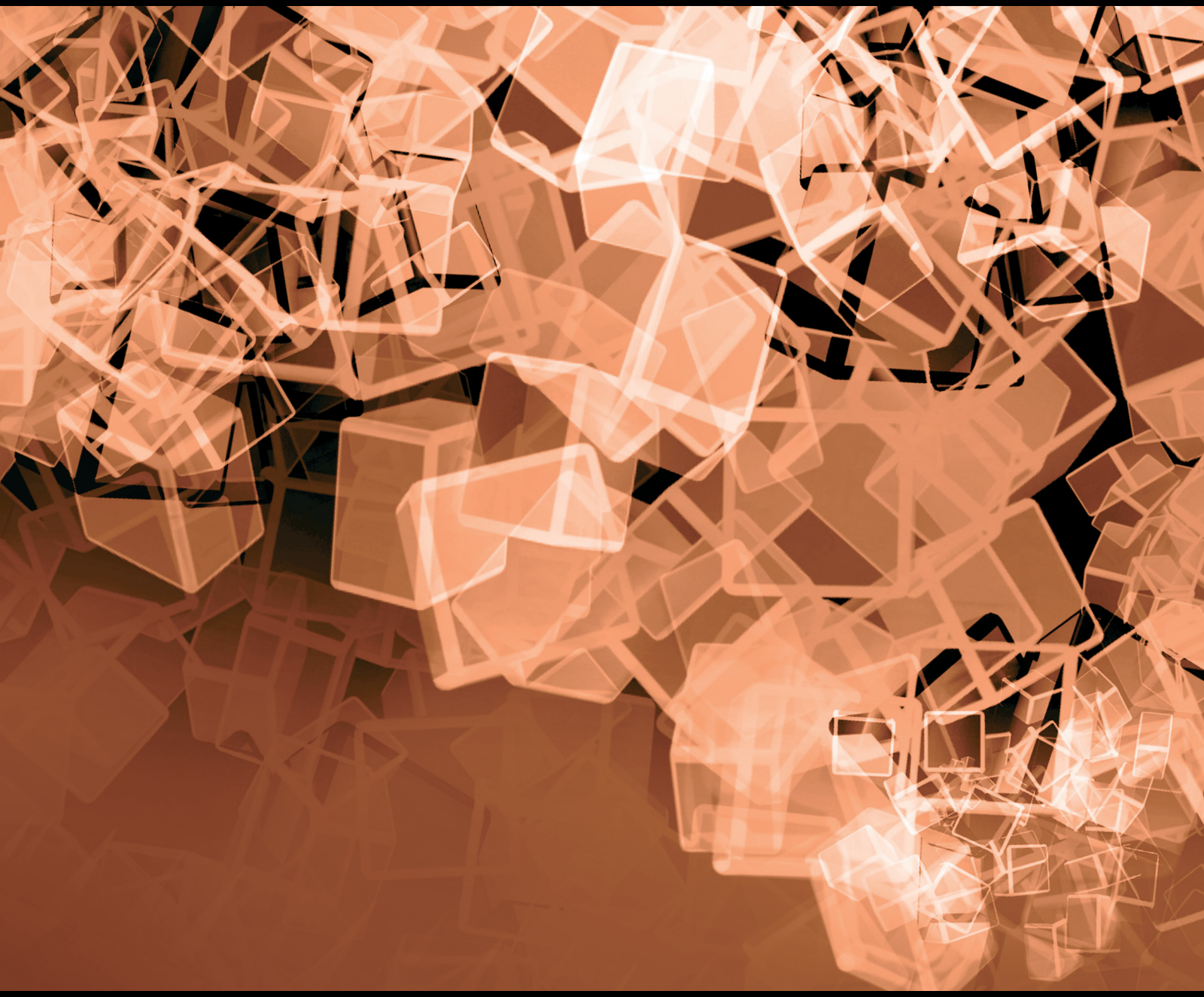
Discrete Dynamics in Nature and Society

Qualitative Analysis on Differential, Fractional Differential, and Dynamic Equations and Related Topics

Guest Editors: Said R. Grace, Taher S. Hassan, Shurong Sun, and Elvan Akin



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Editorial Board

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Editorial

Qualitative Analysis on Differential, Fractional Differential, and Dynamic Equations and Related Topics

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This issue on qualitative analysis on differential, fractional differential, and dynamic equations and related topics aims at an all-around research and the state-of-the-art theoretical, numerical, and practical achievements that contribute to this field. This issue contains the following features.

Oscillation and Asymptotic Behavior. S. R. Grace and E. Akin investigate the asymptotic behavior of nonoscillatory solutions of certain forced integrodifferential equations. From the obtained results, they drive a technique which can be applied to some related integrodifferential as well as integral equations.

T. S. Hassan and S. R. Grace consider the higher-order functional dynamic equations with mixed nonlinearities and study their oscillatory behavior via comparison with some equations whose oscillatory characters are known and studied extensively in the literature.

Stochastic Delay Differential Equations, Dynamics of Stochastic Coral Reefs Model, Stochastic Predator-Prey System Subject to Lévy Jumps, and Stochastic Resonance in a Multistable System Driven by Gaussian Noise. H. Yuan et al. introduced and analyzed split-step theta (SST) method for nonlinear neutral stochastic differential delay equations (NSDDEs). The asymptotic mean square stability of the split-step theta (SST) method is considered for nonlinear neutral stochastic differential equations. It is proved that, under the one-sided Lipschitz condition and the linear growth condition, for all positive step sizes, the split-step theta method with $\theta \in (1/2, 1]$ is asymptotically mean square stable. The stability for the method with $\theta \in [0, 1/2]$ is also obtained under a stronger assumption. It further studies the mean square dissipativity of the split-step theta method with $\theta \in (1/2, 1]$ and proves that the method possesses a bounded absorbing set in mean square independent of initial data.

Z. Huang work is devoted to discerning asymptotic behavior dynamics through the stochastic coral reefs model with multiplicative nonlinear noise. By support theorem and Hörmander theorem, the Markov semigroup corresponding to the solutions is to prove the Foguel alternative. Based on boundary distributions theory, the required conservative operators related to the solutions are further established to ensure the existence of a stationary distribution. Meanwhile, the density of the distribution of the solutions either converges to a stationary density or weakly converges to some probability measure.

X. Wang and X. Meng investigate a new nonautonomous impulsive stochastic predator-prey system with the omnivorous predator. First, they show that the system has a unique global positive solution for any given initial positive value. Second, the extinction of the system under some appropriate conditions is explored. In addition, they obtain the sufficient conditions for the almost sure permanence in mean and stochastic permanence of the system by using the theory

of impulsive stochastic differential equations. Finally, they discuss the biological implications of the main results and show that the large noise can make the system go extinct. Simulations are also carried out to illustrate our theoretical analysis conclusions.

P. Shi et al. investigated stochastic resonance (SR) in a multistable system driven by Gaussian white noise. Using adiabatic elimination theory and three-state theory, the signal-to-noise ratio (SNR) is derived. They find the effects of the noise intensity and the resonance system parameters b, c, and d on the SNR; the results show that SNR is a nonmonotonic function of the noise intensity; therefore, a multistable SR is found in this system, and the value of the peak changes with changing the system parameters.

Hamiltonian Systems and Dynamic Optimization. F. Pierret and D. F. M. Torres derive the Helmholtz theorem for nondifferentiable Hamiltonian systems in the framework of Cresson's quantum calculus. Precisely, they give a theorem characterizing nondifferentiable equations, admitting a Hamiltonian formulation. Moreover, in the affirmative case, they give the associated Hamiltonian.

D.-S. Wang et al. investigate the effects of terms-of-trade shocks on the spending and current account where households with modified Becker-Mulligan endogenous time preference maximize their utility over an infinite planning period. The results reveal the view that with an endogenous rate of time preference the stability requirements preclude the Harberger-Laursen-Metzler effect in an infinite horizon model. Different from Obstfeld (1982), where households with Uzawa endogenous time preference are considered, deterioration in terms of trade leads to a current increase in expenditure in order to catch the new optimum. These theoretical results are consistent with the empirical evidence by numerical simulations.

Singularly Perturbed Systems and Exponential Attractor for the Boussinesq Equation. H. Xu and Y. Jin investigate a class of semilinear singularly perturbed systems with contrast structures discussed. Firstly, they verify the existence of heteroclinic orbits connecting two equilibrium points about the associated systems for contrast structures in the corresponding phase space. Secondly, the asymptotic solutions of the contrast structures by the method of boundary layer functions and smooth connection are constructed. Finally, the uniform validity of the asymptotic expansion is defined and the existence of the smooth solutions is proved. Singularly perturbed problems are often used as the models of ecology and epidemiology.

F. Geng et al. studied the existence of exponential attractor for the Boussinesq equation with strong damping and clamped boundary condition. The main result is concerned with nonlinearities with supercritical growth. In that case, they construct a bounded absorbing set with further regularity and obtain quasi-stability estimates. Then, the exponential attractor is established in natural energy space.

Bifurcation. N. Wang et al. study a predator-prey model mathematically and numerically. The aim is to explore how some key factors influence dynamic evolutionary mechanism of steady conversion and bifurcation behavior in predator-prey model. The theoretical works have been pursuing the investigation of the existence and stability of the equilibria, as well as the occurrence of bifurcation behaviors (transcritical bifurcation, saddle-node bifurcation, and Hopf bifurcation), which can deduce a standard parameter controlled relationship and in turn provide a theoretical basis for the numerical simulation. Numerical analysis ensures reliability of the theoretical results and illustrates that three stable equilibria will arise simultaneously in the model. It testifies the existence of Bogdanov-Takens bifurcation, too. It should also be stressed that the dynamic evolutionary mechanism of steady conversion and bifurcation behavior mainly depend on a specific key parameter. In a word, all these results are expected to be of use in the study of the dynamic complexity of ecosystems.

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Research Article

Asymptotic Behavior of Certain Integrodifferential Equations

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This paper deals with asymptotic behavior of nonoscillatory solutions of certain forced integrodifferential equations of the form: $(a(t)x'(t))' = e(t) + \int_c^t (t-s)^{\alpha-1} k(t,s) f(s,x(s))ds,\ c > 1,\ 0 < \alpha < 1$. From the obtained results, we derive a technique which can be applied to some related integrodifferential as well as integral equations.

1. Introduction

In this paper, we consider the integrodifferential equation

$$\left(a(t)x'(t)\right)' = e(t) + \int_c^t (t-s)^{\alpha-1} k(t,s) f(s,x(s))\,ds, \qquad c > 1,\ 0 < \alpha < 1. \tag{1}$$

In the sequel, we assume that

(i) $a, e \in C([c,\infty),\mathbb{R}^+)$;

(ii) $k \in C([c,\infty)\times[c,\infty),\mathbb{R})$ and also there exists $b \in C([c,\infty),\mathbb{R}^+)$ such that $|k(t,s)| \le b(t)$ for all $t \ge s \ge c$;

(iii) $f \in C([c,\infty)\times\mathbb{R},\mathbb{R})$ and also there exist $h \in C([c,\infty),\mathbb{R}^+)$ and real numbers λ, $0 < \lambda \le 1$, and γ such that

$$0 \le xf(t,x) \le t^{\gamma-1} h(t)\,|x|^{\lambda+1} \tag{2}$$

for $x \ne 0$ and $t \ge c$.

We only consider solutions of (1) which are continuable and nontrivial in any neighborhood of ∞. Such a solution is said to be *oscillatory* if there exists a sequence $\{t_n\} \subset [c,\infty)$, $t_n \to \infty$, such that $x(t_n) = 0$, and it is *nonoscillatory* otherwise.

In the last few decades, integral, integrodifferential, and fractional differential equations have gained considerable attention due to their applications in many engineering and scientific disciplines as the mathematical models for systems and processes in fields such as physics, mechanics, chemistry, aerodynamics, and the electrodynamics of complex media. For more details one can refer to [1–8].

Oscillation and asymptotic results for integral and integrodifferential equations are scarce; some results can be found in [5, 9–13]. It seems that there are no such results for integral equations of type (1). The main objective of this paper is to establish some new criteria on the oscillatory and the asymptotic behavior of all solutions of (1). From the obtained results, we derive a technique which can be applied to some related integrodifferential as well as integral equations.

2. Main Results

To obtain our main results of this paper, we need the following two lemmas.

Lemma 1 (see [5, 7]). *Let β, γ, and p be positive constants such that $p(\beta-1)+1 > 0$ and $p(\gamma-1)+1 > 0$. Then*

$$\int_0^t (t-s)^{p(\beta-1)} s^{p(\gamma-1)} ds = t^{\theta} B, \quad t \ge 0, \tag{3}$$

where $B := B[p(\gamma-1)+1, p(\beta-1)+1]$, $B[\zeta,\eta] = \int_0^1 s^{\zeta-1}(1-s)^{\eta-1}ds$, $\zeta,\eta > 0$, and $\theta = p(\beta+\gamma-2)+1$.

Lemma 2 (see [14]). *If X and Y are nonnegative, then*

$$X^{\lambda}-(1-\lambda)Y^{\lambda}-\lambda XY^{\lambda-1}\leq 0,\quad 0<\lambda<1, \tag{4}$$

where equality holds if and only if $X=Y$.

In what follows, we let

$$\begin{aligned} g_{\pm}(t) &= e(t)\pm(1-\lambda)\lambda^{\lambda/(1-\lambda)}b(t) \\ &\quad\cdot\int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s)\,ds \end{aligned} \tag{5}$$

and $0<\lambda<1$, $t\geq t_1$ for some $t_1\geq c$, where $m\in C([c,\infty),\mathbb{R}^{+})$.

Now we give sufficient conditions under which any solution x of (1) satisfies $|x(t)|=O(t^2)$ as $t\to\infty$.

Theorem 3. *Let $0<\lambda<1$ and conditions (i)–(iii) hold and suppose that $p>1$, $q=p/(p-1)$, $\alpha>0$, $\gamma=2-\alpha-1/p$, $p(\alpha-1)+1>0$, $p(\gamma-1)+1>0$, and*

$$\frac{t}{a(t)}\ \text{and}\ b(t)\ \text{are bounded on}\ [c,\infty), \tag{6}$$

$$\int_{t_1}^{\infty}\frac{s}{a(s)}ds<\infty, \tag{7}$$

$$\int_{t_1}^{\infty}\left(s^2m(s)\right)^{q}ds<\infty. \tag{8}$$

If

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t^2}\int_{t_1}^{t}\frac{1}{a(u)}\int_{t_0}^{u}g_{-}(s)\,ds\,du<\infty, \\ &\liminf_{t\to\infty}\frac{1}{t^2}\int_{t_1}^{t}\frac{1}{a(u)}\int_{t_0}^{u}g_{+}(s)\,ds\,du>-\infty \end{aligned} \tag{9}$$

for any $t_1\geq c$, then every nonoscillatory solution $x(t)$ of (1) satisfies

$$\limsup_{t\to\infty}\frac{|x(t)|}{t^2}<\infty. \tag{10}$$

Proof. Let x be a nonoscillatory solution of (1). We may assume that $x(t)>0$ for $t\geq t_1$ for some $t_1\geq c$. We let $F(t)=f(t,x(t))$. In view of (i)–(iii) we may then write

$$\left(a(t)x'(t)\right)'\leq e(t)+b(t)\int_{c}^{t}(t-s)^{\alpha-1}f(s,x(s))\,ds \tag{11}$$

and so

$$\begin{aligned} \left(a(t)x'(t)\right)' &\leq e(t)+b(t)\int_{c}^{t_1}(t-s)^{\alpha-1}|F(s)|\,ds \\ &\quad+b(t) \\ &\quad\cdot\int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}\left[h(s)x^{\lambda}(s)-m(s)x(s)\right]ds \\ &\quad+b(t)\int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}m(s)x(s)\,ds. \end{aligned} \tag{12}$$

Applying (4) of Lemma 2 to $h(s)x^{\lambda}(s)-m(s)x(s)$ with $X=h^{1/\lambda}x$ and $Y=((1/\lambda)mh^{-1/\lambda})^{1/(\lambda-1)}$ we have

$$\begin{aligned} &h(s)x^{\lambda}(s)-m(s)x(s) \\ &\quad\leq(1-\lambda)\lambda^{\lambda/(1-\lambda)}m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s), \end{aligned} \tag{13}$$

and hence we obtain

$$\begin{aligned} \left(a(t)x'(t)\right)' &\leq e(t)+b(t)\int_{c}^{t_1}(t-s)^{\alpha-1}|F(s)|\,ds \\ &\quad+(1-\lambda)\lambda^{\lambda/(1-\lambda)}b(t) \\ &\quad\cdot\int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s)\,ds+b(t) \\ &\quad\cdot\int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}m(s)x(s)\,ds \end{aligned} \tag{14}$$

or

$$\begin{aligned} \left(a(t)x'(t)\right)' &\leq b(t)\int_{c}^{t_1}(t_1-s)^{\alpha-1}|F(s)|\,ds+g_{+}(t) \\ &\quad+b(t)\int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}m(s)x(s)\,ds \\ &\leq C_1+g_{+}(t) \\ &\quad+k_1\int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}m(s)x(s)\,ds, \end{aligned} \tag{15}$$

where C_1 and k_1 are the upper bounds of the functions $b(t)\int_{a}^{t_1}(t_1-s)^{\alpha-1}|F(s)|ds$ and $b(t)$, respectively. Integrating inequality (15) from t_1 to t we have

$$\begin{aligned} x'(t) &\leq\frac{a(t_1)x'(t_1)}{a(t)}+\frac{C_1(t-t_1)}{a(t)} \\ &\quad+\frac{1}{a(t)}\int_{t_1}^{t}g_{+}(s)\,ds \\ &\quad+\frac{k_1}{a(t)}\int_{t_1}^{t}\int_{t_1}^{u}(u-s)^{\alpha-1}s^{\gamma-1}m(s)x(s)\,ds\,du. \end{aligned} \tag{16}$$

Interchanging the order of integration in the last integral, we have

$$\begin{aligned} x'(t) &\leq\frac{a(t_1)x'(t_1)}{a(t)}+\frac{C_1(t-t_1)}{a(t)} \\ &\quad+\frac{1}{a(t)}\int_{t_1}^{t}g_{+}(s)\,ds \\ &\quad+k_2\int_{t_1}^{t}(t-s)^{\alpha}s^{\gamma-1}m(s)x(s)\,ds, \end{aligned} \tag{17}$$

where k_2 is the upper bound of the function $k_1/\alpha a(t)$. Integrating (17) from t_1 to t and interchanging the order of integration in the last integral we find

$$\begin{aligned} x(t) \le{}& x(t_1) + a(t_1)x'(t_1)\int_{t_1}^{t}\frac{1}{a(s)}ds \\ &+ \int_{t_1}^{t}\frac{C_1(s-t_1)}{a(s)}ds + \int_{t_1}^{t}\frac{1}{a(u)}\int_{t_1}^{u} g_+(s)\,ds\,du \\ &+ \frac{k_2}{\alpha+1}\int_{t_1}^{t}(t-s)^{\alpha+1}m(s)x(s)\,ds. \end{aligned} \tag{18}$$

Now, one can easily see that

$$\begin{aligned} x(t) \le{}& x(t_1) + a(t_1)x'(t_1)\int_{t_1}^{t}\frac{1}{a(s)}ds \\ &+ \int_{t_1}^{t}\frac{C_1(s-t_1)}{a(s)}ds + \int_{t_1}^{t}\frac{1}{a(u)}\int_{t_1}^{u} g_+(s)\,ds\,du \\ &+ \frac{k_2}{\alpha+1}t^2\int_{t_1}^{t}(t-s)^{\alpha-1}m(s)x(s)\,ds \end{aligned} \tag{19}$$

or

$$z(t) := \frac{x(t)}{t^2} \le 1 + C + k\int_{t_1}^{t}(t-s)^{\alpha-1}m(s)x(s)\,ds, \tag{20}$$

where C is the upper bound of the function

$$\begin{aligned} &\frac{1}{t^2}\left[x(t_1) + a(t_1)x'(t_1)\int_{t_1}^{t}\frac{1}{a(s)}ds\right. \\ &\quad\left. + \int_{t_1}^{t}\frac{C_1(s-t_1)}{a(s)}ds + \int_{t_1}^{t}\frac{1}{a(u)}\int_{t_1}^{u} g_+(s)\,ds\,du\right] \end{aligned} \tag{21}$$

and $k = k_2/(\alpha+1)$. Applying Holder's inequality and Lemma 1 we obtain

$$\begin{aligned} &\int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}m(s)x(s)\,ds \\ &\quad\le \left(\int_{t_1}^{t}(t-s)^{p(\alpha-1)}s^{p(\gamma-1)}ds\right)^{1/p} \\ &\quad\cdot\left(\int_{t_1}^{t}m^q(s)x^q(s)\,ds\right)^{1/q} \\ &\quad\le \left(\int_{0}^{t}(t-s)^{p(\alpha-1)}s^{p(\gamma-1)}ds\right)^{1/p} \\ &\quad\cdot\left(\int_{t_1}^{t}m^q(s)x^q(s)\,ds\right)^{1/q} \le \left(Bt^{\theta}\right)^{1/p} \\ &\quad\cdot\left(\int_{t_1}^{t}m^q(s)x^q(s)\,ds\right)^{1/q}, \end{aligned} \tag{22}$$

where $B = B[p(\gamma-1)+1, p(\alpha-1)+1]$, and $\theta = p(\alpha+\gamma-2)+1 = 0$ and so

$$\begin{aligned} &\int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}m(s)x(s)\,ds \\ &\quad\le B^{1/p}\left(\int_{t_1}^{t}m^q(s)x^q(s)\,ds\right)^{1/q}. \end{aligned} \tag{23}$$

Thus, inequality (20) becomes

$$z(t) := \frac{x(t)}{t^2} \le C + kB^{1/p}\left(\int_{t_1}^{t}m^q(s)x^q(s)\,ds\right)^{1/q}. \tag{24}$$

Using (24) and the elementary inequality

$$(x+y)^q \le 2^{q-1}(x^q+y^q), \quad x, y \ge 0,\ q > 1, \tag{25}$$

we obtain from (24)

$$\begin{aligned} &z^q(t) \\ &\quad\le 2^{q-1}\left((1+C)^q + k^qB^{q/p}\int_{t_1}^{t}s^{2q}m^q(s)z^q(s)\,ds\right). \end{aligned} \tag{26}$$

If we denote $u(t) = z^q(t)$, that is, $z(t) = u^{1/q}(t)$, $P = 2^{q-1}(1+C)^q$, and $Q = 2^{q-1}k^qB^{q/p}$, then

$$u(t) \le P + Q\int_{t_1}^{t}s^{2q}m^q(s)u(s)\,ds, \quad t \ge t_1 \ge c. \tag{27}$$

The conclusion follows from Gronwall's inequality and we conclude that

$$\limsup_{t\to\infty}\frac{x(t)}{t^2} < \infty. \tag{28}$$

If x is eventually negative, we can set $y = -x$ to see that y satisfies (1) with $e(t)$ being replaced by $-e(t)$ and $f(t,x)$ by $-f(t,-y)$. It follows in a similar manner that

$$\limsup_{t\to\infty}\frac{-x(t)}{t^2} < \infty. \tag{29}$$

From (28) and (29) we get (10). This completes the proof. □

Next, by employing Theorem 3 we present the following oscillation result for (1).

Theorem 4. *Let $0 < \lambda < 1$ and conditions (i)–(iii), (6)–(9) hold and suppose that $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\gamma = 2-\alpha-1/p$, $p(\alpha-1)+1 > 0$, and $p(\gamma-1)+1 > 0$. If for every M, $0 < M < 1$,*

$$\begin{aligned} &\limsup_{t\to\infty}\left[Mt^2 + \int_{t_1}^{t}\frac{1}{a(u)}\int_{t_1}^{u}g_-(s)\,ds\,du\right] = \infty, \\ &\liminf_{t\to\infty}\left[Mt^2 + \int_{t_1}^{t}\frac{1}{a(u)}\int_{t_1}^{u}g_+(s)\,ds\,du\right] = -\infty \end{aligned} \tag{30}$$

for all $t_1 \ge c$, then (1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1), say $x(t) > 0$, for $t \geq t_1$ for some $t_1 \geq 0$. The proof when x is eventually negative is similar. Proceeding as in the proof of Theorem 3 we arrive at (19). Therefore,

$$\begin{aligned} x(t) \leq{}& x(t_1) + a(t_1) x'(t_1) \int_{t_1}^{\infty} \frac{1}{a(s)} ds \\ &+ \int_{t_1}^{\infty} \frac{C_1(s-t_1)}{a(s)} ds \\ &+ \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} g_+(s)\, ds\, du \\ &+ kt^2 \left(\int_{t_1}^{\infty} s^{2q} m^q(s) \left(\frac{x(s)}{s^2} \right)^q ds \right)^{1/q}. \end{aligned} \tag{31}$$

Clearly, the conclusion of Theorem 3 holds. This together with (7) and (8) implies that the first, second, and fourth integrals on the above inequality are bounded and hence one can easily see that

$$x(t) \leq M_1 + Mt^2 + \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} g_+(s)\, du, \tag{32}$$

where M_1 and M are positive constants. Note that we make $M < 1$ possible by increasing the size of t_1. Finally, taking lim inf in (32) as $t \to \infty$ as well as using (30) result is a contradiction with the fact that x is eventually positive. □

The following corollary is immediate.

Corollary 5. *Let $0 < \lambda < 1$ and conditions (i)–(iii), (6)–(9) hold for some $t_1 \geq c$. In addition, assume that*

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t^2} \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} e(s)\, ds\, du > \infty, \\ &\liminf_{t\to\infty} \frac{1}{t^2} \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} e(s)\, ds\, du > -\infty, \\ &\lim_{t\to\infty} \frac{1}{t^2} \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} (t-s)^{\alpha-1} s^{\gamma-1} m^{\lambda/(\lambda-1)}(s) \\ &\quad \cdot h^{1/(1-\lambda)}(s)\, ds\, du < \infty. \end{aligned} \tag{33}$$

If for every M, $0 < M < 1$,

$$\begin{aligned} &\limsup_{t\to\infty} \left[Mt^2 + \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} e(s)\, ds\, du \right] = \infty, \\ &\liminf_{t\to\infty} \left[Mt^2 + \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} e(s)\, ds\, du \right] = -\infty, \end{aligned} \tag{34}$$

for all $t_1 > c$, then (1) is oscillatory.

The following example is illustrative.

Example 6. Let $p > 1$, $0 < \alpha = 1 - 1/2p < 1$, $\alpha = \gamma$, and $q = p/(p-1)$. Clearly,

$$\begin{aligned} p(\alpha - 1) + 1 &= p(\gamma - 1) + 1 = p\left(1 - \frac{1}{2p} - 1\right) + 1 \\ &= \frac{1}{2} > 0, \\ \theta &= p(\alpha + \gamma - 2) + 1 = 0. \end{aligned} \tag{35}$$

Let the functions $a(t)$ and $b(t)$ be as in (i) and (ii) with $b(t)$ being a bounded function and let $a(t) = e^t$, $e(t) = te^t \sin t$, and $f(t,x) = t^{\gamma-1} h(t) x^{\lambda}$, where $0 < \lambda < 1$, $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $h(t) = m(t)$, $\int^{\infty} s^{2q} h^q(s) ds < \infty$, and

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_{t_1}^{t} e^{-u} \int_{t_1}^{u} (u-s)^{\alpha-1} s^{\gamma-1} h(s)\, ds\, du < \infty. \tag{36}$$

Condition (34) is also fulfilled. Thus, all conditions of Theorem 3 are satisfied and hence every nonoscillatory solution x of (1) satisfies $\limsup_{t\to\infty}(|x(t)|/t^2) < \infty$.

Now if $e(t) = t^{\delta} e^t \sin t$, $\delta \geq 2$, we see that all the hypotheses of Corollary 5 are satisfied and hence (1) is oscillatory.

Similar reasoning to that in the sublinear case guarantees the following theorems for the integrodifferential equation (1) when $\lambda = 1$.

Theorem 7. *Let $\lambda = 1$ and the hypotheses of Theorems 3 and 4 hold with $m(t) = h(t)$ and $g_{\pm} = e(t)$. Then the conclusion of Theorems 3 and 4 holds, respectively.*

From the obtained results, we apply the employed technique to some related integrodifferential equations.

Now, we consider the integrodifferential equation

$$\begin{aligned} x'(t) &= e(t) + \int_{c}^{t} (t-s)^{\alpha-1} k(t,s) f(s, x(s))\, ds, \\ &\qquad c > 1,\ \alpha \in (0,1). \end{aligned} \tag{37}$$

We will give sufficient conditions under which any nonoscillatory solution x of (37) satisfies $|x(t)| = O(t)$ as $t \to \infty$.

Theorem 8. *Let $0 < \lambda < 1$ and let condition (ii) hold and suppose that $p > 1$, $q = p/(p-1)$, $0 < \alpha < 1$, and $\gamma = 2 - \alpha - 1/p$, $p(\alpha - 1) + 1 > 0$, and $p(\gamma - 1) + 1 > 0$,*

$$\int_{t_1}^{\infty} s^q m^q(s)\, ds < \infty, \tag{38}$$

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t} \int_{c}^{t} g_-(s)\, ds < \infty, \\ &\liminf_{t\to\infty} \frac{1}{t} \int_{c}^{t} g_+(s)\, ds > -\infty \end{aligned} \tag{39}$$

for any $t_1 \geq c$. If x is a nonoscillatory solution of (37), then

$$\limsup_{t\to\infty} \frac{|x(t)|}{t} < \infty. \tag{40}$$

Proof. Let x be a nonoscillatory solution of (37). We may assume that $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. We let $F(t) = f(t, x(t))$. In view of (ii) we may then write

$$\begin{aligned} x'(t) &\leq e(t) + \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds \\ &\quad + \int_{t_1}^{t} (t-s)^{\alpha-1} s^{\gamma-1} \left[h(s) x^{\lambda}(s) - m(s) x(s)\right] ds \\ &\quad + \int_{t_1}^{t} (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s)\, ds. \end{aligned} \tag{41}$$

Proceeding as in the proof of Theorem 3, we obtain

$$\begin{aligned} x'(t) &\leq \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds + e(t) + (1-\lambda) \\ &\quad \cdot \lambda^{\lambda/(1-\lambda)} \int_{t_1}^{t} (t-s)^{\alpha-1} s^{\gamma-1} m^{\lambda/(\lambda-1)}(s) \\ &\quad \cdot h^{1/(1-\lambda)}(s)\, ds + \int_{t_1}^{t} (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s)\, ds. \end{aligned} \tag{42}$$

Integrating inequality (42) from t_1 to t and interchanging the order of integration one can easily obtain

$$\begin{aligned} x(t) &\leq x(t_1) + \int_{t_1}^{t} g_+(s)\, ds \\ &\quad + \int_{t_1}^{t} \int_{t_1}^{u} (u-s)^{\alpha-1} s^{\gamma-1} m(s) x(s)\, ds\, du \\ &\quad + \int_{t_1}^{t} \int_{c}^{t_1} (t_1-s)^{\alpha-1} |F(s)|\, ds\, du. \end{aligned} \tag{43}$$

Interchanging the order of integration in second integral we have

$$\begin{aligned} x(t) &\leq x(t_1) + \int_{t_1}^{t} g_+(s)\, ds \\ &\quad + \frac{t}{\alpha} \int_{t_1}^{t} (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s)\, ds \\ &\quad + \int_{t_1}^{t} \int_{c}^{t_1} (t_1-s)^{\alpha-1} |F(s)|\, ds\, du. \end{aligned} \tag{44}$$

The rest of the proof is similar to that of Theorem 3 and hence is omitted. □

Example 9. Let $p > 1$, $0 < \alpha = 1 - 1/2p < 1$, $\alpha = \gamma$, and $q = p/(p-1)$. Clearly,

$$\begin{aligned} p(\alpha-1) + 1 &= p(\gamma-1) + 1 = p\left(1 - \frac{1}{2p} - 1\right) + 1 \\ &= \frac{1}{2} > 0, \\ \theta &= p(\alpha+\gamma-2) + 1 = 0. \end{aligned} \tag{45}$$

Let the functions $e(t) = t \sin t$ and $f(t, x) = t^{\gamma-1} h(t) x^{\lambda}$, where $0 < \lambda < 1$, $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $h(t) = m(t)$, $\int^{\infty} s^q h^q(s) ds < \infty$, and

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_1}^{t} (t-s)^{\alpha-1} s^{\gamma-1} h(s)\, ds < \infty. \tag{46}$$

Condition (39) is also fulfilled. Thus, all conditions of Theorem 8 are satisfied and hence every nonoscillatory solution x of (37) satisfies $\limsup_{t\to\infty}(|x(t)|/t) < \infty$.

Finally, we consider the integral equation

$$x(t) = e(t) + \int_c^{t} (t-s)^{\alpha-1} k(t,s) f(s, x(s))\, ds, \qquad c > 1,\ \alpha \in (0,1). \tag{47}$$

Now we give sufficient conditions for the boundedness of any nonoscillatory solution of (47).

Theorem 10. *Let $0 < \lambda < 1$ and let condition (ii) hold and suppose that $p > 1$, $q = p/(p-1)$, $0 < \alpha < 1$, and $\gamma = 2 - \alpha - 1/p$, $p(\alpha-1) + 1 > 0$, and $p(\gamma-1) + 1 > 0$,*

$$\int_{t_1}^{\infty} m^q(s)\, ds < \infty, \tag{48}$$

$$\begin{aligned} &\limsup_{t\to\infty} g_-(t) < \infty, \\ &\liminf_{t\to\infty} g_+(t) > -\infty, \end{aligned} \tag{49}$$

where $g(t)$ is defined as in (5) for any $t_1 \geq c$. If x is a nonoscillatory solution of (47), then x is bounded.

Proof. Let x be an eventually positive solution of (47). We may assume that $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. We let $F(t) = f(t, x(t))$. In view of (ii) we may then write

$$\begin{aligned} x(t) &\leq \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds + e(t) \\ &\quad + \int_{t_1}^{t} (t-s)^{\alpha-1} s^{\gamma-1} \left[h(s) x^{\lambda}(s) - m(s) x(s)\right] ds \\ &\quad + \int_{t_1}^{t} (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s)\, ds \end{aligned} \tag{50}$$

or

$$\begin{aligned} x(t) &\leq \int_c^{t} (t-s)^{\alpha-1} |F(s)|\, ds + g_+(t) \\ &\quad + \int_{t_1}^{t} (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s)\, ds. \end{aligned} \tag{51}$$

The rest of the proof is similar to that of Theorem 3 and hence is omitted. □

Example 11. Let $p > 1$, $0 < \alpha = 1 - 1/2p < 1$, $\alpha = \gamma$, and $q = p/(p-1)$. Clearly,

$$\begin{aligned} p(\alpha - 1) + 1 &= p(\gamma - 1) + 1 = p\left(1 - \frac{1}{2p} - 1\right) + 1 \\ &= \frac{1}{2} > 0, \\ \theta &= p(\alpha + \gamma - 2) + 1 = 0. \end{aligned} \tag{52}$$

Let the functions $e(t) = \sin t$ and $f(t,x) = t^{\gamma-1}h(t)x^{\lambda}$, where $0 < \lambda < 1$, $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $h(t) = m(t)$, $\int^{\infty} h^q(s)ds < \infty$, and

$$\limsup_{t\to\infty} \int_{t_1}^{t} (t-s)^{\alpha-1} s^{\gamma-1} h(s)\, ds < \infty. \tag{53}$$

Condition (49) is also fulfilled. Thus, all conditions of Theorem 10 are satisfied and hence every nonoscillatory solution x of (37) is bounded.

Similar reasoning to that in the sublinear case guarantees the following theorems for the integrodifferential equations (37) and (47) when $\lambda = 1$.

Theorem 12. *Let $\lambda = 1$ and the hypotheses of Theorems 8 and 10 hold with $m(t) = h(t)$. Then the conclusion of Theorems 8 and 10 holds.*

We may note that results similar to Theorem 4 can be obtained for (37) and (47). The details are left to the reader.

3. General Remarks

(i) The results of this paper are presented in a form which is essentially new and it can also be employed to investigate the asymptotic and oscillatory behavior of certain integrodifferential equations of higher order $\alpha \in (n-1, n)$, $n \geq 1$. The details are left to the reader.

(ii) It would be of interest to study (1) when f satisfies condition (iii) with $\lambda > 1$.

Competing Interests

The authors declare that they have no competing interests.

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Research Article

On Mean Square Stability and Dissipativity of Split-Step Theta Method for Nonlinear Neutral Stochastic Delay Differential Equations

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A split-step theta (SST) method is introduced and used to solve the nonlinear neutral stochastic delay differential equations (NSDDEs). The mean square asymptotic stability of the split-step theta (SST) method for nonlinear neutral stochastic delay differential equations is studied. It is proved that under the one-sided Lipschitz condition and the linear growth condition, the split-step theta method with $\theta \in (1/2, 1]$ is asymptotically mean square stable for all positive step sizes, and the split-step theta method with $\theta \in [0, 1/2]$ is asymptotically mean square stable for some step sizes. It is also proved in this paper that the split-step theta (SST) method possesses a bounded absorbing set which is independent of initial data, and the mean square dissipativity of this method is also proved.

1. Introduction

Stochastic functional differential equations (SFDEs) play important roles in science and engineering applications, especially for systems whose evolutions in time are influenced by random forces as well as their history information. When the time delays in SFDEs are constants, they turn into stochastic delay differential equations (SDDEs). Both the theory and numerical methods for SDDEs have been well developed in the recent decades; see [1–8]. Recently, many dynamical systems not only depend on the present and the past states but also involve derivatives with delays; they are described as the neutral stochastic delay differential equations (NSDDEs). Compared to the stochastic differential equations and the stochastic delay differential equations, the study of the neutral stochastic delay differential equations has just started. In 1981, Kolmanovskii and Myshkis [9] took the environmental disturbances into account, introduced the neutral stochastic delay differential equations (NSDDEs), and gave their applications in chemical engineering and aeroelasticity. The analytical solutions of NSDDEs are hard to obtain; many authors have to study the numerical methods for NSDDEs. Wu and Mao [10] studied the convergence of the Euler-Maruyama method for neutral stochastic functional differential equations under the one-side Lipschitz conditions and the linear growth conditions. In 2009, Zhou and Wu [11] studied the convergence of the Euler-Maruyama method for NSDDEs with Markov switching under the one-side Lipschitz conditions and the linear growth conditions. The convergence of θ-method and the mean square asymptotic stability of the semi-implicit Euler method for NSDDEs were studied by Gan et al. [12], Zhou and Fang [13], and Yin and Ma [14], respectively. Later, the almost sure exponential stability of Euler-Maruyama method for NSDDEs was studied in [15] with the discrete semimartingale convergence theorem.

To the best of our knowledge, most of these studies have focused on the convergence of numerical solutions for NSDDEs; the stability and dissipativity of numerical solutions for them are rarely concerned.

The aim of this paper is to study the mean square stability and dissipativity of the split-step theta method with some conditions and the step constrained for NSDDEs.

The paper is organized as follows. In Section 2, some stability definitions about the analytic solutions for NSDDEs are introduced; some notations and preliminaries are also presented in this section. In Section 3, the split-step theta method is introduced and used to solve the NSDDEs; the asymptotic stability of the split-step theta method is proved. In Section 4, the long time behavior of numerical solution is studied and the mean square dissipativity result of the method is illustrated. In Section 5, some numerical experiments are given to confirm the theoretical results.

2. Exponential Mean Square Stability of Analytic Solution

Let $|\cdot|$ denote both the Euclidean norm in R^d and the trace (or Frobenius) norm in $R^{d\times l}$ (denoted by $|A| = \sqrt{\operatorname{trace}(A^{\mathrm{T}}A)}$); if A is a vector or matrix, its transpose is denoted by A^{T}. Let $\{\Omega, F, \{F_t\}_{t\geq 0}, \mathrm{P}\}$ define a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ which is increasing and right continuous, and F_0 contain all P-null sets. Let $w(t) = (w_1(t), w_2(t), \ldots, w_l(t))^{\mathrm{T}}$ denote standard l-dimensional Brownian motion on the probability space. In this paper we talk about the d-dimensional NSDDEs with the following form:

$$\begin{aligned} &d\left(y(t) - N\left(y(t-\tau)\right)\right) \\ &\quad = f\left(t, y(t), y(t-\tau)\right) dt \\ &\qquad + g\left(t, y(t), y(t-\tau)\right) dw(t), \quad t \geq 0, \\ &y(t) = \varphi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{1}$$

where $N : R^d \mapsto R^d$, $f : R_+ \times R^d \times R^d \mapsto R^d$, and $g : R_+ \times R^d \times R^d \mapsto R^{d\times l}$ are the Borel measurable functions. τ is a positive constant delay, and $\varphi(t)$ is F_0-measurable, $C([-\tau, 0]; R^d)$-valued random variable which satisfies

$$\sup_{-\tau\leq t\leq 0} \mathrm{E}\left[\varphi^{\mathrm{T}}(t)\varphi(t)\right] < +\infty \tag{2}$$

with the notation E denoting the mathematical expectation with respect to P.

The following conditions (a1) and (a2) are standard for the existence and uniqueness of the solution for (1).

(a1) The Local Lipschitz Condition. There exist constants $K_L > 0$ and $L > 0$ such that

$$\begin{aligned} &\left|f(t, x_1, y_1) - f(t, x_2, y_2)\right|^2 \\ &\quad \vee \left|g(t, x_1, y_1) - g(t, x_2, y_2)\right|^2 \\ &\quad \leq K_L\left(\left|x_1 - x_2\right|^2 + \left|y_1 - y_2\right|^2\right), \end{aligned} \tag{3}$$

for all $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq L$ and $t \in R_+$, where $a \vee b$ represents $\max\{a, b\}$ and $a \wedge b$ represents $\min\{a, b\}$.

(a2) The Linear Growth Condition. There exists a constant $K_G > 0$, such that

$$\begin{aligned} &\left|f(t, x, y)\right|^2 \vee \left|g(t, x, y)\right|^2 \vee |N(x)|^2 \\ &\quad \leq K_G\left(1 + |x|^2 + |y|^2\right), \end{aligned} \tag{4}$$

for all $(t, x, y) \in R_+ \times R \times R$.

As an especial case of Theorem 3.1 in Mao's monograph (see [6]), we can easily know that under hypothesis (a1) and (a2), system (1) has a global unique continuous solution on $t \geq -\tau$, which is denoted by $y(t)$.

Now we recall some stability concepts for the solution of (1).

Definition 1 (see [6]). The trivial solution of (1) is said to be exponentially mean square stable, if there exists a pair of constants $r > 0$ and $C > 0$, such that, whenever $\sup_{-\tau\leq t\leq 0}\mathrm{E}[\varphi^{\mathrm{T}}(t)\varphi(t)] < +\infty$,

$$\mathrm{E}\left[y^{\mathrm{T}}(t)\, y(t)\right] \leq C \sup_{-\tau\leq t\leq 0} \mathrm{E}\left[\varphi^{\mathrm{T}}(t)\varphi(t)\right] e^{-rt}, \quad t \geq 0. \tag{5}$$

Lemma 2. *Assume that there exist a symmetric, positive definite $d \times d$ matrix Q and positive constants μ_1, μ_2, and $\lambda \in (0, 1)$ such that for all $(t, x, y) \in R_+ \times R^d \times R^d$*

$$|N(x)| \leq \lambda |x|, \tag{6}$$

$$\begin{aligned} &(x - N(y))^{\mathrm{T}} Q f(t, x, y) \\ &\quad + \frac{1}{2}\operatorname{trace}\left[g^{\mathrm{T}}(t, x, y) Q g(t, x, y)\right] \leq -\mu_1 x^{\mathrm{T}} Q x \\ &\quad + \mu_2 y^{\mathrm{T}} Q y. \end{aligned} \tag{7}$$

If conditions

$$\begin{aligned} 0 < \lambda < \frac{1}{2}, \\ \mu_1 > \frac{\mu_2}{(1-2\lambda)^2} \end{aligned} \tag{8}$$

hold, then the trivial solution of (1) is exponentially mean square stable.

Remark 3. In general, we require $\lambda \neq 0$. When $\lambda = 0$, (1) becomes a stochastic delay differential equation. Many stability and dissipativity results have been studied in the literature (see [5, 16]).

By Lemma 2, the following result can easily be obtained.

Theorem 4. *Suppose (6) holds. Assume that there are positive constants λ_1, λ_2, and K, such that, for all $x, y \in R^d$,*

$$\begin{aligned} &(x - N(y))^{\mathrm{T}} Q f(t, x, y) \leq -\lambda_1 x^{\mathrm{T}} Q x + \lambda_2 y^{\mathrm{T}} Q y, \\ &\left|f(t, x, y)\right|^2 \vee \left|g(t, x, y)\right|^2 \leq K\left(|x|^2 + |y|^2\right). \end{aligned} \tag{9}$$

If conditions

$$0 < \lambda < \frac{1}{2},$$
$$\lambda_1 > \frac{1}{2}K + \frac{2\lambda_2 + K}{2(1-2\lambda)^2} \tag{10}$$

hold, then the trivial solution of (1) is exponentially mean square stable.

Proof. Consider (9) and the inequality; we get the following inequality:

$$\begin{aligned}
&(x - N(y))^{\mathrm{T}} Qf(t,x,y) \\
&\quad + \frac{1}{2}\operatorname{trace}\left[g^{\mathrm{T}}(t,x,y)Qg(t,x,y)\right] \\
&\le -\lambda_1 x^{\mathrm{T}}Qx + \lambda_2 y^{\mathrm{T}}Qy + \frac{1}{2}K\left(x^{\mathrm{T}}Qx + y^{\mathrm{T}}Qy\right) \\
&\le -\left(\lambda_1 - \frac{1}{2}K\right)x^{\mathrm{T}}Qx + \left(\lambda_2 + \frac{1}{2}K\right)\mu_2 y^{\mathrm{T}}Qy.
\end{aligned} \tag{11}$$

Let $\mu_1 = (\lambda_1 - (1/2)K)$, $\mu_2 = (\lambda_2 + (1/2)K)$; when conditions (10) hold, we get that

$$\mu_1 > \frac{\mu_2}{(1-2\lambda)^2}. \tag{12}$$

Using Lemma 2, we can easily prove that the trivial solution of (1) is exponentially mean square stable. □

3. The Stability of the Split-Step Theta Method

The split-step theta method is proved to be able to keep the mean square asymptotic stability of the exact solution under the sufficient conditions of the asymptotic stability of the exact solution, so in this paper we use the split-step theta method to solve the NSDDE.

Appling the split-step theta (SST) method into problem (1) gives the following form:

$$\begin{aligned}
Y_n - NY_{n-m} &= y_n - Ny_{n-m} \\
&\quad + \theta\Delta t f\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right),
\end{aligned} \tag{13}$$

$$\overline{Y}_n = Y_{n-m}, \tag{14}$$

$$\begin{aligned}
y_{n+1} - Ny_{n+1-m} &= y_n - Ny_{n-m} \\
&\quad + \Delta t f\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) \\
&\quad + g\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right)\Delta w_n,
\end{aligned} \tag{15}$$

where the step size $\Delta t = \tau/m$, m is an integer, y_i is an approximation to $y(t_i)$, $t_i = i\Delta t$, $i = 1, 2, \ldots$, and $y_k = Y_k = \varphi(k\Delta t)$ for $k = -m, -m+1, \ldots, 0$. $\theta \in [0,1]$ is a fixed parameter, and $\Delta w_k := w((k+1)\Delta t) - w(k\Delta t)$ is the Brownian increment.

