk}^* \right\} \oplus a_{nm}^* \odot y_m^* \\
&= \left\{ {}_*\sum_{k=0}^{m-1} \left(\Delta a_{nk}^*\right) \odot \left(y_k^* \ominus s\right) \right\} \\
&\quad \oplus \left[s \odot \left(a_{n0}^* \ominus a_{nm}^*\right) \right] \\
&\quad \oplus a_{nm}^* \odot y_m^* \quad \forall m,n \in \mathbb{N}.
\end{aligned} \tag{54}$$

Therefore, we derive by passing to limit in (54) as $m \to \infty$ that

$$\begin{aligned}
(Ax)_n &= {}_*\sum_k a_{nk}^* \odot x_k^* \\
&= \left\{ {}_*\sum_k b_{nk}^* \odot \left(y_k^* \ominus s\right) \right\} \oplus \left(s \odot a_{n0}^*\right)
\end{aligned} \tag{55}$$

for all $n \in \mathbb{N}$. Since ${}^*\lim_n (Ax)_n$ exists and ${}^*\lim_n a_{n0}^* = \alpha_0^*$, we see by letting $n \to \infty$ in (55) that ${}^*\lim_n {}_*\sum_k b_{nk}^* \odot (y_k^* \ominus s)$ also exist. This yields the fact that $B \in (c_0^* : c^*)$, because $y \ominus s \in c_0^*$ if and only if $x \in cs^*$. Hence, the matrix $B = (b_{nk}^*)$ satisfies the condition (33) which is equivalent to the condition (53); that is the condition (53) is necessary.

Conversely, suppose that conditions (38) and (53) hold. First, (53) implies $A_n^* = (a_{nk}^*)_{k\in\mathbb{N}} \in \{cs^*\}^\beta = bv^* \subset \ell_\infty^*$ for every fixed $n \in \mathbb{N}$; hence, $A \odot x$ exists for every $x \in cs^*$. Also (38) and (53) imply by Corollary 24 that $B = (b_{nk}^*) \in (c_0^* : c_0^*)$. Thus, it follows from (55) that

$${}^*\lim_{n\to\infty} {}_*\sum_k a_{nk}^* \odot x_k^* = s \odot \alpha_0^*. \tag{56}$$

Hence $A = (a_{nk}^*) \in (cs^* : c^*)$. This completes the proof. □

Theorem 30. *$A = (a_{nk}^*) \in (cs^* : cs^*)$ if and only if*

$$\sup_{n\in\mathbb{N}} {}_*\sum_k d^*\left({}_*\sum_{j=0}^n \Delta a_{jk}^*, \theta^* \right) \ddot{<} \infty, \tag{57}$$

$${}^*\lim_{n\to\infty} d^*\left({}_*\sum_n a_{nk}^*, \alpha_k^* \right) = \theta^*, \tag{58}$$

where $\alpha_k^* \in \mathbb{C}^*$ for each $k \in \mathbb{N}$.

Proof. Let $x = (x_k^*) \in cs^*$ and define the matrix $C = (c_{nk}^*)$ by $c_{nk} = {}_*\sum_{j=0}^n a_{jk}^*$ as follows:

$$C = \begin{pmatrix}
a_{00}^* & a_{01}^* & a_{02}^* & \cdots & a_{0k}^* \\
a_{00}^* \oplus a_{10}^* & a_{01}^* \oplus a_{11}^* & a_{02}^* \oplus a_{12}^* & \cdots & a_{0k}^* \oplus a_{1k}^* \\
a_{00}^* \oplus a_{10}^* \oplus a_{20}^* & a_{01}^* \oplus a_{11}^* \oplus a_{21}^* & a_{02}^* \oplus a_{12}^* \oplus a_{22}^* & \cdots & a_{0k}^* \oplus a_{1k}^* \oplus a_{2k}^* \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
a_{00}^* \oplus a_{10}^* \oplus \cdots \oplus a_{n0}^* & a_{01}^* \oplus a_{11}^* \oplus \cdots \oplus a_{n1}^* & a_{02}^* \oplus a_{12}^* \oplus \cdots \oplus a_{n2}^* & \cdots & a_{0k}^* \oplus a_{1k}^* \oplus \cdots \oplus a_{nk}^*
\end{pmatrix} \tag{59}$$

for all $k,n \in \mathbb{N}$. Suppose that $A = (a_{nk}^*) \in (cs^* : cs^*)$. Then, $A \odot x$ exists for every $x = (x_k^*) \in cs^*$ and is in cs^*. This yields for $x = e^{(k)} \in cs^*$ for each fixed $k \in \mathbb{N}$ that the condition (58) is necessary. It is clear that the following equality

$${}_*\sum_{j=0}^n {}_*\sum_{k=0}^m a_{jk}^* \odot x_k^* = {}_*\sum_{k=0}^m {}_*\sum_{j=0}^n a_{jk}^* \odot x_k^* = {}_*\sum_{k=0}^m c_{nk}^* \odot x_k^* \tag{60}$$

derived from nth and mth-partial sums of the double series ${}_*\sum_j {}_*\sum_k a_{jk}^* \odot x_k^*$ holds for all $m,n \in \mathbb{N}$. Therefore, by letting $m \to \infty$ in (60) we have

$${}_*\sum_{j=0}^n (Ax)_j = (Cx)_n \quad \forall n \in \mathbb{N}. \tag{61}$$

Then, since $A \odot x \in cs^*$ by the hypothesis, ${}^*\lim_n \sum_{j=0}^n (Ax)_j$ exists, on the left hand side of (61), which leads us to the

consequence that $C = (c_{nk}^*) \in (cs^* : c^*)$. Therefore, condition (53) of Theorem 29 is satisfied by the matrix $C = (c_{nk}^*)$ which is equivalent to condition (57).

Conversely, suppose that conditions (57) and (58) hold, which imply the existence of the A-transform of $x \in cs^*$. Then, since (61) also holds, the matrix C satisfies the conditions of Theorem 29. Hence, ${}^*\lim_n (Cx)_n$ exists which says by (61) that $A \odot x \in cs^*$, as was desired. □

Theorem 31. *$A = (a_{nk}^*) \in (c^* : cs^*)$ if and only if* (58) *holds and*

$$\sup_{n\in\mathbb{N}} {}_*\sum_k d^* \left(\sum_{j=0}^{n} a_{jk}^*, \theta^* \right) \ddot{<} \infty, \tag{62}$$

$${}_*\sum_n {}_*\sum_k a_{nk}^* \text{ is } *\text{-convergent}, \quad \forall k, n \in \mathbb{N}_0. \tag{63}$$

Proof. Let $x = (x_k^*) \in c^*$ and define the matrix $C = (c_{nk}^*)$ as in the proof of Theorem 30. Suppose that $A = (a_{nk}^*) \in (c^* : cs^*)$. Then, $A \odot x$ exists for every $x \in c^*$ and is in cs^*. This yields for $x = e^{(k)} \in c^*$ and $x = e \in c^*$ which give the necessity of the conditions (58) and (63), respectively. It is clear that we have the relation (61), derived by the same way used in the proof of Theorem 30. Then, since $A \odot x \in cs^*$, that is, the series ${}_*\sum_j (Ax)_j$ $*$-converges by the hypothesis ${}^*\lim_n {}_*\sum_{j=0}^{n} (Ax)_j$ exists, on the left hand side of (61), which leads us to the consequence that $C = (c_{nk}^*) \in (c^* : c^*)$. Therefore, condition (33) in Kojima-Schur theorem, is satisfied by the matrix $C = (c_{nk}^*)$ which is equivalent to condition (62).

Conversely, suppose that conditions (58), (62), and (63) hold, which imply the existence of the A-transform of $x \in c^*$. Then, since (61) also holds, the matrix $C = (c_{nk}^*)$ satisfies the conditions of Kojima-Schur theorem. Hence, ${}^*\lim_n (Cx)_n$ exists which says by (61) that $A \odot x \in cs^*$, as was desired. □

4. Conclusion

At the beginning of 1981, the wage-rate in dollars per hour at a certain company was w_0 and the cost of living index for the United States was c_0. At the end of 1981, the amounts were w_1 and c_1, respectively. Company and union negotiators had agreed at the beginning of 1981 that, thereafter, the wage-rate would be adjusted to reflect changes in the cost of living index. Assuming that the cost of living index is always increasing and that the wage-rate changes "uniformly" and continuously relative to the cost of living index, find the wage-rate w_t at time t in terms of the constants c_0, c_1, w_0, and w_1 the cost of living index c_t at time t. There is no unique solution to this problem; we shall give two reasonable solutions (cf. [14]).

Firstly, since the wage-rate changes "uniformly" relative to the cost of living index, we may reasonably assume that equal differences in the cost of living index give rise to equal differences in the wage-rate. Furthermore, since the changes are "continuously" relative to the cost of living index, it can be proved that

$$w_t = w_0 + \left[\frac{w_1 - w_0}{c_1 - c_0} \right] (c_t - c_0). \tag{64}$$

Secondly, since the changes are "continuously" relative to the cost of living index, it can be given that

$$w_t = w_0 \left[\left(\frac{w_1}{w_0} \right)^{1/(\ln c_1 - \ln c_0)} \right]^{\ln c_t - \ln c_0}. \tag{65}$$

We shall see that the expression within the brackets represents a new gradient that plays a fundamental role in the bigeometric calculus which is a branch of non-Newtonian calculus. On the other hand, in the mathematical solution of many fundamental physical problems we are naturally led to series whose terms contain factors which are the mathematical representations of the damping factors of the physicist. These same factors may be interpreted as convergence factors for summable series, since they satisfy the conditions of the general theorems. Thus, the use of convergence factor theorems and the theory of summable series frequently serves to extend the domain of applicability of the mathematical solution of physical problems.

The table on the characterizations of the matrix transformations between certain spaces of sequences with real or complex terms was given by Stieglitz and Tietz [26]. To prepare the corresponding table for certain sequence spaces over the non-Newtonian complex field $\mathbb{C}^*$, we characterize some classes of infinite matrices. Of course, to complete the table of matrix transformations from the set μ_1^* to the set μ_2^*, there are several open problems depending on the choice of generator functions.

The main results given in final section of the present paper will be based on examining the domain of some matrices in the classical sets of sequences. This is a new development of the matrix transformations between sequence spaces over $\mathbb{C}^*$. Finally, we should note from now on that our next papers will be devoted to the matrix domains of the classical sets of sequences over $\mathbb{C}^*$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Determination of the Köthe-Toeplitz Duals over the Non-Newtonian Complex Field

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The important point to note is that the non-Newtonian calculus is a self-contained system independent of any other system of calculus. Therefore the reader may be surprised to learn that there is a uniform relationship between the corresponding operators of this calculus and the classical calculus. Several basic concepts based on non-Newtonian calculus are presented by Grossman (1983), Grossman and Katz (1978), and Grossman (1979). Following Grossman and Katz, in the present paper, we introduce the sets of bounded, convergent, null series and p-bounded variation of sequences over the complex field $\mathbb{C}^*$ and prove that these are complete. We propose a quite concrete approach based on the notion of Köthe-Toeplitz duals with respect to the non-Newtonian calculus. Finally, we derive some inclusion relationships between Köthe space and solidness.

1. Introduction

It is certainly not unusual to measure deviations by ratios rather than differences. For instance, during the Renaissance, many scholars, including Galileo, discussed the following problem. Two estimates, 10 and 1000, are proposed as the value of a horse, which estimates, if any, deviates more from the true value of 100? The scholars who maintained that deviations should be measured by differences concluded that the estimate of 10 was closer to the true value. However, Galileo eventually maintained that the deviations should be measured by ratios, and he concluded that two estimates deviated equally from the true value. From the story, the question comes out this way, what if we measure by ratios? The answer is the main idea of non-Newtonian calculus which consists of many calculuses such as the classical, geometric, anageometric, and bigeometric calculus.

Bashirov et al. [1, 2] have recently concentrated on the non-Newtonian calculus and gave the results with applications corresponding to the well-known properties of derivatives and integrals in the classical calculus. Quite recently, Uzer [3] has extended the non-Newtonian calculus to the complex-valued functions and was interested in the statements of some fundamental theorems and concepts of multiplicative complex calculus and demonstrated some analogies between the multiplicative complex calculus and classical calculus by theoretical and numerical examples. In particular, Bashirov et al. [2] have studied the multiplicative differentiation for complex-valued functions and established the multiplicative Cauchy-Riemann conditions. Further, Tekin and Başar have introduced some certain sequence spaces over the non-Newtonian complex field by using $*$-calculus in [4] and many authors have introduced multiplicative calculus in biomedical image analysis and have derived non-Newtonian calculus as an alternative to the quantum calculus in [5, 6].

Following Tekin and Başar [4] we can construct the sets bs^*, cs^*, and cs_0^* consisting of the sets of all bounded, convergent, null series based on the non-Newtonian calculus, as follows:

$$bs^* := \left\{ x = (x_k) \in \omega^* : \|x\|_{bs}^* = \sup_{n\in\mathbb{N}} \left\| \sum_{k=0}^{n}{}^{*} x_k \right\| < \infty \right\}, \text{ or}$$

$$bs^* := \left\{x = (x_k) \in \omega^* : \left(\sum_{k=0}^{n}{}_* x_k\right) \in \ell_\infty^*\right\},$$
$$cs^* := \left\{x = (x_k) \in \omega^* : \left(\sum_{k=0}^{n}{}_* x_k\right) \in c^*\right\},$$
$$cs_0^* := \left\{x = (x_k) \in \omega^* : \left(\sum_{k=0}^{n}{}_* x_k\right) \in c_0^*\right\}, \tag{1}$$

where $\omega^* = \{x = (x_k) : x_k \in \mathbb{C}^* \text{for all } k \in \mathbb{N}\}$. One can conclude that the sets bs^*, cs^*, and cs_0^* are complete non-Newtonian metric spaces with the metric d_∞^N defined by

$$d_\infty^N := \sup_{n\in\mathbb{N}}\left\{d^*\left(\sum_{k=0}^{n}{}_* x_k, \sum_{k=0}^{n}{}_* y_k\right)\right\}. \tag{2}$$

Secondly, we introduce several sets bv^*, bv_p^*, and bv_∞^* of bounded variation sequences in the sense of non-Newtonian calculus, as follows:

$$bv^* := \left\{x = (x_k) \in \omega^* : \|x\|_{bv}^* := \sum_{k=0}^{\infty}{}_* \ddot{\|}(\Delta x)_k'\ddot{\|} < \infty\right\},$$
$$d_\Delta(x,y) := \sum_{k=0}^{\infty}{}_* \left\{d^*\left[(\Delta x)_k', (\Delta y)_k'\right]\right\}.$$
$$bv_p^* := \left\{x = (x_k) \in \omega^* : \sum_{k=0}^{\infty}{}_* \ddot{\|}(\Delta x)_k\ddot{\|}^{\ddot{p}} < \infty\right\},$$
$$d_p^\Delta(x,y) := \left\{\sum_{k=0}^{\infty}{}_* d^*\left[(\Delta x)_k, (\Delta y)_k\right]^{\ddot{p}}\right\}^{\ddot{1}\ddot{/}\ddot{p}} \quad (1 \le p < \infty),$$
$$bv_\infty^* := \left\{x = (x_k) \in \omega^* : \sup_{k\in\mathbb{N}}\ddot{\|}(\Delta x)_k\ddot{\|} < \infty\right\},$$
$$d_\infty^\Delta(x,y) := \sup_{k\in\mathbb{N}}\left\{d^*\left[(\Delta x)_k, (\Delta y)_k\right]\right\}. \tag{3}$$

One can easily see that the sets bv^*, bv_p^*, and bv_∞^* are complete with corresponding metrics on the right-hand side with $(\Delta x)_k = x_k \ddot{-} x_{k-1}$, $x_{-1} = \ddot{0}$ and $(\Delta x)_k' = x_k \ddot{-} x_{k+1}$ for all $k \in \mathbb{N}$.

2. β-Arithmetics and Some Related Applications

A *generator* is a one-to-one function whose domain is $\mathbb{R}$ and whose range is a subset of $B \subseteq \mathbb{R}$, the set of real numbers. Each generator generates exactly one arithmetic, and conversely each arithmetic is generated by exactly one generator. As a generator, we choose the function β such that its basic algebraic operations are defined as follows:

β-add

$$x \ddot{+} y = \beta\left\{\beta^{-1}(x) + \beta^{-1}(y)\right\};$$

β-subtraction

$$x \ddot{-} y = \beta\left\{\beta^{-1}(x) - \beta^{-1}(y)\right\},$$

β-multipl.

$$x \ddot{\times} y = \beta\left\{\beta^{-1}(x) \times \beta^{-1}(y)\right\}; \tag{4}$$

β-division $x\ddot{/}y$,

$$\frac{x}{y} := \beta\left\{\beta^{-1}(x) \div \beta^{-1}(y)\right\},$$

β-order

$$x \ddot{<} y \Longleftrightarrow \beta(x) < \beta(y),$$

for all $x, y \in \mathbb{R}(N)$, where the non-Newtonian real field $\mathbb{R}(N) := \{\beta\{x\} : x \in \mathbb{R}\}$ as in [7].

The β-positive real numbers, denoted by $\mathbb{R}^+(N)$, are the numbers x in $\mathbb{R}$ such that $\ddot{0} \ddot{<} x$; the β-negative real numbers, denoted by $\mathbb{R}^-(N)$, are those for which $x \ddot{<} \ddot{0}$. The *beta*-zero, $\ddot{0}$, and the *beta*-one, $\ddot{1}$, turn out to be $\beta(0)$ and $\beta(1)$. Further, $\beta(-1) = \ddot{-}\ddot{1}$. Thus the set of all β-integers turns out to be the following:

$$\begin{aligned}\mathbb{Z}(N) &= \{\ldots, \beta(-2), \beta(-1), \beta(0), \beta(1), \beta(2), \ldots\} \\ &= \{\ldots, \ddot{-}\ddot{2}, \ddot{-}\ddot{1}, \ddot{0}, \ddot{1}, \ddot{2}, \ldots\}.\end{aligned} \tag{5}$$

Definition 1. Let X be a nonempty set and let $d^* : X \times X \to \mathbb{R}(N)$ be a function such that, for all $x, y, z \in X$, the following axioms hold:

(NM1) $d^*(x,y) = \ddot{0}$ if and only if $x = y$,
(NM2) $d^*(x,y) = d^*(y,x)$,
(NM3) $d^*(x,y) \ddot{\le} d^*(x,z) \ddot{+} d^*(z,y)$.

Then, the pair (X, d^*) and d^* are called a non-Newtonian metric space and a non-Newtonian metric on X, respectively.

Definition 2 (see [7]). Let $X = (X, d^*)$ be a non-Newtonian metric space. Then the basic notions can be defined as follows.

(a) A sequence $x = (x_k)$ is a function from the set $\mathbb{N}$ into the set $\mathbb{R}(N)$. The β-real number x_k denotes the value of the function at $k \in \mathbb{N}$ and is called the kth term of the sequence.

(b) A sequence (x_n) in a metric space $X = (X, d^*)$ is said to be $*$-convergent if for every given $\varepsilon \ddot{>} \ddot{0}$ there exist an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $x \in X$ such that $d^*(x_n, x) \ddot{<} \varepsilon$ for all $n > n_0$ and is denoted by ${}^*\lim_{n\to\infty} x_n = x$ or $x_n \overset{*}{\to} x$, as $n \to \infty$.

(c) A sequence (x_n) in $X = (X, d^*)$ is said to be non-Newtonian Cauchy ($*$-Cauchy) if for every $\varepsilon \ddot{>} \ddot{0}$ there is an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $d^*(x_n, x_m) \ddot{<} \varepsilon$ for all $m, n > n_0$.

Remark 3. Let $\ddot{b} \in B \subseteq \mathbb{R}$. Then the number $\ddot{b} \ddot{\times} \ddot{b}$ is called the β-square and is denoted by $b^{\ddot{2}}$. Let $\ddot{b}$ be a nonnegative number in B. Then $\beta\{\sqrt{\beta^{-1}(\ddot{b})}\}$ is called the β-square root of $\ddot{b}$ and is denoted by $\sqrt[\ddot{2}]{\ddot{b}}$. Further for each $\ddot{b} \in B$ we can write $\ddot{b}^{\ddot{2}} = \beta\{[\beta^{-1}(\ddot{b})]^2\} = \beta\{b^2\}$.

The β-absolute value of a number x in $B \subset \mathbb{R}(N)$ is defined as $\beta(|\beta^{-1}(x)|)$ and is denoted by $|x|_\beta$. For each number x in $B \subset \mathbb{R}(N)$, $\sqrt[\ddot{2}]{\ddot{x}^{\ddot{2}}} = |x|_\beta = \beta(|\beta^{-1}(x)|)$. Then we say

$$|x|_\beta = \begin{cases} x, & x \ddot{>} \ddot{0} \\ \ddot{0}, & x = \ddot{0} = \beta\left\{\left|\beta^{-1}(x)\right|\right\} \\ \ddot{0} \ddot{-} x, & x \ddot{<} \ddot{0}. \end{cases} \tag{6}$$

The non-Newtonian distance between two real numbers x_1 and x_2 is defined by $|x_1 \ddot{-} x_2|_\beta = \beta\{|\beta^{-1}(x_1) - \beta^{-1}(x_2)|\} = \beta\{|\beta^{-1}(x_2) - \beta^{-1}(x_1)|\} = |x_2 \ddot{-} x_1|_\beta$. Similarly by taking into account the definition for *alpha*-generator in (6) one can conclude that the equality $|x_1 \dot{-} x_2|_\alpha = |x_2 \dot{-} x_1|_\alpha$ holds for all $x_1, x_2 \in \mathbb{R}$.

Now, we give a new type calculus for non-Newtonian complex terms, denoted by $*$-calculus, which is a branch of non-Newtonian calculus. From now on we will use $*$-calculus type with respect to two arbitrarily selected generator functions.

2.1. $*$-Arithmetics with respect to the Complex Field. Suppose that α and β are two arbitrarily selected generators and ("star-") also are the ordered pair of arithmetics (β-arithmetic and α-arithmetic). The sets $(B, \ddot{+}, \ddot{-}, \ddot{\times}, \ddot{/})$ and $(A, \dot{+}, \dot{-}, \dot{\times}, \dot{/})$ are complete ordered fields and *beta*(*alpha*)-generator generates *beta*(*alpha*)-arithmetics, respectively. Definitions given for β-arithmetic are also valid for α-arithmetic.

The important point to note here is that α-arithmetic is used for arguments and β-arithmetic is used for values; in particular, changes in arguments and values are measured by α-differences and β-differences, respectively. The operators of this calculus type are applied only to functions with arguments in A and values in B. The $*$-limit of a function with two generators α and β is defined by

$$\begin{aligned} {}^* \lim_{x \to a} f(x) = b \Longleftrightarrow \forall \varepsilon \ddot{>} \ddot{0}, \\ \exists \delta \dot{>} \dot{0} \ni \left|f(x) \ddot{-} b\right|_\beta \ddot{\leq} \varepsilon, \\ \forall x \in A, \left|x \dot{-} a\right|_\alpha \dot{<} \delta. \end{aligned} \tag{7}$$

A function f is $*$-continuous at a point a in A if and only if a is an argument of f and ${}^*\lim_{x \to a} f(x) = f(a)$. When α and β are the identity function I, the concepts of $*$-limit and $*$-continuity are identical with those of classical limit and classical continuity.

The isomorphism from α-arithmetic to β-arithmetic is the unique function ι (iota) that possesses the following three properties:

(i) ι is one to one.

(ii) ι is from A onto B.

(iii) For any numbers, u and v in A,

$$\begin{aligned} &\iota(u \dot{+} v) = \iota(u) \ddot{+} \iota(v), \qquad \iota(u \dot{-} v) = \iota(u) \ddot{-} \iota(v) \\ &\iota(u \dot{\times} v) = \iota(u) \ddot{\times} \iota(v), \qquad \iota\left(u \dot{/} v\right) = \iota(u) \ddot{/} \iota(v); \\ &\qquad\qquad v \neq \dot{0}, \quad u \dot{\leq} v \Longleftrightarrow \iota(u) \ddot{\leq} \iota(v). \end{aligned} \tag{8}$$

It turns out that $\iota(x) = \beta\{\alpha^{-1}(x)\}$ for every x in A and that $\iota(\dot{n}) = \ddot{n}$ for every integer n. Since, for example, $u \dot{+} v = \iota^{-1}\{\iota(u) \ddot{+} \iota(v)\}$, it should be clear that any statement in α-arithmetic can readily be transformed into a statement in β-arithmetic.

Let $\alpha(a) = \dot{a} \in (A, \dot{+}, \dot{-}, \dot{\times}, \dot{/})$ and $\beta(b) = \ddot{b} \in (B, \ddot{+}, \ddot{-}, \ddot{\times}, \ddot{/})$ be arbitrarily chosen elements from corresponding arithmetics. Then the ordered pair $(\dot{a}, \ddot{b})$ is called a $*$-point. The set of all $*$-points is called the set of $*$-complex numbers and is denoted by $\mathbb{C}^*$; that is,

$$\mathbb{C}^* := \left\{z^* = \left(\dot{a}, \ddot{b}\right) \mid \dot{a} \in A, \ddot{b} \in B \subseteq \mathbb{R}\right\}. \tag{9}$$

Define the binary operations addition ($\oplus$) and multiplication ($\odot$) of $*$-complex numbers $z_1^* = (\dot{a}_1, \ddot{b}_1)$ and $z_2^* = (\dot{a}_2, \ddot{b}_2)$:

$$\begin{aligned} &\oplus : \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^* \\ &\quad (z_1^*, z_2^*) \longmapsto z_1^* \oplus z_2^* = \left(\alpha\{a_1 + a_2\}, \beta\{b_1 + b_2\}\right) \\ &\qquad\qquad = \left(\dot{a}_1 \dot{+} \dot{a}_2, \ddot{b}_1 \ddot{+} \ddot{b}_2\right) \\ &\odot : \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^* \\ &\quad (z_1^*, z_2^*) \longmapsto z_1^* \odot z_2^* = \left(\alpha\{a_1 a_2 - b_1 b_2\}, \right. \\ &\qquad\qquad \left. \beta\{a_1 b_2 + b_1 a_2\}\right), \end{aligned} \tag{10}$$

where $\dot{a}_1, \dot{a}_2 \in A$ and $\ddot{b}_1, \ddot{b}_2 \in B$.

Theorem 4 (see [4]). *$(\mathbb{C}^*, \oplus, \odot)$ is a field.*

Following Grossman and Katz [8] we can give the definition of $*$-distance and some applications with respect to the $*$-calculus which is a kind of calculi of non-Newtonian calculus.

The $*$-distance d^* between two arbitrary elements $z_1^* = (\dot{a}_1, \ddot{b}_1)$ and $z_2^* = (\dot{a}_2, \ddot{b}_2)$ of the set $\mathbb{C}^*$ is defined by

$$\begin{aligned} d^* : \mathbb{C}^* \times \mathbb{C}^* &\longrightarrow \dddot{[}\dddot{0}, \infty\dddot{)} = B' \subset B \\ (z_1^*, z_2^*) &\longmapsto d^*(z_1^*, z_2^*) \\ &= \sqrt[\ddot{2}]{\left[\iota(\dot{a}_1 \dot{-} \dot{a}_2)\right]^{\ddot{2}} \ddot{+} \left(\ddot{b}_1 \ddot{-} \ddot{b}_2\right)^{\ddot{2}}} \\ &= \beta\left\{\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}\right\}. \end{aligned} \tag{11}$$

Up to now, we know that $\mathbb{C}^*$ is a field and the distance between two points in $\mathbb{C}^*$ is computed by the function d^*, defined by (11).

Definition 5. Given a sequence (z_k^*) of $*$-complex numbers, the formal notation,

$$\sum_{k=0}^{\infty}{}_* z_k^* = z_0^* \oplus z_1^* \oplus z_2^* \oplus \cdots \oplus z_k^* \oplus \cdots \quad \forall k \in \mathbb{N}, \tag{12}$$

is called an infinite series with $*$-complex terms or simply complex N-series. Also, for integers, $n \in \mathbb{N}$, the finite $*$-sums $s_n^* = {}_*\sum_{k=0}^{n} z_k^*$ are called the partial sums of complex N-series. If the sequence $*$-converges to a complex number s^*, then we say that the series $*$-converges and write $s^* = {}_*\sum_{k=0}^{n} z_n^*$. The number s^* is then called the $*$-sum of this series. If (s_n) $*$-diverges, we say that the series $*$-diverges or it is $*$-divergent.

Proposition 6 (see [4]). *For any $z_1^*, z_2^* \in \mathbb{C}^*$, the following statements hold:*

(i) $\ddot{\|} z_1^* \oplus z_2^* \ddot{\|} \ddot{\leq} \ddot{\|} z_1^* \ddot{\|} \ddot{+} \ddot{\|} z_2^* \ddot{\|}$ *($*$-triangle inequality).*

(ii) $\ddot{\|} z_1^* \odot z_2^* \ddot{\|} = \ddot{\|} z_1^* \ddot{\|} \ddot{\times} \ddot{\|} z_2^* \ddot{\|}$.

(iii) *Let $p \ddot{>} \ddot{1}$ and $z_k^*, t_k^* \in \mathbb{C}^*$ for $k \in \{0, 1, 2, 3, \ldots, n\}$. Then,*

$$\left({}_*\sum_{k=0}^{n} \ddot{\|} z_k^* \oplus t_k^* \ddot{\|}^p\right)^{\ddot{1}/p} \ddot{\leq} \left({}_*\sum_{k=0}^{n} \ddot{\|} z_k^* \ddot{\|}^p\right)^{\ddot{1}/p} \ddot{+} \left({}_*\sum_{k=0}^{n} \ddot{\|} t_k^* \ddot{\|}^p\right)^{\ddot{1}/p} \left(Minkowski's\ inequality\right). \tag{13}$$

Following Tekin and Başar [4], we can give the $*$-norm and next derive some required inequalities in the sense of non-Newtonian complex calculus.

Let $z^* \in \mathbb{C}^*$ be an arbitrary element. The distance function $d^*(z^*, \theta^*)$ is called $*$-norm of z^* and is denoted by $\ddot{\|} \cdot \ddot{\|}$. In other words,

$$\ddot{\|} z^* \ddot{\|} = d^*(z^*, \theta^*) = \sqrt[\ddot{2}]{\left[\iota(\dot{a} \dot{-} \dot{0})\right]^{\ddot{2}} \ddot{+} \left(\ddot{b} \ddot{-} \ddot{0}\right)^{\ddot{2}}} = \beta\left\{\sqrt{a^2 + b^2}\right\}, \tag{14}$$

where $z^* = (\dot{a}, \ddot{b})$ and $\theta^* = (\dot{0}, \ddot{0})$. Moreover, since for all $z_1^*, z_2^* \in \mathbb{C}^*$ we have $d^*(z_1^*, z_2^*) = \ddot{\|} z_1^* \ominus z_2^* \ddot{\|}$, d^* is the induced metric from $\ddot{\|} \cdot \ddot{\|}$ norm.

Definition 7 (complex conjugate). Let $z^* = (\dot{a}, \ddot{b}) \in \mathbb{C}^*$. We define the $*$-complex conjugate $\overline{z}^*$ of z^* by $\overline{z}^* = (\alpha\{a\}, \beta\{-\beta^{-1}(\ddot{b})\}) = (\dot{a}, \ddot{-}\ddot{b})$. Conjugation changes the sign of the imaginary part of z^* but leaves the real part the same. Thus

$$\begin{aligned} \mathscr{R}e(\overline{z}^*) &= \mathscr{R}e(z^*) = (z^* \oplus \overline{z}^*) \dot{/} \dot{2} = \dot{a}, \\ \mathscr{I}m(\overline{z}^*) &= \ddot{-}\mathscr{I}m(z^*) = (z^* \ominus \overline{z}^*) \ddot{/} \ddot{2} = \ddot{b}. \end{aligned} \tag{15}$$

Remark 8. (i) Let $z^* = (\dot{a}, \ddot{b})$, $w^* = (\dot{c}, \ddot{d}) \in \mathbb{C}^*$. We can give the $*$-division as

$$\begin{aligned} z \oslash w &= (\dot{a}, \ddot{b}) \oslash (\dot{c}, \ddot{d}) \\ &= \left(\alpha\left\{\frac{ac + bd}{c^2 + d^2}\right\}, \beta\left\{\frac{bc - ad}{c^2 + d^2}\right\}\right). \end{aligned} \tag{16}$$

(ii) Let α and β be the same generator functions and $z^* \in \mathbb{C}^*$. Then the following condition holds

$$\begin{aligned} z^* \odot \overline{z^*} &= (\dot{a}, \ddot{b}) \odot (\dot{a}, \ddot{-}\ddot{b}) = \left(\alpha\left\{a^2 + b^2\right\}, \beta(0)\right) \\ &= \beta\left\{a^2 + b^2\right\} = \beta\left\{\left(\beta^{-1}\beta\sqrt{a^2 + b^2}\right)^2\right\} = \left(\ddot{\|} z^* \ddot{\|}\right)^{\ddot{2}}. \end{aligned} \tag{17}$$

Theorem 9 (see [4]). *$(\mathbb{C}^*, d^*)$ is a complete metric space, where d^* is defined by* (11).

Corollary 10 (see [4]). *$\mathbb{C}^*$ is a Banach space with the $*$-norm $\ddot{\|} \cdot \ddot{\|}$ defined by $\ddot{\|} z^* \ddot{\|} = \sqrt[\ddot{2}]{\iota(\dot{a})^{\ddot{2}} \ddot{+} (\ddot{b})^{\ddot{2}}}$; $z^* = (\dot{a}, \ddot{b}) \in \mathbb{C}^*$.*

3. Completeness of the Sets of Bounded, Convergent, and Null Series over the Geometric Complex Field

Quite recently Tekin and Başar [4] have introduced the sets ℓ_∞^*, c^*, c_0^*, and ℓ_p^* of all bounded, convergent, null, and absolutely p-summable sequences over the complex field $\mathbb{C}^*$ which correspond to the sets ℓ_∞, c, c_0, and ℓ_p over the complex field $\mathbb{C}$, respectively. That is to say,

$$\begin{aligned} \ell_\infty^* &= \left\{z^* = (z_k^*) \in \omega^* : \sup_{k \in \mathbb{N}} \ddot{\|} z_k^* \ddot{\|} < \infty\right\}, \\ c^* &= \left\{z^* = (z_k^*) \in \omega^* : \exists l \in \mathbb{C}^* \ni {}^*\lim_{k \to \infty} z_k^* = l\right\}, \\ c_0^* &= \left\{z^* = (z_k^*) \in \omega^* : {}^*\lim_{k \to \infty} z_k^* = \theta^*\right\}, \\ \ell_p^* &= \left\{z^* = (z_k^*) \in \omega^* : {}_*\sum_{k=1}^{\infty} \ddot{\|} z_k^* \ddot{\|}^{\ddot{p}} < \infty\right\}, \quad (1 \leq p < \infty). \end{aligned} \tag{18}$$

It is not hard to show that the sets ℓ_∞^*, c^*, c_0^*, and ℓ_p^* are the subspaces of the space ω^*. This means that ℓ_∞^*, c^*, c_0^*, and ℓ_p^* are classical sequence spaces over the field $\mathbb{C}^*$ and complete metric spaces with corresponding metrics.

Theorem 11 (see [4]). *The following statements hold.*

(a) *The sets ℓ_∞^*, c^*, c_0^*, and ℓ_p^*, $p \geq 1$, are sequence spaces.*

(b) *Let λ^* denote any of the spaces ℓ_∞^*, c^*, and c_0^* and $z = (z_k^*)$, $t = (t_k^*) \in \lambda^*$. Define d_∞^* on the space λ^* by $d_\infty^*(z,t) = \sup_{k\in\mathbb{N}} \|z_k^* \ominus t_k^*\|$. Then, (λ^*, d_∞^*) is a complete metric space.*

(c) *The spaces ℓ_∞^*, c^*, and c_0^* are Banach spaces with the norm $\|z\|_\infty^*$ defined by*

$$\|z\|_\infty^* := \sup_{k\in\mathbb{N}} \|z_k^*\|; \quad z = (z_k^*) \in \lambda^*, \quad \lambda \in \{\ell_\infty, c, c_0\}. \tag{19}$$

(d) *The space ℓ_p^* is Banach spaces with the norm $\|z\|_p^*$ defined by*

$$\|z\|_p^* := \left({}_*\sum_{k=0}^{\infty} \|z_k^*\|^p \right)^{1/p}; \quad z = (z_k^*) \in \ell_p^*. \tag{20}$$

In the present section, we introduce the sets bs^*, cs^*, cs_0^* and bv^*, bv_p^*, bv_∞^* consisting of all bounded, convergent, null series and the sets of bounded variation sequences in the sense of non-Newtonian calculus which correspond to the sets bs, cs, cs_0 and bv, bv_p, bv_∞ over the complex field $\mathbb{C}$, respectively.

Theorem 12. *Let μ^* denote any of the spaces bs^*, cs^*, and cs_0^* and $z = (z_k^*)$, $t = (t_k^*) \in \mu^*$. Define d_∞^N on the space μ^* by*

$$d_\infty^N : \mu^* \times \mu^* \longrightarrow \mathbb{R}(N)$$
$$(z,t) \longrightarrow d_\infty^N(z,t) := \sup_{n\in\mathbb{N}} \left\{ d^* \left({}_*\sum_{k=0}^{n} z_k^*, {}_*\sum_{k=0}^{n} t_k^* \right) \right\} \tag{21}$$

for arbitrarily chosen α, β operators and corresponding function $\iota = \beta\alpha^{-1}$. Then, (μ^*, d_∞^N) is a complete metric space.

Proof. Since the proof is similar to the spaces cs^* and cs_0^*, we prove the theorem only for the space bs^*. Let the $*$-sums ${}_*\sum_{k=0}^n z_k^*$, ${}_*\sum_{k=0}^n t_k^* \in \mathbb{C}^*$, where $z = (z_k^*)$, $t = (t_k^*) \in \mathbb{C}^*$. Then the following metric axioms in Definition 1 are valid.

(NM1) From (11) it can be easily obtained that

$$d_\infty^N(z,t) = \ddot{0} \Longleftrightarrow \sup_{n\in\mathbb{N}} \left\| {}_*\sum_{k=0}^{n} z_k^* \ominus {}_*\sum_{k=0}^{n} t_k^* \right\| = \ddot{0} \Longleftrightarrow {}_*\sum_{k=0}^{n} z_k^* = {}_*\sum_{k=0}^{n} t_k^* \Longleftrightarrow z = t. \tag{22}$$

(NM2) It is trivial that the condition $d_\infty^N(z,t) = d_\infty^N(t,z)$ holds.

(NM3) We show that $*$triangle inequality in Definition 1 holds for $z = (z_k^*)$, $t = (t_k^*)$, $w = (w_k^*) \in \mathbb{C}^*$. In fact by taking into account Proposition 6 (i)

$$\begin{aligned} & d_\infty^N(z,t) \\ &= \sup_{n\in\mathbb{N}} \left\| {}_*\sum_{k=0}^{n} z_k^* \ominus {}_*\sum_{k=0}^{n} w_k^* \oplus {}_*\sum_{k=0}^{n} w_k^* \ominus {}_*\sum_{k=0}^{n} t_k^* \right\| \\ &\ddot{\leq} \sup_{n\in\mathbb{N}} \left\| {}_*\sum_{k=1}^{n} z_k^* \ominus {}_*\sum_{k=0}^{n} w_k^* \right\| \ddot{+} \sup_{n\in\mathbb{N}} \left\| {}_*\sum_{k=0}^{n} w_k^* \ominus {}_*\sum_{k=0}^{n} t_k^* \right\|, \\ & d_\infty^N(z,t) \ddot{\leq} d_\infty^N(z,w) \ddot{+} d_\infty^N(w,t). \end{aligned} \tag{23}$$

Since the axioms (NM1)–(NM3) are satisfied, (bs^*, d_∞^N) is a non-Newtonian metric space. It remains to prove the completeness of the space bs^*.

