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Application and Theory of Functional Analytical Methods in Summability

Lead Guest Editor: Mikail Et
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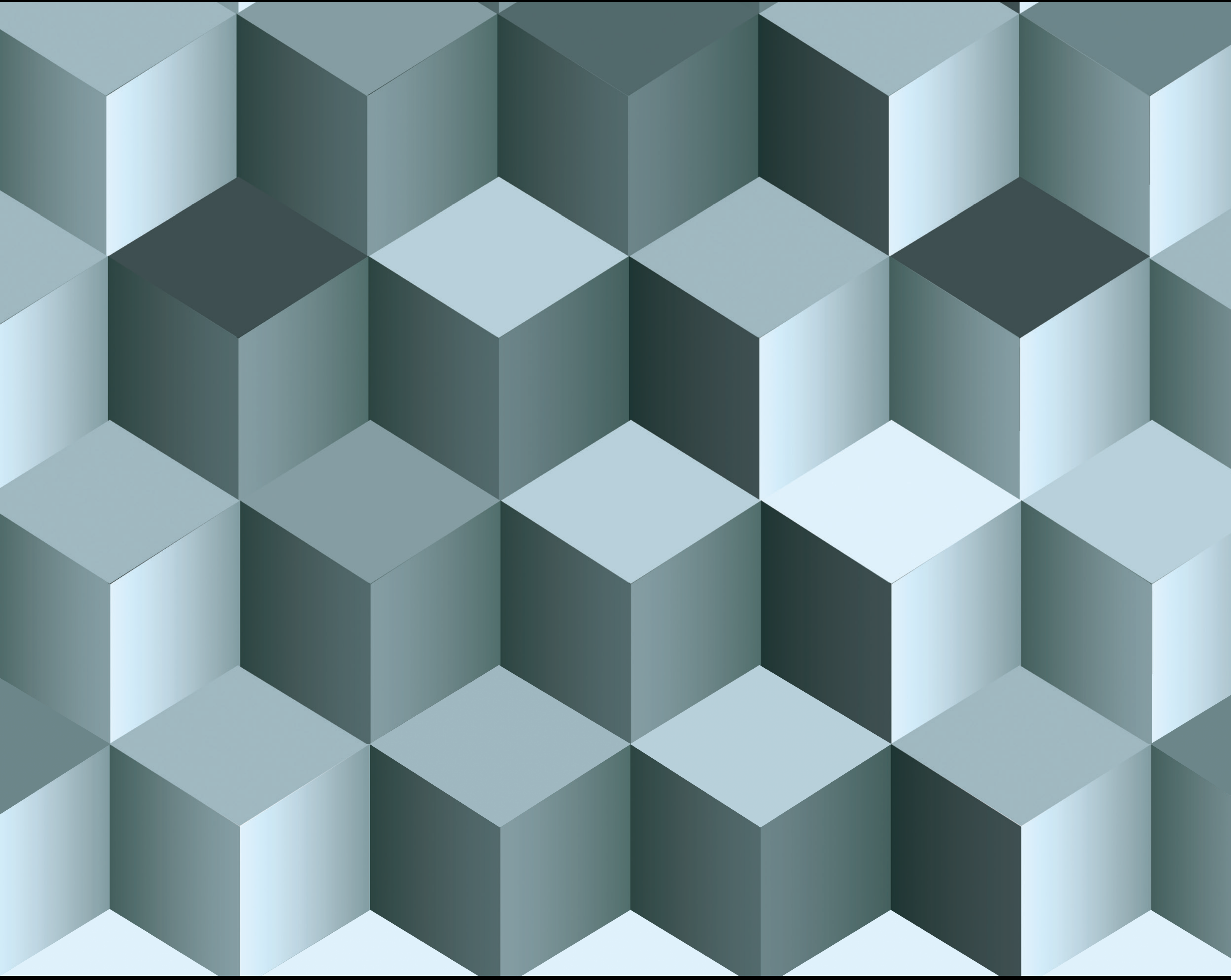

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Research Article

Examination of Generalized Statistical Convergence of Order α on Time Scales

Muhammed Çinar [1] and Mahmut Işik [2]

[1] *Department of Mathematics Education, Muş Alparslan University, Muş, Turkey*
[2] *Department of Mathematics Education, Harran University, Sanliurfa, Turkey*

Correspondence should be addressed to Muhammed Çinar; m.cinar@alparslan.edu.tr

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In this paper, we introduce the concepts of deferred statistical convergence of order α and strongly deferred Cesàro summable functions (real valued) of order α on time scales and give some relationships between deferred statistical convergence of order α and strongly deferred Cesàro summable functions (real valued) of order α on time scales.

1. Introduction

In mathematics, the concept of convergence has been of great importance for many years. This concept has been studied theoretically by many mathematicians in many different fields. Many types of convergence have been defined so far, and then, very valuable results and concepts have been presented to the mathematical community. One of these ideas is the converging statistics. In 1935, Zygmund [1] introduced the concept of statistical convergence to the mathematical community, Steinhaus [2] and Fast [3] independently introduced the concept of statistical convergence, and Schoenberg [4] reintroduced it in the year 1959. Then, it has been addressed under various titles including Fourier analysis, Ergodic theory, Number theory, Turnpike theory, Measure theory, Trigonometric series, and Banach spaces. The concept was later applied to summability theory by various authors such as Çinar et al. ([5, 6]), Çolak [7], Connor [8], Fridy [9], Altay et al. [10], Garcia and Kama [11], Isik et al. ([12–15]), Kucukaslan and Yılmazturk ([16, 17]), Šalát [18], Ercan et al. ([19–21]), and Parida et al. [22], and this concept has been extended to sequence spaces, accordingly, to the notions such as summability theory.

The natural density of a subset A of $\mathbb{N}$ is defined as

$$\delta(A) = \lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k), \tag{1}$$

provided that limit exists, where χ_A is the characteristic function of A. If

$$\delta(k \in \mathbb{N} : |x_k - L| \geq \varepsilon) = 0, \tag{2}$$

for each $\varepsilon > 0$, then $x = (x_k)$ is said to be statistically convergent to ℓ writing $S - \lim_{k \longrightarrow \infty} x_k = \ell$. Over the years, that notion has been presented in a variety of ways, and its relationship to aggregation has been investigated in several domains. In recent years, researchers have attempted to apply the relationship between statistical convergence and summability theory in applicable disciplines.

Now is the time to recall the key notions of our study deferred Cesàro mean and deferred statistical convergence.

Deferred Cesàro mean, defined by Agnew [23] in 1932, is a generalization of Cesàro mean, and its definition can be given as follows:

$$\left(D_{p,q}x\right)_m = \frac{1}{q(m)-p(m)} \sum_{p(m)+1}^{q(m)} x_k, \quad (3)$$

where $p = (p(m))$ and $q = (q(m))$ are the sequences of nonnegative integers satisfying

$$p(m) < q(m) \text{ and } \lim_{m \longrightarrow \infty} q(m) = \infty. \quad (4)$$

Küçükaslan and Yılmaztürk [16, 17] defined the concepts of derferred density and deferred statistical convergence by using the deferred Cesàro mean.

The deferred density of a subset A of the natural numbers $\mathbb{N}$ is defined by

$$\delta_{p,q}(A) = \lim_{m \longrightarrow \infty} \frac{1}{q(m)-p(m)} \left|A_{p,q}(m)\right|, \quad (5)$$

provided the limit exists, where $A_{p,q}(m) = \{p(m) < k \leq q(m) : k \in A\}$.

A real valued sequence $x = (x_k)$ is said to be deferred statistical convergent to L, if

$$\lim_{m \longrightarrow \infty} \frac{1}{q(m)-p(m)} \left|\{p(m) < k \leq q(m) : |x_k - L| \geq \varepsilon\}\right| = 0, \quad (6)$$

for each $\varepsilon > 0$ [16, 17].

Throughout the paper, we assume that the sequences $(p(m))$ and $(q(m))$ satisfy the following conditions

$$p(m) < q(m) \text{ and } \lim_{m \longrightarrow \infty} q(m) = \infty, \quad (7)$$

and additionally, $\lim_{m \longrightarrow \infty} (q(m) - p(m)) = \infty$.

In 1988, Hilger [24] proposed the time scale hypothesis. In 2001, Bohner and Peterson [25] published the first detailed explanation of the time scale theory. In 2003, Guseinov [26] developed a Measure theory on time scales. Cabada and Vivero [27] presented the Lebesque integral on time scales in 2006. These findings provide the foundation for time scale summability theory research. Many mathematicians in various domains have investigated the time scale calculus over the years [28]. As a result, it seems natural to generalize convergence on time scales in light of recent applications of time scales to real-world situations. Numerous writers in the literature have used statistical convergence to apply to time scales for various purposes (see [29–33]).

A nonempty closed subset of real numbers is called a time scale. Two basic concepts on the time scale are the forward jump operator and the backward jump operator. These can be given as follows:

(i) $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$, $\sigma(s) = \inf \{t \in \mathbb{T} : t > s\}$

(ii) $\rho(s) = \sup \{t \in \mathbb{T} : t < s\}$ for $s \in \mathbb{T}$

A Lebesque Δ-measure is defined on the family of intervals $[x, y)_{\mathbb{T}} = \{s \in \mathbb{T} : x \leq s \leq y\}$ on an arbitrary time scale $\mathbb{T}$ with the help of forward and backward jump operators. This defined measure is denoted by ν_Δ and provides the following properties:

(i) If $x \in \mathbb{T} \setminus \max\{\mathbb{T}\}$, then the set $\{x\}$ is Δ-measurable and $\nu_\Delta(x) = \sigma(x) - x$

(ii) If $x, y \in \mathbb{T}$ and $x \leq y$, then $\nu_\Delta([x, y)_{\mathbb{T}}) = y - x$ and $\nu_\Delta((x, y)_{\mathbb{T}}) = y - \sigma(x)$

(iii) If $x, y \in \mathbb{T} \setminus \max\{\mathbb{T}\}$ and $x \leq y$, then $\nu_\Delta((x, y]_{\mathbb{T}}) = \sigma(y) - \sigma(x)$ and $\nu_\Delta([x, y]_{\mathbb{T}}) = \sigma(y) - x$

Let Ω be a Δ-measurable subset of $\mathbb{T}$, and for $t \in \mathbb{T}$, write

$$\Omega(t) = \{s \in [t_0, t] : s \in \Omega\}. \quad (8)$$

Turan and Duman [32, 34] were defined the density and statistical convergence on time scales as follows:

$$\delta_{\mathbb{T}}(\Omega) = \lim_{t \longrightarrow \infty} \frac{\nu_\Delta(\Omega(t))}{\nu_\Delta([t_0, t])}, \quad (9)$$

provided that limit exists.

Let $f : \mathbb{T} \longrightarrow R$ be a Δ-measurable function. On $\mathbb{T}$, f is said to be statistically convergent to L if

$$\delta_{\mathbb{T}}(t \in \mathbb{T} : |f(t) - L| \geq \varepsilon) = 0, \quad (10)$$

for every $\varepsilon > 0$. In this case, we write $st_{\mathbb{T}} - \lim_{t \longrightarrow \infty} f(t) = L$.

2. Main Results

The aim of this study is to define the concepts of deferred Cesàro summability of order α and deferred statistical convergence of order α on the time scale and examine the relationships between them.

Definition 1. Let f be a Δ-delta measurable function on the time scale $\mathbb{T}$ and $\alpha \in (0, 1]$, and we say that f is deferred statistically convergent of order α on $\mathbb{T}$ to the number L if for every $\varepsilon > 0$,

$$\lim_{m \longrightarrow \infty} \frac{\nu_\Delta\{k \in (p(m), q(m)]_{\mathbb{T}} : |f(k) - L| \geqslant \varepsilon\}}{\nu_\Delta^\alpha((p(m), q(m)]_{\mathbb{T}})} = 0. \quad (11)$$

We will show this convergence with $D(st_{\mathbb{T}}^\alpha) - \lim f(t) = L$. We will denote by $DS_{\mathbb{T}}^\alpha[p, q]$ the set of all functions that deferred statistically convergent of order α on the time scale $\mathbb{T}$.

Obviously,

(i) If we get $q(s) = s$ and $p(s) = s_0$ and $\alpha = 1$, then we get the definition of statistical convergence in time scale [32]

(ii) If we take $q(m) = k(m)$, $p(m) = k(m-1)$, and $\alpha = 1$, then we get lacunary statistical convergence in time scale [34]

(iii) If we take $q(t) = t$, $p(t) = t - \lambda_t + t_0$, and $\alpha = 1$, then we get λ statistical convergence in time scale [35]

(iv) If we get $\alpha = 1$, then we get the definition of deferred statistical convergence in time scale [29]

Example 1. Let f be defined as in [36]. It is seen from the following inequality that the function f is deferred statistical convergence in $\mathbb{T}$ for $\alpha > 1/2$

$$\begin{aligned}&\frac{\nu_\Delta\{k \in (p(m), q(m)]_\mathbb{T} : |f(k)| \geq \varepsilon\}}{\nu_\Delta^\alpha((p(m), q(m)]_\mathbb{T})} \\ &\quad\leq \frac{\nu_\Delta\{(p(m), \sigma p(m) + [\sqrt{\nu_m}])\}}{\nu(m)^\alpha} \\ &\quad= \frac{[\sqrt{\nu_m}]}{\nu(m)^\alpha} \longrightarrow 0(m \longrightarrow \infty).\end{aligned} \tag{12}$$

Theorem 2. *If $f, g : \mathbb{T} \longrightarrow \mathbb{R}$ with $D(st_\mathbb{T}^\alpha) - \lim f(t) = L_1$ and $D(st_\mathbb{T}^\alpha) - \lim g(t) = L_2$, then the following statements hold*

(i) $D(st_\mathbb{T}^\alpha) - \lim(f(t) + g(t)) = L_1 + L_2$

(ii) $D(st_\mathbb{T}^\alpha) - \lim(cf(t)) = cL_1$

Proof. (i) Let $D(st_\mathbb{T}^\alpha) - \lim f(t) = L_1$ and $D(st_\mathbb{T}^\alpha) - \lim g(t) = L_2$. We write

$$\begin{aligned}&\frac{\nu_\Delta\{k \in (p(m), q(m)]_\mathbb{T} : |(f(k) + g(k)) - (L_1 + L_2)| \geq \varepsilon\}}{\nu_\Delta^\alpha((p(m), q(m)]_\mathbb{T})} \\ &\quad= \frac{\nu_\Delta\{k \in (p(m), q(m)]_\mathbb{T} : |f(k) - L_1| \geq \varepsilon\}}{\nu_\Delta^\alpha((p(m), q(m)]_\mathbb{T})} \\ &\qquad+ \frac{\nu_\Delta\{k \in (p(m), q(m)]_\mathbb{T} : |g(k) - L_2| \geq \varepsilon\}}{\nu_\Delta^\alpha((p(m), q(m)]_\mathbb{T})},\end{aligned} \tag{13}$$

for every $\varepsilon > 0$. Taking limit as $m \longrightarrow \infty$, (i) will be proved.

(ii) Let $D(st_\mathbb{T}^\alpha) - \lim f(t) = L$. Assume that $c \neq 0$; then, the proof of (ii) follows from

$$\begin{aligned}&\frac{\nu_\Delta\{k \in (p(m), q(m)]_\mathbb{T} : |cf(k) - cL| \geq \varepsilon\}}{\nu_\Delta^\alpha((p(m), q(m)]_\mathbb{T})} \\ &\quad= \frac{\nu_\Delta\{k \in (p(m), q(m)]_\mathbb{T} : |f(k) - L| \geq (\varepsilon/|c|)\}}{\nu_\Delta^\alpha((p(m), q(m)]_\mathbb{T})}.\end{aligned} \tag{14}$$

□

Definition 3. Let f be a Δ-delta measurable function on the time scale $\mathbb{T}$. Then, f is strongly deferred Cesàro summable of order α to L if

$$\lim_{m \longrightarrow \infty} \frac{1}{\nu_\Delta^\alpha((p(m), q(m)]_\mathbb{T})} \int_{(p(m),q(m)]_\mathbb{T}} |f(k) - L| \Delta k = 0. \tag{15}$$

By $D_\mathbb{T}^\alpha[p, q]$, we denote all strongly deferred Cesàro summable functions of order α on $\mathbb{T}$.

If we take the function f as follows, for $\alpha > 1/2$,

$$f(t) = \begin{cases} 1, \text{if } s \in (p(m), \sigma(p(m)) + 1),_\mathbb{T} \\ 1, \text{if } s \in [\sigma(p(m)) + 1, \sigma(p(m)) + 2)_\mathbb{T}, \\ \cdots \\ 1, \text{if } s \in [\sigma(p(m)) + [\sqrt{\nu_m}] - 1, \sigma(p(m)) + [\sqrt{\nu_m}])_\mathbb{T}, \\ 0, \text{otherwise}, \end{cases} \tag{16}$$

which is Cesàro summable.

Theorem 4. *Let f be a Δ-delta measurable function on the time scale $\mathbb{T}$. If $f \in D_\mathbb{T}^\alpha[p, q]$, then $f \in DS_\mathbb{T}^\alpha[p, q]$.*

Proof. Let $\varepsilon > 0$ and $D(\varepsilon) = \{s \in (p(m), q(m)]_\mathbb{T} : |f(s) - L| \geq \varepsilon\}$. The proof is obtained from the following inequality:

$$\int_{(p(m),q(m)]_\mathbb{T}} |f(s) - L| \Delta s \geq \int_{D(\varepsilon)} |f(s) - L| \Delta s \geq \varepsilon \mu_\Delta\{D(\varepsilon)\}. \tag{17}$$

□

Corollary 5. *Let f be a Δ-delta measurable function on the time scale $\mathbb{T}$ and $\alpha, \beta \in (0, 1]$ such that $\alpha < \beta$. If $f \in D_\mathbb{T}^\alpha[p, q]$, then $f \in DS_\mathbb{T}^\beta[p, q]$.*

To show that the inverse of Theorem 4 and Corollary 5 is not true, we can consider the example on page 3 of Çolak's article [7].

The converse of Theorem 4 and Corollary 5 is usually not satisfied, but provided that f is bounded, by taking $\alpha = 1$, we can give the following result.

Theorem 6. *Let f be a Δ-delta measurable function on the time scale $\mathbb{T}$. If $f \in DS_{\mathbb{T}}[p, q]$ and f is bounded, then $f \in D_{\mathbb{T}}[p, q]$.*

Proof. Suppose that $f \in DS_{\mathbb{T}}[p, q]$ and f is bounded. In this case, there is $K > 0$ such that $|f(k)| \leqslant K$ and also

$$\lim_{m \longrightarrow \infty} \frac{1}{\nu_\Delta((p(m), q(m)])} \nu_\Delta(\{k \in (p(m), q(m)]: |f(k) - L| \geqslant \varepsilon\}) = 0. \tag{18}$$

Therefore, we have

$$\begin{aligned} &\frac{1}{\nu_\Delta((p(m), q(m)])} \int_{(p(m),q(m)]_{\mathbb{T}}} |f(k) - L| \Delta k \\ &\quad = \frac{1}{\nu_\Delta(D(p(m), q(m)])} \int_{D(\varepsilon)} |f(k) - L| \Delta k \\ &\qquad + \frac{1}{\nu_\Delta((p(m), q(m)])} \int_{(p(m),q(m)]_{\mathbb{T}} \setminus D(\varepsilon)} |f(k) - L| \Delta k \\ &\quad \leqslant \frac{K}{\nu_\Delta((p(m), q(m)])} \int_{D(\varepsilon)} \Delta k \\ &\qquad + \frac{\varepsilon}{\nu_\Delta((p(m), q(m)])} \int_{(p(m),q(m)]_{\mathbb{T}}} \Delta k \\ &\quad = \frac{K \nu_\Delta(D(\varepsilon))}{\nu_\Delta((p(m), q(m)])} + \varepsilon. \end{aligned} \tag{19}$$

It is obvious that for $m \longrightarrow \infty$, the theorem is proved. □

Theorem 7. *Let f be a Δ-delta measurable function on the time scale $\mathbb{T}$ and $\nu_\Delta^\alpha((p(m), q(m)])/(\sigma(q(m)))^\alpha$ is bounded. If $st_{\mathbb{T}}^\alpha - \lim f(t) = L$, then $D(st_{\mathbb{T}}^\alpha) - \lim f(t) = L$.*

Proof. Since $st_{\mathbb{T}}^\alpha - \lim f(t) = L$, we write

$$\begin{aligned} &\frac{1}{\nu_\Delta^\alpha((t_0, q(m)])} \mu_\Delta\left(\left\{k \in (t_0, q(m)]_{\mathbb{T}} : |f(k) - L| \geqslant \varepsilon\right\}\right) \\ &\quad \geqslant \frac{\nu_\Delta(D(\varepsilon))}{(\sigma(q(m)) - t_0)^\alpha} \\ &\quad \geqslant \frac{(\sigma(q(m)) - \sigma(p(m)))^\alpha}{(\sigma(q(m)))^\alpha} \frac{\nu_\Delta(D(\varepsilon))}{(\sigma(q(m)) - \sigma(p(m)))^\alpha}. \end{aligned} \tag{20}$$

Clearly, for $m \longrightarrow \infty$, the theorem is proved. □

Corollary 8. *Let $(q(m))$ be an arbitrary sequence with $q(m) \in t_0, t]$, and $\nu_\Delta^\alpha(t_0, t])/\nu_\Delta^\alpha((p(m), q(m)])$ is bounded. Then, f is statistical convergence of order α to L on $\mathbb{T}$ implies f is deferred statistical convergence of order α to L on $\mathbb{T}$.*

Let the four sequences $(p(m))$, $(q(m))$, $(p'(m))$, and $(q'(m))$ are nonnegative real numbers such that

$$p(m) \leqslant p'(m) < q'(m) \leqslant q(m), \tag{21}$$

for all $m \in \mathbb{N}$.

Theorem 9. *Let $(p(m))$, $(q(m))$, $(p'(m))$, and $(q'(m))$ be given as in (21). If*

$$\lim_{m \longrightarrow \infty} \frac{\nu_\Delta^\alpha\left(\left(p'(m), q'(m)\right]\right)}{\nu_\Delta^\alpha((p(m), q(m)])} > 0, \tag{22}$$

then $f \in DS_{\mathbb{T}}^\alpha[p, q]$ implies $f \in DS_{\mathbb{T}}^\alpha[p', q']$.

Proof. Let $D'(\varepsilon) = \{k \in (p'(m), q'(m)]: |f(k) - L| \geqslant \varepsilon\}$. The proof is obtained from the following inequality:

$$\begin{aligned} &\frac{1}{\nu_\Delta^\alpha((p(m), q(m)])} \nu_\Delta(D(\varepsilon)) \\ &\quad \geqslant \frac{\nu_\Delta^\alpha\left(\left(p'(m), q'(m)\right]\right)}{\nu_\Delta^\alpha((p(m), q(m)])} \frac{1}{\nu_\Delta^\alpha\left(\left(p'(m), q'(m)\right]\right)} \nu_\Delta\left(D'(\varepsilon).\right). \end{aligned} \tag{23}$$

□

Corollary 10. *Let $(p(m))$, $(q(m))$, $(p'(m))$, and $(q'(m))$ be given as in (21) and $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$. If (22) holds, then $f \in DS_{\mathbb{T}}^\alpha[p, q]$ implies $f \in DS_{\mathbb{T}}^\beta[p', q']$.*

Theorem 11. *Let $(p(m))$, $(q(m))$, $(p'(m))$, and $(q'(m))$ be given as in (21). If*

$$\lim_{m \longrightarrow \infty} \frac{\nu_\Delta^\alpha((p(m), q(m)])}{\nu_\Delta^\alpha\left(\left(p'(m), q'(m)\right]\right)} > 0, \tag{24}$$

then f is deferred Cesàro summable to L of order α on $[p(m), q(m)]$ implies f is deferred Cesàro summable to L of order α on $[p'(m), q'(m)]$.

Proof. Proof follows from the following inequality:

$$\begin{aligned} &\frac{1}{\nu_\Delta^\alpha((p(m), q(m)])} \int_{(p(m),q(m)]_{\mathbb{T}}} |f(k) - L| \Delta k \\ &\quad \geqslant \frac{\nu_\Delta^\alpha\left(\left(p'(m), q'(m)\right]\right)}{\nu_\Delta^\alpha((p(m), q(m)])} \frac{1}{\nu_\Delta^\alpha\left(\left(p'(m), q'(m)\right]\right)} \int_{(p'(m),q'(m)]_{\mathbb{T}}} |f(k) - L| \Delta k. \end{aligned} \tag{25}$$

□

Corollary 12. *Let $(p(m))$, $(q(m))$, $(p'(m))$, and $(q'(m))$ be given as (21) and $\alpha, \beta \in (0, 1]$, $\alpha \leq \beta$. If (24) holds, then f is deferred Cesàro summable to L of order α on $[p(m), q(m)]$ implies f is deferred Cesàro summable to L of order α on $[p'(m), q'(m)]$.*

Theorem 13. *Let* $(p(m))$, $(q(m))$, $(p'(m))$, *and* $(q'(m))$ *be given as (21). If*

$$\lim_{m\longrightarrow\infty}\frac{\nu_\Delta\left(\left(p(m),p'(m)\right]\right)}{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)}>0,\ \lim_{m\longrightarrow\infty}\frac{\nu_\Delta\left(\left(q(m),q'(m)\right]\right)}{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)}>0. \tag{26}$$

If $f\in DS_{\mathbb{T}}^\alpha[p',q']$ *and* f *is bounded, then* $f\in D_{\mathbb{T}}^\alpha[p,q]$.

Proof. Suppose that $f\longrightarrow LDS_{\mathbb{T}}^\alpha[p',q']$. Since f is bounded, there exists a positive number K such that $|f(s)|\leqslant K$. Then, we may write

$$\begin{aligned}
&\frac{1}{\nu_\Delta^\alpha((p(m),q(m)])}\int_{[p(m),q(m)]_{\mathbb{T}}}|f(k)-L|\Delta k\\
&\quad=\frac{1}{\nu_\Delta^\alpha((p(m),q(m)])}\\
&\qquad\cdot\left[\int_{\left[p(m),p'(m)\right]_{\mathbb{T}}}|f(k)-L|\Delta k+\int_{\left[p'(m),q'(m)\right]_{\mathbb{T}}}|f(k)-L|\Delta k\right.\\
&\qquad\left.+\int_{\left[q'(m),q(m)\right]_{\mathbb{T}}}|f(k)-L|\Delta s\right]\\
&\quad\leqslant\frac{1}{\nu_\Delta^\alpha((p(m),q(m)])}\left[\int_{\left[p(m),p'(m)\right]_{\mathbb{T}}}K\Delta k\right.\\
&\qquad\left.+\int_{\left[p'(m),q'(m)\right]_{\mathbb{T}}}|f(k)-L|\Delta k+K\int_{\left[q'(m),q(m)\right]_{\mathbb{T}}}\Delta k\right]\\
&\quad\leqslant\frac{1}{\nu_\Delta^\alpha((p(m),q(m)])}\left(\sigma\left(p'(m)-\sigma(p(m))\right)\right.\\
&\qquad+\left(\sigma\left(q(m)-\sigma\left(q'(m)\right)\right)\right)+\frac{1}{\nu_\Delta^\alpha((p(m),q(m)])}\\
&\qquad\cdot\left[\int_{\left\{[p(m),q(m)]_{\mathbb{T}}:|f(k)-L|\geqslant\varepsilon\right\}}|f(k)-L|\Delta k\right.\\
&\qquad\left.+\int_{\left\{[p(m),q(m)]_{\mathbb{T}}:|f(k)-L|<\varepsilon\right\}}|f(k)-L|\Delta k\right.\\
&\quad\leqslant\frac{\mu_\Delta\left(\left(p(m),p'(m)\right]\right)+\nu_\Delta\left(\left(q(m),q'(m)\right]\right)}{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)}.K\\
&\qquad+\frac{K}{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)}\nu_\Delta\left(\left\{p'(m)<k_1\leqslant q'(m):|f(k)\right.\right.\\
&\qquad\left.\left.-L|\geqslant\varepsilon\right\}\right)+\frac{\varepsilon}{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)}.
\end{aligned} \tag{27}$$

This completes the proof. □

3. Conclusion

Various variations of statistical convergence have been studied throughout the years, yielding some extremely important conclusions. Deferred statistical convergence of order α is one of these versions. This variant of statistical convergence is investigated on arbitrary time scales in this paper, and a significant generalization is made. As a result, the current results constitute a particular case of our findings. Then, on temporal scales, strongly postponed Cesàro summability of order α is built. Finally, various inclusion relations for the newly obtained spaces are investigated. The concepts and theorems mentioned will vary as the time scale changes. This will have a significant impact on applications employing the notion of summability in numerous ways.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally, and they have read and approved the final manuscript for publication.

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Research Article

Solving Fractional-Order Diffusion Equations in a Plasma and Fluids via a Novel Transform

Pongsakorn Sunthrayuth[1] **Haifa A. Alyousef,**[2] **S. A. El-Tantawy**[3,4] **Adnan Khan,**[5] **and Noorolhuda Wyal**[6]

[1]*Department of Mathematics and Computer Science, Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT), Thanyaburi, Pathumthani, Thailand*
[2]*Department of Physics, College of Science, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia*
[3]*Department of Physics, Faculty of Science, Port Said University, Port Said 42521, Egypt*
[4]*Research Center for Physics (RCP), Department of Physics, Faculty of Science and Arts, Al-Mikhwah, Al-Baha University, Al Bahah 1988, Saudi Arabia*
[5]*Department of Mathematics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan*
[6]*Department of Mathematics, Polytechnical University of Kabul, Kabul, Afghanistan*

Correspondence should be addressed to Noorolhuda Wyal; noorolhuda.wyal@kpu.edu.af

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Motivated by the importance of diffusion equations in many physical situations in general and in plasma physics in particular, therefore, in this study, we try to find some novel solutions to fractional-order diffusion equations to explain many of the ambiguities about the phenomena in plasma physics and many other fields. In this article, we implement two well-known analytical methods for the solution of diffusion equations. We suggest the modified form of homotopy perturbation method and Adomian decomposition methods using Jafari-Yang transform. Furthermore, illustrative examples are introduced to show the accuracy of the proposed methods. It is observed that the proposed method solution has the desire rate of convergence toward the exact solution. The suggested method's main advantage is less number of calculations. The proposed methods give series form solution which converges quickly towards the exact solution. To show the reliability of the proposed method, we present some graphical representations of the exact and analytical results, which are in strong agreement with each other. The results we showed through graphs and tables for different fractional-order confirm that the results converge towards exact solution as the fractional-order tends towards integer-order. Moreover, it can solve physical problems having fractional order in different areas of applied sciences. Also, the proposed method helps many plasma physicists in modeling several nonlinear structures such as solitons, shocks, and rogue waves in different plasma systems.

1. Introduction

The integer-order differentiation operators are used to study local phenomena, whereas fractional-order operators are used to studying nonlocal phenomena [1]. The mathematical groundwork for fractional-order derivatives was laid by the collective struggles of various mathematicians, such as Riemann, Liouville, Caputo, Podlubny, Miller, and Ross. Afterward, numerous mathematicians dedicated their efforts to this area. Fractional calculus (FC) can be described as very successful in many phenomena in applied sciences, fluid mechanics, physics of plasmas [2, 3], and other biology utilising mathematical tools of FC. [4, 5]. Other numerous applications of FC in the field of science and technology are related to solid mechanics [6, 7], anomalous transport [8], continuum and statistical mechanics [9], economics

[10], relaxation electrochemistry [11], diffusion procedures [12], complex networks [13, 14], and optimal control problems [15, 16].

Fractional differential equations (FDEs) have gotten a lot of attention from researchers in the last decade due to their ability to improve real-world challenges in different areas of engineering and physics. Mathematicians used various methods described based on the above applications to solve different important fractional-order differential equations (FDEs), mainly partial differential equations of fractional-orders (FPDEs). FPDEs are fundamental mathematical approaches that can be utilized to model different physical models more accurately than integer-order models. Nonlinear FPDEs define various phenomena in engineering, plasma physics, and applied sciences. Especially, nonlinear FPDEs are the preeminent tools to be used in various areas, for example, electrochemistry [17], mathematical social dynamics [18, 19], signal processing [20], informatics [21], traffic model [22], theory of solitons in plasma physics [23], biology [24], and much more [25–28]. Moreover, many authors reduced the fluid plasma equations to FDEs for studying the impact of derivative time-frational on the profile of nonlinear structures in a plasma [2, 3]. For instance, El-Wakil et al. [2] reduced the basic equations of a collisionaless unmagnetized nonthermal plasma having cold inertial electrons and inertialess nonthermal electrons as well as stationary positive ions to a normal KdV equation. After that, the authors use a suitable transformation to convert the normal KdV equation to a time-fractional KdV equation in order to investigate the time-fractional on the profile of electron-acoustic (EA) solitons. Furthermore, the basic equations of an ultrarelativistic plasma were reduced to the normal cylindrical Kadomtsev–Petviashvili (CKP) and cylindrical modified KP (CmKP) equations using a reductive perturbation techniques [3]. Posteriorly, the mentioned equations were converted into space-time fractional CKP and CmKP equations using one of the proper transformations in order to study the influence of space-time fractional domain of the characteristics of the ion-acoustic waves (IAWs) in the ultrarelativistic plasma. The authors made a comparison between the integer- and fractional-order models and found that the fractional-order model gives description to the IAWs in the ultrarelativistic plasmas better than the integer-order model [3]. In literature, different methods are implemented for solving FDEs, such as Iterative Laplace Transform method (ILTM) [29, 30], Approximate-analytical method (AAM) [31], Homotopy Analysis method (HAM) [32], Variational Iteration method (VIM) [33], Elzaki Transform Decomposition method (ETDM) [34], the Differential Transformation method (DTM) [35], and the homotopy perturbation method (HPM) [36, 37].

In literature, there is lot of transformations [38–40], but in this article, we implement the Homotopy Perturbation Jafari-Yang Transform Method (HPYTM) and Jafari-Yang transform decomposition method (YTDM) for the analysis of fractional-order diffusion equations. Xiao-Jun Jafari-Yang introduce the Jafari-Yang transformation and applied for the analysis of different differential problems with constant coefficients, while the Adomian decomposition method [41, 42], on the other hand, is a renowned method to solve linear and nonlinear and nonhomogeneous and homogeneous differential, integro, ordinary, and partial differential equations. It gives analysis in the form of series that converges towards the exact solutions quickly. In 1998, He introduced the homotopy perturbation technique [43, 44]. Later, the nonlinear nonhomogeneous partial differential equations are solved using the HPM (homotopy perturbation method), a semianalytical technique [45–48]. The solution is assumed to be the sum of an infinite sequence that converges rapidly to the exact results. This approach was investigated to analyze both linear and nonlinear problems. In the current work, we proposed a novel approximate analytical method known as (HPYTM). The newly developed technique is the combination of Jafari-Yang transform and HPM. It is investigated that the present methods are very effective in finding fractional diffusion equations analytical solution. The fractional problem results using the proposed methods are also devoted to the fractional view analysis of the problems. It is confirmed that the current techniques can be modified to solve other fractional partial differential equations.

A type of PDE that expresses the phenomenon of atoms or molecule's movement from a region of higher concentration to an area of lower concentration is known as diffusion equations. Adolf Fick, a physiologist, was the first to present Fick's law of diffusion. Fick's law was then transformed into the diffusion equation. Scholar modified diffusion equations such as slow diffusion and the hybrid classical wave equation by generalizing the classical diffusion law [49]. Several implementations of diffusion equation are phase transition, electromagnetism, filtration, biochemistry, geochemistry, dynamics of biological groups, cosmology, plasma physics, and acoustics [50]. There are many investigations related to the Diffusion-type equation and its applications in plasma physics and fluids [51–53]. Motivated by these investigation, in this article, we implement YTDM and HPYTM for solving diffusion equations of the form.

(1) Fractional-order diffusion equation in one dimension as

$$\frac{\partial^\delta \nu}{\partial \tau^\delta} = \frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu \, 0 < \delta \leq 1, \tau \geq 0 \tag{1}$$

having initial values

$$\nu(\psi, 0) = g(\psi) \tag{2}$$

(2) Fractional-order diffusion equation in two dimension as

$$\frac{\partial^\delta \nu}{\partial \tau^\delta} = \frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2} 0 < \delta \leq 1, \tau \geq 0 \tag{3}$$

having initial values

$$\nu(\psi, \phi, 0) = g(\psi, \phi) \tag{4}$$

(3) Fractional-order diffusion equation in three dimension as

$$\frac{\partial^\delta \nu}{\partial \tau^\delta} = \frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2} + \frac{\partial^2 \nu}{\partial \varphi^2} 0 < \delta \leq 1, \tau \geq 0 \tag{5}$$

having initial values

$$\nu(\psi, \phi, \varphi, 0) = g(\psi, \phi, \varphi) \tag{6}$$

2. Preliminaries

We covered several fundamental fractional calculus definitions as well as Laplace transform theory properties in this section.

Definition 1. In Caputo manner, the fractional derivative is given as [54].

$$D_\varphi^\sigma \nu(\psi, \varphi) = \frac{1}{\Gamma(k-\sigma)} \int_0^\varphi (\varphi - \rho)^{k-\sigma-1} \nu^{(k)}(\psi, \rho) d\rho, k - 1 < \sigma \leq k, k \in N. \tag{7}$$

Definition 2. Yang and Jafari introduced the Jafari-Yang Laplace transform in 2018. $Y(.)$ determines the Jafari-Yang transform for a function $\xi(\tau)$ and is given as [55].

$$Y\{\nu(\tau)\} = T(u) = \int_0^\infty e^{-\tau/u} \nu(\tau) d\tau, \tau > 0, u \in (-\tau_1, \tau_2). \tag{8}$$

The inverse transform is given as

$$Y^{-1}\{T(u)\} = \nu(\tau). \tag{9}$$

Definition 3. For derivative of order n, the Jafari-Yang transform is determined as [55].

$$Y\{\nu^n(\tau)\} = \frac{T(u)}{u^n} - \sum_{k=0}^{n-1} \frac{u^k(0)}{\nu^{k-n-1}}, \forall n = 1, 2, 3, \cdots. \tag{10}$$

Definition 4. For fractional-order derivatives, the Jafari-Yang transform is given as [55].

$$T\{u^\sigma(\tau)\} = \frac{T(\nu)}{u^\sigma} - \sum_{k=0}^{n-1} \frac{\nu^k(0)}{u^{k-(\sigma+1)}}, 0 < \sigma \leq n. \tag{11}$$

Definition 5. The Mittag-Leffler function, a generalisation of the exponential function, is as [54].

$$E_\sigma(\varphi) = \sum_{q=0}^\infty \frac{\varphi^q}{\Gamma(\sigma q + 1)} (\sigma \in C, \mathrm{Re}\,(\sigma) > 0). \tag{12}$$

Equation (12) further generalization is of the form

$$E_{\sigma,\beta}(\varphi) = \sum_{q=0}^\infty \frac{\varphi^q}{\Gamma(\sigma q + \beta)}\ ; (\sigma, \beta \in C, \mathrm{Re}\,(\sigma) > 0), \mathrm{Re}\,(\beta) > 0). \tag{13}$$

3. Homotopy Perturbation Jafari-Yang Transform Method

To explain the basic ideas of this approach the following equation is considered:

$$ {}_\tau^\sigma \nu(\psi, \tau) = L[\psi]\nu(\psi, \tau) + N[\psi]\nu(\psi, \tau), 0 < \sigma \leq 2, \tag{14}$$

with some initial sources

$$\nu(\psi, 0) = \xi(\psi), \frac{\partial}{\partial \tau} \nu(\psi, 0) = \zeta(\psi), \tag{15}$$

where $D_\tau^\sigma = \partial^\sigma / \partial \tau^\sigma$ Caputo's derivative, and $L[\psi]$ and $N[\psi]$ are the linear and nonlinear operators respectively.

Implement Jafari-Yang transform to (14), we have

$$[D_\tau^\sigma \nu(\psi, \tau)] = Y[L[\psi]\nu(\psi, \tau) + N[\psi]\nu(\psi, \tau)], \tag{16}$$

$$\frac{1}{u^\sigma}\left\{T(u) - u\nu(0) - u^2\nu'(0)\right\} = Y[L[\psi]\nu(\psi, \tau) + N[\psi]\nu(\psi, \tau)]. \tag{17}$$

Equation (17) implies that

$$T(\nu) = u\nu(0) + u^2\nu'(0) + u^\sigma Y[L[\psi]\nu(\psi, \tau) + N[\psi]\nu(\psi, \tau)]. \tag{18}$$

We now have by using the inverse Jafari-Yang transform

$$\nu(\psi, \tau) = \nu(0) + \nu'(0) + Y^{-1}[u^\sigma Y[L[\psi]\nu(\psi, \tau) + N[\psi]\nu(\psi, \tau)]]. \tag{19}$$

Now, perturbation technique having parameter ε is given as

$$\nu(\psi, \tau) = \sum_{k=0}^\infty \varepsilon^k \nu_k(\psi, \tau), \tag{20}$$

where perturbation parameter is ε and $\varepsilon \in [0, 1]$.

The decomposition of nonlinear terms is defined as

$$N[\psi]\nu(\psi, \tau) = \sum_{k=0}^\infty \varepsilon^k H_n(\nu), \tag{21}$$

where H_n are of the form $\nu_0, \nu_1, \nu_2, \cdots, \nu_n$, and can be determined as

$$H_n(\nu_0, \nu_1, \cdots, \nu_n) = \frac{1}{\gamma(n+1)} D_\varepsilon^k \left[N\left(\sum_{k=0}^{\infty} \varepsilon^i \nu_i \right) \right]_{\varepsilon=0}, \quad (22)$$

where $D_\varepsilon^k = \partial^k / \partial \varepsilon^k$.

Using (21) and (22) in (19) and making the homotopy, we get

$$\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau) = \nu(0) + \nu'(0) + \varepsilon \times \left(Y^{-1}\left[u^\sigma Y\left\{ L \sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau) + \sum_{k=0}^{\infty} \varepsilon^k H_k(\nu) \right\} \right] \right). \quad (23)$$

When the coefficient of ε on both sides is compared, we get

$$\begin{aligned} \varepsilon^0 &: \nu_0(\psi, \tau) = \nu(0) + \nu'(0), \\ \varepsilon^1 &: \nu_1(\psi, \tau) = Y^{-1}[u^\sigma Y(L[\psi]\nu_0(\psi, \tau) + H_0(\nu))], \\ \varepsilon^2 &: \nu_2(\psi, \tau) = Y^{-1}[u^\sigma Y(L[\psi]\nu_1(\psi, \tau) + H_1(\nu))], \\ &\vdots \\ \varepsilon^k &: \nu_n(\psi, \tau) = Y^{-1}[u^\sigma Y(L[\psi]\nu_{k-1}(\psi, \tau) + H_{k-1}(\nu))]. \end{aligned} \quad (24)$$

As a result, we can quickly determine component $\nu_k(\psi, \tau)$, which leads us to the convergent series. We get by taking $\varepsilon \longrightarrow 1$,

$$\nu(\psi, \tau) = \lim_{M \longrightarrow \infty} \sum_{k=1}^{M} \nu_k(\psi, \tau). \quad (25)$$

The result is in the form of a series that rapidly converges to the exact solution of the problem.

4. Idea of YTDM

Consider the fractional order partial differential equation

$$D_\tau^\sigma \nu(\psi, \tau) = \bar{\mathscr{G}}_1(\nu, \varphi) + \mathscr{N}_1(\nu, \varphi), 0 < \sigma \leq 2, \quad (26)$$

with some initial sources

$$\nu(\psi, 0) = \xi(\psi), \frac{\partial}{\partial \tau} \nu(\psi, 0) = \zeta(\psi), \quad (27)$$

where $D_\tau^\sigma = \partial^\sigma / \partial \tau^\sigma$ denotes the derivative having fractional-order σ in Caputo manner, and $\bar{\mathscr{G}}_1$ and $\mathscr{N}_1$ denote linear and nonlinear functions.

On taking Jafari-Yang transformation of (26), we get

$$Y[D_\tau^\sigma \nu(\psi, \tau)] = Y\left[\bar{\mathscr{G}}_1(\nu, \varphi) + \mathscr{N}_1(\nu, \varphi)\right]. \quad (28)$$

By using Jafari-Yang transform property of differentiation, we have

$$\frac{1}{u^\sigma}\left\{ T(u) - u\nu(0) - u^2 \nu'(0) \right\} = Y\left[\bar{\mathscr{G}}_1(\nu, \varphi) + \mathscr{N}_1(\nu, \varphi)\right]. \quad (29)$$

Equation (29) implies that

$$T(\nu) = u\nu(0) + u^2 \nu'(0) + u^\sigma Y\left[\bar{\mathscr{G}}_1(\nu, \varphi) + \mathscr{N}_1(\nu, \varphi)\right]. \quad (30)$$

We now have by using the inverse Jafari-Yang transform

$$\nu(\psi, \tau) = \nu(0) + \nu'(0) + Y^{-1}\left[u^\sigma Y\left[\bar{\mathscr{G}}_1(\nu, \varphi) + \mathscr{N}_1(\nu, \varphi)\right]\right]. \quad (31)$$

The infinite series of $\nu(\psi, \tau)$

$$\nu(\psi, \tau) = \sum_{k=0}^{\infty} \nu_k(\psi, \tau). \quad (32)$$

The nonlinear term decomposition of $\mathscr{N}_1$ by Adomian polynomials is expressed as

$$\mathscr{N}_1(\nu, \varphi) = \sum_{k=0}^{\infty} \mathscr{A}_k. \quad (33)$$

All nonlinear terms can be denoted by means of Adomian polynomials as

$$\mathscr{A}_k = \frac{1}{k!}\left[\frac{\partial^k}{\partial \ell^k}\left\{ \mathscr{N}_1\left(\sum_{j=0}^{\infty} \ell^j \nu_j, \sum_{j=0}^{\infty} \ell^j \varphi_j \right) \right\} \right]_{\ell=0}, \quad (34)$$

putting Equations (32) and (34) into (31), gives

$$\sum_{k=0}^{\infty} \nu_k(\psi, \tau) = \nu(0) + \nu'(0) + Y^{-1} u^\sigma \cdot \left[Y\left\{ \bar{\mathscr{G}}_1\left(\sum_{k=0}^{\infty} \nu_k, \sum_{k=0}^{\infty} \varphi_k \right) + \sum_{k=0}^{\infty} \mathscr{A}_k \right\} \right]. \quad (35)$$

The following terms are described as

$$\begin{aligned} \nu_0(\psi, \tau) &= \nu(0) + \tau \nu'(0), \\ \nu_1(\psi, \tau) &= Y^{-}\left[u^\sigma Y^{+}\left\{\bar{\mathscr{G}}_1(\nu_0, \varphi_0) + \mathscr{A}_0\right\}\right]. \end{aligned} \quad (36)$$

The general for $k \geq 1$ is determined as

$$\nu_{k+1}(\psi, \tau) = Y^{-}\left[u^\sigma Y^{+}\left\{\bar{\mathscr{G}}_1(\nu_k, \varphi_k) + \mathscr{A}_k\right\}\right]. \quad (37)$$

5. Applications

The solutions to various fractional order diffusion equations are obtained by implementing HPYTM and YTDM in this section.

Example 1. Consider one-dimension fractional-order diffusion equation [54]

$$\frac{\partial^\sigma \nu}{\partial \tau^\sigma} = \frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu \, 0 < \sigma \leq 1, \tau > 0, \quad (38)$$

with initial source

$$\nu(\psi, 0) = e^\psi. \quad (39)$$

On taking Jafari-Yang transformation of Equation (38), we get

$$Y\left(\frac{\partial^\sigma \nu}{\partial \tau^\sigma}\right) = Y\left(\frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu\right). \quad (40)$$

We get by using the Jafari-Yang transform's differential property

$$\frac{1}{u^\sigma}\{T(u) - u\nu(0)\} = Y\left(\frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu\right), \quad (41)$$

$$T(u) = u\nu(0) + u^\sigma Y\left(\frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu\right). \quad (42)$$

By applying inverse Jafari-Yang transform to Equation (42)

$$\begin{aligned} \nu(\psi, \tau) &= \nu(0) + Y^{-1}\left[u^\sigma\left\{Y\left(\frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu\right)\right\}\right], \\ \nu(\psi, \tau) &= e^\psi + Y^{-1}\left[u^\sigma\left\{Y\left(\frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu\right)\right\}\right]. \end{aligned} \quad (43)$$

Now, applying the abovementioned homotopy perturbation technique as in (23), we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau) &= e^\psi + \varepsilon\left(Y^{-1}\left[u^\sigma Y\left[\left(\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau)\right)_{\psi\psi}\right.\right.\right. \\ &\quad - \left(\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau)\right)_{\psi} + \left(\sum_{k=0}^{\infty} \varepsilon^k H_k(\nu)\right) \\ &\quad \left.\left.\left. + \sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau)\right]\right]\right). \end{aligned} \quad (44)$$

where $H_k(\nu)$ are He's polynomial that represents the nonlinear terms. He's polynomials first few components are given by

$$\begin{aligned} H_0(\nu) &= \nu_0(\nu_0)_{\psi\psi} - \nu_{0,}^2 \\ H_1(\nu) &= \left(\nu_1(\nu_0)_{\psi\psi} + \nu_0(\nu_1)_{\psi\psi}\right) - (2\nu_0\nu_1), \\ H_2(\nu) &= \left(\nu_2(\nu_0)_{\psi\psi} + \nu_1(\nu_1)_{\psi\psi} + \nu_0(\nu_2)_{\psi\psi}\right) \\ &\quad - \left(2\nu_0\nu_2 + (\nu_2)^2\right). \\ &\vdots \end{aligned} \quad (45)$$

Comparing the same power coefficient of ε, we obtain

$$\begin{aligned} \varepsilon^0 &: \nu_0(\psi, \tau) = e^\psi, \\ \varepsilon^1 &: \nu_1(\psi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_0}{\partial \psi^2} - \frac{\partial \nu_0}{\partial \psi} + \nu_0 \frac{\partial^2 \nu_0}{\partial \psi^2} - \nu_0^2 + \nu_0\right]\right) = e^\psi \frac{\tau^\sigma}{\Gamma(\sigma+1)}, \\ \varepsilon^2 &: \nu_2(\psi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_1}{\partial \psi^2} - \frac{\partial \nu_1}{\partial \psi} + \nu_1 \frac{\partial^2 \nu_0}{\partial \psi^2} + \nu_0 \frac{\partial^2 \nu_1}{\partial \psi^2} - 2\nu_0\nu_1 + \nu_1\right]\right) = e^\psi \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}, \\ \varepsilon^3 &: \nu_3(\psi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_2}{\partial \psi^2} - \frac{\partial \nu_2}{\partial \psi} + \nu_2 \frac{\partial^2 \nu_0}{\partial \psi^2} + \nu_1 \frac{\partial^2 \nu_1}{\partial \psi^2} + \nu_0 \frac{\partial^2 \nu_2}{\partial \psi^2} - 2\nu_0\nu_2 - (\nu_2)^2 + \nu_2\right]\right) \\ \varepsilon^3 &: \nu_3(\psi, \tau) = e^\psi \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}. \\ &\vdots \end{aligned} \quad (46)$$

By taking $\varepsilon \longrightarrow 1$, we obtain the convergence series type result is given as

$$\begin{aligned}\nu(\psi,\tau)&=\nu_0+\nu_1+\nu_2+\nu_3+\nu_4+\cdots\\&=e^{\psi}+e^{\psi}\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}+e^{\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}+e^{\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}+\cdots,\\\nu(\psi,\tau)&=e^{\psi}\left(1+\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}+\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}+\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}+\cdots\right),\\\nu(\psi,\tau)&=e^{\psi}\sum_{k=0}^{\infty}\frac{(t^{\sigma})^k}{\Gamma(k\sigma+1)}=e^{\psi}E_{\sigma}(\tau^{\sigma}).\end{aligned}\tag{47}$$

When $\sigma=1$, the HPYTM solution is

$$\nu(\psi,\tau)=e^{\psi}\sum_{k=0}^{\infty}\frac{(t)^k}{k!},\tag{48}$$

The analytical results by YTDM.

On taking Jafari-Yang transformation of Equation (38), we obtain

$$Y\left\{\frac{\partial^{\sigma}\nu}{\partial\tau^{\sigma}}\right\}=Y\left[\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\nu\frac{\partial^2\nu}{\partial\psi^2}-\nu^2+\nu\right].\tag{49}$$

Using the Jafari-Yang transform the differential property, we get

$$\frac{1}{u^{\sigma}}\{T(u)-u\nu(0)\}=Y\left[\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\nu\frac{\partial^2\nu}{\partial\psi^2}-\nu^2+\nu\right],\tag{50}$$

$$T(u)=u\nu(0)+u^{\sigma}Y\left[\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\nu\frac{\partial^2\nu}{\partial\psi^2}-\nu^2+\nu\right].\tag{51}$$

By inverse Jafari-Yang transform of Equation (51)

$$\begin{aligned}\nu(\psi,\tau)&=\nu(0)+Y^{-1}\left[u^{\sigma}\left\{Y\left(\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\nu\frac{\partial^2\nu}{\partial\psi^2}-\nu^2+\nu\right)\right\}\right],\\\nu(\psi,\tau)&=e^{\psi}+Y^{-1}\left[u^{\sigma}\left\{Y\left(\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\nu\frac{\partial^2\nu}{\partial\psi^2}-\nu^2+\nu\right)\right\}\right],\end{aligned}\tag{52}$$

Assuming the unknown $\nu(\psi,\tau)$ function has the following infinite series form solution:

$$\nu(\psi,\tau)=\sum_{k=0}^{\infty}\nu_k(\psi,\tau).\tag{53}$$

The Adomian polynomials $\nu(\nu)_{\psi\psi}=\sum_{k=0}^{\infty}\mathscr{A}_k$ and $\nu^2=\sum_{k=0}^{\infty}\mathscr{B}_k$, as well as the nonlinear term, have been described.

Applying specific function, Equation (52) may be determined as

$$\begin{aligned}\sum_{k=0}^{\infty}\nu_k(\psi,\tau)&=\nu(\psi,0)+Y^{-}\left[u^{\sigma}Y\left[\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\sum_{k=0}^{\infty}\mathscr{A}_k-\sum_{k=0}^{\infty}\mathscr{B}_k+\nu\right]\right],\\\sum_{k=0}^{\infty}\nu_k(\psi,\tau)&=e^{\psi}+Y^{-}\left[u^{\sigma}Y\left[\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\sum_{k=0}^{\infty}\mathscr{A}_k-\sum_{k=0}^{\infty}\mathscr{B}_k+\nu\right]\right].\end{aligned}\tag{54}$$

According to Equation (34), the nonlinear function can be find with the help of Adomian polynomials is given as

$$\begin{aligned}&\mathscr{A}_0=\nu_0(\nu_0)_{\psi\psi},\mathscr{A}_1=\nu_1(\nu_0)_{\psi\psi}+\nu_0(\nu_1)_{\psi\psi},\\&\mathscr{A}_2=\nu_2(\nu_0)_{\psi\psi}+\nu_1(\nu_1)_{\psi\psi}+\nu_0(\nu_2)_{\psi\psi},\\&\mathscr{B}_0=\nu_0^2,\mathscr{B}_1=2\nu_0\nu_1,\mathscr{B}_2=2\nu_0\nu_2+(\nu_2)^2.\end{aligned}\tag{55}$$

Thus, on comparing both sides of Equation (54)

$$\nu_0(\psi,\tau)=e^{\psi},1\tag{56}$$

For $k=0$

$$\nu_1(\psi,\phi,\varphi,\tau)=-3\sin\psi\sin\phi\sin\varphi\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}.\tag{57}$$

For $k=1$

$$\nu_2(\psi,\tau)=e^{\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}.\tag{58}$$

For $k=2$

$$\nu_3(\psi,\tau)=e^{\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}.\tag{59}$$

The remaining YTDM solution elements ρ_k for $(k\geq 2)$ are similarly simple to get. As a result, the solution in series form is as

$$\begin{aligned}\nu(\psi,\tau)=\sum_{k=0}^{\infty}\nu_k(\psi,\tau)&=\nu_0(\psi,\tau)+\nu_1(\psi,\tau)\\&+\nu_2(\psi,\tau)+\nu_3(\psi,\tau)+\cdots\nu(\psi,\tau)\\&=e^{\psi}+e^{\psi}\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}+e^{\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}\\&+e^{\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}+\cdots.\end{aligned}\tag{60}$$

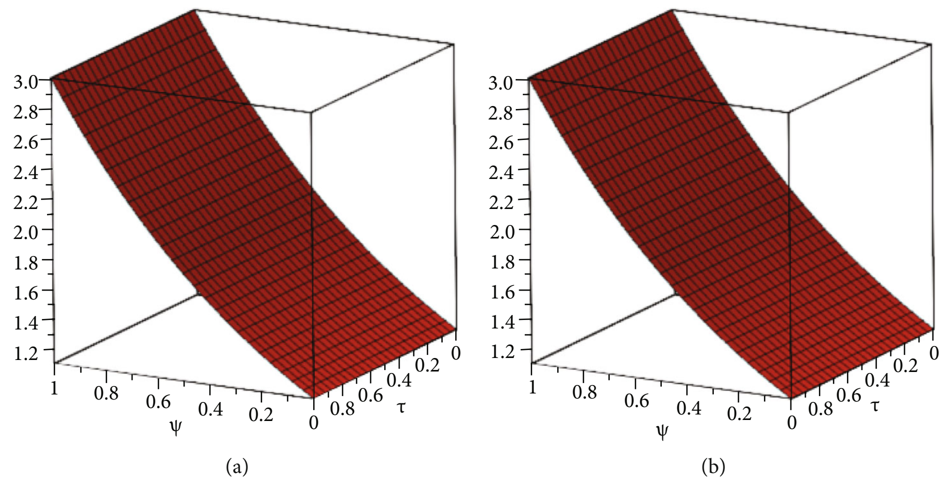

Figure 1: Example 1 solution graph. (a) Exact solution and (b) analytical solution at $\sigma = 1$.

When $\sigma = 1$, the solution by YTDM is

$$\nu(\psi, \tau) = e^{\psi} \sum_{k=0}^{\infty} \frac{(t)^k}{k!}. \tag{61}$$

In closed form, the exact solution is

$$\nu(\psi, \tau) = e^{(\psi+\tau)}. \tag{62}$$

The exact solution shown in Figures 1(a) and 1(b) shows HPYTM and YTDM solution at $\sigma = 1$ and $0 \le \psi \ge 1$. Figure 2(a) shows the solution graph for different values of $\sigma = 1,0.9,0.8,0.7$ and $0 \le \psi \ge 1$ of Example 1 and Figure 2(b), respectively, at $\tau \in [0, 1]$ and $0 \le \psi \ge 1$ while Figure 2(c) shows the error graph. Also, Table 1 shows the comparison of the exact solution and our methods solution with the aid of absolute error at various fractional order. From the figures and table, it is clear that HPYTM and YTDM solution shows strong contact with the exact solutions of the problem.

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Example 2. Consider one-dimension fractional-order gas dynamic equation [56]

$$\frac{\partial^\sigma \nu}{\partial \tau^\sigma} + \nu \frac{\partial \nu}{\partial \psi} + \nu^2 - \nu = 0 \; 0 < \sigma \le 1, \tau > 0, \tag{63}$$

with initial source

$$\nu(\psi, 0) = e^{-\psi}. \tag{64}$$

On taking Jafari-Yang transformation of Equation (63), we get

$$Y\left(\frac{\partial^\sigma \nu}{\partial \tau^\sigma}\right) = Y\left(-\nu \frac{\partial \nu}{\partial \psi} + \nu^2 - \nu\right). \tag{65}$$

We get by using the Jafari-Yang transform's differential property

$$\frac{1}{u^\sigma}\{T(u) - u\nu(0)\} = Y\left(-\nu \frac{\partial \nu}{\partial \psi} + \nu^2 - \nu\right), \tag{66}$$

$$T(u) = u\nu(0) + u^\sigma Y\left(-\nu \frac{\partial \nu}{\partial \psi} + \nu^2 - \nu\right). \tag{67}$$

By applying inverse Jafari-Yang transform to Equation (67)

$$\begin{aligned} \nu(\psi, \tau) &= \nu(0) - Y^{-1}\left[u^\sigma \left\{ Y\left(\nu \frac{\partial \nu}{\partial \psi} - \nu^2 + \nu\right)\right\}\right], \\ \nu(\psi, \tau) &= e^{-\psi} - Y^{-1}\left[u^\sigma \left\{ Y\left(\nu \frac{\partial \nu}{\partial \psi} - \nu^2 + \nu\right)\right\}\right]. \end{aligned} \tag{68}$$

Now, applying the abovementioned homotopy perturbation technique as in (23), we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau) = e^{-\psi} - \varepsilon \Bigg(Y^{-1} \Bigg[u^\sigma Y \Bigg(\Bigg(\sum_{k=0}^{\infty} \varepsilon^k H_k(\nu) \Bigg) \\ + \sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau) \Bigg] \Bigg] \Bigg). \end{aligned} \tag{69}$$

where $H_k(\nu)$ are He's polynomial that represents the nonlinear terms. He's polynomials first few components are given by

$$\begin{aligned} H_0(\nu) &= \nu_0 (\nu_0)_\psi - \nu_0^2, \\ H_1(\nu) &= \left(\nu_1 (\nu_0)_\psi + \nu_0 (\nu_1)_\psi\right) - (2\nu_0 \nu_1), \\ H_2(\nu) &= \left(\nu_2 (\nu_0)_\psi + \nu_1 (\nu_1)_\psi + \nu_0 (\nu_2)_\psi\right) - \left(2\nu_0 \nu_2 + (\nu_2)^2\right). \\ &\vdots \end{aligned} \tag{70}$$

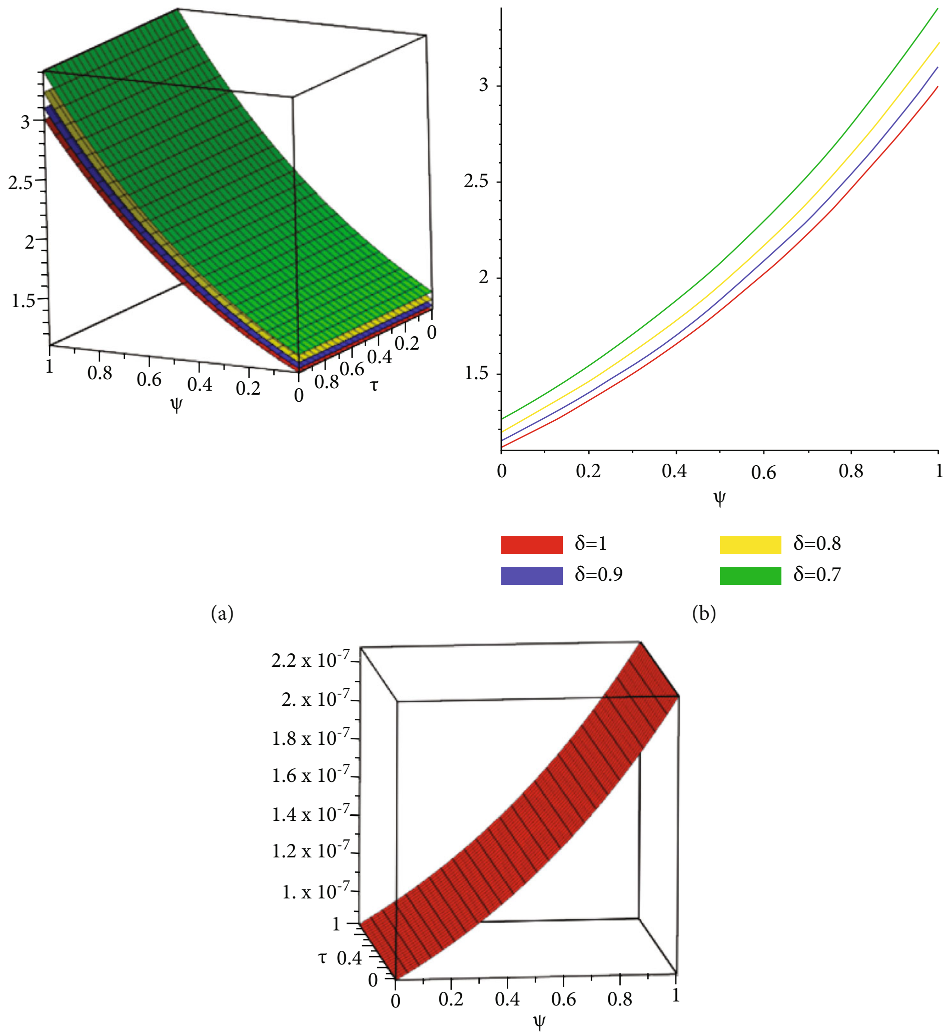


Figure 2: Example 1. (a) Analytical solution at various fractional orders of σ (b) $\phi = 0.5$ and (c) error.

Table 1: $v(\psi, \tau)$ comparison of exact solution, our methods' solution, and Absolute Error (AE) of Example 1.

$\tau = 0.01$ ψ	Absolute error $\sigma = 0.8$	Absolute error $\sigma = 0.9$	Exact solution $\sigma = 1$	Our methods' solution $\sigma = 1$	Absolute error $\sigma = 1$
0	5.2118400000E-04	5.0962000000E-05	1.0100501670000	1.0100501670000	0.00000000E+00
0.1	5.7599800000E-04	5.6322000000E-05	1.1162780700000	1.1162780700000	0.00000000E+00
0.2	6.3657500000E-04	6.2245000000E-05	1.2336780600000	1.2336780600000	0.00000000E+00
0.3	7.0352500000E-04	6.8792000000E-05	1.3634251140000	1.3634251140000	0.00000000E+00
0.4	7.7751600000E-04	7.6027000000E-05	1.5068177850000	1.5068177850000	0.00000000E+00
0.5	8.5928700000E-04	8.4022000000E-05	1.6652911950000	1.6652911950000	0.00000000E+00
0.6	9.4965800000E-04	9.2858000000E-05	1.8404313990000	1.8404313980000	1.00000000E-09
0.7	1.0495350000E-03	1.0262400000E-04	2.0339912590000	2.0339912580000	1.00000000E-09
0.8	1.1599150000E-03	1.1341700000E-04	2.2479079870000	2.2479079860000	1.00000000E-09
0.9	1.2819060000E-03	1.2534600000E-04	2.4843225330000	2.4843225330000	0.00000000E+00
1.0	1.4167240000E-03	1.3852800000E-04	2.7456010150000	2.7456010140000	1.00000000E-09

Comparing the same power coefficient of ε, we obtain

$$\begin{aligned}
\varepsilon^0 &: \nu_0(\psi,\tau) = e^{-\psi},\\
\varepsilon^1 &: \nu_1(\psi,\tau) = Y^{-1}\left(u^\sigma Y\left[\nu_0\frac{\partial \nu_0}{\partial\psi} - \nu_0^2 + \nu_0\right]\right)\\
&= e^{-\psi}\frac{\tau^\sigma}{\Gamma(\sigma+1)},\\
\varepsilon^2 &: \nu_2(\psi,\tau) = Y^{-1}\left(u^\sigma Y\left[\nu_1\frac{\partial \nu_0}{\partial\psi} + \nu_0\frac{\partial \nu_1}{\partial\psi}\right.\right.\\
&\quad\left.\left. - 2\nu_0\nu_1 + \nu_1\right]\right) = e^{-\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)},\\
\varepsilon^3 &: \nu_3(\psi,\tau) = Y^{-1}\left(u^\sigma Y\left[\nu_2\frac{\partial \nu_0}{\partial\psi} + \nu_1\frac{\partial \nu_1}{\partial\psi}\right.\right.\\
&\quad\left.\left. + \nu_0\frac{\partial \nu_2}{\partial\psi} - 2\nu_0\nu_2 - (\nu_2)^2\nu_2\right]\right),\\
\varepsilon^3 &: \nu_3(\psi,\tau) = e^{-\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}\\
&\vdots
\end{aligned} \tag{71}$$

By taking $\varepsilon \longrightarrow 1$, we obtain the convergence series type result is given as

$$\begin{aligned}
\nu(\psi,\tau) &= \nu_0 + \nu_1 + \nu_2 + \nu_3 + \nu_4 + \cdots\\
&= e^{-\psi} + e^{-\psi}\frac{\tau^\sigma}{\Gamma(\sigma+1)} + e^{-\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}\\
&\quad + e^{-\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} + \cdots \nu(\psi,\tau)\\
&= e^{-\psi}\left(1 + \frac{\tau^\sigma}{\Gamma(\sigma+1)} + \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} + \cdots\right),\\
\nu(\psi,\tau) &= e^{-\psi}\sum_{k=0}^{\infty}\frac{(t^\sigma)^k}{\Gamma(k\sigma+1)} = e^{\psi}E_\sigma(\tau^\sigma).
\end{aligned} \tag{72}$$

When $\sigma = 1$, the HPYTM solution is

$$\nu(\psi,\tau) = \exp^{-\psi}\sum_{k=0}^{\infty}\frac{(t)^k}{k!}. \tag{73}$$

The analytical results by YTDM

On taking Jafari-Yang transformation of Equation (63), we obtain

$$Y\left\{\frac{\partial^\sigma \nu}{\partial \tau^\sigma}\right\} = Y\left[-\nu\frac{\partial \nu}{\partial \psi} + \nu^2 - \nu\right], \tag{74}$$

using the Jafari-Yang transform the differential property, we get

$$\frac{1}{u^\sigma}\{T(u) - u\nu(0)\} = Y\left[-\nu\frac{\partial \nu}{\partial \psi} + \nu^2 - \nu\right], \tag{75}$$

$$T(u) = u\nu(0) + u^\sigma Y\left[-\nu\frac{\partial \nu}{\partial \psi} + \nu^2 - \nu\right]. \tag{76}$$

By inverse Jafari-Yang transform of Equation (76)

$$\begin{aligned}
\nu(\psi,\tau) &= \nu(0) - Y^{-1}\left[u^\sigma\left\{Y\left(\nu\frac{\partial \nu}{\partial \psi} - \nu^2 + \nu\right)\right\}\right],\\
\nu(\psi,\tau) &= e^{-\psi} - Y^{-1}\left[u^\sigma\left\{Y\left(\nu\frac{\partial \nu}{\partial \psi} - \nu^2 + \nu\right)\right\}\right].
\end{aligned} \tag{77}$$

Assuming the unknown $\nu(\psi,\tau)$ function has the following infinite series form solution:

$$\nu(\psi,\tau) = \sum_{k=0}^{\infty}\nu_k(\psi,\tau). \tag{78}$$

The Adomian polynomials $\nu(\nu)_{\psi\psi} = \sum_{k=0}^{\infty}\mathscr{A}_k$ and $\nu^2 = \sum_{k=0}^{\infty}\mathscr{B}_k$, as well as the non-linear term, have been described. Applying specific function, Equation (77) may be determined as

$$\begin{aligned}
\sum_{k=0}^{\infty}\nu_k(\psi,\tau) &= \nu(\psi,0) - Y^{-}\left[u^\sigma Y\left[\sum_{k=0}^{\infty}\mathscr{A}_k - \sum_{k=0}^{\infty}\mathscr{B}_k + \nu\right]\right],\\
\sum_{k=0}^{\infty}\nu_k(\psi,\tau) &= e^{-\psi} - Y^{-}\left[u^\sigma Y\left[\sum_{k=0}^{\infty}\mathscr{A}_k - \sum_{k=0}^{\infty}\mathscr{B}_k + \nu\right]\right].
\end{aligned} \tag{79}$$

According to Equation (34), the nonlinear function can be found with the help of Adomian polynomials is given as

$$\begin{aligned}
&\mathscr{A}_0 = \nu_0(\nu_0)_\psi, \mathscr{A}_1 = \nu_1(\nu_0)_{\psi\psi} + \nu_0(\nu_1)_\psi,\\
&\mathscr{A}_2 = \nu_2(\nu_0)_\psi + \nu_1(\nu_1)_\psi + \nu_0(\nu_2)_\psi,\\
&\mathscr{B}_0 = \nu_0^2, \mathscr{B}_1 = 2\nu_0\nu_1, \mathscr{B}_2 = 2\nu_0\nu_2 + (\nu_2)^2.
\end{aligned} \tag{80}$$

Thus, on comparing both sides of Equation (79)

$$\nu_0(\psi,\tau) = e^{-\psi}. \tag{81}$$

For $k = 0$

$$\nu_1(\psi,\tau) = e^{-\psi}\frac{\tau^\sigma}{\Gamma(\sigma+1)} \tag{82}$$

For $k = 1$

$$\nu_2(\psi,\tau) = e^{-\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}. \tag{83}$$

For $k = 2$

$$\nu_3(\psi,\tau) = e^{-\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}. \tag{84}$$

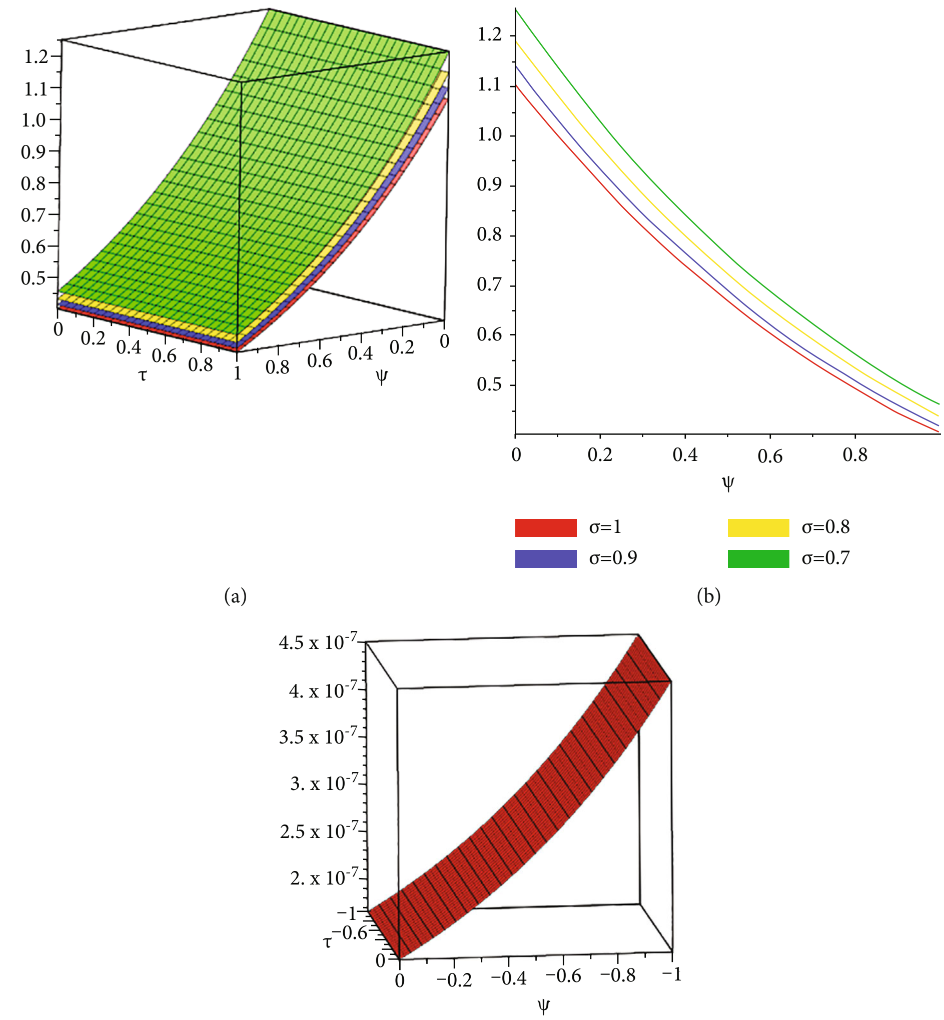


Figure 3: Example 2. (a) Analytical solution at various fractional orders of σ (b) $\phi = 0.5$ and (c) error.

The remaining YTDM solution elements ρ_k for $(k \geq 3)$ are similarly simple to get. As a result, the solution in series form is as

$$\begin{aligned} v(\psi,\tau) &= \sum_{k=0}^{\infty} v_k(\psi,\tau) = v_0(\psi,\tau) + v_1(\psi,\tau) \\ &\quad + v_2(\psi,\tau) + v_3(\psi,\tau) \\ v(\psi,\tau) &= e^{-\psi} + e^{-\psi}\frac{\tau^{\sigma}}{\Gamma(\sigma+1)} + \exp^{-\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)} \\ &\quad + e^{-\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} + \cdots. \end{aligned} \tag{85}$$

When $\sigma = 1$, the solution by YTDM is

$$v(\psi,\tau) = e^{-\psi}\sum_{k=0}^{\infty}\frac{(t)^k}{k!}. \tag{86}$$

In closed form, the exact solution is

$$v(\psi,\tau) = e^{(-\psi+\tau)}. \tag{87}$$

Figure 3(a) shows the solution graph for different values of $\sigma = 1,\ 0.9,\ 0.8,\ 0.7$ and $0 \leq \psi, \phi \geq 1$ of Example 2 and Figure 3(b), respectively, at $\phi = 0.5, \tau \in [0,1]$ and $0 \leq \psi \geq 1$ while Figure 3(c) shows the error graph. It is verified from the figures that HPYTM and YTDM solution is closely related with the exact solution.

Example. 3. Consider two-dimension fractional-order diffusion equation [54]

$$\frac{\partial^{\sigma} v}{\partial \tau^{\sigma}} = \frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} 0 < \sigma \leq 1, \tau \geq 0, \tag{88}$$

with initial source

$$v(\psi,\phi,0) = (1-\phi)e^{\psi}. \tag{89}$$

Taking Jafari-Yang transformation to Equation (88), we get

$$Y\left(\frac{\partial^\sigma \nu}{\partial \tau^\sigma}\right) = Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right), \tag{90}$$

We get by using the Jafari-Yang transform's differential property

$$\begin{aligned} &\frac{1}{u^\sigma}\{T(u) - u\nu(0)\} = Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right), \\ &T(u) = u\nu(0) + u^\sigma Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right). \end{aligned} \tag{91}$$

By applying inverse Jafari-Yang transform to Equation (91)

$$\begin{aligned} \nu(\psi, \phi, \tau) &= \nu(0) + Y^{-1}\left[u^\sigma\left\{Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right)\right\}\right], \\ \nu(\psi, \phi, \tau) &= (1-\phi)e^\psi + Y^{-1}\left[u^\sigma\left\{Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right)\right\}\right]. \end{aligned} \tag{92}$$

Now, using the abovementioned homotopy perturbation technique as in (23), we get

$$\begin{aligned} &\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \phi, \tau) \\ &\quad = (1-\phi)e^\psi + \varepsilon\left(Y^{-1}\left[u^\sigma Y\left[\left(\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \phi, \tau)\right)_{\psi\psi}\right.\right.\right. \\ &\qquad \left.\left.\left. + \left(\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \phi, \tau)\right)_{\phi\phi}\right]\right]\right). \end{aligned} \tag{93}$$

Comparing the same power coefficient of ε, we obtain

$$\begin{aligned} \varepsilon^0 &: \nu_0(\psi, \phi, \tau) = (1-\phi)e^\psi, \\ \varepsilon^1 &: \nu_1(\psi, \phi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_0}{\partial \psi^2} + \frac{\partial^2 \nu_0}{\partial \phi^2}\right]\right) \\ &\qquad = (1-\phi)e^\psi \frac{\tau^\sigma}{\Gamma(\sigma+1)}, \\ \varepsilon^2 &: \nu_2(\psi, \phi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_1}{\partial \psi^2} + \frac{\partial^2 \nu_1}{\partial \phi^2}\right]\right) \\ &\qquad = (1-\phi)e^\psi \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}, \\ \varepsilon^3 &: \nu_3(\psi, \phi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_2}{\partial \psi^2} + \frac{\partial^2 \nu_2}{\partial \phi^2}\right]\right) \\ &\qquad = (1-\phi)e^\psi \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}, \\ \varepsilon^4 &: \nu_4(\psi, \phi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_3}{\partial \psi^2} + \frac{\partial^2 \nu_3}{\partial \phi^2}\right]\right) \\ &\qquad = (1-\phi)e^\psi \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}. \\ &\qquad \vdots \end{aligned} \tag{94}$$

By taking $\varepsilon \longrightarrow 1$, we obtain convergence series type result is

$$\begin{aligned} \nu(\psi, \phi, \tau) &= \nu_0 + \nu_1 + \nu_2 + \nu_3 + \nu_4 + \cdots \\ &= (1-\phi)e^\psi + (1-\phi)e^\psi \frac{\tau^\sigma}{\Gamma(\sigma+1)} \\ &\quad + (1-\phi)e^\psi \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)} + (1-\phi)e^\psi \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} \\ &\quad + (1-\phi)e^\psi \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)} + \cdots \nu(\psi, \phi, \tau) \\ &= (1-\phi)e^\psi\left(1 + \frac{\tau^\sigma}{\Gamma(\sigma+1)} + \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}\right. \\ &\quad \left. + \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} + \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)} + \cdots\right), \\ \nu(\psi, \phi, \tau) &= (1-\phi)e^\psi \sum_{k=0}^{\infty} \frac{(t^\sigma)^k}{\Gamma(k\sigma+1)} = (1-\phi)e^\psi E_\sigma(t^\sigma). \end{aligned} \tag{95}$$

When $\sigma = 1$, the HPYTM solution is

$$\nu(\psi, \phi, \tau) = (1-\phi)e^\psi \sum_{k=0}^{\infty} \frac{(t)^k}{k!}. \tag{96}$$

The analytical results by YTDM

Taking Jafari-Yang transformation of Equation (88), we obtain

$$Y\left\{\frac{\partial^\sigma \nu}{\partial \tau^\sigma}\right\} = Y\left[\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right]. \tag{97}$$

We get by using the Jafari-Yang transform's differential property

$$\frac{1}{u^\sigma}\{T(u) - u\nu(0)\} = Y\left[\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right], \tag{98}$$

$$T(u) = u\nu(0) + u^\sigma Y\left[\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right]. \tag{99}$$

By inverse Jafari-Yang transform of Equation (99)

$$\begin{aligned}v(\psi,\phi,\tau)&=v(0)+Y^{-1}\left[u^{\sigma}\left\{Y\left(\frac{\partial^2 v}{\partial\psi^2}+\frac{\partial^2 v}{\partial\phi^2}\right)\right\}\right],\\ v(\psi,\phi,\tau)&=(1-\phi)e^{\psi}+Y^{-1}\left[u^{\sigma}\left\{Y\left(\frac{\partial^2 v}{\partial\psi^2}+\frac{\partial^2 v}{\partial\phi^2}\right)\right\}\right].\end{aligned} \tag{100}$$

Assuming the unknown $v(\psi, \phi, \tau)$ function has the following infinite series form solution:

$$\begin{aligned}v(\psi,\phi,\tau)&=\sum_{k=0}^{\infty}v_k(\psi,\phi,\tau),\\ \sum_{k=0}^{\infty}v_k(\psi,\phi,\tau)&=e^{\psi}+Y^{-}\left[u^{\sigma}Y\left[\frac{\partial^2 v}{\partial\psi^2}+\frac{\partial^2 v}{\partial\phi^2}\right]\right].\end{aligned} \tag{101}$$

Thus, on comparing both sides of Equation (101)

$$v_0(\psi,\phi,\tau)=(1-\phi)e^{\psi}, \tag{102}$$

For $k = 0$

$$v_1(\psi,\phi,\tau)=(1-\phi)e^{\psi}\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}. \tag{103}$$

For $k = 1$

$$v_2(\psi,\phi,\tau)=(1-\phi)e^{\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}. \tag{104}$$

For $k = 2$

$$v_3(\psi,\phi,\tau)=(1-\phi)e^{\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}. \tag{105}$$

For $k = 3$

$$v_4(\psi,\phi,\tau)=(1-\phi)e^{\psi}\frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}. \tag{106}$$

The remaining YTDM solution elements ρ_k for $(k \geq 3)$ are similarly simple to get. As a result, the solution in series form is as

$$\begin{aligned}v(\psi,\phi,\tau)&=\sum_{k=0}^{\infty}v_k(\psi,\phi,\tau)=v_0(\psi,\phi,\tau)+v_1(\psi,\phi,\tau)\\ &\quad+v_2(\psi,\phi,\tau)+v_3(\psi,\phi,\tau)+v_4(\psi,\phi,\tau)+\cdots\\ v(\psi,\phi,\tau)&=(1-\phi)e^{\psi}+(1-\phi)e^{\psi}\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}\\ &\quad+(1-\phi)e^{\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}+(1-\phi)e^{\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}\\ &\quad+(1-\phi)e^{\psi}\frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}+\cdots\\ v(\psi,\phi,\tau)&=(1-\phi)e^{\psi}\left(1+\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}+\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}\right.\\ &\quad\left.+\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}+\frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}+\cdots\right),\\ v(\psi,\phi,\tau)&=(1-\phi)e^{\psi}\sum_{k=0}^{\infty}\frac{(t^{\sigma})^k}{\Gamma(k\sigma+1)}=(1-\phi)e^{\psi}E_{\sigma}(t^{\sigma}).\end{aligned} \tag{107}$$

When $\sigma = 1$, the solution by YTDM is

$$v(\psi,\phi,\tau)=(1-\phi)e^{\psi}\sum_{k=0}^{\infty}\frac{(t)^k}{k!}. \tag{108}$$

In closed form, the exact solution is:

$$v(\psi,\phi,\tau)=(1-\phi)e^{(\psi+\tau)}. \tag{109}$$

The exact solution shown in Figures 4(a) and 4(b) shows HPYTM and YTDM solutions at $\sigma = 1$ and $0 \leq \psi, \phi \geq 1$. Figure 5(a) shows the error graph for different values of $\sigma = 1,0.8,0.6,0.5$ and $0 \leq \psi, \phi \geq 1$ of Example 3 and Figure 5(d) respectively, at $\phi = 0.5, \tau \in [0, 1]$ and $0 \leq \psi \geq 1$ while Figure 5(e) shows the error graph. From the figures and Table 2, it is clear that HPYTM and YTDM solution shows strong agreement with the exact solutions of the problem.

Example 4. Consider three-dimension fractional-order diffusion equation [54]

$$\frac{\partial^{\sigma}v}{\partial\tau^{\sigma}}=\frac{\partial^2 v}{\partial\psi^2}+\frac{\partial^2 v}{\partial\phi^2}+\frac{\partial^2 v}{\partial\varphi^2},0<\sigma\leq 1,\tau\geq 0, \tag{110}$$

with initial source

$$v(\psi,\phi,\varphi,0)=\sin\psi\sin\phi\sin\varphi. \tag{111}$$

Taking Jafari-Yang transformation of Equation (110), we get

$$Y\left(\frac{\partial^{\sigma}v}{\partial\tau^{\sigma}}\right)=Y\left(\frac{\partial^2 v}{\partial\psi^2}+\frac{\partial^2 v}{\partial\phi^2}+\frac{\partial^2 v}{\partial\varphi^2}\right). \tag{112}$$

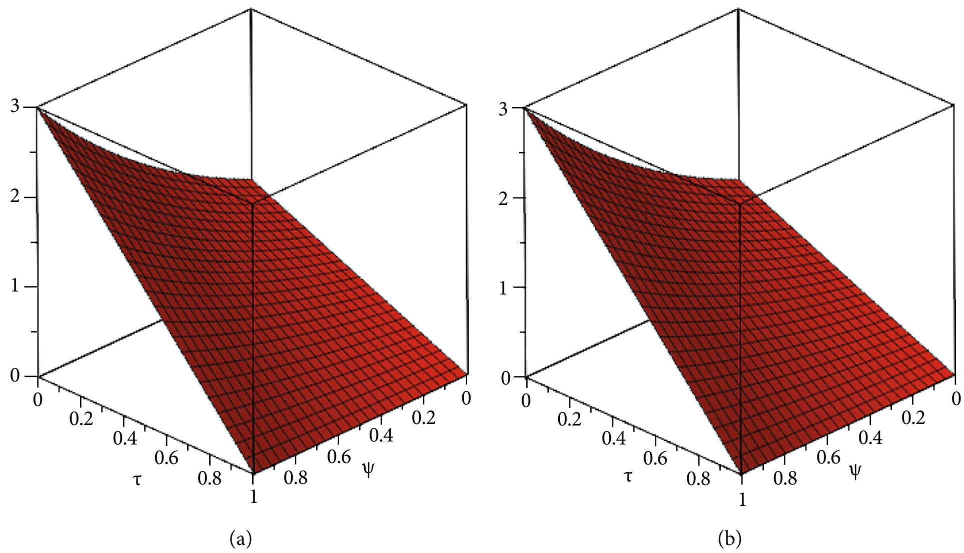
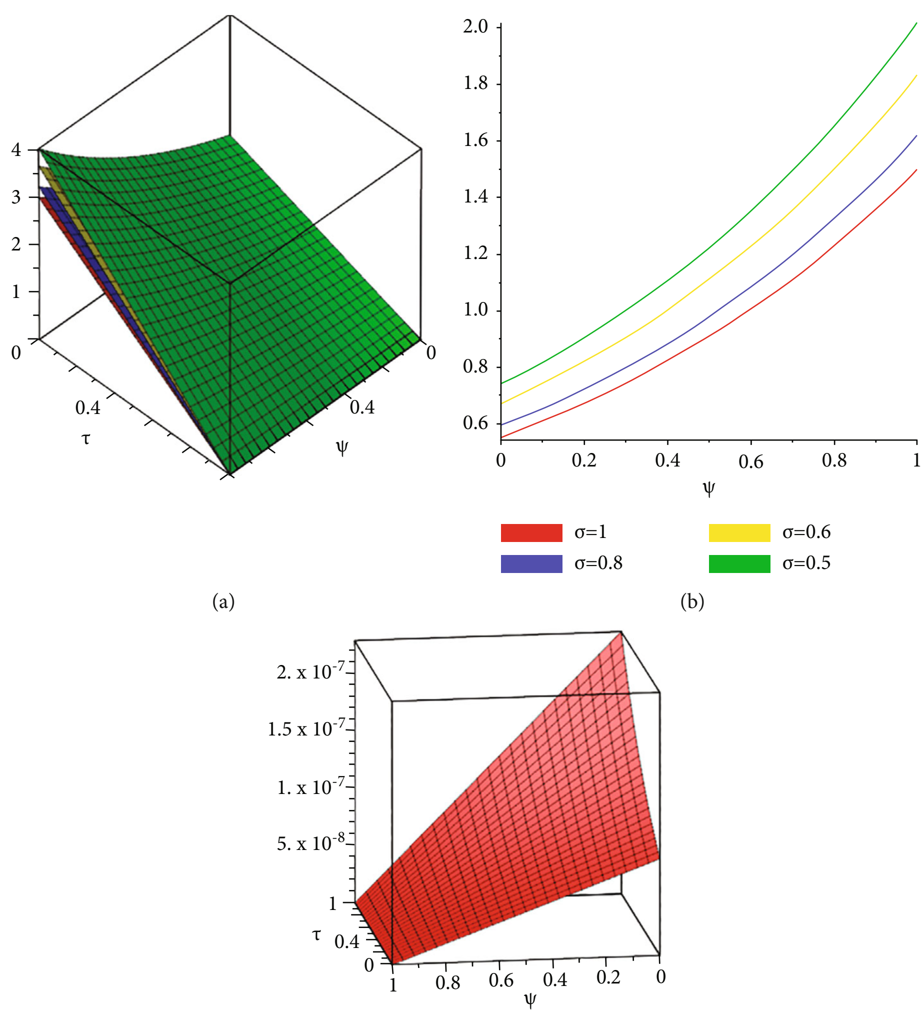


FIGURE 4: Example 3solufiguretion graph. (a) Exact solution and (b) analytical solution at $\sigma = 1$.



(c)

FIGURE 5: Example 3. (a) Analytical solution at various fractional orders of σ (b) $\phi = 0.5$ and (c) error.

Table 2: $v(\psi, \phi, \tau)$ comparison of exact solution, our methods' solution, and Absolute Error (AE) of Example 3.

$\tau = 0.01$ ψ	Exact solution $\sigma = 1$	Our methods' solution $\sigma = 1$	AE of our methods $\sigma = 1$	AE of our methods $\sigma = 0.9$	AE of our methods $\sigma = 0.8$
0	0.505025083500000	0.505025083500000	0.0000000000E+00	2.5480900000E-05	2.6059160000E-04
0.1	0.558139035000000	0.558139035100000	1.0000000000E-10	2.8160900000E-05	2.8799840000E-04
0.2	0.616839030000000	0.616839029800000	2.0000000000E-10	3.1122300000E-05	3.1828710000E-04
0.3	0.681712557000000	0.681712557200000	2.0000000000E-10	3.4395800000E-05	3.5176210000E-04
0.4	0.753408892500000	0.753408892700000	2.0000000000E-10	3.8013200000E-05	3.8875720000E-04
0.5	0.832645597500000	0.832645597600000	1.0000000000E-10	4.2011000000E-05	4.2964300000E-04
0.6	0.920215699500000	0.920215699100000	4.0000000000E-10	4.6428800000E-05	4.7482850000E-04
0.7	1.016995630000000	1.016995629000000	1.0000000000E-09	5.1311000000E-05	5.2476600000E-04
0.8	1.123953994000000	1.123953993000000	1.0000000000E-09	5.6708000000E-05	5.7995600000E-04
0.9	1.242161266000000	1.242161267000000	1.0000000000E-09	6.2673000000E-05	6.4095200000E-04
1.0	1.372800508000000	1.372800507000000	1.0000000000E-09	6.9263000000E-05	7.0836100000E-04

We get by using the Jafari-Yang transform's differential property

$$\frac{1}{u^\sigma}\{T(u) - uv(0)\} = Y\left(\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right), \qquad (113)$$

$$T(u) = uv(0) + u^\sigma Y\left(\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right), \qquad (114)$$

By applying inverse Jafari-Yang transform to Equation (113)

$$\begin{aligned} v(\psi, \phi, \varphi, \tau) &= v(0) + Y^{-1}\left[u^\sigma\left\{Y\left(\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right)\right\}\right], \\ v(\psi, \phi, \varphi, \tau) &= \sin\psi \sin\phi \sin\varphi + Y^{-1} \\ &\cdot \left[u^\sigma\left\{Y\left(\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right)\right\}\right]. \end{aligned} \qquad (115)$$

Now, using the abovementioned homotopy perturbation technique as in (23), we obtain

$$\begin{aligned} &\sum_{k=0}^{\infty} \varepsilon^k v_k(\psi, \phi, \varphi, \tau) \\ &= \sin\psi \sin\phi \sin\varphi \\ &\quad + \varepsilon\left(Y^{-1}\left[u^\sigma Y\left[\left(\sum_{k=0}^{\infty} \varepsilon^k v_k(\psi, \phi, \varphi, \tau)\right)_{\psi\psi}\right.\right.\right. \\ &\quad + \left(\sum_{k=0}^{\infty} \varepsilon^k v_k(\psi, \phi, \varphi, \tau)\right)_{\phi\phi} \\ &\quad \left.\left.\left. + \left(\sum_{k=0}^{\infty} \varepsilon^k v_k(\psi, \phi, \varphi, \tau)\right)_{\varphi\varphi}\right]\right]\right). \end{aligned} \qquad (116)$$

Comparing the same power coefficient of ε, we get

$$\begin{aligned} \varepsilon^0 &: v_0(\psi, \phi, \varphi, \tau) = \sin\psi \sin\phi \sin\varphi, \\ \varepsilon^1 &: v_1(\psi, \phi, \varphi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 v_0}{\partial \psi^2} + \frac{\partial^2 v_0}{\partial \phi^2}\right]\right) \\ &\quad = -3 \sin\psi \sin\phi \sin\varphi \frac{\tau^\sigma}{\Gamma(\sigma+1)}, \\ \varepsilon^2 &: v_2(\psi, \phi, \varphi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 v_1}{\partial \psi^2} + \frac{\partial^2 v_1}{\partial \phi^2}\right]\right) \\ &\quad = (-3)^2 \sin\psi \text{ssin}\phi \sin\varphi \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}, \\ \varepsilon^3 &: v_3(\psi, \phi, \varphi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 v_2}{\partial \psi^2} + \frac{\partial^2 v_2}{\partial \phi^2}\right]\right) \\ &\quad = (-3)^3 \sin\psi \sin\phi \sin\varphi \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}, \\ \varepsilon^4 &: v_4(\psi, \phi, \varphi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 v_3}{\partial \psi^2} + \frac{\partial^2 v_3}{\partial \phi^2}\right]\right) \\ &\quad = (-3)^4 \sin\psi \sin\phi \sin\varphi \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}. \\ &\vdots \end{aligned} \qquad (117)$$

By taking $\varepsilon \longrightarrow 1$, we obtain the convergence series type result is given as

$$\begin{aligned} v(\psi, \phi, \varphi, \tau) &= v_0 + v_1 + v_2 + v_3 + v_4 + \cdots = \sin\psi \sin\phi \sin\varphi \\ &\quad - 3 \sin\psi \sin\phi \sin\varphi \frac{\tau^\sigma}{\Gamma(\sigma+1)} \\ &\quad + (-3)^2 \sin\psi \sin\phi \sin\varphi \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)} \end{aligned}$$

$$+(-3)^3 \sin\psi \sin\phi \sin\varphi \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} + (-3)^4 \sin\psi \sin\phi \sin\varphi \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)} + \cdots v(\psi,\phi,\varphi,\tau) = \sin\psi \sin\phi \sin\varphi \cdot \left(1 - \frac{3\tau^\sigma}{\Gamma(\sigma+1)} + \frac{(-3\tau^\sigma)^2}{\Gamma(2\sigma+1)} + \frac{(-3\tau^\sigma)^3}{\Gamma(3\sigma+1)} + \frac{(-3\tau^\sigma)^4}{\Gamma(4\sigma+1)} + \cdots\right). \tag{118}$$

When $\sigma = 1$, then the HPYTM solution in a closed form:

$$v(\psi,\phi,\varphi,\tau) = \sin\psi \sin\phi \sin\varphi \left(1 - 3\tau + \frac{(-3\tau)^2}{2!} + \frac{(-3\tau)^3}{3!} + \frac{(-3\tau)^4}{4!} + \cdots,\right). \tag{119}$$

The analytical results by YTDM

Taking Jafari-Yang transformation of Equation (110), we obtain

$$Y\left\{\frac{\partial^\sigma v}{\partial \tau^\sigma}\right\} = Y\left[\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right]. \tag{120}$$

We get by using the Jafari-Yang transform's differential property

$$\frac{1}{u^\sigma}\{T(u) - uv(0)\} = Y\left[\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right], \tag{121}$$

$$T(u) = uv(0) + u^\sigma Y\left[\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right], \tag{122}$$

by inverse Jafari-Yang transform of Equation (121)

$$v(\psi,\phi,\varphi,\tau) = v(0) + Y^{-1}\left[u^\sigma\left\{Y\left(\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right)\right\}\right],$$
$$v(\psi,\phi,\varphi,\tau) = \exp(\psi+\phi) + Y^{-1} \cdot \left[u^\sigma\left\{Y\left(\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right)\right\}\right]. \tag{123}$$

Assuming the unknown $v(\psi,\phi,\varphi,\tau)$ function has the following infinite series form solution:

$$v(\psi,\phi,\varphi,\tau) = \sum_{k=0}^{\infty} v_k(\psi,\phi,\varphi,\tau), \tag{124}$$

$$\sum_{k=0}^{\infty} v_k(\psi,\phi,\varphi,\tau) = \sin\psi \sin\phi \sin\varphi + Y^{-} \cdot \left[u^\sigma Y\left[\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right]\right]. \tag{125}$$

Thus, on comparing both sides of Equation (124)

$$v_0(\psi,\phi,\varphi,\tau) = \sin\psi \sin\phi \sin\varphi. \tag{126}$$

For $k = 0$

$$v_1(\psi,\phi,\varphi,\tau) = -3 \sin\psi \sin\phi \sin\varphi \frac{\tau^\sigma}{\Gamma(\sigma+1)}, \tag{127}$$

For $k = 1$

$$v_2(\psi,\phi,\varphi,\tau) = (-3)^2 \sin\psi \sin\phi \sin\varphi \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}. \tag{128}$$

For $k = 2$

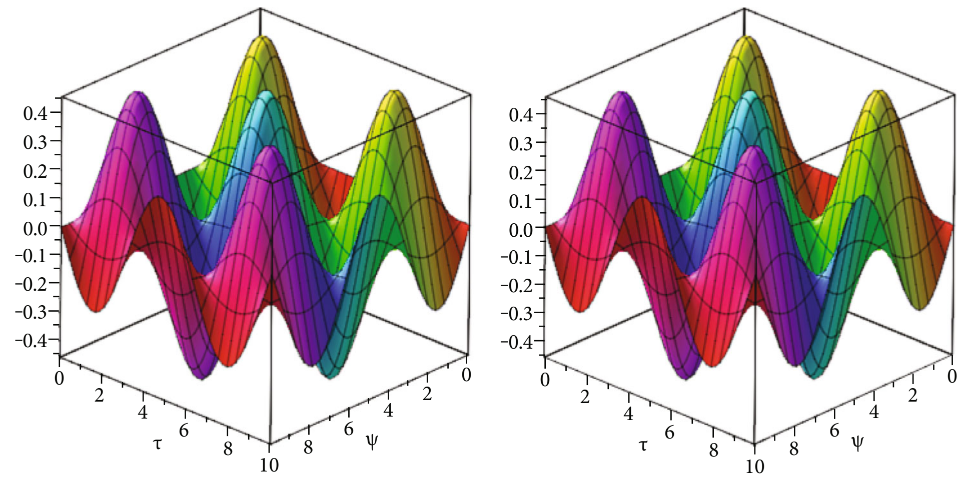
$$v_3(\psi,\phi,\varphi,\tau) = (-3)^3 \sin\psi \sin\phi \sin\varphi \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}. \tag{129}$$

For $k = 3$

$$v_4(\psi,\phi,\varphi,\tau) = (-3)^4 \sin\psi \sin\phi \sin\varphi \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}. \tag{130}$$

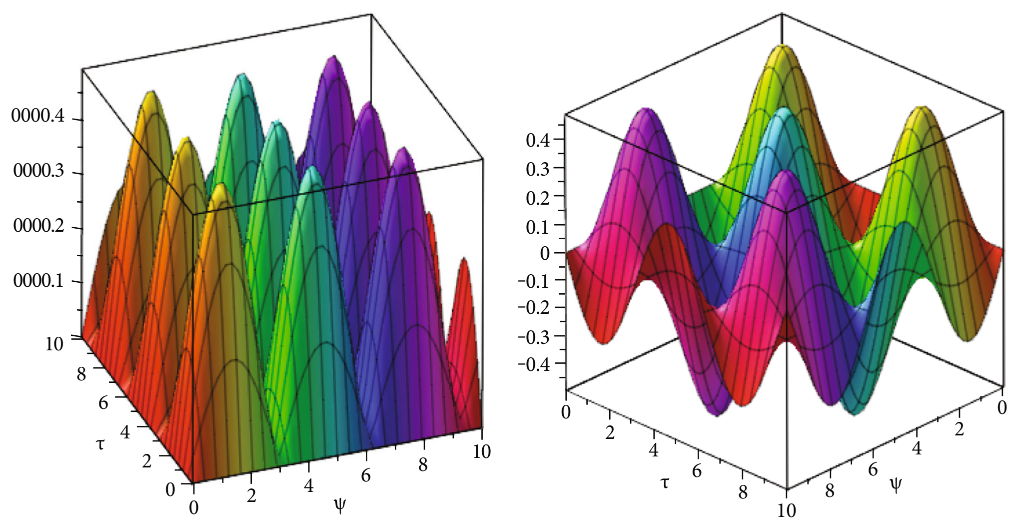
The remaining YTDM solution elements ρ_k for $(k \geq 3)$ are similarly simple to get. As a result, the solution in series form is as

$$\begin{aligned} v(\psi,\phi,\varphi,\tau) &= v_0(\psi,\phi,\varphi,\tau) + v_1(\psi,\phi,\varphi,\tau) + v_2(\psi,\phi,\varphi,\tau) + v_3(\psi,\phi,\varphi,\tau) + \cdots v(\psi,\phi,\varphi,\tau) \\ &= \sin\psi \sin\phi \sin\varphi - 3 \sin\psi \sin\phi \sin\varphi \frac{\tau^\sigma}{\Gamma(\sigma+1)} + (-3)^2 \sin\psi \sin\phi \sin\varphi \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)} \\ &\quad + (-3)^3 \sin\psi \sin\phi \sin\varphi \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} + (-3)^4 \sin\psi \sin\phi \sin\varphi \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)} \\ &\quad + \cdots v(\psi,\phi,\varphi,\tau) = \sin\psi \sin\phi \sin\varphi \cdot \left(1 - \frac{3\tau^\sigma}{\Gamma(\sigma+1)} + \frac{(-3\tau^\sigma)^2}{\Gamma(2\sigma+1)} + \frac{(-3\tau^\sigma)^3}{\Gamma(3\sigma+1)} + \frac{(-3\tau^\sigma)^4}{\Gamma(4\sigma+1)} + \cdots\right). \end{aligned} \tag{131}$$



(a) (b)

Figure 6: Example 4 solution graph. (a) Exact solution. (b) Analytical solution at $\sigma = 1$.



(a) (b)

Figure 7: (a) Error graph and solution graph (b) at $\sigma = 0.5$ of Example 4.

Table 3: $v(\psi, \phi, ,\varphi, \tau)$ comparison of exact solution, our methods' solution, and Absolute Error (AE) of Example 4.

$\tau = 0.01$ ψ	Exact solution $\sigma = 1$	Our methods' solution $\sigma = 1$	AE of our methods $\sigma = 1$	AE of our methods $\sigma = 0.9$	AE of our methods $\sigma = 0.8$
0	0.00000000000000	0.00000000000000	0.000000000E+00	0.000000000E+00	0.000000000E+00
0.1	0.039084756460000	0.039084756470000	8.4147098480E-12	5.8916348320E-06	6.0179783350E-05
0.2	0.077778990960000	0.077778990980000	1.6829419700E-11	1.1724408770E-05	1.1975827680E-04
0.3	0.115696083500000	0.115696083500000	0.0000000000E+00	1.7440075190E-05	1.7814016480E-04
0.4	0.152457179000000	0.152457179000000	0.0000000000E+00	2.2981414070E-05	2.3474221550E-04
0.5	0.187694972800000	0.187694972800000	8.4147098480E-11	2.8293115510E-05	2.8899866580E-04
0.6	0.221057380400000	0.221057380400000	0.0000000000E+00	3.3322251000E-05	3.4036760840E-04
0.7	0.252211055800000	0.252211055900000	8.4147098480E-11	3.8018332270E-05	3.8833574600E-04
0.8	0.280844721700000	0.280844721700000	8.4147098480E-11	4.2334573540E-05	4.3242369270E-04
0.9	0.306672279900000	0.306672280000000	8.4147098480E-11	4.6227807340E-05	4.7219102240E-04
1.0	0.329435670100000	0.329435670200000	8.4147098480E-11	4.9659157730E-05	5.0724039260E-04

When $\sigma = 1$, then the closed form solution by YTDM

$$\nu(\psi, \phi, \varphi, \tau) = \sin \psi \sin \phi \sin \varphi \left(1 - 3\tau + \frac{(-3\tau)^2}{2!} + \frac{(-3\tau)^3}{3!} + \frac{(-3\tau)^4}{4!} + \cdots\right). \quad (132)$$

In closed form, the exact solution is

$$\nu(\psi, \phi, \varphi, \tau) = \exp^{-3\tau} \sin \psi \sin \phi \sin \varphi. \quad (133)$$

In Figures 6(a) and 6(b), we consider fixed order $\sigma = 1$ for piecewise approximation values of ψ, ϕ in the domain $0 \leq \psi, \phi \geq 10$ and $= 1$. Figure 7(a) shows error graph and Figure 7(b) represents HPYTM and YTDM solution at $\sigma = 0.6$ of Example 4. It is verified from the Figures 6(a) and 6(b) and Table 3 that HPYTM and YTDM solution is closely related with the exact solution.

6. Conclusion

In the present article, different analytical techniques are used to show the fractional view analysis of diffusion equations. In Caputo, manner fractional derivative is considered. The suggested techniques are tested to solve fractional-order diffusion equations. It is observed that the suggested techniques are the best tool for investigating fractional partial differential equations. The close relation between the exact and analytical results is confirmed by the plotted graphs. The given methods give series form solution which have higher convergence rate towards the exact results. It is also shown that both methods give same solution for the proposed problems. Finally, both proposed methods are very methodical and efficient and may be used to investigate nonlinear physical problems related to physics of plasmas such as modeling nonlinear unmodulated and modulated structures. Moreover, the obtained results/solutions can be useful in investigating the diffusion characteristics of some plasmas and fluids.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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Research Article

Cesaŕo Summable Relative Uniform Difference Sequence of Positive Linear Functions

Awad A. Bakery[1,2] **O. M. Kalthum S. K. Mohamed**[1,3] **Kshetrimayum Renubebeta Devi,**[4] **and Binod Chandra Tripathy**[4]

[1]*Department of Mathematics, University of Jeddah, College of Science and Arts at Khulis, Jeddah, Saudi Arabia*
[2]*Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Abbassia, Egypt*
[3]*Department of Mathematics, Academy of Engineering and Medical Sciences, Khartoum, Sudan*
[4]*Department of Mathematics, Tripura University, Agartala, Tripura, 799022, India*

Correspondence should be addressed to O. M. Kalthum S. K. Mohamed; om_kalsoom2020@yahoo.com

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In this article, we introduce the class of sequence of functions $C_1(\Delta, ru)$ of Cesaŕo summable relative uniform difference sequence of functions. We have studied the topological properties of $C_1(\Delta, ru)$. We also obtain the necessary and sufficient condition to characterize the matrix classes $(C_1(\Delta, ru), \ell_\infty(ru)), (C_1(\Delta, ru), c(ru), P)$.

1. Introduction

Throughout the study, $\omega(ru)$, $C_1(\Delta, ru)$, and $\ell_\infty(ru)$ denote the classes of all relative uniform sequence space, Cesaŕo summable relative uniform difference sequence space, and bounded relative uniform sequence space, respectively.

Moore in 1910 introduced the notion of uniform convergence of a sequence of functions relative to a scale function. Chittenden [1–3] gave the detailed definition of the notion as follows.

Definition 1 (see [1]). A sequence $(f_i(x))$ of single-valued, real-valued functions $f_i(x)$ of a variable x ranging over a compact subset D of real numbers is said to be relatively uniformly convergent on D w. r. t. a scale function $\sigma(x)$ in case there exist a limiting function $f(x)$ and scale function $\sigma(x)$ defined on D and for every ε, an integer $n_0 = n_0(\varepsilon)$ such that for every $n \geq n_0$ and for all $x \in D$,

$$|f_i(x) - f(x)| < \varepsilon|\sigma(x)|. \tag{1}$$

The notion was further discussed from various aspects by Demirci et al. [4], Demirci and Orhan [5], Devi and Tripathy [6], and many others.

Example 1. Let $0 < a < 1$ be a real number. Consider the sequence of functions $(f_i(x)), f_i : [a, 1] \longrightarrow R$, for all $i \in N$ defined by

$$f_i(x) = \frac{1}{ix}, \text{for all } x \in [a, 1], i \in N. \tag{2}$$

This sequence of functions does not converge uniformly to 0 on $[a, 1]$. However, $(f_i(x))$ converges to 0 uniformly with respect to the scale function $\sigma(x)$ defined by

$$\sigma(x) = \frac{1}{x}, \text{for all } x \in [a, 1]. \tag{3}$$

Kizmaz [7] defined the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ as follows:

$$Z(\Delta) = \{x = (x_i)\colon (\Delta x_i) \in Z\}, \quad (4)$$

for $Z = \ell_\infty, c, c_0$ where $\Delta x_i = x_i - x_{i+1}, i \in N$.