When $\theta = 0$ the split-step theta method is simplified into the split-step forward Euler method and when $\theta = 1$ the split-step theta method is simplified into the split-step backward Euler method. They were discussed for stochastic differential equations in [17–20]. In order to consider the stability property of scheme (13)–(15) we should give some stability concepts for numerical methods firstly.

Definition 5 (see [16]). For a given step size Δt, a numerical method is said to be exponentially mean square stable if there is a pair of positive constants γ and C such that for any initial data $\varphi(t)$ the numerical solution y_n produced by the method satisfies

$$\mathrm{E}\left[y_n^{\mathrm{T}} y_n\right] \le Ce^{-\gamma t_n} \sup_{-\tau\le t\le 0} \mathrm{E}\left[\varphi^{\mathrm{T}}(t)\varphi(t)\right], \quad \forall n \ge 0. \tag{16}$$

Definition 6 (see [16]). For a given step size Δt, a numerical method is said to be asymptotically mean square stable if for any initial data $\varphi(t)$ the numerical solution y_n produced by the method satisfies

$$\lim_{n\to\infty} \mathrm{E}\left[y_n^{\mathrm{T}} y_n\right] = 0. \tag{17}$$

Theorem 7. *Assume that system (1) satisfies (7) with $-\mu_1 + \mu_2 < 0$; then the SST method (13)–(15) with $\theta \in (1/2, 1]$ is asymptotically mean square stable for all $\Delta t > 0$. If we further assume that there exist constants K_1 and K_2 such that*

$$\begin{aligned}
f^{\mathrm{T}}(t,x,y)Qf(t,x,y) &\le K_1 x^{\mathrm{T}}Qx + K_2 y^{\mathrm{T}}Qy, \\
&\qquad (t,x,y) \in R_+ \times R^d \times R^d,
\end{aligned} \tag{18}$$

then, for any $\theta \in [0, 1/2)$, there exists a constant Δt_0 depending on θ such that the method is asymptotically mean square stable for $\Delta t \in (0, \Delta t_0)$.

Proof. From (15) it follows that

$$\begin{aligned}
&(y_{n+1} - Ny_{n+1-m})^{\mathrm{T}} Q(y_{n+1} - Ny_{n+1-m}) \\
&\quad = (y_n - Ny_{n-m})^{\mathrm{T}} Q(y_n - Ny_{n-m}) \\
&\quad + \Delta t^2 f^{\mathrm{T}}\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) Qf\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) \\
&\quad + \Delta w_n^{\mathrm{T}} g^{\mathrm{T}}\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) Qg\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) \\
&\quad \cdot \Delta w_n + 2(y_n - Ny_{n-m})^{\mathrm{T}} \Delta t Qf\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) \\
&\quad + 2(y_n - Ny_{n-m})^{\mathrm{T}} Qg\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right)\Delta w_n \\
&\quad + 2\Delta t f^{\mathrm{T}}\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) Qg\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) \\
&\quad \cdot \Delta w_n.
\end{aligned} \tag{19}$$

Since $w(t) = (w_1(t), w_2(t), \ldots, w_l(t))^{\mathrm{T}}$ is a standard l-dimensional Brownian motion we have that

$$\begin{aligned} &\mathrm{E}(\Delta w_i) = 0, \\ &\mathrm{E}\left[(\Delta w_i)^2\right] = \Delta t, \\ &\mathrm{E}\left[\Delta w_n^{\mathrm{T}} g^{\mathrm{T}}\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) Q g\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right)\right. \\ &\quad \left.\cdot \Delta w_n\right] = \Delta t \mathrm{E}\left[\mathrm{trace} g^{\mathrm{T}}\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right)\right. \\ &\quad \left.\cdot Q g\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right)\right]. \end{aligned} \tag{20}$$

Let $x_n = y_n - N y_{n-m}$, $X_n = Y_n - N Y_{n-m}$, $n = 0, 1, \ldots$, substitute the designation into (19) and then, taking expectation on both sides, one receives

$$\begin{aligned} &\mathrm{E}\left[x_{n+1}^{\mathrm{T}} Q x_{n+1}\right] \le \mathrm{E}\left[x_n^{\mathrm{T}} Q x_n\right] + (1-2\theta)\Delta t^2 f^{\mathrm{T}}\left(t_n\right. \\ &\quad \left. + \theta\Delta t, Y_n, \overline{Y}_n\right) Q f\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) + 2\Delta t \mathrm{E}\left(Y_n\right. \\ &\quad \left. - N Y_{n-m}\right)^{\mathrm{T}} \Delta t Q f\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) \\ &\quad + \Delta t \mathrm{E}\left[\mathrm{trace} g^{\mathrm{T}}\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right)\right. \\ &\quad \left.\cdot Q g\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right)\right], \end{aligned} \tag{21}$$

which, combined with (7), gives

$$\begin{aligned} &\mathrm{E}\left[x_{n+1}^{\mathrm{T}} Q x_{n+1}\right] \le \mathrm{E}\left[x_n^{\mathrm{T}} Q x_n\right] \\ &\quad + 2\Delta t \mathrm{E}\left(-\mu_1 Y_n^{\mathrm{T}} Q Y_n + \mu_2 \overline{Y}_n^{\mathrm{T}} Q \overline{Y}_n\right) + (1-2\theta) \\ &\quad \cdot \Delta t^2 f^{\mathrm{T}}\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) Q f\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right). \end{aligned} \tag{22}$$

In the case of $\theta > 1/2$, using

$$\begin{aligned} &\Delta t f\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) = \frac{1}{\theta}\left(X_n - x_n\right), \\ &2 X_n^{\mathrm{T}} Q x_n \\ &\quad \le \frac{2\theta - 1 - (-\mu_1 + \mu_2)\Delta t \theta^2}{2\theta - 1} X_n^{\mathrm{T}} Q X_n \\ &\quad + \frac{2\theta - 1}{2\theta - 1 - (-\mu_1 + \mu_2)\Delta t \theta^2} x_n^{\mathrm{T}} Q x_n, \end{aligned} \tag{23}$$

then we have

$$\begin{aligned} &\mathrm{E}\left[x_{n+1}^{\mathrm{T}} Q x_{n+1}\right] \\ &\quad \le \left(1 + \frac{(-\mu_1 + \mu_2)\Delta t (2\theta - 1)}{2\theta - 1 - (-\mu_1 + \mu_2)\Delta t \theta^2}\right) \mathrm{E}\left[x_n^{\mathrm{T}} Q x_n\right] \\ &\quad - 2\Delta t \mu_2 \mathrm{E}\left[Y_n^{\mathrm{T}} Q Y_n\right] \\ &\quad + 2\Delta t\left(\left(1 - \lambda^2\right)\mu_2 + \lambda^2 \mu_1\right) \mathrm{E}\left[\overline{Y}_n^{\mathrm{T}} Q \overline{Y}_n\right]. \end{aligned} \tag{24}$$

Let

$$\begin{aligned} k = \max\Bigg\{&1 + \frac{(-\mu_1 + \mu_2)\Delta t (2\theta - 1)}{2\theta - 1 - (-\mu_1 + \mu_2)\Delta t \theta^2}, \\ &\left(\frac{\mu_2}{\left(\left(1 - \lambda^2\right)\mu_2 + \lambda^2 \mu_1\right)}\right)^{1/m}\Bigg\}; \end{aligned} \tag{25}$$

we can deduce that $0 < k < 1$.

By induction, the following results are obtained from (24):

$$\begin{aligned} &\mathrm{E}\left[x_{n+1}^{\mathrm{T}} Q x_{n+1}\right] \\ &\quad \le k^{n+1} \mathrm{E}\left[x_0^{\mathrm{T}} Q x_0\right] - 2\Delta t \mu_2 \sum_{j=0}^{n} k^{n-j} \mathrm{E}\left[Y_j^{\mathrm{T}} Q Y_j\right] \\ &\quad + 2\Delta t\left(\left(1 - \lambda^2\right)\mu_2 + \lambda^2 \mu_1\right) \sum_{j=0}^{n} k^{n-j} \mathrm{E}\left[\overline{Y}_j^{\mathrm{T}} Q \overline{Y}_j\right]. \end{aligned} \tag{26}$$

Using condition (14), we can get the following inequality:

$$\begin{aligned} \sum_{j=0}^{n} k^{n-j} \mathrm{E}\left[\overline{Y}_j^{\mathrm{T}} Q \overline{Y}_j\right] &\le m k^{n-m+1} \max_{-m \le j \le -1} \mathrm{E}\left[Y_j^{\mathrm{T}} Q Y_j\right] \\ &\quad + k^{-m} \sum_{j=0}^{n-m+1} k^{n-j} \mathrm{E}\left[Y_j^{\mathrm{T}} Q Y_j\right]. \end{aligned} \tag{27}$$

Therefore,

$$\begin{aligned} &\mathrm{E}\left[x_{n+1}^{\mathrm{T}} Q x_{n+1}\right] \le k^{n+1}\Bigg(\mathrm{E}\left[x_0^{\mathrm{T}} Q x_0\right] \\ &\quad + 2\tau\left(\left(1 - \lambda^2\right)\mu_2 + \lambda^2 \mu_1\right) k^{-m} \max_{-m \le j \le -1} \mathrm{E}\left[Y_j^{\mathrm{T}} Q Y_j\right]\Bigg) \\ &\quad - 2\Delta t\left(\mu_2 - \left(\left(1 - \lambda^2\right)\mu_2 + \lambda^2 \mu_1\right) k^{-m}\right) \\ &\quad \cdot \sum_{j=0}^{n-m+1} k^{n-j} \mathrm{E}\left[Y_j^{\mathrm{T}} Q Y_j\right]. \end{aligned} \tag{28}$$

It can be deduced from (28) and (25) that $-(\mu_2 - ((1-\lambda^2)\mu_2 + \lambda^2\mu_1)k^{-m}) \le 0$, so, we can have the following inequality:

$$\begin{aligned} &\mathrm{E}\left[x_{n+1}^{\mathrm{T}} Q x_{n+1}\right] \le k^{n+1}\Bigg(\mathrm{E}\left[x_0^{\mathrm{T}} Q x_0\right] \\ &\quad + 2\tau\left(\left(1 - \lambda^2\right)\mu_2 + \lambda^2 \mu_1\right) k^{-m} \max_{-m \le j \le -1} \mathrm{E}\left[Y_j^{\mathrm{T}} Q Y_j\right]\Bigg). \end{aligned} \tag{29}$$

On the other hand, we know that

$$\begin{aligned} \left\|y_{n+1}\right\| &= \left\|y_{n+1} - N y_{n+1-m} + N y_{n+1-m}\right\| \\ &\le \left\|x_{n+1}\right\| + \left\|N y_{n+1-m}\right\|, \end{aligned} \tag{30}$$

then we get

$$\begin{aligned} \mathrm{E}\left[y_{n+1}^{\mathrm{T}} Q y_{n+1}\right] &\le 2\mathrm{E}\left[x_{n+1}^{\mathrm{T}} Q x_{n+1}\right] \\ &\quad + 2\lambda^2 \mathrm{E}\left[y_{n+1-m}^{\mathrm{T}} Q y_{n+1-m}\right]; \end{aligned} \tag{31}$$

define

$$\varepsilon_0 = k^{n+1}\left(\mathrm{E}\left[x_0^{\mathrm{T}}Qx_0\right] + 2\tau\left(\left(1-\lambda^2\right)\mu_2 + \lambda^2\mu_1\right)k^{-m} \cdot \max_{-m\le j\le -1}\mathrm{E}\left[Y_j^{\mathrm{T}}QY_j\right]\right); \tag{32}$$

the following inequality could be deduced from (31):

$$\mathrm{E}\left[y_{n+1}^{\mathrm{T}}Qy_{n+1}\right] \le \frac{2}{1-2\lambda^2}\varepsilon_0 + \left(2\lambda^2\right)^{\lfloor n/m\rfloor+1}\max_{-m\le j\le -1}\mathrm{E}\left[y_j^{\mathrm{T}}Qy_j\right], \tag{33}$$

which implies that the method is asymptotically mean square stable.

For the case that $\theta \in [0, 1/2)$, with the hypothesis (18) and (22) we can obtain the following inequality:

$$\begin{aligned}\mathrm{E}\left[x_{n+1}^{\mathrm{T}}Qx_{n+1}\right] &\le \mathrm{E}\left[x_n^{\mathrm{T}}Qx_n\right] \\ &\quad + \Delta t\left((1-2\theta)\Delta tK_1 - 2\mu_1\right)\mathrm{E}\left[Y_n^{\mathrm{T}}QY_n\right] \\ &\quad + \Delta t\left((1-2\theta)\Delta tK_2 + 2\mu_2\right)\mathrm{E}\left[\overline{Y}_n^{\mathrm{T}}Q\overline{Y}\right].\end{aligned} \tag{34}$$

A combination of (13) and (18) gives

$$x_n^{\mathrm{T}}Qx_n \le L_1Y_n^{\mathrm{T}}QY_n + L_2\overline{Y}_n^{\mathrm{T}}Q\overline{Y}_n, \tag{35}$$

where $L_1 = (1+\theta\Delta t)(2+\theta\Delta tK_1)$, $L_2 = (1+\theta\Delta t)(2\lambda^2+\theta\Delta tK_2)$.

Let

$$\Delta t_0 = \begin{cases} +\infty, & \theta = \dfrac{1}{2}, \\ \dfrac{-2\left(-\mu_1+\mu_2\right)}{(1-2\theta)\left(K_1+K_2\right)}, & \theta \in \left[0, \dfrac{1}{2}\right); \end{cases} \tag{36}$$

then, for any fixed $\Delta t \in (0, \Delta t_0)$, $2(-\mu_1+\mu_2)+\Delta t(1-2\theta)(K_1+K_2) < 0$, there exists a small positive number ε such that

$$2\left(-\mu_1+\mu_2\right) + \Delta t\left(1-2\theta\right)\left(K_1+K_2\right) + \frac{L_1+L_2}{\Delta t}\varepsilon < 0. \tag{37}$$

Therefore,

$$\begin{aligned}\mathrm{E}\left[x_{n+1}^{\mathrm{T}}Qx_{n+1}\right] &\le (1-\varepsilon)\,\mathrm{E}\left[x_n^{\mathrm{T}}Qx_n\right] \\ &\quad + \Delta t\left((1-2\theta)\Delta tK_1 - 2\mu_1 + \frac{L_1}{\Delta t}\varepsilon\right)\mathrm{E}\left[Y_n^{\mathrm{T}}QY_n\right] \\ &\quad + \Delta t\left((1-2\theta)\Delta tK_2 + 2\mu_2 + \frac{L_2}{\Delta t}\varepsilon\right)\mathrm{E}\left[\overline{Y}_n^{\mathrm{T}}Q\overline{Y}_n\right].\end{aligned} \tag{38}$$

Let $\widetilde{k} = \max\{1-\varepsilon, (((1-2\theta)\Delta tK_2 + 2\mu_2 + (L_2/\Delta t)\varepsilon)/-((1-2\theta)\Delta tK_1 - 2\mu_1 + (L_1/\Delta t)\varepsilon))^{1/m}\}$; then $0 < \widetilde{k} < 1$. Similar to the derivation of the first part, the following inequality can be proved from (38):

$$\mathrm{E}\left[x_{n+1}^{\mathrm{T}}Qx_{n+1}\right] \le k^{n+1}\left(\mathrm{E}\left[x_0^{\mathrm{T}}Qx_0\right] + \widetilde{L}\widetilde{k}^{-m}\max_{-m\le j\le -1}\mathrm{E}\left[Y_j^{\mathrm{T}}QY_j\right]\right), \tag{39}$$

where $\widetilde{L} = \tau((1-2\theta)\Delta tK_2 + 2\mu_2 + (L_2/\Delta t)\varepsilon)$.

Similar to the proof of (31), we can prove that when $\Delta t \in (0, \Delta t_0)$ the method is asymptotically mean square stable; the proof of theorem is completed. □

Remark 8. For system (1) with $N = 0$, it becomes a stochastic delay differential equation; the mean square stability of the theta method has been studied in [16]; Theorem 7 can be regarded as an extension of Theorem 3.4 presented in [16].

Remark 9. For the NSDDEs, the mean square asymptotic stability of the BEM method has been studied by Wang and Chen in [15]; it has shown that BEM method can reproduce the mean square stability of the exact solutions; Theorem 7 improves the result in [15].

4. Mean Square Dissipativity

The numerical solutions' long time dynamic behavior will be studied in this section. Before it, we make the following hypothesis: assume that there exist a symmetric, positive definite $d \times d$ matrix Q and positive constants μ_1, μ_2, and γ such that, for all $(t, x, y) \in R_+ \times R^d \times R^d$, the following inequality exists:

$$\begin{aligned}&\left[x - N(y)\right]^{\mathrm{T}}Qf(t,x,y) \\ &\quad + \frac{1}{2}\mathrm{trace}\left[g^{\mathrm{T}}(t,x,y)Qg(t,x,y)\right] \\ &\le \gamma - \mu_1x^{\mathrm{T}}Qx + \mu_2y^{\mathrm{T}}Qy.\end{aligned} \tag{40}$$

Now we state and prove some conclusions.

Definition 10 (see [16]). Assume that system (1) satisfies (40). The numerical method is said to be dissipative if when the method is applied to problem (1) with constraint $\tau = mh$, there exists a constant C such that, for any initial values, there exists n_0, depending only on initial values $\varphi(t)$, such that

$$\mathrm{E}\left[y_n^{\mathrm{T}}Qy_n\right] \le C, \quad n \ge n_0. \tag{41}$$

Theorem 11. *Assume that system (1) satisfies (40); there exists a constant C such that, for any initial values, there exists n_0 depending only on the initial values $\varphi(t)$, when $n \ge n_0$, the numerical solution y_n generated by the SST method (13)–(15) with $\theta \in (1/2, 1]$, such that*

$$\mathrm{E}\left[y_n^{\mathrm{T}}Qy_n\right] \le C. \tag{42}$$

Proof. Consider (21) and (40); the following inequality can be obtained:

$$\begin{aligned} \mathrm{E}\left[x_{n+1}^{\mathrm{T}} Q x_{n+1}\right] &\le \mathrm{E}\left[x_n^{\mathrm{T}} Q x_n\right] + 2\Delta t\gamma \\ &+ 2\Delta t \mathrm{E}\left(-\mu_1 Y_n^{\mathrm{T}} Q Y_n + \mu_2 \overline{Y}_n^{\mathrm{T}} Q \overline{Y}_n\right) + (1-2\theta) \\ &\cdot \Delta t^2 f^{\mathrm{T}}\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right) Q f\left(t_n + \theta\Delta t, Y_n, \overline{Y}_n\right). \end{aligned} \tag{43}$$

We can get the following inequality the same as the derivation of (28):

$$\begin{aligned} \mathrm{E}\left[x_{n+1}^{\mathrm{T}} Q x_{n+1}\right] &\le k^{n+1}\Bigg(\mathrm{E}\left[x_0^{\mathrm{T}} Q x_0\right] \\ &+ 2\tau\left(\left(1-\lambda^2\right)\mu_2 + \lambda^2\mu_1\right) k^{-m} \max_{-m\le j\le -1} \mathrm{E}\left[Y_j^{\mathrm{T}} Q Y_j\right]\Bigg) \\ &- 2\Delta t\left(\mu_2 - \left(\left(1-\lambda^2\right)\mu_2 + \lambda^2\mu_1\right) k^{-m}\right) \\ &\cdot \sum_{j=0}^{n-m+1} k^{n-j} \mathrm{E}\left[Y_j^{\mathrm{T}} Q Y_j\right] + 2\Delta t\gamma \sum_{j=0}^{n} k^j, \end{aligned} \tag{44}$$

where $0 < k < 1$ is the same as defined in (25).

Because $-(\mu_2 - ((1-\lambda^2)\mu_2 + \lambda^2\mu_1)k^{-m}) \le 0$, we have the following inequality:

$$\begin{aligned} \mathrm{E}\left[x_{n+1}^{\mathrm{T}} Q x_{n+1}\right] &\le k^{n+1}\Bigg(\mathrm{E}\left[x_0^{\mathrm{T}} Q x_0\right] \\ &+ 2\tau\left(\left(1-\lambda^2\right)\mu_2 + \lambda^2\mu_1\right) k^{-m} \max_{-m\le j\le -1} \mathrm{E}\left[Y_j^{\mathrm{T}} Q Y_j\right]\Bigg) \\ &+ \frac{2\gamma\Delta t}{1-k}. \end{aligned} \tag{45}$$

Let

$$\begin{aligned} \varepsilon_1 = k^{n+1}\Bigg(&\mathrm{E}\left[x_0^{\mathrm{T}} Q x_0\right] + 2\tau\left(\left(1-\lambda^2\right)\mu_2 + \lambda^2\mu_1\right) k^{-m} \\ &\cdot \max_{-m\le j\le -1} \mathrm{E}\left[Y_j^{\mathrm{T}} Q Y_j\right]\Bigg) + \frac{2\gamma\Delta t}{1-k}; \end{aligned} \tag{46}$$

the following inequality could be deduced from (31):

$$\begin{aligned} \mathrm{E}\left[y_{n+1}^{\mathrm{T}} Q y_{n+1}\right] &\le 2\mathrm{E}\left[x_{n+1}^{\mathrm{T}} Q x_{n+1}\right] \\ &+ 2\lambda^2 \mathrm{E}\left[y_{n+1-m}^{\mathrm{T}} Q y_{n+1-m}\right] \\ &\le 2\varepsilon_1 + 2\lambda^2 \mathrm{E}\left[y_{n+1-m}^{\mathrm{T}} Q y_{n+1-m}\right] \le C, \end{aligned} \tag{47}$$

where $C = 2\varepsilon_1/(1-2\lambda^2) + \varepsilon$. The theorem is completed. □

Theorem 11 means that the discrete system possesses a bounded absorbing set in the sense of mean square. The numerical solution trajectory from any initial date will enter the set in a finite time and thereafter remain inside. It is called mean square dissipativity.

Remark 12. For the study of the dissipativity of numerical methods for deterministic delay differential equations with constant delays, Huang and Chang studied the dissipativity of Runge-Kutta methods and multistep Runge-Kutta methods in [21, 22].

5. The Numerical Experiment

In this section, we will give a numerical experiment to illustrate the stability and dissipativity result obtained in Sections 3 and 4. Consider the following nonlinear scalar neutral stochastic delay differential equation:

$$\begin{aligned} &d\left[y(t) - 0.25\sin\left(y(t-1)\right)\right] \\ &\quad = \left[-8y(t) + \sin\left(y(t-1)\right)\right] dt + y(t-1)\, dW(t), \\ &\qquad\qquad t \ge 0, \\ &y(t) = t+1, \quad -1 \le t \le 0. \end{aligned} \tag{48}$$

It is easy to verify that nonlinear neutral stochastic delay differential equation (48) satisfies the conditions of Theorem 7; the corresponding parameters are given as follows:

$$\begin{aligned} \lambda &= \frac{1}{4}, \\ \mu_1 &= 8, \\ \mu_2 &= \frac{1}{2}, \\ K_1 &= 64, \\ K_2 &= 1, \\ \Delta t_0 &= \begin{cases} +\infty, & \theta = \dfrac{1}{2} \\ 0.2884, & \theta \in \left(0, \dfrac{1}{2}\right). \end{cases} \end{aligned} \tag{49}$$

The initial condition is given by $y(t) = t+1$, $t \in [-1, 0]$, where we take $\tau = 1$. In the following tests, we show the influence of step size Δt and the parameter θ on M-S stability of the SST method; the data used in all figures are obtained by the mean square of data by 200 trajectories; that is, $\mathrm{E}y_n^2 \approx (1/200)\sum_{i=1}^{200}[y_n^{(i)}]^2$, $\mathrm{E}y_n \approx (1/200)\sum_{i=1}^{200} y_n^{(i)}$, where $y_n^{(i)}$ denotes the numerical solution of $y(t_n)$ in the ith trajectory.

Taking step sizes $\Delta t = 0.1$, $\Delta t = 0.2$, $\Delta t = 0.3$, and $\Delta t = 0.6$, we obtain the numerical solutions of (48), and the numerical solutions are displayed in Figures 1–6, respectively. We can see that when $\theta = 0.6$, the SST method is asymptotically mean square stable for all the step sizes selected, but when $\theta = 0.1$, the SST method is asymptotically mean square stable only for the step sizes $\Delta t \le 0.2884$; it is not mean square stable for the step sizes $\Delta t > 0.2884$.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

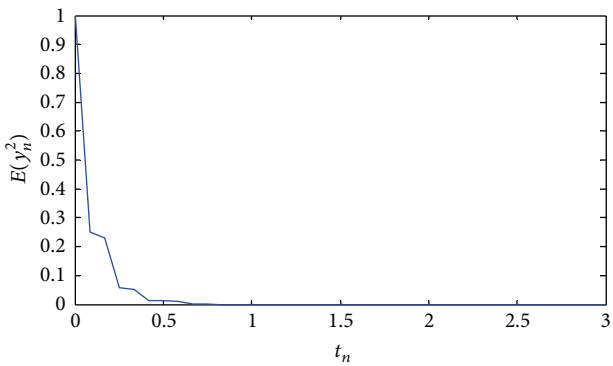

Figure 1: Mean square stability of SST method with $\theta = 0.6$ and $\Delta t = 0.1$.

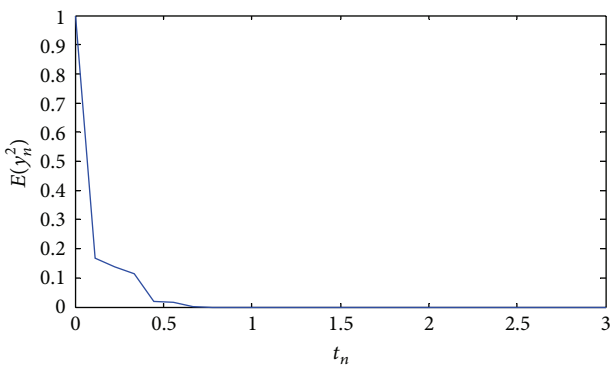

Figure 2: Mean square stability of SST method with $\theta = 0.6$ and $\Delta t = 0.6$.

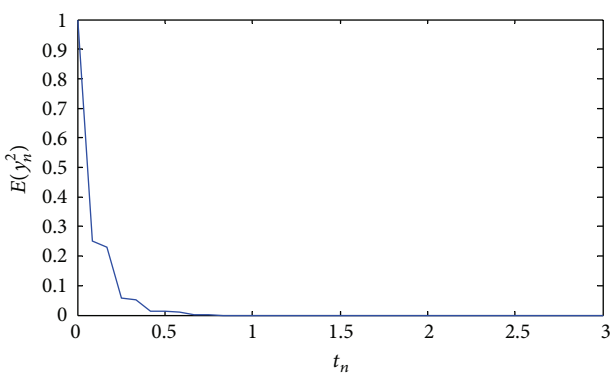

Figure 3: Mean square stability of SST method with $\theta = 0.1$ and $\Delta t = 0.1$.

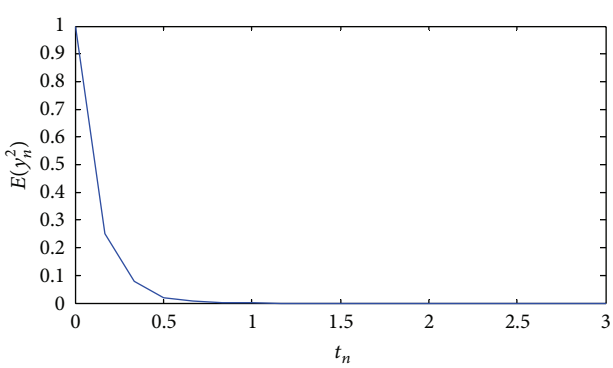

Figure 4: Mean square stability of SST method with $\theta = 0.1$ and $\Delta t = 0.2$.

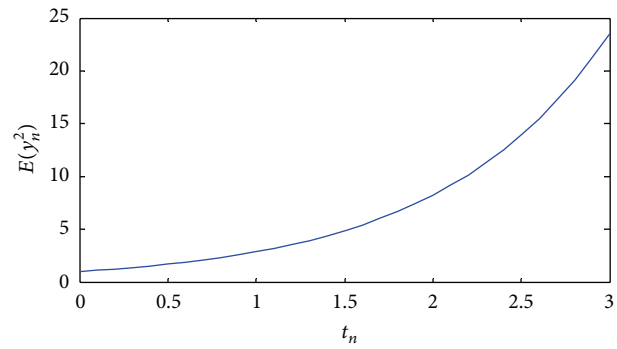

Figure 5: Unstable test for SST method with $\theta = 0.1$ and $\Delta t = 0.3$.

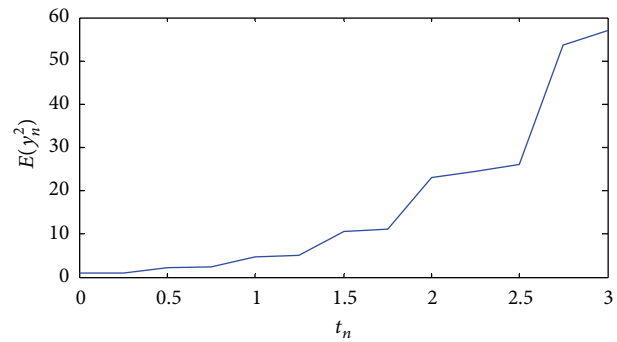

Figure 6: Unstable test for SST method with $\theta = 0.1$ and $\Delta t = 0.6$.

Acknowledgments

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Research Article

Dynamics of Stochastic Coral Reefs Model with Multiplicative Nonlinear Noise

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Little seems to be known about the ergodicity of random dynamical systems with multiplicative nonlinear noise. This paper is devoted to discern asymptotic behavior dynamics through the stochastic coral reefs model with multiplicative nonlinear noise. By support theorem and Hörmander theorem, the Markov semigroup corresponding to the solutions is to prove the Foguel alternative. Based on boundary distributions theory, the required conservative operators related to the solutions are further established to ensure the existence a stationary distribution. Meanwhile, the density of the distribution of the solutions either converges to a stationary density or weakly converges to some probability measure.

1. Introduction

The coral reefs equation is one of the most famous ecosystem models [1]:

$$\begin{aligned} \dot{X}(t) &= X\left(\gamma - \gamma X + (\sigma - \gamma) Y - \frac{g}{1-Y}\right), \\ \dot{Y}(t) &= Y(r - d - (\sigma + r) X - rY), \end{aligned} \tag{1}$$

where $\dot{X}(t)$ represents the cover of macroalgae; $\dot{Y}(t)$ represents the cover of corals.

(i) r is the rate that corals recruit to and overgrow algal turfs;

(ii) d is the natural mortality rate of corals;

(iii) σ is the rate that corals are overgrown by macroalgae;

(iv) γ is the rate that macroalgae spread vegetatively over algal turfs;

(v) g is the grazing rate that parrotfish graze macroalgae without distinction from algal turfs.

By the results in Li et al. [2], they discuss all kinds of dynamical behaviors. Recently, system (1) was studied extensively that it exhibits complex dynamical phenomena, including chaos, bifurcation, stability, and attractiveness [1–10].

However, ecosystem in the real world is very often subject to environmental noise due to uncertainty and unknown factors [11–16]. From a biological point of view and the generality of the models considered [17–21], these systems can appear very formal. This paper studies a stochastic coral reefs model where the intrinsic growth rate of the cover of macroalgae, γ, and the one of the cover of corals, r, are perturbed stochastically $\gamma \to \gamma + \alpha \dot{W}_t$ and $r \to r + \alpha \dot{W}_t$. In the paper, we only consider that stochastic coral reefs model can be described by

$$\begin{aligned} dX &= X\left(\gamma - \gamma X + (\sigma - \gamma) Y - \frac{g}{1-Y}\right) dt \\ &\quad + \alpha X^2 dW(t), \\ dY &= Y(r - d - (\sigma + r) X - rY)\, dt + \beta Y^2 dW(t), \end{aligned} \tag{2}$$

where $W(t)$ is a two-sided canonical Brownian motion. α and β represent the intensity of random noise and the differential $X^2dW(t)$ and $Y^2dW(t)$ are to be understood in the sense of multiplicative nonlinear noise. Since the drift and diffusion coefficients of (2) satisfy locally Lipschitz continuous condition, we can apply standard theorems that provide both existence and uniqueness of the positive solution of (2) (see [20]), for any given initial value. However, since the diffusion term of (2) is not linear but nonlinear, the existing powerful classical results [11–21] fail to work here. Nevertheless, this paper discusses the asymptotic behavior of the stochastic coral reefs model with multiplicative nonlinear noise using Fokker-Planck equations. To overcome the difficulty from the diffusion term, based on boundary distributions theory and conservative operators, we show that the density of the distribution of the solutions either converges to a stationary density or weakly converges to some probability measure.

This paper is organized as follows. In Section 2, we study the global attractiveness of the solution for stochastic coral reefs model with multiplicative nonlinear noise. In Section 3, we discuss the ergodicity of the solution for stochastic coral reefs model with multiplicative nonlinear noise.

2. Global Attractiveness

In the section, we study the global attractiveness of the solution for stochastic coral reefs model with multiplicative nonlinear noise.

Proposition 1. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$ hold.*

(I) *If the cover of macroalgae is absent, then the cover of corals dies with probability one.*

(II) *If the cover of corals is absent, the quantity of the preys oscillates between 0 and ∞, and there exists a unique stationary distribution with the density $f_*(x)$*

$$f_*(x) = c\exp\left\{-3x + \frac{2\gamma}{\alpha^2}e^{-x} - \frac{\gamma}{\alpha^2}e^{-2x}\right\}, \tag{3}$$

where c is a constant.

Proof. Denoting $X = e^{\xi_t}$, $Y = e^{\eta_t}$, we replace system (2) by

$$\begin{aligned} d\xi &= \left(\gamma - \frac{\alpha^2}{2}e^{2\xi_t} - \gamma e^{\xi_t} - (\gamma-\sigma)e^{\eta_t} - \frac{g}{1-e^{\eta_t}}\right)dt \\ &\quad + \alpha e^{\xi_t}dW, \\ d\eta &= \left(r - d - \frac{\beta^2}{2}e^{2\eta_t} - (\sigma+r)e^{\xi_t} - re^{\eta_t}\right)dt \\ &\quad + \beta e^{\eta_t}dW. \end{aligned} \tag{4}$$

Let

$$\begin{aligned} f_1(x,y) &= \gamma - \frac{\alpha^2}{2}e^{2x_t} - \gamma e^{x_t} - (\gamma-\sigma)e^{y_t} - \frac{g}{1-e^{y_t}}, \\ f_2 &= r - d - \frac{\beta^2}{2}e^{2y_t} - (\sigma+r)e^{x_t} - re^{y_t}. \end{aligned} \tag{5}$$

Then, system (4) becomes

$$\begin{aligned} d\xi &= f_1(\xi_t,\eta_t)\,dt + \alpha e^{\xi_t}dW, \\ d\eta &= f_2(\xi_t,\eta_t)\,dt + \beta e^{\eta_t}dW, \end{aligned} \tag{6}$$

or Stratonovich stochastic differential equation

$$\begin{aligned} d\xi &= \left(\gamma - \alpha^2e^{2\xi_t} - \gamma e^{\xi_t} - (\gamma-\sigma)e^{\eta_t} - \frac{g}{1-e^{\eta_t}}\right)dt \\ &\quad + \alpha e^{\xi_t}\circ dW_t, \\ d\eta &= \left(r - d - \beta^2e^{2\eta_t} - (\sigma+r)e^{\xi_t} - re^{\eta_t}\right)dt + \beta e^{\eta_t} \\ &\quad \circ dW_t. \end{aligned} \tag{7}$$

Let $\mathscr{L}$ denote the generator of diffusion (4); that is,

$$\begin{aligned} \mathscr{L}v &= \frac{1}{2}\alpha^2e^{2x}\frac{\partial^2 v}{\partial x^2} + \alpha\beta e^{x+y}\frac{\partial^2 v}{\partial x\partial y} + \frac{1}{2}\beta^2e^{2y}\frac{\partial^2 v}{\partial y^2} \\ &\quad + f_1\frac{\partial v}{\partial x} + f_2\frac{\partial v}{\partial y}. \end{aligned} \tag{8}$$

Then Fokker-Planck equation (FPE) of (4) can be described by

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\alpha^2\frac{\partial^2\left(e^{2x}u\right)}{\partial x^2} + \alpha\beta\frac{\partial^2\left(e^{x+y}u\right)}{\partial x\partial y} + \frac{1}{2}\beta^2\frac{\partial^2\left(e^{2y}u\right)}{\partial y^2} \\ &\quad - \frac{\partial(f_1u)}{\partial x} - \frac{\partial(f_2u)}{\partial y}. \end{aligned} \tag{9}$$

(I) If the cover of macroalgae is absent, the quantity $Y = e^{\eta_t}$ of the cover of corals satisfies

$$d\eta = \left(r - d - \frac{\beta^2}{2}e^{2\eta_t} - re^{\eta_t}\right)dt + \beta e^{\eta_t}dW. \tag{10}$$

Fix

$$\begin{aligned} &s_1(x) \\ &= \int_0^x \exp\left\{-\int_0^y \frac{2\left(r - d - \left(\beta^2/2\right)e^{2u_t} - re^{u_t}\right)}{\beta^2e^{2u_t}}du\right\}dy \\ &= \int_0^x \exp\left\{\frac{r+d}{\beta^2} - \frac{2r}{\beta^2e^{y_t}} + \frac{r-d}{\beta^2e^{2y_t}} + y\right\}dy. \end{aligned} \tag{11}$$

It is easy to see that $\lim_{x\to+\infty}s_1(x) = +\infty$ and $\lim_{x\to-\infty}s_1(x) > -\infty$. Then, we can obtain

$$\begin{gathered} \lim_{t\to+\infty}\eta_t = -\infty \\ \text{or equivalently } \lim_{t\to+\infty}Y_t = 0 \quad \text{a.s.} \end{gathered} \tag{12}$$

It implies that without the cover of macroalgae, the cover of corals dies with probability one.

(II) If the cover of corals is absent, the quantity $X = e^{\xi_t}$ of the cover of macroalgae satisfies

$$d\xi = \left(\gamma - \frac{\alpha^2}{2}e^{2\xi_t} - \gamma e^{\xi_t}\right)dt + \alpha e^{\xi_t}dW. \tag{13}$$

Let

$$\begin{aligned} s_2(x) &= \int_0^x \exp\left\{-\int_0^y \frac{2\left(\gamma - \left(\alpha^2/2\right)e^{2u_t} - \gamma e^{u_t}\right)}{\alpha^2 e^{2u_t}}du\right\}dy \\ &= \int_0^x \exp\left\{\frac{\gamma}{\alpha^2} + \frac{\gamma}{\alpha^2 e^{2y_t}} - \frac{2\gamma}{\alpha^2 e^{y_t}} + y\right\}dy. \end{aligned} \tag{14}$$

It easily shows that $\lim_{x\to+\infty} s_2(x) = +\infty$ and $\lim_{x\to-\infty} s_2(x) > -\infty$. We have

$$\begin{aligned} &\limsup_{t\to+\infty} \xi_t = \infty; \\ &\liminf_{t\to+\infty} \xi_t = -\infty \\ \text{or equivalently } &\limsup_{t\to+\infty} X_t = \infty; \\ &\liminf_{t\to+\infty} X_t = 0 \quad \text{a.s.} \end{aligned} \tag{15}$$

It implies that, without the predators, the quantity of the preys oscillates between 0 and ∞. Furthermore, there exists a stationary distribution of system (13) with the density $f_*(x)$ satisfying the FPE

$$\begin{aligned} &\frac{1}{2}\frac{d^2}{dx^2}\left[\alpha^2 e^{2x} f_*(x)\right] \\ &\quad - \frac{d}{dx}\left[\left(\gamma - \frac{\alpha^2}{2}e^{2x_t} - \gamma e^{x_t}\right) f_*(x)\right] = 0. \end{aligned} \tag{16}$$

Solving (16), then we get

$$\begin{aligned} y(x) &= \exp\left\{-3x + \frac{2\gamma}{\alpha^2}e^{-x} - \frac{\gamma}{\alpha^2}e^{-2x}\right\} \\ &\quad \cdot \left[c + k\int \exp\left\{x - \frac{2\gamma}{\alpha^2}e^{-x} + \frac{\gamma}{\alpha^2}e^{-2x}\right\}dx\right], \end{aligned} \tag{17}$$

where c and k are real numbers. With the conditions $y(x) \geq 0$ and

$$\int_{-\infty}^{+\infty} y(x)\,dx = 1, \tag{18}$$

it means that

$$\begin{aligned} K &= 0, \\ \frac{1}{c} &= \int_{-\infty}^{+\infty} \exp\left\{-3x + \frac{2\gamma}{\alpha^2}e^{-x} - \frac{\gamma}{\alpha^2}e^{-2x}\right\}dx \\ &= \int_0^{+\infty} y^2 \exp\left\{\frac{2\gamma}{\alpha^2}y - \frac{\gamma}{\alpha^2}y^2\right\}dy. \end{aligned} \tag{19}$$

It easily shows that

$$\begin{aligned} \frac{1}{c} &= \exp\left\{\frac{\gamma}{\alpha}\right\}\frac{\alpha}{2\gamma\sqrt{2\gamma}} \\ &\quad \cdot \int_{-\sqrt{2\gamma}/\alpha}^{\infty} \left(\alpha u + \sqrt{2\gamma}\right)^2 \exp\left\{-\frac{u^2}{2}\right\}du = \frac{\alpha^2}{2\gamma} \\ &\quad + \frac{\sqrt{\pi}\alpha\left(2\gamma + \alpha^2\right)}{2\gamma\sqrt{\gamma}}\left[1 - \Phi\left(-\frac{\sqrt{2\gamma}}{\alpha}\right)\right]\exp\left\{\frac{\gamma}{\alpha^2}\right\}, \end{aligned} \tag{20}$$

where $\Phi(x)$ is standard normal distribution function. Thus, we get

$$f_*(x) = c\exp\left\{-3x + \frac{2\gamma}{\alpha^2}e^{-x} - \frac{\gamma}{\alpha^2}e^{-2x}\right\}. \tag{21}$$

By ergodic theorem, if ξ_t is a solution of system (13), then we have

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t h(\xi_s)\,ds = \int_{\infty}^{\infty} h(x) f_*(x)\,dx \quad \text{a.s.} \tag{22}$$

Furthermore, ξ_t converges in probability to $f_*(x)$ when $t \to +\infty$. Define

$$\begin{aligned} m &= \int_{\infty}^{\infty} e^x f_*(x)\,dx \\ &= x\int_0^{+\infty} y\exp\left\{\frac{2\gamma}{\alpha^2}y - \frac{\gamma}{\alpha^2}y^2\right\}dy. \end{aligned} \tag{23}$$

It is easy to see that

$$\begin{aligned} m &= \frac{c\alpha^2}{2\gamma} + \frac{\sqrt{\pi}c\alpha}{\sqrt{\gamma}}\left[1 - \Phi\left(-\frac{\sqrt{2\gamma}}{\alpha}\right)\right]\exp\left\{\frac{\gamma}{\alpha^2}\right\} \\ &= \frac{\gamma\sqrt{\gamma}\alpha^2 + 2\sqrt{\pi}\gamma^2\alpha\left[1 - \Phi\left(-\sqrt{2\gamma}/\alpha\right)\right]\exp\left\{\gamma/\alpha^2\right\}}{\gamma\sqrt{\gamma}\alpha^2 + \sqrt{\pi}\alpha\left(2\gamma^2 + \gamma\alpha^2\right)\left[1 - \Phi\left(-\sqrt{2\gamma}/\alpha\right)\right]\exp\left\{\gamma/\alpha^2\right\}}. \end{aligned} \tag{24}$$

□

Theorem 2. *Suppose that $\alpha > 0$ and $\beta > 0$ hold. Then there exists a constant λ such that*

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t}\left[\frac{4}{\lambda^2}\ln(x+y) + \int_0^t (x+y)^2\,dt\right] \\ &\quad \leq \frac{4\max\{\gamma, r-d\}}{\lambda^2} \end{aligned} \tag{25}$$

holds with probability 1, where x and y are the solution of (2) with the initial condition $(x_0, y_0) \in \mathfrak{R}^2$.

Proof. Define the Lyapunov function on $\mathfrak{R}^2$

$$V(x, y) = \ln(x + y). \tag{26}$$

Applying Itô's formula to function (26), we get

$$\begin{aligned} dV(x,y) = &\left[\frac{x}{x+y}\left(\gamma-\gamma x+(\sigma-\gamma)y-\frac{g}{1-y}\right)\right. \\ &\left.-\frac{1}{2}\alpha^2\frac{x^4}{(x+y)^2}-\alpha\beta\frac{x^2y^2}{(x+y)^2}\right]dt \\ &+\left[\frac{y}{x+y}(r-d-(\sigma+r)x-ry)\right. \\ &\left.-\frac{1}{2}\beta^2\frac{y^4}{(x+y)^2}\right]dt+\frac{\alpha x^2+\beta y^2}{x+y}dW. \end{aligned} \tag{27}$$

That is,

$$\begin{aligned} V(x,y) = &V(0) \\ &+\int_0^t\left[\frac{x}{x+y}\left(\gamma-\gamma x+(\sigma-\gamma)y-\frac{g}{1-y}\right)\right. \\ &\left.-\frac{1}{2}\alpha^2\frac{x^4}{(x+y)^2}-\frac{\alpha\beta x^2y^2}{(x+y)^2}\right]ds \\ &+\int_0^t\left[\frac{y}{x+y}(r-d-(\sigma+r)x-ry)\right. \\ &\left.-\frac{1}{2}\beta^2\frac{y^4}{(x+y)^2}\right]ds+M(t), \end{aligned} \tag{28}$$

where

$$M(t)=\int_0^t\frac{\alpha x^2+\beta y^2}{x+y}dW \tag{29}$$

is a local martingale with quadratic form:

$$\langle M(t)\rangle=\int_0^t\left(\frac{\alpha x^2+\beta y^2}{x+y}\right)^2dt. \tag{30}$$

Fix $0<\varepsilon<1$. For any $\kappa\geq 1$, by martingale inequality, we have

$$P\left\{\sup_{0\leq t\leq\kappa}\left[M(t)-\frac{\varepsilon}{4}\langle M(t)\rangle\right]>\frac{4\ln\kappa}{\varepsilon}\right\}\leq\frac{1}{\kappa^2}. \tag{31}$$

By using Borel-Cantelli theorem, we can choose a set $\Omega'\subset\Omega$ with $P(\Omega')=1$ and for any $\omega\in\Omega$ there is a $\kappa_0(\omega)$ such that $\forall\kappa\geq\kappa_0(\omega)$

$$\sup_{0\leq t\leq\kappa}\left[M(t)-\frac{\varepsilon}{4}\langle M(t)\rangle\right]\leq\frac{4\ln\kappa}{\varepsilon}. \tag{32}$$

It implies that

$$M(t)\leq\frac{\varepsilon}{4}\langle M(t)\rangle+\frac{4\ln\kappa}{\varepsilon}\quad\forall 0\leq t\leq\kappa, \tag{33}$$

for $\omega\in\Omega'$ and $\kappa\geq\kappa_0(\omega)$. Substituting (33) into (28), we get

$$\begin{aligned} &V(x,y)+\frac{1}{4}\int_0^t\frac{(\alpha x+\beta y)^2}{(x+y)^2}dt \\ &\quad\leq V(0) \\ &\qquad+\int_0^t\left[\frac{x}{x+y}\left(\gamma-\gamma x+(\sigma-\gamma)y-\frac{g}{1-y}\right)\right]ds \\ &\qquad+\int_0^t\left[\frac{y}{x+y}(r-d-(\sigma+r)x-ry)\right]ds \\ &\qquad-\int_0^t\frac{(1-\varepsilon)\left(\alpha x^2+\beta y^2\right)^2}{4(x+y)^2}ds+\frac{4\ln\kappa}{\varepsilon}, \end{aligned} \tag{34}$$

for $0\leq t\leq\kappa$ and for almost ω and $\kappa\geq\kappa_0(\omega)$. Moreover, there exists a constant λ satisfying

$$\alpha x^2+\beta y^2\geq\lambda(x+y)^2. \tag{35}$$

By inequality (35), we get

$$\frac{\left(\alpha x^2+\beta y^2\right)^2}{4(x+y)^2}dt\geq\lambda^2\frac{(x+y)^2}{4}dt, \tag{36}$$

it means that

$$\begin{aligned} &V(x,y)+\frac{1}{4}\int_0^t\frac{(\alpha x+\beta y)^2}{(x+y)^2}dt \\ &\quad\geq V(x,y)+\frac{\lambda^2}{4}\int_0^t(x+y)^2dt. \end{aligned} \tag{37}$$

Moreover, there exists a positive constant $\mu=\max\{\gamma,r-d\}$ satisfying

$$\begin{aligned} \gamma x+(r-d)y&\leq\mu(x+y), \\ -\gamma x^2-(r+\gamma)xy-ry^2&\leq 0. \end{aligned} \tag{38}$$

Therefore, we get

$$\begin{aligned} &\frac{1}{x+y}\left(\gamma x-\gamma x^2+(\sigma-\gamma)xy-\frac{gx}{1-y}+(r-d)y\right. \\ &\quad\left.-(\sigma+r)xy-ry^2\right)\leq\mu. \end{aligned} \tag{39}$$

Then, we have

$$\begin{aligned} &V(x,y)+\frac{1}{4}\int_0^t\frac{(\alpha x+\beta y)^2}{(x+y)^2}dt \\ &\quad\leq V(0)+\int_0^t\mu ds+\frac{4\ln\kappa}{\varepsilon}\leq V(0)+\mu t+\frac{4\ln\kappa}{\varepsilon}, \end{aligned} \tag{40}$$

for any $\omega\in\Omega'$, $\kappa\geq\kappa_0(\omega)$, and $0\leq t\leq\kappa$.