Let (x^m) be a $*$-Cauchy sequence in bs^*, where $x^m = \{x_1^{(m)}, x_2^{(m)}, \ldots\}$. Then, for every $\varepsilon \ddot{>} \ddot{0}$, there is an element n_0 such that, for all $m, r > n_0$,

$$d_\infty^N(x^m, x^r) = \sup_{n\in\mathbb{N}} \left\{ d^* \left({}_*\sum_{k=0}^{n} x_k^{(m)}, {}_*\sum_{k=0}^{n} x_k^{(r)} \right) \right\} \ddot{<} \varepsilon. \tag{24}$$

A fortiori, for every fixed $k \in \mathbb{N}$ and for all $m, r > n_0$

$$d^* \left({}_*\sum_{k=0}^{n} x_k^{(m)}, {}_*\sum_{k=0}^{n} x_k^{(r)} \right) \ddot{<} \varepsilon. \tag{25}$$

Hence, for every fixed $k \in \mathbb{N}$, the sequence $(x_m) = \{x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(m)}, \ldots\}$ is a $*$-Cauchy sequence. Before that, by using the completeness of $\mathbb{C}^*$ in Theorem 9, it $*$-converges; that is, $x_k^{(m)} \overset{*}{\to} x_k$ as $m \to \infty$. Using these infinitely many limits $x_1, x_2, \ldots$, we define $x = (x_1, x_2, \ldots)$ and show that $x \in bs^*$. From (25) letting $r \to \infty$ and $m > n_0$ we have

$$d^* \left({}_*\sum_{k=0}^{n} x_k^{(m)}, {}_*\sum_{k=0}^{n} x_k \right) \ddot{\leq} \varepsilon. \tag{26}$$

Since $(x^m) \in bs^*$, there exists $\ddot{p} \in \mathbb{R}(N)$ such that $\|x_k^{(m)}\| \ddot{\leq} \ddot{p}$ for all $k \in \mathbb{N}$. Thus, (26) gives together with the $*$-triangle inequality for $m > n_0$

$$\begin{aligned} & d^* \left({}_*\sum_{k=0}^{n} x_k, \ddot{0} \right) \\ & \ddot{\leq} d^* \left({}_*\sum_{k=0}^{n} x_k, {}_*\sum_{k=0}^{n} x_k^{(m)} \right) \ddot{+} d^* \left({}_*\sum_{k=0}^{n} x_k^{(m)}, \ddot{0} \right) \ddot{\leq} \varepsilon \ddot{+} \ddot{p}. \end{aligned} \tag{27}$$

It is clear that (26) holds for every $k \in \mathbb{N}$ whose right-hand side does not involve k. Hence (x_k) is a bounded sequence of geometric complex numbers; that is, $x = (x_k) \in bs^*$. Also from (26) we obtain for $m > n_0$

$$d_\infty^N(x^m, x) = \sup_{n\in\mathbb{N}} \left\{ d^* \left({}_*\sum_{k=0}^{n} x_k^{(m)}, {}_*\sum_{k=0}^{n} x_k \right) \right\} \ddot{\leq} \varepsilon. \quad (28)$$

Hence, as we have seen above that $x^m \xrightarrow{*} x$, as $m \to \infty$ for an arbitrary $*$-Cauchy sequence (x^m). Hence bs^* is complete. □

Thus it is known by Theorem 12 that the spaces bs^*, cs^*, and cs_0^* are complete metric spaces with the metric d_∞^N induced by the norm $\|\cdot\|_{bs}^*$ or $\|\cdot\|_{cs}^*$. Now, as a consequence of Theorem 12, the following corollary presents for these spaces to be Banach space.

Corollary 13. *The spaces bs^*, cs^*, and cs_0^* are Banach spaces with the norm $\|x\|_{bs}^*$ defined by*

$$\|x\|_{bs}^* = \|x\|_{cs}^* = \sup_{n\in\mathbb{N}} \left\| {}_*\sum_{k=0}^{n} x_k \right\|; \quad x = (x_k) \in \lambda^*, \ \lambda \in \{bs, cs, cs_0\}. \quad (29)$$

To avoid undue repetition in the statements we give the next theorem which is on the complete metric space bv^* without proof since the proof can be obtained similarly as Theorem 12.

Theorem 14. *Let d_Δ be defined on the space bv^* by*

$$d_\Delta : bv^* \times bv^* \longrightarrow \mathbb{R}(N)$$
$$(x, y) \longrightarrow d_\Delta(x, y) := {}_*\sum_{k=0}^{\infty} \left\{ d^* \left[(\Delta x)_k', (\Delta y)_k' \right] \right\}, \quad (30)$$

where $x = (x_k)$, $y = (y_k) \in bv^*$, and $(\Delta x)_k' = x_k \ominus x_{k+1}$. Then, (bv^*, d_Δ) is a complete metric space. Similarly, one can conclude that the other bounded variation sets (bv_p^*, d_p^Δ) and $(bv_\infty^*, d_\infty^\Delta)$ are also complete.

Corollary 15. *The space bv^* is a Banach space with the norm $\|x\|_{bv}^*$ defined by*

$$\|x\|_{bv}^* := {}_*\sum_{k=0}^{\infty} \ddot{\|}(\Delta x)_k'\ddot{\|}, \quad (\Delta x)_k' = (x_k \ominus x_{k+1}), \forall k \in \mathbb{N}. \quad (31)$$

4. The Duals of the Sets of Sequences over the Geometric Complex Field

The idea of dual sequence space which plays an important role in the representation of linear functionals and the characterization of matrix transformations between sequence spaces was introduced by Köthe and Toeplitz [9], whose main results concerned alpha-duals. An account of the duals of sequence spaces can be found in Köthe [10]. One can also know about different types of duals of sequence spaces in Maddox [11].

In this section, we focus on the alpha-, beta-, and gamma-duals of the classical sequence spaces over non-Newtonian complex field. For $\lambda^*, \mu^* \in \mathbb{C}^*$, the set $S(\lambda^*, \mu^*)$, defined by

$$S(\lambda^*, \mu^*) := \{ w = (w_k) \in \omega^* : w \odot z = (w_k \odot z_k) \in \mu^*, \ \forall z = (z_k) \in \lambda^* \}, \quad (32)$$

is called the multiplier space of λ^* and μ^* for all $k \in \mathbb{N}$. One can easily observe for a sequence space ν^* of $*$-complex numbers that the inclusions $S(\lambda^*, \mu^*) \subset S(\nu^*, \mu^*)$ if $\nu^* \subset \lambda^*$ and $S(\lambda^*, \mu^*) \subset S(\lambda^*, \nu^*)$ if $\mu^* \subset \nu^*$ hold.

Firstly, we define the alpha-, beta-, and gamma-duals of a set $\lambda^* \subset \omega^*$ which are, respectively, denoted by $\{\lambda^*\}^\alpha$, $\{\lambda^*\}^\beta$, and $\{\lambda^*\}^\gamma$, as follows:

$$\begin{aligned} \{\lambda^*\}^\alpha &:= \{ w = (w_k) \in \omega^* : w \odot z = (w_k \odot z_k) \in \ell_1^*, \ \forall z = (z_k) \in \lambda^* \}, \\ \{\lambda^*\}^\beta &:= \{ w = (w_k) \in \omega^* : w \odot z = (w_k \odot z_k) \in cs^*, \ \forall z = (z_k) \in \lambda^* \}, \\ \{\lambda^*\}^\gamma &:= \{ w = (w_k) \in \omega^* : w \odot z = (w_k \odot z_k) \in bs^*, \ \forall z = (z_k) \in \lambda^* \}, \end{aligned} \quad (33)$$

where $(w_k \odot z_k)$ is the coordinatewise product of $*$-complex numbers w and z for all $k \in \mathbb{N}$. Then $\{\lambda^*\}^\beta$ is called beta-dual of λ^* or the set of all convergence factor sequences of λ^* in cs^*. Firstly, we give a remark concerning the $*$-convergence factor sequences.

Throughout the text, we also use the notation "$<$" for a non-Newtonian linear subspace which was created in [7].

Remark 16. Let $\emptyset \neq \lambda^* \subset \omega^*$. Then the following statements are valid.

(a) $\{\lambda^*\}^\beta$ is a sequence space if $\varphi^* < \{\lambda^*\}^\beta < \omega^*$, where $\varphi^* := \{x = (x_k) : \exists N \in \mathbb{N}, \forall k \geq N, x_k = \ddot{0}\}$.

(b) If $\lambda^* \subset \mu^* \subset \omega^*$, then $\{\mu^*\}^\beta < \{\lambda^*\}^\beta$.

(c) $\lambda^* \subset \{\lambda^*\}^{\beta\beta} := (\{\lambda^*\}^\beta)^\beta$.

(d) $\{\varphi^*\}^\beta = \omega^*$ and $\{\omega^*\}^\beta = \varphi^*$.

Proof. Since the proof is trivial for the conditions (b) and (c), we prove only (a) and (d). Let $w = (w_k)$, $m = (m_k)$, and $n = (n_k) \in \{\lambda^*\}^\beta$.

(a) It is trivial that $\{\lambda^*\}^\beta < \omega^*$ holds from the hypothesis. We show that $m \oplus n \in \{\lambda^*\}^\beta$ for $m, n \in \{\lambda^*\}^\beta$. Suppose that $l \in \lambda^*$. Then $(m_k \odot l_k) \in cs^*$ and $(n_k \odot l_k) \in cs^*$ for all $l \in \lambda^*$. We can deduce that

$$((m_k \oplus n_k) \odot l_k) = (m_k \odot l_k) \oplus (n_k \odot l_k) \in cs^*, \quad \forall l \in \lambda^*. \quad (34)$$

Hence, $m \oplus n \in \{\lambda^*\}^\beta$. Now, we show that $t \odot w \in \{\lambda^*\}^\beta$ for any $t \in \mathbb{C}^*$ and $w = (w_k) \in \{\lambda^*\}^\beta$, since $(w_k \odot l_k) \in cs^*$ for all $l \in \lambda^*$. Combining this with $((t_k \odot w_k) \odot l_k) = t_k \odot (w_k \odot l_k) \in cs^*$ for all $l \in \lambda^*$ we get $t \odot w \in \{\lambda^*\}^\beta$. Therefore, we have proved that $\{\lambda^*\}^\beta$ is a subspace of the space ω^*.

(d) Using (a) we need only to show $\{\omega^*\}^\beta \subset \varphi^*$. Suppose that $w = (w_n) \in \{\omega^*\}^\beta$ and $z = (z_n)$ are given with $*$-division by $z_n \odot w_n = 1^*$ if $w_n \neq 0^*$ and $z_n := 1^*$ otherwise. By taking into account the set φ^* from inclusion (a), then there exists an integer $N \in \mathbb{N}$ for all $n \geq N$ such that $w_n = \ddot{0}$. Thus, we have

$$\begin{aligned} {}_*\sum_{n=1}^{\infty} w_n \odot z_n &= \left[(w_1 \odot z_1) \oplus \cdots \oplus (w_1 \odot z_1)\right] \\ &\quad \oplus (w_N \odot z_N) \oplus (w_{N+1} \odot z_{N+1}) \\ &\quad \oplus \theta^* \oplus \theta^* \oplus \cdots \\ &= {}_*\sum_{n=1}^{\infty} 1^* < \infty. \end{aligned} \tag{35}$$

Further, $w \odot z \in cs^*$ implies that $w \in \varphi^*$. The rest is an immediate consequence of this part and we omitted the details. □

Theorem 17. *The following statements hold:*

(a) $\{c_0^*\}^\beta = \{c^*\}^\beta = \{\ell_\infty^*\}^\beta = \ell_1^*$.

(b) $\{\ell_1^*\}^\beta = \ell_\infty^*$.

Proof. (a) Obviously $\{\ell_\infty^*\}^\beta \subset \{c^*\}^\beta \subset \{c_0^*\}^\beta$ by Remark 16 (b). Then we must show that $\ell_1^* \subset \{\ell_\infty^*\}^\beta$ and $\{c_0^*\}^\beta \subset \ell_1^*$. Now, consider that $w = (w_k) \in \ell_1^*$ and $z = (z_k) \in \ell_\infty^*$ are given. By taking into account the cases (c)-(d) of Theorem 11, we have

$$ {}_*\sum_{k=0}^{n} \ddot{\|} w_k \odot z_k \ddot{\|} \ddot{\leq} \|z\|_\infty^* \ddot{\times} \|w\|_1^* < \infty, \quad \forall k \in \mathbb{N}, \quad (\zeta \in \{\alpha, \beta, \gamma\}) \tag{36}$$

which implies that $w \odot z \in cs^*$. So the condition $\ell_1^* \subset \{\ell_\infty^*\}^\beta$ holds.

Conversely, for a given $y = (y_k) \in \omega^* \setminus \ell_1^*$, we prove the existence of an $x \in c_0^*$ with $y \odot x \notin cs^*$. According to $y \notin \ell_1^*$ we may confirm an index sequence (n_p) which is strictly increasing with $n_0 = \ddot{0}$ and ${}_*\sum_{k=n_{p-1}}^{n_p-1} \ddot{\|} y_k \ddot{\|} \ddot{>} p; (p \in \mathbb{N})$. By taking into account Remark 8 (i), if we define $x = (x_k) \in c_0^*$ by $x_k := (\mathrm{sgn}^* y_k \oslash p)$, the non-Newtonian complex signum function is defined by

$$\mathrm{sgn}^*(y) := \begin{cases} \overline{y} \oslash \ddot{\|} y \ddot{\|}, & y \neq \ddot{0}, \\ \ddot{0}, & y = \ddot{0}, \end{cases} \tag{37}$$

where $\overline{y}$ is given complex conjugate in Definition 7 for all $y = (y_k) \in \mathbb{C}^*$. Finally, by using Remark 8 (ii) taking the generators $\alpha = \beta$, we get

$$\begin{aligned} {}_*\sum_{k=n_{p-1}}^{n_p-1} y_k \odot x_k &= {}_*\sum_{k=n_{p-1}}^{n_p-1} \left[y_k \odot (\mathrm{sgn}^* y_k \oslash p)\right] \\ &= \frac{1}{p} \ddot{\times} {}_*\sum_{k=n_{p-1}}^{n_p-1} \ddot{\|} y_k \ddot{\|} \ddot{\geq} \ddot{1} \end{aligned} \tag{38}$$

for all $n_{p-1} \leq k < n_p$. Therefore $y \odot x \notin cs^*$ and thus $y \notin \{c_0^*\}^\beta$. Hence $\{c_0^*\}^\beta \subset \ell_1^*$.

(b) From the condition (c) of Remark 16 we have $\ell_\infty^* \subset (\{\ell_\infty^*\}^\beta)^\beta = \{\ell_1^*\}^\beta$ since $\{\ell_\infty^*\}^\beta = \ell_1^*$. Now we assume the existence of a $w = (w_n) \in \{\ell_1^*\}^\beta \setminus \ell_\infty^*$. Since w is unbounded there exists a subsequence (w_{n_k}) of (w_n) and we can find a real number $(k+1)^2$ such that $\ddot{\|} w_{n_k} \ddot{\|} \ddot{\geq} (k+1)^2$ for all $k \in \mathbb{N}_1$. The sequence (x_n) is defined by $x_n := (\mathrm{sgn}^*(w_{n_k}) \oslash (k+1)^2)$ if $n = n_k$ and $\ddot{0}$ otherwise. Then $x \in \ell_1^*$. However,

$$ {}_*\sum_n w_n \odot x_n = {}_*\sum_k \frac{1}{(k+1)^2} \odot \ddot{\|} w_{n_k} \ddot{\|} \ddot{\geq} {}_*\sum_k \ddot{1} = \infty. \tag{39}$$

Hence, $w \notin \{\ell_1^*\}^\beta$, which contradicts our assumption and $\{\ell_1^*\}^\beta \subset \ell_\infty^*$. This step completes the proof. □

In addition to the statements in Remark 16 we make the following remarks which are immediate consequences of the definition of the ζ-duals ($\zeta \in \{\alpha, \beta, \gamma\}$).

Remark 18. Let $\emptyset \neq \lambda^* \subset \omega^*$. Then the following statements are valid:

(a) $\varphi^* < \{\lambda^*\}^\alpha < \{\lambda^*\}^\beta < \{\lambda^*\}^\gamma < \omega^*$; in particular, $\{\lambda^*\}^\zeta$ is a sequence space over $\mathbb{C}^*$;

(b) if $\lambda^* < \mu^* < \omega^*$, then $\{\mu^*\}^\zeta < \{\lambda^*\}^\zeta$;

(c) I is an index set, if λ_i^* are sequence spaces, and if $\lambda^* := \bigcup_{i \in I} \lambda_i^*$, then $\langle \lambda^* \rangle^\zeta = \bigcap_{i \in I} \{\lambda_i^*\}^\zeta$, where the notation "$\langle\,\rangle$" stands for the span of linear subspace over $\mathbb{C}^*$;

(d) $\lambda^* \subset \{\lambda^*\}^{\zeta\zeta} := (\{\lambda^*\}^\zeta)^\zeta$.

Proof. The case (b) obviously is true, and (a) follows from $\ell_\infty^* < cs^* < bs^*$. We only show the cases (c) and (d) taking $\zeta = alpha$. The rest of the parts can be obtained in a similar way.

(c) Now, as an immediate consequence $\lambda_i^* \subset \langle \lambda^* \rangle$ the following $\langle \lambda^* \rangle^\alpha \subset \{\lambda_i^*\}^\alpha$ and $\langle \lambda^* \rangle^\alpha \subset \bigcap_{i \in I} \{\lambda_i^*\}^\alpha$ hold by (b). On the other hand, if $y \in \bigcap_{i \in I} \{\lambda_i^*\}^\alpha$, that is, $y \in \{\lambda_i^*\}^\alpha$, then $x \odot y \in \ell_1^*$ for all $x \in \lambda^*$ and therefore $y \in \{\lambda^*\}^\alpha \subset \langle \lambda^* \rangle^\alpha$.

(d) We can deduce $\lambda^* \subset \{\lambda^*\}^{\zeta\zeta}$. Let $w \in \lambda^*$; then $w \odot z \in \ell_1^*$ for all $z \in \{\lambda^*\}^{\alpha}$; thus $w \in \{\lambda^*\}^{\zeta\zeta}$ and $\lambda^* \subset \{\lambda^*\}^{\zeta\zeta}$ by (a). □

Here $\lambda^* \neq \{\lambda^*\}^{\zeta\zeta}$ as we get from Theorem 17 (a) in the case of $\zeta = \beta$ and $\lambda^* := c_0^*$. We have $\{c_0^*\}^{\beta\beta} = \ell_\infty^* \neq c_0^*$. This remark gives rise to the following definition.

Definition 19 (ζ-space, Köthe space). Let $\zeta \in \{\alpha, \beta, \gamma\}$ and let λ^* be a sequence space over the field $\mathbb{C}^*$. λ^* is called ζ-space if $\lambda^* = \{\lambda^*\}^{\zeta\zeta}$. Further, an α-space is also called a Köthe space or perfect sequence space.

From Remark 18 (d) and (b) we obtain immediately the following remark.

Remark 20. If λ^* is a sequence space over the field $\mathbb{C}^*$ and ($\zeta \in \{\alpha, \beta, \gamma\}$), then $\{\lambda^*\}^{\zeta}$ is a ζ-space; that is, $\{\lambda^*\}^{\zeta} = \{\lambda^*\}^{\zeta\zeta\zeta}$.

Now we look for sufficient conditions for $\{\lambda^*\}^{\alpha} = \{\lambda^*\}^{\beta} = \{\lambda^*\}^{\gamma}$. This gives rise to the notion of solidity.

Definition 21 (solid sequence space). Let X be a sequence space. Then X is called solid if

$$\left\{u = (u_k) \in \omega^* : \exists (x_k) \in X \ \forall k \in \mathbb{N} : \ddot{\|}u_k\ddot{\|} \ddot{\leq} \ddot{\|}x_k\ddot{\|}\right\} \subset X. \tag{40}$$

Theorem 22. *Let λ^* be a sequence space over the field $\mathbb{C}^*$. Then λ^* is solid if and only if*

$$\left\{y = (y_k) \in \omega^* : \exists (x_k) \in \lambda^* \ \forall k \in \mathbb{N} : \ddot{\|}y_k\ddot{\|} = \ddot{\|}x_k\ddot{\|}\right\} \subset \lambda^*. \tag{41}$$

Proof. Let $u = (u_k) \in \omega^*$ and $x = (x_k) \in \lambda^*$ with $\ddot{\|}u_k\ddot{\|} \ddot{\leq} \ddot{\|}x_k\ddot{\|}$ be given. Further, let $(x_k) = (\dot{\xi}_k, \ddot{\eta}_k) \in \omega^*$, where $\xi_k, \eta_k \in \mathbb{R}$. Obviously the condition $\overline{x} = (\overline{x_k}) = (\dot{\xi}_k, \ddot{-}\ddot{\eta}_k) \in \lambda^*$ holds because $\ddot{\|}x_k\ddot{\|} = \ddot{\|}\overline{x_k}\ddot{\|}$. Let $u_k = (\dot{\psi}_k, \ddot{\delta}_k) \in \omega^*$, where $\psi_k, \delta_k \in \mathbb{R}$. We obtain $\beta\{\sqrt{\psi_k^2 + \delta_k^2}\} \leq \beta\{\sqrt{\xi_k^2 + \eta_k^2}\}$ so $\psi_k^2 + \delta_k^2 \leq \xi_k^2 + \eta_k^2$ since $\ddot{\|}u_k\ddot{\|} \ddot{\leq} \ddot{\|}x_k\ddot{\|}$ for all $k \in \mathbb{N}$. We may choose a real number $\gamma_k \in \mathbb{R}$ with $\psi_k^2 + \gamma_k^2 = \xi_k^2 + \eta_k^2$ and consider $w = (w_k)$ defined by $(w_k) = (\dot{\psi}_k, \ddot{\gamma}_k)$. Then $\ddot{\|}w_k\ddot{\|} = \ddot{\|}x_k\ddot{\|}$; thus $w \in \lambda^*$ and $\overline{w} = (\overline{w_k}) \in \lambda^*$. From Definition 7 we get $\dot{\psi}_k = (w \oplus \overline{w})\dot{/}\dot{2} \in \lambda^*$. Determining $\zeta_k \in \mathbb{R}$ such that $\zeta_k^2 + \delta_k^2 = \xi_k^2 + \eta_k^2$ it follows similarly that $\mu = (\mu_k) = (\dot{\zeta}_k, \ddot{\delta}_k) \in \omega^*$ and $\overline{\mu} = (\overline{\mu_k}) \in \lambda^*$, which imply $\ddot{\delta}_k = (\mu \ominus \overline{\mu})\ddot{/}\ddot{2} \in \lambda^*$. Hence $u_k = (\dot{\psi}_k, \ddot{\delta}_k) \in \lambda^*$. This step completes the proof. □

Theorem 23. *Consider that $\lambda^* < \omega^*$ is any sequence space over the field $\mathbb{C}^*$; then the following statements hold.*

(a) *If λ^* is a Köthe space, then λ^* is solid.*

(b) *If λ^* is solid, then $\{\lambda^*\}^{\alpha} = \{\lambda^*\}^{\beta} = \{\lambda^*\}^{\gamma}$.*

(c) *If λ^* is a Köthe space, then λ^* is a ζ-space.*

Proof. Let $\lambda^* < \omega^*$ be a sequence space over the field $\mathbb{C}^*$.

(a) If λ^* is a Köthe space and $u \in \omega^*$, then $u \in \{\lambda^*\}^{\alpha\alpha}$ if and only if the condition $u \odot z \in \ell_1^*$ holds for all $z \in \{\lambda^*\}^{\alpha}$. Besides this we obtain $\ddot{\|}v_k\ddot{\|} = \ddot{\|}u_k\ddot{\|}$ for $u = (u_k) \in \lambda^*$ and $v = (v_k) \in \omega^*$ and the statement

$$*\sum_k \ddot{\|}v_k \odot z_k\ddot{\|} = *\sum_k \ddot{\|}u_k \odot z_k\ddot{\|} < \infty \tag{42}$$

holds for each $z \in \{\lambda^*\}^{\alpha}$. Therefore $v \odot z \in \ell_1^*$. Hence $w \in \lambda^*$ and λ^* is solid over the field $\mathbb{C}^*$.

(b) Consider that λ^* is a solid sequence space over $\mathbb{C}^*$. To show $\{\lambda^*\}^{\alpha} = \{\lambda^*\}^{\beta} = \{\lambda^*\}^{\gamma}$, it is sufficient condition to verify $\{\lambda^*\}^{\gamma} < \{\lambda^*\}^{\alpha}$ in Remark 18 (a). So, let $v = (v_k) \in \{\lambda^*\}^{\gamma}$; that is,

$$\sup_{n\in\mathbb{N}} \ddot{\left\|} *\sum_{k=0}^{n} (u_k \odot v_k) \ddot{\right\|} < \infty \quad \text{for every } u = (u_k) \in \lambda^*. \tag{43}$$

By taking into account solidness of λ^*, for $z = (z_k) \in \lambda^*$, the condition $\ddot{\|}z_k\ddot{\|} = \ddot{\|}u_k\ddot{\|}$ holds and there exists a sequence $u = (u_k) \in \lambda^*$ for all $k \in \mathbb{N}$. Therefore by combining this with the inclusion (43) we deduce that the condition

$$*\sum_{k=0}^{N} \ddot{\|}u_k \odot v_k\ddot{\|} = *\sum_{k=0}^{N} \ddot{\|}z_k \odot v_k\ddot{\|} = \ddot{\left\|} *\sum_{k=0}^{N} z_k \odot v_k \ddot{\right\|} \ddot{\leq} \sup_{n\in\mathbb{N}} \ddot{\left\|} *\sum_{k=0}^{n} z_k \odot v_k \ddot{\right\|} < \infty \tag{44}$$

holds and $u \odot v \in \ell_1^* < cs^*$. Hence $v \in \{\lambda^*\}^{\alpha}$ and $\{\lambda^*\}^{\gamma} < \{\lambda^*\}^{\alpha}$.

(c) This is an obvious consequence of Remark 20 and the cases (a)-(b) in Theorem 23. □

Theorem 24. *The following statements hold.*

(a) *The sets φ^*, ω^*, ℓ_p^*, c_0^*, and ℓ_∞^* of sequences are solid.*

(b) *The sets c^* and bv^* of sequences are not solid; therefore none of them is a Köthe space.*

(c) *For each $\zeta \in \{\alpha, \beta, \gamma\}$, then*

(i) $\{\ell_1^*\}^{\zeta} = \ell_\infty^*$ *and* $\{\ell_\infty^*\}^{\zeta} = \ell_1^*$;

(ii) $\{\omega^*\}^{\zeta} = \varphi^*$ *and* $\{\varphi^*\}^{\zeta} = \omega^*$.

(d) *If $\zeta \in \{\alpha, \beta, \gamma\}$ and $c_0^* < \mu^* < \ell_\infty^*$, then $\{\mu^*\}^{\zeta} = \ell_1^*$ and $\mu^* \subset \{\mu^*\}^{\zeta\zeta} = \ell_\infty^*$. In particular, $\{c_0^*\}^{\zeta} = \{c^*\}^{\zeta} = \ell_1^*$, and each of c^*, c_0^* is not a ζ-space.*

Proof. That the specified spaces in cases (a-b) are solid is an immediate consequence of their definition. Additionally, the cases (i-ii) of (c) can be obtained Theorem 17 and Remark 16 (d). Since c_0^* and ℓ_∞^* are solid, we know that $\{c_0^*\}^\zeta = \{\ell_\infty^*\}^\zeta = \ell_1^*$. So the statements in (d) are obtained from Remark 18 (b). □

Next, we determine the ζ-duals of the spaces cs^*, bs^*, bv^*, and bv_0^*. We will find that none of these sequence spaces is solid; in particular, none of them is a Köthe space.

Theorem 25. *The following statements hold:*

(a) $\{cs^*\}^\alpha = \{bv^*\}^\alpha = \{bv_0^*\}^\alpha = \ell_1^*; \{bv_0^*\} = bv^* \cap c_0^*$,

(b) $\{cs^*\}^\beta = bv^*$, $\{bv^*\}^\beta = cs^*$, $\{bv_0^*\}^\beta = bs^*$, $\{bs^*\}^\beta = bv_0^*$;

(c) $\{cs^*\}^\gamma = bv^*$, $\{bv^*\}^\gamma = bs^*$, $\{bv_0^*\}^\gamma = bs^*$, $\{bs^*\}^\gamma = bv^*$.

In particular, the sets cs^*, bs^*, bv^*, and bv_0^* of sequences are β-spaces (ζ = beta), but they are not Köthe spaces. Moreover, the sets bs^* and bv^* of sequences are γ-spaces (ζ = gamma), whereas both cs^* and bv_0^* are not γ-spaces. None of the spaces cs^*, bs^*, bv^*, and bv_0^* is solid.

Proof. We prove the cases for the spaces $\{cs^*\}^\zeta$, $\zeta \in \{\alpha, \beta, \gamma\}$, and the proofs of all other cases are quite similar.

(a) Let $x = (x_k) \in cs^*$ and $y = (y_k) \in \ell_1^*$. Then,

$$ {}_*\sum_k \ddot{\|} y_k \odot x_k \ddot{\|} \ddot{\leq} \|x\|_{cs}^* \ddot{\times} {}_*\sum_k \ddot{\|} y_k \ddot{\|} < \infty, \quad \forall k \in \mathbb{N}, \tag{45} $$

where $\|x\|_{cs}^*$ is defined by (27). Therefore, $y \in \{cs^*\}^\alpha$ which gives the fact that $\ell_1^* \subset \{cs^*\}^\alpha$.

Conversely, suppose that $y = (y_k) \in \{cs^*\}^\alpha \setminus \ell_1^*$. Then to every natural number p we can construct an index sequence (n_p) with $n_p < n_{p+1}$ and ${}_*\sum_{k=n_p+1}^{n_{p+1}} \ddot{\|} y_k \ddot{\|} \ddot{<} \ddot{4}^{\ddot{p}}$ for all $p \in \mathbb{N}$. Define $x = (x_k)$ by

$$ x_k := \begin{cases} (\ddot{-}\ddot{1})^{\ddot{k}} \ddot{/} \ddot{2}^{\ddot{p}}, & n_p < k \leq n_{p+1}, \\ \ddot{0}, & \text{others.} \end{cases} \tag{46} $$

By using the algebraic operations in (4) and Remark 3 the β-division $(\ddot{-}\ddot{1})^{\ddot{k}} \ddot{/} \ddot{2}^{\ddot{p}}$ can be evaluated as $\beta\{(-1)^k 2^{-p}\}$ for $n_p < k \leq n_{p+1}$. Then $x = (x_k) \in cs^*$. According to the choice of n_p, the inequalities

$$ {}_*\sum_k \ddot{\|} y_k \odot x_k \ddot{\|} \ddot{\geq} {}_*\sum_p \ddot{2}^{\ddot{-}\ddot{p}} \ddot{\times} \ddot{\sum_{k=n_p+1}^{n_{p+1}}} \ddot{\|} y_k \ddot{\|} \ddot{\geq} {}_*\sum_p \ddot{2}^{\ddot{p}} = \ddot{\infty} \tag{47} $$

hold, where the sum ${}_*\sum_p \ddot{2}^{\ddot{p}} = \beta\{\sum_p \beta^{-1}(\ddot{2}^{\ddot{p}})\} = \beta\{\sum_p 2^p\}$ $*$-diverges since the classical geometric series $\sum_p 2^p$ diverges. Thus, $x \odot y \notin \ell_1^*$, which implies $y \notin \{cs^*\}^\alpha$. This contradicts the fact that $y \in \{cs^*\}^\alpha$. Therefore $\{cs^*\}^\alpha \subset \ell_1^*$.

The condition $\{bv_0^*\}^\alpha \subset \ell_1^*$ holds as well if we take the sequence (x_k) by

$$ x_k := \begin{cases} (\ddot{2})^{\ddot{-}\ddot{p}}, & n_p < k \leq n_{p+1}, \\ \ddot{0}, & \text{others.} \end{cases} \tag{48} $$

(b) Let $u = (u_k) \in \{cs^*\}^\beta$ and $w = (w_k) \in c_0^*$. Define the sequence $v = (v_k) \in cs^*$ by $v_k = (w_k \ominus w_{k+1})$ for all $k \in \mathbb{N}$. Therefore, ${}_*\sum_k \odot v_k$ $*$-converges, but

$$ \begin{aligned} {}_*\sum_{k=0}^{n} & (w_k \ominus w_{k+1}) \odot u_k \\ &= \left[{}_*\sum_{k=0}^{n-1} w_k \odot (u_k \ominus u_{k-1}) \right] \ominus w_{n+1} \odot u_n \end{aligned} \tag{49} $$

and the inclusion $\ell_1^* \subset cs^*$ yields $(u_k) \in \{cs^*\}^\beta \subset \{\ell_1^*\}^\beta = \ell_\infty^*$. Then we derive by passing to the $*$-limit in (49) as $n \to \infty$ which implies that

$$ {}_*\sum_{k=0}^{\infty} (w_k \ominus w_{k+1}) \odot u_k = {}_*\sum_{k=0}^{\infty} w_k \odot (u_k \ominus u_{k-1}) \tag{50} $$

for every $k \in \mathbb{N}$. Hence, $(u_k \ominus u_{k-1}) \in \{c_0^*\}^\beta = \{c_0^*\}^\alpha = \ell_1^*$; that is, $u \in bv^*$. Therefore, $\{cs^*\}^\beta \subseteq bv^*$.

Conversely, suppose that $u = (u_k) \in bv^*$. Then, $(u_k \ominus u_{k-1}) \in \ell_1^*$. Further, if $v = (v_k) \in cs^*$, the sequence (w_n) defined by $w_n = {}_*\sum_{k=0}^n v_k$ for all $k \in \mathbb{N}$ is an element of the space c^*. Since $\{c^*\}^\alpha = \ell_1^*$, the N-series ${}_*\sum_k w_k \odot (u_k \ominus u_{k-1})$ is $*$-convergent. Also, we have

$$ \begin{aligned} & {}_*\sum_{k=m}^{n} (w_k \ominus w_{k+1}) \odot u_k \\ & \ddot{\leq} \left[{}_*\sum_{k=m}^{n-1} w_k \odot (u_k \ominus u_{k-1}) \right] \oplus (w_n \odot u_n) \ominus (w_{m-1} \odot u_m). \end{aligned} \tag{51} $$

Since $(w_n) \in c^*$ and $(u_k) \in bv^* \subset c^*$, the right-hand side of inequality (51) $*$-converges to $\ddot{0}$ as $m, n \to \infty$. Hence, the series ${}_*\sum_{k=0}^{\infty} (w_k \ominus w_{k+1}) \odot u_k$ or ${}_*\sum_{k=0}^{\infty} u_k \odot v_k$ $*$-converges. Hence $bv^* \subseteq \{cs^*\}^\beta$.

(c) By using (a), it is known that $bv^* \subseteq \{cs^*\}^\beta$ and since $\{cs^*\}^\beta \subset \{cs^*\}^\gamma$, $bv^* \subset \{cs^*\}^\gamma$. We need to show that $\{cs^*\}^\gamma \subset bv^*$. Let $u = (u_n) \in \{cs^*\}^\gamma$ and $v = (v_n) \in c_0^*$. Then, for the sequence $(w_n) \in cs^*$ defined by $w_n = (v_n \ominus v_{n+1})$ for all $n \in \mathbb{N}$, we can find a non-Newtonian real number $\ddot{K} \ddot{>} \ddot{0}$ such that $\ddot{\|}\, {}_*\sum_{k=0}^n u_k \odot w_k \ddot{\|} \ddot{\leq} \ddot{K}$ for all $n \in \mathbb{N}$. Since $(v_n) \in c_0^*$ and $(u_n) \in \{cs^*\}^\gamma \subset \ell_\infty^*$, there exists a non-Newtonian real number $\ddot{M} \ddot{>} \ddot{0}$ such that $\ddot{\|} u_n \odot v_n \ddot{\|} \ddot{\leq} \ddot{M}$ for all $n \in \mathbb{N}$. Therefore,

$$ \begin{aligned} & \ddot{\left\| {}_*\sum_{k=0}^{n} (u_k \ominus u_{k-1}) \odot v_k \right\|} \\ & \ddot{\leq} \ddot{\left\| {}_*\sum_{k=0}^{n+1} u_k \odot (v_k \ominus v_{k+1}) \right\|} \ddot{+} \ddot{\|} v_{n+2} \odot u_{n+1} \ddot{\|} \ddot{\leq} \ddot{K} \ddot{+} \ddot{M}. \end{aligned} \tag{52} $$

Hence $(u_k \ominus u_{k-1}) \in \{c_0^*\}^\gamma = \{c_0^*\}^\alpha = \ell_1^*$; that is, $(u_n) \in bv^*$. Therefore, since the inclusion $\{cs^*\}^\gamma \subset bv^*$ holds, we conclude that $\{cs^*\}^\gamma = bv^*$, as desired. □

5. Concluding Remarks

Although all arithmetics are isomorphic, only by distinguishing among them, we do obtain suitable tools for constructing all the non-Newtonian calculi. But the usefulness of arithmetics is not limited to the construction of calculi; we believe there is a more fundamental reason for considering alternative arithmetics; they may also be helpful in developing and understanding new systems of measurement that could yield simpler physical laws.

We trust that the basic ideas of the non-Newtonian calculus have been presented in sufficient detail to enable interested persons to develop the theory in various directions. For example, consider the concept the *average speed*. The definition "distance traveled per unit time" is incomplete because it fails to provide a method of determining the average speed of an accelerated particle. The definition "distance divided by time," though not incorrect, is a gross oversimplification that fails to reveal the underlying issues. Fortunately there is a completely satisfactory definition, which undoubtedly was known to Galileo. Then we isolate a constant in each given uniform motion by defining speed to be the distance traveled in any unit time-interval. Finally, for a particle that moves nonuniformly in a distance d in time t, we define the average speed to be the speed that a particle in uniform motion must have in order to travel a distance d in time t. In our opinion, neither the simplicity nor the obviousness of the answer, d/t, justifies its use as the definition of average speed (cf. [12]).

Of course, we can only speculate as to future applications of the non-Newtonian calculi. Perhaps they can be used to define new scientific concepts, to yield new or simpler scientific laws, to solve heretofore unsolved problems, or to formulate and solve new problems. For example, one constructs non-Newtonian calculi of functions of two or more real variables by choosing an arithmetic for each axis. It might even be profitable to seek deeper connections among the corresponding operators of the calculi. Like as, some of them are postponed to our future works.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Implicit Contractive Mappings in Modular Metric and Fuzzy Metric Spaces

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The notion of modular metric spaces being a natural generalization of classical modulars over linear spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, and Calderon-Lozanovskii spaces was recently introduced. In this paper we investigate the existence of fixed points of generalized α-admissible modular contractive mappings in modular metric spaces. As applications, we derive some new fixed point theorems in partially ordered modular metric spaces, Suzuki type fixed point theorems in modular metric spaces and new fixed point theorems for integral contractions. In last section, we develop an important relation between fuzzy metric and modular metric and deduce certain new fixed point results in triangular fuzzy metric spaces. Moreover, some examples are provided here to illustrate the usability of the obtained results.

1. Introduction and Basic Definitions

Chistyakov introduced the notion of modular metric spaces in [1, 2]. The main idea behind this new concept is the physical interpretation of the modular. Informally speaking, whereas a metric on a set represents nonnegative finite distances between any two points of the set, a modular on a set attributes a nonnegative (possibly, infinite valued) "field of (generalized) velocities": to each "time" $\lambda > 0$ the absolute value of an average velocity $\omega_\lambda(x, y)$ is associated in such a way that in order to cover the "distance" between points $x, y \in X$ it takes time λ to move from x to y with velocity $\omega_\lambda(x, y)$. But the way we approached the concept of modular metric spaces is different. Indeed we look at these spaces as the nonlinear version of the classical modular spaces introduced by Nakano [3] on vector spaces and modular function spaces introduced by Musielak [4] and Orlicz [5, 6].

For the study of electrorheological fluids (for instance lithium polymethacrylate), modeling with sufficient accuracy using classical Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, where p is a fixed constant is not adequate, but rather the exponent p should be able to vary [7]. One of the most interesting problems in this setting is the famous Dirichlet energy problem [8, 9]. The classical technique used so far in studying this problem is to convert the energy functional, naturally defined by a modular, to a convoluted and complicated problem which involves a norm (the Luxemburg norm). The modular metric approach is more natural and has not been used extensively. In recent years, there was a strong interest to study the fixed point property in modular function spaces after the first paper [10] was published in 1990. For more on metric fixed point theory, the reader may consult the book [11] and for modular function spaces [12, 13].

Let X be a nonempty set. Throughout this paper for a function $\omega : (0, \infty) \times X \times X \to [0, \infty]$, we will write

$$\omega_\lambda(x, y) = \omega(\lambda, x, y), \tag{1}$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1 (see [1, 2]). A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be modular metric on X if it satisfies the following axioms:

(i) $x = y$ if and only if $\omega_\lambda(x, y) = 0$, for all $\lambda > 0$;

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$, for all $\lambda > 0$, and $x, y \in X$;

(iii) $\omega_{\lambda+\mu}(x,y) \le \omega_\lambda(x,z) + \omega_\mu(z,y)$, for all $\lambda,\mu > 0$ and $x,y,z \in X$.