These sequence spaces are Banach space under the norm

$$\|x_i\|_\Delta = |x_1| + \sup_{i\in N}|\Delta x_i|. \quad (5)$$

The notion was further studied from different aspects by many others [8–13].

The Cesaŕo sequence space $\mathrm{Ces}_\infty, \mathrm{Ces}_p (1 < p < \infty)$ were introduced by Shiue [14], and it has been shown that $\ell_\infty \subset \mathrm{Ces}_p$; the inclusion is strict for $1 < p < \infty$. Further, the Cesaŕo sequence spaces X_p and X_∞ of nonabsolute type were defined by Ng and Lee [15, 16]. For a detail account of Cesaŕo difference sequence space, one may refer to [17–20].

Let $A = (a_{ni})$ be an infinite matrix of real or complex numbers. Then, A transforms from the sequence space λ into the sequence space F, if $Ax \in F$ for each sequence $(x_i) \in \lambda$ that is $Ax = (A_n x) \in F$, where $A_n x = \sum_{i=1}^{\infty} a_{ni} x_i$, provided that the infinite series converges for each $n \in N$.

Matrix transformation between sequence space was studied from different aspects by many others [21–25].

2. Definitions and Preliminaries

Definition 2. A sequence space λ is said to be solid or normal if $(x_i) \in \lambda$ implies $(\alpha_i x_i) \in \lambda$, for all (α_i) with $|\alpha_i| \le 1$, for all $i \in N$.

Definition 3. A sequence space λ is said to be *monotone* if it contains the canonical preimages of all its step spaces.

Remark 4. A sequence space λ is solid then, λ is monotone.

Definition 5. A sequence space λ is said to be *symmetric* if $(x_i) \in \lambda \Rightarrow (x_{\pi(i)}) \in \lambda$, for all $i \in N$, where π is a permutation of N, the set of natural numbers.

Definition 6. A sequence space λ is said to be *convergence free* if $(x_i) \in \lambda$ and $x_i = 0 \Rightarrow y_i = 0$ together with $(y_i) \in \lambda$, for all $i \in N$.

Definition 7. A sequence space λ is said to be a *sequence algebra* if $(x_i . y_i) \in \lambda$ whenever (x_i) and (y_i) belongs to λ, for all $i \in N$.

In this article we introduce the sequence space $C_1(\Delta, ru)$ of Cesaŕo summable relative uniform difference sequence of functions and it is defined as follows:

$$C_1(\Delta, ru) = \{f = (f_i(x)) \in \omega(ru)\colon (\Delta f_i(x)) \in C_1 \ w.r.t. \text{the scale function } \sigma(x)\}, \quad (6)$$

where $\Delta f_i(x) \in C_1$ (ru) and $\Delta f_i(x) = f_i(x) - f_{i+1}(x)$.

3. Main Results

We state the following result without proof.

Theorem 8. *The sequence space $C_1(\Delta, ru)$ is a normed linear space.*

Theorem 9. *The sequence space $C_1(\Delta, ru)$ is a Banach space normed by*

$$\|f\|_{(\Delta,\sigma)} = \sup_{\|x\|\le 1} \frac{\|f_1(x)\|\|\sigma(x)\|}{\|x\|} + \sup_{p\ge 1} \sup_{\|x\|\le 1} \frac{1}{p} \frac{\sum_{i=1}^{p}\|\Delta f_i(x)\|\|\sigma(x)\|}{\|x\|}. \quad (7)$$

Proof. Let $(f^n(x))$ be a Cauchy sequence in $C_1(\Delta, ru)$ where

$$(f^n(x)) = (f_i^n(x)) = (f_1^n(x), f_2^n(x), \cdots) \in C_1(\Delta, ru), \text{for each } i \in N. \quad (8)$$

□

Then,

$$\|f^n(x) - f^m(x)\|_{(\Delta,\sigma)} = \sup_{\|x\|\le 1} \frac{\|f_1^n(x) - f_1^m(x)\|\|\sigma(x)\|}{\|x\|} + \sup_{p\ge 1} \sup_{\|x\|\le 1} \frac{1}{p} \cdot \frac{\sum_{i=1}^{p}\|\Delta f_i^n(x) - \Delta f_i^m(x)\|\|\sigma(x)\|}{\|x\|} \longrightarrow 0. \quad (9)$$

For all $n, m \ge n_0$,

$$\|f^n(x) - f^m(x)\|_{(\Delta,\sigma)} = \sup_{\|x\|\le 1} \frac{\|f_1^n(x) - f_1^m(x)\|\|\sigma(x)\|}{\|x\|} + \sup_{p\ge 1} \sup_{\|x\|\le 1} \frac{1}{p} \cdot \frac{\sum_{i=1}^{p}\|\Delta f_i^n(x) - \Delta f_i^m(x)\|\|\sigma(x)\|}{\|x\|} < \frac{\varepsilon}{2}. \quad (10)$$

$(f_1^n(x))$ is a Cauchy sequence in D w.r.t. $\sigma(x)$ for all $x \in D$.

$\Rightarrow (f_1^n(x))$ is convergent in D w.r.t. $\sigma(x)$ for all $x \in D$.

Let $\lim_{n\longrightarrow\infty} f_1^n(x) = f_1(x), x \in D$.

Similarly, $\lim_{n\longrightarrow\infty}(1/p)\sum_{i=1}^{p}\Delta f_i^n(x) = (1/p)\sum_{i=1}^{p}\Delta f_i(x), x \in D$.

From the above equations we get,

$$\lim_{n\longrightarrow\infty} f_i^n(x) = f_i(x), \quad (11)$$

for all $x \in D$, for all $i \in N$.

From (10) we have

$$\lim_{m\longrightarrow\infty}\frac{\|(f_1^n(x)-f_1^m(x))\sigma(x)\|}{\|x\|}=\frac{\|(f_1^n(x)-f_1(x))\sigma(x)\|}{\|x\|}<\frac{\varepsilon}{2}, \quad (12)$$

for all $x \in D$.

Similarly,

$$\begin{aligned}\lim_{m\longrightarrow\infty}\frac{1}{p}\frac{\sum_{i=1}^{p}\|(\Delta f_i^n(x)-\Delta f_i^m(x))\sigma(x)\|}{\|x\|}\\ =\frac{1}{p}\frac{\sum_{i=1}^{p}\|(\Delta f_i^n(x)-\Delta f_i(x))\sigma(x)\|}{\|x\|}<\frac{\varepsilon}{2},\end{aligned} \quad (13)$$

for all $x \in D$ and $i \in N$.

Since $\varepsilon/2$ is not dependent on i, we have

$$\begin{aligned}\sup_{\|x\|\le 1}\frac{\|(f_1^n(x)-f_1(x))\sigma(x)\|}{\|x\|}<\frac{\varepsilon}{2},\\ \sup_{p\ge 1}\sup_{\|x\|\le 1}\frac{1}{p}\frac{\sum_{i=1}^{p}\|(\Delta f_i^n(x)-\Delta f_i(x))\sigma(x)\|}{\|x\|}<\frac{\varepsilon}{2}.\end{aligned} \quad (14)$$

Evidently,

$$\begin{aligned}\|f^n(x)-f(x)\|_{(\Delta,\sigma)}=\sup_{\|x\|\le 1}\frac{\|f_1^n(x)-f_1(x)\|\|\sigma(x)\|}{\|x\|}\\ +\sup_{p\ge 1}\sup_{\|x\|\le 1}\frac{1}{p}\\ \cdot\frac{\sum_{i=1}^{p}\|\Delta f_i^n(x)-\Delta f_i(x)\|\|\sigma(x)\|}{\|x\|}\le\varepsilon.\end{aligned} \quad (15)$$

$\Rightarrow \|f^n(x)-f(x)\|_{(\Delta,\sigma)} \le \varepsilon$, for all $n \ge n_0$, for all $x \in D$.

Therefore, $(f^n(x)-f(x)) \in C_1(\Delta, ru)$, for all $n \ge n_0$, for all $x \in D$.

Then, $f(x)=f^n(x)-(f^n(x)-f(x)) \in C_1(\Delta, ru)$ since $C_1(\Delta, ru)$ is a linear space.

Theorem 10. *The inclusion $C_1(ru) \subset C_1(\Delta, ru)$ strictly holds.*

Proof. The proof of the theorem is obvious and the strictness of the inclusion is shown in the following example. □

Example 1. Let $0<a<1$ be a real number and $D=[a,1]$. Consider the sequence of real valued functions $(f_i(x)), f_i : [a,1] \longrightarrow R$, for all $i \in N$, defined by

$$\begin{aligned}f_i(x)&=ix, \text{for all } x\in[a,1],\\ \Delta f_i(x)&=f_i(x)-f_{i+1}(x)=x, \text{for } all\, x\in[a,1].\end{aligned} \quad (16)$$

$(f_i(x)) \in C_1(\Delta, ru)$ w.r.t. the scale function $\sigma(x)=1$, for all xin D, but $(f_i(x)) \notin \ell_\infty(ru)$.

Hence, the inclusion is strict.

Theorem 11. *The inclusion $c(\Delta, ru) \subset C_1(\Delta, ru)$ strictly holds.*

Proof. The proof is obvious and the strictness of the inclusion is shown in the following example. □

Example 2. Let $0<a<1$ be a real number and $D=[a,1]$. Consider the sequence of real valued functions $(f_i(x)), f_i : [a,1] \longrightarrow R$, for all $i \in N$, defined by

$$\begin{aligned}f_i(x)&=\begin{cases}x, & \text{for } i \text{ is odd},\\ 0, & \text{otherwise},\end{cases}\\ \Delta f_i(x)&=\begin{cases}x, & \text{for } i \text{ is odd},\\ -x, & \text{otherwise}.\end{cases}\end{aligned} \quad (17)$$

We have $(f_i(x)) \in C_1(\Delta, ru)$ w.r.t. the scale function $\sigma(x)$ defined by

$$\sigma(x)=\left\{\frac{1}{x}, \text{for all } x\in[a,1],\right. \quad (18)$$

but $(f_i(x)) \notin c(\Delta, ru)$. Hence, the inclusion is strict.

Theorem 12. *The sequence space $C_1(\Delta, ru)$ is not monotone.*

Proof. The proof is shown in the following example. □

Example 3. Let $0<a<1$ be a real number and $D=[a,1]$. Consider the sequence of real valued functions $(f_i(x)), f_i : [a,1] \longrightarrow R$, for all $i \in N$, defined by

$$\begin{aligned}f_i(x)&=ix, \text{for all } x\in[a,1],\\ \Delta f_i(x)&=f_i(x)-f_{i+1}(x)=x, \text{for all } x\in[a,1].\end{aligned} \quad (19)$$

$(f_i(x)) \in C_1(\Delta, ru)$ w.r.t. the scale function defined on D by $\sigma(x)=1$.

Let $(g_i(x))$ be the preimage of $(f_i(x))$ defined by

$$g_i(x)=\begin{cases}i^2x, & \text{for } i=k^2, k\in N,\\ 0, & \text{otherwise}.\end{cases} \quad (20)$$

One cannot get a scale function that makes $(g_i(x)) \in C_1(\Delta, ru)$.

Hence $C_1(\Delta, ru)$ is not monotone.

Remark 13. The sequence space $C_1(\Delta, ru)$ is not solid since $C_1(\Delta, ru)$ is not monotone.

Theorem 14. *The sequence space $C_1(\Delta, ru)$ is not symmetric.*

Proof. The proof of the theorem is shown with the help of the following example. □

Example 4. Let us consider the sequence of functions $(f_i(x))$ considered in Example 3. Let $(g_i(x))$ be the rearrangement sequence of functions of $(f_i(x))$ defined by

$$g_i(x)=\begin{cases}(i+1)x, & \text{for } i=2j-1, j\in N,\\ (i-1)x, & \text{for } i=2j, j\in N.\end{cases} \quad (21)$$

One cannot get a scale function that makes $(g_i(x))\in C_1(\Delta, ru)$.

Hence, $C_1(\Delta, ru)$ is not symmetric.

Theorem 15. *The sequence space $C_1(\Delta, ru)$ is not sequence algebra.*

Proof. The proof of the theorem is shown in the following example. □

Example 5. Let $0<a<1$ be a real number and $D=[a,1]$. Consider the sequences of real valued functions $(f_i(x)), f_i : [a,1]\longrightarrow R$, and $(g_i(x)), g_i : [a,1]\longrightarrow R$, for all $i\in N$, defined by

$$\begin{aligned} f_i(x)=g_i(x)=ix, \text{for all } i\in N.\\ f_i(x).g_i(x)=i^2x^2. \end{aligned} \quad (22)$$

We get that $(f_i(x)), (g_i(x))\in C_1(\Delta, ru)$ but one cannot get a scale function that makes $(f_i(x).g_i(x))\in C_1(\Delta, ru)$.

Hence, $C_1(\Delta, ru)$ is not sequence algebra.

Theorem 16. *The sequence space $C_1(\Delta, ru)$ is not convergence free.*

Proof. The proof of the theorem follows from example below. □

Example 6. Let $0<a<1$ be a real number and $D=[a,1]$. Consider the sequence of real valued functions $(f_i(x)), f_i : [a,1]\longrightarrow R$, for all $i\in N$, defined by

$$\begin{aligned} (f_i(x))=x, \text{for all } x\in[a,1],\\ \Delta f_i(x)=0. \end{aligned} \quad (23)$$

Therefore, $(f_i(x))\in C_1(\Delta, ru)$ w.r.t. the constant scale function defined on D by $\sigma(x)=1$.

Let us consider another sequence of functions $(g_i), g_i$: $[a,1]\longrightarrow R$ defined by

$$g_i(x)=\begin{cases} g_1(x)=x\,;\\ g_{i+1}(x)=g_{m-1}(x)+(n+1)x, & \text{for } m\geq 2, m, n, i\in N,\end{cases}$$
$$C_1(\Delta, ru)=-nx, n\geq 2, n\in N. \quad (24)$$

One cannot find a scale function that makes $(g_i(x))\in C_1(\Delta, ru)$.

Hence, the sequence space $C_1(\Delta, ru)$ is not convergence free.

3.1. Matrix Transformation between Sequence of Functions. In this section, we give certain matrix classes between the sequence of functions.

Theorem 17. $A\in(C_1(\Delta, ru), \ell_\infty(ru))$ *if and only if* $\sup_{n\geq 1}\sum_{i=2}^{\infty}(i-1)|a_{ni}|<\infty$.

Proof. Let $(f_i(x))\in C_1(\Delta, ru)$ and $\sup_{n\geq1}\sum_{i=2}^{\infty}(i-1)|a_{ni}|<\infty$.

$$\begin{aligned}\sum_{i=1}^{\infty}a_{ni}f_i(x)\sigma(x)&=-\sum_{i=2}^{\infty}(i-1)a_{ni}\left(\frac{1}{i-1}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x)\right)\\ &\quad+f_1(x)\sigma(x)\sum_{i=1}^{\infty}a_{ni}=\sum_{i=2}^{\infty}a_{ni}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x).\end{aligned} \quad (25)$$

□

From the relation between dual and matrix map, we know that $c^\alpha=\ell_1$. Hence,

$$\sum_{i=2}^{\infty}a_{ni}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) \text{ converges absolutely w.r.t.} \sigma(x). \quad (26)$$

For $p\in N$, we have, $\sum_{i-1}^{p}a_{ni}f_i(x)\sigma(x)=-\sum_{i=1}^{p}a_{ni}(\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x))+f_1(x)\sigma(x)\sum_{i=1}^{p}a_{ni}$.

By the argument (10), we know that $\sum_{i=1}^{p}a_{ni}f_i(x)\sigma(x)$ is absolutely convergent w.r.t. scale function $\sigma(x)$.

Then,

$$\begin{aligned}\sum_{i=1}^{\infty}|a_{ni}f_i(x)\sigma(x)|&\leq\left(\sup_n\sum_{i=2}^{\infty}(i-1)|a_{ni}|\right)\\ &\quad\cdot\left(\sup_{x\leq1}\sup_{i\geq2}\frac{1}{i-1}\sum_{t=1}^{i-1}|\Delta f_t(x)\sigma(x)|\right)\\ &\quad+|f_1(x)\sigma(x)|\sup_n\sum_{i=2}^{\infty}(i-1)|a_{ni}|.\end{aligned} \quad (27)$$

We have, $\sum_{i=1}^{\infty}a_{ni}f_i(x)\sigma(x)<\infty$ since $\sup_{n\geq1}\sum_{i=2}^{\infty}(i-1)|a_{ni}|<\infty$.

Conversely, we know that A is a bounded linear function from $C_1(\Delta, ru)$ to $\ell_\infty(ru)$, so we can write,

$$\begin{aligned}\sum_{i=1}^{\infty}|a_{ni}f_i(x)\sigma(x)|&=|(A_nf)\sigma(x)|\leq\sup_n|(A_nf)\sigma(x)|\\ &=\|(Af)\sigma(x)\|_\infty\leq\|A\|\|f\|_{(\Delta,\sigma).}\end{aligned} \quad (28)$$

We choose a sequence of functions $(f_i), f_i: [0,1]\longrightarrow R$ defined by

$$f_i(x)=\begin{cases}(i-1)xsgna_{ni}, & \text{for } 1<i\leq r,\\ 0, & \text{otherwise.}\end{cases} \quad (29)$$

We get $(f_i(x)) \in C_1(\Delta, ru)$ w.r.t. scale function $\sigma(x) = 1/x$ with the norm $\|f\|_{(\Delta,\sigma)} = 1$.

Putting the value of $(f_i(x))$ in equation (26) and letting the limit $r \longrightarrow \infty$, we get

$$\sum_{i=2}^{\infty}(i-1)|a_{ni}| \le \|A\|. \tag{30}$$

Theorem 18. *$A \in (C_1(\Delta, ru), c(ru), P)$ if and only if*

(i) $\sup_n \sum_{i=2}^{\infty}(i-1)|a_{ni}| < \infty$

(ii) $\lim_n \sum_{i=2}^{\infty}(i-1)a_{ni} = -1$

(iii) $\lim_n a_{ni} = 0$ for each i

(iv) $\lim_n \sum_i a_{ni} = 0$.

Proof. Let us assume that the conditions (i)-(iv) hold true and let $(f_i(x)) \in C_1(\Delta, ru)$, i.e., $\lim_i (1/i)\sum_{t=1}^{i}\Delta f_t(x)\sigma(x) = f(x)$ (say).

We know from (i) that for each $x \in D$ and $n \in N$, $\sum_i (i-1)|a_{ni}|$ converges. It follows that $\sum_{i=2}^{\infty}(i-1)a_{ni}((1/(i-1)i-1)\sum_{i=2}^{\infty}\Delta f_t(x)\sigma(x))$ converges.

$$\sum_i a_{ni} f_i(x)\sigma(x) = -\sum_{i=2}^{\infty}(i-1)a_{ni}\left(\frac{1}{i-1}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)\right) - f(x)\sum_i (i-1)a_{ni} + f_1(x)\sigma(x)\sum_i a_{ni}. \tag{31}$$

For any $n_0 \in N$, we have

$$\begin{aligned}&\left|\sum_{i=2}^{\infty}(i-1)a_{ni}\left(\frac{1}{i-1}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)\right)\right| \\ &\le \sup_{x\le 1}\sup_i\left(\frac{1}{i-1}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)\right)\sum_{i=2}^{n_0}(i-1)|a_{ni}| \\ &\quad + \sup_n \sum_{i=2}^{\infty}(i-1)a_{ni}\sup_{x\le 1}\sup_{i>n_0} \cdot\left(\frac{1}{i-1}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)\right)\end{aligned}$$

$$\begin{aligned}&\lim_n \sup\left|\sum_{i=2}^{\infty}(i-1)a_{ni}\left(\frac{1}{i-1}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)\right)\right| \\ &\le \sup_n \sum_{i=2}^{\infty}(i-1)a_{ni}\sup_{x\le 1}\sup_{i>n_0}\left(\frac{1}{i-1}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)\right).\end{aligned} \tag{32}$$

Let $n_0 = 0$, we get $\sum_{i=2}^{\infty}(i-1)a_{ni}((1/(i-1)i-1)\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)) \longrightarrow 0$.

Substituting this value in equation (31) and using conditions (ii) and (iv), we get $\sum_i a_{ni} f_i(x)\sigma(x)$ converges to $f(x)$ w.r.t. the scale function $\sigma(x)$. Conversely, let $A \in (C_1(\Delta, ru), c(ru), P)$. Then, $(\sum_i a_{ni} f_i(x)\sigma(x)) \in c(ru)$, for all

$$(f_i(x)) \in C_1(\Delta, ru). \tag{33}$$

Condition (i) can be proceed as same as shown in Theorem 17.

(ii) Let $(f_i), f_i$: $[0,1] \longrightarrow R$ defined by

$$f_i(x) = ix, \text{for all } i \in N. \tag{34}$$

Then, $f_i(x) \in C_1(\Delta, ru)\, w.r.t. \sigma(x) = 1/x$.
Since $\Delta f_i(x) \longrightarrow -1$, we have $\lim_n \sum_i (i-1)a_{ni} = -1$.

(iii) Let $(f_i), f_i$: $[0,1] \longrightarrow R$ defined by

$$f_i(x) = e_i x, \text{for all } i \in N. \tag{35}$$

Then, $f_i(x) \in C_1(\Delta, ru)\, w.r.t. \sigma(x) = 1/x$.
Since $\Delta f_i(x) \longrightarrow 0$, we have $\lim_n a_{ni} = 0$.

(iv) Let $(f_i), f_i$: $[0,1] \longrightarrow R$ defined by

$$f_i(x) = x, \text{for all } i \in N. \tag{36}$$

Then, $f_i(x) \in C_1(\Delta, ru)\, w.r.t. \sigma(x) = 1/x$.
Since $\Delta f_i(x) \longrightarrow 0$, we have $\lim_n \sum_i (i-1)a_{ni} = 0$.

The following theorem is stated without proof and can be proceeded the same as in Theorem 18. □

Theorem 19. *$A \in (C_1(\Delta, ru), c_0(ru))$ if and only if*

(i) $\sup_n \sum_{i=2}^{\infty}(i-1)|a_{ni}| < \infty$

(ii) $\lim_n \sum_{i=2}^{\infty}(i-1)a_{ni} = 0$

(iii) $\lim_n a_{ni} = 0$ for each i

(iv) $\lim_n \sum_i a_{ni} = 0$.

4. Conclusions

In this article, we studied the concept of Cesaŕo summability from the aspects of relative uniform convergence of difference sequence of positive linear functions w.r.t. a scale function $\sigma(x)$ on a compact domain D. The class of difference sequence of functions $C_1(\Delta, ru)$ is introduced, and its properties like solid, monotone, symmetric, sequence algebra, and convergence free are discussed. We have also further introduced characterization of matrix classes of $(C_1(\Delta, ru), \ell_\infty(ru)), (C_1(\Delta, ru), c(ru), P)$ and $(C_1(\Delta, ru), c_0(ru))$.

Data Availability

No data were used to support to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

On New Banach Sequence Spaces Involving Leonardo Numbers and the Associated Mapping Ideal

Taja Yaying[1], **Bipan Hazarika**[2], **O. M. Kalthum S. K. Mohamed**[3,4] **and Awad A. Bakery**[3,5]

[1]*Department of Mathematics, Dera Natung Government College, Itanagar 791113, India*
[2]*Department of Mathematics, Gauhati University, Gauhati 781014, India*
[3]*University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia*
[4]*Academy of Engineering and Medical Sciences, Department of Mathematics, Khartoum, Sudan*
[5]*Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Egypt*

Correspondence should be addressed to O. M. Kalthum S. K. Mohamed; om_kalsoom2020@yahoo.com

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In the present study, we have constructed new Banach sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$,$c(\mathfrak{L})$, and $\ell_\infty(\mathfrak{L})$, where $\mathfrak{L} = (\mathfrak{l}_{\nu,k})$ is a regular matrix defined by $\mathfrak{l}_{\nu,k} = \begin{cases} (\mathfrak{l}_k/\mathfrak{l}_{\nu+2} - (\nu+2)), & 0 \le k \le \nu, \\ 0, & k > \nu, \end{cases}$ for all $\nu, k = 0, 1, 2, \cdots$, where $\mathfrak{l} = (\mathfrak{l}_k)$ is a sequence of Leonardo numbers. We study their topological and inclusion relations and construct Schauder bases of the sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$, and $c(\mathfrak{L})$. Besides, α-, β- and γ-duals of the aforementioned spaces are computed. We state and prove results of the characterization of the matrix classes between the sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$,$c(\mathfrak{L})$, and $\ell_\infty(\mathfrak{L})$ to any one of the spaces ℓ_1,c_0,c, and ℓ_∞. Finally, under a definite functional ρ and a weighted sequence of positive reals r, we introduce new sequence spaces $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$. We present some geometric and topological properties of these spaces, as well as the eigenvalue distribution of ideal mappings generated by these spaces and s-numbers.

1. Introduction and Preliminaries

Let ω denote the set of all real- or complex-valued sequences. A linear subspace of ω is called a sequence space. Some of the well-known examples of sequence spaces are the space of absolutely p-summable sequences, the space of null sequences, the space of convergent sequences, and the space of bounded sequences, denoted by ℓ_p,c_0,c, and ℓ_∞, respectively. Here and afterwards, $1 \le p < \infty$, unless stated otherwise. Let bs and cs denote the spaces of all bounded and convergent series, respectively. A Banach sequence space with continuous coordinates is called a BK-space. The spaces $\mathfrak{Z}$ and ℓ_p are BK-spaces equipped with the supremum norm $\|\mathfrak{z}\|_{\ell_\infty} = \sup_{k\in\mathbb{N}_0} |\mathfrak{z}_k|$ and the ℓ_p norm $\|\mathfrak{z}\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |\mathfrak{z}_k|^p\right)^{1/p}$, respectively, where $\mathbb{N}_0$ is the set of nonnegative integers and $\mathfrak{Z}$ is any one of the spaces c_0,c, or ℓ_∞.

Let $\mathfrak{A} = (\mathfrak{a}_{\nu,k})$ be an infinite matrix over the complex field $\mathbb{C}$. The $\mathfrak{A}$-transform of a sequence $\mathfrak{z} = (\mathfrak{z}_k)$ is a sequence $\mathfrak{A}\mathfrak{z} = \{(\mathfrak{A}\mathfrak{z})_\nu\} = \{\sum_{k=0}^{\infty} \mathfrak{a}_{\nu,k}\mathfrak{z}_k\}$, provided that the series $\sum_{k=0}^{\infty} \mathfrak{a}_{\nu,k}\mathfrak{z}_k$ exists, for each $\nu \in \mathbb{N}_0$. In addition, if $\mathfrak{Z}$ and $\mathfrak{U}$ are two sequence spaces and $\mathfrak{A}\mathfrak{z} \in \mathfrak{U}$, for every sequence $\mathfrak{z} \in \mathfrak{Z}$, then the matrix $\mathfrak{A}$ is said to define a matrix mapping from $\mathfrak{Z}$ to $\mathfrak{U}$. The notation $(\mathfrak{Z}, \mathfrak{U})$ represents the family of all matrices that map from $\mathfrak{Z}$ to $\mathfrak{U}$. Furthermore, the matrix $\mathfrak{A} = (\mathfrak{a}_{v,k})$ is called a triangle if $\mathfrak{a}_{v,v} \neq 0$ and $\mathfrak{a}_{v,k} = 0$, for $v < k$. For any $\mathfrak{Z} \subset \omega$, define $\mathfrak{Z}_{\mathfrak{A}} = \{\mathfrak{z} \in \omega : \mathfrak{A}\mathfrak{z} \in \omega\}$. Then, $\mathfrak{Z}_{\mathfrak{A}}$ is a sequence space and is called the matrix domain of $\mathfrak{A}$ in the space $\mathfrak{Z}$. It is well known that if $\mathfrak{Z}$ is a BK-space and $\mathfrak{A}$ is a triangle, then, the matrix domain $\mathfrak{Z}_{\mathfrak{A}}$ is also a BK-space under the norm $\|\mathfrak{z}\|_{\mathfrak{Z}_{\mathfrak{A}}} = \|\mathfrak{A}\mathfrak{z}\|_{\mathfrak{Z}}$. We refer to [1–13] for papers related to theory of sequence spaces and summability.

1.1. Some Special Integer Sequences and the Associated Sequence Spaces. We shall briefly highlight the literature concerning special integer sequences and the construction of the associated sequence spaces.

Let $(\mathfrak{f}_k)_{k=0}^{\infty}$ be the sequence of Fibonacci numbers defined by the recurrence relation $\mathfrak{f}_v = \mathfrak{f}_{v-1} + \mathfrak{f}_{v-2}, v \geq 2$, with $\mathfrak{f}_0 = 1$ and $\mathfrak{f}_1 = 1$. Several authors constructed different types of sequence spaces involving Fibonacci numbers. For instance, Kara [3] studied the Fibonacci sequence spaces $\ell_p(\mathfrak{F}) := (\ell_p)_{\mathfrak{F}}$ and $\ell_\infty(\mathfrak{F}) := (\ell_\infty)_{\mathfrak{F}}$ and examined certain topological and geometrical structures of these Banach sequence spaces, where $\mathfrak{F} = (\mathfrak{f}_{v,k})$ is a double band matrix of Fibonacci numbers defined by

$$\mathfrak{f}_{v,k} = \begin{cases} -\dfrac{\mathfrak{f}_{v+1}}{\mathfrak{f}_v}, & k = v-1, \\ \dfrac{\mathfrak{f}_v}{\mathfrak{f}_{v+1}}, & k = v, \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

$v, k \in \mathbb{N}_0$. Besides, Basarir et al. [7] studied the sequence spaces $\ell_p(\mathfrak{F}) := (\ell_p)_{\mathfrak{F}}, c_0(\mathfrak{F}) := (c_0)_{\mathfrak{F}}$ and $c(\mathfrak{F}) := (c)_{\mathfrak{F}}$, where $0 < p < 1$. The studies on Fibonacci sequence spaces are further strengthened by Kara and Basarir [4] by introducing the matrix domain $\mathfrak{Z}(\tilde{\mathfrak{F}}) := (\mathfrak{Z})_{\tilde{\mathfrak{F}}}$, where $\mathfrak{Z}$ represents any one of the sequence spaces ℓ_p, c_0, c, or ℓ_∞ and $\tilde{\mathfrak{F}} = (\tilde{\mathfrak{f}}_{v,k})$ is a regular matrix of Fibonacci numbers defined by

$$\tilde{\mathfrak{f}}_{v,k} = \begin{cases} \dfrac{\mathfrak{f}_k^2}{\mathfrak{f}_v \mathfrak{f}_{v+1}}, & 0 \leq k \leq v, \\ 0, & k > v, \end{cases} \tag{2}$$

for all $v, k \in \mathbb{N}_0$. Furthermore, another regular matrix $\bar{\mathfrak{F}} = (\bar{\mathfrak{f}}_{v,k})$ of Fibonacci numbers is defined by Debnath and Saha [1] as follows:

$$\bar{\mathfrak{f}}_{v,k} = \begin{cases} \dfrac{\mathfrak{f}_k}{\mathfrak{f}_{v+2} - 1}, & 0 \leq k \leq v, \\ 0, & k > v, \end{cases} \tag{3}$$

for all $v, k \in \mathbb{N}_0$. By using this matrix, Debnath and Saha [1] and Ercan and Bektas [2] defined and studied the matrix domains $\ell_p(\bar{\mathfrak{F}}) := (\ell_p)_{\bar{\mathfrak{F}}}, c_0(\bar{\mathfrak{F}}) := (c_0)_{\bar{\mathfrak{F}}}, c(\bar{\mathfrak{F}}) := (c)_{\bar{\mathfrak{F}}}$, and $\ell_\infty(\bar{\mathfrak{F}}) := (\ell_\infty)_{\bar{\mathfrak{F}}}$. More studies concerning construction of Banach sequence spaces involving Fibonacci numbers can be tracked in the literature that are generalization or extension of any one of the above discussed Fibonacci sequence spaces. We refer to [5, 6, 8, 9], for such studies.

The number sequence $(\mathfrak{t}_v)_{v=0}^{\infty} := (1, 1, 2, 4, 7, 13, 24, \cdots)$ defined by the recurrence relation $\mathfrak{t}_v = \mathfrak{t}_{v-1} + \mathfrak{t}_{v-2} + \mathfrak{t}_{v-3}$, $v \geq 3$, with $\mathfrak{t}_0 = \mathfrak{t}_1 = 1$ and $\mathfrak{t}_2 = 2$, is called tribonacci sequence. Recently, Yaying and Hazarika [10] introduced tribonacci sequence spaces $\ell_p(\mathfrak{T}) := (\ell_p)_{\mathfrak{T}}$ and $\ell_\infty(\mathfrak{T}) := (\ell_\infty)_{\mathfrak{T}}$, where $\mathfrak{T} = (\bar{\mathfrak{t}}_{v,k})$ is an infinite matrix of tribonacci numbers defined by

$$\bar{\mathfrak{t}}_{v,k} = \begin{cases} \dfrac{2\mathfrak{t}_k}{\mathfrak{t}_{v+2} + \mathfrak{t}_v - 1}, & 0 \leq k \leq v, \\ 0, & k > v, \end{cases} \tag{4}$$

for all $v, k \in \mathbb{N}_0$. Quite recently, Yaying and Kara [11] studied the matrix domains $c_0(\mathfrak{T}) := (c_0)_{\mathfrak{T}}$ and $c(\mathfrak{T}) := (c)_{\mathfrak{T}}$. Moreover, Yaying et al. [12] studied Banach sequence spaces defined by the sequence of Padovan numbers $(\mathfrak{p}_v)_{v=0}^{\infty} = (1, 1, 1, 2, 2, 3, 4, 5, 7, \cdots)$. Besides, A. M. Karakas and M. Karakas [13] also constructed BK-sequence spaces defined by using Lucas numbers $(\mathfrak{l}'_v)_{v=0}^{\infty} = (2, 1, 3, 4, 7, 11, 18, 29, 47, \cdots)$.

1.2. Leonardo Numbers. The number sequence 1, 1, 3, 5, 9, 15, 25, 41, 67, $\cdots$ is termed as Leonardo sequence. Let $\mathfrak{l}_v$, $v = 0, 1, 2, \cdots$, denote the v^{th} Leonardo number. Then, the Leonardo numbers are defined by the following recurrence relation:

$$\mathfrak{l}_v = \mathfrak{l}_{v-1} + \mathfrak{l}_{v-2} + 1, v \geq 2, \quad \text{with } \mathfrak{l}_0 = \mathfrak{l}_1 = 1. \tag{5}$$

It is believed that Leonardo sequence is invented by Leonardo de Pisa, also known as Leonardo Fibonacci. But not much studies related to Leonardo numbers can be traced in the literature due to scarcity of research related to this integer sequence. Leonardo sequence has a very close relationship with the well-known Fibonacci sequence $(\mathfrak{f}_v)_{v=0}^{\infty}$ and the Lucus sequence $(\mathfrak{l}'_v)_{v=0}^{\infty}$:

$$\mathfrak{l}_v = 2\mathfrak{f}_{v+1} - 1, \mathfrak{l}_v = \frac{2}{5}\left(\mathfrak{l}'_v + \mathfrak{l}'_{v+2}\right) - 1, \quad v \geq 0. \tag{6}$$

Quite recently, Catarino and Borges [14] studied basic properties of Leonardo numbers and established several interesting identities, some of which are listed below:

$$\begin{aligned} &\sum_{k=0}^{\nu} \mathfrak{l}_\nu = \mathfrak{l}_{\nu+2} - (\nu+2), \quad \nu \in \mathbb{N}_0, \\ &\mathfrak{l}_\nu^2 - \mathfrak{l}_{\nu-r}\mathfrak{l}_{\nu+r} = \mathfrak{l}_{\nu-r} + \mathfrak{l}_{\nu+r} - 2\mathfrak{l}_\nu - (-1)^{\nu-r}(\mathfrak{l}_{\nu-1}+1)^2, \quad \nu > r, r \geq 1 \left(\text{Catalan}'\text{s identity}\right), \\ &\mathfrak{l}_\nu^2 - \mathfrak{l}_{\nu-1}\mathfrak{l}_{\nu+1} = \mathfrak{l}_{\nu-1} + \mathfrak{l}_{\nu-2} + 4(-1)^\nu, \quad \nu \geq 2 \left(\text{Cassini}'\text{s identity}\right), \\ &\mathfrak{l}_\nu = 2\left(\frac{\mathfrak{x}^{\nu+1} - \mathfrak{y}^{\nu+1}}{\mathfrak{x} - \mathfrak{y}}\right) - 1, \nu \geq 0, \quad \text{where } \mathfrak{x} = \frac{1+\sqrt{5}}{2}, \mathfrak{y} = \frac{1-\sqrt{5}}{2}. \end{aligned} \tag{7}$$

Besides, Alp and Koçer [15] also established interesting relationships between Fibonacci, Lucus, and Leonardo numbers. Vieira et al. [16] worked in the matrix form of the Leonardo numbers and established several interesting relations. Moreover, Shannon [17] also worked on the extension and generalization of the Leonardo numbers.

Inspired by the above studies, we define an infinite matrix $\mathfrak{L} = (\mathfrak{l}_{\nu,k})$ involving Leonardo numbers and construct sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$,$c(\mathfrak{L})$, and $\ell_\infty(\mathfrak{L})$. We study their topological and inclusion properties and obtain Schauder bases of the sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$, and $c(\mathfrak{L})$. In Section 3, α-, β-, and γ-duals of these new spaces are determined. In Section 4, matrix classes from the space $\mathfrak{Z} \in \{\ell_p(\mathfrak{L}), c_0(\mathfrak{L}), c(\mathfrak{L}), \ell_\infty(\mathfrak{L})\}$ to any one of the spaces ℓ_1, c_0,c, and ℓ_∞ are characterized. In Sections 5 and 6, we introduce new sequence spaces $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$ under a definite functional ρ and weighted sequence of positive reals r and discuss certain geometric and topological properties of $(\ell_p(\mathfrak{L}, r))_\rho$ and the eigenvalue distribution of mappings ideally generated by these spaces and s-numbers are presented.

2. Leonardo Sequence Spaces

Define an infinite matrix $\mathfrak{L} = (\mathfrak{l}_{\nu,k})$ by

$$\mathfrak{l}_{\nu,k} = \begin{cases} \dfrac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)}, & 0 \leq k \leq \nu, \\ 0, & k > \nu, \end{cases} \tag{8}$$

for all $\nu, k \in \mathbb{N}_0$. Equivalently,

$$\mathfrak{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 & \cdots \\ \frac{1}{10} & \frac{1}{10} & \frac{3}{10} & \frac{5}{10} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{9}$$

The inverse of the matrix $\mathfrak{L} = (\mathfrak{l}_{\nu,k})$ is given by the matrix $\mathfrak{L}^{-1} = (\mathfrak{l}_{\nu,k}^{-1})$ defined by

$$\mathfrak{l}_{\nu,k}^{-1} = \begin{cases} (-1)^{\nu-k}\dfrac{\mathfrak{l}_{k+2} - (k+2)}{\mathfrak{l}_\nu}, & \nu \leq k \leq \nu+1, \\ 0, & k > \nu, \end{cases} \tag{10}$$

for all $\nu, k \in \mathbb{N}_0$.

Now, we define the following sequence spaces:

$$\begin{aligned} \ell_p(\mathfrak{L}) &:= \left\{ \mathfrak{z} = (\mathfrak{z}_k) \in \omega : \sum_{\nu=0}^{\infty} \left| \sum_{k=0}^{\nu} \frac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k \right|^p < \infty \right\}, \\ c_0(\mathfrak{L}) &:= \left\{ \mathfrak{z} = (\mathfrak{z}_k) \in \omega : \lim_{\nu \longrightarrow \infty} \sum_{k=0}^{\nu} \frac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k = 0 \right\}, \\ c(\mathfrak{L}) &:= \left\{ \mathfrak{z} = (\mathfrak{z}_k) \in \omega : \lim_{\nu \longrightarrow \infty} \sum_{k=0}^{\nu} \frac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k \text{ exists} \right\}, \\ \ell_\infty(\mathfrak{L}) &:= \left\{ \mathfrak{z} = (\mathfrak{z}_k) \in \omega : \sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k \right| < \infty \right\}, \end{aligned} \tag{11}$$

where the sequence $\mathfrak{w} = (\mathfrak{w}_k)$ defined by

$$\mathfrak{w}_\nu = (\mathfrak{L}\mathfrak{z})_\nu = \sum_{k=0}^{\nu} \frac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k, \tag{12}$$

for each $\nu \in \mathbb{N}_0$, which is known as the $\mathfrak{L}$-transform of the sequence $\mathfrak{z} = (\mathfrak{z}_k)$. In what follows, the sequences $\mathfrak{z}$ and $\mathfrak{w}$ are related by (12). It is trivial that the above defined sequence spaces can be expressed in the form $\mathfrak{Z}(\mathfrak{L}) = (\mathfrak{Z})_{\mathfrak{L}}$, where $\mathfrak{Z}$ represents any one of the spaces ℓ_p,c_0,c, and ℓ_∞. That is, $\mathfrak{Z}(\mathfrak{L})$ is the domain of the matrix $\mathfrak{L}$ in the sequence space $\mathfrak{Z}$.

We observe by the definition of the matrix $\mathfrak{L} = (\mathfrak{l}_{\nu,k})$ that $\sum_{k=0}^{\nu}\mathfrak{l}_{\nu,k} = 1$. That is, $\sup_{\nu \in \mathbb{N}_0} \sum_{k=0}^{\nu}\mathfrak{l}_{\nu,k} < \infty$ and $\lim_{\nu \longrightarrow \infty} \sum_{k=0}^{\infty}\mathfrak{l}_{\nu,k} = 1$.

Additionally, $\lim_{\nu\longrightarrow\infty}\mathfrak{l}_{\nu,k}=0$, for each $k\in\mathbb{N}_0$. Thus, we conclude that the matrix $\mathfrak{L}$ is regular.

Theorem 1. *The following inclusion relations hold:*

(i) $\mathfrak{Z}\subset\mathfrak{Z}(\mathfrak{L})$, *where* $\mathfrak{Z}$ *is any one of the spaces* ℓ_p,c_0,c *or* ℓ_∞

(ii) $\ell_p(\mathfrak{L})\subset c_0(\mathfrak{L})\subset c(\mathfrak{L})\subset\ell_\infty(\mathfrak{L})$

(iii) $\ell_p(\mathfrak{L})\subset\ell_q(\mathfrak{L})$ *for* $1\le p<q$

Proof.

(i) The inclusion part is trivial. Assume that $\mathfrak{Z}\coloneqq c$ and consider the sequence $\mathfrak{g}=(1,0,1,0,\cdots)$. We observe that $\mathfrak{g}\notin c$. However

$$(\mathfrak{L}\mathfrak{g})_\nu=\sum_{k=0}^{\nu}\frac{\mathfrak{l}_k}{\mathfrak{l}_{\nu+2}-(\nu+2)}\mathfrak{g}_k=\frac{\mathfrak{l}_0+\mathfrak{l}_2+\cdots+\mathfrak{l}_\nu}{\mathfrak{l}_{\nu+2}-(\nu+2)}\quad(\nu\in\mathbb{N}_0),\tag{13}$$

which converges. Thus, $\mathfrak{g}\in c(\mathfrak{L})\setminus c$. In the similar manner strictness can be established for the other inclusions.

(ii) It is known that the matrix $\mathfrak{L}$ is regular and the inclusion $\ell_p\subset c_0\subset c\subset\ell_\infty$ holds. These imply that the inclusion part holds. Now, consider the sequence $\mathfrak{h}=(1,1,1,1,\cdots)$. Then, $(\mathfrak{L}\mathfrak{h})_\nu=\sum_{k=0}^{\nu}(\mathfrak{l}_k/\mathfrak{l}_{\nu+2}-(\nu+2))\mathfrak{h}_k=1$, for all $\nu\in\mathbb{N}_0$. Thus, $\mathfrak{L}\mathfrak{h}\in c\setminus c_0$. That is, $\mathfrak{h}\in c(\mathfrak{L})\setminus c_0(\mathfrak{L})$. This verifies the strictness of the inclusion $c_0(\mathfrak{L})\subset c(\mathfrak{L})$. In the similar fashion, strictness of other inclusions can be established.

(iii) Assume that $1\le p<q$. Since $\mathfrak{L}$ is regular and the inclusion $\ell_p\subset\ell_q$ holds, therefore the desired inclusion holds. To prove the strictness part, we consider a sequence $\mathfrak{g}=(\mathfrak{g}_k)\in\ell_q\setminus\ell_p$. Define a sequence $\mathfrak{h}=(\mathfrak{h}_k)$ by $\mathfrak{h}_k=((\mathfrak{g}_k(\mathfrak{l}_{k+2}-(k+2))-\mathfrak{g}_{k-1}(\mathfrak{l}_{k+1}-(k+1)))/\mathfrak{l}_k)$,$k\in\mathbb{N}_0$. Then, we get

$$\begin{aligned}(\mathfrak{L}\mathfrak{h})_\nu&=\frac{1}{\mathfrak{l}_{\nu+2}-(\nu+2)}\sum_{k=0}^{\nu}\mathfrak{l}_k\mathfrak{h}_k\\&=\frac{1}{\mathfrak{l}_{\nu+2}-(\nu+2)}\sum_{k=0}^{\nu}\{\mathfrak{g}_k(\mathfrak{l}_{k+2}-(k+2))\\&\quad-\mathfrak{g}_{k-1}(\mathfrak{l}_{k+1}-(k+1))\}\\&=\frac{1}{\mathfrak{l}_{\nu+2}-(\nu+2)}\mathfrak{g}_\nu(\mathfrak{l}_{\nu+2}-(\nu+2))\\&=\mathfrak{g}_\nu,\end{aligned}\tag{14}$$

for each $\nu\in\mathbb{N}_0$, where the terms with negative subscripts are considered to be zero. Thus, we deduce that $\mathfrak{L}\mathfrak{h}=\mathfrak{g}\in\ell_q\setminus\ell_p$ which implies $\mathfrak{h}\in\ell_q(\mathfrak{L})\setminus\ell_p(\mathfrak{L})$. Thus there exists at least one sequence that is contained in $\ell_q(\mathfrak{L})$ but not in $\ell_p(\mathfrak{L})$. Hence, the desired inclusion is strict. This completes the proof. □

Theorem 2. *We have the following results:*

(i) The sequence spaces $\ell_\infty(\mathfrak{L})$,$c_0(\mathfrak{L})$, *and* $c(\mathfrak{L})$ *are BK-spaces equipped with the bounded norm* $\|\mathfrak{z}\|_{\ell_\infty}=\sup_{k\in\mathbb{N}_0}|(\mathfrak{L}\mathfrak{z})_k|$

(ii) The sequence space $\ell_p(\mathfrak{L})$ *is a BK-space equipped with the norm* $\|\mathfrak{z}\|_{\ell_p}=(\sum_{\nu=0}^{\infty}|(\mathfrak{L}\mathfrak{z})_\nu|^p)^{1/p}$

Proof. The proof is a routine exercise and so omitted. □

Theorem 3. $\mathfrak{Z}(\mathfrak{L})\cong\mathfrak{Z}$, *where* $\mathfrak{Z}$ *is any one of the spaces* ℓ_p,c_0, c, *or* ℓ_∞.

Proof. We present the proof for the space ℓ_p. Define the mapping $\varphi:\ell_p(\mathfrak{L})\longrightarrow\ell_p$ by $\mathfrak{w}=\varphi\mathfrak{z}=\mathfrak{L}\mathfrak{z}$, for all $\mathfrak{z}\in\ell_p(\mathfrak{L})$. We observe that the mapping φ is linear and injective.

In view of the relation (12), we write

$$\mathfrak{z}_k=\sum_{j=k-1}^{k}(-1)^{k-j}\frac{\mathfrak{l}_{j+2}-(j+2)}{\mathfrak{l}_k}\mathfrak{w}_j⊠,\tag{15}$$

for each $k\in\mathbb{N}_0$ and $\mathfrak{w}=(\mathfrak{w}_k)\in\ell_p$. Then,

$$\begin{aligned}\|\mathfrak{z}\|_{\ell_p(\mathfrak{L})}^p&=\sum_{\nu=0}^{\infty}\left|\sum_{j=0}^{k}\frac{\mathfrak{l}_j}{\mathfrak{l}_{k+2}-(k+2)}\mathfrak{z}_j\right|^p\\&=\sum_{\nu=0}^{\infty}\left|\sum_{j=0}^{k}\frac{\mathfrak{l}_j}{\mathfrak{l}_{k+2}-(k+2)}\right.\\&\quad\left.\cdot\left(\sum_{u=j-1}^{j}(-1)^{j-u}\frac{\mathfrak{l}_{u+2}-(u+2)}{\mathfrak{l}_j}\mathfrak{w}_u\right)\right|^p\\&=\sum_{\nu=0}^{\infty}|\mathfrak{w}|^p=\|\mathfrak{w}\|_{\ell_p}^p.\end{aligned}\tag{16}$$

Thus, $\|\mathfrak{z}\|_{\ell_p(\mathfrak{L})}=\|\mathfrak{w}\|_{\ell_p}<\infty$. Thus, $\mathfrak{z}\in\ell_p(\mathfrak{L})$, and this implies that φ is surjective and norm preserving. Thus, $\ell_p(\mathfrak{L})\cong\ell_p$. In the similar manner, we can prove the existence of isomorphism between other given spaces. This completes the proof. □

Let us consider the following sequences:

$$\mathfrak{g}=\left(1,1,-\frac{2}{3},0,0,\cdots\right),\mathfrak{h}=\left(1,-3,\frac{2}{3},0,0,\cdots\right).\tag{17}$$

Observe that $\mathfrak{L}\mathfrak{g}=(1,1,0,0,0,\cdots)$ and $\mathfrak{L}\mathfrak{h}=(1,-1,0,0,0,\cdots)$. Since $\mathfrak{L}$ is linear, so $\mathfrak{L}(\mathfrak{g}+\mathfrak{h})=(2,0,0,0,\cdots)$ and

$\mathfrak{L}(\mathfrak{g}-\mathfrak{h})=(0,2,0,0,\cdots)$. With some elementary calculation, we deduce that

$$\begin{aligned}\|\mathfrak{g}+\mathfrak{h}\|^2_{\ell_p(\mathfrak{L})}+\|\mathfrak{g}-\mathfrak{h}\|^2_{\ell_p(\mathfrak{L})}&=8\neq 2^{2(1+(1/p))}\\&=2\left(\|\mathfrak{g}\|^2_{\ell_p(\mathfrak{L})}+\|\mathfrak{h}\|^2_{\ell_p(\mathfrak{L})}\right).\end{aligned}\tag{18}$$

Thus, we realize that the norm $\|\cdot\|_{\ell_p(\mathfrak{L})}$ violates the parallelogram identity for $p\neq 2$. This immediately allows us to write the following result.

Theorem 4. *The sequence space $\ell_p(\mathfrak{L})$ is not a Hilbert space for $p\neq 2$.*

Proof. The proof is immediate from the above discussion. □

We are well awarded that a matrix domain $\mathfrak{Z}_{\mathfrak{A}}$, where $\mathfrak{A}$ is a triangle, has a basis, if and only if, $\mathfrak{Z}$ has a basis (cf. [18]). Thus, in the light of Theorem 3, we have the following result:

Theorem 5. *Define the sequence $\mathfrak{b}^{(k)}=(\mathfrak{b}^{(k)}_v)$ by*

$$\mathfrak{b}^{(k)}_v=\begin{cases}(-1)^{v-k}\dfrac{\mathfrak{l}_{k+2}-(k+2)}{\mathfrak{l}_v}, & v-1\leq k\leq v+1,\\ 0, \quad k>v\end{cases}\tag{19}$$

for each fixed $k\in\mathbb{N}_0$. Then,

(i) *The sequence $(\mathfrak{b}^{(k)})_{k\in\mathbb{N}_0}$ is the Schauder basis of the sequence spaces $\ell_p(\mathfrak{L})$ and $c_0(\mathfrak{L})$, and every $\mathfrak{z}$ in $\ell_p(\mathfrak{L})$ or $c_0(\mathfrak{L})$ is expressed uniquely in the form $\mathfrak{z}=\sum_{k=0}^{\infty}\mathfrak{b}^{(k)}\mathfrak{w}_k$, where $\mathfrak{w}=(\mathfrak{w}_k)$ is the $\mathfrak{L}$-transform of the sequence $\mathfrak{z}=(\mathfrak{z}_k)$*

(ii) *The sequence $(\mathfrak{e},\mathfrak{b}^{(0)},\mathfrak{b}^{(1)},\mathfrak{b}^{(2)},\cdots)$ is the Schauder basis of the sequence space $c(\mathfrak{L})$, and every $\mathfrak{z}$ in $c(\mathfrak{L})$ is expressed uniquely in the form $\mathfrak{z}=\tau\mathfrak{e}+\sum_{k=0}^{\infty}(\mathfrak{w}_k-\tau)\mathfrak{b}^{(k)}$, where $\tau=\lim_{v\longrightarrow\infty}\mathfrak{w}_v$ and $\mathfrak{e}$ is the unit sequence*

(iii) *The sequence space $\ell_\infty(\mathfrak{L})$ has no Schauder basis*

Corollary 6. *The sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$, and $c(\mathfrak{L})$ are separable spaces.*

3. α-, β-, and γ-Duals

In this section, we obtain the α-, β-, and γ-duals of the sequence spaces $\ell_p(\mathfrak{L})$,$c_0(\mathfrak{L})$,$c(\mathfrak{L})$, and $\ell_\infty(\mathfrak{L})$. Before proceeding, we recall the definitions of α-, β-, and γ-duals. Define the multiplier sequence space $\mathscr{M}(\mathfrak{Z},\mathfrak{U})$ by

$$\mathscr{M}(\mathfrak{Z},\mathfrak{U}):=\{\mathfrak{d}=(\mathfrak{d}_k)\in\omega:\mathfrak{d}\mathfrak{z}=(\mathfrak{d}_k\mathfrak{z}_k)\in\mathfrak{U},\forall\mathfrak{z}=(\mathfrak{z}_k)\in\mathfrak{U}\}.\tag{20}$$

In particular, if $\mathfrak{U}$ is ℓ_1,cs or bs, then, the sets

$$\mathfrak{Z}^\alpha=\mathscr{M}(\mathfrak{Z},\ell_1),\ \mathfrak{Z}^\beta=\mathscr{M}(\mathfrak{Z},cs),\ \mathfrak{Z}^\gamma=\mathscr{M}(\mathfrak{Z},bs)\tag{21}$$

are, respectively, termed as α-, β-, and γ-dual of the sequence space $\mathfrak{Z}$.

We present Lemma 7 which is essential to compute the dual spaces. In what follows, we denote the collection of all finite subsets of $\mathbb{N}_0$ by $\mathscr{N}$ and $q=p/p-1$.

Lemma 7 (see [19]). *The following statements hold:*

(i) *$\mathfrak{A}=(\mathfrak{a}_{v,k})\in(\ell_1,\ell_1)$, if and only if, $\sup_{k\in\mathbb{N}_0}\sum_{v=0}^{\infty}|\mathfrak{a}_{v,k}|<\infty$*

(ii) *$\mathfrak{A}=(\mathfrak{a}_{v,k})\in(\ell_p,\ell_1)$, if and only if*

$$\sup_{K\in\mathscr{N}}\sum_{v=0}^{\infty}\left|\sum_{k\in K}\mathfrak{a}_{v,k}\right|^q<\infty\tag{22}$$

(iii) *$\mathfrak{A}=(\mathfrak{a}_{v,k})\in(c_0,\ell_1)=(c,\ell_1)=(\ell_\infty,\ell_1)$, if and only if, (22) holds with $q=1$*

(iv) *$\mathfrak{A}=(\mathfrak{a}_{v,k})\in(\ell_1,\ell_\infty)$, if and only if, $\sup_{v,k\in\mathbb{N}_0}|\mathfrak{a}_{v,k}|<\infty$*

(v) *$\mathfrak{A}=(\mathfrak{a}_{v,k})\in(\ell_p,\ell_\infty)$, if and only if,*

$$\sup_{v\in\mathbb{N}_0}\sum_{k=0}^{\infty}|\mathfrak{a}_{v,k}|^q<\infty\tag{23}$$

(vi) *$\mathfrak{A}=(\mathfrak{a}_{v,k})\in(c_0,\ell_\infty)=(c,\ell_\infty)=(\ell_\infty,\ell_\infty)$, if and only if (23) holds with $q=1$*

Theorem 8. *Consider the following sets:*

$$\begin{aligned}\mathfrak{D}^{(q)}_1&:=\left\{\mathfrak{d}=(\mathfrak{d}_k)\in\omega:\sum_{k=0}^{\infty}\left|\frac{\mathfrak{l}_{k+2}-(k+2)}{\mathfrak{l}_k}\mathfrak{d}_k\right|^q<\infty\right\},\\ \mathfrak{D}_1&:=\left\{\mathfrak{d}=(\mathfrak{d}_k)\in\omega:\sup_{k\in\mathbb{N}_0}\left|\frac{\mathfrak{l}_{k+2}-(k+2)}{\mathfrak{l}_k}\mathfrak{d}_k\right|<\infty\right\}.\end{aligned}\tag{24}$$

Then, $[\ell_1(\mathfrak{L})]^\alpha:=\mathfrak{D}_1$, $[\ell_p(\mathfrak{L})]^\alpha:=\mathfrak{D}^{(q)}_1$, and $[c_0(\mathfrak{L})]^\alpha=[c(\mathfrak{L})]^\alpha=[\ell_\infty(\mathfrak{L})]^\alpha:=\mathfrak{D}^{(1)}_1$, where $1<p<\infty$.

Proof. We observe that

$$\mathfrak{d}_v\mathfrak{z}_v=\sum_{k=v-1}^{v}(-1)^{v-k}\frac{\mathfrak{l}_{k+2}-(k+2)}{\mathfrak{l}_v}\mathfrak{d}_v\mathfrak{w}_k=(\mathfrak{B}\mathfrak{w})_v,\tag{25}$$

for each $\nu \in \mathbb{N}_0$ and $\mathfrak{d} = (\mathfrak{d}_k) \in \omega$, where the matrix $\mathfrak{B} = (\mathfrak{b}_{\nu,k})$ is defined by

$$\mathfrak{b}_{\nu,k} = \begin{cases} (-1)^{\nu-k} \dfrac{\mathfrak{l}_{k+2} - (k+2)}{\mathfrak{l}_\nu} \mathfrak{d}_\nu, & \nu - 1 \le k \le \nu, \\ 0, & k > \nu, \end{cases} \tag{26}$$

for all $\nu, k \in \mathbb{N}_0$. We notice that the sequence $\mathfrak{d}\mathfrak{z} = (\mathfrak{d}_\nu \mathfrak{z}_\nu) \in \ell_1$ whenever $\mathfrak{z} = (\mathfrak{z}_\nu) \in c_0(\mathfrak{L})$, if and only if, the sequence $\mathfrak{B}\mathfrak{w} \in \ell_1$ whenever the sequence $\mathfrak{w} = (\mathfrak{w}_\nu) \in c_0$. We realize that the sequence $\mathfrak{d} = (\mathfrak{d}_\nu) \in [c_0(\mathfrak{L})]^\alpha$, if and only if, $\mathfrak{B} \in (c_0, \ell_1)$. Thus, by employing Part (iii) of Lemma 7, we get that

$$\sup_{\mathfrak{B} \in \mathcal{N}} \sum_{k=0}^{\infty} \left| \sum_{\nu \in \mathfrak{B}} \mathfrak{b}_{\nu,k} \right| < \infty. \tag{27}$$

Moreover, for any $\mathfrak{B} \in \mathcal{N}$, (27) holds, if and only if,

$$\sum_{k=0}^{\infty} \left| \frac{\mathfrak{l}_{k+2} - (k+2)}{\mathfrak{l}_k} \mathfrak{d}_k \right| < \infty. \tag{28}$$

Consequently, $[c_0(\mathfrak{L})]^\alpha \coloneqq \mathfrak{D}_1^{(1)}$.

In the similar manner, α-dual of the other sequence spaces can be obtained by employing Part (i), Part (ii), and Part (iii) of Lemma 7. □

Theorem 9. *Consider the following sets:*

$$\begin{aligned} \mathfrak{D}_2^{(q)} &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \sum_{k=0}^{\infty} \left| \Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right|^q < \infty \right\}, \\ \mathfrak{D}_2 &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \sup_{k \in \mathbb{N}_0} \left| \Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right| < \infty \right\}, \\ \mathfrak{D}_3 &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \lim_{k \to \infty} \frac{\mathfrak{l}_{k+2} - (k+2)}{\mathfrak{l}_k} \mathfrak{d}_k = 0 \right\}, \\ \mathfrak{D}_4 &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \lim_{k \to \infty} \frac{\mathfrak{l}_{k+2} - (k+2)}{\mathfrak{l}_k} \mathfrak{d}_k \text{ exists} \right\}, \end{aligned} \tag{29}$$

where $\Delta(\mathfrak{d}_k/\mathfrak{l}_k) = (\mathfrak{d}_k/\mathfrak{l}_k) - (\mathfrak{d}_{k+1}/\mathfrak{l}_{k+1})$. Then, $[\ell_1(\mathfrak{L})]^\beta \coloneqq \mathfrak{D}_1 \cap \mathfrak{D}_2$, $[\ell_p(\mathfrak{L})]^\beta \coloneqq \mathfrak{D}_1 \cap \mathfrak{D}_2^{(q)}$, $[c_0(\mathfrak{L})]^\beta \coloneqq \mathfrak{D}_1 \cap \mathfrak{D}_2^{(1)}$, $[c(\mathfrak{L})]^\beta \coloneqq \mathfrak{D}_2^{(1)} \cap \mathfrak{D}_4$, and $[\ell_\infty(\mathfrak{L})]^\beta \coloneqq \mathfrak{D}_2^{(1)} \cap \mathfrak{D}_3$, where $1 < p < \infty$.

Proof. Let $\mathfrak{d} = (\mathfrak{d}_k) \in \omega$. Then, we have

$$\begin{aligned} \sum_{k=0}^{\nu} \mathfrak{d}_k \mathfrak{z}_k &= \sum_{k=0}^{\nu} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\mathfrak{l}_{j+2} - (j+2)}{\mathfrak{l}_k} \mathfrak{w}_j \mathfrak{d}_k \\ &= \sum_{k=0}^{\nu-1} \left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k} - \frac{\mathfrak{d}_{k+1}}{\mathfrak{l}_{k+1}} \right) (\mathfrak{l}_{k+2} - (k+2)) \mathfrak{w}_k + \frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \mathfrak{d}_\nu \mathfrak{w}_\nu \\ &= \sum_{k=0}^{\nu-1} \Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right) (\mathfrak{l}_{k+2} - (k+2)) \mathfrak{w}_k + \frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \mathfrak{d}_\nu \mathfrak{w}_\nu, \end{aligned} \tag{30}$$

for each $\nu \in \mathbb{N}_0$. By employing Theorem 2 and Corollary 1 of Malkowsky and Savas [20], we get

$$\begin{aligned} [\ell_1(\mathfrak{L})]^\beta &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \left(\Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right) \in \ell_\infty \text{ and} \left(\frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \right) \in \ell_\infty \right\}, \\ [\ell_p(\mathfrak{L})]^\beta &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \left(\Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right) \in \ell_q, \left(\frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \right) \in \ell_\infty \right\}, \\ [c_0(\mathfrak{L})]^\beta &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \left(\Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right) \in \ell_1, \left(\frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \right) \in \ell_\infty \right\}, \\ [c(\mathfrak{L})]^\beta &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \left(\Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right) \in \ell_1, \left(\frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \right) \in c \right\}, \\ [\ell_\infty(\mathfrak{L})]^\beta &\coloneqq \left\{ \mathfrak{d} = (\mathfrak{d}_k) \in \omega : \left(\Delta\left(\frac{\mathfrak{d}_k}{\mathfrak{l}_k}\right)(\mathfrak{l}_{k+2} - (k+2)) \right) \in \ell_1, \left(\frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \right) \in c_0 \right\}. \end{aligned} \tag{31}$$

This completes the proof. □

Theorem 10. *We have the following results:*

(i) $[\ell_1(\mathfrak{L})]^\gamma \coloneqq \mathfrak{D}_1 \cap \mathfrak{D}_2$.

(ii) $[\ell_p(\mathfrak{L})]^\gamma \coloneqq \mathfrak{D}_1 \cap \mathfrak{D}_2^{(q)}$, *where* $1 < p < \infty$

(iii) $[c_0(\mathfrak{L})]^\gamma = [c(\mathfrak{L})]^\gamma = [\ell_\infty(\mathfrak{L})]^\gamma \coloneqq \mathfrak{D}_1 \cap \mathfrak{D}_2^{(1)}$

Proof. It can be obtained by using relation (30) and Parts (iv), (v), and (vi) of Lemma 7, respectively. □

4. Characterization of Matrix Classes

Let $\mathfrak{A} = (\mathfrak{a}_{\nu,k})$ be an infinite matrix over the field of complex numbers. Denote

$$\begin{aligned} \tilde{\mathfrak{A}}_\nu &\coloneqq \frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \mathfrak{A}_\nu \\ &= \left(\frac{\mathfrak{l}_{\nu+2} - (\nu+2)}{\mathfrak{l}_\nu} \mathfrak{a}_{\nu,k} \right)_{k=0}^{\infty}, \quad \text{for each } \nu \in \mathbb{N}_0, \end{aligned} \tag{32}$$

$$\mathfrak{m}_{v,k} := (\mathfrak{l}_{k+2} - (k+2))\Delta\left(\frac{\mathfrak{a}_{v,k}}{\mathfrak{l}_k}\right), \quad \text{for all } v, k \in \mathbb{N}_0. \tag{33}$$

Now, we state the following result:

Lemma 11. *Let $\mathfrak{Z}$ denote either of the spaces ℓ_p or c_0, $1 \le p \le \infty$. Then, $\mathfrak{A} \in (\mathfrak{Z}_{\mathfrak{L}}, \mathfrak{U})$, if and only if, $\tilde{\mathfrak{A}} \in \mathcal{M}(\mathfrak{Z}, c_0)$, for each $v \in \mathbb{N}_0$, and $\mathfrak{M} \in (\mathfrak{Z}, \mathfrak{U})$, where $\tilde{\mathfrak{A}}$ and $\mathfrak{M} = (\mathfrak{m}_{v,k})$ are defined in (32) and (33), respectively.*

Proof. It follows straightly from ([20], Theorem 3]). □

Lemma 12. $\mathfrak{A} \in (c(\mathfrak{L}), \mathfrak{U})$*, if and only if,*

$$\tilde{\mathfrak{A}}_v \in c, \text{ that is } \lim_{k\longrightarrow\infty} \frac{\mathfrak{l}_{v+2} - (v+2)}{\mathfrak{l}_v} \mathfrak{a}_{v,k} = \alpha_v (v \in \mathfrak{N}_0), \tag{34}$$

$$\mathfrak{M} \in (c_0, \mathfrak{U}) = (\ell_\infty, \mathfrak{U}), \tag{35}$$

$$(\alpha_v)_{v=0}^{\infty} \in \mathfrak{U}. \tag{36}$$

Proof. Assume that $\mathfrak{A} \in (c(\mathfrak{L}), \mathfrak{U})$. Then, $\mathfrak{A}_v \in [c(\mathfrak{L})]^\beta$ which immediately shows the necessity of condition (35). Also, it is known that $((c)_{\mathfrak{L}}, \mathfrak{U}) \subset ((c_0)_{\mathfrak{L}}, \mathfrak{U})$. Hence, by employing Lemma 11, we get that $\mathfrak{M} \in (c_0, \mathfrak{U})$. Thus, in the light of (30), we get that

$$\sum_{k=0}^{\infty} \mathfrak{a}_{v,k}\mathfrak{z}_k = \sum_{k=0}^{\infty} \mathfrak{m}_{v,k}\mathfrak{w}_k + \mathfrak{z}\alpha_v. \tag{37}$$

It is clear from the assumptions that $\mathfrak{A}\mathfrak{z} \in \mathfrak{U}$ and $\mathfrak{M}\mathfrak{w} \in \mathfrak{U}$. These together yield $(\alpha_v)_{v=0}^{\infty} \in \mathfrak{U}$.

Conversely, we assume that conditions (34), (35), and (36) hold. We realize that conditions (34) and (35) together imply that $\mathfrak{A}_v \in [c(\mathfrak{L})]^\beta$. Again condition (34) implies (37). By condition (35), $\mathfrak{M}\mathfrak{w} \in \mathfrak{U}$, for all $\mathfrak{w} \in c$. This together with condition (36) implies that $\mathfrak{A}\mathfrak{z} \in \mathfrak{U}$, for all $\mathfrak{z} \in c(\mathfrak{L})$. This proves that $\mathfrak{A} \in (c(\mathfrak{L}), \mathfrak{U})$. □

Now, using Lemmas 11 and 12 together with the properties $\mathcal{M}(\ell_p, c_0) = \mathcal{M}(c_0, c_0) = \ell_\infty (1 \le p < \infty)$,$\mathcal{M}(c, c) = c$, and $\mathcal{M}(\ell_\infty, c_0) = c_0$, we deduce the following results:

Corollary 13. *The following statements hold:*

(i) $\mathfrak{A} \in (\ell_1(\mathfrak{L}), \ell_\infty)$*, if and only if*

$$\sup_{k\in\mathbb{N}_0} \left| \frac{\mathfrak{l}_{v+2} - (v+2)}{\mathfrak{l}_v} \mathfrak{a}_{v,k} \right| < \infty, \tag{38}$$

$$\sup_{v,k\in\mathbb{N}_0} \left| \mathfrak{m}_{v,k} \right| < \infty \tag{39}$$

(ii) *Let $1 < p < \infty$. Then $\mathfrak{A} \in (\ell_p(\mathfrak{L}), \ell_\infty)$, if and only if, (38) holds, and*

$$\sup_{v\in\mathbb{N}_0} \sum_{k=0}^{\infty} \left| \mathfrak{m}_{v,k} \right|^q < \infty \tag{40}$$

(iii) $\mathfrak{A} \in (c_0(\mathfrak{L}), \ell_\infty)$*, if and only if, (38) holds, and (40) holds with $q = 1$*

(iv) $\mathfrak{A} \in (c(\mathfrak{L}), \ell_\infty)$*, if and only if, (34) and (40) hold with $q = 1$, and*

$$\sup_{v\in\mathbb{N}_0} |\alpha_v| < \infty, \tag{41}$$

also holds

(v) $\mathfrak{A} \in (\ell_\infty(\mathfrak{L}), \ell_\infty)$*, if and only if,*

$$\lim_{k\longrightarrow\infty} \frac{\mathfrak{l}_{v+2} - (v+2)}{\mathfrak{l}_v} \mathfrak{a}_{v,k} = 0, \quad \text{for all } v \in \mathbb{N}_0, \tag{42}$$

and (40) holds with $q = 1$

Corollary 14. *The following statements hold:*

(i) $\mathfrak{A} \in (\ell_1(\mathfrak{L}), c_0)$*, if and only if, (38) and (39) hold, and*

$$\lim_{v\longrightarrow\infty} \mathfrak{m}_{v,k} = 0, \quad \text{for all } k \in \mathbb{N}_0, \tag{43}$$

also holds

(ii) *Let $1 < p < \infty$. Then, $\mathfrak{A} \in (\ell_p(\mathfrak{L}), c_0)$, if and only if, (38), (40), and (43) hold*

(iii) $\mathfrak{A} \in (c_0(\mathfrak{L}), c_0)$*, if and only if, (38) and (40) hold with $q = 1$, and (43) also holds*

(iv) $\mathfrak{A} \in (c(\mathfrak{L}), c_0)$*, if and only if, (34), (40) with $q = 1$ and (43) hold, and*

$$\lim_{v\longrightarrow\infty} \alpha_v = 0, \tag{44}$$

also holds

(v) $\mathfrak{A} \in (\ell_\infty(\mathfrak{L}), c_0)$*, if and only if, (42) holds, and*

$$\lim_{v\longrightarrow\infty} \sum_{k=0}^{\infty} \left| \mathfrak{m}_{v,k} \right| = 0, \tag{45}$$

also holds

Corollary 15. *The following statements hold:*

(i) $\mathfrak{A} \in (\ell_1(\mathfrak{L}), c)$*, if and only if, (38), (39), and*

$$\lim_{v\longrightarrow\infty} \mathfrak{m}_{v,k} \text{ exists}, \quad \text{for all } k \in \mathbb{N}_0, \tag{46}$$

also holds

(ii) *Let $1 < p < \infty$. Then $\mathfrak{A} \in (\ell_p(\mathfrak{L}), c)$, if and only if, (38), (40) and (46) holds*

(iii) $\mathfrak{A} \in (c_0(\mathfrak{L}), c)$, *if and only if, (38) holds, (40) with* $q = 1$ *and (46) hold*

(iv) $\mathfrak{A} \in (c(\mathfrak{L}), c)$, *if and only if, (34), (40) with* $q = 1$ *and (46) hold, and*

$$\lim_{v \longrightarrow \infty} \alpha_v \text{ exists} \quad (47)$$

also holds

(v) $\mathfrak{A} \in (\ell_\infty(\mathfrak{L}), c)$, *if and only if, (42) and (46) hold, and*

$$\sum_{k=0}^{\infty} \left| \mathfrak{m}_{v,k} \right| \text{ converges uniformly in } v \quad (48)$$

also holds

Corollary 16. *The following statements hold:*

(i) $\mathfrak{A} \in (\ell_1(\mathfrak{L}), \ell_1)$, *if and only if, (38) holds and*

$$\sup_{v \in \mathbb{N}_0} \sum_{v=0}^{\infty} \left| \mathfrak{m}_{v,k} \right| < \infty, \quad (49)$$

also holds

(ii) *Let* $1 < p < \infty$. *Then* $\mathfrak{A} \in (\ell_p(\mathfrak{L}), \ell_1)$, *if and only if, (38) holds, and*

$$\sup_{\mathfrak{B} \in \mathcal{N}} \sum_{k=0}^{\infty} \left| \sum_{v \in \mathfrak{B}} \mathfrak{m}_{v,k} \right|^q < \infty, \quad (50)$$

also holds

(iii) $\mathfrak{A} \in (c_0(\mathfrak{L}), \ell_1)$, *if and only if, (38) holds, and (50) holds with* $q = 1$

(iv) $\mathfrak{A} \in (c(\mathfrak{L}), \ell_1)$, *if and only if, (34) and (50) hold with* $q = 1$, *and*

$$\sum_{v=0}^{\infty} \left| \alpha_v \right| < \infty, \quad (51)$$

also holds

(v) $\mathfrak{A} \in (\ell_\infty(\mathfrak{L}), \ell_1)$, *if and only if, (42) and (50) hold with* $q = 1$

5. Mapping Ideal

In this section, we construct s-type mapping ideals on Leonardo sequence spaces $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$. By $\mathscr{B}$, we denote the class of all bounded linear mappings between any two Banach spaces. In particular, $\mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ denote the class of all bounded linear mappings acting from Banach space $\mathfrak{X}$ to Banach space $\mathfrak{Y}$. We note down certain notations and definitions before moving to our results:

Definition 17 (see [21, 22]). Let ω^+ represent the set of non-negative real sequences. Then, s-number is a mapping $s : \mathscr{B}(\mathfrak{X}, \mathfrak{Y}) \longrightarrow \omega^+$ that satisfies the following settings:

(i) $\|\phi\| = s_0(\phi) \geq s_1(\phi) \geq s_2(\phi) \geq \cdots \geq 0$, for each $\phi \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$

(ii) $s_{a+b-1}(\phi + \psi) \leq s_a(\phi) + s_b(\psi)$, for each $\phi, \psi \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ and $a, b \in \mathbb{N}_0$

(iii) $s_a(\phi\theta\psi) \leq \|\phi\| s_a(\theta) \|\psi\|$, for all $\phi \in \mathscr{B}(\mathfrak{X}_0, \mathfrak{X})$, $\theta \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$, and $\psi \in \mathscr{B}(\mathfrak{Y}, \mathfrak{Y}_0)$, where $\mathfrak{X}_0$ and $\mathfrak{Y}_0$ are any two Banach sequence spaces

(iv) Let $\phi \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ and $\nu \in \mathbb{C}$. Then, $s_a(\nu\phi) = |\nu| s_a(\phi)$

(v) If rank $(\phi) \leq a$, then $s_a(\phi) = 0$ for all $\phi \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$

(vi) $s_v(\mathfrak{I}_a) = 0$ for $v \geq a$ or $s_v(\mathfrak{I}_a) = 1$ for $v < a$, where $\mathfrak{I}_a$ denotes the identity mapping on the a-dimensional Hilbert space ℓ_2^a

In an assorted illustration of s-numbers, we intimate the next settings:

(1) The a-th Kolmogorov number, denoted by $d_a(X)$, is defined as

$$d_a(X) = \inf_{\dim J \leq a} \sup_{\|f\| \leq 1} \inf_{g \in J} \|Xf - g\| \quad (52)$$

(2) The a-th approximation number, denoted by $\alpha_a(X)$, is defined as

$$\alpha_a(X) = \inf \{ \|X - Y\| : Y \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y}), \text{rank } (Y) \leq a \} \quad (53)$$

Definition 18 (see [23]). Let $\mathscr{W} \subset \mathscr{B}$ and denote $\mathscr{W}(\mathfrak{X}, \mathfrak{Y}) = \mathscr{W} \cap \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$. Then, $\mathscr{W}$ is known as a mapping ideal if it satisfies the following settings:

(i) $\mathfrak{I}_{\mathfrak{D}} \in \mathscr{W}$, where $\mathfrak{D}$ is a Banach sequence space of one dimension

(ii) $\mathscr{W}(\mathfrak{X}, \mathfrak{Y})$ is a linear space over $\mathbb{C}$

(iii) If $\psi \in \mathscr{B}(\mathfrak{X}_0, \mathfrak{X})$, $\theta \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ and $\phi \in \mathscr{B}(\mathfrak{Y}_0, \mathfrak{Y})$, then $\phi\theta\psi \in \mathscr{B}(\mathfrak{X}_0, \mathfrak{Y}_0)$, where $\mathfrak{X}_0$ and $\mathfrak{Y}_0$ are any two normed spaces

Definition 19 (see [24]). A prequasi norm on the ideal $\mathscr{W}$ is a mapping $\mu : \mathscr{W} \longrightarrow \omega^+$ satisfying the following settings:

(i) $\mu(\phi) \geq 0$ and $\mu(\phi) = 0$ if and only if $\phi = 0$, for all $\phi \in \mathscr{W}(\mathfrak{X}, \mathfrak{Y})$

(ii) There exists $m_0 \geq 1$ such that $\mu(\varsigma\phi) \leq m_0|\varsigma|\mu(\phi)$, for all $\phi \in \mathscr{W}(\mathfrak{X}, \mathfrak{Y})$

(iii) There exists $n_0 \geq 1$ such that $\mu(\phi + \psi) \leq n_0(\mu(\phi) + \mu(\psi))$, for all $\phi, \psi \in \mathscr{W}(\mathfrak{X}, \mathfrak{Y})$

(iv) There exists $p_0 \geq 1$ such that $\mu(\phi\theta\psi) \leq p_0\|\phi\|\mu(\theta)\|\psi\|$ whenever $\psi \in \mathscr{B}(\mathfrak{X}_0, \mathfrak{X})$, $\theta \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ and $\phi \in \mathscr{B}(\mathfrak{Y}_0, \mathfrak{Y})$

Definition 20 (see [24]). The subspace $\mathfrak{Z} \subset \omega$ is said to be a private sequence space (or in short pss) if it satisfies the following settings:

(i) $\mathfrak{e}_\nu \in \mathfrak{Z}$, for each $\nu \in \mathbb{N}_0$, where $\mathfrak{e}_\nu$ denotes the sequence with 1 in the ν^{th} position and 0 elsewhere

(ii) If $\mathfrak{g} = (\mathfrak{g}_\nu) \in \omega, |\mathfrak{h}| = (|\mathfrak{h}_\nu|) \in \mathfrak{Z}$ and $|\mathfrak{g}_\nu| \leq |\mathfrak{h}_\nu|$, for $\nu \in \mathbb{N}_0$, then $|\mathfrak{g}| \in \mathfrak{Z}$

(iii) $(|g_{[\nu/2]}|) \in \mathfrak{Z}$ whenever $(|g_\nu|) \in \mathfrak{Z}$, where $[\nu/2]$ denotes the integral part of $\nu/2$

Definition 21 (see [24]). A subspace of the pss is said to be a premodular pss, if there is a function $\upsilon : \mathfrak{Z} \longrightarrow [0,\infty)$ satisfying the following conditions:

(i) For every $j \in \mathfrak{Z}$, $j = 0 \Leftrightarrow \upsilon(|j|) = 0$, and $\upsilon(j) \geq 0$, with 0 is the zero vector of $\mathfrak{Z}$

(ii) If $j \in \mathfrak{Z}$ and $\rho \in \mathbb{C}$, then there are $E_0 \geq 1$ with $\upsilon(\rho j) \leq |\rho| E_0 \upsilon(j)$

(iii) $\upsilon(h + j) \leq G_0(\upsilon(h) + \upsilon(j))$ holds for some $G_0 \geq 1$, with $f, g \in \mathfrak{Z}$

(iv) Assume $x \in \mathbb{N}_0$, $|h_x| \leq |j_x|$, we have $\upsilon((|h_x|)) \leq \upsilon((|j_x|))$

(v) The inequality, $\upsilon((|j_x|)) \leq \upsilon((|j_{[x/2]}|)) \leq D_0\upsilon((|j_x|))$ verifies, for $D_0 \geq 1$

(vi) $\bar{\mathfrak{C}} = \mathfrak{Z}_\upsilon$, where $\bar{\mathfrak{C}}$ denotes the closure of the space of all sequences with infinite zero coordinates

(vii) We have $\eta > 0$ such that $\upsilon(\rho, 0, 0, 0, \cdots) \geq \eta|\rho|\upsilon(1, 0, 0, 0, \cdots)$, with $\rho \in \mathbb{C}$

Definition 22 (see [24]). The *pss* $\mathfrak{Z}_\upsilon$ is said to be a prequasi normed *pss*, if υ confirms the setups (i)-(iii) of Definition 21. If $\mathfrak{Z}$ is complete equipped with υ, then $\mathfrak{Z}_\upsilon$ is called a prequasi Banach *pss*.

Lemma 23 (see [24]). *Every premodular pss is a prequasi normed pss.*

In what follows, we will use the following inequality:

$$|\mathfrak{g} + \mathfrak{h}|^p \leq 2^{p-1}\left(|\mathfrak{g}|^p + |\mathfrak{h}|^p\right), \tag{54}$$

where $1 \leq p < \infty$ and $\mathfrak{g}, \mathfrak{h} \in \mathbb{C}$. For detailed studies concerning s-numbers and mapping ideals, we refer to [23–28].

Definition 24. We define the following sequence spaces:

$$\begin{aligned}
&\left(\ell_p(\mathfrak{L}, r)\right)_{\rho_1} \\
&\quad := \left\{ \mathfrak{z} = (\mathfrak{z}_k) \in \omega : \rho_1(\mathfrak{z}) = \sum_{\nu=0}^{\infty} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k \right|^p < \infty \right\}, \\
&\left(c_0(\mathfrak{L}, r)\right)_{\rho_2} \\
&\quad := \left\{ \mathfrak{z} = (\mathfrak{z}_k) \in \omega : \lim_{\nu \longrightarrow \infty} \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{z}_k = 0 \right\},
\end{aligned} \tag{55}$$

where $r = (r_k) \in \omega^+$ and $\rho_2(\mathfrak{z}) = \sup_{\nu \in \mathbb{N}_0} |\sum_{k=0}^{\nu} (r_k \mathfrak{l}_k / (\mathfrak{l}_{\nu+2} - (\nu+2)))\mathfrak{z}_k|$.

By $\mathfrak{I}_\nearrow$ and $\mathfrak{I}_\searrow$, we will denote the space of all monotonic increasing and decreasing sequences of positive reals, respectively.

Theorem 25. *$c_0(\mathfrak{L}, r)$ is a pss, whenever $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_\searrow$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_\nearrow \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$.*

Proof.

(i) Let $\mathfrak{g}, \mathfrak{h} \in c_0(\mathfrak{L}, r)$, we obtain

$$\begin{aligned}
&\sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} (\mathfrak{g}_k + \mathfrak{h}_k) \right| \\
&\quad \leq \sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{g}_k \right| \\
&\quad + \sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{h}_k \right| < \infty
\end{aligned} \tag{56}$$

Thus, $\mathfrak{g} + \mathfrak{h} \in c_0(\mathfrak{L}, r)$.

Assume that $\varsigma \in \mathbb{C}$ and $\mathfrak{g} \in c_0(\mathfrak{L}, r)$. Then, we have

$$\sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} (\varsigma \mathfrak{g}_k) \right| = |\varsigma| \sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} \mathfrak{g}_k \right| < \infty. \tag{57}$$

Thus $c_0(\mathfrak{L}, r)$ is a linear space. Moreover

$$\sup_{\nu \in \mathbb{N}_0} \left| \sum_{k=0}^{\nu} \frac{r_k \mathfrak{l}_k}{\mathfrak{l}_{\nu+2} - (\nu+2)} (\mathfrak{e}_a)_k \right| = r_a \mathfrak{l}_a \sup_{\nu \in \mathbb{N}_0} \left(\frac{1}{\mathfrak{l}_{\nu+2} - (\nu+2)} \right) < \infty. \tag{58}$$

This implies $\mathbf{e}_a \in c_0(\mathfrak{L}, r)$, for each $a \in \mathbb{N}_0$.

(ii) Assume that $|\mathfrak{g}_k| \leq |\mathfrak{h}_k|$, for all $k \in \mathbb{N}_0$ and $|\mathfrak{h}| \in c_0(\mathfrak{L}, r)$. Then, we have

$$\sup_{v\in\mathbb{N}_0}\left|\sum_{k=0}^{v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{v+2}-(v+2)}|\mathfrak{g}_k|\right| \leq \sup_{v\in\mathbb{N}_0}\left|\sum_{k=0}^{v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{v+2}-(v+2)}|\mathfrak{h}_k|\right| < \infty \tag{59}$$

This concludes that $|\mathfrak{g}| \in c_0(\mathfrak{L}, r)$.

(iii) Let $(|\mathfrak{g}_k|) \in c_0(\mathfrak{L}, r)$, $(r_k\mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k\mathfrak{l}_k$. Then, we have

$$\begin{aligned}
&\sup_{v\in\mathbb{N}_0}\left(\sum_{k=0}^{v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{v+2}-(v+2)}\left|\mathfrak{g}_{[\frac{k}{2}]}\right|\right)\\
&\leq \sup_{v\in\mathbb{N}_0}\left(\sum_{k=0}^{2v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{2v+2}-(2v+2)}\left|\mathfrak{g}_{[\frac{k}{2}]}\right|\right) + \sup_{v\in\mathbb{N}_0}\left(\sum_{k=0}^{2v+1}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{2v+3}-(2v+3)}\left|\mathfrak{g}_{[\frac{k}{2}]}\right|\right)\\
&\leq \sup_{v\in\mathbb{N}_0}\left[\frac{1}{\mathfrak{l}_{2v+2}-(2v+2)}\cdot\left\{r_{2v}\mathfrak{l}_{2v}|\mathfrak{g}_v| + \sum_{k=0}^{v}\left(r_{2k}\mathfrak{l}_{2k}|\mathfrak{g}_k| + r_{2k+1}\mathfrak{l}_{2k+1}|\mathfrak{g}_k|\right)\right\}\right]\\
&\quad + \sup_{v\in\mathbb{N}_0}\left[\frac{1}{\mathfrak{l}_{2v+3}-(2v+3)}\cdot\sum_{k=0}^{v}\left(r_{2k}\mathfrak{l}_{2k}|\mathfrak{g}_k| + r_{2k+1}\mathfrak{l}_{2k+1}|\mathfrak{g}_k|\right)\right]\\
&\leq \sup_{v\in\mathbb{N}_0}\left(\frac{C}{\mathfrak{l}_{v+2}-(v+2)}\sum_{k=0}^{v}r_k\mathfrak{l}_k|\mathfrak{g}_k|\right) + \sup_{v\in\mathbb{N}_0}\left(\frac{2C}{\mathfrak{l}_{v+2}-(v+2)}\sum_{k=0}^{v}r_k\mathfrak{l}_k|\mathfrak{g}_k|\right)\\
&\quad + \sup_{v\in\mathbb{N}_0}\left(\frac{2C}{\mathfrak{l}_{v+2}-(v+2)}\sum_{k=0}^{v}r_k\mathfrak{l}_k|\mathfrak{g}_k|\right)\\
&\leq 5C\sup_{v\in\mathbb{N}_0}\left(\frac{1}{\mathfrak{l}_{v+2}-(v+2)}\sum_{k=0}^{v}\mathfrak{l}_k|\mathfrak{g}_k|\right) < \infty
\end{aligned} \tag{60}$$

Thus, $(\mathfrak{g}_{[k/2]}) \in c_0(\mathfrak{L}, r)$.
This completes the proof. □

Theorem 26. *$\ell_p(\mathfrak{L}, r)$ is a pss, whenever $1 < p < \infty$, $(r_k\mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\nearrow}$ or $(r_k\mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k\mathfrak{l}_k$.*

Proof.

(i) Let $\mathfrak{g}, \mathfrak{h} \in \ell_p(\mathfrak{L}, r)$. By using (54), we obtain

$$\begin{aligned}
&\sum_{v=0}^{\infty}\left|\sum_{k=0}^{v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{v+2}-(v+2)}(\mathfrak{g}_k+\mathfrak{h}_k)\right|^p\\
&\leq 2^{p-1}\left\{\sum_{v=0}^{\infty}\left|\sum_{k=0}^{v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{v+2}-(v+2)}\mathfrak{g}_k\right|^p + \sum_{v=0}^{\infty}\left|\sum_{k=0}^{v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{v+2}-(v+2)}\mathfrak{h}_k\right|^p\right\} < \infty
\end{aligned} \tag{61}$$

Thus, $\mathfrak{g} + \mathfrak{h} \in \ell_p(\mathfrak{L}, r)$.
Assume that $\varsigma \in \mathbb{C}$ and $\mathfrak{g} \in \ell_p(\mathfrak{L}, r)$. Then, we have

$$\sum_{v=0}^{\infty}\left|\sum_{k=0}^{v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{v+2}-(v+2)}(\varsigma\mathfrak{g}_k)\right|^p = |\varsigma|^p\sum_{v=0}^{\infty}\left|\sum_{k=0}^{v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{v+2}-(v+2)}\mathfrak{g}_k\right|^p < \infty. \tag{62}$$

Thus, $\ell_p(\mathfrak{L}, r)$ is a linear space. Moreover,

$$\sum_{v=0}^{\infty}\left|\sum_{k=0}^{v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{v+2}-(v+2)}(\mathfrak{e}_a)_k\right|^p = r_a^p\mathfrak{l}_a^p\sum_{v=r}^{\infty}\left(\frac{1}{\mathfrak{l}_{v+2}-(v+2)}\right)^p < \infty. \tag{63}$$

This implies $\mathfrak{e}_a \in \ell_p(\mathfrak{L}, r)$ for each $a \in \mathbb{N}_0$.

(ii) Assume that $|\mathfrak{g}_k| \leq |\mathfrak{h}_k|$, for all $k \in \mathbb{N}_0$ and $|\mathfrak{h}| \in \ell_p(\mathfrak{L}, r)$. Then, we have

$$\sum_{v=0}^{\infty}\left|\sum_{k=0}^{v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{v+2}-(v+2)}|\mathfrak{g}_k|\right|^p \leq \sum_{v=0}^{\infty}\left|\sum_{k=0}^{v}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{v+2}-(v+2)}|\mathfrak{h}_k|\right|^p < \infty \tag{64}$$

This concludes that $|\mathfrak{g}| \in \ell_p(\mathfrak{L}, r)$.

(iii) Let $(|\mathfrak{g}_k|) \in \ell_p(\mathfrak{L}, r)$, $(r_k\mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq Cr_k\mathfrak{l}_k$. Then, we have

$$\begin{aligned}
&\sum_{\nu=0}^{\infty}\left(\sum_{k=0}^{\nu}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{\nu+2}-(\nu+2)}\left|\mathfrak{g}_{[k/2]}\right|\right)^p \\
&= \sum_{\nu=0}^{\infty}\left(\sum_{k=0}^{2\nu}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{2\nu+2}-(2\nu+2)}\left|\mathfrak{g}_{[k/2]}\right|\right)^p + \sum_{\nu=0}^{\infty}\left(\sum_{k=0}^{2\nu+1}\frac{r_k\mathfrak{l}_k}{\mathfrak{l}_{2\nu+3}-(2\nu+3)}\left|\mathfrak{g}_{[k/2]}\right|\right)^p \\
&\leq \sum_{\nu=0}^{\infty}\left[\frac{1}{\mathfrak{l}_{2\nu+2}-(2\nu+2)}\cdot\left\{r_{2\nu}\mathfrak{l}_{2\nu}|\mathfrak{g}_\nu| + \sum_{k=0}^{\nu}\left(r_{2k}\mathfrak{l}_{2k}|\mathfrak{g}_k| + r_{2k+1}\mathfrak{l}_{2k+1}|\mathfrak{g}_k|\right)\right\}\right]^p \\
&\quad + \sum_{\nu=0}^{\infty}\left[\frac{1}{\mathfrak{l}_{2\nu+3}-(2\nu+3)}\sum_{k=0}^{\nu}\left(r_{2k}\mathfrak{l}_{2k}|\mathfrak{g}_k| + r_{2k+1}\mathfrak{l}_{2k+1}|\mathfrak{g}_k|\right)\right]^p \\
&\leq 2^{p-1}\left[\sum_{\nu=0}^{\infty}\left(\frac{C}{\mathfrak{l}_{\nu+2}-(\nu+2)}\sum_{k=0}^{\nu}r_k\mathfrak{l}_k|\mathfrak{g}_k|\right)^p + \sum_{\nu=0}^{\infty}\left(\frac{2C}{\mathfrak{l}_{\nu+2}-(\nu+2)}\sum_{k=0}^{\nu}r_k\mathfrak{l}_k|\mathfrak{g}_k|\right)^p\right] \\
&\quad + \sum_{\nu=0}^{\infty}\left(\frac{2C}{\mathfrak{l}_{\nu+2}-(\nu+2)}\sum_{k=0}^{\nu}r_k\mathfrak{l}_k|\mathfrak{g}_k|\right)^p \\
&\leq \left(2^{2p-1}+2^p+2^{p-1}\right)C^p\cdot\sum_{\nu=0}^{\infty}\left(\frac{1}{\mathfrak{l}_{\nu+2}-(\nu+2)}\sum_{k=0}^{\nu}\mathfrak{l}_k|\mathfrak{g}_k|\right)^p < \infty
\end{aligned} \tag{65}$$

Thus, $(\mathfrak{g}_{[k/2]}) \in \ell_p(\mathfrak{L}, r)$.
This completes the proof. □

Define the sets $\mathscr{B}^s_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y})$, $\mathscr{B}^{\alpha}_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y})$, and $\mathscr{B}^d_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y})$ by

$$\begin{aligned}
\mathscr{B}^s_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y}) &:= \{\phi \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y}) : (s_a(\phi)) \in \mathfrak{Z}\}, \\
\mathscr{B}^{\alpha}_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y}) &:= \{\phi \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y}) : (\alpha_a(\phi)) \in \mathfrak{Z}\}, \\
\mathscr{B}^d_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y}) &:= \{\phi \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y}) : (d_a(\phi)) \in \mathfrak{Z}\},
\end{aligned} \tag{66}$$

where $\mathfrak{X}$ and $\mathfrak{Y}$ are any two Banach sequence spaces. We denote $\mathscr{B}^s_{\mathfrak{Z}} := \{\mathscr{B}^s_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y})\}$., $\mathscr{B}^{\alpha}_{\mathfrak{Z}} := \{\mathscr{B}^{\alpha}_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y})\}$, and $\mathscr{B}^d_{\mathfrak{Z}} := \{\mathscr{B}^d_{\mathfrak{Z}}(\mathfrak{X}, \mathfrak{Y})\}$, respectively.

Lemma 27 (see [24]). *Let the linear sequence space $\mathfrak{Z}$ be a pss. Then, $\mathscr{B}^s_{\mathfrak{Z}}$ is a mapping ideal.*

Theorem 28. *Let $(r_k\mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k\mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\nearrow} \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq Cr_k\mathfrak{l}_k$. Then, $\mathscr{B}^s_{c_0(\mathfrak{L},r)}$ is a mapping ideal.*

Proof. It follows straightly from Lemma 27. □

Theorem 29. *Let $1 < p < \infty$, $(r_k\mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k\mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\nearrow} \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq Cr_k\mathfrak{l}_k$. Then, $\mathscr{B}^s_{\ell_p(\mathfrak{L},r)}$ is a mapping ideal.*

Proof. It follows straightly from Lemma 27. □

Theorem 30. *Let $(r_k\mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k\mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\nearrow} \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq Cr_k\mathfrak{l}_k$. Then, $(c_0(\mathfrak{L}, r))_\rho$ is a premodular pss.*

Proof.

(i) Clearly, for all $\mathfrak{g} \in (c_0(\mathfrak{L}, r))_\rho$ that $\rho(\mathfrak{g}) \geq 0$ and $\rho(|\mathfrak{g}|) = 0$, if and only if, $\mathfrak{g} = 0$

(ii) For any $\varepsilon \geq 1$. Then $\rho(\alpha\mathfrak{g}) \leq \varepsilon|\alpha|\rho(\mathfrak{g})$, for all $\mathfrak{g} \in c_0(\mathfrak{L}, r)$ and $\alpha \in \mathbb{C}$

(iii) Observe that $\rho(\mathfrak{g} + \mathfrak{h}) \leq \rho(\mathfrak{g}) + \rho(\mathfrak{h})$, for all $\mathfrak{g}, \mathfrak{h} \in c_0(\mathfrak{L}, r)$

(iv) We have $\rho((|\mathfrak{g}_k|)) \leq \rho((|\mathfrak{h}_k|))$, whenever $|\mathfrak{g}_k| \leq |\mathfrak{h}_k|$ (see Proof Part (ii), Theorem 25).

(v) It is immediate from Proof Part (iii) of Theorem 25 that $\rho((|\mathfrak{g}_k|)) \leq \rho((|\mathfrak{g}_{[k/2]}|)) \leq \delta\rho((|\mathfrak{g}_k|))$ with $\delta = 5C$.

$$\bar{\mathfrak{C}} = c_0(\mathfrak{L}, r) \tag{67}$$

(vi) We have, when $\alpha \neq 0$ then $0 < \gamma \leq 1$, for $\rho(\alpha, 0, 0, \cdots) \geq \gamma|\alpha|\rho(1, 0, 0, \cdots)$ and when $\alpha = 0$ then $\gamma > 0$

This completes the proof. □

Theorem 31. *Let $1 < p < \infty$, $(r_k\mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k\mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow} \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq Cr_k\mathfrak{l}_k$. Then, $(\ell_p(\mathfrak{L}, r))_\rho$ is a premodular pss.*

Proof.

(i) Clearly, for all $\mathfrak{g} \in (\ell_p(\mathfrak{L}, r))_\rho$ that $\rho(\mathfrak{g}) \geq 0$ and $\rho(|\mathfrak{g}|) = 0$, if and only if, $\mathfrak{g} = 0$

(ii) Let $\varepsilon = \max\{1, |\alpha|^{p-1}\} \geq 1$. Then, $\rho(\alpha\mathfrak{g}) \leq \varepsilon|\alpha|\rho(\mathfrak{g})$, for all $\mathfrak{g} \in \ell_p(\mathfrak{L}, r)$ and $\alpha \in \mathbb{C}$

(iii) Observe that $\rho(\mathfrak{g} + \mathfrak{h}) \leq 2^{p-1}(\rho(\mathfrak{g}) + \rho(\mathfrak{h}))$, for all $\mathfrak{g}, \mathfrak{h} \in \ell_p(\mathfrak{L}, r)$

(iv) We have $\rho((|\mathfrak{g}_k|)) \leq \rho((|\mathfrak{h}_k|))$, whenever $|\mathfrak{g}_k| \leq |\mathfrak{h}_k|$ (see Proof Part (ii), Theorem 26).

(v) It is immediate from Proof Part (iii) of Theorem 26 that $\rho((|\mathfrak{g}_k|)) \leq \rho((|\mathfrak{g}_{[k/2]}|)) \leq \delta\rho((|\mathfrak{g}_k|))$ with $\delta = (2^{2p-1} + 2^p + 2^{p-1})C^p$

$$\bar{\mathfrak{C}} = \ell_p(\mathfrak{L}, r) \tag{68}$$

(vi) We have, when $\alpha \neq 0$ then $0 < \gamma < |\alpha|^{p-1}$, for $\rho(\alpha, 0, 0, \cdots) \geq \gamma|\alpha|\rho(1, 0, 0, \cdots)$, and when $\alpha = 0$ then $\gamma > 0$

This completes the proof. □

Theorem 32. *Assume that $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow} \cap \ell_{\infty}$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$. Then, $(c_0(\mathfrak{L}, r))_{\rho}$ is a prequasi Banach pss.*

Proof. In view of Theorem 30 and Lemma 23, it is enough to prove that every Cauchy sequence in $(c_0(\mathfrak{L}, r))_{\rho}$ is convergent in $(c_0(\mathfrak{L}, r))_{\rho}$. We assume that $\mathfrak{g}^{(m)} = (\mathfrak{g}_k^{(m)})$ is a Cauchy sequence in $(c_0(\mathfrak{L}, r))_{\rho}$. Then, for all $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}_0$ such that

$$\rho\left(\mathfrak{g}^{(m)} - \mathfrak{g}^{(n)}\right) = \sup_{v=0}^{\infty} \left| \frac{1}{\mathfrak{l}_{v+2} - (v+2)} \sum_{k=0}^{v} \mathfrak{l}_k \left(\mathfrak{g}_k^{(m)} - \mathfrak{g}^{(n)} \right)_k \right| < \varepsilon, \tag{69}$$

for all $m, n \geq n_0$. This implies that $(\mathfrak{g}^{(m)} - \mathfrak{g}^{(n)}) < \varepsilon$, for all $m, n \geq n_0$. Thus, $(\mathfrak{g}^{(m)})$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, $\lim_{m \longrightarrow \infty} \mathfrak{g}_k^{(m)} = \mathfrak{g}_k$, for a fixed $k \in \mathbb{N}_0$. This yields, by using (69), that $\rho(\mathfrak{g}^{(m)} - \mathfrak{g}) < \varepsilon$, for all $m \geq n_0$. Besides, we have $\rho(\mathfrak{g}) \leq \rho(\mathfrak{g}^{(m)} - \mathfrak{g}) + \rho(\mathfrak{g}^{(m)}) < \infty$. This concludes that $\mathfrak{g} \in (c_0(\mathfrak{L}, r))_{\rho}$. Thus, $(c_0(\mathfrak{L}, r))_{\rho}$ is a prequasi Banach *pss*. □

Theorem 33. *Assume that $1 < p < \infty, (r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_{\infty}$, and there exists $C \geq 1$ such that $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$. Then, $(\ell_p(\mathfrak{L}, r))_{\rho}$ is a prequasi Banach pss.*

Proof. In view of Theorem 31 and Lemma 23, it is enough to prove that every Cauchy sequence in $(\ell_p(\mathfrak{L}, r))_{\rho}$ is convergent in $(\ell_p(\mathfrak{L}, r))_{\rho}$. We assume that $\mathfrak{g}^{(m)} = (\mathfrak{g}_k^{(m)})$ is a Cauchy sequence in $(\ell_p(\mathfrak{L}, r))_{\rho}$. Then, for all $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}_0$ such that

$$\rho\left(\mathfrak{g}^{(m)} - \mathfrak{g}^{(n)}\right) = \sum_{v=0}^{\infty} \left| \frac{1}{\mathfrak{l}_{v+2} - (v+2)} \sum_{k=0}^{v} \mathfrak{l}_k \left(\mathfrak{g}_k^{(m)} - \mathfrak{g}^{(n)} \right)_k \right| < \varepsilon^p, \tag{70}$$

for all $m, n \geq n_0$. This implies that $(\mathfrak{g}^{(m)} - \mathfrak{g}^{(n)}) < \varepsilon$, for all $m, n \geq n_0$. Thus, $(\mathfrak{g}^{(m)})$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, $\lim_{m \longrightarrow \infty} \mathfrak{g}_k^{(m)} = \mathfrak{g}_k$, for a fixed $k \in \mathbb{N}_0$. This yields, by using (70), that $\rho(\mathfrak{g}^{(m)} - \mathfrak{g}) < \varepsilon^p$, for all $m \geq n_0$. Besides, we have $\rho(\mathfrak{g}) \leq 2^{p-1}(\rho(\mathfrak{g}^{(m)} - \mathfrak{g}) + \rho(\mathfrak{g}^{(m)})) < \infty$. This concludes that $\mathfrak{g} \in (\ell_p(\mathfrak{L}, r))_{\rho}$. Thus $(\ell_p(\mathfrak{L}, r))_{\rho}$ is a prequasi Banach *pss*. □

Theorem 34 (see [27]). *Suppose $s-type\,\mathscr{E}_{\rho} \coloneqq \{h = (s_x(H)) \in \omega^+ : H \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y}) and\, \rho(h) < \infty\}$. If $\mathbb{B}^s_{\mathscr{E}_{\rho}}$ is a mapping ideal, then the following conditions are verified:*

(1) $\mathfrak{C} \subset s-type\,\mathscr{E}_{\rho}$

(2) Suppose $(s_x(H_1))_{x=0}^{\infty} \in s-type\,\mathscr{E}_{\rho}$ and $(s_x(H_2))_{x=0}^{\infty} \in s-type\,\mathscr{E}_{\rho}$; then $(s_x(H_1 + H_2))_{x=0}^{\infty} \in s-type\,\mathscr{E}_{\rho}$

(3) Assume $\lambda \in \mathbb{C}$ and $(s_x(H))_{x=0}^{\infty} \in s-type\,\mathscr{E}_{\rho}$; then $|\lambda|(s_x(H))_{x=0}^{\infty} \in s-type\,\mathscr{E}_{\rho}$

(4) The sequence space $\mathscr{E}_{\rho}$ is solid; i.e., if $(s_x(J))_{x=0}^{\infty} \in s-type\,\mathscr{E}_{\rho}$ and $s_x(H) \leq s_x(J)$, for all $x \in \mathbb{N}_0$ and $H, J \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$; then $(s_x(H))_{x=0}^{\infty} \in s-type\,\mathscr{E}_{\rho}$

In view of Theorem 34, we construct the next properties of the $s-type\,(c_0(\mathfrak{L}, r))_{\rho}$ and the $s-type\,(\ell_p(\mathfrak{L}, r))_{\rho}$.

Theorem 35. *Let $s-type\,(c_0(\mathfrak{L}, r))_{\rho} \coloneqq \{f = (s_n(X)) \in \omega^+ : X \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ and $\rho(f) < \infty\}$. The next conditions are established:*

(1) One has $s-type\,(c_0(\mathfrak{L}, r))_{\rho} \supset \mathfrak{C}$

(2) Suppose $(s_r(X_1))_{r=0}^{\infty} \in s-type\,(c_0(\mathfrak{L}, r))_{\rho}$ and $(s_r(X_2))_{r=0}^{\infty} \in s-type\,(c_0(\mathfrak{L}, r))_{\rho}$; then $(s_r(X_1 + X_2))_{r=0}^{\infty} \in s-type\,(c_0(\mathfrak{L}, r))_{\rho}$

(3) Assume $\lambda \in \mathbb{C}$ and $(s_r(X))_{r=0}^{\infty} \in s-type\,(c_0(\mathfrak{L}, r))_{\rho}$; hence $|\lambda|(s_r(X))_{r=0}^{\infty} \in s-type\,(c_0(\mathfrak{L}, r))_{\rho}$

(4) The $s-type\,(c_0(\mathfrak{L}, r))_{\rho}$ is solid

Theorem 36. *Let $s-type\,(\ell_p(\mathfrak{L}, r))_{\rho} \coloneqq \{f = (s_n(X)) \in \omega^+ : X \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ and $\rho(f) < \infty\}$. The next conditions are established:*

(1) One has $s-type\,(\ell_p(\mathfrak{L}, r))_{\rho} \supset \mathfrak{C}$

(2) Suppose $(s_r(X_1))_{r=0}^{\infty} \in s-type\,(\ell_p(\mathfrak{L}, r))_{\rho}$ and $(s_r(X_2))_{r=0}^{\infty} \in s-type\,(\ell_p(\mathfrak{L}, r))_{\rho}$; then $(s_r(X_1 + X_2))_{r=0}^{\infty} \in s-type\,(\ell_p(\mathfrak{L}, r))_{\rho}$

(3) Assume $\lambda \in \mathbb{C}$ and $(s_r(X))_{r=0}^{\infty} \in s-type\,(\ell_p(\mathfrak{L}, r))_{\rho}$; hence $|\lambda|(s_r(X))_{r=0}^{\infty} \in s-type\,(\ell_p(\mathfrak{L}, r))_{\rho}$

(4) The $s-type\,(\ell_p(\mathfrak{L}, r))_{\rho}$ is solid

6. Characteristics of the Prequasi Ideal

The conventions listed below will be followed throughout the article; if the species is preowned, we will give it to you.

Conventions 1. Please see the following conventions:

$\mathbb{F}$: the ideal of finite rank mappings between any arbitrary Banach spaces

$\mathscr{A}$: the ideal of approximable mappings between any arbitrary Banach spaces

$\mathscr{K}$: the ideal of compact mappings between any arbitrary Banach spaces

$\mathbb{F}(\mathfrak{X}, \mathfrak{Y})$: the space of finite rank mappings from a Banach space $\mathfrak{X}$ into a Banach space $\mathfrak{Y}$

$\mathbb{F}(\mathfrak{X})$: the space of finite rank mappings from a Banach space $\mathfrak{X}$ into itself

$\mathscr{A}(\mathfrak{X}, \mathfrak{Y})$: the space of approximable mappings from a Banach space $\mathfrak{X}$ into a Banach space $\mathfrak{Y}$

$\mathscr{A}(\mathfrak{X})$: the space of approximable mappings from a Banach space $\mathfrak{X}$ into itself

$\mathscr{K}(\mathfrak{X}, \mathfrak{Y})$: the space of compact mappings from a Banach space $\mathfrak{X}$ into a Banach space $\mathfrak{Y}$

$\mathscr{K}(\mathfrak{X})$: the space of compact mappings from a Banach space $\mathfrak{X}$ into itself

Lemma 37 (see [28]). *If $M \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ and $M \notin \mathscr{A}(\mathfrak{X}, \mathfrak{Y})$, then there are operators $Q \in \mathscr{B}(\mathfrak{X})$ and $L \in \mathscr{B}(\mathfrak{Y})$ so that $LMQe_x = e_x$, for all $x \in \mathbb{N}_0$.*

Definition 38 (see [28]). A Banach space $\mathscr{E}$ is called simple if the algebra $\mathscr{B}(\mathscr{E})$ includes one and only one nontrivial closed ideal.