If $\kappa - 1 \le t \le \kappa$ with $\kappa \ge \kappa_0(\omega)$, then we get

$$\begin{aligned} &\frac{1}{t}\left[V(x,y) + \frac{\lambda^2}{4}\int_0^t (x+y)^2\,dt\right] \\ &\quad \le \mu + \frac{1}{t}\left[V(0) + \frac{4\ln\kappa}{\varepsilon}\right]. \end{aligned} \tag{41}$$

It means that

$$\limsup_{t\to\infty}\frac{1}{t}\left[V(x,y) + \frac{\lambda^2}{4}\int_0^t (x+y)^2\,dt\right] \le \mu, \tag{42}$$

or

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t}\left[\frac{4}{\lambda^2}V(x,y) + \int_0^t (x+y)^2\,dt\right] \\ &\quad \le \frac{4\max\{\gamma, r-d\}}{\lambda^2}. \end{aligned} \tag{43}$$

The proof is completed. □

Theorem 3. *Suppose that $\alpha > 0$ and $\beta > 0$ hold. Then, with probability 1, we have*

$$\limsup_{t\to+\infty}\frac{\ln(x+y)}{\ln t} \le 1. \tag{44}$$

Proof. Define C^2-function:

$$V(x,y) = e^t \ln(x+y). \tag{45}$$

Applying Itô's formula to function (44), we have

$$\begin{aligned} dV(x,y) = &\left[e^t \ln(x+y)\right. \\ &\left. + \frac{xe^t}{x+y}\left(\gamma - \gamma x + (\sigma-\gamma)y - \frac{g}{1-y}\right)\right]dt \\ &- \frac{e^t\left(\alpha x^2 + \beta y^2\right)^2}{2(x+y)^2}dt \\ &+ \left[\frac{ye^t}{x+y}(r - d - (\sigma+r)x - ry)\right]dt + e^t \\ &\cdot \frac{\alpha x^2 + \beta y^2}{x+y}dW. \end{aligned} \tag{46}$$

That is,

$$\begin{aligned} V(x,y) = V(0) &+ \int_0^t \left[e^s \ln(x+y)\right. \\ &\left. + \frac{xe^s}{x+y}\left(\gamma - \gamma x + (\sigma-\gamma)y - \frac{g}{1-y}\right)\right]ds \\ &+ \int_0^t \left[\frac{ye^s}{x+y}(r - d - (\sigma+r)x - ry)\right]ds \\ &- \int_0^t \frac{e^t\left(\alpha x^2 + \beta y^2\right)^2}{2(x+y)^2}ds + M(t), \end{aligned} \tag{47}$$

where

$$M(t) = \int_0^t e^s \frac{\alpha x^2 + \beta y^2}{x+y}dW \tag{48}$$

is a local martingale with quadratic form:

$$\langle M(t)\rangle = \int_0^t e^{2s}\left(\frac{\alpha x^2 + \beta y^2}{x+y}\right)^2 ds. \tag{49}$$

By using Borel-Cantelli theorem and martingale inequality, for $\varepsilon < 1$, $\theta > 1$, and $\rho > 0$, for almost $\omega \in \Omega$, there is a $\kappa_0(\omega)$ such that $\forall\kappa \ge \kappa_0(\omega)$, and we have

$$M(t) \le \frac{\varepsilon e^{-\kappa\rho}}{2}\langle M(t)\rangle + \frac{\theta e^{\kappa\rho}\ln\kappa}{\varepsilon}, \quad 0 \le t \le \kappa\rho. \tag{50}$$

By (47) and (50), we get

$$\begin{aligned} V(x,y) \le V(0) &+ \int_0^t \left[e^s \ln(x+y)\right. \\ &\left. + \frac{xe^s}{x+y}\left(\gamma - \gamma x + (\sigma-\gamma)y - \frac{g}{1-y}\right)\right]ds \\ &+ \int_0^t \left[\frac{ye^s}{x+y}(r - d - (\sigma+r)x - ry)\right. \\ &\left. - \frac{e^s\left(\alpha x^2 + \beta y^2\right)^2}{2(x+y)^2}ds\right]ds \\ &+ \int_0^t \frac{\varepsilon e^{-\kappa\rho}}{2}e^{2s}\left(\frac{\alpha x^2 + \beta y^2}{x+y}\right)^2 ds + \frac{4\theta e^{\kappa\rho}\ln\kappa}{\varepsilon}. \end{aligned} \tag{51}$$

Moreover, there exists a constant λ satisfying

$$\alpha x^2 + \beta y^2 \ge \lambda(x+y)^2. \tag{52}$$

By inequality (52), it easily shows that

$$\frac{\left(\alpha x^2 + \beta y^2\right)^2}{(x+y)^2}dt \ge \lambda^2(x+y)^2\,dt. \tag{53}$$

Moreover, there exists a positive constant μ satisfying

$$\begin{aligned} \gamma x + (r-d)y &\le \mu(x+y), \\ -\gamma x^2 - (r+\gamma)xy - ry^2 - \frac{gx}{1-y} &\le 0. \end{aligned} \tag{54}$$

Combining (51), (52), and (53), we get

$$\begin{aligned} V(x,y) \le V(0) &+ \int_0^t \left[e^s \ln(x+y) + e^s\mu(x+y)\right]ds \\ &- \int_0^t e^s\lambda^2(x+y)^2\frac{1-\varepsilon e^{-\kappa\rho+s}}{2}ds \\ &+ \frac{\theta e^{\kappa\rho}\ln\kappa}{\varepsilon} \end{aligned} \tag{55}$$

for any $0 \le s \le \kappa\rho$.

It is easy to see that there exists a real number K independent of κ satisfying

$$\ln(x+y)+\mu(x+y)-\lambda^2(x+y)^2\frac{1-\varepsilon e^{-\kappa\rho+s}}{2}\le K. \quad (56)$$

From (55) and (56), we get

$$\begin{aligned} V(x,y) &\le V(0)+\int_0^t e^s K ds+\frac{\theta e^{\kappa\rho}\ln\kappa}{\varepsilon} \\ &= V(0)+K\left(e^t-1\right)+\frac{\theta e^{\kappa\rho}\ln\kappa}{\varepsilon} \end{aligned} \quad (57)$$

for any $0\le s\le\kappa\rho$. Then, we have

$$\ln(x+y)\le e^{-t}V(0)+K\left(1-e^{-t}\right)+e^{-t}\frac{\theta e^{\kappa\rho}\ln\kappa}{\varepsilon}. \quad (58)$$

If $(\kappa-1)\rho\le t\le\kappa\rho$ and $\kappa\ge\kappa_0(\omega)$, we obtain

$$\begin{aligned} \frac{\ln(x+y)}{\ln t} &\le \frac{e^{-(\kappa-1)\rho}}{\ln(\kappa-1)\rho}(V(0)-K)+\frac{K}{\ln(\kappa-1)\rho} \\ &\quad+\frac{\theta e^{\rho}\ln\kappa}{\varepsilon\ln(\kappa-1)\rho}. \end{aligned} \quad (59)$$

Letting $k\to\infty$, we get

$$\limsup_{t\to\infty}\frac{\ln(x+y)}{\ln t}\le\frac{\theta e^{\rho}}{\varepsilon}. \quad (60)$$

By (60), for every $\rho>0$, $\varepsilon<1$, and $\theta>1$, then by letting $\rho\to0$, $\theta\to1$, and $\varepsilon\to1$, we obtain

$$\limsup_{t\to\infty}\frac{\ln(x+y)}{\ln t}\le 1. \quad (61)$$

The proof is completed. □

Theorem 4. *Suppose that $\sigma<d<\gamma<r<2\gamma$, $0<g<\gamma$ hold. Let $\overline{\xi}_t$ denote the solution of the following the equation:*

$$d\overline{\xi}_t=\left(\gamma-\frac{\alpha^2}{2}e^{2\overline{\xi}_t}-\gamma e^{\overline{\xi}_t}\right)dt+\alpha e^{\overline{\xi}_t}dW \quad (62)$$

with the initial value $\overline{\xi}_0=\xi_0$. Let $\overline{\eta}_t$ denote the solution of the following the equation:

$$d\overline{\eta}_t=\left(r-d-\frac{\beta^2}{2}e^{2\overline{\eta}_t}-re^{\overline{\eta}_t}\right)dt+\beta e^{\overline{\eta}_t}dW. \quad (63)$$

with the initial value $\overline{\eta}_0=\eta_0$.

Then, with probability 1, there exist $\overline{\xi}_t\ge\xi_t$ and $\overline{\eta}_t\ge\eta_t$.

Proof. Let $Z=e^{-\xi_t}$, $\overline{Z}=e^{-\overline{\xi}_t}$. By Itô's formula, we have

$$\begin{aligned} dZ_t &= \left(\gamma-\gamma Z_t+\alpha^2Z_t^{-1}+(\gamma-\sigma)Z_te^{\eta_t}+\frac{gZ_t}{1-e^{\eta_t}}\right)dt \\ &\quad-\alpha dW. \\ d\overline{Z}_t &= \left(\gamma-\gamma\overline{Z}_t+\alpha^2\overline{Z}_t^{-1}\right)dt-\alpha dW. \end{aligned} \quad (64)$$

By using comparison theorem, we have $Z_t\ge\overline{Z}_t$ for all $t\ge0$ a.s. It implies that $\mathbb{P}\{\overline{\xi}_t\ge\xi_t\}=1$ for all $t\ge0$. It is easy to see that we show the second assertion $\overline{\eta}_t\ge\eta_t$ by a similar way for all $t\ge0$. The proof is completed. □

Theorem 5. *Suppose that $\sigma<d<\gamma<r<2\gamma$, $0<g<\gamma$ hold. Then the following assertions are true:*

$$\begin{aligned} &\text{(I)}\ \liminf_{t\to+\infty}\frac{1}{t}\int_0^t\left(\frac{\beta^2}{2}e^{2\eta_t}+(\sigma+r)e^{\xi_t}+re^{\eta_t}\right)\ge r-d \quad a.s. \\ &\text{(II)}\ \liminf_{t\to+\infty}\frac{1}{t}\int_0^t\left(\frac{\alpha^2}{2}e^{2\xi_t}+\gamma e^{\xi_t}+(\gamma-\sigma)e^{\eta_t}+\frac{g}{1-e^{\eta_t}}\right) \\ &\quad\ge\gamma \quad a.s. \end{aligned} \quad (65)$$

Proof. From Theorem 3, we get

$$\limsup_{t\to\infty}\frac{\eta_t}{\ln t}\le 1 \quad \text{a.s.} \quad (66)$$

Therefore,

$$\begin{aligned} &r-d-\limsup_{t\to\infty} \\ &\quad\cdot\frac{1}{t}\left[\int_0^t\left(\frac{\beta^2}{2}e^{2\eta_s}+(\sigma+r)e^{\xi_t}+re^{\eta_s}\right)ds\right. \\ &\quad\left.-\int_0^t\beta e^{\eta_s}dW_s\right], \\ &y=\limsup_{t\to\infty}\frac{\eta_t}{t}\le 0. \end{aligned} \quad (67)$$

For any $\varepsilon>0$, by martingale inequality, we have

$$\begin{aligned} &\mathbb{P}\left\{\sup_{0\le t\le k}\left\{\int_0^t-\beta e^{\eta_s}dW_s-\frac{\varepsilon}{2}\int_0^t\beta^2e^{2\eta_s}ds\right\}\ge\frac{2\ln k}{\varepsilon}\right\} \\ &\quad\le\frac{1}{k^2}. \end{aligned} \quad (68)$$

By using Borel-Cantelli theorem, for almost all ω, there is a real constant $k=k(\omega)$ satisfying, for all $n>k$ and $0\le t\le n$,

$$\int_0^t-\beta e^{\eta_s}dW_s-\frac{\varepsilon}{2}\int_0^t\beta^2e^{2\eta_s}ds\le\frac{2\ln k}{\varepsilon}. \quad (69)$$

It means that for $k-1\le t\le k$

$$\frac{1}{t}\left(\int_0^t-\beta e^{\eta_s}dW_s-\frac{\varepsilon}{2}\int_0^t\beta^2e^{2\eta_s}ds\right)\le\frac{2\ln k}{(k-1)\varepsilon}. \quad (70)$$

Then, we get

$$\liminf_{t\to\infty}\frac{1}{t}\left(\int_0^t\beta e^{\eta_s}dW_s+\frac{\varepsilon}{2}\int_0^t\beta^2e^{2\eta_s}ds\right)\ge 0. \quad (71)$$

Combining (67) and (71), it yields

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t}\int_0^t\left(-\frac{\beta^2(1+\varepsilon)}{2}e^{2\eta_s}-(\sigma+r)e^{\xi_t}-re^{\eta_s}\right)ds \\ &\quad\le d-r. \end{aligned} \quad (72)$$

Moreover, by Theorem 4, we get

$$0 < \limsup_{t\to\infty} \frac{1}{t} \int_0^t (\sigma + r) e^{\xi_t} ds \le (\sigma + r) m < \infty \quad \text{a.s.} \tag{73}$$

Therefore,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \int_0^t \left(-\frac{\beta^2}{2} e^{2\eta_s} - (\sigma + r) e^{\xi_t} - r e^{\eta_s} \right) ds \\
&\quad \le \frac{1}{1+\varepsilon} \limsup_{t\to\infty} \frac{1}{t} \\
&\quad \cdot \int_0^t \left(-\frac{\beta^2 (1+\varepsilon)}{2} e^{2\eta_s} - (\sigma + r) e^{\xi_t} - r e^{\eta_s} \right) ds \\
&\quad + \frac{\varepsilon}{1+\varepsilon} \limsup_{t\to\infty} \frac{1}{t} \int_0^t \left(-(\sigma + r) e^{\xi_t} - r e^{\eta_s} \right) ds \\
&\quad \le \frac{1}{1+\varepsilon} (d - r).
\end{aligned} \tag{74}$$

Letting $\varepsilon \to 0$, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \int_0^t \left(-\frac{\beta^2}{2} e^{2\eta_s} - (\sigma + r) e^{\xi_t} - r e^{\eta_s} \right) ds \\
&\quad \le (d - r).
\end{aligned} \tag{75}$$

The first assertion is proven. Using similar way, the second assertion is also proven. The proof is completed. □

3. Ergodicity

In the section, we discuss the ergodicity of the solution for stochastic coral reefs model with multiplicative nonlinear noise.

Theorem 6. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$ hold.*

(I) *Then, the transition probability function $\mathscr{P}(t, x_0, y_0, \cdot)$ of system (4), that is, $\mathscr{P}(t, x_0, y_0, A) = \mathbb{P}\{(\xi_t, \eta_t) \in A\}$ for an $A \in \mathscr{B}(\mathfrak{R}^2)$, results in a density $k(t, x, y, x_0, y_0)$ for all $t > 0$.*

(II) *Then, system (4) results in integral Markov semigroup.*

Proof. Let $a(x)$ and $b(x)$ denote vector fields on $\mathfrak{R}^d$; then the Lie bracket $[a, b]$ denotes a vector field:

$$[a, b]_j (x) = \sum_{k=1}^{d} \left(a_k \frac{\partial b_j}{\partial x_k} (x) - b_k \frac{\partial a_j}{\partial x_k} (x) \right). \tag{76}$$

Denote

$$\begin{aligned}
&a_0 (x, y) \\
&\quad = \begin{pmatrix} \gamma - \alpha^2 e^{2x} - \gamma e^{x_t} + (\sigma - \gamma) e^{y_t} - \dfrac{g}{1 - e^{y_t}} \\ (r - d) - \beta^2 e^{2y} - (\sigma + r) e^{x_t} - r e^{y_t} \end{pmatrix}, \\
&a_1 (x, y) = \begin{pmatrix} \alpha e^x \\ \beta e^y \end{pmatrix}.
\end{aligned} \tag{77}$$

Then it is easy to see that

$$\begin{aligned}
a_2 &= [a_0, a_1] = \begin{pmatrix} \alpha e^x \left(\gamma + \alpha^2 e^{2x} - (\gamma - \sigma) e^{y_t} + \dfrac{g}{1 - e^{y_t}} \right) + \beta (\gamma - \sigma) e^{2y} + \dfrac{\beta g e^{2y}}{(1 - e^y)^2} \\ \beta e^y \left((r - d) + \beta^2 e^{2y} - (\sigma + r) e^{x_t} \right) + \beta (\sigma + r) e^{2x} \end{pmatrix}, \\
a_3 &= [a_1, [a_0, a_1]] = \begin{pmatrix} 2\alpha^4 e^{4y} + 2\beta (r - \sigma) e^{3y} + \alpha\beta e^{x+y} \left[(\sigma - r) + \dfrac{g}{(1 - e^y)^2} \right] + \dfrac{2\beta g e^{2y} (2 - e^y)}{(1 - e^y)^3} \\ 2\beta^4 e^{4y} + 2\alpha\beta (\sigma + r) e^{3x} - \beta (\alpha + \beta) (\sigma + r) e^{2x+y} \end{pmatrix}, \\
a_4 &= [a_1, [a_1, [a_0, a_1]]] = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix},
\end{aligned} \tag{78}$$

where

$$\begin{aligned}
\Lambda_1 &= -\alpha e^x \left(2\alpha^4 e^{4y} + 2\beta (r - \sigma) e^{3y} + \frac{2\beta g e^{2y} (2 - e^y)}{(1 - e^t)^3} \right) + \beta e^y \left(8\alpha^4 e^{4y} + 6\beta (r - \sigma) e^{3y} + \alpha\beta e^{x+y} \left[(\sigma - r) + \frac{g (3 - e^y)}{(1 - e^y)^3} \right] + \frac{2\beta g e^{2y} \left[3 (1 - e^y)^2 - 1 \right]}{(1 - e^y)^4} \right), \\
\Lambda_2 &= \alpha e^x \left[6\alpha\beta (\sigma + r) e^{3x} - 2\beta (\alpha + \beta) (\sigma + r) e^{2x+y} \right] + \beta e^y \left[6\beta^4 e^{4y} - 2\alpha\beta (\sigma + r) e^{3x} \right].
\end{aligned} \tag{79}$$

Assume that there is a point (x, y) and so the vectors $a_1(x, y)$, $a_3(x, y)$, and $a_4(x, y)$ can not span the space $\mathfrak{R}^2$. Then, the vectors $a_1(x, y)$ and $a_3(x, y)$ are parallel; the vectors $a_1(x, y)$ and $a_4(x, y)$ are also parallel. Therefore, we get

$$\begin{aligned}
&\alpha e^x \left(2\beta^4 e^{4y} + 2\alpha\beta(\sigma + r) e^{3x}\right. \\
&\quad \left. - \beta(\alpha + \beta)(\sigma + r) e^{2x+y}\right) = \beta e^y \left(2\alpha^4 e^{4y}\right. \\
&\quad + 2\beta(r - \sigma) e^{3y} + \alpha\beta e^{x+y}\left[(\sigma - r) + \frac{g}{(1 - e^y)^2}\right] \\
&\quad \left. + \frac{2\beta g e^{2y}(2 - e^y)}{(1 - e^y)^3}\right) \\
&\alpha e^x \Lambda_2 = \beta e^y \Lambda_1.
\end{aligned} \tag{80}$$

It is easy to check that equality (80) is impossible.

It implies that the vectors a_1, a_3, a_4 span $\mathfrak{R}^2$ at any point (x, y). Therefore, we obtain the Hörmander condition.

($\mathscr{H}$) For every $(x, y) \in \mathfrak{R}^2$, the vectors

$$\begin{gathered}
a_1(x, y), \\
\left[a_i, a_j\right](x, y)_{0 \le i, j \le 1}, \\
\left[a_i, \left[a_j, a_k\right]\right](x, y)_{0 \le i, j, k \le 1}, \ldots
\end{gathered} \tag{81}$$

span the space $\mathfrak{R}^2$.

By Hypothesis ($\mathscr{H}$) and Hörmander theorem [22–29], then the transition probability function $\mathscr{P}(t, x_0, y_0, \cdot)$ results in a density $k(t, x, y, x_0, y_0)$ and $k \in C^\infty((0, \infty) \times \mathfrak{R}^2 \times \mathfrak{R}^2)$. Thus, the first assertion has been proved.

From the first assertion, it easily shows that for any $t > 0$, (ξ_t, η_t) of system (4) results in the density $u(t, x, y)$ satisfying the FPE (9). Furthermore, we get

$$u(t, x, y) = \iint_{-\infty}^{\infty} k(t, x, y, x_1, y_1)\, v(x_1, y_1)\, dx_1 dy_1. \tag{82}$$

Defining the operator $\{P(t)\}_{t \ge 0}$ is

$$\begin{aligned}
P(t)\, v(x, y) &= u(t, x, y) \\
&= \iint_{-\infty}^{\infty} k(t, x, y, x_1, y_1)\, v(x_1, y_1)\, dx_1 dy_1
\end{aligned} \tag{83}$$

for any $t > 0$, $v \in D$. By using continuation theorem of operator and assertion (I), it easily shows that the operator $\{P(t)\}_{t \ge 0}$ is an integral Markov semigroup. The proof is completed. □

Theorem 7. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$ hold. Then there is no more than three solution curves such that $x = g(y)$ satisfying rank $D_{x_0, y_0, \phi} = 2$ if $x_\phi(T) \ne g(y_\phi(T))$.*

Proof. Let $(x_0, y_0) \in \mathfrak{R}^2$ and $\phi \in L^2([0, T]; \mathfrak{R})$. We consider the following system:

$$\begin{aligned}
x'_\phi &= \gamma - \alpha^2 e^{2x_t} + (\alpha\phi(t) - \gamma) e^{x_t} - (\gamma - \sigma) e^{y_t} \\
&\quad - \frac{g}{1 - e^{y_t}}, \\
y'_\phi &= r - d - \beta^2 e^{2y_t} - (\sigma + r) e^{x_t} + (\beta\phi(t) - r) e^{y_t}.
\end{aligned} \tag{84}$$

Denote

$$\begin{aligned}
\overline{f}_1 &= \gamma - \alpha^2 e^{2x_t} + (\alpha\phi(t) - \gamma) e^{x_t} - (\gamma - \sigma) e^{y_t} \\
&\quad - \frac{g}{1 - e^{y_t}}, \\
\overline{f}_2 &= r - d - \beta^2 e^{2y_t} - (\sigma + r) e^{x_t} + (\beta\phi(t) - r) e^{y_t}.
\end{aligned} \tag{85}$$

System (84) becomes

$$\begin{aligned}
x'_\phi &= \overline{f}_1(x_\phi, y_\phi), \\
y'_\phi &= \overline{f}_2(x_\phi, y_\phi).
\end{aligned} \tag{86}$$

We show that (x_ϕ, y_ϕ) denote the solution of system (84) with the initial value $x_\phi(0) = x_0$, $y_\phi(0) = y_0$ and $F : C([0, T], \mathfrak{R}) \to \mathfrak{R}^2$ defined as $F(h) = (x_{\phi+H}(T), y_{\phi+H}(T))$. By using the perturbation method, we get the Frechet derivative $D_{x_0, y_0, \phi}$ of F. Let $f = (\overline{f}_1, \overline{f}_2)$ and $\Lambda(t) = f'(\overline{f}_1, \overline{f}_2)$. $Q(t, s)$ denote the fundamental matrix of the following differential equation:

$$\dot{Y} = \Lambda(t) Y. \tag{87}$$

That is, $\partial Q(t, s)/\partial t = \Lambda(t) Q(t, s)$ and $Q(ts, s) = I$ for $s \le t \le T$. Then, we have

$$D_{x_0, y_0; \phi} h = \int_0^T Q(T, s)\, q(x_\phi, y_\phi)\, h(s)\, ds, \tag{88}$$

where $q(x_\phi, y_\phi) = (\alpha e^y, \beta e^\beta)^\top$. Let $\varepsilon \in (0, T)$ and $h(t) = 0$ if $0 \le t \le T - \varepsilon$ and $h(t) = (1/\epsilon)(t - T + \varepsilon)$ if $T - \varepsilon \le t \le T$. By using Taylor formula, we get $Q(T, s) = I - \Lambda(T)(T - s) + o(T - s)$ as $s \to T$. Then, we have

$$\begin{aligned}
D_{x_0, y_0; \phi} h &= \int_{T-\varepsilon}^T \frac{s - T + \varepsilon}{\varepsilon} q(x_\phi, y_\phi)\, ds + \Lambda(T) \\
&\quad \cdot \int_{T-\varepsilon}^T \frac{s - T + \varepsilon}{\varepsilon} (s - T)\, q(x_\phi, y_\phi)\, ds + o(\varepsilon^2).
\end{aligned} \tag{89}$$

Let $\overline{x} = x_\phi(T)$, $\overline{y} = y_\phi(T)$, and $c = \phi(T)$. By using mean value theorem of integration, we get

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{T-\varepsilon}^T \frac{s - T + \varepsilon}{\varepsilon} q(x_\phi(s), y_\phi(s))\, ds = \frac{1}{2} q(\overline{x}, \overline{y}), \\
&\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \Lambda(T) \int_{T-\varepsilon}^T \frac{s - T + \varepsilon}{\varepsilon} (s - T)\, q(x_\phi(s), y_\phi(s))\, ds \\
&\quad = -\frac{1}{6} \Lambda(T)\, q(\overline{x}, \overline{y}).
\end{aligned} \tag{90}$$

It is easy to calculate that we show

$$\Lambda(T) = \begin{pmatrix} -2\alpha^2 e^{2\overline{x}_t} + (\alpha c - \gamma) e^{\overline{x}_t} & -(\gamma - \sigma) e^{\overline{y}_t} - \dfrac{g e^{\overline{y}_t}}{(1 - e^{\overline{y}_t})^2} \\ -(\sigma + r) e^{\overline{x}_t} & -2\beta^2 e^{2\overline{y}_t} + (\beta c - r) e^{\overline{y}_t} \end{pmatrix}. \quad (91)$$

Therefore, we obtain

$$\Lambda(T) q(\overline{x}, \overline{y}) = \begin{pmatrix} -2\alpha^3 e^{3\overline{x}_t} + \alpha(\alpha c - \gamma) e^{2\overline{x}_t} - \beta(\gamma - \sigma) e^{2\overline{y}_t} - \dfrac{\beta g e^{2\overline{y}_t}}{(1 - e^{\overline{y}_t})^2} \\ -\alpha(\sigma + r) e^{2\overline{x}_t} - 2\beta^3 e^{3\overline{y}_t} + \beta(\beta c - r) e^{2\overline{y}_t} \end{pmatrix}. \quad (92)$$

If the two vectors $q(\overline{x}, \overline{y})$ and $\Lambda(T)$ are not linearly dependent, then we get the rank $D_{x_0, y_0, \phi} = 2$. Since the between $q(\overline{x}, \overline{y})$ and $\Lambda(T) q(\overline{x}, \overline{y})$ is linear dependence, thus, it is easy to see that $\overline{x}$ and $\overline{y}$ denote the solution of the following differential equation:

$$\det \begin{pmatrix} \alpha e^{\overline{x}} & -2\alpha^3 e^{3\overline{x}_t} + \alpha(\alpha c - \gamma) e^{2\overline{x}_t} - \beta(\gamma - \sigma) e^{2\overline{y}_t} - \dfrac{\beta g e^{2\overline{y}_t}}{(1 - e^{\overline{y}_t})^2} \\ \beta e^{\overline{y}} & -\alpha(\sigma + r) e^{2\overline{x}_t} - 2\beta^3 e^{3\overline{y}_t} + \beta(\beta c - r) e^{2\overline{y}_t} \end{pmatrix} = 0. \quad (93)$$

That is,

$$\begin{aligned} &\alpha^2 \left(2\alpha\beta e^{\overline{y}} - (\sigma + r)\right) e^{3\overline{x}_t} - \left(\alpha\beta(\alpha c - \gamma) e^{\overline{y}}\right) e^{2\overline{x}_t} \\ &\quad + \left(\alpha\beta(\beta c - r) e^{2\overline{y}_t} - 2\alpha\beta^3 e^{3\overline{y}_t}\right) e^{\overline{x}} \\ &\quad + \left(\beta^2(\gamma - \sigma) + \frac{\beta^2 g}{(1 - e^{\overline{y}_t})^2}\right) e^{3\overline{y}} = 0. \end{aligned} \quad (94)$$

It easily knows that there is no more than three solution curves satisfying (94), and the graph of a function $\overline{x} = g(\overline{y})$ represents each solution curve. The proof is completed. □

Theorem 8. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r + g < d + \gamma$ hold. Hypothesis condition $\mathscr{H}_1$: there is a point z^* satisfying $g_2(y, z^*) \le 0$ for all $y \in \mathfrak{R}$ and $(y_\phi, z_\phi) \in E$. The following assertions are true.*

(I) *If Hypothesis condition $\mathscr{H}_1$ does not hold, then system (84) is controllable in E.*

(II) *If Hypothesis condition $\mathscr{H}_1$ holds, then system (84) is controllable in E_i $(i = 1, 2)$, where*

$$\begin{aligned} g_2(y_\phi, z_\phi) &= -\gamma z + \alpha\beta^{-1}(r - d - \gamma) e^{-y} + \frac{\alpha^2 - \alpha\beta^{-1} e^{-y}}{z + \beta^{-1}\alpha e^{-y(t)}} \\ &\quad + \frac{g\left(z + \beta^{-1}\alpha e^{-y(t)}\right)}{1 - e^y} + (z(\gamma - \sigma) - \alpha\beta) e^y \\ &\quad + \frac{\alpha(\gamma - \sigma) + \alpha\beta + \beta r}{\beta}, \\ E &= \{(y, z) \mid z > -\alpha\beta e^{-y}\}, \\ E_1 &= \left\{(x, y) \mid e^{-x} - \beta\alpha^{-1} e^{-y} < a^*\right\} \\ &= \{(y, z) \in E \mid z \le a^*\}, \\ E_2 &= \left\{(x, y) \mid e^{-x} - \beta\alpha^{-1} e^{-y} > (\gamma - \sigma)^{-1} \alpha\beta\right\} \\ &= \left\{(y, z) \mid z > (\gamma - \sigma)^{-1} \alpha\beta\right\}. \end{aligned} \quad (95)$$

Proof. Let

$$z_\phi(t) = e^{-x_\phi(t)} - \beta^{-1}\alpha e^{-y_\phi(t)}. \quad (96)$$

Then, system (84) can be replaced by the following differential equations:

$$\begin{aligned} y'_\phi &= \beta e^{y_\phi}\phi + g_1(y_\phi, z_\phi), \\ z'_\phi &= g_2(y_\phi, z_\phi), \end{aligned} \quad (97)$$

where

$$\begin{aligned} g_1(y_\phi, z_\phi) &= r - d - \frac{\sigma + r}{z + \alpha\beta^{-1} e^{-y}} - r e^y - \beta^2 e^{2y}, \\ g_2(y_\phi, z_\phi) &= -\gamma z + \alpha\beta^{-1}(r - d - \gamma) e^{-y} \\ &\quad + \frac{\alpha^2 - \alpha\beta^{-1} e^{-y}}{z + \beta^{-1}\alpha e^{-y(t)}} \\ &\quad + \frac{g\left(z + \beta^{-1}\alpha e^{-y(t)}\right)}{1 - e^y} \\ &\quad + (z(\gamma - \sigma) - \alpha\beta) e^y \\ &\quad + \frac{\alpha(\gamma - \sigma) + \alpha\beta + \beta r}{\beta}. \end{aligned} \quad (98)$$

Denote

$$E = \{(y, z) \mid z > -\alpha\beta e^{-y}\}. \quad (99)$$

By (96), it is easy to see that $(y_\phi, z_\phi) \in E$ for any $\phi \in C([0, T], \mathfrak{R})$. There is a point z^* satisfying $g_2(y, z^*) \le 0$ for

all $y \in \mathfrak{R}$ and $(y_\phi, z_\phi) \in E$. It is easily know that there exists $z^* \in [0, (\gamma-\sigma)^{-1}\alpha\beta]$; let

$$a^* = \inf\{z^* \mid g_2(y, z^*) \le 0 \ \forall y \in \mathfrak{R},\ (y, z^*) \in E\}, \quad (100)$$

$$\begin{aligned} E_1 &= \left\{(x, y) \mid e^{-x} - \beta\alpha^{-1}e^{-y} < a^*\right\} \\ &= \{(y, z) \in E \mid z \le a^*\}, \\ E_2 &= \left\{(x, y) \mid e^{-x} - \beta\alpha^{-1}e^{-y} > (\gamma-\sigma)^{-1}\alpha\beta\right\} \\ &= \left\{(y, z) \mid z > (\gamma-\sigma)^{-1}\alpha\beta\right\}. \end{aligned} \quad (101)$$

Step 1. Fix $z_0 > z_1$. Due to $\lim_{y\to-\infty} g(y, z)_2 = -\infty$ uniformly in $z \in [z_1, z_0]$, it shows that there is y_0 satisfying $g_2(y_0, z) \le -1$ for any $z \in [z_1, z_0]$. We take

$$\phi = -\frac{g_1(y_0, z_1)}{\beta e^{y_0}}, \quad (102)$$

where $z(t)$ denotes the solution of

$$\begin{aligned} z'_\phi &= g_2(y_0, z_1), \\ z(0) &= z_0. \end{aligned} \quad (103)$$

It implies that system (97) results in the solution $(y_\phi, z_\phi) = (y_0, z_1)$ and $z_\phi(0) = z_0$. Due to $z'_\phi = g_2(y_0, z_1) \le -1$ whenever z_ϕ, we can choose a $T > 0$ satisfying $z_\phi(T) = z_1$.

Step 2. Fix $(\gamma-\sigma)^{-1}\alpha\beta < z_0 < z_1$. Due to $\lim_{y\to\infty} g_2(y, z) = \infty$ uniformly in $z \in [z_1, z_0]$, we can choose y_0 satisfying $g_2(y_0, z) \ge 1$ for any $z \in [z_1, z_0]$. By using similar way to Step 1, it easily shows that there is a function $\phi(t)$ and $T > 0$ such that (97) results in a solution $y_\phi(0) \equiv y_0$, $z_\phi(0) \equiv z_0$, and $z_\phi(T) = z_1$.

Step 3. Let $z_0 < 0$ and

$$\overline{z} = \ln\left[\frac{-\alpha}{\beta z_0}\right]. \quad (104)$$

Since $\lim_{y\to\overline{z}^{-1}} g_2(y, z_0) = \infty$, it easily knows that there is real number c_1 and $y_0 \in \mathfrak{R}$ satisfying $g(y_0, z)_2 \ge 1$ for any $z \in [z_0 - c_1, z_0 + c_1]$ and $(y_0, z) \in E$. In the case, we can choose a control function ϕ satisfying $y_\phi \equiv y_0$, $z_\phi(0) = z_0$, and $z_\phi(T) = z_0 + c_1$ for some $T > 0$.

Step 4. If Hypothesis condition $\mathscr{H}_1$ holds. It is easy to see that $z^* \in [0, (\gamma-\sigma)^{-1}\alpha\beta]$. Based on the definition of a^*, for any $\varepsilon > 0$, there are $\mu_1 > 0$ and $\mu_2 > 0$ satisfying the property: if (z_0, z_1) such that $z_1 - \mu_1 < z_0 < z_1 < a^* - \varepsilon$, one can choose a y_0 such that $g(y_0, z)_2 > \mu_2$ for $z \in [z_0, z_1]$ and $(y_0, z) \in E$. Hence, we can choose a control function ϕ satisfying $y_\phi \equiv y_0$, $z_\phi(0) = z_0$, and $z_\phi(T) = z_1$ for some $T > 0$.

Step 5. If Hypothesis condition $\mathscr{H}_1$ does not hold. Then there are μ_1 and $\mu_2 > 0$ satisfying the property: if (z_0, z_1) such that $z_1 - \mu_1 < z_0 < z_1$ and $z_0, z_1 \in [0, (\gamma-\sigma)^{-1}\alpha\beta]$, one can choose a y_0 such that $g(y_0, z)_2 > \mu_2$ for $z \in [z_0, z_1]$ and $(y_0, z) \in E$. By the same way as before, we can choose a control function ϕ satisfying $y_\phi(0) = y_0$, $z_\phi(0) = z_0$, and $z_\phi(T) = z_1$ for some $T > 0$.

Step 6. Let $y_0 \in \mathfrak{R}$, $K > 0$, $L_1 > L_0$, and $0 < \varepsilon < \min(K/4, (L_1 - L_0)/4)$ and $[y_0, y_0 + K] \times [L_0, L_1] \subset E$. Denote

$$\begin{aligned} m^* = \max\{&|g_1(y, z)| + |g_2(y, z)| \mid (y, z) \\ &\in [y_0, y_0 + K] \times [L_0, L_1]\}, \end{aligned} \quad (105)$$

and $t_0 = \varepsilon/m^*$. For every $z_0 \in [L_0 + \varepsilon, L_1 - \varepsilon] \times [L_0, L_1]$, it easily knows that system (97) with $y_\phi(0) = y_0$, $z_\phi(0) = z_0$ results in the solution such that, for all $t \in [0, t_0]$,

$$\begin{aligned} y_\phi(t_0) &\in \left(y_0 + \frac{K}{2}, y_0 + K\right), \\ z_\phi(t) &\in [z_0 - \varepsilon, z_0 + \varepsilon]. \end{aligned} \quad (106)$$

From (97), we get

$$\beta\phi e^{y_\phi} - m^* \le y'_\phi \le \beta\phi e^{y_\phi} - m^*. \quad (107)$$

It is easy to calculate that the solution of the equation $y' = ay + b$ is

$$y = y_0 + bt + \ln\frac{b}{ae^{y_0} + b - ae^{y_0+bt}}. \quad (108)$$

By using comparison theorem, we have, for any $t \in [0, t_0]$,

$$\begin{aligned} y_\phi &\ge y_0 - m^*t + \ln m^* \\ &\quad - \ln\left[\beta\phi e^{y_0}\left(e^{-m^*t} - 1\right) + m^*\right], \\ y_\phi &\le y_0 + m^*t + \ln m^* \\ &\quad - \ln\left[\beta\phi e^{y_0}\left(1 - e^{m^*t}\right) + m^*\right]. \end{aligned} \quad (109)$$

Thus,

$$\begin{aligned} y_\phi &\ge y_0 - \varepsilon + \ln m^* - \ln\left[\beta\phi e^{y_0}\left(e^{-\varepsilon} - 1\right) + m^*\right], \\ y_\phi &\le y_0 + \varepsilon + \ln m^* - \ln\left[\beta\phi e^{y_0}\left(1 - e^{\varepsilon}\right) + m^*\right]. \end{aligned} \quad (110)$$

Therefore, (I) term of (106) can be proven if we find a constant ϕ satisfying

$$\begin{aligned} &y_0 - \varepsilon + \ln m^* - \ln\left[\beta\phi e^{y_0}\left(e^{-\varepsilon} - 1\right) + m^*\right] \\ &\quad > y_0 + \frac{K}{2}, \end{aligned} \quad (111)$$

$$y_0 + \varepsilon + \ln m^* - \ln\left[\beta\phi e^{y_0}\left(1 - e^{\varepsilon}\right) + m^*\right] < y_0 + K. \quad (112)$$

It is obvious that (111) is equivalent to

$$\phi \in \left(\frac{m^*\left(1 - e^{-\varepsilon-K/2}\right)}{\beta e^{y_0}\left(1 - e^{-\varepsilon}\right)}; \frac{m^*}{\beta e^{y_0}\left(1 - e^{-\varepsilon}\right)}\right), \quad (113)$$

and (112) is equivalent to

$$\phi < \frac{m^* \left(1 - e^{-\varepsilon - K}\right)}{\beta e^{y_0} \left(e^{\varepsilon} - 1\right)}. \tag{114}$$

Obviously, for ε is small enough, we get

$$\frac{m^* \left(1 - e^{-\varepsilon - K/2}\right)}{\beta e^{y_0} \left(1 - e^{-\varepsilon}\right)} < \frac{m^* \left(1 - e^{-\varepsilon - K}\right)}{\beta e^{y_0} \left(e^{\varepsilon} - 1\right)}. \tag{115}$$

In other words, we can find ϕ such that (I) term of (106) is true. For the second assertion, we get, for all $0 \le t \le t_0$,

$$\begin{aligned} \left|z_\phi(t) - z_\phi(0)\right| &= \left|\int_0^t g_2\left(y_\phi(s), z_\phi(s)\right) ds\right| \\ &\le \int_0^{t_0} m^* ds = m^* t_0 = \varepsilon. \end{aligned} \tag{116}$$

From (106), it easily shows that for $(y_1, z_1) \in (y_0, y_0 + K/2] \times [L_0 + 2\varepsilon, L_1 - 2\varepsilon]$ there is a $z_0 \in [z_1 - \varepsilon, z_1 + \varepsilon]$ and a $T \in (0, t_0)$ satisfying $y_\phi(T) = y_1$ and $z_\phi(T) = z_1$. By using similar proofs, $(y_0 - K/2, y_0]$. Therefore, for any $(y_1, z_1) \in (y_0 - K/2, y_0 + K/2] \times [L_0 + 2\varepsilon, L_1 - 2\varepsilon]$, we obtain that there is a $z_0 \in [z_1 - \varepsilon, z_1 + \varepsilon]$ and a $T \in (0, t_0)$ satisfying $y_\phi(T) = y_1$ and $z_\phi(T) = z_1$. The proof is completed. □

Theorem 9. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r + g < d + \gamma$ hold. The following assertions are true.*

(I) *If Hypothesis condition $\mathscr{H}_1$ does not hold, then there is a constant $T > 0$ satisfying $k(T, x, y, x_0, y_0) > 0$ for each $(x_0, y_0) \in \mathfrak{R}^2$ and for almost every $(x, y) \in \mathfrak{R}^2$.*

(II) *If Hypothesis condition $\mathscr{H}_1$ holds, then there is a constant $T > 0$ such that $k(T, x, y, x_0, y_0) > 0$ for any point $(x_0, y_0) \in \mathfrak{R}^2$ and almost every $(x, y) \in \mathfrak{R}^2$ such that (x_0, y_0) and (x, y) in E_i $(i = 1, 2)$, where $z = e^{-x} - \beta^{-1}\alpha e^{-y}$, $k(T, x, y, x_0, y_0) > 0$ is a measurable function.*

Proof. By using continuity theory, we can find a continuity function $\phi \in C(0, T; \mathfrak{R})$. If there exists $\phi \in C(0, T; \mathfrak{R})$ such that the derivative $D_{x_0, y_0, \phi}$ is the rank 2, then we have $k(T, x, y, x_0, y_0) > 0$. Therefore, the proof is completed. □

Theorem 10. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r + g < d + \gamma$ hold. If Hypothesis condition $\mathscr{H}_1$ holds. Then, $\limsup_{t\to\infty} \zeta_t \le a^*$, where a^* is defined in (100).*

Proof. System (7) becomes

$$\begin{aligned} d\eta_t &= g_1\left(\eta_t, \zeta_t\right) dt + \beta e^{y_t} \circ dWt, \\ d\zeta_t &= g_2\left(\eta_t, \zeta_t\right) dt. \end{aligned} \tag{117}$$

To prove $\inf_{t\ge 0}\zeta_t < a^*$ s.a, we assume that $\inf_{t\ge 0}\zeta_t \ge a^*$ for all $\omega \in \Omega$ with $\mathbb{P}(\Omega) > 0$. Due to $\zeta_t \ge a^*$ for all $t \ge 0$, we get $g_1(y_t, \zeta_t) \le g_1(y_t, a^*)$ for $\omega \in \Omega$ and $y \in \mathfrak{R}$. Generally, with $\mathbb{P}(\Omega) < 1$, the comparison theorem fails to work here since the diffusion term is nonlinear. By $\lambda(t) = e^{-\eta_t}$, we have

$$d\lambda(t) = -\lambda(t)\, g_1\left(-\ln \lambda(t), \zeta_t\right) dt - \beta dW_t, \tag{118}$$

where $\overline{\lambda}$ denote the solution of

$$d\overline{\lambda}(t) = -\overline{\lambda}(t)\, g_1\left(-\ln \overline{\lambda}(t), \zeta_t\right) dt - \beta dW_t, \tag{119}$$

with $\overline{\lambda}(0) = \lambda(0)$. Since $-y g_1(\ln y, \zeta_t) \ge -y g_1(\ln y, a^*)$ on Ω for all $t > 0$, it is easy to see that $\lambda(t, \omega) \ge \overline{\lambda}(t, \omega)$. Hence, we get $\eta_t \le \eta_t^*(\omega)$ for $\omega \in \Omega$, where $\eta_t^* = -\ln \overline{\lambda}$ is the solution of

$$\begin{aligned} d\eta_t^* &= g_1\left(\eta_t^*, a^*\right) dt + \beta e^{\eta_t^*} \circ dW_t, \\ \eta_0^* &= \eta_0. \end{aligned} \tag{120}$$

Denote

$$\begin{aligned} s_3(x) = \int_0^x \exp\left\{-\int_0^y \frac{2\left(r - d - \left(\beta^2/2\right) e^{2\eta_t} - (\sigma + r)/\left(a^* + \alpha\beta^{-1}e^{-u}\right) - re^{\eta_t}\right)}{\beta^2 e^{2u}} du\right\} dy = \int_0^x \exp\left\{\frac{r + d}{\beta^2} + \frac{r - d}{\beta^2 e^{2y}}\right. \\ \left. - \frac{2r}{\beta^2 e^y} + y + \int_0^y \frac{2(\sigma + r)}{\left(a^* + \alpha\beta^{-1}e^{-u}\right)\beta^2 e^{2u}} du\right\} dy. \end{aligned} \tag{121}$$

It is obvious that $\lim_{t\to\infty} s_3(x) = \infty$, $\lim_{t\to-\infty} s_3(x) > -\infty$. Thus, we get $\lim_{t\to\infty}\eta_t^* = -\infty$ a.s. Hence, we have $\lim_{t\to\infty}\eta_t = -\infty$ on Ω. Moreover, by the definition of g_2, there exists a $M_1 > 0$ satisfying $g_2(y, z) \le -1$ for all $y \le -M_1$, $z \ge 0$. Therefore, for any $\omega \in \Omega$, there is a $M_2 > 0$ such that $\eta_t \le -M_1$ for $t \ge M_2$, it means that $g_2(\eta_t(\omega)\zeta_t(\omega)) \le -1$. By using the second equation of (117), we get $\lim_{t\to\infty}\zeta_t(\omega) = -\infty$, which contradicts our assumption that $\inf_{t\ge 0}\zeta_t \ge a^*$ on Ω.