If instead of (i), we have only the condition (i′)

$$\omega_\lambda(x,x) = 0, \quad \forall \lambda > 0,\ x \in X, \tag{2}$$

then ω is said to be a pseudomodular (metric) on X. A modular metric ω on X is said to be regular if the following weaker version of (i) is satisfied:

$$x = y \quad \text{iff } \omega_\lambda(x,y) = 0, \text{ for some } \lambda > 0. \tag{3}$$

Finally, ω is said to be convex if for $\lambda,\mu > 0$ and $x,y,z \in X$, it satisfies the inequality

$$\omega_{\lambda+\mu}(x,y) \le \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y). \tag{4}$$

Note that for a metric pseudomodular ω on a set X and any $x,y \in X$, the function $\lambda \to \omega_\lambda(x,y)$ is nonincreasing on $(0,\infty)$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda(x,y) \le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y). \tag{5}$$

Following example presented by Abdou and Khamsi [14] is an important motivation of the concept of modular metric spaces.

Example 2. Let X be a nonempty set and Σ a nontrivial σ-algebra of subsets of X. Let $\mathscr{P}$ be a δ-ring of subsets of X, such that $E \cap A \in \mathscr{P}$ for any $E \in \mathscr{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathscr{P}$ such that $X = \bigcup K_n$. By $\mathscr{E}$ we denote the linear space of all simple functions with supports from $\mathscr{P}$. By $\mathscr{M}_\infty$ we will denote the space of all extended measurable functions; that is, all functions $f : X \to [-\infty,\infty]$ such that there exists a sequence $\{g_n\} \subset \mathscr{E}$, $|g_n| \le |f|$, and $g_n(x) \to f(x)$ for all $x \in X$. By 1_A we denote the characteristic function of the set A. Let $\rho : \mathscr{M}_\infty \to [0,\infty]$ be a nontrivial, convex, and even function. We say that ρ is a regular convex function pseudomodular if

(i) $\rho(0) = 0$;

(ii) ρ is monotone; that is, $|f(x)| \le |g(x)|$ for all $x \in X$ implies $\rho(f) \le \rho(g)$, where $f,g \in \mathscr{M}_\infty$;

(iii) ρ is orthogonally subadditive; that is, $\rho(f1_{A\cup B}) \le \rho(f1_A) + \rho(f1_B)$ for any $A,B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathscr{M}$;

(iv) ρ has the Fatou property; that is, $|f_n(x)| \uparrow |f(x)|$ for all $x \in X$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathscr{M}_\infty$;

(v) ρ is order continuous in $\mathscr{E}$; that is, $g_n \in \mathscr{E}$ and $|g_n(x)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

Similarly, as in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathscr{E}$. We say that a property holds ρ-almost everywhere if the exceptional set is ρ-null. As usual we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define

$$\mathscr{M}(X,\Sigma,\mathscr{P},\rho) = \{f \in \mathscr{M}_\infty; |f(x)| < \infty \rho\text{-a.e}\}, \tag{6}$$

where each $f \in \mathscr{M}(X,\Sigma,\mathscr{P},\rho)$ is actually an equivalence class of functions equal to ρ-a.e. rather than an individual function. Where no confusion exists we will write $\mathscr{M}$ instead of $\mathscr{M}(X,\Sigma,\mathscr{P},\rho)$. Let ρ be a regular function pseudomodular.

(a) We say that ρ is a regular function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0$ ρ-a.e.

(b) We say that ρ is a regular function modular if $\rho(f) = 0$ implies $f = 0$ ρ-a.e.

The class of all nonzero regular convex function modulars defined on X will be denoted by $\mathfrak{R}$. Let us denote $\rho(f,E) = \rho(f1_E)$ for $f \in \mathscr{M}, E \in \Sigma$. It is easy to prove that $\rho(f,E)$ is a function pseudomodular in the sense of Definition 2.1.1 in [13] (see also [15, 16]). Let ρ be a convex function modular.

(a) The associated modular function space is the vector space $L_\rho(X,\Sigma)$, or briefly L_ρ, defined by

$$L_\rho = \{f \in \mathscr{M}; \rho(\lambda f) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0\}. \tag{7}$$

(b) The following formula defines a norm in L_ρ (frequently called Luxemburg norm):

$$\|f\|_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{f}{\alpha}\right) \le 1\right\}. \tag{8}$$

A modular function space furnishes a wonderful example of a modular metric space. Indeed, let L_ρ be a modular function space. Define the function modular ω by

$$\omega_\lambda(f,g) = \rho\left(\frac{f-g}{\lambda}\right) \tag{9}$$

for all $\lambda > 0$, and $f,g \in L_\rho$. Then ω is a modular metric on L_ρ. Note that ω is convex if and only if ρ is convex. Moreover we have

$$\|f-g\|_\rho = d^*_\omega(f,g), \tag{10}$$

for any $f,g \in L_\rho$.

Other easy examples may be found in [1, 2].

Definition 3. Let X_ω be a modular metric space.

(1) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is said to be ω-convergent to $x \in X_\omega$ if and only if for each $\lambda > 0$, $\omega_\lambda(x_n,x) \to 0$, as $n \to \infty$. x will be called the ω-limit of (x_n).

(2) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is said to be ω-Cauchy if for each $\lambda > 0$, $\omega_\lambda(x_m,x_n) \to 0$, as $m,n \to \infty$.

(3) A subset M of X_ω is said to be ω-closed if the ω-limit of a ω-convergent sequence of M always belongs to M.

(4) A subset M of X_ω is said to be ω-complete if any ω-Cauchy sequence in M is a ω-convergent sequence and its ω-limit is in M.

(5) A subset M of X_ω is said to be ω-bounded if we have

$$\delta_\omega(M) = \sup\{\omega_\lambda(x,y); x,y \in M\} < \infty. \tag{11}$$

In 2012, Samet et al. [17] introduced the concepts of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings defined on complete metric spaces. Afterwards Salimi et al. [18] and Hussain et al. [19–21] modified the notions of α-ψ-contractive and α-admissible mappings and established certain fixed point theorems.

Definition 4 (see [17]). Let T be self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ a function. One says that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{12}$$

Definition 5 (see [18]). Let T be self-mapping on X and $\alpha, \eta : X \times X \to [0, +\infty)$ two functions. One says that T is an α-admissible mapping with respect to η if

$$\begin{aligned} x, y \in X, \quad \alpha(x,y) \geq \eta(x,y) &\Longrightarrow \alpha(Tx, Ty) \\ &\geq \eta(Tx, Ty). \end{aligned} \tag{13}$$

Note that if we take $\eta(x, y) = 1$ then this definition reduces to Definition 4. Also, if we take, $\alpha(x, y) = 1$ then we say that T is an η-subadmissible mapping.

Definition 6 (see [20]). Let (X, d) be a metric space. Let $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$ be functions. One says T is an α-η-continuous mapping on (X, d), if, for given $x \in X$ and sequence $\{x_n\}$ with

$$\begin{aligned} x_n \longrightarrow x \text{ as } n \longrightarrow \infty, \quad \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \\ \forall n \in \mathbb{N} \Longrightarrow Tx_n \longrightarrow Tx. \end{aligned} \tag{14}$$

Example 7 (see [20]). Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ be a metric on X. Assume $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ are defined by

$$\begin{aligned} Tx &= \begin{cases} x^5, & \text{if } x \in [0,1], \\ \sin \pi x + 2, & \text{if } (1, \infty), \end{cases} \\ \alpha(x,y) &= \begin{cases} x^2 + y^2 + 1, & \text{if } x, y \in [0,1], \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{15}$$

and $\eta(x, y) = x^2$. Clearly, T is not continuous, but T is α-η-continuous on (X, d).

A mapping $T : X \to X$ is called orbitally continuous at $p \in X$ if $\lim_{n\to\infty} T^n x = p$ implies that $\lim_{n\to\infty} TT^n x = Tp$. The mapping T is orbitally continuous on X if T is orbitally continuous for all $p \in X$.

Remark 8 (see [20]). Let $T : X \to X$ be self-mapping on an orbitally T-complete metric space X. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 3, & \text{if } x, y \in O(w) \\ 0, & \text{otherwise,} \end{cases} \qquad \eta(x,y) = 1, \tag{16}$$

where $O(w)$ is an orbit of a point $w \in X$. If $T : X \to X$ is an orbitally continuous map on (X, d), then T is α-η-continuous on (X, d).

In this paper, we investigate existence and uniqueness of fixed points of generalized α-admissible modular contractive mappings in modular metric spaces. As applications, we derive some new fixed point theorems in partially ordered modular metric spaces, Suzuki type fixed point theorems in modular metric spaces and new fixed point theorems for integral contractions. At the end, we develop an important relation between fuzzy metric and modular metric and deduce certain new fixed point results in triangular fuzzy metric spaces. Moreover, some examples are provided here to illustrate the usability of the obtained results.

2. Fixed Point Results for Implicit Contractions

Let us first start this section with a definition of a family of functions.

Definition 9. Assume that $\Delta_{\mathscr{H}}$ denotes the collection of all continuous functions $\mathscr{H} : \mathbb{R}^{+^6} \to \mathbb{R}$ satisfying the following.

($\mathscr{H}$1) $\mathscr{H}$ is increasing in its 1th variable and nonincreasing in its 5th variable.

($\mathscr{H}$2) if $u, v \in \mathbb{R}^+$ with $u, v > 0$ and $\mathscr{H}(u, v, v, u, v+u, 0) \leq 0$, then, there exists $\psi \in \Psi$ such that

$$u \leq \psi(v). \tag{17}$$

Notice that here we denote with Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

Example 10. Let $\mathscr{H}(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi(\max\{t_2, (t_3 + t_4)/2, (t_5 + t_6)/2\})$, where $\psi \in \Psi$; then $\mathscr{H} \in \Delta_{\mathscr{H}}$.

Example 11. Let $\mathscr{H}(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi(\max\{t_2, (1 + t_3)t_4/(1 + t_2)\})$, where $\psi \in \Psi$; then $\mathscr{H} \in \Delta_{\mathscr{H}}$.

Theorem 12. *Let X_ω be a complete modular metric space and $T : X_\omega \to X_\omega$ self-mapping satisfying the following assertions:*

(i) *T is an α-admissible mapping with respect to η;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(iii) *T is an α-η-continuous function;*

(iv) *assume that there exists* $\mathscr{H} \in \Delta_{\mathscr{H}}$ *such that for all* $x, y \in X_\omega$ *and* $\lambda > 0$ *with* $\eta(x, Tx) \leq \alpha(x, y)$ *we have*

$$\mathscr{H}\left(\omega_{\lambda/c}(Tx, Ty), \omega_{\lambda/l}(x, y), \omega_{\lambda/l}(x, Tx), \omega_{\lambda/l}(y, Ty), \omega_{2\lambda/l}(x, Ty), \omega_{\lambda/l}(y, Tx)\right) \leq 0, \tag{18}$$

where $0 < l < c$.

Then T *has a fixed point. Moreover, if for all* $x, y \in \mathrm{Fix}(T)$ *we have* $\eta(x, x) \leq \alpha(x, y)$ *and* $\mathscr{H}(u, u, 0, 0, u, u) > 0$ *for all* $u > 0$, *then* T *has a unique fixed point.*

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. For $x_0 \in X$, we define the sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_n$. Now since T is an α-admissible mapping with respect to η then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. By continuing this process we have

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n) \tag{19}$$

for all $n \in \mathbb{N}$. Also, let there exists $n_0 \in X$ such that $x_{n_0} = x_{n_0+1}$. Then x_{n_0} is fixed point of T and we have nothing to prove. Hence, we assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Now by taking $x = x_{n-1}$ and $y = x_n$ in (iv) we get

$$\mathscr{H}\left(\omega_{\lambda/c}(Tx_{n-1}, Tx_n), \omega_{\lambda/l}(x_{n-1}, x_n), \omega_{\lambda/l}(x_{n-1}, Tx_{n-1}), \omega_{\lambda/l}(x_n, Tx_n), \omega_{2\lambda/l}(x_{n-1}, Tx_n), \omega_{\lambda/l}(x_n, Tx_{n-1})\right) \leq 0 \tag{20}$$

which implies

$$\mathscr{H}\left(\omega_{\lambda/c}(x_n, x_{n+1}), \omega_{\lambda/l}(x_{n-1}, x_n), \omega_{\lambda/l}(x_{n-1}, x_n), \omega_{\lambda/l}(x_n, x_{n+1}), \omega_{2\lambda/l}(x_{n-1}, x_{n+1}), 0\right) \leq 0. \tag{21}$$

On the other hand,

$$\begin{aligned} \omega_{2\lambda/l}(x_{n-1}, x_{n+1}) &\leq \omega_{\lambda/l}(x_{n-1}, x_n) + \omega_{\lambda/l}(x_n, x_{n+1}) \\ &\leq \omega_{\lambda/l}(x_{n-1}, x_n) + \omega_{\lambda/c}(x_n, x_{n+1}), \\ \omega_{\lambda/l}(x_n, x_{n+1}) &\leq \omega_{\lambda/c}(x_n, x_{n+1}). \end{aligned} \tag{22}$$

Now since $\mathscr{H}$ is nonincreasing in its 5th variable, so by (21) and (22) we obtain

$$\mathscr{H}\left(\omega_{\lambda/c}(x_n, x_{n+1}), \omega_{\lambda/l}(x_{n-1}, x_n), \omega_{\lambda/l}(x_{n-1}, x_n), \omega_{\lambda/c}(x_n, x_{n+1}), \omega_{\lambda/l}(x_{n-1}, x_n) + \omega_{\lambda/c}(x_n, x_{n+1}), 0\right) \leq 0. \tag{23}$$

From ($\mathscr{H}1$) we deduce that

$$\omega_{\lambda/c}(x_n, x_{n+1}) \leq \psi(\omega_{\lambda/l}(x_{n-1}, x_n)) \leq \psi(\omega_{\lambda/c}(x_{n-1}, x_n)). \tag{24}$$

Inductively, we obtain

$$\omega_{\lambda/c}(x_n, x_{n+1}) \leq \psi^n(\omega_{\lambda/c}(x_0, x_1)). \tag{25}$$

Taking limit as $n \to \infty$ in the above inequality we get

$$\lim_{n\to\infty} \omega_{\lambda/c}(x_n, x_{n+1}) = 0. \tag{26}$$

Suppose $m, n \in \mathbb{N}$ with $m > n$ and $\epsilon > 0$ be given. Then there exists $n_{\lambda/(m-n)} \in \mathbb{N}$ such that

$$\omega_{\lambda/c(m-n)}(x_n, x_{n+1}) < \frac{\epsilon}{c(m-n)} \tag{27}$$

for all $n \geq n_{\lambda/(m-n)}$. Therefore we get

$$\begin{aligned} \omega_{\lambda/c}(x_n, x_m) &\leq \omega_{\lambda/c(m-n)}(x_n, x_{n+1}) \\ &\quad + \omega_{\lambda/c(m-n)}(x_{n+1}, x_{n+2}) + \cdots \\ &\quad + \omega_{\lambda/c(m-n)}(x_{m-1}, x_m) \\ &< \frac{\epsilon}{c(m-n)} + \frac{\epsilon}{c(m-n)} + \frac{\epsilon}{c(m-n)} + \cdots \\ &\quad + \frac{\epsilon}{c(m-n)} \\ &= \frac{\epsilon}{c} \end{aligned} \tag{28}$$

for all $m, n \geq n_{\lambda/(m-n)}$. This shows that $\{x_n\}$ is a Cauchy sequence. Since X_ω is complete, so there exists $x^* \in X_\omega$ such that $\lim_{n\to\infty} x_n = x^*$. Now since T is an α-η-continuous mapping, so $Tx_n \to Tx^*$ as $n \to \infty$. Therefore,

$$Tx^* = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*. \tag{29}$$

Thus T has a fixed point. Let all $x, y \in \mathrm{Fix}(T)$ we have $\eta(x, x) \leq \alpha(x, y)$ and $\mathscr{H}(u, u, 0, 0, u, u) > 0$ for all $u > 0$. Then by (iv)

$$\begin{aligned} &\mathscr{H}\left(\omega_{\lambda/l}(x, y), \omega_{\lambda/l}(x, y), 0, 0, \omega_{\lambda/l}(x, y), \omega_{\lambda/l}(y, x)\right) \\ &\leq \mathscr{H}\left(\omega_{\lambda/c}(x, y), \omega_{\lambda/l}(x, y), 0, 0, \omega_{2\lambda/l}(x, y), \omega_{\lambda/l}(y, x)\right) \\ &= \mathscr{H}\left(\omega_{\lambda/c}(Tx, Ty), \omega_{\lambda/l}(x, y), \omega_{\lambda/l}(x, Tx), \omega_{\lambda/l}(y, Ty), \omega_{2\lambda/l}(x, Ty), \omega_{\lambda/l}(y, Tx)\right) \leq 0. \end{aligned} \tag{30}$$

Now if $\omega_{\lambda/l}(x, y) > 0$, then

$$0 < \mathscr{H}\left(\omega_{\lambda/l}(x, y), \omega_{\lambda/l}(x, y), 0, 0, \omega_{\lambda/l}(x, y), \omega_{\lambda/l}(y, x)\right) \leq 0 \tag{31}$$

which is a contradiction. Hence, $\omega_{\lambda/l}(x, y) = 0$. That is, $x = y$. Thus T has a unique fixed point. □

Corollary 13. *Let X_ω be a complete modular metric space and $T : X_\omega \to X_\omega$ a self-mapping satisfying the following assertions:*

(i) *T is an α-admissible mapping;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is an α-continuous function;*

(iv) *assume that there exists $\mathscr{H} \in \Delta_{\mathscr{H}}$ such that for all $x, y \in X_\omega$ and $\lambda > 0$ with $\alpha(x, y) \geq 1$ we have*

$$\mathscr{H}\left(\omega_{\lambda/c}(Tx, Ty), \omega_{\lambda/l}(x, y), \omega_{\lambda/l}(x, Tx), \omega_{\lambda/l}(y, Ty), \omega_{2\lambda/l}(x, Ty), \omega_{\lambda/l}(y, Tx)\right) \leq 0, \tag{32}$$

where $0 < l < c$.

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $1 \leq \alpha(x, y)$ and $\mathscr{H}(u, u, 0, 0, u, u) > 0$ for all $u > 0$, then T has a unique fixed point.

Theorem 14. *Let X_ω be a complete modular metric space and $T : X_\omega \to X_\omega$ self-mapping satisfying the following assertions:*

(i) *T is an α-admissible mapping with respect to η;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then either*

$$\eta\left(Tx_n, T^2x_n\right) \leq \alpha\left(Tx_n, x\right) \text{ or } \eta\left(T^2x_n, T^3x_n\right) \leq \alpha\left(T^2x_n, x\right) \tag{33}$$

holds for all $n \in \mathbb{N}$;

(iv) *condition (iv) of Theorem 12 holds.*

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $\eta(x, x) \leq \alpha(x, y)$ and $\mathscr{H}(u, u, 0, 0, u, u) > 0$ for all $u > 0$, then T has a unique fixed point.

Proof. As in proof of Theorem 12, we can deduce a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and there exists $x^* \in X_\omega$ such that $x_n \to x^*$ as $n \to \infty$. From (iii) either

$$\eta\left(x_n, Tx_n\right) \leq \alpha\left(x_n, x^*\right) \text{ or } \eta\left(x_{n+1}, Tx_{n+1}\right) \leq \alpha\left(x_{n+1}, x^*\right) \tag{34}$$

holds for all $n \in \mathbb{N}$. Let $\eta(x_n, Tx_n) \leq \alpha(x_n, x^*)$. Then by taking $x = x_n$ and $y = x^*$ in (iv) we have

$$\mathscr{H}\left(\omega_{\lambda/c}(Tx_n, Tx^*), \omega_{\lambda/l}(x_n, x^*), \omega_{\lambda/l}(x_n, Tx_n), \omega_{\lambda/l}(x^*, Tx^*), \omega_{2\lambda/l}(x_n, Tx^*), \omega_{\lambda/l}(x^*, Tx_n)\right) \leq 0 \tag{35}$$

which implies

$$\mathscr{H}\left(\omega_{\lambda/c}(x_{n+1}, Tx^*), \omega_{\lambda/l}(x_n, x^*), \omega_{\lambda/l}(x_n, x_{n+1}), \omega_{\lambda/l}(x^*, Tx^*), \omega_{2\lambda/l}(x_n, Tx^*), \omega_{\lambda/l}(x^*, x_{n+1})\right) \leq 0. \tag{36}$$

Taking limit as $n \to \infty$ in the above inequality we obtain

$$\mathscr{H}\left(\omega_{\lambda/c}(x^*, Tx^*), 0, 0, \omega_{\lambda/l}(x^*, Tx^*), \omega_{2\lambda/l}(x^*, Tx^*), 0\right) \leq 0. \tag{37}$$

Now since, $\omega_{2\lambda/l}(x^*, Tx^*) < \omega_{\lambda/l}(x^*, Tx^*)$, $\omega_{\lambda/l}(x^*, Tx^*) < \omega_{\lambda/c}(x^*, Tx^*)$, $\mathscr{H}$ is increasing in its 1th variable and nonincreasing in its 5th variable, so we obtain

$$\mathscr{H}\left(\omega_{\lambda/l}(x^*, Tx^*), 0, 0, \omega_{\lambda/l}(x^*, Tx^*), \omega_{\lambda/l}(x^*, Tx^*), 0\right) \leq 0 \tag{38}$$

which is a contradiction. Now by taking $u = \omega_{\lambda/l}(x^*, Tx^*)$ and $v = 0$, from ($\mathscr{H}2$) we have

$$\omega_{\lambda/l}\left(x^*, Tx^*\right) \leq \psi(0) = 0. \tag{39}$$

Hence, $\omega_{\lambda/l}(x^*, Tx^*) = 0$; that is, $x^* = Tx^*$. Similarly we can deduce that $Tx^* = x^*$ when $\eta_\lambda(x_{n+1}, Tx_{n+1}) \leq \alpha_\lambda(x_{n+1}, x^*)$. □

By using Example 10 and Theorem 14 we can obtain the following corollary.

Corollary 15. *Let X_ω be a complete modular metric space. Let $T : X_\omega \to X_\omega$ be self-mapping satisfying the following assertions:*

(i) *T is an α-admissible mapping with respect to η;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then either*

$$\eta\left(Tx_n, T^2x_n\right) \leq \alpha\left(Tx_n, x\right) \text{ or } \eta\left(T^2x_n, T^3x_n\right) \leq \alpha\left(T^2x_n, x\right) \tag{40}$$

holds for all $n \in \mathbb{N}$;

(iv) *for all $x, y \in X_\omega$ and $\lambda > 0$ with $\eta(x, Tx) \leq \alpha(x, y)$ we have*

$$\omega_{\lambda/c}(Tx, Ty) \leq \psi\left(\max\left\{\omega_{\lambda/l}(x, y), \frac{\omega_{\lambda/l}(x, Tx) + \omega_{\lambda/l}(y, Ty)}{2}, \frac{\omega_{2\lambda/l}(x, Ty) + \omega_{\lambda/l}(y, Tx)}{2}\right\}\right), \tag{41}$$

where $\psi \in \Psi$ and $0 < l < c$.

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $\eta(x, x) \leq \alpha(x, y)$, then T has a unique fixed point.

Corollary 16. *Let X_ω be a complete modular metric space. Let $T : X_\omega \to X_\omega$ be self-mapping satisfying the following assertions:*

(i) *T is an α-admissible mapping with respect to η;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then either*

$$\eta\left(Tx_n, T^2x_n\right) \leq \alpha\left(Tx_n, x\right) \text{ or } \eta\left(T^2x_n, T^3x_n\right) \leq \alpha\left(T^2x_n, x\right) \tag{42}$$

holds for all $n \in \mathbb{N}$;

(iv) *for all $x, y \in X_\omega$ and $\lambda > 0$ with $\eta(x, Tx) \leq \alpha(x, y)$ we have*

$$\omega_{\lambda/c}(Tx, Ty) \leq \psi\left(\max\left\{\omega_{\lambda/l}(x, y), \frac{\left[1 + \omega_{\lambda/l}(x, Tx)\right]\omega_{\lambda/l}(y, Ty)}{1 + \omega_{\lambda/l}(x, y)}\right\}\right), \tag{43}$$

where $\psi \in \Psi$ and $0 < l < c$.

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $\eta(x, x) \leq \alpha(x, y)$, then T has a unique fixed point.

Corollary 17. *Let X_ω be a complete modular metric space. Let $T : X_\omega \to X_\omega$ be self-mapping satisfying the following assertions:*

(i) *T is an α-admissible mapping with respect to η;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then either*

$$\eta\left(Tx_n, T^2x_n\right) \leq \alpha\left(Tx_n, x\right) \text{ or } \eta\left(T^2x_n, T^3x_n\right) \leq \alpha\left(T^2x_n, x\right) \tag{44}$$

holds for all $n \in \mathbb{N}$;

(iv) *for all $x, y \in X_\omega$ and $\lambda > 0$ with $\eta(x, Tx) \leq \alpha(x, y)$ we have*

$$\omega_{\lambda/c}(Tx, Ty) \leq a\omega_{\lambda/l}(x, y) + b\frac{\left[1 + \omega_{\lambda/l}(x, Tx)\right]\omega_{\lambda/l}(y, Ty)}{1 + \omega_{\lambda/l}(x, y)}, \tag{45}$$

where $a + b < 1$ and $0 < l < c$.

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $\eta(x, x) \leq \alpha(x, y)$, then T has a unique fixed point.

Example 18. Let $X_\omega = [0, +\infty)$ and $\omega_\lambda(x, y) = (1/\lambda)|x - y|$. Define $T : X_\omega \to X_\omega$, $\alpha, \eta : X_\omega \times X_\omega \to [0, \infty)$, and $\psi : [0, \infty) \to [0, \infty)$ by

$$Tx = \begin{cases} \dfrac{1}{16}x^2, & \text{if } x \in [0, 1], \\ \dfrac{\sin^2 x + \cos x + 1}{\sqrt{x^2 + 1}}, & \text{if } x \in (1, 2], \\ 7x + \ln x, & \text{if } x \in (2, \infty), \end{cases}$$

$$\alpha(x, y) = \begin{cases} \dfrac{1}{2}, & \text{if } x, y \in [0, 1], \\ \dfrac{1}{8}, & \text{otherwise,} \end{cases} \tag{46}$$

$$\eta(x, y) = \frac{1}{4}, \qquad \psi(t) = \frac{1}{2}t.$$

Let $\alpha(x, y) \geq \eta(x, y)$; then $x, y \in [0, 1]$. On the other hand, $Tw \in [0, 1]$ for all $w \in [0, 1]$. Then, $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. That is, T is an α-admissible mapping with respect to η. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then $Tx_n, T^2x_n, T^3x_n \in [0, 1]$ for all $n \in \mathbb{N}$. That is,

$$\eta\left(Tx_n, T^2x_n\right) \leq \alpha\left(Tx_n, x\right), \quad \eta\left(T^2x_n, T^3x_n\right) \leq \alpha\left(T^2x_n, x\right) \tag{47}$$

hold for all $n \in \mathbb{N}$. Clearly, $\alpha(0, T0) \geq \eta(0, T0)$. Let $\alpha(x, y) \geq \eta(x, Tx)$. Now, if $x \notin [0, 1]$ or $y \notin [0, 1]$, then $1/8 \geq 1/4$, which is a contradiction. So, $x, y \in [0, 1]$. Therefore,

$$\begin{aligned} \omega_{\lambda/2}(Tx, Ty) &= \frac{2}{\lambda}\frac{1}{16}\left|x^2 - y^2\right| \\ &= \frac{1}{8\lambda}\left|x - y\right|\left|x + y\right| \leq \frac{1}{4\lambda}\left|x - y\right| \\ &= \frac{1}{2}\frac{1}{\lambda/(1/2)}\left|x - y\right| \\ &= \frac{1}{2}\omega_{\lambda/(1/2)}(x, y) = \psi\left(\omega_{\lambda/(1/2)}(x, y)\right) \\ &\leq \psi\left(\max\left\{\omega_{\lambda/l}(x, y), \frac{\left[1 + \omega_{\lambda/l}(x, Tx)\right]\omega_{\lambda/l}(y, Ty)}{1 + \omega_{\lambda/l}(x, y)}\right\}\right). \end{aligned} \tag{48}$$

Hence all conditions of Corollary 16 hold and T has a unique fixed point.

Let $(X_\omega, \preceq)$ be a partially ordered modular metric space. Recall that $T : X_\omega \to X_\omega$ is nondecreasing if for all $x, y \in X$, $x \preceq y \Rightarrow T(x) \preceq T(y)$. Fixed point theorems for monotone operators in ordered metric spaces are widely

investigated and have found various applications in differential and integral equations (see [20, 22–25] and references therein). From results proved above, we derive the following new results in partially ordered modular metric spaces.

Theorem 19. *Let $(X_\omega, \preceq)$ be a complete partially ordered modular metric space and $T : X_\omega \to X_\omega$ self-mapping satisfying the following assertions:*

(i) *T is nondecreasing;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(iii) *T is continuous function;*

(iv) *assume that there exists $\mathscr{H} \in \Delta_{\mathscr{H}}$ such that for all $x, y \in X_\omega$ and $\lambda > 0$ with $x \preceq y$ we have*

$$\mathscr{H}\left(\omega_{\lambda/c}(Tx,Ty), \omega_{\lambda/l}(x,y), \omega_{\lambda/l}(x,Tx), \omega_{\lambda/l}(y,Ty), \omega_{2\lambda/l}(x,Ty), \omega_{\lambda/l}(y,Tx)\right) \le 0, \tag{49}$$

where $0 < l < c$.

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $x \preceq y$ and $\mathscr{H}(u,u,0,0,u,u) > 0$ for all $u > 0$, then T has a unique fixed point.

Theorem 20. *Let $(X_\omega, \preceq)$ be a complete partially ordered modular metric space and $T : X_\omega \to X_\omega$ self-mapping satisfying the following assertions:*

(i) *T is nondecreasing;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$, then either*

$$Tx_n \preceq x \quad \text{or} \quad T^2 x_n \preceq x \tag{50}$$

holds for all $n \in \mathbb{N}$;

(iv) *assume that there exists $\mathscr{H} \in \Delta_{\mathscr{H}}$ such that for all $x, y \in X_\omega$ and $\lambda > 0$ with $x \preceq y$ we have*

$$\mathscr{H}\left(\omega_{\lambda/c}(Tx,Ty), \omega_{\lambda/l}(x,y), \omega_{\lambda/l}(x,Tx), \omega_{\lambda/l}(y,Ty), \omega_{2\lambda/l}(x,Ty), \omega_{\lambda/l}(y,Tx)\right) \le 0, \tag{51}$$

where $0 < l < c$.

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $x \preceq y$ and $\mathscr{H}(u,u,0,0,u,u) > 0$ for all $u > 0$, then T has a unique fixed point.

Corollary 21. *Let $(X_\omega, \preceq)$ be a complete partially ordered modular metric space and $T : X_\omega \to X_\omega$ self-mapping satisfying the following assertions:*

(i) *T is nondecreasing;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$, then either*

$$Tx_n \preceq x \quad \text{or} \quad T^2 x_n \preceq x \tag{52}$$

holds for all $n \in \mathbb{N}$;

(iv) *assume that for all $x, y \in X_\omega$ and $\lambda > 0$ with $x \preceq y$ we have*

$$\omega_{\lambda/c}(Tx,Ty) \le \psi\left(\max\left\{\omega_{\lambda/l}(x,y), \frac{\omega_{\lambda/l}(x,Tx) + \omega_{\lambda/l}(y,Ty)}{2}, \frac{\omega_{2\lambda/l}(x,Ty) + \omega_{\lambda/l}(y,Tx)}{2}\right\}\right), \tag{53}$$

where $\psi \in \Psi$ and $0 < l < c$.

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $x \preceq y$, then T has a unique fixed point.

Corollary 22. *Let $(X_\omega, \preceq)$ be a complete partially ordered modular metric space and $T : X_\omega \to X_\omega$ self-mapping satisfying the following assertions:*

(i) *T is nondecreasing;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$, then either*

$$Tx_n \preceq x \quad \text{or} \quad T^2 x_n \preceq x \tag{54}$$

holds for all $n \in \mathbb{N}$;

(iv) *assume that for all $x, y \in X_\omega$ and $\lambda > 0$ with $x \preceq y$ we have*

$$\omega_{\lambda/c}(Tx,Ty) \le \psi\left(\max\left\{\omega_{\lambda/l}(x,y), \frac{\left[1+\omega_{\lambda/l}(x,Tx)\right]\omega_{\lambda/l}(y,Ty)}{1+\omega_{\lambda/l}(x,y)}\right\}\right), \tag{55}$$

where $\psi \in \Psi$ and $0 < l < c$.

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $x \preceq y$, then T has a unique fixed point.

Corollary 23. *Let $(X_\omega, \preceq)$ be a complete partially ordered modular metric space and $T : X_\omega \to X_\omega$ self-mapping satisfying the following assertions:*

(i) *T is nondecreasing;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$, then either*

$$Tx_n \preceq x \quad \text{or} \quad T^2 x_n \preceq x \tag{56}$$

holds for all $n \in \mathbb{N}$;

(iv) *assume that for all $x, y \in X_\omega$ and $\lambda > 0$ with $x \preceq y$ we have*

$$\omega_{\lambda/c}(Tx,Ty) \le a\omega_{\lambda/l}(x,y) + b\frac{\left[1+\omega_{\lambda/l}(x,Tx)\right]\omega_{\lambda/l}(y,Ty)}{1+\omega_{\lambda/l}(x,y)}, \tag{57}$$

where $a + b < 1$ and $0 < l < c$.

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $x \preceq y$, then T has a unique fixed point.

3. Suzuki Type Fixed Point Results in Modular Metric Spaces

In 2008, Suzuki proved a remarkable fixed point theorem, that is, a generalization of the Banach contraction principle and characterizes the metric completeness. Consequently, a number of extensions and generalizations of this result appeared in the literature (see [26–30] and references therein). As an application of our results proved above we deduce Suzuki type fixed point theorems in the setting of modular metric spaces.

Theorem 24. *Let X_ω be a complete modular metric space and T continuous self-mapping on X. Assume that*

$$\begin{aligned}\omega_{\lambda/l}(x,Tx) \le \omega_{\lambda/l}(x,y) &\Longrightarrow \mathscr{H}\left(\omega_{\lambda/c}(Tx,Ty), \omega_{\lambda/l}(x,y), \right.\\ &\quad \omega_{\lambda/l}(x,Tx), \omega_{\lambda/l}(y,Ty),\\ &\quad \left.\omega_{2\lambda/l}(x,Ty), \omega_{\lambda/l}(y,Tx)\right) \le 0\end{aligned} \tag{58}$$

for all $x, y \in X_\omega$ and $\lambda > 0$, where $0 < l < c$. Then T has a fixed point. Moreover, if $\mathscr{H}(u,u,0,0,u,u) > 0$ for all $u > 0$, then T has a unique fixed point.

Proof. Define $\alpha, \eta : (0,\infty) \times X_\omega \times X_\omega \to [0,\infty)$ by

$$\alpha(x,y) = \omega_{\lambda/l}(x,y), \qquad \eta(x,y) = \omega_{\lambda/l}(x,y). \tag{59}$$

Clearly, $\eta(x,y) \le \alpha(x,y)$ for all $x, y \in X_\omega$ and $\lambda > 0$. Since T is continuous, T is α-η-continuous. Thus conditions (i)–(iii) of Theorem 12 hold. Let $\eta(x,Tx) \le \alpha(x,y)$. Then $\omega_{\lambda/l}(x,Tx) \le \omega_{\lambda/l}(x,y)$. So from (58) we obtain

$$\begin{aligned}&\mathscr{H}\left(\omega_{\lambda/c}(Tx,Ty), \omega_{\lambda/l}(x,y), \omega_{\lambda/l}(x,Tx), \omega_{\lambda/l}(y,Ty),\right.\\ &\quad \left.\omega_{2\lambda/l}(x,Ty), \omega_{\lambda/l}(y,Tx)\right) \le 0.\end{aligned} \tag{60}$$

Therefore all conditions of Theorem 12 hold and T has a unique fixed point. □

Theorem 25. *Let X_ω be a complete modular metric space and T self-mapping on X. Assume that*

$$\begin{aligned}\frac{1-b}{1-b+a}\omega_{\lambda/l}(x,Tx) \le \omega_{\lambda/2l}(x,y) &\Longrightarrow \omega_{\lambda/c}(Tx,Ty) \le a\omega_{\lambda/l}(x,y)\\ &\quad + b\frac{\left[1+\omega_{\lambda/l}(x,Tx)\right]\omega_{\lambda/l}(y,Ty)}{1+\omega_{\lambda/l}(x,y)}\end{aligned} \tag{61}$$

for all $x, y \in X_\omega$ and $\lambda > 0$, where $0 < l < c$ and $a + b < 1$. Then T has a unique fixed point.

Proof. Define $\alpha, \eta : X_\omega \times X_\omega \to [0,\infty)$ by

$$\alpha(x,y) = \omega_{\lambda/2l}(x,y), \qquad \eta(x,y) = k\omega_{\lambda/l}(x,y), \tag{62}$$

where $k = (1-b)/(1-b+a)$. Clearly, $\eta(x,y) \le \alpha(x,y)$ for all $x, y \in X_\omega$ and $\lambda > 0$. Then conditions (i)-(ii) of Corollary 17 hold. Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$. Since $k\omega_{\lambda/l}(Tx_n, T^2x_n) \le \omega_{\lambda/2l}(Tx_n, T^2x_n)$ for all $n \in \mathbb{N}$, from (61) we obtain

$$\begin{aligned}&\omega_{\lambda/l}\left(T^2x_n, T^3x_n\right)\\ &\quad \le \omega_{\lambda/c}\left(T^2x_n, T^3x_n\right)\\ &\quad \le a\omega_{\lambda/l}\left(Tx_n, T^2x_n\right)\\ &\qquad + b\frac{\left[1+\omega_{\lambda/l}\left(Tx_n, T^2x_n\right)\right]\omega_{\lambda/l}\left(T^2x_n, T^3x_n\right)}{1+\omega_{\lambda/l}\left(Tx_n, T^2x_n\right)}\\ &\quad \le a\omega_{\lambda/l}\left(Tx_n, T^2x_n\right) + b\omega_{\lambda/l}\left(T^2x_n, T^3x_n\right)\end{aligned} \tag{63}$$

which implies

$$\omega_{\lambda/l}\left(T^2x_n, T^3x_n\right) \le \frac{a}{1-b}\omega_{\lambda/l}\left(Tx_n, T^2x_n\right). \tag{64}$$

Suppose there exists $n_0 \in \mathbb{N}$, such that

$$\begin{aligned}&\eta\left(Tx_{n_0}, T^2x_{n_0}\right) > \alpha\left(Tx_{n_0}, x\right) \text{ or}\\ &\eta\left(T^2x_{n_0}, T^3x_{n_0}\right) > \alpha\left(T^2x_{n_0}, x\right);\end{aligned} \tag{65}$$

then

$$\begin{aligned}&k\omega_{\lambda/l}\left(Tx_{n_0}, T^2x_{n_0}\right) > \omega_{\lambda/2l}\left(Tx_{n_0}, x\right) \text{ or}\\ &k\omega_{\lambda/l}\left(T^2x_{n_0}, T^3x_{n_0}\right) > \omega_{\lambda/2l}\left(T^2x_{n_0}, x\right).\end{aligned} \tag{66}$$

Therefore,

$$\begin{aligned}&\omega_{\lambda/l}\left(Tx_{n_0}, T^2x_{n_0}\right)\\ &\quad \le \omega_{\lambda/2l}\left(Tx_{n_0}, x\right) + \omega_{\lambda/2l}\left(T^2x_{n_0}, x\right)\\ &\quad < k\omega_{\lambda/l}\left(Tx_{n_0}, T^2x_{n_0}\right) + k\omega_{\lambda/l}\left(T^2x_{n_0}, T^3x_{n_0}\right)\\ &\quad \le k\omega_{\lambda/l}\left(Tx_{n_0}, T^2x_{n_0}\right) + k\frac{a}{1-b}\omega_{\lambda/l}\left(Tx_{n_0}, T^2x_{n_0}\right)\\ &\quad \le \left(k + k\frac{a}{1-b}\right)\omega_{\lambda/l}\left(Tx_{n_0}, T^2x_{n_0}\right)\\ &\quad = \omega_{\lambda/l}\left(Tx_{n_0}, T^2x_{n_0}\right)\end{aligned} \tag{67}$$

which is a contradiction. Hence, (iii) of Corollary 17 holds. Thus all conditions of Corollary 17 hold and T has a unique fixed point. □

Corollary 26. *Let X_ω be a complete modular metric space and T self-mapping on X. Assume that*

$$\begin{aligned}\frac{1}{1+a}\omega_{\lambda/l}(x,Tx) \le \omega_{\lambda/2l}(x,y) &\Longrightarrow \omega_{\lambda/c}(Tx,Ty) \le a\omega_{\lambda/l}(x,y)\end{aligned} \tag{68}$$

for all $x, y \in X_\omega$ and $\lambda > 0$, where $0 < l < c$ and $a < 1$. Then T has a unique fixed point.