Theorem 39 (see [28]). *Suppose $\mathscr{E}$ is a Banach space with* dim $(\mathscr{E}) = \infty$*; then*

$$\mathbb{F}(\mathscr{E}) \subset \mathscr{A}(\mathscr{E}) \subset \mathscr{K}(\mathscr{E}) \subset \mathscr{B}(\mathscr{E}). \tag{71}$$

In this section, firstly, we introduce the enough setups (not necessary) on $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$ such that $\bar{\mathbb{F}} = \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ and $\bar{\mathbb{F}} = \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$. This investigates a negative answer of Rhoades [29] open problem about the linearity of $s-$type $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$ spaces. Secondly, for which conditions on $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$, are $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ and $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$ closed and complete? Thirdly, we explain the enough setups on $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$ such that $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ and $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$ are strictly contained for different weights and powers. We offer the setups so that $\mathscr{B}^\alpha_{(\ell_p(\mathfrak{L},r))_\rho}$ is minimum. Fourthly, we introduce the conditions so that the Banach prequasi ideal $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ and $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$ are simple Banach spaces. Fifthly, we investigate the enough conditions on $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$ such that the space of all bounded linear operators which sequence of eigenvalues in $(c_0(\mathfrak{L}, r))_\rho$ and $(\ell_p(\mathfrak{L}, r))_\rho$ equal $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ and $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$, respectively.

6.1. Finite Rank Prequasi Ideal

Theorem 40. $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y}) = \overline{\mathbb{F}(\mathfrak{X}, \mathfrak{Y})}$*; suppose the setups $(r_k \mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_\searrow$ or $(r_k \mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ are satisfied. But the converse is not necessarily true.*

Proof. To investigate that $\overline{\mathbb{F}(\mathfrak{X}, \mathfrak{Y})} \subseteq \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$, as $e_l \in (c_0(\mathfrak{L}, r))_\rho$, for every $l \in \mathbb{N}_0$,$(c_0(\mathfrak{L}, r))_\rho$ is a linear space. Let $Z \in \mathbb{F}(\mathfrak{X}, \mathfrak{Y})$, one gets $(s_l(Z))_{l=0}^\infty \in \mathfrak{C}$. To explain that $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y}) \subseteq \overline{\mathbb{F}(\mathfrak{X}, \mathfrak{Y})}$, assume $Z \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$, we obtain $(s_l(Z))_{l=0}^\infty \in (c_0(\mathfrak{L}, r))_\rho$. Since $\rho(s_l(Z))_{l=0}^\infty < \infty$, let $\rho \in (0,1)$; hence, there is $l_0 \in \mathbb{N}_0 - \{0\}$ with $\rho((s_l(Z))_{l=l_0}^\infty) < \rho/16d$, for some $d \geq 1$. Since $s_l(Z) \in \mathfrak{I}_\rceil$, we get

$$\begin{aligned}\sup_{l=l_0+1}^{2l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_{2l_0}(Z)}{\mathfrak{l}_{l+2} - (l+2)} &\leq \sup_{l=l_0+1}^{2l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2} - (l+2)} \\ &\leq \sup_{l=l_0}^{\infty} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2} - (l+2)} < \frac{\rho}{16d}.\end{aligned} \tag{72}$$

Hence, there is $Y \in \mathbb{F}_{2l_0}(\mathfrak{X}, \mathfrak{Y})$ so that $\operatorname{rank}(Y) \leq 2l_0$ and

$$\sup_{l=2l_0+1}^{3l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j \|Z - Y\|}{\mathfrak{l}_{l+2} - (l+2)} \leq \sup_{l=l_0+1}^{2l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j \|Z - Y\|}{\mathfrak{l}_{l+2} - (l+2)} < \frac{\rho}{16d}, \tag{73}$$

we have

$$\sum_{j=0}^{l_0} r_j \mathfrak{l}_j \|Z - Y\| < \frac{\rho}{16}. \tag{74}$$

Therefore, one has

$$\sup_{l=0}^{l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j \|Z - Y\|}{\mathfrak{l}_{l+2} - (l+2)} < \frac{\rho}{16d}. \tag{75}$$

In view of inequalities (72)–(75), and $(r_k \mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_\rceil$, one gets

$$\begin{aligned}d(Z, Y) &= \rho(s_l(Z - Y))_{l=0}^\infty \\ &\leq \sup_{l=0}^{3l_0-1} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_j(Z - Y)}{\mathfrak{l}_{l+2} - (l+2)} + \sup_{l=3l_0}^{\infty} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_j(Z - Y)}{\mathfrak{l}_{l+2} - (l+2)} \\ &\leq \sup_{l=0}^{3l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_j(Z - Y)}{\mathfrak{l}_{l+2} - (l+2)} + \sup_{l=l_0}^{\infty} \frac{\sum_{j=0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z - Y)}{\mathfrak{l}_{l+2l_0+2} - (l+2l_0+2)} \\ &\leq \sup_{l=0}^{3l_0} \frac{\sum_{j=0}^l r_j \mathfrak{l}_j s_j(Z - Y)}{\mathfrak{l}_{l+2} - (l+2)} + \sup_{l=l_0}^{\infty} \frac{\sum_{j=0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z - Y)}{\mathfrak{l}_{l+2} - (l+2)}\end{aligned}$$

$$\begin{aligned}
&\le 3\sup_{l=0}^{l_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)} \\
&\quad+\sup_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r_j \mathfrak{l}_j s_j(Z-Y)+\sum_{j=2l_0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right) \\
&\le 3\sup_{l=0}^{l_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}+\sup_{l=l_0}^{\infty}\frac{\sum_{j=0}^{2l_0-1} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)} \\
&\quad+\sup_{l=l_0}^{\infty}\frac{\sum_{j=2l_0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)} \\
&\le 3\sup_{l=0}^{l_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}+\sup_{l=l_0}^{\infty}\frac{\sum_{j=0}^{2l_0-1} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)} \\
&\quad+\sup_{l=l_0}^{\infty}\frac{\sum_{j=0}^{l} r_{j+2l_0} \mathfrak{l}_{j+2l_0} s_{j+2l_0}(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)} \\
&\le 3\sup_{l=0}^{l_0}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}+2\sum_{j=0}^{l_0}\mathfrak{l}_j r_j\|Z-Y\| \\
&\quad+\sup_{l=l_0}^{\infty}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}<\rho.
\end{aligned} \tag{76}$$

On the opposite side, one has a negative example as $I_3 \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$, where $r=(0,0,0,1,0,1,0,)$. This shows the proof. □

Theorem 41. $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})=\overline{\mathbb{F}(\mathfrak{X},\mathfrak{Y})}$*; suppose the setups* $1<p<\infty,(r_k\mathfrak{l}_k)_{k=0}^{\infty}\in\mathfrak{I}_{\searrow}$ *or* $(r_k\mathfrak{l}_k)_{k=0}^{\infty}\in\mathfrak{I}_Z\cap\ell_\infty$*, and there exists* $C\ge 1$ *such that* $r_{2k+1}\mathfrak{l}_{2k+1}\le Cr_k\mathfrak{l}_k$ *are confirmed. But the converse is not necessarily true.*

Proof. To investigate that $\overline{\mathbb{F}(\mathfrak{X},\mathfrak{Y})}\subseteq\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$, as $e_l\in(\ell_p(\mathfrak{L},r))_\rho$, for every $l\in\mathbb{N}_0$, $(\ell_p(\mathfrak{L},r))_\rho$ is a linear space. Let $Z\in\mathbb{F}(\mathfrak{X},\mathfrak{Y})$; one gets $(s_l(Z))_{l=0}^{\infty}\in\mathfrak{C}$. To explain that $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})\subseteq\overline{\mathbb{F}(\mathfrak{X},\mathfrak{Y})}$, assume $Z\in\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$; we obtain $(s_l(Z))_{l=0}^{\infty}\in(\ell_p(\mathfrak{L},r))_\rho$. Since $\rho(s_l(Z))_{l=0}^{\infty}<\infty$, let $\rho\in(0,1)$, hence, there is $l_0\in\mathbb{N}_0-\{0\}$ with $\rho((s_l(Z))_{l=l_0}^{\infty})<(\rho/2^{p+3}\eta d)$, for some $d\ge 1$, where $\eta=\max\{1,\sum_{l=l_0}^{\infty}(1/(\mathfrak{l}_{l+2}-(l+2)))^p\}$. Since $s_l(Z)\in\mathfrak{I}_{]}$, we get

$$\begin{aligned}
\sum_{l=l_0+1}^{2l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{2l_0}(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p &\le \sum_{l=l_0+1}^{2l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\le \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&< \frac{\rho}{2^{p+3}\eta d}.
\end{aligned} \tag{77}$$

Hence, there is $Y\in\mathbb{F}_{2l_0}(\mathfrak{X},\mathfrak{Y})$ so that $\operatorname{rank}(Y)\le 2l_0$ and

$$\begin{aligned}
\sum_{l=2l_0+1}^{3l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p &\le \sum_{l=l_0+1}^{2l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&< \frac{\rho}{2^{p+3}\eta d}.
\end{aligned} \tag{78}$$

Since $1<p<\infty$, we have

$$\left(\sum_{j=0}^{l_0} r_j \mathfrak{l}_j\|Z-Y\|\right)^p<\frac{\rho}{2^{2p+2}\eta}. \tag{79}$$

Therefore, one has

$$\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p<\frac{\rho}{2^{p+3}\eta d}. \tag{80}$$

In view of inequalities (54), (77)–(80), and $(r_k\mathfrak{l}_k)_{k=0}^{\infty}\in\mathfrak{I}_{]}$, one gets

$$\begin{aligned}
d(Z,Y)&=\rho(s_l(Z-Y))_{l=0}^{\infty} \\
&=\sum_{l=0}^{3l_0-1}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\quad+\sum_{l=3l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\le\sum_{l=0}^{3l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\quad+\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2l_0+2}-(l+2l_0+2)}\right)^p \\
&\le\sum_{l=0}^{3l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\quad+\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\le 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\quad+\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r_j \mathfrak{l}_j s_j(Z-Y)+\sum_{j=2l_0}^{l+2l_0} r_j \mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p \\
&\le 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p
\end{aligned}$$

$$
\begin{aligned}
&+2^{p-1}\left[\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r_j\mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\right.\\
&\left.+\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=2l_0}^{l+2l_0} r_j\mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\right]\\
&\le 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j\|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\\
&+2^{p-1}\left[\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r_j\mathfrak{l}_j s_j(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\right.\\
&\left.+\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_{j+2l_0}\mathfrak{l}_{j+2l_0} s_{j+2l_0}(Z-Y)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\right]\\
&\le 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j\|Z-Y\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p+2^p\left(\sum_{j=0}^{l_0}\mathfrak{l}_j r_j\|Z-Y\|\right)^p\\
&\cdot\sum_{l=l_0}^{\infty}\left(\frac{1}{\mathfrak{l}_{l+2}-(l+2)}\right)^p+2^{p-1}\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p<\rho.
\end{aligned}
\tag{81}
$$

On the opposite side, one has a negative example as $I_4\in\mathscr{B}^s_{(\ell_{0.5}(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$, where $r=(0,0,0,0,1,1,)$. This shows the proof. □

6.2. Banach and Closed Prequasi Ideal

Theorem 42 (see [24]). *The function Ψ is a prequasi norm on $\mathscr{B}^s_{(\mathscr{E})_\rho}$, where $\Psi(Y)=\rho(s_b(Y))_{b=0}^{\infty}$, for every $Y\in\mathscr{B}^s_{(\mathscr{E})_\rho}(\mathfrak{X},\mathfrak{Y})$, if $(\mathscr{E})_\rho$ is a premodular pss.*

Theorem 43. *If the setups $(r_k\mathfrak{l}_k)_{k=0}^{\infty}\in\mathfrak{I}_\searrow$ or $(r_k\mathfrak{l}_k)_{k=0}^{\infty}\in\mathfrak{I}_Z\cap\ell_\infty$, and there exists $C\ge 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1}\le Cr_k\mathfrak{l}_k$ with $r_0>0$ are satisfied, then $(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho},\Psi)$ is a prequasi Banach ideal, where $\psi(X)=\rho((s_l(X))_{l=0}^{\infty})$.*

Proof. As $(c_0(\mathfrak{L},r))_\rho$ is a premodular *pss*, hence from Theorem 42, Ψ is a prequasi norm on $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$. Suppose $(X_b)_{b\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$. As $\mathscr{B}(\mathfrak{X},\mathfrak{Y})\supseteq\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$, one obtains

$$
\Psi(X_a-X_b)=\sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j s_j(X_a-X_b)}{\mathfrak{l}_{l+2}-(l+2)}\ge r_0\,\|X_a-X_b\|.
\tag{82}
$$

Hence, $(X_b)_{b\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathscr{B}(\mathfrak{X},\mathfrak{Y})$. Since $\mathscr{B}(\mathfrak{X},\mathfrak{Y})$ is a Banach space, then there is $X\in\mathscr{B}(\mathfrak{X},\mathfrak{Y})$ with $\lim_{b\longrightarrow\infty}\|X_b-X\|=0$. Since $(s_l(X_b))_{l=0}^{\infty}\in(c_0(\mathfrak{L},r))_\rho$, every $b\in\mathbb{N}_0$. According to Definition 21 setups (ii), (iii), and (v), one gets

$$
\begin{aligned}
\Psi(X)&=\sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j s_j(X)}{\mathfrak{l}_{l+2}-(l+2)}\\
&\le\sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j s_{[j/2]}(X-X_b)}{\mathfrak{l}_{l+2}-(l+2)}+\sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j s_{[j/2]}(X_b)}{\mathfrak{l}_{l+2}-(l+2)}\\
&\le\sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j\|X-X_b\|}{\mathfrak{l}_{l+2}-(l+2)}+D_0\sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j s_j(X_b)}{\mathfrak{l}_{l+2}-(l+2)}<\infty.
\end{aligned}
\tag{83}
$$

Therefore, $(s_l(X))_{l=0}^{\infty}\in(c_0(\mathfrak{L},r))_\rho$; then $X\in\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$. □

Theorem 44. *If the setups $1<p<\infty,(r_k\mathfrak{l}_k)_{k=0}^{\infty}\in\mathfrak{I}_\searrow$ or $(r_k\mathfrak{l}_k)_{k=0}^{\infty}\in\mathfrak{I}_Z\cap\ell_\infty$, and there exists $C\ge 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1}\le Cr_k\mathfrak{l}_k$ with $r_0>0$ are confirmed; then $(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho},\Psi)$ is a prequasi Banach ideal, where $\psi(X)=\rho((s_l(X))_{l=0}^{\infty})$.*

Proof. As $(\ell_p(\mathfrak{L},r))_\rho$ is a premodular *pss*, hence from Theorem 42, Ψ is a prequasi norm on $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$. Suppose $(X_b)_{b\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$. As $\mathscr{B}(\mathfrak{X},\mathfrak{Y})\supseteq\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$, one obtains

$$
\Psi(X_a-X_b)=\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j s_j(X_a-X_b)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\ge(r_0\|X_a-X_b\|)^p.
\tag{84}
$$

Hence, $(X_b)_{b\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathscr{B}(\mathfrak{X},\mathfrak{Y})$. Since $\mathscr{B}(\mathfrak{X},\mathfrak{Y})$ is a Banach space, then there is $X\in\mathscr{B}(\mathfrak{X},\mathfrak{Y})$ with $\lim_{b\longrightarrow\infty}\|X_b-X\|=0$.Since $(s_l(X_b))_{l=0}^{\infty}\in(\ell_p(\mathfrak{L},r))_\rho$, every $b\in\mathbb{N}_0$. According to Definition 21 setups (ii), (iii), and (v), one gets

$$
\begin{aligned}
\Psi(X)&=\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j s_j(X)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\\
&\le 2^{p-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j s_{[j/2]}(X-X_b)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\\
&\quad+2^{p-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j s_{[j/2]}(X_b)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\\
&\le 2^{p-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j\|X-X_b\|}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\\
&\quad+2^{p-1}D_0\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j\mathfrak{l}_j s_j(X_b)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p<\infty.
\end{aligned}
\tag{85}
$$

Therefore, $(s_l(X))_{l=0}^{\infty} \in (\ell_p(\mathfrak{L}, r))_\rho$; then $X \in \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$. □

Theorem 45. *Assume $\mathfrak{X}$, $\mathfrak{Y}$ are normed spaces; the setups $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$; and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ with $r_0 > 0$ are satisfied; then $(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}, \Psi)$ is a prequasi closed ideal, where $\Psi(X) = \rho((s_l(X))_{l=0}^{\infty})$.*

Proof. As $(c_0(\mathfrak{L}, r))_\rho$ is a premodular *pss*, by using Theorem 42, Ψ is a prequasi norm on $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$. Assume $X_b \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$, every $b \in \mathbb{N}_0$ and $\lim_{b\longrightarrow\infty}\Psi(X_b - X) = 0$. As $\mathscr{B}(\mathfrak{X}, \mathfrak{Y}) \supseteq \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$, we have

$$\Psi(X - X_b) = \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X - X_b)}{\mathfrak{l}_{l+2} - (l+2)} \geq r_0 \|X - X_b\|. \quad (86)$$

Hence, $(X_b)_{b\in\mathbb{N}_0}$ is a convergent sequence in $\mathscr{B}(\mathfrak{X}, \mathfrak{Y})$. Since $(s_l(X_b))_{l=0}^{\infty} \in (c_0(\mathfrak{L}, r))_\rho$, for every $b \in \mathbb{N}_0$. In view of Definition 21 setups (ii), (iii), and (v), one has

$$\begin{aligned}\Psi(X) &= \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X)}{\mathfrak{l}_{l+2} - (l+2)} \\ &\leq \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{[j/2]}(X - X_b)}{\mathfrak{l}_{l+2} - (l+2)} + \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{[j/2]}(X_b)}{\mathfrak{l}_{l+2} - (l+2)} \\ &\leq \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|X - X_b\|}{\mathfrak{l}_{l+2} - (l+2)} + D_0 \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X_b)}{\mathfrak{l}_{l+2} - (l+2)} < \infty.\end{aligned} \quad (87)$$

We get $(s_l(X))_{l=0}^{\infty} \in (c_0(\mathfrak{L}, r))_\rho$, so $X \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$. □

Theorem 46. *Assume $\mathfrak{X}$, $\mathfrak{Y}$ are normed spaces; the setups $1 < p < \infty$, $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$; and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ with $r_0 > 0$ are satisfied; hence, $(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}, \Psi)$ is a prequasi closed ideal, where $\Psi(X) = \rho((s_l(X))_{l=0}^{\infty})$.*

Proof. As $(\ell_p(\mathfrak{L}, r))_\rho$ is a premodular *pss*, by using Theorem 42, Ψ is a prequasi norm on $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$. Assume $X_b \in \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$, for every $b \in \mathbb{N}_0$ and $\lim_{b\longrightarrow\infty}\Psi(X_b - X) = 0$. As $\mathscr{B}(\mathfrak{X}, \mathfrak{Y}) \supseteq \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$, we have

$$\Psi(X - X_b) = \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X - X_b)}{\mathfrak{l}_{l+2} - (l+2)}\right)^p \geq (r_0 \|X - X_b\|)^p. \quad (88)$$

Hence, $(X_b)_{b\in\mathbb{N}_0}$ is a convergent sequence in $\mathscr{B}(\mathfrak{X}, \mathfrak{Y})$. Since $(s_l(X_b))_{l=0}^{\infty} \in (\ell_p(\mathfrak{L}, r))_\rho$, every $b \in \mathbb{N}_0$. In view of Definition 21 setups (ii), (iii), and (v), one has

$$\begin{aligned}\Psi(X) &= \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X)}{\mathfrak{l}_{l+2} - (l+2)}\right)^p \\ &\leq 2^{p-1} \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{[j/2]}(X - X_b)}{\mathfrak{l}_{l+2} - (l+2)}\right)^p \\ &\quad + 2^{p-1} \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_{[j/2]}(X_b)}{\mathfrak{l}_{l+2} - (l+2)}\right)^p \\ &\leq 2^{p-1} \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j \|X - X_b\|}{\mathfrak{l}_{l+2} - (l+2)}\right)^p \\ &\quad + 2^{p-1} D_0 \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X_b)}{\mathfrak{l}_{l+2} - (l+2)}\right)^p < \infty.\end{aligned} \quad (89)$$

We get $(s_l(X))_{l=0}^{\infty} \in (\ell_p(\mathfrak{L}, r))_\rho$, so $X \in \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$. □

6.3. Minimum Prequasi Ideal

Theorem 47. *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces with $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, and the setups $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{]}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1}\mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ are confirmed with $0 < r_l^{(2)} \leq r_l^{(1)}$, for all $l \in \mathbb{N}_0$, hence*

$$\mathscr{B}^s_{\left(c_0\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X}, \mathfrak{Y}) \subset \mathscr{B}^s_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X}, \mathfrak{Y}) \subset \mathscr{B}(\mathfrak{X}, \mathfrak{Y}). \quad (90)$$

Proof. Let $Z \in \mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X}, \mathfrak{Y})$; then $(s_l(Z)) \in (c_0(\mathfrak{L}, (r_l^{(1)})))_\rho$. One obtains

$$\sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2} - (l+2)} \leq \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2} - (l+2)} < \infty. \quad (91)$$

Then, $Z \in \mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X}, \mathfrak{Y})$. Next, if we choose $(s_l(Z))_{l=0}^{\infty}$ with $\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z) = \mathfrak{l}_{l+2} - (l+2)$ and $\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z) = (\mathfrak{l}_{l+2} - (l+2))/(l+1)(\mathfrak{l}_{l+2} - (l+2)/l+1)$, one gets $Z \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ such that $Z \notin \mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X}, \mathfrak{Y})$ and $Z \in \mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X}, \mathfrak{Y})$.

Clearly, $\mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X}, \mathfrak{Y}) \subset \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$. Next, if we put $(s_l(Z))_{l=0}^{\infty}$ such that $\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(Z) = \mathfrak{l}_{l+2} - (l+2)$. We have $Z \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ such that $Z \notin \mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X}, \mathfrak{Y})$. This explains the proof. □

Theorem 48. *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$*, and the setups* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ *or* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$*, and there exists* $C \geq 1$ *such that* $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ *are confirmed with* $1 < h^{(1)} < h^{(2)}$ *and* $0 < r_l^{(2)} \leq r_l^{(1)}$ *for all* $l \in \mathbb{N}_0$*; hence,*

$$\mathscr{B}^s_{\left(\ell_{h^{(1)}}\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}) \subset \mathscr{B}^s_{\left(\ell_{h^{(2)}}\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}) \subset \mathscr{B}(\mathfrak{X},\mathfrak{Y}). \quad (92)$$

Proof. Let $Z \in \mathscr{B}^s_{(\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y})$; then $(s_l(Z)) \in (\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho$. One obtains

$$\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^{h^{(2)}} < \sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^{h^{(1)}} < \infty. \quad (93)$$

Then $Z \in \mathscr{B}^s_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y})$. Next, if we choose $(s_l(Z))_{l=0}^{\infty}$ with $\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z) = (\mathfrak{l}_{l+2}-(l+2))/\sqrt[h1]{l+1}(\mathfrak{l}_{l+2} - (l+2)/\sqrt[h1]{l+1})$, one gets $Z \in \mathscr{B}(\mathfrak{X},\mathfrak{Y})$ such that

$$\begin{aligned}\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^{h^{(1)}} &= \sum_{l=0}^{\infty}\frac{1}{l+1} = \infty,\\ \sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^{h^{(2)}} &\leq \sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(Z)}{\mathfrak{l}_{l+2}-(l+2)}\right)^{h^{(2)}}\\ &= \sum_{l=0}^{\infty}\left(\frac{1}{l+1}\right)^{\frac{h^{(2)}}{h^{(1)}}} < \infty.\end{aligned} \quad (94)$$

Therefore, $Z \notin \mathscr{B}^s_{(\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y})$ and $Z \in \mathscr{B}^s_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y})$.

Clearly, $\mathscr{B}^s_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}) \subset \mathscr{B}(\mathfrak{X},\mathfrak{Y})$. Next, if we put $(s_l(Z))_{l=0}^{\infty}$ such that $\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(Z) = (\mathfrak{l}_{l+2}-(l+2))/\sqrt[h2]{l+1}$. We have $Z \in \mathscr{B}(\mathfrak{X},\mathfrak{Y})$ such that $Z \notin \mathscr{B}^s_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y})$. This explains the proof. □

Theorem 49. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$*, and the setups* $1 < p < \infty, (r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ *or* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$*, and there exists* $C \geq 1$ *such that* $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ *are established with* $((\sum_{j=0}^{l} r_j \mathfrak{l}_j)/(\mathfrak{l}_{l+2}-(l+2)))_{l\in\mathbb{N}_0} \notin \ell_p$*; then,* $\mathscr{B}^{\alpha}_{(\ell_p(\mathfrak{L},r))_\rho}$ *is minimum.*

Proof. Suppose the enough setups are confirmed; then $(\mathscr{B}^{\alpha}_{(\ell_p(\mathfrak{L},r))_\rho}, \Psi)$, where $\Psi(Z) = \sum_{l=0}^{\infty}((\sum_{j=0}^{l} r_j \mathfrak{l}_j \alpha_j(Z))/(\mathfrak{l}_{l+2}-(l+2)))^p$, is a prequasi Banach ideal. Suppose $\mathscr{B}^{\alpha}_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}) = \mathscr{B}(\mathfrak{X},\mathfrak{Y})$; hence, there is $\eta > 0$ with $\Psi(Z) \leq \eta\|Z\|$, for every $Z \in \mathscr{B}(\mathfrak{X},\mathfrak{Y})$. According to Dvoretzky's Theorem [23], for every $b \in \mathbb{N}_0$, one obtains quotient spaces $\mathfrak{X}/Y_b$ and subspaces M_b of $\mathfrak{Y}$ which can be mapped onto ℓ_2^b by isomorphisms V_b and X_b with $\|V_b\|\|V_b^{-1}\| \leq 2$ and $\|X_b\|\|X_b^{-1}\| \leq 2$. Let I_b be the identity operator on ℓ_2^b and T_b be the quotient operator from $\mathfrak{X}$ onto $\mathfrak{X}/Y_b$, and J_b is the natural embedding operator from M_b into $\mathfrak{Y}$. Suppose m_z is the Bernstein numbers [26]; then

$$\begin{aligned}1 = m_z(I_b) &= m_z\left(X_b X_b^{-1} I_b V_b V_b^{-1}\right)\\ &\leq \|X_b\| m_z\left(X_b^{-1} I_b V_b\right)\left\|V_b^{-1}\right\|\\ &= \|X_b\| m_z\left(J_b X_b^{-1} I_b V_b\right)\left\|V_b^{-1}\right\|\\ &\leq \|X_b\| d_z\left(J_b X_b^{-1} I_b V_b\right)\left\|V_b^{-1}\right\|\\ &= \|X_b\| d_z\left(J_b X_b^{-1} I_b V_b T_b\right)\left\|V_b^{-1}\right\|\\ &\leq \|X_b\| \alpha_z\left(J_b X_b^{-1} I_b V_b T_b\right)\left\|V_b^{-1}\right\|,\end{aligned} \quad (95)$$

for $0 \leq l \leq b$. We have

$$\begin{aligned}\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)} &\leq \frac{\sum_{z=0}^{l} \|X_b\| r_z \mathfrak{l}_z \alpha_z\left(J_b X_b^{-1} I_b V_b T_b\right)\left\|V_b^{-1}\right\|}{\mathfrak{l}_{l+2}-(l+2)}\\ &\Longrightarrow \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\\ &\leq \left(\|X_b\|\left\|V_b^{-1}\right\|\right)^p \left(\frac{\sum_{z=0}^{l} r_z \mathfrak{l}_z \alpha_z\left(J_b X_b^{-1} I_b V_b T_b\right)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p.\end{aligned} \quad (96)$$

Hence, for some $\rho \geq 1$, one gets

$$\begin{aligned}\sum_{l=0}^{b}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)}\right)^p &\leq \rho\|X_b\|\left\|V_b^{-1}\right\|\\ &\cdot \sum_{l=0}^{b}\left(\frac{\sum_{z=0}^{l} r_z \mathfrak{l}_z \alpha_z\left(J_b X_b^{-1} I_b V_b T_b\right)}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\\ &\Longrightarrow \sum_{l=0}^{b}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\\ &\leq \rho\|X_b\|\left\|V_b^{-1}\right\|\Psi\left(J_b X_b^{-1} I_b V_b T_b\right)\\ &\Longrightarrow \sum_{l=0}^{b}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\\ &\leq \rho\eta\|X_b\|\left\|V_b^{-1}\right\|\left\|J_b X_b^{-1} I_b V_b T_b\right\|\\ &\Longrightarrow \sum_{l=0}^{b}\left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)}\right)^p\\ &\leq \rho\eta\|X_b\|\left\|V_b^{-1}\right\|\left\|J_b X_b^{-1}\right\|\|I_b\|\|V_b T_b\|\\ &= \rho\eta\|X_b\|\left\|V_b^{-1}\right\|\left\|X_b^{-1}\right\|\|I_b\|\|V_b\| \leq 4\rho\eta.\end{aligned} \quad (97)$$

Therefore, we have a contradiction, if $b \longrightarrow \infty$. Then, $\mathfrak{X}$ and $\mathfrak{Y}$ both cannot be infinite dimensional if $\mathscr{B}^{\alpha}_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}) = \mathscr{B}(\mathfrak{X},\mathfrak{Y})$. This shows the proof. □

By the same manner, we can easily conclude the next theorem.

Theorem 50. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups* $1 < p < \infty, (r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ *or* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, *and there exists* $C \geq 1$ *such that* $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ *are established with* $((\sum_{j=0}^{l} r_j \mathfrak{l}_j)/(\mathfrak{l}_{l+2} - (l+2)))_{l \in \mathbb{N}_0} \notin \ell_p$; *then* $\mathscr{B}^{d}_{(\ell_p(\mathfrak{L},r))_\rho}$ *is minimum.*

6.4. Simple Banach Prequasi Ideal

Theorem 51. *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ *or* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, *and there exists* $C \geq 1$ *such that* $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ *are confirmed with* $0 < r_l^{(2)} \leq r_l^{(1)}$, *for all* $l \in \mathbb{N}_0$; *then*

$$\mathscr{B}\left(\mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right) = \mathscr{A}\left(\mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right). \tag{98}$$

Proof. Let $X \in \mathscr{B}(\mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ and $X \notin \mathscr{A}(\mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$. In view of Lemma 37, there are $Y \in \mathscr{B}(\mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ and $Z \in \mathscr{B}(\mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ with $ZXYI_b = I_b$. Therefore, for every $b \in \mathbb{N}_0$, we get

$$\begin{aligned} \|I_b\|_{\mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})} &= \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(I_b)}{\mathfrak{l}_{l+2} - (l+2)} \\ &\leq \|ZXY\| \|I_b\|_{\mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})} \\ &\leq \sup_{l\in\mathbb{N}_0} \frac{\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(I_b)}{\mathfrak{l}_{l+2} - (l+2)}. \end{aligned} \tag{99}$$

This contradicts Theorem 47. Then $X \in \mathscr{A}(\mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{(c_0(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$, which finishes the proof. □

Corollary 52. *Assume $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ *or* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, *and there exists* $C \geq 1$ *such that* $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ *are established with* $0 < r_l^{(2)} \leq r_l^{(1)}$, *for all* $l \in \mathbb{N}_0$; *then*

$$\mathscr{B}\left(\mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right) = \mathscr{K}\left(\mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{\left(c_0\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right). \tag{100}$$

Proof. Clearly, $\mathscr{A} \subset \mathscr{K}$. □

Theorem 53. *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ *or* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, *and there exists* $C \geq 1$ *such that* $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ *are confirmed with* $1 < h^{(1)} < h^{(2)}$ *and* $0 < r_l^{(2)} \leq r_l^{(1)}$, *for all* $l \in \mathbb{N}_0$; *then*

$$\mathscr{B}\left(\mathscr{B}^{s}_{\left(\ell_{h^{(2)}}\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{\left(\ell_{h^{(1)}}\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right) = \mathscr{A}\left(\mathscr{B}^{s}_{\left(\ell_{h^{(2)}}\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{\left(\ell_{h^{(1)}}\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right). \tag{101}$$

Proof. Let $X \in \mathscr{B}(\mathscr{B}^{s}_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{(\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ and $X \notin \mathscr{A}(\mathscr{B}^{s}_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{(\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$. In view of Lemma 37, there are $Y \in \mathscr{B}(\mathscr{B}^{s}_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ and $Z \in \mathscr{B}(\mathscr{B}^{s}_{(\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ with $ZXYI_b = I_b$. Therefore, for every $b \in \mathbb{N}_0$, we get

$$\begin{aligned} \|I_b\|_{\mathscr{B}^{s}_{\left(\ell_{h^{(1)}}\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})} &= \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j^{(1)} \mathfrak{l}_j s_j(I_b)}{\mathfrak{l}_{l+2} - (l+2)}\right)^{h^{(1)}} \\ &\leq \|ZXY\| \|I_b\|_{\mathscr{B}^{s}_{\left(\ell_{h^{(2)}}\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})} \\ &\leq \sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j^{(2)} \mathfrak{l}_j s_j(I_b)}{\mathfrak{l}_{l+2} - (l+2)}\right)^{h^{(2)}}. \end{aligned} \tag{102}$$

This contradicts Theorem 48. Then $X \in \mathscr{A}(\mathscr{B}^{s}_{(\ell_{h^{(2)}}(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}), \mathscr{B}^{s}_{(\ell_{h^{(1)}}(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$, which finishes the proof. □

Corollary 54. *Assume $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ *or* $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, *and there exists* $C \geq 1$ *such that* r_{2k+1}

$\mathfrak{l}_{2k+1} \leq Cr_k\mathfrak{l}_k$ *are established with* $1 < h^{(1)} < h^{(2)}$ *and* $0 < r_l^{(2)} \leq r_l^{(1)}$, *for all* $l \in \mathbb{N}_0$; *then*

$$\mathscr{B}\left(\mathscr{B}^s_{\left(\ell_{h^{(2)}}\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{\left(\ell_{h^{(1)}}\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right) = \mathscr{K}\left(\mathscr{B}^s_{\left(\ell_{h^{(2)}}\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{\left(\ell_{h^{(1)}}\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right). \tag{103}$$

Proof. Clearly, $\mathscr{A} \subset \mathscr{K}$. □

Theorem 55. *Suppose* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *are Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_\searrow$ *or* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_Z \cap \ell_\infty$, *and there exists* $C \geq 1$ *such that* $r_{2k+1}\mathfrak{l}_{2k+1} \leq Cr_k\mathfrak{l}_k$ *are confirmed with* $1 < h < \infty$ *and* $0 < r_l^{(2)} \leq r_l^{(1)}$, *for all* $l \in \mathbb{N}_0$, *then*

$$\mathscr{B}\left(\mathscr{B}^s_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{\left(\ell_h\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right) = \mathscr{A}\left(\mathscr{B}^s_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{\left(\ell_h\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})\right). \tag{104}$$

Proof. Let $X \in \mathscr{B}(\mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{(\ell_h(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ and $X \notin \mathscr{A}(\mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{(\ell_h(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$. In view of Lemma 37, there are $Y \in \mathscr{B}(\mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ and $Z \in \mathscr{B}(\mathscr{B}^s_{(\ell_h(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$ with $ZXYI_b = I_b$. Therefore, for every $b \in \mathbb{N}_0$, we get

$$\begin{aligned}\|I_b\|_{\mathscr{B}^s_{\left(\ell_h\left(\mathfrak{L},\left(r_l^{(1)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})} &= \sum_{l=0}^\infty\left(\frac{\sum_{j=0}^l r_j^{(1)}\mathfrak{l}_j s_j(I_b)}{\mathfrak{l}_{l+2}-(l+2)}\right)^h \\ &\leq \|ZXY\|\|I_b\|_{\mathscr{B}^s_{\left(c_0\left(\mathfrak{L},\left(r_l^{(2)}\right)\right)\right)_\rho}(\mathfrak{X},\mathfrak{Y})} \\ &\leq \sup_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^l r_j^{(2)}\mathfrak{l}_j s_j(I_b)}{\mathfrak{l}_{l+2}-(l+2)}.\end{aligned} \tag{105}$$

This contradicts $\ell_h(\mathfrak{L},(r_l^{(1)})) \subset c_0(\mathfrak{L},(r_l^{(2)}))$. Then $X \in \mathscr{A}(\mathscr{B}^s_{(c_0(\mathfrak{L},(r_l^{(2)})))_\rho}(\mathfrak{X},\mathfrak{Y}),\mathscr{B}^s_{(\ell_h(\mathfrak{L},(r_l^{(1)})))_\rho}(\mathfrak{X},\mathfrak{Y}))$, which finishes the proof. □

Theorem 56. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_\searrow$ *or* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_Z \cap \ell_\infty$, *and there exists* $C \geq 1$ *such that* $r_{2k+1}\mathfrak{l}_{2k+1} \leq Cr_k\mathfrak{l}_k$ *are satisfied; hence,* $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ *is simple.*

Proof. Assume the closed ideal $\mathscr{K}(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$ includes an operator $X \notin \mathscr{A}(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$. In view of Lemma 37, we have $Y, Z \in \mathscr{B}(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$ with $ZXYI_b = I_b$. This gives that $I_{\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})} \in \mathscr{K}(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$. Then, $\mathscr{B}(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})) = \mathscr{K}(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$. Hence, $\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}$ is simple Banach space. □

Theorem 57. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups* $1 < p < \infty$, $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_\searrow$ *or* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_Z \cap \ell_\infty$, *and there exists* $C \geq 1$ *such that* $r_{2k+1}\mathfrak{l}_{2k+1} \leq Cr_k\mathfrak{l}_k$ *are satisfied; hence,* $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$ *is simple.*

Proof. Assume the closed ideal $\mathscr{K}(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$ includes an operator $X \notin \mathscr{A}(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$. In view of Lemma 37, we have $Y, Z \in \mathscr{B}(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$ with $ZXYI_b = I_b$. This gives that $I_{\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})} \in \mathscr{K}(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$. Then, $\mathscr{B}(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})) = \mathscr{K}(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y}))$. Hence, $\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}$ is simple Banach space. □

6.5. Eigenvalues of S-Type Operators

Conventions 2. Please see the following conventions:

$$\begin{aligned}(\mathscr{B}^s_{\mathscr{E}})^\rho &:= \{(\mathscr{B}^s_{\mathscr{E}})^\rho(\mathfrak{X},\mathfrak{Y})\,;\,\mathfrak{X}\text{ and }\mathfrak{Y}\text{ are Banach Spaces}\},\text{ where} \\ (\mathscr{B}^s_{\mathscr{E}})^\rho(\mathfrak{X},\mathfrak{Y}) &:= \big\{X \in \mathscr{B}(\mathfrak{X},\mathfrak{Y}) \colon ((\rho_l(X))_{n=0}^\infty \\ &\quad\in \mathscr{E}\text{ and }\|X-\rho_l(X)I\|\text{ is not invertible, for all }l\in\mathbb{N}_0\big\}\end{aligned} \tag{106}$$

Theorem 58. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_\searrow$ *or* $(r_k\mathfrak{l}_k)_{k=0}^\infty \in \mathfrak{I}_Z \cap \ell_\infty$, *and there exists* $C \geq 1$ *such that* $r_{2k+1}\mathfrak{l}_{2k+1} \leq Cr_k\mathfrak{l}_k$ *are verified with* $\inf_{l\in\mathbb{N}_0}((\sum_{j=0}^l r_j\mathfrak{l}_j)/(\mathfrak{l}_{l+2}-(l+2))) > 0$; *then* $(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho})^\rho(\mathfrak{X},\mathfrak{Y}) = \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$.

Proof. Let $X \in (\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho})^\rho(\mathfrak{X},\mathfrak{Y})$; hence, $(\rho_l(X))_{l=0}^\infty \in (c_0(\mathfrak{L},r))_\rho$ and $\|X-\rho_l(X)I\| = 0$, for all $l \in \mathbb{N}_0$. We have $X = \rho_l(X)I$, for all $l \in \mathbb{N}_0$; hence, $s_l(X) = s_l(\rho_l(X)I) = |\rho_l(X)|$, for every $l \in \mathbb{N}_0$. Therefore, $(s_l(X))_{l=0}^\infty \in (c_0(\mathfrak{L},r))_\rho$; then $X \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$.

Secondly, suppose $X \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X},\mathfrak{Y})$. Then $(s_l(X))_{l=0}^\infty \in (c_0(\mathfrak{L},r))_\rho$. Hence, we have

$$\lim_{l\longrightarrow\infty}\frac{\sum_{j=0}^l r_j\mathfrak{l}_j s_j(X)}{\mathfrak{l}_{l+2}-(l+2)} \geq \inf_{l\in\mathbb{N}_0}\frac{\sum_{j=0}^l r_j\mathfrak{l}_j}{\mathfrak{l}_{l+2}-(l+2)}\lim_{l\longrightarrow\infty}s_l(X). \tag{107}$$

Therefore, $\lim_{l\longrightarrow\infty}s_l(X) = 0$. Assume $\|X-s_l(X)I\|^{-1}$ exists, for every $l \in \mathbb{N}_0$. Hence, $\|X-s_l(X)I\|^{-1}$ exists and bounded, for every $l \in \mathbb{N}_0$. Then, $\lim_{l\longrightarrow\infty}\|X-s_l(X)I\|^{-1} = \|X\|^{-1}$ exists and bounded. As $(\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}, \Psi)$ is a prequasi operator ideal, we get

$$I = XX^{-1} \in \mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y}) \Longrightarrow (s_l(I))_{l=0}^{\infty} \in (c_0(\mathfrak{L}, r))_\rho \Longrightarrow \lim_{l \longrightarrow \infty} s_l(I) = 0. \tag{108}$$

So we have a contradiction, since $\lim_{l \longrightarrow \infty} s_l(I) = 1$. Hence, $\|X - s_l(X)I\| = 0$, for every $l \in \mathbb{N}_0$. This gives $X \in (\mathscr{B}^s_{(c_0(\mathfrak{L},r))_\rho})^\rho(\mathfrak{X}, \mathfrak{Y})$. This shows the proof. □

Theorem 59. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces with* $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) = \infty$, *and the setups $1 < p < \infty$, $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_{\searrow}$ or $(r_k \mathfrak{l}_k)_{k=0}^{\infty} \in \mathfrak{I}_Z \cap \ell_\infty$, and there exists $C \geq 1$ such that $r_{2k+1} \mathfrak{l}_{2k+1} \leq C r_k \mathfrak{l}_k$ are satisfied with $\inf_l ((\sum_{j=0}^{l} r_j \mathfrak{l}_j)/(\mathfrak{l}_{l+2} - (l+2)))^p > 0$; then $(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho})^\rho(\mathfrak{X}, \mathfrak{Y}) = \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$.*

Proof. Assume $X \in (\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho})^\rho(\mathfrak{X}, \mathfrak{Y})$; hence, $(\rho_l(X))_{l=0}^{\infty} \in (\ell_p(\mathfrak{L}, r))_\rho$ and $\|X - \rho_l(X)I\| = 0$, for all $l \in \mathbb{N}_0$. We have $X = \rho_l(X)I$, for all $l \in \mathbb{N}_0$; hence, $s_l(X) = s_l(\rho_l(X)I) = |\rho_l(X)|$, for every $l \in \mathbb{N}_0$. Therefore, $(s_l(X))_{l=0}^{\infty} \in (\ell_p(\mathfrak{L}, r))_\rho$; then $X \in \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$.

Secondly, suppose $X \in \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y})$. Then $(s_l(X))_{l=0}^{\infty} \in (\ell_p(\mathfrak{L}, r))_\rho$. Hence, we have

$$\sum_{l=0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j s_j(X)}{\mathfrak{l}_{l+2} - (l+2)} \right)^p \geq \inf_l \left(\frac{\sum_{j=0}^{l} r_j \mathfrak{l}_j}{\mathfrak{l}_{l+2} - (l+2)} \right)^p \sum_{l=0}^{\infty} [s_l(X)]^p. \tag{109}$$

Therefore, $\lim_{l \longrightarrow \infty} s_l(X) = 0$. Assume $\|X - s_l(X)I\|^{-1}$ exists, for every $l \in \mathbb{N}_0$. Hence, $\|X - s_l(X)I\|^{-1}$ exists and bounded, for every $l \in \mathbb{N}_0$. Then, $\lim_{l \longrightarrow \infty} \|X - s_l(X)I\|^{-1} = \|X\|^{-1}$ exists and bounded. As $(\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}, \Psi)$ is a prequasi operator ideal, we get

$$I = XX^{-1} \in \mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho}(\mathfrak{X}, \mathfrak{Y}) \Longrightarrow (s_l(I))_{l=0}^{\infty} \in (\ell_p(\mathfrak{L}, r))_\rho \Longrightarrow \lim_{l \longrightarrow \infty} s_l(I) = 0. \tag{110}$$

So we have a contradiction, since $\lim_{l \longrightarrow \infty} s_l(I) = 1$. Hence, $\|X - s_l(X)I\| = 0$, for every $l \in \mathbb{N}_0$. This gives $X \in (\mathscr{B}^s_{(\ell_p(\mathfrak{L},r))_\rho})^\rho(\mathfrak{X}, \mathfrak{Y})$. This shows the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

On Solutions of Fractional-Order Gas Dynamics Equation by Effective Techniques

Naveed Iqbal,[1] **Ali Akgül**,[2] **Rasool Shah**,[3] **Abdul Bariq**,[4] **M. Mossa Al-Sawalha**,[1] **and Akbar Ali**[1]

[1]*Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il 2440, Saudi Arabia*
[2]*Siirt University, Art and Science Faculty, Department of Mathematics, Siirt 56100, Turkey*
[3]*Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan*
[4]*Department of Mathematics, Laghman University, Mehtarlam 2701, Laghman, Afghanistan*

Correspondence should be addressed to Abdul Bariq; abdulbariq.maths@lu.edu.af

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In this work, the novel iterative transformation technique and homotopy perturbation transformation technique are used to calculate the fractional-order gas dynamics equation. In this technique, the novel iteration method and homotopy perturbation method are combined with the Elzaki transformation. The current methods are implemented with four examples to show the efficacy and validation of the techniques. The approximate solutions obtained by the given techniques show that the methods are accurate and easy to apply to other linear and nonlinear problems.

1. Introduction

Fractional calculus (FC) has been there since classical calculus, but it has recently gained much attention due to its reaction to the requirements as mentioned above. The framework of Liouville and Riemann is used to analyze FC using differential and integral operators. Following that, it was widely used to investigate a variety of phenomena. Many academics, however, pointed out some limitations in engaging this operator, in particular, the physical meaning of the initial condition and the nonzero derivative of a constant. Caputo then presented a unique and new fractional operator that incorporated all of the abovementioned constraints. The Caputo operator is used to study most of the models studied and analyzed under the FC framework. For many ideas of FC, senior academics propose many pioneering directions, and they are the ones who provide the groundwork for the concept [1–5]. The theory and core ideas of FC have been applied to a variety of real-world problems, including biomathematics, financial models, chaos theory, optics, and other fields [6–12].

Gas dynamic equations are mathematical representations defined as the physical laws of energy conservation, mass conservation, momentum conservation, etc. Nonlinear fractional-order gas dynamics equations are applied in shock fronts, unusual factions, and connection discontinuities. Gas dynamics is a study in the field of fluid dynamics that studies gas motion and its effect on physical constructions based on the concept of fluid mechanics and fluid dynamics. The science emerges from research of gas flows, mostly around or within human minds, several instances for these research involve, and not restricted to, choked flows in nozzles and pipes, gas fuel streams in a rocket engine, aerodynamic heating on atmospheric reentry cars, and shock waves around aircraft [13, 14].

Consider the nonlinear fractional-order gas dynamics equation:

$$\frac{\partial^{\varphi} \upsilon}{\partial \eta^{\varphi}} + \upsilon \frac{\partial \upsilon}{\partial \eta} - \upsilon(1-\upsilon) = 0, \quad \eta \in \mathrm{R},\ 0 < \delta \leq 1. \tag{1}$$

The initial condition is $\Phi(\mathfrak{I}, 0) = g(\mathfrak{I})$, where δ is a parameter that describes the fractional-order derivatives.

When $\varphi = 1$, (1) improves the equation of classical gas dynamics.

Since certain physical processes, both in engineering and applied sciences, can be successfully explained by the creation of models with the aid of fractional calculus theory. The response of the fractional-order equations eventually converges to the equations of the integer order, attracting particular interest nowadays. Due to a broad variety of applications for mathematical modeling of real-world problems, fractional differentiations are very efficient, e.g., traffic flow models, earthquake modeling, regulation, diffusion model, and relaxation processes [15–17]. In the past decade, the gas dynamic equations are obtained by using different numerical analytical techniques [13, 14]. The homogeneous and nonhomogeneous nonlinear gas dynamics equations have been used the differential transformation technique [18]. Many techniques have been applied to the gas dynamics model such as fractional reduced differential transform technique [19], Elzaki transform homotopy perturbation technique [20], q-homotopy analysis technique [21], Adomian decomposition technique [22], variational iteration technique [23, 24], fractional homotopy analysis transform technique [25], homotopy perturbation algorithm using Laplace transform [26], and natural decomposition technique [27].

The goal of this study is to show how, applying the novel iterative technique and the homotopy perturbation technique, the Elzaki transform can be used to obtain approximate solutions for linear and nonlinear fractional-order differential equations. The homotopy perturbation technique was developed by Chinese mathematician J.H. In 1998, he played an important role [28]. This approach is equitable, efficient, and effective, as it eliminates an unconditioned matrix, infinite series, and complicated integrals. This technique does not necessitate the use of any unique problem parameter. Tarig Elzaki, in 2010, develops a new transformation known as the Elzaki transform (E.T). E.T. is a new transform of Laplace and Sumudu transformations [29–32]. Many other researchers use HPTM to solve various equations, such as heat-like models [33], Navier–Stokes models [34], hyperbolic and Fisher's equations [35], and gas dynamic problem [36].

Jafari and Daftardar-Gejji presented a new iterative approach for solving nonlinear equations in 2006 [37]. Jafari et al. first apply the iterative technique and Laplace transformation and combine it. They developed an iterative Laplace transformation method, which is a modified straightforward method [38] to solve the FPDE system [39, 40].

2. Basic Definitions

2.1. Definition. The fractional-order Riemann operator D^{φ} of order φ is defined as [29]

$$D^{\varphi}\nu(\zeta) = \begin{cases} \dfrac{d^{\ell}}{d\zeta^{\ell}}\nu(\zeta), & \varphi = \ell, \\[2ex] \dfrac{1}{\Delta(\ell-\varphi)}\dfrac{d}{d\zeta^{\ell}}\displaystyle\int_0^{\zeta}\dfrac{\nu(\zeta)}{(\zeta-\psi)^{\varphi-\ell+1}}d\psi, & \ell-1<\varphi<\ell, \end{cases} \quad (2)$$

where $\ell \in Z^{+}$, $\varphi \in R^{+}$, and

$$D^{-\varphi}\nu(\zeta) = \frac{1}{\Gamma(\varphi)}\int_0^{\zeta}(\zeta-\psi)^{\varphi-1}\nu(\psi)d\psi, \quad 0<\varphi\leq 1. \quad (3)$$

2.2. Definition. The Riemann fractional-order integral operator J^{ψ} is presented by [29]

$$J^{\varphi}\nu(\zeta) = \frac{1}{\Gamma(\varphi)}\int_0^{\zeta}(\zeta-\psi)^{\varphi-1}\nu(\zeta)d\zeta, \quad \zeta>0,\ \varphi>0. \quad (4)$$

The basic properties of the operator are presented as

$$\begin{aligned} J^{\varphi}\zeta^{\ell} &= \frac{\Gamma(\ell+1)}{\Gamma(\ell+\varphi+1)}\zeta^{\ell+\psi}, \\ D^{\varphi}\zeta^{\ell} &= \frac{\Gamma(\ell+1)}{\Gamma(\ell-\varphi+1)}\zeta^{\ell-\psi}. \end{aligned} \quad (5)$$

2.3. Definition. The Caputo fractional operator D^{φ} of φ is defined as [29]

$$CD^{\varphi}\nu(\zeta) = \begin{cases} \dfrac{1}{\Gamma(\ell-\varphi)}\displaystyle\int_0^{\zeta}\dfrac{\nu^{\ell}(\psi)}{(\zeta-\psi)^{\varphi-\ell+1}}d\psi, & \ell-1<\varphi<\ell, \\[2ex] \dfrac{d^{\ell}}{d\zeta^{\ell}}\nu(\zeta), & \ell=\varphi. \end{cases} \quad (6)$$

2.4. Definition. The Elzaki transformation Caputo fractional-order operator is defined as

$$\mathrm{E}\left[D_{\zeta}^{\varphi}g(\zeta)\right] = s^{-\varphi}\left[D_{\zeta}^{\varphi}g(\zeta)\right] - \sum_{k=0}^{\ell-1}s^{2-\varphi+k}g^{(k)}(0), \quad (7)$$

where $\ell-1<\varphi<\ell$.

3. New Iterative Transformation Technique

We consider

$$D_\eta^\varphi v(\mathfrak{I},\eta) + Mv(\mathfrak{I},\eta) + Nv(\mathfrak{I},\eta) = h(\mathfrak{I},\eta), \quad \ell \in N,\ \ell - 1 < \varphi \le \ell, \tag{8}$$

where M and N are linear and nonlinear terms. We consider the initial condition as

$$v^\ell(\mathfrak{I},0) = g_\ell(\mathfrak{I}). \tag{9}$$

Using the Elzaki transformation of (8), we obtain

$$E\left[D_\eta^\varphi v(\mathfrak{I},\eta)\right] + E[Mv(\mathfrak{I},\eta) + Nv(\mathfrak{I},\eta)] = E[h(\mathfrak{I},\eta)]. \tag{10}$$

Implement the Elzaki differentiation property:

$$E[v(\mathfrak{I},\eta)] = \sum_{\ell=0}^{m} s^{2-\varphi+\ell} v^{(\ell)}(\mathfrak{I},0) + s^\varphi E[h(\mathfrak{I},\eta)] - s^\varphi E[Mv(\mathfrak{I},\eta) + Nv(\mathfrak{I},\eta)]. \tag{11}$$

Using the inverse Elzaki transformation (11),

$$v(\mathfrak{I},\eta) = E^{-1}\left[\left\{\sum_{\ell=0}^{m} s^{2-\varphi+\ell} v^\ell(\mathfrak{I},0) + s^\varphi E[h(\mathfrak{I},\eta)]\right\}\right] - E^{-1}\left[s^\varphi E[Mv(\mathfrak{I},\eta) + Nv(\mathfrak{I},\eta)]\right]. \tag{12}$$

Then, we reach

$$v(\mathfrak{I},\eta) = \sum_{\ell=0}^{\infty} v_\ell(\mathfrak{I},\eta), \tag{13}$$

$$N\left(\sum_{\ell=0}^{\infty} v_\ell(\mathfrak{I},\eta)\right) = \sum_{\ell=0}^{\infty} N[v_\ell(\mathfrak{I},\eta)]. \tag{14}$$

We consider the nonlinear term N by

$$N\left(\sum_{\ell=0}^{\infty} v_\ell(\mathfrak{I},\eta)\right) = v_0(\mathfrak{I},\eta) + N\left(\sum_{\ell=0}^{m} v_\ell(\mathfrak{I},\eta)\right) - M\left(\sum_{\ell=0}^{m} v_\ell(\mathfrak{I},\eta)\right). \tag{15}$$

Replacing equations (12), (13), and (15) in (12) yields

$$\sum_{\ell=0}^{\infty} v_\ell(\mathfrak{I},\eta) = E^{-1}\left[s^\varphi\left(\sum_{\ell=0}^{m} s^{2-\mathfrak{I}+\ell} v^\ell(\mathfrak{I},\eta) + E[h(\mathfrak{I},\eta)]\right)\right] - E^{-1}\left[s^\varphi E\left[M\left(\sum_{\ell=0}^{m} v_\ell(\mathfrak{I},\eta)\right) - N\left(\sum_{\ell=0}^{m} v_\ell(\mathfrak{I},\eta)\right)\right]\right]. \tag{16}$$

We describe the iterative method:

$$\begin{aligned}
v_0(\mathfrak{I},\eta) &= E^{-1}\left[s^\varphi\left(\sum_{\ell=0}^{\ell} s^{2-\mathfrak{I}+\ell} v^\ell(\mathfrak{I},0) + s^\varphi E(g(\mathfrak{I},\eta))\right)\right],\\
v_1(\mathfrak{I},\eta) &= -E^{-1} s^\varphi E[M[v_0(\mathfrak{I},\eta)] + N[v_0(\mathfrak{I},\eta)]],\\
v_{\ell+1}(\mathfrak{I},\eta) &= -E^{-1}\left[s^\varphi E\left[-M\left(\sum_{\ell=0}^{\ell} v_\ell(\mathfrak{I},\eta)\right) - N\left(\sum_{\ell=0}^{\ell} v_\ell(\mathfrak{I},\eta)\right)\right]\right], \quad \ell \ge 1.
\end{aligned} \tag{17}$$

Finally, we can write as

$$v(\mathfrak{I},\eta) \cong v_0(\mathfrak{I},\eta) + v_1(\mathfrak{I},\eta) + v_2(\mathfrak{I},\eta) + \cdots + v_\ell(\mathfrak{I},\eta), \quad m = 1, 2, \ldots. \tag{18}$$

4. Homotopy Perturbation Transform Method

In this section, we give the general solution of FPDEs via the homotopy perturbation method:

$$\begin{gathered} vD_\eta^\varphi v(\mathfrak{I},\eta) + Mv(\mathfrak{I},\eta) + Nv(\mathfrak{I},\eta) = h(\mathfrak{I},\eta), \quad \eta > 0,\ 0 < \varphi \le 1, \\ v(\mathfrak{I},0) = g(\mathfrak{I}), \quad \nu \in \mathfrak{R}. \end{gathered} \tag{19}$$

Applying Elzaki transformation of (16),

$$\begin{gathered} E\left[D_\eta^\varphi v(\mathfrak{I},\eta) + Mv(\mathfrak{I},\eta) + Nv(\mathfrak{I},\eta)\right] = E[h(\mathfrak{I},\eta)], \quad \eta > 0,\ 0 < \varphi \le 1, \\ v(\mathfrak{I},\eta) = s^2 g(\mathfrak{I}) + s^\varphi E[h(\mathfrak{I},\eta)] - s^\varphi E[Mv(\mathfrak{I},\eta) + Nv(\mathfrak{I},\eta)]. \end{gathered} \tag{20}$$

Now, we use Elzaki inverse transformation, and we obtain

$$v(\mathfrak{I},\eta) = F(x,\eta) - E^{-1}\left[s^\varphi E\{Mv(\mathfrak{I},\eta) + Nv(\mathfrak{I},\eta)\}\right], \tag{21}$$

where

$$F(\mathfrak{I},\eta) = E^{-1}\left[s^2 g(\mathfrak{I}) + s^\varphi E[h(\mathfrak{I},\eta)]\right] = g(\nu) + E^{-1}\left[s^\varphi E[h(\mathfrak{I},\eta)]\right]. \tag{22}$$

Now, the parameter p shows the producer of perturbation:

$$v(\mathfrak{I},\eta) = \sum_{\ell=0}^{\infty} p^m v_\ell(\mathfrak{I},\eta). \tag{23}$$

The nonlinear term can be defined as

$$Nv(\mathfrak{I},\eta) = \sum_{\ell=0}^{\infty} p^\ell H_\ell(v_\ell), \tag{24}$$

where H_ℓ are He's polynomial in terms of $v_0, v_1, v_2, \ldots, v_\ell$ and can be calculated as

$$H_n(v_0, v_1, \ldots, v_n) = \frac{1}{\varphi(n+1)} D_p^\ell \left[N\left(\sum_{\ell=0}^{\infty} p^\ell v_\ell\right)\right]_{p=0}. \tag{25}$$

Substituting (24) and (25) in (21), we achieve as

$$\sum_{\ell=0}^{\infty} p^\ell v_\ell(\mathfrak{I},\eta) = F(\mathfrak{I},\eta) - p \times \left[E^{-1}\left\{s^\varphi E\left\{M\sum_{\ell=0}^{\infty} p^\ell v_\ell(\mathfrak{I},\eta) + \sum_{\ell=0}^{\infty} p^\ell H_\ell(v_\ell)\right\}\right\}\right]. \tag{26}$$

Comparison of coefficients p on both sides, we obtain

$$\begin{aligned}&p^0: v_0(\mathfrak{I},\eta) = F(\mathfrak{I},\eta),\\&p^1: v_1(\mathfrak{I},\eta) = E^{-1}\left[s^\varphi E\left(Mv_0(\mathfrak{I},\eta) + H_0(v)\right)\right],\\&p^2: v_2(\mathfrak{I},\eta) = E^{-1}\left[s^\varphi E\left(Mv_1(\mathfrak{I},\eta) + H_1(v)\right)\right],\\&\vdots\\&p^\ell: v_\ell(\mathfrak{I},\eta) = E^{-1}\left[s^\varphi E\left(Mv_{\ell-1}(\mathfrak{I},\eta) + H_{\ell-1}(v)\right)\right], \quad \ell > 0,\ \ell \in N.\end{aligned} \tag{27}$$

The $v_\ell(\mathfrak{I},\eta)$ components can be calculated easily which is a fast convergence series. We can obtain $p \longrightarrow 1$:

$$v(\mathfrak{I},\eta) = \lim_{M\longrightarrow\infty}\sum_{\ell=1}^{M} v_\ell(\mathfrak{I},\eta). \tag{28}$$

4.1. Example. Consider the fractional-order gas dynamics equation:

$$\frac{\partial^\varphi v}{\partial\eta^\varphi} + v\frac{\partial v}{\partial\eta} - v(1-v) = 0, \quad 0 < \varphi \le 1, \tag{29}$$

with initial condition,

$$v(\mathfrak{I},0) = e^{-\mathfrak{I}}. \tag{30}$$

First on both sides apply Elzaki transformation in (29), we have

$$E[v(\mathfrak{I},\eta)] = s^2\left(e^{-\mathfrak{I}}\right) - s^\varphi E\left[v\frac{\partial v}{\partial\eta} - v(1-v)\right]. \tag{31}$$

Using inverse Elzaki transform on the above equation,

$$v(\mathfrak{I},\eta) = e^{-\mathfrak{I}} - E^{-1}\left[s^\varphi E\left\{v\frac{\partial v}{\partial\eta} - v(1-v)\right\}\right]. \tag{32}$$

We use the NITM:

$$\begin{aligned}&v_0(\mathfrak{I},\eta) = e^{-\mathfrak{I}},\\&v_1(\mathfrak{I},\eta) = -E^{-1}\left[s^\varphi E\left\{v_0\frac{\partial v_0}{\partial\eta} - v_0(1-v_0)\right\}\right] = \frac{e^{-\mathfrak{I}}\eta^\varphi}{\Gamma(\varphi+1)},\\&v_2(\mathfrak{I},\eta) = -E^{-1}\left[s^\varphi E\left\{v_1\frac{\partial v_1}{\partial\eta} - v_1(1-v_1)\right\}\right] = \frac{e^{-\mathfrak{I}}\eta^{2\varphi}}{\Gamma(2\varphi+1)},\\&v_3(\mathfrak{I},\eta) = -E^{-1}\left[s^\varphi E\left\{v_2\frac{\partial v_2}{\partial\eta} - v_2(1-v_2)\right\}\right] = \frac{e^{-\mathfrak{I}}\eta^{3\varphi}}{\Gamma(3\varphi+1)},\\&\vdots\\&v_{\ell+1}(\mathfrak{I},\eta) = -E^{-1}\left[s^\varphi E\left\{v_\ell\frac{\partial v_\ell}{\partial\eta} - v_\ell(1-v_l)\right\}\right] = \frac{e^{-\mathfrak{I}}\eta^{\ell\varphi}}{\Gamma(l\varphi+1)}.\end{aligned} \tag{33}$$

The series solution form is given as

$$\begin{aligned}v(\mathfrak{I},\eta) &= v_0(\mathfrak{I},\eta) + v_1(\mathfrak{I},\eta) + v_2(\mathfrak{I},\eta) + v_3(\mathfrak{I},\eta)\\&\quad + \cdots v_\ell(\mathfrak{I},\eta).\end{aligned} \tag{34}$$

The approximate solution is achieved as

$$\begin{aligned}v(\mathfrak{I},\eta) &= e^{-\mathfrak{I}} + \frac{e^{-\mathfrak{I}}\eta^\varphi}{\Gamma(\varphi+1)} + \frac{e^{-\mathfrak{I}}\eta^{2\varphi}}{\Gamma(2\varphi+1)} + \frac{e^{-\mathfrak{I}}\eta^{3\varphi}}{\Gamma(3\varphi+1)}\\&\quad + \cdots + \frac{e^{-\mathfrak{I}}\eta^{\ell\varphi}}{\Gamma(l\varphi+1)}.\end{aligned} \tag{35}$$

Now, we apply the HPTM, and we obtain

$$\sum_{\ell=0}^{\infty} p^\ell v_\ell(\mathfrak{I},\eta) = \left(e^{-\mathfrak{I}}\right) + p\left\{E^{-1}\left(s^\varphi E\left[\sum_{\ell=0}^{\infty} p^\ell H_\ell(v)\right]\right)\right\}. \tag{36}$$

Then, we have

$$\begin{aligned}&p^0: v_0(\mathfrak{I},\eta) = e^{-\mathfrak{I}},\\&p^1: v_1(\mathfrak{I},\eta) = \left[E^{-1}\{s^\varphi E(H_0(v))\}\right] = \frac{e^{-\mathfrak{I}}\eta^\varphi}{\Gamma(\varphi+1)},\\&p^2: v_2(\mathfrak{I},\eta) = \left[E^{-1}\{s^\varphi E(H_1(v))\}\right] = \frac{e^{-\mathfrak{I}}\eta^{2\varphi}}{\Gamma(2\varphi+1)},\\&p^3: v_3(\mathfrak{I},\eta) = \left[E^{-1}\{s^\varphi E(H_2(v))\}\right] = \frac{e^{-\mathfrak{I}}\eta^{3\varphi}}{\Gamma(3\varphi+1)},\\&\vdots\\&p^n: v_n(\mathfrak{I},\eta) = \left[E^{-1}\{s^\varphi E(H_{n-1}(v))\}\right] = \frac{e^{-\mathfrak{I}}\eta^{n\varphi}}{\Gamma(\eta\varphi+1)}.\end{aligned} \tag{37}$$

Then, the series form solution of HPTM is presented:

$$v(\mathfrak{I},\eta) = \sum_{n=0}^{\infty} p^n v_n(\mathfrak{I},\eta). \tag{38}$$

The approximate solution of Example 1 is given by

$$\begin{aligned}v(\mathfrak{I},\eta) &= e^{-\mathfrak{I}} + \frac{e^{-\mathfrak{I}}\eta^\varphi}{\Gamma(\varphi+1)} + \frac{e^{-\mathfrak{I}}\eta^{2\varphi}}{\Gamma(2\varphi+1)} + \frac{e^{-\mathfrak{I}}\eta^{3\varphi}}{\Gamma(3\varphi+1)}\\&\quad + \cdots + \frac{e^{-\mathfrak{I}}\eta^{m\varphi}}{\Gamma(m\varphi+1)},\\v(\mathfrak{I},\eta) &= e^{-\mathfrak{I}}\sum_{n=0}^{\infty}\frac{(\eta^\varphi)^n}{\Gamma(n\varphi+1)} = e^{-\mathfrak{I}}E_\varphi(\eta^\varphi).\end{aligned} \tag{39}$$

The exact result of (29):

$$v(\mathfrak{I}, \eta) = e^{-\mathfrak{I}+\eta}. \tag{40}$$

In Figure 1, the actual and analytical solutions are proved at $\varphi = 1$ of Example 4.1. In Figure 2, the three-dimensional figure for numerous fractional orders are described which demonstrates that the modified decomposition technique and new iterative transform technique approximated obtained results are in close contact with the analytical and the exact results. In Figure 3, the analytical solution graph of fractional order $\varphi = 0.4$ of problem 3.1. This comparative shows a strong connection among the homotopy perturbation transform method and actual solutions. Consequently, the homotopy perturbation transform method and new iterative transformation technique are accurate innovative techniques which need less calculation time and is very simple and more flexible as compared to other methods.

4.2. Example. We take into consideration

$$\frac{\partial^{\varphi} v}{\partial \eta^{\varphi}} + v\frac{\partial v}{\partial \eta} - v(1-v)\log b = 0, \quad b > 0,\ 0 < \varphi \leq 1, \tag{41}$$

with initial condition,

$$v(\mathfrak{I}, 0) = b^{-\mathfrak{I}}. \tag{42}$$

Applying the Elzaki transformation in (40) gives

$$E[v(\mathfrak{I}, \eta)] = s^2\left(b^{-\mathfrak{I}}\right) - s^{\varphi} E\left[v\frac{\partial v}{\partial \eta} - v(1-v)\log b\right]. \tag{43}$$

Using inverse Elzaki transform on the above equation,

$$v(\mathfrak{I}, \eta) = b^{-\mathfrak{I}} - E^{-1}\left[s^{\varphi} E\left\{v\frac{\partial v}{\partial \eta} - v(1-v)\log b\right\}\right]. \tag{44}$$

We use the NITM:

$$\begin{aligned}
v_0(\mathfrak{I}, \eta) &= b^{-\mathfrak{I}}, \\
v_1(\mathfrak{I}, \eta) &= -E^{-1}\left[s^{\varphi} E\left\{v_0\frac{\partial v_0}{\partial \eta} - v_0(1-v_0)\log b\right\}\right] = b^{-\mathfrak{I}}\frac{\log b\eta^{\varphi}}{\Gamma(\varphi+1)}, \\
v_2(\mathfrak{I}, \eta) &= -E^{-1}\left[s^{\varphi} E\left\{v_1\frac{\partial v_1}{\partial \eta} - v_1(1-v_1)\log b\right\}\right] = b^{-\mathfrak{I}}\frac{(\log b)^2\eta^{2\varphi}}{\Gamma(2\varphi+1)}, \\
v_3(\mathfrak{I}, \eta) &= -E^{-1}\left[s^{\varphi} E\left\{v_2\frac{\partial v_2}{\partial \eta} - v_2(1-v_2)\log b\right\}\right] = b^{-\mathfrak{I}}\frac{(\log b)^2\eta^{3\varphi}}{\Gamma(3\varphi+1)}, \\
&\vdots \\
v_{n+1}(\mathfrak{I}, \eta) &= -E^{-1}\left[s^{\varphi} E\left\{v_{\ell}\frac{\partial v_n}{\partial \eta} - v_n(1-v_n)\log b\right\}\right] = b^{-\mathfrak{I}}\frac{(\log b)^2\eta^{n\varphi}}{\Gamma(n\varphi+1)}.
\end{aligned} \tag{45}$$

The series solution form is presented by

$$v(\mathfrak{I}, \eta) = v_0(\mathfrak{I}, \eta) + v_1(\mathfrak{I}, \eta) + v_2(\mathfrak{I}, \eta) + v_3(\mathfrak{I}, \eta) + \cdots v_n(\mathfrak{I}, \eta). \tag{46}$$

The approximate solution is achieved as

$$\begin{aligned}
v(\mathfrak{I}, \eta) &= b^{-\mathfrak{I}} + b^{-\mathfrak{I}}\frac{\log b\eta^{\varphi}}{\Gamma(\varphi+1)} + b^{-\mathfrak{I}}\frac{(\log b)^2\eta^{2\varphi}}{\Gamma(2\varphi+1)} + b^{-\mathfrak{I}}\frac{(\log b)^3\eta^{3\varphi}}{\Gamma(3\varphi+1)} + \cdots + b^{-\mathfrak{I}}\frac{\left((\log b\eta^{\varphi})^m\right)}{\Gamma(m\varphi+1)}, \\
v(\mathfrak{I}, \eta) &= b^{-\mathfrak{I}}\sum_{m=0}^{\infty}\frac{(\log b\eta^{\varphi})^m}{\Gamma(m\varphi+1)} = b^{-\mathfrak{I}}E_{\varphi}(\log b\eta^{\varphi}).
\end{aligned} \tag{47}$$

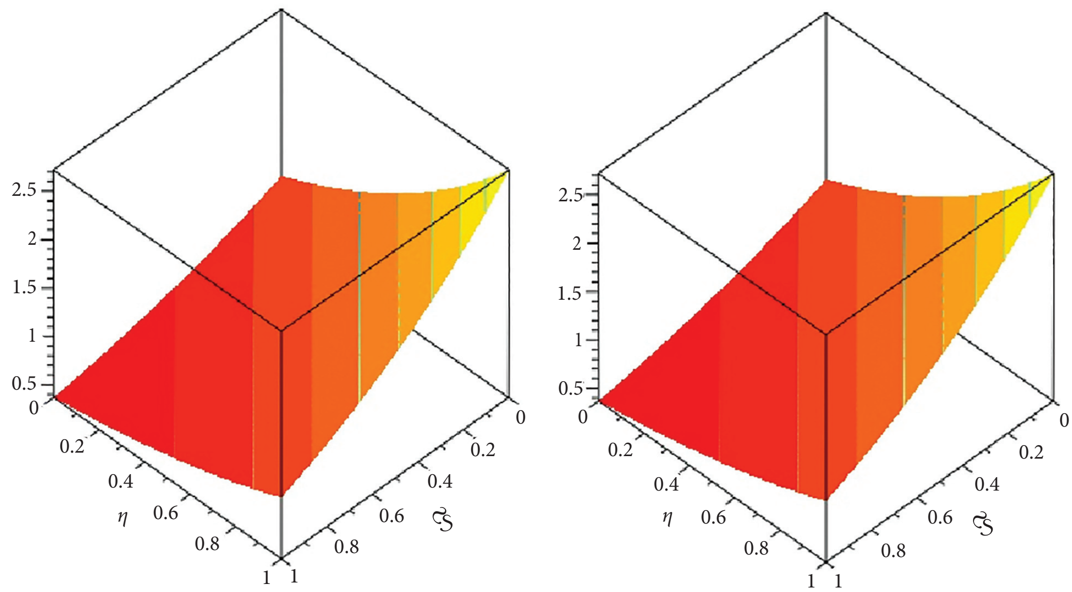

Figure 1: Simulations of the solutions of problem 3.1.

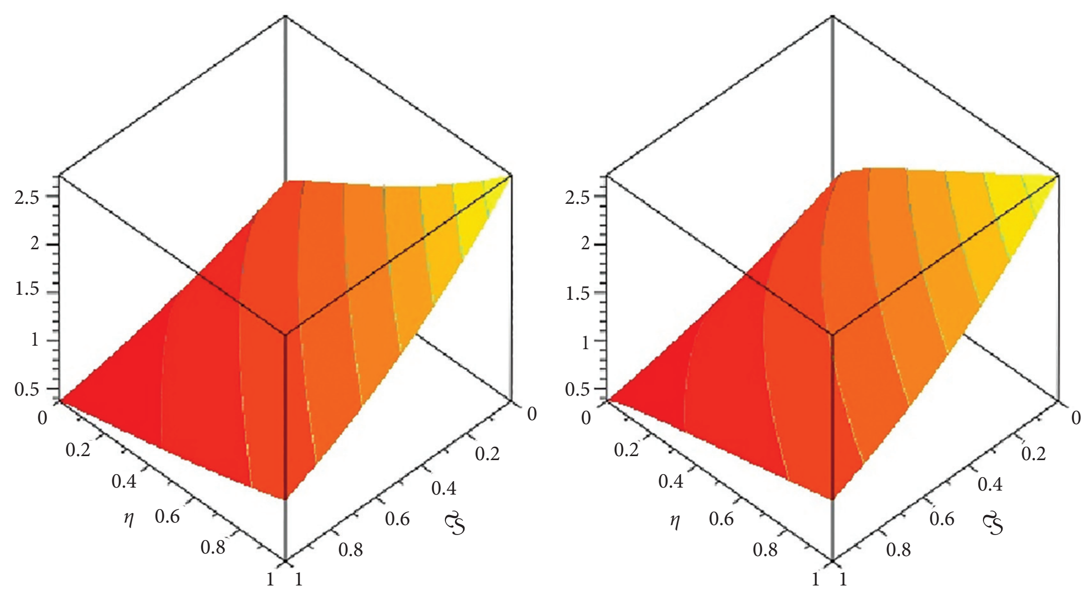

Figure 2: The fractional order of $\varphi = 0.8$ and 0.6 of problem 3.1.

Now, we apply the HPTM; we obtain

$$\sum_{\ell=0}^{\infty} p^{\ell} v_{\ell}(\mathfrak{J}, \eta) = \left(b^{-\mathfrak{J}}\right) + p\left\{E^{-1}\left(s^{\varphi} E\left[\sum_{\ell=0}^{\infty} p^{\ell} H_{\ell}(v)\right]\right)\right\}, \tag{48}$$

where the polynomial signifying the nonlinear expressions is $H_{\ell}(v)$. For instance, the components of He's polynomials are obtained through the recursive correlation $H_{\ell}(v) = v_{\ell}(\partial v_{\ell}/\partial \eta) - v_{\ell}(1 - v_{\ell})\log b,\ \forall \ell \in N$. Now, both sides as the equivalent power coefficient of p are compared; the following calculation is obtain by

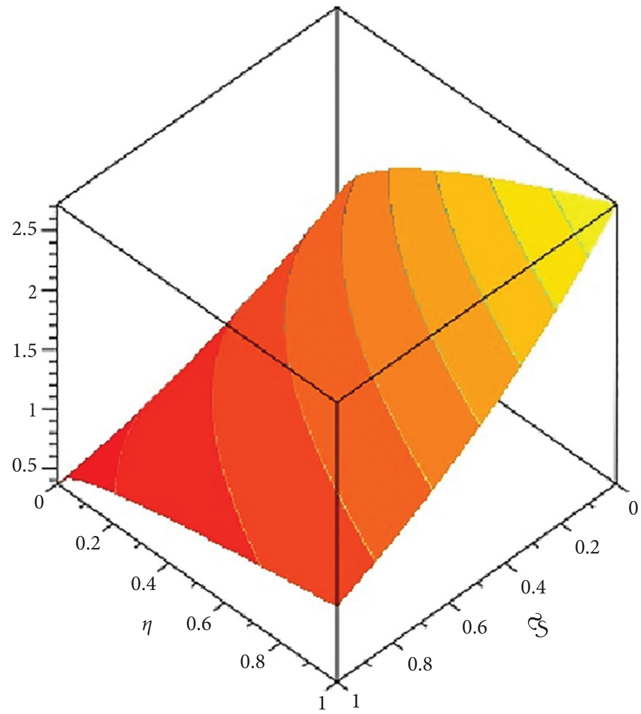

Figure 3: The fractional order of $\varphi = 0.4$ of problem 3.1.

$$p^0: v_0(\mathfrak{I},\eta) = b^{-\mathfrak{I}},$$

$$p^1: v_1(\mathfrak{I},\eta) = \left[E^{-1}\{s^{\varphi}E(H_0(v))\}\right] = b^{-\mathfrak{I}}\frac{\log b\eta^{\varphi}}{\Gamma(\varphi+1)},$$

$$p^2: v_2(\mathfrak{I},\eta) = \left[E^{-1}\{s^{\varphi}E(H_1(v))\}\right] = b^{-\mathfrak{I}}\frac{(\log b)^2\eta^{2\varphi}}{\Gamma(2\varphi+1)},$$

$$p^3: v_3(\mathfrak{I},\eta) = \left[E^{-1}\{s^{\varphi}E(H_2(v))\}\right] = b^{-\mathfrak{I}}\frac{(\log b)^3\eta^{3\varphi}}{\Gamma(3\varphi+1)},$$

$$\vdots$$

$$p^n: v_n(\mathfrak{I},\eta) = \left[E^{-1}\{s^{\varphi}E(H_{n-1}(v))\}\right] = b^{-\mathfrak{I}}\frac{(\log b)^l\eta^{n\varphi}}{\Gamma(l\varphi+1)}. \tag{49}$$

Thus, we obtain

$$v(\mathfrak{I},\eta) = \sum_{\ell=0}^{\infty} p^{\ell} v_{\ell}(\mathfrak{I},\eta). \tag{50}$$

The approximate solution of Example 2 is given as

$$v(\mathfrak{I},\eta) = b^{-\mathfrak{I}} + b^{-\mathfrak{I}}\frac{\log b\eta^{\varphi}}{\Gamma(\varphi+1)} + b^{-\mathfrak{I}}\frac{(\log b)^2\eta^{2\varphi}}{\Gamma(2\varphi+1)} + b^{-\mathfrak{I}}\frac{(\log b)^3\eta^{3\varphi}}{\Gamma(3\varphi+1)} + \cdots + b^{-\mathfrak{I}}\frac{((\log b\eta^{\varphi})^m)}{\Gamma(m\varphi+1)}, \tag{51}$$

$$v(\mathfrak{I},\eta) = b^{-\mathfrak{I}}\sum_{m=0}^{\infty}\frac{(\log b\eta^{\varphi})^m}{\Gamma(m\varphi+1)} = b^{-\mathfrak{I}}E_{\varphi}(\log b\eta^{\varphi}).$$

The exact result of (40) is

$$v(\mathfrak{I},\eta) = b^{-\mathfrak{I}+\eta}. \tag{52}$$

In Figure 4, the actual and analytical solutions are proved at $\varphi = 1$ of Example 4.2. In Figure 5, the three-dimensional figure for numerous fractional order is described which demonstrates that the modified decomposition technique and new iterative transform technique approximated obtained results are in close contact with the analytical and the exact results. In Figure 6, the analytical solution graph of fractional order $\varphi = 0.4$ of problem 3.2. This comparative result shows a strong connection between the homotopy perturbation transform method and actual solutions. Consequently, the homotopy perturbation transform

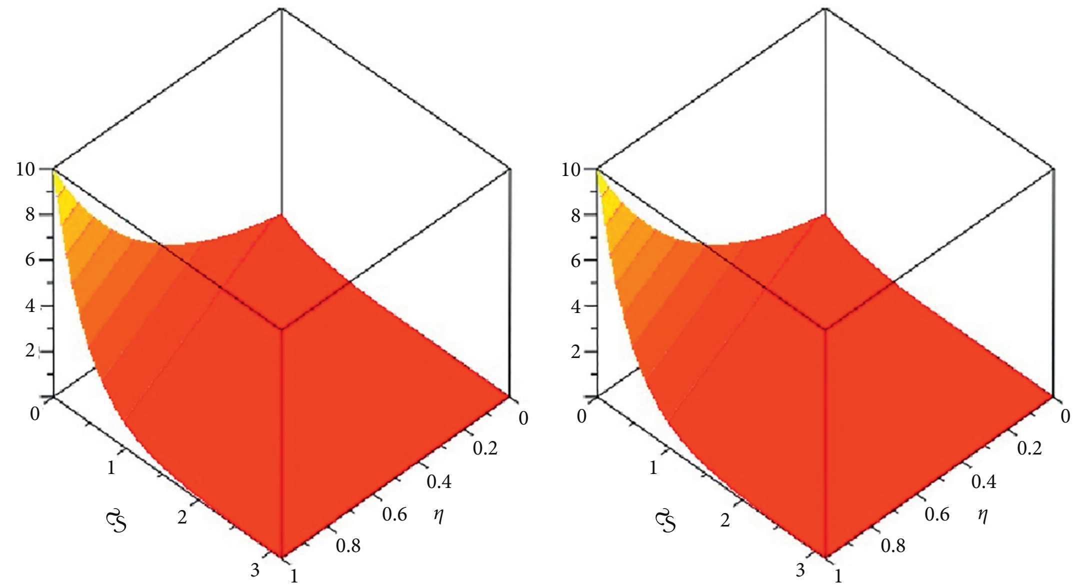

FIGURE 4: Simulations of the solutions of problem 3.2.

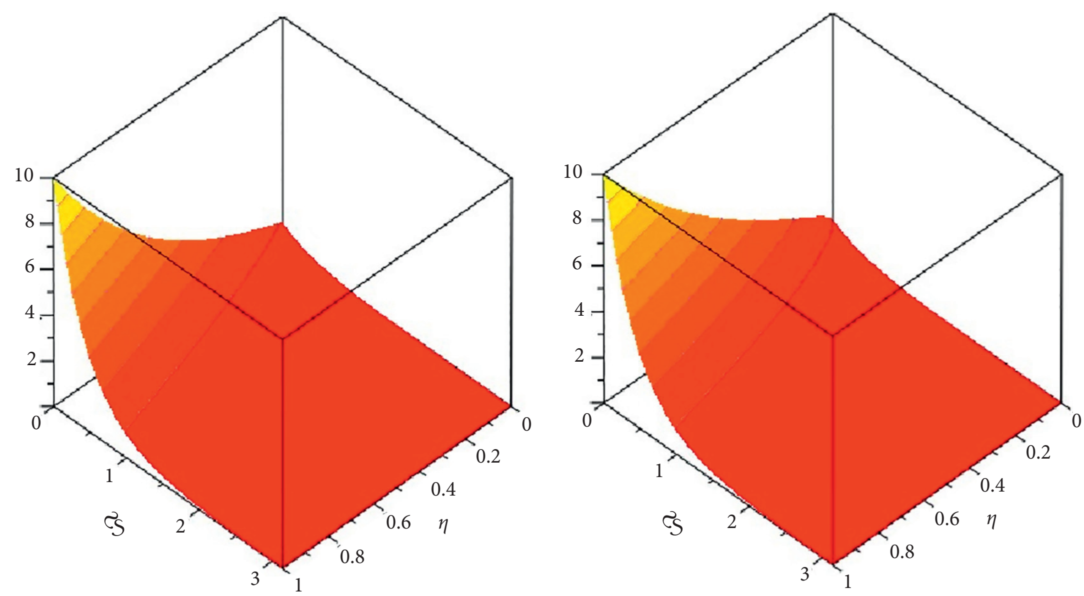

FIGURE 5: The fractional order of $\varphi = 0.8$ and 0.6 of problem 3.2.

method and new iterative transformation technique are accurate innovative techniques which needs less calculation time and is very simple and more flexible as compare to other methods.

4.3. Example. We take into consideration the fractional-order nonlinear homogeneous gas dynamics equation:

$$\frac{\partial^{\varphi} v}{\partial \eta^{\varphi}} + v\frac{\partial v}{\partial \eta} - v(1-v) + e^{-\mathfrak{I}+\eta} = 0, \quad 0 < \varphi \leq 1, \tag{53}$$

with initial condition,

$$v(\mathfrak{I}, 0) = 1 - e^{-\mathfrak{I}}. \tag{54}$$

Applying the Elzaki transformation in (52) yields

$$E[v(\mathfrak{I}, \eta)] = s^2\left(1 - e^{-\mathfrak{I}}\right) - s^{\varphi}E\left[v\frac{\partial v}{\partial \eta} - v(1-v) + e^{-\mathfrak{I}+\eta}\right]. \tag{55}$$

Using inverse Elzaki transform on the above equation,

$$v(\mathfrak{I}, \eta) = 1 - e^{-\mathfrak{I}} - E^{-1}\left[s^{\varphi}E\left\{v\frac{\partial v}{\partial \eta} - v(1-v) + e^{-\mathfrak{I}+\eta}\right\}\right]. \tag{56}$$

We use the NITM:

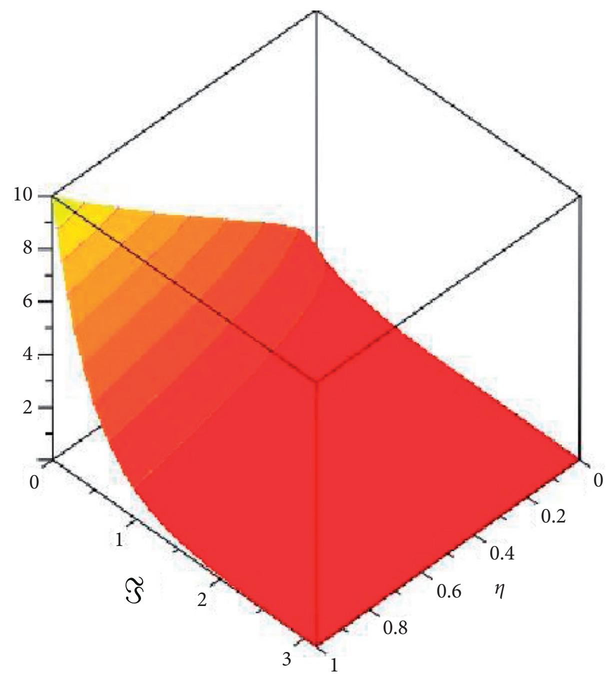

FIGURE 6: The fractional order of $\varphi = 0.4$ of problem 3.2.

$$
\begin{aligned}
v_0(\mathfrak{I},\eta) &= 1 - e^{-\mathfrak{I}},\\
v_1(\mathfrak{I},\eta) &= -E^{-1}\left[s^{\varphi}E\left\{v_0\frac{\partial v_0}{\partial \eta} - v_0(1-v_0) + e^{-\mathfrak{I}+\eta}\right\}\right] = -e^{-\mathfrak{I}}\frac{\eta^{\varphi}}{\Gamma(\varphi+1)} - e^{-\mathfrak{I}}\frac{\eta^{2\varphi}}{\Gamma(2\varphi+1)},\\
v_2(\mathfrak{I},\eta) &= -E^{-1}\left[s^{\varphi}E\left\{v_1\frac{\partial v_1}{\partial \eta} - v_1(1-v_1) + e^{-\mathfrak{I}+\eta}\right\}\right] = e^{-\mathfrak{I}}\frac{\eta^{3\varphi}}{\Gamma(3\varphi+1)},\\
v_3(\mathfrak{I},\eta) &= -E^{-1}\left[s^{\varphi}E\left\{v_2\frac{\partial v_2}{\partial \eta} - v_2(1-v_2) + e^{-\mathfrak{I}+\eta}\right\}\right] = b^{-\mathfrak{I}}\frac{\eta^{4\varphi}}{\Gamma(4\varphi+1)},\\
&\vdots\\
v_{\ell+1}(\mathfrak{I},\eta) &= -E^{-1}\left[s^{\varphi}E\left\{v_\ell\frac{\partial v_\ell}{\partial \eta} - v_\ell(1-v_\ell) + e^{-\mathfrak{I}+\eta}\right\}\right].
\end{aligned}
\tag{57}
$$

The series solution form is given as

$$
\begin{aligned}
v(\mathfrak{I},\eta) &= v_0(\mathfrak{I},\eta) + v_1(\mathfrak{I},\eta) + v_2(\mathfrak{I},\eta)\\
&\quad + v_3(\mathfrak{I},\eta) + \cdots v_n(\mathfrak{I},\eta).
\end{aligned}
\tag{58}
$$

The approximate solution is achieved as

$$
\begin{aligned}
v(\mathfrak{I},\eta) &= 1 - e^{-\mathfrak{I}} - e^{-\mathfrak{I}}\frac{\eta^{\varphi}}{\Gamma(\varphi+1)} - e^{-\mathfrak{I}}\frac{\eta^{2\varphi}}{\Gamma(2\varphi+1)} - e^{-\mathfrak{I}}\frac{\eta^{3\varphi}}{\Gamma(3\varphi+1)} - \cdots - e^{-\mathfrak{I}}\frac{\eta^{m\varphi}}{\Gamma(m\varphi+1)},\\
v(\mathfrak{I},\eta) &= e^{-\mathfrak{I}}\sum_{m=0}^{\infty}\frac{(\eta^{\varphi})^m}{\Gamma(m\varphi+1)} = e^{-\mathfrak{I}}E_{\varphi}(\eta^{\varphi}).
\end{aligned}
\tag{59}
$$

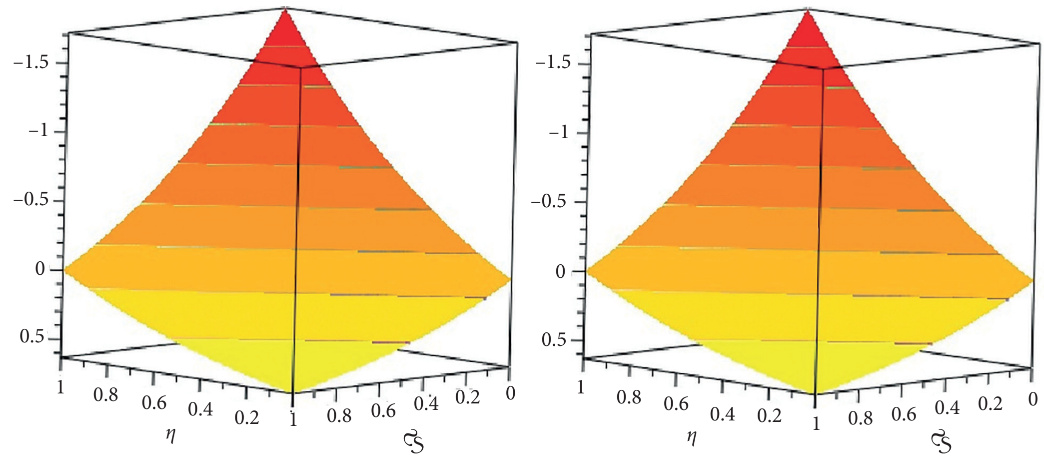

FIGURE 7: Simulations of the solutions of Example 4.3.

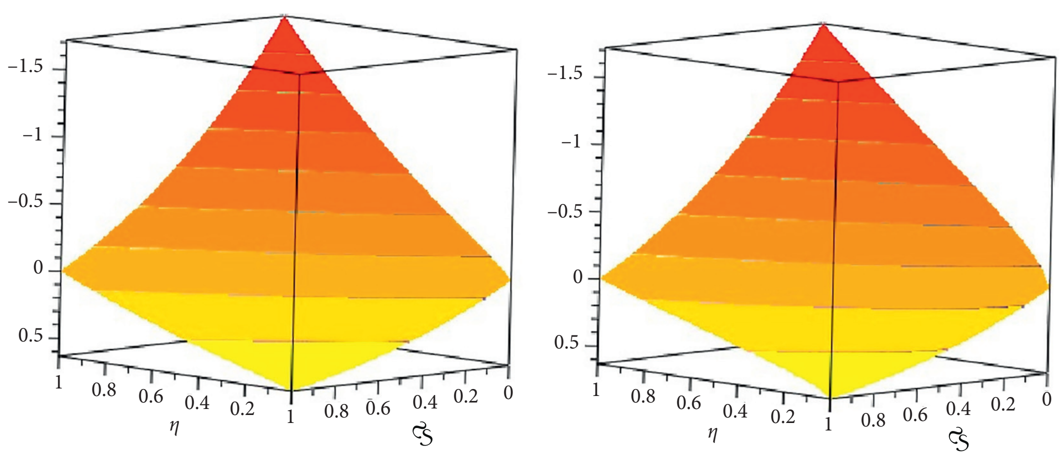

FIGURE 8: The fractional order of $\varphi = 0.8$ and 0.6 of problem 3.4.

Now, we apply the HPTM, and we obtain

$$\sum_{\ell=0}^{\infty} p^{\ell} v_{\ell}(\mathfrak{I},\eta) = 1 - e^{-\mathfrak{I}} + p\left\{E^{-1}\left(s^{\varphi}E\left[\sum_{\ell=0}^{\infty} p^{\ell} H_{\ell}(v) + e^{-\mathfrak{I}+\eta}\right)\right)\right\}, \tag{60}$$

where the polynomial signifying the nonlinear expressions is $H_{\ell}(v)$. For instance, the components of He's polynomials are obtained through the recursive correlation $H_{\ell}(v) = v_{\ell}(\partial v_{\ell}/\partial \eta) - v_{\ell}(1 - v_{\ell})\log b,\ \forall \ell \in N$. Now, both sides of the equivalent power coefficient of p is compared; the following calculation is obtain by

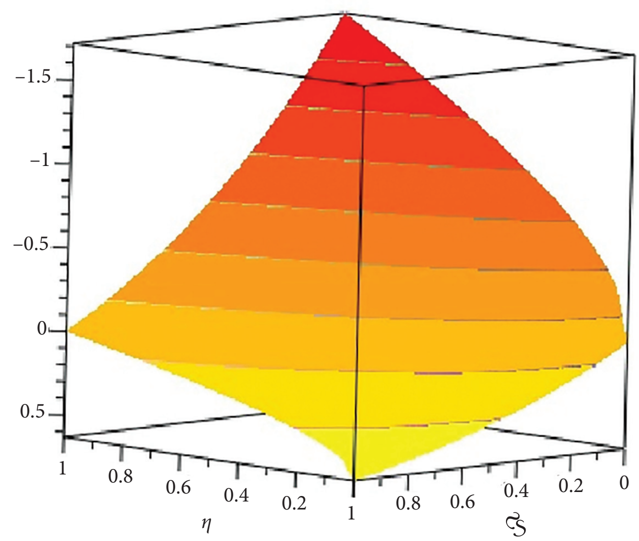

FIGURE 9: The fractional order of $\varphi = 0.4$ of problem 3.4.

$$\begin{aligned}
&p^0: v_0(\Im,\eta) = 1 - e^{-\Im},\\
&p^1: v_1(\Im,\eta) = \left[E^{-1}\left\{s^{\varphi}E(H_0(v)) + e^{-\Im+\eta}\right\}\right] = -e^{-\Im}\frac{\eta^{\varphi}}{\Gamma(\varphi+1)},\\
&p^2: v_2(\Im,\eta) = \left[E^{-1}\left\{s^{\varphi}E(H_1(v)) + e^{-\Im+\eta}\right\}\right] = -e^{-\Im}\frac{\eta^{2\varphi}}{\Gamma(2\varphi+1)},\\
&p^3: v_3(\Im,\eta) = \left[E^{-1}\left\{s^{\varphi}E(H_2(v)) + e^{-\Im+\eta}\right\}\right] = -e^{-\Im}\frac{\eta^{3\varphi}}{\Gamma(3\varphi+1)},\\
&\vdots\\
&p^n: v_n(\Im,\eta) = \left[E^{-1}\left\{s^{\varphi}E(H_{n-1}(v)) + e^{-\Im+\eta}\right\}\right] = -e^{-\Im}\frac{\eta^{n\varphi}}{\Gamma(\eta\varphi+1)}.
\end{aligned} \tag{61}$$

Then, the series-form solution of HPTM is given as

$$v(\Im,\eta) = \sum_{\ell=0}^{\infty} p^{\ell} v_{\ell}(\Im,\eta). \tag{62}$$

The approximate solution of example in this section is given as

$$\begin{aligned}
v(\Im,\eta) &= 1 - e^{-\Im} - e^{-\Im}\frac{\eta^{\varphi}}{\Gamma(\varphi+1)} - e^{-\Im}\frac{\eta^{2\varphi}}{\Gamma(2\varphi+1)} - e^{-\Im}\frac{\eta^{3\varphi}}{\Gamma(3\varphi+1)} - \cdots - e^{-\Im}\frac{\eta^{m\varphi}}{\Gamma(m\varphi+1)},\\
v(\Im,\eta) &= e^{-\Im}\sum_{\ell=0}^{\infty}\frac{(\eta^{\varphi})^{\ell}}{\Gamma(\ell\varphi+1)} = e^{-\Im}E_{\varphi}(\eta^{\varphi}).
\end{aligned} \tag{63}$$

The exact result of (52) is

$$u(\Im,\eta) = 1 - e^{-\Im+\eta}. \tag{64}$$

In Figure 7, the actual and analytical solutions are proved at $\varphi = 1$ of Example 4.3. In Figure 8, the three-dimensional figure for numerous fractional order is described, which demonstrates that the modified decomposition technique and new iterative transform technique approximated obtained results are in close contact with the analytical and the exact results. In Figure 9, the analytical solution graph of fractional order $\varphi = 0.4$ of problem 3.3. This comparative shows a strong connection among the homotopy perturbation transform method and actual solutions. Consequently, the homotopy perturbation transform method and new iterative transformation technique are accurate innovative techniques which need less calculation time and is very simple and more flexible as compared to other methods.

5. Conclusion

In this paper, we analyzed the time factional of gas dynamics equation by applying two analytical techniques. It is also used that the suggested methods' rate of convergence is sufficient for the solution of fractional-order partial differential equations. The computations of these methods are very straightforward and simple. Therefore, these methods can be applied to fractional partial differential equations.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this study.

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Research Article

A Note on Lacunary Sequence Spaces of Fractional Difference Operator of Order (α, β)

Qing-Bo Cai,[1] **Sunil K. Sharma**,[2] **and Mohammad Ayman Mursaleen**[3,4]

[1]*Fujian Provincial Key Laboratory of Data-Intensive Computing, Key Laboratory of Intelligent Computing and Information Processing, School of Mathematics and Computer Science, Quanzhou Normal University, Quanzhou 362000, China*
[2]*Department of Mathematics, Central University of Jammu, Bagla Suchani, Samba 181143, Jammu & Kashmir, India*
[3]*School of Information and Physical Sciences, The University of Newcastle, Callaghan, New South Wales 2308, Australia*
[4]*Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*

Correspondence should be addressed to Mohammad Ayman Mursaleen; mohdaymanm@gmail.com

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In the present paper, we defined lacunary sequence spaces of fractional difference operator of order (α, β) over n-normed spaces via Musielak-Orlicz function $\mathcal{M} = (\mathfrak{I}_k)$. Our aim in this paper is to study some topological properties and inclusion relation between the spaces $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$, $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)$, and $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_\infty$.

1. Introduction and Preliminaries

The concept of statistical convergence was introduced by Fast [1] and Schoenberg [2] independently. Many authors studied the concept of statistical convergence from the past few years we may refer to ([3–19]) and references therein.

The sequence $\xi = (\xi_k)$ is statistically convergent of order α to ℓ (see Çolak) if there is a complex number ℓ such that

$$\lim_{n\longrightarrow\infty} \frac{1}{n^\alpha} |\{k \le n : |\xi_k - \ell| \ge \varepsilon\}| = 0. \quad (1)$$

Let $0 < \alpha \le \beta \le 1$. We define the (α, β)-density of the subset E of $\mathbb{N}$ by

$$\delta_\alpha^\beta(E) = \lim_n \frac{1}{n^\alpha} |\{k \le n : k \in E\}|^\beta, \quad (2)$$

provided the limit exists, where $|\{k \le n : k \in E\}|^\beta$ denotes the βth power of number of elements of E not exceeding n ([20–22]).

By a lacunary sequence $\theta = (\theta_r)$, we mean a sequence of positive integers such that $\theta_0 = 0$, $0 < \theta_r < \theta_{r+1}$, and $\phi_r = \theta_r - \theta_{r-1} \longrightarrow \infty$ as $r \longrightarrow \infty$. The intervals determined by θ will be denoted by $J_r = (\theta_{r-1}, \theta_r]$ and $t_r = \theta_r/\theta_{r-1}$. Freedman et al. [23] defined the space of lacunary strongly convergent sequences by

$$N_\theta = \left\{\xi \in w : \lim_{r\longrightarrow\infty} \frac{1}{\phi_r} \sum_{k\in J_r} |\xi_k - l| = 0, \text{for some } l\right\}. \quad (3)$$

Definition 1. Let $\theta = (\theta_r)$ be a lacunary sequence. The sequence $\xi = (\xi_k)$ is $S_\alpha^\beta(\theta)$-statistically convergent (or lacunary statistically convergent of order (α, β)) (see [20]) if there is a real number L such that

$$\lim_{r\longrightarrow\infty} \frac{1}{\phi_r^\alpha} |\{k \in J_r : |\xi_k - L| \ge \varepsilon\}|^\beta = 0, \quad (4)$$

where $J_r = (\theta_{r-1}, \theta_r]$ and ϕ_r^α denotes the αth power $(\phi_r)^\alpha$ of ϕ_r, that is, $\phi^\alpha = (\phi_r^\alpha) = (\phi_1^\alpha, \phi_2^\alpha, \cdots, \phi_r^\alpha, \cdots)$. In this case, we write $S_\alpha^\beta(\theta) - \lim \xi_k = L$. The set of all $S_\alpha^\beta(\theta)$-statistically convergent sequences is denoted by $S_\alpha^\beta(\theta)$. If $\alpha=\beta=1$ and $\theta=(2^r)$, then, we will write S instead of $S_\alpha^\beta(\theta)$.

A family $\mathscr{I}\subset 2^X$ of subsets of a nonempty set X is said to be an ideal in X if

(1) $\phi\in\mathscr{I}$

(2) $A,B\in\mathscr{I}$ imply $A\cup B\in\mathscr{I}$

(3) $A\in\mathscr{I}$, $B\subset A$ imply $B\in\mathscr{I}$

while an admissible ideal $\mathscr{I}$ of X further satisfies $\{\xi\}\in\mathscr{I}$ for each $\xi\in X$ (see [24]).

A sequence $(\xi_n)_{n\in\mathbb{N}}$ in X is said to be $\mathscr{I}$-convergent to $\xi\in X$ (see [24]), if $A(\varepsilon)=\{n\in\mathbb{N}:\|\xi_n-\xi\|\geq\varepsilon\}\in\mathscr{I}$, for each $\varepsilon>0$.

A sequence $(\xi_n)_{n\in\mathbb{N}}$ in X is said to be $\mathscr{I}$-bounded to $\xi\in X$ if there exists an $K>0$ such that $\{n\in\mathbb{N}:|\xi_n|>K\}\in\mathscr{I}$. Many authors studied the topological properties and applications of ideal, we refer to ([25–37]) and references therein.

The concept of difference sequence spaces was introduced in [38] and further generalized in [39].

In [40], Baliarsingh defined the fractional difference operator as follows:

Let $\xi=(\xi_k)\in w$ and γ be a real number, then, the fractional difference operator $\Delta^{(\gamma)}$ is defined by

$$\Delta^{(\gamma)}\xi_k=\sum_{i=0}^{k}\frac{(-\gamma)_i}{i!}\xi_{k-i}, \tag{5}$$

where $(-\gamma)_i$ denotes the Pochhammer symbol defined as

$$(-\gamma)_i=\begin{cases}1, & \text{if } \gamma=0 \text{ or } i=0,\ \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+i-1), \text{ otherwise},\\ \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+i-1), & \text{otherwise}.\end{cases} \tag{6}$$

The concept of difference sequences, Orlicz function, Musielak-Orlicz function, and n-normed spaces was used by many authors and proves some topological properties (see [41–50]) and references therein. For details about n-normed spaces, we refer to ([51–55]), difference sequence spaces ([38, 39]), Orlicz function ([56–58]). Ideal convergence and fractional difference operator Δ^α has been studied in [59, 60]. We continue in this connection and construct new sequence spaces as follows.

Let $M=(\mathfrak{I}_k)$ be a Musielak-Orlicz function, $u=(u_k)$ be a bounded sequence of positive real numbers, and $0<\alpha\leq\beta\leq 1$. We define the following sequence spaces in the present paper

$$\mathscr{I}-N_\alpha^\beta\left(A,\mathscr{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|\right)_0=\left\{\xi\in w:\mathscr{I}-\lim_r\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta=0,\text{ for some }\rho>0\right\}, \tag{7}$$

$$\mathscr{I}-N_\alpha^\beta\left(A,\mathscr{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|\right)=\left\{\xi\in w:\mathscr{I}-\lim_r\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)-L}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta=0,\text{ for some }L\text{ and }\rho>0\right\}, \tag{8}$$

$$\mathscr{I}-N_\alpha^\beta\left(A,\mathscr{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|\right)_\infty=\left\{\xi\in w:\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\text{ is bounded, for some }\rho>0\right\}. \tag{9}$$

If we take $\mathscr{M}(\xi)=\xi$, the above spaces reduces to $\mathscr{I}-N_\alpha^\beta(A,u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_0$, $\mathscr{I}-N_\alpha^\beta(A,u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)$, and $\mathscr{I}-N_\alpha^\beta(A,u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_\infty$.

If we take $u=(u_k)=1$, the above spaces reduces to $\mathscr{I}-N_\alpha^\beta(A,\mathscr{M},\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_0$, $\mathscr{I}-N_\alpha^\beta(A,\mathscr{M},\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)$, and $\mathscr{I}-N_\alpha^\beta(A,\mathscr{M},\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_\infty$.

The following inequality will be used in the proceeding results. If $0 \le u_k \le \sup u_k = H$, $D = \max(1, 2^{H-1})$, then

$$|r_k + s_k|^{u_k} \le D\{|r_k|^{u_k} + |s_k|^{u_k}\}, \quad (10)$$

for all k and $r_k, s_k \in \mathbb{C}$. Also $|r|^{u_k} \le \max(1, |r|^H)$ for all $r \in \mathbb{C}$.

2. Main Results

In this section, we study topological properties and prove some inclusion relations. In what follows, we will take $M = (\mathfrak{I}_k)$ a Musielak-Orlicz function and $u = (u_k)$ a bounded sequence of positive real numbers.