Next, we prove that $\limsup_{t\to\infty}\zeta_t \le a^*$ a.s. It easily knows that

$$g_2(y, z) = \frac{G(y, z)}{e^{-y}\left(z + \alpha\beta^{-1}e^{-y}\right)}, \tag{122}$$

where

$$\begin{aligned} G(y, z) = z\left(z(\gamma - \sigma) - \alpha\beta\right) + \alpha^2\beta^{-2}(r - d - \gamma) \\ \cdot e^{-3y} + \left[\alpha\beta^{-1}(r - d - \gamma) z - \gamma\alpha\beta^{-1} z - \alpha\beta^{-1}\right. \end{aligned}$$

$$+\alpha\beta^{-2}\left(\alpha\left(\gamma-\sigma\right)+\alpha\beta+\beta r\right)\Big]e^{-2y}+\Big[\alpha^2-\gamma z^2 +\alpha\beta^{-1}\left(z\left(\gamma-\sigma\right)-\alpha\beta\right) +\left(\alpha\left(\gamma-\sigma\right)+\alpha\beta+\beta r\right)\beta^{-1}z\Big]e^{-y} \tag{123}$$

is a polynomial of order 3 of the variable e^{-y}. Based on the definition of a^*, there is no more than one point $c_0 \in \mathfrak{R}$ satisfying $g_2(c_0, a^*) = 0$. Then, we have $g_2(y, a^*) \le 0$ for all $y \in \mathfrak{R}$ that for every $\tau > 0$, $g_2(y, a^*) < 0$ for all $y > c_0 + \tau$ or $y < c_0 - \tau$. By continuity theorem, we can choose an $\varepsilon > 0$ and a "rectangle" such that

$$B = \left(-\infty, c_0 - \tau\right] \times \left[a^*, a^* + k\right] \cup \left[c_0 + \tau, \infty\right) \times \left[a^* + k\right] \tag{124}$$

with $k > 0$ satisfying $g_2(y, z) < -\varepsilon$ for all $(y, z) \in A$. By using the Markov property, we get $\limsup_{t\to\infty}\zeta_t \le a^*$ a.s. The proof is complete. □

Theorem 11. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r + g < d + \gamma$ hold. If Hypothesis condition $\mathscr{H}_1$ holds, then E_1 is an invariant set; namely, if $(\eta_0, \zeta_0) \in E_1$, then $(\eta_t, \zeta_t) \in E_1$ for all $t > 0$.*

Proof. System (117) results in a solution (η_t, ζ_t) satisfying $(\eta_0, \zeta_0) \in E_1$ and $(\eta_{t_1}(\omega), \zeta_{t_1}(\omega)) \in E_1$ with some $t_1 > 0$, $\omega \in \Omega$. Based on the continuity of the path $\zeta_t(\omega)$, it is easy to show that there is $0 \le t_0 < \bar{t}_0 < t_1$ satisfying

$$\zeta_{t_0}(\omega) = a^*, \quad \zeta_s(\omega) < \zeta_t(\omega) \quad \forall t_0 < s < t < \bar{t}_0. \tag{125}$$

The Property (P). For any $\tau > 0$ there are two constants $k > 0$ and $\varepsilon > 0$ satisfying $g_2(y, z) < -\varepsilon$ for all $(y, z) \in B$, where

$$B = \left(-\infty, c_0 - \tau\right] \times \left[a^*, a^* + k\right] \cup \left[c_0 + \tau, \infty\right) \times \left[a^* + k\right]. \tag{126}$$

The property (P) has been proven in Theorem 10. Since the equation $d\zeta_t = g_2(\eta_t(\omega), \zeta_t(\omega))dt$, (125) and the property (P) that $\eta_t(\omega) = c_0$ for any $t \in [t_0, \bar{t}_0]$, we get $d\zeta_t/dt = g_2(c_0, \zeta_t(\omega))$. From (125), it easily shows that there is a decreasing sequence $\{s_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} s_n = t_0$ and $g_2(c_0, \zeta_{s_n}(\omega)) > 0$ for any $n = 1, 2, \ldots$. It is obvious that the result contradicts the property (P). Therefore, the proof is complete. □

Theorem 12. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r + g < d + \gamma$, $(\sigma + r)m > r - d$ hold, where m is defined by (23). Then $\lim_{t\to+\infty}\eta_t = -\infty$; the distribution of the Markov process ξ_t weakly converges to the probability measure with the density f_* when $t \to \infty$, where the Markov process (ξ_t, η_t) denotes a solution of (4) with $(\xi_0, \eta_0) \in \mathfrak{R}^2$.*

Proof. By Theorem 4, it is easy to see that $\xi_s \le \bar{\xi}_s$; then we get

$$\eta_t = \eta_0 + \int_0^t \left(r - d - \frac{\beta^2}{2}e^{2\eta_s} - (\sigma + r)e^{\xi_t} - re^{\eta_s}\right)ds + \int_0^t \beta e^{\eta_s} dW_s. \tag{127}$$

Therefore,

$$\frac{\eta_t}{t} \le \frac{\eta_0}{t} + \frac{\beta\int_0^t e^{\eta_s}dW_s - \left(\beta^2/2\right)\int_0^t e^{2\eta_s}ds}{t} - \frac{(\sigma + r)}{t}\int_0^t e^{\xi_t}ds + r - d. \tag{128}$$

Based on the proof of Theorem 5, we have

$$\limsup_{t\to\infty}\frac{\beta\int_0^t e^{\eta_s}dW_s - \left(\beta^2/2\right)\int_0^t e^{2\eta_s}ds}{t} \le 0 \quad \text{a.a.} \tag{129}$$

Moreover, by ergodic theorem, we get

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{\xi_t}ds = \int_{-\infty}^{+\infty} e^x f_*(x)\,dx = m. \tag{130}$$

From (23) and (128)–(130), we have

$$\limsup_{t\to\infty}\frac{\eta_t}{t} \le r - d - (\sigma + r)m < 0. \tag{131}$$

Then

$$\lim_{t\to+\infty}\eta_t = -\infty. \tag{132}$$

Thus, for sufficiently small $\varepsilon > 0$, it is easy to see that there exist t_0 and a set Ω_ε satisfying $\mathrm{Prob}(\Omega_\varepsilon) > 1-\varepsilon$ and $(\gamma-\sigma)e^{y_t} + g/(1 - e^{y_t}) \le \varepsilon$ for $t \ge t_0$ and $\omega \in \Omega_\varepsilon$. From the inequalities

$$\alpha dW + \left(\gamma - \frac{\alpha^2}{2} - \gamma e^{\xi_t} - \varepsilon\right)dt \le d\xi_t \le \alpha dW + \left(\gamma - \frac{\alpha^2}{2} - \gamma e^{\xi_t}\right)dt, \tag{133}$$

it easily shows that the distribution of the Markov process ξ_t weakly converges to the probability measure which possesses the density f_*. The proof is complete. □

Theorem 13. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r + g < d + \gamma$, $(\sigma + r)m < r - d$ hold; then there is a stationary distribution in system (4).*

Proof. By using the former random variables X_t and Y_t. It is easy to see that (X_t, Y_t) is a Markov process on $\mathfrak{R}_+^2$ and $\gamma = m\gamma + m'(\alpha^2/2)$, where

$$m' = \int_{\mathfrak{R}} e^{2x} f_*(x)\,dx. \tag{134}$$

Moreover,

$$\limsup_{t\to\infty}\int_0^t X_s^2 ds = \limsup_{t\to\infty}\int_0^t e^{2\xi_s} ds \le \lim_{t\to\infty} e^{2\bar{\xi}_s} ds = m'. \tag{135}$$

Therefore, by using the second assertion of Theorem 5, we get

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left(Y_s + \frac{g(\gamma-\sigma)^{-1}}{1-e^{Y_s}} + \frac{\gamma}{\gamma-\sigma}X_s\right)ds \ge \frac{\gamma}{\gamma-\sigma}m. \tag{136}$$

Furthermore, by Theorem 5, we have

$$\frac{\gamma}{(\gamma-\sigma)(\sigma+r)}\liminf_{t\to\infty}\frac{1}{t} \cdot\int_0^t\left(-(\sigma+r)X_s - rY_s - \frac{\beta^2}{2}Y_s^2\right)ds \ge -\frac{\gamma(r-d)}{(\gamma-\sigma)(\sigma+r)}. \tag{137}$$

From (136) and (137), we get

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left(-\frac{\gamma(r+\sigma-\gamma)}{(\gamma-\sigma)(\sigma+r)}Y_t - \frac{\beta^2\gamma}{2(\gamma-\sigma)(\sigma+r)}Y_2^2\right)ds \ge \frac{\gamma((\sigma+r)m-(r-d))}{(\gamma-\sigma)(\sigma+r)}. \tag{138}$$

By Theorem 5 again, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left(rY_s + \frac{\beta^2}{2}Y_t^2\right)ds \ge (r-d) - m(\sigma+r) > 0. \tag{139}$$

Moreover, we get that the inequality

$$\int_0^t Y_s ds \le \sqrt{t\int_0^t Y_s^2 ds}. \tag{140}$$

Then there are two constants $M_1 > 0$ and $M_2 > 0$ such that

$$2M_1 \le \liminf_{t\to\infty}\frac{1}{t}\int_0^t Y_s^2 ds \le \limsup_{t\to\infty}\frac{1}{t}\int_0^t Y_s^2 ds \le M_2. \tag{141}$$

From the inequality

$$2M_1 \le \liminf_{t\to\infty}\frac{1}{t}\int_0^t Y_s^2 ds, \tag{142}$$

it is easy to see that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t 1_{\{Y_s^2>M_1\}}Y_s^2 ds > M_1 > 0. \tag{143}$$

By Holder's inequality, we get

$$\frac{1}{t}\int_0^t \mathbb{E}\left[1_{\{Y_s^2>M_1\}}Y_s^2\right]ds \le \left(\frac{1}{t}\int_0^t \mathbb{E}\left[1_{\{Y_s^2>M_1\}}\right]^{(p+1)/p}ds\right)^{p/(p+1)} \cdot\left(\frac{1}{t}\int_0^t \mathbb{E}\left[Y_s\right]^{2+2p}ds\right)^{1/(p+1)}. \tag{144}$$

For $0 < p < 1/2$, there is a constant M_3 satisfying

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\left[Y_s\right]^{2+2p}ds < M_3. \tag{145}$$

Thus, we get

$$M_1 \le \liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\left[1_{\{Y_s^2>M_1\}}Y_s^2\right]ds \le \left(\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}\left\{Y_s^2 > M_1\right\}ds\right)^{p/(p+1)} \cdot\left(\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\left[Y_s\right]^{2+2p}ds\right)^{1/(p+1)}. \tag{146}$$

That is,

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}\left\{Y_s^2 > M_1\right\}ds \ge \varepsilon_0 = \frac{M_1^{(p+1)/p}}{M_3^{1/p}}. \tag{147}$$

Furthermore, there is a constant $H > 0$ such that

$$\liminf_{t\to\infty}\mathbb{P}\left\{X_t + Y_t \le H\right\} \ge 1 - \frac{\varepsilon_0}{2}. \tag{148}$$

It means that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t Q_t(A)ds \ge \frac{\varepsilon_0}{2} > 0, \tag{149}$$

where Q_t is the semigroup related to the random variable (X_t, Y_t) and

$$A = \left\{y \ge \sqrt{M_1}, x + y \le H\right\}. \tag{150}$$

From (149) and Theorem 1 in [29, pp 36], there is a stationary distribution λ in $\mathfrak{R}_+^2$ for the random variable (X_t, Y_t) satisfying $\lambda(A) > 0$. Because the boundary $A_1 = \{x = 0\} \times \mathfrak{R}_+$ is invariant under Q_t and $\lim_{t\to\infty}Y_t = 0$ if $X_0 = 0$, then we get that $\lambda(A_1) > 0$. Thus, there is a stationary distribution λ on int $\mathfrak{R}^2$; that is, there is a stationary distribution related to the random variable (ξ_t, η_t). The proof is complete. □

Theorem 14. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r+g < d+\gamma$ hold. Then the distribution of the random variable (ξ_t, η_t) exists a density $u(t,x,y)$ satisfying (8). Furthermore, the following assertions are true*

(I) *Assume* $(\sigma + r)m < r - d$; *if Hypothesis condition* $\mathscr{H}_1$ *does not hold, then the Markov semigroup* $\{P(t)\}_{t\geq 0}$ *is asymptotically stable on* $\mathfrak{R}^2$; *that is, there is a stationary density* $u_*(t, x, y)$ *of (8) satisfying*

$$\lim_{t\to\infty} \iint_{\mathfrak{R}^2} \left|u(t, x, y) - u_*(x, y)\right| dx\, dy = 0. \tag{151}$$

(II) *Assume* $(\sigma + r)m < r - d$; *if Hypothesis condition* $\mathscr{H}_1$ *holds, then* E_2 *is a transient set and* E_1 *is an invariant set. Furthermore, the integral Markov semigroup* $\{P(t)\}_{t\geq 0}$ *is asymptotically stable on* E_1. *It implies that support* $u_* \subset E_1$ *and*

$$\lim_{t\to\infty} \iint_{E_1} \left|u(t, x, y) - u_*(x, y)\right| dx\, dy = 0. \tag{152}$$

(III) *Assume* $(\sigma+r)m > r-d$; *then* $\lim_{t\to\infty}\eta_t = -\infty$ *a.s. and the distribution of random variable* ξ_t *weakly converges to the probability measure with the density* $f_*(x)$ *when* $t \to \infty$.

Proof. From Theorem 6, then the distribution of the random variable (ξ_t, η_t) results in a density $u(t, x, y)$ satisfying (8). By Theorem 9, for every $f \in D$, we get

$$\int_0^\infty P(t)\, f dt > 0 \quad \text{a.e. on } \mathfrak{R}^2 \text{ or on } E_1. \tag{153}$$

By Corollary 1 in [23, pp 248], it is easy to see that the integral Markov semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable or sweeping. Based on Theorems 10, 11, and 13, it is easy to see that (I) assertion and (II) assertion are directly proven. By Theorem 12, (III) assertion is directly proven. The proof is complete. □

Competing Interests

The author declares that he has no competing interests.

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Research Article

Comparison Criteria for Nonlinear Functional Dynamic Equations of Higher Order

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We will consider the higher order functional dynamic equations with mixed nonlinearities of the form $x^{[n]}(t) + \sum_{j=0}^{N} p_j(t)\phi_{\gamma_j}(x(\varphi_j(t))) = 0$, on an above-unbounded time scale $\mathbb{T}$, where $n \geq 2$, $x^{[i]}(t) := r_i(t)\phi_{\alpha_i}[(x^{[i-1]})^{\Delta}(t)]$, $i = 1, \ldots, n-1$, with $x^{[0]} = x$, $\phi_\beta(u) := |u|^\beta \operatorname{sgn} u$, and $\alpha[i,j] := \alpha_i \cdots \alpha_j$. The function $\varphi_i : \mathbb{T} \to \mathbb{T}$ is a rd-continuous function such that $\lim_{t\to\infty}\varphi_i(t) = \infty$ for $j = 0, 1, \ldots, N$. The results extend and improve some known results in the literature on higher order nonlinear dynamic equations.

1. Introduction

In this paper, we consider comparison criteria for higher order nonlinear dynamic equation with mixed nonlinearities of the form

$$x^{[n]}(t) + \sum_{j=0}^{N} p_j(t)\phi_{\gamma_j}\left(x\left(\varphi_j(t)\right)\right) = 0, \tag{1}$$

on an above-unbounded time scale $\mathbb{T}$, where

(i) $n \geq 2$ is an integer, and $x^{[i]}(t) := r_i(t)\phi_{\alpha_i}[(x^{[i-1]})^{\Delta}(t)]$, $i = 1, 2, \ldots, n-1$, $t \in \mathbb{T}$, with $r_n = 1$, $\alpha_n = 1$, and $x^{[0]} = x$;

(ii) $\phi_\beta(u) := |u|^\beta \operatorname{sgn} u$ for $\beta > 0$.

Without loss of generality we assume $t_0 \in \mathbb{T}$. For $A \subset \mathbb{T}$ and $B \subset \mathbb{R}$, we denote by $C_{\mathrm{rd}}(A, B)$ the space of right-dense continuous functions from A to B and by $C^1_{\mathrm{rd}}(A, B)$ the set of functions in $C_{\mathrm{rd}}(A, B)$ with right-dense continuous Δ-derivatives. Throughout this paper we make the following assumptions:

(iii) $\alpha_i, \gamma_j > 0$, $i = 1, 2, \ldots, n-1$ and $j = 0, 1, \ldots, N$, are constants and $r_i \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ for $i = 1, 2, \ldots, n-1$, such that

$$\int_{t_0}^{\infty} r_i^{-1/\alpha_i}(s)\,\Delta s = \infty, \quad i = 1, 2, \ldots, n-1; \tag{2}$$

$$\begin{aligned} &\gamma_j > \gamma_0, \quad j = 1, 2, \ldots, l; \\ &\gamma_j < \gamma_0, \quad j = l+1, l+2, \ldots, N. \end{aligned} \tag{3}$$

(iv) $p_j \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ such that $p_j \not\equiv 0$, $j = 0, 1, \ldots, N$ on $[t_0, \infty)_{\mathbb{T}}$.

(v) $\varphi_j \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$ rd-continuous function such that $\lim_{t\to\infty}\varphi_j(t) = \infty$, $j = 0, 1, \ldots, N$, and we let $\varphi(t) := \inf\{\varphi_0(t), \varphi_1(t), \ldots, \varphi_N(t)\}$ be a nondecreasing function on $[t_0, \infty)_{\mathbb{T}}$.

Recall that the knowledge and understanding of time scales and time scale notation are assumed. For an excellent introduction to the calculus on time scales, see [1–3]. By a solution of (1) we mean a nontrivial real-valued function

$x \in C^1_{\mathrm{rd}}[T_x, \infty)_{\mathbb{T}}$ for some $T_x \geq t_0$ such that $x^{[i]} \in C^1_{\mathrm{rd}}[T_x, \infty)_{\mathbb{T}}$, $i = 1, 2, \ldots, n-1$, and $x(t)$ satisfies (1) on $[T_x, \infty)_{\mathbb{T}}$, where C_{rd} is the space of right-dense continuous functions. An extendable solution x of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is said to be nonoscillatory.

In the last few years, there has been an increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations; we refer the reader to [4–13] and the references cited therein. Special cases of (1) have been studied by many authors. When $\alpha_i = r_i = N = 1$ and $\varphi_i(t) = t$ for $i = 1, 2, \ldots, n$, $p(t) = q_0(t) = 0$, and $q_1(t) \geq 0$, Grace et al. [14] established some oscillation criteria for higher order nonlinear dynamic equation of the form

$$x^{\Delta^n}(t) + p(t)\left(x^{\sigma}(t)\right)^{\gamma} = 0, \tag{4}$$

where γ is the ratio of positive odd integers. In paper by Grace [15], some new criteria for the oscillation of the even order dynamic equation

$$\left[r(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right]^{\Delta} + p(t)\left(x^{\sigma}(t)\right)^{\gamma} = 0, \tag{5}$$

where α and γ are the ratios of positive odd integers, were given. Recently, Hassan and Kong [16] obtained asymptotics and oscillation criteria for the nth-order half-linear dynamic equation with deviating argument

$$\left(x^{[n-1]}\right)^{\Delta}(t) + p(t)\,\phi_{\alpha[1,n-1]}\left(x\left(g(t)\right)\right) = 0, \tag{6}$$

and Grace and Hassan [17] establish oscillation criteria for more general higher order dynamic equation

$$x^{[n]}(t) + p(t)\,\phi_{\gamma}\left(x^{\sigma}\left(\varphi(t)\right)\right) = 0. \tag{7}$$

The purpose of this paper is to derive comparison criteria for higher order nonlinear dynamic equation with mixed nonlinearities (1).

2. Preliminaries

We will employ the following lemmas. Consider the inequality

$$\left(r(t)\,\phi_{\alpha}\left(x^{\Delta}(t)\right)\right)^{\Delta} + Q(t)\,\phi_{\gamma}\left(x\left(\varphi(t)\right)\right) \leq 0, \tag{8}$$

where r and Q are positive real-valued, rd-continuous functions on $\mathbb{T}$ and r satisfies condition (2), $\varphi : \mathbb{T} \to \mathbb{T}$ is a rd-continuous function and $\varphi(t) \to \infty$ as $t \to \infty$, and α and γ are positive real numbers.

Now, we present the following lemma.

Lemma 1. *If inequality (8) has an eventually positive solution, then the equation*

$$\left(r(t)\,\phi_{\alpha}\left(x^{\Delta}(t)\right)\right)^{\Delta} + Q(t)\,\phi_{\gamma}\left(x\left(\varphi(t)\right)\right) = 0 \tag{9}$$

has also an eventually positive solution.

Proof. Let $x(t)$ be an eventually positive solution of inequality (8). It is easy to see that $x^{\Delta} > 0$ eventually. Let t_0 be sufficiently large so that $x(t) > 0$, $x(\varphi(t)) > 0$, and $y(t) := r(t)\phi_{\alpha}(x^{\Delta}(t))$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Then, in view of

$$x(t) = x(t_0) + \int_{t_0}^{t} \phi_{\alpha}^{-1}\left(\frac{y(s)}{r(s)}\right)\Delta s, \tag{10}$$

there is $t_1 \geq t_0$ such that $\varphi(t) \geq t_0$, for $t \geq t_1$. Inequality (8) becomes

$$y^{\Delta}(t) + Q(t)\,\phi_{\gamma}\left(x(t_0) + \int_{t_0}^{\varphi(t)} \phi_{\alpha}^{-1}\left(\frac{y(s)}{r(s)}\right)\Delta s\right) \leq 0. \tag{11}$$

Integrating (11) from t to $v \geq t \geq t_1$ and letting $v \to \infty$, we have

$$y(t) \geq G(t, y(t)), \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \tag{12}$$

where

$$\begin{aligned} &G(t, y(t)) \\ &\quad := \int_{t}^{\infty} Q(v)\,\phi_{\gamma}\left(x(t_0) + \int_{t_0}^{\varphi(v)} \phi_{\alpha}^{-1}\left(\frac{y(s)}{r(s)}\right)\Delta s\right)\Delta v. \end{aligned} \tag{13}$$

Now, we define a sequence of successive approximations $\{w_j(t)\}$ as follows:

$$\begin{aligned} w_0(t) &:= y(t), \\ w_{j+1}(t) &:= G\left(t, w_j(t)\right), \quad j = 0, 1, 2, \ldots. \end{aligned} \tag{14}$$

It is easy to show that

$$\begin{aligned} 0 < w_j(t) &\leq y(t), \\ w_{j+1}(t) &\leq w_j(t), \\ &\qquad j = 0, 1, 2, \ldots. \end{aligned} \tag{15}$$

Then, the sequence $\{w_j(t)\}$ is nonincreasing and bounded for each $t \geq t_1$. This means that we may define $w(t) := \lim_{j\to\infty} w_j(t) \geq 0$. Since

$$0 \leq w(t) \leq w_j(t) \leq y(t), \quad \forall j \geq 0, \tag{16}$$

we find that

$$\int_{t_1}^{t} w_j(s)\,\Delta s \leq \int_{t_1}^{t} y(s)\,\Delta s. \tag{17}$$

By Lebesgue's dominated convergence theorem on time scale, one can easily find

$$w(t) = G(t, w(t)). \tag{18}$$

Therefore

$$\begin{aligned} w^{\Delta}(t) &= -Q(t)\,\phi_{\gamma}\left(x(t_0) + \int_{t_0}^{\varphi(t)} \phi_{\alpha}^{-1}\left(\frac{w(s)}{r(s)}\right)\Delta s\right) \\ &= -Q(t)\,\phi_{\gamma}\left(m\left(\varphi(t)\right)\right), \end{aligned} \tag{19}$$

where

$$m(t) := x(t_0) + \int_{t_0}^{t} \phi_\alpha^{-1}\left(\frac{w(s)}{r(s)}\right)\Delta s. \tag{20}$$

Then

$$\begin{gathered} m(t) > 0, \\ r(t)\,\phi_\alpha\left(m^\Delta(t)\right) = w(t), \\ \text{for } t \geq t_1. \end{gathered} \tag{21}$$

Equation (19) then gives

$$\left(r(t)\,\phi_\alpha\left(m^\Delta(t)\right)\right)^\Delta + Q(t)\,\phi_\gamma\left(m(\varphi(t))\right) = 0. \tag{22}$$

Hence (9) has a positive solution $m(t)$. This completes the proof. □

The second one is cited from [18, 19].

Lemma 2. *Assume that (3) holds. Then there exists an N-tuple $(\eta_1, \eta_2, \ldots, \eta_N)$ with $\eta_j > 0$ satisfying*

$$\begin{gathered} \sum_{j=1}^{N} \gamma_j \eta_j = \gamma_0, \\ \sum_{j=1}^{N} \eta_j = 1. \end{gathered} \tag{23}$$

The next lemma is cited from [17] and improves the well-known lemma of Kiguradze.

Lemma 3. *Assume that (2) holds. If (1) has an eventually positive solution x, then there exists an integer $m \in [0, n]$ with $m + n$ being odd such that*

$$\begin{gathered} m \geq 1 \\ \text{implies } x^{[k]} > 0 \quad \text{for } k = 0, 1, \ldots, m-1, \end{gathered} \tag{24}$$

eventually, and

$$\begin{gathered} m \leq n \\ \text{implies } (-1)^{m+k} x^{[k]} > 0 \quad \text{for } k = m, \ldots, n, \end{gathered} \tag{25}$$

eventually.

3. Main Results

In the following main theorem, we will use the following notations: $\alpha[h,k] := \alpha_h \cdots \alpha_k$ for $1 \leq h \leq k \leq n-1$ and $\alpha[h,k] = 1$ for $h > k$ and, for any $u, v \in \mathbb{T}$, define $R_j(v,u)$, $P_j(t)$, and $\overline{P}_j(t)$, $j = 0, \ldots, n-1$, by the following recurrence formulas:

$$\begin{aligned} R_j(v,u) &:= \begin{cases} \left[\displaystyle\int_u^v \frac{R_{j-1}(s,u)}{r_{m-j+1}(s)}\Delta s\right]^{1/\alpha_{m-j+1}}, & j = 1, \ldots, m, \\ 1, & j = 0; \end{cases} \\ P_j(t) &:= \begin{cases} \left[\displaystyle\frac{1}{r_{n-j}(t)} \int_t^\infty P_{j-1}(s)\,\Delta s\right]^{1/\alpha_{n-j}}, & j = 1, \ldots, n-1, \\ p(t), & j = 0; \end{cases} \\ \overline{P}_j(t) &:= \begin{cases} \left[\displaystyle\frac{1}{r_{n-j}(t)} \int_t^\infty \overline{P}_{j-1}(s)\,\Delta s\right]^{1/\alpha_{n-j}}, & j = 1, \ldots, n-1, \\ \overline{p}(t), & j = 0, \end{cases} \end{aligned} \tag{26}$$

with $p(t) := p_0(t) + \prod_{j=1}^{N} [p_j(t)/\eta_j]^{\eta_j}$ and $\overline{p}(t) := \sum_{j=0}^{N} p_j(t)$, provided the improper integrals involved are convergent.

Theorem 4. *Assume that for sufficiently large $T \in [t_0, \infty)_\mathbb{T}$ the first-order dynamic equation*

$$\begin{gathered} z^\Delta(t) + K_m(t)\,\phi_{\gamma_0/\alpha[1,n]}\left(z(\varphi(t))\right) = 0, \\ \text{for } \varphi(t) \in [T, \infty)_\mathbb{T}, \end{gathered} \tag{27}$$

is oscillatory, where

$$K_m(t) := P_{n-m-1}(t)\left[R_m(\varphi(t), T)\right]^{\gamma_0/\alpha[m+1,n]}, \tag{28}$$

for every number $m \in \{1, \ldots, n-1\}$ with $m+n$ being odd. Then

(1) *if n is even, every solution of (1) is oscillatory,*

(2) *if n is odd and*

$$\int_T^\infty \overline{P}_{n-1}(t)\,\Delta t = \infty, \tag{29}$$

then every solution of (1) either is oscillatory or tends to zero eventually.

Proof. Assume that (1) has a nonoscillatory solution x on $[t_0, \infty)_\mathbb{T}$. Then, without loss of generality, $x(t) > 0$ and $x(\varphi_j(t)) > 0$, for $t \in [t_0, \infty)_\mathbb{T}$ and $j = 0, 1, \ldots, N$. By Lemma 3, there exists an integer m, $0 \leq m < n$, with $n + m$ being odd such that (24) and (25) hold for $t \geq T_1 \in [t_0, \infty)_\mathbb{T}$.

(I) When $m \geq 1$, from (1), we get

$$\begin{aligned} -x^{[n]}(t) &= \sum_{j=0}^{N} p_j(t)\,\phi_{\gamma_j}\left(x\left(\varphi_j(t)\right)\right) \\ &\geq \sum_{j=0}^{N} p_j(t)\,\phi_{\gamma_j}\left(x(\varphi(t))\right) \\ &= \phi_{\gamma_0}\left(x(\varphi(t))\right) \sum_{j=0}^{N} p_j(t)\left[x(\varphi(t))\right]^{\gamma_j - \gamma_0}. \end{aligned} \tag{30}$$

From (23), we have

$$\sum_{j=1}^{N}\gamma_j\eta_j-\gamma_0\sum_{j=1}^{N}\eta_j=0. \tag{31}$$

Using the arithmetic-geometric mean inequality (see [20, Page 17]), we have

$$\sum_{j=1}^{N}\eta_j v_j\geq\prod_{j=1}^{N}v_j^{\eta_j},\quad \text{for any } v_j\geq 0,\quad j=1,\ldots,N. \tag{32}$$

Then for $t\geq T_1$,

$$\begin{aligned}&\sum_{j=0}^{N}p_j(t)\left[x(\varphi(t))\right]^{\gamma_j-\gamma_0}\\&\quad=p_0(t)+\sum_{j=1}^{N}\eta_j\frac{p_j(t)}{\eta_j}\left[x(\varphi(t))\right]^{\gamma_j-\gamma_0}\\&\quad\geq p_0(t)+\prod_{j=1}^{N}\left[\frac{p_j(t)}{\eta_j}\right]^{\eta_j}\left[x(\varphi(t))\right]^{\eta_j(\gamma_j-\gamma_0)}\\&\quad=p_0(t)+\prod_{j=1}^{N}\left[\frac{p_j(t)}{\eta_j}\right]^{\eta_j}=p(t).\end{aligned} \tag{33}$$

This together with (30) shows that

$$-x^{[n]}(t)\geq p(t)\,\phi_{\gamma_0}(x(\varphi(t))),\quad \text{for } t\geq T_1. \tag{34}$$

Integrating above inequality from $t\geq T_1$ to $v\in[t,\infty)_{\mathbb{T}}$ and then using the fact that x is strictly increasing and φ is nondecreasing, we get

$$\begin{aligned}-x^{[n-1]}(v)+x^{[n-1]}(t)&\geq\int_t^v p(s)\,\phi_{\gamma_0}(x(\varphi(s)))\,\Delta s\\&\geq\phi_{\gamma_0}(x(\varphi(t)))\int_t^v p(s)\,\Delta s,\end{aligned} \tag{35}$$

and by (25) we see that $x^{[n-1]}(v)>0$. Hence by taking limits as $v\to\infty$ we have

$$x^{[n-1]}(t)\geq\phi_{\gamma_0}(x(\varphi(t)))\int_t^{\infty}p(s)\,\Delta s, \tag{36}$$

which implies

$$\begin{aligned}&\left[x^{[n-2]}(t)\right]^{\Delta}\\&\quad\geq\phi_{\alpha_{n-1}}^{-1}\left[\phi_{\gamma_0}(x(\varphi(t)))\right]\left[\frac{1}{r_{n-1}(t)}\int_t^{\infty}p(s)\,\Delta s\right]^{1/\alpha_{n-1}}\\&\quad=\phi_{\alpha[n-1,n]}^{-1}\left[\phi_{\gamma_0}(x(\varphi(t)))\right]P_1(t).\end{aligned} \tag{37}$$

Integrating above inequality (37) from $t\geq T_1$ to $v\in[t,\infty)_{\mathbb{T}}$ and letting $v\to\infty$, we get

$$\begin{aligned}&-x^{[n-2]}(t)\geq\phi_{\alpha[n-1,n]}^{-1}\left[\phi_{\gamma_0}(x(\varphi(t)))\right]\int_t^{\infty}P_1(s)\,\Delta s,\\&-\left[x^{[n-3]}(t)\right]^{\Delta}\geq\phi_{\alpha[n-2,n]}^{-1}\left[\phi_{\gamma_0}(x(\varphi(t)))\right]\\&\quad\cdot\left[\frac{1}{r_{n-2}(t)}\int_t^{\infty}P_1(s)\,\Delta s\right]^{1/\alpha_{n-2}}\\&\quad=\phi_{\alpha[n-2,n]}^{-1}\left[\phi_{\gamma_0}(x(\varphi(t)))\right]P_2(t).\end{aligned} \tag{38}$$

Continuing this process $(n-m-3)$ times, we find

$$-\left[x^{[m]}(t)\right]^{\Delta}\geq\phi_{\alpha[m+1,n]}^{-1}\left[\phi_{\gamma_0}(x(\varphi(t)))\right]P_{n-m-1}(t),\quad \text{for } t\geq T_1. \tag{39}$$

Also, from (24) and (25), we get

$$\begin{aligned}x^{[m-1]}(t)&=x^{[m-1]}(T_1)\\&\quad+\int_{T_1}^{t}\phi_{\alpha_m}^{-1}\left(x^{[m]}(s)\right)\left[\frac{1}{r_m(s)}\right]^{1/\alpha_m}\Delta s\\&\geq\phi_{\alpha_m}^{-1}\left(x^{[m]}(t)\right)\int_{T_1}^{t}\left[\frac{1}{r_m(s)}\right]^{1/\alpha_m}\Delta s\\&=\phi_{\alpha_m}^{-1}\left(x^{[m]}(t)\right)R_1(t,T_1).\end{aligned} \tag{40}$$

It follows that

$$\begin{aligned}&\left(x^{[m-2]}(t)\right)^{\Delta}\\&\quad\geq\phi_{\alpha[m-1,m]}^{-1}\left(x^{[m]}(t)\right)\left[\frac{R_1(t,T_1)}{r_{m-1}(t)}\right]^{1/\alpha_{m-1}}.\end{aligned} \tag{41}$$

Then for $t\in[T_1,\infty)_{\mathbb{T}}$,

$$\begin{aligned}x^{[m-2]}(t)&\geq x^{[m-2]}(t)-x^{[m-2]}(T_1)\\&\geq\int_{T_1}^{t}\phi_{\alpha[m-1,m]}^{-1}\left(x^{[m]}(s)\right)\left[\frac{R_1(s,T_1)}{r_{m-1}(s)}\right]^{1/\alpha_{m-1}}\Delta s\\&\geq\phi_{\alpha[m-1,m]}^{-1}\left(x^{[m]}(t)\right)\int_{T_1}^{t}\left[\frac{R_1(s,T_1)}{r_{m-1}(s)}\right]^{1/\alpha_{m-1}}\Delta s\\&=\phi_{\alpha[m-1,m]}^{-1}\left(x^{[m]}(t)\right)R_2(t,T_1).\end{aligned} \tag{42}$$

Analogously, we have

$$x(t)\geq\phi_{\alpha[1,m]}^{-1}\left(x^{[m]}(t)\right)R_m(t,T_1). \tag{43}$$

Then for $\varphi(t)\in[T_1,\infty)_{\mathbb{T}}$

$$x(\varphi(t))\geq\phi_{\alpha[1,m]}^{-1}\left(x^{[m]}(\varphi(t))\right)R_m(\varphi(t),T_1). \tag{44}$$

From (39) and (44), we get

$$-\left[x^{[m]}(t)\right]^{\Delta} \geq P_{n-m-1}(t)\left[R_m(\varphi(t),T_1)\right]^{\gamma_0/\alpha[m+1,n]} \cdot \phi_{\gamma_0/\alpha[1,n]}\left(x^{[m]}(\varphi(t))\right) = K_m(t) \cdot \phi_{\gamma_0/\alpha[1,n]}\left(x^{[m]}(\varphi(t))\right). \tag{45}$$

Let $z(t) := x^{[m]}(t) > 0$; we get

$$-z^{\Delta}(t) \geq K_m(t)\,\phi_{\gamma_0/\alpha[1,n]}\left(z(\varphi(t))\right), \tag{46}$$

or

$$z^{\Delta}(t) + K_m(t)\,\phi_{\gamma_0/\alpha[1,n]}\left(z(\varphi(t))\right) \leq 0. \tag{47}$$

In view of Corollary 2.3.5 in [21], there exists a positive solution of (27) which contradicts the assumption of the theorem.

(II) When $m = 0$ (in this case n is odd), therefore

$$(-1)^k x^{[k]} > 0 \quad \text{for } k = 0,\ldots,n. \tag{48}$$

Since $x^{\Delta} < 0$ eventually, then $\lim_{t\to\infty} x(t) = l \geq 0$. Assume that $l > 0$. Then for sufficiently large $T_2 \in [T_1,\infty)_{\mathbb{T}}$, we have $x(\varphi_j(t)) \geq l$ for $t \geq T_2$ and $j = 0,1,\ldots,N$. It follows that

$$\phi_{\gamma_j}\left(x\left(\varphi_j(t)\right)\right) \geq l^{\gamma_j} \geq L \quad \text{for } t \in [T_2,\infty)_{\mathbb{T}}, \tag{49}$$

where $L := \inf_{0\leq j\leq N}\{l^{\gamma_j}\} > 0$. Then from (1), we have

$$-x^{[n]}(t) = \sum_{j=0}^{N} p_j(t)\,\phi_{\gamma_j}\left(x\left(\varphi_j(t)\right)\right) \geq L\sum_{j=0}^{N} p_j(t) = L\overline{p}(t). \tag{50}$$

Integrating from t to $v \in [t,\infty)_{\mathbb{T}}$, we get

$$-x^{[n-1]}(v) + x^{[n-1]}(t) = L\int_t^v \overline{p}(s)\,\Delta s. \tag{51}$$

And by (48) we see that $x^{[n-1]}(v) > 0$. Hence by taking limits as $v \to \infty$ we have

$$x^{[n-1]}(t) \geq L\int_t^{\infty} \overline{p}(s)\,\Delta s, \tag{52}$$

which implies

$$\left(x^{[n-2]}(t)\right)^{\Delta} \geq L^{1/\alpha_{n-1}}\left[\frac{1}{r_{n-1}(t)}\int_t^{\infty}\overline{p}(s)\,\Delta s\right]^{1/\alpha_{n-1}} = L^{1/\alpha_{n-1}}\overline{P}_1(t). \tag{53}$$

Again integrating above inequality from t to $v \in [t,\infty)_{\mathbb{T}}$ and then taking $v \to \infty$, we get

$$-x^{[n-2]}(t) \geq L^{1/\alpha_{n-1}}\int_t^{\infty}\overline{P}_1(s)\,\Delta s, \tag{54}$$

which implies

$$-\left(x^{[n-3]}(t)\right)^{\Delta} \geq L^{1/\alpha[n-2,n-1]}\left[\frac{1}{r_{n-2}(t)}\int_t^{\infty}\overline{P}_1(s)\,\Delta s\right]^{1/\alpha_{n-2}} = L^{1/\alpha[n-2,n-1]}\overline{P}_2(t). \tag{55}$$

Continuing this process $(n-3)$ times, we find

$$-(x(t))^{\Delta} \geq L^{1/\alpha[1,n-1]}\overline{P}_{n-1}(t) \quad \text{for } t \in [T_2,\infty)_{\mathbb{T}}. \tag{56}$$

Integrating above inequality T_2 to $t \in [T_2,\infty)_{\mathbb{T}}$, we get

$$-x(t) + x(T_2) \geq L^{1/\alpha[1,n-1]}\int_{T_2}^{t}\overline{P}_{n-1}(s)\,\Delta s. \tag{57}$$

Hence by (29), we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x > 0$ eventually. This shows that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Theorem 5. *Assume that for sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$ the second-order dynamic equation*

$$\left[r_m(t)\,\phi_{\alpha_m}\left(z^{\Delta}(t)\right)\right]^{\Delta} + Q_m(t)\,\phi_{\alpha_m\gamma_0/\alpha[1,n]}\left(z(\varphi(t))\right) = 0, \quad \text{for } \varphi(t) \in (T,\infty)_{\mathbb{T}}, \tag{58}$$

is oscillatory, where

$$Q_m(t) := \frac{P_{n-m-1}(t)\left[R_m(\varphi(t),T_1)\right]^{\gamma_0/\alpha[m+1,n]}}{\left[R_1(\varphi(t),T_1)\right]^{\alpha_m\gamma_0/\alpha[1,n]}}, \tag{59}$$

for every integer number $m \in \{1,\ldots,n-1\}$ with $m+n$ being odd. Then

(1) *if n is even, then every solution of (1) is oscillatory,*

(2) *if n is odd and (29) holds, then every solution of (1) either is oscillatory or tends to zero eventually.*

Proof. Assume that (1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, $x(t) > 0$ and $x(\varphi_j(t)) > 0$, $j = 0,1,2,\ldots,N$ on $[t_0,\infty)_{\mathbb{T}}$. By Lemma 3, there exists an integer m, $0 \leq m < n$, with $n+m$ being odd such that (24) and (25) hold for $t \geq T_1 \in [t_0,\infty)_{\mathbb{T}}$.

(I) When $m \geq 1$, as seen in the proof of Theorem 4, we obtain, for $t \in [T_1,\infty)_{\mathbb{T}}$,

$$-\left[x^{[m]}(t)\right]^{\Delta} \geq \phi^{-1}_{\alpha[m+1,n]}\left[\phi_{\gamma_0}\left(x(\varphi(t))\right)\right]P_{n-m-1}(t), \tag{60}$$

$$x^{[m-1]}(t) \geq \phi^{-1}_{\alpha_m}\left(x^{[m]}(t)\right)R_1(t,T_1). \tag{61}$$

Hence, we have

$$\left[\frac{x^{[m-1]}(t)}{R_1(t,T_1)}\right]^{\Delta} \leq 0, \quad \text{for } t \in (T_1,\infty)_{\mathbb{T}}. \tag{62}$$

Since for $t \in (T_1, \infty)_{\mathbb{T}}$,

$$x^{[m-1]}(t) = \frac{x^{[m-1]}(t)}{R_1(t, T_1)} R_1(t, T_1), \tag{63}$$

we have

$$\begin{aligned} &\left(x^{[m-2]}(t)\right)^{\Delta} \\ &\quad = \phi_{\alpha_{m-1}}^{-1}\left(\frac{x^{[m-1]}(t)}{R_1(t, T_1)}\right)\left[\frac{R_1(t, T_1)}{r_{m-1}(t)}\right]^{1/\alpha_{m-1}}. \end{aligned} \tag{64}$$

It follows from (62) that we have, for $t \in (T_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned} x^{[m-2]}(t) &\geq x^{[m-2]}(t) - x^{[m-2]}(T_1) \\ &= \int_{T_1}^{t} \phi_{\alpha_{m-1}}^{-1}\left(\frac{x^{[m-1]}(s)}{R_1(s, T_1)}\right)\left[\frac{R_1(s, T_1)}{r_{m-1}(s)}\right]^{1/\alpha_{m-1}} \Delta s \\ &\geq \phi_{\alpha_{m-1}}^{-1}\left(\frac{x^{[m-1]}(t)}{R_1(t, T_1)}\right)\int_{T_1}^{t}\left[\frac{R_1(s, T_1)}{r_{m-1}(s)}\right]^{1/\alpha_{m-1}} \Delta s \\ &= \phi_{\alpha_{m-1}}^{-1}\left(\frac{x^{[m-1]}(t)}{R_1(t, T_1)}\right) R_2(t, T_1). \end{aligned} \tag{65}$$

Continuing this process, we have

$$x(t) \geq \phi_{\alpha[1,m-1]}^{-1}\left(\frac{x^{[m-1]}(t)}{R_1(t, T_1)}\right) R_m(t, T_1). \tag{66}$$

Then for $\varphi(t) \in (T_1, \infty)_{\mathbb{T}}$

$$x(\varphi(t)) \geq \phi_{\alpha[1,m-1]}^{-1}\left(\frac{x^{[m-1]}(\varphi(t))}{R_1(\varphi(t), T_1)}\right) R_m(\varphi(t), T_1). \tag{67}$$

From (60) and (67), we get

$$\begin{aligned} -\left[x^{[m]}(t)\right]^{\Delta} &\geq \phi_{\alpha_m \gamma_0/\alpha[1,n]}\left(\frac{x^{[m-1]}(\varphi(t))}{R_1(\varphi(t), T_1)}\right) \\ &\quad \cdot P_{n-m-1}(t)\left[R_m(\varphi(t), T_1)\right]^{\gamma_0/\alpha[m+1,n]} = Q_m(t) \\ &\quad \cdot \phi_{\alpha_m \gamma_0/\alpha[1,n]}\left(x^{[m-1]}(\varphi(t))\right). \end{aligned} \tag{68}$$

Set $z(t) := x^{[m-1]}(t) > 0$; we have, for $t \geq T_1$,

$$\begin{aligned} &-\left[r_m(t)\phi_{\alpha_m}\left(z^{\Delta}(t)\right)\right]^{\Delta} \\ &\quad \geq Q_m(t)\phi_{\alpha_m \gamma_0/\alpha[1,n]}(z(\varphi(t))), \end{aligned} \tag{69}$$

or

$$\begin{aligned} &\left[r_m(t)\phi_{\alpha_m}\left(z^{\Delta}(t)\right)\right]^{\Delta} + Q_m(t)\phi_{\alpha_m \gamma_0/\alpha[1,n]}(z(\varphi(t))) \\ &\quad \leq 0. \end{aligned} \tag{70}$$

In view of Lemma 1, there exists a positive solution of (58) which contradicts the assumption of the theorem.