4. Fixed Point Results for Integral Type Contractions

Recently, Azadifar et al. [31] and Razani and Moradi [32] proved common fixed point theorems of integral type in modular metric spaces. In this section we present more general fixed point theorems for integral type contractions.

Theorem 27. *Let X_ω be a complete modular metric space and $T : X_\omega \to X_\omega$ self-mapping satisfying the following assertions:*

(i) *T is an α-admissible mapping with respect to η;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(iii) *T is an α-η-continuous function;*

(iv) *assume that there exists $\mathscr{H} \in \Delta_{\mathscr{H}}$ such that for all $x, y \in X_\omega$ and $\lambda > 0$ with $\eta(x, Tx) \leq \alpha(x, y)$ we have*

$$\mathscr{H}\left(\int_0^{\omega_{\lambda/c}(Tx,Ty)} \rho(t)\,dt, \int_0^{\omega_{\lambda/l}(x,y)} \rho(t)\,dt, \int_0^{\omega_{\lambda/l}(x,Tx)} \rho(t)\,dt, \int_0^{\omega_{\lambda/l}(y,Ty)} \rho(t)\,dt, \int_0^{\omega_{2\lambda/l}(x,Ty)} \rho(t)\,dt, \int_0^{\omega_{\lambda/l}(y,Tx)} \rho(t)\,dt\right) \leq 0, \tag{69}$$

where $0 < l < c$, $\rho : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^\varepsilon \rho(t)dt > 0$ for $\varepsilon > 0$.

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $\eta(x, x) \leq \alpha(x, y)$ and $\mathscr{H}(u, u, 0, 0, u, u) > 0$ for all $u > 0$, then T has a unique fixed point.

Theorem 28. *Let X_ω be a complete modular metric space and $T : X_\omega \to X_\omega$ self-mapping satisfying the following assertions:*

(i) *T is an α-admissible mapping with respect to η;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then either*

$$\eta\left(Tx_n, T^2x_n\right) \leq \alpha\left(Tx_n, x\right) \text{ or } \eta\left(T^2x_n, T^3x_n\right) \leq \alpha\left(T^2x_n, x\right) \tag{70}$$

holds for all $n \in \mathbb{N}$;

(iv) *condition (iv) of Theorem 27 holds.*

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $\eta(x, x) \leq \alpha(x, y)$ and $\mathscr{H}(u, u, 0, 0, u, u) > 0$ for all $u > 0$, then T has a unique fixed point.

Theorem 29. *Let X_ω be a complete modular metric space and T continuous self-mapping on X. Assume that*

$$\begin{aligned} &\int_0^{\omega_{\lambda/l}(x,Tx)} \rho(t)\,dt \\ &\leq \int_0^{\omega_{\lambda/l}(x,y)} \rho(t)\,dt \\ &\Longrightarrow \mathscr{H}\left(\int_0^{\omega_{\lambda/c}(Tx,Ty)} \rho(t)\,dt, \int_0^{\omega_{\lambda/l}(x,y)} \rho(t)\,dt, \int_0^{\omega_{\lambda/l}(x,Tx)} \rho(t)\,dt, \int_0^{\omega_{\lambda/l}(y,Ty)} \rho(t)\,dt, \int_0^{\omega_{2\lambda/l}(x,Ty)} \rho(t)\,dt, \int_0^{\omega_{\lambda/l}(y,Tx)} \rho(t)\,dt\right) \leq 0 \end{aligned} \tag{71}$$

for all $x, y \in X_\omega$ and $\lambda > 0$, where $0 < l < c$, $\rho : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^\varepsilon \rho(t)dt > 0$ for $\varepsilon > 0$. Then T has a fixed point. Moreover, if $\mathscr{H}(u, u, 0, 0, u, u) > 0$ for all $u > 0$, then T has a unique fixed point.

Theorem 30. *Let X_ω be a complete modular metric space and T self-mapping on X. Assume that*

$$\begin{aligned} &\frac{1-b}{1-b+a} \int_0^{\omega_{\lambda/l}(x,Tx)} \rho(t)\,dt \\ &\leq \int_0^{\omega_{\lambda/2l}(x,y)} \rho(t)\,dt \Longrightarrow \int_0^{\omega_{\lambda/c}(Tx,Ty)} \rho(t)\,dt \\ &\leq a \int_0^{\omega_{\lambda/l}(x,y)} \rho(t)\,dt \\ &\quad + b\frac{\left[1 + \int_0^{\omega_{\lambda/l}(x,Tx)} \rho(t)\,dt\right] \int_0^{\omega_{\lambda/l}(y,Ty)} \rho(t)\,dt}{1 + \int_0^{\omega_{\lambda/l}(x,y)} \rho(t)\,dt} \end{aligned} \tag{72}$$

for all $x, y \in X_\omega$ and $\lambda > 0$, where $0 < l < c$, $a + b < 1$ and $\rho : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^\varepsilon \rho(t)dt > 0$ for $\varepsilon > 0$. Then T has a unique fixed point.

5. Modular Metric Spaces to Fuzzy Metric Spaces

In 1988, Grabiec [33] defined contractive mappings on a fuzzy metric space and extended fixed point theorems of Banach and Edelstein in such spaces. Successively, George and Veeramani [34] slightly modified the notion of a fuzzy metric space introduced by Kramosil and Michálek and then defined a Hausdorff and first countable topology on it. Since then, the notion of a complete fuzzy metric space presented by George and Veeramani has emerged as another characterization of completeness, and many fixed point theorems have also been proved (see for more details [35–39] and the references therein). In this section we develop an important relation

between modular metric and fuzzy metric and deduce in fixed point results in a triangular fuzzy metric space.

Definition 31. A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s > 0$:

(i) $M(x, y, t) > 0$;

(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(iii) $M(x, y, t) = M(y, x, t)$;

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(v) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

The function $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Definition 32 (see [36]). Let $(X, M, *)$ be a fuzzy metric space. The fuzzy metric M is called triangular whenever

$$\frac{1}{M(x, y, t)} - 1 \leq \frac{1}{M(x, z, t)} - 1 + \frac{1}{M(z, y, t)} - 1 \tag{73}$$

for all $x, y, z \in X$ and all $t > 0$.

Lemma 33 (see [33, 35]). *For all $x, y \in X$, $M(x, y, \cdot)$ is nondecreasing on $(0, \infty)$.*

As an application of Lemma 33, we establish the following important fact that each triangular fuzzy metric on X induces a modular metric.

Lemma 34. *Let $(X, M, *)$ be a triangular fuzzy metric space. Define*

$$\omega_\lambda(x, y) = \frac{1}{M(x, y, \lambda)} - 1 \tag{74}$$

for all $x, y \in X$ and all $\lambda > 0$. Then ω_λ is a modular metric on X.

Proof. Let $s, t > 0$. Then we get

$$\begin{aligned} M(x, y, s) &= M(x, y, s) * 1 \\ &= M(x, y, s) * M(x, x, t) \leq M(x, y, s + t) \end{aligned} \tag{75}$$

for all $x, y \in X$ and $s, t > 0$. Now, since $(X, M, *)$ is triangular, then we get

$$\begin{aligned} \omega_{\lambda+\mu}(x, y) &= \frac{1}{M(x, y, \mu + \lambda)} - 1 \\ &\leq \frac{1}{M(x, z, \mu + \lambda)} - 1 + \frac{1}{M(z, y, \mu + \lambda)} - 1 \\ &\leq \frac{1}{M(x, z, \lambda)} - 1 + \frac{1}{M(z, y, \mu)} - 1 \\ &= \omega_\lambda(x, z) + \omega_\mu(z, y). \end{aligned} \tag{76}$$

□

As an application of Lemma 34 and the results proved above we deduce following new fixed point theorems in triangular fuzzy metric spaces.

Theorem 35. *Let $(X, M, *)$ be a complete triangular fuzzy metric space and $T : X \to X$ self-mapping satisfying the following assertions:*

(i) *T is an α-admissible mapping with respect to η;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(iii) *T is an α-η-continuous function;*

(iv) *assume that there exists $\mathscr{H} \in \Delta_{\mathscr{H}}$ such that for all $x, y \in X$ and $\lambda > 0$ with $\eta(x, Tx) \leq \alpha(x, y)$ we have*

$$\begin{aligned} \mathscr{H}\Bigg(&\frac{1}{M(Tx, Ty, \lambda/c)} - 1, \frac{1}{M(x, y, \lambda/l)} - 1, \\ &\frac{1}{M(x, Tx, \lambda/l)} - 1, \frac{1}{M(y, Ty, \lambda/l)} - 1, \\ &\frac{1}{M(x, Ty, 2\lambda/l)} - 1, \frac{1}{M(y, Tx, \lambda/l)} - 1 \Bigg) \leq 0, \end{aligned} \tag{77}$$

where $0 < l < c$.

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $\eta(x, x) \leq \alpha(x, y)$ and $\mathscr{H}(u, u, 0, 0, u, u) > 0$ for all $u > 0$, then T has a unique fixed point.

Theorem 36. *Let $(X, M, *)$ be a complete triangular fuzzy metric space and $T : X \to X$ self-mapping satisfying the following assertions:*

(i) *T is an α-admissible mapping with respect to η;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then either*

$$\begin{aligned} &\eta\left(Tx_n, T^2x_n\right) \leq \alpha\left(Tx_n, x\right) \text{ or} \\ &\eta\left(T^2x_n, T^3x_n\right) \leq \alpha\left(T^2x_n, x\right) \end{aligned} \tag{78}$$

holds for all $n \in \mathbb{N}$;

(iv) *condition (iv) of Theorem 35 holds.*

Then T has a fixed point. Moreover, if for all $x, y \in \mathrm{Fix}(T)$ we have $\eta(x, x) \leq \alpha(x, y)$ and $\mathscr{H}(u, u, 0, 0, u, u) > 0$ for all $u > 0$, then T has a unique fixed point.

Theorem 37. *Let $(X, M, *, \preceq)$ be a partially ordered complete triangular fuzzy metric space and $T : X \to X$ self-mapping satisfying the following assertions:*

(i) *T is nondecreasing;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(iii) *T is continuous function;*

(iv) *assume that there exists* $\mathscr{H} \in \Delta_{\mathscr{H}}$ *such that for all* $x, y \in X$ *and* $\lambda > 0$ *with* $x \preceq y$ *we have*

$$\mathscr{H}\left(\frac{1}{M(Tx,Ty,\lambda/c)}-1, \frac{1}{M(x,y,\lambda/l)}-1, \frac{1}{M(x,Tx,\lambda/l)}-1, \frac{1}{M(y,Ty,\lambda/l)}-1, \frac{1}{M(x,Ty,2\lambda/l)}-1, \frac{1}{M(y,Tx,\lambda/l)}-1\right) \leq 0, \tag{79}$$

where $0 < l < c$.

Then T *has a fixed point. Moreover, if for all* $x, y \in \mathrm{Fix}(T)$ *we have* $x \preceq y$ *and* $\mathscr{H}(u,u,0,0,u,u) > 0$ *for all* $u > 0$, *then* T *has a unique fixed point.*

Theorem 38. *Let* $(X, M, *, \preceq)$ *be a partially ordered complete triangular fuzzy metric space and* $T : X \to X$ *self-mapping satisfying the following assertions:*

(i) T *is nondecreasing;*

(ii) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$;

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $x_n \preceq x_{n+1}$ *with* $x_n \to x$ *as* $n \to \infty$, *then either*

$$Tx_n \preceq x \quad \text{or} \quad T^2 x_n \preceq x \tag{80}$$

holds for all $n \in \mathbb{N}$;

(iv) *assume that there exists* $\mathscr{H} \in \Delta_{\mathscr{H}}$ *such that for all* $x, y \in X_\omega$ *and* $0\lambda > 0$ *with* $x \preceq y$ *we have*

$$\mathscr{H}\left(\frac{1}{M(Tx,Ty,\lambda/c)}-1, \frac{1}{M(x,y,\lambda/l)}-1, \frac{1}{M(x,Tx,\lambda/l)}-1, \frac{1}{M(y,Ty,\lambda/l)}-1, \frac{1}{M(x,Ty,2\lambda/l)}-1, \frac{1}{M(y,Tx,\lambda/l)}-1\right) \leq 0, \tag{81}$$

where $0 < l < c$.

Then T *has a fixed point. Moreover, if for all* $x, y \in \mathrm{Fix}(T)$ *we have* $x \preceq y$ *and* $\mathscr{H}(u,u,0,0,u,u) > 0$ *for all* $u > 0$, *then* T *has a unique fixed point.*

Theorem 39. *Let* $(X, M, *)$ *be a complete triangular fuzzy metric space and* T *continuous self-mapping on* X. *Assume that*

$$\frac{1}{M(x,Tx,\lambda/l)}-1 \leq \frac{1}{M(x,y,\lambda/l)}-1 \Longrightarrow \mathscr{H}\left(\frac{1}{M(Tx,Ty,\lambda/c)}-1, \frac{1}{M(x,y,\lambda/l)}-1, \frac{1}{M(x,Tx,\lambda/l)}-1, \frac{1}{M(y,Ty,\lambda/l)}-1, \frac{1}{M(x,Ty,2\lambda/l)}-1, \frac{1}{M(y,Tx,\lambda/l)}-1\right) \leq 0 \tag{82}$$

for all $x, y \in X$ *and* $\lambda > 0$, *where* $0 < l < c$. *Then* T *has a unique fixed point.*

Theorem 40. *Let* $(X, M, *)$ *be a complete triangular fuzzy metric space and* T *continuous self-mapping on* X. *Assume that*

$$\frac{1-b}{1-b+a}\left(\frac{1}{M(x,Tx,\lambda/l)}-1\right) \leq \frac{1}{M(x,y,\lambda/2l)}-1 \Longrightarrow \frac{1}{M(Tx,Ty,\lambda/c)}-1 \leq a\left(\frac{1}{M(x,y,\lambda/l)}-1\right) + b\frac{(1/M(x,Tx,\lambda/l))\left[\left(1/M(y,Ty,\lambda/l)\right)-1\right]}{1/M(x,y,\lambda/l)} \tag{83}$$

for all $x, y \in X_\omega$ *and* $\lambda > 0$, *where* $0 < l < c$ *and* $a + b < 1$. *Then* T *has a unique fixed point.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Some Common Fixed Point Theorems in Complex Valued b-Metric Spaces

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Azam et al. (2011), introduce the notion of complex valued metric spaces and obtained common fixed point result for mappings in the context of complex valued metric spaces. Rao et al. (2013) introduce the notion of complex valued b-metric spaces. In this paper, we generalize the results of Azam et al. (2011), and Bhatt et al. (2011), by improving the conditions of contraction to establish the existence and uniqueness of common fixed point for two self-mappings on complex valued b-metric spaces. Some examples are given to illustrate the main results.

1. Introduction

Banach contraction principle in [1] was the starting point for many researchers during last decades in the field of nonlinear analysis. In 1989, Bakhtin [2] introduced the concept of b-metric space as a generalization of metric spaces. The concept of complex valued b-metric spaces was introduced in 2013 by Rao et al. [3], which was more general than the well-known complex valued metric spaces that were introduced in 2011 by Azam et al. [4]. The main purpose of this paper is to present common fixed point results of two self-mappings satisfying a rational inequality on complex valued b-metric spaces. The results presented in this paper are generalization of work done by Azam et al. in [4] and Bhatt et al in [5].

Definition 1 (see [6]). Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

Example 2 (see [7]). Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then (X, ρ) is a b-metric space with $s = 2^{p-1}$.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \quad \text{iff } \operatorname{Re}(z_1) \leq \operatorname{Re}(z_1), \quad \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2). \tag{1}$$

Thus $z_1 \precsim z_2$ if one of the following holds:

(1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,

(2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,

(3) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,

(4) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.

We will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (2), (3), and (4) is satisfied; also we will write $z_1 \prec z_2$ if only (4) is satisfied.

Remark 3. We can easily check that the following statements are held:

(i) if $a, b \in \mathbb{R}$ and $a \leq b$, then $az \precsim bz$ for all $z \in \mathbb{C}$;

(ii) if $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$;

(iii) if $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 4 (see [4]). Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \precsim d(x, z) + d(z, y)$.

The pair (X, d) is called a complex valued metric space.

Example 5 (see [8]). Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(x, y) = i|x - y|, \quad \forall x, y \in X. \tag{2}$$

Then (X, d) is a complex valued metric space.

Example 6 (see [9]). Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(x, y) = e^{ik}|x - y|, \quad \text{where } k \in \mathbb{R}, \ \forall x, y \in X. \tag{3}$$

Then (X, d) is a complex valued metric space.

Definition 7 (see [3]). Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \precsim s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a complex valued b-metric space.

Example 8 (see [3]). Let $X = [0, 1]$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(x, y) = |x - y|^2 + i|x - y|^2, \quad \forall x, y \in X. \tag{4}$$

Then (X, d) is a complex valued b-metric space with $s = 2$.

Definition 9 (see [3]). Let (X, d) be a complex valued b-metric space. Consider the following.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) := \{y \in X : d(x, y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set A whenever, for every $0 \prec r \in \mathbb{C}$, $B(x, r) \cap (A - X) \neq \emptyset$.

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.

(iv) A subset $A \subseteq X$ is called closed whenever each element of A belongs to A.

(v) A subbasis for a Hausdorff topology τ on X is a family $F = \{B(x, r) : x \in X \text{ and } 0 \prec r\}$.

Definition 10 (see [3]). Let (X, d) be a complex valued b-metric space and $\{x_n\}$ a sequence in X and $x \in X$. Consider the following.

(i) If for every $c \in \mathbb{C}$, with $0 \prec r$, there is $N \in \mathbb{N}$ such that, for all $n > \mathbb{N}$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x, and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $\{x_n\} \to x$ as $n \to \infty$.

(ii) If for every $c \in \mathbb{C}$, with $0 \prec r$, there is $N \in \mathbb{N}$ such that, for all $n > \mathbb{N}$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then (X, d) is said to be a complete complex valued b-metric space.

Lemma 11 (see [3]). *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 12 (see [3]). *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

Theorem 13 (see [4]). *Let (X, d) be a complete complex valued metric space and let λ, μ be nonnegative real numbers such that $\lambda + \mu < 1$. Suppose that $S, T : X \to X$ are mappings satisfying*

$$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Sx)\, d(y, Ty)}{1 + d(x, y)}, \tag{5}$$

for all $x, y \in X$. Then S, T have a unique common fixed point in X.

Theorem 14 (see [5]). *Let (X, d) be a complete complex valued metric space and let $S, T : X \to X$ be mappings satisfying*

$$d(Sx, Ty) \precsim \frac{a[d(x, Sx)\, d(x, Ty) + d(y, Ty)\, d(y, Sx)]}{d(x, Ty) + d(y, Sx)}, \tag{6}$$

for all $x, y \in X$, where $a \in [0, 1)$. Then S, T have a unique common fixed point in X.

2. Main Result

Our next theorem is a generalization of Theorem 13 in complex valued b-metric spaces.

Theorem 15. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ be mappings satisfying*

$$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Sx)\, d(y, Ty)}{1 + d(x, y)}, \tag{7}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $s\lambda + \mu < 1$. Then S, T have a unique common fixed point in X.

Proof. For any arbitrary point, $x_0 \in X$. Define sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}, \quad \text{for } n = 0, 1, 2, 3, \ldots. \tag{8}$$

Now, we show that the sequence $\{x_n\}$ is Cauchy. Let $x = x_{2n}$ and $y = x_{2n+1}$ in (7); we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\precsim \lambda d(x_{2n}, x_{2n+1}) + \frac{\mu d(x_{2n}, Sx_{2n})\, d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \\ &= \lambda d(x_{2n}, x_{2n+1}) + \frac{\mu d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})}, \end{aligned} \tag{9}$$

which implies that

$$\begin{aligned} &|d(x_{2n+1}, x_{2n+2})| \\ &\quad \le \lambda |d(x_{2n}, x_{2n+1})| + \frac{\mu |d(x_{2n}, x_{2n+1})|\, |d(x_{2n+1}, x_{2n+2})|}{|1 + d(x_{2n}, x_{2n+1})|}. \end{aligned} \tag{10}$$

Since $|1 + d(x_{2n}, x_{2n+1})| > |d(x_{2n}, x_{2n+1})|$, we get

$$\begin{aligned} &|d(x_{2n+1}, x_{2n+2})| \\ &\quad \le \lambda |d(x_{2n}, x_{2n+1})| + \mu |d(x_{2n+1}, x_{2n+2})|, \end{aligned} \tag{11}$$

and hence

$$|d(x_{2n+1}, x_{2n+2})| \le \frac{\lambda}{1-\mu} |d(x_{2n}, x_{2n+1})|. \tag{12}$$

Similarly, we obtain

$$|d(x_{2n+2}, x_{2n+3})| \le \frac{\lambda}{1-\mu} |d(x_{2n+1}, x_{2n+2})|. \tag{13}$$

Since $s\lambda + \mu < 1$ and $s \ge 1$, we get $\lambda + \mu < 1$.

Therefore, with $\delta = \lambda/(1-\mu) < 1$, and for all $n \ge 0$, consequently, we have

$$\begin{aligned} &|d(x_{2n+1}, x_{2n+2})| \\ &\quad \le \delta |d(x_{2n}, x_{2n+1})| \le \delta^2 |d(x_{2n-1}, x_{2n})| \\ &\quad \le \cdots \le \delta^{2n+1} |d(x_0, x_1)|. \end{aligned} \tag{14}$$

Thus for any $m > n$, $m, n \in \mathbb{N}$, and since $s\delta = s\lambda/(1-\mu) < 1$, we get

$$\begin{aligned} &|d(x_n, x_m)| \\ &\le s|d(x_n, x_{n+1})| + s|d(x_{n+1}, x_m)| \\ &\le s|d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| + s^2 |d(x_{n+2}, x_m)| \\ &\le s|d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| + s^3 |d(x_{n+2}, x_m)| \\ &\quad + s^3 |d(x_{n+2}, x_m)| \\ &\le s|d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| + s^3 |d(x_{n+2}, x_m)| \\ &\quad + \cdots + s^{m-n-2} |d(x_{m-3}, x_{m-2})| \\ &\quad + s^{m-n-1} |d(x_{m-2}, x_{m-1})| + s^{m-n-1} |d(x_{m-1}, x_m)| \\ &\le s|d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| \\ &\quad + s^3 |d(x_{n+2}, x_m)| \\ &\quad + \cdots + s^{m-n-2} |d(x_{m-3}, x_{m-2})| \\ &\quad + s^{m-n-1} |d(x_{m-2}, x_{m-1})| + s^{m-n} |d(x_{m-1}, x_m)|. \end{aligned} \tag{15}$$

By using (14) we get

$$\begin{aligned} &|d(x_n, x_m)| \\ &\le s\delta^n |d(x_0, x_1)| + s^2\delta^{n+1} |d(x_0, x_1)| \\ &\quad + s^3\delta^{n+2} |d(x_0, x_1)| \\ &\quad + \cdots + s^{m-n-2}\delta^{m-3} |d(x_0, x_1)| \\ &\quad + s^{m-n-1}\delta^{m-2} |d(x_0, x_1)| + s^{m-n}\delta^{m-1} |d(x_0, x_1)| \\ &= \sum_{i=1}^{m-n} s^i \delta^{i+n-1} |d(x_0, x_1)|. \end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned} &|d(x_n, x_m)| \\ &\quad \le \sum_{i=1}^{m-n} s^{i+n-1}\delta^{i+n-1} |d(x_0, x_1)| \\ &\quad = \sum_{t=n}^{m-1} s^t \delta^t |d(x_0, x_1)| \\ &\quad \le \sum_{t=n}^{\infty} (s\delta)^t |d(x_0, x_1)| \\ &\quad = \frac{(s\delta)^n}{1 - s\delta} |d(x_0, x_1)|, \end{aligned} \tag{17}$$

and hence

$$|d(x_n, x_m)| \le \frac{(s\delta)^n}{1 - s\delta} |d(x_0, x_1)| \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty. \tag{18}$$

Thus, $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Assuming not, then there exist $z \in X$ such that

$$|d(u, Su)| = |z| > 0. \tag{19}$$

So by using the triangular inequality and (7), we get

$$\begin{aligned} z = d(u, Su) &\precsim sd(u, x_{2n+2}) + sd(x_{2n+2}, Su) \\ &= sd(u, x_{2n+2}) + sd(Tx_{2n+1}, Su) \\ &\precsim sd(u, x_{2n+2}) + s\lambda d(u, x_{2n+2}) \\ &\quad + \frac{s\mu d(u, Su)\, d(x_{2n+1}, Tx_{2n+1})}{1 + d(u, x_{2n+2})} \end{aligned}$$

$$= sd\left(u, x_{2n+2}\right) + s\lambda d\left(u, x_{2n+2}\right) + \frac{s\mu d\left(u, Su\right) d\left(x_{2n+1}, x_{2n+2}\right)}{1 + d\left(u, x_{2n+2}\right)}, \tag{20}$$

which implies that

$$|z| = |d\left(u, Su\right)| \leq s\left|d\left(u, x_{2n+2}\right)\right| + \frac{s\mu \left|d\left(u, Su\right)\right| \left|d\left(x_{2n+1}, x_{2n+2}\right)\right|}{\left|1 + d\left(u, x_{2n+2}\right)\right|}. \tag{21}$$

Taking the limit of (21) as $n \to \infty$, we obtain that $|z| = |d(u, Su)| \leq 0$, a contradiction with (19). So $|z| = 0$. Hence $Su = u$. Similarly, we obtain $Tu = u$.

Now we show that S and T have unique common fixed point of S and T. To show this, assume that u^* is another common fixed point of S and T. Then

$$d\left(u, u^*\right) = d\left(Su, Tu^*\right) \precsim \lambda d\left(u, u^*\right) + \frac{\mu d\left(u, Su\right) d\left(u^*, Tu^*\right)}{1 + d\left(u, u^*\right)} \prec d\left(u, u^*\right). \tag{22}$$

This implies that $|d(x, x^*)| < |d(u, u^*)|$, a contradiction. So $u = u^*$ which proves the uniqueness of common fixed point in X. This completes the proof. □

Corollary 16. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying*

$$d\left(Tx, Ty\right) \precsim \lambda d\left(x, y\right) + \frac{\mu d\left(x, Tx\right) d\left(y, Ty\right)}{1 + d\left(x, y\right)}, \tag{23}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $s\lambda + \mu < 1$. Then T has a unique fixed point in X.

Proof. We can prove this result by applying Theorem 15 with $S = T$. □

Corollary 17. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying*

$$d\left(T^n x, T^n y\right) \precsim \lambda d\left(x, y\right) + \frac{\mu d\left(x, T^n x\right) d\left(y, T^n y\right)}{1 + d\left(x, y\right)}, \tag{24}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $s\lambda + \mu < 1$. Then T has a unique fixed point in X.

Proof. From Corollary 20, we obtain $u \in X$ such that

$$T^n u = u. \tag{25}$$

The uniqueness follows from

$$\begin{aligned} d\left(Tu, u\right) &= d\left(TT^n u, T^n u\right) = d\left(T^n Tu, T^n u\right) \\ &\precsim \lambda d\left(Tu, u\right) + \frac{\mu d\left(Tu, T^n Tu\right) d\left(u, T^n u\right)}{1 + d\left(Tu, u\right)} \\ &\precsim \lambda d\left(Tu, u\right) + \frac{\mu d\left(Tu, T^n Tu\right) d\left(u, u\right)}{1 + d\left(Tu, u\right)} \\ &= \lambda d\left(Tu, u\right). \end{aligned} \tag{26}$$

By taking modulus of (26) and since $\lambda < 1$, we obtain $|d(Tu, u)| \leq \lambda |d(Tu, u)| < |d(Tu, u)|$, a contradiction. So, $Tu = u$. Hence

$$Tu = T^n u = u. \tag{27}$$

Therefore, the fixed point of T is unique. This completes the proof. □

Example 18. Let $X = \mathbb{C}$. Define a function $d : X \times X \to \mathbb{C}$ such that

$$d\left(z_1, z_2\right) = \left|x_1 - x_2\right|^2 + i\left|y_1 - y_2\right|^2, \tag{28}$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

To verify that (X, d) is a complete complex valued b-metric space with $s = 2$, it is enough to verify the triangular inequality condition.

Let z_1, z_2, and $z_3 \in X$; then,

$$\begin{aligned} &d\left(z_1, z_2\right) \\ &= \left|x_1 - x_2\right|^2 + i\left|y_1 - y_2\right|^2 \\ &= \left|x_1 - x_3 + x_3 - x_2\right|^2 + i\left|y_1 - y_3 + y_3 - y_2\right|^2 \\ &\precsim \left|x_1 - x_3\right|^2 + \left|x_3 - x_2\right|^2 + 2\left|x_1 - x_3\right|\left|x_3 - x_2\right| \\ &\quad + i\left[\left|y_1 - y_3\right|^2 + \left|y_3 - y_2\right|^2 + 2\left|y_1 - y_3\right|\left|y_3 - y_2\right|\right] \\ &\precsim \left|x_1 - x_3\right|^2 + \left|x_3 - x_2\right|^2 + \left|x_1 - x_3\right|^2 + \left|x_3 - x_2\right|^2 \\ &\quad + i\left[\left|y_1 - y_3\right|^2 + \left|y_3 - y_2\right|^2 + \left|y_1 - y_3\right|^2 + \left|y_3 - y_2\right|^2\right] \\ &= 2\left\{\left|x_1 - x_3\right|^2 + \left|x_3 - x_2\right|^2 + i\left[\left|y_1 - y_3\right|^2 + \left|y_3 - y_2\right|^2\right]\right\} \\ &= 2\left[d\left(z_1, z_3\right) + d\left(z_3, z_2\right)\right]. \end{aligned} \tag{29}$$

Therefore, $s = 2$.

Now, define two self-mappings $S, T : X \to X$ as follows:

$$Tz = T\left(x + iy\right) = \begin{cases} 0 & \text{if } x, y \in \mathbb{Q} \\ 2 & \text{if } x \in \mathbb{Q}^c, y \in \mathbb{Q} \\ 2i & \text{if } x \in \mathbb{Q}^c, y \in \mathbb{Q}^c \\ 2 + 2i & \text{if } x \in \mathbb{Q}, y \in \mathbb{Q}^c \end{cases} \tag{30}$$

such that $S = T$ and $z = x + iy$. Let $x = 1/\pi$ and $y = 0$, and since $\lambda \in [0, 1)$, we have

$$\begin{aligned} d\left(Tx, Ty\right) &= d\left(T\frac{1}{\pi}, T0\right) \\ &= d\left(2, 0\right) = 4 > \lambda\frac{1}{\pi^2} \\ &= \lambda d\left(\frac{1}{\pi}, 0\right) + \frac{\mu d\left(1/\pi, T\left(1/\pi\right)\right) d\left(0, T0\right)}{1 + d\left(1/\pi, 0\right)}. \end{aligned} \tag{31}$$

Note that $T^n z = 0$ for $n > 1$, so

$$d\left(T^n x, T^n y\right) = 0 \precsim \lambda d\left(x, y\right) + \frac{\mu d\left(x, T^n x\right) d\left(y, T^n y\right)}{1 + d\left(x, y\right)}, \tag{32}$$

for all $x, y \in X$ and $\lambda, \mu \geq 0$ with $2\lambda + \mu < 1$. So all conditions of Corollary 21 are satisfied to get a unique fixed point 0 of T.

Our next theorem is a generalization of Theorem 14 in complex valued b-metric spaces.

Theorem 19. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ be mappings satisfying*

$$d\left(Sx, Ty\right) \precsim \frac{a\left[d\left(x, Sx\right) d\left(x, Ty\right) + d\left(y, Ty\right) d\left(y, Sx\right)\right]}{d\left(x, Ty\right) + d\left(y, Sx\right)}, \tag{33}$$

for all $x, y \in X$, where $sa \in [0, 1)$. Then S, T have a unique common fixed point in X.

Proof. For any arbitrary point, $x_0 \in X$. Define sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}, \quad \text{for } k = 0, 1, 2, 3, \ldots. \tag{34}$$

Now, we show that the sequence $\{x_n\}$ is Cauchy. Let $x = x_{2n}$ and $y = x_{2n+1}$ in (33); we have

$$\begin{aligned} d\left(x_{2n+1}, x_{2n+2}\right) &= d\left(Sx_{2n}, Tx_{2n+1}\right) \\ &\precsim a\left[d\left(x_{2n}, Sx_{2n}\right) d\left(x_{2n}, Tx_{2n+1}\right)\right. \\ &\qquad \left. + d\left(x_{2n+1}, Tx_{2n+1}\right) d\left(x_{2n+1}, Sx_{2n}\right)\right] \\ &\quad \times \left(d\left(x_{2n}, Tx_{2n+1}\right) + d\left(x_{2n+1}, Sx_{2n}\right)\right)^{-1} \\ &= a\left[d\left(x_{2n}, x_{2n+1}\right) d\left(x_{2n}, x_{2n+2}\right)\right. \\ &\qquad \left. + d\left(x_{2n+1}, x_{2n+2}\right) d\left(x_{2n+1}, x_{2n+1}\right]\right. \\ &\quad \times \left(d\left(x_{2n}, x_{2n+2}\right) + d\left(x_{2n+1}, x_{2n+1}\right)\right)^{-1}, \end{aligned} \tag{35}$$

which implies that

$$\begin{aligned} &\left|d\left(x_{2n+1}, x_{2n+2}\right)\right| \\ &\quad \leq a\left[\left|d\left(x_{2n}, x_{2n+1}\right)\right| \left|d\left(x_{2n}, x_{2n+2}\right)\right|\right. \\ &\qquad \left. + \left|d\left(x_{2n+1}, x_{2n+2}\right)\right| \left|d\left(x_{2n+1}, x_{2n+1}\right)\right|\right] \\ &\qquad \times \left(\left|d\left(x_{2n}, x_{2n+2}\right) + d\left(x_{2n+1}, x_{2n+1}\right)\right|\right)^{-1} \\ &\quad = \frac{a\left|d\left(x_{2n}, x_{2n+1}\right)\right| \left|d\left(x_{2n}, x_{2n+2}\right)\right|}{\left|d\left(x_{2n}, x_{2n+2}\right)\right|} \\ &\quad = a\left|d\left(x_{2n}, x_{2n+1}\right)\right|, \end{aligned} \tag{36}$$

and hence

$$\left|d\left(x_{2n+1}, x_{2n+2}\right)\right| \leq a\left|d\left(x_{2n}, x_{2n+1}\right)\right|. \tag{37}$$

Similarly, we can see that

$$\left|d\left(x_{2n+2}, x_{2n+3}\right)\right| \leq a\left|d\left(x_{2n+1}, x_{2n+2}\right)\right|. \tag{38}$$

Since $sa < 1$ and $s \geq 1$, we get $a < 1$.

Therefore, for all $n \geq 0$, consequently, we have

$$\begin{aligned} \left|d\left(x_{2n+1}, x_{2n+2}\right)\right| &\leq a\left|d\left(x_{2n}, x_{2n+1}\right)\right| \\ &\leq a^2\left|d\left(x_{2n-1}, x_{2n}\right)\right| \\ &\leq \cdots \leq a^{2n+1}\left|d\left(x_0, x_1\right)\right|. \end{aligned} \tag{39}$$

Thus for any $m > n$, $m, n \in \mathbb{N}$, we have

$$\begin{aligned} &\left|d\left(x_n, x_m\right)\right| \\ &\quad \leq s\left|d\left(x_n, x_{n+1}\right)\right| + s\left|d\left(x_{n+1}, x_m\right)\right| \\ &\quad \leq s\left|d\left(x_n, x_{n+1}\right)\right| + s^2\left|d\left(x_{n+1}, x_{n+2}\right)\right| + s^2\left|d\left(x_{n+2}, x_m\right)\right| \\ &\quad \leq s\left|d\left(x_n, x_{n+1}\right)\right| + s^2\left|d\left(x_{n+1}, x_{n+2}\right)\right| \\ &\qquad + s^3\left|d\left(x_{n+2}, x_m\right)\right| + s^3\left|d\left(x_{n+2}, x_m\right)\right| \\ &\quad \leq s\left|d\left(x_n, x_{n+1}\right)\right| + s^2\left|d\left(x_{n+1}, x_{n+2}\right)\right| + s^3\left|d\left(x_{n+2}, x_m\right)\right| \\ &\qquad + \cdots + s^{m-n-2}\left|d\left(x_{m-3}, x_{m-2}\right)\right| \\ &\qquad + s^{m-n-1}\left|d\left(x_{m-2}, x_{m-1}\right)\right| + s^{m-n-1}\left|d\left(x_{m-1}, x_m\right)\right| \\ &\quad \leq s\left|d\left(x_n, x_{n+1}\right)\right| + s^2\left|d\left(x_{n+1}, x_{n+2}\right)\right| + s^3\left|d\left(x_{n+2}, x_m\right)\right| \\ &\qquad + \cdots + s^{m-n-2}\left|d\left(x_{m-3}, x_{m-2}\right)\right| \\ &\qquad + s^{m-n-1}\left|d\left(x_{m-2}, x_{m-1}\right)\right| + s^{m-n}\left|d\left(x_{m-1}, x_m\right)\right|. \end{aligned} \tag{40}$$

By using (39), we get

$$\begin{aligned} &\left|d\left(x_n, x_m\right)\right| \\ &\quad \leq sa^n\left|d\left(x_0, x_1\right)\right| + s^2 a^{n+1}\left|d\left(x_0, x_1\right)\right| + s^3 a^{n+2}\left|d\left(x_0, x_1\right)\right| \\ &\qquad + \cdots + s^{m-n-2}a^{m-3}\left|d\left(x_0, x_1\right)\right| \\ &\qquad + s^{m-n-1}a^{m-2}\left|d\left(x_0, x_1\right)\right| + s^{m-n}a^{m-1}\left|d\left(x_0, x_1\right)\right| \\ &\quad = \sum_{i=1}^{m-n} s^i a^{i+n-1}\left|d\left(x_0, x_1\right)\right|. \end{aligned} \tag{41}$$

Therefore,

$$\begin{aligned} \left|d\left(x_n, x_m\right)\right| &\leq \sum_{i=1}^{m-n} s^{i+n-1} a^{i+n-1}\left|d\left(x_0, x_1\right)\right| \\ &= \sum_{t=n}^{m-1} s^t a^t\left|d\left(x_0, x_1\right)\right| \\ &\leq \sum_{t=n}^{\infty} (sa)^t\left|d\left(x_0, x_1\right)\right| \\ &= \frac{(sa)^n}{1 - sa}\left|d\left(x_0, x_1\right)\right|. \end{aligned} \tag{42}$$

Now, since $sa < 1$, we deduce

$$\left|d\left(x_n, x_m\right)\right| \leq \frac{(sa)^n}{1 - sa}\left|d\left(x_0, x_1\right)\right| \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty. \tag{43}$$

Thus, $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Assuming not, then there exist $z \in X$ such that

$$|d\left(u, Su\right)| = |z| > 0. \tag{44}$$

So by using the triangular inequality and (33), we get

$$\begin{aligned} z &= d(u, Su) \\ &\precsim sd(u, x_{2n+2}) + sd(x_{2n+2}, Su) \\ &= sd(u, x_{2n+2}) + sd(Tx_{2n+1}, Su) \\ &\precsim sd(u, x_{2n+2}) + (sad(u, Su)\, d(u, Tx_{2n+1}) \\ &\qquad + sad(x_{2n+1}, Tx_{2n+1})\, d(x_{2n+1}, Su)) \\ &\qquad \times (d(u, Tx_{2n+1}) + d(x_{2n+1}, Su))^{-1} \\ &= sd(u, x_{2n+2}) + (sad(u, Su)\, d(u, x_{2n+2}) \\ &\qquad + sad(x_{2n+1}, x_{2n+2})\, d(x_{2n+1}, Su)) \\ &\qquad \times (d(u, x_{2n+2}) + d(x_{2n+1}, Su))^{-1}, \end{aligned} \tag{45}$$

which implies that

$$\begin{aligned} |z| = |d(u, Su)| &\le s|d(u, x_{2n+2})| \\ &\quad + (sa|d(u, Su)|\,|d(u, x_{2n+2})| \\ &\qquad + sa|d(x_{2n+1}, x_{2n+2})|\,|d(x_{2n+1}, Su)|) \\ &\quad \times (|d(u, x_{2n+2}) + d(x_{2n+1}, Su)|)^{-1}. \end{aligned} \tag{46}$$

Taking the limit of (48) as $n \to \infty$, we obtain that $|z| = |d(u, Su)| \le 0$, a contradiction with (44). So $|z| = 0$. Hence $Su = u$. Similarly, we obtain $Tu = u$.