Theorem 2. *The spaces* $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|$, $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)$, *and* $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_\infty$ *are linear spaces.*

Proof. Let $\xi_1, \xi_2 \in \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$ and let μ, ν be scalars. Then, there exist two positive numbers ρ_1 and ρ_2 for $\varepsilon > 0$

$$D_1 = \left\{ r \in \mathbb{N} : \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi_1\right)}{\rho_1}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \ge \frac{\varepsilon}{2D} \right\} \in \mathcal{I}, \quad (11)$$

$$D_2 = \left\{ r \in \mathbb{N} : \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi_2\right)}{\rho_2}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \ge \frac{\varepsilon}{2D} \right\} \in \mathcal{I}. \quad (12)$$

Let $\rho_3 = \max\{2|\mu|\rho_1, 2|\nu|\rho_2\}$ and by inequality (1), we have

$$\begin{aligned}
&\frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}(\mu\xi_1 + \nu\xi_2)\right)}{\rho_3}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \\
&\quad \le \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{\mu A_k\left(\Delta^{(\gamma)}\xi_1\right)}{\rho_3}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \\
&\qquad + \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{\nu A_k\left(\Delta^{(\gamma)}\xi_2\right)}{\rho_3}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \\
&\quad \le \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi_1\right)}{\rho_1}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \\
&\qquad + \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi_2\right)}{\rho_2}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta.
\end{aligned} \quad (13)$$

Now by (11) and (12), we get

$$\left\{ r \in \mathbb{N} : \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}(\mu\xi_1 + \nu\xi_2)\right)}{\rho_3}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta > \varepsilon \right\} \subset D_1 \cup D_2. \quad (14)$$

Therefore, $\mu\xi_1 + \nu\xi_2 \in \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Hence, $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$ is a linear space. On a similar way, we can prove that $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)$ and $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_\infty$ are linear spaces. □

Theorem 3. *The inclusions* $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0 \subset \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|) \subset \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_\infty$ *hold.*

Proof. The inclusion $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0 \subset \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)$ is obvious. We prove $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|) \subset \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_\infty$. For this, let $\xi \in \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)$. Then, there exists $\rho_1 > 0$ such that for every $\varepsilon > 0$

$$B_1 = \left\{ r \in \mathbb{N} : \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi\right) - L}{\rho_1}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \ge \varepsilon \right\} \in \mathcal{I}. \quad (15)$$

We put $\rho = 2\rho_1$ and $\mathcal{M} = (\mathfrak{I}_k)$ is a Musielak-Orlicz function, we have

$$\begin{aligned}
&\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1, \cdots, x_{n-1} \right\| \right) \\
&\quad \le \mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi\right) - L}{\rho_1}, x_1, \cdots, x_{n-1} \right\| \right) \\
&\qquad + \mathfrak{I}_k \left(\left\| \frac{L}{\rho_1}, x_1, \cdots, x_{n-1} \right\| \right).
\end{aligned} \quad (16)$$

Suppose that $r \notin B_1$. Hence by above inequality and (1), we have

$$\begin{aligned}
&\frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \\
&\quad \le D \left\{ \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi\right) - L}{\rho_1}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \right.
\end{aligned}$$

$$+\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{L}{\rho_1},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\Bigg\}$$

$$< D\left\{\varepsilon+\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{L}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\right\}. \quad (17)$$

By using $[\mathfrak{I}_k(\|(L/\rho_1),x_1,\cdots,x_{n-1}\|)]^{u_k}\le\max\{1,[\mathfrak{I}_k(\|(L/\rho_1),x_1,\cdots,x_{n-1}\|)]^H\}$, we have

$$\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{L}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta<\infty. \quad (18)$$

Put $K=D\{\varepsilon+1/\phi_r^\alpha[\sum_{k\in J_r}[\mathfrak{I}_k(\|(L/\rho),x_1,\cdots,x_{n-1}\|)]^{u_k}]^\beta\}$. It follows that

$$\left\{r\in\mathbb{N}:\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta>K\right\}\in\mathcal{I}. \quad (19)$$

This shows that $\xi\in\mathcal{I}-N_\alpha^\beta(A,\mathcal{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_\infty$, which completes the proof. □

Theorem 4. *The space $\mathcal{I}-N_\alpha^\beta(A,\mathcal{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_\infty$ is a paranormed space with paranorm defined by*

$$g(x)=\inf\left\{\rho>0:\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\le 1\right\}. \quad (20)$$

Proof. Since $g(\xi)=g(-\xi)$ and $\mathfrak{I}_k(0)=0$, we have $g(0)=0$. Let $\xi_1,\xi_2\in\mathcal{I}-N_\alpha^\beta(A,\mathcal{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_\infty$. Let

$$B(\xi_1)=\left\{\rho>0:\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi_1\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\le 1\right\}, \quad (21)$$

$$B(\xi_2)=\left\{\rho>0:\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi_2\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\le 1\right\}. \quad (22)$$

Let $\rho_1\in B(\xi_1)$ and $\rho_2\in B(\xi_2)$ and $\rho=\rho_1+\rho_2$, we have

$$\begin{aligned}&\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}(\xi_1+\xi_2)\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^\beta\\&\le\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi_1\right)}{\rho_1},x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^\beta\\&\quad+\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi_2\right)}{\rho_2},x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^\beta.\end{aligned} \quad (23)$$

Thus, $1/\phi_r^\alpha[\sum_{k\in J_r}[\mathfrak{I}_k(\|(A_k(\Delta^{(\gamma)}(\xi_1+\xi_2))/\rho_1+\rho_2),x_1,\cdots,x_{n-1}\|)]^{u_k}]^\beta\le 1$ and

$$\begin{aligned}g(\xi_1+\xi_2)&\le\inf\{(\rho_1+\rho_2)>0:\rho_1\in B(\xi_1),\rho_2\in B(\xi_2)\}\\&\le\inf\{\rho_1>0:\rho_1\in B(\xi_1)\}\\&\quad+\inf\{\rho_2>0:\rho_2\in B(\xi_2)\}\\&=g(\xi_1)+g(\xi_2).\end{aligned} \quad (24)$$

Let $\sigma^s\longrightarrow\sigma$ where $\sigma,\sigma^s\in\mathbb{C}$ and let $g(\xi^s-\xi)\longrightarrow 0$ as $s\longrightarrow\infty$. We have to show that $g(\sigma^s\xi^s-\sigma\xi)\longrightarrow 0$ as $s\longrightarrow\infty$. Let

$$B(\xi^s)=\left\{\rho_s>0:\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi^s\right)}{\rho_s},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\le 1\right\}, \quad (25)$$

$$B(\xi^s-\xi)=\left\{\rho_s'>0:\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}(\xi^s-\xi)\right)}{\rho_s'},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\le 1\right\}. \quad (26)$$

If $\rho_s\in B(\xi^s)$ and $\rho_s'\in B(\xi^s-\xi)$; then, we have

$$\begin{aligned}&\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\left(\sigma^s\xi^s-\sigma\xi\right)\right)}{\rho_s\,|\sigma^s-\sigma|+\rho_s'\,|\sigma|},x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^\beta\\&\le\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\left(\sigma^s\xi^s-\sigma\xi^s\right)\right)}{\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|}\right.\right.\right.\right.\\&\quad\left.\left.\left.\left.+\frac{\left|A_k\left(\Delta^{(\gamma)}\left(\sigma\xi^s-\sigma\xi\right)\right)\right|}{\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|},x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^\beta\end{aligned}$$

$$\leq \frac{|\sigma^s - \sigma|\rho_s}{\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|}\frac{1}{\phi_r^\alpha} \cdot \left[\sum_{k\in J_r}\left[\mathfrak{J}_k\left(\left\|\frac{\left(A_k\left(\Delta^{(\gamma)}\left(\xi^s\right)\right)\right)}{\rho_s}, x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^\beta + \frac{|\sigma|\rho_s'}{\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|}\frac{1}{\phi_r^\alpha} \cdot \left[\sum_{k\in J_r}\left[\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\left(\xi^s - \xi\right)\right)}{\rho_s'}, x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^\beta. \tag{27}$$

From the above inequality, it follows that

$$\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\left(\sigma^s\xi^s - \sigma\xi\right)\right)}{\rho_s \mid \sigma^s - \sigma\mid + \rho_s' \mid \sigma \mid}, x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \leq 1, \tag{28}$$

and consequently,

$$\begin{aligned} g\left(\sigma^s\xi^s - \sigma\xi\right) &\leq \inf\left\{\left(\rho_s|\sigma^s - \sigma| + \rho_s'|\sigma|\right) > 0 : \rho_s \in B\left(\xi^s\right), \rho_s' \in B\left(\xi^s - \xi\right)\right\} \\ &\leq (|\sigma^s - \sigma|) > 0 \inf\left\{\rho > 0 : \rho_s \in B\left(\xi^s\right)\right\} + (|\sigma|) \\ &> 0 \inf\left\{\left(\rho_s'\right)^{u_n/H} : \rho_s' \in B\left(\xi^s - \xi\right)\right\} \longrightarrow 0\, as\, s \longrightarrow \infty, \end{aligned} \tag{29}$$

which completes the proof. □

Theorem 5. *Let $\mathcal{M} = (\mathfrak{J}_k)$ and $\mathcal{M}' = (\mathfrak{J}_k')$ be Musielak-Orlicz functions that satisfy the Δ_2-condition. Then*

$$\begin{aligned} &\mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0 \\ &\quad\subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}' \circ \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0, \end{aligned} \tag{30}$$

$$\begin{aligned} &\mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right) \\ &\quad\subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}' \circ \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right), \end{aligned} \tag{31}$$

$$\begin{aligned} &\mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_\infty \\ &\quad\subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}' \circ \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_\infty. \end{aligned} \tag{32}$$

Proof. (i) Let $\xi \in \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Then, there exists $K_1 > 0$ such that

$$B_1 = \left\{r \in \mathbb{N} : \frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \geq K_1\right\} \in \mathcal{I}, \tag{33}$$

for $\rho > 0$. Since $\mathcal{M}'$ is a Musielak-Orlicz function which satisfies Δ_2-condition, we have

$$\begin{aligned} &\frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r \\ \mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1,\cdots x_{n-1}\right\|\right) > \delta}}\left[\mathfrak{J}_k'\left(\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1,\cdots,x_{n-1}\right\|\right)\right)\right]^{u_k}\right]^\beta \\ &\leq \max\left\{1, \left(K\frac{1}{\delta}\mathfrak{J}_k'(2)\right)^H\right\}\frac{1}{\phi_r^\alpha} \left[\sum_{\substack{k\in J_r \\ \mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1,\cdots x_{n-1}\right\|\right) > \delta}}\left(\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1,\cdots,x_{n-1}\right\|\right)\right)^{u_k}\right]^\beta, \end{aligned} \tag{34}$$

for $K \geq 1$. By continuity of $\mathcal{M}'$, we have

$$\begin{aligned} &\frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r \\ \mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1,\cdots x_{n-1}\right\|\right) \leq \delta}}\left[\mathfrak{J}_k'\left(\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1,\cdots,x_{n-1}\right\|\right)\right)\right]^{u_k}\right]^\beta \\ &\leq \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r \\ \mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1,\cdots x_{n-1}\right\|\right) \leq \delta}}\varepsilon^{u_k}\right]^\beta \\ &\leq \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r \\ \mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1,\cdots x_{n-1}\right\|\right) \leq \delta}}\max\left\{\varepsilon^h, \varepsilon^H\right\}\right]^\beta. \end{aligned} \tag{35}$$

Suppose $r \notin B_1$. Then, by using (34) and (35), we have

$$\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k'\left(\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right)\right]^{u_k}\right]^\beta = \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ \mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)>\delta}}\left(\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right)^{u_k}\right]^\beta + \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ \mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\le\delta}}\left(\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right)^{u_k}\right]^\beta \le \max\left\{1,\left(K\frac{1}{\delta}\mathfrak{I}_k'(2)\right)^H\right\}K_1+\max\left\{\varepsilon^h,\varepsilon^H\right\}=K_2. \tag{36}$$

Hence, $r \notin B_2 = \{r \in \mathbb{N} : 1/\phi_r^\alpha[\sum_{k\in J_r}[\mathfrak{I}_k'(\mathfrak{I}_k(\|A_k(\Delta^{(\gamma)}\xi)/\rho, x_1,\cdots,x_{n-1}\|))]^{u_k}]^\beta > K_2\}$ and so $B_2 \subset B_1$ which implies $B_2 \in \mathcal{I}$. This shows that $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}' \circ \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Hence, $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0 \subseteq \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}' \circ \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Similarly, we can prove (ii) and (iii) part. □

Corollary 6. *Let $\mathcal{M} = (\mathfrak{I}_k)$ satisfy Δ_2-condition. Then,*

$$\mathcal{I} - N_\alpha^\beta\left(A, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0 \subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0, \tag{37}$$

$$\mathcal{I} - N_\alpha^\beta\left(A, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right) \subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right), \tag{38}$$

$$\mathcal{I} - N_\alpha^\beta\left(A, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_\infty \subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_\infty. \tag{39}$$

Proof. If we put $\mathfrak{I}_k(x) = x$ and $\mathfrak{I}_k'(x) = \mathfrak{I}_k(x) \forall x \in [0,\infty)$ in Theorem 5, the result follows. □

Theorem 7. *Let $\mathcal{M} = (\mathfrak{I}_k)$ and $\mathcal{M}' = (\mathfrak{I}_k')$ be Musielak-Orlicz functions that satisfy the Δ_2-condition. Then,*

$$\mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0 \cap \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}', u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0 \subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}' + \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0, \tag{40}$$

$$\mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right) \cap \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}', u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right) \subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}' + \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right), \tag{41}$$

$$\mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_\infty \cap \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_\infty \subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}' + \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_\infty. \tag{42}$$

Proof. (i) Let $\xi \in \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0 \cap \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}', u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Then, there exists $K_1 > 0$ and $K_2 > 0$ such that

$$B_1 = \left\{r \in \mathbb{N} : \frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \ge K_1\right\} \in \mathcal{I}, \tag{43}$$

$$B_2 = \left\{r \in \mathbb{N} : \frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k'\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \ge K_2\right\} \in \mathcal{I}, \tag{44}$$

for some $\rho > 0$. Let $r \notin B_1 \cup B_2$. Then, we have

$$\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k+\mathfrak{I}_k'\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \le D\left\{\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left(\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right)^{u_k}\right]^\beta + \frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left(\mathfrak{I}_k'\left(\left\|\frac{A_k\left(\Delta^{(\alpha)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right)^{u_k}\right]^\beta\right\} < \{K_1+K_2\}. \tag{45}$$

$r \notin B = \{r \in \mathbb{N} : 1/\phi_r^\alpha[\sum_{k\in J_r}[(\mathfrak{I}_k' + \mathfrak{I}_k)(A_k(\Delta^{(\gamma)}\xi)/\rho, x_1,\cdots, x_{n-1})]^{u_k}]^\beta > K\}$. We have $B_1 \cup B_2 \in \mathcal{I}$ and so $B \subset B_1 \cup B_2$ which implies $B \in \mathcal{I}$. This shows that $x \in \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}' + \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Hence, $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0 \cap \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}', u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0 \subseteq \mathcal{I} -$

$N_{\alpha}^{\beta}(A, \mathcal{M}' + \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Similarly, we can prove (ii) and (iii) part of the theorem. □

Theorem 8. *Let $0 < u_k \leq v_k$ and (v_k/u_k) be bounded. Then, the following inclusions hold*

$$\begin{aligned}&\mathcal{I} - N_{\alpha}^{\beta}\left(A, \mathcal{M}, v, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0\\&\quad\subseteq \mathcal{I} - N_{\alpha}^{\beta}\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0,\end{aligned} \tag{46}$$

$$\begin{aligned}&\mathcal{I} - N_{\alpha}^{\beta}\left(A, \mathcal{M}, v, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)\\&\quad\subseteq \mathcal{I} - N_{\alpha}^{\beta}\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right).\end{aligned} \tag{47}$$

Proof. (i) Let $\xi \in \mathcal{I} - N_{\alpha}^{\beta}(A, \mathcal{M}, v, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Write $s_k = [\mathfrak{I}_k(\|A_k(\Delta^{(\gamma)}\xi)/\rho, x_1,\cdots,x_{n-1}\|)]^{v_k}$ and $\lambda_k = u_k/v_k$, so that $0 < \lambda < \lambda_k \leq 1$. By using Hölder inequality, we have

$$\begin{aligned}\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}(s_k)^{\lambda_k}\right]^{\beta} &= \frac{1}{\phi_r^{\alpha}}\left[\sum_{\substack{k\in J_r\\ s_k\geq 1}}(s_k)^{\lambda_k}\right]^{\beta} + \frac{1}{\phi_r^{\alpha}}\left[\sum_{\substack{k\in J_r\\ s_k<1}}(s_k)^{\lambda_k}\right]^{\beta}\\
&\leq \frac{1}{\phi_r^{\alpha}}\left[\sum_{\substack{k\in J_r\\ s_k\geq 1}}(s_k)\right]^{\beta} + \frac{1}{\phi_r^{\alpha}}\left[\sum_{\substack{k\in J_r\\ s_k<1}}(s_k)^{\lambda}\right]^{\beta}\\
&= \frac{1}{\phi_r^{\alpha}}\left[\sum_{\substack{k\in J_r\\ s_k\geq 1}}(s_k)\right]^{\beta} + \left[\sum_{\substack{k\in J_r\\ s_k<1}}\left(\frac{1}{\phi_r^{\alpha}}s_k\right)^{\lambda}\left(\frac{1}{\phi_r^{\alpha}}\right)^{1-\lambda}\right]^{\beta}\\
&\leq \frac{1}{\phi_r^{\alpha}}\left[\sum_{\substack{k\in J_r\\ s_k\geq 1}}(s_k)\right]^{\beta} + \left[\left(\sum_{\substack{k\in J_r\\ s_k<1}}\left[\left(\frac{1}{\phi_r^{\alpha}}s_k\right)^{\lambda}\right]^{1/\lambda}\right)^{\lambda}\right]^{\beta}\\
&\quad\cdot\left[\left(\sum_{\substack{k\in J_r\\ s_k<1}}\left[\left(\frac{1}{\phi_r^{\alpha}}\right)^{1-\lambda}\right]^{1/\lambda-1}\right)^{1-\lambda}\right]^{\beta}\\
&\leq \frac{1}{\phi_r^{\alpha}}\left[\sum_{\substack{k\in J_r\\ s_k\geq 1}}(s_k)\right]^{\beta} + \frac{1}{\phi_r^{\alpha}}\left[\sum_{\substack{k\in J_r\\ s_k<1}}(s_k)^{\lambda}\right]^{\beta}.\end{aligned} \tag{48}$$

Hence, for every $\varepsilon > 0$, we have

$$\begin{aligned}&\left\{r\in\mathbb{N} : \frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\geq\varepsilon\right\}\\
&\subset\left\{r\in\mathbb{N} : \frac{1}{\phi_r^{\alpha}}\left[\sum_{\substack{k\in J_r\\ s_k\geq 1}}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\geq\frac{\varepsilon}{2}\right\}\\
&\cup\left\{r\in\mathbb{N} : \frac{1}{\phi_r^{\alpha}}\left[\sum_{\substack{k\in J_r\\ s_k<1}}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1,\cdots,x_{n-1}\right\|\right)\right]^{v_k}\right]^{\beta}\geq\left(\frac{\varepsilon}{2}\right)^{1/\lambda}\right\}.\end{aligned} \tag{49}$$

This implies that $\{r\in\mathbb{N} : 1/\phi_r^{\alpha}[\sum_{k\in I_r}[\mathfrak{I}_k(\|A_k(\Delta^{(\gamma)}\xi)/\rho, x_1,\cdots,x_{n-1}\|)]^{u_k}]^{\beta}\geq\varepsilon\}\in\mathcal{I}$ and so $\xi\in\mathcal{I} - N_{\alpha}^{\beta}(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Hence, $\mathcal{I} - N_{\alpha}^{\beta}(A, \mathcal{M}, v, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0 \subseteq \mathcal{I} - N_{\alpha}^{\beta}(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Similarly, we can prove $\mathcal{I} - N_{\alpha}^{\beta}(A, \mathcal{M}, v, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|) \subseteq \mathcal{I} - N_{\alpha}^{\beta}(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)$. □

Corollary 9. *If $0 < \inf u_k \leq 1$. Then, the following inclusions hold:*

$$\begin{aligned}&\mathcal{I} - N_{\alpha}^{\beta}\left(A, \mathcal{M}, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0\\&\quad\subseteq \mathcal{I} - N_{\alpha}^{\beta}\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0\end{aligned} \tag{50}$$

$$\begin{aligned}&\mathcal{I} - N_{\alpha}^{\beta}\left(A, \mathcal{M}, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)\\&\quad\subseteq \mathcal{I} - N_{\alpha}^{\beta}\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right).\end{aligned} \tag{51}$$

Proof. The proof follows from Theorem 8. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to writing this paper. All authors read and approved the manuscript.

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Research Article

Spectral Radius Formulas Involving Generalized Aluthge Transform

Zhiqiang Zhang,[1] **Muhammad Saeed Akram**,[2] **Javariya Hyder**,[3] **Ghulam Farid**,[4] **and Tao Yan**[1]

[1]*School of Computer Science, Chengdu University, Chengdu, China*
[2]*Department of Mathematics, Faculty of Science, Ghazi University, 32200 Dera Ghazi Khan, Pakistan*
[3]*Department of Mathematics, Khwaja Fareed University of Engineering and Information Technology, Rahim Yar Khan, Pakistan*
[4]*Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan*

Correspondence should be addressed to Ghulam Farid; faridphdsms@hotmail.com

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In this paper, we aim to develop formulas of spectral radius for an operator S in terms of generalized Aluthge transform, numerical radius, iterated generalized Aluthge transform, and asymptotic behavior of powers of S. These formulas generalize some of the formulas of spectral radius existing in literature. As an application, these formulas are used to obtain several characterizations of normaloid operators.

1. Introduction

Generally, in mathematical analysis and particularly in functional analysis, the spectral analysis of operators is an essential research topic. It is useful to study the properties of operators, including spectrum and the spectral radius of operators (see [1]). The spectrum of an operator is connected with an invariant subspace problem on a complex Hilbert space (see [2]), and the important property of spectrum is the expression of spectral radius in various formulas (see [3–5]). These formulas help to obtain several characterizations of operators, including normaloid and spectraloid operators (see [6]). Since the advent of various transformations of bounded linear operators, including Aluthge transform and its generalizations, the study of spectral properties of operators has become the center point for many researchers (see [7–9]).

An operator can be decomposed into two Hermitian operators being its real and imaginary parts, and this decomposition is known as Cartesian decomposition. Clearly, Hermitian operators are self-adjoint and hence symmetric operators. The symmetric operators involved in Cartesian decomposition are helpful to develop the spectral radius formulas and numerical radius inequalities involving Aluthge transform [10–12].

This paper is aimed at studying the generalization of spectral radius formulas involving generalized Aluthge transform. Henceforward, we will give the notions to proceed with the results of this paper.

Let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators on complex Hilbert space H. Let $S = U|S|$ be the polar decomposition of $S \in \mathscr{B}(\mathscr{H})$, where $|S|$ is the square root of an operator defined as $|S| = \sqrt{S^* S}$ and U is a partial isometry.

In [13], Aluthge introduced a transform to study the properties of hyponormal operators that were connected with the invariant subspace problem in operator theory. This transform is called Aluthge transform, which is defined as

$$\Delta_{1/2} S = |S|^{1/2} U |S|^{1/2}, \tag{1}$$

and its nth iterated Aluthge transform is defined as

$$\begin{aligned} \Delta_{1/2}^{n}(S) &= \Delta\left(\Delta_{1/2}^{n-1}(S)\right), \\ \Delta_{1/2}^{1}(S) &= \Delta(S), \forall n \in \mathbb{N}. \end{aligned} \tag{2}$$

Yamazaki, in [3], gave the formula of spectral radius for bounded linear operator involving iterated Aluthge transform, i.e.,

$$r(S) = \lim_n \|\Delta^n_{1/2}(S)\|. \tag{3}$$

In [14], a generalization of Aluthge transform was introduced that is called λ-Aluthge transform which is defined as

$$\Delta_\lambda S = |S|^\lambda U |S|^{1-\lambda}, \lambda \in [0,1]. \tag{4}$$

Tam [4] gave a formula of spectral radius involving iterated λ-Aluthge transform for invertible operators using unitarily invariant norm, i.e.,

$$r(S) = \lim_n \|\Delta^n_\lambda(S)\|, \lambda \in (0,1). \tag{5}$$

Chabbabi and Mbekhta [12] gave various expressions for spectral radius formulas involving λ-Aluthge transform, iterated λ-Aluthge transform, asymptotic behavior of powers of an operator, and numerical radius. The expression of spectral radius involving λ-Aluthge transform is given by

$$r(S) = \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\|\Delta_\lambda\left(YSY^{-1}\right)\right\| = \inf_{\substack{A \in \mathscr{B}(\mathscr{H}) \\ \text{selfadjoint}}} \left\|\Delta_\lambda\left(e^A S e^{-A}\right)\right\|, \tag{6}$$

and the expressions of spectral radius involving iterated λ-Aluthge transform and the asymptotic behavior of powers of S are given by

$$r(S) = \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\|\Delta^n_\lambda\left(YSY^{-1}\right)\right\| = \inf_{\substack{A \in \mathscr{B}(\mathscr{H}) \\ \text{self-adjoint}}} \left\|\Delta^n_\lambda\left(e^A S e^{-A}\right)\right\|, \tag{7}$$

$$r(S) = \lim_k \left\|\Delta^n_\lambda\left(S^k\right)\right\|^{1/k} = \lim_k \left\|\Delta_\lambda\left(S^k\right)\right\|^{1/k}, \tag{8}$$

for each $n \geq 0$.

The expressions of spectral radius involving iterated λ-Aluthge transform, numerical radius, and the asymptotic behavior of powers of S are given by

$$r(S) = \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\|w\left(\Delta^n_\lambda\left(YSY^{-1}\right)\right)\right\| = \inf_{\substack{A \in \mathscr{B}(\mathscr{H}) \\ \text{self-adjoint}}} \left\|w\left(\Delta^n_\lambda\left(e^A S e^{-A}\right)\right)\right\|, \tag{9}$$

$$r(S) = \lim_k \left\|w\left(\Delta^n_\lambda\left(S^k\right)\right)\right\|^{1/k} = \lim_k \left\|w\left(\Delta_\lambda\left(S^k\right)\right)\right\|^{1/k}, \tag{10}$$

for each $n \geq 0$. With the help of the above formulas, the author [14] gave a characterization of normaloid operators.

In [15], Shebrawi and Bakherad introduced a new generalization of Aluthge transform, called generalized Aluthge transform. This transform is defined as

$$\Delta_{f,g} S = f(|S|) U g(|S|), \tag{11}$$

where f and g both are continuous functions such that $g(x)f(x) = x, x \geq 0$. The iterated generalized Aluthge transform is defined as

$$\Delta^n_{f,g}(S) = \Delta\left(\Delta^{n-1}_{f,g}(S)\right), \forall n \in \mathbb{N}. \tag{12}$$

In this paper, we establish the formulas of spectral radius for operator S by assuming that $\|\Delta_{f,g}(S)\| \leq \|S\|$. These formulas generalize the spectral radius formulas (6)–(10).

The paper is organized as follows. In Section 2, we give the properties of the generalized Aluthge transform. In Section 3, spectral radius formulas involving generalized Aluthge transform and asymptotic behavior of powers of the bounded operator S are given. In Section 4, we develop spectral radius formulas of bounded linear operators involving numerical radius of generalized Aluthge transform. Furthermore, some characterizations of normaloid operators are established.

2. Preliminaries and Some Auxiliary Results

We start this section with some basic definitions and properties of generalized Aluthge transform which will be useful in establishing the main results of this paper. An operator T is *similar* to S if there exists an invertible operator Y such that $S = Y^{-1}TY$ (see [16]). If $r(S) = \|S\|$, then the operator is said to be *normaloid*. An operator S is said to be a *contraction* if $\|S\| \leq 1$. The *spectral radius* of an operator S is defined as

$$r(S) = \sup \{|\lambda| : \lambda \in \sigma(S)\}, \tag{13}$$

where $\sigma(S)$ is the spectrum of the operator S.

To prove spectral radius formulas, we recall some properties of generalized Aluthge transform.

Proposition 1 [7]. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, we have*

(i) $\sigma(S) = \sigma(\Delta_{f,g}(S))$

(ii) $r(S) = r(\Delta_{f,g}(S))$

Proposition 2. *Let $T, S \in \mathscr{B}(\mathscr{H})$. If T is similar to S, then*

(i) $\sigma(S) = \sigma(T)$

(ii) $\sigma(\Delta_{f,g}(S)) = \sigma(\Delta_{f,g}(T))$

(iii) $r(\Delta_{f,g}(S)) = r(\Delta_{f,g}(T))$

Proof. The proofs of parts (i) and (iii) are trivial. The proof of part (ii) follows from part (i) and Proposition 1 (i). □

Proposition 3. *Let $S \in \mathscr{B}(\mathscr{H})$ and f be any continuous function on $\sigma(S)$. Then,*

$$f(U|S|U^*) = Uf(|S|)U^*, \tag{14}$$

for any unitary $U \in \mathscr{B}(\mathscr{H})$.

Proof. Since $U^* U|S| = |S|$, we have

$$(U|S|U^*)^n = U|S|^n U^*, \tag{15}$$

for each $n \in \mathbb{N}$, which implies

$$P(U|S|U^*) = UP(|S|)U^*, \tag{16}$$

for any polynomial $P(t)$. Since f is a continuous, so there exist a sequence of polynomial $\{P_n(t)\}_{n=1}^{\infty}$ such that $P_n(0) = 0$ for each $n \in \mathbb{N}$, and $\{P_n(t)\}_{n=1}^{\infty}$ converges uniformly to $f(t)$ on the interval $[0, \||T|\|]$. Then, from Equation (16), we have

$$\begin{aligned} f(U|S|U^*) &= \lim_{n\longrightarrow\infty} P_n(U|S|U^*) = \lim_{n\longrightarrow\infty} (UP_n(|S|)U^*) \\ &= U \lim_{n\longrightarrow\infty} P_n(|S|)U^* = Uf(|S|)U^*, \end{aligned} \tag{17}$$

as required. □

Proposition 4. *Let $S, U \in \mathscr{B}(\mathscr{H})$ such that U is unitary. Then, we have*

$$\Delta_{f,g}(USU^*) = U\Delta_{f,g}(S)U^*. \tag{18}$$

Proof. Let $S = V|S|$ be the polar decomposition of S. Then, we have

$$|USU^*| = U|S|U^*. \tag{19}$$

Now by using Proposition 3, we have

$$f(|USU^*|) = Uf(|S|)U^*. \tag{20}$$

The polar decomposition of operator USU^* is as follows:

$$\begin{aligned} USU^* &= UV|S|U^*, \\ USU^* &= (UVU^*)(U|S|U^*), \end{aligned} \tag{21}$$

where UVU^* is partial isometry. Therefore,

$$\begin{aligned} \Delta_{f,g}(USU^*) &= f((USU^*))UVU^* g((USU^*)) \\ &= Uf((S))Vg((S))U^* = U\Delta_{f,g}(S)U^*. \end{aligned} \tag{22}$$

The second equality holds by Proposition 3 and by the fact that $U^* U = I$. □

Proposition 5. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, the sequence $\{\|\Delta_{f,g}^n(S)\|\}_{n=1}^{\infty}$ is nonincreasing.*

Proof. The proof follows from the repeated application of the inequality

$$\left\|\Delta_{f,g}(S)\right\| \le \|S\|. \tag{23}$$

□

3. Formulas of Spectral Radius Involving Generalized Aluthge Transform

In this section, we give formulas of the spectral radius by using Rota's theorem [16] and the properties of generalized Aluthge transform.

Theorem 6. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, we have*

$$r(S) = \inf_{\substack{Y\in\mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| = \inf_{\substack{A\in\mathscr{B}(\mathscr{H}) \\ \text{self-adjoint}}} \left\|\Delta_{f,g}\left(e^A S e^{-A}\right)\right\|. \tag{24}$$

Proof. From Propositions 1 and 2, we have

$$r(S) = r\left(\Delta_{f,g}\left(YSY^{-1}\right)\right). \tag{25}$$

It follows that

$$r(S) = r\left(\Delta_{f,g}\left(YSY^{-1}\right)\right) \le \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| \text{ for invertible } Y \in \mathscr{B}(\mathscr{H}). \tag{26}$$

Hence,

$$r(S) \le \inf_{\substack{Y\in\mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\|. \tag{27}$$

Let $Y = U|Y|$ be the polar decomposition of Y. Since Y is an invertible operator, then U is unitary and $|Y|$ invertible. Therefore, there exists $\beta > 0$ such that $\sigma(|Y|) \subseteq [\beta,\infty)$. Consequently, $A = \ln(|Y|)$ exists and self-adjoint; then, we have

$$\begin{aligned} |Y| &= e^A, \\ |Y|^{-1} &= e^{-A}. \end{aligned} \tag{28}$$

Therefore,

$$\begin{aligned} \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| &= \left\|\Delta_{f,g}(U|Y|)S(U|Y|)^{-1}\right)\right\| \\ &= \left\|U\left(\Delta_{f,g}|Y|S|Y|^{-1}\right)U^*\right\| \\ &= \left\|U\left(\Delta_{f,g}\left(e^A S e^{-A}\right)U^*\right\| \\ &= \left\|\Delta_{f,g}\left(e^A S e^{-A}\right)\right\|. \end{aligned} \tag{29}$$

The second equality holds by Proposition 4. Hence,

$$r(S) \leq \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| \leq \inf_{\substack{A\in\mathscr{B}(\mathscr{H})\\ \text{self-adjoint}}} \left\|\Delta_{f,g}\left(e^{A}Se^{-A}\right)\right\|. \tag{30}$$

To prove above inequality in other direction, for an arbitrary $\varepsilon > 0$, we define an operator

$$S_{\varepsilon} = \frac{S}{r(S)+\varepsilon}. \tag{31}$$

For operator S_{ε}, we have

$$r(S_{\varepsilon}) = r\left(\frac{S}{r(S)+\varepsilon}\right) = \frac{r(S)}{r(S)+\varepsilon} < 1. \tag{32}$$

From [16], Theorem 2, the spectrum of operator S_{ε} lies in the unit disk; thus, the operator S_{ε} is similar to contraction for which there exists an invertible operator $Y_{\varepsilon} \in \mathscr{B}(\mathscr{H})$ such that

$$\left\|\frac{Y_{\varepsilon}SY_{\varepsilon}^{-1}}{r(S)+\varepsilon}\right\| < 1, \tag{33}$$

and this implies that

$$\left\|\Delta_{f,g}\left(e^{A_{\varepsilon}}Se^{-A_{\varepsilon}}\right)\right\| \leq \left\|Y_{\varepsilon}SY_{\varepsilon}^{-1}\right\| < r(S)+\varepsilon. \tag{34}$$

For $\varepsilon > 0$, we obtain

$$\begin{aligned} r(S) &\leq \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| \leq \inf_{\substack{A\in\mathscr{B}(\mathscr{H})\\ \text{self-adjoint}}} \left\|\Delta_{f,g}\left(e^{A}Se^{-A}\right)\right\| \\ &\leq \inf_{\substack{A_{\varepsilon}\in\mathscr{B}(\mathscr{H})\\ \text{self-adjoint}}} \left\|\Delta_{f,g}\left(e^{A_{\varepsilon}}Se^{-A_{\varepsilon}}\right)\right\| \leq \inf_{\substack{Y_{\varepsilon}\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} \left\|Y_{\varepsilon}SY_{\varepsilon}^{-1}\right\| \leq r(S)+\varepsilon. \end{aligned} \tag{35}$$

Since $\varepsilon > 0$ is arbitrary, therefore

$$r(S) = \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| = \inf_{\substack{A\in\mathscr{B}(\mathscr{H})\\ \text{self-adjoint}}} \left\|\Delta_{f,g}\left(e^{A}Se^{-A}\right)\right\|. \tag{36}$$

□

The next Corollary is the direct result of Theorem 6 involving iterated generalized Aluthge transform.

Corollary 7. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, for each $n \in \mathbb{N}$, we have*

$$r(S) = \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \textit{invertible}}} \left\|\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right\| = \inf_{\substack{A\in\mathscr{B}(\mathscr{H})\\ \textit{self-adjoint}}} \left\|\Delta_{f,g}^{n}\left(e^{A}Se^{-A}\right)\right\|. \tag{37}$$

Proof. From Propositions 1 and 2, we can easily obtain

$$r\left(\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right) = r(S), \forall n \in \mathbb{N}. \tag{38}$$

From above equality and by using Proposition 5, we have

$$r(S) \leq \left\|\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right\| \leq \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\|, \tag{39}$$

for all invertible $Y \in \mathscr{B}(\mathscr{H})$. Therefore,

$$\begin{aligned} r(S) &\leq \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} \left\|\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right\| \leq \inf_{\substack{A\in\mathscr{B}(\mathscr{H})\\ \text{self-adjoint}}} \left\|\Delta_{f,g}^{n}\left(e^{A}Se^{-A}\right)\right\| \\ &\leq \inf_{\substack{A\in\mathscr{B}(\mathscr{H})\\ \text{self-adjoint}}} \left\|\Delta_{f,g}\left(e^{A}Se^{-A}\right)\right\| = r(S). \end{aligned} \tag{40}$$

The third inequality holds by Proposition 5, and the last equality holds by Theorem 6, which completes the proof. □

The next Corollary is the direct result of Corollary 7 that is the characterization of normaloid operators.

Corollary 8. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, the following assertions are equivalent*

(i) S is normaloid

(ii) $\|S\| \leq \|YSY^{-1}\|$, for invertible $Y \in \mathscr{B}(\mathscr{H})$

Proof. Assume that S is normaloid. Then,

$$\|S\| = r\left(YSY^{-1}\right) \leq \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| \leq \left\|YSY^{-1}\right\|, \tag{41}$$

for all invertible $Y \in \mathscr{B}(\mathscr{H})$. The first equality holds by Proposition 2. The first inequality holds because the spectral radius is less than the operator norm, and the second inequality holds by Proposition 5.

Assume that assertion (ii) holds. Then, we have

$$r(S) \leq \|S\| \leq \left\|YSY^{-1}\right\| \leq \left\|Y\varepsilon SY\varepsilon^{-1}\right\| \leq r(S)+\varepsilon, \tag{42}$$

for all invertible $Y \in \mathscr{B}(\mathscr{H})$. The last inequality holds by inequality (33) in Theorem 6. Since $\varepsilon > 0$ is arbitrary, hence S is normaloid. □

Corollary 9. *Let $S \in \mathscr{B}(\mathscr{H})$. Then the following assertions are equivalent.*

(i) S is normaloid;

(ii) $\|S\| \leq \|\Delta_{f,g}(YSY^{-1})\|$ for invertible $Y \in \mathscr{B}(\mathscr{H})$;

(iii) $\|S\| \leq \|\Delta_{f,g}^{n}(YSY^{-1})\|$ for invertible $Y \in \mathscr{B}(\mathscr{H})$ and every $n \in \mathbb{N}$.

Proof. (i)⇒(iii)⇒(ii). Since S is normaloid, therefore

$$\|S\| = r\left(\Delta_{f,g}^{n} YSY^{-1}\right) \le \left\|\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right\| \le \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\|, \tag{43}$$

for all invertible $Y \in \mathscr{B}(\mathscr{H})$. The first inequality holds because the spectral radius is less than the operator norm, and the second inequality holds by Proposition 5. Hence,

$$\begin{aligned} &\|S\| \le \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| \text{ for invertible } Y \in \mathscr{B}(\mathscr{H}), \\ &\|S\| \le \left\|\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right\| \text{ for invertible } Y \in \mathscr{B}(\mathscr{H}). \end{aligned} \tag{44}$$

(ii)⇒(i)

Since spectral radius is less than operator norm and by assertion (ii), we have

$$r(S) \le \|S\| \le \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| \le \left\|\Delta_{f,g}\left(Y_\varepsilon SY_\varepsilon^{-1}\right)\right\| \le \left\|Y_\varepsilon SY_\varepsilon^{-1}\right\| \le r(S) + \varepsilon, \tag{45}$$

for all invertible $Y \in \mathscr{B}(\mathscr{H})$. The third inequality holds by inequality (34) of Theorem 6. Since $\varepsilon > 0$ is arbitrary, therefore S is normaloid. □

Now, we will give a formula of spectral radius involving iterated generalized Aluthge transform and asymptotic behavior of powers of S.

Theorem 10. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, we have*

$$r(S) = \lim_k \left\|\Delta_{f,g}^{n}\left(S^k\right)\right\|^{1/k}, \forall n \in \mathbb{N} = \lim_k \left\|\Delta_{f,g}\left(S^k\right)\right\|^{1/k}. \tag{46}$$

Proof.

$$r(S) = r\left(\Delta_{f,g}^{n}(S)\right) \le \left\|\Delta_{f,g}^{n}(S)\right\| \le \left\|\Delta_{f,g}(S)\right\| \le \|S\|, \forall n \in \mathbb{N}. \tag{47}$$

The first equality holds by Proposition 1, second inequality holds by $r(S) \le \|S\|$, and third inequality holds by Proposition 5. Thus, for kth power of an operator, we have

$$\begin{aligned} r(S)^k = r\left(S^k\right) = r\left(\Delta_{f,g}^{n}\left(S^k\right)\right) &\le \left\|\Delta_{f,g}^{n}\left(S^k\right)\right\| \\ &\le \left\|\Delta_{f,g}\left(S^k\right)\right\| \le \left\|S^k\right\|, \forall n, k \in \mathbb{N}, \\ r(S) \le \left\|\Delta_{f,g}^{n}\left(S^k\right)\right\|^{1/k} \le \left\|\Delta_{f,g}\left(S^k\right)\right\|^{1/k} &\le \left\|S^k\right\|^{1/k}, \forall n, k \in \mathbb{N}, \\ r(S) \le \lim_k \left\|\Delta_{f,g}^{n}\left(S^k\right)\right\|^{1/k} \le \lim_k \left\|\Delta_{f,g}\left(S^k\right)\right\|^{1/k} &\le \lim_k \left\|S^k\right\|^{1/k}, \forall n \in \mathbb{N}. \end{aligned} \tag{48}$$

Since

$$r(S) = \lim_k \left\|S^k\right\|^{1/k}. \tag{49}$$

Thus,

$$\begin{aligned} r(S) &\le \lim_k \left\|\Delta_{f,g}^{n}\left(S^k\right)\right\|^{1/k} \le \lim_k \left\|\Delta_{f,g}\left(S^k\right)\right\|^{1/k} \\ &\le \lim_k \left\|S^k\right\|^{1/k} = r(S), \forall n \in \mathbb{N}, \end{aligned} \tag{50}$$

which completes the proof. □

The next Corollary is obtain in the consequence of Theorem 10.

Corollary 11. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, the following assertions are equivalent.*

(i) S is normaloid

(ii) $\|S\|^k = \|\Delta_{f,g}(S^k)\|$, $\forall k \in \mathbb{N}$

(iii) $\|S\|^k = \|\Delta_{f,g}^{n}(S^k)\|$, $\forall n, k \in \mathbb{N}$

Proof. (i)⇒(ii).

$$\begin{aligned} \|S\| &= \lim_k \left\|\Delta_{f,g}(S)^k\right\|^{1/k}, \\ \|S\|^k &= \left(\lim_k \left\|\Delta_{f,g}(S)^k\right\|^{1/k}\right)^k, \\ \|S\|^k &= \left\|\Delta_{f,g}(S)^k\right\|, \forall k \in \mathbb{N}. \end{aligned} \tag{51}$$

The first equality holds by assertion (i) and Theorem 10.
(i)⇒(iii)

$$\begin{aligned} \|S\| &= \lim_k \left\|\Delta_{f,g}^{n}(S)^k\right\|^{1/k}, \forall n \in \mathbb{N}, \\ \|S\|^k &= \left\|\Delta_{f,g}^{n}(S)^k\right\|, \forall n, k \in \mathbb{N}. \end{aligned} \tag{52}$$

The first equality holds by assertion (i) and Theorem 10.
(ii)⇒(i)

$$\begin{aligned} \|S\|^k &= \left\|\Delta_{f,g}(S)^k\right\|, \forall k \in \mathbb{N}, \\ \left(\|S\|^k\right)^{1/k} &= \left\|\Delta_{f,g}(S)^k\right\|^{1/k}, \forall k \in \mathbb{N}, \\ \lim_k \|S\| &= \lim_k \left\|\Delta_{f,g}(S)^k\right\|^{1/k}, \\ r(S) &= \|S\|. \end{aligned} \tag{53}$$

The last equality holds by Theorem 10.

(iii)⇒(i)

$$\begin{aligned} \|S\|^k &= \left\| \Delta^n_{f,g}(S)^k \right\|, \forall n, k \in \mathbb{N}, \\ \lim_k \|S\| &= \lim_k \left\| \Delta^n_{f,g}(S)^k \right\|^{1/k}, \forall n \in \mathbb{N}, \\ \|S\| &= r(S). \end{aligned} \tag{54}$$

The last equality holds by Theorem 10. Hence, S is normaloid. □

4. Formulas of Spectral Radius Involving Generalized Aluthge Transform and Numerical Radius

This section gives spectral radius formulas for the bounded linear operator in terms of numerical radius and iterated generalized Aluthge transform. The numerical radius is defined as

$$w(S) = \sup \{|\lambda|: \lambda \in W(S)\}, \tag{55}$$

where $W(S)$ is the numerical range.

Theorem 12. *Let $S \in \mathcal{B}(\mathcal{H})$. Then, for all $n \in \mathbb{N}$, we have*

$$r(S) = \inf_{\substack{Y \in \mathcal{B}(\mathcal{H}) \\ \text{invertible}}} w\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) = \inf_{\substack{A \in \mathcal{B}(\mathcal{H}) \\ \text{self-adjoint}}} w\left(\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right). \tag{56}$$

Proof. As we know that

$$r(S) \le w(S) \le \|S\|. \tag{57}$$

Thus, for every invertible operator $Y \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned} r(S) &= r\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) \le w\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) \\ &\le \left\|\Delta^n_{f,g}\left(YSY^{-1}\right)\right\|, \forall n \in \mathbb{N}. \end{aligned} \tag{58}$$

Let Y be any bounded linear invertible operator with polar decomposition $Y = U|Y|$. Since Y is an invertible operator, then U is unitary and $|Y|$ is also invertible and positive. Thus, there exists $\beta > 0$ such that $\sigma(|Y|) \subseteq [\beta, \infty)$. So, $A = \ln(|Y|)$ exists and self-adjoint. Thus, we have

$$\begin{aligned} |Y| &= e^A, \\ |Y|^{-1} &= e^{-A}. \end{aligned} \tag{59}$$

Therefore,

$$\begin{aligned} W\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) &= \left\langle \Delta^n_{f,g}\left(YSY^{-1}\right)x, x \right\rangle \\ &= \left\langle \Delta^n_{f,g}\left((U|Y|)S(U|Y|)^{-1}\right)x, x \right\rangle \\ &= \left\langle \Delta^n_{f,g}\left((U|Y|)S|Y|^{-1}U^*\right)x, x \right\rangle \\ &= \left\langle \Delta^n_{f,g}\left(|Y|S|Y|^{-1}\right)U^*x, U^*x \right\rangle \\ &= \left\langle \Delta^n_{f,g}\left(e^A S e^{-A}\right)\frac{U^*x}{\|U^*x\|}, \frac{U^*x}{\|U^*x\|} \right\rangle \cdot \langle UU^*x, x\rangle. \end{aligned} \tag{60}$$

The second equality holds by $Y = U|Y|$, third equality holds because U is unitary, and fourth equality holds by Proposition 4. Thus,

$$W\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) \subseteq W\left(\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right) W(UU^*). \tag{61}$$

In the above equation, U is unitary. This implies that

$$w\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) \le w\left(\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right). \tag{62}$$

It follows that

$$\begin{aligned} r(S) &= r\left(\Delta^n_{f,g} YSY^{-1}\right) \le w\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right), \text{for invertible } Y \in \mathcal{B}(\mathcal{H}) \\ &\le w\left(\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right), \text{for self-adjoint } A \in \mathcal{B}(\mathcal{H}) \\ &\le \left\|\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right\|, \text{for self-adjoint } A \in \mathcal{B}(\mathcal{H}). \end{aligned} \tag{63}$$

For every invertible $Y \in \mathcal{B}(\mathcal{H})$, all above inequalities are satisfied; thus, we have

$$\begin{aligned} r(S) &\le \inf_{\substack{Y \in \mathcal{B}(\mathcal{H}) \\ \text{invertible}}} w\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) \\ &\le \inf_{\substack{A \in \mathcal{B}(\mathcal{H}) \\ \text{self-adjoint}}} w\left(\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right) \\ &\le \inf_{\substack{A \in \mathcal{B}(\mathcal{H}) \\ \text{self-adjoint}}} \left\|\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right\| = r(S). \end{aligned} \tag{64}$$

The last equality holds by Corollary 7, which completes the proof. □

Let A be any bounded linear operator with cartesian decomposition

$$A = \frac{A + A^*}{2} + \frac{A - A^*}{2i}. \tag{65}$$

In this decomposition $1/2(A + A^*)$ is the real part and $1/2i(A - A^*)$ is the imaginary part.

In [17], the spectrum of a bounded linear operator is contained in the closure of the numerical range.

Theorem 13. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, for all $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$, we have*

$$\begin{aligned} r(S) &= \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} w\left(\operatorname{Re}\left(e^{i\theta}\left(\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right)\right)\right) \\ &= \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\|\operatorname{Re}\left(e^{i\theta}\left(\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right)\right)\right\|. \end{aligned} \tag{66}$$

Proof. Let $r(S) \in \sigma(\mathcal{S})$. Then,

$$r(S) \in \operatorname{Re}\left(\sigma(S)\right) = \operatorname{Re}\left(\sigma\left(\Delta_{f,g}^{n} YSY^{-1}\right)\right), \text{for invertible operator } Y \in \mathscr{B}(\mathscr{H}). \tag{67}$$

Thus,

$$\begin{aligned} r(S) &\in \operatorname{Re}\left(\sigma\left(\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right)\right) \subseteq \operatorname{Re}\left(\bar{W}\left(\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right)\right) \\ &= \bar{W}\left(\operatorname{Re}\left(\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right)\right), \end{aligned} \tag{68}$$

which implies

$$r(S) \leq w\left(\operatorname{Re}\left(\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right)\right) \leq \left\|\operatorname{Re}\left(\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right)\right\| \leq \left\|\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right\|, \tag{69}$$

for all invertible $Y \in \mathscr{B}(\mathscr{H})$. Thus, we have

$$\begin{aligned} r(S) &\leq \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} w\left(\operatorname{Re}\left(\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right)\right) \\ &\leq \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\|\operatorname{Re}\left(\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right)\right\| \\ &\leq \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\|\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right\| = r(S). \end{aligned} \tag{70}$$

The last equality holds by Corollary 7. For $r(S) \in \sigma(\mathcal{S})$, we have proved

$$\begin{aligned} r(S) &= \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} w\left(\operatorname{Re}\left(\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right)\right) \\ &= \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\|\operatorname{Re}\left(\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right)\right\|. \end{aligned} \tag{71}$$

If S is an arbitrary operator, then there exists $z \in \sigma(S)$ such that $|z| = r(S)$. Put $\theta = -\arg(z)$. Then, $r(S) = ze^{i\theta} \in \sigma(e^{i\theta}S)$. Hence, by the first part of the proof, we conclude that

$$\begin{aligned} r(S) = r\left(e^{i\theta}S\right) &\leq \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} w\left(\operatorname{Re}\left(\Delta_{f,g}^{n}\left(e^{i\theta}\left(YSY^{-1}\right)\right)\right)\right) \\ &\leq \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\|\operatorname{Re}\left(\Delta_{f,g}^{n}\left(e^{i\theta}\left(YSY^{-1}\right)\right)\right)\right\| \\ &\leq \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\|\Delta_{f,g}^{n}\left(e^{i\theta}\left(YSY^{-1}\right)\right)\right\| = r\left(e^{i\theta}S\right). \end{aligned} \tag{72}$$

The last inequality holds by Corollary 7, which completes the proof. □

The next Corollary is the characterization of normaloid operators.

Corollary 14. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, for each $n \in \mathbb{N}$, the following assertions are equivalent:*

(i) S is normaloid

(ii) There exists $\theta \in \mathbb{R}$ such that for any invertible $Y \in \mathscr{B}(\mathscr{H})$

$$\|S\| \leq w\left(\operatorname{Re}\left(\Delta_{f,g}^{n}\left(e^{i\theta}YSY^{-1}\right)\right)\right) \tag{73}$$

(iii) There exists $\theta \in \mathbb{R}$ such that for any invertible $Y \in \mathscr{B}(\mathscr{H})$

$$\|S\| \leq \left\|\operatorname{Re}\left(\Delta_{f,g}^{n}\left(e^{i\theta}YSY^{-1}\right)\right)\right\| \tag{74}$$

Theorem 15. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, we have*

$$r(S) = \lim_{k} w\left(\Delta_{f,g}^{n}\left(S^{k}\right)\right)^{1/k}, \forall n \in \mathbb{N}. \tag{75}$$

Proof. Since $r(S) \leq w(S) \leq \|S\|$, therefore

$$\begin{aligned} r(S)^{k} = r\left(S^{k}\right) = r\left(\Delta_{f,g}^{n}\left(S^{k}\right)\right) &\leq w\left(\Delta_{f,g}^{n}\left(S^{k}\right)\right) \\ &\leq \left\|\Delta_{f,g}^{n}\left(S^{k}\right)\right\|, \forall n, k \in \mathbb{N}. \end{aligned}$$

$$r(S) \leq \left(w\left(\Delta_{f,g}^{n}\left(S^{k}\right)\right)\right)^{1/k} \leq \left\|\Delta_{f,g}^{n}\left(S^{k}\right)\right\|^{1/k}, \forall n, k \in \mathbb{N}. \tag{76}$$

By Theorem 10, we obtain

$$r(S) \le \lim_k \left(w\left(\Delta^n_{f,g}\left(S^k\right)\right)\right)^{1/k} \le \lim_k \left\| \Delta^n_{f,g}\left(S^k\right)\right\|^{1/k} = r(S), \forall n \in \mathbb{N}, \tag{77}$$

which completes the proof. □

Data Availability

There is no any data required for this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Ulam Stability and Non-Stability of Additive Functional Equation in IFN-Spaces and 2-Banach Spaces by Different Methods

N. Uthirasamy,[1] **K. Tamilvanan**,[2] **and Masho Jima Kabeto**[3]

[1] *Department of Mathematics, K.S. Rangasamy College of Technology, Tiruchengode 637 215, Tamil Nadu, India*
[2] *Department of Mathematics, School of Advanced Sciences, Kalasalingam Academy of Research and Education, Srivilliputhur 626 126, Tamil Nadu, India*
[3] *Department of Mathematics, College of Natural Sciences, Jimma University, Jimma, Ethiopia*

Correspondence should be addressed to Masho Jima Kabeto; masho.jima@ju.edu.et

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This paper introduces a new dimension of an additive functional equation and obtains its general solution. The main goal of this study is to examine the Ulam stability of this equation in IFN-spaces (intuitionistic fuzzy normed spaces) with the help of direct and fixed point approaches and 2-Banach spaces. Also, we use an appropriate counterexample to demonstrate that the stability of this equation fails in a particular case.

1. Introduction

The study of stability problems for functional equations is one of the essential research areas in mathematics, which originated in issues related to applied mathematics. The first question concerning the stability of homomorphisms was given by Ulam [1] as follows.

Given a group $(G, *)$, a metric group $(G', \cdot)$ with the metric d, and a mapping f from G and G', does $\delta > 0$ exist such that

$$d(f(x * y), f(x) \cdot f(y)) \leq \delta, \tag{1}$$

for all $x, y \in G$. If such a mapping exists, then does a homomorphism $h: G \longrightarrow G'$ exist such that

$$d(f(x), h(x)) \leq \varepsilon, \tag{2}$$

for all $x \in G$? Ulam defined such a problem in 1940 and solved it the following year for the Cauchy functional equation

$$\psi(u + v) = \psi(u) + \psi(v), \tag{3}$$

by the way of Hyers [2]. The consequence of Hyers becomes stretched out by Aoki [3] with the aid of assuming the unbounded Cauchy contrasts. Hyers theorem for additive mapping was investigated by Rassias [4], and then Rassias results were generalized by Gavruta [5].

As of late, Nakmahachalasint [6] gave the overall answer and HUR (briefly, Hyers–Ulam–Rassias) stability of finite variable functional equation; furthermore, Khodaei and Rassias [7] examined the stability of generalized additive functions in several variables. The stability result of additive functional equations was examined by means of Najati and Moghimi [8], Shin et al. [9], and Gordji [10]. Stability problems of various functional equations have been investigated by many researchers, and there are various interesting results about this problem (see [11–14]).

Zadeh [15] established the concept of fuzzy sets, which is a tool for demonstrating weakness and ambiguity in several scientific and technological problems. The possibility of IFN-spaces, from the start, has been presented in [16]. Saadati [17] have examined the modified intuitionistic fuzzy metric spaces and proven some fixed point theorems in these spaces.

The IFN-spaces and IF2N-spaces (briefly, intuitionistic fuzzy 2-normed spaces) have been studied by a number of researchers [18–20]. Furthermore, several researchers have discussed the generalized Ulam–Hyers stability of various functional equations in IFN-spaces (see [21–24]).

In this current work, we present a new kind of additive functional equation:

$$\sum_{1\le a<b<c\le s}\phi\left(-v_a-v_b-v_c+\sum_{d=1,d\neq a\neq b\neq c}^{s}v_d\right)-\left(\frac{s^3-9s^2+20s-12}{6}\right)\sum_{a=1}^{s}\left[\frac{\phi(v_a)-\phi(-v_a)}{2}\right]=0, \quad (4)$$

where $s>4$ is a fixed integer, and obtain its general solution. The main goal of this study is to examine the Ulam–Hyers stability of this equation in IFN-spaces with the help of direct and fixed point approaches and 2-Banach spaces by using the direct approach. Also, we use an appropriate counter-example to demonstrate that the stability of equation (4) fails in a particular case.

2. General Solution

Theorem 1. *If a mapping ϕ between two real vector spaces W and F satisfies functional equation (4), then the function ϕ is additive.*

Proof. Setting $v_1=\cdots=v_s=0$ in (4), we have $\phi(0)=0$. Replacing $(v_1,v_2,\ldots,v_s)$ by $(v,\underbrace{0,0,\ldots,0}_{(s-1)-\text{times}})$ in (4), we get $\phi(-v)=-\phi(v)$ for all $v\in W$. Hence, ϕ is an odd function. Replacing $(v_1,v_2,v_3,\ldots,v_s)$ by $(v,v,\underbrace{0,0,\ldots,0}_{(s-2)-\text{times}})$ in (4), we have

$$\phi(2v)=2\phi(v), \quad (5)$$

for all $v\in W$. Replacing v by $2v$ in (5), we have

$$\phi\left(2^2v\right)=2^2\phi(v), \quad (6)$$

for all $v\in W$. Again, replacing v with $2v$ in (6), we get

$$\phi\left(2^3v\right)=2^3\phi(v), \quad (7)$$

for all $v\in W$. In general, for any non-negative integer $a>0$, we have

$$\phi(2^av)=2^a\phi(v), \quad (8)$$

for all $v\in W$. Replacing $(v_1,v_2,v_3,\ldots,v_s)$ by $(s,t,\underbrace{0,0,\ldots,0}_{(s-2)-\text{times}})$ in (4), we obtain (3) for all $s,t\in W$. □

Remark 1. If a mapping ϕ between two real vector spaces W and F satisfies functional equation (3), then the function ϕ satisfies additive functional equation (4), for all $v_1,v_2,v_3,\ldots,v_s\in W$.

For our notational handiness, we define a mapping $\phi\colon W\longrightarrow F$ by

$$D\phi(v_1,v_2,\ldots,v_s)=\sum_{1\le a<b<c\le s}\phi\left(-v_a-v_b-v_c+\sum_{d=1,d\neq a\neq b\neq c}^{s}v_d\right)-\left(\frac{s^3-9s^2+20s-12}{6}\right)\sum_{a=1}^{s}\left[\frac{\phi(v_a)-\phi(-v_a)}{2}\right], \quad (9)$$

for all $v_1,v_2,\ldots,v_s\in W$.

3. Stability Results in IFN-Spaces

We can recall some basic notions and preliminaries from [25] and using the alternative fixed point theorem which are important results in fixed point theory [26].

Definition 1 (see [25]). Consider a membership degree μ and non-membership degree ν of an intuitionistic fuzzy set from $W\times(0,+\infty)$ to $[0,1]$ such that $\mu_v(t)+\nu_v(t)\le 1$ for all $v\in W$ and $t>0$. The triple $(W,I_{\mu,\nu},\Upsilon)$ is called as an Intuitionistic Fuzzy Normed-space (briefly, IFN-space) if a vector space W, a continuous t-representable Υ and $I_{\mu,\nu}\colon W\times(0,+\infty)\longrightarrow L^*$ satisfying $v_1,v_2\in W$ and $t,s>0$,

(IFN1) $I_{\mu,\nu}(v_1,0)=0_{L^*}$.

(IFN2) $I_{\mu,\nu}(v_1,t)=1_{L^*}$ if and only if $v_1=0$.

(IFN3) $I_{\mu,\nu}(\alpha v_1,t)=I_{\mu,\nu}(v_1,(t/|\alpha|))$, for all $\alpha\neq 0$.

(IFN4) $I_{\mu,\nu}(v_1+v_2,t+s)\ge_{L^*}Y(I_{\mu,\nu}(v_1,t),I_{\mu,\nu}(v_2,s))$.

In this case, $I_{\mu,\nu}$ is called an intuitionistic fuzzy norm, where $I_{\mu,\nu}(v_1,t)=(\mu_{v_1}(t),\nu_{v_1}(t))$.

Definition 2 (see [25]). A sequence $\{v_m\}$ in W is called as a Cauchy sequence if for every $\epsilon>0$ and $t>0$, there exists m_0 such that

$$I_{\mu,\nu}\left(v_{m+p}-v_m,t\right)>1-\varepsilon,\quad m\ge m_0, \quad (10)$$

for all $p>0$.

Remark 2. In an intuitionistic fuzzy normed space, every convergent sequence is a Cauchy sequence.

If every Cauchy sequence is convergent, then the intuitionistic fuzzy normed space is called as complete.

Definition 3 (see [25]). A mapping ϕ between two IFN-spaces W and F is continuous at v_0 if for every $\{v_m\}$ converging to v_0 in W, the sequence $\phi\{v_m\}$ converges to $\phi\{v_0\}$. If

ϕ is continuous at each point $v_0 \in W$, then the mapping ϕ is called as a continuous mapping on W.

Example 1. Let $(W, \|\cdot\|)$ be a normed space. Let $T(a,b) = (a, b, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$; $b = (b_1, b_2) \in L^*$ and μ, ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$I_{\mu,\nu}(v,t) = (\mu_v(t), \nu_v(t)) = \left(\frac{t}{t + \|v\|}, \frac{\|v\|}{t + \|v\|}\right), \ t \in R^+. \quad (11)$$

Then, $(W, I_{\mu,\nu}, T)$ is an IFN-space.

Theorem 2 (see [26]). *Let (W, d) be a generalized complete metric space and a strictly contractive mapping $M: W \longrightarrow W$ with Lipschitz constant $L < 1$. Then, for all $v_1 \in W$, either*

$$d\left(M^m v_1, M^{m+1} v_1\right) = \infty, \quad m \geq m_0, \quad (12)$$

or there exists a positive integer m_0 such that

(i) $d(M^m v_1, M^{m+1} v_1) < \infty$, $m \geq m_0$.

(ii) The sequence $\{M^m v_1\}_{m\in\mathbb{N}}$ converges to a fixed point v_1^* of M.

(iii) v_1^* is the unique fixed point of M in $W^* = \{v_2 \in W | d(M^{m_0} v_1, v_2) < \infty\}$.

(iv) $d(v_2, v_1^*) \leq (1/1 - L) d(M v_2, v_2)$, for all $v_2 \in W^*$.

3.1. Stability Results: Direct Technique. In this section, we assume that W, $(Z, I'_{\mu,\nu}, Y)$, and $(F, I_{\mu,\nu}, Y)$ are linear space, IFN-space, and complete IFN-space, respectively.

Theorem 3. *If a mapping $\varphi: W^s \longrightarrow Z$ with $0 < (\varsigma/2) < 1$,*

$$I'_{\mu,\nu}(\varphi(2v, 2v, 0, \ldots, 0), \varepsilon) \geq_{L^*} I'_{\mu,\nu}(\varsigma\varphi(v, v, 0, \ldots, 0), \varepsilon), \quad (13)$$

$$\lim_{k\longrightarrow\infty} I'_{\mu,\nu}\left(\varphi\left(2^k v_1, 2^k v_2, \ldots, 2^k v_s\right), 2^k \varepsilon\right) = 1_{L^*}, \quad (14)$$

for all $v, v_1, v_2, \ldots, v_s \in W$ and all $\epsilon > 0$. If a mapping $\phi: W \longrightarrow F$ satisfies

$$I_{\mu,\nu}(D\phi(v_1, v_2, \ldots, v_s), \varepsilon) \geq_{L^*} I'_{\mu,\nu}(\varphi(v_1, v_2, \ldots, v_s), \varepsilon), \quad (15)$$

for all $v_1, v_2, \ldots, v_s \in W$ and all $\epsilon > 0$, then the limit

$$I_{\mu,\nu}\left(A_1(v) - \frac{\phi\left(2^k v\right)}{2^k}, \epsilon\right) \longrightarrow 1_{L^*} \text{ as } k \longrightarrow \infty, \quad (16)$$

exists and there exists a unique additive mapping $A_1: W \longrightarrow F$ satisfying functional equation (4) and

$$I_{\mu,\nu}(\phi(v) - A_1(v), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(\varphi(v, v, 0, \ldots, 0), \left(\frac{s^3 - 9s^2 + 20s - 12}{6}\right)\epsilon(2 - \varsigma)\right), \quad (17)$$

for all $v \in W$ and all $\epsilon > 0$.

Proof. Fix $v \in W$ and all $\epsilon > 0$. Replacing $(v_1, v_2, \ldots, v_s)$ by $(v, v, 0, \ldots, 0)$ in (15), we have

$$I_{\mu,\nu}\left(\left(\frac{s^3 - 9s^2 + 20s - 12}{6}\right)\phi(2v) - \left(\frac{2\left(s^3 - 9s^2 + 20s - 12\right)}{6}\right)\phi(v), \varepsilon\right) \geq_{L^*} I'_{\mu,\nu}(\varphi(v, v, 0, \ldots, 0), \varepsilon). \quad (18)$$

Replacing v by $2^k v$ in (18) and using (IFN3), we obtain

$$I_{\mu,\nu}\left(\frac{\phi\left(2^{k+1} v\right)}{2} - \phi\left(2^k v\right), \left(\frac{6\epsilon}{2\left(s^3 - 9s^2 + 20s - 12\right)}\right)\right) \geq_{L^*} I'_{\mu,\nu}\left(\varphi\left(2^k v, 2^k v, 0, \ldots, 0\right), \epsilon\right). \quad (19)$$

By the inequality (13) and (IFN3) in (19), we have

$$I_{\mu,\nu}\left(\frac{\phi\left(2^{k+1} v\right)}{2} - \phi\left(2^k v\right), \left(\frac{6\epsilon}{2\left(s^3 - 9s^2 + 20s - 12\right)}\right)\right) \geq_{L^*} I'_{\mu,\nu}\left(\varphi(v, v, 0, \ldots, 0), \frac{\epsilon}{\varsigma^k}\right). \quad (20)$$

Clearly, we can show from inequality (20) that

$$I_{\mu,\nu}\left(\frac{\phi(2^{k+1}\nu)}{2^{k+1}}-\frac{\phi(2^k\nu)}{2^k},\left(\frac{6\epsilon}{2^{k+1}(s^3-9s^2+20s-12)}\right)\right)\geq_{L^*} I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\frac{\epsilon}{\varsigma^k}\right). \quad (21)$$

Replacing ϵ by $\varsigma^k\epsilon$ in (21), we get

$$I_{\mu,\nu}\left(\frac{\phi(2^{k+1}\nu)}{2^{k+1}}-\frac{\phi(2^k\nu)}{2^k},\left(\frac{6\varsigma^k\epsilon}{2^{k+1}(s^3-9s^2+20s-12)}\right)\right)\geq_{L^*} I'_{\mu,\nu}(\varphi(\nu,\nu,0,\ldots,0),\epsilon). \quad (22)$$

Clearly,

$$\frac{\phi(2^k\nu)}{2^k}-\phi(\nu)=\sum_{a=0}^{k-1}\frac{\phi(2^{a+1}\nu)}{2^{a+1}}-\frac{\phi(2^a\nu)}{2^a}. \quad (23)$$

It follows from (22) and (23) that

$$\begin{aligned}I_{\mu,\nu}&\left(\frac{\phi(2^k\nu)}{2^k}-\phi(\nu),\sum_{a=0}^{k-1}\frac{6\varsigma^a\epsilon}{2^{a+1}(s^3-9s^2+20s-12)}\right)\\&\geq_{L^*}\Upsilon_{a=0}^{k-1}\left\{I'_{\mu,\nu}\left(\frac{\phi(2^{a+1}\nu)}{2^{a+1}}-\frac{\phi(2^a\nu)}{2^a},\frac{6\varsigma^a\epsilon}{2^{a+1}(s^3-9s^2+20s-12)}\right)\right\}\\&\geq_{L^*}\Upsilon_{a=0}^{k-1}\left\{I'_{\mu,\nu}(\varphi(\nu,\nu,0,\ldots,0),\epsilon)\right\}\\&\geq_{L^*}I'_{\mu,\nu}(\varphi(\nu,\nu,0,\ldots,0),\epsilon),\end{aligned} \quad (24)$$

for all $\nu\in W$ and $\epsilon>0$. Replacing ν by $2^t\nu$ in (24) and with the help of (13), we have

$$I_{\mu,\nu}\left(\frac{\phi(2^{k+t}\nu)}{2^{k+t}}-\frac{\phi(2^t\nu)}{2^t},\sum_{a=0}^{k-1}\frac{6\varsigma^a\epsilon}{2^{a+t}2(s^3-9s^2+20s-12)}\right)\geq_{L^*} I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\frac{\epsilon}{\varsigma^t}\right), \quad (25)$$

for every $t,k\geq 0$. Replacing ϵ by $\varsigma^t\epsilon$ in (25), we have

$$I_{\mu,\nu}\left(\frac{\phi(2^{k+t}\nu)}{2^{k+t}}-\frac{\phi(2^t\nu)}{2^t},\sum_{a=t}^{k+t-1}\frac{6\varsigma^a\epsilon}{2^{a+1}(s^3-9s^2+20s-12)}\right)\geq_{L^*} I'_{\mu,\nu}(\varphi(\nu,\nu,0,\ldots,0),\epsilon). \quad (26)$$

Using (IFN3) in (26), we obtain

$$I_{\mu,\nu}\left(\frac{\phi(2^{k+t}\nu)}{2^{k+t}}-\frac{\phi(2^t\nu)}{2^t},\epsilon\right)\geq_{L^*} I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\frac{\epsilon}{\sum_{a=t}^{k+t-1}\left(6\varsigma^a/2^a 2(s^3-9s^2+20s-12)\right)}\right), \quad (27)$$

for all $t,k\geq 0$. Since $0<\varsigma<2$ and $\sum_{a=0}^{k}(\varsigma/2)^a<\infty$, the Cauchy criterion for convergence in IFNS shows that $\{\phi(2^k\nu)/2^k\}$ is Cauchy sequence in $(F,I_{\mu,\nu},\Upsilon)$. Since $(F,I_{\mu,\nu},\Upsilon)$ is a complete, this sequence converges to some point $A_1(\nu)\in F$. Then, we can define the mapping $A_1: W\longrightarrow F$ by

$$I_{\mu,\nu}\left(A_1(\nu)-\frac{\phi(2^k\nu)}{2^k}\right)\longrightarrow 1_{L^*}\ \text{as}\ k\longrightarrow\infty. \quad (28)$$

Setting $t=0$ in inequality (29), we obtain

$$I_{\mu,\nu}\left(\frac{\phi(2^k\nu)}{2^k}-\phi(\nu),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\frac{\epsilon}{\sum_{a=0}^{k-1}\left(6\varsigma^a/2^a 2\left(s^3-9s^2+20s-12\right)\right)}\right). \tag{29}$$

Taking the limit as $k\longrightarrow\infty$ in (29), we obtain

$$I_{\mu,\nu}(\phi(\nu)-A_1(\nu),\epsilon)\geq_{L^*}I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\left(\frac{s^3-9s^2+20s-12}{6}\right)\epsilon(2-\varsigma)\right). \tag{30}$$

Next, we want to prove that the function A_1 satisfies functional equation (4); replacing $(\nu_1,\nu_2,\ldots,\nu_s)$ by $(2^k\nu_1,2^k\nu_2,\ldots,2^k\nu_s)$ in (15), we have

$$\begin{aligned}I_{\mu,\nu}&\left(\frac{1}{2^k}D\phi\left(2^k\nu_1,\ldots,2^k\nu_s\right),\epsilon\right)\\&\geq_{L^*}I'_{\mu,\nu}\left(\varphi\left(2^k\nu_1,\ldots,2^k\nu_s\right),2^k\epsilon\right),\end{aligned} \tag{31}$$

for all $\nu_1,\nu_2,\ldots,\nu_s\in W$ and all $\epsilon>0$. Since

$$\lim_{k\longrightarrow\infty}I'_{\mu,\nu}\left(\varphi\left(2^k\nu_1,2^k\nu_2,\ldots,2^k\nu_s\right),2^k\varepsilon\right)=1_{L^*}, \tag{32}$$

the function A_1 satisfies functional equation (4). Thus, the function A_1 is additive. Finally, we want to prove that the function A_1 is unique; consider another additive mapping A_2: $W\longrightarrow F$ satisfying functional equations (4) and (17). Hence,

$$\begin{aligned}I_{\mu,\nu}(A_1(\nu)-A_2(\nu),\epsilon)&=I_{\mu,\nu}\left(\frac{A_1(2^k\nu)}{2^k}-\frac{A_2(2^k\nu)}{2^k},\epsilon\right)\geq_{L^*}\\ \Upsilon&\left\{I_{\mu,\nu}\left(\frac{A_1(2^k\nu)}{2^k}-\frac{\phi(2^k\nu)}{2^k},\frac{\epsilon}{2}\right),I_{\mu,\nu}\left(\frac{\phi(2^k\nu)}{2^k}-\frac{A_2(2^k\nu)}{2^k},\frac{\epsilon}{2}\right)\right\}\\&\geq_{L^*}I'_{\mu,\nu}\left(\varphi\left(2^k\nu,2^k\nu,0,\ldots,0\right),\frac{\left(s^3-9s^2+20s-12\right)2^k\epsilon(2-\varsigma)}{12}\right)\\&\geq_{L^*}I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\frac{\left(s^3-9s^2+20s-12\right)2^k\epsilon(2-\varsigma)}{12\varsigma^k}\right),\end{aligned} \tag{33}$$

for all $\nu\in W$ and all $\epsilon>0$. As

$$\lim_{s\longrightarrow\infty}\frac{\left(s^3-9s^2+20s-12\right)2^k\epsilon(2-\varsigma)}{12\varsigma^k}=\infty, \tag{34}$$

we obtain

$$\lim_{k\longrightarrow\infty}I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\frac{\left(s^3-9s^2+20s-12\right)2^k\epsilon(2-\varsigma)}{12\varsigma^k}\right)=1_{L^*}. \tag{35}$$

Thus, $I_{\mu,\nu}(A_1(\nu)-A_2(\nu),\epsilon)=1_{L^*}$.

Therefore, $A_1(\nu)=A_2(\nu)$. Thus, the additive function $A_1(\nu)$ is unique. This ends the proof. □

Theorem 4. *If a mapping* φ: $W^s\longrightarrow Z$ *with* $0<(2/\varsigma)<1$,

$$I'_{\mu,\nu}\left(\varphi\left(2^{-1}\nu,2^{-1}\nu,0,\ldots,0\right),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}\left(\frac{1}{\varsigma}\varphi(\nu,\nu,0,\ldots,0),\epsilon\right), \tag{36}$$

$$\lim_{k\longrightarrow\infty}I'_{\mu,\nu}\left(\varphi\left(2^{-k}\nu_1,2^{-k}\nu_2,\ldots,2^{-k}\nu_s\right),2^{-k}\epsilon\right)=1_{L^*}, \tag{37}$$

for all $v, v_1, v_2, \ldots, v_s \in W$ *and all* $\epsilon > 0$. *If a mapping* $\phi: E \longrightarrow F$ *satisfies (15), then the limit* $I_{\mu,\nu}(A_1(v) - 2^k\phi(v/2^k), \epsilon) \longrightarrow 1_{L^*}$ *as* $k \longrightarrow \infty$ *exists and there exists a unique additive mapping* $A_1: W \longrightarrow F$ *satisfying functional equation (4) and*

$$I_{\mu,\nu}(\phi(v) - A_1(v), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(\varphi(v, v, 0, \ldots, 0), \left(\frac{s^3 - 9s^2 + 20s - 12}{6}\right)\epsilon(\varsigma - 2)\right), \quad (38)$$

for all $v \in W$ *and all* $\epsilon > 0$.

Proof. Fix $v \in W$ and all $\epsilon > 0$. Replacing $(v_1, v_2, \ldots, v_s)$ by $(v, v, 0, \ldots, 0)$ in (15), we have

$$I_{\mu,\nu}\left(\left(\frac{s^3 - 9s^2 + 20s - 12}{6}\right)\phi(2v) - \left(\frac{2(s^3 - 9s^2 + 20s - 12)}{6}\right)\phi(v), \epsilon\right) \geq_{L^*} I'_{\mu,\nu}(\varphi(v, v, 0, \ldots, 0), \epsilon). \quad (39)$$

From (39), we obtain that

$$I_{\mu,\nu}\left(\phi(2v) - 2\phi(v), \frac{6\epsilon}{2(s^3 - 9s^2 + 20s - 12)}\right) \geq_{L^*} I'_{\mu,\nu}(\varphi(v, v, 0, \ldots, 0), \epsilon). \quad (40)$$

Replacing v by $v/2$ in (40), we get

$$I_{\mu,\nu}\left(\phi(v) - 2\phi\left(\frac{v}{2}\right), \frac{6\epsilon}{2(s^3 - 9s^2 + 20s - 12)}\right) \geq_{L^*} I'_{\mu,\nu}\left(\varphi\left(\frac{v}{2}, \frac{v}{2}, 0, \ldots, 0\right), \epsilon\right). \quad (41)$$

Replacing v by $v/2^k$ in (41) and using (IFN3), we have

$$I_{\mu,\nu}\left(\phi\left(\frac{v}{2^k}\right) - 2\phi\left(\frac{v}{2^{k+1}}\right), \frac{6\epsilon}{2(s^3 - 9s^2 + 20s - 12)}\right) \geq_{L^*} I'_{\mu,\nu}\left(\varphi\left(\frac{v}{2^{k+1}}, \frac{v}{2^{k+1}}, 0, \ldots, 0\right)\right). \quad (42)$$

With the help of inequality (36) and (IFN3) in (42), we obtain that

$$I_{\mu,\nu}\left(\phi\left(\frac{v}{2^k}\right) - 2\phi\left(\frac{v}{2^{k+1}}\right), \frac{6\epsilon}{2(s^3 - 9s^2 + 20s - 12)}\right) \geq_{L^*} I'_{\mu,\nu}\left(\varphi(v, v, 0, \ldots, 0), \epsilon\varsigma^{k+1}\right). \quad (43)$$

The remaining part of the proof can be proven in the same way as Theorem 3. □

Corollary 1. *Let* $\theta \in \mathbb{R}^+$. *If a mapping* $\phi: W \longrightarrow F$ *such that*

$$I_{\mu,\nu}(D\phi(v_1, v_2, \ldots, v_s), \epsilon) \geq_{L^*} I'_{\mu,\nu}(\theta, \epsilon), \quad (44)$$

for all $v_1, v_2, \ldots, v_s \in W$ *and all* $\epsilon > 0$, *then there exists a unique additive mapping* $A_1: W \longrightarrow F$ *satisfying*

$$I_{\mu,\nu}(\phi(v) - A_1(v), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(6\theta, |2 - 1|\epsilon(s^3 - 9s^2 + 20s - 12)\right), \quad (45)$$

for all $v \in W$ *and all* $\epsilon > 0$.

Proof. The proof holds from Theorems 3 and 4 by letting $\varphi(v_1, v_2, \ldots, v_s) = \theta$ and $\varsigma = 2^0$. □

Corollary 2. *Let* $\theta, \xi \in \mathbb{R}^+$ *with* $\xi \in (0,1) \cup (1, +\infty)$. *If a mapping* $\phi: W \longrightarrow F$ *such that*

$$I_{\mu,\nu}(D\phi(v_1, v_2, \ldots, v_s), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(\theta\sum_{a=1}^{s}\|v_a\|^{\xi}, \epsilon\right), \quad (46)$$

for all $v_1, v_2, \ldots, v_s \in W$ *and all* $\epsilon > 0$, *then there exists a unique additive mapping* $A_1: W \longrightarrow F$ *satisfying*

$$I_{\mu,\nu}(\phi(v) - A_1(v), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(12\theta\|v\|^{\xi}, \left|2 - 2^{\xi}\right|(s^3 - 9s^2 + 20s - 12)\epsilon\right), \quad (47)$$

for all $v \in W$ *and all* $\epsilon > 0$.

Proof. The proof holds from Theorems 3 and 4 by setting $\varphi(v_1, v_2, \ldots, v_s) = \theta\sum_{a=1}^{s}\|v_a\|^{\xi}$ and $\varsigma = 2^{\xi}$. □

Corollary 3. *Let* $\theta, \xi, \gamma, \tau \in \mathbb{R}^+$ *with* $s\xi, s\tau \in (0,1) \cup (1, +\infty)$. *If a mapping* $\phi: W \longrightarrow F$ *such that*

$$I_{\mu,\nu}(D\phi(v_1, v_2, \ldots, v_s), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(\theta\sum_{a=1}^{s}\|v_a\|^{s\xi} + \gamma\prod_{a=1}^{s}\|v_a\|^{\tau}, \epsilon\right), \quad (48)$$

for all $v_1, v_2, \ldots, v_s \in W$ *and all* $\epsilon > 0$, *then there exists a unique additive mapping* $A_1: W \longrightarrow F$ *satisfying*

$$I_{\mu,\nu}(\phi(v) - A_1(v), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(12\theta\|v\|^{s\xi}, \left|2 - 2^{s\xi}\right|\left(s^3 - 9s^2 + 20s - 12\right)\epsilon\right), \quad (49)$$

for all $v \in W$ and all $\epsilon > 0$.

Proof. The proof holds from Theorems 3 and 4 by setting $\varphi(v_1, v_2, \ldots, v_s) = \theta\sum_{a=1}^{s}\|v_a\|^{s\xi} + \gamma\prod_{a=1}^{s}\|v_a\|^{\tau}$ and $\varsigma = 2^{s\xi}$. □

Corollary 4. *Let $\gamma, \tau \in \mathbb{R}^+$ with $0 < s\tau \neq 1$. If a mapping $\phi: W \longrightarrow F$ such that*

$$I_{\mu,\nu}(D\phi(v_1, v_2, \ldots, v_s), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(\gamma\prod_{a=1}^{s}\|v_a\|^{\tau}, \epsilon\right), \quad (50)$$

for all $v_1, v_2, \ldots, v_s \in W$ and all $\epsilon > 0$, then the mapping ϕ is additive.

Proof. The proof is valid from Theorems 3 and 4 by setting $\varphi(v_1, v_2, \ldots, v_s) = \gamma\prod_{a=1}^{s}\|v_a\|^{\tau}$. □

3.2. Stability Results: Fixed Point Technique. Before we begin, let us consider a constant β_a such that

$$\beta_a = \begin{cases} 2, & \text{if } a = 0, \\ \dfrac{1}{2}, & \text{if } a = 1, \end{cases} \quad (51)$$

and Ψ is the set such that $\Psi = \{n_1 | n_1: W \longrightarrow F, n_1(0) = 0\}$.

Theorem 5. *Consider a mapping $\phi: W \longrightarrow F$ for which there is a mapping $\varphi: W^s \longrightarrow Z$ with*

$$\lim_{l\longrightarrow\infty} I'_{\mu,\nu}\left(\varphi\left(2^l v_1, 2^l v_2, \ldots, 2^l v_s\right), 2^l\epsilon\right) = 1_{L^*}, \quad (52)$$

satisfying functional inequality (15). If there is $L = L(a)$ such that $v \longrightarrow \eta(v) = 6/(s^3 - 9s^2 + 20s - 12)\varphi((v/2), (v/2), 0, \ldots, 0)$ has the property

$$I'_{\mu,\nu}\left(L\frac{1}{\beta_a}\eta(\beta_a v), \epsilon\right) = I'_{\mu,\nu}(\eta(v), \epsilon), \quad (53)$$

then there exists a unique additive mapping $A_1: W \longrightarrow F$ satisfying functional equation (4) and

$$I_{\mu,\nu}(\phi(v) - A_1(v), \epsilon)n \geq_{L^*} I'_{\mu,\nu}\left(\frac{L^{1-a}}{1-L}\eta(v), \epsilon\right), \quad (54)$$

for all $v \in W$ and all $\epsilon > 0$.

Proof. Let ς be a general metric on Ψ:

$$\varsigma(n_1, n_2) = \inf\left\{t \in (0, \infty) | I_{\mu,\nu}(n_1(v) - n_2(v), \epsilon) \geq_{L^*} I'_{\mu,\nu}(t\eta(v), \epsilon), v \in W, \epsilon > 0\right\}. \quad (55)$$

Clearly, (Ψ, ς) is complete. Define a mapping $\Upsilon: \Psi \longrightarrow \Psi$ by $\Upsilon n_1(v) = (1/\beta_a)n_1(\beta_a v)$, for all $v \in W$. For $n_1, n_2 \in \Psi$, we have

$$\begin{aligned}
&\varsigma(n_1, n_2) \leq t, \\
&\quad \Rightarrow I_{\mu,\nu}(n_1(v) - n_2(v), \varepsilon) \geq_{L^*} I'_{\mu,\nu}(t\eta(v), \varepsilon) \\
&\quad \Rightarrow I_{\mu,\nu}\left(\frac{n_1(\beta_a v)}{\beta_a} - \frac{n_2(\beta_a v)}{\beta_a}, \varepsilon\right) \geq_{L^*} I'_{\mu,\nu}\left(\frac{t\eta(\beta_a v)}{\beta_a}, \varepsilon\right) \\
&\quad \Rightarrow I_{\mu,\nu}(\Upsilon n_1(v) - \Upsilon n_2(v), \varepsilon) \geq_{L^*} I_{\mu,\nu}(tL\eta(v), \varepsilon) \\
&\quad \Rightarrow \varsigma(\Upsilon n_1(v), \Upsilon n_2(v)) \leq tL \\
&\quad \Rightarrow \varsigma(\Upsilon n_1, \Upsilon n_2) \leq L\varsigma(n_1, n_2).
\end{aligned} \quad (56)$$

Thus, the function Υ is strictly contractive on Ψ with L (Lipschitz constant). Replacing $(v_1, v_2, \ldots, v_s)$ by $(v, v, 0, \ldots, 0)$ in (15), we have

$$I_{\mu,\nu}\left(\left(\frac{s^3-9s^2+20s-12}{6}\right)\phi(2\nu)-\left(\frac{2\left(s^3-9s^2+20s-12\right)}{6}\right)\phi(\nu),\epsilon\right) \geq_{L^*} I'_{\mu,\nu}(\varphi(\nu,\nu,0,\ldots,0),\epsilon). \tag{57}$$

Using (IFN3) in (57), we have

$$I_{\mu,\nu}\left(\frac{\phi(2\nu)}{2}-\phi(\nu),\epsilon\right)\geq_{L^*} I'_{\mu,\nu}\left(\left(\frac{6}{2\left(s^3-9s^2+20s-12\right)}\right)\varphi(\nu,\nu,0,\ldots,0),\epsilon\right). \tag{58}$$

Using equation (53) for the case $a=0$, we have

$$I_{\mu,\nu}\left(\frac{\phi(2\nu)}{2}-\phi(\nu),\epsilon\right)\geq_{L^*} I'_{\mu,\nu}(L\eta(\nu),\epsilon) \Rightarrow \varsigma(\Upsilon\phi,\phi)\leq L=L^1=L^{1-a}. \tag{59}$$

Replacing ν by $(\nu/2)$ in (57), we have

$$I_{\mu,\nu}\left(\phi(\nu)-2\phi\left(\frac{\nu}{2}\right),\epsilon\right)\geq_{L^*} I'_{\mu,\nu}\left(\left(\frac{6}{\left(s^3-9s^2+20s-12\right)}\right)\varphi\left(\frac{\nu}{2},\frac{\nu}{2},0,\ldots,0\right),\epsilon\right), \tag{60}$$

for all $\nu\in W$ and all $\epsilon>0$; using (53) for the case $a=1$, we have

$$I_{\mu,\nu}\left(\phi(\nu)-2\phi\left(\frac{\nu}{2}\right),\epsilon\right)\geq_{L^*} I'_{\mu,\nu}(\eta(\nu),\epsilon) \Rightarrow \varsigma(\phi,\Upsilon\phi)\leq 1=L^0=L^{1-a}. \tag{61}$$

We can conclude from equations (59) and (61) that

$$\varsigma(\phi,\Upsilon\phi)\leq L^{1-a}<\infty. \tag{62}$$

By the fixed point alternative in both cases, there is a fixed point A_1 of Υ in Ψ such that

$$\lim_{k\longrightarrow\infty} I_{\mu,\nu}\left(\frac{\phi\left(\beta_a^k\nu\right)}{\beta_a^k}-A_1(\nu),\epsilon\right)\longrightarrow 1_{L^*},\quad \nu\in W,\ \epsilon>0. \tag{63}$$

Replacing $(\nu_1,\nu_2,\ldots,\nu_s)$ by $(\beta_a\nu_1,\beta_a\nu_2,\ldots,\beta_a\nu_s)$ in (15), we obtain

$$I_{\mu,\nu}\left(\frac{1}{\beta_a}D\phi(\beta_a\nu_1,\beta_a\nu_2,\ldots,\beta_a\nu_s),\epsilon\right) \geq_{L^*} I'_{\mu,\nu}(\varphi(\beta_a\nu_1,\beta_a\nu_2,\ldots,\beta_a\nu_s),\beta_a\epsilon), \tag{64}$$

for all $\nu_1,\nu_2,\ldots,\nu_s\in W$ and all $\epsilon>0$. By same manner of Theorem 3, we can show that the function A_1 satisfies functional equation (4). By Theorem 2, as A_1 is a unique fixed point of Υ in $\Delta=\{\phi\in\Psi|\varsigma(\phi,A_1)<\infty\}$, the function A_1 is unique such that

$$I_{\mu,\nu}(A_1(\nu)-\phi(\nu),\varepsilon)\geq_{L^*} I'_{\mu,\nu}(t\eta(\nu),\varepsilon),\quad t>0. \tag{65}$$

Using fixed point alternative, we reach

$$\varsigma(\phi,A_1)\leq\frac{1}{1-L}\varsigma(\phi,\Upsilon\phi) \Rightarrow \varsigma(\phi,A_1)\leq\frac{L^{1-a}}{1-L} \Rightarrow I_{\mu,\nu}(\phi(\nu)-A_1(\nu),\epsilon)\geq_{L^*} I'_{\mu,\nu}\left(\frac{L^{1-a}}{1-L}\eta(\nu),\epsilon\right), \tag{66}$$

for all $\nu\in W$ and all $\epsilon>0$. Hence, the proof of the theorem is now completed. □

Corollary 5. *Let $\theta,\xi\in\mathbb{R}_+$ with $\theta>0$. If a mapping $\phi: W\longrightarrow F$ such that*

$$I_{\mu,\nu}(D\phi(\nu_1,\nu_2,\ldots,\nu_s),\epsilon)\geq_{L^*}\begin{cases} I'_{\mu,\nu}(\theta,\epsilon),\\ I'_{\mu,\nu}\left(\theta\sum_{j=1}^{s}\left\|\nu_j\right\|^{\xi},\epsilon\right),\\ I'_{\mu,\nu}\left(\theta\left(\prod_{j=1}^{s}\left\|\nu_j\right\|^{\xi}+\sum_{j=1}^{s}\left\|\nu_j\right\|^{s\xi}\right),\epsilon\right),\end{cases} \tag{67}$$

for all $\nu_1,\nu_2,\ldots,\nu_s\in W$ and $\epsilon>0$, then there exists a unique additive mapping $A_1: W\longrightarrow F$ satisfying

$$I_{\mu,\nu}(\phi(v)-A_1(v),\epsilon) \geq_{L^*} \begin{cases} I'_{\mu,\nu}\left(|6|\theta, \left(s^3-9s^2+20s-12\right)\epsilon\right), \\ I'_{\mu,\nu}\left(12\theta\|v\|^s, \left(s^3-9s^2+20s-12\right)\left|2-2^{\xi}\right|\epsilon\right), \xi<1 \text{ or } \xi>1, \\ I'_{\mu,\nu}\left(12\theta\|v\|^s, \left(s^3-9s^2+20s-12\right)\left|2-2^{s\xi}\right|\epsilon\right), \xi<\dfrac{1}{s} \text{ or } \xi>\dfrac{1}{s}, \end{cases} \tag{68}$$

for all $v \in W$ and all $\epsilon>0$.

Proof. Set

$$\varphi(v_1,v_2,\ldots,v_s) = \begin{cases} \theta, \\ \theta\sum_{j=1}^{s}\left\|v_j\right\|^{\xi}, \\ \theta\left(\prod_{j=1}^{s}\left\|v_j\right\|^{\xi}+\sum_{j=1}^{s}\left\|v_j\right\|^{s\xi}\right). \end{cases} \tag{69}$$

Then,

$$I'_{\mu,\nu}\left(\varphi\left(\beta_a^l v_1,\beta_a^l v_2,\ldots,\beta_a^l v_s\right),\beta_a^l\epsilon\right) = \begin{cases} I'_{\mu,\nu}\left(\theta, (\beta_a)^l\epsilon\right), \\ I'_{\mu,\nu}\left(\theta\sum_{j=1}^{s}\left\|v_j\right\|^{\xi}, \left(\beta_a^{1-\xi}\right)^l\epsilon\right), \\ I'_{\mu,\nu}\left(\theta\left(\prod_{j=1}^{s}\left\|v_j\right\|^{\xi}+\sum_{j=1}^{s}\left\|v_j\right\|^{s\xi}\right), \left(\beta_a^{1-s\xi}\right)^l\epsilon\right), \end{cases}$$
$$= \begin{cases} \longrightarrow 1_{L^*} \text{ as } l \longrightarrow \infty, \\ \longrightarrow 1_{L^*} \text{ as } l \longrightarrow \infty, \\ \longrightarrow 1_{L^*} \text{ as } l \longrightarrow \infty. \end{cases} \tag{70}$$

Thus, (52) holds. But we have that

$$\eta(v) = \frac{6}{\left(s^3-9s^2+20s-12\right)}\varphi\left(\frac{v}{2},\frac{v}{2},0,\ldots,0\right), \tag{71}$$

has the property

$$I'_{\mu,\nu}\left(L\frac{1}{\beta_a}\eta(\beta_a v),\epsilon\right) \geq_{L^*} I'_{\mu,\nu}(\eta(v),\epsilon), \quad v\in W,\ \epsilon>0. \tag{72}$$

Hence,

$$I'_{\mu,\nu}(\eta(v),\epsilon) = I'_{\mu,\nu}\left(\frac{6}{\left(s^3-9s^2+20s-12\right)}\varphi\left(\frac{v}{2},\frac{v}{2},0,\ldots,0\right),\epsilon\right)$$
$$= \begin{cases} I'_{\mu,\nu}\left(6\theta, \left(s^3-9s^2+20s-12\right)\epsilon\right), \\ I'_{\mu,\nu}\left(\dfrac{12\theta}{2^{\xi}}\|v\|^{\xi}, \left(s^3-9s^2+20s-12\right)\epsilon\right), \\ I'_{\mu,\nu}\left(\dfrac{12\theta}{2^{s\xi}}\|v\|^{s\xi}, \left(s^3-9s^2+20s-12\right)\epsilon\right). \end{cases} \tag{73}$$

Now,

$$I'_{\mu,\nu}\left(\frac{1}{\beta_a}\eta(\beta_a v),\epsilon\right)=\begin{cases}I'_{\mu,\nu}\left(\dfrac{6\theta}{\beta_a},\left(s^3-9s^2+20s-12\right)\epsilon\right),\\ I'_{\mu,\nu}\left(\dfrac{12\theta}{2^{\xi}\beta_a}\|\beta_a v\|^{\xi},\left(s^3-9s^2+20s-12\right)\epsilon\right),\\ I'_{\mu,\nu}\left(\dfrac{12\theta}{2^{s\xi}\beta_a}\|\beta_a v\|^{s\xi},\left(s^3-9s^2+20s-12\right)\epsilon\right),\end{cases}$$
$$=\begin{cases}I'_{\mu,\nu}\left(\beta_a^{-1}\eta(v),\epsilon\right),\\ I'_{\mu,\nu}\left(\beta_a^{\xi-1}\eta(v),\epsilon\right),\\ I'_{\mu,\nu}\left(\beta_a^{s\xi-1}\eta(v),\epsilon\right).\end{cases} \quad (74)$$

From inequality (53), we can verify the following cases for conditions of β_a. □

Case 1. $L=2^{-1}$ if $a=0$.

$$I_{\mu,\nu}(\phi(v)-A_1(v),\epsilon)\geq_{L^*}I'_{\mu,\nu}\left(\frac{2^{-1}}{1-2^{-1}}\eta(v),\epsilon\right)=I'_{\mu,\nu}\left(6\theta,\left(s^3-9s^2+20s-12\right)\epsilon\right). \quad (75)$$

Case 2. $L=2$ if $a=1$.

$$I_{\mu,\nu}(\phi(v)-A_1(v),\epsilon)\geq_{L^*}I'_{\mu,\nu}\left(\frac{1}{1-2}\eta(v),\epsilon\right)=I'_{\mu,\nu}\left(-6\theta,\left(s^3-9s^2+20s-12\right)\epsilon\right). \quad (76)$$

Case 3. $L=2^{\xi-1}$ for $\xi<1$ if $a=0$.

$$I_{\mu,\nu}(\phi(v)-A_1(v),\epsilon)\geq_{L^*}I'_{\mu,\nu}\left(\frac{2^{\xi-1}}{1-2^{\xi-1}}\eta(v),\epsilon\right)=I'_{\mu,\nu}\left(12\theta\|v\|^{\xi},\left(s^3-9s^2+20s-12\right)\cdot\left(2-2^{\xi}\right)\epsilon\right). \quad (77)$$

Case 4. $L=2^{1-\xi}$ for $\xi>1$ if $a=1$.

$$I_{\mu,\nu}(\phi(v)-A_1(v),\epsilon)\geq_{L^*}I'_{\mu,\nu}\left(\frac{1}{1-2^{1-\xi}}\eta(v),\epsilon\right)=I'_{\mu,\nu}\left(12\theta\|v\|^{\xi},\left(s^3-9s^2+20s-12\right)\cdot\left(2^{\xi}-2\right)\epsilon\right). \quad (78)$$

Case 5. $L=2^{s\xi-1}$ for $\xi<(1/s)$ if $a=0$.

$$I_{\mu,\nu}(\phi(v)-A_1(v),\epsilon)\geq_{L^*}I'_{\mu,\nu}\left(\frac{2^{s\xi-1}}{1-2^{s\xi-1}}\eta(v),\epsilon\right)=I'_{\mu,\nu}\left(12\theta\|v\|^{s\xi},\left(s^3-9s^2+20s-12\right)\cdot\left(2-2^{s\xi}\right)\epsilon\right). \quad (79)$$

Case 6. $L=2^{1-s\xi}$ for $\xi<(1/s)$ if $a=1$.

$$I_{\mu,\nu}(\phi(v)-A_1(v),\epsilon)\geq_{L^*}I'_{\mu,\nu}\left(\frac{1}{1-2^{1-s\xi}}\eta(v),\epsilon\right)=I'_{\mu,\nu}\left(12\theta\|v\|^{s\xi},\left(s^3-9s^2+20s-12\right)\cdot\left(2^{s\xi}-2\right)\epsilon\right). \quad (80)$$

4. Stability Results in 2-Banach Spaces

In 1960, Gahler [27, 28] developed the concept of linear 2-normed spaces.

Definition 4. Consider a linear space W over $\mathbb{R}$ with dimension $W > 1$ and consider a mapping $\|\cdot,\cdot\|\colon W^2 \longrightarrow \mathbb{R}$ with the following conditions:

(a) $\|r, s\| = 0$ if and only if r and s are linearly dependent.
(b) $\|r, s\| = \|s, r\|$,
(c) $\|\lambda r, s\| = |\lambda|\|r, s\|$,
(d) $\|r, s + w\| \le \|r, s\| + \|r, w\|$,for all $r, s, w \in W$ and $\lambda \in \mathbb{R}$.

Then, the function $\|\cdot,\cdot\|$ is called as a 2-norm on W and the pair $(W, \|\cdot,\cdot\|)$ is called as a linear 2-normed space. A typical example of 2-normed space is $\mathbb{R}^2$ with 2-norm defined as $|r, s|$ = the area of the triangle with the vertices 0, r, and s is a typical example of a 2-normed space.

As a result of (d), it follows that

$$\|r + s, w\| \le \|r, w\| + \|s, w\| \text{ and } |\, \|r, w\| - \|s, w\| \,| \le \|r - s, w\|. \tag{81}$$

Thus, $r \longrightarrow \|r, s\|$ are continuous mappings of W into $\mathbb{R}$ for any fixed $s \in W$.

Definition 5. A sequence $\{r_j\}$ in a linear 2-normed space W is known as a Cauchy sequence if there exist two points $s, w \in W$ such that s and w are linearly independent.

$$\begin{aligned} \lim_{i,j\longrightarrow\infty} \left\| r_i - r_j, s \right\| &= 0, \\ \lim_{i,j\longrightarrow\infty} \left\| r_i - r_j, w \right\| &= 0. \end{aligned} \tag{82}$$

Definition 6. A sequence $\{r_j\}$ in a linear 2-normed space W is called as a convergent sequence if there exists an element $r \in W$ such that

$$\lim_{i,j\longrightarrow\infty} \left\| r_j - r, s \right\| = 0, \tag{83}$$

for all $s \in W$. If $\{r_j\}$ converges to r, then we denote $r_j \longrightarrow r$ as $j \longrightarrow \infty$ and say that r is the limit point of $\{r_j\}$. We also write in this instance

$$\lim_{j\longrightarrow\infty} r_j = r. \tag{84}$$

Definition 7. A 2-Banach space is a linear 2-normed space in which every Cauchy sequence is convergent.

Lemma 1 (see [29]). *Let $(W, \|\cdot,\cdot\|)$ be a linear 2-normed space. If $r \in W$ and $\|r, s\| = 0$ for every $s \in W$, then $r = 0$.*

Lemma 2 (see [29]). *For a convergent sequence $\{r_j\}$ in a linear 2-normed space W,*

$$\lim_{j\longrightarrow\infty} \left\| r_j, w \right\| = \left\| \lim_{j\longrightarrow\infty} r_j, s \right\|, \tag{85}$$

for every $s \in W$.

Park studied approximate additive mappings, approximate Jensen mappings, and approximate quadratic mappings in 2-Banach spaces in his paper [29]. In [30], Park examined the superstability of the Cauchy functional inequality and the Cauchy–Jensen functional inequality in 2-Banach spaces under certain conditions.

In this section, we let W be a normed linear space and F be a 2-Banach space.

Theorem 6. *Let $\varphi\colon W \longrightarrow [0, +\infty)$ be a function such that*

$$\lim_{i\longrightarrow\infty} \frac{1}{2^i} \varphi\left(2^i v_1, 2^i v_2, \ldots, 2^i v_s, w\right) = 0, \tag{86}$$

for all $v_1, v_2, \ldots, v_s, w \in W$. If a mapping $\phi\colon W \longrightarrow F$ such that $\phi(0) = 0$ and

$$\left\| D\phi(v_1, v_2, \ldots, v_s), w \right\| \le \varphi(v_1, v_2, \ldots, v_s, w), \tag{87}$$

$$\widehat{\varphi}(v, w) =: \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left(2^j v, 2^j v, 0, \ldots, 0, w\right) < \infty, \tag{88}$$

for all $v_1, v_2, \ldots, v_s, w \in W$. Then, there exists a unique additive mapping $A_1\colon W \longrightarrow F$ satisfying

$$\left\| \phi(v) - A_1(v), w \right\| \le \frac{6}{2\left(s^3 - 9s^2 + 20s - 12\right)} \widehat{\varphi}(v, w), \tag{89}$$

for all $v, w \in W$.

Proof. Replacing $(v_1, v_2, \ldots, v_s)$ by $(v, v, 0, \ldots, 0)$ in (87), we get

$$\begin{aligned} &\left\| \frac{\left(s^3 - 9s^2 + 20s - 12\right)}{6} \phi(2v) - \frac{2\left(s^3 - 9s^2 + 20s - 12\right)}{6} \phi(v), w \right\| \\ &\quad \le \varphi(v, v, 0, \ldots, 0, w), \end{aligned} \tag{90}$$

for all $v, w \in W$. Replacing v by $2^n v$ in (90) and dividing both sides by 2^{n-1}, we have

$$\begin{aligned} &\left\| \frac{1}{2^{(n+1)}} \phi\left(2^{n+1} v\right) - \frac{1}{2^n} \phi\left(2^n v\right), w \right\| \\ &\quad \le \frac{6}{2^{n+1}\left(s^3 - 9s^2 + 20s - 12\right)} \varphi\left(2^i v, 2^i v, 0, \ldots, 0, w\right), \end{aligned} \tag{91}$$

for all $v, w \in W$ and all non-negative integers i. Hence,

$$\begin{aligned} &\left\| \frac{1}{2^{n+1}} \phi\left(2^{n+1} v\right) - \frac{1}{2^m} \phi\left(2^m v\right), w \right\| \\ &\quad \le \sum_{j=m}^{i} \left\| \frac{1}{2^{j+1}} \phi\left(2^{j-1} v\right) - \frac{1}{2^j} \phi\left(2^j v\right), w \right\| \\ &\quad \le \frac{6}{2\left(s^3 - 9s^2 + 20s - 12\right)} \sum_{j=m}^{i} \frac{1}{2^j} \varphi\left(2^j v, 2^j v, 0, \ldots, 0, w\right), \end{aligned} \tag{92}$$

for all $v, w \in W$ and all non-negative integers m and i with $i \geq m$. Therefore, it follows from (15) and (19) that the sequence $\{(1/2^i)\phi(2^i v)\}$ is Cauchy in F for every $v \in W$. Since F is complete, the sequence $\{(1/2^i)\phi(2^i v)\}$ converges in F for all $v \in W$. Thus, we may define a mapping $A_1: W \longrightarrow F$ by

$$A_1(v): \lim_{i\longrightarrow\infty} \frac{1}{2^i}\phi\left(2^i v\right), \tag{93}$$

for all $v \in W$. Therefore,

$$\lim_{i\longrightarrow\infty}\left\|\frac{1}{2^i}\phi\left(2^i v\right) - A_1(v), w\right\| = 0, \tag{94}$$

for all $v, w \in W$. Letting $m = 0$ and taking the limit as $i \longrightarrow \infty$ in (94), we have (89). Next, we want to prove that the function A_1 is additive. From inequalities (86), (87), and (94) and Lemma 2,

$$\begin{aligned}\left\|D\phi(v_1, v_2, \ldots, v_s), w\right\| &= \lim_{i\longrightarrow\infty}\left\|D\phi\left(2^i v_1, 2^i v_2, \ldots, 2^i v_s\right), w\right\| \\ &\leq \lim_{i\longrightarrow\infty}\frac{1}{2^i}\varphi\left(2^i v_1, 2^i v_2, \ldots, 2^i v_s, w\right) = 0,\end{aligned} \tag{95}$$

for all $v_1, v_2, \ldots, v_s, w \in W$. By Lemma 1,

$$DA_1(v_1, v_2, \ldots, v_s) = 0, \tag{96}$$

for all $v_1, v_2, \ldots, v_s \in W$. Hence, according to Theorem 1, the mapping $A_1: W \longrightarrow F$ is additive.

To prove that the function A_1 is unique, we consider another additive mapping $A_1': W \longrightarrow F$ satisfying (89). Then,

$$\begin{aligned}\left\|A_1(v) - A_1'(v), w\right\| &= \lim_{i\longrightarrow\infty}\frac{1}{2^i}\left\|A_1\left(2^i v\right) - \phi\left(2^i v\right) + \phi\left(2^i v\right) - A_1'\left(2^i v\right), w\right\| \\ &\leq \frac{6}{\left(s^3 - 9s^2 + 20s - 12\right)}\lim_{i\longrightarrow\infty}\frac{1}{2^i}\widehat{\varphi}\left(2^i v, w\right) = 0,\end{aligned} \tag{97}$$

for all $v, w \in W$. By Lemma 1, $A_1(v) - A_1'(v) = 0$ for all $v \in W$. Therefore, $A_1 = A_1'$. □

Remark 3. A theorem analogous to (93) can be formulated, in which the sequence

$$A_1(v) := \lim_{i\longrightarrow\infty} 2^i\phi\left(\frac{v}{2^i}\right) \tag{98}$$

is defined with appropriate assumptions for φ.

Corollary 6. *Let $\lambda: [0,\infty) \longrightarrow [0,\infty)$ be a mapping such that $\lambda(0) = 0$ and*

(i) $\lambda(pq) \leq \lambda(p)\lambda(s)$.

(ii) $\lambda(p) < p$ for all $p > 1$.

If a mapping $\phi: W \longrightarrow F$ with $\phi(0) = 0$ and

$$\left\|D\phi(v_1, v_2, \ldots, v_s), w\right\| \leq \sum_{i=1}^{s}\lambda\left(\left\|v_i\right\|\right) + \lambda(\|w\|), \tag{99}$$

for all $v_1, v_2, \ldots, v_s, w \in W$, then there exists a unique additive mapping $A_1: W \longrightarrow F$ satisfying

$$\left\|\phi(v) - A_1(v), w\right\| \leq \frac{6}{\left(s^3 - 9s^2 + 20s - 12\right)}\left[\frac{2\lambda(\|v\|)}{2 - \lambda(2)} + \lambda(\|w\|)\right], \tag{100}$$

for all $v, w \in W$.

Proof. Let

$$\varphi(v_1, v_2, \ldots, v_s, w) = \sum_{i=1}^{s}\lambda\left(\left\|v_i\right\|\right) + \lambda(\|w\|), \tag{101}$$

for all $v_1, v_2, \ldots, v_s, w \in W$. It follows from (i) that

$$\lambda\left(2^i\right) \leq (\lambda(2))^i,$$

$$\varphi\left(2^i v_1, 2^i v_2, \ldots, 2^i v_s, w\right) \leq (\lambda(2))^i\left(\sum_{i=1}^{s}\lambda\left(\left\|v_i\right\|\right)\right) + \lambda(\|w\|). \tag{102}$$

By using Theorem 6, we obtain (96). □

Corollary 7. *Let q be a positive real number such that $q < 1$ and let $H: [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ be a homogeneous mapping with degree q. If a mapping $\phi: W \longrightarrow F$ with $\phi(0) = 0$ and*

$$\left\|D\phi(v_1, v_2, \ldots, v_s), w\right\| \leq H\left(\left\|v_1\right\|, \left\|v_2\right\|, \ldots, \left\|v_s\right\|\right) + \|w\|, \tag{103}$$

for all $v_1, v_2, \ldots, v_s, w \in W$, then there exists a unique additive mapping $A_1: W \longrightarrow F$ satisfying

$$\left\|\phi(v)-A_1(v),w\right\| \le \frac{6}{\left(s^3-9s^2+20s-12\right)}\frac{H(\|v\|,\|v\|,0,\ldots,0)+\|w\|}{2-q}, \tag{104}$$

for all $v, w \in W$.

Proof. Let

$$\varphi(v_1,v_2,\ldots,v_s,w)=H\left(\|v_1\|,\|v_2\|,\ldots,\|v_s\|\right)+\|w\|, \tag{105}$$

for all $v_1,v_2,\ldots,v_s,w\in W$. By using Theorem 6, we have (104). □

Corollary 8. *Let $q\in\mathbb{R}^+$ such that $q<1$ and let H: $[0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$ be a homogeneous mapping with degree q. If a mapping ϕ: $W\longrightarrow F$ with $\phi(0)=0$ and*

$$\left\|D\phi(v_1,v_2,\ldots,v_s),w\right\|\le H\left(\|v_1\|,\|v_2\|,\ldots,\|v_s\|\right)\|w\|, \tag{106}$$

for all $v_1,v_2,\ldots,v_s,w\in W$, then there exists a unique additive mapping A_1: $W\longrightarrow F$ satisfying

$$\left\|\phi(v)-A_1(v),w\right\|\le\frac{6}{2\left(s^3-9s^2+20s-12\right)}\cdot\frac{H(\|v\|,\|v\|,0,\ldots,0)\|w\|}{2-2^q}, \tag{107}$$

for all $v, w \in W$.

Proof. Let

$$\varphi(v_1,v_2,\ldots,v_s,w)=H\left(\|v_1\|,\|v_2\|,\ldots,\|v_s\|\right)\|w\|, \tag{108}$$

for all $v_1,v_2,\ldots,v_s,w\in W$. By using Theorem 6, we have (110). □

Corollary 9. *Let $p\in\mathbb{R}^+$ such that $p<1$. If a mapping ϕ: $W\longrightarrow F$ with $\phi(0)=0$ and*

$$\left\|D\phi(v_1,v_2,\ldots,v_s),w\right\|\le\sum_{i=1}^{s}\|v_i\|^p+\|w\|, \tag{109}$$

for all $v_1,v_2,\ldots,v_s,w\in W$, then there exists a unique additive mapping A_1: $W\longrightarrow F$ satisfying

$$\left\|\phi(v)-A_1(v),w\right\|\le\frac{6}{\left(s^3-9s^2+20s-12\right)}\frac{2\|v\|^p+\|w\|}{2-p}, \tag{110}$$

for all $v, w \in W$.