(II) When $m = 0$, as shown in the proof of Theorem 4, we show that if (29) holds, then $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Remark 6. The conclusion of Theorems 4 and 5 remains intact if assumption (29) is replaced by one of the following conditions:

$$\begin{gathered} \int_T^{\infty} \overline{P}_0(t)\,\Delta t = \infty, \\ \int_T^{\infty} \overline{P}_1(t)\,\Delta t = \infty, \\ \vdots \\ \text{or } \int_T^{\infty} \overline{P}_{n-2}(t)\,\Delta t = \infty. \end{gathered} \tag{71}$$

Theorem 7. *Assume that*

$$\int_{t_0}^{\infty} \overline{P}_2(t)\,\Delta t = \infty. \tag{72}$$

And for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$, the first-order dynamic equation

$$z^{\Delta}(t) + K_{n-1}(t)\phi_{\gamma_0/\alpha[1,n]}(z(\varphi(t))) = 0, \quad \text{for } \varphi(t) \in [T, \infty)_{\mathbb{T}}, \tag{73}$$

is oscillatory, where

$$K_{n-1}(t) := p(t)\left[R_{n-1}(\varphi(t), T)\right]^{\gamma_0}. \tag{74}$$

Then

(1) *if n is even, every solution of (1) is oscillatory,*

(2) *if n is odd, then every solution of (1) either is oscillatory or tends to zero eventually.*

Theorem 8. *Assume that (72) holds and, for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$, the second-order dynamic equation*

$$\begin{aligned} &\left[r_{n-1}(t)\phi_{\alpha_{n-1}}\left(z^{\Delta}(t)\right)\right]^{\Delta} \\ &\quad + Q_{n-1}(t)\phi_{\gamma_0/\alpha[1,n-2]}(z(\varphi(t))) = 0, \\ &\qquad \text{for } \varphi(t) \in (T, \infty)_{\mathbb{T}}, \end{aligned} \tag{75}$$

is oscillatory, where

$$Q_{n-1}(t) := \frac{p(t)\left[R_{n-1}(\varphi(t), T_1)\right]^{\gamma_0}}{\left[R_1(\varphi(t), T_1)\right]^{\gamma_0/\alpha[1,n-2]}}. \tag{76}$$

Then

(i) *if n is even, every solution of (1) is oscillatory,*

(ii) *if n is odd, every solution of (1) either is oscillatory or tends to zero eventually.*

Proof of Theorems 7 and 8. Assume that (1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, it is sufficiently large, such that $x(t) > 0$ and $x(\varphi_j(t)) > 0$, for $t \in [t_0, \infty)_{\mathbb{T}}$ and $j = 0, 1, \ldots, N$. By Lemma 3, there exists

an integer m, $0 \le m < n$, with $n+m$ being odd such that (24) and (25) hold for $t \ge T_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) When $m \ge 1$, we claim that (72) implies that $m = n-1$. In fact, if $1 \le m \le n-3$, then for $t \ge T_1$

$$\begin{aligned} x^{[n]}(t) &< 0, \\ x^{[n-1]}(t) &> 0, \\ x^{[n-2]}(t) &< 0, \\ x^{[n-3]}(t) &> 0. \end{aligned} \tag{77}$$

Since $x^{\Delta}(t) > 0$ on $[T_1, \infty)_{\mathbb{T}}$, then $x(t) > x(T_1) := c_1 > 0$ for $t \ge T_1$. Then there exists $T_2 \in [T_1, \infty)_{\mathbb{T}}$ such that $x(\varphi_j(t)) \ge l$ for $t \ge T_2$ and $j = 0, 1, \ldots, N$. It follows that

$$\phi_{\gamma_j}\left(x\left(\varphi_j(t)\right)\right) \ge l^{\gamma_j} \ge L, \quad \text{for } t \in [T_2, \infty)_{\mathbb{T}}, \tag{78}$$

where $L := \inf_{0 \le j \le N}\{l^{\gamma_j}\} > 0$. Then from (1), we have

$$\begin{aligned} -x^{[n]}(t) &= \sum_{j=0}^{N} p_j(t)\, \phi_{\gamma_j}\left(x\left(\varphi_j(t)\right)\right) \ge L \sum_{j=0}^{N} p_j(t) \\ &= L\overline{p}(t). \end{aligned} \tag{79}$$

Integrating from t to $v \in [t, \infty)_{\mathbb{T}}$, we get

$$-x^{[n-1]}(v) + x^{[n-1]}(t) = L \int_t^v \overline{p}(s)\, \Delta s. \tag{80}$$

And by (48) we see that $x^{[n-1]}(v) > 0$. Hence by taking limits as $v \to \infty$ we have

$$x^{[n-1]}(t) \ge L \int_t^{\infty} \overline{p}(s)\, \Delta s, \tag{81}$$

which implies

$$\begin{aligned} \left(x^{[n-2]}(t)\right)^{\Delta} &\ge L^{1/\alpha_{n-1}} \left[\frac{1}{r_{n-1}(t)} \int_t^{\infty} \overline{p}(s)\, \Delta s\right]^{1/\alpha_{n-1}} \\ &= L^{1/\alpha_{n-1}} \overline{P}_1(t). \end{aligned} \tag{82}$$

Integrating above inequality from t to $v \in [t, \infty)_{\mathbb{T}}$ and then taking $v \to \infty$, we get

$$-x^{[n-2]}(t) \ge L^{1/\alpha_{n-1}} \int_t^{\infty} \overline{P}_1(s)\, \Delta s, \tag{83}$$

which implies

$$\begin{aligned} &-\left(x^{[n-3]}(t)\right)^{\Delta} \\ &\quad \ge L^{1/\alpha[n-2,n-1]} \left[\frac{1}{r_{n-2}(t)} \int_t^{\infty} \overline{P}_1(s)\, \Delta s\right]^{1/\alpha_{n-2}} \\ &\quad = L^{1/\alpha[n-2,n-1]} \overline{P}_2(t). \end{aligned} \tag{84}$$

Again, integrating above inequality from T_2 to $t \in [T_2, \infty)_{\mathbb{T}}$ and noting that $x^{[n-3]} > 0$ eventually, we get

$$x^{[n-3]}(T_2) - x^{[n-3]}(t) \ge L^{1/\alpha[n-2,n-1]} \int_{T_2}^{t} \overline{P}_2(s)\, \Delta s. \tag{85}$$

Then by (72), we have $\lim_{t\to\infty} x^{[n-3]}(t) = -\infty$, which contradicts the fact that $x^{[n-3]} > 0$ on $[T_2, \infty)_{\mathbb{T}}$. This shows that if (72) holds, then $m = n-1$. The rest of proof of (I) is similar to proof (I) of Theorems 4 and 5, respectively, with $m = n-1$ and hence can be omitted.

(II) When $m = 0$ (in this case n is odd), therefore

$$(-1)^k x^{[k]} > 0 \quad \text{for } k = 0, \ldots, n. \tag{86}$$

Since $x^{\Delta} < 0$ eventually, then $\lim_{t\to\infty} x(t) = l \ge 0$. Assume that $l > 0$. Then for sufficiently large $T_2 \in [T_1, \infty)_{\mathbb{T}}$, we have $x(\varphi_j(t)) \ge l$ for $t \ge T_2$ and $j = 1, \ldots, N$. It follows that

$$\phi_{\gamma_j}\left(x\left(\varphi_j(t)\right)\right) \ge l^{\gamma_j} \ge L \quad \text{for } t \in [T_2, \infty)_{\mathbb{T}}, \tag{87}$$

where $L := \inf_{0 \le j \le N}\{l^{\gamma_j}\} > 0$. Then from (1), we have

$$\begin{aligned} -x^{[n]}(t) &= \sum_{j=0}^{N} p_j(t)\, \phi_{\gamma_j}\left(x\left(\varphi_j(t)\right)\right) \ge L \sum_{j=0}^{N} p_j(t) \\ &= L\overline{p}(t). \end{aligned} \tag{88}$$

Integrating from t to $v \in [t, \infty)_{\mathbb{T}}$, we get

$$-x^{[n-1]}(v) + x^{[n-1]}(t) = L \int_t^v \overline{p}(s)\, \Delta s. \tag{89}$$

And using (48) we see that $x^{[n-1]}(v) > 0$. Hence by taking limits as $v \to \infty$ we have

$$x^{[n-1]}(t) \ge L \int_t^{\infty} \overline{p}(s)\, \Delta s, \tag{90}$$

which implies

$$\begin{aligned} \left(x^{[n-2]}(t)\right)^{\Delta} &\ge L^{1/\alpha_{n-1}} \left[\frac{1}{r_{n-1}(t)} \int_t^{\infty} \overline{p}(s)\, \Delta s\right]^{1/\alpha_{n-1}} \\ &= L^{1/\alpha_{n-1}} \overline{P}_1(t). \end{aligned} \tag{91}$$

Integrating above inequality from t to $v \in [t, \infty)_{\mathbb{T}}$ and then taking $v \to \infty$, we get

$$-x^{[n-2]}(t) \ge L^{1/\alpha_{n-1}} \int_t^{\infty} \overline{P}_1(s)\, \Delta s, \tag{92}$$

which implies

$$\begin{aligned} &-\left(x^{[n-3]}(t)\right)^{\Delta} \\ &\quad \ge L^{1/\alpha[n-2,n-1]} \left[\frac{1}{r_{n-2}(t)} \int_t^{\infty} \overline{P}_1(s)\, \Delta s\right]^{1/\alpha_{n-2}} \\ &\quad = L^{1/\alpha[n-2,n-1]} \overline{P}_2(t). \end{aligned} \tag{93}$$

Again, integrating above inequality from T_2 to $t \in [T_2, \infty)_{\mathbb{T}}$ and noting that $x^{[n-3]} > 0$ eventually, we get

$$x^{[n-3]}(T_2) - x^{[n-3]}(t) \geq L^{1/\alpha[n-2,n-1]} \int_{T_2}^{t} \overline{P}_2(s)\,\Delta s. \quad (94)$$

Then by (72), we have $\lim_{t\to\infty} x^{[n-3]}(t) = -\infty$, which contradicts the fact that $x^{[n-3]} > 0$ on $[T_2, \infty)_{\mathbb{T}}$. This shows that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Remark 9. The conclusion of Theorems 7 and 8 remains intact if assumption (72) is replaced by one of the following conditions:

$$\int_{T}^{\infty} \overline{P}_0(t)\,\Delta t = \infty \quad \text{or} \quad \int_{T}^{\infty} \overline{P}_1(t)\,\Delta t = \infty. \quad (95)$$

Theorem 10. *Assume that*

$$\int_{T}^{\infty} \overline{P}_0(t)\,\Delta t = \infty. \quad (96)$$

Then

(1) *if n is even, every solution of (1) is oscillatory,*

(2) *if n is odd, then every solution of (1) either is oscillatory or tends to zero eventually.*

Proof. Assume that (1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, it is sufficiently large, such that $x(t) > 0$ and $x(\varphi_j(t)) > 0$, for $t \in [t_0, \infty)_{\mathbb{T}}$ and $j = 0, 1, \ldots, N$. By Lemma 3, there exists an integer m, $0 \leq m < n$, with $n + m$ being odd such that (24) and (25) hold for $t \geq T_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) When $m \geq 1$, this implies that $x(t)$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Then for sufficiently large $T_2 \in [T_1, \infty)_{\mathbb{T}}$, we have $x(\varphi_j(t)) \geq l$ for $t \geq T_2$. It follows that

$$\phi_{\gamma_j}\left(x\left(\varphi_j(t)\right)\right) \geq l^{\gamma_j} \geq L \quad \text{for } t \in [T_2, \infty)_{\mathbb{T}}, \quad (97)$$

where $L := \inf_{0\leq j\leq N}\{l^{\gamma_j}\} > 0$. Equation (1) becomes

$$\begin{aligned} -\left(x^{[n-1]}(t)\right)^{\Delta} &= \sum_{j=0}^{N} p_j(t)\,\phi_{\gamma_j}\left(x\left(\varphi_j(t)\right)\right) \geq L\sum_{j=0}^{N} p_j(t) \\ &= L\overline{p}(t) \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \end{aligned} \quad (98)$$

Replacing t by s in (98), integrating from t_2 to $t \in [t_2, \infty)_{\mathbb{T}}$, we obtain

$$-x^{[n-1]}(t) + x^{[n-1]}(t_2) \geq L\int_{t_2}^{t} \overline{p}(s)\,\Delta s. \quad (99)$$

Hence by (96), we have $\lim_{t\to\infty} x^{[n-1]}(t) = -\infty$, which contradicts the fact that $x^{[n-1]}(t) > 0$ eventually.

(II) When $m = 0$ (in this case n is odd), therefore

$$(-1)^k x^{[k]} > 0 \quad \text{for } k = 0, \ldots, n. \quad (100)$$

This implies that $x(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. Then $\lim_{t\to\infty} x(t) = l \geq 0$. Assume that $l > 0$. Then for sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$, we have $x(\varphi_j(t)) \geq l$ for $t \geq t_2$. It follows that

$$\phi_{\gamma_j}\left(x\left(\varphi_j(t)\right)\right) \geq l^{\gamma_j} \geq L \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}, \quad (101)$$

where $L := \inf_{0\leq j\leq N}\{l^{\gamma_j}\} > 0$. As is case (I), we get a contradiction with the fact that $x^{[n-1]}(t) > 0$ eventually. This shows that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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Research Article

Helmholtz Theorem for Nondifferentiable Hamiltonian Systems in the Framework of Cresson's Quantum Calculus

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We derive the Helmholtz theorem for nondifferentiable Hamiltonian systems in the framework of Cresson's quantum calculus. Precisely, we give a theorem characterizing nondifferentiable equations, admitting a Hamiltonian formulation. Moreover, in the affirmative case, we give the associated Hamiltonian.

1. Introduction

Several types of quantum calculus are available in the literature, including Jackson's quantum calculus [1, 2], Hahn's quantum calculus [3–5], the time-scale q-calculus [6, 7], the power quantum calculus [8], and the symmetric quantum calculus [9–11]. Cresson introduced in 2005 his quantum calculus on a set of Hölder functions [12]. This calculus attracted attention due to its applications in physics and the calculus of variations and has been further developed by several different authors (see [13–16] and references therein). Cresson's calculus of 2005 [12] presents, however, some difficulties, and in 2011 Cresson and Greff improved it [17, 18]. Indeed, the quantum calculus of [12] let a free parameter, which is present in all the computations. Such parameter is certainly difficult to interpret. The new calculus of [17, 18] bypasses the problem by considering a quantity that is free of extra parameters and reduces to the classical derivative for differentiable functions. It is this new version of 2011 that we consider here, with a brief review of it being given in Section 2. Along the text, by *Cresson's calculus* we mean this quantum version of 2011 [17, 18]. For the state of the art on the quantum calculus of variations we refer the reader to the recent book [19]. With respect to Cresson's approach, the quantum calculus of variations is still in its infancy: see [13, 17, 18, 20–22]. In [17] nondifferentiable Euler-Lagrange equations are used in the study of PDEs. Euler-Lagrange equations for variational functionals with Lagrangians containing multiple quantum derivatives, depending on a parameter or containing higher-order quantum derivatives, are studied in [20]. Variational problems with constraints, with one and more than one independent variable, of first and higher-order type are investigated in [21]. Recently, problems of the calculus of variations and optimal control with time delay were considered [22]. In [18], a Noether type theorem is proved but only with the momentum term. This result is further extended in [23] by considering invariance transformations that also change the time variable, thus obtaining not only the generalized momentum term of [18] but also a new energy term. In [13], nondifferentiable variational problems with a free terminal point, with or without constraints, of first and higher-order are investigated. Here, we continue to develop Cresson's quantum calculus in obtaining a result for Hamiltonian systems and by considering the so-called inverse problem of the calculus of variations.

A classical problem in analysis is the well-known *Helmholtz's inverse problem of the calculus of variations*: find a necessary and sufficient condition under which a (system of) differential equation(s) can be written as an Euler-Lagrange or a Hamiltonian equation and, in the affirmative case, find all possible Lagrangian or Hamiltonian formulations. This condition is usually called the *Helmholtz condition*. The

Lagrangian Helmholtz problem has been studied and solved by Douglas [24], Mayer [25], and Hirsch [26, 27]. The Hamiltonian Helmholtz problem has been studied and solved, up to our knowledge, by Santilli in his book [28]. Generalization of this problem in the *discrete calculus* of variations framework has been done in [29, 30], in the discrete Lagrangian case. In the case of *time-scale calculus*, that is, a mixing between continuous and discrete subintervals of time, see [31] for a necessary condition for a dynamic integrodifferential equation to be an Euler-Lagrange equation on time scales. For the Hamiltonian case it has been done for the discrete calculus of variations in [32] using the framework of [33] and in [34] using a discrete embedding procedure derived in [35]. In the case of *time-scale calculus* it has been done in [36]; for the *Stratonovich stochastic calculus* see [37]. Here we give the Helmholtz theorem for Hamiltonian systems in the case of *nondifferentiable Hamiltonian systems* in the framework of Cresson's quantum calculus. By definition, the nondifferentiable calculus extends the differentiable calculus. Such as in the discrete, time-scale, and stochastic cases, we recover the same conditions of existence of a Hamiltonian structure.

The paper is organized as follows. In Section 2, we give some generalities and notions about the nondifferentiable calculus introduced in [17], the so-called *Cresson's quantum calculus*. In Section 3, we remind definitions and results about classical and nondifferentiable Hamiltonian systems. In Section 4, we give a brief survey of the classical Helmholtz Hamiltonian problem and then we prove the main result of this paper—the nondifferentiable Hamiltonian Helmholtz theorem. Finally, we give two applications of our results in Section 5, and we end in Section 6 with conclusions and future work.

2. Cresson's Quantum Calculus

We briefly review the necessary concepts and results of the quantum calculus [17].

2.1. Definitions. Let $\mathbb{X}^d$ denote the set $\mathbb{R}^d$ or $\mathbb{C}^d$, $d \in \mathbb{N}$, and let I be an open set in $\mathbb{R}$ with $[a,b] \subset I$, $a < b$. We denote by $\mathcal{F}(I,\mathbb{X}^d)$ the set of functions $f : I \to \mathbb{X}^d$ and by $\mathcal{C}^0(I,\mathbb{X}^d)$ the subset of functions of $\mathcal{F}(I,\mathbb{X}^d)$ which are continuous.

Definition 1 (Hölderian functions [17]). Let $f \in \mathcal{C}^0(I,\mathbb{R}^d)$. Let $t \in I$. Function f is said to be α-Hölderian, $0 < \alpha < 1$, at point t if there exist positive constants $\epsilon > 0$ and $c > 0$ such that $|t - t'| \leqslant \epsilon$ implies $\|f(t) - f(t')\| \leqslant c|t - t'|^\alpha$ for all $t' \in I$, where $\|\cdot\|$ is a norm on $\mathbb{R}^d$.

The set of Hölderian functions of Hölder exponent α, for some α, is denoted by $H^\alpha(I,\mathbb{R}^d)$. The quantum derivative is defined as follows.

Definition 2 (the ϵ-left and ϵ-right quantum derivatives [17]). Let $f \in \mathcal{C}^0(I,\mathbb{R}^d)$. For all $\epsilon > 0$, the ϵ-left and ϵ-right quantum derivatives of f, denoted, respectively, by $d_\epsilon^- f$ and $d_\epsilon^+ f$, are defined by

$$d_\epsilon^- f(t) = \frac{f(t) - f(t-\epsilon)}{\epsilon}, \qquad d_\epsilon^+ f(t) = \frac{f(t+\epsilon) - f(t)}{\epsilon}. \tag{1}$$

Remark 3. The ϵ-left and ϵ-right quantum derivatives of a continuous function f correspond to the classical derivative of the ϵ-mean function f_ϵ^σ defined by

$$f_\epsilon^\sigma(t) = \frac{\sigma}{\epsilon}\int_t^{t+\sigma\epsilon} f(s)\,ds, \quad \sigma = \pm. \tag{2}$$

The next operator generalizes the classical derivative.

Definition 4 (the ϵ-scale derivative [17]). Let $f \in \mathcal{C}^0(I,\mathbb{R}^d)$. For all $\epsilon > 0$, the ϵ-scale derivative of f, denoted by $\square_\epsilon f/\square t$, is defined by

$$\frac{\square_\epsilon f}{\square t} = \frac{1}{2}\left[\left(d_\epsilon^+ f + d_\epsilon^- f\right) + i\mu\left(d_\epsilon^+ f - d_\epsilon^- f\right)\right], \tag{3}$$

where i is the imaginary unit and $\mu \in \{-1, 1, 0, -i, i\}$.

Remark 5. If f is differentiable, then one can take the limit of the scale derivative when ϵ goes to zero. We then obtain the classical derivative df/dt of f.

We also need to extend the scale derivative to complex valued functions.

Definition 6 (see [17]). Let $f \in \mathcal{C}^0(I,\mathbb{C}^d)$ be a continuous complex valued function. For all $\epsilon > 0$, the ϵ-scale derivative of f, denoted by $\square_\epsilon f/\square t$, is defined by

$$\frac{\square_\epsilon f}{\square t} = \frac{\square_\epsilon \mathrm{Re}(f)}{\square t} + i\frac{\square_\epsilon \mathrm{Im}(f)}{\square t}, \tag{4}$$

where $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ denote the real and imaginary part of f, respectively.

In Definition 4, the ϵ-scale derivative depends on ϵ, which is a free parameter related to the smoothing order of the function. This brings many difficulties in applications to physics, when one is interested in particular equations that do not depend on an extra parameter. To solve these problems, the authors of [17] introduced a procedure to extract information independent of ϵ but related with the mean behavior of the function.

Definition 7 (see [17]). Let $\mathcal{C}^0_{\mathrm{conv}}(I \times]0,1],\mathbb{R}^d) \subseteq \mathcal{C}^0(I \times]0,1],\mathbb{R}^d)$ be such that for any function $f \in \mathcal{C}^0_{\mathrm{conv}}(I \times]0,1],\mathbb{R}^d)$ the $\lim_{\epsilon\to 0} f(t,\epsilon)$ exists for any $t \in I$. We denote by E a complementary space of $\mathcal{C}^0_{\mathrm{conv}}(I \times]0,1],\mathbb{R}^d)$ in $\mathcal{C}^0(I \times]0,1],\mathbb{R}^d)$. We define the projection map π by

$$\begin{gathered}\pi : \mathcal{C}^0_{\mathrm{conv}}\left(I \times]0,1],\mathbb{R}^d\right) \oplus E \longrightarrow \mathcal{C}^0_{\mathrm{conv}}\left(I \times]0,1],\mathbb{R}^d\right) \\ f_{\mathrm{conv}} + f_E \longmapsto f_{\mathrm{conv}}\end{gathered} \tag{5}$$

and the operator $\langle\cdot\rangle$ by

$$\begin{aligned}\langle\cdot\rangle &: \mathscr{C}^0\left(I\times]0,1],\mathbb{R}^d\right)\longrightarrow\mathscr{C}^0\left(I,\mathbb{R}^d\right)\\ f &\longmapsto \langle f\rangle : t\longmapsto \lim_{\epsilon\to 0}\pi\left(f\right)(t,\epsilon).\end{aligned}\tag{6}$$

The quantum derivative of f without the dependence of ϵ is introduced in [17].

Definition 8 (see [17]). The quantum derivative of f in the space $\mathscr{C}^0(I,\mathbb{R}^d)$ is given by

$$\frac{\square f}{\square t}=\left\langle\frac{\square_\epsilon f}{\square t}\right\rangle.\tag{7}$$

The quantum derivative (7) has some nice properties. Namely, it satisfies a Leibniz rule and a version of the fundamental theorem of calculus.

Theorem 9 (the quantum Leibniz rule [17]). *Let $\alpha+\beta>1$. For $f\in H^\alpha(I,\mathbb{R}^d)$ and $g\in H^\beta(I,\mathbb{R}^d)$, one has*

$$\frac{\square}{\square t}\left(f\cdot g\right)(t)=\frac{\square f(t)}{\square t}\cdot g(t)+f(t)\cdot\frac{\square g(t)}{\square t}.\tag{8}$$

Remark 10. For $f\in\mathscr{C}^1(I,\mathbb{R}^d)$ and $g\in\mathscr{C}^1(I,\mathbb{R}^d)$, one obtains from (8) the classical Leibniz rule: $(f\cdot g)'=f'\cdot g+f\cdot g'$.

Definition 11. We denote by $\mathscr{C}^1_\square$ the set of continuous functions $q\in\mathscr{C}^0([a,b],\mathbb{R}^d)$ such that $\square q/\square t\in\mathscr{C}^0(I,\mathbb{R}^d)$.

Theorem 12 (the quantum version of the fundamental theorem of calculus [17]). *Let $f\in\mathscr{C}^1_\square([a,b],\mathbb{R}^d)$ be such that*

$$\lim_{\epsilon\to 0}\int_a^b\left(\frac{\square_\epsilon f}{\square t}\right)_E(t)\,dt=0.\tag{9}$$

Then,

$$\int_a^b\frac{\square f}{\square t}(t)\,dt=f(b)-f(a).\tag{10}$$

2.2. Nondifferentiable Calculus of Variations. In [17] the calculus of variations with quantum derivatives is introduced and respective Euler-Lagrange equations derived without the dependence of ϵ.

Definition 13. An admissible Lagrangian L is a continuous function $L:\mathbb{R}\times\mathbb{R}^d\times\mathbb{C}^d\to\mathbb{C}$ such that $L(t,x,v)$ is holomorphic with respect to v and differentiable with respect to x. Moreover, $L(t,x,v)\in\mathbb{R}$ when $v\in\mathbb{R}^d$; $L(t,x,v)\in\mathbb{C}$ when $v\in\mathbb{C}^d$.

An admissible Lagrangian function $L:\mathbb{R}\times\mathbb{R}^{\mathrm{d}}\times\mathbb{C}^d\to\mathbb{C}$ defines a functional on $\mathscr{C}^1(I,\mathbb{R}^d)$, denoted by

$$\begin{aligned}\mathscr{L}&:\mathscr{C}^1\left(I,\mathbb{R}^d\right)\longrightarrow\mathbb{R}\\ q&\longmapsto\int_a^b L\left(t,q(t),\dot q(t)\right)dt.\end{aligned}\tag{11}$$

Extremals of the functional $\mathscr{L}$ can be characterized by the well-known Euler-Lagrange equation (see, e.g., [38]).

Theorem 14. *The extremals $q\in\mathscr{C}^1(I,\mathbb{R}^d)$ of $\mathscr{L}$ coincide with the solutions of the Euler-Lagrange equation*

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}\left(t,q(t),\dot q(t)\right)\right]=\frac{\partial L}{\partial x}\left(t,q(t),\dot q(t)\right).\tag{12}$$

The nondifferentiable embedding procedure allows us to define a natural extension of the classical Euler-Lagrange equation in the nondifferentiable context.

Definition 15 (see [17]). The nondifferentiable Lagrangian functional $\mathscr{L}_\square$ associated with $\mathscr{L}$ is given by

$$\begin{aligned}\mathscr{L}_\square&:\mathscr{C}^1_\square\left(I,\mathbb{R}^d\right)\longrightarrow\mathbb{R}\\ q&\longmapsto\int_a^b L\left(s,q(s),\frac{\square q(s)}{\square t}\right)ds.\end{aligned}\tag{13}$$

Let $H_0^\beta:=\{h\in H^\beta(I,\mathbb{R}^d),\ h(a)=h(b)=0\}$ and $q\in H^\alpha(I,\mathbb{R}^d)$ with $\alpha+\beta>1$. A H_0^β-variation of q is a function of the form $q+h$, where $h\in H_O^\beta$. We denote by $D\mathscr{L}_\square(q)(h)$ the quantity

$$\lim_{\epsilon\to 0}\frac{\mathscr{L}_\square\left(q+\epsilon h\right)-\mathscr{L}_\square\left(q\right)}{\epsilon}\tag{14}$$

if there exists the so-called Fréchet derivative of $\mathscr{L}_\square$ at point q in direction h.

Definition 16 (nondifferentiable extremals). A H_0^β-extremal curve of the functional $\mathscr{L}_\square$ is a curve $q\in H^\alpha(I,\mathbb{R}^d)$ satisfying $D\mathscr{L}_\square(q)(h)=0$ for any $h\in H_0^\beta$.

Theorem 17 (nondifferentiable Euler-Lagrange equations [17]). *Let $0<\alpha,\beta<1$ with $\alpha+\beta>1$. Let L be an admissible Lagrangian of class $\mathscr{C}^2$. We assume that $\gamma\in H^\alpha(I,\mathbb{R}^d)$, such that $\square\gamma/\square t\in H^\alpha(I,\mathbb{R}^d)$. Moreover, we assume that $L(t,\gamma(t),\square\gamma(t)/\square t)h(t)$ satisfies condition (9) for all $h\in H_0^\beta(I,\mathbb{R}^d)$. A curve γ satisfying the nondifferentiable Euler-Lagrange equation*

$$\frac{\square}{\square t}\left[\frac{\partial L}{\partial v}\left(t,\gamma(t),\frac{\square\gamma(t)}{\square t}\right)\right]=\frac{\partial L}{\partial x}\left(t,\gamma(t),\frac{\square\gamma(t)}{\square t}\right)\tag{15}$$

is an extremal curve of functional (13).

3. Reminder about Hamiltonian Systems

We now recall the main concepts and results of both classical and Cresson's nondifferentiable Hamiltonian systems.

3.1. Classical Hamiltonian Systems. Let L be an admissible Lagrangian function. If L satisfies the so-called Legendre property, then we can associate to L a Hamiltonian function denoted by H.

Definition 18. Let L be an admissible Lagrangian function. The Lagrangian L is said to satisfy the Legendre property if the mapping $v \mapsto (\partial L/\partial v)(t, x, v)$ is invertible for any $(t, q, v) \in I \times \mathbb{R}^d \times \mathbb{C}^d$.

If we introduce a new variable

$$p = \frac{\partial L}{\partial v}(t, q, v) \tag{16}$$

and L satisfies the Legendre property, then we can find a function f such that

$$v = f(t, q, p). \tag{17}$$

Using this notation, we have the following definition.

Definition 19. Let L be an admissible Lagrangian function satisfying the Legendre property. The Hamiltonian function H associated with L is given by

$$\begin{gathered} H : \mathbb{R} \times \mathbb{R}^d \times \mathbb{C}^d \longrightarrow \mathbb{C} \\ (t, q, p) \longmapsto H(t, q, p) = pf(t, q, p) - L(t, q, f(t, q, p)). \end{gathered} \tag{18}$$

We have the following theorem (see, e.g., [38]).

Theorem 20 (Hamilton's least-action principle). *The curve $(q, p) \in \mathscr{C}(I, \mathbb{R}^d) \times \mathscr{C}(I, \mathbb{C}^d)$ is an extremal of the Hamiltonian functional*

$$\mathscr{H}(q, p) = \int_a^b p(t)\dot{q}(t) - H(t, q(t), p(t))\, dt \tag{19}$$

if and only if it satisfies the Hamiltonian system associated with H given by

$$\begin{aligned} \dot{q}(t) &= \frac{\partial H(t, q(t), p(t))}{\partial p}, \\ \dot{p}(t) &= -\frac{\partial H(t, q(t), p(t))}{\partial q} \end{aligned} \tag{20}$$

called the Hamiltonian equations.

A vectorial notation is obtained for the Hamiltonian equations in posing $z = (q, p)^\top$ and $\nabla H = (\partial H/\partial q, \partial H/\partial p)^\top$, where $\top$ denotes the transposition. The Hamiltonian equations are then written as

$$\frac{dz(t)}{dt} = J \cdot \nabla H(t, z(t)), \tag{21}$$

where

$$J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \tag{22}$$

denotes the symplectic matrix with I_d being the identity matrix on $\mathbb{R}^d$.

3.2. Nondifferentiable Hamiltonian Systems. The nondifferentiable embedding induces a change in the phase space with respect to the classical case. As a consequence, we have to work with variables (x, p) that belong to $\mathbb{R}^d \times \mathbb{C}^d$ and not only to $\mathbb{R}^d \times \mathbb{R}^d$, as usual.

Definition 21 (nondifferentiable embedding of Hamiltonian systems [17]). The nondifferentiable embedded Hamiltonian system (20) is given by

$$\begin{aligned} \frac{\square q(t)}{\square t} &= \frac{\partial H(t, q(t), p(t))}{\partial p}, \\ \frac{\square p(t)}{\square t} &= -\frac{\partial H(t, q(t), p(t))}{\partial q} \end{aligned} \tag{23}$$

and the embedded Hamiltonian functional $\mathscr{H}_\square$ is defined on $H^\alpha(I, \mathbb{R}^d) \times H^\alpha(I, \mathbb{C}^d)$ by

$$\mathscr{H}_\square(q, p) = \int_a^b \left(p(t) \frac{\square q(t)}{\square t} - H(t, q(t), p(t)) \right) dt. \tag{24}$$

The nondifferentiable calculus of variations allows us to derive the extremals for $\mathscr{H}_\square$.

Theorem 22 (nondifferentiable Hamilton's least-action principle [17]). *Let $0 < \alpha, \beta < 1$ with $\alpha + \beta > 1$. Let L be an admissible $\mathscr{C}^2$-Lagrangian. We assume that $\gamma \in H^\alpha(I, \mathbb{R}^d)$, such that $\square\gamma/\square t \in H^\alpha(I, \mathbb{R}^d)$. Moreover, we assume that $L(t, \gamma(t), \square\gamma(t)/\square t)h(t)$ satisfies condition (9) for all $h \in H_0^\beta(I, \mathbb{R}^d)$. Let H be the corresponding Hamiltonian defined by (18). A curve $\gamma \mapsto (t, q(t), p(t)) \in I \times \mathbb{R}^d \times \mathbb{C}^d$ solution of the nondifferentiable Hamiltonian system (23) is an extremal of functional (24) over the space of variations $V = H_0^\beta(I, \mathbb{R}^d) \times H_0^\beta(I, \mathbb{C}^d)$.*

4. Nondifferentiable Helmholtz Problem

In this section, we solve the inverse problem of the nondifferentiable calculus of variations in the Hamiltonian case. We first recall the usual way to derive the Helmholtz conditions following the presentation made by Santilli [28]. Two main derivations are available:

(i) The first is related to the characterization of Hamiltonian systems via the *symplectic two-differential form* and the fact that by duality the associated one-differential form to a Hamiltonian vector field is closed—the so-called *integrability conditions*.

(ii) The second uses the characterization of Hamiltonian systems via the self-adjointness of the Fréchet derivative of the differential operator associated with the equation—the so-called *Helmholtz conditions*.

Of course, we have coincidence of the two procedures in the classical case. As there is no analogous of differential form in the framework of Cresson's quantum calculus, we follow the second way to obtain the nondifferentiable analogue of

the Helmholtz conditions. For simplicity, we consider a time-independent Hamiltonian. The time-dependent case can be done in the same way.

4.1. Helmholtz Conditions for Classical Hamiltonian Systems. In this section we work on $\mathbb{R}^{2d}$, $d \geq 1$, $d \in \mathbb{N}$.

4.1.1. Symplectic Scalar Product. The *symplectic scalar product* $\langle \cdot, \cdot \rangle_J$ is defined by

$$\langle X, Y \rangle_J = \langle X, J \cdot Y \rangle \tag{25}$$

for all $X, Y \in \mathbb{R}^{2d}$, where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product and J is the symplectic matrix (22). We also consider the L^2 symplectic scalar product induced by $\langle \cdot, \cdot \rangle_J$ defined for $f, g \in \mathscr{C}^0([a,b], \mathbb{R}^{2d})$ by

$$\langle f, g \rangle_{L^2,J} = \int_a^b \langle f(t), g(t) \rangle_J \, dt. \tag{26}$$

4.1.2. Adjoint of a Differential Operator. In the following, we consider first-order differential equations of the form

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} X_q(q,p) \\ X_p(q,p) \end{pmatrix}, \tag{27}$$

where the vector fields X_q and X_p are $\mathscr{C}^1$ with respect to q and p. The associated differential operator is written as

$$O_X(q,p) = \begin{pmatrix} \dot{q} - X_q(q,p) \\ \dot{p} - X_p(q,p) \end{pmatrix}. \tag{28}$$

A *natural* notion of adjoint for a differential operator is then defined as follows.

Definition 23. Let $A : \mathscr{C}^1([a,b], \mathbb{R}^{2d}) \to \mathscr{C}^1([a,b], \mathbb{R}^{2d})$. We define the adjoint A_J^* of A with respect to $\langle \cdot, \cdot \rangle_{L^2,J}$ by

$$\langle A \cdot f, g \rangle_{L^2,J} = \langle A_J^* \cdot g, f \rangle_{L^2,J}. \tag{29}$$

An operator A will be called *self-adjoint* if $A = A_J^*$ with respect to the L^2 symplectic scalar product.

4.1.3. Hamiltonian Helmholtz Conditions. The Helmholtz conditions in the Hamiltonian case are given by the following result (see Theorem 3.12.1, p. 176-177 in [28]).

Theorem 24 (Hamiltonian Helmholtz theorem). *Let $X(q,p)$ be a vector field defined by $X(q,p)^\top = (X_q(q,p), X_p(q,p))$. The differential equation (27) is Hamiltonian if and only if the associated differential operator O_X given by (28) has a self-adjoint Fréchet derivative with respect to the L^2 symplectic scalar product. In this case the Hamiltonian is given by*

$$H(q,p) = \int_0^1 \left[p \cdot X_q(\lambda q, \lambda p) - q \cdot X_p(\lambda q, \lambda p) \right] d\lambda. \tag{30}$$

The conditions for the self-adjointness of the differential operator can be made *explicit*. They coincide with the *integrability conditions* characterizing the exactness of the one-form associated with the vector field by duality (see [28], Theorem 2.7.3, p. 88).

Theorem 25 (integrability conditions). *Let $X(q,p)^\top = (X_q(q,p), X_p(q,p))$ be a vector field. The differential operator O_X given by (28) has a self-adjoint Fréchet derivative with respect to the L^2 symplectic scalar product if and only if*

$$\frac{\partial X_q}{\partial q} + \left(\frac{\partial X_p}{\partial p}\right)^\top = 0, \quad \frac{\partial X_q}{\partial p}, \frac{\partial X_p}{\partial q} \text{ are symmetric.} \tag{31}$$

4.2. Helmholtz Conditions for Nondifferentiable Hamiltonian Systems. The previous scalar products extend naturally to complex valued functions. Let $0 < \alpha < 1$ and let $(q,p) \in H^\alpha(I, \mathbb{R}^d) \times H^\alpha(I, \mathbb{C}^d)$, such that $\square q/\square t \in H^\alpha(I, \mathbb{C}^d)$ and $\square p/\square t \in H^\alpha(I, \mathbb{C}^d)$. We consider first-order nondifferential equations of the form

$$\frac{\square}{\square t}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} X_q(q,p) \\ X_p(q,p) \end{pmatrix}. \tag{32}$$

The associated quantum differential operator is written as

$$O_{\square,X}(q,p) = \begin{pmatrix} \dfrac{\square q}{\square t} - X_q(q,p) \\ \dfrac{\square p}{\square t} - X_p(q,p) \end{pmatrix}. \tag{33}$$

A *natural* notion of adjoint for a quantum differential operator is then defined.

Definition 26. Let $A : \mathscr{C}^1_\square([a,b], \mathbb{C}^{2d}) \to \mathscr{C}^1_\square([a,b], \mathbb{C}^{2d})$. We define the adjoint A_J^* of A with respect to $\langle \cdot, \cdot \rangle_{L^2,J}$ by

$$\langle A \cdot f, g \rangle_{L^2,J} = \langle A_J^* \cdot g, f \rangle_{L^2,J}. \tag{34}$$

An operator A will be called *self-adjoint* if $A = A_J^*$ with respect to the L^2 symplectic scalar product. We can now obtain the adjoint operator associated with $O_{\square,X}$.

Proposition 27. *Let β be such that $\alpha + \beta > 1$. Let $(u,v) \in H_0^\beta(I, \mathbb{R}^d) \times H_0^\beta(I, \mathbb{C}^d)$, such that $\square u/\square t \in H^\alpha(I, \mathbb{C}^d)$ and $\square v/\square t \in H^\alpha(I, \mathbb{C}^d)$. The Fréchet derivative $DO_{\square,X}$ of (33) at (q,p) along (u,v) is then given by*

$$DO_{\square,X}(q,p)(u,v) = \begin{pmatrix} \dfrac{\square u}{\square t} - \dfrac{\partial X_q}{\partial q} \cdot u - \dfrac{\partial X_q}{\partial p} \cdot v \\ \dfrac{\square v}{\square t} - \dfrac{\partial X_p}{\partial q} \cdot u - \dfrac{\partial X_p}{\partial p} \cdot v \end{pmatrix}. \tag{35}$$

Assume that $u \cdot h$ and $v \cdot h$ satisfy condition (9) for any $h \in H_0^\beta(I, \mathbb{C}^d)$. In consequence, the adjoint $DO^*_{\square,X}$ of $DO_{\square,X}(q,p)$ with respect to the L^2 symplectic scalar product is given by

$$DO^*_{\square,X}(q,p)(u,v) = \begin{pmatrix} \frac{\square u}{\square t} + \left(\frac{\partial X_p}{\partial p}\right)^\top \cdot u - \left(\frac{\partial X_q}{\partial p}\right)^\top \cdot v \\ \frac{\square v}{\square t} - \left(\frac{\partial X_p}{\partial q}\right)^\top \cdot u + \left(\frac{\partial X_q}{\partial q}\right)^\top \cdot v \end{pmatrix}. \tag{36}$$

Proof. The expression for the Fréchet derivative of (33) at (q,p) along (u,v) is a simple computation. Let $(w,x) \in H_0^\beta(I,\mathbb{R}^d) \times H_0^\beta(I,\mathbb{C}^d)$ be such that $\square w/\square t \in H^\alpha(I,\mathbb{C}^d)$ and $\square x/\square t \in H^\alpha(I,\mathbb{C}^d)$. By definition, we have

$$\begin{aligned} \langle DO_{\square,X}(q,p)(u,v),(w,x)\rangle_{L^2,J} = \int_a^b \Bigg[& \frac{\square u}{\square t}\cdot x \\ & - \left(\frac{\partial X_q}{\partial q}\cdot u\right)\cdot x - \left(\frac{\partial X_q}{\partial p}\cdot v\right)\cdot x - \frac{\square v}{\square t}\cdot w \\ & + \left(\frac{\partial X_p}{\partial q}\cdot u\right)\cdot w + \left(\frac{\partial X_p}{\partial p}\cdot v\right)\cdot w \Bigg] dt. \end{aligned} \tag{37}$$

As $u \cdot h$ and $v \cdot h$ satisfy condition (9) for any $h \in H_0^\beta(I,\mathbb{C}^d)$, using the quantum Leibniz rule and the quantum version of the fundamental theorem of calculus, we obtain

$$\begin{aligned} \int_a^b \frac{\square u}{\square t}\cdot b\,dt &= \int_a^b -u\cdot\frac{\square b}{\square t}dt, \\ \int_a^b \frac{\square v}{\square t}\cdot a\,dt &= \int_a^b -u\cdot\frac{\square b}{\square t}dt. \end{aligned} \tag{38}$$

Then,

$$\begin{aligned} \langle DO_{\square,X}(q,p)(u,v),(w,x)\rangle_{L^2,J} = \int_a^b \Bigg[& -u \\ & \cdot\left(\frac{\square x}{\square t} - \left(\frac{\partial X_p}{\partial q}\right)^\top\cdot w + \left(\frac{\partial X_q}{\partial q}\right)^\top\cdot x\right) + v \\ & \cdot\left(\frac{\square w}{\square t} + \left(\frac{\partial X_p}{\partial p}\right)^\top\cdot w - \left(\frac{\partial X_q}{\partial p}\right)^\top\cdot x\right)\Bigg] dt. \end{aligned} \tag{39}$$

By definition, we obtain the expression of the adjoint $DO^*_{\square,X}$ of $DO_{\square,X}(q,p)$ with respect to the L^2 symplectic scalar product. □

In consequence, from a direct identification, we obtain the nondifferentiable self-adjointess conditions called *Helmholtz's conditions*. As in the classical case, we call these conditions *nondifferentiable integrability conditions*.

Theorem 28 (nondifferentiable integrability conditions). *Let $X(q,p)^\top = (X_q(q,p), X_p(q,p))$ be a vector field. The differential operator $O_{\square,X}$ given by (33) has a self-adjoint Fréchet derivative with respect to the symplectic scalar product if and only if*

$$\frac{\partial X_q}{\partial q} + \left(\frac{\partial X_p}{\partial p}\right)^\top = 0, \tag{HC1}$$

$$\frac{\partial X_q}{\partial p}, \frac{\partial X_p}{\partial q} \text{ are symmetric.} \tag{HC2}$$

Remark 29. One can see that the Helmholtz conditions are the same as in the classical, discrete, time-scale, and stochastic cases. We expected such a result because Cresson's quantum calculus provides a quantum Leibniz rule and a quantum version of the fundamental theorem of calculus. If such properties of an underlying calculus exist, then the Helmholtz conditions will always be the same up to some conditions on the working space of functions.

We now obtain the main result of this paper, which is the *Helmholtz theorem for nondifferentiable Hamiltonian systems.*

Theorem 30 (nondifferentiable Hamiltonian Helmholtz theorem). *Let $X(q,p)$ be a vector field defined by $X(q,p)^\top = (X_q(q,p), X_p(q,p))$. The nondifferentiable system of (32) is Hamiltonian if and only if the associated quantum differential operator $O_{\square,X}$ given by (33) has a self-adjoint Fréchet derivative with respect to the L^2 symplectic scalar product. In this case, the Hamiltonian is given by*

$$H(q,p) = \int_0^1 \left[p\cdot X_q(\lambda q,\lambda p) - q\cdot X_p(\lambda q,\lambda p)\right] d\lambda. \tag{40}$$

Proof. If X is Hamiltonian, then there exists a function $H : \mathbb{R}^d \times \mathbb{C}^d \to \mathbb{C}$ such that $H(q,p)$ is holomorphic with respect to v and differentiable with respect to q and $X_q = \partial H/\partial p$ and $X_p = -\partial H/\partial q$. The nondifferentiable integrability conditions are clearly verified using Schwarz's lemma. Reciprocally, we assume that X satisfies the nondifferentiable integrability conditions. We will show that X is Hamiltonian with respect to the Hamiltonian

$$H(q,p) = \int_0^1 \left[p\cdot X_q(\lambda q,\lambda p) - q\cdot X_p(\lambda q,\lambda p)\right] d\lambda; \tag{41}$$

that is, we must show that

$$\begin{aligned} X_q(q,p) &= \frac{\partial H(q,p)}{\partial p}, \\ X_p(q,p) &= -\frac{\partial H(q,p)}{\partial q}. \end{aligned} \tag{42}$$

We have

$$\begin{aligned} \frac{\partial H(q,p)}{\partial q} = \int_0^1 \Bigg[& p\cdot\lambda\frac{\partial X_q(\lambda q,\lambda p)}{\partial q} - X_p(\lambda q,\lambda p) \\ & - q\cdot\lambda\frac{\partial X_p(\lambda q,\lambda p)}{\partial q}\Bigg] d\lambda, \end{aligned}$$

$$\frac{\partial H(q,p)}{\partial p} = \int_0^1 \left[X_p(\lambda q, \lambda p) + p \cdot \lambda \frac{\partial X_q(\lambda q, \lambda p)}{\partial p} - q \cdot \lambda \frac{\partial X_p(\lambda q, \lambda p)}{\partial p} \right] d\lambda. \tag{43}$$

Using the nondifferentiable integrability conditions, we obtain

$$\begin{aligned} \frac{\partial H(q,p)}{\partial q} &= \int_0^1 -\frac{\partial\left(\lambda X_p(\lambda q, \lambda p)\right)}{\partial \lambda} d\lambda = -X_p(q,p), \\ \frac{\partial H(q,p)}{\partial q} &= \int_0^1 \frac{\partial\left(\lambda X_q(\lambda q, \lambda p)\right)}{\partial \lambda} d\lambda = X_q(q,p), \end{aligned} \tag{44}$$

which concludes the proof. □

5. Applications

We now provide two illustrative examples of our results: one with the formulation of dynamical systems with linear parts and another with Newton's equation, which is particularly useful to study partial differentiable equations such as the Navier-Stokes equation. Indeed, the Navier-Stokes equation can be recovered from a Lagrangian structure with Cresson's quantum calculus [17]. For more applications see [34].