Now we show that S and T have unique common fixed point of S and T. To show this, assume that u^* is another common fixed point of S and T. Then

$$\begin{aligned} d(u, u^*) &= d(Su, Tu^*) \\ &\precsim \frac{a[d(u, Su)\, d(u, Tu^*) + d(u^*, Tu^*)\, d(u^*, Su)]}{d(u, Tu^*) + d(u^*, Su)} \\ &\prec d(u, u^*). \end{aligned} \tag{47}$$

This implies that $|d(x, x^*)| \le 0$, and then $u = u^*$ which proves the uniqueness of common fixed point in X. This completes the proof. □

Corollary 20. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \ge 1$ and let $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \precsim \frac{a[d(x, Tx)\, d(x, Ty) + d(y, Ty)\, d(y, Tx)]}{d(x, Ty) + d(y, Tx)}, \tag{48}$$

for all $x, y \in X$, where $sa \in [0, 1)$. Then T has a unique fixed point in X.

Corollary 21. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \ge 1$ and let $T : X \to X$ be a mapping satisfying*

$$\begin{aligned} &d(T^n x, T^n y) \\ &\precsim \frac{a[d(x, T^n x)\, d(x, T^n y) + d(y, T^n y)\, d(y, T^n x)]}{d(x, T^n y) + d(y, T^n x)}, \end{aligned} \tag{49}$$

for all $x, y \in X$, where $sa \in [0, 1)$ and $n \in \mathbb{N}$. Then T has a unique fixed point in X.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

On Generalized Difference Hahn Sequence Spaces

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We construct some generalized difference Hahn sequence spaces by mean of sequence of modulus functions. The topological properties and some inclusion relations of spaces $h_p(\mathscr{F}, u, \Delta^r)$ are investigated. Also we compute the dual of these spaces, and some matrix transformations are characterized.

1. Introduction and Preliminaries

By a sequence space, we understand a linear subspace of the space $w = \mathbb{C}^{\mathbb{N}}$ of all real or complex-valued sequences, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = 0, 1, 2, \ldots$. For $x = (x_k) \in w$, we write l_∞, c, and c_0 for the classical spaces of all bounded, convergent, and null sequences, respectively. Also by bs, cs, and l_p we denote the space of all bounded, convergent, and p-absolutely convergent series, which are Banach spaces with the following norms: $\|x\|_{bs} = \|x\|_{cs} = \sup_n |\sum_{k=1}^{n} x_k|$ and $\|x\|_{l_p} = (\sum_k |x_k|^p)^{1/p}$, respectively. Additionally, the spaces bv^p and $\int\lambda$ are defined by

$$bv^p = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} |x_k - x_{k-1}|^p < \infty \right\},$$
$$\int\lambda = \{ x = (x_k) \in w : (kx_k) \in \lambda \}. \tag{1}$$

A coordinate space (or a K-space) is a vector space of numerical sequences, where addition and scalar multiplication are defined pointwise. That is, a sequence space λ with a linear topology is called a K-space provided that each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A BK-space is a K-space, which is also a Banach space with continuous coordinate functionals $f_k(x) = x_k$, $(k = 1, 2, \ldots)$. A K-space λ is called an FK-space provided that λ is a complete linear metric space. An FK-space whose topology is normable is called a BK-space. If a normed sequence space λ contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_n b_n)\| = 0, \tag{2}$$

then (b_n) is called Schauder basis (or briefly basis) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum \alpha_k b_k$. An FK-space λ is said to have AK property, if $\phi \subset \lambda$ and $\{e^k\}$ is a basis for λ, where e^k is a sequence whose only nonzero term is 1 in kth place for each $k \in \mathbb{N}$ and $\phi = \text{span}\{e^k\}$, the set of all finitely nonzero sequences. If ϕ is dense in λ, then λ is called an AD-space, and thus AK implies AD.

The notion of difference sequence spaces was introduced by Kizmaz [1], who defined the sequence spaces as follows:

$$Z(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in Z \} \quad \text{for } Z = c, c_0, l_\infty, \tag{3}$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. The notion was further generalized by Et and Çolak [2] by introducing the spaces. Let r be a nonnegative integer; then,

$$Z(\Delta^r) = \{ x = (x_k) \in w : (\Delta^r x_k) \in Z \} \quad \text{for } Z = c, c_0, l_\infty, \tag{4}$$

where $\Delta^r x = (\Delta^r x_k) = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$. The generalized difference sequence has the following binomial representation:

$$\Delta^r x_k = \sum_{m=0}^{r} (-1)^m \binom{r}{m} x_{k+m}. \tag{5}$$

Later concept have been studied by Bektaş et al. [3] and Et and Esi [4]. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [5] who studied the spaces $l_\infty(\Delta_\nu)$, $c(\Delta_\nu)$, and $c_0(\Delta_\nu)$. Recently, Esi et al. [6] and Tripathy et al. [7] have introduced a new type of generalized difference operators and unified those as follows.

Let r, ν be nonnegative integers; then, for Z a given sequence space, we have

$$Z(\Delta_\nu^r) = \{x = (x_k) \in w : (\Delta_\nu^r x_k) \in Z\} \tag{6}$$

for $Z = c$, c_0 and l_∞, where $\Delta_\nu^r x = (\Delta_\nu^r x_k) = (\Delta_\nu^{r-1} x_k - \Delta_\nu^{r-1} x_{k+\nu})$ and $\Delta_\nu^0 x_k = x_k$ for all $k \in \mathbb{N}$.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$ for all $x \in X$,

(2) $p(-x) = p(x)$ for all $x \in X$,

(3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,

(4) (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [8], Theorem 10.4.2, pp. 183). For more details about sequence spaces (see [9, 10]) and the references therein.

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,

(2) $f(x + y) \leq f(x) + f(y)$, for all $x, y \geq 0$,

(3) f is increasing,

(4) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = x/(x + 1)$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus function $f(x)$ is unbounded. Subsequentially, modulus function has been discussed in ([11–14]) and references therein.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers, where $k, n \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$ for every sequence $x = (x_k) \in \lambda$. The sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k \quad \text{for each } n \in \mathbb{N}. \tag{7}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (7) converges for each $n \in \mathbb{N}$ and each $x \in \lambda$ and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to l if Ax converges to l which is called the A-limit of x.

The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\} \tag{8}$$

which is a sequence space (for several examples of matrix domains, see [15] p. 49–176). In [16], Başar and Altay have defined the sequence space bv_p which consists of all sequences such that Δ-transforms of them are in l_p, where Δ denotes the matrix $\Delta = (\delta_{nk})$ as follows:

$$\Delta = \delta_{nk} = \begin{cases} (-1)^{n-k}, & (n-1 \leq k \leq n); \\ 0, & (0 \leq k < n-1 \text{ or } k > n) \end{cases} \tag{9}$$

for all $k, n \in \mathbb{N}$. The space $[l(p)]_{A^u} = bv(u, p)$ has been studied by Başar et al. [17], where

$$A^u = a_{nk}^u = \begin{cases} (-1)^{n-k} u_k, & (n-1 \leq k \leq n); \\ 0, & (0 \leq k < n-1 \text{ or } k > n). \end{cases} \tag{10}$$

Hahn [18] introduced the BK-space h of all sequences $x = (x_k)$ such that

$$h = \left\{ x : \sum_{k=1}^{\infty} k \left| \Delta x_k \right| < \infty, \ \lim_{k \to \infty} x_k = 0 \right\}, \tag{11}$$

where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$. The following norm:

$$\|x\|_h = \sum_k k \left| \Delta x_k \right| + \sup_k \left| x_k \right| \tag{12}$$

was defined on the space h by Hahn [18] (and also [19]). Rao ([20], Proposition 2.1) defined a new norm on h as $\|x\| = \sum_k k|\Delta x_k|$. G. Goes and S. Goes [19] proved that the space h is a BK-space.

Hahn proved the following properties of the space h.

Lemma 1. *(i) h is a Banach space.*

(ii) $h \subset l_1 \cap \int c_0$.

(iii) $h^\beta = \sigma_\infty$.

In [19], G. Goes and S. Goes studied functional analytic properties of the BK-space $bv_0 \cap dl_1$. Additionally, G. Goes and S. Goes considered the arithmetic means of sequences in bv_0 and $bv_0 \cap dl_1$ and used an important fact which the sequence of arithmetic means $(n^{-1} \sum_{k=1}^{n} x_k)$ of $x \in bv_0$ is a quasiconvex null sequence. And also G. Goes and S. Goes proved that $h = l_1 \cap \int bv = l_1 \cap \int bv_0$.

Rao [20] studied some geometric properties of Hahn sequence space and gave the characterizations of some classes of matrix transformations. Balasubramanian and Pandiarani

[21] defined the new sequence space $h(F)$ called the Hahn sequence space of fuzzy numbers and proved that β and γ duals of $h(F)$ is the Cesàro space of the set of all fuzzy bounded sequences. Kirişci [22] compiled to studies on Hahn sequence space and defined a new Hahn sequence space by Cesàro mean in [23].

In [24], Kirişci introduce the sequence space h_p by

$$h_p = \left\{ x : \sum_{k=1}^{\infty} k|\Delta x_k|^p < \infty, \lim_{k\to\infty} x_k = 0 \right\}, \quad (1 < p < \infty), \tag{13}$$

where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$. If we take $p = 1$, $h_p = h$ which are called Hahn sequence spaces. We denote the collection of all finite subsets of $\mathbb{N}$ by F.

Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers, and $u = (u_k)$ be a sequence of strictly positive real numbers. In the present paper we defined the following sequence space:

$$h_p(\mathcal{F}, u, \Delta^r) = \left\{ x : \sum_{k=1}^{\infty} f_k\left[\left(k\left|u_k\Delta^r x_k\right|\right)^{p_k}\right] < \infty, \lim_{k\to\infty} x_k = 0 \right\}, \quad (1 < p < \infty), \tag{14}$$

where $\Delta^r x = (\Delta^r x_k) = (\Delta^{r-1}x_k - \Delta^{r-1}x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$. Define the sequence $y = (y_k)$, which will be frequently used, by M-transform of a sequence $x = (x_k)$; that is,

$$y_k = (Mx)_k = f_k u_k k\left(\Delta^{r-1}x_k - \Delta^{r-1}x_{k+1}\right), \tag{15}$$

where $M = (m_{nk})$ with

$$m_{nk} = \begin{cases} n, & (n = k); \\ -n, & (n+1 = k); \\ 0, & \text{other} \end{cases} \tag{16}$$

for all $k, n \in \mathbb{N}$.

If we take $u = (u_k) = 1$, $r = 1$ and $\mathcal{F} = f_k(x) = x$ for all $k \in \mathbb{N}$, then we get the sequence space h_p defined by [24] Kirişci. By taking $u = (u_k) = 1$, $r = 1$, $\mathcal{F} = f_k(x) = x$, and $p = (p_k) = 1$ for all $k \in \mathbb{N}$, we obtained a Hahn sequence space defined by Hanh [18].

The following inequality will be used throughout the paper.

Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \le \sup_k p_k = H$, and let $K = \max\{1, 2^{H-1}\}$. Then, for the factorable sequences (a_k) and (b_k) in the complex plane, we have

$$\left|a_k + b_k\right|^{p_k} \le K\left(\left|a_k\right|^{p_k} + \left|b_k\right|^{p_k}\right). \tag{17}$$

The main purpose of this paper is to study some difference Hahn sequence spaces by mean of sequence of modulus functions. We will study some topological and algebraic properties of the sequence spaces $h_p(\mathcal{F}, u, \Delta^r)$ in Section 2. In Section 3 we will determine the α-, β-, and γ-duals of the spaces $h_p(\mathcal{F}, u, \Delta^r)$. Finally, we also made an attempt to characterize some matrix transformations on the spaces $h_p(\mathcal{F}, u, \Delta^r)$.

2. Main Results

The purpose of this section is to study the properties like linearity, paranorm, and relevant inclusion relations in the spaces $h_p(\mathcal{F}, u, \Delta^r)$.

Theorem 2. *The sequence space $h_p(\mathcal{F}, u, \Delta^r)$ is a linear space over the complex field $\mathbb{C}$.*

Proof. Let $x = (x_k)$, $y = (y_k) \in h_p(\mathcal{F}, u, \Delta^r)$ and $\rho, \varrho \in \mathbb{C}$. Then their exist integers M_ρ and N_ϱ such that $|\rho| \le M_\rho$ and $|\varrho| \le N_\varrho$. By using the inequality (17) and the properties of modulus function, we have

$$\begin{aligned} &\sum_{k=1}^{\infty} f_k\left[\left(k\left|u_k\Delta^r\left(\rho x_k + \varrho y_k\right)\right|\right)^{p_k}\right] \\ &\le \sum_{k=1}^{\infty} f_k\left[\left(k\left|u_k\left(\rho\Delta^r x_k + \varrho\Delta^r y_k\right)\right|\right)^{p_k}\right] \\ &\le K\sum_{k=1}^{\infty} f_k\left[M_\rho\left(k\left|u_k\Delta^r x_k\right|\right)^{p_k}\right] \\ &\quad + K\sum_{k=1}^{\infty} f_k\left[N_\varrho\left(k\left|u_k\Delta^r y_k\right|\right)^{p_k}\right] \\ &\le KM_\rho^H\sum_{k=1}^{\infty} f_k\left[\left(k\left|u_k\Delta^r x_k\right|\right)^{p_k}\right] \\ &\quad + KN_\varrho^H\sum_{k=1}^{\infty} f_k\left[\left(k\left|u_k\Delta^r y_k\right|\right)^{p_k}\right] < \infty. \end{aligned} \tag{18}$$

Thus, $\rho x + \varrho y \in h_p(\mathcal{F}, u, \Delta^r)$. This proves that $h_p(\mathcal{F}, u, \Delta^r)$ is a linear space over the field of complex number $\mathbb{C}$. □

Theorem 3. *Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $u = (u_k)$ be any sequence of strictly positive real numbers. Then $h_p(\mathcal{F}, u, \Delta^r)$ is a paranormed space with the paranorm defined by*

$$g(x) = \sup_k\left\{f_k\left[\left(k\left|u_k\Delta^r x_k\right|\right)^{p_k}\right]\right\}^{1/M}, \tag{19}$$

where $H = \sup_k p_k < \infty$ and $M = \max(1, H)$.

Proof. Clearly $g(x) = g(-x)$ for all $x \in h_p(\mathscr{F}, u, \Delta^r)$. It is trivial that $u_k \Delta^r x_k = 0$ for $x = 0$. Since $p_k/M \leq 1$ using Minkowski's inequality, we have

$$\begin{aligned}
&\left\{f_k\left[\left(k\left|u_k\Delta^r\left(x_k+y_k\right)\right|\right)^{p_k}\right]\right\}^{1/M} \\
&\quad\leq \left\{f_k\left[\left(k\left|\left(u_k\Delta^r x_k+u_k\Delta^r y_k\right)\right|\right)^{p_k}\right]\right\}^{1/M} \\
&\quad\leq \left\{f_k\left[\left(k\left|u_k\Delta^r x_k\right|\right)^{p_k}\right]\right\}^{1/M}+\left\{f_k\left[\left(k\left|u_k\Delta^r y_k\right|\right)^{p_k}\right]\right\}^{1/M}.
\end{aligned} \tag{20}$$

Hence $g(x+y) \leq g(x)+g(y)$. Finally to check the continuity of scalar multiplication, let us take a complex number δ by definition, we have

$$g(\delta x) = \sup_k\left\{f_k\left[\left(k\left|u_k\Delta^r\delta x_k\right|\right)^{p_k}\right]\right\}^{1/M} \leq K_\delta^{H/M} g(x), \tag{21}$$

where K_δ is a positive integer such that $|\delta| \leq K_\delta$. Let $\delta \to 0$ for any fixed x with $g(x) = 0$. By definition for $|\delta| < 1$, we have

$$\sup_k\left\{f_k\left[\left(k\left|u_k\Delta^r x_k\right|\right)^{p_k}\right]\right\}^{1/M} < \epsilon \quad \text{for } n > N(\epsilon). \tag{22}$$

Also for $1 \leq n \leq N$, taking δ small enough, since f_k is continuous for each k, we have

$$\sup_k\left\{f_k\left[\left(k\left|u_k\Delta^r x_k\right|\right)^{p_k}\right]\right\}^{1/M} < \epsilon. \tag{23}$$

Equations (22) and (23) imply that $g(\delta x) \to 0$ as $\delta \to 0$. This completes the proof. □

Theorem 4. *Let $\mathscr{F} = (f_k)$ be a sequence of modulus functions and $\phi = \lim_{t\to\infty}(f_k(t)/t) > 0$. Then $h_p(\mathscr{F}, u, \Delta^r) \subset h_p(u, \Delta^r)$.*

Proof. Let $\phi > 0$. By definition of ϕ, we have $f_k(t) \geq \phi \cdot t$, for all $t \geq 0$. Since $\phi > 0$, we have $t \leq (1/\phi) f_k(t)$ for all $t \geq 0$. Let $x = (x_k) \in h_p(\mathscr{F}, u, \Delta^r)$. Thus, we have

$$\sum_{k=1}^{\infty}\left[\left(k\left|u_k\Delta^r x_k\right|\right)^{p_k}\right] \leq \frac{1}{\phi}\sum_{k=1}^{\infty} f_k\left[\left(k\left|u_k\Delta^r x_k\right|\right)^{p_k}\right] < \infty. \tag{24}$$

Which implies that $x = (x_k) \in h_p(u, \Delta^r)$. This completes the proof. □

Theorem 5. $h_p(\mathscr{F}, u, \Delta^r) = l_p(\mathscr{F}, u, \Delta^r) \cap \int bv^p(\mathscr{F}, u, \Delta^r) = l_p(\mathscr{F}, u, \Delta^r) \cap \int bv_0^p(\mathscr{F}, u, \Delta^r)$.

Proof. We consider

$$f_k u_k\left(k\Delta^r x_k\right) \leq f_k u_k\left(x_k\right) + f_k u_k\left(\Delta^r\left(kx_k\right)\right). \tag{25}$$

Then, for $x \in l_p(\mathscr{F}, u, \Delta^r) \cap \int bv^p(\mathscr{F}, u, \Delta^r)$,

$$\sum_{k=1}^{n} f_k u_k\left(k\left|\Delta^r x_k\right|\right) \leq \sum_{k=1}^{n} f_k u_k\left|x_k\right| + \sum_{k=1}^{n} f_k u_k\left|\Delta^r\left(kx_k\right)\right|, \tag{26}$$

and from $|a+b|^{p_k} \leq 2^{p_k}(|a|^{p_k} + |b|^{p_k})$, $(1 \leq p_k < \infty)$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n} f_k u_k\left[k^{p_k}\left|\Delta^r x_k\right|^{p_k}\right] \\
&\qquad\leq 2^{p_k} f_k u_k\left[\sum_{k=1}^{n}\left|x_k\right|^{p_k} + \sum_{k=1}^{n}\left|\Delta^r\left(kx_k\right)\right|^{p_k}\right].
\end{aligned} \tag{27}$$

For each positive integer s, we get

$$\begin{aligned}
&\sum_{k=1}^{s} f_k u_k\left[k^{p_k}\Delta^r x_k|^{p_k}\right] \\
&\qquad\leq 2^{p_k} f_k u_k\left[\sum_{k=1}^{s}\left|x_k\right|^{p_k} + \sum_{k=1}^{s}\left|\Delta^r kx_k\right|^{p_k}\right],
\end{aligned} \tag{28}$$

and as $s \to \infty$,

$$\begin{aligned}
&\sum_{k=1}^{\infty} f_k u_k\left[k^{p_k}\left|\Delta^r x_k\right|^{p_k}\right] \\
&\qquad\leq 2^{p_k} f_k u_k\left[\sum_{k=1}^{\infty}\left|x_k\right|^{p_k} + \sum_{k=1}^{\infty}\left|\Delta^r\left(kx_k\right)\right|^{p_k}\right]
\end{aligned} \tag{29}$$

and $\lim_{k\to\infty} x_k = 0$. Then $x \in h_p(\mathscr{F}, u, \Delta^r)$ and

$$l_p(\mathscr{F}, u, \Delta^r) \cap \int bv^p(\mathscr{F}, u, \Delta^r) \subset h_p(\mathscr{F}, u, \Delta^r). \tag{30}$$

Let $x \in h_p(\mathscr{F}, u, \Delta^r)$, and we consider

$$\begin{aligned}
&\sum_{k=1}^{\infty} f_k u_k\left|x_{k+1}\right|^{p_k} - \sum_{k=1}^{\infty} f_k u_k\left|\Delta^r\left(kx_k\right)\right|^{p_k} \\
&\qquad\leq \sum_{k=1}^{\infty} f_k u_k k^{p_k}\left|\Delta^r\left(kx_k\right)\right|^{p_k}.
\end{aligned} \tag{31}$$

Then the series $\sum_{k=1}^{\infty} f_k u_k |x_{k+1}|^{p_k}$ is convergent from the definition of l_p. Also, $\sum_{k=1}^{\infty} f_k u_k |\Delta^r(kx_k)|^{p_k} < \infty$, and therefore $x \in l_p(\mathscr{F}, u, \Delta^r) \cap \int bv^p(\mathscr{F}, u, \Delta^r)$.

Then,

$$h_p(\mathscr{F}, u, \Delta^r) \subset l_p(\mathscr{F}, u, \Delta^r) \cap \int bv^p(\mathscr{F}, u, \Delta^r). \tag{32}$$

From (30) and (32), we have

$$h_p(\mathscr{F}, u, \Delta^r) = l_p(\mathscr{F}, u, \Delta^r) \cap \int bv^p(\mathscr{F}, u, \Delta^r). \tag{33}$$

□

Theorem 6. *The sequence space $h_p(\mathscr{F}, u, \Delta^r)$ is a BK-space with AK.*

Proof. If x is any sequence, we write $\sigma_n(x) = M_n x$. Let $\epsilon > 0$ and $x \in h_p(\mathscr{F}, u, \Delta^r)$. Then, their exists N such that

$$\sigma_n(x) < \frac{\epsilon}{2} \tag{34}$$

for all $n \geq N$. Now let $m \geq N$ be given. Then, we have for all $n \geq m+1$ by (34)

$$\left|\sigma_n\left(x - x^{[m]}\right)\right| \leq \left[\sum_{k=m+1}^{\infty} f_k\left(\left|ku_k\Delta^r x_k\right|^{p_k}\right)\right]^{1/p_k} \leq \left|\sigma_n(x)\right| + \left|\sigma_m(x)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \tag{35}$$

whence $\|x - x^{[m]}\|_{h_p(\mathcal{F},u,\Delta^r)} \leq \epsilon$ for all $m > N$. This shows that $x = \lim_{m\to\infty} x^{[m]}$. □

Since $h_p(\mathcal{F}, u, \Delta^r)$ is an AK-space and every AK-space is AD, we can give the following corollary.

Corollary 7. *The sequence space $h_p(\mathcal{F}, u, \Delta^r)$ is AD.*

3. Duals of Hahn Sequence Space $h_p(\mathcal{F}, u, \Delta^r)$

In this section, we determining the α, β, and γ-duals of the sequence space $h_p(\mathcal{F}, u, \Delta^r)$. Let x and y be sequences, X and Y be subsets of w, and $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers. We write $xy = (x_k y_k)_{k=0}^{\infty}$, $x^{-1} * Y = \{a \in w : ax \in Y\}$, and $M(X,Y) = \cap_{x\in X} x^{-1} * Y = \{a \in w : ax \in Y \text{ for all } x \in X\}$ for the multiplier space of X and Y. In the special cases of $Y = \{l_1, cs, bs\}$, we write $x^{\alpha} = x^{-1} * l_1$, $x^{\beta} = x^{-1} * cs$, $x^{\gamma} = x^{-1} * bs$ and $X^{\alpha} = M(X, l_1)$, $X^{\beta} = M(X, cs)$, $X^{\gamma} = M(X, bs)$ for the α-dual, β-dual, and γ-dual of X. By $A_n = (a_{nk})_{k=0}^{\infty}$, we denote the sequence in the nth-row of A, and we write $A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k$ $n = (0, 1, \ldots)$ and $A(x) = (A_n(x))_{n=0}^{\infty}$, provided that $A_n \in x^{\beta}$ for all n.

Given an FK-space X containing ϕ, its conjugate is denoted by X' and its f-dual or sequential dual is denoted by X^f and is given by $X^f = \{\text{all sequences } (f(e^k)) : f \in X'\}$. Let λ be a sequence space. Then λ is called perfect if $\lambda = \lambda^{\alpha\alpha}$, normal if $y \in \lambda$ whenever $|y_k| \leq |x_k|$, $k \geq 1$ for some $x \in \lambda$, and monotone if λ contains the canonical preimages of all its stepspace.

Lemma 8. *(i) $A \in (h : l_1)$ if and only if*

$$\sum_{n=1}^{\infty} |a_{nk}| \text{ converges}, \quad (k = 1, 2, 3, \ldots), \qquad \sup_k \frac{1}{k}\sum_{n=1}^{\infty}\left|\sum_{v=1}^{k} a_{nv}\right| < \infty. \tag{36}$$

(ii) $A \in (l_p : l_1)$ if and only if

$$\sup_{K\in F}\sum_k \left|\sum_{n\in K} a_{nk}\right|^q < \infty. \tag{37}$$

Lemma 9. *(i) $A \in (h : C)$ if and only if*

$$\sup_{n,k}\frac{1}{k}\left|\sum_{v=1}^{k} a_{nv}\right| < \infty, \tag{38}$$

$$\lim_{n\to\infty} a_{nk} \text{ exists}, \quad (k = 1, 2, 3, \ldots). \tag{39}$$

(ii) $A \in (l_p : C)$ if and only if (39) *holds and*

$$\sup_n \sum_k |a_{nk}|^q < \infty \quad 1 < p \leq \infty. \tag{40}$$

Lemma 10. *(i) $A \in (h : l_\infty)$ if and only if* (38) *holds.*
(ii) $A \in (l_p : l_\infty)$ if and only if (40) *holds with $1 < p \leq \infty$.*

Lemma 11. *(i) $A \in (h : C_0)$ if and only if* (38) *holds and*

$$\lim_{n\to\infty} a_{nk} = 0. \tag{41}$$

Lemma 12. *(i) $A \in (h : h)$ if and only if* (41) *holds and*

$$\sum_{n=1}^{\infty} n\left|a_{nk} - a_{n+1,k}\right| \text{ converges}, \quad (k = 1, 2, 3, \ldots), \qquad \sup_k \frac{1}{k}\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}\left(a_{nv} - a_{n+1,v}\right)\right| < \infty. \tag{42}$$

Theorem 13. *We define the set*

$$d_1 = \left\{a = (a_k) \in w : \sup_{K\in F}\sum_k \left|\sum_{n\in K}\frac{1}{k} f_k\left(\left|u_k\Delta^r x_k\right|\right) a_n\right|^{p_k} < \infty\right\}. \tag{43}$$

Then, $[h_p(\mathcal{F}, u, \Delta^r)]^{\alpha} = d_1$.

Proof. Let us take any $a = (a_k) \in w$. We define the matrix $D = d_{nk}$ by

$$d_{nk} = \begin{cases} \dfrac{1}{k} f_k\left(\left|u_k\Delta^r\right|\right) a_n, & k \geq n; \\ 0, & k < n \end{cases} \tag{44}$$

for all, $k, n \in \mathbb{N}$.

Consider the equation

$$\frac{1}{k} f_k\left(\left|u_k\Delta^r\right|\right) a_n x_n = \sum_{j=n}^{\infty}\frac{1}{j} f_j\left(\left|u_j\Delta^r\right|\right) a_n y_j = (Dy)_n \quad (n \in \mathbb{N}). \tag{45}$$

It follows from (45) with Lemma 8(ii) that $ax = (a_n x_n) \in l_1$ whenever $x = (x_k) \in h_p(\mathcal{F}, u, \Delta^r)$ if and only if $Dy \in l_1$, whenever $y = (y_k) \in l_p$. This means that $a = (a_n) \in [h_p(\mathcal{F}, u, \Delta^r)]^{\alpha}$, whenever $x = (x_k) \in h_p(\mathcal{F}, u, \Delta^r)$ if and only if $D \in (h_p(\mathcal{F}, u, \Delta^r) : l_1)$. This gives the result that $[h_p(\mathcal{F}, u, \Delta^r)]^{\alpha} = d_1$. □

Theorem 14. *Let $1 < p < \infty$. Then $[h_p(\mathcal{F}, u, \Delta^r)]^{\beta} = d_2$, where*

$$d_2 = \left\{a = (a_k) \in w : \sup_{n\in\mathbb{N}}\left(n^{-1}\right)^q \times \sum_k f_k\left(\left|u_k\Delta^r x_k\right|\right)\left|\sum_{j=k}^{n} a_j\right|^q < \infty\right\}. \tag{46}$$

Proof. Consider the equation

$$\begin{aligned}
&\sum_{k=1}^{n}\frac{1}{k}f_k\left(\left|u_k\Delta^r\right|\right)a_kx_k\\
&\quad=\sum_{k=1}^{n}\frac{1}{k}f_k\left(\left|u_k\Delta^r\right|\right)a_k\left(\sum_{j=1}^{k}\frac{y_j}{j}\right)\\
&\quad=\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{1}{k}f_j\left(\left|u_j\Delta^r\right|\right)a_j\right)y_k\\
&\quad=\left(By\right)_n\quad(n\in\mathbb{N}),
\end{aligned}\tag{47}$$

where $B=(b_{nk})$ are defined by

$$b_{nk}\begin{cases}\displaystyle\sum_{j=1}^{k}\frac{1}{k}f_j\left(\left|u_j\Delta^r\right|\right)a_j, & k\geq n;\\ 0, & k<n\end{cases}\tag{48}$$

for all, $k,n\in\mathbb{N}$. Thus, we deduce from Lemma 9(ii) with (47) that $ax=(a_kx_k)\in cs$ whenever $x=(x_k)\in h_p(\mathcal{F},u,\Delta^r)$ if and only if $By\in c$ whenever $y=(y_k)\in l_p$. Thus $(a_k)\in cs$ and $(a_k)ind_2$ by (39) and (40), respectively. Nevertheless, the inclusion $d_2\subset cs$ holds, and thus, we have $(a_k)\in d_2$, whence $[h_p(\mathcal{F},u,\Delta^r)]^\beta=d_2$. □

Lemma 15. *Let X be FK-space with $X\supset\varphi$. Then,*

(i) $X^\beta\subset X^\gamma\subset X^f$;

(ii) *If X has AK, $X^\beta=X^f$;*

(iii) *If X has AD, $X^\beta=X^\gamma$.*

From Theorem 6, Corollary 7, and Lemma 15, we can write the following corollary.

Corollary 16. *(i)* $[h_p(\mathcal{F},u,\Delta^r)]^\beta=[h_p(\mathcal{F},u,\Delta^r)]^f$;
(ii) $[h_p(\mathcal{F},u,\Delta^r)]^\beta=[h_p(\mathcal{F},u,\Delta^r)]^\gamma$.

Lemma 17. *Let λ be a sequence space. Then, the following assertions are true:*

(i) *λ is perfect $\Rightarrow$ λ is normal $\Rightarrow$ λ is monotone;*

(ii) *λ is normal $\Rightarrow$ $\lambda^\alpha=\lambda^\gamma$;*

(iii) *λ is monotone $\Rightarrow$ $\lambda^\alpha=\lambda^\beta$.*

Combining Theorem 13, Theorem 14, and Lemma 17, we can give the following corollary.

Corollary 18. *The space $h_p(\mathcal{F},u,\Delta^r)$ is not monotone and so it is neither normal nor perfect.*

4. Matrix Transformations

In this section we characterize some matrix transformations on the space $h_p(\mathcal{F},u,\Delta^r)$.

Lemma 19. *Let λ, μ be any two sequence spaces, A be an infinite matrix, and U be a triangle matrix. Then, $A\in(\lambda:\mu_U)$ if and only if $UA\in(\lambda:U)$.*

If we define $\widetilde{a}_{nk}=n(a_{nk}-a_{n+1,k})$, then we can give the following corollary from Lemma 19 with $U=M$ defined by (16).

Corollary 20. *(i) $A\in(l_1:h)$ if and only if*

$$\sup_k\sum_n\left|\widetilde{a}_{nk}\right|<\infty.\tag{49}$$

(ii) $A\in(c:h)=(c_0:h)=(l_\infty:h)$ if and only if

$$\sup_{K\in F}\sum_n\left|\sup_{k\in K}\widetilde{a}_{nk}\right|<\infty.\tag{50}$$

Theorem 21. *Suppose that the entries of the infinite matrices $A=(a_{nk})$ and $E=(e_{nk})$ are connected with the relation*

$$e_{nk}=\overline{a}_{nk}\tag{51}$$

for all $k,n\in\mathbb{N}$, where $\overline{a}_{nk}=\sum_{j=k}^{\infty}(a_{nj}/j)[f_j(|u_j\Delta^rx_j|)]$ and μ is any sequence space. Then $A\in(h_p(\mathcal{F},u,\Delta^r):\mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}}\in[h_p(\mathcal{F},u,\Delta^r)]^\beta$, for all $n\in\mathbb{N}$ and $E\in(h:\mu)$.

Proof. Let μ be any given sequence spaces. Suppose that (51) holds between $A=(a_{nk})$ and $E=(e_{nk})$, and take into account that the spaces $h_p(\mathcal{F},u,\Delta^r)$ and h are norm isomorphic. Let $A\in(h_p(\mathcal{F},u,\Delta^r):\mu)$ and take any $y=y_k\in h$. Then, EM exists and $\{a_{nk}\}_{k\in\mathbb{K}}\in[h_p(\mathcal{F},u,\Delta^r)]^\beta$, which yields that $\{e_{nk}\}_{k\in\mathbb{N}}$ for each $n\in\mathbb{N}$. Hence, Ey exists, and thus,

$$\sum_k f_k\left(\left|u_k\Delta^r\right|\right)e_{nk}y_k=\sum_k f_k\left(\left|u_k\Delta^r\right|\right)a_{nk}x_k\tag{52}$$

for all $n\in\mathbb{N}$. We have $Ey=Ax$ which leads us to the consequence $E\in(h:\mu)$. Conversely, let $\{a_{nk}\}_{k\in\mathbb{K}}\in d_1$ for all $n\in\mathbb{N}$, and $E\in(h:\mu)$ hold, and take any $x=x_k\in h_p(\mathcal{F},u,\Delta^r)$. Then, Ax exists. Therefore, we obtain from the equality that

$$\sum_k f_k\left(\left|u_k\Delta^r\right|\right)a_{nk}x_k=\left[\sum_{j=k}^{\infty}\frac{a_{nj}}{j}\right]\sum_k f_k\left(\left|u_k\Delta^r\right|\right)y_k\tag{53}$$

for all $n\in\mathbb{N}$. Thus, $Ax=Ey$ and this shows that $A\in(h_p(\mathcal{F},u,\Delta^r):\mu)$. □

If we use the Corollary 20 and change the roles of the spaces $h_p(\mathcal{F},u,\Delta^r)$ with μ in Theorem 21, we can give the following theorem.

Theorem 22. *Suppose that the entries of the infinite matrices $A=(a_{nk})$ and $\widetilde{A}=(\widetilde{a}_{nk})$ are connected with the relation $\widetilde{a}_{nk}=nf_k(|u_k\Delta^rx_k|)(a_{nk}-a_{n+1,k})$ for all $k,n\in\mathbb{N}$ and μ is any sequence space. Then $A\in(\mu:h_p(\mathcal{F},u,\Delta^r))$ if and only if $\widetilde{A}\in(\mu:h)$.*

Proof. Let $z = z_k \in \mu$ and consider the following equality:

$$\sum_{k=0}^{m} \widetilde{a}_{nk} z_k = \sum_{k=0}^{m} n f_k \left(\left|u_k \Delta^r\right|\right) \left(a_{nk} - a_{n+1,k}\right) z_k, \quad \forall m, n \in \mathbb{N} \tag{54}$$

which yields that as $m \to \infty$ $(\widetilde{A}z)_n = (M(Az))_n$ for all $n \in \mathbb{N}$. Therefore, one can observe from here that $Az \in h_p(\mathcal{F}, u, \Delta^r)$ whenever $z \in \mu$ if and only if $\widetilde{A}z \in h$ whenever $z \in \mu$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Fixed Point Theorems for Generalized α-β-Weakly Contraction Mappings in Metric Spaces and Applications

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We extend the notion of generalized weakly contraction mappings due to Choudhury et al. (2011) to generalized α-β-weakly contraction mappings. We show with examples that our new class of mappings is a real generalization of several known classes of mappings. We also establish fixed point results for such mappings in metric spaces. Applying our new results, we obtain fixed point results on ordinary metric spaces, metric spaces endowed with an arbitrary binary relation, and metric spaces endowed with graph.

1. Introduction

The well-known Banach's contraction principle has been generalized in many ways over the years [1–6]. One of the most interesting studies is the extension of Banach's contraction principle to a case of weakly contraction mappings which was first given by Alber and Guerre-Delabriere [7] in Hilbert spaces. In 2001, Rhoades [8] has shown that the result of Alber and Guerre-Delabriere [7] is also valid in complete metric spaces. Fixed point problems involving weak contractions and mappings satisfying weak contractive type inequalities have been considered in [9–13] and references therein.

On the other hand, the concept of the altering distance function was introduced by Khan et al. [14]. In 2011, Choudhury et al. [15] generalized weakly contraction mappings by using an altering distance control function and proved fixed point theorem for a pair of these mappings. Some generalizations of this function of fixed point problems in metric and probabilistic metric spaces have been studied [16–18].

Recently, Samet et al. [19] introduced the concepts of α-ψ-contraction mappings and α-admissible mappings and established various fixed point theorems for such mappings in complete metric spaces. Afterwards, many fixed point results via the concepts of α-admissible mappings occupied a prominent place in many aspects (see [20–25] and references therein).

From the mentioned above, we introduce the concept of generalized α-β-weakly contraction mappings and give some examples to show the real generality of these mappings. We also obtain fixed point results for such mappings. Our result improves and complements several results in the literatures. As an application of our results, fixed point results on ordinary metric spaces, metric spaces endowed with an arbitrary binary relation, and metric spaces endowed with graph are also derived from our results.

2. Preliminaries

In this section, we give some notations and basic knowledge. Throughout this paper, $\mathbb{N}$ denotes the set of positive integers.

Definition 1 (see [14]). A function $\psi : [0,\infty) \to [0,\infty)$ is called an *altering distance function* if the following properties are satisfied:

(i) ψ is monotone increasing and continuous;

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2 (see [26]). Let (X, d) be a metric space and let T be a self-mapping on X. A mapping T is said to be *contraction* if, for each $x, y \in X$, one has

$$d(Tx, Ty) \le kd(x, y), \tag{1}$$

where $k \in [0, 1)$.

Definition 3 (see [8]). Let (X, d) be a metric space and let T be a self-mapping on X. A mapping T is said to be *weak contraction* if, for each $x, y \in X$, one has

$$d(Tx, Ty) \le d(x, y) - \phi(d(x, y)), \tag{2}$$

where $\phi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

In fact, if we take $\phi(t) = (1 - k)t$ for all $t \ge 0$, where $0 \le k < 1$, then the condition (2) becomes (1).

In 2011, Choudhury et al. [15] introduced the concept of a generalized weakly contractive condition as follows.

Definition 4 (see [15]). Let (X, d) be a metric space and let T be a self-mapping on X. A mapping T is said to be a *generalized weakly contraction*, if, for each $x, y \in X$, one has

$$\begin{aligned}\psi(d(Tx, Ty)) \le{}& \psi(m(x, y)) \\ &- \phi(\max\{d(x, y), d(y, Ty)\}),\end{aligned} \tag{3}$$

where

$$\begin{aligned}m(x, y) = \max\Big\{&d(x, y), d(x, Tx), d(y, Ty), \\ &\frac{1}{2}(d(x, Ty) + d(y, Tx))\Big\},\end{aligned} \tag{4}$$

$\psi : [0,\infty) \to [0,\infty)$ is altering distance function, and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$.

Remark 5. It is easy to see that a generalized weakly contractive condition (3) is more general than several generalized contractive conditions. The following conditions are an example of a special case of a generalized weakly contractive condition (3):

(i) $d(Tx, Ty) \le km(x, y)$ for all $x, y \in X$, where $k \in [0, 1)$;

(ii) $d(Tx, Ty) < m(x, y)$ for all $x, y \in X$;

(iii) $d(Tx, Ty) \le m(x, y) - (1 - k)\max\{d(x, y), d(y, Ty)\}$ for all $x, y \in X$, where $k \in [0, 1)$.

Moreover, the contractive condition (1) is also a special case of condition (3).