We use an appropriate example to demonstrate that the stability of the functional equation (4) fails in the singular case. We provide the following counterexample, which shows the instability in a particular condition $p=2$ in Corollary 9 of functional equation (4), inspired by Gajda's excellent example in [31].

Remark 4. If a mapping ϕ: $\mathbb{R}\longrightarrow W$ satisfies (4), then the following assertions hold:

(1) $\phi(m^c v)=m^c\phi(v), v\in\mathbb{R}, m\in\mathbb{Q}$, and $c\in\mathbb{Z}$.

(2) $\phi(v)=v\phi(1), v\in\mathbb{R}$ if the function ϕ is continuous.

Example 2. Let a mapping ϕ: $\mathbb{R}\longrightarrow\mathbb{R}$ be defined by

$$\phi(v)=\sum_{p=0}^{\infty}\frac{\psi(2^p v)}{2^p}, \tag{111}$$

where

$$\psi(v)=\begin{cases}\lambda v, & -1<v<1,\\ \lambda, & \text{else}.\end{cases} \tag{112}$$

Then, the mapping ϕ: $\mathbb{R}\longrightarrow\mathbb{R}$ satisfies

$$\left|D\phi(v_1,v_2,\ldots,v_s)\right|\le\left(\frac{n^4-8n^3+5n^2+34n-32}{4}\right)\cdot\left(\frac{4}{3}\right)\lambda\left(\sum_{j=1}^{s}\left|v_j\right|\right), \tag{113}$$

for all $v_1,v_2,\ldots,v_s\in\mathbb{R}$, but there does not exist an additive mapping A_1: $\mathbb{R}\longrightarrow\mathbb{R}$ satisfying

$$\left|\phi(v)-A_1(v)\right|\le\delta|v|,\quad v\in\mathbb{R}, \tag{114}$$

where λ and δ are constants.

5. Conclusion

In this work, a new dimensional additive functional (equation (4)) has been introduced. We primarily found its solution and examined Hyers–Ulam stability in IFN-spaces using the direct approach in Section 3.1 and the fixed point approach in Section 3.2. In Section 4, we investigated the Hyers–Ulam stability in 2-Banach space by using the direct method. Also, we provided the counterexample, which shows the instability in a particular condition $p=2$ in Corollary 9 of equation (4), by the way of Gajda.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this study. All authors have read and approved the final version of the manuscript.

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Research Article

Fractional Analysis of Coupled Burgers Equations within Yang Caputo-Fabrizio Operator

Nehad Ali Shah,[1] **Essam R. El-Zahar**,[2,3] **and Jae Dong Chung**[1]

[1]*Department of Mechanical Engineering, Sejong University, Seoul 05006, Republic of Korea*
[2]*Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, P.O. Box 83, Al-Kharj 11942, Saudi Arabia*
[3]*Department of Basic Engineering Science, Faculty of Engineering, Menoufia University, Shebin El-Kom 32511, Egypt*

Correspondence should be addressed to Jae Dong Chung; jdchung@sejong.ac.kr

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This work applies a novel analytical technique to the fractional view analysis of coupled Burgers equations. The proposed problems have been fractionally analyzed in the Caputo-Fabrizio sense. The Yang transformation was initially applied to the specified problem in the current approach. The series form solution is then obtained using the Adomian decomposition technique. The desired analytical solution is obtained after performing the inverse transform. Specific examples of fractional Burgers couple systems are used to validate the proposed technique. The current strategy has been found to be a useful methodology with a close match to actual solutions. The proposed method offers a lower computing cost and a faster convergence rate. As a result, the suggested technique can be applied to a variety of fractional order problems.

1. Introduction

The branch of mathematics, which deals with the study of derivatives and integrals of non-integer orders, is known as fractional calculus (FC). It was born in 1695 on September 30 due to an important question asked by L'Hospital in a letter to Leibniz. The answer of Leibniz [1] gives motivation to a series of interesting results during the last 325 years [2–4]. In the last decades, FC has been used as a powerful tool by many researchers in various fields of science and engineering, for example, the fractional control theory [2, 5], anomalous diffusion, fractional neutron point kinetic model, fractional filters, soft matter mechanics, non-Fourier heat conduction, notably control theory, Levy statistics, nonlocal phenomena, fractional signal and image processing, porous media, fractional Brownian motion, relaxation, groundwater problems, rheology, acoustic dissipation, creep, fractional phase-locked loops, and fluid dynamics [6–10].

In recent years, fractional partial differential equations (FPDEs) have gained considerable interest because of their applications in various fields such as finance, biological processes and systems, fluid flow [11, 12], chaotic dynamics, electrochemistry, diffusion processes, material science, electromagnetic, turbulent flow [13–18], elastoplastic indentation problems [19], dynamics of van der Pol equation [20], and statistical mechanics model [21].

To find the solution of FPDEs is a hard task, however, many mathematicians devoted their sincere work and developed numerical and analytical techniques to solve FPDEs. Some of these techniques include homotopy analysis method (HAM) [22], operational matrix [23], Adomian decomposition method (ADM) [24], homotopy perturbation method (HPM) [25], meshless method [26], variational iteration method (VIM) [27], tau method [28], Bernstein polynomials [29], the Haar wavelet method [30], the Laplace transform method [31], the Legendre base method [32], Laplace variational iteration method [33], G'/G-expansion method [34], Jacobi spectral collocation method [35], Yang-Laplace transform [36], new spectral algorithm [37], fractional complex transform method [38], cylindrical-coordinate method [39], and spectral Legendre-Gauss-Lobatto collocation method [40].

The Burgers equation was initially introduced by Harry Bateman in the year 1915 [41]. They have many applications in various fields, especially in equations having nonlinear form. This equation describes many phenomena such as acoustic waves, heat conduction, dispersive water, shock waves [42], continuous stochastic processes [43], and modeling of dynamics [44–46]. The one-dimensional Burgers equations have many applications in plasma physics, gas dynamics, etc. [47]. Various techniques were developed by mathematicians to find the numerical and analytical solutions of Burgers equations. Some of these methods are a direct variational iteration method by Ozis and Ozdes [48]. Jaiswal [49] solved the equations numerically by finite difference method. Group explicit method was used by Evans and Abdullah [50]. Singhal and Mittal applied the Galerkin method [51] to solve these equations numerically. A weighted residue method was applied by Caldwell et al. [52]. Fractional Riccati expansion method was applied by Kurt et al. [53], and variational iteration method was applied by Inc [54] to solve space-time fractional Burgers equation. Esen et al. [55] used HAM to solve time-fractional Burgers equation. The cubic B-spline finite elements method was applied by Esen and Tasbozan to solve these equations [56].

Yang decomposition method (YDM) is one of the straightforward and effective techniques to solve nonlinear FPDEs. YDM possesses the combined behavior of Yang transformation and Adomian decomposition method (ADM). It is observed that the suggested method require no predefined declaration size like RK4. Laplace Adomian decomposition method required less number of parameters, no discritization, and linerization as compared to other analytical technique. Laplace Adomian decomposition method is also compared with ADM to analyze the solution of FPDEs given in [57]. The solution of Kundu-Eckhaus equation is discussed in [58], via Laplace Adomian decomposition method. Multistep Laplace Adomian decomposition method is implemented to solve FPDEs in [59]. Laplace Adomian decomposition method is also used for the solution of fractional Navier-Stokes and smoke models [60–62].

In the current study, we implemented YDM for the solution of coupled Burgers equations. The desired degree of accuracy is achieved. The procedure of the suggested technique is very simple and straightforward. The accuracy is calculated in terms of absolute error. The results have shown the present method has the desired accuracy as compared to other analytical techniques.

2. Preliminary Concepts

We provide the fundamental definitions that will be used throughout the article. For the purpose of simplification, we write the exponential decay kernel as, $K(\Psi, \varrho) = e^{[-\wp(\Psi-\varrho/1-\wp)]}$.

Definition 1. If the Caputo-Fabrizio derivative is given as follows [63]:

$$ {}^{CF}D_{\Psi}^{\wp}[\mathbb{P}(\Psi)] = \frac{N(\wp)}{1-\wp}\int_0^{\Psi}\mathbb{P}'(\varrho)K(\Psi,\varrho)d\varrho, \quad n-1<\wp\leq n. \tag{1}$$

$N(\wp)$ is the normalization function with $N(0) = N(1) = 1$.

$$ {}^{CF}D_{\Psi}^{\wp}[\mathbb{P}(\Psi)] = \frac{N(\wp)}{1-\wp}\int_0^{\Psi}[\mathbb{P}(\Psi)-\mathbb{P}(\varrho)]K(\Psi,\varrho)d\varrho. \tag{2}$$

Definition 2. The fractional integral Caputo-Fabrizio is given as [63]

$$ {}^{CF}I_{\Psi}^{\wp}[\mathbb{P}(\Psi)] = \frac{1-\wp}{N(\wp)}\mathbb{P}(\Psi) + \frac{\wp}{N(\wp)}\int_0^{\Psi}\mathbb{P}(\varrho)d\varrho, \quad \Psi\geq 0, \wp\in(0,1]. \tag{3}$$

Definition 3. For $N(\wp) = 1$, the following result shows the Caputo-Fabrizio derivative of Laplace transformation [63]:

$$ L\left[{}^{CF}D_{\Psi}^{\wp}[\mathbb{P}(\Psi)]\right] = \frac{\nu L[\mathbb{P}(\Psi)-\mathbb{P}(0)]}{\nu+\wp(1-\nu)}. \tag{4}$$

Definition 4. The Yang transformation of $\mathbb{P}(\Psi)$ is expressed as [64].

$$ \mathbb{Y}[\mathbb{P}(\Psi)] = \chi(\nu) = \int_0^{\infty}\mathbb{P}(\Psi)e^{-\frac{\Psi}{\nu}}d\Psi. \quad \Psi>0. \tag{5}$$

Remarks 5. Yang transformation of few useful functions is defined as below.

$$ \begin{aligned} \mathbb{Y}[1] &= \nu, \\ \mathbb{Y}[\Psi] &= \nu^2, \\ \mathbb{Y}\left[\Psi^i\right] &= \Gamma(i+1)\nu^{i+1}. \end{aligned} \tag{6}$$

Lemma 6 (Laplace-Yang duality). *Let the Laplace transformation of $\mathbb{P}(\Psi)$ is $F(\nu)$, then $\chi(\nu) = F(1/\nu)$ [65].*

Proof. From equation (5), we can achieve another type of the Yang transformation by putting $\Psi/\nu = \zeta$ as

$$ L[\mathbb{P}(\Psi)] = \chi(\nu) = \nu\int_0^{\infty}\mathbb{P}(\nu\zeta)e^{\zeta}d\zeta. \quad \zeta>0, \tag{7}$$

Since $L[\mathbb{P}(\Psi)] = F(\nu)$, this implies that

$$ F(\nu) = L[\mathbb{P}(\Psi)] = \int_0^{\infty}\mathbb{P}(\Psi)e^{-\nu\Psi}d\Psi. \tag{8}$$

Put $\Psi = \zeta/\nu$ in (8), we have

$$ F(\nu) = \frac{1}{\nu}\int_0^{\infty}\mathbb{P}\left(\frac{\zeta}{\nu}\right)e^{\zeta}d\zeta. \tag{9}$$

Thus, from equation (7), we achieve

$$F(\nu) = \chi\left(\frac{1}{\nu}\right). \tag{10}$$

Also from equations. (5) and (8), we achieve

$$F\left(\frac{1}{\nu}\right) = \chi(\nu). \tag{11}$$

The connections (10) and (11) represent the duality link between the Laplace and Yang transformation. □

Lemma 7. *Let $\mathbb{P}(\Psi)$ be a continuous function; then, the Caputo-Fabrizio derivative Yang transformation of $\mathbb{P}(\Psi)$ is define by [65].*

$$\mathbb{Y}[\mathbb{P}(\Psi)] = \frac{\mathbb{Y}[\mathbb{P}(\Psi) - \nu\mathbb{P}(0)]}{1+\wp(\nu - 1)}. \tag{12}$$

Proof. The Caputo-Fabrizio fractional Laplace transformation is given by

$$L[\mathbb{P}(\Psi)] = \frac{L[\nu\mathbb{P}(\Psi) - \mathbb{P}(0)]}{\nu+\wp(1 - \nu)}. \tag{13}$$

Also, we have that the connection among Laplace and Yang property, i.e., $\chi(\nu) = F(1/\nu)$. To achieve the necessary result, we substitute ν by $1/\nu$ in equation (13), and we get

$$\begin{aligned}\mathbb{Y}[\mathbb{P}(\Psi)] &= \frac{(1/\nu)\mathbb{Y}[\mathbb{P}(\Psi) - \mathbb{P}(0)]}{(1/\nu)+\wp(1 - 1/\nu)},\\ \mathbb{Y}[\mathbb{P}(\Psi)] &= \frac{\mathbb{Y}[\mathbb{P}(\Psi) - \nu\mathbb{P}(0)]}{1+\wp(\nu - 1)}.\end{aligned} \tag{14}$$

The proof is completed. □

3. Implementation of YDM with Caputo-Fabrizio

To explain the fundamental concept of this technique, we consider a particular fractional-order nonlinear partial differential equation:

$$^{CF}D^{\delta}u(\xi,\Psi) + Lu(\xi,\Psi) + Nu(\xi,\Psi) = q(\xi,\Psi), \quad \xi,\Psi \ge 0, \quad m-1 < \delta < m, \tag{15}$$

where the fractional derivative in equation (15) is defined in Caputo-Fabrizio. The operator $\mathscr{R}$ and $\mathscr{N}$ describe the linear and nonlinear operators, respectively, and $g(\zeta,\Psi)$ is the source term.

The initial condition is

$$u(\xi,0) = k(\xi), \tag{16}$$

Using Yang transformation to equation (15), we get

$$\mathscr{Y}\left[D^{\delta}u(\xi,\Psi)\right] + \mathscr{Y}[Lu(\xi,\Psi) + Nu(\xi,\Psi)] = \mathscr{Y}[q(\xi,\Psi)], \tag{17}$$

with the help of fractional derivative Yang property, we have

$$\begin{aligned}&\frac{1}{(1+\delta(s-1))}\mathscr{Y}\{u(\xi,0)\} - su(\xi,0)\\ &\quad = \mathscr{Y}[q(\xi,\Psi)] - \mathscr{Y}[Lu(\xi,\Psi) + Nu(\xi,\Psi)],\end{aligned} \tag{18}$$

$$\begin{aligned}\mathscr{Y}[u(\xi,\Psi)] = sk(\xi) &+ (1+\delta(s-1))\mathscr{Y}[q(\xi,\Psi)]\\ &- (1+\delta(s-1))\mathscr{Y}[Lu(\xi,\Psi) + Nu(\xi,\Psi)].\end{aligned} \tag{19}$$

Using YDM procedure, the solution is expressed as

$$u(\xi,\Psi) = \sum_{j=0}^{\infty} u_j(\xi,\Psi), \tag{20}$$

The nonlinear term can be decomposed as

$$Nu(\xi,\Psi) = \sum_{j=0}^{\infty} A_j, \tag{21}$$

$$A_j = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j}\left[N\sum_{j=0}^{\infty}\left(\lambda^j u_j\right)\right]\right]_{\lambda=0}, \quad j = 0, 1, 2\cdots, \tag{22}$$

substitution (20) and (21) in equation (18), we get

$$\begin{aligned}\mathscr{Y}\left[\sum_{j=0}^{\infty} u(\xi,\Psi)\right] &= sk(\xi) + (1+\delta(s-1))\mathscr{Y}[q(\xi,\Psi)]\\ &\quad - (1+\delta(s-1))\mathscr{Y}\left[L\sum_{j=0}^{\infty} u_j(\xi,\Psi) + \sum_{j=0}^{\infty} A_j\right].\end{aligned} \tag{23}$$

$$\mathscr{Y}[u_0(\xi,\Psi)] = su(\xi,0) + (1+\delta(s-1))\mathscr{Y}[q(\xi,\Psi)], \tag{24}$$

$$\mathscr{Y}[u_1(\xi,\Psi)] = -(1+\delta(s-1))\mathscr{Y}[Lu_0(\xi,\Psi) + A_0]. \tag{25}$$

Generally, we can write

$$\mathscr{Y}\left[u_{j+1}(\xi,\Psi)\right] = -(1+\delta(s-1))\mathscr{Y}\left[Lu_j(\xi,\Psi) + A_j\right], \quad j \ge 1. \tag{26}$$

Taking the inverse Yang transformation of Eq. (26), we get

$$u_0(\xi,\Psi) = k(\xi,\Psi) + \mathscr{Y}^{-1}[(1+\delta(s-1))\mathscr{Y}[q(\xi,\Psi)]], \tag{27}$$

$$u_{j+1}(\xi,\Psi) = -\mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left[Lu_j(\xi,\Psi)+A_j\right]\right]. \tag{28}$$

4. Example

Consider the following fractional-order coupled Burgers equations:

$$\begin{aligned} \frac{{}^{CF}\partial^\delta \mu}{\partial \Psi^\delta} + \frac{\partial^2 \mu}{\partial \zeta^2} - 2\mu\frac{\partial \mu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta} = 0, \\ \frac{{}^{CF}\partial^\delta \nu}{\partial \Psi^\delta} + \frac{\partial^2 \nu}{\partial \zeta^2} - 2\nu\frac{\partial \nu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta} = 0, \quad 0<\delta\le 1, \end{aligned} \tag{29}$$

with initial conditions

$$\mu(\zeta,0) = \sin(\zeta), \quad \nu(\zeta,0) = -\sin(\zeta). \tag{30}$$

Taking Yang transform of (29),

$$\mathscr{Y}\left[\frac{\partial^\delta \mu}{\partial \Psi^\delta}\right] = -\mathscr{Y}\left[\frac{\partial^2 \mu}{\partial \zeta^2} - 2\mu\frac{\partial \mu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right], \tag{31}$$

$$\mathscr{Y}\left[\frac{\partial^\delta \nu}{\partial \Psi^\delta}\right] = -\mathscr{Y}\left[\frac{\partial^2 \nu}{\partial \zeta^2} - 2\nu\frac{\partial \nu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right], \tag{32}$$

$$\frac{1}{(1+\delta(s-1))}\mathscr{Y}\{\mu(\zeta,0)\} - s\mu(\zeta,0) = -\mathscr{Y}\left[\frac{\partial^2 \mu}{\partial \zeta^2} - 2\mu\frac{\partial \mu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right], \tag{33}$$

$$\frac{1}{(1+\delta(s-1))}\mathscr{Y}\{\nu(\zeta,0)\} - s\nu(\zeta,0) = -\mathscr{Y}\left[\frac{\partial^2 \nu}{\partial \zeta^2} - 2\nu\frac{\partial \nu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right]. \tag{34}$$

Applying inverse Yang transform

$$\mu(\zeta,\Psi) = \mathscr{Y}^{-1}\left[s\mu(\zeta,0) - (1+\delta(s-1))\mathscr{Y}\left\{\frac{\partial^2 \mu}{\partial \zeta^2} - 2\mu\frac{\partial \mu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right\}\right], \tag{35}$$

$$\nu(\zeta,\Psi) = \mathscr{Y}^{-1}\left[s\mu(\zeta,0) - (1+\delta(s-1))\mathscr{Y}\left\{\frac{\partial^2 \nu}{\partial \zeta^2} - 2\nu\frac{\partial \nu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right\}\right], \tag{36}$$

$$\mu(\zeta,\Psi) = \sin(\zeta) - \mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\frac{\partial^2 \mu}{\partial \zeta^2} - 2\mu\frac{\partial \mu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right\}\right], \tag{37}$$

$$\nu(\zeta,\Psi) = -\sin(\zeta) - \mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\frac{\partial^2 \nu}{\partial \zeta^2} - 2\nu\frac{\partial \nu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right\}\right]. \tag{38}$$

Using ADM procedure, we get

$$\sum_{j=0}^{\infty}\mu_j(\zeta,\Psi) = \sin(\zeta) - \mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\sum_{j=0}^{\infty}(\mu_{\zeta\zeta})_j - 2\sum_{j=0}^{\infty}A_j(\mu\mu_\zeta) - \sum_{j=0}^{\infty}B_j(\mu\nu)_\zeta\right\}\right], \tag{39}$$

$$\sum_{j=0}^{\infty}\nu_j(\zeta,\Psi) = -\sin(\zeta) - \mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\sum_{j=0}^{\infty}(\nu_{\zeta\zeta})_j - 2\sum_{j=0}^{\infty}C_j(\nu\nu_\zeta) - \sum_{j=0}^{\infty}D_j(\mu\nu)_\zeta\right\}\right], \tag{40}$$

where $A_j(\mu\mu_\zeta)$, $B_j(\mu\nu)_\zeta$, $C_j(\nu\nu_\zeta)$, and $D_j(\mu\nu)_\zeta$ are Adomian polynomials are given below,

$$\begin{aligned} A_0(\mu\mu_\zeta) &= \mu_0\frac{\partial \mu_0}{\partial \zeta}, & B_0(\mu\nu)_\zeta &= \frac{\partial \mu_0}{\partial \zeta}\frac{\partial \nu_0}{\partial \zeta}, \\ A_1(\mu\mu_\zeta) &= \mu_0\frac{\partial \mu_1}{\partial \zeta} + \mu_1\frac{\partial \mu_0}{\partial \zeta}, & B_1(\mu\nu)_\zeta &= \frac{\partial \mu_0}{\partial \zeta}\frac{\partial \nu_1}{\partial \zeta} + \frac{\partial \mu_1}{\partial \zeta}\frac{\partial \nu_0}{\partial \zeta}, \\ A_2(\mu\mu_\zeta) &= \mu_0\frac{\partial \mu_2}{\partial \zeta} + \mu_1\frac{\partial \mu_1}{\partial \zeta} + \mu_2\frac{\partial \mu_0}{\partial \zeta}. & B_2(\mu\nu)_\zeta &= \frac{\partial \mu_0}{\partial \zeta}\frac{\partial \nu_2}{\partial \zeta} + \frac{\partial \mu_1}{\partial \zeta}\frac{\partial \nu_1}{\partial \zeta} + \frac{\partial \mu_2}{\partial \zeta}\frac{\partial \nu_0}{\partial \zeta}. \end{aligned} \tag{41}$$

$$\begin{aligned} C_0(\nu\nu_\zeta) &= \nu_0\frac{\partial \nu_0}{\partial \zeta}, & D_0(\mu\nu)_\zeta &= \frac{\partial \mu_0}{\partial \zeta}\frac{\partial \nu_0}{\partial \zeta}, \\ C_1(\nu\nu_\zeta) &= \nu_0\frac{\partial \nu_1}{\partial \zeta} + \nu_1\frac{\partial \nu_0}{\partial \zeta}, & D_1(\mu\nu)_\zeta &= \frac{\partial \mu_0}{\partial \zeta}\frac{\partial \nu_1}{\partial \zeta} + \frac{\partial \mu_1}{\partial \zeta}\frac{\partial \nu_0}{\partial \zeta}, \\ C_2(\nu\nu_\zeta) &= \nu_0\frac{\partial \nu_2}{\partial \zeta} + \nu_1\frac{\partial \nu_1}{\partial \zeta} + \nu_2\frac{\partial \nu_0}{\partial \zeta}. & D_2(\mu\nu)_\zeta &= \frac{\partial \mu_0}{\partial \zeta}\frac{\partial \nu_2}{\partial \zeta} + \frac{\partial \mu_1}{\partial \zeta}\frac{\partial \nu_1}{\partial \zeta} + \frac{\partial \mu_2}{\partial \zeta}\frac{\partial \nu_0}{\partial \zeta}. \end{aligned} \tag{42}$$

$$\begin{aligned} \mu_0(\zeta,\Psi) &= \sin\zeta, \\ \nu_0(\zeta,\Psi) &= -\sin(\zeta), \end{aligned} \tag{43}$$

$$\mu_{j+1}(\zeta,\Psi) = -\mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\sum_{j=0}^{\infty}(\mu_{\zeta\zeta})_j - 2\sum_{j=0}^{\infty}A_j(\mu\mu_\zeta) - \sum_{j=0}^{\infty}B_j(\mu\nu)_\zeta\right\}\right], \tag{44}$$

$$\nu_{j+1}(\zeta,\Psi) = -\mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\sum_{j=0}^{\infty}(\nu_{\zeta\zeta})_j - 2\sum_{j=0}^{\infty}C_j(\nu\nu_\zeta) - \sum_{j=0}^{\infty}D_j(\mu\nu)_\zeta\right\}\right], \tag{45}$$

for $j = 0, 1, 2 \cdots$

$$\begin{aligned} \mu_1(\zeta,\Psi) &= -\mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\frac{\partial^2 \mu_0}{\partial \zeta^2} - 2\mu_0\frac{\partial \mu_0}{\partial \zeta} - \frac{\partial \mu_0}{\partial \zeta}\frac{\partial \nu_0}{\partial \zeta}\right\}\right], \\ \mu_1(\zeta,\Psi) &= -\mathscr{Y}^{-1}\left[(1+\delta(s-1)) \times \frac{-\sin\zeta}{s}\right] = \sin(\zeta)\{\delta\Psi + (1-\delta)\}, \\ \nu_1(\zeta,\Psi) &= -\mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\frac{\partial^2 \nu_0}{\partial \zeta^2} - 2\nu_0\frac{\partial \nu_0}{\partial \zeta} - \frac{\partial \mu_0}{\partial \zeta}\frac{\partial \nu_0}{\partial \zeta}\right\}\right] \\ \nu_1(\zeta,\Psi) &= -\mathscr{Y}^{-1}\left[(1+\delta(s-1)) \times \frac{\sin(\zeta)}{s}\right] = -\sin(\zeta)\{\delta\Psi + (1-\delta)\}, \end{aligned} \tag{46}$$

The subsequent terms are

$$\mu_2(\zeta,\Psi) = -\mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\frac{\partial^2\mu_1}{\partial\zeta^2} - 2\mu_0\frac{\partial\mu_1}{\partial\zeta} - 2\mu_1\frac{\partial\mu_0}{\partial\zeta} - \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_1}{\partial\zeta} - \frac{\partial\mu_1}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta}\right\}\right],$$
$$\mu_2(\zeta,\Psi) = \sin(\zeta)\left\{(1-\delta)^2 + 2\delta(1-\delta)\Psi + \frac{\delta^2\Psi^2}{2}\right\},$$
$$\nu_2(\zeta,\Psi) = -\mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\frac{\partial^2\nu_1}{\partial\zeta^2} - 2\nu_0\frac{\partial\nu_1}{\partial\zeta} - 2\nu_1\frac{\partial\nu_0}{\partial\zeta} - \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_1}{\partial\zeta} - \frac{\partial\mu_1}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta}\right\}\right]$$
$$\nu_2(\zeta,\Psi) = -\sin(\zeta)\left\{(1-\delta)^2 + 2\delta(1-\delta)\Psi + \frac{\delta^2\Psi^2}{2}\right\}, \tag{47}$$

The YDM solution for example (4) is

$$\mu(\zeta,\Psi) = \mu_0(\zeta,\Psi) + \mu_1(\zeta,\Psi) + \mu_2(\zeta,\Psi) + \mu_3(\zeta,\Psi) + \cdots, \tag{48}$$

$$\nu(\zeta,\Psi) = \nu_0(\zeta,\Psi) + \nu_1(\zeta,\Psi) + \nu_2(\zeta,\Psi) + \nu_3(\zeta,\Psi) + \cdots, \tag{49}$$

$$\mu(\zeta,\Psi) = \sin(\zeta) + \sin(\zeta)\{\delta\Psi + (1-\delta)\} + \sin(\zeta)\left\{(1-\delta)^2 + 2\delta(1-\delta)\Psi + \frac{\delta^2\Psi^2}{2}\right\} + \cdots, \tag{50}$$

$$\nu(\zeta,\Psi) = -\sin(\zeta) - \sin(\zeta)\{\delta\Psi + (1-\delta)\} - \sin(\zeta)\left\{(1-\delta)^2 + 2\delta(1-\delta)\Psi + \frac{\delta^2\Psi^2}{2}\right\} - \cdots, \tag{51}$$

when $\delta = 1$, then YDM solution is

$$\mu(\zeta,\Psi) = \sin(\zeta) + \sin(\zeta)\Psi + \sin(\zeta)\frac{\Psi^2}{2} + \sin(\zeta)\frac{\Psi^3}{6} + \sin(\zeta)\frac{\Psi^4}{24} + \cdots, \tag{52}$$

$$\nu(\zeta,\Psi) = -\sin(\zeta) - \sin(\zeta)\Psi - \sin(\zeta)\frac{\Psi^2}{2} - \sin(\zeta)\frac{\Psi^3}{6} - \sin(\zeta)\frac{\Psi^4}{24} - \cdots. \tag{53}$$

The exact solutions are

$$\mu(\zeta,\Psi) = e^{\Psi}\sin(\zeta),$$
$$\nu(\zeta,\Psi) = -e^{\Psi}\sin(\zeta). \tag{54}$$

5. Example

Consider the following fractional-order couple Burgers equations [17]:

$$\frac{{}^{CF}\partial^\delta\mu}{\partial\Psi^\delta} + \mu\frac{\partial\mu}{\partial\zeta} + \nu\frac{\partial\mu}{\partial\xi} - \frac{\partial^2\mu}{\partial\zeta^2} - \frac{\partial^2\mu}{\partial\xi^2} = 0,$$
$$\frac{{}^{CF}\partial^\delta\nu}{\partial\Psi^\delta} + \mu\frac{\partial\nu}{\partial\zeta} + \nu\frac{\partial\nu}{\partial\xi} - \frac{\partial^2\nu}{\partial\zeta^2} - \frac{\partial^2\nu}{\partial\xi^2} = 0, \quad 0 < \delta \le 1, \tag{55}$$

with initial condition

$$\mu(\zeta,\xi,0) = \zeta + \xi, \quad \nu(\zeta,\xi,0) = \zeta - \xi. \tag{56}$$

Taking Yang transform of (55),

$$\mathscr{Y}\left[\frac{\partial^\delta\mu}{\partial\Psi^\delta}\right] = -\mathscr{Y}\left[\mu\frac{\partial\mu}{\partial\zeta} + \nu\frac{\partial\mu}{\partial\xi} - \frac{\partial^2\mu}{\partial\zeta^2} - \frac{\partial^2\mu}{\partial\xi^2}\right], \tag{57}$$

$$\mathscr{Y}\left[\frac{\partial^\delta\nu}{\partial\Psi^\delta}\right] = -\mathscr{Y}\left[\mu\frac{\partial\nu}{\partial\zeta} + \nu\frac{\partial\nu}{\partial\xi} - \frac{\partial^2\nu}{\partial\zeta^2} - \frac{\partial^2\nu}{\partial\xi^2}\right], \tag{58}$$

$$\frac{1}{(1+\delta(s-1))}\mathscr{Y}\{\mu(\zeta,\xi,0)\} - s\mu(\zeta,\xi,0) = -\mathscr{Y}\left[\mu\frac{\partial\mu}{\partial\zeta} + \nu\frac{\partial\mu}{\partial\xi} - \frac{\partial^2\mu}{\partial\zeta^2} - \frac{\partial^2\mu}{\partial\xi^2}\right], \tag{59}$$

$$\frac{1}{(1+\delta(s-1))}\mathscr{Y}\{\nu(\zeta,\xi,0)\} - s\nu(\zeta,\xi,0) = -\mathscr{Y}\left[\mu\frac{\partial\nu}{\partial\zeta} + \nu\frac{\partial\nu}{\partial\xi} - \frac{\partial^2\nu}{\partial\zeta^2} - \frac{\partial^2\nu}{\partial\xi^2}\right]. \tag{60}$$

Applying inverse Yang transform

$$\mu(\zeta,\xi,\Psi) = \mathscr{Y}^{-1}\left[s\mu(\zeta,\xi,0) - (1+\delta(s-1))\mathscr{Y}\left\{\mu\frac{\partial\mu}{\partial\zeta} + \nu\frac{\partial\mu}{\partial\xi} - \frac{\partial^2\mu}{\partial\zeta^2} - \frac{\partial^2\mu}{\partial\xi^2}\right\}\right], \tag{61}$$

$$\nu(\zeta,\xi,\Psi) = \mathscr{Y}^{-1}\left[s\nu(\zeta,\xi,0) - (1+\delta(s-1))\mathscr{Y}\left\{\mu\frac{\partial\nu}{\partial\zeta} + \nu\frac{\partial\nu}{\partial\xi} - \frac{\partial^2\nu}{\partial\zeta^2} - \frac{\partial^2\nu}{\partial\xi^2}\right\}\right], \tag{62}$$

$$\mu(\zeta,\xi,\Psi) = \zeta + \xi - \mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\mu\frac{\partial\mu}{\partial\zeta} + \nu\frac{\partial\mu}{\partial\xi} - \frac{\partial^2\mu}{\partial\zeta^2} - \frac{\partial^2\mu}{\partial\xi^2}\right\}\right], \tag{63}$$

$$\nu(\zeta,\xi,\Psi) = \zeta - \xi - \mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\mu\frac{\partial\nu}{\partial\zeta} + \nu\frac{\partial\nu}{\partial\xi} - \frac{\partial^2\nu}{\partial\zeta^2} - \frac{\partial^2\nu}{\partial\xi^2}\right\}\right]. \tag{64}$$

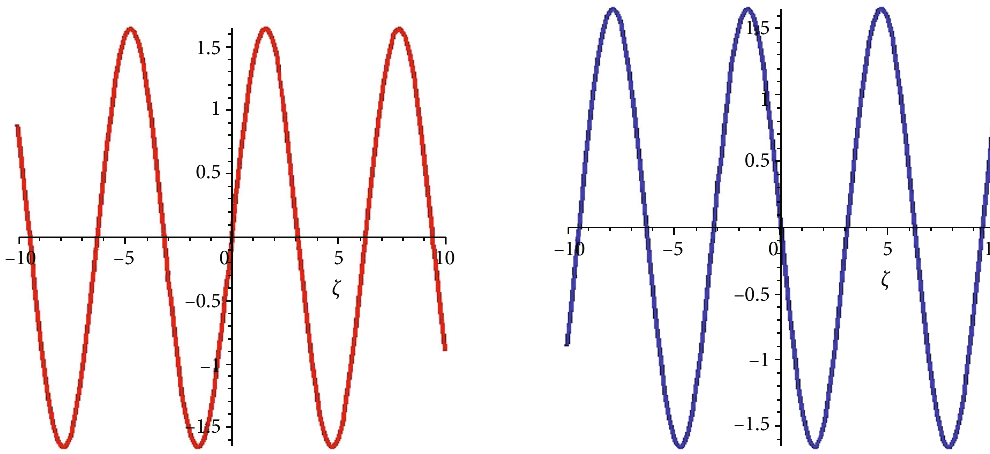

FIGURE 1: YDM solutions of $\mu(\zeta,\Psi)$ and $\nu(\zeta,\Psi)$ for example 1 at $\delta = 1$.

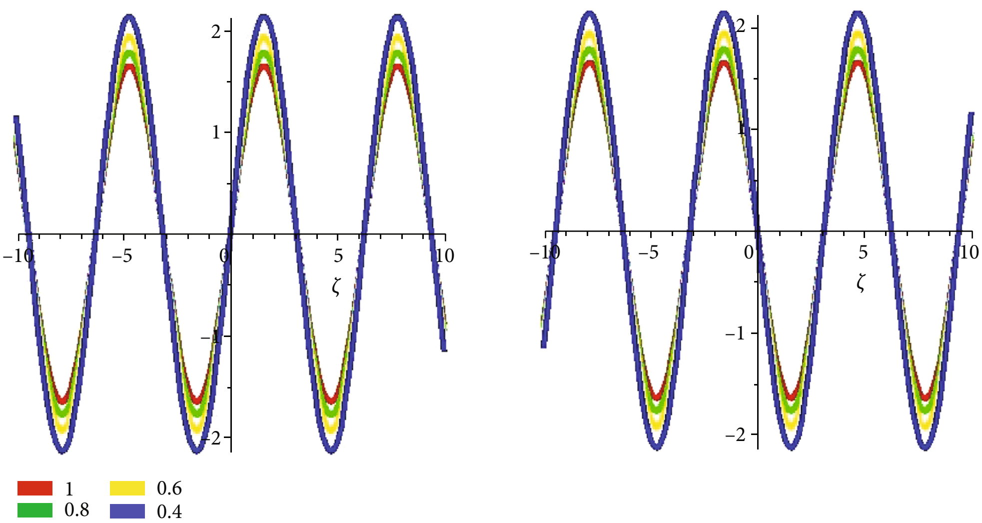

FIGURE 2: YDM solutions of $\mu(\zeta,\Psi)$ and $\nu(\zeta,\Psi)$ for example 1 at different value of δ.

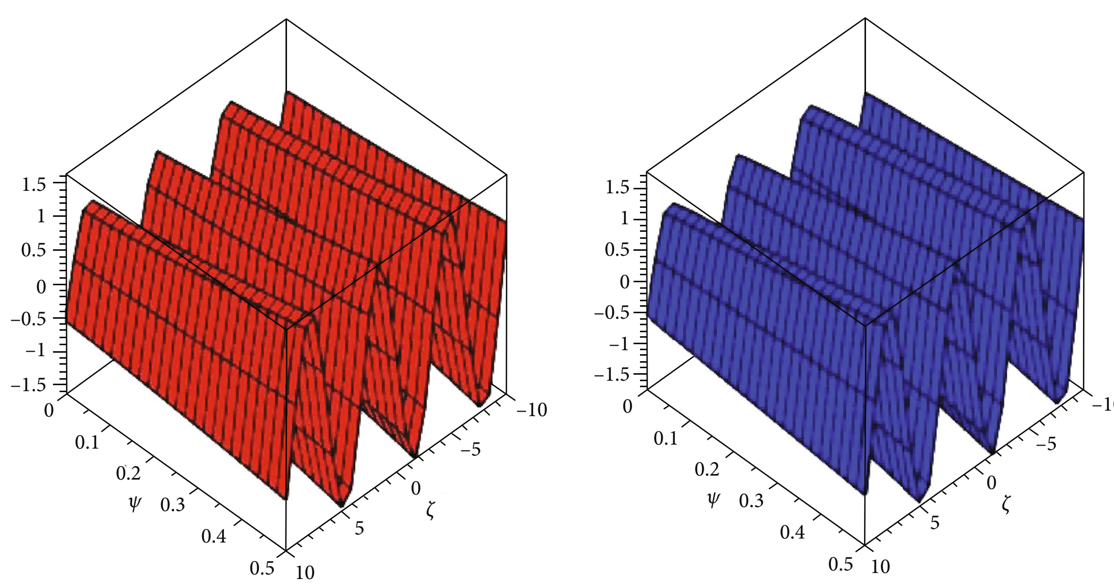

FIGURE 3: The YDM solution of example 1 of $\mu(\zeta,\Psi)$ at $\delta = 1$, and 0.8.

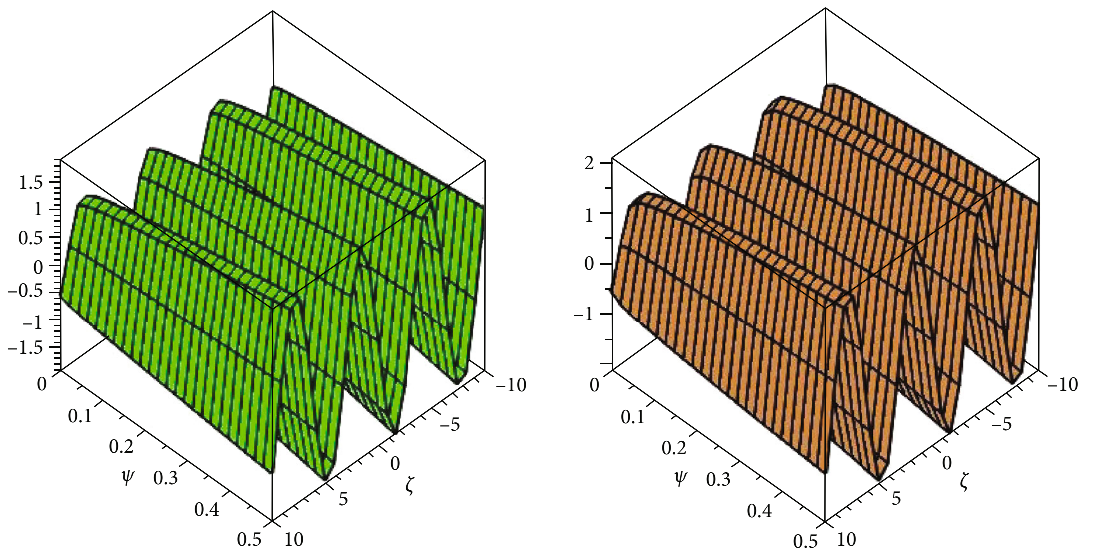

FIGURE 4: The YDM solution of example 1 of $\mu(\zeta,\Psi)$ at $\delta=0.6$, and 0.4.

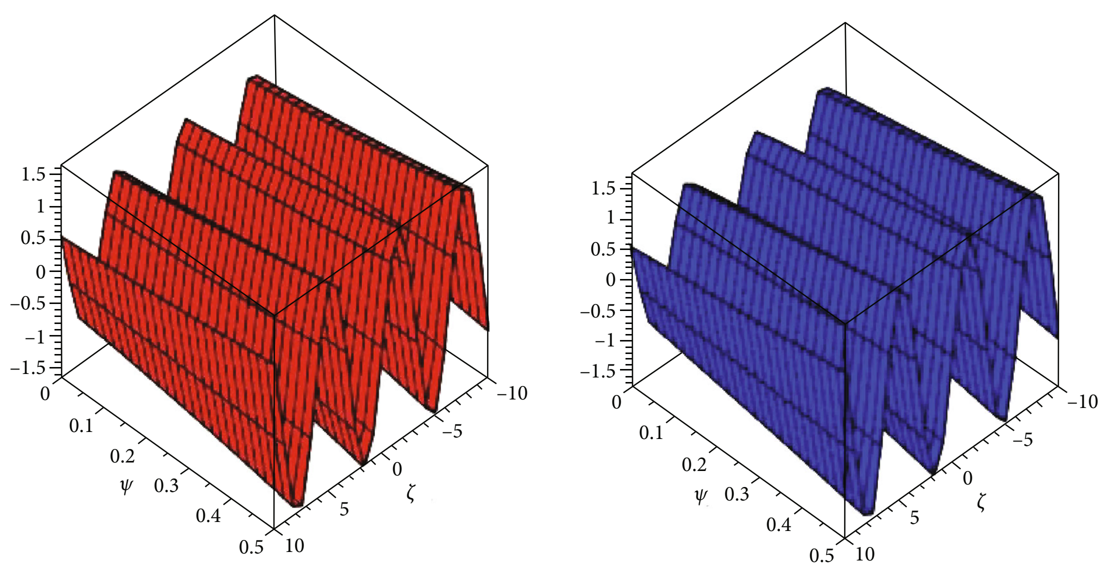

FIGURE 5: The YDM solution of example 1 of $\mu(\zeta,\Psi)$ at $\delta=1$, and 0.8.

Using ADM procedure, we get

$$\sum_{j=0}^{\infty}\mu_j(\zeta,\xi,\Psi)=\zeta+\xi-\mathcal{Y}^{-1}\left[(1+\delta(s-1))\mathcal{Y}\left\{\sum_{j=0}^{\infty}A_j(\mu\mu_\zeta)+\sum_{j=0}^{\infty}B_j(\nu\mu_\xi)-\sum_{j=0}^{\infty}\mu_{\zeta\zeta}-\sum_{j=0}^{\infty}\mu_{\xi\xi}\right\}\right], \tag{65}$$

$$\sum_{j=0}^{\infty}\nu_j(\zeta,\xi,\Psi)=\zeta-\xi-\mathcal{Y}^{-1}\left[(1+\delta(s-1))\mathcal{Y}\left\{\sum_{j=0}^{\infty}C_j(\mu\nu_\zeta)+\sum_{j=0}^{\infty}D_j(\nu\nu_\xi)-\sum_{j=0}^{\infty}\nu_{\zeta\zeta}-\sum_{j=0}^{\infty}\nu_{\xi\xi}\right\}\right], \tag{66}$$

where $A_j(\mu\mu_\zeta)$, $B_j(\nu\mu_\xi)$, $C_j(\mu\nu_\zeta)$, and $D_j(\nu\nu_\xi)$, the Adomian polynomials are given below,

$$\begin{aligned}
A_0(\mu\mu_\zeta)&=\mu_0\frac{\partial\mu_0}{\partial\zeta}, & B_0(\nu\mu_\xi)&=\nu_0\frac{\partial\mu_0}{\partial\xi},\\
A_1(\mu\mu_\zeta)&=\mu_0\frac{\partial\mu_1}{\partial\zeta}+\mu_1\frac{\partial\mu_0}{\partial\zeta}, & B_1(\nu\mu_\xi)&=\nu_0\frac{\partial\mu_1}{\partial\xi}+\nu_1\frac{\partial\mu_0}{\partial\xi},\\
A_2(\mu\mu_\zeta)&=\mu_0\frac{\partial\mu_2}{\partial\zeta}+\mu_1\frac{\partial\mu_1}{\partial\zeta}+\mu_2\frac{\partial\mu_0}{\partial\zeta}. & B_2(\nu\mu_\xi)&=\nu_0\frac{\partial\mu_2}{\partial\xi}+\nu_1\frac{\partial\mu_1}{\partial\xi}+\nu_2\frac{\partial\mu_0}{\partial\xi}.
\end{aligned} \tag{67}$$

$$\begin{aligned}
C_0(\mu\nu_\zeta)&=\mu_0\frac{\partial\nu_0}{\partial\zeta}, & D_0(\nu\nu_\xi)&=\nu_0\frac{\partial\nu_0}{\partial\xi},\\
C_1(\mu\nu_\zeta)&=\mu_0\frac{\partial\nu_1}{\partial\zeta}+\mu_1\frac{\partial\nu_0}{\partial\zeta}, & D_1(\nu\nu_\xi)&=\nu_0\frac{\partial\nu_1}{\partial\xi}+\nu_1\frac{\partial\nu_0}{\partial\xi},\\
C_2(\mu\nu_\zeta)&=\mu_0\frac{\partial\nu_2}{\partial\zeta}+\mu_1\frac{\partial\nu_1}{\partial\zeta}+\mu_2\frac{\partial\nu_0}{\partial\zeta}. & D_2(\nu\nu_\xi)&=\nu_0\frac{\partial\nu_2}{\partial\xi}+\nu_1\frac{\partial\nu_1}{\partial\xi}+\nu_2\frac{\partial\nu_0}{\partial\xi}.
\end{aligned} \tag{68}$$

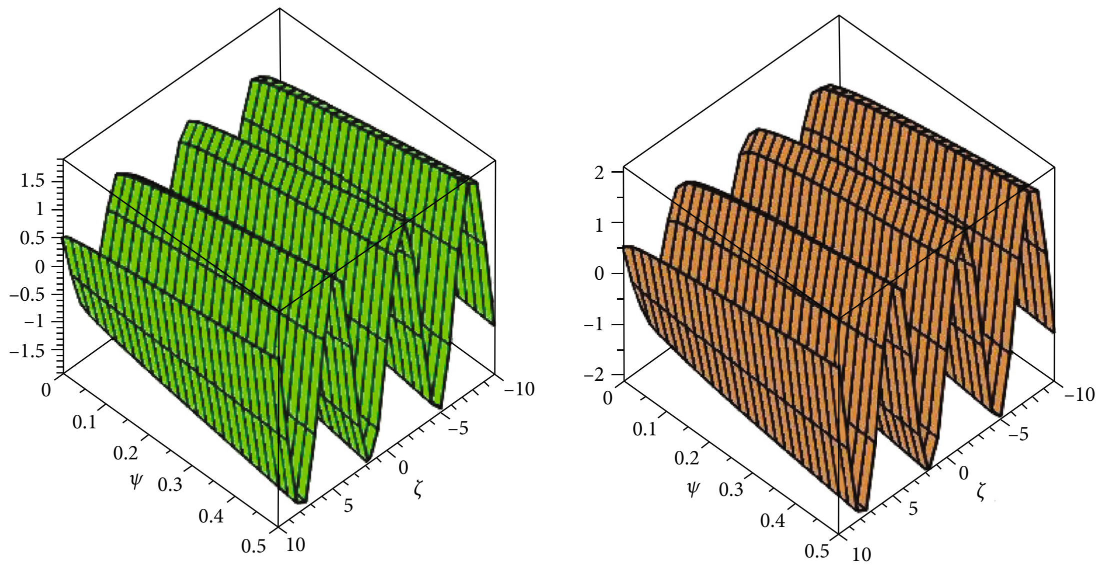

Figure 6: The YDM solution of example 1 of $\nu(\zeta, \Psi)$ at $\delta = 1$, and 0.8.

Table 1: YDM-solutions of example 1 $\mu(\zeta, \Psi)$ and $\nu(\zeta, \Psi)$ different fractional-order of δ.

Ψ	ζ	Absolute error ($\delta = 0.4$)	Absolute error ($\delta = 0.6$)	Absolute error ($\delta = 0.8$)	Absolute error ($\delta = 1$)
	1	$1.6795833810 \times 10^{-02}$	$5.4134672620 \times 10^{-03}$	$7.0667992140 \times 10^{-04}$	$7.0683562720 \times 10^{-09}$
	2	$1.8149655480 \times 10^{-02}$	$5.8498176890 \times 10^{-03}$	$7.6364158210 \times 10^{-04}$	$7.6380983850 \times 10^{-09}$
0.1	3	$2.8167675970 \times 10^{-03}$	$9.0787271070 \times 10^{-04}$	$1.1851469400 \times 10^{-04}$	$1.1854080680 \times 10^{-09}$
	4	$1.5105843420 \times 10^{-02}$	$4.8687662520 \times 10^{-03}$	$6.3557405730 \times 10^{-04}$	$6.3571409610 \times 10^{-09}$
	5	$1.9140211660 \times 10^{-02}$	$6.1690839760 \times 10^{-03}$	$8.0531895140 \times 10^{-04}$	$8.0549639070 \times 10^{-09}$
	1	$2.2895909420 \times 10^{-02}$	$8.0513685670 \times 10^{-03}$	$2.3663474260 \times 10^{-04}$	$2.3207769760 \times 10^{-07}$
	2	$2.4741425310 \times 10^{-02}$	$8.7003460040 \times 10^{-03}$	$2.5570859420 \times 10^{-04}$	$2.5078423030 \times 10^{-07}$
0.2	3	$3.8397888720 \times 10^{-03}$	$1.3502654490 \times 10^{-03}$	$3.9685143520 \times 10^{-05}$	$3.8920898230 \times 10^{-08}$
	4	$2.0592131750 \times 10^{-02}$	$7.2412429330 \times 10^{-03}$	$2.1282464510 \times 10^{-04}$	$2.0872612820 \times 10^{-07}$
	5	$2.6091741410 \times 10^{-02}$	$9.1751859580 \times 10^{-03}$	$2.6966443650 \times 10^{-04}$	$2.6447131500 \times 10^{-07}$
	1	$2.7579918610 \times 10^{-02}$	$1.0158028710 \times 10^{-02}$	$3.1243332140 \times 10^{-04}$	$1.7930063740 \times 10^{-06}$
	2	$2.9802987250 \times 10^{-02}$	$1.0976812670 \times 10^{-02}$	$3.3761688790 \times 10^{-04}$	$1.9375309570 \times 10^{-06}$
0.3	3	$4.6253268490 \times 10^{-02}$	$1.7035656840 \times 10^{-03}$	$5.2397044750 \times 10^{-05}$	$3.0069851330 \times 10^{-07}$
	4	$2.4804837720 \times 10^{-02}$	$9.1359317340 \times 10^{-02}$	$2.8099639980 \times 10^{-04}$	$1.6125947570 \times 10^{-06}$
	5	$3.1429548890 \times 10^{-02}$	$1.1575895650 \times 10^{-02}$	$3.5604305020 \times 10^{-04}$	$2.0432758450 \times 10^{-06}$
	1	$3.1561849440 \times 10^{-02}$	$1.1987254260 \times 10^{-02}$	$3.7995256610 \times 10^{-04}$	$7.6880155060 \times 10^{-06}$
	2	$3.4105880060 \times 10^{-02}$	$1.2953482240 \times 10^{-02}$	$4.1057849520 \times 10^{-04}$	$8.3077050100 \times 10^{-06}$
0.4	3	$5.2931218410 \times 10^{-02}$	$2.0103383830 \times 10^{-03}$	$6.3720449280 \times 10^{-05}$	$1.2893288420 \times 10^{-06}$
	4	$2.8386108190 \times 10^{-02}$	$1.0781101310 \times 10^{-02}$	$3.4172188380 \times 10^{-04}$	$6.9144503180 \times 10^{-06}$
	5	$3.5967281260 \times 10^{-02}$	$1.3660446180 \times 10^{-02}$	$4.3298669290 \times 10^{-04}$	$8.7611157430 \times 10^{-06}$
	1	$3.5108679510 \times 10^{-02}$	$1.3639899070 \times 10^{-02}$	$4.4195033070 \times 10^{-04}$	$2.3878506280 \times 10^{-06}$
	2	$3.7938600990 \times 10^{-02}$	$1.4739337840 \times 10^{-02}$	$4.7757356550 \times 10^{-04}$	$2.5803224010 \times 10^{-06}$
0.5	3	$5.8879476850 \times 10^{-02}$	$2.2874973730 \times 10^{-03}$	$7.4117866660 \times 10^{-05}$	$4.0045765820 \times 10^{-06}$
	4	$3.1576057570 \times 10^{-02}$	$1.2267457630 \times 10^{-02}$	$3.9748145700 \times 10^{-04}$	$2.1475860090 \times 10^{-05}$
	5	$4.0009181120 \times 10^{-02}$	$1.5543768660 \times 10^{-02}$	$5.0363816220 \times 10^{-04}$	$2.7211490040 \times 10^{-05}$

Table 2: YDM-solutions of example 1 at $\mu(\zeta, \Psi)$ different fractional-order of δ.

Ψ	ζ	Absolute error ($\delta = 0.4$)	Absolute error ($\delta = 0.6$)	Absolute error ($\delta = 0.8$)	Absolute error ($\delta = 1$)
	1	$5.1388035000 \times 10^{-03}$	$8.5868730000 \times 10^{-04}$	$1.2436850000 \times 10^{-05}$	$1.4585000000 \times 10^{-10}$
	2	$1.0422343800 \times 10^{-02}$	$1.7352281000 \times 10^{-03}$	$2.2053740000 \times 10^{-05}$	$2.1331000000 \times 10^{-09}$
0.1	3	$1.4825705100 \times 10^{-02}$	$2.4804600000 \times 10^{-03}$	$3.3501640000 \times 10^{-05}$	$2.6843000000 \times 10^{-09}$
	4	$2.1340157300 \times 10^{-02}$	$3.3450207000 \times 10^{-03}$	$4.5130530000 \times 10^{-05}$	$3.3374000000 \times 10^{-09}$
	5	$2.5732507500 \times 10^{-02}$	$4.2223515000 \times 10^{-03}$	$5.6568430000 \times 10^{-05}$	$4.1714000000 \times 10^{-09}$
	1	$6.5526269000 \times 10^{-03}$	$1.2845443000 \times 10^{-04}$	$1.9524030000 \times 10^{-05}$	$8.9043500000 \times 10^{-08}$
	2	$1.3585147000 \times 10^{-02}$	$2.5823037000 \times 10^{-03}$	$3.9081030000 \times 10^{-05}$	$1.2243480000 \times 10^{-08}$
0.2	3	$2.0617667100 \times 10^{-03}$	$3.8800631000 \times 10^{-03}$	$5.8638020000 \times 10^{-05}$	$1.5582620000 \times 10^{-08}$
	4	$2.7650187200 \times 10^{-02}$	$5.1778225000 \times 10^{-03}$	$7.8195020000 \times 10^{-05}$	$1.8921750000 \times 10^{-08}$
	5	$3.4682707200 \times 10^{-02}$	$6.4755819000 \times 10^{-03}$	$9.7752020000 \times 10^{-05}$	$2.2260880000 \times 10^{-08}$
	1	$7.5239217000 \times 10^{-03}$	$1.6203247000 \times 10^{-04}$	$2.6503270000 \times 10^{-05}$	$9.9570730000 \times 10^{-08}$
	2	$1.5715901300 \times 10^{-02}$	$3.2621458000 \times 10^{-03}$	$5.3069340000 \times 10^{-05}$	$1.2801951000 \times 10^{-08}$
0.3	3	$2.3907880800 \times 10^{-02}$	$4.9039669000 \times 10^{-03}$	$7.9635420000 \times 10^{-05}$	$1.5646829000 \times 10^{-08}$
	4	$3.2099860300 \times 10^{-02}$	$6.5457880000 \times 10^{-03}$	$1.0620149000 \times 10^{-05}$	$1.8491707000 \times 10^{-08}$
	5	$4.0291839800 \times 10^{-02}$	$8.1876092000 \times 10^{-03}$	$1.3276756000 \times 10^{-05}$	$2.1336585000 \times 10^{-08}$
	1	$8.2762123000 \times 10^{-03}$	$1.9075950000 \times 10^{-03}$	$3.2874570000 \times 10^{-05}$	$5.7825882000 \times 10^{-09}$
	2	$1.7398405800 \times 10^{-02}$	$3.8455493000 \times 10^{-03}$	$6.5848290000 \times 10^{-05}$	$6.7463529000 \times 10^{-08}$
0.4	3	$2.6520599300 \times 10^{-02}$	$5.7835036000 \times 10^{-03}$	$9.8822010000 \times 10^{-05}$	$7.7101176000 \times 10^{-08}$
	4	$3.5642792800 \times 10^{-02}$	$7.7214580000 \times 10^{-03}$	$1.3179574000 \times 10^{-05}$	$8.6738823000 \times 10^{-08}$
	5	$4.4764986200 \times 10^{-02}$	$9.6594122000 \times 10^{-03}$	$1.6476946000 \times 10^{-05}$	$9.6376470000 \times 10^{-08}$
	1	$8.8947364000 \times 10^{-03}$	$2.1627817000 \times 10^{-03}$	$3.8817930000 \times 10^{-04}$	$2.3900000000 \times 10^{-08}$
	2	$1.8806520800 \times 10^{-02}$	$4.3652454000 \times 10^{-03}$	$7.7777130000 \times 10^{-04}$	$3.5300000000 \times 10^{-08}$
0.5	3	$2.8718305200 \times 10^{-02}$	$6.5677091000 \times 10^{-03}$	$1.1673633000 \times 10^{-05}$	$4.6600000000 \times 10^{-08}$
	4	$3.8630089800 \times 10^{-02}$	$8.7701729000 \times 10^{-03}$	$1.5569554000 \times 10^{-04}$	$5.8000000000 \times 10^{-08}$
	5	$4.8541874400 \times 10^{-02}$	$1.0972636700 \times 10^{-03}$	$1.9465475000 \times 10^{-04}$	$6.9300000000 \times 10^{-08}$

$$\mu_0(\zeta, \xi, \Psi) = \zeta + \xi, \quad \nu_0(\zeta, \xi, \Psi) = \zeta - \xi, \tag{69}$$

$$\mu_{j+1}(\zeta, \xi, \Psi) = -\mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\sum_{j=0}^{\infty} A_j(\mu\mu_\zeta) + \sum_{j=0}^{\infty} B_j(\nu\mu_\xi) - \sum_{j=0}^{\infty}\mu_{\zeta\zeta} - \sum_{j=0}^{\infty}\mu_{\xi\xi}\right\}\right], \tag{70}$$

$$\nu_{j+1}(\zeta, \xi, \Psi) = -\mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left\{\sum_{j=0}^{\infty} C_j(\mu\nu_\zeta) + \sum_{j=0}^{\infty} D_j(\nu\nu_\xi) - \sum_{j=0}^{\infty}\nu_{\zeta\zeta} - \sum_{j=0}^{\infty}\nu_{\xi\xi}\right\}\right], \tag{71}$$

for $j = 0, 1, 2 \cdots$

$$\mu_1(\zeta, \xi, \Psi) = -\mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left[\mu_0\frac{\partial \mu_0}{\partial \zeta} + \nu_0\frac{\partial \mu_0}{\partial \xi} - \frac{\partial^2 \mu_0}{\partial \zeta^2} - \frac{\partial^2 \mu_0}{\partial \xi^2}\right]\right] = -2\zeta\{\delta\Psi + (1-\delta)\},$$

$$\nu_1(\zeta, \xi, \Psi) = -\mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left[\mu_0\frac{\partial \nu_0}{\partial \zeta} + \nu_0\frac{\partial \nu_0}{\partial \xi} - \frac{\partial^2 \nu_0}{\partial \zeta^2} - \frac{\partial^2 \nu_0}{\partial \xi^2}\right]\right] = -2\xi\{\delta\Psi + (1-\delta)\}. \tag{72}$$

The subsequent terms are

$$\mu_2(\zeta, \xi, \Psi) = -\mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left[\mu_0\frac{\partial \mu_1}{\partial \zeta} + \mu_1\frac{\partial \mu_0}{\partial \zeta} + \nu_0\frac{\partial \mu_1}{\partial \xi} + \nu_1\frac{\partial \mu_0}{\partial \xi} - \frac{\partial^2 \mu_1}{\partial \zeta^2} - \frac{\partial^2 \mu_1}{\partial \xi^2}\right]\right],$$

$$\mu_2(\zeta, \xi, \Psi) = 2(\zeta + \xi)\left\{(1-\delta)^2 + 2\delta(1-\delta)\Psi + \frac{\delta^2\Psi^2}{2}\right\},$$

Table 3: YDM-solutions of example 2 at $\nu(\zeta,\Psi)$ different fractional-order of δ.

Ψ	ζ	Absolute error ($\delta=0.4$)	Absolute error ($\delta=0.6$)	Absolute error ($\delta=0.8$)	Absolute error ($\delta=1$)
	1	$5.6770988000\times10^{-04}$	$8.7119340000\times10^{-05}$	$1.1548840000\times10^{-07}$	$1.6326000000\times10^{-10}$
	2	$5.3834512000\times10^{-03}$	$8.4544080000\times10^{-04}$	$9.5379000000\times10^{-07}$	$2.0407510200\times10^{-09}$
0.1	3	$5.0898036000\times10^{-03}$	$8.1968820000\times10^{-04}$	$7.5269600000\times10^{-07}$	$4.0814857200\times10^{-09}$
	4	$4.7961560000\times10^{-03}$	$7.9393560000\times10^{-04}$	$5.5160200000\times10^{-07}$	$6.1222204100\times10^{-09}$
	5	$4.5025084000\times10^{-03}$	$7.6818300000\times10^{-04}$	$3.5050800000\times10^{-07}$	$8.1629551100\times10^{-09}$
	1	$7.5124132000\times10^{-04}$	$1.3109744000\times10^{-05}$	$1.9589970000\times10^{-07}$	$2.2260880000\times10^{-08}$
	2	$6.9925200000\times10^{-03}$	$1.2577593000\times10^{-04}$	$1.5557000000\times10^{-07}$	$4.3444869600\times10^{-08}$
0.2	3	$6.4726268000\times10^{-03}$	$1.2045442000\times10^{-04}$	$1.1524030000\times10^{-07}$	$8.6867478200\times10^{-08}$
	4	$5.9527336000\times10^{-03}$	$1.1513291000\times10^{-04}$	$7.4910600000\times10^{-06}$	$1.3029008690\times10^{-08}$
	5	$5.4328404000\times10^{-03}$	$1.0981140000\times10^{-04}$	$3.4580900000\times10^{-06}$	$1.7371269560\times10^{-08}$
	1	$8.8600372900\times10^{-04}$	$1.6633175900\times10^{-05}$	$2.6628869000\times10^{-06}$	$4.2673170000\times10^{-08}$
	2	$8.1319795000\times10^{-03}$	$1.5818212000\times10^{-04}$	$2.0566070000\times10^{-06}$	$7.2886243900\times10^{-08}$
0.3	3	$7.4039217000\times10^{-03}$	$1.5003248000\times10^{-04}$	$1.4503270000\times10^{-06}$	$1.4534575610\times10^{-08}$
	4	$6.6758639000\times10^{-03}$	$1.4188284000\times10^{-04}$	$8.4404700000\times10^{-06}$	$2.1780526830\times10^{-08}$
	5	$5.9478061000\times10^{-03}$	$1.3373320000\times10^{-04}$	$2.3776700000\times10^{-06}$	$2.9026478050\times10^{-08}$
	1	$9.9681746700\times10^{-04}$	$1.9683136700\times10^{-04}$	$3.3072887000\times10^{-06}$	$3.8550588000\times10^{-09}$
	2	$9.0421934000\times10^{-03}$	$1.8579543000\times10^{-04}$	$2.4973720000\times10^{-06}$	$1.1668329410\times10^{-08}$
0.4	3	$8.1162122000\times10^{-03}$	$1.7475950000\times10^{-04}$	$1.6874560000\times10^{-06}$	$2.2951152940\times10^{-08}$
	4	$7.1902310000\times10^{-03}$	$1.6372357000\times10^{-04}$	$8.7754000000\times10^{-06}$	$3.4233976470\times10^{-08}$
	5	$6.2642497000\times10^{-03}$	$1.5268763000\times10^{-04}$	$6.7623000000\times10^{-06}$	$4.5516800000\times10^{-08}$
	1	$1.0928832650\times10^{-04}$	$2.2421458500\times10^{-04}$	$3.9100485000\times10^{-06}$	$1.2600000000\times10^{-08}$
	2	$9.8117844000\times10^{-03}$	$2.1024637000\times10^{-04}$	$2.8959200000\times10^{-06}$	$1.0050239900\times10^{-08}$
0.5	3	$8.6947361000\times10^{-03}$	$1.9627815000\times10^{-04}$	$1.8817910000\times10^{-06}$	$2.0100478600\times10^{-08}$
	4	$7.5776879000\times10^{-03}$	$1.8230994000\times10^{-04}$	$8.6766300000\times10^{-06}$	$3.0150717200\times10^{-08}$
	5	$6.4606397000\times10^{-03}$	$1.6834173000\times10^{-04}$	$1.4646500000\times10^{-06}$	$4.0200955900\times10^{-08}$

$$\nu_2(\zeta,\xi,\Psi)=-\mathscr{Y}^{-1}\left[(1+\delta(s-1))\mathscr{Y}\left[\mu_0\frac{\partial\nu_1}{\partial\zeta}+\mu_1\frac{\partial\nu_0}{\partial\zeta}+\nu_0\frac{\partial\nu_1}{\partial\xi}+\nu_1\frac{\partial\nu_0}{\partial\xi}-\frac{\partial^2\nu_0}{\partial\zeta^2}-\frac{\partial^2\nu_0}{\partial\xi^2}\right]\right],$$

$$\nu_2(\zeta,\xi,\Psi)=2(\zeta-\xi)\left\{(1-\delta)^2+2\delta(1-\delta)\Psi+\frac{\delta^2\Psi^2}{2}\right\}. \tag{73}$$

The YDM solution for example (5) is

$$\mu(\zeta,\xi,\Psi)=\mu_0(\zeta,\xi,\Psi)+\mu_1(\zeta,\xi,\Psi)+\mu_2(\zeta,\xi,\Psi)+\mu_3(\zeta,\xi,\Psi)+\cdots, \tag{74}$$

$$\nu(\zeta,\xi,\Psi)=\nu_0(\zeta,\xi,\Psi)+\nu_1(\zeta,\xi,\Psi)+\nu_2(\zeta,\xi,\Psi)+\nu_3(\zeta,\xi,\Psi)+\cdots, \tag{75}$$

$$\mu(\zeta,\xi,\Psi)=\zeta+\xi-2\zeta\{\delta\Psi+(1-\delta)\}+2(\zeta+\xi)\cdot\left\{(1-\delta)^2+2\delta(1-\delta)\Psi+\frac{\delta^2\Psi^2}{2}\right\}+\cdots, \tag{76}$$

$$\nu(\zeta,\xi,\Psi)=\zeta-\xi-2\xi\{\delta\Psi+(1-\delta)\}+2(\zeta-\xi)\cdot\left\{(1-\delta)^2+2\delta(1-\delta)\Psi+\frac{\delta^2\Psi^2}{2}\right\}+\cdots, \tag{77}$$

when $\delta=1$, then YDM solution is

$$\mu(\zeta,\xi,\Psi)=\zeta+\xi-2\zeta\Psi+2(\zeta+\xi)\Psi^2-4\Psi^3\zeta+4(\zeta+\xi)\Psi^4+\cdots, \tag{78}$$

$$\nu(\zeta,\xi,\Psi)=\zeta-\xi-2\xi\Psi+2(\zeta-\xi)\Psi^2-4\Psi^3\xi+4(\zeta-\xi)\Psi^4+\cdots. \tag{79}$$

The exact solutions are

$$\mu(\zeta,\xi,\Psi)=\frac{\zeta-2\zeta\Psi+\xi}{1-2\Psi^2},$$
$$\nu(\zeta,\xi,\Psi)=\frac{\zeta-2\xi\Psi-\xi}{1-2\Psi^2}. \tag{80}$$

6. Results and Discussion

In this section, we analyze the solution-figures of problem which have been investigated by applying Yang decomposition method in the sense of Caputo-Fabrizio operator. Figure 1 represents the two-dimensional solution-figures for variables $\mu(\zeta,\Psi)$ and $\nu(\zeta,\Psi)$ of example 1 at fractional order $\delta=1$, respectively, in Figure 2 at different fractional-order of $\wp$. It is observed that Yang method solution-figures are identical and close contact with each other. In a similar way in Figures 3 and 4 represent the three-dimensional solution-figures for variables $\mu(\zeta,\Psi)$ of example 1 at fractional order $\delta=1$, 0.8, 0.6, and 0.4. Figure 5 shows that the three dimensional figure of $\mu(\zeta,\Psi)$ of fractional order $\delta=1$ and 0.8 of example 2 and Figure 6, approximate solution graphs of example 2 with respect to $\nu(\zeta,\Psi)$ at $\delta=1$ and 0.8. Tables 1–3 show the absolute error of different fractional order of δ with respect to $\mu(\zeta,\Psi)$ and $\nu(\zeta,\Psi)$ of examples 1 and 2. The same graphs of the suggested methods attained and confirmed the applicability of the present technique. The convergence phenomenon of the fractional-solutions towards integer-solution is observed. The same accuracy is achieved by using the present techniques.

7. Conclusion

In this paper, Yang Adomian decomposition method is implemented for the solution of dynamic systems of fractional Burger equations. The derived results have been graphed and tables. The analytical solutions for some numerical problems represent the validity of the suggested technique. It is also analyzed that the fractional-order solution is convergence to the actual result for the problem as fractional-order approach integer-order. The higher accuracy of the suggested procedure is clearly demonstrated by this representation of the acquired results. The results for fractional systems that are closely akin to their actual solutions are obtained. It has been demonstrated that fractional solutions converge to integer-order solutions. The present method's valuable themes include fewer calculations and improved precision. The researchers modified it to solve fractional partial differential equations in various systems.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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Research Article

Existence and Uniqueness of Common Fixed Point on Complex Partial *b*-Metric Space

Arul Joseph Gnanaprakasam,[1] **Gunaseelan Mani**,[2] **M. Aphane**,[3] **and Yaé U. Gaba**[3,4]

[1]*Department of Mathematics, College of Engineering and Technology, Faculty of Engineering and Technology, SRM Institute of Science and Technology, SRM Nagar, Kattankulathur 603203, Kanchipuram, Chennai, Tamil Nadu, India*
[2]*Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602 105, Tamil Nadu, India*
[3]*Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, GA-Rankuwa, South Africa*
[4]*Quantum Leap Africa (QLA), AIMS Rwanda Centre, Remera Sector, Kigali KN 3, Rwanda*

Correspondence should be addressed to Yaé U. Gaba; yaeulrich.gaba@gmail.com

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In this study, we establish the existence and uniqueness of common fixed point on complex partial *b*-metric space. An example and application to support our result is presented.

The fourth author Y. U. Gaba would like to dedicate this publication to his wife, Clémence A. Epse G., in celebration of her 33rd birthday

1. Introduction

In 1989, Backhtin [1] and Czerwik [2] introduced the concept of b-metric spaces and provided a framework to extend the results in the classical setting of metric spaces which are known already. Azam et al. [3]introduced complex-valued metric spaces in 2011 and proved some common fixed-point theorems under the contraction condition. Then, in 2013, Rao et al. [4]introduced the definition of complex valued b-metric space and provided a method to extend the results. Later, in 2017, the concept of complex partial metric space was introduced by Dhivya and Marudai [5], and they proved common fixed-point theorems. Recently, Gunaseelan [6]introduced the concept of complex partial b-metric space in 2019. Many authors have discussed significant results and application on complex metric spaces [7–23]. In this study, we establish common fixed-point theorems on complex partial b-metric space using continuity property.

2. Preliminaries

Let C be the set of complex numbers and $\zeta_1, \zeta_2, \zeta_3 \in C$. Define a partial order $\lesssim$ on C as follows:

$\zeta_1 \lesssim \zeta_2$ if and only if $R(\zeta_1) \leq R(\zeta_2)$ and $I(\zeta_1) \leq I(\zeta_2)$.

Then, $\zeta_1 \lesssim \zeta_2$ if one of the following properties is fulfilled:

(i) $R(\zeta_1) = R(\zeta_2)$, $I(\zeta_1) < I(\zeta_2)$

(ii) $R(\zeta_1) < R(\zeta_2)$, $I(\zeta_1) = I(\zeta_2)$

(iii) $R(\zeta_1) < R(\zeta_2)$, $I(\zeta_1) < I(\zeta_2)$

(iv) $R(\zeta_1) = R(\zeta_2)$, $I(\zeta_1) = I(\zeta_2)$

In particular, we write $\zeta_1 \precnsim \zeta_2$ if $\zeta_1 \neq \zeta_2$ and one of (i), (ii), and (iii) is fulfilled, and we write $\zeta_1 \prec \zeta_2$ if only (iii) is fulfilled.

Definition 1 (see [4]). Let H be a nonvoid set and let $s \geq 1$ be a given real number. A function $\ell: H \times H \longrightarrow C$ is called a complex valued b-metric on H if, for all $\phi, \mu, \lambda \in H$, the following conditions are fulfilled:

(i) $0 \le \ell(\phi,\mu)$ and $\ell(\phi,\mu)=0$ if and only if $\phi=\mu$

(ii) $\ell(\phi,\mu)=\ell(\mu,\phi)$

(iii) $\ell(\phi,\mu) \le s[\ell(\phi,\lambda)+\ell(\lambda,\mu)]$

The pair (H,ℓ) is called a complex valued b-metric space.

Definition 2 (see [5]). A complex partial metric on a nonvoid set H is a function $\eta_{c\mathfrak{b}}: H\times H \longrightarrow C^+$ such that, for all $\phi,\mu,\lambda \in H$,

(i) $0 \le \eta_{c\mathfrak{b}}(\phi,\phi) \le \eta_{c\mathfrak{b}}(\phi,\mu)$ (small self − distances)

(ii) $\eta_{c\mathfrak{b}}(\phi,\mu)=\eta_{c\mathfrak{b}}(\mu,\phi)$ (symmetry)

(iii) $\eta_{c\mathfrak{b}}(\phi,\phi)=\eta_{c\mathfrak{b}}(\phi,\mu)=\eta_{c\mathfrak{b}}(\mu,\mu)$ *if and only if* $\phi=\mu$ (equality)

(iv) $\eta_{c\mathfrak{b}}(\phi,\mu) \le \eta_{c\mathfrak{b}}(\phi,\lambda)+\eta_{c\mathfrak{b}}(\lambda,\mu)-\eta_{c\mathfrak{b}}(\lambda,\lambda)$ (triangularity)

A complex partial metric space is a pair $(H,\eta_{c\mathfrak{b}})$ such that H is a nonvoid set and $\eta_{c\mathfrak{b}}$ is the complex partial metric on H.

Definition 3 (see [6]). A complex partial b-metric on a nonvoid set H is a function $\ell_{c\mathfrak{b}}: H\times H \longrightarrow C^+$ such that, for all $\phi,\mu,\lambda \in H$,

(i) $0 \le \ell_{c\mathfrak{b}}(\phi,\phi) \le \ell_{c\mathfrak{b}}(\phi,\mu)$ (small self − distances)

(ii) $\ell_{c\mathfrak{b}}(\phi,\mu)=\ell_{c\mathfrak{b}}(\mu,\phi)$ (symmetry)

(iii) $\ell_{c\mathfrak{b}}(\phi,\phi)=\ell_{c\mathfrak{b}}(\phi,\mu)=\ell_{c\mathfrak{b}}(\mu,\mu) \Leftrightarrow \phi=\mu$ (equality)

(iv) $\exists s\ge 1$ *such that* $\ell_{c\mathfrak{b}}(\phi,\mu) \le s[\ell_{c\mathfrak{b}}(\phi,\lambda)+\ell_{c\mathfrak{b}}(\lambda,\mu)]-\ell_{c\mathfrak{b}}(\lambda,\lambda)$ (triangularity)

A complex partial b-metric space (b-CPMS) is a pair $(H,\ell_{c\mathfrak{b}})$ such that H is a nonvoid set and $\ell_{c\mathfrak{b}}$ is the complex partial b-metric on H. The number s is called the coefficient of $(H,\ell_{c\mathfrak{b}})$.

Definition 4 (see [6]). Let $(H,\ell_{c\mathfrak{b}})$ be a complex partial b-metric space with coefficient s. Let $\{\phi_\alpha\}$ be any sequence in H and $\phi \in H$. Then,

(i) The sequence $\{\phi_\alpha\}$ is said to be convergent with respect to $\ell_{c\mathfrak{b}}$ and converges to ϕ if $\lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi_\alpha,\phi)=\ell_{c\mathfrak{b}}(\phi,\phi)$

(ii) The sequence $\{\phi_\alpha\}$ is said to be Cauchy sequence in $(H,\ell_{c\mathfrak{b}})$ if
$\lim_{\alpha,m\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_m)$ exists and is finite

(iii) $(H,\ell_{c\mathfrak{b}})$ is said to be a complete complex partial b-metric space if, for every Cauchy sequence $\{\phi_\alpha\}$ in H, there exists $\phi\in H$ such that $\lim_{\alpha,\, m\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_m)=\lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi_\alpha,\phi)=\ell_{c\mathfrak{b}}(\phi,\phi)$

In 2019, Gunaseelan [6] proved some fixed-point theorems on complex partial b-metric space as follows.

Theorem 1. *Let $(H,\ell_{c\mathfrak{b}})$ be any complete complex partial b-metric space with coefficient $s\ge 1$ and $\Delta: H\longrightarrow H$ be a mapping satisfying*

$$\ell_{c\mathfrak{b}}(\Delta\phi,\Delta\mu) \le \vartheta \max\{\ell_{c\mathfrak{b}}(\phi,\mu),\ell_{c\mathfrak{b}}(\phi,\Delta\phi),\ell_{c\mathfrak{b}}(\mu,\Delta\mu)\}, \quad (1)$$

for all $\phi,\mu\in H$, where $\vartheta\in[0,1/s)$. Then, Δ has a unique fixed point $\phi_*\in H$ and $\ell_{c\mathfrak{b}}(\phi_*,\phi_*)=0$.

We prove the existence and uniqueness of common fixed point on complex partial b-metric space, inspired by his work.

3. Main Results

Theorem 2. *Let $(H,\ell_{c\mathfrak{b}})$ be a complete b-CPMS with the coefficient $s\ge 1$ and $\Delta,\Omega: H\longrightarrow H$ be two continuous mappings such that*

$$\ell_{c\mathfrak{b}}(\Delta\phi,\Omega\mu) \le \vartheta \max\left\{\ell_{c\mathfrak{b}}(\phi,\mu),\ell_{c\mathfrak{b}}(\phi,\Delta\phi),\ell_{c\mathfrak{b}}(\mu,\Omega\mu),\frac{1}{2}\left(\ell_{c\mathfrak{b}}(\phi,\Omega\mu)+\ell_{c\mathfrak{b}}(\mu,\Delta\phi)\right)\right\}, \quad (2)$$

for all $\phi,\mu\in H$, where $0\le\vartheta<1/s$. Then, Δ and Ω have a unique common fixed point $\phi^*\in H$ and $\ell_{c\mathfrak{b}}(\phi^*,\phi^*)=0$.

Proof. Let $\phi_0\in H$. Define

$$\phi_{2\alpha+1}=\Delta\phi_{2\alpha} \quad \text{and} \quad \phi_{2\alpha+2}=\Omega\phi_{2\alpha+1}, \alpha=0,1,2,\ldots. \quad (3)$$

Then, by (1) and (2), we obtain

$$\begin{aligned}\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}) &= \ell_{c\mathfrak{b}}(\Delta\phi_{2\alpha},\Omega\phi_{2\alpha+1}),\\
&\lesssim \vartheta\max\left\{\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{c\mathfrak{b}}(\phi_{2\alpha},\Delta\phi_{2\alpha}),\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\Omega\phi_{2\alpha+1}),\frac{1}{2}(\ell_{c\mathfrak{b}}(\phi_{2\alpha},\Omega\phi_{2\alpha+1}))+\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\Delta\phi_{2\alpha})\right\},\\
&\lesssim \vartheta\max\left\{\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}),\frac{1}{2}(\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+2}))+\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+1})\right\},\\
&\lesssim \vartheta\max\left\{\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}),\ \frac{1}{2}(s(\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1}))+\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2})-\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+1})\right.\\
&\quad\left.+\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+1}))\right\},\\
&= \vartheta\max\left\{\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}),\frac{s}{2}(\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1})+\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}))\right\}.\end{aligned}\tag{4}$$

□

Case 1. If $\max\{\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}),s/2(\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1})+\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}))\}=\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2})$, then we have

$$\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2})\lesssim\vartheta\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}).\tag{5}$$

This implies $\vartheta\geq 1$, which is a reductio ad absurdum.

Case 2. If $\max\{\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}),s/2(\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1})+\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}))\}=\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1})$, then we have

$$\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2})\lesssim\vartheta\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1}).\tag{6}$$

From the next step, we have

$$\ell_{c\mathfrak{b}}(\phi_{2\alpha+2},\phi_{2\alpha+3})\lesssim\vartheta\max\left\{\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}),\ell_{c\mathfrak{b}}(\phi_{2\alpha+2},\phi_{2\alpha+3}),\frac{s}{2}(\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2})+\ell_{c\mathfrak{b}}(\phi_{2\alpha+2},\phi_{2\alpha+3}))\right\}.\tag{7}$$

We consider three cases.

Case 3.

$$\ell_{c\mathfrak{b}}(\phi_{2\alpha+2},\phi_{2n+3})\lesssim\vartheta\ell_{c\mathfrak{b}}(\phi_{2\alpha+2},\phi_{2n+3}),\tag{8}$$

which implies $\vartheta\geq 1$, is a reductio ad absurdum.

Case 4.

$$\ell_{c\mathfrak{b}}(\phi_{2\alpha+2},\phi_{2n+3})\lesssim\vartheta\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}).\tag{9}$$

From (6) and (9), $\forall\alpha=0,1,2,\ldots$, we obtain

$$\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\alpha+2})\lesssim\vartheta\ell_{c\mathfrak{b}}(\phi_{\alpha},\phi_{\alpha+1})\lesssim\cdots\lesssim\vartheta^{\alpha+1}\ell_{c\mathfrak{b}}(\phi_0,\phi_1).\tag{10}$$

For $\mathfrak{q},\alpha\in\mathbb{N}$, with $\mathfrak{q}>\alpha$, we have

$$\begin{aligned}\ell_{c\mathfrak{b}}(\phi_{\alpha},\phi_{\mathfrak{q}})&\lesssim s\left[\ell_{c\mathfrak{b}}(\phi_{\alpha},\phi_{\alpha+1})+\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\mathfrak{q}})-\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\alpha+1})\right],\\
&\lesssim s\left[\ell_{c\mathfrak{b}}(\phi_{\alpha},\phi_{\alpha+1})+\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\mathfrak{q}})\right],\\
&\lesssim s(\ell_{c\mathfrak{b}}(\phi_{\alpha},\phi_{\alpha+1}))+s^2\left(\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\alpha+2})+\ell_{c\mathfrak{b}}(\phi_{\alpha+2},\phi_{\mathfrak{q}})\right)-\ell_{c\mathfrak{b}}(\phi_{\alpha+2},\phi_{\alpha+2}),\\
&\lesssim s(\ell_{c\mathfrak{b}}(\phi_{\alpha},\phi_{\alpha+1})+s^2\left[\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\alpha+2})+\ell_{c\mathfrak{b}}(\phi_{\alpha+2},\phi_{\mathfrak{q}})\right]\\
&\lesssim s(\ell_{c\mathfrak{b}}(\phi_{\alpha},\phi_{\alpha+1}))+s^2(\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\alpha+2}))+s^3(\ell_{c\mathfrak{b}}(\phi_{\alpha+2},\phi_{\alpha+3})),\\
&\quad+\cdots+s^{\mathfrak{q}-\alpha-1}\left(\ell_{c\mathfrak{b}}(\phi_{\mathfrak{q}-2},\phi_{\mathfrak{q}-1})\right)+s^{\mathfrak{q}-\alpha}\left(\ell_{c\mathfrak{b}}(\phi_{\mathfrak{q}-1},\phi_{\mathfrak{q}})\right).\end{aligned}\tag{11}$$

Moreover, by using (9), we obtain

$$\begin{aligned}\ell_{c\mathfrak{b}}(\phi_{\alpha},\phi_{\mathfrak{q}})&\lesssim s\vartheta^{\alpha}(\ell_{c\mathfrak{b}}(\phi_0,\phi_1))+s^2\vartheta^{\alpha+1}(\ell_{c\mathfrak{b}}(\phi_0,\phi_1))+s^3\vartheta^{\alpha+2}(\ell_{c\mathfrak{b}}(\phi_0,\phi_1))\\
&\quad+\cdots+s^{\mathfrak{q}-\alpha-1}\vartheta^{\mathfrak{q}-2}(\ell_{c\mathfrak{b}}(\phi_0,\phi_1))+s^{\mathfrak{q}-\alpha}\vartheta^{\mathfrak{q}-1}(\ell_{c\mathfrak{b}}(\phi_0,\phi_1))=\sum_{i=1}^{\mathfrak{q}-\alpha}s^i\vartheta^{i+\alpha-1}(\ell_{c\mathfrak{b}}(\phi_0,\phi_1)).\end{aligned}\tag{12}$$

Therefore,

$$\begin{aligned}\left|\ell_{c\mathfrak{b}}\left(\phi_{\alpha},\phi_{\mathfrak{q}}\right)\right| &\leq \sum_{i=1}^{\mathfrak{q}-\alpha} s^{i+\alpha-1}\vartheta^{i+\alpha-1}\left|\ell_{c\mathfrak{b}}\left(\phi_0,\phi_1\right)\right| = \sum_{t=\alpha}^{\mathfrak{q}-1} s^{t}\vartheta^{t}\left|\ell_{c\mathfrak{b}}\left(\phi_0,\phi_1\right)\right|,\\ &\leq \sum_{i=\alpha}^{\infty}(s\vartheta)^{t}\left|\ell_{c\mathfrak{b}}\left(\phi_0,\phi_1\right)\right|,\\ &= \frac{(s\vartheta)^{\alpha}}{1-s\vartheta}\left|\ell_{c\mathfrak{b}}\left(\phi_0,\phi_1\right)\right|.\end{aligned} \tag{13}$$

Then, we have

$$\left|\ell_{c\mathfrak{b}}\left(\phi_{\alpha},\phi_{\mathfrak{q}}\right)\right| \leq \frac{(s\vartheta)^{\alpha}}{1-s\vartheta}\left|\ell_{c\mathfrak{b}}\left(\phi_0,\phi_1\right)\right|| \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow \infty. \tag{14}$$

Hence, $\{\phi_{\alpha}\}$ is a Cauchy sequence in H.

Case 5.

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2},\phi_{2\alpha+3}\right) \lesssim \vartheta\frac{s}{2}\left(\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1},\phi_{2\alpha+2}\right)+\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2},\phi_{2\alpha+3}\right)\right). \tag{15}$$

This implies that

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2},\phi_{2n+3}\right) \lesssim \frac{\vartheta s}{(2-\vartheta s)}\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1},\phi_{2\alpha+2}\right). \tag{16}$$

Since $a: = \vartheta s/(2-\vartheta s) < 1$, we get $\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\alpha+2}) \lesssim a\ell_{c\mathfrak{b}}(\phi_{\alpha},\phi_{\alpha+1})$. Therefore, $\{\phi_{\alpha}\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in H.

Case 6. If $\max\{\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+}\ 1), \ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}), s/2(\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1})+\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}))\} = s/2\,(\ell_{c\mathfrak{b}}(\phi_{2\alpha},\phi_{2\alpha+1})+\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+2}))$, then we have

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1},\phi_{2\alpha+2}\right) \lesssim \vartheta s/2\left(\ell_{c\mathfrak{b}}\left(\phi_{2\alpha},\phi_{2\alpha+1}\right)+\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1},\phi_{2\alpha+2}\right)\right). \tag{17}$$

Hence,

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1},\phi_{2\alpha+2}\right) \lesssim \frac{\vartheta s}{(2-\vartheta s)}\ell_{c\mathfrak{b}}\left(\phi_{2\alpha},\phi_{2\alpha+1}\right). \tag{18}$$

For the next step, we have

$$\begin{aligned}\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2},\phi_{2\alpha+3}\right) \lesssim \vartheta\max\Big\{&\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1},\phi_{2\alpha+2}\right), \ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2},\phi_{2\alpha+3}\right),\\ &\frac{s}{2}\left(\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1},\phi_{2\alpha+2}\right)+\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2},\phi_{2\alpha+3}\right)\right)\Big\}.\end{aligned} \tag{19}$$

Then, we consider three cases.

Case 7.

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2},\phi_{2\alpha+3}\right) \lesssim \vartheta\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2},\phi_{2\alpha+3}\right), \tag{20}$$

which implies $\vartheta \geq 1$ and is a reductio ad absurdum.

Case 8.

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2},\phi_{2n+3}\right) \lesssim \vartheta\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1},\phi_{2\alpha+2}\right). \tag{21}$$

Then, by (18) and (21), we get $\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\alpha+2}) \lesssim \gamma\ell_{c\mathfrak{b}}(\phi_{\alpha},\phi_{\alpha+1})$, where $\gamma = \max\{\vartheta, \vartheta s/2-\vartheta s\} < 1$. Hence, $\{\phi_{\alpha}\}_{\alpha\in N}$ is a Cauchy sequence in H.

Case 9.

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2},\phi_{2n+3}\right) \lesssim \frac{s}{2}\left(\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1},\phi_{2\alpha+2}\right)\right)+\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2},\phi_{2\alpha+3}\right). \tag{22}$$

Hence, we obtain

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2},\phi_{2\alpha+3}\right) \lesssim \frac{\vartheta s}{(2-\vartheta s)}\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1},\phi_{2\alpha+2}\right). \tag{23}$$

By using (18) and (21) yields

$$\ell_{c\mathfrak{b}}\left(\phi_{\alpha+1},\phi_{\alpha+2}\right) \lesssim \wr\,\ell_{c\mathfrak{b}}\left(\phi_{\alpha},\phi_{\alpha+1}\right), \tag{24}$$

where $0 \leq \wr = \vartheta s/(2-\vartheta s) < 1$.

Then, $\forall\alpha = 0, 1, 2, \ldots$, we obtain

$$\ell_{c\mathfrak{b}}\left(\phi_{\alpha+1},\phi_{\alpha+2}\right) \lesssim \wr\,\ell_{c\mathfrak{b}}\left(\phi_{\alpha},\phi_{\alpha+1}\right) \lesssim \cdots \lesssim \wr^{\alpha+1}\ell_{c\mathfrak{b}}\left(\phi_0,\phi_1\right). \tag{25}$$

For $\mathfrak{q}, \alpha \in \mathbb{N}$, with $\mathfrak{q} > \alpha$,

$$\begin{aligned}
\ell_{cb}(\phi_\alpha, \phi_q) &\lesssim s[\ell_{cb}(\phi_\alpha, \phi_{\alpha+1}) + \ell_{cb}(\phi_{\alpha+1}, \phi_q) - \ell_{cb}(\phi_{\alpha+1}, \phi_{\alpha+1})],\\
&\lesssim s[\ell_{cb}(\phi_\alpha, \phi_{\alpha+1}) + \ell_{cb}(\phi_{\alpha+1}, \phi_q)],\\
&\lesssim s(\ell_{cb}(\phi_\alpha, \phi_{\alpha+1})) + s^2(\ell_{cb}(\phi_{\alpha+1}, \phi_{\alpha+2}) + \ell_{cb}(\phi_{\alpha+2}, \phi_q) - \ell_{cb}(\phi_{\alpha+2}, \phi_{\alpha+2})),\\
&\lesssim s(\ell_{cb}(\phi_\alpha, \phi_{\alpha+1})) + s^2[\ell_{cb}(\phi_{\alpha+1}, \phi_{\alpha+2}) + \ell_{cb}(\phi_{\alpha+2}, \phi_q)],\\
&\lesssim s(\ell_{cb}(\phi_\alpha, \phi_{\alpha+1})) + s^2(\ell_{cb}(\phi_{\alpha+1}, \phi_{\alpha+2})) + s^3(\ell_{cb}(\phi_{\alpha+2}, \phi_{\alpha+3})),\\
&\quad + \cdots + s^{q-\alpha-1}(\ell_{cb}(\phi_{q-2}, \phi_{q-1})) + s^{q-\alpha}(\ell_{cb}(\phi_{q-1}, \phi_q)).
\end{aligned} \tag{26}$$

Using (24), we obtain

$$\begin{aligned}
\ell_{cb}(\phi_\alpha, \phi_q) &\lesssim s\wr^\alpha(\ell_{cb}(\phi_0, \phi_1)) + s^2\wr^{\alpha+1}(\ell_{cb}(\phi_0, \phi_1)) + s^3\wr^{\alpha+2}(\ell_{cb}(\phi_0, \phi_1))\\
&\quad + \cdots + s^{q-\alpha-1}\wr^{q-2}(\ell_{cb}(\phi_0, \phi_1)) + s^{q-\alpha}\wr^{q-1}(\ell_{cb}(\phi_0, \phi_1))\\
&= \sum_{i=1}^{q-\alpha} s^i \wr^{i+\alpha-1}(\ell_{cb}((\phi_0, \phi_1)).
\end{aligned} \tag{27}$$

Therefore,

$$\begin{aligned}
|\ell_{cb}(\phi_\alpha, \phi_q)| &\leq \sum_{i=1}^{q-\alpha} s^{i+\alpha-1}\wr^{i+\alpha-1}|\ell_{cb}(\phi_0, \phi_1)| = \sum_{t=\alpha}^{q-1} s^t \wr^t |\ell_{cb}(\phi_0, \phi_1)|\\
&\leq \sum_{i=\alpha}^{\infty} (s\wr)^t |\ell_{cb}(\phi_0, \phi_1)|,\\
&= \frac{(s\wr)^\alpha}{1 - s\wr}|\ell_{cb}(\phi_0, \phi_1)|.
\end{aligned} \tag{28}$$

Hence, we have

$$|\ell_{cb}(\phi_\alpha, \phi_q)| \leq \frac{(s\wr)^\alpha}{1 - s\wr}|\ell_{cb}(\phi_0, \phi_1)| \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow \infty. \tag{29}$$

Hence, $\{\phi_\alpha\}$ is a Cauchy sequence in H. In all cases, $\{\phi_\alpha\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence. Since H is complete, there exists $\phi^* \in H$ such that $\phi_\alpha \longrightarrow \phi^*$ as $\alpha \longrightarrow \infty$ and

$$\ell_{cb}(\phi^*, \phi^*) = \lim_{\alpha\longrightarrow\infty}\ell_{cb}(\phi^*, \phi_\alpha) = \lim_{\alpha\longrightarrow\infty}\ell_{cb}(\phi_\alpha, \phi_\alpha) = 0. \tag{30}$$

By the continuity of Δ, $\phi_{2\alpha+1} = \Delta\phi_{2\alpha} \longrightarrow \Delta\phi^*$ as $\alpha \longrightarrow \infty$:

$$\text{i.e.}\, \ell_{cb}(\Delta\phi^*, \Delta\phi^*) = \lim_{\alpha\longrightarrow\infty}\ell_{cb}(\Delta\phi^*, \Delta\phi_{2\alpha}) = \lim_{\alpha\longrightarrow\infty}\ell_{cb}(\Delta\phi_{2\alpha}, \Delta\phi_{2\alpha}). \tag{31}$$

However,

$$\ell_{cb}(\Delta\phi^*, \Delta\phi^*) = \lim_{\alpha\longrightarrow\infty}\ell_{cb}(\Delta\phi_{2\alpha}, \Delta\phi_{2\alpha}) = \lim_{\alpha\longrightarrow\infty}\ell_{cb}(\phi_{2\alpha+1}, \phi_{2\alpha+1}) = 0. \tag{32}$$

Next, we prove that ϕ^* is a fixed point of Δ:

$$\ell_{c\mathfrak{b}}(\Delta\phi^*, \phi^*) \lesssim \ell_{c\mathfrak{b}}(\Delta\phi^*, \Delta\phi_{2\alpha}) + \ell_{c\mathfrak{b}}(\Delta\phi_{2\alpha}, \phi^*) - \ell_{c\mathfrak{b}}(\Delta\phi_{2\alpha}, \Delta\phi_{2\alpha}). \tag{33}$$

As $\alpha \longrightarrow \infty$, we obtain $|\ell_{c\mathfrak{b}}(\Delta\phi^*, \phi^*)| \leq 0$. Thus, $\ell_{cp}(\Delta\phi^*, \phi^*) = 0$. Hence, $\ell_{c\mathfrak{b}}(\phi^*, \phi^*) = \ell_{c\mathfrak{b}}(\phi^*, \Delta\phi^*) = \ell_{c\mathfrak{b}}(\Delta\phi^*, \Delta\phi^*) = 0$ and $\Delta\phi^* = \phi^*$. In the same way, we have $\phi^* \in H$ such that $\phi_\alpha \longrightarrow \phi^*$ as $\alpha \longrightarrow \infty$ and

$$\ell_{c\mathfrak{b}}(\phi^*, \phi^*) = \lim_{\alpha\longrightarrow\infty} \ell_{c\mathfrak{b}}(\phi^*, \phi_\alpha) = \lim_{\alpha\longrightarrow\infty} \ell_{c\mathfrak{b}}(\phi_\alpha, \phi_\alpha) = 0. \tag{34}$$

By the continuity of $\Delta\,\phi_{2\alpha+2} = \Omega\phi_{2\alpha+1} \longrightarrow \Omega\phi^*$ as $\alpha \longrightarrow \infty$,

$$\begin{aligned} \text{i.e.}\, \ell_{c\mathfrak{b}}(\Omega\phi^*, \Omega\phi^*) &= \lim_{\alpha\longrightarrow\infty} \ell_{c\mathfrak{b}}(\Omega\phi^*, \Omega\phi_{2\alpha+1}) \\ &= \lim_{\alpha\longrightarrow\infty} \ell_{c\mathfrak{b}}(\Omega\phi_{2\alpha+1}, \Omega\phi_{2\alpha+1}). \end{aligned} \tag{35}$$

However,

$$\begin{aligned} \ell_{c\mathfrak{b}}(\Omega\phi^*, \Omega\phi)^* &= \lim_{\alpha\longrightarrow\infty} \ell_{c\mathfrak{b}}(\Omega\phi_{2\alpha+1}, \Omega\phi_{2\alpha+1}) \\ &= \lim_{\alpha\longrightarrow\infty} \ell_{c\mathfrak{b}}(\phi_{2\alpha+2}, \phi_{2\alpha+2}) = 0. \end{aligned} \tag{36}$$

Next, we prove that ϕ^* is a fixed point of Ω:

$$\begin{aligned} \ell_{c\mathfrak{b}}(\Omega\phi^*, \phi^*) \leq{}& \ell_{c\mathfrak{b}}(\Omega\phi^*, \Omega\phi_{2\alpha+1}) + \ell_{c\mathfrak{b}}(\Omega\phi_{2\alpha+1}, \phi^*) \\ &- \ell_{c\mathfrak{b}}(\Omega\phi_{2\alpha+1}, \Delta\phi_{2\alpha+1}). \end{aligned} \tag{37}$$

As $\alpha \longrightarrow \infty$, we obtain $|\ell_{c\mathfrak{b}}(\Omega\phi^*, \phi^*)| \leq 0$. Thus, $\ell_{cp}(\Omega\phi^*, \phi^*) = 0$. Hence, $\ell_{c\mathfrak{b}}(\phi^*, \phi^*) = \ell_{c\mathfrak{b}}(\phi^*, \Omega\phi^*) = \ell_{c\mathfrak{b}}(\Omega\phi^*, \Omega\phi^*) = 0$ and $\Omega\phi^* = \phi^*$. Therefore, Δ and Ω have a common fixed point ϕ^*.

Let $\mu^* \in H$ be another common fixed point for the mappings Δ and Ω. Then,

$$\begin{aligned} \ell_{c\mathfrak{b}}(\phi^*, \mu^*) = \ell_{c\mathfrak{b}}(\Delta\phi^*, \Omega\mu^*) &\lesssim \vartheta \max\Big\{\ell_{c\mathfrak{b}}(\phi^*, \mu^*), \ell_{c\mathfrak{b}}(\phi^*, \Delta\phi^*), \ell_{c\mathfrak{b}}(\mu^*, \Omega\mu^*), \\ &\quad \frac{1}{2}\left(\ell_{c\mathfrak{b}}(\phi^*, \Omega\mu^*)\right) + \ell_{c\mathfrak{b}}(\mu^*, \Delta\phi^*)\Big\}, \\ &\lesssim \vartheta \max\Big\{\ell_{c\mathfrak{b}}(\phi^*, \mu^*), \ell_{c\mathfrak{b}}(\phi^*, \phi^*), \ell_{c\mathfrak{b}}(\mu^*, \mu^*) \\ &\quad \frac{1}{2}\left(\ell_{c\mathfrak{b}}(\phi^*, \mu^*)\right) + \ell_{c\mathfrak{b}}(\mu^*, \phi^*)\Big\} \\ &\lesssim \vartheta \ell_{c\mathfrak{b}}(\phi^*, \mu^*). \end{aligned} \tag{38}$$

This implies that $\phi^* = \mu^*$.

Theorem 3. *Let $(H, \ell_{c\mathfrak{b}})$ be a complete b-CPMS with the coefficient $s \geq 1$ and $\Delta, \Omega: H \longrightarrow H$ be two continuous mappings such that*

$$\begin{aligned} \ell_{c\mathfrak{b}}(\Delta\phi, \Omega\mu) \lesssim \vartheta \max\Big\{&\ell_{c\mathfrak{b}}(\phi, \mu), \ell_{c\mathfrak{b}}(\phi, \Delta\phi), \ell_{c\mathfrak{b}}(\mu, \Omega\mu), \\ &\frac{1}{2}\left(\ell_{c\mathfrak{b}}(\phi, \Omega\mu) + \ell_{c\mathfrak{b}}(\mu, \Delta\phi)\right)\Big\}, \end{aligned} \tag{39}$$

for all $\phi, \mu \in H$, where $0 \leq \vartheta < 1/s$. Then, Δ and Ω have a unique common fixed point $\phi^* \in H$ and $\ell_{c\mathfrak{b}}(\phi^*, \phi^*) = 0$.

Proof. Following from Theorem 2, we can easily prove $\{\phi_\alpha\}$ is a Cauchy sequence. Since H is complete, there exists $\phi^* \in H$ such that $\phi_\alpha \longrightarrow \phi^*$ as $\alpha \longrightarrow \infty$.

Suppose that $\ell_{c\mathfrak{b}}(\phi^*, \Delta\phi^*) = \lambda > 0$.

Then, we estimate

$$\begin{aligned}
\lambda &= \ell_{c\mathfrak{b}}(\phi^*, \Delta\phi^*),\\
&\precsim s\{\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + \ell_{c\mathfrak{b}}(\phi_{2i+2}, \Delta\phi^*) - \ell_{c\mathfrak{b}}(\phi_{2i+2}, \phi_{2i+2})\},\\
&\precsim s\{\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + \ell_{c\mathfrak{b}}(\phi_{2i+2}, \Delta\phi^*)\},\\
&\precsim s\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + s\ell_{c\mathfrak{b}}(\Omega\phi_{2i+1}, \Delta\phi^*),\\
&\precsim s\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + \vartheta s \max\{\ell_{c\mathfrak{b}}(\phi_{2i+1}, \phi^*), \ell_{c\mathfrak{b}}(\phi_{2i+1}, \Omega\phi_{2i+1}), \ell_{c\mathfrak{b}}(\phi^*, \Delta\phi^*)\\
&\quad \frac{1}{2}((\ell_{c\mathfrak{b}}(\phi_{2i+1}, \Delta\phi^*) + \ell_{c\mathfrak{b}}(\phi^*, \Omega\phi_{2i+1}))\Big\},\\
&\precsim s\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + \vartheta s \max\{\ell_{c\mathfrak{b}}(\phi_{2i+1}, \phi^*), \ell_{c\mathfrak{b}}(\phi_{2i+1}, \phi_{2i+2}), \ell_{c\mathfrak{b}}(\phi^*, \Delta\phi^*)\\
&\quad \frac{1}{2}((\ell_{c\mathfrak{b}}(\phi_{2i+1}, \Delta\phi^*) + \ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}))\Big\},\\
&\precsim s\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + s\vartheta\ell_{c\mathfrak{b}}(\phi^*, \Delta\phi^*),\\
&\precsim s\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + s\vartheta\lambda.
\end{aligned} \tag{40}$$

This yields

$$|\lambda| \leq s|\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2})| + s\vartheta|\lambda|. \tag{41}$$

Hence, $\vartheta \geq 1$, which is a reductio ad absurdum. Then, $\phi^* = \Delta\phi^*$. Similarly, we derive that $\phi^* = \Omega\phi^*$. Therefore, Δ and Ω have a common fixed point ϕ^*. Following from Theorem 2, we can easily prove uniqueness part. ☐

Theorem 4. *Let $(H, \ell_{c\mathfrak{b}})$ be a complete b-CPMS with the coefficient $s \geq 1$ and $\Delta, \Omega: H \longrightarrow H$ be two continuous mappings such that*

$$\ell_{c\mathfrak{b}}(\Delta\phi, \Omega\mu) \precsim \vartheta \max\left\{\ell_{c\mathfrak{b}}(\phi, \mu), \frac{\ell_{c\mathfrak{b}}(\phi, \Delta\phi)\ell_{c\mathfrak{b}}(\mu, \Omega\mu)}{1 + \ell_{c\mathfrak{b}}(\phi, \mu)}, \frac{\ell_{c\mathfrak{b}}(\phi, \Delta\phi)\ell_{c\mathfrak{b}}(\Delta\phi, \Omega\mu)}{1 + \ell_{c\mathfrak{b}}(\phi, \mu)}\right\}, \tag{42}$$

for all $\phi, \mu \in H$, where $0 \leq \vartheta < 1/s$. Then, Δ and Ω have a unique common fixed point $\phi^* \in H$ and $\ell_{c\mathfrak{b}}(\phi^*, \phi^*) = 0$.