Let $0 < \alpha < 1$ and let $(q,p) \in H^\alpha(I, \mathbb{R}^d) \times H^\alpha(I, \mathbb{C}^d)$ be such that $\square q/\square t \in H^\alpha(I, \mathbb{C}^d)$ and $\square q/\square t \in H^\alpha(I, \mathbb{C}^d)$.

5.1. The Linear Case. Let us consider the discrete nondifferentiable system

$$\begin{aligned} \frac{\square q}{\square t} &= \alpha q + \beta p, \\ \frac{\square p}{\square t} &= \gamma q + \delta p, \end{aligned} \tag{45}$$

where α, β, γ, and δ are constants. The Helmholtz condition (HC2) is clearly satisfied. However, system (45) satisfies the condition (HC1) if and only if $\alpha + \delta = 0$. As a consequence, linear Hamiltonian nondifferentiable equations are of the form

$$\begin{aligned} \frac{\square q}{\square t} &= \alpha q + \beta p, \\ \frac{\square p}{\square t} &= \gamma q - \alpha p. \end{aligned} \tag{46}$$

Using formula (40), we compute explicitly the Hamiltonian, which is given by

$$H(q,p) = \frac{1}{2}\left(\beta p^2 - \gamma q^2\right) + \alpha q \cdot p. \tag{47}$$

5.2. Newton's Equation. Newton's equation (see [38]) is given by

$$\begin{aligned} \dot{q} &= \frac{p}{m}, \\ \dot{p} &= -U'(q), \end{aligned} \tag{48}$$

with $m \in \mathbb{R}^+$ and $q, p \in \mathbb{R}^d$. This equation possesses a natural Hamiltonian structure with the Hamiltonian given by

$$H(q,p) = \frac{1}{2m} p^2 + U(q). \tag{49}$$

Using Cresson's quantum calculus, we obtain a natural nondifferentiable system given by

$$\begin{aligned} \frac{\square q}{\square t} &= \frac{p}{m}, \\ \frac{\square p}{\square t} &= -U'(q). \end{aligned} \tag{50}$$

The Hamiltonian Helmholtz conditions are clearly satisfied.

Remark 31. It must be noted that Hamiltonian (49) associated with (50) is recovered by formula (40).

6. Conclusion

We proved a Helmholtz theorem for nondifferentiable equations, which gives necessary and sufficient conditions for the existence of a Hamiltonian structure. In the affirmative case, the Hamiltonian is given. Our result extends the results of the classical case when restricting attention to differentiable functions. An important complementary result for the nondifferentiable case is to obtain the Helmholtz theorem in the Lagrangian case. This is nontrivial and will be subject of future research.

Competing Interests

The authors declare that they have no competing interests.

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Research Article

Harberger-Laursen-Metzler Effect with Modified Becker-Mulligan Preference by Dynamic Optimization

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We investigate the effects of terms-of-trade shocks on the spending and current account where households with the modified Becker-Mulligan endogenous time preference maximize their utility over an infinite planning period. Our results show that, with the modified Becker-Mulligan preference, the effect of the deterioration in terms of trade on the current account depends on people's characters. However, with the second preference we have considered, the deterioration in terms of trade will result in a current account deficit, which is the same as Obstfeld (1982), where households with Uzawa endogenous time preference are considered; deterioration in terms of trade leads to a decline in the current account. These theoretical results are consistent with the empirical evidence by numerical simulations.

1. Introduction

There are many fluctuations in small open economies, since they tend to be easily disturbed by external shocks through international trade. The effects of deterioration in terms of trade faced by small open economies have caused much attention. Harberger [1] and Laursen and Metzler [2] put forward that a decrease in current income arising from an adverse terms-of-trade shock would lower both private savings and the current account balance. Meeting the conditions of Harberger-Laursen-Metzler effect (H-L-M effect), income effect caused by the terms-of-trade degradation will bring decreases of current income and total savings and then finally lead to the deterioration of the current account.

So far, the H-L-M effect has caused lots of academic discussions, and many researchers have gotten support for this effect or the contrary results, by establishing different models or modifying the parameters in the model, especially the time preference. Obstfeld [3] proposes a model to investigate the H-L-M effect, assuming that households maximize their utility over an infinite planning period. It is found that an economy specialized in production (means that the country is small and open) must experience a fall in aggregate spending and a current surplus as a result of an unanticipated, permanent worsening in its terms of trade. He provides a setting in which the current account deficit predicted by Harberger [1] and Laursen and Metzler [2] fails to materialize, where an endogenous Uzawa time preference is used. Following Uzawa [4], it is supposed that time preferences factor is a function of the utility of consumption level, and the consumer utility and the future patient degree changed inversely. Svensson and Razin [5] reexamine the issues on the basis of Obstfeld's propositions. They set up a two-period, two-good framework. A new discovery is that there are deteriorations in the current account and savings as a result of temporary deterioration in terms of trade, but this might go either way as a result of any permanent deterioration. After more than 20 years, Huang and Meng [6] investigate the effects of a permanent terms-of-trade change on a dynamic small open economy facing an imperfect world capital market as studied in Obstfeld [3] under the assumption that households' subjective discount rate is a decreasing function of instantaneous

utility. They show that an unanticipated permanent terms-of-trade deterioration leads to an increase in aggregate expenditure and a current account deficit, which is in stark contrast to those obtained in Obstfeld [3].

Sen and Turnovsky [7] use fixed time preferences in the indefinite model to discuss the H-L-M effect. They consider the choice between leisure (labor) and capital accumulation. In their model, worsening terms of trade would lead to the current account deficit or do not depend on people's investment behavior. Based on the habit fixed model, Mansoorian [8] also considers the demonstration in the discussions of the indefinite model. In the model, the changes of people habits depend on the past consumption. If time preferences change as people's consumption slowly changes, the deteriorating terms of trade will gradually reduce savings and eventually lead to the current account deficit. More recently, Angyridis and Mansoorian [9] investigate the influences of terms-of-trade deterioration on the current account when the representative agent has Marshallian preference. With this preference, the rate of time preference is a decreasing function of savings and the permanent income of the representative agent is reduced by the terms-of-trade deterioration. As savings fall, the country experiences a current account deficit. Various evaluations of the model imply that the H-L-M effect is applied in an infinite horizon model with an endogenous rate of time preference, with standard functional forms and reasonable parameter values.

Becker and Mulligan [10] propose a time preference that is not a by-product of other choices. The presupposition is that consumers often make efforts and carry out activities in order to influence the discount on future utilities. According to the viewpoint, the resources are spent on imagining future pleasures, which is termed as "future oriented capital," determining the rate of a time preference. Thus, Becker and Mulligan determine the time preference rate by relating it to resources spent on imagining future pleasures, where the larger the resources are spent the more patient the individual is. Such resources may be spent on schooling, newspapers, membership in a Christmas Club, and so on. In what follows, for simplicity we call this Becker-Mulligan endogenous time preference to be B-M preference.

Recently, there are a vast and ever-growing number of studies on the B-M preference. For example, Gong and Zou [11] discuss the effects of government expenditure, income tax rate, and consumption tax rate on the steady-state capital stock, consumption level, in the Ramsey model with B-M preference, and find that, with the increasing of the government expenditure, the steady-state capital stock and the spending on imaging the future will be decrease, but the effect on the consumption level is ambiguous. Recently, Gong [12] investigates the effects of monetary growth in an infinitely lived, representative agent model with B-M preference. He finds that an increase in the inflation rate reduces the resources spent on imagining the future, which increases the rate of time preference and decreases the steady-state value of capital stock.

This paper studies the H-L-M effect with modified B-M preference which has been the academic focus for more than a half century. We present the model based on the modified B-M preference in Section 2. In the model, the representative consumer strengthens his expectation of future consumption with certain endowment, and the model has the utility maximization function and restrictive conditions about capital increase. Using the Pontryagin Maximum Principle [13], we get the optimality conditions when the objective function model is maximized and the conditions which guarantee that the economy becomes stable. Section 3 studies the saddle point stability of the economy system and points out the existence and inexistence of the H-L-M effect which depend on people's characters. The empirical evidence is explored by numerical simulations in Section 4. We conclude this paper in the final section.

2. The Model

Consider an open and small economy consisting of identical households, each maximizing its utility over an infinite lifetime. Assume that the instantaneous utility function U is composed of two goods: one is imported from abroad, c^f, and the other is available at home, c^h, in fixed supply. Households may save by accumulating an internationally traded bond, b, which is assumed, without loss of generality, to be indexed to the foreign good.

2.1. Assumptions. The instantaneous utility function U of the representative household is taken to be nonnegative, strictly increasing in both its arguments, strictly concave, and twice continuously differentiable. To avoid noninterior solutions to the household's lifetime consumption problem, we assume that

$$\lim_{c_t^f \to 0} \frac{\partial U}{\partial c_t^f} = \lim_{c_t^h \to 0} \frac{\partial U}{\partial c_t^h} = \infty. \tag{1}$$

The household's objective is to maximize the discounted sum of future instantaneous utility:

$$\int_0^\infty U\left(c^f, c^h\right) e^{-\Delta_t} dt, \tag{2}$$

where

$$\Delta_t = \int_0^t \rho\left(s\left(v\right)\right) dv. \tag{3}$$

Following Becker and Mulligan [10], assume that people have the option to increase their appreciation of the future. This effort can be modeled by allowing the consumer to make future pleasures less remote by spending resources on imagination, which is denoted as s. Suppose that the time preference is a function of the spending resources s, mathematically $\rho(s)$, and we make two assumptions as follows:

Case 1:

$$\begin{aligned} \rho\left(s\right) &> 0, \\ \rho'\left(s\right) &< 0, \\ \rho''\left(s\right) &< 0. \end{aligned} \tag{4a}$$

Case 2:

$$\rho(s) > 0,$$
$$\rho'(s) > 0, \tag{4b}$$
$$\rho''(s) > 0,$$

for all $s \geq 0$.[1] With the assumption in (4a), the time preference in maximization problem (3) is called the modified B-M preference. The assumption in (4b) is another case we have considered.

From (4a), we know that, with the increasing of the resources spent on imagination, the propinquity of future pleasures will increase, and thus $\rho(s)$ is decreasing. The concavity assumption $\rho''(s) < 0$ requires that the resources spent on imagining future utilities become increasingly more effective in decreasing their remoteness.

At each instant, the representative family is bound by a flow constraint linking any divergence between its income and its expenditure to its accumulation of claims on future units of the foreign good. Letting $b = b(t)$ denote bond holdings at time t, letting p be the price of foreign goods in terms of domestic goods, letting y be the family's (fixed) endowment of the home good, and letting r be the international rate of interest, we can write this budget constraint as

$$\dot{b} = \frac{y}{p} - c^f - \frac{c^h}{p} + rb - \frac{s}{p}, \tag{5}$$

which implies that, for the economy as a whole, the capital account deficit must be equal to the excess of income over expenditure and resources spent on imagination.

The household is also bound by a second constraint on its program of saving and spending. The discounted integral of lifetime expenditure (measured in domestic goods) and resources spent on imagination must be not greater than the capitalized value of lifetime output plus initial bond holdings; that is,

$$\frac{y}{r} + pb_0 \geq \int_0^\infty e^{-r}\left[pc^f + c^h + s\right] dt. \tag{6}$$

The importance of this constraint can be appreciated by contrasting the present infinite horizon planning problem with a finite-horizon problem whose planning period ends at time T. In the latter setting, the family's budget constraint clearly implies that $b(T)$ is nonnegative, and for any lifetime, borrowing must be repaid before death. But an infinitely lived family facing a perfect capital market may borrow and consume arbitrarily large amounts while it is always meeting its interest payments through further borrowing. Its lifetime utility is unbounded; that is, no optimal program exists unless such a condition is imposed.

To rule out this "paradox of borrowing," we impose the feasibility constraint that, at each moment, the family's capitalized future output must exceed its indebtedness:

$$\frac{y}{r} + pb \geq 0. \tag{7}$$

The household's problem is to choose paths for consumptions on c^f, c^h, and s to

$$\text{maximize} \quad \int_0^\infty U\left(c^f, c^h\right) e^{-\Delta_t} dt \tag{8}$$

$$\begin{aligned} \text{subject to} \quad & \text{(i)} \ \Delta_t = \int_0^t \rho(s(v))\, dv, \\ & \text{(ii)} \ \dot{b} = \frac{y}{p} - c^f - \frac{c^h}{p} + rb - \frac{s}{p}, \\ & \text{(iii)} \ c^f, c^h \geq 0, \\ & \text{(iv)} \ \frac{y}{r} + pb \geq 0, \end{aligned} \tag{9}$$

given an initial stock b_0 of net claims on foreigners. The number of households is for convenience taken to be 1, so that the consumption and saving paths chosen by the representative household may be identified with those of the economy as a whole.

2.2. Two Simplifications. Two simplifications will facilitate our derivation of the necessary conditions for a solution to problem (8), just as what Obstfeld [3] has used. In order to maximize the lifetime welfare, at each moment the household must maximize its instantaneous utility, given relative prices and its chosen level of expenditure on consumption goods in general. The first simplification is to replace the utility function with the indirect utility function as

$$V(p, z) \equiv \sup\left\{U\left(c^f, c^h\right) \mid pc^f + c^h = z\right\}, \tag{10}$$

which both reduces the dimensionality of our problem and allows us to focus on a variable of primary interest, expenditure measured in terms of the domestic good z.[2]

The second simplification is achieved by changing variables in the maximization problem from t to Δ_t, using the fact that

$$d\Delta_t = \rho(s)\, dt, \tag{11}$$

which reduces the household's problem to that of choosing a path for z and s to

$$\begin{aligned} \text{maximize} \quad & \int_0^\infty \frac{V(p, z)}{\rho(s)} e^{-\Delta_t} d\Delta_t, \\ \text{subject to} \quad & \frac{db}{d\Delta_t} = \frac{(y - z - s)/p + rb}{\rho(s)}, \end{aligned} \tag{12}$$

and the given initial bond holdings b_0.

2.3. Solutions of the Model. Necessary conditions for a solution to such a maximization problem are readily derived using the Pontryagin Maximum Principle.[3] To apply this principle, we introduce the costate variable $\lambda = \lambda_\Delta$, which may be interpreted as the imputed value or shadow price of saving, measured in utility terms. According to the Maximum

Principle, an optimal program must maximize the following Hamiltonian for each λ:

$$H(b,z,s,\lambda) = \frac{V(p,z) + \lambda\left[(y-z-s)/p + rb\right]}{\rho(s)}, \tag{13}$$

with respect to b, z, and s.

The solutions to the first-order conditions $\partial H/\partial z = 0$ and $\partial H/\partial s = 0$ are

$$\lambda = pV_z, \tag{14}$$

$$\lambda = \frac{-pV}{y-z-s+\rho/\rho'(s)+rpb}. \tag{15}$$

The Euler equation reversing initial change of variables ($d\Delta_t = \rho(s)dt$) is

$$\dot{\lambda} = \lambda(\rho(s) - r), \tag{16}$$

which denotes that the marginal rate of time preference must at each instant equal the rate of return on bonds plus "capital gains," that is, $\dot{\lambda}/\lambda$. Moreover, from the previous simplification, the feasibility budget condition is

$$\dot{b} = \frac{(y-z-s)}{p} + rb. \tag{17}$$

While the four equations (14)–(17) are necessarily satisfied by an optimal program, they are not in themselves sufficient to guarantee optimality. The stock of bonds inherited from the past is predetermined at time $t = 0$ and this fact provides an initial condition for the differential equation system defined by the four equations (14)–(17). But an initial value of shadow price λ is also required, and unless this shadow price is chosen correctly, the path will be suboptimal. The problem is again due to our assumption that the planning unit has an infinite lifetime. If this lifetime instead ends at time T, the terminal condition $b(T) = 0$ would enable us to solve backward for λ_0. But in the present context, no such restriction is available.

2.4. Stationary Conditions. It is natural to consider the steady state $(\bar{z},\bar{b},\bar{s})$ of the economy. Firstly, the condition

$$\dot{\lambda} = \lambda(\rho(s) - r) = 0 \tag{18}$$

guarantees that the marginal rate of time preference equals the rate of interest, and the condition

$$\dot{b} = \frac{(y-z-s)}{p} + rb = 0 \tag{19}$$

gives the external balance. The reason for doing so is that the paths which converge to the steady state satisfy the sufficiency condition for optimal problem

$$\lim_{\Delta\to\infty}\left[\frac{y}{rp} + b_\Delta\right]e^{-\Delta}\lambda_\Delta = 0 \tag{20}$$

and so are optimal. In the next section, we show that there exists a unique convergent path, which will guarantee that the economy's response to a permanent change in the parameters that it faces is uniquely determined.

3. The H-L-M Effect with Modified B-M Preference

We are interested in the situation that there is a steady state and an optimal path moving towards to it. Just as shown above, the optimal path must be in accordance with (14)–(17), and the steady state will be determined by (18) and (19).

3.1. The Steady State. Firstly, transform the differential equation (16), an equation of shadow price λ, into the equation of expenditure z. From (14), we know

$$\dot{\lambda} = pV_{zz}\dot{z}. \tag{21}$$

Putting (14) and (21) into (16), we have

$$\dot{z} = \frac{V_z}{V_{zz}}(\rho(s) - r). \tag{22}$$

From (14) and (15), we get

$$s - \frac{\rho(s)}{\rho'(s)} = y - z + \frac{V}{V_z} + rpb, \tag{23}$$

from which and the implicit function theorem we can represent s as functions of b and z; that is,

$$s = s(z,b). \tag{24}$$

Thus the steady state $(\bar{z},\bar{b},\bar{s})$ of the economy with $\bar{s} = \bar{s}(\bar{z},\bar{b})$ reached when $\dot{z} = \dot{b} = 0$ is characterized by

$$\begin{gathered}\frac{V_z}{V_{zz}}(\rho(\bar{s}) - r) = 0,\\ \frac{\left(y - \bar{z} - \bar{s}\left(\bar{z},\bar{b}\right)\right)}{p} + r\bar{b} = 0.\end{gathered} \tag{25}$$

The linearized system associated with the dynamic system (19) and (22) around the steady state is

$$\begin{pmatrix}\dot{z}\\ \dot{b}\end{pmatrix} = \begin{pmatrix}\frac{V_z}{V_{zz}}\rho'(\bar{s})\bar{s}_z & \frac{V_z}{V_{zz}}\rho'(\bar{s})\bar{s}_b\\ \frac{(-1-\bar{s}_z)}{p} & \frac{-\bar{s}_b}{p} + r\end{pmatrix}\begin{pmatrix}z-\bar{z}\\ b-\bar{b}\end{pmatrix}. \tag{26}$$

The determinate of the coefficient matrix of the above linear system is given by

$$\begin{aligned}&\det\begin{pmatrix}\frac{V_z}{V_{zz}}\rho'(\bar{s})\bar{s}_z & \frac{V_z}{V_{zz}}\rho'(\bar{s})\bar{s}_b\\ \frac{(-1-\bar{s}_z)}{p} & \frac{-\bar{s}_b}{p} + r\end{pmatrix}\\ &= \frac{V_z}{V_{zz}}\rho'(\bar{s})\left(r\bar{s}_z + \frac{\bar{s}_b}{p}\right).\end{aligned} \tag{27}$$

From (23), we have

$$s_z = -\frac{VV_{zz}}{V_z^2} \cdot \frac{\rho'^2(s)}{\rho\rho''(s)},$$
$$s_b = rp \cdot \frac{\rho'^2(s)}{\rho\rho''(s)}. \tag{28}$$

In the next two subsections, we analyze the dynamic system (19) and (22) above near the steady state and investigate the existence of the H-L-M effect.

3.2. The H-L-M Effect in Case 1. Under the assumption in Case 1, we have

$$s_z = -\frac{VV_{zz}}{V_z^2} \cdot \frac{\rho'^2(s)}{\rho\rho''(s)} < 0,$$
$$s_b = rp \cdot \frac{\rho'^2(s)}{\rho\rho''(s)} < 0; \tag{29}$$

thus the determinant of the coefficient matrix of (26), that is, (27), is negative. So the dynamic system has one negative and one positive characteristic root, which is crucial for stability. The dynamic system has a perfect foresight saddle point path near the steady state. The phase diagram analysis is done to show the changes of current account and expenditure in response to terms-of-trade shocks. Firstly, consider the $\dot{z} = 0$ locus. Any point on this locus must satisfy $\rho(s(z,b)) = r$, so differentiating both sides with respect to z, we have

$$\rho'(s)\left(s_z + s_b \cdot \frac{db}{dz}\right) = 0. \tag{30}$$

Hence we have

$$\frac{db}{dz} = -\frac{s_z}{s_b} < 0, \tag{31}$$

which shows that $\dot{z} = 0$ locus is downward in the diagram. For the $\dot{b} = 0$ locus, in the same way, we have

$$\frac{db}{dz} = \frac{s_z + 1}{pr - s_b}, \tag{32}$$

whose slope depends on the sign of the numerator. In what follows, the sign of db/dz is considered in two cases.

Case 1. (a) If $s_z + 1 > 0$, we have $db/dz > 0$ and the $\dot{b} = 0$ curve is upward. When doing the long-run equilibrium analysis, $\dot{b} = 0$ shows the $prb + y = z + s(z,b)$. Thus when the price p increases, this curve must move downward. Figure 1 displays the dynamic behaviors of the system described by (19) and (22) in response to the increase of price p. It is shown that when facing with the deterioration in terms of trade (i.e., the price p increases), the current account of the small open economy will decrease and the expenditure will increase to get to a new saddle point path. This supports the H-L-M presumption of a current account deficit when deterioration of terms-of-trade occurs.

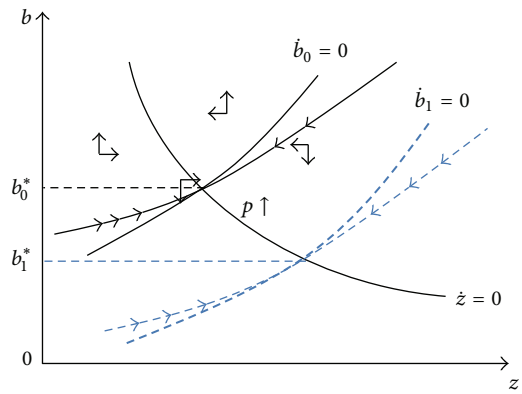

Figure 1: The phase diagram and the effect of a permanent terms-of-trade deterioration for Case 1(a).

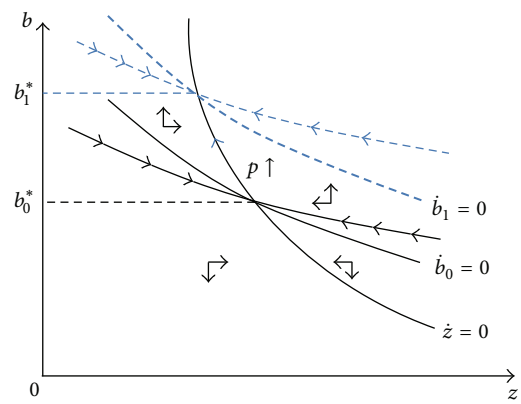

Figure 2: The phase diagram and the effect of a permanent terms-of-trade deterioration for Case 1(b).

(b) If $s_z + 1 < 0$, we have $db/dz < 0$. In this case, one should firstly compare the slope of the curves $\dot{b} = 0$ and $\dot{z} = 0$, since

$$\frac{s_z + 1}{pr - s_b} - \left(-\frac{s_z}{s_b}\right) = \frac{s_b + prs_z}{(pr - s_b)\, s_b} > 0. \tag{33}$$

The $\dot{b} = 0$ curve is flatter than the $\dot{z} = 0$ curve. Figure 2 displays the dynamic behaviors of the system described by (19) and (22) in response to the increase of price p. It is observed that the deterioration in terms of trade leads to the increase of the current account and decrease of the expenditure, which is a result contrary to the H-L-M presumption of a current account deficit when worsening of terms-of-trade occurs.

3.3. The H-L-M Effect in Case 2. Under the assumption in Case 2, we have

$$s_z = -\frac{VV_{zz}}{V_z^2} \cdot \frac{\rho'^2(s)}{\rho\rho''(s)} > 0,$$
$$s_b = rp \cdot \frac{\rho'^2(s)}{\rho\rho''(s)} > 0, \tag{34}$$

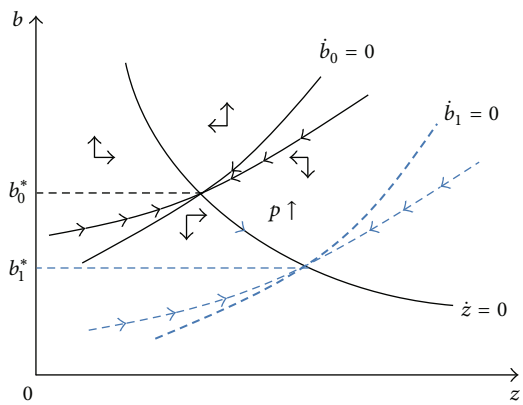

Figure 3: The phase diagram and the effect of a permanent terms-of-trade deterioration for Case 2(a).

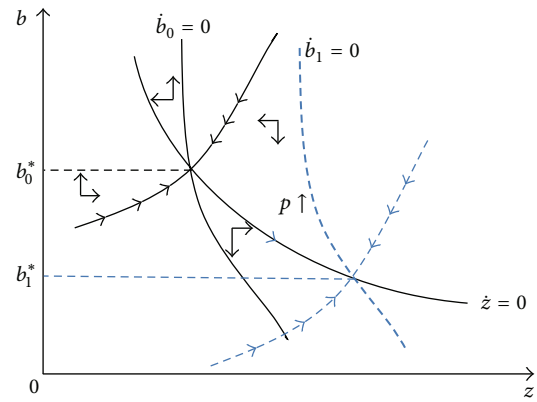

Figure 4: The phase diagram and the effect of a permanent terms-of-trade deterioration for Case 2(b).

so the determinant of the coefficient matrix of (26), that is, (27), is also negative. So the dynamic system has one negative and one positive characteristic root, which has a perfect foresight saddle point path near the steady state. In the following, the phase diagram analysis is done to show the changes of current account and expenditure in response to terms-of-trade shocks. The $\dot{z} = 0$ locus can be easily drawn as what we have done in Section 3.2. Since

$$\frac{db}{dz} = \frac{s_z + 1}{pr - s_b}, \tag{35}$$

the sign of $pr - s_b$ is key to know the direction of the curve $\dot{b} = 0$. We should also consider two cases to determine the sign of db/dz.

Case 2. (a) When $pr > s_b$, that is, $\rho'(s)^2/(\rho(s) \cdot \rho(s)'') < 1$, we have

$$\frac{db}{dz} = \frac{s_z + 1}{pr - s_b} > 0. \tag{36}$$

So the $\dot{b} = 0$ curve is upward. As done in Section 3.2, when the price p increases, the $\dot{b} = 0$ curve moves downward. Figure 3 displays the phase diagram and effect of a permanent terms-of-trade deterioration. It is shown that when facing with the deterioration in terms of trade, the current account decreases and the expenditure increases to get to a new saddle point path, which supports the H-L-M presumption of a current account deficit when deterioration of terms-of-trade occurs.

(b) When $pr < s_b$, that is, $\rho'(s)^2/(\rho(s) \cdot \rho(s)'') > 1$, we have

$$\frac{db}{dz} = \frac{s_z + 1}{pr - s_b} < 0. \tag{37}$$

So the $\dot{b} = 0$ curve is downward. We should compare the slope of this two curves to determine their location in the diagram. Since

$$\frac{s_z + 1}{pr - s_b} - \left(-\frac{s_z}{s_b}\right) = \frac{s_b + prs_z}{(pr - s_b)\, s_b} < 0, \tag{38}$$

the $\dot{b} = 0$ curve must be steeper than $\dot{z} = 0$. Figure 4 also displays the phase diagram and effect of a permanent terms-of-trade deterioration. We see that when facing with the deterioration in terms of trade, the current account decreases and the expenditure increases to get to a new saddle point path, which supports the H-L-M presumption.

4. Empirical Evidence by Numerical Simulations

To illustrate the results above, the analytical model is simulated numerically. As is known, this model poses a two-point boundary value problem in a continuous setting and the deterioration in terms of trade is permanent; we refer to the simulation method used by Trimborn et al. [14].

For the numerical implication, the instantaneous utility function is taken to be Cobb-Douglas, so that the import content of consumption is independent of relative price, which is the same as Serven [15] and the form is

$$U\left(c^f, c^h\right) = \frac{\left(\left(c^f\right)^{\theta} + \left(c^h\right)^{1-\theta}\right)^{1-\delta}}{1 - \delta}, \tag{39}$$

where the parameter θ satisfies $0 < \theta < 1$.

Our choice of preference parameters is standard, the same as Eicher et al. [16]. The time preference function is set as quadratic with a parameter ρ_0; that is,

$$\rho(s) = \rho_0 s^2. \tag{40}$$

The world real interest rate is $r = 0.06$ and the constant income is $y = 0.5$. The terms of trade is 0.95 initially and rises to 1, as shown in Table 1. Thus the transitional paths begin at the initial steady state and then come to the new steady state after the terms of trade deteriorates.

With the parameterization in Table 1, Figure 5 presents the simulated trajectories of bond holdings, consumptions, and resources on the imagination following a permanent unanticipated increase in the price.

Table 1: The data in empirical analysis.

Preference parameters	$\theta = 0.5, \delta = 2.5, \rho_0 = 0.01$
Constant income	$y = 0.5$
World interest rate	$r = 0.06$
Terms of trade	$p_0 = 0.95, p_1 = 1$

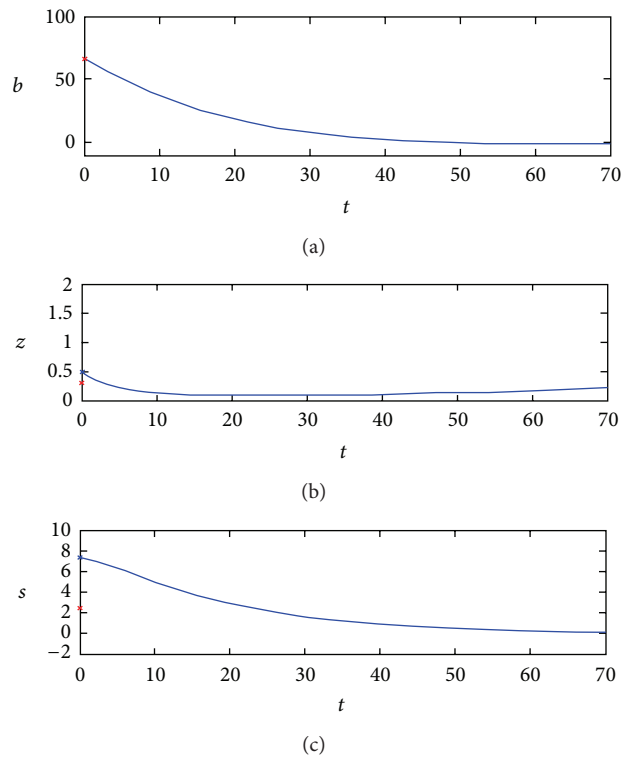

Figure 5: Time evolutions of bond holdings, consumptions, and resources.

It is shown in Figure 5 that starting from the steady-state equilibrium at time $t = 0$, both consumption and resources spent on imagination jump to a higher level, from the red point to the blue point, in the short run. Bond holdings decrease in the short run and in the long run, which means that the current account is deficit and the result is the same as the outcome predicted by Laursen and Metzler [2] and Obstfeld [3], which is consistent with theoretical results in Section 3, where the H-L-M effect exists.

5. Conclusion

Obstfeld [3] finds that, in a context of explicit, intertemporal optimization, the well-known H-L-M relationship predicts a decline in saving and thus a current deficit is invalid. Deterioration in the terms of trade, in his paper, leads to a current surplus as households acquire interest-bearing claims on foreigners in order to restore their steady-state utility to its original level. He assumes that people have Uzawa preference, which holds the point that time preferences factor is a function of the utility of consumption level. However, according to Marshall [17], agents derive direct utility from the act of savings. It is demonstrated by Angyridis and Mansoorian [9] that, with this preference, a terms-of-trade deterioration, by lowering the permanent income of the representative agent, reduces savings and leads to a current account deficit. This supports the H-L-M effect.

Becker and Mulligan [10] believe that consumers often make efforts and carry out activities to influence the discount on future utilities. According to their viewpoint, the resource spent on imagining future pleasures, which is called s, is to increase their appreciation of the future. By modifying this time preference, we study in this paper the H-L-M effect which has been the academic discussion for a half century.

We have studied the H-L-M effect on an open economy peopled by infinitely lived, utility-maximizing families, and it is found that whether the H-L-M effect exists depends on people's characters, such as sagacity and far-sight. The validity of the H-L-M prediction depends in general on the economy's intertemporal utility forms and its time preferences. We have analyzed some possible cases and have focused on resources on imagination that are freely chosen, but it is clearly desirable to extend our analysis to more general resources changing paths, for example, to the expenditure. In the future work, we will develop a two-country world economy with modified Becker-Mulligan endogenous time preference to investigate the macroeconomic dynamics and H-L-M effect [18]. Moreover, in order to associate with the recent empirical evidence, we will incorporate investment behavior explicitly in the model. The inclusion of investment is important because the current account should be the difference between saving and investment, and hence investment should play an important role in explaining the current account adjustment. There are numerous studies that argue that the current account fluctuations are largely driven by investment rather than by saving [19, 20]. The recent empirical evidence also indicates that capital goods in trade flows actually represent the leading import item for many countries [21, 22]. Thus the empirical studies [23–25] provide a way for us to improve the theoretical model [26].

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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Endnotes

1. This may be different to common sense of economy but ensure our calculation in further discussion, with the affirmation that $\rho'''(s) > 0$.

2. We can easily affirm that $\partial V/\partial z > 0$ and $\partial^2 V/\partial z^2 < 0$, which means that the household's utility will increase if the expenditure grows up, but its effect will decrease. And $\partial^2 V/\partial z \partial p > 0$ can be deduced by the concavity of V and will help us calculate the model in Section 2.3.

3. See Arrow and Kurz [27] or Kamien and Schwartz [13], in their book *Dynamic Optimization*, where the Pontryagin Maximum Principle is referred to. Suppose that the problem is to maximize $\int_{t_0}^{t_1} f(t, x(t), u(t))dt$, subject to $\dot{x}(t) = g(t, x(t), u(t))$, given $x(t_0)$. Then define Hamilton function:

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u), \qquad (*)$$

where λ is a Hamilton multiplier. The solutions to the problem must satisfy the necessary conditions as

$$\begin{aligned} \frac{\partial H}{\partial u} &= \frac{\partial f}{\partial u} + \lambda \frac{\partial g}{\partial u} = 0, \\ \frac{d\lambda}{dt} &= -\frac{\partial H}{\partial x} = -\frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x}. \end{aligned} \qquad (**)$$

The second function is just the Euler equation and the traversal condition is $\lambda(t_1) = 0$.

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Research Article

Stochastic Predator-Prey System Subject to Lévy Jumps

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This paper investigates a new nonautonomous impulsive stochastic predator-prey system with the omnivorous predator. First, we show that the system has a unique global positive solution for any given initial positive value. Second, the extinction of the system under some appropriate conditions is explored. In addition, we obtain the sufficient conditions for almost sure permanence in mean and stochastic permanence of the system by using the theory of impulsive stochastic differential equations. Finally, we discuss the biological implications of the main results and show that the large noise can make the system go extinct. Simulations are also carried out to illustrate our theoretical analysis conclusions.

1. Introduction

Omnivory is considered as a common ecological phenomenon in the natural world. Omnivorous predator feeds on both animal prey and plant, so the intrinsic growth rate for predator should be positive. For example, the giant panda is omnivorous animal, since it can eat both meat and plant such as bamboo. With the development of the economy, pollution is becoming more and more serious. Thus pollution models have widely attracted the focus of the people [1–5]. A deterministic predator-prey system with omnivorous predator in an impulsive polluted environment takes the following form:

$$\begin{aligned}
&dx(t) \\
&\quad = x(t)\left(r_1(t) - d_1 c_0(t) - a_1(t)\,x(t) - \beta(t)\,y(t)\right)dt, \\
&dy(t) \\
&\quad = y(t)\left(r_2(t) - d_2 c_0(t) - a_2(t)\,y(t) + \beta(t)\,x(t)\right)dt, \\
&\dot{c}_0(t) = k c_e(t) - g c_0(t) - m c_0(t), \\
&\dot{c}_e(t) = -h c_e(t), \\
&\qquad\qquad\qquad\qquad t \neq \tau_k, \\
&\Delta x(\tau_k) = \Delta y(\tau_k) = \Delta c_0(\tau_k) = 0, \\
&\Delta c_e(\tau_k) = u, \\
&\qquad\qquad\qquad\qquad \tau_k = k\tau,\ k = 0, 1, \ldots,
\end{aligned} \tag{1}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k)$, $\Delta y(\tau_k) = y(\tau_k^+) - y(\tau_k)$, $\Delta c_0(\tau_k) = c_0(\tau_k^+) - c_0(\tau_k)$, $c_e(\tau_k) = c_e(\tau_k^+) - c_e(\tau_k)$, δ_i $(i = 1, 2)$, k, g, m, h, and u are positive constants, $x(t)$ and $y(t)$ denote prey and omnivorous predator densities, and $c_0(t)$ and $c_e(t)$ denote the concentrations of the toxicant in the organism and in the environment, respectively. $r_i(t)$, $a_i(t)$ $(i = 1, 2)$, and $\beta(t)$ are all positive bounded continuous functions on $R_+ := [0, +\infty)$. $r_1(t)$ and $r_2(t)$ represent the prey intrinsic growth rate and the predator intrinsic growth rate, respectively, $a_i(t)$ $(i = 1, 2)$ are the density-dependent coefficients of the prey and the predator, $\beta(t)$ is the capturing rate of the predator, d_1 and d_2 are damage rates of the prey and predator by the toxicant, respectively, k represents environmental toxicant uptake rate per unit mass organism, g and m are organismal net ingestion and depuration rates of toxicant, respectively, h denotes the loss rate of toxicant from the environment itself by volatilization, and u is the amount of pulsed input concentration of the toxicant at each τ.

System (1) is a deterministic model, where all parameters in the model are deterministic. However, there are some limitations in mathematical model from a biological viewpoint. Therefore, it is significant to study the effects of noises on population systems [6–9]. There are many kinds of environmental noises. First, we assume that the toxicant uptake rates are disturbed by white noise. If we still let $d_i c_0(t)$ $(i = 1, 2)$ represent the toxicant uptake rates, then $d_1 c_0(t)$ and $d_2 c_0(t)$ can be replaced by

$$\begin{aligned} d_1 c_0(t) &\longrightarrow d_1 c_0(t) + \sigma_1(t)\dot{B}_1(t), \\ d_2 c_0(t) &\longrightarrow d_2 c_0(t) + \sigma_2(t)\dot{B}_2(t), \end{aligned} \tag{2}$$

where $\dot{B}_i(t)$ $(i = 1, 2)$ are white noises and $\sigma_i(t)$ $(i = 1, 2)$ are the intensities of the white noises, which are bounded continuous functions on $[0, +\infty)$. Then we obtain the following stochastic system with impulsive toxicant input in a polluted environment:

$$\begin{aligned} dx(t) &= x(t)\left(r_1(t) - d_1 c_0(t) - a_1(t)x(t) - \beta(t)y(t)\right)dt \\ &\quad - \sigma_1(t)x(t)\,dB_1(t), \\ dy(t) &= y(t)\left(r_2(t) - d_2 c_0(t) - a_2(t)y(t) + \beta(t)x(t)\right)dt \\ &\quad - \sigma_2(t)y(t)\,dB_2(t), \\ \dot{c}_0(t) &= kc_e(t) - gc_0(t) - mc_0(t), \\ \dot{c}_e(t) &= -hc_e(t), \\ &\qquad t \neq \tau_k, \\ \Delta x(\tau_k) &= \Delta y(\tau_k) = \Delta c_0(\tau_k) = 0, \\ \Delta c_e(\tau_k) &= u, \\ &\qquad \tau_k = k\tau,\ k = 0, 1, \ldots, \end{aligned} \tag{3}$$

where $B_i(t)$ are mutually independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathcal{P})$.

On the other hand, populations may be affected by sudden environmental fluctuations, such as severe weather, earthquakes, floods, and epidemics. Brownian motion cannot describe these phenomena better, so it is very important to introduce Lévy noise into the population system [10]. There are many researches about autonomous stochastic predator-prey system with Lévy jumps [11, 12].

Inspired by these, we focus on nonautonomous impulsive stochastic predator-prey system with white noises and Lévy jumps

$$\begin{aligned} dx(t) &= x(t)\left(r_1(t) - d_1 c_0(t) - a_1(t)x(t) - \beta(t)y(t)\right)dt \\ &\quad - \sigma_1(t)x(t)\,dB_1(t) \\ &\quad + \int_{\mathbb{Y}} x(t^-)\gamma_1(t,u)\widetilde{N}(dt,du), \\ dy(t) &= y(t)\left(r_2(t) - d_2 c_0(t) - a_2(t)y(t) + \beta(t)x(t)\right)dt \\ &\quad - \sigma_2(t)y(t)\,dB_2(t) \\ &\quad + \int_{\mathbb{Y}} y(t^-)\gamma_2(t,u)\widetilde{N}(dt,du), \\ \dot{c}_0(t) &= kc_e(t) - gc_0(t) - mc_0(t), \\ \dot{c}_e(t) &= -hc_e(t), \\ &\qquad t \neq \tau_k, \\ \Delta x(\tau_k) &= \Delta y(\tau_k) = \Delta c_0(\tau_k) = 0, \\ \Delta c_e(\tau_k) &= u, \\ &\qquad \tau_k = k\tau,\ k = 0, 1, \ldots, \end{aligned} \tag{4}$$

where $x(t^-)$ and $y(t^-)$ represent the left limit of $x(t)$ and $y(t)$, respectively, $N(dt, du)$ is a Poisson counting measure with characteristic measure ν on a measurable bounded subset $\mathbb{Y}$ of $(0, \infty)$ with $\nu(\mathbb{Y}) < \infty$, and B_i $(i = 1, 2)$ are independent of N. The Poisson counting measure is represented by $\widetilde{N}(dt, du) := N(dt, du) - \nu(du)dt$; $\gamma_i > -1$, $\gamma_i : \mathbb{R}_+ \times \mathbb{Y} \to \mathbb{R}$ $(i = 1, 2)$ are continuous functions on $[0, +\infty)$, which are assumed to be periodic with period $\tau > 0$. Other parameters are defined as in system (1).

The paper is arranged as follows. In Section 2, we prove that system (4) has a global positive solution. Section 3 shows the main result; in Section 3.1 we prove the extinction of system (4). We also examine almost sure permanence in mean and the stochastic permanence of the system in Sections 3.2 and 3.3. Finally we present some simulations and conclusions to close the paper in Section 4.

2. Notations and Global Positive Solution

For the purpose of convenience, we introduce some notions and some lemmas which will be used for our main results. We throughout this paper assume that $x(t)$, $y(t)$, and $c_0(t)$ are continuous at $t = k\tau$, and $c_e(t)$ is left continuous at $t = k\tau$ and $c_e(k\tau^+) = \lim_{t\to k\tau^+} c_e(t)$ and let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets). Further assume that $B(t)$ is a scalar Brownian motion defined on the complete probability space Ω.

We denote $\mathbb{R}^2_+$ as the positive cone in $\mathbb{R}^2$; that is, $\mathbb{R}^2_+ = \{z = (z_1, z_2) \in \mathbb{R}^2 \mid z_i > 0,\ i = 1, 2\}$. If $f(t)$ is a bounded continuous function on $[0, +\infty)$, we define

$$\langle f(t)\rangle = \frac{1}{t}\int_0^t f(t)\,dt,$$

$$f^M = \max_{t\in[0,+\infty)} f(t),$$

$$f^L = \min_{t\in[0,+\infty)} f(t),$$
$$f^* = \limsup_{t\to+\infty} f(t),$$
$$f_* = \liminf_{t\to+\infty} f(t). \tag{5}$$

Now we give some basic properties of the following subsystem of systems (1) and (4):

$$\begin{aligned} dc_0(t) &= \left(kc_e(t) - gc_0(t) - mc_0(t)\right)dt, \\ dc_e(t) &= -hc_e(t)\,dt, \\ &\qquad\qquad t \neq k\tau,\ k\in Z^+, \\ \Delta c_0(t) &= 0, \\ \Delta c_e(t) &= u, \\ &\qquad\qquad t = k\tau,\ k\in Z^+. \end{aligned} \tag{6}$$

Lemma 1 (see [3]). *System (6) has a unique positive τ-periodic solution $(c_0^*(t), c_e^*(t))^T$ which is globally asymptotically stable. Moreover, $c_0(t) > c_0^*(t), c_e(t) > c_e^*(t)$ for all $t \geq 0$ if $c_0(0) > c_0^*(0), c_e(0) > c_e^*(0)$, where*

$$\begin{aligned} c_0^*(t) &= c_0^*(0)\,e^{-(g+m)(t-n\tau)} \\ &\quad + \frac{ku\left(e^{-(g+m)(t-n\tau)} - e^{-h(t-n\tau)}\right)}{(h-g-m)\left(1-e^{-h\tau}\right)}, \\ c_e^*(t) &= \frac{ue^{-h(t-n\tau)}}{1-e^{-h\tau}}, \\ c_0^*(0) &= \frac{ku\left(e^{-(g+m)\tau} - e^{-h\tau}\right)}{(h-g-m)\left(1-e^{-(g+m)\tau}\right)\left(1-e^{-h\tau}\right)}, \\ c_e^*(0) &= \frac{u}{1-e^{-h\tau}}, \end{aligned} \tag{7}$$

for $t \in (k\tau, (k+1)\tau]$ and $k \in Z^+$.

Lemma 2. *For any positive solution $(c_0(t), c_e(t))$ of system (1) or (4) with initial value $(c_0(0), c_e(0^+)) \in \mathbb{R}_+^2$, one has*

$$\lim_{t\to+\infty} \langle c_0(t)\rangle = \frac{ku}{h(g+m)\tau} \triangleq \overline{c_0}. \tag{8}$$

Proof. Through a simple calculation, we can get

$$\int_{n\tau}^{(n+1)\tau} c_0^*(t)\,dt = \frac{ku}{h(g+m)}. \tag{9}$$

Moreover, since $c_0^*(t)$ is a periodic function, we have

$$\begin{aligned} \lim_{t\to+\infty} \langle c_0(t)\rangle^* &\leq \lim_{n\to+\infty} \frac{1}{n\tau}\int_0^{(n+1)\tau} c_0^*(t)\,dt \\ &= \lim_{n\to+\infty} \frac{n+1}{n\tau}\int_{n\tau}^{(n+1)\tau} c_0^*(t)\,dt \\ &= \frac{ku}{h(g+m)\tau}, \\ \lim_{t\to+\infty} \langle c_0(t)\rangle_* &\geq \lim_{n\to+\infty} \frac{1}{(n+1)\tau}\int_0^{n\tau} c_0^*(t)\,dt \\ &= \lim_{n\to+\infty} \frac{n}{(n+1)\tau}\int_{(n-1)\tau}^{n\tau} c_0^*(t)\,dt \\ &= \frac{ku}{h(g+m)\tau}. \end{aligned} \tag{10}$$

Hence one can observe that

$$\lim_{t\to+\infty} \langle c_0(t)\rangle = \frac{ku}{h(g+m)\tau} \triangleq \overline{c_0}. \tag{11}$$

□

Then we show an assumption which will be used in the following proof.