Definition 6 (see [19]). Let X be a nonempty set and let $\alpha : X \times X \to [0,\infty)$ be a mapping. A self-mapping $T : X \to X$ is said to be *α-admissible* if the following condition holds:

$$x, y \in X, \quad \alpha(x, y) \ge 1 \Longrightarrow \alpha(Tx, Ty) \ge 1. \tag{5}$$

Example 7 (see [19]). Let $X = [0,\infty)$ and define $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by

$$\begin{aligned}&Tx = \sqrt{x}, \quad \forall x \in X, \\ &\alpha(x, y) = \begin{cases} e^{x-y}; & x \ge y, \\ 0; & x < y. \end{cases}\end{aligned} \tag{6}$$

Then T is α-admissible.

3. Main Results

In this section, we introduce the concept of generalized α-β-weakly contraction mappings and prove the fixed point theorems for such mappings.

Definition 8. Let (X, d) be a metric space, $\alpha, \beta : X \times X \to [0,\infty)$ two given mappings, and T a self-mapping on X. A mapping T is said to be a *generalized α-β-weakly contraction type A* if, for each $x, y \in X$, one has

$$\begin{aligned}\psi(d(Tx, Ty)) \le{}& \beta(x, y)\psi(m(x, y)) \\ &- \alpha(x, y)\phi(\max\{d(x, y), d(y, Ty)\}),\end{aligned} \tag{7}$$

where

$$\begin{aligned}m(x, y) = \max\Big\{&d(x, y), d(x, Tx), d(y, Ty), \\ &\frac{1}{2}(d(x, Ty) + d(y, Tx))\Big\},\end{aligned} \tag{8}$$

$\psi : [0,\infty) \to [0,\infty)$ is altering distance function, and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$.

Definition 9. Let (X, d) be a metric space, $\alpha, \beta : X \times X \to [0,\infty)$ two given mappings, and T a self-mapping on X. A mapping T is said to be a *generalized α-β-weakly contraction type B* if, for each $x, y \in X$, one has

$$\begin{aligned}\alpha(x, y)\psi(d(Tx, Ty)) \le{}& \beta(x, y)\psi(m(x, y)) \\ &- \phi(\max\{d(x, y), d(y, Ty)\}),\end{aligned} \tag{9}$$

where

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\left(d(x,Ty) + d(y,Tx)\right)\right\}, \tag{10}$$

ψ is altering distance function, and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$.

If we take $\alpha(x,y) = \beta(x,y) = 1$ for all $x, y \in X$, then generalized α-β-weakly contraction mappings type A and type B become generalized weakly contraction mappings due to Choudhury et al. [15]. Therefore, classes of generalized α-β-weakly contraction mappings type A and type B are larger than the class of generalized weakly contraction mappings. Next, we give some examples to show the real generality of classes of generalized α-β-weakly contraction mappings.

Example 10. Let $X = [0,1] \cup \{2,3,4,\ldots\}$. From [27], X is a complete metric space with metric defined by

$$d(x,y) = \begin{cases} |x-y|; & x, y \in [0,1], \\ x+y; & \text{one of } x, y \notin [0,1],\ x \neq y, \\ 0; & x = y. \end{cases} \tag{11}$$

Let a mapping $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{x}{2}; & x \in [0,1], \\ x-1; & x \in \{2,3,4,\ldots,10\}, \\ x^2; & x \in \{11,12,13,\ldots\}. \end{cases} \tag{12}$$

First, we show that T is generalized α-β-weakly contractive type A with the functions $\psi, \phi : [0,\infty) \to [0,\infty)$ and $\alpha, \beta : X \times X \to [0,\infty)$ defined by

$$\psi(t) = \begin{cases} t; & 0 \le t \le 1, \\ t^2; & t > 1, \end{cases}$$

$$\phi(t) = \begin{cases} \frac{t}{2}; & 0 \le t \le 1, \\ \frac{1}{2}; & t > 1, \end{cases}$$

$$\alpha(x,y) = \begin{cases} 1; & x, y \in [0,1], \\ 2; & \text{one of } x, y \notin [0,1],\ x, y \le 10, \\ 0; & \text{otherwise}, \end{cases}$$

$$\beta(x,y) = \begin{cases} 1; & x, y \in [0,1] \cup \{2,3,4,\ldots,10\}, \\ \psi(d(Tx,Ty)); & \text{otherwise}. \end{cases} \tag{13}$$

Next we show that T is a generalized α-β-weakly contraction mapping type A. For $x, y \in X$, we distinguish the following cases.

Case 1 ($x, y \in [0,1]$). Without loss of generality, we may assume that $x \ge y$. Now we obtain that

$$\begin{aligned} m(x,y) &= \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\left(d(x,Ty) + d(y,Tx)\right)\right\} \\ &= \max\left\{x-y, \frac{x}{2}, \frac{y}{2}, \frac{1}{2}\left(x - \frac{y}{2} + \left|y - \frac{x}{2}\right|\right)\right\} \\ &= \begin{cases} x-y; & 0 \le y \le \frac{x}{2}, \\ \frac{x}{2}; & \frac{x}{2} < y \le x. \end{cases} \end{aligned} \tag{14}$$

In case of $0 \le y \le x/2$, we have $\phi(\max\{d(x,y), d(y,Ty)\}) = (1/2)(x-y)$ and then

$$\begin{aligned} \psi(d(Tx,Ty)) &= \frac{1}{2}(x-y) \\ &= (x-y) - \frac{1}{2}(x-y) \\ &= (x-y) - \phi(\max\{d(x,y), d(y,Ty)\}) \\ &\le \beta(x,y)\psi(m(x,y)) \\ &\quad - \alpha(x,y)\phi(\max\{d(x,y), d(y,Ty)\}). \end{aligned} \tag{15}$$

In case of $x/2 < y \le x$, we have $\max\{d(x,y), d(y,Ty)\} = \max\{x-y, y/2\}$.

If $\max\{d(x,y), d(y,Ty)\} = y/2$, then we have

$$\begin{aligned} \psi(d(Tx,Ty)) &= \frac{1}{2}(x-y) \\ &\le \frac{x}{2} - \frac{y}{4} \\ &= \beta(x,y)\psi(m(x,y)) \\ &\quad - \alpha(x,y)\phi(\max\{d(x,y), d(y,Ty)\}). \end{aligned} \tag{16}$$

If $\max\{d(x,y), d(y,Ty)\} = x-y$, then we have

$$\begin{aligned} \psi(d(Tx,Ty)) &= \frac{1}{2}(x-y) < \frac{x}{2} - \frac{x}{4} \le \frac{x}{4} < \frac{y}{2} \\ &= \frac{x}{2} - \frac{1}{2}(x-y) \\ &= \beta(x,y)\psi(m(x,y)) \\ &\quad - \alpha(x,y)\phi(\max\{d(x,y), d(y,Ty)\}). \end{aligned} \tag{17}$$

Therefore, for $x, y \in [0,1]$, we get that T satisfies condition (7).

Case 2 ($x \in \{2,3,4,\ldots,10\}$ and $y \in [0,1]$). In this case, we obtain that

$$-x^2 + xy < 0, \quad -x^2 + \left(\frac{y^2}{4} - y\right) < 0, \quad -x^2 + 1 < 0, \tag{18}$$

$$\begin{aligned} m(x,y) &= \max\left\{d(x,y), d(x,Tx), d(y,Ty), \right. \\ &\qquad \left. \frac{1}{2}\left(d(x,Ty) + d(y,Tx)\right)\right\} \\ &= \max\left\{x+y, 2x-1, \frac{3y}{2}, \frac{1}{2}\left(x + \frac{y}{2} + y + x - 1\right)\right\} \\ &= \max\left\{x+y, 2x-1, \frac{3y}{2}, \frac{1}{2}\left(2x + \frac{3y}{2} - 1\right)\right\} \\ &= 2x - 1. \end{aligned} \tag{19}$$

From (18), we obtain that

$$\begin{aligned} \psi(d(Tx,Ty)) &= \psi\left(x - 1 + \frac{y}{2}\right) \\ &= \left(x + \frac{y}{2} - 1\right)^2 \\ &= x^2 + xy + \frac{y^2}{4} - 2x - y + 1 \\ &= x^2 + xy + \frac{y^2}{4} - 2x - y + 1 + 3x^2 - 3x^2 \\ &= 4x^2 + \left(-x^2 + xy\right) + \left(-x^2 + \frac{y^2}{4} - y\right) \\ &\quad + \left(-x^2 + 1\right) - 2x \\ &\le 4x^2 - 2x \\ &= 4x^2 - 2x + 1 - 1 \\ &= (2x-1)^2 - (2)\left(\frac{1}{2}\right) \\ &= \beta(x,y)\,\psi(m(x,y)) \\ &\quad - \alpha(x,y)\,\phi\left(\max\left\{d(x,y), d(y,Ty)\right\}\right). \end{aligned} \tag{20}$$

Therefore, we conclude that T satisfies condition (7) in this case.

Case 3 ($x \in [0,1]$ and $y \in \{2,3,4,\ldots,10\}$). This case is similar to Case 2.

Case 4 ($x \in \{2,3,4,\ldots,10\}$ and $y \in \{2,3,4,\ldots,10\}$). Now we obtain that

$$\begin{aligned} m(x,y) &= \max\left\{d(x,y), d(x,Tx), d(y,Ty), \right. \\ &\qquad \left. \frac{1}{2}\left(d(x,Ty) + d(y,Tx)\right)\right\} \\ &= \begin{cases} \max\left\{0, 2x-1, 2y-1, \right. \\ \qquad \left. \frac{1}{2}\left(x + y - 1 + y + x - 1\right)\right\}; & x = y, \\ \max\left\{x+y, 2x-1, 2y-1, \right. \\ \qquad \left. \frac{1}{2}\left(x + y - 1 + y + x - 1\right)\right\}; & x \ne y. \end{cases} \\ &= \begin{cases} 2x - 1; & x = y, \\ \max\{x+y, 2x-1, 2y-1, \\ \qquad x + y - 1\}; & x \ne y. \end{cases} \\ &= \begin{cases} 2x - 1; & y \le x, \\ 2y - 1; & y > x. \end{cases} \end{aligned} \tag{21}$$

If $x = y$, we have $\phi(\max\{d(x,y), d(y,Ty)\}) = 2x - 1$ and so

$$\begin{aligned} \psi(d(Tx,Ty)) &= \psi(0) \\ &= 0 \\ &\le (2x-1)^2 - 2\left(\frac{1}{2}\right) \\ &= \psi(2x-1) - \alpha(x,y)\,\phi(2x-1) \\ &= \beta(x,y)\,\psi(m(x,y)) \\ &\quad - \alpha(x,y)\,\phi\left(\max\left\{d(x,y), d(y,Ty)\right\}\right). \end{aligned} \tag{22}$$

If $y < x$, we have $\phi(\max\{d(x,y), d(y,Ty)\}) = 2x - 1$, and hence

$$\begin{aligned} \psi(d(Tx,Ty)) &= \psi(x + y - 2) \\ &= (x + y - 2)^2 \\ &< (2x-2)^2 \\ &= (2x-1)^2 + (3 - 4x) \end{aligned}$$

$$
\begin{aligned}
&< (2x-1)^2 - 1 \\
&= (2x-1)^2 - 2\left(\frac{1}{2}\right) \\
&= \psi(2x-1) - \alpha(x,y)\,\phi(2x-1) \\
&= \beta(x,y)\,\psi(m(x,y)) \\
&\quad - \alpha(x,y)\,\phi(\max\{d(x,y), d(y,Ty)\}).
\end{aligned} \tag{23}
$$

If $y > x$, we have $\phi(\max\{d(x,y), d(y,Ty)\}) = 2y - 1$, and hence

$$
\begin{aligned}
\psi(d(Tx,Ty)) &= \psi(x+y-2) \\
&= (x+y-2)^2 \\
&< (2y-2)^2 \\
&= (2y-1)^2 + (3-4y) \\
&< (2y-1)^2 - 1 \\
&= (2y-1)^2 - 2\left(\frac{1}{2}\right) \\
&= \psi(2y-1) - \alpha(x,y)\,\phi(2y-1) \\
&= \beta(x,y)\,\psi(m(x,y)) \\
&\quad - \alpha(x,y)\,\phi(\max\{d(x,y), d(y,Ty)\}).
\end{aligned} \tag{24}
$$

Now we conclude that T satisfies condition (7) in this case.

Case 5 (one of $x, y \notin [0,1] \cup \{2,3,4,\ldots,10\}$). If $x = y$, we obtain that

$$
\begin{aligned}
\psi(d(Tx,Ty)) &= \psi(0) \\
&= 0 \\
&= \psi(d(Tx,Ty))\,\psi(m(x,y)) \\
&\quad - \alpha(x,y)\,\phi(\max\{d(x,y), d(y,Ty)\}) \\
&= \beta(x,y)\,\psi(m(x,y)) \\
&\quad - \alpha(x,y)\,\phi(\max\{d(x,y), d(y,Ty)\}).
\end{aligned} \tag{25}
$$

If $x \neq y$, we get $\psi(m(x,y)) \geq 1$ and so

$$
\begin{aligned}
\psi(d(Tx,Ty)) &\leq \psi(d(Tx,Ty))\,\psi(m(x,y)) \\
&= \beta(x,y)\,\psi(m(x,y)) \\
&\quad - \alpha(x,y)\,\phi(\max\{d(x,y), d(y,Ty)\}).
\end{aligned} \tag{26}
$$

Now we conclude that T satisfies condition (7) in this case.

From all cases, we get that T is generalized α-β-weakly contraction mapping type A.

Remark 11. From Example 10, we can see that T is not a generalized weakly contraction mapping. Indeed, putting $x = 11$ and $y = 12$, we get

$$
\begin{aligned}
m(11,12) = \max\Big\{&d(11,12), d(11,T(11)), d(12,T(12)), \\
&\frac{1}{2}(d(11,T(12)) + d(12,T(11)))\Big\} = 156,
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
\psi(d(T(11),T(12))) &= \psi(121+144) \\
&= 165^2 \\
&> 156^2 \\
&> 156^2 - \frac{1}{2} \\
&= \psi(m(11,12)) \\
&\quad - \phi(\max\{d(11,12), \\
&\qquad d(11,T(12))\}).
\end{aligned} \tag{28}
$$

Before presenting the main results in this paper, we introduce the following concept, which will be used in our results.

Definition 12. Let X be a nonempty set and $\beta : X \times X \to [0,\infty)$. A self-mapping $T : X \to X$ is said to be β_0-*subadmissible* if the following condition holds:

$$
x, y \in X, \quad 0 < \beta(x,y) \leq 1 \Longrightarrow 0 < \beta(Tx,Ty) \leq 1. \tag{29}
$$

Definition 13. Let X be a nonempty set and $\alpha : X \times X \to [0,\infty)$ a mapping.

(i) α is said to be *forward transitive* if for each $x, y, z \in X$ for which $\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1$ one has $\alpha(x,z) \geq 1$;

(ii) α is said to be 0-*backward transitive* if for each $x, y, z \in X$ for which $0 < \alpha(x,y) \leq 1$ and $0 < \alpha(y,z) \leq 1$ one has $0 < \alpha(x,z) \leq 1$.

3.1. Generalized α-β-Weakly Contraction Mappings TypeA. In this subsection, we give the fixed point results for generalized α-β-weakly contraction mappings type A.

Theorem 14. *Let (X,d) be a complete metric space, $\alpha, \beta : X \times X \to [0,\infty)$ two given mappings, and $T : X \to X$ a generalized α-β-weakly contraction type A; then the following conditions hold:*

(a) *T is continuous;*

(b) *T is α-admissible and β_0-subadmissible;*

(c) α *is forward transitive and* β *is* 0*-backward transitive;*

(d) *there exists* $x_0 \in X$ *such that* $0 < \beta(x_0, Tx_0) \leq 1 \leq \alpha(x_0, Tx_0)$.

Then T *has a fixed point in* X.

Proof. Starting from $x_0 \in X$ in assumption (d) and letting $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$, if there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T. This finishes the proof. Therefore, we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since $0 < \beta(x_0, Tx_0) \leq 1 \leq \alpha(x_0, Tx_0)$, we get

$$0 < \beta(x_0, x_1) \leq 1 \leq \alpha(x_0, x_1). \tag{30}$$

It follows from T is α-admissible and β_0-subadmissible that

$$0 < \beta(Tx_0, Tx_1) \leq 1 \leq \alpha(Tx_0, Tx_1), \tag{31}$$

and then

$$0 < \beta(x_1, x_2) \leq 1 \leq \alpha(x_1, x_2). \tag{32}$$

By repeating this process, we get that $\{x_n\}$ is a sequence in X such that $x_{n+1} = Tx_n$ and

$$0 < \beta(x_n, x_{n+1}) \leq 1 \leq \alpha(x_n, x_{n+1}), \tag{33}$$

for all $n \in \mathbb{N} \cup \{0\}$. By using the generalized α-β-weakly contractive condition type A of T, we have

$$\begin{aligned}
&\psi(d(x_{n+1}, x_{n+2})) \\
&\quad = \psi(d(Tx_n, Tx_{n+1})) \\
&\quad \leq \beta(x_n, x_{n+1})\, \psi(m(x_n, x_{n+1})) - \alpha(x_n, x_{n+1}) \\
&\qquad \times \phi(\max\{d(x_n, x_n), d(x_{n+1}, Tx_{n+1})\}) \\
&\quad \leq \psi(m(x_n, x_{n+1})) \\
&\qquad - \phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}),
\end{aligned} \tag{34}$$

for all $n \in \mathbb{N} \cup \{0\}$. Now we obtain that

$$\begin{aligned}
&m(x_n, x_{n+1}) \\
&\quad = \max\left\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \right. \\
&\qquad \left. \frac{1}{2}(d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n))\right\} \\
&\quad = \max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \right. \\
&\qquad \left. \frac{1}{2}(d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1}))\right\} \\
&\quad = \max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \right. \\
&\qquad \left. \frac{1}{2}(d(x_n, x_{n+2}))\right\} \\
&\quad \leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\},
\end{aligned} \tag{35}$$

for all $n \in \mathbb{N} \cup \{0\}$. From (34) and (35), we get

$$\begin{aligned}
&\psi(d(x_{n+1}, x_{n+2})) \\
&\quad \leq \psi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) \\
&\qquad - \phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}),
\end{aligned} \tag{36}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Suppose that $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N} \cup \{0\}$. Then we have

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &\leq \psi(d(x_{n+1}, x_{n+2})) \\
&\quad - \phi(d(x_{n+1}, x_{n+2})) \\
&< \psi(d(x_{n+1}, x_{n+2}))
\end{aligned} \tag{37}$$

which is a contradiction. Hence $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. This means that $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence. Since $\{d(x_n, x_{n+1})\}$ is bounded below, there exists $r \geq 0$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = r. \tag{38}$$

Using (36), we get

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &\leq \psi(d(x_n, x_{n+1})) \\
&\quad - \phi(d(x_n, x_{n+1})),
\end{aligned} \tag{39}$$

for all $n \in \mathbb{N} \cup \{0\}$. Taking $n \to \infty$ in the above inequality, we have

$$\psi(r) \leq \psi(r) - \phi(r). \tag{40}$$

This implies that $r = 0$; that is,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{41}$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \tag{42}$$

for all $n_k > m_k \geq k$, where $k \in \mathbb{N}$. Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k \geq k$ satisfying (42). Then we have

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon, \qquad d(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{43}$$

By using (43) and triangular inequality, we get

$$\begin{aligned}
\varepsilon &\leq d(x_{m_k}, x_{n_k}) \\
&\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\
&\leq \varepsilon + d(x_{n_k-1}, x_{n_k}).
\end{aligned} \tag{44}$$

From (41) and (44), we have

$$\lim_{k \to \infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \tag{45}$$

From the triangular inequality, we get

$$\begin{aligned} d\left(x_{m_k}, x_{n_k}\right) &\leq d\left(x_{m_k}, x_{m_k+1}\right) + d\left(x_{m_k+1}, x_{n_k+1}\right) \\ &\quad + d\left(x_{n_k+1}, x_{n_k}\right) \\ &= d\left(x_{m_k+1}, x_{n_k+1}\right) + d\left(x_{m_k}, x_{m_k+1}\right) \\ &\quad + d\left(x_{n_k+1}, x_{n_k}\right) \\ &\leq d\left(x_{m_k+1}, x_{m_k}\right) + d\left(x_{m_k}, x_{n_k}\right) \\ &\quad + d\left(x_{n_k}, x_{n_k+1}\right) \\ &\quad + d\left(x_{m_k}, x_{m_k+1}\right) + d\left(x_{n_k+1}, x_{n_k}\right) \\ &\leq 2d\left(x_{m_k+1}, x_{m_k}\right) + d\left(x_{m_k}, x_{n_k}\right) \\ &\quad + 2d\left(x_{n_k}, x_{n_k+1}\right). \end{aligned} \tag{46}$$

Using (45) and (46), we get

$$\lim_{k\to\infty} d\left(x_{m_k+1}, x_{n_k+1}\right) = \varepsilon. \tag{47}$$

Again, by the triangular inequality, we get

$$\begin{aligned} d\left(x_{m_k}, x_{n_k}\right) &\leq d\left(x_{m_k}, x_{n_k+1}\right) + d\left(x_{n_k+1}, x_{n_k}\right), \\ d\left(x_{m_k}, x_{n_k+1}\right) &\leq d\left(x_{m_k}, x_{n_k}\right) + d\left(x_{n_k}, x_{n_k+1}\right). \end{aligned} \tag{48}$$

Using (48), we obtain that

$$\lim_{k\to\infty} d\left(x_{m_k}, x_{n_k+1}\right) = \varepsilon. \tag{49}$$

Similarly, we can prove

$$\lim_{k\to\infty} d\left(x_{n_k}, x_{m_k+1}\right) = \varepsilon. \tag{50}$$

Since α is forward transitive, β is 0-backward transitive, and $n_k > m_k$, we can conclude that

$$0 < \beta\left(x_{m_k}, x_{n_k}\right) \leq 1 \leq \alpha\left(x_{m_k}, x_{n_k}\right). \tag{51}$$

In view of the fact that T is generalized α-β-weakly contractive type A mapping and (51), we have

$$\begin{aligned} &\psi\left(d\left(x_{m_k+1}, x_{n_k+1}\right)\right) \\ &\quad = \psi\left(d\left(Tx_{m_k}, Tx_{n_k}\right)\right) \\ &\quad \leq \beta\left(x_{m_k}, x_{n_k}\right)\psi\left(m\left(x_{m_k}, x_{n_k}\right)\right) \\ &\qquad - \alpha\left(x_{m_k}, x_{n_k}\right)\phi\left(\max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{n_k}, Tx_{n_k}\right)\right\}\right) \\ &\quad \leq \psi\left(m\left(x_{m_k}, x_{n_k}\right)\right) \\ &\qquad - \phi\left(\max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{n_k}, Tx_{n_k}\right)\right\}\right) \\ &\quad \leq \psi\left(\max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{m_k}, Tx_{m_k}\right), d\left(x_{n_k}, Tx_{n_k}\right), \right.\right. \\ &\qquad \left.\left.\frac{1}{2}\left(d\left(x_{m_k}, Tx_{n_k}\right) + d\left(x_{n_k}, Tx_{m_k}\right)\right)\right\}\right) \\ &\qquad - \phi\left(\max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{n_k}, Tx_{n_k}\right)\right\}\right) \\ &\quad = \psi\left(\max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{m_k}, x_{m_k+1}\right), d\left(x_{n_k}, x_{n_k+1}\right), \right.\right. \\ &\qquad \left.\left.\frac{1}{2}\left(d\left(x_{m_k}, x_{n_k+1}\right) + d\left(x_{n_k}, x_{m_k+1}\right)\right)\right\}\right) \\ &\qquad - \phi\left(\max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{n_k}, x_{n_k+1}\right)\right\}\right). \end{aligned} \tag{52}$$

Letting $k \to \infty$, by using (45), (47), (49), and (50), we obtain that

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon) \tag{53}$$

which is a contradiction. Then, we deduce that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, then there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. From the continuity of T, it follows that

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*. \tag{54}$$

Using the uniqueness of limit of the sequence, we conclude that $Tx^* = x^*$ and the proof is complete. □

In the next theorem, the continuity of a generalized α-β-weakly contraction mapping type A in Theorem 14 is replaced by the following condition:

($\mathscr{C}$) if $\{x_n\}$ is a sequence in X such that $0 < \beta(x_n, x_{n+1}) \leq 1 \leq \alpha(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $0 < \beta(x_n, x) \leq 1 \leq \alpha(x_n, x)$ for all $n \in \mathbb{N}$.

Theorem 15. *Let (X, d) be a complete metric space, $\alpha, \beta : X \times X \to [0, \infty)$ two given mappings, and $T : X \to X$ a generalized α-β-weakly contraction type A; then the following conditions hold:*

(a) *condition ($\mathscr{C}$) holds;*

(b) *T is α-admissible and β_0-subadmissible;*

(c) *α is forward transitive and β is 0-backward transitive;*

(d) *there exists $x_0 \in X$ such that $0 < \beta(x_0, Tx_0) \leq 1 \leq \alpha(x_0, Tx_0)$.*

Then T has a fixed point in X.

Proof. As in the proof of Theorem 14, we can find a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ is Cauchy sequence which converges to some point x^* in X. Moreover, we have

$$0 < \beta(x_n, x_{n+1}) \le 1 \le \alpha(x_n, x_{n+1}), \tag{55}$$

for all $n \in \mathbb{N}$. By using condition $(\mathscr{C})$, we obtain that

$$0 < \beta(x_n, x) \le 1 \le \alpha(x_n, x), \tag{56}$$

for all $n \in \mathbb{N}$. Now, let us claim that $Tx^* = x^*$. Supposing the contrary, from the fact that T is a generalized α-β-weakly contractive type A and (55), we get

$$\begin{aligned}
&\psi(d(x_{n+1}, Tx^*)) \\
&\quad = \psi(d(Tx_n, Tx^*)) \\
&\quad \le \beta(x_n, x^*)\psi(m(x_n, x^*)) \\
&\qquad - \alpha(x_n, x^*)\phi(\max\{d(x_n, x^*), d(x^*, Tx^*)\}) \\
&\quad \le \beta(x_n, x^*)\psi(m(x_n, x^*)) \\
&\qquad - \phi(\max\{d(x_n, x^*), d(x^*, Tx^*)\}) \\
&\quad \le \psi(m(x_n, x^*)) \\
&\qquad - \phi(\max\{d(x_n, x^*), d(x^*, Tx^*)\}).
\end{aligned} \tag{57}$$

On the other hand, we obtain that

$$\begin{aligned}
m(x_n, x^*) &= \max\left\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \right. \\
&\qquad \left. \frac{1}{2}(d(x_n, Tx^*) + d(x^*, Tx_n))\right\} \\
&= \max\left\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \right. \\
&\qquad \left. \frac{1}{2}(d(x_n, Tx^*) + d(x^*, x_{n+1}))\right\},
\end{aligned} \tag{58}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (57), by using (58) and the continuity of ψ, we get

$$\begin{aligned}
\psi(d(x^*, Tx^*)) &\le \psi(d(x^*, Tx^*)) \\
&\quad - \phi(d(x^*, Tx^*)) \\
&< \psi(d(x^*, Tx^*))
\end{aligned} \tag{59}$$

which is a contradiction. Therefore $Tx^* = x^*$ and the proof is complete. □

We obtain that Theorems 14 and 15 cannot claim the uniqueness of fixed point. To assure the uniqueness of the fixed point, we will add the following condition:

$(\mathscr{C}')$ for all $x, y \in X$ there exists $z \in X$ such that

$$\begin{aligned}
0 < \beta(x, z) \le 1 \le \alpha(x, z), \\
0 < \beta(y, z) \le 1 \le \alpha(y, z).
\end{aligned} \tag{60}$$

Definition 16. Let X be a nonempty set, $T : X \to X$ a mapping, and $x_0 \in X$. The orbit of T at x_0 is denoted by $\mathcal{O}(T, x_0)$ and defined by

$$\mathcal{O}(T, x_0) := \left\{x_0, T(x_0), T^2(x_0), T^3(x_0), \ldots, T^n(x_0), \ldots\right\}, \tag{61}$$

where $T^{n+1}x_0 = T(T^n x_0)$ and $T^0 x_0 = x_0$.

Theorem 17. *By adding condition $(\mathscr{C}')$ to the hypotheses of Theorem 14 (or Theorem 15) and the limit of orbit $\mathcal{O}(T, z)$ exists, where z is an element in X satisfying* (60)*. Then T has a unique fixed point.*

Proof. Suppose that x and x^* are two fixed points of T. By condition $(\mathscr{C}')$ there exists $z \in X$ such that

$$\begin{aligned}
0 < \beta(x, z) \le 1 \le \alpha(x, z), \\
0 < \beta(x^*, z) \le 1 \le \alpha(x^*, z).
\end{aligned} \tag{62}$$

It follows from T is α-admissible and β_0-subadmissible that

$$\begin{aligned}
0 < \beta(x, T^n z) \le 1 \le \alpha(x, T^n z), \\
0 < \beta(x^*, T^n z) \le 1 \le \alpha(x^*, T^n z),
\end{aligned} \tag{63}$$

for all $n \in \mathbb{N}$. Since the limit of $\mathcal{O}(T, z)$ exists, we get that $\{T^n z\}$ converges to some element in X. Let us claim that $T^n z \to x$ as $n \to \infty$. Suppose the contrary; that is

$$\lim_{n \to \infty} d(T^n z, x) := r > 0. \tag{64}$$

By the generalized α-β-weakly contractive condition type A of T, we have

$$\begin{aligned}
&\psi\left(d\left(x, T^{n+1}z\right)\right) \\
&\quad = \psi(d(Tx, T(T^n z))) \\
&\quad \le \beta(x, T^n z)\psi(m(x, T^n z)) - \alpha(x, T^n z) \\
&\qquad \times \phi(\max\{d(x, T^n z), d(T^n z, T(T^n z))\}) \\
&\quad \le \beta(x, T^n z)\psi(m(x, T^n z)) \\
&\qquad - \phi(\max\{d(x, T^n z), d(T^n z, T(T^n z))\}) \\
&\quad \le \psi(m(x, T^n z)) \\
&\qquad - \phi\left(\max\left\{d(x, T^n z), d\left(T^n z, T^{n+1}z\right)\right\}\right),
\end{aligned} \tag{65}$$

for all $n \in \mathbb{N}$. On the other hand, we have

$$\begin{aligned}
m(x, T^n z) &= \max\left\{d(x, T^n z), d(x, Tx), d(T^n z, T(T^n z)), \right. \\
&\qquad \left. \frac{1}{2}(d(x, T(T^n z)) + d(T^n z, Tx))\right\} \\
&= \max\left\{d(x, T^n z), d\left(T^n z, T^{n+1}z\right), \right. \\
&\qquad \left. \frac{1}{2}\left(d\left(x, T^{n+1}z\right) + d(T^n z, x)\right)\right\},
\end{aligned} \tag{66}$$

for all $n \in \mathbb{N}$. From (64), (65), (66), and the property of ψ and ϕ, we obtain that

$$\psi(r) \leq \psi(r) - \phi(r) < \psi(r) \tag{67}$$

which is a contradiction, and hence $T^n z \to x$ as $n \to \infty$. Similarly, we can show that $T^n z \to x^*$ as $n \to \infty$. Using the uniqueness of limit of the sequence, we conclude that $x = x^*$ and the proof is complete. □

3.2. Generalized α-β-Weakly Contraction Mappings Type B. In this subsection, we obtain the existence and uniqueness of fixed point theorems for generalized α-β-weakly contraction mappings type B.

Theorem 18. *Let (X, d) be a complete metric space, $\alpha, \beta : X \times X \to [0, \infty)$ two given mappings, and $T : X \to X$ a generalized α-β-weakly contraction type B; then the following conditions hold:*

(a) *T is continuous;*

(b) *T is α-admissible and β_0-subadmissible;*

(c) *α is forward transitive and β is 0-backward transitive;*

(d) *there exists $x_0 \in X$ such that $0 < \beta(x_0, Tx_0) \leq 1 \leq \alpha(x_0, Tx_0)$.*

Then T has a fixed point in X.

Proof. As in the proof of Theorem 14, we can find a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$, $x_n \neq x_{n+1}$, and

$$0 < \beta(x_n, x_{n+1}) \leq 1 \leq \alpha(x_n, x_{n+1}), \tag{68}$$

for all $n \in \mathbb{N} \cup \{0\}$. Moreover, for each $n \in \mathbb{N} \cup \{0\}$, we have

$$m(x_n, x_{n+1}) \leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \tag{69}$$

Since T is a generalized α-β-weakly contraction mapping type B, for each $n \in \mathbb{N} \cup \{0\}$, we get

$$\begin{aligned}
&\psi(d(x_{n+1}, x_{n+2})) \\
&\quad = \psi(d(Tx_n, Tx_{n+1})) \\
&\quad \leq \alpha(x_n, x_{n+1})\, \psi(d(Tx_n, Tx_{n+1})) \\
&\quad \leq \beta(x_n, x_{n+1})\, \psi(m(x_n, x_{n+1})) \\
&\qquad - \phi(\max\{d(x_n, x_n), d(x_{n+1}, x_{n+2})\}) \\
&\quad \leq \psi(m(x_n, x_{n+1})) \\
&\qquad - \phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) \\
&\quad \leq \psi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) \\
&\qquad - \phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}).
\end{aligned} \tag{70}$$

Suppose that $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N} \cup \{0\}$. From (70), we get

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &\leq \psi(d(x_{n+1}, x_{n+2})) \\
&\quad - \phi(d(x_{n+1}, x_{n+2})) \\
&< \psi(d(x_{n+1}, x_{n+2}))
\end{aligned} \tag{71}$$

which is a contradiction. Therefore $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. This means that $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence. It follows from a sequence $\{d(x_n, x_{n+1})\}$ bounded below that there exists $r \geq 0$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = r. \tag{72}$$

From (70), we get

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &\leq \psi(d(x_n, x_{n+1})) \\
&\quad - \phi(d(x_n, x_{n+1})),
\end{aligned} \tag{73}$$

for all $n \in \mathbb{N} \cup \{0\}$. Taking $n \to \infty$ in the above inequality, we have

$$\psi(r) \leq \psi(r) - \phi(r). \tag{74}$$

This implies that $r = 0$; that is,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{75}$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon, \tag{76}$$

for all $n_k > m_k \geq k$, where $k \in \mathbb{N}$. Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k \geq k$ satisfying (76). Then we have

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon, \qquad d(x_{m_k}, x_{n_k - 1}) < \varepsilon. \tag{77}$$

As the same argument in Theorem 14, we have

$$\begin{aligned}
\lim_{k \to \infty} d(x_{m_k}, x_{n_k}) &= \lim_{n \to \infty} d(x_{m_k}, x_{n_k + 1}) \\
&= \lim_{n \to \infty} d(x_{n_k}, x_{m_k + 1}) \\
&= \varepsilon.
\end{aligned} \tag{78}$$

Moreover, we have

$$0 < \beta(x_{m_k}, x_{n_k}) \leq 1 \leq \alpha(x_{m_k}, x_{n_k}). \tag{79}$$

In view of the fact that T is generalized α-β-weakly contraction mapping type B, we have

$$\begin{aligned}
&\psi(d(x_{m_k + 1}, x_{n_k + 1})) \\
&\quad = \psi(d(Tx_{m_k}, Tx_{n_k})) \\
&\quad \leq \alpha(x_{m_k}, x_{n_k})\, \psi(d(x_{m_k + 1}, x_{n_k + 1})) \\
&\quad \leq \beta(x_{m_k}, x_{n_k})\, \psi(m(x_{m_k}, x_{n_k}))
\end{aligned}$$

$$
\begin{aligned}
&- \phi\left(\max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{n_k}, Tx_{n_k}\right)\right\}\right) \\
&\leq \psi\left(\max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{m_k}, Tx_{m_k}\right), d\left(x_{n_k}, Tx_{n_k}\right),\right.\right. \\
&\qquad \left.\left.\frac{1}{2}\left(d\left(x_{m_k}, Tx_{n_k}\right) + d\left(x_{n_k}, Tx_{m_k}\right)\right)\right\}\right) \\
&\quad - \phi\left(\max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{n_k}, Tx_{n_k}\right)\right\}\right) \\
&= \psi\left(\max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{m_k}, x_{m_k+1}\right), d\left(x_{n_k}, x_{n_k+1}\right),\right.\right. \\
&\qquad \left.\left.\frac{1}{2}\left(d\left(x_{m_k}, x_{n_k+1}\right) + d\left(x_{n_k}, x_{m_k+1}\right)\right)\right\}\right) \\
&\quad - \phi\left(\max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{n_k}, x_{n_k+1}\right)\right\}\right),
\end{aligned} \tag{80}
$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$ in the above relation, we obtain that

$$
\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon) \tag{81}
$$

which is a contradiction. Therefore, we deduce that $\{x_n\}$ is a Cauchy sequence and so it converges to some element $x^* \in X$. By the continuity of T, we get

$$
\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*, \tag{82}
$$

and hence $Tx^* = x^*$. Therefore the proof is complete. □

Theorem 19. *Let (X, d) be a complete metric space, $\alpha, \beta : X \times X \to [0, \infty)$ two given mappings, and $T : X \to X$ a generalized α-β-weakly contraction type B; then the following conditions hold:*

(a) *condition $(\mathscr{C})$ holds;*

(b) *T is α-admissible and β_0-subadmissible;*

(c) *α is forward transitive and β is 0-backward transitive;*

(d) *there exists $x_0 \in X$ such that $0 < \beta(x_0, Tx_0) \leq 1 \leq \alpha(x_0, Tx_0)$.*

Then T has a fixed point in X.

Proof. As in the proof of Theorem 18, we can find a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ is a Cauchy sequence converging to some point x^* in X.

Also, as in the proof of Theorem 15, for each $n \in \mathbb{N}$, we get

$$
0 < \beta\left(x_n, x^*\right) \leq 1 \leq \alpha\left(x_n, x^*\right), \tag{83}
$$

$$
\lim_{n\to\infty} m\left(x_n, x^*\right) = d\left(x^*, Tx^*\right). \tag{84}
$$

Now, let us claim that $Tx^* = x^*$. On contrary, assume that $x^* \neq Tx^*$ that is $d(x^*, Tx^*) \neq 0$. By using (83) and a generalized α-β-weakly contractive condition type B, we get

$$
\begin{aligned}
\psi\left(d\left(x_{n+1}, Tx^*\right)\right) &= \psi\left(d\left(Tx_n, Tx^*\right)\right) \\
&\leq \alpha\left(x_n, x^*\right)\psi\left(d\left(Tx_n, Tx^*\right)\right) \\
&\leq \beta\left(x_n, x^*\right)\psi\left(m\left(x_n, x^*\right)\right) \\
&\quad - \phi\left(\max\left\{d\left(x_n, x^*\right), d\left(x^*, Tx^*\right)\right\}\right) \\
&\leq \psi\left(m\left(x_n, x^*\right)\right) \\
&\quad - \phi\left(\max\left\{d\left(x_n, x^*\right), d\left(x^*, Tx^*\right)\right\}\right).
\end{aligned} \tag{85}
$$

Letting $n \to \infty$ in (85), we get

$$
\begin{aligned}
\psi\left(d\left(x^*, Tx^*\right)\right) &\leq \psi\left(d\left(x^*, Tx^*\right)\right) \\
&\quad - \phi\left(d\left(x^*, Tx^*\right)\right) \\
&< \psi\left(d\left(x^*, Tx^*\right)\right)
\end{aligned} \tag{86}
$$

which is a contradiction. Therefore $Tx^* = x^*$ and the proof is complete. □

Theorem 20. *By adding condition $(\mathscr{C}')$ to the hypotheses of Theorem 18 (or Theorem 19) and the limit of orbit $\mathcal{O}(T, z)$ exists, where z is an element in A satisfying* (60)*. Then T has a unique fixed point.*

Proof. Apply the proof of Theorem 18 (or Theorem 19) and Theorem 17. □

4. Applications

In this section, we give the several fixed point results which are obtained by our results in Section 3.

4.1. Fixed Point Results on an Ordinary Metric Space. Setting $\alpha(x, y) = \beta(x, y) = 1$ for all $x, y \in X$ in Theorem 14 (or Theorem 18), we get the following result.

Corollary 21. *Let (X, d) be a complete metric space and $T : X \to X$ a continuous generalized weakly contraction mapping. Then T has a fixed point in X.*

By using Remark 5, we obtain the following results.

Corollary 22. *Let (X, d) be a complete metric space and $T : X \to X$ a continuous mapping and*

$$
d\left(Tx, Ty\right) \leq k m\left(x, y\right), \tag{87}
$$

for all $x, y \in X$, where $k \in [0, 1)$ and

$$
\begin{aligned}
m\left(x, y\right) = \max\left\{d\left(x, y\right), d\left(x, Tx\right), d\left(y, Ty\right),\right. \\
\left.\frac{1}{2}\left(d\left(x, Ty\right) + d\left(y, Tx\right)\right)\right\}.
\end{aligned} \tag{88}
$$

Then T has a fixed point in X.