Proof. Let $\phi_0 \in H$. Define

$$\phi_{2\alpha+1} = \Delta\phi_{2\alpha} \quad \text{and} \quad \phi_{2\alpha+2} = \Omega\phi_{2\alpha+1}, \alpha = 0, 1, 2, \ldots. \tag{43}$$

Then, by (42) and (43), we obtain

$$\begin{aligned}
\ell_{c\mathfrak{b}}(\phi_{2\alpha+1}, \phi_{2\alpha+2}) &= \ell_{c\mathfrak{b}}(\Delta\phi_{2\alpha}, \Omega\phi_{2\alpha+1}),\\
&\precsim \vartheta \max\left\{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1}), \frac{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})\ell_{c\mathfrak{b}}(\Omega\phi_{2\alpha+1}, \Delta\phi_{2\alpha})}{1 + \ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})},\right.\\
&\qquad \left.\frac{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \Delta\phi_{2\alpha})\ell_{c\mathfrak{b}}(\Delta\phi_{2\alpha}, \Omega\phi_{2\alpha+1})}{1 + \ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})}\right\},\\
&\precsim \vartheta \max\left\{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1}), \frac{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})\ell_{c\mathfrak{b}}(\phi_{2\alpha+1}, \phi_{2\alpha+2})}{1 + \ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})},\right.\\
&\qquad \left.\frac{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})\ell_{c\mathfrak{b}}(\phi_{2\alpha+1}, \phi_{2\alpha+2})}{1 + \ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})}\right\},\\
&\precsim \vartheta \max\{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1}), \ell_{c\mathfrak{b}}(\phi_{2\alpha+1}, \phi_{2\alpha+2})\}.
\end{aligned} \tag{44}$$

If $\max\{\ell_{cb}(\phi_{2\alpha}, \phi_{2\alpha+1}), \ell_{cb}(\phi_{2\alpha+1}, \phi_{2\alpha+2})\} = \ell_{cb}(\phi_{2\alpha+1}, \phi_{2\alpha+2})$, then

$$\ell_{cb}(\phi_{2\alpha+1}, \phi_{2\alpha+2}) \lesssim \vartheta \ell_{cb}(\phi_{2\alpha+1}, \phi_{2\alpha+2}). \tag{45}$$

This shows that $\vartheta \geq 1$, which is a reductio ad absurdum. Therefore,

$$\ell_{cb}(\phi_{2\alpha+1}, \phi_{2\alpha+2}) \lesssim \vartheta \ell_{cb}(\phi_{2\alpha}, \phi_{2\alpha+1}). \tag{46}$$

Similarly, we obtain

$$\ell_{cb}(\phi_{2\alpha+2}, \phi_{2\alpha+3}) \lesssim \vartheta \ell_{cb}(\phi_{2\alpha+1}, \phi_{2\alpha+2}). \tag{47}$$

From (46) and (47), $\forall \alpha = 0, 1, 2, \ldots$, we obtain

$$\ell_{cb}(\phi_{\alpha+1}, \phi_{\alpha+2}) \lesssim \vartheta \ell_{cb}(\phi_{\alpha}, \phi_{\alpha+1}) \lesssim \cdots \lesssim \vartheta^{\alpha+1} \ell_{cb}(\phi_0, \phi_1). \tag{48}$$

For $\mathfrak{q}, \alpha \in \mathbb{N}$, with $\mathfrak{q} > \alpha$, we have

$$\begin{aligned}
\ell_{cb}(\phi_\alpha, \phi_{\mathfrak{q}}) &\lesssim s\left[\ell_{cb}(\phi_\alpha, \phi_{\alpha+1}) + \ell_{cb}(\phi_{\alpha+1}, \phi_{\mathfrak{q}}) - \ell_{cb}(\phi_{\alpha+1}, \phi_{\alpha+1})\right],\\
&\lesssim s\left[\ell_{cb}(\phi_\alpha, \phi_{\alpha+1}) + \ell_{cb}(\phi_{\alpha+1}, \phi_{\mathfrak{q}})\right],\\
&\lesssim s(\ell_{cb}(\phi_\alpha, \phi_{\alpha+1})) + s^2\left(\ell_{cb}(\phi_{\alpha+1}, \phi_{\alpha+2}) + \ell_{cb}(\phi_{\alpha+2}, \phi_{\mathfrak{q}}) - \ell_{cb}(\phi_{\alpha+2}, \phi_{\alpha+2})\right),\\
&\lesssim s(\ell_{cb}(\phi_\alpha, \phi_{\alpha+1})) + s^2\left[\ell_{cb}(\phi_{\alpha+1}, \phi_{\alpha+2}) + \ell_{cb}(\phi_{\alpha+2}, \phi_{\mathfrak{q}})\right],\\
&\lesssim s(\ell_{cb}(\phi_\alpha, \phi_{\alpha+1})) + s^2(\ell_{cb}(\phi_{\alpha+1}, \phi_{\alpha+2})) + s^3(\ell_{cb}(\phi_{\alpha+2}, \phi_{\alpha+3})),\\
&\quad + \cdots + s^{\mathfrak{q}-\alpha-1}\left(\ell_{cb}(\phi_{\mathfrak{q}-2}, \phi_{\mathfrak{q}-1})\right) + s^{\mathfrak{q}-\alpha}\left(\ell_{cb}(\phi_{\mathfrak{q}-1}, \phi_{\mathfrak{q}})\right).
\end{aligned} \tag{49}$$

By using (48), we obtain

$$\begin{aligned}
\ell_{cb}(\phi_\alpha, \phi_{\mathfrak{q}}) &\lesssim s\vartheta^\alpha(\ell_{cb}(\phi_0, \phi_1)) + s^2\vartheta^{\alpha+1}(\ell_{cb}(\phi_0, \phi_1)) + s^3\vartheta^{\alpha+2}(\ell_{cb}(\phi_0, \phi_1))\\
&\quad + \cdots + s^{\mathfrak{q}-\alpha-1}\vartheta^{\mathfrak{q}-2}(\ell_{cb}(\phi_0, \phi_1)) + s^{\mathfrak{q}-\alpha}\vartheta^{\mathfrak{q}-1}(\ell_{cb}(\phi_0, \phi_1)),\\
&= \sum_{i=1}^{\mathfrak{q}-\alpha} s^i \vartheta^{i+\alpha-1}(\ell_{cb}(\phi_0, \phi_1)).
\end{aligned} \tag{50}$$

Therefore,

$$\begin{aligned}
\left|\ell_{cb}(\phi_\alpha, \phi_{\mathfrak{q}})\right| &\leq \sum_{i=1}^{\mathfrak{q}-\alpha} s^{i+\alpha-1}\vartheta^{i+\alpha-1}\left|\ell_{cb}(\phi_0, \phi_1)\right| = \sum_{i=1}^{\mathfrak{q}-\alpha} s^t\vartheta^t\left|\ell_{cb}(\phi_0, \phi_1)\right|\\
&\leq \sum_{i=\alpha}^{\infty} (s\vartheta)^t\left|\ell_{cb}(\phi_0, \phi_1)\right|,\\
&= \frac{(s\vartheta)^\alpha}{1 - s\vartheta}\left|\ell_{cb}(\phi_0, \phi_1)\right|.
\end{aligned} \tag{51}$$

Hence, we have

$$\left|\ell_{cb}(\phi_\alpha, \phi_{\mathfrak{q}})\right| \leq \frac{(s\vartheta)^\alpha}{1 - s\vartheta}\left|\ell_{cb}(\phi_0, \phi_1)\right| \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow \infty. \tag{52}$$

Hence, $\{\phi_\alpha\}$ is a Cauchy sequence in H. Since H is complete, there exists $\phi^* \in H$ such that $\phi_\alpha \longrightarrow \phi^*$ as $\alpha \longrightarrow \infty$ and

$$\ell_{cb}(\phi^*, \phi^*) = \lim_{\alpha\longrightarrow\infty}\ell_{cb}(\phi^*, \phi_\alpha) = \lim_{\alpha\longrightarrow\infty}\ell_{cb}(\phi_\alpha, \phi_\alpha) = 0. \tag{53}$$

Since Ω is continuous, we obtain

$$\phi^* = \lim_{\alpha\longrightarrow\infty}\phi_{2\alpha+2} = \lim_{\alpha\longrightarrow\infty}\Omega\phi_{2\alpha+1} = \Omega\lim_{\alpha\longrightarrow\infty}\phi_{2\alpha+1} = \Omega\phi^*. \tag{54}$$

Similarly, we derive that $\phi^* = \Delta\phi^*$. Then, Δ and Ω have a common fixed point. Let $\mu^* \in H$ be another common fixed point for the mappings Δ and Ω. Then,

$$\begin{aligned}\ell_{c\mathfrak{b}}(\phi^*,\mu^*) &= \ell_{c\mathfrak{b}}(\Delta\phi^*,\Omega\mu^*),\\ &\lesssim \vartheta \max\left\{\ell_{c\mathfrak{b}}(\phi^*,\mu)^*, \frac{\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*)\ell_{c\mathfrak{b}}(\mu^*,\Omega\mu^*)}{1+\ell_{c\mathfrak{b}}(\phi^*,\mu^*)},\right.\\ &\left.\frac{\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*)\ell_{c\mathfrak{b}}(\Omega\mu^*,\Delta\phi^*)}{1+\ell_{c\mathfrak{b}}(\phi^*,\mu^*)}\right\},\\ &\lesssim \vartheta \ell_{c\mathfrak{b}}(\phi^*,\mu^*).\end{aligned} \tag{55}$$

This implies that $\phi^* = \mu^*$. □

Theorem 5. *Let $(H,\ell_{c\mathfrak{b}})$ be a complete b-CPMS with the coefficient $s \geq 1$ and $\Delta,\Omega\colon H \longrightarrow H$ be two continuous mappings such that*

$$\ell_{c\mathfrak{b}}(\Delta\phi,\Omega\mu) \lesssim \vartheta \max\left\{\ell_{c\mathfrak{b}}(\phi,\mu), \frac{\ell_{c\mathfrak{b}}(\phi,\Delta\phi)\ell_{c\mathfrak{b}}(\mu,\Omega\mu)}{1+\ell_{c\mathfrak{b}}(\phi,\mu)}, \frac{\ell_{c\mathfrak{b}}(\phi,\Delta\phi)\ell_{c\mathfrak{b}}(\Delta\phi,\Omega\mu)}{1+\ell_{c\mathfrak{b}}(\phi,\mu)}\right\}, \tag{56}$$

for all $\phi,\mu \in H$, where $0 \leq \vartheta < 1/s$. Then, Δ and Ω have a unique common fixed point $\phi^* \in H$ and $\ell_{c\mathfrak{b}}(\phi^*,\phi^*) = 0$.

Proof. Following from Theorem 5, we can easily prove $\{\phi_\alpha\}$ is a Cauchy sequence. Since H is complete, there exists $\phi^* \in H$ such that $\phi_\alpha \longrightarrow \phi^*$ as $\alpha \longrightarrow \infty$ and

$$\ell_{c\mathfrak{b}}(\phi^*,\phi^*) = \lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi^*,\phi_\alpha) = \lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_\alpha) = 0. \tag{57}$$

Suppose that $\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*) = \lambda > 0$.
Then, we estimate

$$\begin{aligned}\lambda &= \ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*),\\ &\lesssim s\{\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2}) + \ell_{c\mathfrak{b}}(\phi_{2i+2},\Delta\phi^*) - \ell_{c\mathfrak{b}}(\phi_{2i+2},\phi_{2i+2})\},\\ &\lesssim s\{\{\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2}) + \ell_{c\mathfrak{b}}(\Delta\phi^*,\phi_{2i+2})\},\\ &\lesssim s\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2}) + s\ell_{c\mathfrak{b}}(\Delta\phi^*,\Omega\phi_{2i+1}),\\ &\lesssim s\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2}) + \vartheta s \max\left\{\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+1}), \frac{\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*)\ell_{c\mathfrak{b}}(\phi_{2i+1},\Omega\phi_{2i+1})}{1+\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+1})},\right.\\ &\left.\frac{\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi_*)\ell_{c\mathfrak{b}}(\Delta\phi^*,\Omega\phi_{2i+1})}{1+\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+1})}\right\},\\ &\lesssim s\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2}) + \vartheta s \max\left\{\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+1}), \frac{\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*)\ell_{c\mathfrak{b}}(\phi_{2i+1},\phi_{2i+2})}{1+\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+1})}\right\}\\ &\frac{\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi_*)\ell_{c\mathfrak{b}}(\Delta\phi^*,\phi_{2i+2})}{1+\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+1})},\\ &\lesssim s\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2}) + s\vartheta\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*)^2,\\ &\lesssim s\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2}) + s\vartheta\lambda^2.\end{aligned} \tag{58}$$

This yields

$$|\lambda| \leq s\left|\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2})\right| + s\vartheta|\lambda|^2. \tag{59}$$

Hence, $\vartheta \geq 1$, which is a reductio ad absurdum. Then, $\phi^* = \Delta\phi^*$. Similarly, we derive that $\phi^* = \Omega\phi^*$. Therefore, Δ and Ω have a common fixed point ϕ^*. Following from Theorem 5, we can easily prove the uniqueness part. □

Example 3.5. Let $H = \{1,2,3,4\}$ be endowed with the partial order $\phi \lesssim \mu$ iff $\mu \leq \phi$. We define $\ell_{c\mathfrak{b}}\colon H \times H \longrightarrow C^+$ in Tables 1 and 2.

It is easy to verify that $(H,\ell_{c\mathfrak{b}})$ is a complete b-CPMS with the coefficient $s \geq 1$ for $x \in [0,\pi/2]$. Define $\Delta,\Omega\colon H \longrightarrow H$ by $\Delta\phi = 1$:

Table 1: Example of unique common fixed point.

(ϕ,μ)	$\ell_{cb}(\phi,\mu)$
(1,1), (3,3)	0
(1,2), (2,1), (1,3), (3,1), (2,3), (3,2), (2,2)	e^{2ix}
(1,4), (4,1), (2,4), (4,2), (3,4), (4,3), (4,4)	$9e^{2ix}$

Table 2: Example of no common fixed point.

(ϕ,μ)	$\ell_{cb}(\phi,\mu)$
(1,1), (3,3)	0
(1,2), (2,1), (1,3), (3,1), (2,3), (3,2), (2,2)	e^{2ix}
(1,4), (4,1), (2,4), (4,2), (3,4), (4,3), (4,4)	e^{2ix}

$$\Omega((\phi) = \begin{cases} 1 & \text{if } \phi \in \{1,2,3\}, \\ 2 & \text{if } \phi = 4. \end{cases} \tag{60}$$

Clearly, Δ and Ω are continuous functions. Now, for $\vartheta = 1/9$, we consider the following cases:

(A) If $\phi = 1$ and $\mu \in H - \{4\}$, then $\Delta(\phi) = \Omega(\mu) = 1$. Hence, all the conditions of Theorem 2 are fulfilled.

(B) If $\phi = 1$ and $\mu = 4$, then $\Delta\phi = 1$ and $\Omega\mu = 2$:

$$\begin{aligned} \ell_{cb}(\Delta\phi,\Omega\mu) &= e^{2ix} \precsim 9\vartheta e^{i2x} \\ &= \vartheta \max\left\{9e^{i2x}, 0, 9e^{i2x}, \frac{1}{2}\left(e^{i2x} + 9e^{i2x}\right)\right\} \\ &= \vartheta \max\{\ell_{cb}(\phi,\mu), \ell_{cb}(\phi,\Delta\phi), \ell_{cb}(\mu,\Omega\mu) \\ &\quad \frac{1}{2}\left(\ell_{cb}(\phi,\Omega\mu)\right) + \ell_{cb}(\mu,\Delta\phi)\bigg\}. \end{aligned} \tag{61}$$

(C) *If* $\phi = 2$ and $\mu = 4$, *then* $\Delta\phi = 1$ and $\Omega\mu = 2$:

$$\begin{aligned} \ell_{cb}(\Delta\phi,\Omega\mu) &= e^{2ix} \precsim 9\vartheta e^{i2x} \\ &= \vartheta \max\left\{9e^{i2x}, e^{i2x}, 9e^{i2x}, \frac{1}{2}\left(0 + 9e^{i2x}\right)\right\} \\ &= \vartheta \max\{\ell_{cb}(\phi,\mu), \ell_{cb}(\phi,\Delta\phi), \ell_{cb}(\mu,\Omega\mu) \\ &\quad \frac{1}{2}\left(\ell_{cb}(\phi,\Omega\mu)\right) + \ell_{cb}(\mu,\Delta\phi)\bigg\}. \end{aligned} \tag{62}$$

(D) If $\phi = 3$ and $\mu = 4$, then $\Delta\phi = 1$ and $\Omega\mu = 2$:

$$\begin{aligned} \ell_{cb}(\Delta\phi,\Omega\mu) &= e^{2ix} \precsim 9\vartheta e^{i2x} \\ &= \vartheta \max\left\{9e^{i2x}, e^{i2x}, 9e^{i2x}, \frac{1}{2}\left(e^{i2x} + 9e^{i2x}\right)\right\} \\ &= \vartheta \max\{\ell_{cb}(\phi,\mu), \ell_{cb}(\phi,\Delta\phi), \ell_{cb}(\mu,\Omega\mu) \\ &\quad \frac{1}{2}\left(\ell_{cb}(\phi,\Omega\mu) + \ell_{cb}(\mu,\Delta\phi)\right)\bigg\}. \end{aligned} \tag{63}$$

(E) If $\phi = 4$ and $\mu = 4$, then $\Delta\phi = 2$ and $\Omega\mu = 2$:

$$\begin{aligned} \ell_{cb}(\Delta\phi,\Omega\mu) &= e^{2ix} \precsim 9\vartheta e^{i2x} \\ &= \vartheta \max\left\{9e^{i2x}, 9e^{i2x}, 9e^{i2x}, \frac{1}{2}\left(9e^{i2x} + 9e^{i2x}\right)\right\} \\ &= \vartheta \max\{\ell_{cb}(\phi,\mu), \ell_{cb}(\phi,\Delta\phi), \ell_{cb}(\mu,\Omega\mu) \\ &\quad \frac{1}{2}\left(\ell_{cb}(\phi,\Omega\mu)\right) + \ell_{cb}(\mu,\Delta\phi)\bigg\}. \end{aligned} \tag{64}$$

All the conditions of Theorem 1, with $\vartheta = 1/9 < 1$, are fulfilled. Therefore, Δ and Ω have a unique common fixed point 1.

Example 3.6. Let $H = P \cup Q$, where $P = [1,2]$ and $Q = \{3,4\}$ be endowed with the partial order $\phi \precsim \mu$ iff $\mu \leq \phi$. Define $\ell_{cb}: H \times H \longrightarrow C^+$ by $\ell_{cb}(\phi,\mu) = (|\phi-\mu|^2 + 2)e^{2ix}$, for all $\phi, \mu \in P$ or $\phi \in P$ and $\mu \in Q$ or $\phi \in Q, \mu \in P$.

It is easy to verify that (H, ℓ_{cb}) is a complete b-CPMS with the coefficient $s \geq 1$ for $x \in [0, \pi/2]$. Define $\Delta, \Omega: H \longrightarrow H$ by

$$\Delta(\phi) = \begin{cases} 1 & \text{if } \phi \in \left[1, \frac{3}{2}\right], \\ 2 & \text{if } \phi \in \left(\frac{3}{2}, 2\right] \cup Q, \end{cases} \tag{65}$$

$$\Omega(\phi) = \begin{cases} 2 & \text{if } \phi \in \left[1, \frac{3}{2}\right], \\ 4 & \text{if } \phi \in \left(\frac{3}{2}, 2\right] \cup Q. \end{cases} \tag{66}$$

Clearly, Δ and Ω are not continuous functions. Now, we consider the following cases:

(A) *If* $\phi, \mu \in [1, 3/2]$, *then* $\Delta(\phi) = 1$ and $\Omega(\mu) = 2$:

$$\ell_{c\mathfrak{b}}(\Delta\phi, \Omega\mu) = e^{2ix} \lesssim \frac{1}{3} 3e^{i2x}$$
$$= \vartheta \max\{\ell_{c\mathfrak{b}}(\phi,\mu), \ell_{c\mathfrak{b}}(\phi,\Delta\phi), \ell_{c\mathfrak{b}}(\mu,\Omega\mu)$$
$$\frac{1}{2}\left(\ell_{c\mathfrak{b}}(\phi,\Omega\mu) + \ell_{c\mathfrak{b}}(\mu,\Delta\phi)\right)\Big\}. \quad (67)$$

(B) If $\phi \in [1, 3/2]$ and $\mu \in (3/2, 2] \cup Q$, then $\Delta\phi = 1$ and $\Omega\mu = 4$:

$$\ell_{c\mathfrak{b}}(\Delta\phi, \Omega\mu) = e^{2ix} \lesssim \frac{1}{3} 3e^{i2x}$$
$$= \vartheta \max\{\ell_{c\mathfrak{b}}(\phi,\mu), \ell_{c\mathfrak{b}}(\phi,\Delta\phi), \ell_{c\mathfrak{b}}(\mu,\Omega\mu)$$
$$\frac{1}{2}\left(\ell_{c\mathfrak{b}}(\phi,\Omega\mu) + \ell_{c\mathfrak{b}}(\mu,\Delta\phi)\right)\Big\}. \quad (68)$$

(C) *If* $\mu \in [1, 3/2]$ and $\phi \in (3/2] \cup Q$, *then* $\Delta\phi = 2$ and $\Omega\mu = 2$:

$$\ell_{c\mathfrak{b}}(\Delta\phi, \Omega\mu) = e^{2ix} \lesssim 3\frac{1}{3} e^{i2x}$$
$$= \vartheta \max\{\ell_{c\mathfrak{b}}(\phi,\mu), \ell_{c\mathfrak{b}}(\phi,\Delta\phi), \ell_{c\mathfrak{b}}(\mu,\Omega\mu)$$
$$\frac{1}{2}\left(\ell_{c\mathfrak{b}}(\phi,\Omega\mu) + \ell_{c\mathfrak{b}}(\mu,\Delta\phi)\right)\Big\}. \quad (69)$$

(D) If $\mu, \phi \in (3/2, 2] \cup Q$, then $\Delta\phi = 2$ and $\Omega\mu = 4$:

$$\ell_{c\mathfrak{b}}(\Delta\phi, \Omega\mu) = e^{2ix} \lesssim 3\frac{1}{3} e^{i2x}$$
$$= \vartheta \max\{\ell_{c\mathfrak{b}}(\phi,\mu), \ell_{c\mathfrak{b}}(\phi,\Delta\phi), \ell_{c\mathfrak{b}}(\mu,\Omega\mu) \quad (70)$$
$$\frac{1}{2}\left(\ell_{c\mathfrak{b}}(\phi,\Omega\mu) + \ell_{c\mathfrak{b}}(\mu,\Delta\phi)\right)\Big\}.$$

All the conditions of Theorems 2 and 3, with $\vartheta < 1$, are fulfilled except continuous mapping. Therefore, Δ and Ω have no common fixed point.

Remark 1. In view of the fact in Theorems 2 and 3, we cannot drop the continuous mapping.

4. Application

Consider the following systems of integral equations:

$$v(s) = \int_c^d T_1(s, \aleph, v(\aleph)) d\aleph, \quad (71)$$

$$\varrho(s) = \int_c^d T_2(s, \aleph, \varrho(\aleph)) d\aleph, \quad (72)$$

where

(i) $v(s)$ and $\varrho(s)$ are unknown variables for each $s \in J = [c, d]$, $d > c \geq 0$

(ii) $T_1(s, \aleph)$ and $T_2(s, \aleph)$ are deterministic kernels defined for $s, \aleph \in J = [c, d]$

Let $H = (C(J), R^\alpha)$ be the set of continuous functions defined on J. Define $\ell_{c\mathfrak{b}}: H \times H \longrightarrow C^+$ by

$$\ell_{c\mathfrak{b}}(v, \varrho) = |v(s) - \varrho(s)|^2 + 2, \quad (73)$$

$\forall\, v, \varrho \in H$. Then, $\ell_{c\mathfrak{b}}$ is a complete b-CPMS. Define partial order $\lesssim$ given by

$$v, \varrho \in H, v \lesssim \varrho \text{ if and only } v(s) \geq \varrho(s), \ \forall s \in J. \quad (74)$$

Theorem 6. *Assume that*

(A) $T_1, T_2: J \times J \times R^\alpha \longrightarrow R^\alpha$ *are continuous functions satisfying*

$$\left|T_1(s, \aleph, v(\aleph)) - T_2(s, \aleph, \varrho(s))\right| \lesssim \sqrt{\frac{S(v,\varrho)}{(b-a)e^t} - \frac{2}{b-a}}, \quad \forall t > 0, \quad (75)$$

where

$$S(v,\varrho) = \max\{\ell_{c\mathfrak{b}}(v,\varrho), \ell_{c\mathfrak{b}}(v,\Delta v), \ell_{c\mathfrak{b}}(\varrho,\Omega\varrho)$$
$$\frac{1}{2}\left(\ell_{c\mathfrak{b}}(v,\Omega\varrho) + \ell_{c\mathfrak{b}}(\varrho,\Delta v)\right)\Big\}. \quad (76)$$

Then, systems (71) and (72) have a unique common solution.

Proof. For $v, \varrho \in (C(J), R^\alpha)$ and $s \in J$, define the continuous mappings $\Delta, \Omega: H \longrightarrow H$ by

$$\Delta v(s) = \int_c^d T_1(s, \aleph, v(\aleph)) \mathrm{d}\aleph, \quad (77)$$

$$\Omega\varrho(s) = \int_c^d T_2(s, \aleph, \varrho(\aleph)) d\aleph. \quad (78)$$

Then,

$$\ell_{c\mathfrak{b}}(\Delta v(s), \Omega\varrho(s)) = |\Delta v(s) - \Omega\varrho(s)|^2 + 2$$
$$= \int_c^d \left|T_1(s, \aleph, v(\aleph)) - T_2(s, \aleph, \rangle(s))\right|^2 d\aleph + 2$$
$$\lesssim \int_c^d \left(\frac{S(v,\varrho)}{(b-a)e^t} - \frac{2}{b-a}\right) d\aleph + 2$$
$$= \frac{S(v,\varrho)}{e^t}$$
$$= \vartheta S(v,\varrho)$$
$$= \vartheta \max\{\ell_{c\mathfrak{b}}(v,\varrho), \ell_{c\mathfrak{b}}(v,\Delta v), \ell_{c\mathfrak{b}}(\varrho,\Omega\varrho)$$
$$\frac{1}{2}\left(\ell_{c\mathfrak{b}}(v,\Omega\varrho) + \ell_{c\mathfrak{b}}(\varrho,\Delta v)\right)\Big\}. \quad (79)$$

Hence, all the conditions of Theorem 2 are fulfilled for $0 < \vartheta = 1/e^t < 1/s$ with $t > 0$. Therefore, integrals (71) and (72) have a unique common solution. □

5. Conclusion

In this paper, we proved common fixed-point theorems on complex partial b-metric space. An illustrative example and application on complex partial b-metric space is given. Recently, Khalehoghli et al. [24, 25] introduced R-metric spaces and obtained a generalization of Banach fixed-point theorem. It is an interesting open problem to study the relation R instead of complex partial b-metric space and obtain common fixed-point results on R-complete complex partial b-metric spaces.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Novel Investigation of Fractional-Order Cauchy-Reaction Diffusion Equation Involving Caputo-Fabrizio Operator

Meshari Alesemi,[1] Naveed Iqbal,[2] and Mohammed S. Abdo[3]

[1]*Department of Mathematics, College of Science, University of Bisha, P. O. Box 511, Bisha 61922, Saudi Arabia*
[2]*Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il 2440, Saudi Arabia*
[3]*Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen*

Correspondence should be addressed to Naveed Iqbal; n.iqbal@uoh.edu.sa and Mohammed S. Abdo; msabdo@hoduniv.net.ye

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In this article, the new iterative transform technique and homotopy perturbation transform method are applied to calculate the fractional-order Cauchy-reaction diffusion equation solution. Yang transformation is mixed with the new iteration method and homotopy perturbation method in these methods. The fractional derivative is considered in the sense of Caputo-Fabrizio operator. The convection-diffusion models arise in physical phenomena in which energy, particles, or other physical properties are transferred within a physical process via two processes: diffusion and convection. Four problems are evaluated to demonstrate, show, and verify the present methods' efficiency. The analytically obtained results by the present method suggest that the method is accurate and simple to implement.

1. Introduction

The convection-diffusion equation is a mixture of convection and diffusion equations and identifies physical processes where energy, particles, or other physical properties are transmitted inside a physical process due to two process steps: diffusion and convection. In standard form, the convection-diffusion model is written as follows:

$$\frac{\partial \mathbb{U}}{\partial \mathfrak{J}} = \nabla \cdot (D \cdot \nabla \mathbb{U}) - \nabla \cdot \left(\vec{v}\mathbb{U}\right) + R, \tag{1}$$

where D is the diffusivity, $\mathbb{U}$ is the variable term, such as thermal diffusivity for heat flow or mass diffusion coefficient for particle, and $\vec{v}$ is the average velocity that the volume is travelling. For instance, in convection, u might be the density of river in salt and then the flow velocity of water $\vec{v}$. For example, in a calm lake, $\vec{v}$ would be the average velocity of bubbles rising to the surface due to buoyancy, and $\mathbb{U}$ would be the concentration of small bubbles. R defines "sinks" or "sources" of the quantity $\mathbb{U}$. For a chemical species, $R > 0$ indicates that a chemical reaction is increasing the number of the species, while $R > 0$ indicates that a chemical reaction is decreasing the number of the species. If thermal energy is generated by friction, $R > 0$ may occur in heat transport. ∇ denotes gradient, while $\nabla\cdot$ denotes divergence. Previously, different techniques have been applied to investigate these models such as Adomian's decomposition technique [1], variational iteration technique [2], Bessel collocation technique [3], and homotopy perturbation technique [4].

In recent decades, fractional derivatives have been used to interpret many physical problems mathematically, and these representations have produced excellent results in modelling real-world issues. Many basic definitions of fractional operators were given by Riesz, Riemann-Liouville, Hadamard, Weyl, Grunwald-Letnikov, Liouville-Caputo, Caputo-Fabrizio, and Atangana-Baleanu, among others [5–8]. Over the last few years, many nonlinear equations have been developed and widely used in nonlinear physical sciences like chemistry, biology, mathematics, and different branches of physics like plasma physics, condensed matter physics, fluid mechanics, field theory, and nonlinear optics. The exact outcome of nonlinear equations is crucial in determining the characteristics and behaviour of physical processes. Still, it is impossible to find exact results when dealing with linear equations. Many useful methods have been applied to investigate nonlinear fractional partial differential equations, for example, analytical solutions with the help of natural decomposition method of fractional-order heat and wave equations [9], fractional-order partial differential equations with proportional delay [10], fractional-order hyperbolic telegraph equation [11] and fractional-order diffusion equations [12], the variational iterative transform method [13], the homotopy perturbation transform method [14, 15], the homotopy analysis transform method [16, 17], reduced differential transform method [18, 19], q-homotopy analysis transform method [20–24], the finite element technique [25], the finite difference technique [26], and so on [27–30].

Daftardar-Gejji and Jafari developed a new iterative method of analysis for solving nonlinear equations in 2006 [31, 32]. It is the first application of Laplace transformation in iterative technique by Jafari et al. Iterative Laplace transformation method [33] was introduced as a simple method for estimating approximate effects of the fractional partial differential equation system. Iterative Laplace transformation method (NITM) is used to solve linear and nonlinear partial differential equations such as fractional-order Fornberg Whitham equations [34], time-fractional Zakharov Kuznetsov equation [35], and fractional-order Fokker Planck equations [36].

In 1999, He developed the homotopy perturbation method (HPM) [37], which combines the homotopy technique, and the standard perturbation method has been broadly utilized to both linear and nonlinear models [38–40]. The homotopy perturbation method is important because it eliminates the need for a small parameter in the model, eliminating the disadvantages of traditional perturbation techniques. The main goal of this paper is to use HPM to solve nonlinear fractional-order Cauchy-reaction diffusion equation using a newly introduced integral transformation known as the "Yang transform" [41]. The suggested technique is applied to analyse two well-known nonlinear partial differential equations. In the context of a quickly convergent series, we obtain a power series solution, and only a few iterations are required to obtain very efficient solutions. There is no need for a discretization technique or linearization for the nonlinear equations, and just a few few can yield a result that can be quickly estimated to utilize these methods.

2. Basic Definitions

We provide the fundamental definitions that will be used throughout the article. For the purpose of simplification, we write the exponential decay kernel as, $K(\mathfrak{I}, \varrho) = e^{[-\wp(\mathfrak{I}-\varrho/1-\wp)]}$.

Definition 1. The Caputo-Fabrizio derivative is given as follows [42]:

$$ {}^{\mathrm{CF}}D_{\mathfrak{I}}^{\wp}[\mathbb{P}(\mathfrak{I})] = \frac{N(\wp)}{1-\wp}\int_0^{\mathfrak{I}} \mathbb{P}'(\varrho)K(\mathfrak{I},\varrho)d\varrho, n-1 < \wp \le n. \tag{2} $$

$N(\wp)$ is the normalization function with $N(0) = N(1) = 1$.

$$ {}^{\mathrm{CF}}D_{\mathfrak{I}}^{\wp}[\mathbb{P}(\mathfrak{I})] = \frac{N(\wp)}{1-\wp}\int_0^{\mathfrak{I}} [\mathbb{P}(\mathfrak{I}) - \mathbb{P}(\varrho)]K(\mathfrak{I},\varrho)d\varrho. \tag{3} $$

Definition 2. The fractional integral Caputo-Fabrizio is given as [42]

$$ {}^{\mathrm{CF}}I_{\mathfrak{I}}^{\wp}[\mathbb{P}(\mathfrak{I})] = \frac{1-\wp}{N(\wp)}\mathbb{P}(\mathfrak{I}) + \frac{\wp}{N(\wp)}\int_0^{\mathfrak{I}} \mathbb{P}(\varrho)d\varrho, \mathfrak{I} \ge 0, \wp \in (0,1]. \tag{4} $$

Definition 3. For $N(\wp) = 1$, the following result shows the Caputo-Fabrizio derivative of Laplace transformation [42]:

$$ L\left[{}^{\mathrm{CF}}D_{\mathfrak{I}}^{\wp}[\mathbb{P}(\mathfrak{I})]\right] = \frac{\nu L[\mathbb{P}(\mathfrak{I}) - \mathbb{P}(0)]}{\nu + \wp(1-\nu)}. \tag{5} $$

Definition 4. The Yang transformation of $\mathbb{P}(\mathfrak{I})$ is expressed as [42]

$$ \mathbb{Y}[\mathbb{P}(\mathfrak{I})] = \chi(\nu) = \int_0^{\infty} \mathbb{P}(\mathfrak{I})e^{-\frac{\mathfrak{I}}{\nu}}d\mathfrak{I}. \mathfrak{I} > 0, \tag{6} $$

Remark 5. Yang transformation of few useful functions is defined as below.

$$ \begin{aligned} \mathbb{Y}[1] &= \nu, \\ \mathbb{Y}[\mathfrak{I}] &= \nu^2, \\ \mathbb{Y}\left[\mathfrak{I}^i\right] &= \Gamma(i+1)\nu^{i+1}. \end{aligned} \tag{7} $$

Lemma 6 (Laplace-Yang duality). *Let the Laplace transformation of $\mathbb{P}(\mathfrak{I})$ be $F(\nu)$, and then, $\chi(\nu) = F(1/\nu)$ [43].*

Proof. From Equation (5), we can achieve another type of the Yang transformation by putting $\mathfrak{I}/\nu = \zeta$ as

$$ L[\mathbb{P}(\mathfrak{I})] = \chi(\nu) = \nu\int_0^{\infty} \mathbb{P}(\nu\zeta)e^{\zeta}d\zeta. \zeta > 0. \tag{8} $$

Since $L[\mathbb{P}(\mathfrak{I})] = F(\nu)$, this implies that

$$ F(\nu) = L[\mathbb{P}(\mathfrak{I})] = \int_0^{\infty} \mathbb{P}(\mathfrak{I})e^{-\nu\mathfrak{I}}d\mathfrak{I}. \tag{9} $$

Put $\mathfrak{I} = \zeta/\nu$ in (8), and we have

$$ F(\nu) = \frac{1}{\nu}\int_0^{\infty} \mathbb{P}\left(\frac{\zeta}{\nu}\right)e^{\zeta}d\zeta. \tag{10} $$

Thus, from Equation (7), we achieve

$$F(\nu) = \chi\left(\frac{1}{\nu}\right). \tag{11}$$

Also from Equations (5) and (8), we achieve

$$F\left(\frac{1}{\nu}\right) = \chi(\nu). \tag{12}$$

The connections (10) and (11) represent the duality link between the Laplace and Yang transformation. □

Lemma 7. *Let $\mathbb{P}(\mathfrak{I})$ be a continuous function; then, the Caputo-Fabrizio derivative Yang transformation of $\mathbb{P}(\mathfrak{I})$ is define by [43]*

$$\mathbb{Y}[\mathbb{P}(\mathfrak{I})] = \frac{\mathbb{Y}[\mathbb{P}(\mathfrak{I}) - \nu\mathbb{P}(0)]}{1+\wp(\nu - 1)}. \tag{13}$$

Proof. The Caputo-Fabrizio fractional Laplace transformation is given by

$$L[\mathbb{P}(\mathfrak{I})] = \frac{L[\nu\mathbb{P}(\mathfrak{I}) - \mathbb{P}(0)]}{\nu+\wp(1 - \nu)}. \tag{14}$$

Also, we have that the connection among Laplace and Yang property, i.e., $\chi(\nu) = F(1/\nu)$. To achieve the necessary result, we substitute ν by $1/\nu$ in Equation (13), and we get

$$\begin{aligned} \mathbb{Y}[\mathbb{P}(\mathfrak{I})] &= \frac{1/\nu\mathbb{Y}[\mathbb{P}(\mathfrak{I}) - \mathbb{P}(0)]}{1/\nu+\wp(1 - 1/\nu)}, \\ \mathbb{Y}[\mathbb{P}(\mathfrak{I})] &= \frac{\mathbb{Y}[\mathbb{P}(\mathfrak{I}) - \nu\mathbb{P}(0)]}{1+\wp(\nu - 1)}. \end{aligned} \tag{15}$$

The proof is completed. □

3. Algorithm of the HPTM

The procedure of general nonlinear Caputo-Fabrizio fractional partial differential equations is through HPTM. Let us take a general nonlinear Caputo-Fabrizio partial differential equations with nonlinear function $N(\mathbb{U}(\varphi, \mathfrak{I}))$ and linear fractional $L(\mathbb{U}(\varphi, \mathfrak{I}))$ as [43]

$$\begin{cases} \mathrm{CF}D_{\mathfrak{I}}^{\rho}\mathbb{V}(\varphi, \mathfrak{I}) + L(\mathbb{V}(\varphi, \mathfrak{I})) + N(\mathbb{V}(\varphi, \mathfrak{I})) = g(\varphi, \mathfrak{I}), \\ \mathbb{V}(\varphi, 0) = h(\varphi), \end{cases} \tag{16}$$

where the term $g(\varphi, \mathfrak{I})$ shows the source function. Using Yang transformation to Equation (16), one can obtain

$$\begin{aligned} &\frac{\mathbb{Y}[\mathbb{V}(\varphi, \mathfrak{I}) - \nu\mathbb{V}(\varphi, 0)]}{1+\wp(\nu - 1)} \\ &\quad = -\mathbb{Y}[L(\mathbb{V}(\varphi, \mathfrak{I})) + N(\mathbb{V}(\varphi, \mathfrak{I}))] + \mathbb{Y}[g(\varphi, \mathfrak{I})], \end{aligned}$$

$$\begin{aligned} \mathbb{Y}[\mathbb{V}(\varphi, \mathfrak{I})] &= \nu h(\varphi) - (1+\wp(\nu - 1)) \\ &\quad \cdot [\mathbb{Y}[L(\mathbb{V}(\varphi, \mathfrak{I})) + N(\mathbb{V}(\varphi, \mathfrak{I}))] + \mathbb{Y}[g(\varphi, \mathfrak{I})]. \end{aligned} \tag{17}$$

Implementing inverse Yang transformation, we obtain

$$\begin{aligned} \mathbb{V}(\varphi, \mathfrak{I}) &= \mathbb{V}(\varphi, 0) - \mathbb{Y}^{-1}[1+\wp(\nu - 1) \\ &\quad \cdot [\mathbb{Y}[L(\mathbb{V}(\varphi, \mathfrak{I})) + N(\mathbb{V}(\varphi, \mathfrak{I}))] + \mathbb{Y}[g(\varphi, \mathfrak{I})]], \end{aligned} \tag{18}$$

where the term $\mathbb{V}(\varphi, \mathfrak{I})$ shows the source function and with the initial condition. Now, we apply homoptopy perturbation method.

$$\mathbb{V}(\varphi, \mathfrak{I}) = \sum_{i=0}^{\infty} \rho^i\mathbb{V}_i(\varphi, \mathfrak{I}). \tag{19}$$

We decompose the nonlinear term $N(\mathbb{V}(\varphi, \mathfrak{I}))$ as

$$N(\mathbb{V}(\varphi, \mathfrak{I})) = \sum_{i=0}^{\infty} \rho^i H_i(\mathbb{V}), \tag{20}$$

where $H_i(\mathbb{V})$ represents the He's polynomial and is calculated through the following formula:

$$H_i(\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \cdots, \mathbb{V}_i) = \frac{1}{\Gamma(i+1)}\frac{\partial^i}{\partial\rho^i}\left[N\left(\sum_{i=0}^{\infty} \rho^i\mathbb{V}_i\right)\right]_{\rho=0}, i = 1, 2, 3. \tag{21}$$

Substituting Equations (19) and (20) in Equation (18), we obtain

$$\begin{aligned} \sum_{i=0}^{\infty} \rho^i\mathbb{V}_i(\varphi, \mathfrak{I}) &= \mathbb{V}(\varphi, \mathfrak{I}) - \rho\left(\mathbb{Y}^{-1}\left[(1+\wp(\nu - 1))\mathbb{Y}\right.\right. \\ &\quad \left.\left. \cdot \left[L\sum_{i=0}^{\infty} \rho_i^{i\mathbb{V}}(\varphi, \mathfrak{I}) + N\sum_{i=0}^{\infty} \rho^i H_i(\mathbb{V})\right]\right]\right). \end{aligned} \tag{22}$$

We obtain the following terms by coefficients comparing of ρ in (22):

$$\begin{aligned} \rho^0 &: \mathbb{V}_0(\varphi, \mathfrak{I}) = \mathbb{V}(\varphi, \mathfrak{I}), \\ \rho^1 &: \mathbb{V}_1(\varphi, \mathfrak{I}) = \mathbb{Y}^{-1}[(1+\wp(\nu - 1))\mathbb{Y}[L(\mathbb{V}_0(\varphi, \mathfrak{I})) + H_0(\mathbb{V})]], \\ \rho^2 &: \mathbb{V}_2(\varphi, \mathfrak{I}) = \mathbb{Y}^{-1}[(1+\wp(\nu - 1))\mathbb{Y}[L(\mathbb{V}_1(\varphi, \mathfrak{I})) + H_1(\mathbb{V})]], \\ \rho^3 &: \mathbb{V}_3(\varphi, \mathfrak{I}) = \mathbb{Y}^{-1}[(1+\wp(\nu - 1))\mathbb{Y}[L(\mathbb{V}_2(\varphi, \mathfrak{I})) + H_2(\mathbb{V})]], \\ &\vdots \\ \rho^i &: \mathbb{V}_i(\varphi, \mathfrak{I}) = \mathbb{Y}^{-1}[(1+\wp(\nu - 1))\mathbb{Y}[L(\mathbb{V}_i(\varphi, \mathfrak{I})) + H_i(\mathbb{V})]]. \end{aligned} \tag{23}$$

As a result, the obtained solution of Equation (16) can be written as follows:

$$\mathbb{V}(\varphi, \mathfrak{J}) = \mathbb{V}_0(\varphi, \mathfrak{J}) + \mathbb{V}_1(\varphi, \mathfrak{J}) + \cdots. \tag{24}$$

4. Error Analysis and Convergence

The following theorems are fundamental on the techniques address the original models [16] error analysis and convergence.

Theorem 8. *Let $\mathbb{V}(\varphi, \mathfrak{J})$ be the actual result of (16), and let $\mathbb{V}_i(\varphi, \mathfrak{J}) \in H$ and $\sigma \in (0, 1)$, where H denotes the Hilbert space. Then, the achieved result $\sum_{i=0}^{\infty} \mathbb{V}_i(\varphi, \mathfrak{J})$ will converge $\mathbb{V}(\varphi, \mathfrak{J})$ if $\mathbb{V}_i(\varphi, \mathfrak{J}) \leq \mathbb{V}_{i-1}(\varphi, \mathfrak{J}) \forall i > A$, i.e., for any $\omega > 0 \exists A > 0$, such that $\|\mathbb{V}_{i+n}(\varphi, \mathfrak{J})\| \leq \beta, \forall i, n \in N$ [43].*

Proof. We make a sequence of $\sum_{i=0}^{\infty} \mathbb{V}_i(\varphi, \mathfrak{J})$.

$$\begin{aligned}
C_0(\varphi, \mathfrak{J}) &= \mathbb{V}_0(\varphi, \mathfrak{J}), \\
C_1(\varphi, \mathfrak{J}) &= \mathbb{V}_0(\varphi, \mathfrak{J}) + \mathbb{V}_1(\varphi, \mathfrak{J}), \\
C_2(\varphi, \mathfrak{J}) &= \mathbb{V}_0(\varphi, \mathfrak{J}) + \mathbb{V}_1(\varphi, \mathfrak{J}) + \mathbb{V}_2(\varphi, \mathfrak{J}), \\
C_3(\varphi, \mathfrak{J}) &= \mathbb{V}_0(\varphi, \mathfrak{J}) + \mathbb{V}_1(\varphi, \mathfrak{J}) + \mathbb{V}_2(\varphi, \mathfrak{J}) + \mathbb{V}_3(\varphi, \mathfrak{J}), \\
&\vdots \\
C_i(\varphi, \mathfrak{J}) &= \mathbb{V}_0(\varphi, \mathfrak{J}) + \mathbb{V}_1(\varphi, \mathfrak{J}) + \mathbb{V}_2(\varphi, \mathfrak{J}) + \cdots + \mathbb{V}_i(\varphi, \mathfrak{J}).
\end{aligned} \tag{25}$$

To provide the correct outcome, we have to demonstrate that $C_i(\varphi, \mathfrak{J})$ forms a "Cauchy sequence." Take, for example,

$$\begin{aligned}
\|C_{i+1}(\varphi, \mathfrak{J}) - C_i(\varphi, \mathfrak{J})\| &= \|\mathbb{V}_{i+1}(\varphi, \mathfrak{J})\| \leq \sigma \|\mathbb{V}_i(\varphi, \mathfrak{J})\| \\
&\leq \sigma^2 \|\mathbb{V}_{i-1}(\varphi, \mathfrak{J})\| \leq \sigma^3 \|\mathbb{V}_{i-2}(\varphi, \mathfrak{J})\| \cdots \\
&\leq \sigma_{i+1} \|\mathbb{V}_0(\varphi, \mathfrak{J})\|.
\end{aligned} \tag{26}$$

For $i, n \in N$, we acquire

$$\begin{aligned}
&\|C_i(\varphi, \mathfrak{J}) - C_n(\varphi, \mathfrak{J})\| = \|\mathbb{V}_{i+n}(\varphi, \mathfrak{J})\| \\
&\quad = \|C_i(\varphi, \mathfrak{J}) - C_{i-1}(\varphi, \mathfrak{J}) + (C_{i-1}(\varphi, \mathfrak{J}) - C_{i-2}(\varphi, \mathfrak{J})) \\
&\qquad + (C_{i-2}(\varphi, \mathfrak{J}) - C_{i-3}(\varphi, \mathfrak{J})) + \cdots + (C_{n+1}(\varphi, \mathfrak{J}) - C_n(\varphi, \mathfrak{J}))\| \\
&\quad \leq \|C_i(\varphi, \mathfrak{J}) - C_{i-1}(\varphi, \mathfrak{J})\| + \|C_{i-1}(\varphi, \mathfrak{J}) - C_{i-2}(\varphi, \mathfrak{J})\| \\
&\qquad + \|C_{i-2}(\varphi, \mathfrak{J}) - C_{i-3}(\varphi, \mathfrak{J})\| + \cdots + \|C_{n+1}(\varphi, \mathfrak{J}) - C_n(\varphi, \mathfrak{J})\| \\
&\quad \leq \sigma^i \|\mathbb{V}_0(\varphi, \mathfrak{J})\| + \sigma^{i-1} \|\mathbb{V}_0(\varphi, \mathfrak{J})\| + \cdots + \sigma^{i+1} \|\mathbb{V}_0(\varphi, \mathfrak{J})\| \\
&\quad = \|\mathbb{V}_0(\varphi, \mathfrak{J})\| \left(\sigma^i + \sigma^{i-1} + \sigma^{i+1}\right) \\
&\quad = \|\mathbb{V}_0(\varphi, \mathfrak{J})\| \frac{1 - \sigma^{i-n}}{1 - \sigma^{i+1}} \sigma^{n+1}.
\end{aligned} \tag{27}$$

Since $0 < \sigma < 1$, and $\mathbb{V}_0(\varphi, \mathfrak{J})$ is bounded, let us take $\beta = 1 - \sigma/(1 - \sigma_{i-n}) \sigma^{n+1} \|\mathbb{V}_0(\varphi, \mathfrak{J})\|$. Thus, $\{\mathbb{V}_i(\varphi, \mathfrak{J})\}_{i=0}^{\infty}$ forms a "Cauchy sequence" in H. It follows that the sequence $\{\mathbb{V}_i(\varphi, \mathfrak{J})\}_{i=0}^{\infty}$ is a convergent sequence with the limit $\lim_{i \longrightarrow \infty} \mathbb{V}_i(\varphi, \mathfrak{J}) = \mathbb{V}(\varphi, \mathfrak{J})$ for $\exists \mathbb{V}(\varphi, \mathfrak{J}) \in \mathscr{H}$. Hence, this ends the proof. □

Theorem 9. *Let $\sum_{h=0}^{k} \mathbb{V}_h(\varphi, \mathfrak{J})$ is finite and $\mathbb{V}(\varphi, \mathfrak{J})$ represents the obtained series solution. Let $\sigma > 0$ such that $\|\mathbb{V}_{h+1}(\varphi, \mathfrak{J})\| \leq \|\mathbb{V}_h(\varphi, \mathfrak{J})\|$; then, the following relation gives the maximum absolute error [43].*

$$\left\| \mathbb{V}(\varphi, \mathfrak{J}) - \sum_{h=0}^{k} \mathbb{V}_h(\varphi, \mathfrak{J}) \right\| < \frac{\sigma^{k+1}}{1 - \sigma} \|\mathbb{V}_0(\varphi, \mathfrak{J})\|. \tag{28}$$

Proof. Since $\sum_{h=0}^{k} \mathbb{V}_h(\varphi, \mathfrak{J})$ is finite, this implies that $\sum_{h=0}^{k} \mathbb{V}_h(\varphi, \mathfrak{J}) < \infty$.

Consider

$$\begin{aligned}
\left\| \mathbb{V}(\varphi, \mathfrak{J}) - \sum_{h=0}^{k} \mathbb{V}_h(\varphi, \mathfrak{J}) \right\| &= \left\| \sum_{h=k+1}^{\infty} \mathbb{V}_h(\varphi, \mathfrak{J}) \right\| \\
&\leq \sum_{h=k+1}^{\infty} \|\mathbb{V}_h(\varphi, \mathfrak{J})\| \\
&\leq \sum_{h=k+1}^{\infty} \sigma^h \|\mathbb{V}_0(\varphi, \mathfrak{J})\| \\
&\leq \sigma^{k+1} \left(1 + \sigma + \sigma^2 + \cdots\right) \|\mathbb{V}_0(\varphi, \mathfrak{J})\| \\
&\leq \frac{\sigma^{k+1}}{1 - \sigma} \|\mathbb{V}_0(\varphi, \mathfrak{J})\|.
\end{aligned} \tag{29}$$

This ends the theorem's proof. □

5. The General Procedure of NITM

The general solution of fractional-order partial differential equation is as follows:

$$\begin{aligned}
&{}^{\mathrm{CF}}D_{\mathfrak{J}}^{\wp} \mathbb{V}(\varphi, \mathfrak{J}) + N\mathbb{V}(\varphi, \mathfrak{J}) + M\mathbb{V}(\varphi, \mathfrak{J}) \\
&\quad = h(\varphi, \mathfrak{J}), i \in N, i - 1 < \wp \leq i,
\end{aligned} \tag{30}$$

where N is nonlinear and M linear functions.

With the initial condition

$$\mathbb{V}^k(\varphi, 0) = g_k(\varphi), k = 0, 1, 2, \cdots, i - 1, \tag{31}$$

implementing the Yang transformation of Equation (30), we get

$$\mathbb{Y}\left[D_{\mathfrak{J}}^{\wp} \mathbb{V}(\varphi, \mathfrak{J})\right] + \mathbb{Y}[N\mathbb{V}(\varphi, \mathfrak{J}) + M\mathbb{V}(\varphi, \mathfrak{J})] = \mathbb{Y}[h(\varphi, \mathfrak{J})]. \tag{32}$$

Applying the Yang differentiation is given to

$$\begin{aligned}
\mathbb{Y}[\mathbb{V}(\varphi, \mathfrak{J})] &= \nu \mathbb{V}(\varphi, 0) + (1 + \wp(\nu - 1)) \mathbb{Y}[h(\varphi, \mathfrak{J})] \\
&\quad - (1 + \wp(\nu - 1)) \mathbb{Y}[N\mathbb{V}(\varphi, \mathfrak{J}) + M\mathbb{V}(\varphi, \mathfrak{J})].
\end{aligned} \tag{33}$$

Using inverse Yang transformation Equation (32), we get

$$\mathbb{V}(\varphi,\mathfrak{J}) = \mathbb{Y}^{-1}[\{\nu\mathbb{V}(\varphi,0) + (1+\wp(\nu-1))\mathbb{Y}[h(\varphi,\mathfrak{J})]\}] - \mathbb{Y}^{-1}[(1+\wp(\nu-1))\mathbb{Y}[N\mathbb{V}(\varphi,\mathfrak{J}) + M\mathbb{V}(\varphi,\mathfrak{J})]]. \quad (34)$$

By iterative method, we get

$$\mathbb{V}(\varphi,\mathfrak{J}) = \sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J}), \quad (35)$$

$$N\left(\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J})\right) = \sum_{i=0}^{\infty}N[\mathbb{V}_i(\varphi,\mathfrak{J})]. \quad (36)$$

The nonlinear term N is identified as

$$N\left(\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J})\right) = \mathbb{V}_0(\varphi,\mathfrak{J}) + N\left(\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J})\right) - M\left(\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J})\right). \quad (37)$$

Putting Equations (35)–(37) in Equation (34), we have obtain the following solution:

$$\begin{aligned}
\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J}) &= \mathbb{Y}^{-1}\left[\left((1+\wp(\nu-1))\left(\sum_{i=0}^{\infty}s^{2-\varphi+i}u^i(\varphi,0) + \mathbb{Y}[h(\varphi,\mathfrak{J})]\right)\right] - \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\right.\right.\\
&\quad\left.\left.\cdot\left[N\left(\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J})\right) - M\left(\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J})\right)\right]\right]\right],\\
\mathbb{V}_0(\varphi,\mathfrak{J}) &= \mathbb{Y}^{-1}[\nu\mathbb{V}(\varphi,0) + (1+\wp(\nu-1))\mathbb{Y}(g(\varphi,\mathfrak{J}))],\\
\mathbb{V}_1(\varphi,\mathfrak{J}) &= -\mathbb{Y}^{-1}[(1+\wp(\nu-1))\mathbb{Y}[N[\mathbb{V}_0(\varphi,\mathfrak{J})] + M[\mathbb{V}_0(\varphi,\mathfrak{J})]]],\\
\mathbb{V}_{m+1}(\varphi,\mathfrak{J}) &= -\mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left[-N\left(\sum_{i=0}^{i}\mathbb{V}_i(\varphi,\mathfrak{J})\right) - M\left(\sum_{i=0}^{i}\mathbb{V}_i(\varphi,\mathfrak{J})\right)\right]\right], m \geq 1.
\end{aligned} \quad (38)$$

Lastly, Equations (30) and (31) provide the i-term solution in series form which is expressed as

$$\mathbb{V}(\varphi,\mathfrak{J}) \cong \mathbb{V}_0(\varphi,\mathfrak{J}) + \mathbb{V}_1(\varphi,\mathfrak{J}) + \mathbb{V}_2(\varphi,\mathfrak{J}) + \cdots.., + \mathbb{V}_i(\varphi,\mathfrak{J}), i = 1, 2, \cdots. \quad (39)$$

Example 10. Consider fractional-order Cauchy-reaction diffusion equation as [44]

$${}^{CF}D_{\mathfrak{J}}^{\wp}\mathbb{V}(\varphi,\mathfrak{J}) = D_{\mathfrak{J}}^{2}\mathbb{V}(\varphi,\mathfrak{J}) - \mathbb{V}(\varphi,\mathfrak{J}), 0 < \wp \leq 1, (\varphi,\mathfrak{J}) \in \Omega \subset R^2, \quad (40)$$

with initial and boundary conditions

$$\begin{aligned}
&\mathbb{V}(\varphi,0) = e^{-\varphi} + \varphi = g(\varphi), \mathbb{V}(0,\mathfrak{J}) = 1 = f_0(\mathfrak{J}),\\
&\frac{\partial\mathbb{V}(0,\mathfrak{J})}{\partial\mathfrak{J}} = e^{-\mathfrak{J}} - 1 = f_1(\mathfrak{J}), \varphi, \mathfrak{J} \in R.
\end{aligned} \quad (41)$$

The methodology consists of applying Yang transformation first on both side in (40) and utilizing the differentiation property of Yang transformation, and we have

$$\mathbb{Y}[\mathbb{V}(\varphi,\mathfrak{J})] = \nu(e^{-\varphi} + \varphi) + (1+\wp(\nu-1))\mathbb{Y}\left[D_{\mathfrak{J}}^{2}\mathbb{V} - \mathbb{V}\right]. \quad (42)$$

Using Yang inverse transform, we get

$$\mathbb{V}(\varphi,\mathfrak{J}) = (e^{-\varphi} + \varphi) + \mathbb{Y}^{-1}\left((1+\wp(\nu-1))\mathbb{Y}\left[D_{\mathfrak{J}}^{2}\mathbb{V} - \mathbb{V}\right]\right). \quad (43)$$

Now, we apply the new iterative transform method

$$\begin{aligned}
\mathbb{V}_0(\varphi,\mathfrak{J}) &= e^{-\varphi} + \varphi,\\
\mathbb{V}_1(\varphi,\mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{J}}^{2}\mathbb{V}_0 - \mathbb{V}_0\right\}\right] = -\varphi\{1+\wp\mathfrak{J}-\wp\},\\
\mathbb{V}_2(\varphi,\mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{J}}^{2}\mathbb{V}_1 - \mathbb{V}_1\right\}\right]\\
&= \varphi\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\},\\
\mathbb{V}_3(\varphi,\mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{J}}^{2}\mathbb{V}_2 - \mathbb{V}_2\right\}\right]\\
&= -\varphi\left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\}\\
&\vdots
\end{aligned} \quad (44)$$

The series type solution is given as

$$\mathbb{V}(\varphi,\mathfrak{J}) = \mathbb{V}_0(\varphi,\mathfrak{J}) + \mathbb{V}_1(\varphi,\mathfrak{J}) + \mathbb{V}_2(\varphi,\mathfrak{J}) + \mathbb{V}_3(\varphi,\mathfrak{J}) + \cdots\mathbb{V}_i(\varphi,\mathfrak{J}). \quad (45)$$

The approximate solution is achieved as

$$\mathbb{V}(\varphi,\mathfrak{I}) = e^{-\varphi} + \varphi\left\{1 - \{1+\wp\mathfrak{I}-\wp\} + \left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\} - \left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\} + \cdots\right\}. \tag{46}$$

Now applying the HPTM, we get

$$\sum_{i=0}^{\infty} p^i \mathbb{V}_i(\varphi,\mathfrak{I}) = (e^{-\varphi} + \varphi) + p \cdot \left\{\mathbb{Y}^{-1}\left((1+\wp(\nu-1))\mathbb{Y}\left[\sum_{i=0}^{\infty} p^i H_i(\mathbb{V})\right]\right)\right\}, \tag{47}$$

where the polynomials represent the nonlinear functions are $H_i(\mathbb{V})$. For instance, the terms of He's polynomials are achieved through the recursive relationship $H_i(\mathbb{V}) = D_{\mathfrak{I}}^2\mathbb{V}_i - \mathbb{V}_i$, $\forall n \in N$. Now, as the correspond power coefficients of p is comparison on both sides, the following solution is obtained as follows:

$$\begin{aligned}
p^0 &: \mathbb{V}_0(\varphi,\mathfrak{I}) = e^{-\varphi} + \varphi,\\
p^1 &: \mathbb{V}_1(\varphi,\mathfrak{I}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_0(\mathbb{V}))\}\right] = -\varphi\{1+\wp\mathfrak{I}-\wp\},\\
p^2 &: \mathbb{V}_2(\varphi,\mathfrak{I}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_1(\mathbb{V}))\}\right] = \varphi\left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\},\\
p^3 &: \mathbb{V}_3(\varphi,\mathfrak{I}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_2(\mathbb{V}))\}\right] = -\varphi\left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\},\\
&\vdots
\end{aligned} \tag{48}$$

Then, the homotopy perturbation method series form solution is defined as

$$\mathbb{V}(\varphi,\mathfrak{I}) = \sum_{i=0}^{\infty} p^i \mathbb{V}_i(\varphi,\mathfrak{I}). \tag{49}$$

The analytical result of the above equation is defined as

$$\begin{aligned}
\mathbb{V}(\varphi,\mathfrak{I}) &= e^{-\varphi} + \varphi\left\{1 - \{1+\wp\mathfrak{I}-\wp\} + \left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\} - \left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\} + \cdots\right\} = \varphi\sum_{i=0}^{\infty}\frac{(\mathfrak{I}^{\wp})^i}{\Gamma(i\wp+1)},\\
\mathbb{V}(\varphi,\mathfrak{I}) &= e^{-\varphi} + \varphi E_{\wp}(\mathfrak{I}^{\wp}).
\end{aligned} \tag{50}$$

The exact result of the above equation is

$$\mathbb{V}(\varphi,\mathfrak{I}) = e^{-\varphi} + \varphi e^{-\mathfrak{I}}. \tag{51}$$

Figure 1 shows the analytical solution of two methods at different fractional-order $\wp = 1$ and 0.8, and Figure 2 shows separate fractional-order at $\wp = 0.6$ and 0.4 with close contact with each other. In Figure 3, the graph shows the different fractional-order $\wp$ of Example 10.

Example 11. Consider fractional-order Cauchy-reaction diffusion equation as [44]

$${}^{CF}D_{\mathfrak{I}}^{\wp}\mathbb{V}(\varphi,\mathfrak{I}) = D_{\mathfrak{I}}^2\mathbb{V}(\varphi,\mathfrak{I}) - \left(1+4\varphi^2\right)\mathbb{V}(\varphi,\mathfrak{I}),\ 0<\wp\le 1,\ (\varphi,t) \in \Omega \subset R^2, \tag{52}$$

with initial condition

$$\mathbb{V}(\varphi,0) = e^{\varphi^2}. \tag{53}$$

and the exact result is given as

$$\mathbb{V}(\varphi,\mathfrak{I}) = e^{\varphi^2+1}. \tag{54}$$

Now, we apply the new iterative transform method

$$\begin{aligned}
\mathbb{V}_0(\varphi,\mathfrak{I}) &= e^{\varphi^2},\\
\mathbb{V}_1(\varphi,\mathfrak{I}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{I}}^2\mathbb{V}_0(\varphi,\mathfrak{I}) - \left(1+4\varphi^2\right)\mathbb{V}_0(\varphi,\mathfrak{I})\right\}\right] = e^{\varphi^2}\{1+\wp\mathfrak{I}-\wp\},\\
\mathbb{V}_2(\varphi,\mathfrak{I}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{I}}^2\mathbb{V}_1(\varphi,\mathfrak{I}) - \left(1+4\varphi^2\right)\mathbb{V}_1(\varphi,\mathfrak{I})\right\}\right] = e^{\varphi^2}\left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\},\\
\mathbb{V}_3(\varphi,\mathfrak{I}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{I}}^2\mathbb{V}_2(\varphi,\mathfrak{I}) - \left(1+4\varphi^2\right)\mathbb{V}_2(\varphi,\mathfrak{I})\right\}\right] = e^{\varphi^2}\left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\},\\
&\vdots
\end{aligned} \tag{55}$$

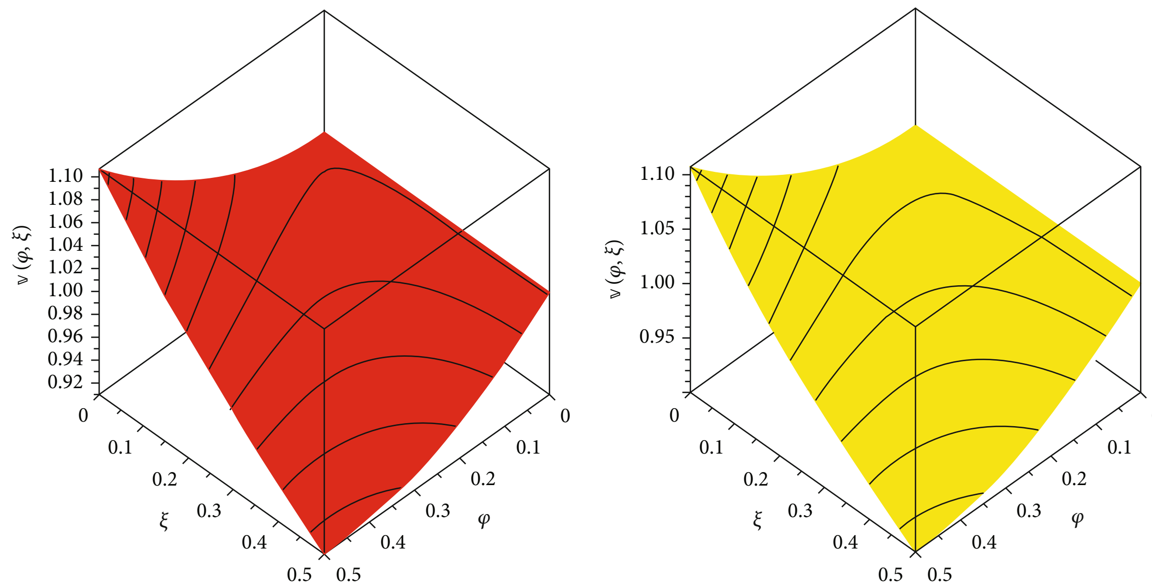

Figure 1: (a) $\wp = 1$ and (b) the fractional-order $\wp = 0.8$ of Example 10.

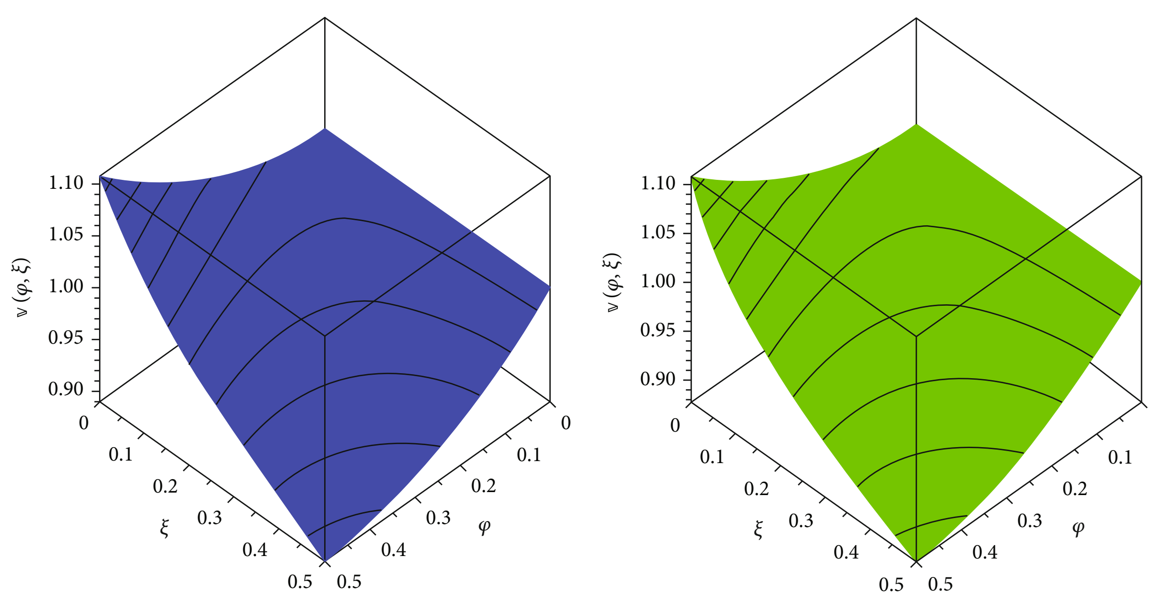

Figure 2: Different fractional-order of $\wp = 0.6$ and 0.4 of Example 10.

The series type solution is given as

$$\mathbb{V}(\varphi, \mathfrak{I}) = \mathbb{V}_0(\varphi, \mathfrak{I}) + \mathbb{V}_1(\varphi, \mathfrak{I}) + \mathbb{V}_2(\varphi, \mathfrak{I}) + \mathbb{V}_3(\varphi, \mathfrak{I}) + \cdots \mathbb{V}_i(\varphi, \mathfrak{I}). \tag{56}$$

The approximate solution of the above equation is defined as

$$\begin{aligned}\mathbb{V}(\varphi, \mathfrak{I}) = e^{\varphi^2}\Bigg\{ & 1 + \{1 + \wp\mathfrak{I} - \wp\} + \left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\} \\ & + \left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\} + \cdots\Bigg\},\end{aligned}$$

$$\mathbb{V}(\varphi, \mathfrak{I}) = e^{\varphi^2} E_{\wp}(\mathfrak{I}^{\wp}). \tag{57}$$

Now by applying homotopy perturbation transform method, we get

$$\begin{aligned}&\sum_{i=0}^{\infty} p^i \mathbb{V}_i(\varphi, \mathfrak{I}) \\ &\quad = e^{\varphi^2} + p\left\{\mathbb{Y}^{-1}\left((1 + \wp(\nu - 1))\mathbb{Y}\left[\sum_{i=0}^{\infty} p^i H_i(w)\right]\right)\right\}.\end{aligned} \tag{58}$$

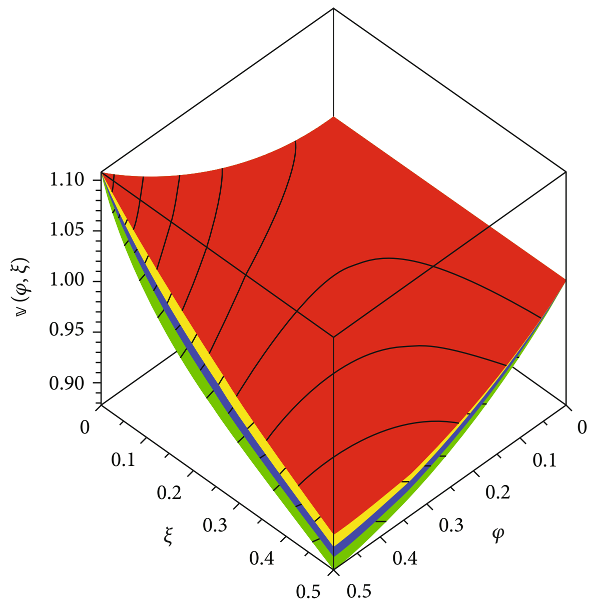

Figure 3: The different fractional-order $\wp$ of Example 10.

Comparing the coefficients of power p, we get

$$
\begin{aligned}
p^0 &: \mathbb{V}_0(\varphi, \mathfrak{I}) = e^{\varphi^2},\\
p^1 &: \mathbb{V}_1(\varphi, \mathfrak{I}) = \left\{\mathbb{Y}^{-1}((1+\wp(\nu-1))\mathbb{Y}[H_0(w)])\right\} = e^{\varphi^2}\{1+\wp\mathfrak{I}-\wp\},\\
p^2 &: \mathbb{V}_2(\varphi, \mathfrak{I}) = \left\{\mathbb{Y}^{-1}((1+\wp(\nu-1))\mathbb{Y}[H_1(w)])\right\}\\
&\qquad = e^{\varphi^2}\left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\},\\
p^3 &: \mathbb{V}_3(\varphi, \mathfrak{I}) = \left\{\mathbb{Y}^{-1}((1+\wp(\nu-1))\mathbb{Y}[H_2(w)])\right\}\\
&\qquad = e^{\varphi^2}\left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\},\\
&\quad\vdots
\end{aligned}
\tag{59}
$$

The HPTM series solution is given as

$$
\begin{aligned}
\mathbb{V}(\varphi, \mathfrak{I}) &= \sum_{i=0}^{\infty} p^i \mathbb{V}_i(\varphi, \mathfrak{I}),\\
\mathbb{V}(\varphi, \mathfrak{I}) &= e^{\varphi^2}\left\{1 + \{1+\wp\mathfrak{I}-\wp\} + \left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\} + \left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\} + \cdots\right\},\\
\mathbb{V}(\varphi, \mathfrak{I}) &= e^{\varphi^2} E_{\wp}(\mathfrak{I}^{\wp}).
\end{aligned}
\tag{60}
$$

Now $\wp = 1$; then, the actual result of Equation (52) is $\mathbb{V}(\varphi, \mathfrak{I}) = e^{\varphi^2+\mathfrak{I}}$.

Figure 4 shows the analytical solution of two methods at different fractional-order $\wp = 1$ and 0.8, and Figure 5 shows the separate fractional-order at $\wp = 0.6$ and 0.4 with close contact with each other. In Figure 6, the graph shows the different fractional-order $\wp$ of Example 11.

Example 12. Consider fractional-order Cauchy-reaction diffusion equation [44]

$$
{}^{CF}D_{\mathfrak{I}}^{\wp}\mathbb{V}(\varphi, \mathfrak{I}) = D_{\mathfrak{I}}^{2}\mathbb{V}(\varphi, \mathfrak{I}) + 2\mathfrak{I}\mathbb{V}(\varphi, \mathfrak{I}), 0 < \wp \le 1, (\varphi, \mathfrak{I}) \in \Omega \subset R^2,
\tag{61}
$$

with initial condition

$$
\mathbb{V}(\varphi, 0) = e^{\varphi}.
\tag{62}
$$

The exact result is

$$
\mathbb{V}(\varphi, \mathfrak{I}) = e^{\varphi+\mathfrak{I}+\mathfrak{I}^2}.
\tag{63}
$$

By using the Yang transformation, we get

$$
\mathbb{V}(\varphi, \mathfrak{I}) = (e^{\varphi}) + \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left(D_{\mathfrak{I}}^{2}\mathbb{V}(\varphi, \mathfrak{I}) + 2\mathfrak{I}\mathbb{V}(\varphi, \mathfrak{I})\right)\right].
\tag{64}
$$

Now, we apply the new iterative transform method

$$
\begin{aligned}
\mathbb{V}_0(\varphi, \mathfrak{I}) &= e^{\varphi},\\
\mathbb{V}_1(\varphi, \mathfrak{I}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{I}}^{2}\mathbb{V}_0(\varphi, \mathfrak{I}) + 2\mathfrak{I}\mathbb{V}_0(\varphi, \mathfrak{I})\right\}\right]\\
&= e^{\varphi}\left(\{1+\wp\mathfrak{I}-\wp\} + \left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\}\right),\\
\mathbb{V}_2(\varphi, \mathfrak{I}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{I}}^{2}\mathbb{V}_1(\varphi, \mathfrak{I}) + 2\mathfrak{I}\mathbb{V}_1(\varphi, \mathfrak{I})\right\}\right]\\
&= e^{\varphi}\left(\left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\} + \left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\}\right),\\
&\ \vdots
\end{aligned}
\tag{65}
$$

The series type solution is given as

$$
\mathbb{V}(\varphi, \mathfrak{I}) = \mathbb{V}_0(\varphi, \mathfrak{I}) + \mathbb{V}_1(\varphi, \mathfrak{I}) + \mathbb{V}_2(\varphi, \mathfrak{I}) + \mathbb{V}_3(\varphi, \mathfrak{I}) + \cdots \mathbb{V}_i(\varphi, \mathfrak{I}).
\tag{66}
$$

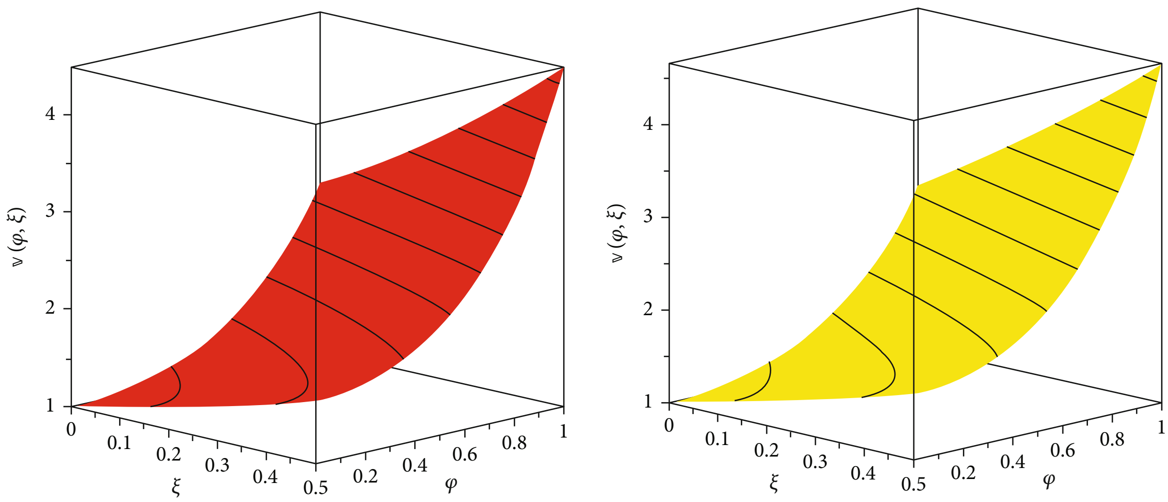

Figure 4: (a) $\wp = 1$ and (b) the fractional-order $\wp = 0.8$ of Example 10.

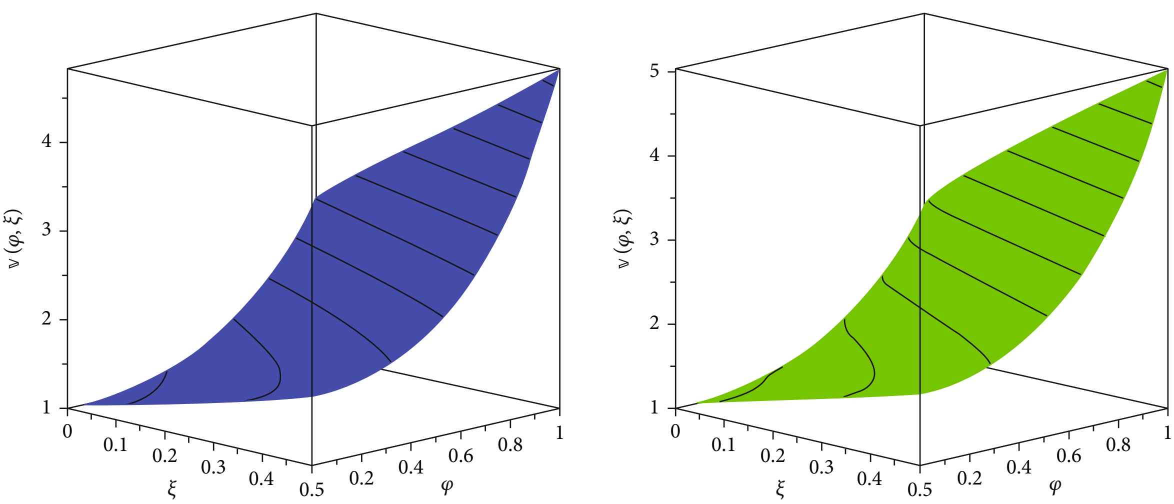

Figure 5: The different fractional-order of $\wp = 0.6$ and 0.4 of Example 10.

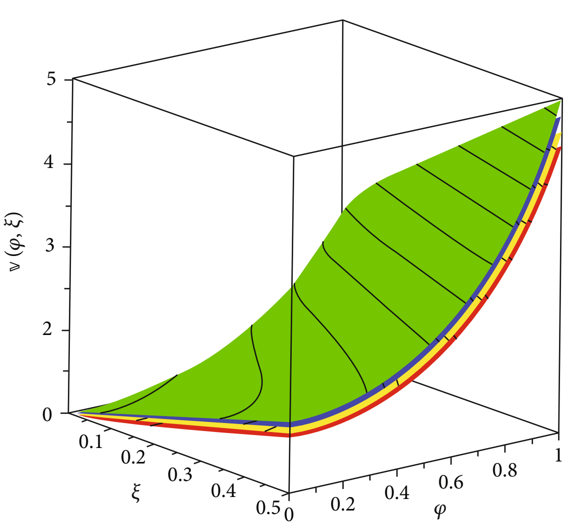

Figure 6: The different fractional-order $\wp$ of Example 10.

The approximate solution of the above equation is defined as

$$\begin{aligned}\mathbb{V}(\varphi, \mathfrak{I}) = {} & e^{\varphi} + e^{\varphi}\left(\{1+\wp\mathfrak{I}-\wp\} + \left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\}\right. \\ & + e^{\varphi}\left(\left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\}\right. \\ & \left.\left. + \left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\}\right) + \cdots.\end{aligned} \tag{67}$$

Now, using HPM, we get

$$\sum_{i=0}^{\infty} p^i \mathbb{V}_i(\varphi, \mathfrak{I}) = e^{\varphi} + p\left[\mathbb{Y}^{-1}\left\{(1+\wp(\nu-1))\mathbb{Y}\left(\sum_{i=0}^{\infty} p^i H_i(w)\right)\right\}\right]. \tag{68}$$

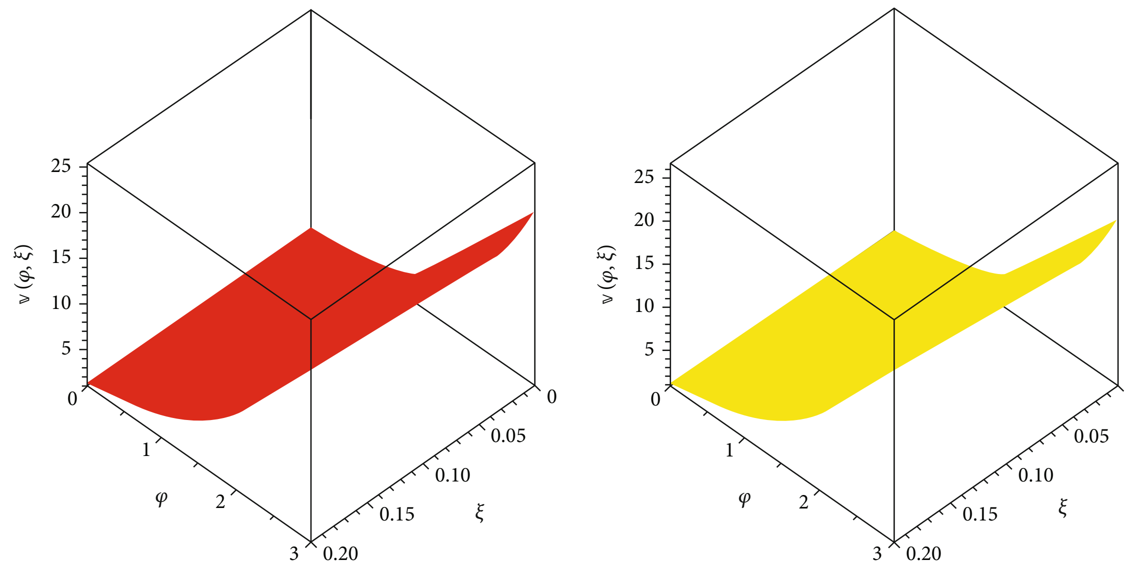

FIGURE 7: (a) $\wp = 1$ and (b) the fractional-order $\wp = 0.8$ of Example 10.

Comparing the coefficients of power p, we get

$$
\begin{aligned}
p^0 &: \mathbb{V}_0(\varphi, \mathfrak{J}) = e^{\varphi},\\
p^1 &: \mathbb{V}_1(\varphi, \mathfrak{J}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_0(w))\}\right]\\
&= e^{\varphi}\left(\{1+\wp\mathfrak{J}-\wp\} + \left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\}\right),\\
p^2 &: \mathbb{V}_2(\varphi, \mathfrak{J}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_1(w))\}\right]\\
&= e^{\varphi}\left(\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\}\right.\\
&\quad \left. + \left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\}\right).
\end{aligned} \tag{69}
$$

Proceeding in this path, the rest of the $\mathbb{V}n(\varphi, \mathfrak{J})$ for $n \geq 3$ component can be completely recovered and the series solution can therefore be absolutely determined. Eventually, we approximate the numerical solution $\mathbb{V}(\varphi, \mathfrak{J})$ to the truncated series.

$$
\begin{aligned}
\mathbb{V}(\varphi, \mathfrak{J}) &= \lim_{N\longrightarrow\infty} \sum_{n=1}^{N} \mathbb{V}_i(\varphi, \mathfrak{J}),\\
\mathbb{V}(\varphi, \mathfrak{J}) &= e^{\varphi} + e^{\varphi}\left(\{1+\wp\mathfrak{J}-\wp\} + \left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\}\right)\\
&\quad + e^{\varphi}\left(\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\}\right.\\
&\quad \left. + \left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\}\right) + \cdots.
\end{aligned} \tag{70}
$$

Now for $\wp = 1$, the closed form of the above series is

$$\mathbb{V}(\varphi, \mathfrak{J}) = e^{\varphi+\mathfrak{J}+\mathfrak{J}^2}. \tag{71}$$

Figure 7 shows the analytical solution of two methods at different fractional-order $\wp = 1$ and 0.8, and Figure 8 shows the separate fractional-order at $\wp = 0.6$ and 0.4 with close contact with each other. In Figure 9, the graph shows the different fractional-order $\wp$ of Example 12.

Example 13. Consider fractional-order Cauchy-reaction diffusion equation as [44]

$$
\begin{aligned}
{}^{CF}D_{\mathfrak{J}}^{\wp}\mathbb{V}(\varphi, \mathfrak{J}) &= D_{\mathfrak{J}}^{2}\mathbb{V}(\varphi, \mathfrak{J}) - \left(4\varphi^2 - 2\mathfrak{J} + 2\right)\mathbb{V}\\
&\quad \cdot (\varphi, \mathfrak{J}), 0 < \wp \leq 1, (\varphi, \mathfrak{J}) \in \Omega \subset R^2,
\end{aligned} \tag{72}
$$

with initial condition

$$\mathbb{V}(\varphi, 0) = e^{\varphi^2}. \tag{73}$$

The exact result is

$$\mathbb{V}(\varphi, \mathfrak{J}) = e^{\varphi^2+\mathfrak{J}^2}. \tag{74}$$

Now, we apply the new iterative transform method

$$
\begin{aligned}
\mathbb{V}_0(\varphi, \mathfrak{J}) &= e^{\varphi^2},\\
\mathbb{V}_1(\varphi, \mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{J}}^{2}\mathbb{V}_0(\varphi, \mathfrak{J})\right.\right.\\
&\quad \left.\left. - \left(4\varphi^2 - 2\mathfrak{J} + 2\right)\mathbb{V}_0(\varphi, \mathfrak{J})\right\}\right] = e^{\varphi^2}\{1+\wp\mathfrak{J}-\wp\},\\
\mathbb{V}_2(\varphi, \mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{J}}^{2}\mathbb{V}_1(\varphi, \mathfrak{J})\right.\right.\\
&\quad \left.\left. - \left(4\varphi^2 - 2\mathfrak{J} + 2\right)\mathbb{V}_1(\varphi, \mathfrak{J})\right\}\right]\\
&= e^{\varphi^2}\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\},
\end{aligned}
$$

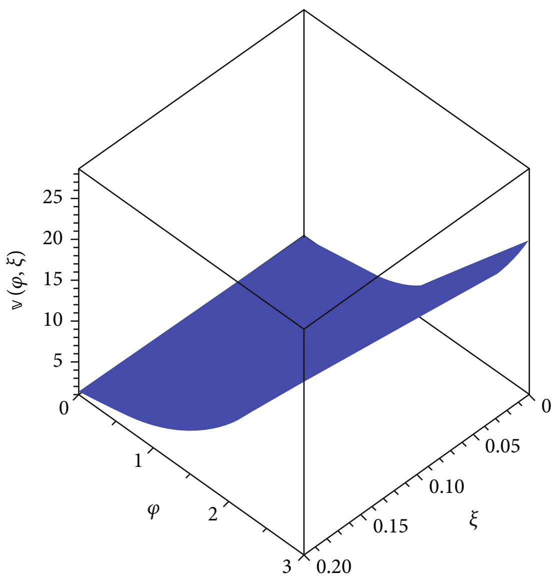

Figure 8: The different fractional-order of $\wp = 0.6$ and 0.4 of Example 10.

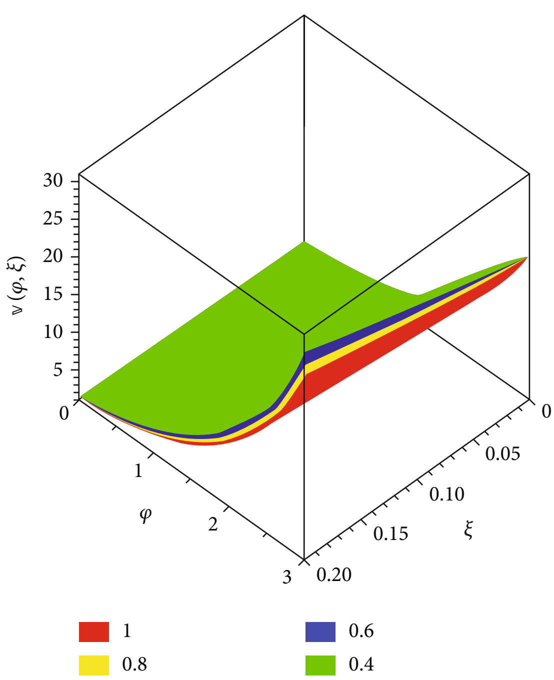

Figure 9: The different fractional-order $\wp$ of Example 10.

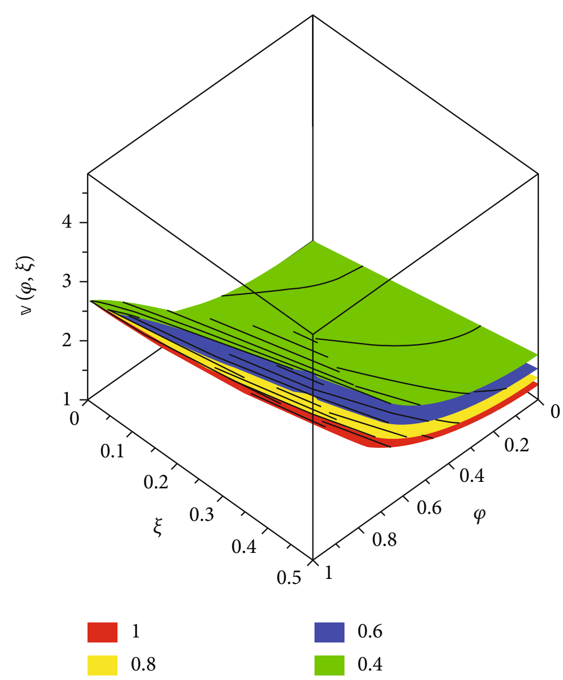

Figure 10: The different fractional-order $\wp$ of Example 13.

$$\begin{aligned}\mathbb{V}_3(\varphi, \mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{D^2_{\mathfrak{J}}\mathbb{V}_2(\varphi, \mathfrak{J}) - \left(4\varphi^2 - 2\mathfrak{J} + 2\right)\mathbb{V}_2(\varphi, \mathfrak{J})\right\}\right] \\ &= e^{\varphi^2}\left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\}, \\ &\vdots\end{aligned} \tag{75}$$

The series type solution is given as

$$\mathbb{V}(\varphi, \mathfrak{J}) = \mathbb{V}_0(\varphi, \mathfrak{J}) + \mathbb{V}_1(\varphi, \mathfrak{J}) + \mathbb{V}_2(\varphi, \mathfrak{J}) + \mathbb{V}_3(\varphi, \mathfrak{J}) + \cdots \mathbb{V}_i(\varphi, \mathfrak{J}). \tag{76}$$

The approximate solution of the above example is

$$\mathbb{V}(\varphi, \mathfrak{J}) = e^{\varphi^2} + e^{\varphi^2}\{1+\wp\mathfrak{J}-\wp\} + e^{\varphi^2} \cdot \left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\} + e^{\varphi^2}\left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\} + \cdots. \tag{77}$$

Now, using the HPM, we get

$$\sum_{i=0}^{\infty} p^i \mathbb{V}_i(\varphi, \mathfrak{J}) = e^{\varphi^2} + p\left[\mathbb{Y}^{-1}\left\{(1+\wp(\nu-1))\mathbb{Y}\left(\sum_{i=0}^{\infty} p^i H_i(w)\right)\right\}\right]. \tag{78}$$

Comparing the coefficients of power p, we have

$$p^0 : \mathbb{V}_0(\varphi, \mathfrak{J}) = e^{\varphi^2},$$

$$p^1 : \mathbb{V}_1(\varphi, \mathfrak{J}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_0(w))\}\right] = e^{\varphi^2}\{1+\wp\mathfrak{J}-\wp\},$$

$$p^2 : \mathbb{V}_2(\varphi, \mathfrak{J}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_1(w))\}\right] = e^{\varphi^2}\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\},$$

$$p^3 : \mathbb{V}_3(\varphi, \mathfrak{J}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_2(w))\}\right] = e^{\varphi^2}\left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\}. \tag{79}$$

Similarly, the remainder of the $\mathbb{V}_i(\varphi, \mathfrak{J})$ components for $n \geq 4$ can be completely achieved, thereby fully evaluating the series solutions. Finally, we estimate the approximate result $\mathbb{V}(\varphi, \mathfrak{J})$ by truncated sequence

$$\mathbb{V}(\varphi, \mathfrak{J}) = \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} \mathbb{V}_i(\varphi, \mathfrak{J}),$$

$$\mathbb{V}(\varphi, \mathfrak{J}) = e^{\varphi^2} + e^{\varphi^2}\{1+\wp\mathfrak{J}-\wp\} + e^{\varphi^2}\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\} + e^{\varphi^2}\left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\} + \cdots. \tag{80}$$

Figure 10 shows the analytical solution of two methods at different fractional-order $\wp = 1$, 0.8, 0.6, and 0.4 of Example 13. The special case for $\wp = 1$, and the above problem close form is given as

$$\mathbb{V}(\varphi, \mathfrak{J}) = e^{\varphi^2 + \mathfrak{J}^2}. \tag{81}$$

6. Conclusion

The homotopy perturbation transform technique and the Iterative transform method are used in this article to obtain numerical solutions for the fractional-order Cauchy-reaction diffusion equation, which is broadly used in applied sciences as a problem for spatial effects. In physical models, the techniques produce a series of form results that converge quickly. The obtained results in this article are expected to be useful for further analysis of complicated nonlinear physical problems. The calculations for these techniques are very simple and straightforward. As a result, we can conclude that these techniques can be used to solve a variety of nonlinear fractional-order partial differential equation schemes.

Data Availability

The numerical data used to support the findings of this study are included in the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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Research Article

Approximation Properties and q-Statistical Convergence of Stancu-Type Generalized Baskakov-Szász Operators

Qing-Bo Cai,[1] **Adem Kilicman**,[2] **and Mohammad Ayman Mursaleen**[2,3]

[1]Fujian Provincial Key Laboratory of Data-Intensive Computing, Key Laboratory of Intelligent Computing and Information Processing, School of Mathematics and Computer Science, Quanzhou Normal University, Quanzhou 362000, China
[2]Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia
[3]School of Information and Physical Sciences, The University of Newcastle, Callaghan, NSW 2308, Australia

Correspondence should be addressed to Mohammad Ayman Mursaleen; mohdaymanm@gmail.com

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In this article, we introduce Stancu-type modification of generalized Baskakov-Szász operators. We obtain recurrence relations to calculate moments for these new operators. We study several approximation properties and q-statistical approximation for these operators.

1. Introduction

In 1912, Bernstein [1] proposed the famous polynomial known as the Bernstein polynomial to give a simple, short, and most elegant proof of the Weierstrass approximation theorem. Since then, several papers have appeared to study approximation properties in different settings and spaces. Many new operators were constructed, e.g., Szász [2], Mirakjan [3], Kantorovic [4], Durrmeyer [5], Stancu [6], and many more [7–9]. These operators provide the improvement of approximating functions of different classes and give better and better estimates. For example, the Baskakov operators were given in [10]:

$$V_p(\mathfrak{h};\mathfrak{v}) == \sum_{l=0}^{\infty}\binom{p+l-1}{l}\frac{\mathfrak{v}^l}{(1+\mathfrak{v})^{p+l}}\mathfrak{h}\left(\frac{l}{p}\right). \tag{1}$$

For $\mathfrak{h}\in C[0,\infty)$, the space of all continuous functions on $[0,\infty)$ normed with standard sup-norm $\|\cdot\|_\infty$.

Devore and Lorentz [11] introduced a generalization of operators (1) dependent on a constant $a>0$ and independent of p as follows:

$$B_p(\mathfrak{h};u) = \sum_{j=0}^{\infty} W_{p,j}^a(u)\mathfrak{h}\left(\frac{j}{p}\right), \tag{2}$$

where

$$W_{p,j}^a(u) = e^{-au/(1+u)}\frac{G_j(p,a)}{j!}\frac{u^j}{(1+u)^{p+j}}, \sum_{j=0}^{\infty} W_{p,j}^a(u) = 1,$$
$$G_j(p,a) = \sum_{i=0}^{j}\binom{j}{i}(p)_i a^{j-i}, \tag{3}$$

and $(p)_i = p(p+1)\cdots(p+i-1), (p)_0 = 1$.

Recently, Agrawal et al. [12] studied the following operators (2):

$$L_{p,a}^*(\mathfrak{h};u) = p\sum_{j=0}^{\infty} W_{p,j}^a(u)\int_0^\infty s_{p,j}(t)\mathfrak{h}(t)dt, \tag{4}$$

for $\mathfrak{h}\in C_\gamma[0,\infty) \coloneqq \{\mathfrak{h}\in C[0,\infty): |\mathfrak{h}(t)|\le M_{\mathfrak{h}}e^{\gamma t}$, for some $\gamma>0, M_{\mathfrak{h}}>0\}$, where $s_{p,j}(t) = e^{-pt}((pt)^j/j!)$.

Inspired by Stancu's work [6], we have studied recently the Stancu-type generalization in [13]. Now, we propose the Stancu-type generalization of operators (4) as follows:

$$\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v}) = p\sum_{k=0}^{\infty} W_{p,k}^{a}(\mathfrak{v})\int_0^\infty s_{p,k}(t)\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}\right)dt, \quad (5)$$

for any bounded and integrable function $\mathfrak{h}$ defined on $[0,\infty)$, where $0 \le \lambda \le \mu$. For $\lambda = \mu = 0$, the operators (5) reduce to operators (4).

We establish recurrence relations to find moments and central moments. We study some approximation properties and the Voronovskaja-type asymptotic formula. We also study weighted approximation.

2. Auxiliary Results

Our first result is the recurrence formula for moments.

Theorem 1. *The m^{th} order moment for (5) is defined by*

$$\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}) \coloneqq \mathfrak{L}_{p,a}^{(\lambda,\mu)}(t^m;\mathfrak{v}) = p\sum_{k=0}^{\infty} W_{p,k}^{a}(\mathfrak{v})\int_0^\infty s_{p,k}(t)\mathfrak{h} \cdot \left(\frac{pt+\lambda}{p+\mu}\right)^m dt. \quad (6)$$

$m = 0, 1, 2, \cdots$. Then, $\mathfrak{T}_{p,a,0}^{(\lambda,\mu)}(\mathfrak{v}) = 1$, and

$$(m+1)(1+\mathfrak{v})\mathfrak{T}_{p,a,m+1}^{(\lambda,\mu)}(\mathfrak{v}) = \mathfrak{v}(1+\mathfrak{v})^2\left[\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v})\right]' + \{(1+\mathfrak{v})(\lambda+p\mathfrak{v}+m+1)+a\mathfrak{v}\}\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}) - \frac{\lambda m}{p+\mu}(1+\mathfrak{v})\mathfrak{T}_{p,a,m-1}^{(\lambda,\mu)}(\mathfrak{v}). \quad (7)$$

Proof. We use the identity

$$\mathfrak{v}(1+\mathfrak{v})^2\left(W_{p,k}^{a}(\mathfrak{v})\right)' = [(k-p\mathfrak{v})(1+\mathfrak{v})-a\mathfrak{v}]W_{p,k}^{a}(\mathfrak{v}). \quad (8)$$

Then,

$$\begin{aligned}\mathfrak{v}(1+\mathfrak{v})^2\left[\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v})\right]' &= p\sum_{k=0}^{\infty}\mathfrak{v}(1+\mathfrak{v})^2\left(W_{p,k}^{a}(\mathfrak{v})\right)'\int_0^\infty s_{p,k}(t)\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= p\sum_{k=0}^{\infty}[(k-p\mathfrak{v})(1+\mathfrak{v})-a\mathfrak{v}]W_{p,k}^{a}(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= p(1+\mathfrak{v})\sum_{k=0}^{\infty}(k-p\mathfrak{v})W_{p,k}^{a}(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &\quad - a\mathfrak{v}.p\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt,\end{aligned} \quad (9)$$

$$\mathfrak{v}(1+\mathfrak{v})^2\left[\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v})\right]' = I - ax\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}), \quad (10)$$

where

$$\begin{aligned}I &= p(1+\mathfrak{v})\sum_{k=0}^{\infty}(k-p\mathfrak{v})W_{p,k}^{a}(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= p(1+\mathfrak{v})\sum_{k=0}^{\infty}[k-(pt+\lambda)+(pt+\lambda)-p\mathfrak{v}] \cdot W_{p,k}^{a}(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= p(1+\mathfrak{v})\sum_{k=0}^{\infty}[(k-pt)+(pt+\lambda)-(\lambda+p\mathfrak{v})] \cdot W_{p,k}^{a}(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= p(1+\mathfrak{v})\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^\infty (k-pt)s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &\quad + p(1+\mathfrak{v})(p+\mu)\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^\infty (k-pt)s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^{m+1} \cdot dt - p(1+\mathfrak{v})(\lambda+p\mathfrak{v})\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt,\end{aligned} \quad (11)$$

$$I = \sum_1 + \sum_2 + \sum_3, \text{ say} \quad (12)$$

where

$$\sum_2 = (1+\mathfrak{v})(p+\mu)\mathfrak{T}_{p,a,m+1}^{(\lambda,\mu)}(\mathfrak{v}), \quad (13)$$

$$\sum_3 = -(1+\mathfrak{v})(\lambda+p\mathfrak{v})\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}), \quad (14)$$

$$\begin{aligned}\sum_1 &= p(1+\mathfrak{v})\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^\infty (k-pt)s_{n,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= (1+\mathfrak{v})\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^\infty pt\left(s_{p,k}(t)\right)' \cdot \left(\frac{pt+\lambda}{p+\mu}\right)^m dt, \left(\text{usingt}\left(s_{p,k}(t)\right)' = \int_0^\infty (k-pt)s_{p,k}(t)\right),\\ &= (p+\mu)(1+\mathfrak{v})\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^\infty\left(\frac{pt+\lambda}{p+\mu}-\frac{\lambda}{p+\mu}\right) \cdot \left(s_{p,k}(t)\right)'\left(\frac{pt+\lambda}{p+\mu}\right)^m dt = (p+\mu)(1+\mathfrak{v})\\ &\quad \cdot \sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^\infty \left(s_{p,k}(t)\right)'\left(\frac{pt+\lambda}{p+\mu}\right)^{m+1} dt\\ &\quad - \lambda(1+\mathfrak{v})\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^\infty \left(s_{p,k}(t)\right)'\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= \mathcal{J}_1 + \mathcal{J}_2, \text{ say}\end{aligned} \quad (15)$$

where

$$\begin{aligned}\mathcal{J}_1 &= (p+\mu)(1+\mathfrak{v})W_{p,k}^a(\mathfrak{v})\int_0^\infty \left(s_{p,k}(t)\right)'\left(\frac{pt+\lambda}{p+\mu}\right)^{m+1}dt\\ &= -(1+\mathfrak{v})(m+1)p\sum_{k=0}^\infty W_{p,k}^a(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &\quad -(1+\mathfrak{v})(m+1)\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}),\\ \mathcal{J}_2 &= -\lambda(1+\mathfrak{v})\sum_{k=0}^\infty W_{p,k}^a(\mathfrak{v})\int_0^\infty \left(s_{p,k}(t)\right)'\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= -\lambda(1+\mathfrak{v})\left(\frac{-mn}{p+\mu}\right)\sum_{k=0}^\infty W_{p,k}^a(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^{m-1}dt\\ &= \left(\frac{-m\lambda}{p+\mu}\right)(1+\mathfrak{v})\mathfrak{T}_{p,a,m-1}^{(\lambda,\mu)}(\mathfrak{v}).\end{aligned} \tag{16}$$

Therefore,

$$\sum_1 = \left(\frac{-m\lambda}{p+\mu}\right)(1+\mathfrak{v})\mathfrak{T}_{p,a,m-1}^{(\lambda,\mu)}(\mathfrak{v}) - (1+\mathfrak{v})(m+1)\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}). \tag{17}$$

Substituting (17), (14), and (13) in (12), we get

$$\begin{aligned}I &= \frac{-m\lambda}{p+\mu}(1+\mathfrak{v})\mathfrak{T}_{p,a,m-1}^{(\lambda,\mu)}(\mathfrak{v}) - (1+\mathfrak{v})(p\mathfrak{v}+\lambda\\ &\quad + m+1)\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}) + (1+\mathfrak{v})(p+\mu)\mathfrak{T}_{p,a,m+1}^{(\lambda,\mu)}(\mathfrak{v}).\end{aligned} \tag{18}$$

Further, substituting (18) in (10), we get the result. □

Corollary 2. *From the above theorem, we get*

(i) $\mathfrak{L}_{p,a}^{(\lambda,\mu)}(1;\mathfrak{v}) = 1$

(ii) $\mathfrak{L}_{p,a}^{(\lambda,\mu)}(t;\mathfrak{v}) = 1/(p+\mu)(p\mathfrak{v} + (a\mathfrak{v}/(1+\mathfrak{v})) + 1 + \lambda)$

(iii) $\mathfrak{L}_{p,a}^{(\lambda,\mu)}(t^2;\mathfrak{v}) = 1/(p+\mu)^2[\{p^2 + p + (a^2/(1+\mathfrak{v})^2) + (2ap/(1+\mathfrak{v}))\}\mathfrak{v}^2 + \{4p + (4a/(1+\mathfrak{v})) + 2\lambda p + (2a\lambda/(1+\mathfrak{v}))\}\mathfrak{v} + \{\lambda^2 + 2\lambda + 2\}]$

Theorem 3. *The m^{th} order central moment is defined by*

$$\begin{aligned}\mathfrak{M}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}) &:= \mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v})^m;\mathfrak{v})\\ &= p\sum_{k=0}^\infty W_{p,k}^a(\mathfrak{v})\int_0^\infty s_{p,k}(t)\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}-u\right)^m dt.\end{aligned} \tag{19}$$

$m = 0, 1, 2, \cdots$. The following recurrence relation holds:

$$\begin{aligned}&\mathfrak{v}(1+\mathfrak{v})^2\left(\mathfrak{M}_{p,a,m+1}^{(\lambda,\mu)}(\mathfrak{v})\right)'\\ &\quad = \left(\frac{1+\mathfrak{v}}{p+\mu}\right)(\lambda - p\mathfrak{v} - \mu\mathfrak{v} - mp\mathfrak{v} - m\mu\mathfrak{v} - mp\mathfrak{v}^2\\ &\quad - m\mu\mathfrak{v}^2)\mathfrak{M}_{p,a,m-1}^{(\lambda,\mu)}(\mathfrak{v}) - (m + m\mathfrak{v} + 1 + \mathfrak{v}\\ &\quad - \mu\mathfrak{v} + \lambda - \mu\mathfrak{v}^2 + \lambda\mathfrak{v} + a\mathfrak{v})\mathfrak{M}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v})\\ &\quad + (1+\mathfrak{v})(p+\mu)\mathfrak{M}_{p,a,m+1}^{(\lambda,\mu)}(\mathfrak{v}).\end{aligned} \tag{20}$$

Corollary 4. *From the above theorem, we get*

(a) $\mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v});\mathfrak{v}) = 1/(p+\mu)(-\mu\mathfrak{v} + (a\mathfrak{v}/(1+\mathfrak{v})) + \lambda + 1)$

(b) $\mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v})^2;\mathfrak{v}) = 1/(p+\mu)^2(p + \mu^2 + (a^2/(1+\mathfrak{v})^2) - (2a\mu/(1+\mathfrak{v})))\mathfrak{v}^2 + (2/(p+\mu)^2)(p - \mu - \lambda\mu + ((2+\lambda)a/(1+\mathfrak{v})))\mathfrak{v} + (1/(p+\mu)^2)(\lambda^2 + 2\lambda + 2)$

Corollary 5. *We further get*

(a) $\lim_{p\longrightarrow\infty} p\mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v});\mathfrak{v}) = -\mu u + (a\mathfrak{v}/1+\mathfrak{v}) + \lambda + 1$

(b) $\lim_{p\longrightarrow\infty} p\mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v})^2;\mathfrak{v}) = \mathfrak{v}(\mathfrak{v}+2)$

3. Main Results

Peetre's K-functional is defined as

$$K_2(\mathfrak{h},\delta) := \inf\left\{\|\mathfrak{h}-\mathfrak{g}\| + \delta\|\mathfrak{g}''\| : \mathfrak{g}\in C_B^2[0,\infty)\right\}, \tag{21}$$

for $\mathfrak{h}\in C_B[0,\infty)$, $\delta > 0$, where $C_B[0,\infty) := \{\mathfrak{h}\in C_B[0,\infty)$: $\mathfrak{h}$ is bounded on $[0,\infty)\}$ and $C_B^2[0,\infty) := \{\mathfrak{g}\in C_B[0,\infty)$: $\mathfrak{g}', \mathfrak{g}'' \in C_B[0,\infty)\}$. Note that

$$K_2(\mathfrak{h};\delta) \le M\omega_2\left(\mathfrak{h};\sqrt{\delta}\right), M > 0, \tag{22}$$

where $\omega_2(\mathfrak{h};\delta)$ is the second-order modulus of continuity [11].

$$\omega_2(\mathfrak{h},\delta) = \sup_{0<l\le\delta}\sup_{\mathfrak{v}\in 0,\infty)} |\mathfrak{h}(\mathfrak{v}+2l) - 2\mathfrak{h}(\mathfrak{v}+l) + \mathfrak{h}(\mathfrak{v})|, \quad \delta > 0. \tag{23}$$

The usual modulus of continuity of $\mathfrak{h}\in C_B[0,\infty)$ is defined as

$$\omega_1(\mathfrak{h},\delta) = \sup_{0<l\le\delta}\sup_{\mathfrak{v}\in 0,\infty)} |\mathfrak{h}(\mathfrak{v}+l) - \mathfrak{h}(\mathfrak{v})|. \tag{24}$$

Theorem 6. *For* $\mathfrak{h} \in C_B[0,\infty)$,

$$\left|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})\right| \leq M\omega_2\left(\mathfrak{h};\sqrt{\phi_{p,a}^{(\lambda,\mu)}}\right)+\omega_1\left(\mathfrak{h};\frac{1}{p+\mu}\cdot\left(1+\lambda-\mu\mathfrak{v}+\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)\right), \tag{25}$$

where $M>0$ *and*

$$\phi_{p,a}^{(\lambda,\mu)}=\frac{1}{(p+\mu)^2}\left\{\left(p^2+\mu^2+\frac{a^2}{(1+\mathfrak{v})^2}-\frac{2a\mu}{(1+\mathfrak{v})}\right)\mathfrak{v}^2+\left(2p-2\mu-2\lambda\mu+\frac{2a(2+\lambda)}{(1+\mathfrak{v})}\right)\mathfrak{v}\right\}+\frac{1}{(p+\mu)^2}\left\{\left(1+\lambda-\mu\mathfrak{v}+\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)^2+\lambda^2+2\lambda+2\right\}. \tag{26}$$

Proof. Put

$$\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})=\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})+\mathfrak{h}(\mathfrak{v})-\mathfrak{h}\left(\frac{1+\lambda}{p+\mu}+\frac{p\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right). \tag{27}$$

Note that $\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(1;\mathfrak{v})=1$ and $\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(t;\mathfrak{v})=\mathfrak{v}$. Let $\mathfrak{E}\in C_B^2[0,\infty)$. Then, by using Taylor's theorem, we may write

$$\mathfrak{E}(t)=\mathfrak{E}(\mathfrak{v})+(t-\mathfrak{v})\mathfrak{E}'(x)\int_{\mathfrak{v}}^{t}(t-y)\mathfrak{E}''(y)dy, \tag{28}$$

which gives

$$\begin{aligned}\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{E};\mathfrak{v})-\mathfrak{E}(\mathfrak{v})&=\mathfrak{E}'(\mathfrak{v})\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v});\mathfrak{v})+\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}\cdot\left(\int_{\mathfrak{v}}^{t}(t-y)\mathfrak{E}''(y)dy;\mathfrak{v}\right)\\&=\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}\left(\int_{\mathfrak{v}}^{t}(t-y)\mathfrak{E}''(y)dy;\mathfrak{v}\right)\\&=\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(\int_{\mathfrak{v}}^{t}(t-y)\mathfrak{E}''(y)dy;\mathfrak{v}\right)-\int_{\mathfrak{v}}^{1/(p+\mu)(1+\lambda+p\mathfrak{v}+(a\mathfrak{v}/1+\mathfrak{v}))}\left(\frac{1+\lambda}{p+\mu}+\frac{p\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}-\mathfrak{v}\right)\mathfrak{E}''(y)dy.\end{aligned} \tag{29}$$

Hence,

$$\begin{aligned}\left|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{E};\mathfrak{v})-\mathfrak{E}(\mathfrak{v})\right|&\leq\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(\left|\int_{\mathfrak{v}}^{t}(t-y)\mathfrak{E}''(y)dy\right|;\mathfrak{v}\right)\\&+\left|\int_{\mathfrak{v}}^{1/(p+\mu)(1+\lambda+p\mathfrak{v}+(a\mathfrak{v}/(1+\mathfrak{v})))}\left(\frac{1+\lambda}{p+\mu}+\frac{p\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}-y\right)\mathfrak{E}''(y)dy\right|.\end{aligned} \tag{30}$$

Since $|\int_{\mathfrak{v}}^{t}(t-y)\mathfrak{E}''(y)dy|\leq(t-\mathfrak{v})^2\|\mathfrak{E}''\|$ and

$$\left|\int_{\mathfrak{v}}^{1/(p+\mu)(1+\lambda+p\mathfrak{v}+(a\mathfrak{v}/(1+\mathfrak{v})))}\left(\frac{1+\lambda}{p+\mu}+\frac{p\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}-y\right)\mathfrak{E}''(y)dy\right|\leq\left(\frac{1+\lambda}{p+\mu}-\frac{\mu\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)^2\|\mathfrak{E}''\|, \tag{31}$$

we have

$$\left|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{E};\mathfrak{v})-\mathfrak{E}(\mathfrak{v})\right|\leq\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right)+\left(\frac{1+\lambda}{p+\mu}-\frac{\mu\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)^2\|\mathfrak{E}''\|. \tag{32}$$

Now, by Corollary 4 (b), we get

$$\left|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{E};\mathfrak{v})-\mathfrak{E}(\mathfrak{v})\right|\leq\phi_{p,a}^{(\lambda,\mu)}(\mathfrak{v})\|\mathfrak{E}''\|. \tag{33}$$

By (27), we get

$$\begin{aligned}|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})|&\leq\left|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{h}-\mathfrak{E};\mathfrak{v})\right|+|(\mathfrak{h}-\mathfrak{E})(\mathfrak{v})|+\left|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{E};\mathfrak{v})-\mathfrak{E}(\mathfrak{v})\right|\\&+\left|\mathfrak{h}\left(\frac{1+\lambda}{p+\mu}+\frac{p\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)-\mathfrak{h}(\mathfrak{v})\right|.\end{aligned} \tag{34}$$

Since $|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})|\leq 3\|\mathfrak{h}\|$, we get

$$\begin{aligned}|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})|&\leq 4\|\mathfrak{h}-\mathfrak{E}\|+\left|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{E};\mathfrak{v})-\mathfrak{E}(\mathfrak{v})\right|\\&+\left|\mathfrak{h}\left(\frac{1+\lambda}{p+\mu}+\frac{p\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)-\mathfrak{h}(\mathfrak{v})\right|.\end{aligned} \tag{35}$$

From (33), we get

$$|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})|\leq 4\|\mathfrak{h}-\mathfrak{E}\|+\phi_{p,a}^{(\lambda,\mu)}(\mathfrak{v})\|\mathfrak{E}''\|+\omega_1\left(\mathfrak{h}:\frac{1+\lambda}{p+\mu}-\frac{\mu\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right). \tag{36}$$

Now, taking the infimum over all $\mathfrak{E}\in C_B^2[0,\infty)$, we obtain

$$|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})|\leq 4K_2\left(\mathfrak{h};\phi_{p,a}^{(\lambda,\mu)}(\mathfrak{v})\right)+\omega_1\left(f:\frac{1+\lambda}{p+\mu}-\frac{\mu\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right). \tag{37}$$

Hence, by using (22), we get the result. □

For our next result, we consider the functions belonging to the Lipschitz class:

$$\mathrm{lip}_{\mathscr{M}}(\gamma)=\left\{\mathfrak{h}\in C_B[0,\infty):\ |\mathfrak{h}(t)-\mathfrak{h}(\mathfrak{v})|\leq\mathscr{M}\frac{|t-\mathfrak{v}|^{\gamma}}{(t+\mathfrak{v})^{\gamma/2}}\right\}, \tag{38}$$

where $\mathscr{M}>0$ and $0<\gamma\leq 1$; $\mathfrak{v},t\in 0,\infty)$.