Assumption 3. There exists a bounded continuous function $c_i(t)$ such that

$$\int_{\mathbb{Y}} \left[\gamma_i - \ln\left(1+\gamma_i\right)\right] \nu(du) \leq c_i(t), \quad (i = 1, 2). \tag{12}$$

Theorem 4. *For any given initial value $(x(0), y(0)) \in \mathbb{R}_+^2$, there is a unique solution $X(t) = (x(t), y(t))$ of (4) and the solution will remain in $\mathbb{R}_+^2$ with probability 1; that is, $X(t) \in \mathbb{R}_+^2$ for all $t \geq 0$ almost surely.*

Proof. The coefficient of (4) is locally Lipschitz continuous, and so, for any given initial value $(x(0), y(0)) \in \mathbb{R}_+^2$, there is a unique local solution $(x(t), y(t))$ for $t \in [0, \tau_e]$, where τ_e is the explosion time. To demonstrate that this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $k_0 > 0$ be sufficiently large for $x_0 \in [1/k_0, k_0]$ and $y_0 \in [1/k_0, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\begin{aligned} &\tau_k \\ &\quad = \inf\left\{t \in [0, \tau_e] : x(t) \bar{\in} \left(\frac{1}{k}, k\right) \text{ or } y(t) \bar{\in} \left(\frac{1}{k}, k\right)\right\}, \end{aligned} \tag{13}$$

where we set $\inf \emptyset = \infty$ ($\emptyset$ denotes the empty set). Obviously, τ_k is increasing as $k \to +\infty$. Set $\tau_\infty = \lim_{k\to+\infty}\tau_k$; thus $\tau_\infty \leq \tau_e$ a.s., so we just need to demonstrate that $\tau_\infty = \infty$. If $\tau_\infty = \infty$ is not true, then there exist two constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \varepsilon$. Thus there is an integer $k_1 \geq k_0$ such that $P\{\tau_k \leq T\} \geq \varepsilon$ for all $k \geq k_1$.

Define a function $V : \mathbb{R}_+^2 \to \mathbb{R}_+$ as follows:

$$V(x, y) = x - 1 - \ln x + y - 1 - \ln y. \tag{14}$$

Let $T > 0$ be arbitrary. Applying Itô's formula and Assumption 3 leads to

$$\begin{aligned} dV(x,y) &= LVdt - (x-1)\sigma_1(t)\,dB_1(t) - (y-1) \\ &\quad \cdot \sigma_2(t)\,dB_2(t) \\ &\quad + \int_{\mathbb{Y}} \left[x(t^-)\gamma_1(t,u) - \ln(1+\gamma_1(t,u))\right]\widetilde{N}(dt,du) \\ &\quad + \int_{\mathbb{Y}} \left[y(t^-)\gamma_2(t,u) - \ln(1+\gamma_2(t,u))\right]\widetilde{N}(dt,du), \end{aligned} \tag{15}$$

where

$$\begin{aligned} LV &= (x-1)\left(r_1(t) - d_1c_0(t) - a_1(t)x - \beta(t)y\right) \\ &\quad + \frac{1}{2}\sigma_1^2(t) + (y-1)\left(r_2(t) - d_2c_0(t) - a_1(t)y\right. \\ &\quad \left. + \beta(t)x\right) + \frac{1}{2}\sigma_2^2(t) \\ &\quad + \int_{\mathbb{Y}} \left[\gamma_1(t,u) - \ln(1+\gamma_1(t,u))\right]\nu(du) \\ &\quad + \int_{\mathbb{Y}} \left[\gamma_2(t,u) - \ln(1+\gamma_2(t,u))\right]\nu(du) \le -a_1(t) \\ &\quad \cdot x^2 + \left(a_1(t) + r_1(t) - d_1c_0(t) - \beta(t)\right)x - r_1(t) \\ &\quad + d_1c_0(t) + \frac{1}{2}\sigma_1^2(t) + c_1(t) + \Big[-a_2(t)y^2 \\ &\quad + \left(a_2(t) + r_2(t) - d_2c_0(t) + \beta(t)\right)y - r_2(t) \\ &\quad + d_2c_0(t) + \frac{1}{2}\sigma_2^2(t) + c_2(t)\Big] \le K_1, \end{aligned} \tag{16}$$

where

$$K_1 = \max_{0\le t<+\infty} \sum_{i=1}^{2} \frac{4a_i(t)\left[-r_i(t) + d_ic_0(t) + (1/2)\sigma_i^2(t) + c_i(t)\right] + \left[a_i(t) + r_i(t) - d_ic_0(t) + \beta(t)\right]^2}{4a_i(t)}. \tag{17}$$

So

$$\begin{aligned} dV(x,y) &\le K_1dt - (x-1)\sigma_1(t)\,dB_1(t) - (y-1) \\ &\quad \cdot \sigma_2(t)\,dB_2(t) \\ &\quad + \int_{\mathbb{Y}} \left[x(t^-)\gamma_1(t,u) - \ln(1+\gamma_1(t,u))\right]\widetilde{N}(dt,du) \\ &\quad + \int_{\mathbb{Y}} \left[y(t^-)\gamma_2(t,u) - \ln(1+\gamma_2(t,u))\right]\widetilde{N}(dt,du). \end{aligned} \tag{18}$$

Integrating (18) from 0 to $\tau_k \wedge T$ and taking the expectations for both sides result in

$$\mathbb{E}\left[V(x(\tau_k\wedge T), y(\tau_k\wedge T))\right] \le V(x_0,y_0) + K_1T. \tag{19}$$

Let $\Omega_k = \{\tau_k \le T\}$ for $k \ge k_1$; we have $P(\Omega_k) \ge \varepsilon$. Then

$$\begin{aligned} &V(x(\tau_k\wedge T,\omega), y(\tau_k\wedge T,\omega)) \\ &\quad \ge [k-1-\ln k] \wedge \left[\frac{1}{k} - 1 + \ln k\right]. \end{aligned} \tag{20}$$

It is inferred from (19) and (20) that

$$\begin{aligned} V(x_0,y_0) + K_1T &\ge \mathbb{E}\left[1_\Omega V(x(\tau_k\wedge T), y(\tau_k\wedge T))\right] \\ &\ge \varepsilon[k-1-\ln k] \wedge \left[\frac{1}{k} - 1 + \ln k\right], \end{aligned} \tag{21}$$

where 1_Ω is the indicator function of Ω_k. Let $k \to +\infty$; we have that

$$+\infty > V(x_0,y_0) + K_1T \ge +\infty \tag{22}$$

is a contradiction; then we have $\tau_\infty = \infty$.

This completes the proof of Theorem 4. □

3. Main Results

3.1. Extinction. For convenience, we prepare the following lemma.

Lemma 5. *Suppose that $\eta(t) \in C(\Omega \times [0,\infty), \mathbb{R}_+)$ and let Assumption 3 hold.*

(I) If there exist two positive constants T and δ_0 such that

$$\begin{aligned} \ln\eta(t) &\le \int_0^t \delta(s)\,ds - \delta_0\int_0^t \eta(s)\,ds + \alpha B(t) \\ &\quad + \sum_{i=1}^{2}\delta_i \int_0^t\int_{\mathbb{Y}} \ln(1+\gamma_i(u))\,\widetilde{N}(ds,du) \\ &\quad a.s. \end{aligned} \tag{23}$$

for all $t \ge T$, where α, δ_1, and δ_2 are constants, then

$$\begin{aligned} &\langle\eta\rangle^* \le \frac{\langle\delta\rangle^*}{\delta_0} \quad a.s.,\ if\ \langle\delta\rangle^* \ge 0; \\ &\lim_{t\to\infty}\eta(t) = 0 \quad a.s.,\ if\ \langle\delta\rangle^* < 0. \end{aligned} \tag{24}$$

(II) If there exist two positive constants T and δ_0 such that

$$\begin{aligned} \ln\eta(t) &\ge \int_0^t \delta(s)\,ds - \delta_0\int_0^t \eta(s)\,ds + \alpha B(t) \\ &\quad + \sum_{i=1}^{2}\delta_i \int_0^t\int_{\mathbb{Y}} \ln(1+\gamma_i(u))\,\widetilde{N}(ds,du) \quad a.s. \end{aligned} \tag{25}$$

for all $t \ge T$, then $\langle\eta\rangle_* \ge \langle\delta\rangle_*/\delta_0$ a.s. provided that $\langle\delta\rangle_* \ge 0$.

Lemma 5 is proved in Appendix by using a similar method in [13].

Now, we will prove the extinction of system (4).

Define

$$\begin{aligned} b_1(t) &= r_1(t) - \frac{1}{2}\sigma_1^2(t) \\ &\quad + \int_{\mathbb{Y}} \left(\ln\left(1+\gamma_1(t,u)\right) - \gamma_1(t,u)\right)\nu(du), \\ b_2(t) &= r_2(t) - \frac{1}{2}\sigma_2^2(t) \\ &\quad + \int_{\mathbb{Y}} \left(\ln\left(1+\gamma_2(t,u)\right) - \gamma_2(t,u)\right)\nu(du), \\ \overline{c_0} &= \frac{ku}{h(g+m)\tau}. \end{aligned} \tag{26}$$

Theorem 6. *If*

$$\langle b_1 \rangle_* < d_1\overline{c_0}, \tag{27}$$

$$\langle b_2 \rangle_* < d_2\overline{c_0}, \tag{28}$$

then

$$\begin{aligned} &\lim_{t\to\infty} x(t) = 0, \\ &\lim_{t\to\infty} y(t) = 0 \end{aligned} \tag{29}$$

a.s.

Proof. By Assumption 3 and the strong law of large numbers for local martingales, one has

$$\begin{aligned} &\lim_{t\to+\infty} \frac{1}{t}\int_0^t \int_{\mathbb{Y}} \ln\left(1+\gamma_i(s,u)\right)\widetilde{N}(ds,du) = 0, \\ &\lim_{t\to+\infty} \frac{\int_0^t \sigma_i(s)\,dB_i(s)}{t} = 0 \\ &\qquad\qquad \text{a.s.,}\ \ i = 1,2. \end{aligned} \tag{30}$$

Define

$$\begin{aligned} V_1(x) &= \ln x(t), \\ V_2(x) &= \ln y(t). \end{aligned} \tag{31}$$

Applying Itô's formula yields

$$\begin{aligned} &d\ln x(t) \\ &\quad = \left(b_1(t) - d_1c_0(t) - a_1(t)x(t) - \beta(t)y(t)\right)dt \\ &\qquad - \sigma_1(t)\,dB_1(t) \\ &\qquad + \int_{\mathbb{Y}} \ln\left(1+\gamma_1(t,u)\right)\widetilde{N}(dt,du), \end{aligned} \tag{32}$$

$$\begin{aligned} &d\ln y(t) \\ &\quad = \left(b_2(t) - d_2c_0(t) - a_2(t)y(t) + \beta(t)x(t)\right)dt \\ &\qquad - \sigma_2(t)\,dB_2(t) \\ &\qquad + \int_{\mathbb{Y}} \ln\left(1+\gamma_2(t,u)\right)\widetilde{N}(dt,du). \end{aligned} \tag{33}$$

Integrating both sides of (32) and (33) from 0 to t, respectively, we have

$$\begin{aligned} &\ln x(t) - \ln x(0) \\ &\quad = \int_0^t \left(b_1(s) - d_1c_0(s) - a_1(s)x(s) - \beta(s)y(s)\right)dt \\ &\qquad - \int_0^t \sigma_1(s)\,dB_1(s) \\ &\qquad + \int_0^t \int_{\mathbb{Y}} \ln\left(1+\gamma_1(t,u)\right)\widetilde{N}(dt,du), \end{aligned} \tag{34}$$

$$\begin{aligned} &\ln y(t) - \ln y(0) \\ &\quad = \int_0^t \left(b_2(s) - d_2c_0(s) - a_2(s)y(s) + \beta(s)x(s)\right)ds \\ &\qquad - \int_0^t \sigma_2(s)\,dB_2(s) \\ &\qquad + \int_0^t \int_{\mathbb{Y}} \ln\left(1+\gamma_2(t,u)\right)\widetilde{N}(dt,du). \end{aligned} \tag{35}$$

From (34), we can obtain that

$$\begin{aligned} &\frac{\ln x(t) - \ln x(0)}{t} \\ &\quad \le \frac{\int_0^t b_1(s)\,ds}{t} - d_1\frac{\int_0^t c_0(s)\,ds}{t} - \frac{\int_0^t \sigma_1(s)\,dB_1(s)}{t} \\ &\qquad + \frac{\int_0^t \int_{\mathbb{Y}} \ln\left(1+\gamma_1(t,u)\right)\widetilde{N}(dt,du)}{t}. \end{aligned} \tag{36}$$

Taking the limit superior results in

$$\left[\frac{\ln x(t)}{t}\right]^* \le \limsup_{t\to\infty}\left[\frac{\int_0^t b_1(s)\,ds}{t} - d_1\frac{\int_0^t c_0(s)\,ds}{t}\right] \tag{37}$$

a.s.

From (27) we can know that $[\ln x(t)/t]^* < 0$. Thus we have

$$\lim_{t\to\infty} x(t) = 0 \quad \text{a.s.} \tag{38}$$

Applying (35) leads to

$$\begin{aligned}\frac{\ln y(t)-\ln y(0)}{t} &\le \frac{\int_0^t b_2(s)\,ds}{t}+\frac{\int_0^t \beta(s)x(s)\,ds}{t}-d_2\frac{\int_0^t c_0(s)\,ds}{t}\\ &\quad-\frac{\int_0^t \sigma_2(s)\,dB_2(s)}{t}\\ &\quad+\frac{\int_0^t\int_{\mathbb{Y}}\ln(1+\gamma_2(t,u))\widetilde{N}(dt,du)}{t}.\end{aligned} \tag{39}$$

Taking the limit superior, together with (28) and (38), we can see that

$$\left[\frac{\ln y(t)}{t}\right]^* < 0 \quad \text{a.s.} \tag{40}$$

Therefore,

$$\lim_{t\to\infty} y(t) = 0 \quad \text{a.s.} \tag{41}$$

This completes the proof of Theorem 6. □

For simplicity, we define

$$R^* = \frac{\langle b_2\rangle^* + \beta^M/a_1^L\cdot\langle b_1\rangle^*}{(d_2+d_1(\beta^M/a_1^L))\overline{c_0}}. \tag{42}$$

Theorem 7. *If $\langle b_1\rangle_* > d_1\overline{c_0}$ and $R^* < 1$, then*

$$\begin{aligned}&\frac{\langle b_1\rangle_* - d_1\overline{c_0}}{a_1^M} \le \langle x(t)\rangle_* \le \langle x(t)\rangle^* \le \frac{\langle b_1\rangle^* - d_1\overline{c_0}}{a_1^L},\\ &\lim_{t\to\infty} y(t) = 0\end{aligned} \tag{43}$$

a.s.

Proof. According to (34), we can obtain that

$$\begin{aligned}&\ln x(t)-\ln x(0)\\ &\quad\le \int_0^t b_1(s)\,ds - d_1\int_0^t c_0(s)\,ds - \int_0^t a_1(s)x(s)\,ds\\ &\qquad-\int_0^t \sigma_1(s)\,dB_1(s)\\ &\qquad+\int_0^t\int_{\mathbb{Y}}\ln(1+\gamma_1(t,u))\widetilde{N}(dt,du)\\ &\quad\le \int_0^t b_1(s)\,ds - d_1\int_0^t c_0(s)\,ds - a_1^L\int_0^t x(s)\,ds\\ &\qquad-\int_0^t \sigma_1(s)\,dB_1(s)\\ &\qquad+\int_0^t\int_{\mathbb{Y}}\ln(1+\gamma_1(t,u))\widetilde{N}(dt,du).\end{aligned} \tag{44}$$

It follows from Lemma 5 that

$$\langle x(t)\rangle^* \le \frac{\langle b_1\rangle^* - d_1\overline{c_0}}{a_1^L} \quad \text{a.s.} \tag{45}$$

By (35), we have

$$\begin{aligned}\ln y(t)-\ln y(0) &\le \int_0^t b_2(s)\,ds + \int_0^t \beta(s)x(s)\,ds\\ &\quad- d_2\int_0^t c_0(s)\,ds - \int_0^t a_2(s)y(s)\,ds\\ &\quad-\int_0^t \sigma_2(s)\,dB_2(s)\\ &\quad+\int_0^t\int_{\mathbb{Y}}\ln(1+\gamma_2(t,u))\widetilde{N}(dt,du)\\ &\le \left(\frac{\int_0^t b_2(s)\,ds}{t}+\beta^M\frac{\langle b_1\rangle^*-d_1\overline{c_0}}{a_1^L}\right.\\ &\quad\left.-d_2\frac{\int_0^t c_0(s)\,ds}{t}\right)t-\int_0^t\sigma_2(s)\,dB_2(s)\\ &\quad+\int_0^t\int_{\mathbb{Y}}\ln(1+\gamma_2(t,u))\widetilde{N}(dt,du),\end{aligned} \tag{46}$$

and then

$$\begin{aligned}y(t) &\le y(0)\exp\left\{\left(\frac{\int_0^t b_2(t)\,dt}{t}+\beta^M\frac{\langle b_1\rangle^*-d_1\overline{c_0}}{a_1^L}\right.\right.\\ &\quad\left.-d_2\frac{\int_0^t c_0(s)\,ds}{t}\right)t-\int_0^t\sigma_2(t)\,dB_2(t)\\ &\quad\left.+\int_0^t\int_{\mathbb{Y}}\ln(1+\gamma_2(t,u))\widetilde{N}(dt,du)\right\}.\end{aligned} \tag{47}$$

Since $R^* < 1$, then we know that

$$\lim_{t\to\infty} y(t) = 0 \quad \text{a.s.,} \tag{48}$$

which combined with (34) and the property of the limit superior shows that, for any $\varepsilon > 0$, there exists a random number $T > 0$ for $t > T$ such that

$$\begin{aligned}\ln\frac{x(t)}{x(0)} &\ge \int_0^t (b_1(s)-d_1c_0(s))\,ds - \beta^M\varepsilon\\ &\quad- a_1^M\int_0^t x(t)\,dt - \int_0^t\sigma_1(t)\,dB_1(t)\\ &\quad+\int_0^t\int_{\mathbb{Y}}\ln(1+\gamma_1(t,u))\widetilde{N}(dt,du).\end{aligned} \tag{49}$$

By Lemma 5, we have that

$$\langle x(t)\rangle_* \ge \frac{\langle b_1\rangle_* - d_1\overline{c_0}}{a_1^M} \quad \text{a.s.} \tag{50}$$

This completes the proof of Theorem 7. □

3.2. Permanence in Mean. In this section, we need to show the permanence in mean. Note that $\langle b_1\rangle_* < d_1\overline{c_0}$ implies that productiveness of the prey is less than its death loss rate; then the prey can go extinct. Naturally, we assume $\langle b_1\rangle_* > d_1\overline{c_0}$ in the rest of this paper.

Define

$$\begin{aligned}
\Delta_1 &= \beta^L \langle b_1\rangle_* + a_1^M \langle b_2\rangle_* - \left(\beta^L d_1 + a_1^M d_2\right)\overline{c_0},\\
\Delta_2 &= \beta^L\beta^M + a_1^M a_2^M,\\
\Delta_3 &= \langle b_1\rangle_* - d_1\overline{c_0} - \beta^L\frac{\Delta_1}{\Delta_2},\\
\Delta_4 &= \langle b_1\rangle_* - d_1\overline{c_0} - \beta^M\frac{\Delta_5}{a_2^L},\\
\Delta_5 &= \langle b_2\rangle^* - d_2\overline{c_0} + \beta^M\frac{\Delta_3}{a_1^L},\\
R_* &= \frac{\langle b_2\rangle_* + \beta^L/a_1^M\cdot\langle b_1\rangle_*}{\left(d_2 + d_1\left(\beta^L/a_1^M\right)\right)\overline{c_0}}.
\end{aligned} \tag{51}$$

Theorem 8. *If*

$$\begin{aligned}
\langle b_2\rangle^* &> d_2\overline{c_0},\\
R_* &> 1,\\
\Delta_3 &> 0,\\
\Delta_4 &> 0,
\end{aligned} \tag{52}$$

then

$$\begin{aligned}
\frac{\Delta_4}{a_1^M} &\le \langle x(t)\rangle_* \le \langle x(t)\rangle^* \le \frac{\Delta_3}{a_1^L},\\
\frac{\Delta_1}{\Delta_2} &\le \langle y(t)\rangle_* \le \langle y(t)\rangle^* \le \frac{\Delta_5}{a_2^L}.
\end{aligned} \tag{53}$$

Proof. Define $u(t)$ such that $x(t) \le u(t)$, where $u(t)$ satisfies

$$\begin{aligned}
du(t) = {} & u(t)\left(r_1(t) - a_1(t)u(t)\right)dt\\
& - \sigma_1(t)u(t)\,dB_1(t)\\
& + \int_{\mathbb{Y}} u(t^-)\gamma_1(t,u)\widetilde{N}(dt,du).
\end{aligned} \tag{54}$$

In [9], we know that

$$\lim_{t\to\infty}\frac{\ln u(t)}{t} = 0, \quad \text{a.s.} \tag{55}$$

It can be inferred from comparison theorem that

$$\left(\frac{\ln x(t)}{t}\right)^* \le \lim_{t\to\infty}\frac{\ln u(t)}{t} = 0. \tag{56}$$

From (34) and (35), we know that

$$\begin{aligned}
&\frac{\ln x(t) - \ln x(0)}{t}\\
&\quad = \frac{\int_0^t b_1(s)\,ds}{t} - \frac{\int_0^t d_1c_0(s)\,ds}{t} - \frac{\int_0^t a_1(s)x(s)\,ds}{t}\\
&\qquad - \frac{\int_0^t \beta(s)y(s)\,ds}{t} - \frac{\int_0^t \sigma_1(s)\,dB_1(s)}{t}\\
&\qquad + \frac{\int_0^t\int_{\mathbb{Y}}\ln\left(1+\gamma_1(t,u)\right)\widetilde{N}(dt,du)}{t},
\end{aligned} \tag{57}$$

$$\begin{aligned}
&\frac{\ln y(t) - \ln y(0)}{t}\\
&\quad = \frac{\int_0^t b_2(s)\,ds}{t} - \frac{\int_0^t d_2c_0(s)\,ds}{t} - \frac{\int_0^t a_2(s)y(s)\,ds}{t}\\
&\qquad + \frac{\int_0^t \beta(s)x(s)\,ds}{t} - \frac{\int_0^t \sigma_2(s)\,dB_2(s)}{t}\\
&\qquad + \frac{\int_0^t\int_{\mathbb{Y}}\ln\left(1+\gamma_2(t,u)\right)\widetilde{N}(dt,du)}{t}.
\end{aligned} \tag{58}$$

By $\beta_L \times (57) + a_1^M \times (58)$, we obtain that

$$\begin{aligned}
&\beta^L \times \frac{\ln x(t) - \ln x(0)}{t} + a_1^M \times \frac{\ln y(t) - \ln y(0)}{t}\\
&\quad \ge \Delta_1 - \Delta_2\langle y(t)\rangle - \beta_L\frac{\int_0^t \sigma_1(s)\,dB_1(s)}{t}\\
&\qquad + \beta_L\frac{\int_0^t\int_{\mathbb{Y}}\ln\left(1+\gamma_1(t,u)\right)\widetilde{N}(dt,du)}{t}\\
&\qquad - a_1^M\frac{\int_0^t \sigma_2(s)\,dB_2(s)}{t}\\
&\qquad + a_1^M\frac{\int_0^t\int_{\mathbb{Y}}\ln\left(1+\gamma_2(t,u)\right)\widetilde{N}(dt,du)}{t}.
\end{aligned} \tag{59}$$

When $R_* > 1$ is used in (59), applying Lemma 5 and (56) leads to

$$\langle y(t)\rangle_* \ge \frac{\Delta_1}{\Delta_2}. \tag{60}$$

According to (57) and (60), we can have

$$\begin{aligned}
&\frac{\ln x(t) - \ln x(0)}{t}\\
&\quad \le \langle b_1\rangle^* - d_1\overline{c_0} - \beta^L\frac{\Delta_1}{\Delta_2} - a_1^L\langle x(t)\rangle\\
&\qquad - \frac{\int_0^t \sigma_1(s)\,dB_1(s)}{t}\\
&\qquad + \frac{\int_0^t\int_{\mathbb{Y}}\ln\left(1+\gamma_1(t,u)\right)\widetilde{N}(dt,du)}{t}.
\end{aligned} \tag{61}$$

From the conditions of Theorem 8, we know that $\Delta_3 > 0$ and applying Lemma 5 results in

$$\langle x(t)\rangle^* \le \frac{\Delta_3}{a_1^L}. \tag{62}$$

By (58) and (62), we obtain that

$$\begin{aligned}\frac{\ln y(t) - \ln y(0)}{t} &\le \langle b_2\rangle^* - d_2\overline{c_0} + \beta^M \frac{\Delta_3}{a_1^L} - a_2^L \langle y(t)\rangle \\ &\quad - \frac{\int_0^t \sigma_1(s)\, dB_1(s)}{t} \\ &\quad + \frac{\int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_1(t,u))\, \widetilde{N}(dt,du)}{t}.\end{aligned} \tag{63}$$

From the conditions of Theorem 8, we see that $\Delta_5 > 0$ and applying Lemma 5 leads to

$$\langle y(t)\rangle^* \le \frac{\Delta_5}{a_2^L}. \tag{64}$$

By (58) and (64), we can have

$$\begin{aligned}\frac{\ln x(t) - \ln x(0)}{t} &\ge \langle b_1\rangle_* - d_1\overline{c_0} - \beta^M \frac{\Delta_5}{a_2^L} - a_1^M \langle x(t)\rangle \\ &\quad - \frac{\int_0^t \sigma_1(s)\, dB_1(s)}{t} \\ &\quad + \frac{\int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_1(t,u))\, \widetilde{N}(dt,du)}{t}.\end{aligned} \tag{65}$$

According to the conditions of Theorem 8, we see that $\Delta_4 > 0$ and applying Lemma 5 yields

$$\langle x(t)\rangle_* \ge \frac{\Delta_4}{a_1^M}. \tag{66}$$

This completes the proof of Theorem 8. □

3.3. Stochastic Permanence. For the sake of proving stochastic permanence, we should examine the pth moment boundedness firstly.

Define

$$\begin{aligned}\lambda_1 &= \max_{t\in[0,+\infty)} \left\{ pr_1(t) - pd_1c_0(t) + \frac{p(p-1)}{2}\sigma_1^2(t) + c_1(t) \right\}, \\ \lambda_1' &= \max_{t\in[0,+\infty)} \left\{ pr_2(t) - pd_2c_0(t) + \frac{p(p-1)}{2}\sigma_2^2(t) + c_2(t) \right\}, \\ \lambda_1'' &= \max\{\lambda_1, \lambda_1'\}, \\ \lambda_2 &= \min_{t\in[0,+\infty)} \left\{ a_1(t)\, p - \frac{p\beta(t)}{p+1} \right\}, \\ \lambda_2' &= \min_{t\in[0,+\infty)} \left\{ a_2(t)\, p - \frac{p^2\beta(t)}{p+1} \right\}, \\ \lambda_2'' &= \min\{\lambda_2, \lambda_2'\}.\end{aligned} \tag{67}$$

Theorem 9. *If there exists a constant p such that*

$$\int_{\mathbb{Y}} \left((1+\gamma_i(t,u))^p - 1 - p\gamma_i(t,u) \right) \nu(du) < \infty, \tag{68}$$

$$\begin{aligned}\lambda_1'' &> 0, \\ \lambda_2'' &> 0,\end{aligned} \tag{69}$$

then

$$\limsup_{t\to\infty} \mathbb{E}\left[\left(x^2(t) + y^2(t) \right)^{p/2} \right] \le K(p) \quad a.s. \tag{70}$$

Proof. Define

$$V(x,y) = x^p(t) + y^p(t). \tag{71}$$

Applying Itô's formula and (68) yields

$$\begin{aligned}dV(x,y) &= LVdt - p\sigma_1(t)\, x^p(t)\, dB_1(t) \\ &\quad - p\sigma_2(t)\, y^p(t)\, dB_2(t) \\ &\quad + \int_{\mathbb{Y}} x^p(t^-) \left((1+\gamma_1(t,u))^p - 1 \right) \widetilde{N}(dt,du) \\ &\quad + \int_{\mathbb{Y}} y^p(t^-) \left((1+\gamma_2(t,u))^p - 1 \right) \widetilde{N}(dt,du),\end{aligned} \tag{72}$$

where

$$\begin{aligned}LV &= px^p(t) \left(r_1(t) - d_1c_0(t) - a_1(t)\, x(t) - \beta(t) \cdot y(t) \right) + py^p(t) \left(r_2(t) - d_2c_0(t) - a_2(t)\, x(t) + \beta(t)\, x(t) \right) + \frac{p(p-1)}{2} \left(\sigma_1^2(t)\, x^p(t) + \sigma_2^2(t) \cdot y^p(t) \right) + \int_{\mathbb{Y}} x^p(t^-) \\ &\quad \cdot \left((1+\gamma_1(t,u))^p - 1 - p\gamma_1(t,u) \right) \nu(du)\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{Y}} y^p(t^-)\left((1+\gamma_1(t,u))^p-1-p\gamma_1(t,u)\right)\\
&\cdot\nu(du)\le\left(pr_1(t)-pd_1c_0(t)+\frac{p(p-1)}{2}\sigma_1^2(t)\right.\\
&\left.+c_1(t)\right)x^p(t)+\left(pr_2(t)-pd_2c_0(t)+\frac{p(p-1)}{2}\right.\\
&\left.\cdot\sigma_2^2(t)+c_2(t)\right)y^p(t)-a_1(t)px^{p+1}(t)-\beta(t)\\
&\cdot px^p(t)y(t)-a_2(t)py^{p+1}(t)+p\beta(t)x(t)y^p(t)\\
&\le\left(pr_1(t)-pd_1c_0(t)+\frac{p(p-1)}{2}\sigma_1^2(t)+c_1(t)\right)\\
&\cdot x^p(t)-\left(a_1(t)p-\frac{p\beta(t)}{p+1}\right)x^{p+1}(t)+\left(pr_2(t)\right.\\
&\left.-pd_2c_0(t)+\frac{p(p-1)}{2}\sigma_2^2(t)+c_2(t)\right)y^p(t)\\
&-\left(a_2(t)p-\frac{p^2\beta(t)}{p+1}\right)y^{p+1}(t),
\end{aligned}
\tag{73}
$$

where $c_i(t)$ $(i=1,2)$ is a bounded continuous function.

Integrating (72) from 0 to t and taking the expectations for both sides yield

$$
\mathbb{E}[V(x,y)]=V(x(0),y(0))+\mathbb{E}\left[\int_0^t LV\,dt\right],
\tag{74}
$$

and then

$$
\begin{aligned}
&\frac{d\mathbb{E}[V(x,y)]}{dt}\\
&\le\lambda_1\mathbb{E}[x^p]-\lambda_2\mathbb{E}\left[x^{p+1}\right]+\lambda_1'\mathbb{E}[y^p]\\
&\quad-\lambda_2'\mathbb{E}\left[y^{p+1}\right]\\
&\le\lambda_1''\mathbb{E}[x^p+y^p]-\lambda_2''\mathbb{E}\left[x^{p+1}+y^{p+1}\right]\\
&\le\lambda_1''\mathbb{E}[x^p+y^p]-\lambda_2''\cdot2^{-1/p}\left(\mathbb{E}[x^p+y^p]\right)^{(p+1)/p}\\
&=\mathbb{E}[x^p+y^p]\left(\lambda_1''-\lambda_2''\cdot2^{-1/p}\left(\mathbb{E}[x^p+y^p]\right)^{1/p}\right).
\end{aligned}
\tag{75}
$$

We know that the auxiliary equation

$$
\dot z(t)=z(t)\left[\lambda_1-\lambda_2(z(t))^{1/p}\right]
\tag{76}
$$

has a globally asymptotically stable positive equilibrium $\overline{z}=(\lambda_1/\lambda_2)^p$. Let $z(t)$ be the solution of (76) with $z(0)=\mathbb{E}[(x(0))^p+(y(0))^p]$, and according to the comparison theorem we have $\mathbb{E}[x^p+y^p]\le z(t)$, $t\ge0$. Hence, we obtain

$$
\limsup_{t\to+\infty}\mathbb{E}[V(x,y)]\le2^{1/p}\cdot\frac{\lambda_1''}{\lambda_2''}:=K_1(p)\quad\text{a.s.}
\tag{77}
$$

We find that

$$
\left(x^2+y^2\right)^{p/2}\le2^{p/2}\cdot\max\{x^p,y^p\}\le2^{p/2}\cdot V(x,y),
\tag{78}
$$

so we have

$$
\limsup_{t\to+\infty}\mathbb{E}\left[\left(x^2+y^2\right)^{p/2}\right]\le\sqrt{2}\cdot\frac{\lambda_1''}{\lambda_2''}:=K(p)\quad\text{a.s.}
\tag{79}
$$

This completes the proof of Theorem 9. □

Define

$$
\begin{aligned}
r(t)&=\min\{r_1(t)-d_1c_0(t),r_2(t)-d_2c_0(t)\},\\
\sigma(t)&=\max\{\sigma_1(t),\sigma_2(t)\},\\
H(t)&=r(t)-\sigma^2(t)-\sum_{i=1}^{2}\int_{\mathbb{Y}}\frac{\gamma_i^2}{1+\gamma_i}\nu(du).
\end{aligned}
\tag{80}
$$

Theorem 10. *If*

$$
\langle H(t)\rangle_*>0,
\tag{81}
$$

then system (4) is stochastically permanent.

Proof. Define

$$
V(x,y)=\frac{1}{x(t)+y(t)}.
\tag{82}
$$

By applying Itô's formula, we have

$$
\begin{aligned}
d(V(x,y))&=LVdt+\frac{1}{(x+y)^2}\left(\sigma_1(t)x(t)dB_1(t)\right.\\
&\left.+\sigma_2(t)y(t)dB_2(t)\right)\\
&+\int_{\mathbb{Y}}\left(\frac{1}{x(t^-)+y(t^-)+x(t^-)\gamma_1(t,u)+y(t^-)\gamma_2(t,u)}\right.\\
&\left.-\frac{1}{x(t^-)+y(t^-)}\right)\widetilde{N}(dt,du),
\end{aligned}
\tag{83}
$$

where

$$
\begin{aligned}
LV&=-\frac{1}{(x+y)^2}\left(x(t)(r_1(t)-d_1c_0(t))-a_1(t)x^2(t)\right.\\
&\left.+(r_2(t)-d_2c_0(t))y(t)-a_2(t)y^2(t)\right)\\
&+\frac{\sigma_1^2(t)x^2(t)+\sigma_2^2(t)y^2(t)+2\sigma_1(t)\sigma_2(t)x(t)y(t)}{(x(t)+y(t))^3}\\
&+\int_{\mathbb{Y}}\left(\frac{1}{x+y+x\gamma_1(t,u)+y\gamma_2(t,u)}-\frac{1}{x+y}\right.\\
&\left.+\frac{1}{(x+y)^2}(x\gamma_1(t,u)+y\gamma_2(t,u))\right)\nu(du)\\
&\le-\frac{1}{x(t)+y(t)}\left(r(t)-\sigma^2(t)\right.\\
&\left.-\sum_{i=1}^{2}\int_{\mathbb{Y}}\frac{\gamma_i^2(t,u)}{1+\gamma_i(t,u)}\nu(du)\right)+a(t)=-V\cdot H(t)\\
&+a(t),
\end{aligned}
\tag{84}
$$

where $a(t)=\max\{a_1(t),a_2(t)\}$.

Integrating (83) from 0 to t and taking the expectation on both sides yield

$$\mathbb{E}[V(t)] = V(0) + \mathbb{E}\left[\int_0^t LV(s)\,ds\right]. \tag{85}$$

Therefore,

$$\begin{aligned} d\mathbb{E}[V(t)] &= \mathbb{E}[LV(t)]\,dt \\ &\le (-H(t)\,\mathbb{E}[V(t)] + a(t))\,dt. \end{aligned} \tag{86}$$

By the variation of constants method, we can derive that

$$\begin{aligned} \mathbb{E}[V(t)] \le{}& \int_0^t a(s)\exp\left(-\int_s^t H(\tau)\,d\tau\right)ds \\ &+ V(0)\exp\left(-\int_0^t H(s)\,ds\right), \end{aligned} \tag{87}$$

which together with (81) results in

$$\mathbb{E}[V(x,y)] \le K, \tag{88}$$

where K is a positive constant. Then, for any given $\varepsilon > 0$ and constant $\delta(\varepsilon) = \varepsilon/K$, according to the Chebyshev inequality, we can know that

$$\begin{aligned} P\left\{\frac{1}{V(x,y)} < \delta\right\} &= P\left\{V(x,y) > \frac{1}{\delta}\right\} \\ &\le \frac{\mathbb{E}[V(x,y)]}{1/\delta} = \delta\mathbb{E}[V(x,y)] \\ &\le \varepsilon, \end{aligned} \tag{89}$$

and then

$$P\left\{\frac{1}{V(x,y)} \ge \delta\right\} \ge 1 - \varepsilon. \tag{90}$$

Therefore,

$$\liminf_{t\to\infty} P\{(x(t) + y(t)) \ge \delta\} \ge 1 - \varepsilon \quad \text{a.s.} \tag{91}$$

Then, we have

$$\liminf_{t\to\infty} P\left\{\left(\sqrt{x^2(t) + y^2(t)}\right) \ge \frac{\sqrt{2}}{2}\delta\right\} \ge 1 - \varepsilon \quad \text{a.s.} \tag{92}$$

Using Chebyshev inequality and Theorem 9, we can obtain

$$\limsup_{t\to\infty} P\left\{\left(\sqrt{x^2(t) + y^2(t)}\right) \le \chi\right\} \ge 1 - \varepsilon \quad \text{a.s.,} \tag{93}$$

where χ is a constant. This completes the proof of Theorem 10. □

4. Simulations and Conclusions

In this section, we show some numerical examples, which will demonstrate our results.

Choose the parameters in (4) as follows:

$$\begin{aligned} r_1(t) &= 1.8, \\ r_2(t) &= 1.3, \\ d_1 &= 0.35, \\ d_2 &= 0.4, \\ a_1(t) &= 0.3, \\ a_2(t) &= 0.4, \\ \beta(t) &= 0.1, \\ k &= 0.5, \\ g &= 0.2, \\ h &= 0.2, \\ m &= 0.3, \\ u &= 1, \\ \tau &= 1. \end{aligned} \tag{94}$$

In Figure 1, we choose $\sigma_1(t) = 0.3$, $\sigma_2(t) = 0.4$ and the values of $\gamma_1(t,u)$ and $\gamma_2(t,u)$ are different between (a) and (b).

In Figure 1(a), we choose $\gamma_1(t,u) = 0.1$, $\gamma_2(t,u) = 0.3$; then the conditions in Theorem 7 are satisfied. According to Theorem 7, we know that $x(t)$ is permanent and $y(t)$ is extinct.

In Figure 1(b), we choose $\gamma_1(t,u) = 0.3$, $\gamma_2(t,u) = 0.3$; by Theorem 6, we can obtain that both of $x(t)$ and $y(t)$ are extinct.

Then, we choose $\gamma_1(t,u) = 0.1$, $\gamma_2(t,u) = 0.1$, the values of $\sigma_1(t)$ and $\sigma_2(t)$ are different between Figures 1(c) and 1(d).

In Figure 1(c), we choose $\sigma_1(t) = 0.3$, $\sigma_2(t) = 0.9$; then the conditions in Theorem 7 are satisfied. According to Theorem 7, we know that $x(t)$ is permanent and $y(t)$ is extinct.

In Figure 1(d), we choose $\sigma_1(t) = 0.8$, $\sigma_2(t) = 0.9$; by Theorem 6, we get that both of $x(t)$ and $y(t)$ are extinct.

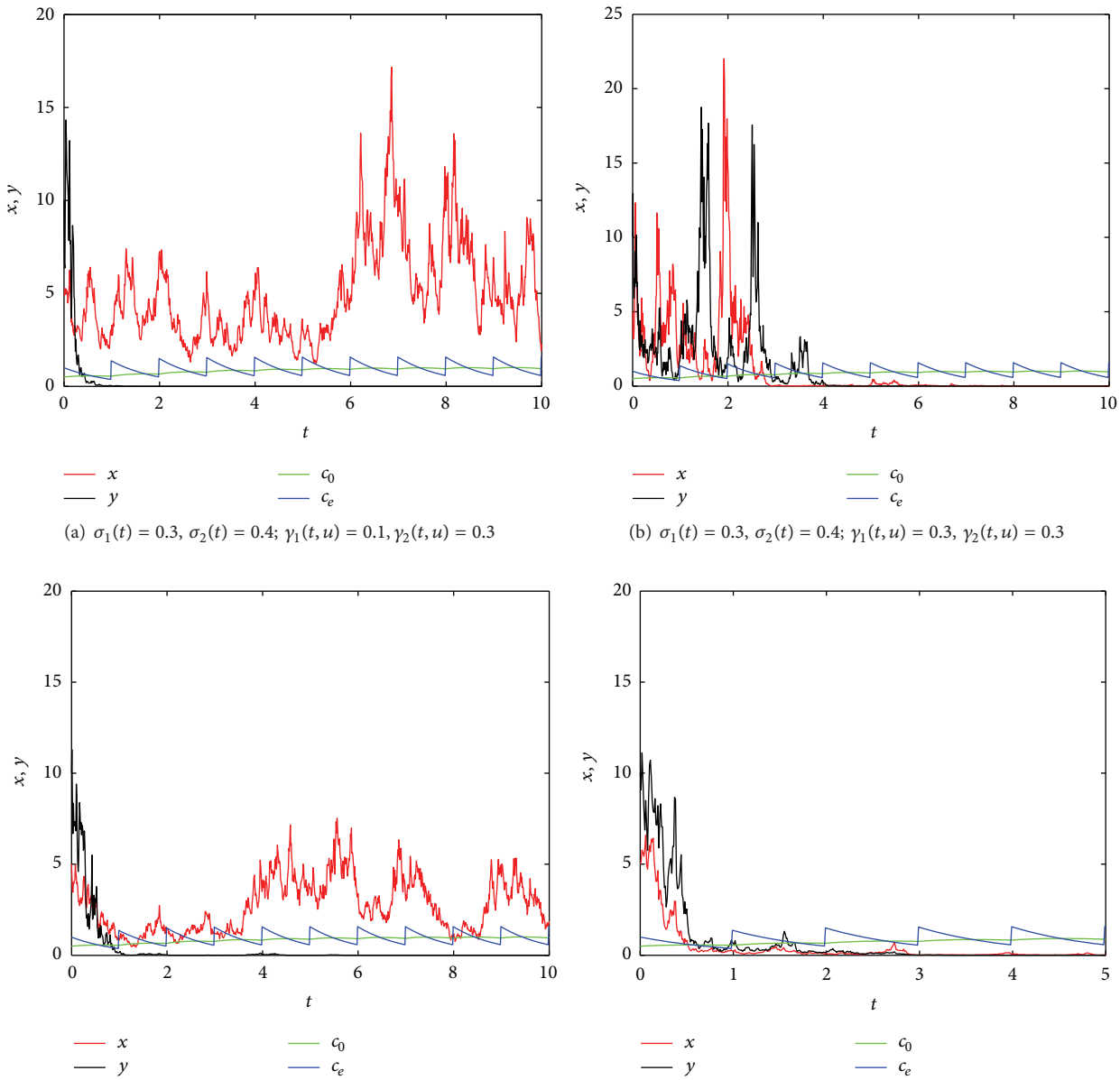
In Figure 2, we choose $\gamma_1 = 2.8$, $\gamma_2 = 2.3$, $\sigma_1(t) = 0.3$, $\sigma_2(t) = 0.4$, $\gamma_1(t,u) = 0.1$, and $\gamma_2(t,u) = 0.1$; other parameters are the same as in Figure 1; according to Theorems 8 and 10, we obtain that system (4) is permanent in mean and stochastically permanent.

In conclusion, permanence and extinction of the predator and the prey for the intensity of white noise and Lévy noise are given in Table 1.

In this paper, we consider a nonautonomous impulsive stochastic predator-prey system with Lévy jumps, which considers the predator is omnivorous. We prove that the system has a unique global positive solution. From Theorem 6 and Figure 1, we can know that if the intensities of white noise and Lévy noise are sufficiently large, then the species will be extinct (see Figures 1(b) and 1(d)). According to Theorems 8 and 10, we know that system (4) is permanent in mean and stochastically permanent under some conditions in Figure 2. The change of permanent conditions shows an

Table 1: Permanence and extinction for different noise parameters.

White noise σ_1	White noise σ_2	Lévy noise γ_1	Lévy noise γ_2	Permanence and extinction
0.3	0.4	0.1	0.3	Permanence for x and extinction for y
0.3	0.4	0.3	0.3	Extinction for x and y
0.3	0.9	0.1	0.1	Permanence for x and extinction for y
0.8	0.9	0.1	0.1	Extinction for x and y
0.3	0.4	0.1	0.1	Permanence for x and y



(a) $\sigma_1(t) = 0.3$, $\sigma_2(t) = 0.4$; $\gamma_1(t,u) = 0.1$, $\gamma_2(t,u) = 0.3$

(b) $\sigma_1(t) = 0.3$, $\sigma_2(t) = 0.4$; $\gamma_1(t,u) = 0.3$, $\gamma_2(t,u) = 0.3$

(c) $\sigma_1(t) = 0.3$, $\sigma_2(t) = 0.9$; $\gamma_1(t,u) = 0.1$, $\gamma_2(t,u) = 0.1$

(d) $\sigma_1(t) = 0.8$, $\sigma_2(t) = 0.9$; $\gamma_1(t,u) = 0.1$, $\gamma_2(t,u) = 0.1$

Figure 1: The red line and the black line represent $x(t)$ and $y(t)$, respectively. The initial value is $x(0) = 5$, $y(0) = 10$, $c_0(0) = 0.5$, and $c_e(0) = 1$. In (a), $x(t)$ is permanent and $y(t)$ is extinct; in (b), both of $x(t)$ and $y(t)$ are extinct. In (c), $x(t)$ is permanent and $y(t)$ is extinct; in (d), both of $x(t)$ and $y(t)$ are extinct.

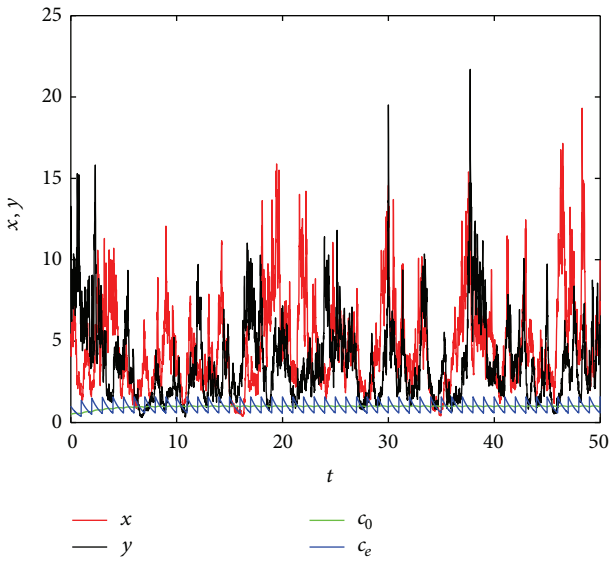


FIGURE 2: The red line and the black line represent $x(t)$ and $y(t)$, respectively. The initial value is $x(0) = 5$, $y(0) = 10$, $c_0(0) = 0.5$, and $c_e(0) = 1$. Both $x(t)$ and $y(t)$ are stochastically permanent when $\sigma_1(t) = 0.3$, $\sigma_2(t) = 0.4$, $\gamma_1(t,u) = 0.1$, and $\gamma_2(t,u) = 0.1$.

important property; that is, the permanence of species has a close relationship with the intensity of white noise and Lévy noise. Obviously, white noises and Lévy noises are harmful to the permanence of populations.