Corollary 23. *Let (X,d) be a complete metric space and $T : X \to X$ a continuous mapping and*

$$d(Tx,Ty) < m(x,y), \tag{89}$$

for all $x, y \in X$, where

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))\right\}. \tag{90}$$

Then T has a fixed point in X.

Corollary 24. *Let (X,d) be a complete metric space and let $T : X \to X$ be a continuous mapping and*

$$\psi(d(Tx,Ty)) \le m(x,y) - \max\{d(x,y), d(y,Ty)\}, \tag{91}$$

for all $x, y \in X$, where

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))\right\}. \tag{92}$$

Then T has a fixed point in X.

4.2. Fixed Point Results on Metric Spaces Endowed with an Arbitrary Binary Relation. In this section, we give the existence of fixed point theorems on a metric space endowed with an arbitrary binary relation. Before presenting our results, we give the following notions and definitions.

Definition 25. Let X be a nonempty set and $\mathcal{R}$ a binary relation over X. One says that $T : X \to X$ is a *comparative mapping with respect to $\mathcal{R}$* if

$$x, y \in X, \quad x\mathcal{R}y \Longrightarrow (Tx)\,\mathcal{R}\,(Ty). \tag{93}$$

Definition 26. Let X be a nonempty set and $\mathcal{R}$ a binary relation over X. One says that X has a *transitive property with respect to $\mathcal{R}$* if

$$x, y, z \in X, \quad x\mathcal{R}y, y\mathcal{R}z \Longrightarrow x\mathcal{R}z. \tag{94}$$

Definition 27. Let (X,d) be a metric space and $\mathcal{R}$ a binary relation over X. A mapping $T : X \to X$ is said to be a *generalized weakly contraction with respect to $\mathcal{R}$* if for each $x, y \in X$ for which $x\mathcal{R}y$ one has

$$\psi(d(Tx,Ty)) \le \psi(m(x,y)) - \phi(\max\{d(x,y), d(y,Ty)\}), \tag{95}$$

where

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))\right\}, \tag{96}$$

$\psi : [0,\infty) \to [0,\infty)$ is altering distance function, and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$.

Theorem 28. *Let (X,d) be a metric space and $\mathcal{R}$ a binary relation over X and $T : X \to X$ a generalized weakly contraction with respect to $\mathcal{R}$; then the following conditions hold:*

(A) *T is continuous;*

(B) *X has a transitive property with respect to $\mathcal{R}$;*

(C) *T is comparative mapping with respect to $\mathcal{R}$;*

(D) *there exists $x_0 \in X$ such that $x_0\mathcal{R}(Tx_0)$.*

Then T has a fixed point in X.

Proof. Consider two mappings $\alpha, \beta : X \times X \to [0,\infty)$ defined by

$$\alpha(x,y) = \beta(x,y) = \begin{cases} 1 & \text{if } x\mathcal{R}y; \\ 0 & \text{otherwise.} \end{cases} \tag{97}$$

From condition (D), we get $\alpha(x_0,Tx_0) = \beta(x_0,Tx_0) = 1$. Since T is comparative mapping with respect to $\mathcal{R}$, we get T is α-admissible and β_0-subadmissible. Also, α is forward transitive and β is 0-backward transitive since X has a transitive property with respect to $\mathcal{R}$. Since T is a generalized weakly contraction with respect to $\mathcal{R}$, we have, for all $x, y \in X$,

$$\begin{aligned} &\psi(d(Tx,Ty)) \\ &\quad \le \beta(x,y)\,\psi(m(x,y)) \\ &\qquad - \alpha(x,y)\,\phi(\max\{d(x,y), d(y,Ty)\}), \\ &\alpha(x,y)\,\psi(d(Tx,Ty)) \\ &\quad \le \beta(x,y)\,\psi(m(x,y)) \\ &\qquad - \phi(\max\{d(x,y), d(y,Ty)\}). \end{aligned} \tag{98}$$

This implies that T is generalized α-β-weakly contraction mapping types A and B. Now all the hypotheses of Theorem 14 (or Theorem 18) are satisfied and thus the existence of the fixed point of T follows from Theorem 14 (or Theorem 18). □

In order to remove the continuity of T, we need the following condition:

($\mathcal{C}_{\mathcal{R}}$) if $\{x_n\}$ is the sequence in X such that $x_n\mathcal{R}x_{n+1}$ for all $n \in \mathbb{N}$ and it converges to the point $x \in X$, then $x_n\mathcal{R}x$ for all $n \in \mathbb{N}$.

Theorem 29. *Let (X,d) be a metric space and $\mathcal{R}$ a binary relation over X and $T : X \to X$ a generalized weakly contraction with respect to $\mathcal{R}$; then the following conditions hold:*

(A) *the condition ($\mathcal{C}_{\mathcal{R}}$) holds on X;*

(B) *X has a transitive property with respect to $\mathcal{R}$;*

(C) *T is comparative mapping with respect to $\mathcal{R}$;*

(D) *there exists $x_0 \in X$ such that $x_0\mathcal{R}(Tx_0)$.*

Then T has a fixed point in X.

Proof. The result follows from Theorem 15 (or Theorem 19) by considering the mappings α and β given by (97) and by observing that condition $(\mathscr{C}_{\mathscr{R}})$ implies property $(\mathscr{C})$. □

To assure the uniqueness of the fixed point, we will add the following condition:

$(\mathscr{C}'_{\mathscr{R}})$ for all $x, y \in X$ there exists $z \in X$ such that

$$x\mathscr{R}z, \qquad y\mathscr{R}z. \tag{99}$$

Theorem 30. *By adding condition $(\mathscr{C}'_{\mathscr{R}})$ to the hypotheses of Theorem 28 (or Theorem 29) and the limit of orbit $\mathcal{O}(T, z)$ exists, where z is an element in X satisfying* (99)*. Then T has a unique fixed point.*

Proof. The result follows from Theorem 17 (or Theorem 20) by considering the mappings α and β given by (97) and by observing that condition $(\mathscr{C}'_{\mathscr{R}})$ implies property $(\mathscr{C}')$. □

4.3. Fixed Point Results on Metric Spaces Endowed with Graph. Throughout this section, let (X, d) be a metric space. A set $\{(x, x) : x \in X\}$ is called a diagonal of the Cartesian product $X \times X$ and is denoted by Δ. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops; that is, $\Delta \subseteq E(G)$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices. A graph G is connected if there is a path between any two vertices.

In this section, we give the existence of fixed point theorems on a metric space endowed with graph. Before presenting our results, we give the following notions and definitions.

Definition 31. Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ mapping. One says that *T preserves edges of G* if

$$x, y \in X, \quad (x, y) \in E(G) \Longrightarrow (Tx, Ty) \in E(G). \tag{100}$$

Definition 32. Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ mapping. One says that X has a *transitive property with respect to graph G* if

$$\begin{aligned} &x, y, z \in X, \quad (x, y) \in E(G), \\ &(y, z) \in E(G) \Longrightarrow (x, z) \in E(G). \end{aligned} \tag{101}$$

Remark 33. It is easy to see that if G is a connected graph, then X has a transitive property with respect to graph G.

Definition 34. Let (X, d) be a metric space endowed with a graph G. A mapping $T : X \to X$ is said to be a *generalized weakly contraction with respect to graph G* if for each $x, y \in X$ for which $(x, y) \in E(G)$ one has

$$\begin{aligned} \psi(d(Tx, Ty)) \le{}& \psi(m(x, y)) \\ &- \phi(\max\{d(x, y), d(y, Ty)\}), \end{aligned} \tag{102}$$

where

$$\begin{aligned} m(x, y) = \max\Big\{&d(x, y), d(x, Tx), d(y, Ty), \\ &\frac{1}{2}(d(x, Ty) + d(y, Tx))\Big\}, \end{aligned} \tag{103}$$

$\psi : [0, \infty) \to [0, \infty)$ is altering distance function, and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$.

Example 35. Let (X, d) be a metric space, $T : X \to X$ a given mapping, $\psi : [0, \infty) \to [0, \infty)$ an arbitrary altering distance function, and $\phi : [0, \infty) \to [0, \infty)$ an arbitrary continuous function with $\phi(t) = 0$ if and only if $t = 0$. If $\phi(d(x, Tx)) \le \psi(d(x, Tx))$ for all $x \in X$, then T is trivially generalized weakly contraction with respect to graph G, where $G = (V(G), E(G)) = (X, \Delta)$.

Theorem 36. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ a generalized weakly contraction with respect to graph G; then the following conditions hold:*

(A) *T is continuous;*

(B) *X has a transitive property with respect to graph G;*

(C) *T preserves edges of G;*

(D) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$.*

Then T has a fixed point in X.

Proof. Consider two mappings $\alpha, \beta : X \times X \to [0, \infty)$ defined by

$$\alpha(x, y) = \beta(x, y) = \begin{cases} 1 & \text{if } (x, y) \in E(G); \\ 0 & \text{otherwise.} \end{cases} \tag{104}$$

From condition (D), we get $\alpha(x_0, Tx_0) = \beta(x_0, Tx_0) = 1$. Since T preserves edges of G, we get T is α-admissible and β_0-subadmissible. Also, α is forward transitive property and β is 0-backward transitive since X has transitive property with respect to graph G. Since T is a generalized weakly contraction with respect to graph G, we get

$$\begin{aligned} &\psi(d(Tx, Ty)) \\ &\quad \le \beta(x, y)\psi(m(x, y)) \\ &\qquad - \alpha(x, y)\phi(\max\{d(x, y), d(y, Ty)\}), \\ &\alpha(x, y)\psi(d(Tx, Ty)) \\ &\quad \le \beta(x, y)\psi(m(x, y)) \\ &\qquad - \phi(\max\{d(x, y), d(y, Ty)\}), \end{aligned} \tag{105}$$

for all $x, y \in X$. This implies that T is generalized α-β-weakly contraction mapping types A and B. Therefore, all the hypotheses of Theorem 14 (or Theorem 18) are satisfied. Now the existence of the fixed point of T follows from Theorem 14 (or Theorem 18). □

In order to remove the continuity of T, we need the following condition.

Definition 37. Let (X, d) be a metric space endowed with a graph G. One says that X has *G-regular property* if $\{x_n\}$ is the sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and it converges to the point $x \in X$; then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.

Theorem 38. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ a generalized α-β-weakly contraction with respect to graph G; then the following conditions hold:*

(A) *X has G-regular property;*

(B) *X has a transitive property with respect to graph G;*

(C) *T preserves edges of G;*

(D) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$.*

Then T has a fixed point in X.

Proof. The result follows from Theorem 15 (or Theorem 19) by considering the mappings α and β given by (104) and by observing that G-regular property implies property $(\mathscr{C})$. □

To assure the uniqueness of the fixed point, we will add the following condition:

$(\mathscr{C}'_{\mathscr{G}})$ for all $x, y \in X$ there exists $z \in X$ such that

$$(x, z) \in E(G), \qquad (y, z) \in E(G). \tag{106}$$

Theorem 39. *By adding condition $(\mathscr{C}'_{\mathscr{G}})$ to the hypotheses of Theorem 36 (or Theorem 38) and the limit of orbit $\mathcal{O}(T, z)$ exists, where z is an element in X satisfying* (106). *Then T has a unique fixed point.*

Proof. The result follows from Theorem 17 (or Theorem 20) by considering the mappings α and β given by (97) and by observing that condition $(\mathscr{C}'_{\mathscr{G}})$ implies property $(\mathscr{C}')$. □

By using Remark 33, we get the following results.

Corollary 40. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ a generalized weakly contraction with respect to graph G; then the following conditions hold:*

(A) *T is continuous;*

(B) *G is connected graph;*

(C) *T preserves edges of G;*

(D) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$.*

Then T has a fixed point in X.

Corollary 41. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ a generalized α-β-weakly contraction with respect to graph G; then the following conditions hold:*

(A) *X has G-regular property;*

(B) *G is connected graph;*

(C) *T preserves edges of G;*

(D) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$.*

Then T has a fixed point in X.

Corollary 42. *By adding condition $(\mathscr{C}'_{\mathscr{G}})$ to the hypotheses of Corollary 40 (or Corollary 41), the limit of orbit $\mathcal{O}(T, z)$ exists, where z is an element in X satisfying* (106). *Then T has a unique fixed point.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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Research Article

Fixed Point Results for G-α-Contractive Maps with Application to Boundary Value Problems

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We unify the concepts of G-metric, metric-like, and b-metric to define new notion of generalized b-metric-like space and discuss its topological and structural properties. In addition, certain fixed point theorems for two classes of G-α-admissible contractive mappings in such spaces are obtained and some new fixed point results are derived in corresponding partially ordered space. Moreover, some examples and an application to the existence of a solution for the first-order periodic boundary value problem are provided here to illustrate the usability of the obtained results.

1. Introduction and Mathematical Preliminaries

The concept of a b-metric space was introduced by Czerwik [1]. After that, several interesting results about the existence of fixed point for single-valued and multivalued operators in (ordered) b-metric spaces have been obtained (see, e.g., [2–11]).

Definition 1 (see [1]). Let X be a (nonempty) set and $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b_1) $d(x, y) = 0$ if and only if $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

The concept of a generalized metric space, or a G-metric space, was introduced by Mustafa and Sims [12].

Definition 2 (see [12]). Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ a function satisfying the following properties:

(G_1) $G(x, y, z) = 0$ if and only if $x = y = z$;
(G_2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
(G_3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;
(G_4) $G(x, y, z) = G(p\{x, y, z\})$, where p is any permutation of x, y, z (symmetry in all three variables);
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 3 (see [13]). A metric-like on a nonempty set X is a mapping $\sigma : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$, the following hold:

(σ_1) $\sigma(x, y) = 0$ implies $x = y$;
(σ_2) $\sigma(x, y) = \sigma(y, x)$;
(σ_3) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

The pair (X, σ) is called a metric-like space.

Below, we give some examples of metric-like spaces.

Example 4 (see [14]). Let $X = [0, 1]$. Then, the mapping $\sigma_1 : X \times X \to \mathbb{R}^+$ defined by $\sigma_1(x, y) = x + y - xy$ is a metric-like on X.

Example 5 (see [14]). Let $X = \mathbb{R}$; then the mappings $\sigma_i : X \times X \to \mathbb{R}^+ (i \in \{2,3,4\})$ defined by

$$\sigma_2(x,y) = |x| + |y| + a,$$
$$\sigma_3(x,y) = |x-b| + |y-b|, \quad (1)$$
$$\sigma_4(x,y) = x^2 + y^2$$

are metric-likes on X, where $a \geq 0$ and $b \in \mathbb{R}$.

Definition 6 (see [15]). Let X be a nonempty set and $s \geq 1$ a given real number. A function $\sigma_b : X \times X \to \mathbb{R}^+$ is a b-metric-like if, for all $x, y, z \in X$, the following conditions are satisfied:

$(\sigma_b 1)$ $\sigma_b(x,y) = 0$ implies $x = y$;
$(\sigma_b 2)$ $\sigma_b(x,y) = \sigma_b(y,x)$;
$(\sigma_b 3)$ $\sigma_b(x,y) \leq s[\sigma_b(x,z) + \sigma_b(z,y)]$.

A b-metric-like space is a pair (X, σ_b) such that X is a nonempty set and σ_b is a b-metric-like on X. The number s is called the coefficient of (X, σ_b).

In a b-metric-like space (X, σ_b) if $x, y \in X$ and $\sigma_b(x,y) = 0$, then $x = y$, but the converse may not be true and $\sigma_b(x,x)$ may be positive for all $x \in X$. It is clear that every b-metric space is a b-metric-like space with the same coefficient s but not conversely in general.

Example 7 (see [8]). Let $X = \mathbb{R}^+$, let $p > 1$ be a constant, and let $\sigma_b : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma_b(x,y) = (x+y)^p \quad \forall x, y \in X. \quad (2)$$

Then, (X, σ_b) is a b-metric-like space with coefficient $s = 2^{p-1}$.

The following propositions help us to construct some more examples of b-metric-like spaces.

Proposition 8 (see [8]). *Let (X, σ) be a metric-like space and $\sigma_b(x,y) = [\sigma(x,y)]^p$, where $p > 1$ is a real number. Then, σ_b is a b-metric-like with coefficient $s = 2^{p-1}$.*

From the above proposition and Examples 4 and 5, we have the following examples of b-metric-like spaces.

Example 9 (see [8]). Let $X = [0,1]$. Then, the mapping $\sigma_{b1} : X \times X \to \mathbb{R}^+$ defined by $\sigma_{b1}(x,y) = (x + y - xy)^p$, where $p > 1$ is a real number, is a b-metric-like on X with coefficient $s = 2^{p-1}$.

Example 10 (see [8]). Let $X = \mathbb{R}$. Then, the mappings $\sigma_{bi} : X \times X \to \mathbb{R}^+ (i \in \{2,3,4\})$ defined by

$$\sigma_{b2}(x,y) = (|x| + |y| + a)^p,$$
$$\sigma_{b3}(x,y) = (|x-b| + |y-b|)^p, \quad (3)$$
$$\sigma_{b4}(x,y) = \left(x^2 + y^2\right)^p$$

are b-metric-like on X, where $p > 1$, $a \geq 0$, and $b \in \mathbb{R}$.

Each b-metric-like σ_b on X generates a topology τ_{σ_b} on X whose base is the family of all open σ_b-balls $\{B_{\sigma_b}(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_{\sigma_b}(x,\varepsilon) = \{y \in X : |\sigma_b(x,y) - \sigma_b(x,x)| < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Now, we introduce the concept of generalized b-metric-like space, or G_{σ_b}-metric space, as a proper generalization of both of the concepts of b-metric-like spaces and G-metric spaces.

Definition 11. Let X be a nonempty set. Suppose that a mapping $G_{\sigma_b} : X \times X \times X \to \mathbb{R}^+$ satisfies the following:

$(G_{\sigma_b} 1)$ $G_{\sigma_b}(x,y,z) = 0$ implies $x = y = z$;

$(G_{\sigma_b} 2)$ $G_{\sigma_b}(x,y,z) = G_{\sigma_b}(p\{x,y,z\})$, where p is any permutation of x, y, z (symmetry in all three variables);

$(G_{\sigma_b} 3)$ $G_{\sigma_b}(x,y,z) \leq s[G_{\sigma_b}(x,a,a) + G_{\sigma_b}(a,y,z)]$ for all $x, y, z, a \in X$ (rectangle inequality).

Then, G_{σ_b} is called a G_{σ_b}-metric and (X, G_{σ_b}) is called a generalized b-metric-like space.

The following proposition will be useful in constructing examples of a generalized b-metric-like space.

Proposition 12. *Let (X, σ_b) be a b-metric-like space with coefficient s. Then,*

$$G^m_{\sigma_b}(x,y,z) = \max\{\sigma_b(x,y), \sigma_b(y,z), \sigma_b(z,x)\},$$
$$G^s_{\sigma_b}(x,y,z) = \sigma_b(x,y) + \sigma_b(y,z) + \sigma_b(z,x) \quad (4)$$

are two generalized b-metric-like functions on X.

Proof. It is clear that $G^m_{\sigma_b}$ and $G^s_{\sigma_b}$ satisfy conditions $G_{\sigma_b}1$ and $G_{\sigma_b}2$ of Definition 11. So, we only show that $G_{\sigma_b}3$ is satisfied by $G^m_{\sigma_b}$ and $G^s_{\sigma_b}$. Let $x, y, z, a \in X$. Then, using the triangular inequality in b-metric-like spaces, we have

$$\begin{aligned} G^m_{\sigma_b}(x,y,z) &= \max\{\sigma_b(x,y), \sigma_b(y,z), \sigma_b(z,x)\} \\ &\leq \max\{s(\sigma_b(x,a) + \sigma_b(a,y)), \\ &\qquad \sigma_b(y,z), s(\sigma_b(z,a) + \sigma_b(a,x))\} \\ &\leq \max\{s(\sigma_b(x,a) + \sigma_b(a,y)), \\ &\qquad s(\sigma_b(y,z) + \sigma_b(a,a)), \\ &\qquad s(\sigma_b(z,a) + \sigma_b(a,x))\} \\ &= s\max\{\sigma_b(x,a), \sigma_b(a,a), \sigma_b(a,x)\} \\ &\quad + s\max\{\sigma_b(a,y), \sigma_b(y,z), \sigma_b(z,a)\} \\ &= s\left(G^m_{\sigma_b}(x,a,a) + G^m_{\sigma_b}(a,y,z)\right). \end{aligned} \quad (5)$$

Also,

$$\begin{aligned} G^s_{\sigma_b}(x,y,z) &= \sigma_b(x,y) + \sigma_b(y,z) + \sigma_b(z,x) \\ &\le s\left(\sigma_b(x,a) + \sigma_b(a,y)\right) \\ &\quad + \sigma_b(y,z) + s\left(\sigma_b(z,a) + \sigma_b(a,x)\right) \\ &\le s\left(\sigma_b(x,a) + \sigma_b(a,y)\right) \\ &\quad + s\left(\sigma_b(y,z) + \sigma_b(a,a)\right) \\ &\quad + s\left(\sigma_b(z,a) + \sigma_b(a,x)\right) \\ &= s\left(\sigma_b(x,a) + \sigma_b(a,a) + \sigma_b(a,x)\right) \\ &\quad + s\left(\sigma_b(a,y), \sigma_b(y,z), \sigma_b(z,a)\right) \\ &= s\left(G^s_{\sigma_b}(x,a,a) + G^s_{\sigma_b}(a,y,z)\right). \end{aligned} \tag{6}$$

□

According to the above proposition, we provide some examples of generalized b-metric-like spaces.

Example 13. Let $X = \mathbb{R}^+$, let $p > 1$ be a constant, and let $G^m_{\sigma_b}, G^s_{\sigma_b} : X \times X \times X \to \mathbb{R}^+$ be defined by

$$\begin{aligned} G^m_{\sigma_b}(x,y,z) &= \max\left\{(x+y)^p, (y+z)^p, (z+x)^p\right\}, \\ G^s_{\sigma_b}(x,y,z) &= (x+y)^p + (y+z)^p + (z+x)^p, \end{aligned} \tag{7}$$

for all $x, y, z \in X$. Then, $(X, G^m_{\sigma_b})$ and $(X, G^s_{\sigma_b})$ are generalized b-metric-like spaces with coefficient $s = 2^{p-1}$. Note that, for $x = y = z > 0, G^m_{\sigma_b}(x,x,x) = (2x)^p > 0$ and $G^s_{\sigma_b}(x,x,x) = 3(2x)^p > 0$.

Example 14. Let $X = [0,1]$. Then, the mappings $G^m_{\sigma_{b1}}, G^s_{\sigma_{b1}} : X \times X \times X \to \mathbb{R}^+$ defined by

$$\begin{aligned} &G^m_{\sigma_{b1}}(x,y,z) \\ &\quad = \max\left\{(x+y-xy)^p, (y+z-yz)^p, (z+x-zx)^p\right\}, \\ &G^s_{\sigma_{b1}}(x,y,z) \\ &\quad = (x+y-xy)^p + (y+z-yz)^p + (z+x-zx)^p, \end{aligned} \tag{8}$$

where $p > 1$ is a real number, are generalized b-metric-like spaces with coefficient $s = 2^{p-1}$.

By some straight forward calculations, we can establish the following.

Proposition 15. *Let X be a G_{σ_b}-metric space. Then, for each $x, y, z, a \in X$, it follows that:*

(1) $G_{\sigma_b}(x,y,y) > 0$ *for* $x \neq y$;

(2) $G_{\sigma_b}(x,y,z) \le s(G_{\sigma_b}(x,x,y) + G_{\sigma_b}(x,x,z))$;

(3) $G_{\sigma_b}(x,y,y) \le 2sG_{\sigma_b}(y,x,x)$;

(4) $G_{\sigma_b}(x,y,z) \le sG_{\sigma_b}(x,a,a) + s^2G_{\sigma_b}(y,a,a) + s^2G_{\sigma_b}(z,a,a)$.

Definition 16. Let (X, G_{σ_b}) be a G_{σ_b}-metric space. Then, for any $x \in X$ and $r > 0$, the G_{σ_b}-ball with center x and radius r is

$$B_{G_{\sigma_b}}(x,r) = \left\{y \in X \mid \left|G_{\sigma_b}(x,x,y) - G_{\sigma_b}(x,x,x)\right| < +r\right\}. \tag{9}$$

The family of all G_{σ_b}-balls

$$\mathcal{F} = \left\{B_{G_{\sigma_b}}(x,r) \mid x \in X, r > 0\right\} \tag{10}$$

is a base of a topology $\tau(G_{\sigma_b})$ on X, which we call it G_{σ_b}-metric topology.

Definition 17. Let (X, G_{σ_b}) be a G_{σ_b}-metric space. Let $\{x_n\}$ be a sequence in X. Consider the following.

(1) A point $x \in X$ is said to be a limit of the sequence $\{x_n\}$, denoted by $x_n \to x$, if $\lim_{n,m\to\infty} G_{\sigma_b}(x, x_n, x_m) = G_{\sigma_b}(x,x,x)$.

(2) $\{x_n\}$ is said to be a G_{σ_b}-Cauchy sequence, if $\lim_{n,m\to\infty} G_{\sigma_b}(x_n, x_m, x_m)$ exists (and is finite).

(3) (X, G_{σ_b}) is said to be G_{σ_b}-complete if every G_{σ_b}-Cauchy sequence in X is G_{σ_b}-convergent to an $x \in X$ such that

$$\lim_{n,m\to\infty} G_{\sigma_b}(x_n, x_m, x_m) = \lim_{n\to\infty} G_{\sigma_b}(x_n, x, x) = G_{\sigma_b}(x,x,x). \tag{11}$$

Using the above definitions, one can easily prove the following proposition.

Proposition 18. *Let (X, G_{σ_b}) be a G_{σ_b}-metric space. Then, for any sequence $\{x_n\}$ in X and a point $x \in X$, the following are equivalent:*

(1) $\{x_n\}$ *is G_{σ_b}-convergent to x;*

(2) $G_{\sigma_b}(x_n, x_n, x) \to G_{\sigma_b}(x,x,x)$, *as* $n \to \infty$;

(3) $G_{\sigma_b}(x_n, x, x) \to G_{\sigma_b}(x,x,x)$, *as* $n \to \infty$;

Definition 19. Let (X, G_{σ_b}) and (X', G'_{σ_b}) be two generalized b-metric like spaces and let $f : (X, G_{\sigma_b}) \to (X', G'_{\sigma_b})$ be a mapping. Then, f is said to be G_{σ_b}-continuous at a point $a \in X$ if, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $|G_{\sigma_b}(a,a,x) - G_{\sigma_b}(a,a,a)| < \delta$ imply that $|G'_{\sigma_b}(f(a), f(a), f(x)) - G'_{\sigma_b}(f(a), f(a), f(a))| < \varepsilon$. The mapping f is G_{σ_b}-continuous on X if it is G_{σ_b}-continuous at all $a \in X$. For simplicity, we say that f is continuous.

Proposition 20. *Let (X, G_{σ_b}) and (X', G'_{σ_b}) be two generalized b-metric like spaces. Then, a mapping $f : X \to X'$ is G_{σ_b}-continuous at a point $x \in X$ if and only if it is G_{σ_b}-sequentially continuous at x; that is, whenever $\{x_n\}$ is G_{σ_b}-convergent to x, $\{f(x_n)\}$ is G_{σ_b}-convergent to $f(x)$.*

We need the following simple lemma about the G_{σ_b}-convergent sequences in the proof of our main results.

Lemma 21. *Let (X, G_{σ_b}) be a G_{σ_b}-metric space and suppose that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are G_{σ_b}-convergent to x, y, and z, respectively. Then, we have*

$$\begin{aligned}
&\frac{1}{s^3}G_{\sigma_b}(x,y,z) - \frac{1}{s^2}G_{\sigma_b}(x,x,x) \\
&\quad - \frac{1}{s}G_{\sigma_b}(y,y,y) - G_{\sigma_b}(z,z,z) \\
&\le \liminf_{n\to\infty} G_{\sigma_b}(x_n,y_n,z_n) \\
&\le \limsup_{n\to\infty} G_{\sigma_b}(x_n,y_n,z_n) \\
&\le s^3 G_{\sigma_b}(x,y,z) + sG_{\sigma_b}(x,x,x) \\
&\quad + s^2 G_{\sigma_b}(y,y,y) + s^3 G_{\sigma_b}(z,z,z).
\end{aligned} \tag{12}$$

In particular, if $\{y_n\} = \{z_n\} = a$ are constant, then

$$\begin{aligned}
&\frac{1}{s}G_{\sigma_b}(x,a,a) - G_{\sigma_b}(x,x,x) \\
&\le \liminf_{n\to\infty} G_{\sigma_b}(x_n,a,a) \le \limsup_{n\to\infty} G_{\sigma_b}(x_n,a,a) \\
&\le sG_{\sigma_b}(x,a,a) + sG_{\sigma_b}(x,x,x).
\end{aligned} \tag{13}$$

Proof. Using the rectangle inequality, we obtain

$$\begin{aligned}
G_{\sigma_b}(x,y,z) &\le sG_{\sigma_b}(x,x_n,x_n) + s^2 G_{\sigma_b}(y,y_n,y_n) \\
&\quad + s^3 G_{\sigma_b}(z,z_n,z_n) + s^3 G_{\sigma_b}(x_n,y_n,z_n), \\
G_{\sigma_b}(x_n,y_n,z_n) &\le sG_{\sigma_b}(x_n,x,x) + s^2 G_{\sigma_b}(y_n,y,y) \\
&\quad + s^3 G_{\sigma_b}(z_n,z,z) + s^3 G_{\sigma_b}(x,y,z).
\end{aligned} \tag{14}$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality, we obtain the desired result.

If $\{y_n\} = \{z_n\} = a$, then

$$\begin{aligned}
G_{\sigma_b}(x,a,a) &\le sG_{\sigma_b}(x,x_n,x_n) + sG_{\sigma_b}(x_n,a,a), \\
G_{\sigma_b}(x_n,a,a) &\le sG_{\sigma_b}(x_n,x,x) + sG_{\sigma_b}(x,a,a).
\end{aligned} \tag{15}$$

Again taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality, we obtain the desired result. □

2. Main Results

Samet et al. [16] defined the notion of α-admissible mappings and proved the following result.

Definition 22. Let T be a self-mapping on X and $\alpha : X \times X \to [0,\infty)$ a function. We say that T is an α-admissible mapping if

$$x, y \in X, \qquad \alpha(x,y) \ge 1 \Longrightarrow \alpha(Tx,Ty) \ge 1. \tag{16}$$

Denote with Ψ' the family of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

Theorem 23. *Let (X,d) be a complete metric space and T an α-admissible mapping. Assume that*

$$\alpha(x,y)\, d(Tx,Ty) \le \psi(d(x,y)), \tag{17}$$

where $\psi \in \Psi'$. Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*

(ii) *either T is continuous or, for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$ such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then, T has a fixed point.

For more details on α-admissible mappings, we refer the reader to [17–20].

Definition 24 (see [21]). Let (X,G) be a G-metric space, let f be a self-mapping on X, and let $\alpha : X^3 \to [0,\infty)$ be a function. We say that f is a G-α-admissible mapping if

$$x, y, z \in X, \qquad \alpha(x,y,z) \ge 1 \Longrightarrow \alpha(fx,fy,fz) \ge 1. \tag{18}$$

Motivated by [22], let $\mathcal{F}$ denote the class of all functions $\beta : [0,\infty) \to [0, 1/s)$ satisfying the following condition:

$$\limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \Longrightarrow \limsup_{n\to\infty} t_n = 0. \tag{19}$$

Definition 25. Let $f : X \to X$ and $\alpha : X \times X \times X \to \mathbb{R}$. We say that f is a rectangular G-α-admissible mapping if

(T1) $\alpha(x,y,z) \ge 1$ implies $\alpha(fx,fy,fz) \ge 1$, $x, y, z \in X$;

(T2) $\begin{cases} \alpha(x,y,y) \ge 1 \\ \alpha(y,z,z) \ge 1 \end{cases}$ implies $\alpha(x,z,z) \ge 1$, $x, y, z \in X$.

From now on, let $\alpha : X^3 \to [0,\infty)$ be a function and

$$\begin{aligned}
&M_s(x,y,z) \\
&= \max\left\{ G_{\sigma_b}(x,y,z), \right. \\
&\qquad \left. \frac{G_{\sigma_b}(x,fx,fx)\, G_{\sigma_b}(y,fy,fy)\, G_{\sigma_b}(z,fz,fz)}{1 + s^2 G^2_{\sigma_b}(fx,fy,fz)} \right\}.
\end{aligned} \tag{20}$$

Theorem 26. *Let (X, G_{σ_b}) be a G_{σ_b}-complete generalized b-metric-like space and let $f : X \to X$ be a rectangular G-α-admissible mapping. Suppose that*

$$\begin{aligned}
&s\alpha(x,y,z)\, G_{\sigma_b}(fx,fy,fz) \\
&\qquad \le \beta\left(G_{\sigma_b}(x,y,z)\right) M_s(x,y,z),
\end{aligned} \tag{21}$$

for all $x, y, z \in X$.

Also, suppose that the following assertions hold:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0, fx_0) \geq 1$;

(ii) f *is continuous and, for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *such that* $x_n \to x$ *as* $n \to \infty$, *we have* $\alpha(x, x, x) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then, f *has a fixed point.*

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, fx_0, fx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0$ for all $n \in \mathbb{N}$. Since f is a G-α-admissible mapping and $\alpha(x_0, x_1, x_1) = \alpha(x_0, fx_0, fx_0) \geq 1$, we deduce that $\alpha(x_1, x_2, x_2) = \alpha(fx_0, fx_1, fx_1) \geq 1$. Continuing this process, we get $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Step I. We will show that $\lim_{n\to\infty} G_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) = 0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n = fx_n$. Thus, x_n is a fixed point of f. Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, then we can apply (21) which yields

$$\begin{aligned} sG_{\sigma_b}&(x_n, x_{n+1}, x_{n+1}) \\ &= sG_{\sigma_b}(fx_{n-1}, fx_n, fx_n) \\ &\leq \beta\left(G_{\sigma_b}(x_{n-1}, x_n, x_n)\right) M_s(x_{n-1}, x_n, x_n) \\ &< \frac{1}{s} G_{\sigma_b}(x_{n-1}, x_n, x_n) \\ &\leq G_{\sigma_b}(x_{n-1}, x_n, x_n), \end{aligned} \tag{22}$$

$$\begin{aligned} M_s&(x_{n-1}, x_n, x_n) \\ &= \max\Big\{G_{\sigma_b}(x_{n-1}, x_n, x_n), \\ &\qquad \Big(G_{\sigma_b}(x_{n-1}, fx_{n-1}, fx_{n-1}) \\ &\qquad\quad \times G_{\sigma_b}(x_n, fx_n, fx_n)\, G_{\sigma_b}(x_n, fx_n, fx_n)\Big) \\ &\qquad\quad \times\Big(1 + s^2 G^2_{\sigma_b}(fx_{n-1}, fx_n, fx_n)\Big)^{-1}\Big\} \\ &= \max\Big\{G_{\sigma_b}(x_{n-1}, x_n, x_n), \\ &\qquad \Big(G_{\sigma_b}(x_{n-1}, x_n, x_n) \\ &\qquad\quad \times G_{\sigma_b}(x_n, x_{n+1}, x_{n+1})\, G_{\sigma_b}(x_n, x_{n+1}, x_{n+1})\Big) \\ &\qquad\quad \times\Big(1 + s^2 G^2_{\sigma_b}(x_n, x_{n+1}, x_{n+1})\Big)^{-1}\Big\} \\ &= G_{\sigma_b}(x_{n-1}, x_n, x_n). \end{aligned} \tag{23}$$

Therefore, $\{G_{\sigma_b}(x_n, x_{n+1}, x_{n+1})\}$ is a decreasing and bounded sequence of nonnegative real numbers. Then, there exists $r \geq 0$ such that $\lim_{n\to\infty} G_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) = r$. Letting $n \to \infty$ in (22), we have

$$sr \leq r. \tag{24}$$

Since $s > 1$, we deduce that $r = 0$, that is

$$\lim_{n\to\infty} G_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) = 0. \tag{25}$$

By Proposition 15(2), we conclude that

$$\lim_{n\to\infty} G_{\sigma_b}(x_n, x_n, x_{n+1}) = 0. \tag{26}$$

Step II. Now, we prove that the sequence $\{x_n\}$ is a G_{σ_b}-Cauchy sequence. For this purpose, we will show that

$$\limsup_{m,n\to\infty} G_{\sigma_b}(x_n, x_m, x_m) = 0. \tag{27}$$

Using the rectangular inequality with (21) (as $\alpha(x_n, x_m, x_m) \geq 1$, since f is a rectangular G-α-admissible mapping), we have

$$\begin{aligned} &G_{\sigma_b}(x_n, x_m, x_m) \\ &\quad \leq sG_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) \\ &\qquad + s^2 G_{\sigma_b}(x_{n+1}, x_{m+1}, x_{m+1}) + s^2 G_{\sigma_b}(x_{m+1}, x_m, x_m) \\ &\quad \leq sG_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) \\ &\qquad + s\beta\left(G_{\sigma_b}(x_n, x_m, x_m)\right) M_s(x_n, x_m, x_m) \\ &\qquad + s^2 G_{\sigma_b}(x_{m+1}, x_m, x_m). \end{aligned} \tag{28}$$

Taking limit as $m, n \to \infty$ in the above inequality and applying (25) and (26), we have

$$\begin{aligned} &\limsup_{m,n\to\infty} G_{\sigma_b}(x_n, x_m, x_m) \\ &\quad \leq s \limsup_{n,m\to\infty} \beta\left(G_{\sigma_b}(x_n, x_m, x_m)\right) \limsup_{n,m\to\infty} M_s(x_n, x_m, x_m). \end{aligned} \tag{29}$$

Here,

$$\begin{aligned} &G_{\sigma_b}(x_n, x_m, x_m) \\ &\quad \leq M_s(x_n, x_m, x_m) \\ &\quad = \max\left\{G_{\sigma_b}(x_n, x_m, x_m), \right. \\ &\qquad \left. \frac{G_{\sigma_b}(x_n, fx_n, fx_n)\left[G_{\sigma_b}(x_m, fx_m, fx_m)\right]^2}{1 + s^2 G^2_{\sigma_b}(fx_n, fx_m, fx_m)}\right\} \\ &\quad = \max\left\{G_{\sigma_b}(x_n, x_m, x_m), \right. \\ &\qquad \left. \frac{G_{\sigma_b}(x_n, x_{n+1}, x_{n+1})\left[G_{\sigma_b}(x_m, x_{m+1}, x_{m+1})\right]^2}{1 + s^2 G^2_{\sigma_b}(x_{n+1}, x_{m+1}, x_{m+1})}\right\}. \end{aligned} \tag{30}$$

Letting $m, n \to \infty$ in the above inequality, we get

$$\limsup_{m,n\to\infty} M_s(x_n, x_m, x_m) = \limsup_{m,n\to\infty} G_{\sigma_b}(x_n, x_m, x_m). \quad (31)$$

Hence, from (29) and (31), we obtain

$$\limsup_{m,n\to\infty} G_{\sigma_b}(x_n, x_m, x_m) \leq s \limsup_{m,n\to\infty} \beta\left(G_{\sigma_b}(x_n, x_m, x_m)\right) \limsup_{m,n\to\infty} G_{\sigma_b}(x_n, x_m, x_m). \quad (32)$$

If $\limsup_{m,n\to\infty} G_{\sigma_b}(x_n, x_m, x_m) \neq 0$, then we get

$$\frac{1}{s} \leq \limsup_{m,n\to\infty} \beta\left(G_{\sigma_b}(x_n, x_m, x_m)\right). \quad (33)$$

Since $\beta \in \mathcal{F}$, we deduce that

$$\limsup_{m,n\to\infty} G_{\sigma_b}(x_n, x_m, x_m) = 0, \quad (34)$$

which is a contradiction. Consequently, $\{x_n\}$ is a G_{σ_b}-Cauchy sequence in X. Since (X, G_{σ_b}) is G_{σ_b}-complete, there exists $u \in X$ such that $x_n \to u$, as $n \to \infty$. Now, from (34) and G_{σ_b}-completeness of X,

$$\lim_{n\to\infty} G_{\sigma_b}(u, x_n, x_n) = \lim_{m,n\to\infty} G_{\sigma_b}(x_n, x_m, x_m) = G_{\sigma_b}(u, u, u) = 0. \quad (35)$$

Step III. Now, we show that u is a fixed point of f.