Theorem 7. *For $\mathfrak{h}\in lip_{\mathscr{M}}(\gamma)$, we have*

$$|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})|\leq\mathscr{M}\left(\frac{\varphi_p^{(\lambda,\mu)}(\mathfrak{v})}{\mathfrak{v}}\right)^{\gamma/2}, \tag{39}$$

where $\varphi_{p,a}^{(\lambda,\mu)}(\mathfrak{v})=\mathfrak{L}_{p,a}^{(\lambda,\mu)}((e_1-\mathfrak{v})^2;\mathfrak{v})$.

Proof. First, we prove for $\gamma=1$. For $\mathfrak{h}\in\mathrm{lip}_{\mathscr{M}}(\gamma)$, we get

$$\begin{aligned}\left|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})\right|&\leq p\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}s_{p,k}(t)\left|\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}\right)-\mathfrak{h}(\mathfrak{v})\right|dt\\&\leq\mathscr{M}p\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}s_{p,k}(t)\frac{|((pt+\lambda)/(p+\mu))-\mathfrak{v}|}{\sqrt{((pt+\lambda)/(p+\mu))+\mathfrak{v}}}dt.\end{aligned} \tag{40}$$

Since $\sqrt{\mathfrak{v}}<\sqrt{((pt+\lambda)/(p+\mu))+\mathfrak{v}}$, we get by the Cauchy-Schwarz inequality:

$$\begin{aligned}\left|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})\right|&\leq\frac{\mathscr{M}}{\sqrt{\mathfrak{v}}}p\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}s_{n,k}(t)\left|\frac{pt+\lambda}{p+\mu}-u\right|dt\\&=\frac{\mathscr{M}}{\sqrt{\mathfrak{v}}}\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((e_1-\mathfrak{v})^2;\mathfrak{v}\right)\leq\mathscr{M}\sqrt{\frac{\varphi_{p,a}^{(\lambda,\mu)}(\mathfrak{v})}{\mathfrak{v}}}.\end{aligned} \tag{41}$$

For $0<\gamma<1$, applying Hölder's inequality, we get

$$\begin{aligned}\left|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})\right|&\leq p\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}s_{p,k}(t)\left|\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}\right)-\mathfrak{h}(\mathfrak{v})\right|dt\\&\leq p\sum_{k=0}^{\infty}\left\{W_{p,k}^{a}(\mathfrak{v})\left(\int_0^{\infty}s_{p,k}(t)\left|\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}\right)-\mathfrak{h}(\mathfrak{v})\right|dt\right)^{1/\gamma}\right\}^{\gamma}\\&\leq p\sum_{k=0}^{\infty}\left\{W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}s_{p,k}(t)\left|\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}\right)-\mathfrak{h}(\mathfrak{v})\right|^{1/\gamma}dt\right\}^{\gamma}.\end{aligned} \tag{42}$$

Since $\mathfrak{h}\in\mathrm{lip}_M(\gamma)$, we have

$$\begin{aligned}|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})&\leq\frac{\mathscr{M}}{\mathfrak{v}^{\gamma/2}}\left\{p\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}s_{p,k}(t)\mid\frac{pt+\lambda}{p+\mu}-\mathfrak{v}\mid dt\right\}^{\gamma}\\&=\frac{\mathscr{M}}{\mathfrak{v}^{\gamma/2}}\mathfrak{L}_{p,a}^{(\lambda,\mu)}(|e_1-\mathfrak{v}|;\mathfrak{v})^{\gamma}=\frac{\mathscr{M}}{\mathfrak{v}^{\gamma/2}}\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((e_1-\mathfrak{v})^2;\mathfrak{v}\right)^{\gamma}\\&\leq\mathscr{M}\left(\sqrt{\frac{\varphi_{p,a}^{(\lambda,\mu)}(\mathfrak{v})}{\mathfrak{v}}}\right)^{\gamma}.\end{aligned} \tag{43}$$

Therefore, we get (39). □

Next, we obtain a Voronovskaja-type asymptotic formula.

Theorem 8. *If $\mathfrak{h}''$ exists at a point $\mathfrak{v}\in 0,\infty)$ for $\mathfrak{h}\in C_{\gamma}[0,\infty)$, then*

$$\begin{aligned}&\lim_{p\longrightarrow\infty}p\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h}\simeq\mathfrak{v})-\mathfrak{h}(\mathfrak{v})\right)\\&\quad=\left(1+\lambda-\mu\mathfrak{v}+\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)\mathfrak{h}'(\mathfrak{v})+\frac{\mathfrak{v}}{2}(2+\mathfrak{v})\mathfrak{h}''(\mathfrak{v}).\end{aligned} \tag{44}$$

Proof. From Taylor's expansion of $\mathfrak{v}$, we may write

$$\mathfrak{h}(t)=\mathfrak{h}(\mathfrak{v})+(t-\mathfrak{v})\mathfrak{h}'(\mathfrak{v})+\frac{1}{2}(t-\mathfrak{v})^2\mathfrak{h}''(\mathfrak{v})+R(t,\mathfrak{v})(t-\mathfrak{v})^2, \tag{45}$$

where $R(t, \mathfrak{v}) \longrightarrow 0(t \longrightarrow \mathfrak{v})$. By operating $\mathfrak{L}_{p,a}^{(\lambda,\mu)}$, we obtain

$$\begin{aligned}\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h} \simeq \mathfrak{v}) - \mathfrak{h}(\mathfrak{v}) = {} & \mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v});\mathfrak{v})\mathfrak{h}'(\mathfrak{v}) \\ & + \mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right)\frac{\mathfrak{h}''(\mathfrak{v})}{2} \\ & + \mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(R(t,\mathfrak{v})(t-\mathfrak{v})^2;\mathfrak{v}\right).\end{aligned} \tag{46}$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(R(t,\mathfrak{v})(t-\mathfrak{v})^2;\mathfrak{v}\right) \\ &\quad \le \left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(R^2(t,\mathfrak{v});\mathfrak{v}\right)\right)^{1/2}\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^4;\mathfrak{v}\right)\right)^{1/2}.\end{aligned} \tag{47}$$

Since $\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h} \simeq \mathfrak{v}) \longrightarrow \mathfrak{h}(\mathfrak{v})$, we get

$$\begin{aligned}&\lim_{p\longrightarrow\infty}\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(R^2(t,\mathfrak{v});\mathfrak{v}\right) = R^2(\mathfrak{v},\mathfrak{v}) = 0, \\ &\lim_{p\longrightarrow\infty} p\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(R(t,\mathfrak{v})(t-\mathfrak{v})^2;\mathfrak{v}\right) = 0.\end{aligned} \tag{48}$$

Now, combining the above equations and using Corollary 5, we get

$$\begin{aligned}&\lim_{p\longrightarrow\infty} p\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h} \simeq \mathfrak{v}) - \mathfrak{h}(\mathfrak{v})\right) \\ &\quad = \lim_{p\longrightarrow\infty} p\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v});\mathfrak{v})\right)\mathfrak{h}'(\mathfrak{v}) \\ &\qquad + \lim_{p\longrightarrow\infty} p\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right)\right)\frac{\mathfrak{h}''(\mathfrak{v})}{2} \\ &\qquad + \lim_{p\longrightarrow\infty} p\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(R(t,\mathfrak{v})(t-\mathfrak{v})^2;\mathfrak{v}\right)\right) \\ &\quad = \left(1+\lambda-\mu\mathfrak{v}+\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)\mathfrak{h}'(\mathfrak{v}) + \frac{\mathfrak{v}}{2}(2+\mathfrak{v})\mathfrak{h}''(\mathfrak{v}).\end{aligned} \tag{49}$$

□

Let $B_\sigma[0,\infty) = \{\mathfrak{h} : [0,\infty) \longrightarrow \mathbb{R} | |\mathfrak{h}(\mathfrak{v})| \le \mathscr{K}_{\mathfrak{h}}\sigma(\mathfrak{v}), \mathfrak{v} \ge 0\}$, where $\mathscr{K}_{\mathfrak{h}}$ is a constant which depends only on $\mathfrak{h}$, and

$$\|\mathfrak{h}\|_\sigma = \sup_{\mathfrak{v}\in 0,\infty)} \frac{|\mathfrak{h}(\mathfrak{v})|}{\sigma(\mathfrak{v})}. \tag{50}$$

Also, let $C_\sigma[0,\infty) = \{\mathfrak{h} \in B_\sigma[0,\infty)$: $\mathfrak{h}$ be continuous on $[0,\infty)\}$, and

$$C_\sigma^0[0,\infty) = \left\{\mathfrak{h} \in C_\sigma[0,\infty): \lim_{\mathfrak{v}\longrightarrow\infty} \frac{|\mathfrak{h}(\mathfrak{v})|}{\sigma(\mathfrak{v})} \text{exists}\right\}, \tag{51}$$

where $\sigma(\mathfrak{v}) = 1 + \mathfrak{v}^2$.

The weighted modulus of continuity [14] is defined by

$$\Omega(l,\delta) \coloneqq \sup_{0<l\le\delta}\sup_{\mathfrak{v}\in 0,\infty)} \frac{|\mathfrak{h}(\mathfrak{v}+l) - \mathfrak{h}(\mathfrak{v})|}{1+(\mathfrak{v}+l)^2}. \tag{52}$$

Lemma 9 (see [14]). *Let $\mathfrak{h} \in C_\sigma^0[0,\infty)$. Then,*

(i) $\Omega(l,\delta)$ is a monotone increasing function of δ

(ii) $\Omega(l,\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$

(iii) $\Omega(l,k\delta) \le k\Omega(l,\delta)$ for each $k \in \mathbb{N}$

(iv) $\Omega(l,\alpha\delta) \le (1+\alpha)\Omega(l,\delta)$ for each $\alpha \in \mathbb{R}^+$

Theorem 10. *For $\mathfrak{h} \in C_\sigma^0[0,\infty)$, we have*

$$\sup_{\mathfrak{v}\in 0,\infty)} \frac{|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v}) - \mathfrak{h}(\mathfrak{v})|}{(1+\mathfrak{v}^2)^{5/2}} \le M\Omega\left(l,\frac{1}{p}\right), M > 0. \tag{53}$$

Proof. By Lemma 9, we have

$$\begin{aligned}|\mathfrak{h}(t) - \mathfrak{h}(\mathfrak{v})| &\le (1+(\mathfrak{v}+|t-\mathfrak{v}|)^2\Omega(l,|t-\mathfrak{v}|) \\ &\le 2\left(1+\mathfrak{v}^2\right)\left(1+(t-\mathfrak{v})^2\right)\left(1+\frac{|t-\mathfrak{v}|}{\delta}\right)\Omega(l,\delta).\end{aligned} \tag{54}$$

Operating $\mathfrak{L}_{p,a}^{(\lambda,\mu)}$, we get

$$\begin{aligned}&\left|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v}) - \mathfrak{h}(\mathfrak{v})\right| \\ &\quad \le 2\left(1+\mathfrak{v}^2\right)\Omega(l,\delta)\left\{1+\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right)\right. \\ &\qquad \left. + \mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(1+(t-\mathfrak{v})^2\frac{|\mathfrak{v}-t|}{\delta};\mathfrak{v}\right)\right\}.\end{aligned} \tag{55}$$

Using a second-order central moment, we get

$$\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right) \le M_1\frac{\left(1+\mathfrak{v}^2\right)}{(p+\mu)} \le M_1\frac{\left(1+\mathfrak{v}^2\right)}{p}, M_1 > 0. \tag{56}$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(1+(t-\mathfrak{v})^2\frac{|\mathfrak{v}-t|}{\delta};u\right) \\ &\quad \le \frac{1}{\delta}\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right)\right)^{1/2} \\ &\qquad + \frac{1}{\delta}\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^4;\mathfrak{v}\right)\right)^{1/2}\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right)\right)^{1/2}.\end{aligned} \tag{57}$$

Again, using the central moment of order 4, we get

$$\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^4;\mathfrak{v}\right)\right)^{1/2} \le M_2\frac{\left(1+\mathfrak{v}^2\right)}{(p+\mu)} \le M_2\frac{\left(1+\mathfrak{v}^2\right)}{p}, M_2 > 0. \tag{58}$$

Combining the estimates (55)–(58) and choosing $M = 2(1+M_1+\sqrt{M_1}+M_2\sqrt{M_1})$, $\delta = 1/\sqrt{p}$, we get the required result. □

4. q-Statistical Convergence

Defining a q-analog of the Cesàro matrix C_1 is not unique (see [15, 16]). Here, we consider the q-Cesàro matrix, $C_1(q) = (c^1_{nk}(q^k))^\infty_{n,k=0}$, defined by

$$c^1_{nk}\left(q^k\right) = \begin{cases} \frac{q^k}{[n+1]_q}, & \text{if } k \le n, \\ 0, & \text{otherwise}, \end{cases} \tag{59}$$

which is regular for $q \ge 1$.

Let $\mathscr{K} \subseteq \mathbb{N}$ (the set of natural numbers). Then, $\delta(\mathscr{K}) = \lim_r (1/r)\#\{k \le r : k \in \mathscr{K}\}$ is called the asymptotic density of $\mathscr{K}$, where # denotes the cardinality of the enclosed set. A sequence $\eta = (\eta_k)$ is called statistically convergent to the number $\mathfrak{s}$ if $\delta(\mathscr{K}_\varepsilon) = 0$ for each $\varepsilon > 0$, where $\mathscr{K}_\varepsilon = \{k \le r : |\eta_k - \mathfrak{s}| > \varepsilon\}$ (see [17]).

Recently, Aktuğlu and Bekar [16] defined q-density and q-statistical convergence. The q-density is defined by

$$\begin{aligned} \delta_q(\mathscr{K}) &= \delta_{C_1^q}(\mathscr{K}) = \lim_{n\longrightarrow\infty} \inf \left(C_1^q \chi_{\mathscr{K}}\right)_n \\ &= \lim_{n\longrightarrow\infty} \inf \sum_{k\in K} \frac{q^{k-1}}{[n]}, q \ge 1. \end{aligned} \tag{60}$$

A sequence $\eta = (\eta_k)$ is said to be q-statistically convergent to the number $\mathscr{L}$ if $\delta_q(\mathscr{K}_\varepsilon) = 0$, where $\mathscr{K}_\varepsilon = \{k \le n : |\eta_k - \mathscr{L}| \ge \varepsilon\}$ for every $\varepsilon > 0$. That is, for each $\varepsilon > 0$,

$$\lim_n \frac{1}{[n]} \#\left\{k \le n : q^{k-1} \mid \eta_k - \mathscr{L}| \ge \varepsilon\right\} = 0, \tag{61}$$

and we write $St_q - \lim \eta_k = \mathscr{L}$.

If $\delta(\mathscr{K}) = 0$ for an infinite set $\mathscr{K}$, then $\delta_q(\mathscr{K}) = 0$; hence, statistical convergence implies q-statistical convergence but not conversely (c.f. [16, Example 15]). Recently in [18], authors proved Korovkin's type theorem via q-statistical convergence. Using the same technique we prove the following theorem.

Theorem 11. *For all $\mathfrak{h} \in C^0_\rho$, we have*

$$St_q - \lim_p \left\| \mathfrak{L}^{(\lambda,\mu)}_{p,a}(\mathfrak{h}; \mathfrak{v}) - \mathfrak{h}(\mathfrak{v}) \right\|_\sigma = 0, \mathfrak{v} \in [0, \infty). \tag{62}$$

Proof. It is sufficient to show that $St_q - \lim_p \|\mathfrak{L}^{(\lambda,\mu)}_{p,a}(e_i; \mathfrak{v}) - e_i\|_\sigma = 0$, for $i = 0, 1, 2$, where $e_i(\mathfrak{v}) = \mathfrak{v}^i$. It is clear that

$$St_q - \lim_p \left\| \mathfrak{L}^{(\lambda,\mu)}_{p,a}(e_0; \mathfrak{v}) - e_0 \right\|_\sigma = 0. \tag{63}$$

By Corollary 2 (ii), we have

$$\begin{aligned} \left\| \mathfrak{L}^{(\lambda,\mu)}_{p,a}(t; \mathfrak{v}) - \mathfrak{v} \right\|_\sigma &= \sup_{u\in 0,\infty)} \frac{1}{1+\mathfrak{v}^2} \left| \frac{1}{p+\mu} \left(p\mathfrak{v} + \frac{a\mathfrak{v}}{1+\mathfrak{v}} \right.\right. \\ &\left.\left. + 1 + \lambda \right) - \mathfrak{v} \right| \le \frac{1}{p+\mu} |-\mu + 1 + \lambda + a|. \end{aligned} \tag{64}$$

For $\varepsilon > 0$, define the sets:

$$\begin{aligned} \mathscr{E}_1 &:= \left\{ p \in \mathbb{N} : \|\mathfrak{L}^{(\lambda,\mu)}_{p,a}(e_1; \mathfrak{v}) - e_1\|_\sigma \ge \varepsilon \right\}, \\ \mathscr{E}_2 &:= \left\{ p \in \mathbb{N} : \left| \frac{1+\lambda-\mu+a}{(p+\mu)} \right| \ge \varepsilon \right\}. \end{aligned} \tag{65}$$

Then,

$$\delta_q(\mathscr{E}_2) = \lim_{p\longrightarrow\infty} \inf \left(C_1^q \chi_{\mathscr{E}_2} \right)_p = \lim_{p\longrightarrow\infty} \inf \sum_{k\in\mathscr{E}_2} \frac{q^{k-1}}{[p]} = 0. \tag{66}$$

Since $\mathscr{E}_1 \subseteq \mathscr{E}_2$, we have $\delta_q(\mathscr{E}_1) \le \delta_q(\mathscr{E}_2)$. Hence,

$$St_q - \lim_p \left\| \mathfrak{L}^{(\lambda,\mu)}_{p,a}(e_1; \mathfrak{v}) - e_1 \right\|_\sigma = 0. \tag{67}$$

Again, by Corollary 2 (iii), we obtain

$$\begin{aligned} &\left\| \mathfrak{L}^{(\lambda,\mu)}_{p,a}(t^2; \mathfrak{v}) - \mathfrak{v}^2 \right\|_\sigma \\ &\le \sup_{\mathfrak{v}\in 0,\infty)} \frac{1}{1+\mathfrak{v}^2} \left| \frac{1}{(p+\mu)^2} \left\{ p + \mu^2 + \frac{a^2}{(1+\mathfrak{v})^2} - \frac{2a\mu}{(1+\mathfrak{v})} \right\} \mathfrak{v}^2 \right| \\ &+ \sup_{\mathfrak{v}\in 0,\infty)} \frac{1}{1+\mathfrak{v}^2} \left| \frac{2}{(p+\mu)^2} \left(p - \mu - \lambda\mu + \frac{(2+\lambda)a}{(1+\mathfrak{v})} \right) \mathfrak{v} \right. \\ &\left. + \frac{1}{(p+\mu)^2} \left(\lambda^2 + 2\lambda + 2 \right) \right| \le \frac{1}{(p+\mu)^2} \left\{ p + \mu^2 + a^2 - 2a\mu \right\} \\ &+ \frac{2}{(p+\mu)^2} (p - \mu - \lambda\mu + (2+\lambda)a) + \frac{1}{(p+\mu)^2} \left(\lambda^2 + 2\lambda + 2 \right). \end{aligned} \tag{68}$$

For $\varepsilon > 0$, define the sets:

$$\begin{aligned} \mathfrak{D}_1 &:= \left\{ p \in \mathbb{N} : \|\mathfrak{L}^{(\lambda,\mu)}_{p,a}(e_2; \mathfrak{v}) - e_2\|_\sigma \ge \varepsilon \right\}, \\ \mathfrak{D}_2 &:= \left\{ p \in \mathbb{N} : \left(\frac{1}{(p+\mu)^2} \left\{ p + \mu^2 + a^2 - 2a\mu \right\} \right) \ge \frac{\varepsilon}{3} \right\}, \\ \mathfrak{D}_3 &:= \left\{ p \in \mathbb{N} : \frac{2}{(p+\mu)^2} (p - \mu - \lambda\mu + (2+\lambda)a) \ge \frac{\varepsilon}{3} \right\}, \\ \mathfrak{D}_4 &:= \left\{ p \in \mathbb{N} : \frac{1}{(p+\mu)^2} \left(\lambda^2 + 2\lambda + 2 \right) \ge \frac{\varepsilon}{3} \right\}. \end{aligned} \tag{69}$$

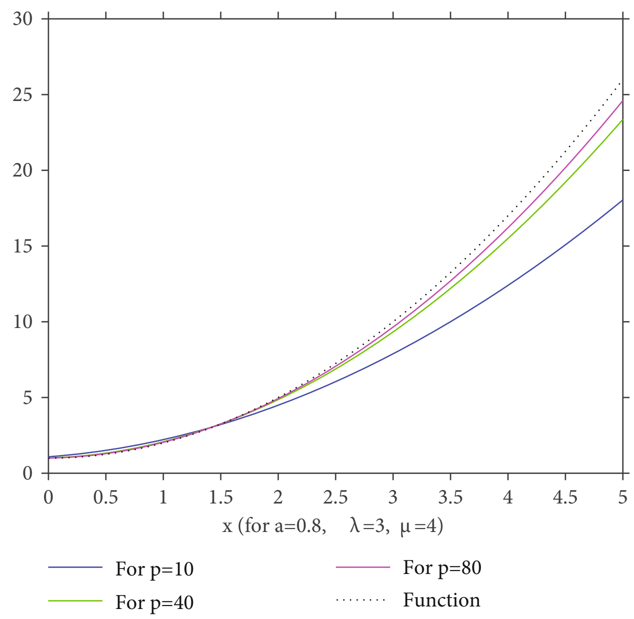

FIGURE 1: Convergence of the operator towards the function $f(x) = x^2 + 1$.

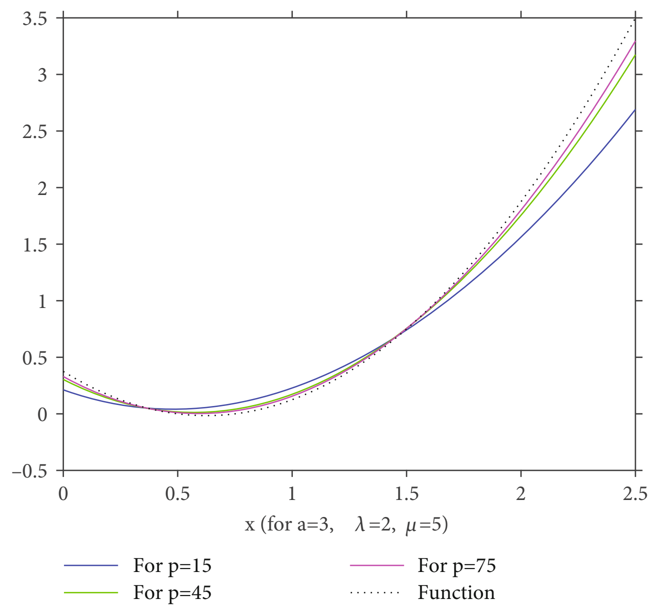

FIGURE 2: Convergence of the operator towards the function $f(x) = (x - (1/2))(x - (3/4))$.

Then, $\delta_q(\mathfrak{D}_2) = 0 = \delta_q(\mathfrak{D}_3) = \delta_q(\mathfrak{D}_4)$. Since $\mathfrak{D}_1 \subseteq \mathfrak{D}_2 \cup \mathfrak{D}_3 \cup \mathfrak{D}_4$ which implies that $\delta_q(\mathfrak{D}_1) \le \delta_q(\mathfrak{D}_2) + \delta_q(\mathfrak{D}_3) + \delta_q(\mathfrak{D}_4)$,

$$St_q - \lim_p \left\| \mathfrak{L}_{p,a}^{(\lambda,\mu)}(e_2\,;\mathfrak{v}) - e_2 \right\|_\rho = 0. \tag{70}$$

Hence, the proof is completed. □

Example 12. Let $\eta = (\eta_p)$ be defined by

$$\eta_p = \begin{pmatrix} 1\left(2^{2n}\text{times}\right) \\ 0\left(2^{2n-1}\text{times}\right) \end{pmatrix} n = 0, 1, 2, \cdots. \tag{71}$$

That is, 1 occurs 2^{2n} times and 0 occurs 2^{2n-1} times ($n = 0, 1, 2, \cdots$), respectively. Let $\mathscr{K} = \{k \in \mathbb{N} : \eta_k = 1\}$. Then, $\lim_{n\longrightarrow\infty}(C_1^q \chi_{\mathscr{K}})_{2^{2n}-1} = 0$, i.e., $St_q - \lim \eta_k = 0$, but $\delta(\mathscr{K})$ does not exist, so η is not statistically convergent.

Define $\mathscr{A}_p^{(\lambda,\mu)} = (1 + \eta_p)\mathfrak{L}_{p,a}^{(\lambda,\mu)}$, where it is defined by (71). Then, obviously $st - \lim_p \|\mathscr{A}_p^{(\lambda,\mu)}(e_i\,;\mathfrak{v}) - e_i\|_\sigma = 0 (i = 0, 1, 2)$. Applying the above theorem, we have

$$St_q - \lim_p \left\| \mathscr{A}_p^{(\lambda,\mu)}(\mathfrak{h}\,;\mathfrak{v}) - \mathfrak{h} \right\|_\sigma = 0 \text{ for all } \mathfrak{h} \in C_\rho^0. \tag{72}$$

On the other hand, since $\eta = (\eta_p)$ is q-statistically convergent but neither convergent nor statistically convergent, the sequence $(\mathscr{A}_p^{(\lambda,\mu)})$ can not be convergent, while it is q-statistically convergent.

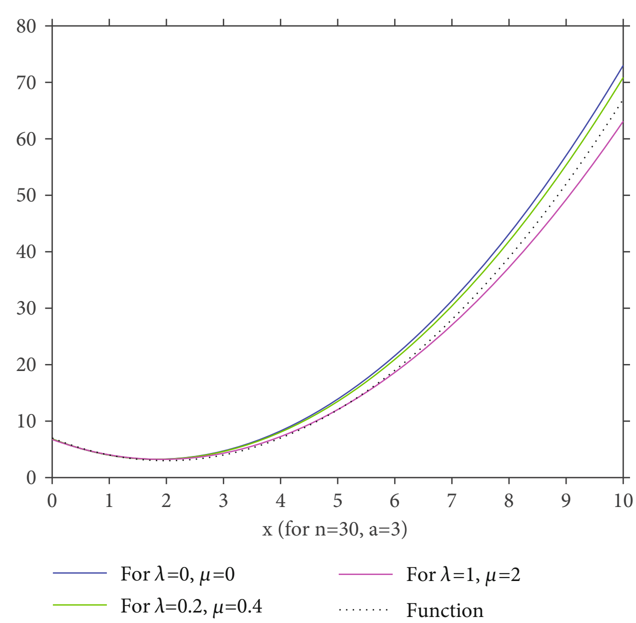

FIGURE 3: Comparison of convergence of the operator.

5. Graphical Analysis

In this section, we will give some numerical examples with illustrative graphics with the help of MATLAB.

Example 13. Let $f(x) = x^2 + 1$, $\lambda = 3$, $\mu = 4$, $a = 0.8$, and $p \in \{10,40,80\}$. The convergence of the operator towards the function $f(x)$ is shown in Figure 1.

Example 14. Let $f(x) = (x - (1/2))(x - (3/4))$, $\lambda = 2$, $\mu = 5$, $a = 3$, and $p \in \{15,45,75\}$. The convergence of the operator towards the function $f(x)$ is shown in Figure 2.

From these examples, we observe that the approximation of function by the operators becomes better when we take larger values of n.

Notice that for $\lambda = \mu = 0$, the operators (5) reduce to operators (4).

Example 15. Let $f(x) = x^2 - 4x + 7$. For $a = 3, p = 30$, comparison of convergence of the constructed operator (5) (green and pink) with the previously defined operator (4) (blue) is shown in Figure 3. From this figure, it is clear that the constructed operator gives a better approximation to $f(x)$ than the previously defined operator.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to writing this paper. All authors read and approved the manuscript.

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Research Article

On Numerical Radius Bounds Involving Generalized Aluthge Transform

Tao Yan,[1] **Javariya Hyder**,[2] **Muhammad Saeed Akram**,[3] **Ghulam Farid**,[4] **and Kamsing Nonlaopon**[5]

[1] *School of Computer Science, Chengdu University, Chengdu, China*
[2] *Department of Mathematics, Khwaja Fareed University of Engineering and Information Technology, Rahim Yar Khan, Pakistan*
[3] *Department of Mathematics, Faculty of Science, Ghazi University, 32200 Dera Ghazi Khan, Pakistan*
[4] *Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan*
[5] *Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand*

Correspondence should be addressed to Kamsing Nonlaopon; nkamsi@kku.ac.th

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In this paper, we establish some upper bounds of the numerical radius of a bounded linear operator S defined on a complex Hilbert space with polar decomposition $S = U|S|$, involving generalized Aluthge transform. These bounds generalize some bounds of the numerical radius existing in the literature. Moreover, we consider particular cases of generalized Aluthge transform and give some examples where some upper bounds of numerical radius are computed and analyzed for certain operators.

1. Introduction

In mathematical analysis, inequalities play a vital role in studying the properties of operators in the form of their upper and lower bounds. Mathematical inequalities provide the best way to describe as well as propose solutions to real-world problems in almost all fields of science and engineering. The boundedness property of different kinds of operators studied in the subjects of analysis, precisely in mathematical and functional analysis, is the key factor in developing the theory and applications. For example, upper and lower bounds are utilized to define the operator norm, which plays significantly in solving related problems. The study of the numerical radius of an operator defined on the Hilbert space is in the focus of researchers in these days in studying perturbation, convergence, iterative solution methods, and integrative methods, etc, see [1–9]. In this regard, the numerical radius inequality stated in (3) is studied extensively by various mathematicians, see [10–21]. Actually, it is interesting for the researchers to get refinements and generalizations of this inequality [22–27]. The goal of this paper is to study generalizations of numerical radius bounds under certain additional conditions. Henceforth, we define the preliminary notions to proceed with the findings of this work.

The polar decomposition is an important feature in the theory of operators. It is defined by $A = UB$, where U is the unitary matrix, and B is the symmetric positive semidefinite matrix. It is interesting to see that when A is nonsingular and symmetric, then B is a good symmetric positive definite approximation to A and $1/2(A + B)$ is the best symmetric positive semidefinite approximation to A, see [6]. Let $\mathscr{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on complex Hilbert space. Let $S = U|S|$ be the unique polar decomposition of $S \in \mathscr{B}(\mathscr{H})$, where U is a partial isometry and $|S|$ is the square root of an operator which is defined as $|S| = \sqrt{S^* S}$. The numerical range of an operator S is defined as

$$W(S) = \{\langle Sx, x\rangle : \|x\| = 1, x \in \mathscr{H}\}, \tag{1}$$

where $W(S)$ denotes the numerical range. The numerical radius of an operator is the radius of the smallest circle centered at the origin and contains the numerical range, i.e.,

$$w(S) = \sup\{|\lambda|: \lambda \in W(S)\}. \tag{2}$$

The numerical radius defines a norm on $\mathscr{B}(\mathscr{H})$ which is equivalent to the usual operator norm, satisfying the following inequality:

$$\frac{1}{2}\|S\| \le w(S) \le \|S\|. \tag{3}$$

If $S^2 = 0$, then the first inequality becomes equality and if S is normal then the second inequality becomes equality. Many authors worked on numerical radius inequalities and developed a number of numerical radius bounds [10, 13–18].

In [17], Kittaneh gave an upper bound of numerical radius as follows:

$$w(S) \le \frac{1}{2}\left(\|S\| + \left\|S^2\right\|^{1/2}\right), \tag{4}$$

and showed that this bound is sharper than the upper bound given in (3).

In [25], Aluthge introduced a transform of an operator $S \in \mathscr{B}(\mathscr{H})$ which is called Aluthge transform that is defined as

$$\Delta(S) = |S|^{1/2} U |S|^{1/2}. \tag{5}$$

In [26], Yamazaki developed an upper bound of the numerical radius involving Aluthge transform as follows:

$$w(S) \le \frac{1}{2}(\|S\| + w(\Delta S)), \tag{6}$$

and proved that it is sharper than the bound given in (4).

In [27], Okubo introduced a new generalization of Aluthge transform, called λ-Aluthge transform defined by

$$\Delta_\lambda S = |S|^\lambda U |S|^{1-\lambda}; \lambda \in [0, 1]. \tag{7}$$

In [23], Abu-Omar and Kittaneh further generalized the bound given in (6) using λ-Aluthge transform as follows:

$$w(S) \le \frac{1}{2}(\|S\| + w(\Delta_\lambda S)). \tag{8}$$

In [19], Bhunia et al. found some bounds of the numerical radius for $S \in \mathscr{B}(\mathscr{H})$. Later, Bag et al. [24] working along the same lines succeeded to get the following upper bounds of the numerical radius:

$$w^2(S) \le \frac{1}{2}\|S\|\|\Delta_\lambda S\| + \frac{1}{4}\|S^*S + SS^*\|, \tag{9}$$

$$w^2(S) \le \frac{1}{4}\left(w\left((\Delta_\lambda S)^2\right) + \|S\|\|\Delta_\lambda S\| + \|S^*S + SS^*\|\right), \tag{10}$$

$$w^2(S) \le \sum_{n=1}^{\infty} \frac{1}{4^n}\left(\|\Delta_\lambda^{n-1} S\|\|\Delta_\lambda^n S\| + \left\|\left(\Delta_\lambda^{n-1} S\right)^*\left(\Delta_\lambda^{n-1} S\right) + \left(\Delta_\lambda^{n-1} S\right)\left(\Delta_\lambda^{n-1} S\right)^*\right\|\right), \tag{11}$$

$$w^4(S) \le \frac{1}{16}\left(w\left((\Delta_\lambda S)^2\right) + \|S\|\|\Delta_\lambda S\|\right)^2 + \frac{1}{8} w\left(S^2 P + P S^2\right) + \frac{1}{16}\|P\|^2, \tag{12}$$

where $P = S^*S + SS^*$ and $\lambda \in [0, 1]$.

In [22], Shebrawi and Bakherad presented a new form of Aluthge transform so called generalized Aluthge transform defined by

$$\Delta_{f,g} S = f(|S|) U g(|S|), \tag{13}$$

where f and g are nonnegative continuous functions such that $f(|S|)g(|S|) = |S|$, $(|S| \ge 0)$. They proved the following upper bound of the numerical radius by using generalized Aluthge transform

$$w(S) \le \frac{1}{2}\left(\|S\| + w\left(\Delta_{f,g} S\right)\right), \tag{14}$$

which is a generalization of the upper bound shown in (6) and (8).

Our aim is to study the upper bounds of the numerical radius by applying generalized Aluthge transform defined in (13) by imposing further certain conditions on continuous functions. The first contribution of this paper is that we develop upper bounds of the numerical radius using generalized Aluthge transform, which extends and generalizes some already existing bounds. Specifically, we extend the inequalities (9)–(12) for generalized Aluthge transform under certain conditions on f and g. As a consequence, the upper bounds of numerical radius involving Aluthge transform and λ-Aluthge transform appear as a special case of our bounds. Another contribution of the paper is that we have presented examples of generalized Aluthge transform in addition to the classical Aluthge transform and λ-Aluthge transform, which are used for computing bounds of numerical radius. More precisely, we have considered five choices of continuous functions f and g in (13) and used them to compute upper bounds of numerical radius for certain operators.

2. Main Results

We start this section by attaining the generalized Aluthge transform $\Delta_{f,g}$ defined in (13) under the following additional conditions:

(i) $g(|S|)f(|S|) = |S|$

(ii) $f(|S|)$ and $g(|S|)$ both are positive operators

Now, we give some results that will be used repeatedly to achieve our goal.

Lemma 1 [26]. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, we have*

$$w(S) = \sup_{\theta\in\mathbb{R}} \|H_\theta\| = \sup_{\theta\in\mathbb{R}} \left\|\mathrm{Re}\left(e^{\iota\theta}S\right)\right\|, \tag{15}$$

where $H_\theta = (\mathrm{Re}\,(e^{\iota\theta}S)) = (e^{\iota\theta}S + e^{-\iota\theta}S^*)/2$ for all $\theta \in \mathbb{R}$.

Lemma 2 [23]. *Let $M_1, M_2, N_1, N_2 \in \mathscr{B}(\mathscr{H})$. Then,*

$$\begin{aligned} r(M_1N_1 + M_2N_2) &\le \frac{1}{2}(w(N_1M_1) + w(N_2M_2)) \\ &+ \frac{1}{2}\sqrt{w(N_1M_1) - w(N_2M_2) + 4\|N_1M_2\|\|N_2M_1\|}, \end{aligned} \tag{16}$$

where r denotes the spectral radius.

Next, we give the numerical radius bound by using the generalized Aluthge transform.

Theorem 3. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, we have*

$$w^2(S) \le \frac{1}{2}\|g(|S|)\|\left\|\Delta_{f,g}S\right\|\|f(|S|)\| + \frac{1}{4}\|S^*S + SS^*\|. \tag{17}$$

Proof. Since

$$H_\theta = \frac{1}{2}\left(e^{\iota\theta}S + e^{-\iota\theta}S^*\right) \text{ for all } \theta \in \mathbb{R}, \tag{18}$$

therefore,

$$\begin{aligned} {H_\theta}^2 &= \frac{1}{4}\left(e^{\iota\theta}S + e^{-\iota\theta}S^*\right)^2 \\ &= \frac{1}{4}\left(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2} + SS^* + S^*S\right) \\ &= \frac{1}{4}\left(e^{2\iota\theta}U|S|U|S| + e^{-2\iota\theta}|S|U^*|S|\,U^* + SS^* + S^*S\right) \\ &= \frac{1}{4}\left(e^{2\iota\theta}Ug(|S|)f(|S|)Ug(|S|)f(|S|)\right. \\ &\quad \left. + e^{-2\iota\theta}f(|S|)g(|S|)U^*f(|S|)g(|S|)U^* + SS^* + S^*S\right) \\ &= \frac{1}{4}\left(e^{2\iota\theta}Ug(|S|)\left(\Delta_{f,g}S\right)f(|S|)\right. \\ &\quad \left. + e^{-2\iota\theta}f(|S|)\left(\Delta_{f,g}S\right)^*g(|S|)U^* + SS^* + S^*S\right). \end{aligned} \tag{19}$$

The third equality is obtained by putting $S = U\,|\,S\,|$ and $S^* = |S\,|\,U^*$ in second equality, the fourth equality holds because $f(|S|)g(|S|) = |S|$ and $g(|S\,|)f(|S\,|) = |S\,|$, and the fifth equality holds because $\Delta_{f,g}S = f(|S\,|)Ug(|S\,|)$ and $(\Delta_{f,g}S)^* = g(|S\,|)U^*f(|S\,|)$. Since $\|ZZ^*\| = \|Z\|^2$ for any $Z \in \mathscr{B}(\mathscr{H})$, therefore

$$\begin{aligned} \|H_\theta\|^2 &= \frac{1}{4}\left(\left\|e^{2\iota\theta}Ug(|S|)\left(\Delta_{f,g}S\right)f(|S|)\right.\right. \\ &\quad \left.\left. + e^{-2\iota\theta}f(|S|)\left(\Delta_{f,g}S\right)^*g(|S|)U^* + SS^* + S^*S\right\|\right) \\ &\le \frac{1}{4}\left(\|g(|S|)\|\left\|\Delta_{f,g}S\right\|\|f(|S|)\|\right. \\ &\quad \left. + \|f(|S|)\|\left\|\left(\Delta_{f,g}S\right)^*\right\|\|g(|S|)\| + \|SS^* + S^*S\|\right) \\ &= \frac{1}{4}\left(2\|g(|S|)\|\left\|\Delta_{f,g}S\right\|\|f(|S|)\| + \|SS^* + S^*S\|\right). \end{aligned} \tag{20}$$

The first inequality holds because $\|S_1S_2\| \le \|S_1\|\|S_2\|$, $\|S_1 + S_2\| \le \|S_1\| + \|S_2\|$ for any $S_1, S_2 \in \mathscr{B}(\mathscr{H})$, U is partial isometry and $|e^{2\iota\theta}| = 1$ and the second equality holds by using the fact that $\|S\| = \|S^*\|$.

Now, by taking supremum of the last inequality over $\theta \in \mathbb{R}$ and then using Lemma 1, we get

$$w^2(S) \le \frac{1}{2}\|g(|S|)\|\left\|\Delta_{f,g}S\right\|\|f(|S|)\| + \frac{1}{4}\|SS^* + S^*S\|, \tag{21}$$

as required. □

The following result is another generalized bound of numerical radius for bounded linear operators on $\mathscr{H}$.

Theorem 4. *Let $S \in \mathscr{B}(\mathscr{H})$. Then,*

$$\begin{aligned} w^2(S) \le \frac{1}{4}&\left(w\left(\left(\Delta_{f,g}S\right)^2\right) + \|g(|S|)\|\left\|\Delta_{f,g}S\right\|\|f(|S|)\|\right. \\ &\left. + \|S^*S + SS^*\|\right). \end{aligned} \tag{22}$$

Proof. Let S be any bounded linear operator with polar decomposition $S = U|S|$. Since

$$H_\theta = \frac{1}{2}\left(e^{\iota\theta}S + e^{-\iota\theta}S^*\right) \text{ for all } \theta \in \mathbb{R}, \tag{23}$$

therefore,

$${H_\theta}^2 = \frac{1}{4}\left(e^{\iota\theta}S + e^{-\iota\theta}S^*\right)^2 = \frac{1}{4}\left(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2} + SS^* + S^*S\right). \tag{24}$$

Using the properties of operator norm $\|\cdot\|$ on $\mathscr{B}(\mathscr{H})$, we have

$$\begin{aligned} \|H_\theta\|^2 &\le \frac{1}{4}\left(\left\|e^{2\iota\theta}Ug(|S|)\left(\Delta_{f,g}S\right)f(|S|)\right.\right. \\ &\quad \left.\left. + e^{-2\iota\theta}f(|S|)\left(\Delta_{f,g}S\right)^*g(|S|)U^*\right\| + \|SS^* + S^*S\|\right) \\ &= \frac{1}{4}\left(r\left(e^{2\iota\theta}Ug(|S|)\left(\Delta_{f,g}S\right)f(|S|)\right.\right. \\ &\quad \left.\left. + e^{-2\iota\theta}f(|S|)\left(\Delta_{f,g}S\right)^*g(|S|)U^*\right) + \|SS^* + S^*S\|\right) \\ &= \frac{1}{4}(r(M_1N_1 + M_2N_2) + \|SS^* + S^*S\|), \end{aligned} \tag{25}$$

where $M_1 = e^{\iota\theta} U g(|S|)(\Delta_{f,g}S)$, $N_1 = f(|S|)$, $M_2 = e^{-2\iota\theta} f(|S|)(\Delta_{f,g}S)^*$, $N_2 = g(|S|)U^*$. The first equality above holds for hermitian operator $A \in \mathscr{B}(\mathscr{H})$ satisfying $r(A) = \|A\|$. Now, an application of Lemma 2 together with $w(S) = w(S^*)$ and $w(\alpha S) = |\alpha| w(S)$ yields

$$\begin{aligned}
\|H_\theta\|^2 &\le \frac{1}{4}\Big(w\big((\Delta_{f,g}S)^2\big) \\
&\quad + \frac{1}{2}\sqrt{4\big\|(f(|S|))^2(\Delta_{f,g}S)^*\big\|\big\|(g(|S|))^2\Delta_{f,g}S\big\|} \\
&\quad + \|SS^* + S^*S\|\Big) \\
&\le \frac{1}{4}\Big(w\big((\Delta_{f,g}S)^2\big) \\
&\quad + \frac{1}{2}\sqrt{4\|f(|S|)\|^2\big\|(\Delta_{f,g}S)^*\big\|\|g(|S|)\|^2\big\|\Delta_{f,g}S\big\|} \\
&\quad + \|SS^* + S^*S\|\Big) \\
&= \frac{1}{4}\Big(w\big((\Delta_{f,g}S)^2\big) + \|f(|S|)\|\|g(|S|)\|\big\|\Delta_{f,g}S\big\| \\
&\quad + \|SS^* + S^*S\|\Big).
\end{aligned} \tag{26}$$

The last equality holds by using the fact $\|S\| = \|S^*\|$. Now, we take supremum over $\theta \in \mathbb{R}$ to get

$$\begin{aligned}
\sup_{\theta\in\mathbb{R}}\|H_\theta\|^2 &\le \sup_{\theta\in\mathbb{R}}\Big(\frac{1}{4}\Big(w\big((\Delta_{f,g}S)^2\big) \\
&\quad + \|g(|S|)\|\big\|\Delta_{f,g}S\big\|\|f(|S|)\| + \|SS^* + S^*S\|\Big)\Big).
\end{aligned} \tag{27}$$

By using Lemma 1 in above inequality, we obtain

$$\begin{aligned}
w^2(S) &\le \frac{1}{4}\Big(w\big((\Delta_{f,g}S)^2\big) + \|g(|S|)\|\big\|\Delta_{f,g}S\big\|\|f(|S|)\| \\
&\quad + \|SS^* + S^*S\|\Big),
\end{aligned} \tag{28}$$

as required. □

The following inequality is another generalized bound of numerical radius.

Theorem 5. *Let $S \in \mathscr{B}(\mathscr{H})$. Then we have*

$$\begin{aligned}
w^4(S) &\le \frac{1}{16}\Big(w\big((\Delta_{f,g}S)^2\big) + \|g(|S|)\|\big\|\Delta_{f,g}S\big\|\|f(|S|)\|\Big)^2 \\
&\quad + \frac{1}{8} w\big(S^2P + PS^2\big) + \frac{1}{16}\|P\|^2,
\end{aligned} \tag{29}$$

where $P = S^*S + SS^*$.

Proof. Since

$$H_\theta = \frac{1}{2}\left(e^{\iota\theta}S + e^{-\iota\theta}S^*\right) \text{ for all } \theta \in \mathbb{R}, \tag{30}$$

therefore,

$$\begin{aligned}
{H_\theta}^2 &= \frac{1}{4}\left(e^{\iota\theta}S + e^{-\iota\theta}S^*\right)^2 \\
&= \frac{1}{4}\left(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2} + SS^* + S^*S\right){H_\theta}^4 \\
&= \frac{1}{16}\left(\left(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\right) + P\right)^2 \\
&= \frac{1}{16}\Big(\left(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\right)^2 \\
&\quad + \left(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\right)P + P\left(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\right) + P^2\Big) \\
&= \frac{1}{16}\Big(\left(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\right)^2 \\
&\quad + \left(e^{2\iota\theta}S^2P + e^{-2\iota\theta}S^{*2}P + Pe^{2\iota\theta}S^2 + Pe^{-2\iota\theta}S^{*2}\right) + P^2\Big) \\
&= \frac{1}{16}\Big(\left(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\right)^2 + e^{2\iota\theta}\left(S^2P + PS^2\right) \\
&\quad + e^{-2\iota\theta}\left(S^{*2}P + PS^{*2}\right) + P^2\Big) \\
&= \frac{1}{16}\Big(\left(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\right)^2 \\
&\quad + 2\left(\operatorname{Re}\left(e^{2\iota\theta}\left(S^2P + PS^2\right)\right)\right) + P^2\Big),
\end{aligned} \tag{31}$$

where

$$\operatorname{Re}\left(e^{2\iota\theta}\left(S^2P + PS^2\right)\right) = \frac{e^{2\iota\theta}\left(S^2P + PS^2\right) + e^{-2\iota\theta}\left(S^2P + PS^2\right)^*}{2}. \tag{32}$$

In third equality $P = S^*S + SS^*$. Now, by using the properties of operator norm $\|\cdot\|$ on $\mathscr{B}(\mathscr{H})$, we have

$$\begin{aligned}
\|H_\theta\|^4 &\le \frac{1}{16}\Big(\left\|e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\right\|^2 \\
&\quad + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\Big) \\
&= \frac{1}{16}\Big(\left\|e^{2\iota\theta}U|S|U|S| + e^{-2\iota\theta}|S|U^*|S|U^*\right\|^2 \\
&\quad + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\Big) \\
&= \frac{1}{16}\Big(\big\|e^{2\iota\theta}Ug(|S|)(\Delta_{f,g}S)f(|S|) \\
&\quad + e^{-2\iota\theta}f(|S|)(\Delta_{f,g}S)^*g(|S|)U^*\big\|^2 \\
&\quad + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{16}\Big(r^2\Big(e^{2\iota\theta}Ug(|S|)(\Delta_{f,g}S)f(|S|) \\
&\quad + e^{-2\iota\theta}f(|S|)(\Delta_{f,g}S)^* g(|S|)U^*\Big) \\
&\quad + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}(S^2P+PS^2)\right)\right\| + \|P\|^2\Big) \\
&= \frac{1}{16}\Big(r^2(M_1N_1+M_2N_2) \\
&\quad + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}(S^2P+PS^2)\right)\right\| + \|P\|^2\Big),
\end{aligned} \tag{33}
$$

where $M_1 = e^{\iota\theta}Ug(|S|)(\Delta_{f,g}S)$, $N_1 = f(|S|)$, $M_2 = e^{-2\iota\theta}f(|S|)((\Delta_{f,g}S)^*)$, and $N_2 = g(|S|)U^*$. The first equality obtained by using $S = U|S|$ and $S^* = |S|U^*$ in first inequality, the second equality obtained by using $f(|S|)g(|S|) = |S|$ and $g(|S|)f(|S|) = |S|$ in third equality, and the fifth equality holds for hermitian operator satisfying $r(A) = \|A\|$. Now, by using Lemma 2 together with $w(S) = w(S^*)$ and $w(\alpha S) = |\alpha| w(S)$, it yields

$$
\begin{aligned}
\|H_\theta\|^4 &\le \frac{1}{16}\Bigg(\Bigg(w\left((\Delta_{f,g}S)^2\right) \\
&\quad + \frac{1}{2}\sqrt{4\left\|(f(|S|))^2 e^{-2\iota\theta}(\Delta_{f,g}S)^*\right\|\left\|(g|S|))^2 e^{\iota\theta}\Delta_{f,g}S\right\|}\Bigg)^2 \\
&\quad + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}(S^2P+PS^2)\right)\right\| + \|P\|^2\Bigg) \\
&\le \frac{1}{16}\Bigg(\Bigg(w\left((\Delta_{f,g}S)^2\right) \\
&\quad + \frac{1}{2}\sqrt{4\|f(|S|)\|^2\left\|(\Delta_{f,g}S)^*\right\|\|g(|S|)\|^2\left\|\Delta_{f,g}S\right\|}\Bigg)^2 \\
&\quad + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}(S^2P+PS^2)\right)\right\| + \|P\|^2\Bigg) \\
&= \frac{1}{16}\Bigg(\Big(w\left((\Delta_{f,g}S)^2\right) + \|f(|S|)\|\|g(|S|)\|\left\|\Delta_{f,g}S\right\|\Big)^2 \\
&\quad + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}(S^2P+PS^2)\right)\right\| + \|P\|^2\Bigg).
\end{aligned} \tag{34}
$$

The last equality holds by using the fact $\|S\| = \|S^*\|$. Now, we take supremum over $\theta \in \mathbb{R}$ to get

$$
\begin{aligned}
\sup_{\theta\in\mathbb{R}}\|H_\theta\|^4 &\le \sup_{\theta\in\mathbb{R}}\frac{1}{16}\Bigg(\Big(w\left((\Delta_{f,g}S)^2\right) + \|f(|S|)\|\|g(|S|)\|\left\|\Delta_{f,g}S\right\|\Big)^2 \\
&\quad + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}(S^2P+PS^2)\right)\right\| + \|P\|^2\Bigg).
\end{aligned} \tag{35}
$$

Applying Lemma 1 on above inequality, we obtain

$$
\begin{aligned}
w^4(S) &\le \frac{1}{16}\Big(w\left((\Delta_{f,g}S)^2\right) + \|f(|S|)\|\|g(|S|)\|\left\|\Delta_{f,g}S\right\|\Big)^2 \\
&\quad + \frac{1}{8}w(S^2P+PS^2) + \frac{1}{16}\|P\|^2.
\end{aligned} \tag{36}
$$

as required. □

To give the next bound of numerical radius, first, we define iterated generalized Aluthge transform. The iterated generalized Aluthge transform is defined as

$$
\Delta^k_{f,g}S = \Delta\left(\Delta^{k-1}_{f,g}S\right); \forall k \in \mathbb{N}, \tag{37}
$$

where f and g both are nonnegative and continuous functions.

By using Theorem 4 repeatedly, we can obtain numerical radius bound in terms of iterated generalized Aluthge transform.

Theorem 6. *Let $S \in \mathscr{B}(\mathscr{H})$ be such that the sequence $\{\|\Delta^n_{f,g}S\|\}_{n=1}^{\infty}$ is convergent then*

$$
\begin{aligned}
w^2(S) &\le \sum_{k=1}^{\infty}\frac{1}{4^k}\Big(\left\|f\left(|\Delta^{k-1}_{f,g}S|\right)\right\|\left\|g\left(|\Delta^{k-1}_{f,g}S|\right)\right\|\left\|\Delta^k_{f,g}S\right\| \\
&\quad + \left\|\left(\Delta^{k-1}_{f,g}S\right)^*\left(\Delta^{k-1}_{f,g}S\right) + \left(\Delta^{k-1}_{f,g}S\right)\left(\Delta^{k-1}_{f,g}S\right)^*\right\|.
\end{aligned} \tag{38}
$$

Proof. In order to prove the theorem, it is sufficient to prove the following assertion

$$
\begin{aligned}
w^2(S) &\le \sum_{k=1}^{n}\frac{1}{4^k}\Big(\left\|f\left(\left|\Delta^{k-1}_{f,g}S\right|\right)\right\|\left\|\Delta^k_{f,g}S\right\|\left\|g\left(\left|\Delta^{k-1}_{f,g}S\right|\right)\right\| \\
&\quad + \left\|\left(\Delta^{k-1}_{f,g}S\right)^*\left(\Delta^{k-1}_{f,g}S\right) + \left(\Delta^{k-1}_{f,g}S\right)\left(\Delta^{k-1}_{f,g}S\right)^*\right\|\Big) \\
&\quad + \frac{1}{4^n}w^2\left(\Delta^n_{f,g}S\right) \text{for all } n \in \mathbb{N}.
\end{aligned} \tag{39}
$$

We use mathematical induction to prove the above assertion. An application of Theorem 4 gives

$$
\begin{aligned}
w^2(S) &\le \frac{1}{4}\left(\|f(|S|)\|\|g(|S|)\|\left\|\Delta_{f,g}S\right\| + \|S^*S + SS^*\|\right) \\
&\quad + \frac{1}{4}w\left((\Delta_{f,g}S)^2\right).
\end{aligned} \tag{40}
$$

The use of the inequality $w(S^2) \le w^2(S)$ gives

$$
\begin{aligned}
w^2(S) &\le \frac{1}{4}\left(\|f(|S|)\|\left\|\Delta_{f,g}S\right\|\|g(|S|)\| + \|S^*S + SS^*\|\right) \\
&\quad + \frac{1}{4}w^2(\Delta_{f,g}S).
\end{aligned} \tag{41}
$$

Table 1: Bounds (14), (17), (22), and (29) for different choices of f and g in (13).

(f, g)	Bound (14)	Bound (17)	Bound (22)	Bound (29)
$\left(e^{\|S\|}, \|S\|e^{-\|S\|}\right)$	22.7288	56.6787	40.1743	40.0320
$\left(e^{\|S\|^{1/2}}, \|S\|e^{-\|S\|^{1/2}}\right)$	3.7695	4.1934	3.6238	3.2146
$\left(\|S\|^{1/2}, \|S\|^{1/2}\right)$	3.4881	3.7078	3.3353	3.0645
$\left(e^{\|S\|^{1/3}}, \|S\|e^{-\|S\|^{1/3}}\right)$	3.3468	3.6354	3.2707	3.0389

Table 2: Bounds (14), (17), (22), and (29) for different choices of f and g in (13).

(f, g)	Bound (14)	Bound (17)	Bound (22)	Bound (29)
$\left(\|S\|^{1/3}, \|S\|^{2/3}\right)$	2.62245	2.37007943	2.27893615	2.1589862
$\left(\|S\|^{1/2}, \|S\|^{1/2}\right)$	2.5	2.2912878	2.1794494	2.0963298
$\left(e^{\|S\|^{1/3}}, \|S\|e^{-\|S\|^{1/3}}\right)$	2.5	2.29120815	2.1794075	2.09630510
$\left(e^{\|S\|^{1/2}}, \|S\|e^{-\|S\|^{1/2}}\right)$	2.5	2.29120547	2.17940617	2.09630427

Thus the preliminary induction step holds. Now, suppose that

$$\begin{aligned} w^2(S) \leq \sum_{k=1}^{m} \frac{1}{4^k} \Big(& \left\| f\left(\left| \Delta_{f,g}^{k-1} S \right| \right) \right\| \left\| \Delta_{f,g}^{k} S \right\| \left\| g\left(\left| \Delta_{f,g}^{k-1} S \right| \right) \right\| \\ & + \left\| \left(\Delta_{f,g}^{k-1} S \right)^* \left(\Delta_{f,g}^{k-1} S \right) + \left(\Delta_{f,g}^{k-1} S \right) \left(\Delta_{f,g}^{k-1} S \right)^* \right\| \Big) \\ & + \frac{1}{4^m} w^2\left(\Delta_{f,g}^{m} S \right) \text{ for some } m \in \mathbb{N}. \end{aligned} \tag{42}$$

Then, another application of Theorem 4 yields

$$\begin{aligned} w^2(S) \leq \sum_{k=1}^{m} \frac{1}{4^k} \Big(& \| f\left(|\Delta_{f,g}^{k} S| \right) \| \| \Delta_{f,g}^{k+1} S \| \| g\left(\left| \Delta_{f,g}^{k} S \right| \right) \| \\ & + \left\| \left(\Delta_{f,g}^{k} S \right)^* \left(\Delta_{f,g}^{k} S \right) + \left(\Delta_{f,g}^{k} S \right) \left(\Delta_{f,g}^{k} S \right)^* \right\| \Big) \\ & + \frac{1}{4} \Big(\| f\left(|\Delta_{f,g}^{m} S| \right) \| \| \Delta_{f,g}^{m+1} S \| \| g\left(\left| \Delta_{f,g}^{m} S \right| \right) \| \\ & + \left\| \left(\Delta_{f,g}^{m} S \right)^* \left(\Delta_{f,g}^{m} S \right) + \left(\Delta_{f,g}^{m} S \right) \left(\Delta_{f,g}^{m} S \right)^* \right\| \Big) \\ & + \frac{1}{4^{m+1}} w\left(\left(\Delta_{f,g}^{m+1} S \right)^2 \right). \end{aligned} \tag{43}$$

Simplifying and using the inequality $w(S^2) \leq w^2(S)$ gives

$$\begin{aligned} w^2(S) \leq \sum_{k=1}^{m+1} \frac{1}{4^k} \Big(& \left\| f\left(\left| \Delta_{f,g}^{k} S \right| \right) \right\| \left\| \Delta_{f,g}^{k+1} S \right\| \left\| g\left(\left| \Delta_{f,g}^{k} S \right| \right) \right\| \\ & + \left\| \left(\Delta_{f,g}^{k} S \right)^* \left(\Delta_{f,g}^{k} S \right) + \left(\Delta_{f,g}^{k} S \right) \left(\Delta_{f,g}^{k} S \right)^* \right\| \Big) \\ & + \frac{1}{4^{m+1}} w^2\left(\Delta_{f,g}^{m+1} S \right). \end{aligned} \tag{44}$$

Hence, the assertion (39) holds for all $n \in \mathbb{N}$.

Now, using the inequality $w(S) \leq \|S\|$ in (39) and then using the hypothesis, we get the desired inequality (38). □

Remark 7. It is easy to observe from the Theorems 3–6 that the upper bounds (17)–(38) are generalized bounds. Indeed, if we take $f(|S|) = |S|^\lambda$ and $g(|S|) = |S|^{1-\lambda}$ for $\lambda \in [0, 1]$, in the bounds (17)–(38), then we obtain bounds (9)–(12).

3. Examples

In this section, we shall consider some choices of f and g in generalized Aluthge transform (13) and use them to compute upper bounds of numerical radius for some matrices.

Example 8. Given $S = \begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$. Then, $S = U|S|$ is a polar decomposition of S, where $|S| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{29} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $U = \begin{pmatrix} 0 & 5/\sqrt{29} & 0 \\ 0 & 0 & 1 \\ 0 & 2/\sqrt{29} & 0 \end{pmatrix}$ is partial isometry. The bounds (14), (17), (22), and (29) are computed for some choices of f and g in (13) for given S in Table 1.

The numerical radius of S is

$$w(S) = 2.9154. \tag{45}$$

Example 9. Let $S = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix}$. Then, $S = U \mid S \mid$ is a polar decomposition of S, where $|S| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, and $U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is partial isometry.

The bounds (14), (17), (22), and (29) are computed for some choices of f and g in (13) for given S in Table 2.

The numerical radius of S is

$$w(S) = 2. \tag{46}$$

4. Conclusion

Summarizing the investigation carried out, we note that generalized Aluthge transform (13) with additional conditions (i) and (ii) is useful in achieving the generalized upper bounds for numerical radius. It is proved in Theorems 3, Theorem 4, Theorem 5, and Theorem 6 that bounds (17), (22), (29), and (38) are upper bounds of numerical radius that generalize the upper bounds (9), (10), (11), and (12) of numerical radius already existing in the literature. Theoretical investigations are supported by examples in which computations are carried out for finding bounds (14), (17), (22), and (29) of numerical radius for some choices of the pair f, g in the generalized Aluthge transform $\Delta_{f,g}$. Examples 8 and 9 demonstrate that generalized Aluthge transform provides a wide range of transforms that may be used as a tool to compute the upper bounds for numerical radius. These results might be helpful in studying perturbation, convergence, iterative solution methods, and integrative methods, which is the subject of future work. In the future, we also have a plan to investigate the lower bounds of numerical radius.

Data Availability

There is no need of any data for this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

A Multiple Fixed Point Result for (θ, ϕ, ψ)-Type Contractions in the Partially Ordered s-Distance Spaces with an Application

Maliha Rashid,[1] Amna Kalsoom,[1] Abdul Ghaffar,[2] Mustafa Inc,[3,4,5] and Ndolane Sene[6]

[1]*Department of Mathematics and Statistics, International Islamic University, Islamabad, Pakistan*
[2]*Department of Mathematics, Ghazi University, D G Khan, Pakistan*
[3]*Department of Computer Engineering, Biruni University, Istanbul, Turkey*
[4]*Department of Mathematics, Science Faculty, Firat University, 23119 Elazig, Turkey*
[5]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan*
[6]*Laboratoire Lmdan, Département de Mathématiques de la Décision, Université Cheikh AntaDiop de Dakar, Faculté des Sciences Economiques et Gestion, BP 5683 Dakar Fann, Senegal*

Correspondence should be addressed to Mustafa Inc; minc@firat.edu.tr and Ndolane Sene; ndolanesene@yahoo.fr

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In this manuscript, the aim is to prove a multiple fixed point (FP) result for partially ordered s-distance spaces under (θ, ϕ, ψ)-type weak contractive condition. The result will generalize some well-known results in literature such as coupled FP (Guo and Lakshmikantham, 1987), triple fixed point (Berinde and Borcut, 2011), and quadruple FP results (Karapinar, 2011). Moreover, to validate the result, an application for the existence of solution of a system of integral equations is also provided.

1. Introduction

In pure mathematics, the Banach fixed-point theorem [1] (contractive mapping theorem or also known as the contraction mapping theorem) is main result in the study of metric spaces; it assurances the uniqueness and existence of FP of certain self-maps of metric spaces and requires a constructive technique to discover those FP. It can be understood as an abstract formulation of Picard's method of successive approximations. The theorem is named after Stefan Banach (1892-1945) who first stated it in 1922. It has numerous applications in different fields such as computer science, physics, engineering, and various branches of mathematics itself. FP theory as a whole got an upward flight after this celebrated result.

Many authors started working in this field, and soon it became a hot field of research. A number of authors have extended this fundamental result to nonlinear analysis [2]. Following this streak, Guo and Lakshmikantham [3] established the idea of doubled FP. This is considered to be the first of its nature and was extended to *triple FP* by Berinde and Borcut [4]. Continuing in this direction, Karapinar [5] used four variable to strengthen the idea of *quadruple FP* in partially ordered metric spaces. In 2012, Berzig and Samet [6] discussed the existence of the fixed point of N-order for m-mixed monotone mappings in complete ordered metric spaces. In the same year, Roldán et al. [7] extended the notion of the FP of N-order to the ϕ-fixed point and obtained some ϕ-fixed point theorems for a mixed monotone mapping in partially ordered complete metric spaces. In [8], Karapinar et al. studied the existence and uniqueness of a FP of the multidimensional operators which satisfy Meir-Keeler type contraction condition. Soon after, a number of articles were published to discussed the concept of a " multidimensional FP" or "an m-tuples fixed point." For applications of such results, we refer the reader to [9] and the references cited therein.

In 2016, Choban and Berinde [10] generalized metric spaces to distance spaces. They established multidimensional FP results for ordered spaces with distance under certain contractive conditions [4, 11]. They pointed out that the concept allowed them to reduce the multidimensional case of FP and coincidence points to the one dimensional case. Recently, Rashid et al. [12] established some multiple FP findings for the C-distance spaces in the existence of various contractive mapping.

The abovementioned ideas serve as motivation of the work in the present paper. We have extended the results of [10] for partially ordered s-distance space, in terms of a significant multiple FP result under (θ, ϕ, ψ)-type contractive condition [13]. In support of this result, an example is also given.

Since, in Section 1, introduction and historical background of generalized metric spaces is given. In Section 2, preliminaries and some basic definitions are stated. In Section 3, main result is stated, and in Section 2, some consequences and examples of our main result will be described. In Section 3, an application is stated to support our main result. In the last section, article is concluded.

2. Preliminaries

Definition 1 [11]. Consider the $\mathbb{M}$, as a nonempty set and a function $\sigma : \mathbb{M} \times \mathbb{M} \longrightarrow \mathbb{R}$ is called s-distance on $\mathbb{M}$ if for all $\varsigma, \xi, \eta \in \mathbb{M}$, σ satisfies the following axioms:

(1) $\sigma(\varsigma, \xi) \geq 0$, $\forall \varsigma, \xi \in \mathbb{M}$

(2) $\sigma(\varsigma, \xi) + \sigma(\xi, \varsigma) = 0$ if and only if $\varsigma = \xi$, $\forall \varsigma, \xi \in \mathbb{M}$

(3) For a positive real number $s > 0$

$$\sigma(\varsigma, \xi) \leq s[\sigma(\varsigma, \eta) + \sigma(\eta, \xi)], \forall \varsigma, \xi, \eta \in \mathbb{M}. \quad (1)$$

An s-distance space $(\mathbb{M}, \sigma)$ is said to be a symmetric s-distance space if $\sigma(\varsigma, \xi) = \sigma(\xi, \varsigma)$, $\forall \varsigma, \xi \in \mathbb{M}$. Now, some remarks and examples as given below.

Remark 2.

(1) Every b-metric space is an s-distance space but not conversely

(2) In s-distance space, distance σ is not necessarily a continuous function, i.e., if $\varsigma_n \longrightarrow \varsigma$ and $\xi_n \longrightarrow \xi$ then $\sigma(\varsigma_n, \xi_n) \nrightarrow \sigma(\varsigma, \xi)$

(3) In an s-distance space, the limit of a convergent sequence may not be unique

Example 3. Let $\mathbb{M} = \{\alpha_1, \alpha_2, \alpha_3\}$, $\sigma : \mathbb{M} \times \mathbb{M} \longrightarrow [0,\infty)$ and

$$\begin{gathered} \sigma(\alpha_1, \alpha_2) = \frac{1}{4}, \sigma(\alpha_2, \alpha_1) = \frac{1}{2}, \\ \sigma(\alpha_1, \alpha_3) = \sigma(\alpha_2, \alpha_3) = 1, \sigma(\alpha_i, \alpha_i) = 0, \\ \sigma(\alpha_3, \alpha_1) = \sigma(\alpha_3, \alpha_2) \frac{1}{3}, \\ \sigma(\alpha_i, \alpha_j) \geq 0, \\ \sigma(\alpha_i, \alpha_j) + \sigma(\alpha_j, \alpha_i) = 0 \Leftrightarrow \alpha_i = \alpha_j, \end{gathered} \quad (2)$$

for all $i, j \in \{1, 2, 3\}$.

$$\begin{gathered} \sigma(\alpha_1, \alpha_2) < [\sigma(\alpha_1, \alpha_3) + \sigma(\alpha_3, \alpha_2)], \\ \sigma(\alpha_1, \alpha_3) < [\sigma(\alpha_1, \alpha_2) + \sigma(\alpha_2, \alpha_3)], \\ \sigma(\alpha_2, \alpha_3) < [\sigma(\alpha_2, \alpha_1) + \sigma(\alpha_1, \alpha_3)], \\ \sigma(\alpha_2, \alpha_1) < [\sigma(\alpha_2, \alpha_3) + \sigma(\alpha_3, \alpha_1)], \\ \sigma(\alpha_3, \alpha_1) < [\sigma(\alpha_3, \alpha_2) + \sigma(\alpha_2, \alpha_1)], \\ \sigma(\alpha_2, \alpha_3) < [\sigma(\alpha_2, \alpha_1) + \sigma(\alpha_1, \alpha_3)]. \end{gathered} \quad (3)$$

Hence, $(\mathbb{M}, \sigma)$ is an s-distance space with $s = 1$. However, it is not a b-metric space.

Fix $r \in \mathbb{N}$ and $\Gamma = (\Gamma_1, \cdots, \Gamma_r)$ is said to be a collection of mappings such that

$$\{\Gamma_{\acute{i}} : \{1, 2, 3, \cdots, r\} \longrightarrow \{1, 2, 3, \cdots, r\} ; 1 \leq i \leq r\}. \quad (4)$$

Let $(\mathbb{M}, \sigma)$ be a distance space and also $G : \mathbb{M}^r \longrightarrow \mathbb{M}$ be a mapping. The mapping $\Gamma G : \mathbb{M}^r \longrightarrow \mathbb{M}^r$ which is a composition of G and Γ is gvien as

$$\begin{gathered} \Gamma G(\varsigma_1, \cdots, \varsigma_r) = (\xi_1, \cdots, \xi_r), \\ \xi_i = G\left(\varsigma_{\Gamma_i(1)}, \cdots, \varsigma_{\Gamma_i(r)}\right), \end{gathered} \quad (5)$$

for any point $(\varsigma_1, \cdots, \varsigma_r) \in \mathbb{M}^r$ and $\acute{i} \in \{1, 2, \cdots, r\}$. A point $a = (a_1, \cdots, a_r) \in \mathbb{M}^r$ is considered as a Γ *-multiple FP* of G if it is a FP of ΓG, i.e., $a = \Gamma G(a)$ and

$$a_{\acute{i}} = G\left(a_{\Gamma_{\acute{i}}(1)}, \cdots, a_{\Gamma_{\acute{i}}(r)}\right) \text{ for any } \acute{i} \in \{1, 2, 3, \cdots, r\}. \quad (6)$$

Let $(\mathbb{M}, \sigma)$ be a distance space, $r \in \mathbb{N} = \{1, 2, \cdots\}$. On $\mathbb{M}^r$, consider the distance

$$\sigma^r((\varsigma_1, \cdots, \varsigma_r), (\xi_1, \cdots, \xi_r)) = \sup \left\{\sigma(\varsigma_{\acute{i}}, \xi_{\acute{i}}) : \acute{i} \leq r\right\}. \quad (7)$$

Obviously, $(\mathbb{M}^r, \sigma^r)$ is a distance space, too.

The following are some basic concepts from [14]:

Consider a partially ordered distance space $(\mathbb{M}, \sigma, \preccurlyeq)$, r be a positive integer and $\{J, K\}$ be a partition of $J_r = \{1, 2, \cdots, r\}$, i.e., $J, K \neq \phi$, $J \cup K = J_r$, and $J \cap K = \phi$. Define $\mathbb{M}^r = \mathbb{M} \times \mathbb{M} \times \ldots \times \mathbb{M}(r \text{ times})$ the Cartesian product of the set $\mathbb{M}$. Define a partial order $\preccurlyeq_r$ over $\mathbb{M}^r$ as follows:

For any $\omega = (\varsigma_1, \varsigma_2, \cdots, \varsigma_r), \nu = (\xi_1, \xi_2, \cdots, \xi_r) \in \mathbb{M}^r$, $\omega \preccurlyeq_r \nu$ if and only if $\varsigma_{\acute{i}} \preccurlyeq_{\acute{i}} \xi_{\acute{i}}$ for all $\acute{i} \in J_r$, where

$$\varsigma \preccurlyeq_{\acute{i}}{}' \xi \text{ iff} \begin{cases} \varsigma \preccurlyeq \xi, & \text{if } \acute{i} \in J, \\ \varsigma \succcurlyeq \xi, & \text{if } \acute{i} \in K. \end{cases} \quad (8)$$

The function $\sigma^r : \mathbb{M}^r \times \mathbb{M}^r \longrightarrow [0,+\infty)$ given by

$$\sigma^r(\varsigma, \xi) = \sup_{1 \leq \acute{i} \leq r} \{\sigma(\varsigma_{\acute{i}}, \xi_{\acute{i}})\}, \quad (9)$$

defines a distance on $\mathbb{M}^r$, where $\varsigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_r)$ and $\xi = (\xi_1, \xi_2, \cdots, \xi_r)$. Obviously, $(\mathbb{M}^r, \sigma^r, \preccurlyeq_r)$ is a partially ordered distance space, so that it inherits the properties of $(\mathbb{M}, \sigma, \preccurlyeq)$ and $\sigma^r(\varsigma^\vartheta, \varsigma) \longrightarrow 0$ if and only if $\sigma(\varsigma_{\acute{i}}^\vartheta, \varsigma_{\acute{i}}) \longrightarrow 0$ as $\vartheta \longrightarrow \infty$ for all $\acute{i} \in J_r$.

Definition 4. [15]. Let g be a self mapping on $\mathbb{M}$. A mapping G has the mixed g -monotone property with respect to the partition $\{J, K\}$, if G is g-monotone nondecreasing in arguments of J and g -monotone nonincreasing in arguments of K, i.e., $\forall \varsigma_1, \varsigma_2, \cdots, \varsigma_r, \xi, \eta \in \mathbb{M}$, $\acute{i} \in J_r$, and

$$g(\xi) \preccurlyeq g(\eta) \Rightarrow G(\varsigma_1, \cdots, \varsigma_{\acute{i}-1}, \xi, \varsigma_{\acute{i}+1}, \cdots, \varsigma_r) \preccurlyeq_{\acute{i}} G(\varsigma_1, \cdots, \varsigma_{\acute{i}-1}, \eta, \varsigma_{\acute{i}+1}, \cdots, \varsigma_r). \quad (10)$$

If g is the identity mapping on $\mathbb{M}$, then the mapping G has the mixed monotone property with respect to the partition $\{J, K\}$. Define a set of mappings by

$$\Omega_{J,K} = \{\Gamma_{\acute{i}} : J_r \longrightarrow J_r : \Gamma_{\acute{i}}(J) \subseteq J, \Gamma_{\acute{i}}(K) \subseteq K\},$$
$$\Omega'_{J,K} = \{\Gamma_{\acute{i}} : J_r \longrightarrow J_r : \Gamma_{\acute{i}}(J) \subseteq K, \Gamma_{\acute{i}}(K) \subseteq J\}, \quad (11)$$

such that $\Gamma_{\acute{i}} \in \Omega_{J,K}$ if $\acute{i} \in J$ and $\Gamma_{\acute{i}} \in \Omega'_{J,K}$ if $\acute{i} \in K$.

Definition 5. If a function $\psi : [0,\infty) \longrightarrow 0,\infty)$ is continuous, nondecreasing, and $\psi(0) = 0$, then it is called an altering distance function.

Definition 6. The metric space $(\mathbb{M}, \sigma, \preccurlyeq)$ is called regular if it satisfies the following properties:

(1) If $\{\varsigma_r\}$ is a nondecreasing sequence and $\varsigma_r \longrightarrow \varsigma$ then $\varsigma_r \preccurlyeq \varsigma$ for all $r \geq 1$

(2) If $\{\varsigma_r\}$ is a nonincreasing sequence and $\varsigma_r \longrightarrow \varsigma$ then $\varsigma_r \pm \varsigma$ for all $r \geq 1$

3. Main Result

Berinde and Borcut [4] extended the concept of multidimensional FP to ordered distance spaces by utilizing the properties of contractive type mappings. Keeping ourselves in touch with all these concepts, we are extending these results to symmetric s-distance spaces by using a combination of altering distance functions. This result will generalize the main theorems of [14], in which the space considered is a metric space. It is also valid for b-metric spaces.

Theorem 7. *Consider a complete partially ordered symmetric s-distance space.*

$(\mathbb{M}, \sigma, \preccurlyeq)$ *and* $\Gamma = (\Gamma_1, \Gamma_2, \cdots, \Gamma_r)$ *be collection of mappings verifying* $\Gamma_{\acute{i}} \in \Omega_{J,K}$ *if* $\acute{i} \in J$ *and* $\Gamma_{\acute{i}} \in \Omega'_{J,K}$ *if* $\acute{i} \in K$. *If the mapping* $G : \mathbb{M}^r \longrightarrow \mathbb{M}$ *satisfies the following conditions:*

(a) For $\mu > 0$

$\psi(\mu) - \theta(\mu) - \varphi(\mu) > 0, (*)$

where ψ *is an altering distance function and* $\theta : [0,\infty) \longrightarrow 0,\infty)$ *and* $\varphi : [0,\infty) \longrightarrow 0,\infty)$ *are upper semicontinuous and increasing functions such that* $\theta(0) = \varphi(0) = 0$ *satisfying*

$$\psi(\sigma(G\varsigma, G\xi)) \leq \theta\left(\frac{\sigma^r(\varsigma, \xi)}{s+1}\right) + \varphi\left(\frac{\sigma^r(\varsigma, \xi)}{s+1}\right), \quad (12)$$

for all $\varsigma, \xi \in \mathbb{M}^r$ *with* $\varsigma \preccurlyeq_r \xi$.

(b) There is $\varsigma^0 = (\varsigma_1^0, \varsigma_2^0, \cdots, \varsigma_r^0)$ *such that* $\varsigma_{\acute{i}}^0 \preccurlyeq_{\acute{i}} G(\varsigma_{\Gamma_{\acute{i}}(1)}^0, \cdots, \varsigma_{\Gamma_{\acute{i}}(r)}^0)$ *for all* $\acute{i} \in J_r$

(c) G has mixed monotone property with respect to $\{J, K\}$

(d) G is continuous, or $(\mathbb{M}, \sigma, \preccurlyeq)$ *is regular; then, G has* Γ*-multiple FP*

(e) Moreover, if for ς *and* ξ *in* $\mathbb{M}^r$, *there is* $\eta \in \mathbb{M}^r$ *such that* $\varsigma \preccurlyeq_r \eta$ *and* $\xi \preccurlyeq_r \eta$; *then, G has a unique* Γ*-multiple FP*

Proof. Step 1. Let $\varsigma^\vartheta = (\Gamma G)^\vartheta(\varsigma^0)$ be the nth Picard iterate of ς^0 under ΓG, i.e., $\varsigma^\vartheta = (\Gamma G)^\vartheta(\varsigma^0) = (\varsigma_1^\vartheta, \varsigma_2^\vartheta, \cdots, \varsigma_r^\vartheta)$, where

$$\begin{aligned} \varsigma_1^\vartheta &= G\left(\varsigma_{\Gamma_1(1)}^{\vartheta-1}, \varsigma_{\Gamma_1(2)}^{\vartheta-1}, \cdots \varsigma_{\Gamma_1(r)}^{\vartheta-1}\right), \\ \varsigma_2^\vartheta &= G\left(\varsigma_{\Gamma_2(1)}^{\vartheta-1}, \varsigma_{\Gamma_2(2)}^{\vartheta-1}, \cdots \varsigma_{\Gamma_2(r)}^{\vartheta-1}\right), \\ &\vdots \\ \varsigma_r^\vartheta &= G\left(\varsigma_{\Gamma_r(1)}^{\vartheta-1}, \varsigma_{\Gamma_r(2)}^{\vartheta-1}, \cdots \varsigma_{\Gamma_r(r)}^{\vartheta-1}\right). \end{aligned} \quad (13)$$

By condition (b) and definition of ΓG, it follows that $\varsigma^0 \preccurlyeq_r \varsigma^1$. Since, G has mixed monotone property so ΓG is monotone nondecreasing [15]; therefore

$$\varsigma^{\vartheta-1} \preccurlyeq_r \varsigma^\vartheta \forall \vartheta \geq 1. \quad (14)$$

Step 2. We need to show that $\lim_{\vartheta\longrightarrow\infty} \sigma^r(\varsigma^{\vartheta-1}, \varsigma^\vartheta) = 0$. Set $D_{\acute{i}}^\vartheta = \sigma(\varsigma_{\acute{i}}^{\vartheta-1}, \varsigma_{\acute{i}}^\vartheta)$ and $D^\vartheta = \sup_{\acute{i}\in J_r}(D_{\acute{i}}^\vartheta) = \sigma^r(\varsigma^{\vartheta-1}, \varsigma^\vartheta)$. If $D^\vartheta = 0$ for some $\vartheta \geq 1$ then $\varsigma^{\vartheta-1} = \Gamma G(\varsigma^{\vartheta-1})$ which means G has Γ-multiple FP which completes the proof. Assume $D^\vartheta > 0$ for all $\vartheta \geq 1$. Since $\varsigma^{\vartheta-1} \preccurlyeq_r \varsigma^\vartheta$ and $\Gamma_{\acute{i}}(J_r) \subseteq J_r$, it follows that

$$\left(\varsigma_{\Gamma_{\acute{i}}(1)}^{\vartheta-1}, \varsigma_{\Gamma_{\acute{i}}(2)}^{\vartheta-1}, \cdots, \varsigma_{\Gamma_{\acute{i}}(r)}^{\vartheta-1}\right) \preccurlyeq_r \left(\varsigma_{\Gamma_{\acute{i}}(1)}^\vartheta, \varsigma_{\Gamma_{\acute{i}}(2)}^\vartheta, \cdots \varsigma_{\Gamma_{\acute{i}}(r)}^\vartheta\right), \quad (15)$$

for any $\acute{i} \in J_r$ and $\vartheta \geq 1$. Now, using condition (12)

$$\psi\left(D_{\acute{i}}^\vartheta\right) = \psi\left(\sigma\left(G\left(\varsigma_{\Gamma_{\acute{i}}(1)}^{\vartheta-2}, \varsigma_{\Gamma_{\acute{i}}(2)}^{\vartheta-2}, \cdots, \varsigma_{\Gamma_{\acute{i}}(r)}^{\vartheta-2}\right), \cdot G\left(\varsigma_{\Gamma_{\acute{i}}(1)}^{\vartheta-1}, \varsigma_{\Gamma_{\acute{i}}(2)}^{\vartheta-1}, \cdots, \varsigma_{\Gamma_{\acute{i}}(r)}^{\vartheta-1}\right)\right)\right),$$

$$\le \theta\left(\frac{\sup_{j\in J_r}\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(j)}\right)}{s+1}\right) + \varphi\left(\frac{\sup_{j\in J_r}\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(j)}\right)}{s+1}\right). \tag{16}$$

Since, θ and φ both are increasing and $s>0$ so that

$$\psi\left(D^{\vartheta}_{\acute{i}}\right) \le \theta\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(j)}\right)\right\}\right) + \varphi\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(j)}\right)\right\}\right), \tag{17}$$

for any $\acute{i}\in J_r$, since J_r is a finite set, there is an index $\acute{i}(\vartheta)\in J_r$ such that $\sup_{\acute{i}\in J_r}\{D^{\vartheta}_{\acute{i}}\} = D^{\vartheta}_{\acute{i}(\vartheta)}$. From above inequality, it follow that

$$\begin{aligned}
&\psi\left(D^{\vartheta}_{\acute{i}(\vartheta)}\right)\\
&= \psi\left(D^{\vartheta}_{\acute{i}(\vartheta)}\right), = \psi\left(\sigma\left(\begin{array}{c} G\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(1)},\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(2)},\cdots,\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(r)}\right),\\ G\left(\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(1)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(2)},\cdots,\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(r)}\right)\end{array}\right)\right),\\
&\le \theta\left(\frac{\sup_{j\in J_r}\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)}{s+1}\right) + \varphi\left(\frac{\sup_{j\in J_r}\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)}{s+1}\right),\\
&\le \theta\left(\sup_{j\in J_r}\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right) + \varphi\left(\sup_{j\in J_r}\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right).
\end{aligned} \tag{18}$$

Since

$$0 < \sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\} \le D^{\vartheta-1}, \tag{19}$$

for all $\vartheta\ge 1$. Therefore, from the inequality $(*)$, we get

$$\begin{aligned}
&\theta\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\}\right) + \varphi\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\}\right)\\
&< \psi\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\}\right)\\
&\le \psi\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{(j)},\varsigma^{\vartheta-1}_{(j)}\right)\right\}\right).
\end{aligned} \tag{20}$$

Combining (19) and (20), we have

$$\psi\left(D^{\vartheta}\right) < \psi\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\}\right) \le \psi\left(D^{\vartheta-1}\right), \tag{21}$$

for all $\vartheta\ge 1$. Since ψ is an altering distance function, it follows that

$$D^{\vartheta} < \sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\} \le D^{\vartheta-1}. \tag{22}$$

Hence, the sequence D^{ϑ} and $\sup_{j\in J_r}\{\sigma(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)})\}$ are monotone decreasing and bounded below. So, we have $\tau\ge 0$ such that

$$\lim_{\vartheta\longrightarrow\infty} D^{\vartheta} = \lim_{\vartheta\longrightarrow\infty}\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)},\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\} = \tau. \tag{23}$$

We need to show that $\tau=0$. Suppose $\tau>0$, then by applying limit as $\vartheta\longrightarrow\infty$ in (19) and utilizing the properties of ψ,θ and φ, we have $\psi(\tau)-\theta(\tau)-\varphi(\tau)\le 0$, which contradicts $(*)$. Hence

$$\lim_{\vartheta\longrightarrow\infty} D^{\vartheta} = \lim_{\vartheta\longrightarrow\infty}\sigma^r\left(\varsigma^{\vartheta-1},\varsigma^{\vartheta}\right) = 0. \tag{24}$$

Step 3. Now to prove that the sequence $\{\varsigma^{\vartheta}\}_{\vartheta\in\mathbb{N}}$ is Cauchy. Suppose on contrary that it is not Cauchy, then there exists $\varepsilon>0$ for which there are subsequences $\{\varsigma^{\vartheta_\zeta}\}$ and $\{\varsigma^{r_\zeta}\}$ of $\{\varsigma^{\vartheta}\}$ with $\vartheta_\zeta > r_\zeta > \zeta$ such that

$$\sigma^r\left(\varsigma^{r_\zeta},\varsigma^{\vartheta_\zeta}\right) \ge \varepsilon. \tag{25}$$

Let ϑ_ζ be the smallest integer satisfying above, then

$$\sigma^r\left(\varsigma^{r_\zeta},\varsigma^{\vartheta_\zeta-1}\right) < \varepsilon. \tag{26}$$

Assume that $s<1$, then consider

$$\begin{aligned}
\sigma^r\left(\varsigma^{r_\zeta},\varsigma^{\vartheta_\zeta}\right) &\le s\left[\sigma^r\left(\varsigma^{r_\zeta},\varsigma^{\vartheta_\zeta-1}\right) + \sigma^r\left(\varsigma^{\vartheta_\zeta-1},\varsigma^{\vartheta_\zeta}\right)\right]\\
&< \left[\varepsilon + \sigma^r\left(\varsigma^{\vartheta_\zeta-1},\varsigma^{\vartheta_\zeta}\right)\right].
\end{aligned} \tag{27}$$

On letting $\zeta\longrightarrow\infty$ and using condition (24), we get

$$\varepsilon \le \lim\sup_{\zeta\longrightarrow\infty}\sigma^r\left(\varsigma^{r_\zeta},\varsigma^{\vartheta_\zeta}\right) < \varepsilon, \tag{28}$$

which leads to a contradiction. Now consider $s\ge 1$, then

$$\begin{aligned}
\sigma^r\left(\varsigma^{r_\zeta},\varsigma^{\vartheta_\zeta}\right) &\le s\left[\sigma^r\left(\varsigma^{r_\zeta},\varsigma^{\vartheta_\zeta-1}\right) + \sigma^r\left(\varsigma^{\vartheta_\zeta-1},\varsigma^{\vartheta_\zeta}\right)\right]\\
&< s\left[\varepsilon + \sigma^r\left(\varsigma^{\vartheta_\zeta-1},\varsigma^{\vartheta_\zeta}\right)\right].
\end{aligned} \tag{29}$$

Again letting $\zeta \longrightarrow \infty$ and using condition (24), we get

$$\varepsilon \le \lim_{\zeta\longrightarrow\infty} \sup \sigma^r\left(\varsigma^{r_\zeta}, \varsigma^{\vartheta_\zeta}\right) < s\varepsilon. \tag{30}$$

Now

$$\begin{gathered}\sigma^r\left(\varsigma^{r_\zeta-1}, \varsigma^{\vartheta_\zeta-1}\right) \le s\left[\sigma^r\left(\varsigma^{r_\zeta-1}, \varsigma^{r_\zeta}\right) + \sigma^r\left(\varsigma^{r_\zeta}, \varsigma^{\vartheta_\zeta-1}\right)\right],\\ \lim_{\zeta\longrightarrow\infty} \sup \sigma^r\left(\varsigma^{r_\zeta-1}, \varsigma^{\vartheta_\zeta-1}\right) < s\varepsilon.\end{gathered} \tag{31}$$

From $\varsigma^{\vartheta-1} \preccurlyeq_r \varsigma^{\vartheta}$, it follows that

$$\varsigma^{r_\zeta} \preccurlyeq_r \varsigma^{r_\zeta+1} \preccurlyeq_r \cdots \preccurlyeq_r \varsigma^{\vartheta_\zeta-1} \preccurlyeq_r \varsigma^{\vartheta_\zeta} \preccurlyeq_r \varsigma^{\vartheta_\zeta+1}. \tag{32}$$

Since $\varsigma^{r_\zeta-1} \preccurlyeq_r \varsigma^{\vartheta_\zeta-1}$ and $\Gamma_{\acute{i}}(J_r) \subseteq J_r$, we have

$$\left(\varsigma^{r_\zeta-1}_{r_i(1)}, \varsigma^{r_\zeta-1}_{r_i(2)}, \cdots, \varsigma^{r_\zeta-1}_{r_i(r)}\right) \preccurlyeq_r \left(\varsigma^{\vartheta_\zeta-1}_{r_i(1)}, \varsigma^{\vartheta_\zeta-1}_{r_i(2)}, \cdots, \varsigma^{\vartheta_\zeta-1}_{\Gamma_{\acute{i}}(r)}\right), \tag{33}$$

for any $\acute{i} \in J_r$ and $\zeta \ge 1$. By condition (a), it follows that

$$\begin{aligned}\psi\left(\sigma\left(\varsigma^{r_\zeta}_{\acute{i}}, \varsigma^{\vartheta_\zeta}_{\acute{i}}\right)\right) &\le \psi\left(\sigma\left(G\left(\varsigma^{r_\zeta-1}_{\Gamma_{\acute{i}}(1)}, \varsigma^{r_\zeta-1}_{\Gamma_{\acute{i}}(2)}, \cdots \varsigma^{r_\zeta-1}_{\Gamma_{\acute{i}}(r)}\right), \cdot G\left(\varsigma^{r_\zeta-1}_{\Gamma_{\acute{i}}(1)}, \varsigma^{r_\zeta-1}_{\Gamma_{\acute{i}}(2)}, \cdots \varsigma^{r_\zeta-1}_{\Gamma_{\acute{i}}(r)}\right)\right)\right),\\ &\le \theta\left(\frac{\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{r_\zeta-1}_{\Gamma_{\acute{i}}(j)}, \varsigma^{\vartheta_\zeta-1}_{\Gamma_{\acute{i}}(j)}\right)\right\}}{s+1}\right) + \varphi\left(\frac{\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{r_\zeta-1}_{\Gamma_{\acute{i}}(j)}, \varsigma^{\vartheta_\zeta-1}_{\Gamma_{\acute{i}}(j)}\right)\right\}}{s+1}\right),\end{aligned} \tag{34}$$

for any $\acute{i} \in J_r$. Since J_r is finite, there will be an index $\acute{i}(\zeta)$ in J_r, so

$$\sup_{\acute{i}\in J_r}\left\{\sigma\left(\varsigma^{r_\zeta}_{\acute{i}}, \varsigma^{\vartheta_\zeta}_{\acute{i}}\right)\right\} = \sigma\left(\varsigma^{r_\zeta}_{\acute{i}(\zeta)}, \varsigma^{\vartheta_\zeta}_{\acute{i}(\zeta)}\right). \tag{35}$$

From inequality (29)

$$\begin{aligned}\psi(\varepsilon) &\le \psi\left(\sigma^r\left(\varsigma^{r_\zeta}, \varsigma^{\vartheta_\zeta}\right)\right),\\ &\le \psi\left(\sigma\left(\begin{matrix}G\left(\varsigma^{r_\zeta-1}_{\Gamma_{i(\zeta)}(1)}, \varsigma^{r_\zeta-1}_{\Gamma_{i(\zeta)}(2)}, \cdots \varsigma^{r_\zeta-1}_{\Gamma_{i(\zeta)}(r)}\right),\\ G\left(\varsigma^{r_\zeta-1}_{\Gamma_{i(\zeta)}(1)}, \varsigma^{r_\zeta-1}_{\Gamma_{i(\zeta)}(2)}, \cdots \varsigma^{r_\zeta-1}_{\Gamma_{i(\zeta)}(r)}\right)\end{matrix}\right)\right),\\ &\le \theta\left(\frac{\sup_{j\in J_r}\sigma\left(\varsigma^{r_\zeta-1}_{\Gamma_{i(\zeta)}(j)}, \varsigma^{\vartheta_\zeta-1}_{\Gamma_{i(\zeta)}(j)}\right)}{s+1}\right) + \varphi\left(\frac{\sup_{j\in J_r}\sigma\left(\varsigma^{r_\zeta-1}_{\Gamma_{i(\zeta)}(j)}, \varsigma^{\vartheta_\zeta-1}_{\Gamma_{i(\zeta)}(j)}\right)}{s+1}\right).\end{aligned} \tag{36}$$

Since θ and φ are increasing functions and $\Gamma_i{}'(J_r) \subseteq J_r$, so

$$\begin{aligned}\psi(\varepsilon) &\le \theta\left(\frac{\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{r_\zeta-1}_{\Gamma_{i(\zeta)}(j)}, \varsigma^{\vartheta_\zeta-1}_{\Gamma_{i(\zeta)}(j)}\right)\right\}}{s}\right) + \varphi\left(\frac{\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{r_\zeta-1}_{\Gamma_{i(\zeta)}(j)}, \varsigma^{\vartheta_\zeta-1}_{\Gamma_{i(\zeta)}(j)}\right)\right\}}{s}\right),\\ &\le \theta\left(\frac{\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{r_\zeta-1}_{(j)}, \varsigma^{\vartheta_\zeta-1}_{(j)}\right)\right\}}{s}\right) + \varphi\left(\frac{\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{r_\zeta-1}_{(j)}, \varsigma^{\vartheta_\zeta-1}_{(j)}\right)\right\}}{s}\right),\\ &\le \theta\left(\frac{\sigma^r\left(\varsigma^{r_\zeta-1}, \varsigma^{\vartheta_\zeta-1}\right)}{s}\right) + \varphi\left(\frac{\sigma^r\left(\varsigma^{r_\zeta-1}, \varsigma^{\vartheta_\zeta-1}\right)}{s}\right).\end{aligned} \tag{37}$$

Applying $\limsup_{\zeta\longrightarrow\infty}$ over above inequality and then using (29), we get

$$\psi(\varepsilon) < \theta\left(\frac{s\varepsilon}{\varepsilon}\right) + \varphi\left(\frac{s\varepsilon}{\varepsilon}\right), < \theta(\varepsilon) + \varphi(\varepsilon), \tag{38}$$

which is a contradiction to the condition $(*)$. Hence, $(\varsigma^\vartheta)_{\vartheta\in\mathbb{N}}$ is a Cauchy sequence.

Step 4. Since the space $(\mathbb{M}, \sigma)$ is complete so $(\mathbb{M}^r, \sigma^r)$ is complete. Therefore, we have $\nu \in \mathbb{M}^r$ such that $(\varsigma^\vartheta)_{\vartheta\in\mathbb{N}} \longrightarrow \nu$, i.e.,

$$\begin{gathered}\lim_{\vartheta\longrightarrow\infty} \varsigma^\vartheta_1 = G\left(\varsigma^{\vartheta-1}_{\Gamma_1(1)}, \varsigma^{\vartheta-1}_{\Gamma_1(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_1(r)}\right) = \nu_1,\\ \lim_{\vartheta\longrightarrow\infty} \varsigma^\vartheta_2 = G\left(\varsigma^{\vartheta-1}_{\Gamma_2(1)}, \varsigma^{\vartheta-1}_{\Gamma_2(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_2(r)}\right) = \nu_2,\\ \vdots\\ \lim_{\vartheta\longrightarrow\infty} \varsigma^\vartheta_r = G\left(\varsigma^{\vartheta-1}_{\Gamma_r(1)}, \varsigma^{\vartheta-1}_{\Gamma_r(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_r(r)}\right) = \nu_r.\end{gathered} \tag{39}$$

Next, we show that $\nu = (\nu_1, \nu_2, \cdots, \nu_r) \in \mathbb{M}^r$ is a Γ-multiple FP of operator G. If condition (c) holds and G is continuous, then

$$\begin{gathered}\lim_{\vartheta\longrightarrow\infty} G\left(\varsigma^{\vartheta-1}_{\Gamma_1(1)}, \varsigma^{\vartheta-1}_{\Gamma_1(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_1(r)}\right) = G\left(\nu_{\Gamma_1(1)}, \nu_{\Gamma_1(2)}, \cdots, \nu_{\Gamma_1(r)}\right),\\ \lim_{\vartheta\longrightarrow\infty} G\left(\varsigma^{\vartheta-1}_{\Gamma_2(1)}, \varsigma^{\vartheta-1}_{\Gamma_2(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_2(r)}\right) = G\left(\nu_{\Gamma_2(1)}, \nu_{\Gamma_2(2)}, \cdots, \nu_{\Gamma_2(r)}\right),\\ \vdots\\ \lim_{\vartheta\longrightarrow\infty} G\left(\varsigma^{\vartheta-1}_{\Gamma_r(1)}, \varsigma^{\vartheta-1}_{\Gamma_r(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_r(r)}\right) = G\left(\nu_{\Gamma_i(1)}, \nu_{\Gamma_i(2)}, \cdots, \nu_{\Gamma_i(r)}\right).\end{gathered} \tag{40}$$

Above shows that

$$\nu_{\acute{i}} = G\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right), \text{for all } \acute{i} \in J_r, \quad (41)$$

which means the point $\nu = (\nu_1, \nu_2, \cdots, \nu_r)$ is a Γ-FP of G. Next, suppose $(\mathbb{M}, \sigma\preccurlyeq)$ is regular. The above relation implies $\varsigma^{\vartheta} = (\varsigma_1^{\vartheta}, \varsigma_2^{\vartheta}, \cdots, \varsigma_r^{\vartheta}) \preccurlyeq_r \nu = (\nu_1, \nu_2, \cdots, \nu_r)$. On the other hand

$$\left(\varsigma^{\vartheta}_{\Gamma_{\acute{i}}(1)}, \varsigma^{\vartheta}_{\Gamma_{\acute{i}}(2)}, \cdots, \varsigma^{\vartheta}_{\Gamma\acute{i}(r)}\right) \preccurlyeq_r \left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right). \quad (42)$$

Since, $\Gamma_{\acute{i}}(J_r) \subseteq J_r$ for any $\acute{i} \in J_r$. By using (a)

$$\begin{aligned}
&\psi\left(\sigma\left(G\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right), G\left(\varsigma^{\vartheta}_{\Gamma_{\acute{i}}(1)}, \varsigma^{\vartheta}_{\Gamma_{\acute{i}}(2)}, \cdots, \varsigma^{\vartheta}_{\Gamma\acute{i}(r)}\right)\right)\right), \\
&\leq \theta\left(\frac{\sup_{j\in J_r}\left\{\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \varsigma^{\vartheta}_{\Gamma_{\acute{i}}(j)}\right)\right\}}{s+1}\right) + \varphi\left(\frac{\max_{j\in J_r}\left\{\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \varsigma^{\vartheta}_{\Gamma_{\acute{i}}(j)}\right)\right\}}{s+1}\right), \\
&\leq \theta\left(\sup_{j\in J_r}\left\{\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \varsigma^{\vartheta}_{\Gamma_{\acute{i}}(j)}\right)\right\}\right) + \varphi\left(\sup_{j\in J_r}\left\{\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \varsigma^{\vartheta}_{\Gamma_{\acute{i}}(j)}\right)\right\}\right).
\end{aligned} \quad (43)$$

Taking into account above and letting $\vartheta \longrightarrow \infty$ in the last inequality, it implies

$$\psi\left(\sigma\left(G\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right), \nu_{\acute{i}}\right)\right) = 0. \quad (44)$$

Hence

$$G\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right) = \nu_{\acute{i}} \text{ for all } \acute{i} \in J_r. \quad (45)$$

This completes the existence of Γ-multiple FP.

Step 5. In this step, we will show that the FP of G is unique.

Suppose $w \in \mathbb{M}^r$ be another FP of G. By condition (e), there exists $\eta = (\eta_1, \eta_2, \cdots, \eta_r) \in \mathbb{M}^r$ such that $\nu \preccurlyeq_r \eta$ and $w \preccurlyeq_r \eta$.

Put $\eta^0 = \eta$ and define

$$\begin{aligned}
\eta_1^{\vartheta} &= G\left(\eta^{\vartheta-1}_{\Gamma_1(1)}, \eta^{\vartheta-1}_{\Gamma_1(2)}, \cdots, \eta^{\vartheta-1}_{\Gamma_1(r)}\right), \\
\eta_2^{\vartheta} &= G\left(\eta^{\vartheta-1}_{\Gamma_2(1)}, \eta^{\vartheta-1}_{\Gamma_2(2)}, \cdots, \eta^{\vartheta-1}_{\Gamma_2(r)}\right), \\
&\vdots \\
\eta_r^{\vartheta} &= G\left(\eta^{\vartheta-1}_{\Gamma_r(1)}, \eta^{\vartheta-1}_{\Gamma_r(2)}, \cdots, \eta^{\vartheta-1}_{\Gamma_r(r)}\right).
\end{aligned} \quad (46)$$

By the induction method, we have

$$\nu \preccurlyeq_r \eta^{\vartheta}, w \preccurlyeq_r \eta^{\vartheta}, \quad (47)$$

for all $\vartheta \geq 0$. By condition (e), $\nu \preccurlyeq_r \eta^0$. Assume that above condition holds for $\vartheta - 1$.

Using the procedure of Step 1, it can be shown that

$$\begin{aligned}
\nu &= G\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right) \\
&\preccurlyeq_{\acute{i}} G\left(\eta^{\vartheta-1}_{\Gamma\acute{i}(1)}, \eta^{\vartheta-1}_{\Gamma_{\acute{i}}(2)}, \cdots, \eta^{\vartheta-1}_{\Gamma\acute{i}(r)}\right) = \eta^{\vartheta}_{\acute{i}},
\end{aligned} \quad (48)$$

for all $\acute{i} \in J_r$; that is, $\nu \preccurlyeq_r \eta^{\vartheta}$. Similarly, we can prove the second inequality. Further, we prove that

$$\lim_{\vartheta\longrightarrow\infty} \sigma^r\left(\nu, \eta^{\vartheta}\right) = 0. \quad (49)$$

For this, we first show that if $\sigma^r(\nu, \eta^{\vartheta}) = 0$ for some ϑ_0 then $\sigma^r(\nu, \eta^{\vartheta}) = 0$, for all $\vartheta \geq \vartheta_0$. Indeed, from above condition, it follows that

$$\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right) \preccurlyeq_r \left(\eta^{\vartheta}_{\Gamma\acute{i}(1)}, \eta^{\vartheta}_{\Gamma_{\acute{i}}(2)}, \cdots, \eta^{\vartheta}_{\Gamma\acute{i}(r)}\right), \quad (50)$$

for all $\acute{i} \in J_r$ and $\vartheta \geq 1$. From (a), it follows that

$$\begin{aligned}
\psi\left(\sigma\left(\nu_{\acute{i}}, \eta^{\vartheta}_{\acute{i}}\right)\right) &= \psi\left(\sigma\left(G\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right),\right.\right. \\
&\quad \left.\left.\cdot\, G\left(\eta^{\vartheta-1}_{\Gamma\acute{i}(1)}, \eta^{\vartheta-1}_{\Gamma_{\acute{i}}(2)}, \cdots, \eta^{\vartheta-1}_{\Gamma\acute{i}(r)}\right)\right)\right), \\
&\leq \theta\left(\frac{\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right)}{s+1}\right) \\
&\quad + \varphi\left(\frac{\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right)}{s+1}\right), \\
&\leq \theta\left(\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma i'(j)}\right)\right)\right) \\
&\quad + \varphi\left(\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right)\right),
\end{aligned} \quad (51)$$

for all $\acute{i} \in J_r$ and $\vartheta \geq 1$. Recall that $\Gamma_{\acute{i}}(J_r) \subseteq J_r$. Hence

$$\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right) \leq \sigma^r\left(\nu, \eta^{\vartheta-1}\right), \quad (52)$$

for all $\vartheta \geq 1$. Taking into account $(*)$, it follows

$$\begin{aligned}
&\theta\left(\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right)\right) + \varphi\left(\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right)\right), \\
&\leq \psi\left(\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right)\right) \leq \psi\left(\sigma^r\left(\nu, \eta^{\vartheta}\right)\right),
\end{aligned} \quad (53)$$

for all $\acute{i} \in J_r$ and $\vartheta \geq 1$. Combining (50) and (52), we get

$$\psi\left(\sup_{j\in J_r}\left\{\sigma\left(\nu_{\acute{i}}, \eta_{\acute{i}}^{\vartheta}\right)\right\}\right) \leq \psi\left(\sigma^r\left(\nu, \eta^{\vartheta-1}\right)\right), \quad (54)$$

for $\vartheta \geq 1$. Since, ψ is an altering distance function

$$\psi\left(\sigma^r\left(\nu, \eta^{\vartheta}\right)\right) \leq \psi\left(\sigma^r\left(\nu, \eta^{\vartheta-1}\right)\right). \quad (55)$$

Now, it is obvious that if $\sigma^r(\nu, \eta^{\vartheta_0}) = 0$, then $\sigma^r(\nu, \eta^{\vartheta}) = 0$ for all $\vartheta \geq \vartheta_0$.

Next, assume that $\sigma^r(\nu, \eta^{\vartheta}) > 0$ for all $\vartheta \geq 1$. Using the same manner adopted in Step 1, it can be proved that $\sigma^r(\nu, \eta^{\vartheta}) \leq \sigma^r(\nu, \eta^{\vartheta-1})$. Hence, there exists $r \geq 0$ such that $\lim_{\vartheta\longrightarrow\infty}\sigma^r(\nu, \eta^{\vartheta}) = r$. In similar way, it can be shown

$$\lim_{\vartheta\longrightarrow\infty} \sigma^r\left(\nu, \eta^{\vartheta}\right) = 0 \text{ and } \lim_{\vartheta\longrightarrow\infty} \sigma^r\left(w, \eta^{\vartheta}\right) = 0. \quad (56)$$

On the other hand,

$$\sigma^r(\nu, w) \leq s\left[\sigma^r\left(\nu, \eta^{\vartheta}\right) + \sigma^r\left(\eta^{\vartheta}, w\right)\right]. \quad (57)$$

Applying $\lim \vartheta \longrightarrow \infty$ over the above inequality, it follows

$$\sigma^r(\nu, w) = 0. \quad (58)$$

This implies $\nu = w$. Hence, G has a unique multidimensional FP.□ □

4. Some Consequences and Example of Theorem 7

In the following section, some important concept regarding the consequences of Theorem 7 are discussed which are in terms of the main results represented in the articles [4, 16]. An illustrative example is also added in this section which will be helpful for the readers to understand the structure of multidimensional FP under the weak contractions for partially ordered s-distance spaces.

Corollary 8. *Let $(\mathbb{M}, \sigma, \preccurlyeq)$ is known as complete partially ordered s-distance space and $\Gamma = (\Gamma_1, \Gamma_2, \cdots, \Gamma_r)$ be a collection of mappings verifying $\Gamma_{\acute{i}} \in \Omega_{J,K}$ if $\acute{i} \in J$ and $\Gamma_{\acute{i}} \in \Omega'_{J,K}$ if $\acute{i} \in K$. Assume that the mapping $G : \mathbb{M}^r \longrightarrow \mathbb{M}$ satisfies the following conditions:*

(1) If there exists $\gamma \in (0, 1)$ such that

$$\sigma(G(\varsigma_1, \varsigma_2, \cdots, \varsigma_r), G(\xi_1, \xi_2, \cdots, \xi_r)) \leq \gamma\sigma^r(\varsigma, \xi), \quad (59)$$

for all $\varsigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_r) \in \mathbb{M}^r$ and $\xi = (\xi_1, \xi_2, \cdots, \xi_r) \in \mathbb{M}^r$ with $\varsigma \preccurlyeq_r \xi$ satisfies conditions.

(2) To (e) of the above theorem then G has a unique multidimensional FP

Proof. We can prove this corollary easily by taking $\psi(\varsigma) = \varsigma$, $\theta(\varsigma) = (s + 1)\gamma\varsigma$, and $\varphi(\varsigma) = 0$ in the above theorem.□ □

Remark 9. Theorems 2.1 and 2.2 of [16], in s-distance spaces, follow from Corollary 8. In [16], contractive condition is

$$\sigma(F(\varsigma_1, \varsigma_2), F(\xi_1, \xi_2)) \leq \frac{\delta}{2}[\sigma(\varsigma_1, \xi_1) + \sigma(\varsigma_2, \xi_2)], \quad (60)$$

and for $\varsigma, \xi \in \mathbb{M}^2$ such that $\varsigma \preccurlyeq_2 \xi$, it implies

$$\sigma(F(\varsigma_1, \varsigma_2), F(\xi_1, \xi_2)) \leq \frac{\delta}{2}[\sigma(\varsigma_1, \xi_1) + \sigma(\varsigma_2, \xi_2)] \leq \delta\sigma^2(\varsigma, \xi). \quad (61)$$

Applying Corollary 8, we get the desired result in s-distance space.

Remark 10. Corollary 8 also generalizes the main triple FP result of [4], in which the space under consideration is a metric space. We generalize it for b -metric space and for an s-distance space. In [4], $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3\}$, A_3 is chosen as $A = \{1, 3\}, B = \{2\}$, collection of mappings is defined as

$$\begin{pmatrix} \Gamma_1(1) & \Gamma_1(2) & \Gamma_1(3) \\ \Gamma_2(1) & \Gamma_2(2) & \Gamma_2(3) \\ \Gamma_3(1) & \Gamma_3(2) & \Gamma_3(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad (62)$$

the contractive condition in [4] is

$$\begin{aligned} &\sigma(F(\varsigma_1, \varsigma_2, \varsigma_3), F(\xi_1, \xi_2, \xi_3)) \\ &\quad \leq \delta_1\sigma(\varsigma_1, \xi_1) + \delta_2\sigma(\varsigma_2, \xi_2) + \delta_3\sigma(\varsigma_3, \xi_3), \end{aligned} \quad (63)$$

for any $\varsigma = (\varsigma_1, \varsigma_2, \varsigma_3), \xi = (\xi_1, \xi_2, \xi_3)$ with $\varsigma \preccurlyeq_i \xi$ and $\delta_1, \delta_2, \delta_3 \geq 0$ and $\delta_1 + \delta_2 + \delta_3 < 1$. Obviously

$$\sigma(F(\varsigma_1, \varsigma_2, \varsigma_3), F(\xi_1, \xi_2, \xi_3)) \leq (\delta_1 + \delta_2 + \delta_3)\sigma^3(\varsigma, \xi). \quad (64)$$

Applying Corollary 8, we get the desired result in s-distance space.

Corollary 11. *Consider $(\mathbb{M}, \sigma, \preccurlyeq)$ as a complete partially ordered b-metric space and $\Gamma = (\Gamma_1, \Gamma_2, \cdots, \Gamma_r)$ be collection of mappings verifying $\Gamma_{\acute{i}} \in \Omega_{J,K}$ if $\acute{i} \in J$ and $\Gamma_{\acute{i}} \in \Omega'_{J,K}$ if $\acute{i} \in K$. Assume that the mapping $G : \mathbb{M}^r \longrightarrow \mathbb{M}$ satisfies conditions (a) to (e), then G has a unique multidimensional FP.*

Example 12. Let $\mathbb{M} = \mathbb{R}$, $\mathbb{M}^r = \mathbb{R}^r$. Define $\sigma^r(\omega, \nu) = \sup_{\acute{i}\in J_r}\{\sigma(\varsigma_{\acute{i}}, \xi_{\acute{i}})\}$ and $\sigma(\varsigma, \xi) = |\varsigma - \xi|$, for all $\omega = (\varsigma_1, \varsigma_2, \cdots, \varsigma_r), \nu = (\xi_1, \xi_2, \cdots, \xi_r) \in \mathbb{M}^r$. Now, define $G : \mathbb{M}^r \longrightarrow \mathbb{M}$, as $G(\varsigma_1, \varsigma_2, \cdots, \varsigma_r) = \varsigma_1/2$.

Since "≤" is a partial order on $\mathbb{M}$, therefore

$$\begin{aligned}&\omega \preccurlyeq_r \nu \text{ if and only if } \varsigma_1 \preccurlyeq_i \xi_i,\\&\varsigma \preccurlyeq_i \xi \text{ iff} \left\{\begin{array}{ll}\varsigma \leq \xi & i \in J = \{1, 2, \cdots, k\}\\ \varsigma \leq \xi & i \in K = \{k+1, k+2, \cdots, r\}\end{array}\right\},\end{aligned} \tag{65}$$

where J and K partitions $J_r = \{1, 2, \cdots, r\}$ such that $J \cup K = J_r$ and $J \cap K = \phi$.

A function $\psi : [0,\infty) \longrightarrow 0,\infty)$ defined as $\psi(t) = (1/3)t$ is increasing and continuous, $\theta : [0,\infty) \longrightarrow [0,\infty)$ defined as $\theta(t) = (1/9)t$, θ is increasing and upper semicontinuous function, and $\varphi : [0,\infty) \longrightarrow [0,\infty)$ defined as $\varphi(t) = (2/9)t$, which is increasing and lower semicontinuous function, for all $t \in [0,\infty)$.

$$\begin{aligned}\sigma(G\omega, G\nu) &= \left|\frac{\varsigma_1 - \xi_1}{2}\right|, \psi(\sigma(G\omega, G\nu))\\&= \left|\frac{\varsigma_1 - \xi_1}{6}\right|, \theta\left(\frac{\sigma^r(\omega, \nu)}{s+1}\right)\\&= \frac{\sigma^r(\omega, \nu)}{9(s+1)}, \varphi\left(\frac{\sigma^r(\omega, \nu)}{s+1}\right) = \frac{2\sigma^r(\omega, \nu)}{9(s+1)}.\end{aligned} \tag{66}$$

Consider

$$\begin{aligned}&\theta\left(\frac{\sigma^r(\omega, \nu)}{s+1}\right) + \varphi\left(\frac{\sigma^r(\omega, \nu)}{s+1}\right)\\&\quad= \frac{\sigma^r(\omega, \nu)}{9(s+1)} + \frac{2\sigma^r(\omega, \nu)}{9(s+1)} = \frac{\sigma^r(\omega, \nu)}{2(s+1)},\\&\quad\leq \frac{\sigma^r(\omega, \nu)}{2} = \frac{\sup_{\acute{i}\in J_r}\{|\varsigma_{\acute{i}} - \xi_{\acute{i}}|\}}{2}.\end{aligned} \tag{67}$$

Thus, we have

$$\left|\frac{\varsigma_1 - \xi_1}{6}\right| \leq \frac{\sup_{\acute{i}\in J_r}\{|\varsigma_{\acute{i}} - \xi_{\acute{i}}|\}}{2}, \tag{68}$$

which follows

$$\psi(\sigma(G\omega, G\nu)) \leq \theta\left(\frac{\sigma^r(\omega, \nu)}{s+1}\right) + \varphi\left(\frac{\sigma^r(\omega, \nu)}{s+1}\right). \tag{69}$$

Now to show G has mixed monotone property, if $\xi \preccurlyeq \eta \Rightarrow \xi \leq \eta$, consider

$$\begin{aligned}G(\varsigma_1, \varsigma_2, \cdots, \varsigma_{i-1}, \xi, \cdots, \varsigma_r) &= \frac{\varsigma_1}{2},\\G(\varsigma_1, \varsigma_2, \cdots, \varsigma_{i-1}, \eta, \cdots, \varsigma_r) &= \frac{\varsigma_1}{2}.\end{aligned} \tag{70}$$

So

$$G(\varsigma_1, \varsigma_2, \cdots, \varsigma_{\acute{i}-1}, \xi, \cdots, \varsigma_r) \preccurlyeq_{\acute{i}} G(\varsigma_1, \varsigma_2, \cdots, \varsigma_{\acute{i}-1}, \eta, \cdots, \varsigma_r), \tag{71}$$

which implies G has mixed monotone property with respect to $\{J, K\}$. Set of mappings is defined as

$$\begin{pmatrix}\Gamma_1(1) & \Gamma_1(2) & \cdots & \Gamma_1(r)\\ \Gamma_2(1) & \Gamma_2(2) & \cdots & \Gamma_2(r)\\ \vdots & \vdots & & \vdots\\ \Gamma_r(1) & \Gamma_r(2) & \cdots & \Gamma_r(r)\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & \cdots & r\\ 2 & 3 & 4 & \cdots & 1\\ 3 & 4 & 5 & \cdots & 2\\ \vdots & \vdots & \vdots & & \vdots\\ r & 1 & 2 & \cdots & r-1\end{pmatrix}. \tag{72}$$

Let $(\varsigma_1^0, \varsigma_2^0, \cdots, \varsigma_r^0) = (\varsigma_1, \varsigma_2, \cdots, \varsigma_r) \in \mathbb{M}^r$ and

$$\begin{aligned}\varsigma_1^\vartheta &= G\left(\varsigma_1^{\vartheta-1}, \varsigma_2^{\vartheta-1}, \cdots, \varsigma_r^{\vartheta-1}\right) = \frac{\varsigma_1^{\vartheta-1}}{2},\\ \varsigma_2^\vartheta &= G\left(\varsigma_2^{\vartheta-1}, \varsigma_3^{\vartheta-1}, \cdots, \varsigma_r^{\vartheta-1}, \varsigma_1^{\vartheta-1}\right) = \frac{\varsigma_2^{\vartheta-1}}{2},\\ &\vdots\\ \varsigma_r^\vartheta &= G\left(\varsigma_r^{\vartheta-1}, \varsigma_1^{\vartheta-1}, \cdots, \varsigma_{r-2}^{\vartheta-1}, \varsigma_{r-1}^{\vartheta-1}\right) = \frac{\varsigma_r^{\vartheta-1}}{2},\end{aligned} \tag{73}$$

for $\vartheta = 1$

$$\begin{aligned}\varsigma_1^1 &= \frac{\varsigma_1^0}{2} = \frac{\varsigma_1}{2},\\ \varsigma_2^1 &= \frac{\varsigma_2^0}{2} = \frac{\varsigma_2}{2},\\ &\vdots\\ \varsigma_r^1 &= \frac{\varsigma_r^0}{2} = \frac{\varsigma_r}{2},\end{aligned} \tag{74}$$

for $\vartheta = 2$

$$\begin{aligned}\varsigma_1^2 &= \frac{\varsigma_1^1}{2} = \frac{\varsigma_1}{2^2},\\ \varsigma_2^2 &= \frac{\varsigma_2^1}{2} = \frac{\varsigma_2}{2^2},\\ &\vdots\\ \varsigma_r^2 &= \frac{\varsigma_r^1}{2} = \frac{\varsigma_r}{2^2}.\end{aligned} \tag{75}$$

Continuing this process, we have

$$\begin{aligned}\lim_{\vartheta\longrightarrow\infty}\left(\varsigma_1^\vartheta, \varsigma_2^\vartheta, \cdots, \varsigma_r^\vartheta\right) &= \lim_{\vartheta\longrightarrow\infty}\left(\frac{\varsigma_1}{2^\vartheta}, \frac{\varsigma_2}{2^\vartheta}, \cdots, \frac{\varsigma_r}{2^\vartheta}\right)\\&= (0, 0, \cdots, 0) = (O_1, \cdots, O_r).\end{aligned} \tag{76}$$

Thus

$$O_i' = G\left(O_{\Gamma_i(1)}, O_{\Gamma_i(2)}, \cdots, O_{\Gamma_i(r)}\right), \tag{77}$$

Hence, O_i' is a unique multidimensional FP of G.

5. An Application

To give the broad impact of FP results to the different areas of study like physics, biological sciences, engineering, game theory, and economics has always been an interest of various mathematicians. The applications can be found in various directions (like [17, 18]). In the following section, we have also included an application of the main result to find the solution of a system of integral equations.

Consider the point $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$. Let the space $\mathbb{M} = C(I)$ of all continuous real valued functions defined on I, which define a partial order $\preccurlyeq$ on $\mathbb{M}$ by

$$\lambda \preccurlyeq \mu \text{ if and only if } \lambda(t) \leq \mu(t), \text{for all } t \in [a, b], \tag{78}$$

and the distance

$$\sigma(\lambda, \mu) = \max_{t\in I} |\lambda(t) - \mu(t)|^p \text{ for all } \lambda, \mu \in \mathbb{M}, p \geq 1, \tag{79}$$

then $(\mathbb{M}, \sigma)$ is a complete partially ordered s-distance space with $s = 2^{p-1}$.

Consider the following system of equations

$$\varsigma_1(t) = \kappa + \int_a^t K(w) L(\varsigma_1(w), \varsigma_2(w), \cdots, \varsigma_r(w)) dw, \tag{80}$$

$$\varsigma_i(t) = \kappa + \int_a^t K(w) L(\varsigma_i(w), \varsigma_{i+1}(w), \cdots, \varsigma_r(w), \cdot \varsigma_1(w), \cdots, \varsigma_{i-1}(w)) dw, \tag{81}$$

for $i = 1, 2, \cdots, r$, where $L : \mathbb{R}^r \longrightarrow \mathbb{R}$ be a mapping verifying

(i) L is continuous

(ii) For all $(e_1, e_2, \cdots, e_r), (h_1, h_2, \cdots, h_r) \in \mathbb{R}^r$

$$|L(e_1, e_2, \cdots, e_r) - L(h_1, h_2, \cdots, h_r)| \leq \frac{\left(\sup_{1\leq i\leq r} |e_i - h_i|^p\right)^{1/p}}{(b-a)(s+1)^{2/p}}, \tag{82}$$

and $K : I \longrightarrow \mathbb{R}$ be a continuous mapping such that $K(t) \geq 0$ and $\int_a^b K(t) dt \leq (b-a)$.

Define a mapping $G : \mathbb{M}^r \longrightarrow \mathbb{M}$ for all $\varsigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_r)$ in $\mathbb{M}^r$ and $\kappa \in \mathbb{R}$ such that

$$G(\varsigma_1, \varsigma_2, \cdots, \varsigma_r)(t) = \kappa + \int_a^t K(w) L(\varsigma_1(w), \varsigma_2(w), \cdots, \varsigma_r(w)) dw. \tag{83}$$

Clearly, $G \in C(I)$. To find solution of the system (79), it is required to show that G has a multiple FP. For this, consider

$$\begin{aligned}
&\sigma(G(\varsigma_1, \varsigma_2, \cdots, \varsigma_r), G(\nu_1, \nu_2, \cdots, \nu_r)) \\
&= \max_{t\in I} |G(\varsigma_1, \varsigma_2, \cdots, \varsigma_r)(t) - G(\nu_1, \nu_2, \cdots, \nu_r)(t)|^p, \\
&= \max_{t\in I} \left| \begin{matrix} \left(\kappa + \int_a^t K(w) L(\varsigma_1(w), \varsigma_2(w), \cdots, \varsigma_r(w)) dw\right) \\ -\left(\kappa + \int_a^t K(w) L(\nu_1(w), \nu_2(w), \cdots, \nu_r(w)) dw\right) \end{matrix} \right|^p, \\
&= \max_{t\in I} \left| \int_a^t K(w) \left(\begin{matrix} L(\varsigma_1(w), \varsigma_2(w), \cdots, \varsigma_r(w)) \\ -L(\nu_1(w), \nu_2(w), \cdots, \nu_r(w)) \end{matrix} \right) dw \right|^p, \\
&\leq \max_{t\in I} \left(\int_a^t |K(w)| \left| \begin{matrix} L(\varsigma_1(w), \varsigma_2(w), \cdots, \varsigma_r(w)), \\ -L(\nu_1(w), \nu_2(w), \cdots, \nu_r(w)) \end{matrix} \right| dw \right)^p, \\
&\leq \max_{t\in I} \left(\int_a^t K(w) \frac{\left(\sup_{1\leq i\leq r} |\varsigma_i(w) - \nu_i(w)|^p\right)^{1/p}}{(b-a)(s+1)^{2/p}} dw \right)^p, \\
&\leq \max_{t\in I} \left(\begin{matrix} \left(\frac{1}{(b-a)(s+1)^{2/p}}\right) \times \\ \sup_{1\leq i\leq r} \int_a^t K(w) \left(\max_{t\in I} |\varsigma_i(w) - \nu_i(w)|^p\right)^{1/p} dw \end{matrix} \right)^p, \\
&= \max_{t\in I} \left(\left(\frac{1}{(b-a)(s+1)^{2/p}}\right) \sup_{1\leq i\leq r} \int_a^t K(w) (\sigma(\varsigma_i, \nu_i))^{1/p} dw \right)^p, \\
&\leq \frac{\sup_{1\leq i\leq r} \{\sigma(\varsigma_i, \nu_i)\}}{(s+1)^2} \left(\frac{1}{(b-a)} \left(\max_{t\in I} \int_a^t K(w) dw \right) \right)^p, \\
&\cdot \sigma(G(\varsigma_1, \varsigma_2, \cdots, \varsigma_r), G(\nu_1, \nu_2, \cdots, \nu_r)) \\
&\leq \frac{\sigma^r((\varsigma_1, \varsigma_2, \cdots, \varsigma_r), (\nu_1, \nu_2, \cdots, \nu_r))}{(s+1)^2}.
\end{aligned} \tag{84}$$

Define functions $\psi, \theta, \varphi : [0,\infty) \longrightarrow [0,\infty)$ by

$$\psi(t) = (s+1)t, \theta(t) = t, \varphi(t) = 0, \tag{85}$$

such that for $t > 0$

$$\psi(t) - \theta(t) - \varphi(t) > 0. \tag{86}$$

Then, from above inequality, we get

$$\psi(\sigma(G\varsigma, G\nu)) \leq \theta\left(\frac{\sigma^r(\varsigma, \nu)}{s+1}\right) + \varphi\left(\frac{\sigma^r(\varsigma, \nu)}{s+1}\right), \tag{87}$$

which by Theorem 7 implies that G has a unique multiple FP, which gives the required solution of system (79).

6. Conclusion

The main idea of this paper was to prove a multiple fixed point (FP) result for partially ordered s-distance spaces under (θ, ϕ, ψ)-type weak contractive condition which is the generalization of some well-known results in literature. Many other related and relevant results can be obtained in the same manner for partially ordered generalized distance spaces such as C-distance space, balanced distance space, and (s, q) distance space.

Data Availability

No data used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

New Results on $\mathcal{I}_2$-Statistically Limit Points and $\mathcal{I}_2$-Statistically Cluster Points of Sequences of Fuzzy Numbers

Ö. Kişi,[1] **M. B. Huban**,[2] **and M. Gürdal**[3]

[1]*Department of Mathematics, Faculty of Science, Bartin University, Bartin, Turkey*
[2]*Isparta University of Applied Sciences, Isparta, Turkey*
[3]*Department of Mathematics, Suleyman Demirel University, 32260 Isparta, Turkey*

Correspondence should be addressed to M. B. Huban; muallahuban@isparta.edu.tr

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In this paper, some existing theories on convergence of fuzzy number sequences are extended to $\mathcal{I}_2$-statistical convergence of fuzzy number sequence. Also, we broaden the notions of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points of a sequence of fuzzy numbers to $\mathcal{I}_2$-statistical limit points and $\mathcal{I}_2$-statistical cluster points of a double sequence of fuzzy numbers. Also, the researchers focus on important fundamental features of the set of all $\mathcal{I}_2$-statistical cluster points and the set of all $\mathcal{I}_2$-statistical limit points of a double sequence of fuzzy numbers and examine the relationship between them.

1. Introduction

The theory of statistical convergence reverts to the first edition of monograph of Zygmund [1]. Statistical convergence of number sequences was given by Fast [2] and then was reissued by Schoenberg [3] independently for real and complex sequences. This conception was studied for the double sequences by Mursaleen and Edely [4]. Fridy [5] considered statistical limit points and statistical cluster points of real number sequences. When we focus on the statistical convergence in the literature, we meet Fridy [6], Temizsu and Mikail [7], Braha et al. [8], Nuray and Ruckle [9], Das et al. [10], and so many other researchers (see [11–13]).

The concept of ideal convergence was given by Kostyrko et al. [14] which generalizes and combines different concepts of convergence of sequences containing usual convergence and statistical convergence. Das et al. [15] presented the concept of $\mathcal{I}$-convergence of double sequences in a metric space. In [16], Savas and Das extended the conception of ideal convergence as studied by Kostyrko et al. [14] to $\mathcal{I}$-statistical convergence and examined remarkable basic features of it. For different studies on these topics, we refer to [17–23].

The theory of fuzzy sets was firstly given by Zadeh [24]. Matloka [25] identified the convergence of a sequence of fuzzy numbers. Nanda [26] worked on the sequences of fuzzy numbers and displayed that the set of all convergent sequences of fuzzy numbers generates a complete metric space. Nuray and Savas [27] generalized ordinary convergence and defined statistically Cauchy and statistical convergent sequences of fuzzy number. Later on, it was studied and advanced by Aytar and Pehlivan [28] and many others. Aytar [29] worked on the conception of statistical limit points and cluster points for sequences of fuzzy number. Kumar et al. [30, 31] worked $\mathcal{I}$-convergence, $\mathcal{I}$-limit points, and $\mathcal{I}$-cluster point for sequence of fuzzy numbers. The concepts of $\mathcal{I}$-statistically convergence for sequences of fuzzy numbers were established by Debnath and Debnath [32]. Later on, $\mathcal{I}$-statistically limit points and $\mathcal{I}$-statistically cluster points of sequences of fuzzy numbers were studied by Tripathy et al. [33].

In this paper, we examine some essential features of $\mathcal{I}_2$-statistically convergent sequence of fuzzy numbers and describe $\mathcal{I}_2$-statistical limit point and $\mathcal{I}_2$-statistical cluster point for fuzzy number sequences.

2. Preliminaries

First, we emphasize some properties of double sequences which are not satisfied by a (single) sequence. This provides a proper motivation for studying double sequences.

The essential deficiency of this kind of convergence is that a convergent sequence does not require to be bounded. Hardy [34] defined the concept of regular sense, which does not have this shortcoming, for double sequence. In regular convergence, both the row-index and the column-index of the double sequence need to be convergent besides the convergent in Pringsheim's sense.

The notion of Cesàro summable double sequences was described by [35]. Note that if a bounded sequence (x_{mn}) is statistically convergent then it is also Cesàro summable but not contrariwise.

Let $(x_{mn}) = (-1)^m, \forall n$; then, $\lim_{p,r}\sum_{m=1}^{p}\sum_{n=1}^{r}x_{mn} = 0$, but apparently x is not statistically convergent.

The convergence of double sequences plays a significant part not only in pure mathematics but also in other subjects of science including computer science, biological science, and dynamical systems, as well. Also, the double sequence can be used in convergence of double trigonometric series and in the opening series of double functions and in the making differential solution.

Now, we remember some notions and fundamental definitions required in this study.

We signify by $\mathscr{D}$ the set of all bounded and closed intervals on $\mathbb{R}$, i.e.,

$$\mathscr{D} = \left\{M \subset \mathbb{R} : M = \left[\underline{M}, \bar{M}\right]\right\}. \tag{1}$$

For $M, N \in \mathscr{D}$, we describe $M \leq N$ iff $\underline{M} \leq \underline{N}$ and $\bar{M} \leq \bar{N}$ and $\bar{d} = \max\{|\underline{M} - \underline{N}|, |\bar{M} - \bar{N}|\}$. $(\mathscr{D}, \bar{d})$ forms a complete metric space.

Definition 1. A fuzzy number is a function X from $\mathbb{R}$ to $[0, 1]$, which satisfies the subsequent conditions:

(i) X is normal

(ii) X is fuzzy convex

(iii) X is upper semicontinuous

(iv) The closure of the set $\{x \in \mathbb{R} : X(x) > 0\}$ is compact

The features (i)-(iv) give that for each $\alpha \in [0, 1]$, the α-level set,

$$X^\alpha = \{x \in \mathbb{R} : X(x) \geq \alpha\} = \left[\underline{X}^\alpha, \bar{X}^\alpha\right], \tag{2}$$

is a nonempty compact convex subset of $\mathbb{R}$. The 0-level set is the class of the strong 0-cut, i.e., $cl(\{x \in \mathbb{R} : X(x) \geq 0\})$. The set of all fuzzy numbers is indicated by $L(\mathbb{R})$. Consider a map $\bar{d}(X, Y) = \sup_{\alpha\in[0,1]} d(X^\alpha, Y^\alpha)$. $(L(\mathbb{R}), \bar{d})$ also forms a complete metric space [36].

Definition 2 (see [25]). A sequence (X_k) of fuzzy numbers is named to be convergent to a fuzzy number X_0 if for each $\varepsilon > 0$ there is $m > 0$ such that $\bar{d}(X_k, X_0) < \varepsilon$ for every $k \geq m$. We write $\lim_{k\longrightarrow\infty} X_k = X_0$.

Definition 3 (see [25]). A fuzzy number X_0 is known as a limit point of a sequence of fuzzy number (X_k) on condition that there is a subsequence of (X_k) that converges to X_0.

Definition 4 (see [27]). A sequence (X_k) of fuzzy numbers is named to be statistically convergent to a fuzzy number X_0 if for each $\varepsilon > 0$ the set

$$A(\varepsilon) = \left\{k \in \mathbb{N} : \bar{d}(X_k, X_0) \geq \varepsilon\right\}, \tag{3}$$

has natural density zero. We write $St - \lim_{k\longrightarrow\infty} X_k = X_0$.

Definition 5 (see [30]). Take $\mathscr{I}$ as a nontrivial ideal. A sequence (X_k) of fuzzy numbers is known as $\mathscr{I}$-convergent to a fuzzy number X_0 provided that each $\xi > 0$

$$A(\xi) = \left\{k \in \mathbb{N} : \bar{d}(X_k, X_0) \geq \xi\right\} \in \mathscr{I}. \tag{4}$$

We write $\mathscr{I} - \lim_{k\longrightarrow\infty} X_k = X_0$.

Definition 6 (see [31]). A fuzzy number X_0 is known as ideal limit point of a sequence of fuzzy number (X_k) provided that there is a subset $M = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ such that $M \notin \mathscr{I}$ and $\lim X_{k_n} = X_0$.

Definition 7 (see [31]). A fuzzy number X_0 is known as ideal cluster point of a sequence of fuzzy number (X_k) provided that for each $\xi > 0$ the set $\{k \in \mathbb{N} : \bar{d}(X_k, X_0) < \xi\} \notin \mathscr{I}$.

The set of all $\mathscr{I}$-limit points and $\mathscr{I}$-cluster points of the sequence X is shown by $\mathscr{I}(\Lambda_X)$ and $\mathscr{I}(\Gamma_X)$, respectively.

Natural density of a subset K of $\mathbb{N} \times \mathbb{N}$ is demonstrated by

$$d(K) = \lim_{m,n\longrightarrow\infty} \frac{K(m, n)}{m.n}, \tag{5}$$

where $K(m, n) = |\{(j, k) \in \mathbb{N} \times \mathbb{N} : j \leq m, k \leq n\}|$.

A nontrivial ideal $\mathscr{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is known as strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathscr{I}_2$ for each $i \in \mathbb{N}$.

It is the proof that a strongly admissible ideal is admissible also.

Throughout the paper, we consider $\mathscr{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

3. Main Results

In this study, the researchers focus on remarkable features of the set of all $\mathscr{I}_2$-statistical cluster points and the set of all $\mathscr{I}_2$-statistical limit points of fuzzy number sequences. We examine interrelationship between them.

Theorem 8. *If (X_{kl}) be a double sequence of fuzzy numbers such that $\mathcal{I}_2 - st\lim X_{kl} = X_0$, then, X_0 identified uniquely.*

Proof. Presume that $\mathcal{I}_2 - st\lim X_{kl} = X_0$ and $\mathcal{I}_2 - st\lim X_{kl} = Y_0$, where $X_0 \neq Y_0$. For any $\varepsilon, \delta > 0$, we get

$$K_1 = \left\{(s,w) \in \mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}_2), \tag{6}$$

$$K_2 = \left\{(s,w) \in \mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, Y_0) \ge \varepsilon\right\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}_2). \tag{7}$$

Therefore, $K_1 \cap K_2 \neq \varnothing$, since $K_1 \cap K_2 \in \mathcal{F}(\mathcal{I}_2)$. Let $(i, j) \in K_1 \cap K_2$ and take $\varepsilon := \bar{d}(X_0, Y_0)/3 > 0$ such that we have

$$\frac{1}{ij}\left|\left\{k \le i, l \le j : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| < \delta, \tag{8}$$

and it goes along with

$$\frac{1}{ij}\left|\left\{k \le i, l \le j : \bar{d}(X_{kl}, Y_0) \ge \varepsilon\right\}\right| < \delta, \tag{9}$$

i.e., for maximum $k \le i$, $l \le j$, we have $\bar{d}(X_{kl}, X_0) < \varepsilon$ and $\bar{d}(X_{kl}, Y_0) < \varepsilon$ for a very small $\delta > 0$. Thus, we have to acquire

$$\left\{k \le i, l \le j : \bar{d}(X_{kl}, X_0) < \varepsilon\right\} \cap \left\{k \le i, l \le j : \bar{d}(X_{kl}, Y_0) < \varepsilon\right\} \neq \varnothing, \tag{10}$$

a contradiction, as the nbd of X_0 and Y_0 are disjoint. Hence, X_0 is uniquely identified. □

Theorem 9. *Let (X_{kl}) be a fuzzy numbers sequence then $st\lim X_{kl} = X_0$ implies $\mathcal{I}_2 - st\lim X_{kl} = X_0$.*

Proof. Let $st\lim X_{kl} = X_0$. Then, for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}, \tag{11}$$

has natural density zero, i.e.,

$$\lim_{s,w\longrightarrow\infty} \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| = 0. \tag{12}$$

Therefore, for every $\varepsilon > 0$ and $\delta > 0$,

$$T(\varepsilon,\delta) = \left\{(s,w) \in \mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| \ge \delta\right\}, \tag{13}$$

is a finite set and so $T(\varepsilon, \delta) \in \mathcal{I}_2$, where $\mathcal{I}_2$ is an admissible ideal. Hence, we get $\mathcal{I}_2 - st\lim X_{kl} = X_0$. □

Theorem 10. *Take (X_{kl}) as a sequence of fuzzy numbers. $\mathcal{I}_2 - \lim X_{kl} = X_0$ implies $\mathcal{I}_2 - st\lim X_{kl} = X_0$.*

Proof. The proof of this theorem is clear. □

But the reverse is not true. For instance, take for $\mathcal{I}_2$ the class $\mathcal{I}_2^f$ of all finite subsets of $\mathbb{N}\times\mathbb{N}$, the fuzzy number (X_{kl}), where

$$X_{kl}(p) := \begin{cases} \dfrac{n+m+p}{n+m}, & -n-m \le p \le 0, \\ \dfrac{n+m-p}{n+m} & 0 \le p \le n+m, \end{cases} \tag{14}$$

for $k = n^2, l = m^2,\ n, m \in \mathbb{N}$, and

$$X_{kl}(p) := \begin{cases} 1 + pnm, & -\dfrac{1}{nm} \le p \le 0, \\ 1 - pnm & 0 \le p \le \dfrac{1}{nm}, \end{cases} \tag{15}$$

for $k \neq n^2, l \neq m^2,\ n, m \in \mathbb{N}$. Then, (X_{kl}) is $\mathcal{I}_2$-statistically convergent, but not $\mathcal{I}_2$-convergent.

Theorem 11. *Let (X_{kl}) and (Y_{kl}) be two fuzzy numbers sequence. Then,*

(i) $\mathcal{I}_2 - st\lim X_{kl} = X_0$, $c \in \mathbb{R}$ implies $\mathcal{I}_2 - st\lim cX_{kl} = cX_0$

(ii) $\mathcal{I}_2 - st\lim X_{kl} = X_0$, $\mathcal{I}_2 - st\lim Y_{kl} = Y_0$ implies $\mathcal{I}_2 - st\lim (X_{kl} + Y_{kl}) = X_0 + Y_0$

Proof. (i) For $c = 0$, there is nothing to prove. So, presume that $c \neq 0$. Now

$$\begin{aligned} &\frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(cX_{kl}, cX_0) \ge \varepsilon\right\}\right| \\ &\quad = \frac{1}{sw}\left|\left\{k \le s, l \le w : |c|\bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| \\ &\quad \le \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \frac{\varepsilon}{|c|}\right\}\right| < \delta. \end{aligned} \tag{16}$$

Therefore, we obtain

$$\left\{(s,w) \in \mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(cX_{kl}, cX_0) \ge \varepsilon\right\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}_2), \tag{17}$$

i.e., $\mathcal{I}_2 - st\lim cX_{kl} = cX_0$.

(ii) We have

$$K_1 = \left\{(s,w) \in \mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \frac{\varepsilon}{2}\right\}\right| < \frac{\delta}{2}\right\} \in \mathcal{F}(\mathcal{I}_2), \tag{18}$$

$$K_2 = \left\{(s,w) \in \mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(Y_{kl}, Y_0) \ge \frac{\varepsilon}{2}\right\}\right| < \frac{\delta}{2}\right\} \in \mathcal{F}(\mathcal{I}_2). \tag{19}$$

Since, $K_1 \cap K_2 \neq \emptyset$, therefore, for all $(s,w) \in K_1 \cap K_2$, we get

$$\begin{aligned} &\frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}+Y_{kl}, X_0+Y_0) \ge \varepsilon\right\}\right| \\ &\quad \le \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \frac{\varepsilon}{2}\right\}\right| \\ &\qquad + \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(Y_{kl}, Y_0) \ge \frac{\varepsilon}{2}\right\}\right| < \delta, \end{aligned} \tag{20}$$

i.e.,

$$\left\{(s,w) \in \mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}+Y_{kl}, X_0+Y_0) \ge \varepsilon\right\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}_2). \tag{21}$$

Hence, we have $\mathcal{I}_2 - st\lim(X_{kl}+Y_{kl}) = X_0 + Y_0$. □

Definition 12. An element $X_0 \in L(\mathbb{N})$ is called to be an $\mathcal{I}_2$-statistical limit point of a fuzzy number sequence $X = (X_{kl})$ provided that for each $\varepsilon > 0$ there is a set

$$M = \{(k_1,l_1) < (k_2,l_2) < \cdots < (k_r,l_s) < \cdots\} \subset \mathbb{N}\times\mathbb{N}, \tag{22}$$

such that $M \notin \mathcal{I}_2$ and $st - \lim X_{k_r,l_s} = X_0$.

$\mathcal{I}_2 - S(\Lambda_X)$ indicates the set of all $\mathcal{I}_2$-statistical limit point of a fuzzy number sequence (X_{kl}).

Theorem 13. *Take (X_{kl}) as a sequence of fuzzy numbers. If $\mathcal{I}_2 - st\lim X_{kl} = X_0$, then $\mathcal{I}_2 - S(\Lambda_X) = \{X_0\}$.*

Proof. Since $\mathcal{I}_2 - st\lim X_{kl} = X_0$, for each $\varepsilon, \delta > 0$, the set

$$K = \left\{(s,w) \in \mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| \ge \delta\right\} \in \mathcal{I}_2, \tag{23}$$

where $\mathcal{I}_2$ is an admissible ideal.

Assume that $\mathcal{I}_2 - S(\Lambda_X)$ involves Y_0 different from X_0, i.e., $Y_0 \in \mathcal{I}_2 - S(\Lambda_X)$. So, there is a $M \subset \mathbb{N}\times\mathbb{N}$ such that $M \notin \mathcal{I}_2$ and $st - \lim X_{k_r,l_s} = Y_0$.

Let

$$P = \left\{(s,w) \in M : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, Y_0) \ge \varepsilon\right\}\right| \ge \delta\right\}. \tag{24}$$

So P is a finite set and therefore $P \in \mathcal{I}_2$. So

$$P^c = \left\{(s,w) \in M : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, Y_0) \ge \varepsilon\right\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}_2). \tag{25}$$

Again let

$$K_1 = \left\{(s,w) \in M : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| \ge \delta\right\}. \tag{26}$$

So $K_1 \subset K \in \mathcal{I}_2$, i.e., $K_1^c \in \mathcal{F}(\mathcal{I}_2)$. Therefore, $K_1^c \cap P^c \neq \emptyset$, since $K_1^c \cap P^c \in \mathcal{F}(\mathcal{I}_2)$.

Let $(i,j) \in K_1^c \cap P^c$ and take $\varepsilon := \bar{d}(X_0, Y_0)/3 > 0$, so

$$\frac{1}{ij}\left|\left\{k \le i, l \le j : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| < \delta \quad \text{and,} \tag{27}$$

$$\frac{1}{ij}\left|\left\{k \le i, l \le j : \bar{d}(X_{kl}, Y_0) \ge \varepsilon\right\}\right| < \delta, \tag{28}$$

i.e., for maximum $k \le i, l \le j$ will satisfy $\bar{d}(X_{kl}, X_0) < \varepsilon$ and $\bar{d}(X_{kl}, Y_0) < \varepsilon$ for a very small $\delta > 0$. Thus, we have to obtain

$$\left\{k \le i, l \le j : \bar{d}(X_{kl}, X_0) < \varepsilon\right\} \cap \left\{k \le i, l \le j : \bar{d}(X_{kl}, Y_0) < \varepsilon\right\} \neq \emptyset, \tag{29}$$

a contradiction, as the nbd of X_0 and Y_0 are disjoint. Hence, $\mathcal{I}_2 - S(\Lambda_X) = \{X_0\}$. □

Definition 14. An element $X_0 \in L(\mathbb{R})$ is known as $\mathcal{I}_2$-statistical cluster point of a fuzzy number sequence $X = (X_{kl})$ if for each $\varepsilon > 0$ and $\delta > 0$, the set

$$\left\{(s,w) \in \mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| < \delta\right\} \notin \mathcal{I}_2. \tag{30}$$

$\mathcal{I}_2 - S(\Gamma_X)$ demonstrates the set of all $\mathcal{I}_2$-statistical cluster point of a fuzzy number sequence (X_{kl}).

Theorem 15. *For any sequence (X_{kl}) of fuzzy numbers $\mathcal{I}_2 - S(\Gamma_X)$ is closed.*

Proof. Let the fuzzy number Y_0 be a limit point of the set $\mathcal{I}_2 - S(\Gamma_X)$. Then, for any $\varepsilon > 0$,

$$\mathcal{I}_2 - S(\Gamma_X) \cap B(Y_0, \varepsilon) \neq \emptyset, \tag{31}$$

where

$$B(Y_0, \varepsilon) = \left\{W \in L(\mathbb{R}) : \bar{d}(W, Y_0) < \varepsilon\right\}. \tag{32}$$

Let $Z_0 \in \mathcal{I}_2 - S(\Gamma_X) \cap B(Y_0, \varepsilon)$ and select $\varepsilon_1 > 0$ such that $B(Z_0, \varepsilon_1) \subseteq B(Y_0, \varepsilon)$. Then, we get

$$\left\{k \le s, l \le w : \bar{d}(X_{kl}, Z_0) \ge \varepsilon_1\right\} \supseteq \left\{k \le s, l \le w : \bar{d}(X_{kl}, Y_0) \ge \varepsilon\right\}, \tag{33}$$

which implies that

$$\frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, Z_0) \ge \varepsilon_1\right\}\right| \ge \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, Y_0) \ge \varepsilon\right\}\right|. \tag{34}$$

Now, for any $\delta > 0$, we obtain

$$\left\{(s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, Z_0) \ge \varepsilon_1\right\}\right| < \delta\right\}, \tag{35}$$

$$\subseteq \left\{(s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, Y_0) \ge \varepsilon\right\}\right| < \delta\right\}. \tag{36}$$

Since $Z_0 \in \mathcal{I}_2 - S(\Gamma_X)$, we have

$$\left\{(s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, Y_0) \ge \varepsilon\right\}\right| < \delta\right\} \notin \mathcal{I}_2, \tag{37}$$

i.e., $Y_0 \in \mathcal{I}_2 - S(\Gamma_X)$. This concludes the proof. □

Theorem 16. *For any fuzzy number sequence* (X_{kl}),

$$\mathcal{I}_2 - S(\Lambda_X) \subseteq \mathcal{I}_2 - S(\Gamma_X). \tag{38}$$

Proof. Let $X_0 \in \mathcal{I}_2 - S(\Lambda_X)$. In that case, there is a set

$$M = \{(k_1, l_1) < (k_2, l_2) < \cdots < (k_r, l_s) < \cdots\} \notin \mathcal{I}_2, \tag{39}$$

such that $st - \lim X_{k_r, l_s} = X_0$. So, we have

$$\lim_{k,l \longrightarrow \infty} \frac{1}{kl}\left|\left\{k_r \le k, l_s \le l : \bar{d}\left(X_{k_r l_s}, X_0\right) \ge \varepsilon\right\}\right| = 0. \tag{40}$$

Take $\delta > 0$, so there is $n_0 \in \mathbb{N}$ such that for $s, w > n_0$, we have

$$\frac{1}{sw}\left|\left\{k_r \le s, l_s \le w : \bar{d}\left(X_{k_r l_s}, X_0\right) \ge \varepsilon\right\}\right| < \delta. \tag{41}$$

Let

$$K = \left\{(s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw}\left|\left\{k_r \le s, l_s \le w : \bar{d}\left(X_{k_r l_s}, X_0\right) \ge \varepsilon\right\}\right| < \delta\right\}. \tag{42}$$

Also, we have

$$K \supset M \setminus \left\{(k_1, l_1), \left((k_2, l_2), \cdots, \left(k_{n_0}, l_{n_0}\right)\right)\right\}. \tag{43}$$

Considering that $\mathcal{I}_2$ is an admissible ideal and $M \notin \mathcal{I}_2$, therefore, $K \notin \mathcal{I}_2$. Hence, according to the definition of $\mathcal{I}_2$-statistical cluster point $X_0 \in \mathcal{I}_2 - S(\Gamma_X)$, this finalizes the proof. □

Theorem 17. *If* (X_{kl}) *and* (Y_{kl}) *are two sequences of fuzzy numbers such that*

$$\{(k, l) \in \mathbb{N} \times \mathbb{N} : X_{kl} \ne Y_{kl}\} \in \mathcal{I}_2, \tag{44}$$

then

$$\mathcal{I}_2 - S(\Lambda_X) = \mathcal{I}_2 - S(\Lambda_Y). \tag{45}$$

$$\mathcal{I}_2 - S(\Gamma_X) = \mathcal{I}_2 - S(\Gamma_Y). \tag{46}$$

Proof. (i) Let $X_0 \in \mathcal{I}_2 - S(\Lambda_X)$. So, according to the definition, there is a set

$$M = \{(k_1, l_1) < (k_2, l_2) < \cdots < (k_r, l_s) < \cdots\} \subset \mathbb{N} \times \mathbb{N}, \tag{47}$$

such that $M \notin \mathcal{I}_2$ and $st - \lim X_{k_r, l_s} = X_0$. Since

$$\{(k, l) \in M : X_{kl} \ne Y_{kl}\} \subseteq \{(k, l) \in \mathbb{N} \times \mathbb{N} : X_{kl} \ne Y_{kl}\} \in \mathcal{I}_2, \tag{48}$$

$$M' = \{(k, l) \in M : X_{kl} = Y_{kl}\} \notin \mathcal{I}_2 \quad \text{and} \quad M' \subseteq M. \tag{49}$$

So, we have $st - \lim Y_{k'_r, l'_s} = X_0$. This denotes that $X_0 \in \mathcal{I}_2 - S(\Lambda_Y)$ and therefore $\mathcal{I}_2 - S(\Lambda_X) \subseteq \mathcal{I}_2 - S(\Lambda_Y)$. By symmetry, $\mathcal{I}_2 - S(\Lambda_Y) \subseteq \mathcal{I}_2 - S(\Lambda_X)$. Hence, we obtain $\mathcal{I}_2 - S(\Lambda_X) = \mathcal{I}_2 - S(\Lambda_Y)$.

(ii) Let $X_0 \in \mathcal{I}_2 - S(\Gamma_X)$. So, by the definition for each $\varepsilon > 0$, we have

$$K = \left\{(s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| < \delta\right\} \notin \mathcal{I}_2. \tag{50}$$

Let

$$L = \left\{(s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(Y_{kl}, X_0) \ge \varepsilon\right\}\right| < \delta\right\}. \tag{51}$$

We have to prove that $L \notin \mathcal{I}_2$. Presume that $L \in \mathcal{I}_2$, So

$$L^c = \left\{(s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(Y_{kl}, X_0) \ge \varepsilon\right\}\right| \ge \delta\right\} \in \mathcal{F}(\mathcal{I}_2). \tag{52}$$

By hypothesis,

$$P = \{(k, l) \in \mathbb{N} \times \mathbb{N} : X_{kl} = Y_{kl}\} \in \mathcal{F}(\mathcal{I}_2). \tag{53}$$

Therefore, $L^c \cap P \in \mathcal{F}(\mathcal{I}_2)$. Also, it is clear that $L^c \cap P \subseteq K^c \in \mathcal{F}(\mathcal{I}_2)$, i.e., $K \in \mathcal{I}_2$, this is a contradiction. Therefore, $L \notin \mathcal{I}_2$ and thus the desired result was achieved. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

A New Application of the Sumudu Transform for the Falling Body Problem

Esra Karatas Akgül[1] **Ali Akgül**[1] **and Rubayyi T. Alqahtani**[2]

[1]Siirt University, Art and Science Faculty, Department of Mathematics, TR-56100 Siirt, Turkey
[2]Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia

Correspondence should be addressed to Ali Akgül; aliakgul00727@gmail.com

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In this study, we investigate the falling body problem with three different fractional derivatives. We acquire the solutions of the model by the Sumudu transform. We show the accuracy of the Sumudu transform by some theoretic results and implementations.

1. Introduction

The Sumudu transform is an important integral transformation. This transformation has been begun with Watugala [1] who has presented a new integral transform to solve differential equations and control engineering problems. Weerakoon [2, 3] has made a big contribution with the application of the Sumudu transform to partial differential equations and complex inversion formula for the Sumudu transform. Then, this transformation has attracted great attention, and lots of work have been done related to this transformation by the authors. Some of them are as follows: Belgacem et al. [4] have searched the analytical investigations of the Sumudu transform. Fundamental properties of the Sumudu transform have been studied by Belgacem and Karaballi [5]. Atangana and Akgül [6] have obtained the transfer function and Bode diagram by the Sumudu transform.

Fractional differential equations have taken much interest recently. Arqub and Maayah [7, 8] have studied a fitted fractional reproducing kernel algorithm for the numerical solutions of ABC-fractional Volterra integro-differential equations and solution of the fractional epidemic model by the homotopy analysis method. Jangid et al. [9] have investigated some fractional calculus findings associated with the incomplete functions. Some new fractional-calculus connections between Mittag-Leffler functions have been studied by Srivastava et al. [10]. Singh et al. [11] have investigated the fractional epidemiological model for computer viruses. Ghanbari and Atangana [12, 13] have presented an efficient numerical approach for fractional diffusion partial differential equations and a new application of fractional Atangana-Baleanu derivatives. Abdeljawad et al. [14, 15] have investigated an efficient sustainable algorithm for numerical solutions of systems of fractional-order differential equations by the Haar wavelet collocation method and more general fractional integration by part formulae and applications. For more details, see [16–25].

In this study, we examine the falling body problem depending on Newton's second law that represents that the acceleration of a particle relied on the mass of the particle and the net force action on the particle. Take into consideration an object of mass m falling through the air from a height h with velocity $v(0)$ in a gravitational field. If we use Newton's second law, we acquire

$$\frac{mdv}{dt} + mkv = -mg, \tag{1}$$

where k is the positive constant rate and g expresses the gravitational constant. The solution of this equation is presented as [26]

$$v(t) = -\frac{g}{k} + \exp(-kt)\left(v(0) + \frac{g}{k}\right), \quad (2)$$

and by integrating for $z(0) = h$, we get

$$z(t) = h - \frac{gt}{k} + \frac{1}{k}(1 - \exp(-kt))\left(v(0) + \frac{g}{k}\right). \quad (3)$$

Considering all the information presented above, we organize this study as follows. In Section 2, some fundamental definitions and lemmas about nonlocal fractional calculus are given. In Section 3, the fractional falling body problem is investigated by means of Caputo, Caputo-Fabrizio sense of Caputo, and ABC. Also, some outstanding consequences are clarified in Section 4.

2. Preliminaries

After giving some introduction, we want to present some significant definition and lemma properties of fractional calculus for setting up a mathematically sound theory that will serve the purpose of the current study.

Definition 1. Over

$$B = \left\{ v(t) | \exists N, \eta_1, \eta_2 > 0, |v(t)| < N \exp\left(|t|/\eta_j, \text{if } t \in (-1)^j \times [0,\infty)\right\}, \quad (4)$$

the Sumudu transform is identified by [17]

$$V(u) = S[v(t)] = \int_0^\infty f(ut)\exp(-t)dt, \quad u \in (-\eta_1, \eta_2). \quad (5)$$

Definition 2. We define the classical Mittag-Leffler function $E_\alpha(z)$ as [27]

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + 1)} (z \in \mathbb{C}, \mathrm{Re}(\alpha) > 0), \quad (6)$$

and the Mittag-Leffler kernel with two parameters is presented by [27]

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)} (z, \beta \in \mathbb{C}, \mathrm{Re}(\alpha) > 0). \quad (7)$$

Definition 3. The generalized Mittag-Leffler function is defined by [28]

$$E^{\rho}_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m (\rho)_m}{\Gamma(\alpha m + \beta) m!} (z, \beta, \rho, \alpha \in \mathbb{C}, \mathrm{Re}(\alpha) > 0), \quad (8)$$

where $(\rho)_m = \rho(\rho + 1) \cdots (\rho + m - 1)$ is the Pochhammer symbol introduced by Prabhakar. As seen clearly, $(1)_m = m$! and $E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$.

Definition 4. Let $\zeta, \psi : [0,\infty) \longrightarrow \mathfrak{R}$; then, the convolution of ζ, ψ is given as [27]

$$(\zeta * \psi) = \int_0^t \zeta(t-u)\psi(u)du, \quad (9)$$

and assume that $\zeta, \psi : [0,\infty) \longrightarrow \mathfrak{R}$; then, we have

$$S\{(\zeta * \psi)(t)\} = uS\{\zeta(t)\}S\{\psi(t)\}. \quad (10)$$

Definition 5. Let $v : [0,\infty] \longrightarrow \mathbb{R}$ be a smooth function. Then, the Caputo fractional derivative is defined as follows [18]:

$${}_0^C D^\alpha v(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-z)^{n-\alpha-1} v^{(n)}(z)dz, \quad (11)$$

where $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ and $n = [\mathrm{Re}(\alpha)] + 1$.

Lemma 6. *The Sumudu transform of the Caputo fractional derivative is presented by [6]*

$$S\left[{}_0^C D_t^\alpha v(t)\right] = \frac{V[u] - v(0)}{u^\alpha}, \quad (12)$$

where $V[u] = S[v(t)]$.

Definition 7. We identify the Caputo-Fabrizio fractional derivative as [29]

$${}_0^{\mathrm{CFC}} D^\alpha v(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t v'(z)\exp(-\lambda(t-z))dz. \quad (13)$$

Lemma 8. *The Sumudu transform of the CFC fractional derivative is acquired by [6]*

$$S\left[{}_0^{CFC} D_t^\alpha v(t)\right] = \frac{M(\alpha)}{(1-\alpha)} \frac{V[u]}{(1 + (\alpha/(1-\alpha))u)} - \frac{M(\alpha)}{(1-\alpha)} \frac{v(0)}{(1 + (\alpha/(1-\alpha))u)}. \quad (14)$$

Definition 9. We describe the Atangana-Baleanu fractional derivative by [27]

$${}_0^{\mathrm{ABC}} D^\alpha v(t) = \frac{\mathrm{AB}(\alpha)}{1-\alpha} \int_0^t v'(z) E_\alpha(-\lambda(t-z)^\alpha)dz. \quad (15)$$

Lemma 10. *The Sumudu transform of the ABC fractional derivative is acquired by [6]*

$$S\left[{}_0^{ABC} D_t^\alpha v(t)\right] = \frac{AB(\alpha)}{(1-\alpha)} \frac{V[u]}{(1 + (\alpha/(1-\alpha))u^\alpha)} - \frac{AB(\alpha)}{(1-\alpha)} \frac{v(0)}{(1 + (\alpha/(1-\alpha))u^\alpha)}. \quad (16)$$

Definition 11. The generalized fractional integral is given by [28]

$${}_0 I^{\alpha,\beta} v(t) = \frac{1}{\Gamma(\alpha)\beta^{\alpha-1}} \int_0^t \left(t^\beta - z^\beta\right)^{\alpha-1} v(z) z^{\beta-1} dz. \quad (17)$$

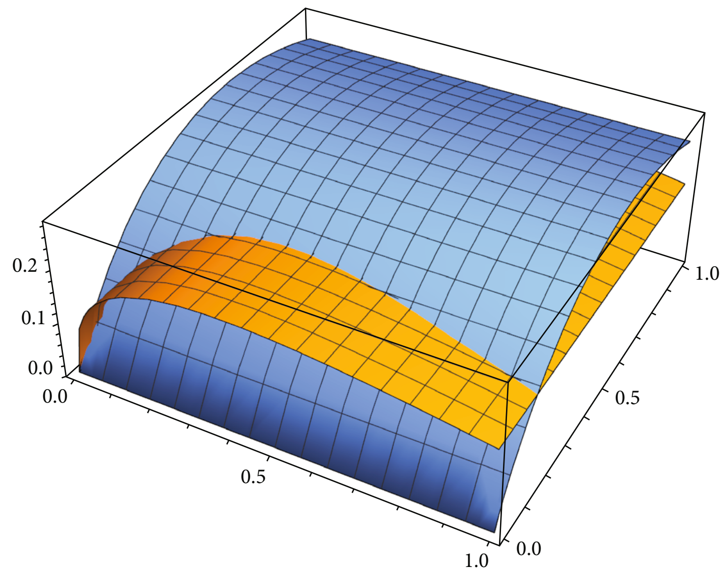

FIGURE 1: Simulations of the solution for $\alpha = 0.5$ and $\alpha = 0.9$.

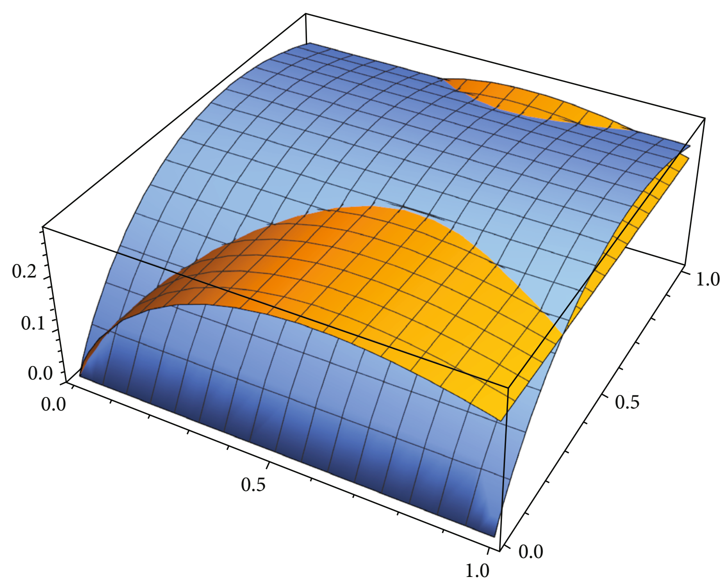

FIGURE 2: Simulations of the solution for $\alpha = 0.8$ and $\alpha = 0.9$.

Definition 12. The generalized fractional derivatives in the Caputo sense are defined, respectively, by [28]

$$
\begin{aligned}
{}_0^C D^{\alpha,\beta} v(t) &= {}_0 I^{n-\alpha,\beta} \left(t^{1-\beta} \frac{d}{dt} \right)^n v(t) \\
&= \frac{1}{\Gamma(n-\alpha)\beta^{n-\alpha-1}} \int_0^t \left(t^\beta - z^\beta \right)^{n-\alpha-1} \left(t^{1-\beta} \frac{d}{dt} \right)^n v(z) z^{\beta-1} dz.
\end{aligned} \tag{18}
$$

Lemma 13. *We have [6]*

$$
S[E_\alpha(-\lambda t^\alpha)] = \frac{1}{1+\lambda u^\alpha}, \tag{19}
$$

$$
S[1 - E_\alpha(-\lambda t^\alpha)] = \frac{\lambda u^\alpha}{1+\lambda u^\alpha}. \tag{20}
$$

3. Main Results

The aim of this section is to obtain the solutions for the fractional falling body problem by means of some nonlocal fractional derivative operators such as Caputo, Caputo-Fabrizio, and ABC.

3.1. The Falling Body Problem in the Sense of Caputo. The falling body problem in the sense of Caputo depended on Newton's second law which is given as follows:

$$
{}_0^C D^\alpha v(t) + k\sigma^{1-\alpha} v(t) = -g\sigma^{1-\alpha}, \tag{21}
$$

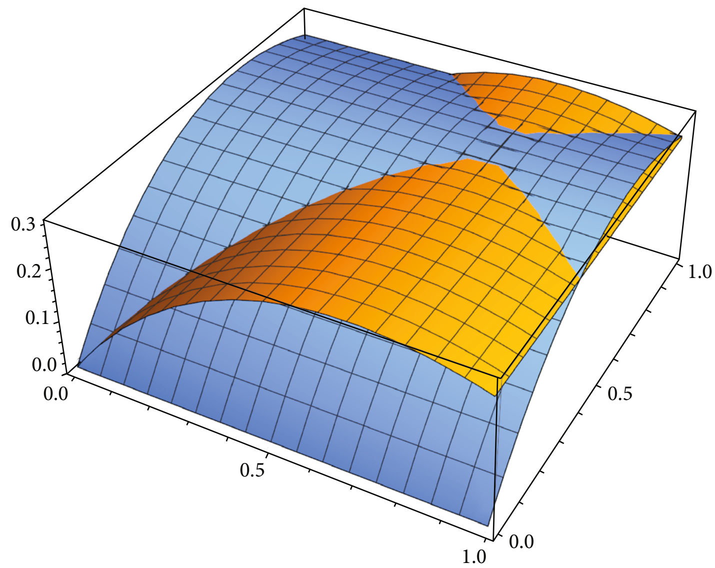

FIGURE 3: Simulations of the solution for $\alpha = 0.98$ and $\alpha = 0.99$.

where the initial velocity $v(0) = v_0$, g is the gravitational constant, and m and k are the positive constant rate which indicates the mass of the body. We have

$$S\{{}_0^C D^\alpha v(t)\} + k\sigma^{1-\alpha} S\{v(t)\} = S\{-g\sigma^{1-\alpha}\}. \tag{22}$$

Using the relation equation (9), we can write

$$\frac{S\{v(t)\} - v(0)}{u^\alpha} + k\sigma^{1-\alpha} S\{v(t)\} = -g\sigma^{1-\alpha}, \tag{23}$$

$$S\{v(t)\} = \frac{v(0)}{1 + k\sigma^{1-\alpha}u^\alpha} - \frac{g\sigma^{1-\alpha}u^\alpha}{1 + k\sigma^{1-\alpha}u^\alpha}. \tag{24}$$

If we apply the inverse Sumudu transform, we will obtain

$$v(t) = v(0)E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right) - \frac{g}{k}\left[1 - E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right)\right]. \tag{25}$$

Because of $\alpha = \sigma k$, $0 < \sigma \leq 1/k$, the velocity $v(t)$ can be put down as follows:

$$v(t) = v(0)E_\alpha\left(-k^\alpha\alpha^{1-\alpha}t^\alpha\right) - \frac{g}{k}\left[1 - E_\alpha\left(-k^\alpha\alpha^{1-\alpha}t^\alpha\right)\right]. \tag{26}$$

Note that we put the condition $v_0 = -g/k$ in order to satisfy the initial condition $v(0) = v_0$. By benefiting from the velocity (33), vertical distance $z(t)$ can be obtained in the following way:

$$ {}_0^C D^\alpha z(t) = v(0)\sigma^{1-\alpha}E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right) - \frac{g\sigma^{1-\alpha}}{k}\left[1 - E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right)\right]. \tag{27}$$

Applying the Sumudu transformation to the above equation, we have

$$\begin{aligned} S\{{}_0^C D^\alpha z(t)\} &= v(0)\sigma^{1-\alpha}S\{E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right)\} - \frac{g\sigma^{1-\alpha}}{k}S\left[1 - E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right)\right], \\ \frac{S\{z(t)\} - z(0)}{u^\alpha} &= v(0)\sigma^{1-\alpha}\frac{1}{1 + k\sigma^{1-\alpha}u^\alpha} - \frac{g\sigma^{1-\alpha}}{k} + \frac{g\sigma^{1-\alpha}}{k}\frac{1}{1 + k\sigma^{1-\alpha}u^\alpha}, \\ S\{z(t)\} &= z(0) + v(0)\sigma^{1-\alpha}\frac{u^\alpha}{1 + k\sigma^{1-\alpha}u^\alpha} - \frac{g\sigma^{1-\alpha}u^\alpha}{k} + \frac{g\sigma^{1-\alpha}}{k}\frac{u^\alpha}{1 + k\sigma^{1-\alpha}u^\alpha}. \end{aligned} \tag{28}$$

Using the inverse Sumudu transformation for the last equation and taking the $z(0) = h$, we acquire the vertical distance $z(t)$ as

$$z(t) = h + \frac{v(0)}{k}\left[1 - E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right)\right] - \frac{g\sigma^{1-\alpha}t^\alpha}{k\Gamma(\alpha+1)} + \frac{g}{k^2}\left[1 - E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right)\right]. \tag{29}$$

Because of $\alpha = \sigma k$, $0 < \sigma \leq 1/k$, the vertical distance $z(t)$ can be put down as

$$\begin{aligned} z(t) = h &+ \frac{v(0)}{k}\left[1 - E_\alpha\left(-k^\alpha\alpha^{1-\alpha}t^\alpha\right)\right] - \frac{g\alpha^{1-\alpha}t^\alpha}{k^{(2-\alpha)}\Gamma(\alpha+1)} \\ &+ \frac{g}{k^2}\left[1 - E_\alpha\left(-k^\alpha\alpha^{1-\alpha}t^\alpha\right)\right]. \end{aligned} \tag{30}$$

We demonstrate the simulations of the above solution for different values of α as shown in Figures 1–3. Similar simulations can be shown easily for other solutions.

3.2. The Falling Body Problem in the Sense of Caputo-Fabrizio. The falling body problem in the sense of Caputo-Fabrizio depended on Newton's second law which is given as follows:

$$ {}_0^{CFC}D^\alpha v(t) + k\sigma^{1-\alpha}v(t) = -g\sigma^{1-\alpha}. \tag{31}$$

where the initial velocity $v(0)=v_0$, g is the gravitational constant, and m and k are the positive constant rate which indicates the mass of the body. We have

$$S\left\{{}_0^{\mathrm{CFC}}D^{\alpha}v(t)\right\}+k\sigma^{1-\alpha}S\{v(t)\}=S\left\{-g\sigma^{1-\alpha}\right\}. \tag{32}$$

If we use the relation equation(19), we can write

$$\frac{M(\alpha)}{(1-\alpha)}\frac{S\{v(t)\}}{(1+(\alpha/1-\alpha)u)}-\frac{M(\alpha)}{(1-\alpha)}\frac{v(0)}{(1+(\alpha/(1-\alpha))u)}+k\sigma^{1-\alpha}S\{v(t)\}=-g\sigma^{1-\alpha}, \tag{33}$$

$$S\{v(t)\}=\frac{M(\alpha)v(0)/(1-\alpha)(1+(\alpha/1-\alpha)u)}{(M(\alpha)/(1-\alpha)(1+(\alpha/(1-\alpha))u))+k\sigma^{1-\alpha}}-\frac{g\sigma^{1-\alpha}}{(M(\alpha)/(1-\alpha)(1+(\alpha/(1-\alpha))u))+k\sigma^{1-\alpha}}. \tag{34}$$

Applying the inverse Sumudu transform yields

$$v(t)=S^{-1}\left\{\frac{M(\alpha)v(0)/(1-\alpha)(1+(\alpha/1-\alpha)u)}{(M(\alpha)/(1-\alpha)(1+(\alpha/1-\alpha)u))+k\sigma^{1-\alpha}}\right\}-S^{-1}\left\{\frac{g\sigma^{1-\alpha}}{(M(\alpha)/(1-\alpha)(1+(\alpha/1-\alpha)u))+k\sigma^{1-\alpha}}\right\}.$$

$$v(t)=\frac{M(\alpha)v(0)}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\exp\left(\frac{-\alpha k\sigma^{1-\alpha}t}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)-\frac{g\left[M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-\alpha k\sigma^{1-\alpha}t/\left(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)\right)\right)\right]}{k(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha))}. \tag{35}$$

Because of $\alpha=\sigma k$, $0<\sigma\leq 1/k$, the velocity $v(t)$ can be put down as follows:

$$v(t)=\frac{M(\alpha)v(0)}{M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\exp\left(\frac{-k^{\alpha}\alpha^{2-\alpha}t}{M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right)-\frac{g\left[M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-k^{\alpha}\alpha^{2-\alpha}t/\left(M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)\right)\right)\right]}{k(M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha))}. \tag{36}$$

Note that we put the condition $v_0=-g/k$ in order to satisfy the initial condition $v(0)=v_0$. By benefiting from the velocity (33), vertical distance $z(t)$ can be obtained in the following way:

$${}_0^{\mathrm{CFC}}D^{\alpha}z(t)=\frac{M(\alpha)\sigma^{1-\alpha}v(0)}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\exp\left(\frac{-\alpha k\sigma^{1-\alpha}t}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)-\frac{g\sigma^{1-\alpha}\left[M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-\alpha k\sigma^{1-\alpha}t/\left(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)\right)\right)\right]}{k(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha))}. \tag{37}$$

If we implement the Sumudu transformation to the last equation, we will obtain

$$S\left\{{}_0^{\mathrm{CFC}}D^{\alpha}z(t)\right\}=\frac{M(\alpha)\sigma^{1-\alpha}v(0)}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}S\left\{\exp\left(\frac{-\alpha k\sigma^{1-\alpha}t}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right\}-S\left\{\frac{g\sigma^{1-\alpha}}{k}\right\}+\frac{g\sigma^{1-\alpha}M(\alpha)}{k(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha))}S\cdot\left\{\exp\left(\frac{-\alpha k\sigma^{1-\alpha}t}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right\},$$

$$\frac{M(\alpha)}{(1-\alpha)}\frac{S\{z(t)\}}{(1+(\alpha/(1-\alpha))u)}-\frac{M(\alpha)}{(1-\alpha)}\frac{z(0)}{(1+(\alpha/(1-\alpha))u)}=\frac{M(\alpha)\sigma^{1-\alpha}v(0)}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u/(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}-\frac{g\sigma^{1-\alpha}}{k}+\frac{g\sigma^{1-\alpha}M(\alpha)}{k(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha))}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u/(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))},$$

$$S\{z(t)\}=z(0)+\sigma^{1-\alpha}v(0)\frac{(1-\alpha+\alpha u)}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha+\alpha u)}\Bigg\}-\frac{g\sigma^{(1-\alpha)}(1-\alpha)}{M(\alpha)k}-\frac{g\sigma^{(1-\alpha)}\alpha u}{M(\alpha)k}+\frac{g\sigma^{1-\alpha}}{k}\frac{(1-\alpha+\alpha u)}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha+\alpha u)}\Bigg\}, \tag{38}$$

If we utilize the inverse Sumudu transformation for equation (13) and take the $z(0)=h$, we acquire the vertical distance $z(t)$ as

$$z(t)=h+\frac{v(0)}{k}\frac{\left[M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-\alpha k\sigma^{1-\alpha}t/\left(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)\right)\right)\right]}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}-\frac{g\sigma^{1-\alpha}}{M(\alpha)k}-\frac{g\sigma^{1-\alpha}\alpha t}{M(\alpha)k}+\frac{g}{k^2}\frac{\left[M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-\alpha k\sigma^{1-\alpha}t/\left(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)\right)\right)\right]}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}. \tag{39}$$

Because of $\alpha=\sigma k$, $0<\sigma\leq 1/k$, the vertical distance $z(t)$ can be put down as

$$z(t)=h+\frac{v(0)}{k}\frac{\left[M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-k^{\alpha}\alpha^{2-\alpha}t/\left(M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)\right)\right)\right]}{M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}-\frac{g\alpha^{1-\alpha}}{M(\alpha)k}-\frac{g\alpha^{2-\alpha}t}{M(\alpha)k}+\frac{g}{k^2}\frac{\left[M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-k^{\alpha}\alpha^{2-\alpha}t/\left(M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)\right)\right)\right]}{M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}. \tag{40}$$

3.3. The Falling Body Problem in the Sense of ABC. The falling body problem in the sense of ABC depended on Newton's second law which is given as follows:

$${}_0^{\mathrm{ABC}}D^{\alpha}v(t)+k\sigma^{1-\alpha}v(t)=-g\sigma^{1-\alpha}, \tag{41}$$

where the initial velocity $v(0)=v_0$, g is the gravitational constant, and m and k are the positive constant rate which indicates the mass of the body. We have

$$S\left\{{}_0^{\mathrm{ABC}}D^{\alpha}v(t)\right\}+k\sigma^{1-\alpha}S\{v(t)\}=S\left\{-g\sigma^{1-\alpha}\right\}. \tag{42}$$

Using the relation equation (15), we can write

$$\frac{\mathrm{AB}(\alpha)}{(1-\alpha)}\frac{S\{v(t)\}}{(1+(\alpha/(1-\alpha))u^\alpha)}-\frac{\mathrm{AB}(\alpha)}{(1-\alpha)}\frac{v(0)}{(1+(\alpha/(1-\alpha))u^\alpha)}+k\sigma^{1-\alpha}S\{v(t)\}=-g\sigma^{1-\alpha},$$
$$S\{v(t)\}=\frac{\mathrm{AB}(\alpha)v(0)-g\sigma^{1-\alpha}(1-\alpha)(1+(\alpha/(1-\alpha))u^\alpha)}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)(1+(\alpha/(1-\alpha))u^\alpha)}. \tag{43}$$

If we apply the inverse Sumudu transform, we will reach

$$v(t)=S^{-1}\left\{\frac{\mathrm{AB}(\alpha)v(0)}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^\alpha/(\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\right\}$$
$$-S^{-1}\left\{\frac{g\sigma^{1-\alpha}(1-\alpha)(1+(\alpha/(1-\alpha))u^\alpha)}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)(1+(\alpha/(1-\alpha))u^\alpha)}\right\}, \tag{44}$$

$$v(t)=\frac{\mathrm{AB}(\alpha)v(0)}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g\sigma^{1-\alpha}(1-\alpha)}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g}{k}\left[1-E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right]. \tag{45}$$

Because $\alpha=\sigma k$, $0<\sigma\le 1/k$, the velocity $v(t)$ can be put down as follows:

$$v(t)=\frac{\mathrm{AB}(\alpha)v(0)}{\mathrm{AB}(\alpha)+k^\alpha\alpha^{1-\alpha}(1-\alpha)}E_\alpha\left(-\frac{\alpha^{2-\alpha}(kt)^\alpha}{\mathrm{AB}(\alpha)+k^\alpha\alpha^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{gk^{\alpha-1}\alpha^{1-\alpha}(1-\alpha)}{\mathrm{AB}(\alpha)+k^\alpha\alpha^{1-\alpha}(1-\alpha)}E_\alpha\left(-\frac{\alpha^{2-\alpha}(kt)^\alpha}{\mathrm{AB}(\alpha)+k^\alpha\alpha^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g}{k}\left[1-E_\alpha\left(-\frac{\alpha^{2-\alpha}(kt)^\alpha}{\mathrm{AB}(\alpha)+k^\alpha\alpha^{1-\alpha}(1-\alpha)}\right)\right]. \tag{46}$$

Note that we put the condition $v_0=-g/k$ in order to satisfy the initial condition $v(0)=v_0$. By benefiting from the velocity (33), vertical distance $z(t)$ can be obtained in the following way:

$${}_0^{\mathrm{ABC}}D^\alpha z(t)=\frac{\mathrm{AB}(\alpha)\sigma^{1-\alpha}v(0)}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g\sigma^2 1-\alpha(1-\alpha)}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g\sigma^{1-\alpha}}{k}\left[1-E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right]. \tag{47}$$

If we apply the Sumudu transformation to the above equation, we have

$$S\left\{{}_0^{\mathrm{ABC}}D^\alpha z(t)\right\}=\frac{\mathrm{AB}(\alpha)\sigma^{1-\alpha}v(0)}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}S\left\{E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right\}$$
$$-\frac{g\sigma^2 1-\alpha(1-\alpha)}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}S\left\{E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right\}$$
$$-S\left\{\frac{g\sigma^{1-\alpha}}{k}\right\}+\frac{g\sigma^{1-\alpha}}{k}S\left\{E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right\},$$

$$\frac{\mathrm{AB}(\alpha)}{(1-\alpha)}\frac{S\{z(t)\}}{(1+(\alpha/(1-\alpha))u^\alpha)}-\frac{\mathrm{AB}(\alpha)}{(1-\alpha)}\frac{z(0)}{(1+(\alpha/(1-\alpha))u^\alpha)}$$
$$=\frac{\mathrm{AB}(\alpha)\sigma^{1-\alpha}v(0)}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^\alpha/(\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$-\frac{g\sigma^2 1-\alpha(1-\alpha)}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^\alpha/(\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$-\frac{g\sigma^{1-\alpha}}{k}+\frac{g\sigma^{1-\alpha}}{k}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^\alpha/(\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\},$$

$$S\{z(t)\}=z(0)+\frac{\sigma^{1-\alpha}(1-\alpha)v(0)}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^\alpha/(\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$+\frac{\sigma^{1-\alpha}v(0)\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{u^\alpha}{1+(\alpha k\sigma^{1-\alpha}u^\alpha/(\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$-\frac{g\sigma^{2(1-\alpha)}(1-\alpha)^2}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^\alpha/(\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$-\frac{g\sigma^{2(1-\alpha)}\alpha(1-\alpha)}{\mathrm{AB}(\alpha)(\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha))}\frac{u^\alpha}{1+(\alpha k\sigma^{1-\alpha}u^\alpha/(\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$-\frac{g\sigma^{1-\alpha}\alpha u^\alpha}{\mathrm{AB}(\alpha)k}-\frac{g\sigma^{1-\alpha}(1-\alpha)}{\mathrm{AB}(\alpha)k}+\frac{g\sigma^{1-\alpha}\alpha}{\mathrm{AB}(\alpha)k}\frac{u^\alpha}{1+(\alpha k\sigma^{1-\alpha}u^\alpha/(\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$+\frac{g\sigma^{1-\alpha}(1-\alpha)}{\mathrm{AB}(\alpha)k}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^\alpha/(\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}. \tag{48}$$

Using the inverse Sumudu transformation for the last equation and taking the $z(0)=h$, we acquire the vertical distance $z(t)$ as

$$z(t)=h+\frac{\sigma^{1-\alpha}(1-\alpha)v(0)}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$+\frac{v(0)}{k}\left[1-E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right]$$
$$-\frac{g\sigma^{2(1-\alpha)}(1-\alpha)^2}{\mathrm{AB}(\alpha)(AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha))}E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g\sigma^{(1-\alpha)}(1-\alpha)}{\mathrm{AB}(\alpha)k}E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g\sigma^{(1-\alpha)}}{\mathrm{AB}(\alpha)k}\left[1-\alpha+\frac{\alpha t^\alpha}{\Gamma(1+\alpha)}\right]$$
$$+\frac{g\sigma^{(1-\alpha)}(1-\alpha)}{\mathrm{AB}(\alpha)k}E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$+\frac{g\mathrm{AB}(\alpha)+gk\sigma^{(1-\alpha)}(1-\alpha)}{\mathrm{AB}(\alpha)k^2}\left[1-E_\alpha\left(-\frac{\alpha k\sigma^{1-\alpha}t^\alpha}{\mathrm{AB}(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right], \tag{49}$$

where $v_0=g\sigma^{1-\alpha}/AB(\alpha)$. Because of $\alpha=\sigma k$, $0<\sigma\le 1/k$, the vertical distance $z(t)$ can be put down as

$$\begin{aligned} z(t) = h &+ \frac{\alpha^{1-\alpha}k^{\alpha-1}(1-\alpha)v(0)}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right) \\ &+ \frac{v(0)}{k}\left[1-E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right)\right] \\ &- \frac{g\alpha^{2(1-\alpha)}k^{2(\alpha-1)}(1-\alpha)^2}{\mathrm{AB}(\alpha)(\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha))}E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right) \\ &- \frac{g\alpha^{(1-\alpha)}k^{(\alpha-1)}(1-\alpha)}{\mathrm{AB}(\alpha)k}E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right) \\ &- \frac{g\alpha^{(1-\alpha)}k^{\alpha}}{\mathrm{AB}(\alpha)k^2}\left[1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(1+\alpha)}\right] \\ &+ \frac{g\sigma^{(1-\alpha)k^{\alpha}}(1-\alpha)}{\mathrm{AB}(\alpha)k}E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right) \\ &+ \frac{gAB(\alpha)+gk^{\alpha}\alpha^{(1-\alpha)}(1-\alpha)}{\mathrm{AB}(\alpha)k^2}\left[1-E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right)\right]. \end{aligned} \tag{50}$$

4. Conclusions

We searched the falling body problem in detail by the Sumudu transform in this study. We obtained the exact solutions of this problem with Caputo, Caputo-Fabrizio, and Atangana-Baleanu derivatives. We demonstrated the effect of the Sumudu transform by these results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they do not have any conflict of interest.

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Research Article

A New Type of Sturm-Liouville Equation in the Non-Newtonian Calculus

Sertac Goktas

Department of Mathematics, Faculty of Science and Letters, Mersin University, Mersin, Turkey

Correspondence should be addressed to Sertac Goktas; srtcgoktas@gmail.com

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In mathematical physics (such as the one-dimensional time-independent Schrödinger equation), Sturm-Liouville problems occur very frequently. We construct, with a different perspective, a Sturm-Liouville problem in multiplicative calculus by some algebraic structures. Then, some asymptotic estimates for eigenfunctions of the multiplicative Sturm-Liouville problem are obtained by some techniques. Finally, some basic spectral properties of this multiplicative problem are examined in detail.

1. Introduction

In the 1960's, Grossman and Katz [1, 2] constructed a comprehensive family of calculus that includes classical calculus as well an infinite subbranches of non-Newtonian calculus. Arithmetics, a complete ordered field on $A \subset \mathbb{R}$, are of great importance in the construction of non-Newtonian calculus. The real number system is a classical arithmetic. Every arithmetic produces one generator, which is one to one on the domain and range of $A \subset \mathbb{R}$. Conversely, every generator produces one arithmetic. For instance, I, exp and $\sigma(x) = (e^x - 1)/(e^x + 1)$ are generators. So, I generates usual arithmetic, exp produces geometric arithmetic, and the function $\sigma(x)$ generates sigmoidal arithmetic mathematically describing the sigmoidal curves that occur in the study of population and biological growth.

Non-Newtonian calculus is divided into many subbranches as geometric, anageometric, biogeometric, quadratic, and harmonic calculus. Geometric calculus, which is one of these, is also defined as multiplicative calculus. Changes of arguments and values of a function are measured by differences and ratios in multiplicative calculus, respectively, while they are measured by differences in the classical case. Multiplicative calculus is especially useful in situations where products and ratios provide the natural methods of combining and comparing magnitudes. There are actually many reasons to study multiplicative calculus. It improves the work of additive calculations indirectly. Problems that are difficult to solve in the usual case can be solved with incredible ease in here.

Many events such as the levels of sound signals, the acidities of chemicals, and the magnitudes of earthquakes change exponentially. For this reason, examining these problems in nature using multiplicative calculus offers great convenience and benefits. It allows the physical properties of the events dealt with physically to be examined from different angles. The problems encountered in the study of these physical properties can be expressed with multiplicative differential equations [3–5]. It has applications in many areas required by mathematical modeling, especially in applied mathematics [6–11], engineering [3], economics [12, 13], business [14], and medicine [15] (see also [16–22]). Different alternative analyses have been developed to solve the problems that arise while working with these problems and to achieve better results in solving the problems. For example, the analytical solution of a differential equation that is very difficult in classical calculus can be obtained more easily in multiplicative calculus. Moreover, one of the importance of this theory is to find positive solutions of nonlinear differential equations. The investigation of various properties of positive solutions has an important place in the spectral theory of differential operators (see [23, 24]). This theory has few applications to spectral analysis. For this reason, we think that the results we will obtain will have very significant reflections in spectral theory and will open new fields.

The concepts and methods developed during the study of the Sturm-Liouville (SL) equation led to the development of many important directions of mathematics and physics. In kindred areas of analysis and the SL theory that studies some properties such as asymptotic behavior of eigenvalues and eigenfunctions, these are a source of new problems and ideas [25]. It is a very important equation used to explain many phenomena in nature. The one-dimensional time-independent Schrödinger equation in quantum mechanics can be given as an example of SL equation. Significant results have been obtained by many mathematicians over the years regarding the SL equation (see [25–36]). This equation has not yet been addressed in multiplicative calculus. The results we will obtain will make important contributions to mathematical physics. Therefore, we examine the multiplicative SL problem other than the Newtonian calculus. Multiplicative analysis techniques can also be applied to different operators that have a significant impact on spectral theory.

2. Preliminaries

In this section, we will express the notions and theorems in multiplicative analysis, which are extremely important in solving the problem and examining its properties. There are many other features of this new theory that are available, other than the ones below. However, expressing the properties of the multiplicative derivative and integral is especially important for the rest of our study. This derivative and integral are structurally quite different from classical derivative and integral. In fact, it makes a great difference in logic. These concepts will make a great difference in physics, biology, spectral theory, and economics.

Definition 1 (see [37]). Let $f : A \subset \mathbb{R} \longrightarrow \mathbb{R}^+$. *Derivative of f is expressed by

$$f^*(x) = \lim_{h \longrightarrow 0} \left[\frac{f(x+h)}{f(x)} \right]^{1/h}, \tag{1}$$

if the above limit exists and is positive. Indeed, *derivative is also called as the multiplicative (or geometric) derivative. Moreover, f is usual differentiable at x, and then,

$$f^*(x) = e^{(\ln \circ f)'(x)}. \tag{2}$$

Theorem 2 (see [37]). *Let f, g be *differentiable and h be classical differentiable at x.*

The following equalities hold for *derivative.

$$\begin{aligned} (cf)^*(x) &= f^*(x), \\ (fg)^*(x) &= f^*(x) g^*(x), \\ \left(\frac{f}{g}\right)^*(x) &= \frac{f^*(x)}{g^*(x)}, \\ \left(f^h\right)^*(x) &= f^*(x)^{h(x)} f(x)^{h'(x)}, \\ (f \circ h)^*(x) &= f^*(h(x))^{h'(x)}, \\ (f+g)^*(x) &= f^*(x)^{\frac{f(x)}{f(x)+g(x)}} g^*(x)^{\frac{g(x)}{f(x)+g(x)}}, \end{aligned} \tag{3}$$

where c is a positive constant.

Definition 3 (see [37]). Let $f \in \mathbb{R}^+$ be bounded on $[a, b]$. Consider the partition $\mathscr{P} = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ and the numbers $\xi_1, \xi_2, \cdots, \xi_n$ associated with the partition $\mathscr{P}$.f is said to be *integrable if there exists a number $\mathbf{P}$ having the following property: for every $\varepsilon > 0$, there exists a partition $\mathscr{P}_\varepsilon$ of $[a, b]$ such that $|\mathbf{P}(f, \mathscr{P}) - \mathbf{P}| < \varepsilon$ for every refinement $\mathscr{P}$ of $\mathscr{P}_\varepsilon$ independently on the selection of the numbers associated with the partition $\mathscr{P}$ where

$$\mathbf{P}(f, \mathscr{P}) = \prod_{i=1}^{n} f(\xi_i)^{(x_i - x_{i-1})}. \tag{4}$$

Then, symbol $\int_a^b f(x)^{dx}$ is called *integral of f on $[a, b]$.

Considering this definition, if $f \in \mathbb{R}^+$ is integrable on $[a, b]$, it is *integrable on $[a, b]$,

$$\int_a^b f(x)^{dx} = \exp\left\{ \int_a^b (\ln \circ f)(x) dx \right\}. \tag{5}$$

Conversely, *integrability of f on $[a, b]$ implies

$$\int_a^b f(x) dx = \ln \int_a^b \left(e^{f(x)}\right)^{dx}. \tag{6}$$

Indeed, *integral is also called as multiplicative integral.

Theorem 4 (see [37]). *Let $f, g \in \mathbb{R}^+$ be bounded and *integrable and $h \in \mathbb{R}^+$ be usual differentiable on $[a, b]$. Then, the following expression holds*

$$\begin{aligned}
&\int_a^b \left[f(x)^k\right]^{dx} = \left[\int_a^b f(x)^{dx}\right]^k, \\
&\int_a^b [f(x)g(x)]^{dx} = \int_a^b f(x)^{dx} . \int_a^b g(x)^{dx}, \\
&\int_a^b \left[\frac{f(x)}{g(x)}\right]^{dx} = \frac{\int_a^b f(x)^{dx}}{\int_a^b g(x)^{dx}}, \\
&\int_a^b f(x)^{dx} = \int_a^c f(x)^{dx} . \int_c^b f(x)^{dx}, \\
&\int_a^b \left[f^*(x)^{g(x)}\right]^{dx} = \frac{f(b)^{g(b)}}{f(a)^{g(a)}} \left\{\int_a^b \left[f(x)^{g'(x)}\right]^{dx}\right\}^{-1}.
\end{aligned} \tag{7}$$

where $k \in \mathbb{R}$ is a constant and $c \in [a, b]$. The expression ν is known as *integration by parts formula.

3. Multiplicative SL Equation

In this section, the multiplicative SL equation will be established by using some algebraic structures and the eigenfunctions of the constructed problem will be obtained.

Firstly, let us express some concepts that form the basis of the SL equation in the multiplicative case. nth-order *multiplicative linear differential expression* is in the form of

$$l(y) = \left[y^{*(n)}\right]^{s_n(x)} \left[y^{*(n-1)}\right]^{s_{n-1}(x)} \cdots y^{s_0(x)}, \tag{8}$$

where $s_n(x), s_{n-1}(x), \cdots, s_0(x)$ are the continuous exponents on $[a, b]$. Let

$$u(y) = \left[y_a^{*(n-1)}\right]^{\alpha_{n-1}} \cdots [y_a^*]^{\alpha_1} y_a^{\alpha_0} \cdot \left[y_b^{*(n-1)}\right]^{\beta_{n-1}} \cdots [y_b^*]^{\beta_1} y_b^{\beta_0} \tag{9}$$

be the linear form when y_a and y_b are the values of y at end points of $[a, b]$. If such forms $u_\nu(y)$ have been specified for $\nu = \overline{1, m}$ and the conditions $u_\nu(y) = 1$ are imposed into $y(x) \in C^{*(n)}$, it must satisfy these boundary conditions, where $C^{*(n)}$ shows the set of the functions which are nth-order multiplicative differentiable and continuous. Let us consider a certain multiplicative differential expression $l(y)$ with $u_\nu(y) = 1$ on $D \subset C^{*(n)}$. Assume that $u = l(y)$ is a function where $y(x) \in D$. This relation is denoted by L whose domain is D. The operator L is called multiplicative differential operator generated by $l(y) = \omega(x)$ and $u_\nu(y) = 1$. The problem of determination $y(x) \in C^{*(n)}$ which satisfies the conditions $l(y) = 1$ and $u_\nu(y) = 1$ is called the homogeneous multiplicative boundary value problem.

Definition 5. Let $Ly = y^\lambda$. $y \neq 1$ is called multiplicative eigenfunction (*eigenfunction) of the operator L. Here, λ is a multiplicative eigenvalue (*eigenvalue) of L. That is, the * eigenvalues of an operator L are the values of λ when the multiplicative boundary value problem

$$\begin{aligned}
&l(y) = y^\lambda, \\
&u_\nu(y) = 1, \quad \nu = \overline{1, n}
\end{aligned} \tag{10}$$

has nontrivial solutions.

We will soon construct the multiplicative SL problem. That way, let us express multiplicative algebraic structures that we will encounter while establishing and solving the multiplicative SL equation. Arithmetic operations created with exponential functions are called multiplicative algebraic operations. Let us show some properties of these operations with a multiplicative arithmetic table for $f, g \in \mathbb{R}^+$ [37].

$$\begin{aligned}
&f \oplus g = fg, \\
&f \ominus g = \frac{f}{g}, \\
&f \odot g = f^{\ln g} = g^{\ln f}.
\end{aligned} \tag{11}$$

These operations create some algebraic structures. If $\oplus : A \times A \longrightarrow A$ is an operation where $A \neq \varnothing$ and $A \subset \mathbb{R}^+$, the algebraic structure $(A, \oplus)$ is called a multiplicative group. Similarly, $(A, \oplus, \odot)$ is a multiplicative ring. This situation gives us the opportunity to use these processes easily and define different structures.

Consider the following multiplicative SL equation for $x \in [a, b]$

$$L[y] = \left(e^{-1} \odot y^{**}(x)\right) \oplus \left(e^{q(x)} \odot y(x)\right) = e^\lambda \odot y(x), \tag{12}$$

with the conditions

$$\begin{aligned}
&\left(e^{\cos\alpha} \odot y(a)\right) \oplus \left(e^{\sin\alpha} \odot y^*(a)\right) = 1, \\
&\left(e^{\cos\beta} \odot y(b)\right) \oplus \left(e^{\sin\beta} \odot y^*(b)\right) = 1,
\end{aligned} \tag{13}$$

where q is real valued on $[a, b]$ and α, β arbitrary real numbers. If we expand and simplify this problem by using the properties of multiplicative calculus, the multiplicative SL problem

$$\begin{aligned}
&\left(y^{**}\right)^{-1} y^{q(x)} = y^\lambda, \\
&(y(a))^{\cos\alpha} (y^*(a))^{\sin\alpha} = 1, \\
&(y(b))^{\cos\beta} (y^*(b))^{\sin\beta} = 1
\end{aligned} \tag{14}$$

is obtained.

In usual case, (12) is equivalent to the following nonlinear equation

$$y''y - \left(y'\right)^2 + [(\lambda - q(x)) \ln y] y^2 = 0. \tag{15}$$

The solutions of this nonlinear equation coincide with the solutions of multiplicative equation (12). This shows how important the multiplicative calculus is.

We assume that $a=0, b=\pi$ throughout this study without the loss of generality. In fact, $[a, b]$ is mapped to $[0, \pi]$ by the substitution $t=(x-a)^{\log_{b-a}\pi}$.

By setting $\cot \alpha=-h$ and $\cot \beta=H$, the boundary conditions in (13) are converted to

$$\begin{aligned} & y^*(0) y^{-h}(0)=1, \\ & y^*(\pi) y^H(\pi)=1 . \end{aligned} \tag{16}$$

Let us denote the solutions of (12) by $u(x, \lambda)$ and $v(x, \lambda)$ which satisfies

$$\begin{aligned} u(0, \lambda) & =e, \\ u_x^*(0, \lambda) & =e^h, \end{aligned} \tag{17}$$

$$\begin{aligned} v(0, \lambda) & =1, \\ v_x^*(0, \lambda) & =e . \end{aligned} \tag{18}$$

In order to avoid any difficulties in expressing the main parts of the study, the multiplicative inner product will be defined and the spaces used throughout the study will be given in the multiplicative case.

Definition 6. Let $S \neq \varnothing$ and $<,>_*: S \times S \longrightarrow \mathbb{R}^{+}$ be a mapping such that the following axioms hold for each $x, y, z \in X$:

(i) $<f, f>_* \geq 1$

(ii) $<f, f>_*=1$ if $f=1$

(iii) $<f \oplus g, h>_*=<f, h>_* \oplus<g, h>_*$

(iv) $<e^\alpha \odot f, g>_*=e^\alpha \odot<f, g>_*, \alpha \in \mathbb{R}$

(v) $<f, g>_*=<g, f>_*$

Here, $(S,<,>_*)$ is called the *inner product space and $<,>_*$ is the *inner product on S.

Lemma 7. *The space* $L_2^*[a, b]=\{f: \int_a^b[f(x) \odot f(x)]^{dx}<\infty\}$ *is an* **inner product space with*

$$<,>_*: L_2^*[a, b] \times L_2^*[a, b] \longrightarrow \mathbb{R}^{+},<f, g>_*=\int_a^b[f(x) \odot g(x)]^{dx}, \tag{19}$$

where $f, g \in L_2^*[a, b]$ *are positive functions.*

Proof. Using the properties of the multiplicative inner product and the definition of the given space, it can be easily proved. □

Theorem 8. *Let* $\lambda=\mu^2$. *The asymptotic formulas of* **eigenfunction of problems (12) and (13) are*

$$\begin{aligned} & u(x, \lambda)=e^{\cos \mu x+(h / \mu) \sin \mu x} \cdot\left[\int_0^x u(t, \lambda)^{q(t) \sin [\mu(x-t)] d t}\right]^{1 / \mu}, \\ & v(x, \lambda)=e^{(1 / \mu) \sin \mu x} \cdot\left[\int_0^x v(t, \lambda)^{q(t) \sin [\mu(x-t)] d t}\right]^{1 / \mu} . \end{aligned} \tag{20}$$

Proof. The first estimate will only be proved because the other can be proved similarly. Since $u(x, \lambda)$ satisfies (12), we get

$$\int_0^x u(t, \lambda)^{q(t) \sin [\mu(x-t)]^{d t}}=\int_0^x u^{**}(t, \lambda)^{\sin [\mu(x-t)]^{d t}}\left[\int_0^x u(t, \lambda)^{\sin [\mu(x-t)]^{d t}}\right]^{\mu^2} . \tag{21}$$

If the *integration by parts method is applied to the first multiplicative integral on the right twice in a row, we get

$$\begin{aligned} \int_0^x u^{**}(t, \lambda)^{\sin [\mu(x-t)]^{d t}}= & \frac{u^\mu(x, \lambda)}{u^*(0, \lambda)^{\sin \mu x} u(0, \lambda)^{\mu \cos \mu x}} \\ & \left[\int_0^x u(t, \lambda)^{\sin [\mu(x-t)]^{d t}}\right]^{-\mu^2} . \end{aligned} \tag{22}$$

Then, by considering the conditions in (17) in (21) with above relation,

$$\int_0^x u(t, \lambda)^{q(t) \sin [\mu(x-t)]^{d t}}=u^\mu(x, \lambda) e^{-(h \sin \mu x+\mu \cos \mu x)} . \tag{23}$$

It completes the proof.□ □

Remark 9. The more general 2nd-order multiplicative differential equation

$$y^{**}\left(y^*\right)^{p(x)} y^{r(x)+\lambda w(x)}=1 \tag{24}$$

can be transformed into the following multiplicative SL equation

$$\left[\left(y^*\right)^{\mu(x)}\right]^* y^{r(x) \mu(x)+\lambda \mu(x) w(x)}=1, \tag{25}$$

where $\mu(x)=\int\left(e^{p(x)}\right)^{d x}$. If the multiplicative Liouville transform $u=y^{\int_a^x(p(t) / 2)^{d t}}$ is used here, the multiplicative differential equation above turns to the following multiplicative SL equation:

$$u^{**} u^{q(x)+\lambda w(x)}=1, \tag{26}$$

where $q(x)=r(x)-(1 / 4)\left(2 p^{\prime}(x)+p^2(x)\right)$.

Example 10. Consider the nonlinear eigenvalue problem in a usual case:

$$y''y - \left(y'\right)^2 + \lambda(\ln y)y^2 = 0, \quad 0 < x < S, \\ y(0) = y(S) = 1. \tag{27}$$

It is quite difficult to solve this problem in the usual case. For this reason, we will obtain the eigenvalues and eigenfunctions of the problem by using multiplicative calculus techniques. By the relation $y^{*(n)} = e^{(\ln y)^{(n)}}$, $n = 1, 2$, (27) turns into the multiplicative linear eigenvalue problem

$$y^{**}y^{\lambda} = 1, \\ y(0) = y(S) = 1. \tag{28}$$

If $\lambda \leq 0$, the trivial eigenfunction $y(x, \lambda) = 1$ is obtained. Suppose that $\lambda > 0$. Then, the solution of (28) is

$$\lambda_n = \left(\frac{n\pi}{S}\right)^2, \\ y_n(x) = e^{\alpha_n \sin\left(\frac{n\pi}{S}x\right)}, \tag{29}$$

where $n = 1, 2, 3, \cdots$, and α_n are constants. This solution is also the solution to nonlinear eigenvalue problem (27). This situation shows how effective multiplicative calculus can be in mathematical physics.

Example 11. Now let us consider the periodic nonlinear eigenvalue problem in the usual case:

$$y''y - \left(y'\right)^2 + \lambda(\ln y)y^2 = 0, \quad -S < x < S, \\ y(-S) = y(S), \\ y'(-S) = y'(S). \tag{30}$$

This problem is very important in mathematical physics and its solution is extremely difficult in the usual case. Now, let us turn this problem into an equation that is more solvable in multiplicative calculus by relation $y^{*(n)} = e^{(\ln y)^{(n)}}$, $n = 1, 2$:

$$y^{**}y^{\lambda} = 1, \\ y(-S) = y(S), \\ y^{*}(-S) = y^{*}(S). \tag{31}$$

This problem is a multiplicative periodic linear eigenvalue problem. Here, we get $y(x, \lambda) = 1$ and $y(x, \lambda) = e$ when $\lambda < 0$ and $\lambda = 0$, respectively. Assume that $\lambda > 0$. If the similar operations to the above solution are performed and periodic conditions are taken into account, we get

$$\lambda_n = \left(\frac{n\pi}{S}\right)^2, \\ y_n(x) = \{e\} \cup \left\{e^{\cos\left(\frac{n\pi}{S}x\right)}\right\} \cup \left\{e^{\sin\left(\frac{n\pi}{S}x\right)}\right\}, \quad n = 0, 1, 2, 3, \cdots, \tag{32}$$

where a_n are constants.

Now, we will establish the above equation with new conditions and examine its solutions with another method. For this solution, the multiplicative Laplace transform will be expressed and all the necessary properties will be given. Then, the multiplicative SL problem will be solved using this multiplicative transformation. The flawless operation of this transformation in multiplicative analysis is important in terms of carrying many concepts and theorems present in the classical case to this field. We can guess from this situation that transformations used for different purposes in mathematical physics can also be carried. This is important in terms of considering many theories in mathematical physics from a different perspective and obtaining different results.

Definition 12 (see [10]). Let $f(t) \in \mathbb{R}^{+}$ on $[0, \infty)$. Multiplicative Laplace transform for f is expressed by

$$\mathscr{L}_m\{f(t)\} = F(s) = e^{\mathscr{L}\{f(t)\}}, \tag{33}$$

where $\mathscr{L}$ denotes usual Laplace transform.

Lemma 13 (see [10]). *The multiplicative Laplace transform is multiplicatively linear. Namely,*

$$\mathscr{L}_m\left\{f_1^{c_1}(t)f_2^{c_2}(t)\right\} = \mathscr{L}_m\{f_1(t)\}^{c_1}\mathscr{L}_m\{f_2(t)\}^{c_2}, \tag{34}$$

where c_1, c_2 are arbitrary exponents.

Definition 14 (see [10]). Let $f, f^{*}, f^{**}, \cdots, f^{*(n-1)}$ be continuous and $f^{*(n)}$ be piece-wise continuous on $0 \leq t \leq A$. Also, assume that there exists positive real numbers K, α, and t_0 such that

$$\left|f^{*(n-1)}(t)\right| \leq Ke^{e^{\alpha t}}, \quad t \geq t_0. \tag{35}$$

Furthermore, $\mathscr{L}_m\{f^{*(n)}(t)\}$ exists and can be calculated by

$$\mathscr{L}_m\left\{f^{*(n)}(t)\right\} = \frac{1}{f(0)^{s^{n-1}}f^{*}(0)^{s^{n-2}}f^{**}(0)^{s^{n-3}}\cdots f^{*(n-1)}(0)}F(s)^{s^n}. \tag{36}$$

Definition 15 (see [10]). Let $F(s)$ be a multiplicative Laplace transform of continuous function f, i.e., $\mathscr{L}_m\{f(t)\} = F(s)$. And $\mathscr{L}_m^{-1}\{F\}$ is called the inverse multiplicative Laplace transform. Here, we have the following relation

$$\mathscr{L}_m\left\{f(t)^{t^n}\right\} = \left(F^{*(n)}(s)\right)^{(-1)^n}. \tag{37}$$

Now, let us solve a nonlinear initial value problem in the usual case by the multiplicative Laplace transform.

Example 16. Consider the below nonlinear IVP.

$$\begin{gathered} y''y - \left(y'\right)^2 + \{(\lambda - q(x)) \ln y\}y^2 = 0, \\ y(0) = e^{\alpha}, \\ y'(0) - \beta y(0) = 0, \end{gathered} \tag{38}$$

where $q(x) = c$ and c is a constant. By substitution $y^{*(n)}(x) = e^{(\ln y)^{(n)}(x)}$, $n = 1, 2$, (38) turns into the following multiplicative IVP.

$$\begin{gathered} (y^{**})^{-1} y^{q(x)} = y^{\lambda}, \\ y(0) = e^{\alpha}, \\ y^*(0) = e^{\beta}. \end{gathered} \tag{39}$$

If the multiplicative Laplace transform is applied to both sides of the obtained equation (8) and necessary adjustments are made, we get

$$Y(s) = e^{\frac{\alpha s + \beta}{s^2 + \lambda - c}}. \tag{40}$$

Finally, using the multiplicative inverse Laplace transform, the solution of (39) is

$$y(x, \lambda) = \begin{cases} e^{\alpha \cos\left(\sqrt{\lambda - c}x\right) + \frac{\beta}{\sqrt{\lambda - c}} \sin\left(\sqrt{\lambda - c}x\right)}, & \lambda > c, \\ e^{\alpha + \beta x}, & \lambda = c, \\ e^{\alpha \cos h\left(\sqrt{c - \lambda}x\right) + \frac{\beta}{\sqrt{c - \lambda}} \sin h\left(\sqrt{c - \lambda}x\right)}, & \lambda = c. \end{cases} \tag{41}$$

4. Some Spectral Properties of the Multiplicative SL Problem

We examine some properties of the multiplicative SL operator as self-adjointness, orthogonality, reality, and simplicity in this section. Especially, the concepts of operator self-adjointness and simplicity of eigenvalues have a very important place in physics. The self-adjointness of an operator provides a great advantage in explaining the problem and event. In addition, the simplicity of the eigenvalues is useful in resolving complex physical structures. Orthogonality of eigenfunctions and realism of eigenvalues also have different and important meanings physically. For all these reasons, these features will be examined mathematically.

Lemma 17. *The multiplicative Sturm-Liouville operator L in (12) is formally self-adjoint on $L_2^*[0, \pi]$.*

Proof. Let us use the notion of the multiplicative inner product on $L_2^*[0, \pi]$. It gives

$$\begin{aligned} \langle Lu, v\rangle_* &= \int_0^{\pi} \left[(Lu)^{\ln v}\right]^{dx} = \int_0^{\pi} \left[\left((u^{**})^{-1} u^{q(x)}\right)^{\ln v}\right]^{dx} \\ &= \int_0^{\pi} \left[(u^{**})^{-\ln v}\right]^{dx} \int_0^{\pi} \left[u^{q(x) \ln v}\right]^{dx}. \end{aligned}$$

Using *integration by the parts formula two times for the first factor of the right side,

$$\int_0^{\pi} \left[(u^{**})^{-\ln v}\right]^{dx} = \left.\frac{u^{\ln v^*}}{(u^*)^{\ln v}}\right|_0^{\pi} \cdot \int_0^{\pi} \left[u^{-\ln v^{**}}\right]^{dx}, \tag{43}$$

where $F(x)|_{\alpha}^{\beta} = F(\beta)/F(\alpha)$. Setting this result in (42) implies that

$$\begin{aligned} \langle Lu, v\rangle_* &= W_m\{u,v\}(x)|_0^{\pi} \cdot \int_0^{\pi} \left[u^{\ln\left((v^{**})^{-1} v^{q(x)}\right)}\right]^{dx} \\ &= W_m\{u, v\}(x)|_0^{\pi} \cdot \int_0^{\pi} \left[u^{\ln (Lv)}\right]^{dx} \\ &= W_m\{u, v\}(x)|_0^{\pi} \cdot \langle Lu, v\rangle_*, \end{aligned} \tag{44}$$

where$W_m\{u, v\}(x) = (u \odot v^*) \ominus (v \odot u^*)$. It follows by the conditions in (13) that $W_m\{u, v\}(0) = W_m\{u, v\}(\pi) = 1$. Therefore, we get

$$\langle Lu, v\rangle_* = \langle u, Lv\rangle_*. \tag{45}$$

It completes the proof of self-adjointness.□ □

Lemma 18. *The *eigenfunctions $\varphi(x, \lambda)$ and $\psi(x, \mu)$ related to distinct eigenvalues λ and μ are orthogonal, i.e.,*

$$\int_a^b \left[\varphi(x, \lambda)^{\ln \psi(x,\mu)}\right]^{dx} = 1. \tag{46}$$

Proof. By the self-adjointness of the SL operator L, we get

$$1 = \frac{\langle Lu, v\rangle_*}{\langle u, Lv\rangle_*} = \frac{\left\langle \varphi^{\lambda}(x, \lambda), \psi(x, \mu)\right\rangle_*}{\left\langle \varphi(x, \lambda), \psi^{\mu}(x, \mu)\right\rangle_*} = \left[\int_0^{\pi} \left[\varphi(x, \lambda)^{\ln \psi(x,\mu)}\right]^{dx}\right]^{\lambda - \mu}, \tag{47}$$

where $u(x) = \varphi(x, \lambda)$, $v(x) = \psi(x, \mu)$. Since $\lambda \neq \mu$ and the right-side multiplicative integral is positive, it gives orthogonality of the *eigenfunctions.□ □

Lemma 19. *All eigenvalues of multiplicative SL problems (12) and (13) are real.*

Proof. Let $\lambda = u + iv$ be a complex eigenvalue for the given problem. $\mu = \bar{\lambda} = u - iv$ is also an eigenvalue for (12) and (13) corresponding to $y(\bar{x}, \lambda)$. By previous the lemma, we acquire

$$\int_a^b \left[y(x,\lambda)^{\ln y(\bar{x},\bar{\lambda})} \right]^{dx} = 1. \quad (48)$$

By definition of the multiplicative integral,

$$e^{\int_a^b \ln y(\bar{x},\bar{\lambda}) \ln y(x,\lambda) dx} = 1,$$
$$\int_a^b |\ln y(x,\lambda)|^2 dx = 0 \Rightarrow y(x,\lambda) = 1. \quad (49)$$

This is a contradiction. It is due to our assumption. So, the chosen eigenvalue is real. Since this eigenvalue is arbitrary, all eigenvalues of the problem are real.

Now, let us examine another important spectral property of this problem. The simplicity of eigenvalues is a very important feature in mathematical physics, and there are many proof techniques for simplicity. The algebraic multiplicity of an eigenvalue is the number of times it repeats as a root of the characteristic polynomial. If the algebraic multiplicity of an eigenvalue is 1, that eigenvalue is called a simple eigenvalue.□

□

Lemma 20. *All eigenvalues of multiplicative SL equations (12) and (13) are simple.*

Proof. Let $y_1(x,\lambda)$, $y_2(x,\lambda)$ be eigenfunctions of (12) and (13) corresponding to λ. Therefore, both of these eigenfunctions satisfy the given equation.

$$(y_1^{**})^{-1} y_1^{q(x)} = y_1^{\lambda},$$
$$(y_2^{**})^{-1} y_2^{q(x)} = y_2^{\lambda}. \quad (50)$$

After some straightforward operations,

$$(y_1^{**})^{\ln y_2} (y_2^{**})^{-\ln y_1} = 1. \quad (51)$$

By using multiplicative integral from a to x,

$$\frac{y_1(x)^{\ln y_2^*(x)}}{(y_1^*(x))^{\ln y_2(x)}} \frac{y_1^*(a)^{\ln y_2(a)}}{(y_1(a))^{\ln y_2^*(a)}} = 1. \quad (52)$$

Since $y_1(x,\lambda)$, $y_2(x,\lambda)$ satisfy the given conditions,

$$y_1^*(a) = y_1(a)^{-\cot \alpha},$$
$$y_2^*(a) = y_2(a)^{-\cot \alpha}. \quad (53)$$

If we use this result in (52), it yields $W_m\{y_1, y_2\}(a) = 1$. It gives that y_1 and y_2 are linearly dependent on $[a, b]$. It completes the proof.□ □

5. Conclusion

In this study, we have constructed the multiplicative SL problem and obtained *eigenfunctions of that problem by using some techniques. Later, this problem was investigated in terms of spectral theory in the multiplicative case. This study shows that multiplicative calculus methods can be applied to problems in spectral analysis and give solutions more effectively. This situation will make great contributions to the theory if many important theorems and problems in spectral theory are dealt with in multiplicative calculus. The foundations of multiplicative analysis in spectral theory established with this study can then be applied to different topics of mathematical physics. For example, inverse problems in spectral theory can be identified in this analysis and quality results can be obtained for applications of inverse problems in engineering and medicine. Problems that are difficult to solve in medicine and engineering and situations that cause time loss during application can be reduced by using multiplicative analysis. The current methods and techniques in medicine and engineering can be developed by multiplicative analysis. These developments can be made in many application areas other than reverse problems. Some numerical computation techniques used in spectral analysis can be reestablished in this new theory, and different evaluations can be made.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no competing interests.

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