Using the rectangle inequality, we get

$$G_{\sigma_b}(u, fu, fu) \leq sG_{\sigma_b}(u, fx_n, fx_n) + sG_{\sigma_b}(fx_n, fu, fu). \quad (36)$$

Letting $n \to \infty$ and using the continuity of f and (35), we obtain

$$G_{\sigma_b}(u, fu, fu) \leq s \lim_{n\to\infty} G_{\sigma_b}(u, fx_n, fx_n) + s \lim_{n\to\infty} G_{\sigma_b}(fx_n, fu, fu) = sG_{\sigma_b}(fu, fu, fu). \quad (37)$$

Note that, from (21), as $\alpha(u, u, u) \geq 1$, we have

$$sG_{\sigma_b}(fu, fu, fu) \leq \beta\left(G_{\sigma_b}(u, u, u)\right) M_s(u, u, u), \quad (38)$$

where, by (37),

$$M_s(u, u, u) = \max\left\{G_{\sigma_b}(u, u, u), \frac{G_{\sigma_b}(u, fu, fu)\, G_{\sigma_b}(u, fu, fu)\, G_{\sigma_b}(u, fu, fu)}{1 + s^2 G^2_{\sigma_b}(fu, fu, fu)}\right\} < G_{\sigma_b}(u, fu, fu). \quad (39)$$

Hence, as $\beta(t) \leq 1$ for all $t \in [0, \infty)$, we have $sG_{\sigma_b}(fu, fu, fu) \leq G_{\sigma_b}(u, fu, fu)$. Thus, by (37), we obtain that $sG_{\sigma_b}(fu, fu, fu) = G_{\sigma_b}(u, fu, fu)$. But then, using (38), we get that

$$G_{\sigma_b}(u, fu, fu) = sG_{\sigma_b}(fu, fu, fu) \leq \beta\left(G_{\sigma_b}(u, u, u)\right) M_s(u, u, u) < \frac{1}{s} G_{\sigma_b}(u, fu, fu), \quad (40)$$

which is a contradiction. Hence, we have $fu = u$. Thus, u is a fixed point of f. □

We replace condition (ii) in Theorem 26 by regularity of the space X.

Theorem 27. *Under the same hypotheses of Theorem 26, instead of condition (ii), assume that whenever $\{x_n\}$ in X is a sequence such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, one has $\alpha(x_n, x, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Then, f has a fixed point.*

Proof. Repeating the proof of Theorem 26, we can construct a sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to u \in X$ for some $u \in X$. Using the assumption on X, we have $\alpha(x_n, u, u) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now, we show that $u = fu$. By Lemma 21 and (35),

$$\begin{aligned} s\left[\frac{1}{s} G_{\sigma_b}(u, fu, fu) - G_{\sigma_b}(u, u, u)\right] &\leq s \limsup_{n\to\infty} G_{\sigma_b}(x_{n+1}, fu, fu) \\ &\leq \limsup_{n\to\infty} \left(\beta\left(G_{\sigma_b}(x_n, u, u)\right) M_s(x_n, u, u)\right) \\ &\leq \frac{1}{s} \limsup_{n\to\infty} M_s(x_n, u, u), \end{aligned} \quad (41)$$

where

$$\begin{aligned} \lim_{n\to\infty} M_s(x_n, u, u) &= \lim_{n\to\infty} \max\left\{G_{\sigma_b}(x_n, u, u), \frac{\left[G_{\sigma_b}(x_n, fx_n, fx_n)\right]\left[G_{\sigma_b}(u, fu, fu)\right]^2}{1 + s^2 G^2_{\sigma_b}(fx_n, fu, fu)}\right\} \\ &= \lim_{n\to\infty} \max\left\{G_{\sigma_b}(x_n, u, u), \frac{\left[G_{\sigma_b}(x_n, x_{n+1}, x_{n+1})\right]\left[G_{\sigma_b}(u, fu, fu)\right]^2}{1 + s^2 G^2_{\sigma_b}(x_{n+1}, fu, fu)}\right\} \\ &= \max\left\{G_{\sigma_b}(u, u, u), 0\right\} = 0 \ \text{(see (25) and (35))}. \end{aligned} \quad (42)$$

Therefore, we deduce that $G_{\sigma_b}(u, fu, fu) \leq 0$. Hence, we have $u = fu$. □

A mapping $\varphi : [0,\infty) \to [0,\infty)$ is called a *comparison function* if it is increasing and $\varphi^n(t) \to 0$, as $n \to \infty$ for any $t \in [0,\infty)$ (see, e.g., [23, 24] for more details and examples).

Definition 28. A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be a (c)-comparison function if

(c_1) φ is increasing,

(c_2) there exists $k_0 \in \mathbb{N}$, $a \in (0,1)$, and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k$ for $k \geq k_0$ and any $t \in [0,\infty)$.

Later, Berinde [5] introduced the notion of (b)-comparison function as a generalization of (c)-comparison function.

Definition 29 (see [5]). Let $s \geq 1$ be a real number. A mapping $\varphi : [0,\infty) \to [0,\infty)$ is called a (b)-comparison function if the following conditions are fulfilled:

(1) φ is increasing;

(2) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$, and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) + v_k$ for $k \geq k_0$ and any $t \in [0,\infty)$.

Let Ψ_b be the class of all (b)-comparison functions $\varphi : [0,\infty) \to [0,\infty)$. It is clear that the notion of (b)-comparison function coincides with (c)-comparison function for $s = 1$.

Lemma 30 (see [25]). *If $\varphi : [0,\infty) \to [0,\infty)$ is a (b)-comparison function, then we have the following:*

(1) *the series $\sum_{k=0}^{\infty} s^k\varphi^k(t)$ converges for any $t \in \mathbb{R}_+$;*

(2) *the function $b_s : [0,\infty) \to [0,\infty)$ defined by $b_s(t) = \sum_{k=0}^{\infty} s^k\varphi^k(t)$, $t \in [0,\infty)$, is increasing and continuous at 0.*

Remark 31. It is easy to see that if $\varphi \in \Psi_b$, then we have $\varphi(0) = 0$ and $\varphi(t) < t$ for each $t > 0$ and φ is continuous at 0.

In the next example, we present a class of (b)-comparison functions.

Example 32. Any function of the form $\psi(t) = \ln((a/s)t+1)$ for all $t \in [0,\infty)$ where $0 < a < 1$ is a (b)-comparison function.

Proof. From the part (1) of Lemma 30, the necessary condition is that the series $\sum_{k=0}^{\infty} s^k\varphi^k(t)$ converges for any $t \in \mathbb{R}_+$. But, for each $t > 0$ and $k \geq 1$, we have

$$\begin{aligned} s^k\varphi^k(t) &= s^k\varphi\left(\varphi^{k-1}(t)\right) \\ &= s^k \ln\left(\frac{a}{s}\varphi^{k-1}(t) + 1\right) \leq s^k\frac{a}{s}\varphi^{k-1}(t) \\ &\leq \cdots \leq s^k\left(\frac{a}{s}\right)^k t = a^k t. \end{aligned} \tag{43}$$

So, according to the comparison test of the series, we should have $a < 1$. On the other hand, we have

$$\begin{aligned} s^{k+1}\varphi^{k+1}(t) &= s^{k+1}\varphi\left(\varphi^k(t)\right) \\ &= s^{k+1}\ln\left(\frac{a}{s}\varphi^k(t) + 1\right) \\ &\leq s^{k+1}\frac{a}{s}\varphi^k(t) = s^k a\varphi^k(t). \end{aligned} \tag{44}$$

Therefore, for any convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ and each $k \geq k_0 = 1$, we have

$$s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) \leq as^k\varphi^k(t) + v_k. \tag{45}$$

□

For example, for $s = 2$ and $a = 1/2$, the function $\psi(t) = \ln(t/4 + 1)$ is a (b)-comparison function.

Theorem 33. *Let (X, G_{σ_b}) be a G_{σ_b}-complete generalized b-metric-like space and let $f : X \to X$ be a G-α-admissible mapping. Suppose that*

$$s\alpha(x,y,z)\,G_{\sigma_b}(fx,fy,fz) \leq \psi\left(M_s(x,y,z)\right), \tag{46}$$

for all $x, y, z \in X$ where $\psi \in \Psi_b$ and

$$\begin{aligned} &M_s(x,y,z) \\ &= \max\left\{G_{\sigma_b}(x,y,z),\right. \\ &\qquad \left.\frac{G_{\sigma_b}(x,x,fx)\,G_{\sigma_b}(y,y,fy)\,G_{\sigma_b}(z,z,fz)}{1 + 2s^2G^2_{\sigma_b}(fx,fy,fz)}\right\}. \end{aligned} \tag{47}$$

Also, suppose that the following assertions hold:

(i) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, f^2x_0) \geq 1$;

(ii)

(a) f is continuous and, for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ such that $x_n \to x$ as $n \to \infty$, one has $\alpha(x,x,x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$;

(b) assume that whenever $\{x_n\}$ in X is a sequence such that $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x$ as $n \to \infty$, one has $\alpha(x_n, x, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, f has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, fx_0, f^2x_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0$ for all $n \in \mathbb{N}$. Since f is a G-α-admissible mapping and $\alpha(x_0, x_1, x_2) = \alpha(x_0, fx_0, f^2x_0) \geq 1$, we deduce that $\alpha(x_1, x_2, x_3) = \alpha(fx_0, fx_1, fx_2) \geq 1$. Continuing this process, we get $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = fx_{n_0}$ and so we have nothing to prove. Hence, for all $n \in \mathbb{N}$, we assume that $x_n \neq x_{n+1}$.

Step I (*Cauchyness of* $\{x_n\}$). As $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all $n \geq 0$, using condition (46), we obtain

$$\begin{aligned} sG_{\sigma_b}(x_n, x_{n+1}, x_{n+2}) &\leq s\alpha(x_{n-1}, x_n, x_{n+1}) G_{\sigma_b}(x_n, x_{n+1}, x_{n+2}) \\ &= s\alpha(x_{n-1}, x_n, x_{n+1}) G_{\sigma_b}(fx_{n-1}, fx_n, fx_{n+1}) \\ &\leq \psi(M_s(x_{n-1}, x_n, x_{n+1})). \end{aligned} \tag{48}$$

Using Proposition 15(2) as $x_n \neq x_{n+1}$, we get

$$\begin{aligned} M_s(x_{n-1}, x_n, x_{n+1}) &= \max\Big\{G_{\sigma_b}(x_{n-1}, x_n, x_{n+1}), \\ &\quad \big(G_{\sigma_b}(x_{n-1}, x_{n-1}, fx_{n-1}) \\ &\quad \times G_{\sigma_b}(x_n, x_n, fx_n) \\ &\quad \times G_{\sigma_b}(x_{n+1}, x_{n+1}, fx_{n+1})\big) \\ &\quad \times\big(1 + 2s^2 G^2_{\sigma_b}(fx_{n-1}, fx_n, fx_{n+1})\big)^{-1}\Big\} \\ &= \max\Big\{G_{\sigma_b}(x_{n-1}, x_n, x_{n+1}), \\ &\quad \big(G_{\sigma_b}(x_{n-1}, x_{n-1}, x_n) \\ &\quad \times G_{\sigma_b}(x_n, x_n, x_{n+1}) \\ &\quad \times G_{\sigma_b}(x_{n+1}, x_{n+1}, x_{n+2})\big) \\ &\quad \times\big(1 + 2s^2 G^2_{\sigma_b}(x_n, x_{n+1}, x_{n+2})\big)^{-1}\Big\} \\ &\leq \max\left\{G_{\sigma_b}(x_{n-1}, x_n, x_{n+1}), \right. \\ &\quad \left. \frac{2sG_{\sigma_b}(x_{n-1}, x_n, x_{n+1}) G^2_{\sigma_b}(x_n, x_{n+1}, x_{n+2})}{1 + 2s^2 G^2_{\sigma_b}(x_n, x_{n+1}, x_{n+2})}\right\} \\ &= G_{\sigma_b}(x_{n-1}, x_n, x_{n+1}). \end{aligned} \tag{49}$$

Hence,

$$\begin{aligned} G_{\sigma_b}(x_n, x_{n+1}, x_{n+2}) &\leq sG_{\sigma_b}(x_n, x_{n+1}, x_{n+2}) \leq \psi\big(G_{\sigma_b}(x_{n-1}, x_n, x_{n+1})\big). \end{aligned} \tag{50}$$

By induction, since $x_n \neq x_{n+1}$, we get that

$$\begin{aligned} G_{\sigma_b}(x_n, x_{n+1}, x_{n+2}) &\leq \psi\big(G_{\sigma_b}(x_{n-1}, x_n, x_{n+1})\big) \\ &\leq \psi^2\big(G_{\sigma_b}(x_{n-2}, x_{n-1}, x_n)\big) \\ &\leq \cdots \leq \psi^n\big(G_{\sigma_b}(x_0, x_1, x_2)\big). \end{aligned} \tag{51}$$

Let $\varepsilon > 0$ be arbitrary. Then, there exists a natural number N such that

$$\sum_{n=N}^{\infty} s^n \psi^n\big(G_{\sigma_b}(x_0, x_1, x_2)\big) < \frac{\varepsilon}{2s}. \tag{52}$$

Let $m > n \geq N$. Then, by the rectangular inequality and Proposition 15(2) as $x_n \neq x_{n+1}$, we get

$$\begin{aligned} &G_{\sigma_b}(x_n, x_m, x_m) \\ &\quad \leq sG_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) + s^2 G_{\sigma_b}(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\qquad + \cdots + s^{m-n-1} G_{\sigma_b}(x_{m-1}, x_m, x_m) \\ &\quad \leq 2s\big(sG_{\sigma_b}(x_n, x_n, x_{n+1}) + s^2 G_{\sigma_b}(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\qquad + \cdots + s^{m-n-1} G_{\sigma_b}(x_{m-1}, x_{m-1}, x_m)\big) \\ &\quad \leq 2s\big(sG_{\sigma_b}(x_n, x_{n+1}, x_{n+2}) + s^2 G_{\sigma_b}(x_{n+1}, x_{n+2}, x_{n+3}) \\ &\qquad + \cdots + s^{m-n-1} G_{\sigma_b}(x_{m-1}, x_m, x_{m+1})\big) \\ &\quad \leq 2s\sum_{k=n}^{m-2} s^{k-n+1}\psi^k\big(G_{\sigma_b}(x_0, x_1, x_2)\big) \\ &\quad \leq 2s\sum_{k=n}^{\infty} s^k\psi^k\big(G_{\sigma_b}(x_0, x_1, x_2)\big) < \varepsilon. \end{aligned} \tag{53}$$

Consequently, $\{x_n\}$ is a G_{σ_b}-Cauchy sequence in X. Since (X, G_{σ_b}) is G_{σ_b}-complete, so there exists $u \in X$ such that

$$\begin{aligned} \lim_{n\to\infty} G_{\sigma_b}(u, x_n, x_n) &= \lim_{m,n\to\infty} G_{\sigma_b}(x_n, x_m, x_m) \\ &= G_{\sigma_b}(u, u, u) = 0. \end{aligned} \tag{54}$$

Step II. Now, we show that u is a fixed point of f. Suppose to the contrary, that is, $fu \neq u$, then, we have $G_{\sigma_b}(u, u, fu) > 0$.

Let the part (a) of (ii) holds.

Using the rectangle inequality, we get

$$G_{\sigma_b}(u, u, fu) \leq sG_{\sigma_b}(fu, fx_n, fx_n) + sG_{\sigma_b}(fx_n, u, u). \tag{55}$$

Letting $n \to \infty$ and using the continuity of f, we get

$$G_{\sigma_b}(u, u, fu) \leq sG_{\sigma_b}(fu, fu, fu). \tag{56}$$

From (46) and part (a) of condition (ii), we have

$$sG_{\sigma_b}(fu, fu, fu) \leq \psi(M_s(u, u, u)), \tag{57}$$

where, by using (56), we have

$$\begin{aligned} M_s(u, u, u) &= \max\left\{G_{\sigma_b}(u, u, u), \right. \\ &\quad \left. \frac{G_{\sigma_b}(u, u, fu) G_{\sigma_b}(u, u, fu) G_{\sigma_b}(u, u, fu)}{1 + 2s^2 G^2_{\sigma_b}(fu, fu, fu)}\right\} \\ &\leq G_{\sigma_b}(u, u, fu). \end{aligned} \tag{58}$$

Hence, from properties of ψ, $sG_{\sigma_b}(fu, fu, fu) \leq G_{\sigma_b}(u, u, fu)$. Thus, by (56), we obtain that

$$G_{\sigma_b}(u, u, fu) = sG_{\sigma_b}(fu, fu, fu). \tag{59}$$

Moreover, (57) yields that $G_{\sigma_b}(u,u,fu) \le \psi(M_s(u,u,u)) \le \psi(G_{\sigma_b}(u,u,fu))$. This is impossible, according to our assumptions on ψ. Hence, we have $fu = u$. Thus, u is a fixed point of f.

Now, let part (b) of (ii) holds.

As $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}, x_{n+2}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x, x) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, we show that $u = fu$. By (46), we have

$$\begin{aligned} sG_{\sigma_b}(fu, fu, x_n) &\le s\alpha(u,u,x_{n-1})\, G_{\sigma_b}(fu, fu, fx_{n-1}) \\ &\le \psi(M_s(u,u,x_{n-1})), \end{aligned} \tag{60}$$

where

$$\begin{aligned} M_s(u,u,x_{n-1}) &= \max\left\{ G_{\sigma_b}(u,u,x_{n-1}), \right. \\ &\qquad \left. \frac{\left[G_{\sigma_b}(u,u,fu)\right]^2 G_{\sigma_b}(x_{n-1}, x_{n-1}, fx_{n-1})}{1 + \left[sG_{\sigma_b}(fu, fu, fx_{n-1})\right]^2} \right\}. \end{aligned} \tag{61}$$

Letting $n \to \infty$ in the above inequality and using (54) and Lemma 21, we get

$$\lim_{n\to\infty} M_s(u,u,x_{n-1}) = 0. \tag{62}$$

Again, taking the upper limit as $n \to \infty$ in (60) and using (62) and Lemma 21, we obtain

$$\begin{aligned} s\left[\frac{1}{s} G_{\sigma_b}(u, fu, fu) - G_{\sigma_b}(u,u,u)\right] &\le s \limsup_{n\to\infty} G_{\sigma_b}(x_n, fu, fu) \\ &\le \limsup_{n\to\infty} \psi(M_s(u,u,x_{n-1})) \\ &= \psi\left(\limsup_{n\to\infty} M_s(u,u,x_{n-1})\right) \\ &= \psi(0) = 0. \end{aligned} \tag{63}$$

So, we get $G_{\sigma_b}(u, fu, fu) = 0$. That is, $fu = u$. □

Let $(X, G_{\sigma_b}, \preceq)$ be a partially ordered G_{σ_b}-metric-like space. We say that T is an increasing mapping on X if $\forall x\ y \in X$, $x \preceq y \Rightarrow T(x) \preceq T(y)$ [26]. Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [27–30] and references therein). From the results proved above, we derive the following new results in partially ordered G_{σ_b}-metric-like space.

Theorem 34. *Let $(X, G_{\sigma_b}, \preceq)$ be a partially ordered G_{σ_b}-complete generalized b-metric-like space and let $f : X \to X$ be an increasing mapping. Suppose that*

$$sG_{\sigma_b}(fx, fy, fz) \le \beta\left(G_{\sigma_b}(x,y,z)\right) M_s(x,y,z), \tag{64}$$

for all $x, y, z \in X$ with $x \preceq y \preceq z$, where

$$\begin{aligned} M_s(x,y,z) &= \max\left\{ G_{\sigma_b}(x,y,z), \right. \\ &\qquad \left. \frac{G_{\sigma_b}(x, fx, fx)\, G_{\sigma_b}(y, fy, fy)\, G_{\sigma_b}(z, fz, fz)}{1 + s^2 G^2_{\sigma_b}(fx, fy, fz)} \right\}. \end{aligned} \tag{65}$$

Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;*

(ii) *f is continuous or assume that whenever $\{x_n\}$ in X is an increasing sequence such that $x_n \to x$ as $n \to \infty$, one has $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then, f has a fixed point.

Proof. Define $\alpha : X \times X \times X \to [0, +\infty)$ by

$$\alpha(x,y,z) = \begin{cases} 1, & \text{if } x \preceq y \preceq z \\ 0, & \text{otherwise.} \end{cases} \tag{66}$$

First, we prove that f is a triangular α-admissible mapping. Hence, we assume that $\alpha(x,y,z) \ge 1$. Therefore, we have $x \preceq y \preceq z$. Since f is increasing, we get $fx \preceq fy \preceq fz$; that is, $\alpha(fx, fy, fz) \ge 1$. Also, let $\alpha(x,y,y) \ge 1$ and $\alpha(y,z,z) \ge 1$; then $x \preceq y$ and $y \preceq z$. Consequently, we deduce that $x \preceq z$; that is, $\alpha(x,z,z) \ge 1$. Thus, f is a triangular α-admissible mapping. Since f satisfies (64) so, by the definition of α, we have

$$\begin{aligned} s\alpha(x,y,z)\, G_{\sigma_b}(fx, fy, fz) &\le \beta\left(G_{\sigma_b}(x,y,z)\right) M_s(x,y,z) \end{aligned} \tag{67}$$

for all $x, y, z \in X$. Therefore, f satisfies the contractive condition (21). From (i), there exists $x_0 \in X$ such that $x_0 \preceq fx_0$; that is, $\alpha(x_0, fx_0, fx_0) \ge 1$. According to (ii), we conclude that all the conditions of Theorems 26 and 27 are satisfied and so f has a fixed point. □

Similarly, using Theorem 33, we can prove following result.

Theorem 35. *Let $(X, G_{\sigma_b}, \preceq)$ be an ordered G_{σ_b}-complete generalized b-metric-like space and let $f : X \to X$ be an increasing mapping. Suppose that*

$$sG_{\sigma_b}(fx, fy, fz) \le \psi(M_s(x,y,z)), \tag{68}$$

for all $x, y, z \in X$ with $x \preceq y \preceq z$ where $\psi \in \Psi_b$ and

$$\begin{aligned} M_s(x,y,z) &= \max\left\{ G_{\sigma_b}(x,y,z), \right. \\ &\qquad \left. \frac{G_{\sigma_b}(x, x, fx)\, G_{\sigma_b}(y, y, fy)\, G_{\sigma_b}(z, z, fz)}{1 + 2s^2 G^2_{\sigma_b}(fx, fy, fz)} \right\}. \end{aligned} \tag{69}$$

Also, suppose that the following assertions hold:

(i) *there exists* $x_0 \in X$ *such that* $x_0 \preceq fx_0$;

(ii)

(a) f *is continuous;*

(b) *assume that whenever* $\{x_n\}$ *in* X *is an increasing sequence such that* $x_n \to x$ *as* $n \to \infty$, *one has* $x_n \preceq x$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then, f *has a fixed point.*

We conclude this section by presenting some examples that illustrate our results.

Example 36. Let $X = [0, (-1+\sqrt{5})/2]$ be endowed with the usual ordering on $\mathbb{R}$. Define the generalized b-metric-like function G_{σ_b} given by

$$G_{\sigma_b}(x,y,z) = \max\left\{(x+y)^2, (y+z)^2, (z+x)^2\right\} \tag{70}$$

with $s = 2$. Consider the mapping $f : X \to X$ defined by $f(x) = (1/4)x\sqrt{e^{-(x+((-1+\sqrt{5})/2))^2}}$ and the function $\beta \in \mathcal{F}$ given by $\beta(t) = (1/2)e^{-t}, t > 0$, and $\beta(0) \in [0, 1/2)$. It is easy to see that f is an increasing function on X. We show that f is G_{σ_b}-continuous on X. By Proposition 20, it is sufficient to show that f is G_{σ_b}-sequentially continuous on X. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} G_{\sigma_b}(x_n, x, x) = G_{\sigma_b}(x, x, x)$, so we have $\max\{(\lim_{n\to\infty} x_n + x)^2, 4x^2\} = 4x^2$ and, equivalently, $\lim_{n\to\infty} x_n = \alpha \le x$. On the other hand, we have

$$\begin{aligned}
&\lim_{n\to\infty} G_{\sigma_b}(fx_n, fx, fx)\\
&= \lim_{n\to\infty} \max\left\{(fx_n + fx)^2, 4f^2x\right\}\\
&= \lim_{n\to\infty} \max\left\{\frac{1}{16}\left(x_n\sqrt{e^{-(x_n+(-1+\sqrt{5})/2)^2}} + x\sqrt{e^{-(x+(-1+\sqrt{5})/2)^2}}\right)^2, \frac{1}{4}x^2e^{-(x+(-1+\sqrt{5})/2)^2}\right\}\\
&= \max\left\{\frac{1}{16}\left(\lim_{n\to\infty} x_n\sqrt{e^{-(\lim_{n\to\infty}x_n+(-1+\sqrt{5})/2)^2}} + x\sqrt{e^{-(x+(-1+\sqrt{5})/2)^2}}\right)^2, \frac{1}{4}x^2e^{-(x+(-1+\sqrt{5})/2)^2}\right\}\\
&= \max\left\{\frac{1}{16}\left(\alpha\sqrt{e^{-(\alpha+(-1+\sqrt{5})/2)^2}} + x\sqrt{e^{-(x+(-1+\sqrt{5})/2)^2}}\right)^2, \frac{1}{4}x^2e^{-(x+(-1+\sqrt{5})/2)^2}\right\}\\
&= \frac{1}{4}x^2e^{-(x+(-1+\sqrt{5})/2)^2}\\
&= G_{\sigma_b}(fx, fx, fx).
\end{aligned} \tag{71}$$

So, f is G_{σ_b}-sequentially continuous on X.

For all elements $x, y, z \in X$, and the fact that $g(x) = x^2e^{-x^2}$ is an increasing function on X, we have

$$\begin{aligned}
&sG_{\sigma_b}(fx, fy, fz)\\
&= 2\max\left\{(fx+fy)^2, (fy+fz)^2, (fz+fx)^2\right\}\\
&= 2\max\left\{\left(\frac{1}{4}x\sqrt{e^{-(x+(-1+\sqrt{5})/2)^2}} + \frac{1}{4}y\sqrt{e^{-(y+(-1+\sqrt{5})/2)^2}}\right)^2, \left(\frac{1}{4}y\sqrt{e^{-(y+(-1+\sqrt{5})/2)^2}} + \frac{1}{4}z\sqrt{e^{-(z+(-1+\sqrt{5})/2)^2}}\right)^2, \left(\frac{1}{4}z\sqrt{e^{-(z+(-1+\sqrt{5})/2)^2}} + \frac{1}{4}x\sqrt{e^{-(x+(-1+\sqrt{5})/2)^2}}\right)^2\right\}\\
&\le 2\max\left\{\left(\frac{1}{4}x\sqrt{e^{-(x+y)^2}} + \frac{1}{4}y\sqrt{e^{-(y+x)^2}}\right)^2, \left(\frac{1}{4}y\sqrt{e^{-(y+z)^2}} + \frac{1}{4}z\sqrt{e^{-(z+y)^2}}\right)^2, \left(\frac{1}{4}z\sqrt{e^{-(z+x)^2}} + \frac{1}{4}x\sqrt{e^{-(x+z)^2}}\right)^2\right\}\\
&= \frac{1}{8}\max\left\{(x+y)^2e^{-(x+y)^2}, (y+z)^2e^{-(y+z)^2}, (z+x)^2e^{-(z+x)^2}\right\}\\
&= \frac{1}{8}e^{-\max\{(x+y)^2,(y+z)^2,(z+x)^2\}} \times \max\left\{(x+y)^2, (y+z)^2, (z+x)^2\right\}\\
&\le \frac{1}{2}e^{-\max\{(x+y)^2,(y+z)^2,(z+x)^2\}} \times \max\left\{(x+y)^2, (y+z)^2, (z+x)^2\right\}\\
&= \beta\left(G_{\sigma_b}(x,y,z)\right)G_{\sigma_b}(x,y,z)\\
&\le \beta\left(G_{\sigma_b}(x,y,z)\right)M_s(x,y,z).
\end{aligned} \tag{72}$$

Hence, f satisfies all the assumptions of Theorem 34 and thus it has a fixed point (which is $u = 0$).

Example 37. Let $X = [0, 1]$ with the usual ordering on $\mathbb{R}$. Define the generalized b-metric-like function G_{σ_b} given by $G_{\sigma_b}(x, y, z) = \max\{(x+y)^2, (y+z)^2, (z+x)^2\}$ with $s = 2$. Consider the mapping $f : X \to X$ defined by $f(x) =$

$\ln(1 + xe^{-x}/4)$ and the function $\psi \in \Psi_b$ given by $\psi(t) = (1/8)t$, $t \geq 0$. It is easy to see that f is increasing function. Now, we show that f is a G_{σ_b}-continuous function on X.

Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} G_{\sigma_b}(x_n, x, x) = G_{\sigma_b}(x, x, x)$, so we have $\max\{(\lim_{n\to\infty} x_n + x)^2, 4x^2\} = 4x^2$ and, equally, $\lim_{n\to\infty} x_n = \alpha \leq x$. On the other hand, we have

$$\begin{aligned}
&\lim_{n\to\infty} G_{\sigma_b}(fx_n, fx, fx)\\
&\quad = \lim_{n\to\infty} \max\left\{(fx_n + fx)^2, 4f^2x\right\}\\
&\quad = \max\left\{\left(\ln\left(1 + \frac{\lim_{n\to\infty} x_n e^{-\lim_{n\to\infty} x_n}}{4}\right)\right.\right.\\
&\qquad \left.\left. + \ln\left(1 + \frac{xe^{-x}}{4}\right)\right)^2,\right.\\
&\qquad \left.4\left(\ln\left(1 + \frac{xe^{-x}}{4}\right)\right)^2\right\}\\
&\quad = \max\left\{\left(\ln\left(1 + \frac{\alpha e^{-\alpha}}{4}\right) + \ln\left(1 + \frac{xe^{-x}}{4}\right)\right)^2,\right.\\
&\qquad \left.4\left(\ln\left(1 + \frac{xe^{-x}}{4}\right)\right)^2\right\}\\
&\quad = 4\left(\ln\left(1 + \frac{xe^{-x}}{4}\right)\right)^2\\
&\quad = G_{\sigma_b}(fx, fx, fx).
\end{aligned} \tag{73}$$

So, f is G_{σ_b}-sequentially continuous on X.

For all elements $x, y, z \in X$, we have

$$\begin{aligned}
&sG_{\sigma_b}(fx, fy, fz)\\
&\quad = 2\max\left\{\left(\ln\left(1 + \frac{xe^{-x}}{4}\right) + \ln\left(1 + \frac{ye^{-y}}{4}\right)\right)^2,\right.\\
&\qquad \left.\left(\ln\left(1 + \frac{ye^{-y}}{4}\right) + \ln\left(1 + \frac{ze^{-z}}{4}\right)\right)^2,\right.\\
&\qquad \left.\left(\ln\left(1 + \frac{ze^{-z}}{4}\right) + \ln\left(1 + \frac{xe^{-x}}{4}\right)\right)^2\right\}\\
&\quad \leq 2\max\left\{\left(\frac{xe^{-x}}{4} + \frac{ye^{-y}}{4}\right)^2, \left(\frac{ye^{-y}}{4} + \frac{ze^{-z}}{4}\right)^2,\right.\\
&\qquad \left.\left(\frac{ze^{-z}}{4} + \frac{xe^{-x}}{4}\right)^2\right\}\\
&\quad \leq \frac{1}{8}\max\left\{(x + y)^2, (y + z)^2, (z + x)^2\right\}\\
&\quad = \psi\left(G_{\sigma_b}(x, y, z)\right)\\
&\quad \leq \psi\left(M_s(x, y, z)\right).
\end{aligned} \tag{74}$$

Hence, f satisfies all the assumptions of Theorem 35 and thus it has a fixed point (which is $u = 0$).

3. Application

In this section, we present an application of our results to establish the existence of a solution to a periodic boundary value problem (see [30, 31]).

Let $X = C([0, T])$ be the set of all real continuous functions on [0,T]. We first endow X with the b-metric-like

$$\sigma_b(u, v) = \sup_{t\in[0,T]} (|u(t)| + |v(t)|)^p \tag{75}$$

for all $u, v \in X$ where $p > 1$ and then we endow it with the generalized b-metric-like G_{σ_b} defined by

$$\begin{aligned}
&G_{\sigma_b}(u, v, w)\\
&\quad = \max\left\{\sup_{t\in[0,T]} (|u(t)| + |v(t)|)^p,\right.\\
&\qquad \sup_{t\in[0,T]} (|v(t)| + |w(t)|)^p,\\
&\qquad \left.\sup_{t\in[0,T]} (|w(t)| + |u(t)|)^p\right\}.
\end{aligned} \tag{76}$$

Clearly, (X, G_{σ_b}) is a complete generalized b-metric-like space with parameter $K = 2^{p-1}$. We equip $C([0, T])$ with a partial order given by

$$x \preceq y \quad \text{iff } x(t) \leq y(t) \quad \forall t \in [0, T]. \tag{77}$$

Moreover, as in [30], it is proved that $(C([0, T]), \preceq)$ is regular; that is, whenever $\{x_n\}$ in X is an increasing sequence such that $x_n \to x$ as $n \to \infty$, we have $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Consider the first-order periodic boundary value problem

$$x'(t) = f(t, x(t)), \qquad x(0) = x(T), \tag{78}$$

where $t \in I = [0, T]$, with $T > 0$, and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

A lower solution for (78) is a function $\alpha \in C^1[0, T]$ such that

$$\begin{aligned}
\alpha'(t) &\leq f(t, \alpha(t)),\\
\alpha(0) &\leq \alpha(T),
\end{aligned} \tag{79}$$

where $t \in I = [0, T]$.

Assume that there exists $\lambda > 0$ such that, for all $x, y \in C[0, T]$, we have

$$\begin{aligned}
&\left|f(t, x(t)) + \lambda x(t)\right| + \left|f(t, y(t)) + \lambda y(t)\right|\\
&\quad \leq \frac{\lambda}{2^{p-1}}\sqrt[p]{\ln\left(\frac{a}{2^{p-1}}\left(|x(t)| + \left|y(t)\right|\right)^p + 1\right)},
\end{aligned} \tag{80}$$

where $0 \leq a < 1$. Then, the existence of a lower solution for (78) provides the existence of a solution of (78).

Problem (78) can be rewritten as

$$\begin{aligned}
x'(t) + \lambda x(t) &= f(t, x(t)) + \lambda x(t),\\
x(0) &= x(T).
\end{aligned} \tag{81}$$

Consider

$$\begin{aligned}
x'(t) + \lambda x(t) &= \delta_x(t) = F(t, x(t)),\\
x(0) &= x(T),
\end{aligned} \tag{82}$$

where $t \in I$.

Using variation of parameters formula, we get

$$x(t) = x(0)\, e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} \delta_x(s)\, ds \tag{83}$$

which yields

$$x(T) = x(0)\, e^{-\lambda T} + \int_0^T e^{-\lambda(T-s)} \delta_x(s)\, ds. \tag{84}$$

Since $x(0) = x(T)$, we get

$$x(0)\left[1 - e^{-\lambda T}\right] = e^{-\lambda T} \int_0^T e^{\lambda(s)} \delta_x(s)\, ds \tag{85}$$

or

$$x(0) = \frac{1}{e^{\lambda T} - 1} \int_0^T e^{\lambda s} \delta_x(s)\, ds. \tag{86}$$

Substituting the value of $x(0)$ in (83), we arrive at

$$x(t) = \int_0^T G(t,s)\, \delta_x(s)\, ds \tag{87}$$

where

$$G(t,s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}, & 0 \le s \le t \le T, \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}, & 0 \le t \le s \le T. \end{cases} \tag{88}$$

Now, define the operator $S : C[0,T] \to C[0,T]$ as

$$Sx(t) = \int_0^T G(t,s)\, F(s, x(s))\, ds. \tag{89}$$

The mapping S is increasing [31]. Note that if $u \in C[0,T]$ is a fixed point of S, then $u \in C^1[0,T]$ is a solution of (78).

Let $x, y, z \in C[0,T]$. Then, we have

$$\begin{aligned} &2^{p-1}\left[|Sx(t)| + |Sy(t)|\right] \\ &\quad = 2^{p-1}\left[\left|\int_0^T G(t,s)\, F(s,x(s))\, ds\right| + \left|\int_0^T G(t,s)\, F(s,y(s))\, ds\right|\right] \\ &\quad \le 2^{p-1} \int_0^T |G(t,s)| \times \left[|F(s,x(s))| + |F(s,y(s))|\right] ds \\ &\quad \le 2^{p-1} \int_0^T |G(t,s)|\, \frac{\lambda}{2^{p-1}} \times \sqrt[p]{\ln\left(\frac{a}{2^{p-1}}\left(|x(t)| + |y(t)|\right)^p + 1\right)}\, ds \\ &\quad \le \lambda \sqrt[p]{\ln\left(\frac{a}{2^{p-1}}\sigma_b(x,y) + 1\right)} \times \left[\int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds\right] \\ &\quad = \lambda \sqrt[p]{\ln\left(\frac{a}{2^{p-1}}\sigma_b(x,y) + 1\right)} \times \left[\frac{1}{\lambda\left(e^{\lambda T} - 1\right)}\left(e^{\lambda(T+s-t)}\Big|_0^t + e^{\lambda(s-t)}\Big|_t^T\right)\right] \\ &\quad = \lambda \sqrt[p]{\ln\left(\frac{a}{2^{p-1}}\sigma_b(x,y) + 1\right)} \times \left[\frac{1}{\lambda\left(e^{\lambda T} - 1\right)}\left(e^{\lambda T} - e^{\lambda(T-t)} + e^{\lambda(T-t)} - 1\right)\right] \\ &\quad = \sqrt[p]{\ln\left(\frac{a}{2^{p-1}}\sigma_b(x,y) + 1\right)} \\ &\quad \le \sqrt[p]{\ln\left(\frac{a}{2^{p-1}}G_{\sigma_b}(x,y,z) + 1\right)} \\ &\quad \le \sqrt[p]{\ln\left(\frac{a}{2^{p-1}}M(x,y,z) + 1\right)}. \end{aligned} \tag{90}$$

Similarly,

$$\begin{aligned} 2^{p-1}\left|Sy(t)\right| + |Sz(t)| &\le \sqrt[p]{\ln\left(\frac{a}{2^{p-1}}M(x,y,z) + 1\right)}, \\ 2^{p-1}|Sx(t)| + |Sz(t)| &\le \sqrt[p]{\ln\left(\frac{a}{2^{p-1}}M(x,y,z) + 1\right)}, \end{aligned} \tag{91}$$

where

$$\begin{aligned} &M(x,y,z) \\ &\quad = \max\left\{G_{\sigma_b}(x,y,z), \frac{G_{\sigma_b}(x,Sx,Sx)\, G_{\sigma_b}(y,Sy,Sy)\, G_{\sigma_b}(z,Sz,Sz)}{1 + 2^{2p-1} G_{\sigma_b}^2(Sx,Sy,Sz)}\right\}. \end{aligned} \tag{92}$$

Equivalently,

$$\begin{aligned} \left(2^{p-1}|Sx(t)| + \left|Sy(t)\right|\right)^p &\le \ln\left(\frac{a}{2^{p-1}}M(x,y,z) + 1\right), \\ \left(2^{p-1}\left|Sy(t)\right| + |Sz(t)|\right)^p &\le \ln\left(\frac{a}{2^{p-1}}M(x,y,z) + 1\right), \\ \left(2^{p-1}|Sx(t)| + |Sz(t)|\right)^p &\le \ln\left(\frac{a}{2^{p-1}}M(x,y,z) + 1\right), \end{aligned} \tag{93}$$

which yields that

$$\begin{aligned} 2^{p-1}\sigma_b(Sx,Sy) &\le \ln\left(\frac{a}{2^{p-1}}M(x,y,z) + 1\right), \\ 2^{p-1}\sigma_b(Sy,Sz) &\le \ln\left(\frac{a}{2^{p-1}}M(x,y,z) + 1\right), \\ 2^{p-1}\sigma_b(Sx,Sz) &\le \ln\left(\frac{a}{2^{p-1}}M(x,y,z) + 1\right). \end{aligned} \tag{94}$$

Finally, it is easy to obtain that

$$2^{p-1}G_{\sigma_b}(Sx, Sy, Sz) \leq \ln\left(\frac{a}{2^{p-1}}M(x, y, z) + 1\right). \quad (95)$$

Finally, since α is a lower solution for (78), so it is easy to show that $\alpha \preceq S(\alpha)$ [31].

Hence, the hypotheses of Theorem 35 are satisfied, with $\psi(t) = \ln((a/2^{p-1})t + 1)$ where $0 \leq a < 1$. Hence, there exists a fixed point $\widehat{x} \in C[0, T]$ such that $S\widehat{x} = \widehat{x}$ which is a solution to periodic boundary value problem (78).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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