There are some investigations for stochastic epidemic models [14, 15]. It is very significant for the natural world. Some interesting questions deserve further investigation. One could study more realistic but more complex models. Also it is interesting to investigate evolutionary dynamics of stochastic evolutionary model, and we leave these for future work.

Appendix

Proof of Lemma 5. (I) If $\langle\delta\rangle^* \geq 0$, $\int_0^t \delta(s)ds > 0$ for t large enough. By Assumption 3 and the strong law of large numbers for local martingales, one has

$$\lim_{t\to+\infty}\frac{1}{t}\sum_{i=1}^{2}\delta_i\int_0^t\int_{\mathbb{Y}}\ln\left(1+\gamma_i(u)\right)\widetilde{N}(ds,du)=0,$$
$$\lim_{t\to+\infty}\frac{B(t)}{t}=0 \quad \text{a.s.} \tag{A.1}$$

Then, for arbitrary $\varepsilon > 0$, there exists a $T_1 > 0$ such that for $t > T_1$

$$\frac{1}{t}\left[\alpha B(t)+\sum_{i=1}^{2}\delta_i\int_0^t\int_{\mathbb{Y}}\ln\left(1+\gamma_i(u)\right)\widetilde{N}(ds,du)\right] < \varepsilon. \tag{A.2}$$

Define

$$\varphi(t)=\int_0^t\eta(s)\,ds, \tag{A.3}$$

for $t \geq T_1$; then

$$\frac{d\varphi(t)}{dt}=\eta(t),$$
$$\ln\frac{d\varphi(t)}{dt}\leq\int_0^t(\delta(s)+\varepsilon)\,ds-\delta_0\varphi(t), \tag{A.4}$$

for $t \geq \max\{T, T_1\} = T_2$.

Integrating the above inequality from T_2 to t yields

$$\begin{aligned}\varphi(t)&\leq\frac{1}{\delta_0}\ln\left(e^{\delta_0\varphi(T_2)}+\delta_0\int_{T_2}^t e^{\int_0^\theta(\delta(s)+\varepsilon)ds}d\theta\right)\\&\leq\frac{1}{\delta_0}\ln\left(e^{\delta_0\varphi(T_2)}+\delta_0 e^{\int_0^t(\delta(s)+\varepsilon)ds}(t-T_2)\right)\\&\leq\frac{1}{\delta_0}\int_0^t(\delta(s)+\varepsilon)\,ds\\&\quad+\frac{1}{\delta_0}\ln\left(\delta_0(t-T_2)+e^{\delta_0\varphi(T_2)}\right).\end{aligned} \tag{A.5}$$

Dividing by t and taking the superior limit of both sides of (A.5) lead to

$$\langle\eta\rangle^*\leq\frac{\langle\delta\rangle^*+\varepsilon}{\delta_0} \quad \text{a.s.} \tag{A.6}$$

By the arbitrariness of ε, one has $\langle\eta\rangle^* \leq \langle\delta\rangle^*/\delta_0$, if $\langle\delta\rangle^* \geq 0$. Moreover, if $\langle\delta\rangle^* < 0$, we get

$$\limsup_{t\to+\infty}\frac{1}{t}\ln\eta(t)\leq\langle\delta\rangle^*+\varepsilon<0, \tag{A.7}$$

for ε sufficiently small. Thus

$$\lim_{t\to\infty}\eta(t)=0 \quad \text{a.s., if } \langle\delta\rangle^*<0. \tag{A.8}$$

(II) The proof of (II) is similar to (I). Here we omits the proof. □

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

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Research Article

Exponential Attractor for the Boussinesq Equation with Strong Damping and Clamped Boundary Condition

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The paper studies the existence of exponential attractor for the Boussinesq equation with strong damping and clamped boundary condition $u_{tt} - \Delta u + \Delta^2 u - \Delta u_t - \Delta g(u) = f(x)$. The main result is concerned with nonlinearities $g(u)$ with supercritical growth. In that case, we construct a bounded absorbing set with further regularity and obtain quasi-stability estimates. Then the exponential attractor is established in natural energy space $V_2 \times H$.

1. Introduction

In this paper, we are concerned with the existence of exponential attractor for the Boussinesq equation with strong damping and clamped boundary condition

$$u_{tt} - \Delta u + \Delta^2 u - \Delta u_t - \Delta g(u) = f(x) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with the smooth boundary $\partial\Omega$, on which we consider the clamped boundary condition

$$\begin{aligned} u|_{\partial\Omega} &= 0, \\ \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} &= 0, \end{aligned} \tag{2}$$

where ν is the unit outward normal on $\partial\Omega$, and the initial condition

$$\begin{aligned} u(x,0) &= u_0(x), \\ u_t(x,0) &= u_1(x), \\ x &\in \Omega, \end{aligned} \tag{3}$$

and the assumptions on $g(u)$ and f will be specified later.

In 1872, Boussinesq [1] established the equation

$$u_{tt} - u_{xx} - \alpha u_{xxxx} = \beta \left(u^2\right)_{xx} \tag{4}$$

to describe the longitudinal displacement of the shallow water wave. Here u and α, β are some constants depending on the depth of the fluid and characteristic velocity of the water wave. When $\alpha < 0$, (4) is called "good" Boussinesq equation, when $\alpha > 0$, (4) is called "bad" Boussinesq equation. There have been lots of research on the well-posedness, blowup, and other properties of solutions for both the "good" and the "bad" Boussinesq equation of type (1) (see [2–14] and references therein). While for the investigation on the global attractor to (1), one can see [15–19] and references therein.

Chueshov and Lasiecka [20, 21] studied the longtime behavior of solutions to the Kirchhoff-Boussinesq plate equation

$$u_{tt} + ku_t + \Delta^2 u = \operatorname{div}\left[f_0(\nabla u)\right] + \Delta\left[f_1(u)\right] - f_2(u) \tag{5}$$

with $\Omega \subset \mathbb{R}^2$ and the clamped boundary condition (2). Here $k > 0$ is the damping parameter and the mapping $f_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and the smooth functions f_1 and f_2 represent (nonlinear) feedback forces acting upon the plate, in particular,

$$\begin{aligned} f_0(\nabla u) &= |\nabla u|^2 \nabla u, \\ f_1(u) &= u^2 + u. \end{aligned} \tag{6}$$

Ignoring both restoring force $f_0(\nabla u)$ and feedback force $f_2(u)$ and replacing the inertial term u_{tt} by ϵu_{tt}, with $\epsilon > 0$ (the

relaxation time) sufficiently small, (5) becomes the modified Cohn-Hilliard equation

$$\epsilon u_{tt} + u_t - \Delta\left(-\Delta u + f(u)\right) = g, \tag{7}$$

which is proposed by Galenko et al. [22–24] to model rapid spinodal decomposition in nonequilibrium phase separation processes. Grasselli et al. [25–27] studied the well-posedness and the longtime dynamics of (7) in both $2D$ and $3D$ cases, with hinged boundary condition. They established the existence of the global and exponential attractor for $\epsilon = 1$ in $2D$ case, and for $\epsilon > 0$ sufficiently small in $3D$ case. Taking $\epsilon = 1$ in (7) or taking $f_0(\nabla u) = \nabla u$, $f_2 = 0$ in (5), and taking into account the inertial force represented by $-\Delta u$ and replacing the weak damping u_t by a strong one $-\Delta u_t$, (1) arises.

In $1D$ case, Dai and Guo [15, 16] studied the "bad" Boussinesq equation with strong damping

$$\begin{aligned} &u_{tt} - u_{xx} + 2ku_{xxt} - \alpha u_{xxxx} = \beta\left(u^n\right)_{xx} \\ &\qquad\qquad \text{in } \mathbb{R}\times(0,+\infty), \\ &u(x,0) = u_0(x), \\ &u_t(x,0) = u_1(x), \\ &\qquad\qquad x\in\mathbb{R}. \end{aligned} \tag{8}$$

They got global solution $u \in C^\infty((0,T];H^\infty(R)) \cap C([0,T];H^1(R)) \cap C([0,T];H^{-1}(R))$ $\forall T > 0$, where $k, \alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}^+$, $n > 2$.

For the multidimensional case, Yang [17] proved the IVP of the Boussinesq equation

$$\begin{aligned} &u_{tt} - \Delta u + \mu\Delta^2 u = \Delta\sigma(u) \quad \text{in } \mathbb{R}^N\times(0,+\infty), \\ &u(x,0) = u_0(x), \\ &u_t(x,0) = u_1(x), \\ &\qquad x\in R^N. \end{aligned} \tag{9}$$

There existed the global weak solution, where $\mu > 0$, $\sigma(u) \in C(R)$, $|\sigma(s)| \le b|s|^p$, $s \in R$, $1 < p \le (N+2)/(N-2)^+$, and $0 \le \sigma(s)s \le \beta\int_0^s \sigma(\tau)d\tau$, $s \in \mathbb{R}$, $\beta > 0$. Here the growth exponent $\widetilde{p} = N/(N-2)$ $(N \ge 3)$ is called critical. The growth exponent $p^* \equiv (N+2)/(N-2)^+$ $(\ge \widetilde{p})$ is called supercritical. However, there is little research on the the higher global regularity of a bounded absorbing set, the global attractor and an exponential attractor in natural energy space for the dynamical system. We try to solve those problems in this paper.

Global attractor is a basic concept in the research studies of the asymptotic behavior of the dissipative system. From the physical point of view, the global attractor of the dissipative equation (1) represents the permanent regime that can be observed when the excitation starts from any point in natural energy space, and its dimension represents the number of degrees of freedom of the related turbulent phenomenon and thus the level of complexity concerning the flow. All the information concerning the attractor and its dimension from the qualitative nature to the quantitative nature then yields valuable information concerning the flows that this physical system can generate. On the physical and numerical sides, this dimension gives one an idea of the number of parameters and the size of the computations needed in numerical simulations. However, the global attractor may possess an essential drawback; namely, the rate of attraction may be arbitrarily slow and it can not be estimated in terms of physical parameters of the system under consideration. While the exponential attractor overcomes the drawback because not only it has finite fractal dimension but also its contractive rate is exponential and measurable in terms of the physical parameters, the purpose of the present paper is to establish the existence of an exponential attractor in supercritical case. Our result (see Theorem 8 below) in this paper extends the corresponding result in [28].

In comparison with the results in [17, 18], the contribution of the paper lies in that

(1) the exponential attractor is established in natural energy space E in supercritical case. See Theorem 8;

(2) the critical case $p = \widetilde{p}$ is solved in E_1. In the concrete, when $1 \le p \le \widetilde{p}$, the global and exponential attractor in E_1 is established, and the higher regularity of the global attractor is obtained. See Theorem 15;

(3) the restriction $N \le 5$ is removed in subcritical case. See Theorem 15.

The plan of the paper is as follows. In Section 2, the global existence of the weak solutions is discussed by the energy method and the existence of global attractor is established. In Section 3, the exponential attractor is established for supercritical case. In Section 4, global attractor and the exponential attractor are established for nonsupercritical case.

2. Global Existence of Weak Solutions

For brevity, we use the following abbreviations:

$$\begin{aligned} L^p &= L^p(\Omega), \\ H^k &= H^k(\Omega), \\ H &= L^2, \\ V_2 &= H_0^2, \\ \|\cdot\| &= \|\cdot\|_{L^2}, \\ \|\cdot\|_p &= \|\cdot\|_{L^p}, \end{aligned} \tag{10}$$

with $p \ge 1$, where H^k are the L^2-based Sobolev spaces and H_0^k are the completion of $C_0^\infty(\Omega)$ in H^k for $k > 0$. The notation $(\cdot,\cdot)$ for the H-inner product will also be used for the notation of duality pairing between dual spaces and $C(\cdots)$ denotes positive constants depending on the quantities appearing in the parenthesis.

We define the operator $A : V_2 \to V_2'$ (the dual space of V_2),

$$(Au, v) = (\Delta u, \Delta v) \quad \text{for any } u, v \in V_2. \tag{11}$$

Then, the operators A^s ($s \in \mathbb{R}$) are strictly positive and the spaces $V_s = D(A^{s/4})$ are Hilbert spaces with the scalar products and the norms

$$(u, v)_s = \left(A^{s/4}u, A^{s/4}v\right), \qquad \|u\|_{V_s} = \left\|A^{s/4}u\right\|, \tag{12}$$

respectively. Obviously,

$$\|u\|_{V_2} = \left\|A^{1/2}u\right\| = \|\Delta u\|, \qquad \|u\|_{V_1} = \left\|A^{1/4}u\right\| = \|\nabla u\|. \tag{13}$$

Rewriting (1) in the operator equation and applying $A^{-1/2}$ to the resulting expression, we get the Cauchy problem equivalent to problem (1), (2), and (3):

$$A^{-1/2}u_{tt} + \left(I + A^{1/2}\right)u + u_t + g(u) = A^{-1/2}f, \tag{14}$$

$$u(0) = u_0, \qquad u_t(0) = u_1. \tag{15}$$

For each $s \in \mathbb{R}$, $V_s = D(A^{s/4})$, we denote the Banach space

$$E = V_2 \times H, \qquad E_1 = V_1 \times H, \tag{16}$$

which is equipped with the usual graph norm,

$$\|(u, v)\|_E^2 = \|u\|_{V_2}^2 + \|v\|^2. \tag{17}$$

Theorem 1. *Assume that* (H_1) $g \in C^1(\mathbb{R})$,

$$\begin{aligned} &|g(s)| \le K_1\left(|s|^p + 1\right), \\ &\left|g'(s)\right| \le K_2\left(|s|^{p-1} + 1\right), \\ &\left|g'(s_1) - g'(s_2)\right| \\ &\quad \le K_3\left(|s_1|^{p-1-\gamma} + |s_2|^{p-1-\gamma} + 1\right)|s_1 - s_2|^{\gamma}, \end{aligned} \tag{18}$$

where

$$0 < \gamma < p - 1, \quad 1 < p < 2, \qquad \gamma = 1, \quad p \ge 2, \tag{19}$$

where $(a)^+ = \max\{0, a\}$, $K_i > 0$, $i = 1, 2, 3$, $1 \le p \le p^* \equiv (N+2)/(N-2)$, *and* $N \ge 3$.

(H_2) *Consider*

$$\liminf_{|s|\to+\infty} \frac{G(s)}{|s|^2} \ge 0, \qquad \liminf_{|s|\to+\infty} \frac{sg(s) - \rho G(s)}{|s|^2} \ge 0, \tag{20}$$

where $G(s) = \int_0^s g(\tau)d\tau$, $0 < \rho < 2$.

(H_3) $(u_0, u_1) \in E$, $f \in V_{-1}$.

Then problem (14) and (15) admits a unique weak solution u, *with* $(u, u_t) \in C_b(R^+, V_2 \times H)$. *More precisely, the solution* u *possesses the following properties:*

(i) *There exists a small positive constant* δ *such that*

$$\|(u, u_t)\|_E^2 + \int_t^{t+1} \|u_t(\tau)\|^2 d\tau \le C\left(\|(u_0, u_1)\|_E\right)e^{-\delta t} + C\left(\|f\|_{V_{-1}}\right), \quad t \ge 0. \tag{21}$$

(ii) *When* $1 \le p < p^*$, *the solution is Lipschitz continuous in the weaker space* E *as* $N \le 5$; *that is,*

$$\|(z(t), z_t(t))\|_E^2 \le C(R)e^{kt}\|(z(0), z_t(0))\|_E^2, \quad t \ge 0, \tag{22}$$

for some $k > 0$, *where* $z = u - v$, u *and* v *are, respectively, the weak solutions of (14) corresponding to initial data* (u_0, u_1) *and* (v_0, v_1).

Remark 2. The formula (20) implies that every $\eta > 0$; there exists $C_\eta > 0$, $\widetilde{C}_\eta > 0$ such that

$$G(s) + \eta|s|^2 \ge -C_\eta, \qquad sg(s) - \rho G(s) + \eta|s|^2 \ge -\widetilde{C}_\eta, \tag{23}$$

where $G(s) = \int_0^s g(\tau)d\tau$.

Lemma 3 (see [29]). *Let* X, B, *and* Y *be the Banach spaces,* $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$,

$$\begin{aligned} &W = \left\{u \in L^p(0, T; X) \mid u_t \in L^1(0, T; Y)\right\}, \quad \text{with } 1 \le p < \infty, \\ &W_1 = \left\{u \in L^\infty(0, T; X) \mid u_t \in L^r(0, T; Y)\right\}, \quad \text{with } r > 1. \end{aligned} \tag{24}$$

Then,

$$W \hookrightarrow\hookrightarrow L^p(0, T; B), \qquad W_1 \hookrightarrow\hookrightarrow C([0, T]; B). \tag{25}$$

Proof of Theorem 1. We first obtain a priori estimate to the solutions of problem (14) and (15).

Let $v = u_t + \epsilon u$ and rewrite (1); we have

$$\begin{aligned} &A^{-1/2}\left(v_t - \epsilon v + \epsilon^2 u\right) + \left(I + A^{1/2}\right)u + v - \epsilon u + g(u) \\ &\quad = A^{-1/2}f, \\ &v(0) = u_1 + \epsilon u_0 = v_0. \end{aligned} \tag{26}$$

Using the multiplier u_t in (26), we get

$$\frac{d}{dt}H_1(u, v) + K_1(u, v) = 0, \quad t > 0, \tag{27}$$

where

$$\begin{aligned} H_1(u,u_t) &= \frac{1}{2}\left(\left\|A^{-1/4}v\right\|^2 + \epsilon^2\left\|A^{-1/4}u\right\|^2 + \left\|A^{1/4}u\right\|^2\right. \\ &\quad \left. + (1-\epsilon)\|u\|^2 + 2\int_\Omega G(u)\,dx - 2\left(A^{-1/2}f,u\right)\right) \\ &\geq C_1\left(\left\|A^{-1/4}u_t\right\|^2 + \left\|A^{-1/4}u\right\|^2 + \left\|A^{1/4}u\right\|^2 + \|u\|^2\right) \\ &\quad - C_0, \\ K_1(u,v) - \epsilon H_1(u,v) &\geq C\left(\|v\|^2 + \left\|A^{-1/4}v\right\|^2\right. \\ &\quad \left. + \left\|A^{1/4}u\right\|^2\right) - C_0. \end{aligned} \tag{28}$$

Hence,

$$\frac{d}{dt}H_1(u,v) + \epsilon H_1(u,v) \leq -C_0, \quad t \geq 0. \tag{29}$$

Applying the Gronwall lemma to (29),

$$\left\|A^{1/4}u\right\|^2 + \left\|A^{-1/4}u_t\right\|^2 \leq C_1 e^{-\delta t} + C_0, \tag{30}$$

where $C_0 = (\|f\|_{V_{-1}})$, $C_1 = (\|(u_0,u_1)\|_{E_1})$, $\delta = \epsilon \in (0,1)$.

Using the multiplier $A^{1/2}u_t$, $A^{1/2}u$ in (26),

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u_t\|^2 + \left\|A^{1/4}u\right\|^2 + \left\|A^{1/2}u\right\|^2 - 2(u,f)\right) \\ &\quad + \left\|A^{1/4}u_t\right\|^2 + \left(A^{1/2}u_t, g(u)\right) = 0, \quad t > 0, \end{aligned} \tag{31}$$

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\left\|A^{1/4}u\right\|^2 + (u_t,u)\right) + \left\|A^{1/4}u\right\|^2 \\ &\quad + \left(A^{1/2}u, g(u)\right) = \|u_t\|^2 + (u,f), \quad t > 0, \end{aligned} \tag{32}$$

(31) + ϵ(32); we have

$$\frac{d}{dt}H_2(u) + K_2(u) = 0, \quad t > 0. \tag{33}$$

Obviously,

$$\begin{aligned} H_2(u) &= \frac{1}{2}\left[\|u_t\|^2 + (1+\epsilon)\left\|A^{1/4}u\right\|^2 + \left\|A^{1/2}u\right\|^2\right] \\ &\quad + \epsilon(u,u_t) - (u,f) \\ &\backsim \left\|A^{1/4}u\right\|^2 + \|u_t\|^2 + \left\|A^{1/2}u\right\|^2, \\ K_2(u) &= \left\|A^{1/4}u_t\right\|^2 - \epsilon\|u_t\|^2 + \left(A^{1/4}g(u), A^{1/4}u_t\right) \\ &\quad + \epsilon\left(\left\|A^{1/2}u\right\|^2 + \left(A^{1/2}u, g(u)\right) - (u,f)\right) \\ &\geq C\left(\left\|A^{1/4}u_t\right\|^2 + \left\|A^{1/2}u\right\|^2\right) - C_1 e^{-\delta t} - C_0. \end{aligned} \tag{34}$$

Indeed,

$$\begin{aligned} \left|\left(A^{1/4}g(u), A^{1/4}u_t\right)\right| &\leq \frac{1}{4}\left\|A^{1/4}u_t\right\|^2 + \frac{\varepsilon}{4}\left\|A^{1/2}u\right\|^2 \\ &\quad + C, \\ \left|\left(A^{1/2}u, g(u)\right)\right| &\leq \frac{1}{4}\left\|A^{1/2}u\right\|^2 + C, \end{aligned} \tag{35}$$

where $1 \leq p < N/(N-2)^+$, $H^s \hookrightarrow L^{2p}$; $N/(N-2)^+ \leq p < (N+2)/(N-2)^+$, $V_{2-\delta} \hookrightarrow L^{2(p+1)}$, and $H^1 \hookrightarrow L^{p+1}$

$$\frac{d}{dt}H_2(u) + \delta H_2(u) + \delta\left\|A^{1/4}u_t(t)\right\|^2 \leq C_1 e^{-\delta t} + C_0. \tag{36}$$

Thus

$$\left\|A^{1/2}u\right\|^2 + \left\|A^{1/4}u\right\|^2 + \|u_t\|^2 \leq C_2 e^{-\delta t} + C_0, \quad t > 0. \tag{37}$$

That is,

$$\begin{aligned} \|(u,u_t)\|_E + \int_t^{t+1}\left\|A^{1/4}u_t(\tau)\right\|^2 d\tau &\leq C_2 e^{-\delta t} + C_0, \\ &\quad t > 0. \end{aligned} \tag{38}$$

It follows from (14) and (38) that

$$\begin{aligned} u_{tt} &= f - A^{1/2}u - A^{1/2}u_t - Au - A^{1/2}g(u) \\ &\in L^\infty\left(R^+, V_{-2}\right). \end{aligned} \tag{39}$$

Now, we look for the approximate solutions u^n of problem (14) and (15) of the form

$$u^n(t) = \sum_{j=1}^n T_{jn}(t)\,w_j, \tag{40}$$

where $Aw_j = \lambda_j w_j$, $j = 1,2,\ldots,\{w_j\}$ is an orthonormal basis in H, and at the same time an orthogonal one in V_2, and $T_{jn}(t) = (u^n, w_j)$ with

$$\begin{aligned} &\left(A^{-1/2}u^n_{tt}, w_j\right) + \left(\left(I + A^{1/2}\right)u^n, w_j\right) \\ &\quad + 2\eta\left(A^{1/4}u^n_t, w_j\right) + \left(u^n_t, w_j\right) + \left(g(u^n), w_j\right) \\ &\quad = \left(A^{-1/2}f, w_j\right), \quad t > 0,\ j = 1,\ldots,n, \\ &\left(u^n(0), u^n_t(0)\right) = \left(u_{0n}, u_{1n}\right) \longrightarrow \left(u_0, u_1\right) \quad \text{in } E. \end{aligned} \tag{41}$$

Obviously, the estimate (38) is valid for u^n. So we can extract a subsequence, still denoted by $\{u^n\}$, such that

$$\begin{aligned} &\left(u^n, u^n_t\right) \longrightarrow (u, u_t) \\ &\qquad \text{weakly}^* \text{ in } L^\infty_{\text{loc}}\left(\mathbb{R}^+; E\right); \\ &u^n_{tt} \longrightarrow u_{tt} \quad \text{weakly}^* \text{ in } L^\infty_{\text{loc}}\left(\mathbb{R}^+; V_{-2}\right); \\ &\left(u^n(t), u^n_t(t)\right) \rightharpoonup \left(u(t), u_t(t)\right) \quad \text{for } t \geq 0. \end{aligned} \tag{42}$$

Applying Lemma 3 to (33), we have

$$(u^n, u_t^n) \longrightarrow (u, u_t) \quad \text{in } C_w([0,T]; E). \tag{43}$$

Indeed, when $1 \le p \le N/(N-2)^+$, we have $H_0^1 \hookrightarrow L^{2p}$; hence

$$\begin{aligned}\int_0^t \left|\left(g(u^n) - g(u), w_j\right)\right| d\tau \\ \le C\int_0^t \left(\|u^n - u\| + \left\|A^{1/4}u^n - A^{1/4}u\right\|\right) d\tau \longrightarrow 0,\end{aligned} \tag{44}$$

when $N/(N-2)^+ < p < (N+2)/(N-2)^+$, $V_{2-\delta} \hookrightarrow L^{2(p+1)}$ and by virtue of the interpolation theorem,

$$\begin{aligned}\int_0^t \left|\left(g(u^n) - g(u), w_j\right)\right| d\tau \le C\int_0^t \Big(\|u^n - u\| \\ + \left\|A^{1/4}(u^n - u)\right\|^{\theta} \left\|A^{1/2}(u^n - u)\right\|^{1-\theta}\Big) d\tau \\ \longrightarrow 0.\end{aligned} \tag{45}$$

Letting $n \to \infty$ in (41) we see that u is a weak solution of problem (14) and (15), with $(u, u_t) \in C_w(\mathbb{R}^+; E)$.

Integrating (31) over (t_0, t),

$$\begin{aligned}&\left(\|u_t(t)\|^2 + \left\|A^{1/4}u(t)\right\|^2 + \left\|A^{1/2}u(t)\right\|^2\right) \\ &- \left(\|u_t(t_0)\|^2 + \left\|A^{1/4}u(t_0)\right\|^2 + \left\|A^{1/2}u(t_0)\right\|^2\right) \\ &= 2\int_{t_0}^t (u, f)\, dt + 2\int_{t_0}^t \left\|A^{1/4}u_t\right\|^2 dt \\ &+ 2\int_{t_0}^t \left(A^{1/4}g(u), A^{1/4}u_t\right) dt \longrightarrow 0, \quad \text{as } t \longrightarrow t_0.\end{aligned} \tag{46}$$

Obviously,

$$\begin{aligned}\|(u, u_t)(t)\|_{V_2\times H} - \|(u, u_t)(t_0)\|_{V_2\times H} \longrightarrow 0, \\ t \longrightarrow t_0;\end{aligned} \tag{47}$$

we prove that $(u, u_t) \in C([0,T], V_2 \times H)$, $u_{tt} \in C([0,T], V_{-1})$ and (22).

(ii) Now, we show that (u, u_t) is Lipschitz continuous in the weak space E.

In fact, let u, v be two solutions of problem (14) and (15) as shown above corresponding to initial data u_0, u_1 and v_0, v_1, respectively. Then $z = u - v$ solves

$$\begin{aligned}&A^{-1/2}z_{tt} + \left(I + A^{1/2}\right)z + z_t + g(u) - g(v) = 0, \\ &z(0) = u_0 - v_0 \equiv z_0, \\ &z_t(0) = u_1 - v_1 \equiv z_1.\end{aligned} \tag{48}$$

Using the multiplier $A^{1/2}z_t$ in (48),

$$\begin{aligned}H_3(z, z_t) &= \frac{1}{2}\left(\|z_t\|^2 + \left\|A^{1/4}z\right\|^2 + \left\|A^{1/2}z\right\|^2\right) \\ &\quad - \left(A^{1/2}z_t, g(u) - g(v)\right) \\ &\le \frac{1}{2}\left\|A^{1/4}z_t\right\|^2 + C(R)\left\|A^{1/2}z\right\|^2.\end{aligned} \tag{49}$$

We get

$$\begin{aligned}&\frac{d}{dt}H_3(z, z_t) + \kappa H_3(z, z_t) \le C(R)\|z\|^2, \\ &\|(z, z_t)(t)\|_E^2 \le C(R,T)\|(z, z_t)(0)\|_E^2, \\ &\qquad 0 \le t \le T, (22) \\ &\|(z, z_t)(t)\|_E^2 \\ &\quad \le C(R)e^{-\kappa t}\|(z, z_t)(0)\|_E^2 + \int_0^t e^{-\kappa(t-\tau)}\|z(\tau)\|^2 d\tau, \\ &\qquad t > 0.\end{aligned} \tag{50}$$

Theorem 1 is proved. Under the assumptions of Theorem 1, with $1 \le p < p^*$, $N \le 5$, we can define the solution operator $S(t) : E \to E$,

$$\begin{aligned}S(t)(u_0, u_1) = (u(t), u_t(t)) \\ \text{for every } (u_0, u_1) \in E,\ t \ge 0,\end{aligned} \tag{51}$$

where u is the weak solution of problem (14) and (15). Theorem 1 shows that $\{T(t)\}$ constitutes a semigroup on E, which is Lipschitz continuous in E. □

Theorem 4 (existence of the global attractor). *Under the same assumptions of Theorem 1, with $1 \le p < p^*$, $\{T(t)\}$ has a global attractor $\mathscr{A}$ in E and $\mathscr{A} \subset K$, where K is bounded set in $E_\sigma = V_{\sigma+2} \times V_\sigma$, $0 < \sigma < 1$.*

Proof of Theorem 4. Estimate (38) implies that the ball

$$B_0 = \left\{(u, v) \in E \mid \|(u, v)\|_E^2 \le C_0\right\} \tag{52}$$

is an absorbing set of the semigroup $S(t)$ in E. For every bound B in E,

$$\operatorname{dist}(S(t)B, B_0) \le C(B)e^{-\delta t}, \quad \forall B \subset E. \tag{53}$$

Let : $S(t) = S_1(t) + S_2(t)$, where $S_2(t) : V_2 \times H \to V_2 \times H$, $S_2(t)(u_0, u_1) = (\overline{u}, \overline{u}_t)$ and

$$\begin{aligned}&A^{-1/2}\overline{u}_{tt} + \left(I + A^{1/2}\right)\overline{u} + \overline{u}_t = 0, \\ &\qquad t > 0,\ \overline{u}(0) = u_0,\ \overline{u}_t(0) = u_1.\end{aligned} \tag{54}$$

It is easy to get

$$\|S_2(t)\|_{\mathscr{L}(E)} \le Ce^{-\delta t}. \tag{55}$$

$S_1(t) = S(t) - S_2(t)$, $S_1(t) : E \to E$, $S_1(t)(u_0, u_1) = (\widehat{u}, \widehat{u}_t)$ solves

$$\begin{aligned}&A^{-1/2}\widehat{u}_{tt} + \left(I + A^{1/2}\right)\widehat{u} + \widehat{u}_t = A^{-1/2}f - g(u), \\ &\qquad t > 0,\ \widehat{u}(0) = 0,\ \widehat{u}_t(0) = 0,\end{aligned} \tag{56}$$

where $u \in C_b(R^+, V_2)$.

Let $w = \widehat{u}_t$,

$$A^{-1/2}w_{tt} + \left(I + A^{1/2}\right)w + w_t = -g'(u)u_t,$$
$$w(0) = 0 = w_0, \tag{57}$$
$$w_t(0) = f - A^{1/2}g(u_0) = w_1,$$

because $f \in V_{\sigma-2}$, $A^{1/2}g(u_0) \in V_{\sigma-2}$, $w_1 \in V_{\sigma-2}$, $g'(u)u_t \in C_b(R^+, V_{\sigma-2})$.

Therefore

$$(\widehat{u}_t, \widehat{u}_{tt}) \in C_b\left(R^+, V_\sigma \times V_{\sigma-2}\right). \tag{58}$$

We know $(I + A^{1/2})\widehat{u} = A^{-1/2}f - g(u) - A^{-1/2}\widehat{u}_{tt} - \widehat{u}_t \in C_b(R^+, V_\sigma)$, $\widehat{u} \in C_b(R^+, V_{\sigma+2})$ so

$$(\widehat{u}, \widehat{u}_t) \in C_b\left(R^+, V_{\sigma+2} \times V_\sigma\right). \tag{59}$$

Thus $\bigcup_{t\geq 0, \|(\widehat{u},\widehat{u}_t)\|_{E_\sigma} \leq R}(\widehat{u}, \widehat{u}_t)$ is bounded in $V_{\sigma+2} \times V_\sigma$. $V_{\sigma+2} \times V_\sigma \hookrightarrow\hookrightarrow V_2 \times H$,

$$K = \overline{\bigcup_{t\geq 0, \|(\widehat{u},\widehat{u}_t)\|_{E_\sigma} \leq R} (\widehat{u}, \widehat{u}_t)^E} \tag{60}$$

is compact in $E = V_2 \times H$. For every bounded set $B \subset E$,

$$\begin{aligned} &\operatorname{dist}\{S(t)B, K\} \\ &\quad = \sup_{(u_0,u_1)\in B} \inf_{(\widehat{u},\widehat{u}_t)\in K} \operatorname{dist}\{S(t)(u_0,u_1), (\widehat{u},\widehat{u}_t)\} \\ &\quad \leq \sup_{(u_0,u_1)\in B} \|(\overline{u}, \overline{u}_t)\|_E \longrightarrow 0. \end{aligned} \tag{61}$$

Therefore $S(t)$ has the global attractor $\mathscr{A} = \omega(B_0)$ and

$$\operatorname{dist}\{\mathscr{A}, K\} = \operatorname{dist}\{S(t)\mathscr{A}, K\} \longrightarrow 0 \quad t \longrightarrow \infty; \tag{62}$$

that is $\mathscr{A} \subset K$. This completes the proof. □

3. Exponential Attractor

Definition 5. The set $\mathbb{A}_{\exp} \subset E$ is called an exponential attractor for the solution semigroup $S(t)$ of acting on the energy space E if

(i) the set $\mathbb{A}_{\exp}$ is a compact set in E;

(ii) $\mathbb{A}_{\exp}$ is forward invariant set; that is, $S(t)\mathscr{A} \subset \mathscr{A}, t \geq 0$;

(iii) $\mathbb{A}_{\exp}$ attracts exponentially the images of all bounded set in E; that is,

$$\operatorname{dist}_E\left\{S(t)B, \mathscr{A}_{\exp}\right\} \leq Q\left(\|B\|_E\right)e^{-\gamma t}; \tag{63}$$

for all bounded set $B \subset E$;

(iv) it has finite fractal dimension in E; that is, $\dim_f\{\mathscr{A}_{\exp}, E\} < +\infty$.

From Theorem 1, estimate (38) implies that the ball

$$B_R = \{\zeta \in E \mid \|\zeta\|_E \leq R\} \tag{64}$$

is an absorbing set of the semigroup $T(t)$ in E for $R > C_0$. Without loss of generality we assume that B_R is a forward invariant set. Let

$$\mathscr{B}_R = \left[\bigcup_{t\geq t_0+1} T(t)B_R\right]_E, \tag{65}$$

where $t_0 > 0$ is chosen such that $T(t)B_R \subset B_R$ for $t \geq t_0$ and $[\,]_X$ stands for the closure in space X. Obviously, the set $\mathscr{B}_R$ is bounded closed set in $E, T(t)\mathscr{B}_R \subset \mathscr{B}_R$, $t \geq 0$, and it is also an absorbing set of $T(t)$. $\mathscr{B}_R$ constitutes a complete metric space (with the E norm) and one sees from (22) that the solution semigroup $T(t)$ is continuous on $\mathscr{B}_R$, and the system $(T(t), \mathscr{B}_R)$ constitutes a dissipative dynamical system.

Lemma 6 (see [19]). *Let X be a Banach space and M a bounded closed set in X. Assume that the mapping $V : M \to M$ possesses the following properties:*

(i) *V is Lipschitz on M; that is, there exists $L > 0$ such that*

$$\|Vv_1 - Vv_2\| \leq L\|v_1 - v_2\|, \quad v_1, v_2 \in M; \tag{66}$$

(ii) *there exist compact seminorms $n_1(x)$ and $n_2(x)$ on X such that*

$$\begin{aligned} \|Vv_1 - Vv_2\| &\leq \eta\|v_1 - v_2\| \\ &\quad + K\left[n_1(v_1 - v_2) + n_2(Vv_1 - Vv_2)\right], \end{aligned} \tag{67}$$

for any $v_1, v_2 \in M$, where $0 < \eta < 1$ and $K > 0$ are constants. Then for any $\kappa > 0$ and $\delta \in (0, 1-\eta)$, there exists a positively invariant compact set $A_{\kappa,\delta} \subset M$ of finite fractal dimension such that

$$\operatorname{dist}\left(V^k M, A_{\kappa,\delta}\right) \leq q^k, \quad k = 1, 2, \ldots, \tag{68}$$

where $q = \eta + \delta < 1$, and

$$\begin{aligned} \dim_f A_{\kappa,\delta} &\leq \left(\ln\frac{1}{\delta+\eta}\right)^{-1} \\ &\quad \cdot \left(\ln m_0\left(\frac{2K\left(1+L^2\right)^{1/2}}{\delta}\right) + \kappa\right), \end{aligned} \tag{69}$$

where $m_0(R)$ is the maximal number of pairs (x_i, y_i) in $X \times X$ possessing the properties

$$\begin{aligned} \|x_i\|^2 + \|y_i\|^2 &\leq R^2, \\ n_1(x_i - x_j) + n_2(y_i - y_j) &> 1, \\ i &\neq j. \end{aligned} \tag{70}$$

Lemma 7. *Let X, Y be the metric spaces and let the mapping $h : X \to Y$ be θ-Hölder continuous on the set $B \subset X$. Then*

$$\dim_f\{h(B), Y\} \leq \frac{1}{\theta}\dim_f\{B, X\}. \tag{71}$$

Theorem 8. *Let the assumptions of Theorem 1 be in force, with $1 \le p < p^*$. Then the solution semigroup $T(t)$ has an exponential attractor $\mathscr{A}_{exp}$ in E.*

Proof. Define the operator

$$V^k = S(kT) : \mathscr{B}_R \longrightarrow \mathscr{B}_R, \quad k \in \mathbb{Z}^+. \tag{72}$$

We show that the discrete system $(V^k, \mathscr{B}_R)$ has an exponential attractor. □

Definition 9. We introduce the functional space

$$\begin{aligned} W(0,T) = \{ & z \in L^2(0,T;V_2) \mid z_t \\ & \in L^2(0,T;H), \|\xi_z\|_W^2 < \infty \}, \end{aligned} \tag{73}$$

equipped with the norm

$$\|\xi_z\|_W^2 = \int_0^T \|(z(t), z_t(t))\|_E^2 \, dt \tag{74}$$

and the functional space

$$H_T = E \times W(0,T), \tag{75}$$

equipped with the usual graph norm; that is

$$\|U\|_{H_T}^2 = \|\eta\|_E^2 + \|\xi_z\|_W^2, \quad \forall U = (\eta, \xi_z) \in H_T. \tag{76}$$

Obviously, the spaces $W(0,T)$ and H_T are Banach spaces. Let the set

$$\begin{aligned} B_T = \{ & (\xi_u(0), \xi_u(t), \ t \in [0,T]) \mid \xi_u(0) \in \mathscr{B}_r, \ \xi_u(t) \\ & = S(t)\xi_u(0)\}. \end{aligned} \tag{77}$$

Define the operator

$$\begin{aligned} & \mathbb{V} : B_T \longrightarrow H_T, \\ & \mathbb{V}U = (S(T)\xi_u(0), \xi_u(T+\cdot)) \\ & \quad = (S(T)\xi_u(0), S(t+T)\xi_u(0), \ t \in [0,T]), \end{aligned} \tag{78}$$

where $U = (\xi_u(0), \xi_u(\cdot)) \in B_T$ and in the following $\xi_u(\cdot)$ means $\xi_u(t)$, $t \in [0,T]$.

Lemma 10. *The set B_T is a bounded closed set in H_T.*

Proof. Obviously, B_T is bounded in H_T. For any sequence $\{U^n\} \subset B_T$,

$$U^n = (\xi_{u^n}(0), \xi_{u^n}(\cdot)) \longrightarrow U = (\xi_u(0), \xi_v(\cdot)) \text{ in } B_T. \tag{79}$$

Since $\xi_{u^n}(0) \in \mathscr{B}_R$ and $\mathscr{B}_R$ is closed in E, $\xi_u(0) \in \mathscr{B}_0$. By the Lipschitz continuity of $S(t)$ in E,

$$\begin{aligned} \|\xi_{u^n}(t) - \xi_u(t)\|_E^2 &= \|S(t)\xi_{u^n}(0) - S(t)\xi_u(0)\|_E^2 \\ &\le C(R,T)\|\xi_{u^n}(0) - \xi_u(0)\|_E^2 \\ &\longrightarrow 0, \end{aligned} \tag{80}$$

so by the uniqueness of the limit; $\xi_v(\cdot) = \xi_u(t)$, that is, $U = (\xi_u(0), \xi_v(\cdot)) \in B_T$, where B_T is closed in H_T. □

Lemma 10 implies that B_T is complete with respect to the topology of H_T, and the dynamical system $(\mathbb{V}^k, B_T)$ constitutes a discrete dissipative dynamical system.

Lemma 11. *Under the same assumptions of Theorem 1, then discrete dissipative dynamical system $(\mathbb{V}^k, B_T)$ has an exponential attractor $\mathbb{A}$.*

Proof. Obviously, $\mathbb{V}B_T \subset B_T$

$$\begin{aligned} \forall U_1 &= (\xi_{u_1}(0), \xi_{u_1}(\cdot)), \\ U_2 &= (\xi_{u_2}(0), \xi_{u_2}(\cdot)), \\ z &= u_1 - u_2; \end{aligned} \tag{81}$$

the inequality (93) holds; then integrating (93) over $(T, 2T)$ we get

$$\begin{aligned} & \int_T^{2T} \left(\|z(t)\|_{V_2}^2 + \|z_t(t)\|_H^2 \right) dt \\ & \quad \le C \int_T^{2T} e^{-\kappa t} dt \left(\|z_0\|_{V_2}^2 + \|z_1\|_H^2 \right) \\ & \quad + CT \int_0^{2T} \|z(\tau)\|^2 \, d\tau. \end{aligned} \tag{82}$$

Hence,

$$\begin{aligned} & \|\mathbb{V}U_1 - \mathbb{V}U_2\|_{H_T}^2 \\ & \quad = \|S(T)\xi_{u_1}(0) - S(T)\xi_{u_2}(0)\|_E^2 + \|\xi_z(t+T)\|_W^2 \\ & \quad \le Ce^{-\kappa T} \|\xi_{u_1}(0) - \xi_{u_2}(0)\|_E^2 \\ & \qquad + C \int_0^T e^{-\kappa(T-\tau)} \|z(\tau)\|^2 \, d\tau + \int_T^{2T} \|\xi_z(t)\|_E^2 \, dt \\ & \quad \le \eta_T \|\xi_{u_1}(0) - \xi_{u_2}(0)\|_E^2 + K_T \int_0^{2T} \|z(\tau)\|^2 \, d\tau \\ & \quad \le \eta_T \|U_1 - U_2\|_{H_T}^2 \\ & \qquad + K_T (n_1(U_1 - U_2) + n_1(\mathbb{V}U_1 - \mathbb{V}U_2)), \end{aligned} \tag{83}$$

where $\xi_z(t) = \xi_{u_1}(t) - \xi_{u_2}(t)$,

$$\begin{aligned} & \eta_T = 2Ce^{-\kappa T} + C \int_T^{2T} e^{-\kappa t} dt, \quad K_T = C(T+2), \\ & n_1(U) = \int_0^T \|u(t)\|^2 \, dt, \quad U = (\xi_u(0), \xi_u(\cdot)) \in B_T. \end{aligned} \tag{84}$$

It follows from (83) that

$$\|\mathbb{V}U_1 - \mathbb{V}U_2\|_{H_T}^2 \le a_T \|U_1 - U_2\|_{H_T}^2, \tag{85}$$

where $a_T = \eta_T + CK_T \int_0^{2T} e^{\kappa t} dt$. Since $W(0,T) \hookrightarrow\hookrightarrow L^2(0,T;H)$, the seminorm $n_1(U)$ is compact in H_T. Taking

$T : 0 < \eta_T < 1$ and making use of Lemma 6, we get the conclusion of Lemma 11. That is, the discrete dynamical system (V^k, M) possesses an exponential attractor $A_{\kappa,\delta}$. Define the project operator

$$\begin{aligned} &\Pi : B_T \longrightarrow \mathscr{B}_R, \\ &\Pi U = \xi_u(0), \\ &U = (\xi_u(0), \xi_u(\cdot)) \in B_T. \end{aligned} \tag{86}$$

□

Lemma 12. *$A = \Pi\mathbb{A}$ is an exponential attractor of the discrete dynamical system $(V^k, \mathscr{B}_R)$.*

Proof. (1) A is compact because A is the image of the compact set $\mathbb{A}$ under the continuous mapping Π.

(2) $\mathbb{V}^k A \subset A$; we have $\mathbb{V}^k\mathbb{A} \subset \mathbb{A}$; thus $V^k A = \Pi\mathbb{V}^k\mathbb{A} \subset \Pi\mathbb{A} = A$.

(3) Obviously,

$$\operatorname{dist}_E\left\{V^k\mathscr{B}_R, A\right\} \le \operatorname{dist}_{H_T}\left\{\mathbb{V}^k B_T, \mathbb{A}\right\} \le Cq^k, \quad 0 < q < 1; \tag{87}$$

for some $0 < q < 1$ (see Lemma 11).

(4) $\dim_f\{A, E\} \le \dim_f\{\mathbb{A}, H_T\} < \infty$.

Hence, A is a desired exponential attractor. Lemma 12 is proved.

Let

$$\mathbb{A}_{\exp} = \bigcup_{0\le t\le S} S(t) A. \tag{88}$$

By the method used in [11], one easily knows that $\mathbb{A}_{\exp}$ is an exponential attractor of $(S(t), \mathscr{B}_R)$, with E topology. So by the definition of the exponential attractor, there exists a constant $\kappa > 0$, such that

$$\operatorname{dist}_E\left\{S(t)\mathscr{B}_R, \mathbb{A}_{\exp}\right\} \le Ce^{-\kappa t}, \quad t > 0. \tag{89}$$

Since the set $\mathbb{A}_{\exp} \subset \mathscr{B}_0$ is bounded in E, we claim that $\mathbb{A}_{\exp}$ is an exponential attractor of the system $(S(t), E)$. Indeed, (i) obviously, $\mathbb{A}_{\exp}$ is forward invariant; (ii) define the project operator

$$\begin{aligned} &F : [0,T] \times A \longrightarrow \mathscr{B}_R, \\ &F(t, \xi_u) = \xi_u(t), \quad \xi_u(t) \in \mathscr{B}_R, \ t \in [0,T], \\ &\left\|F(t_1, \xi_u) - F(t_2, \xi_u)\right\|_E \\ &\quad \le C\left(\int_{t_1}^{t_2} \left\|\xi_u'(\tau)\right\|_E^2 d\tau\right)^{1/2} |t_1 - t_2|^{1/2} \\ &\quad \le C|t_2 - t_1|^{1/2}, \\ &\left\|F(t, \xi_{u_1}) - F(t, \xi_{u_2})\right\|_{E_1} \le C(R) e^{\kappa T} \left\|\xi_{u_1} - \xi_{u_2}\right\|_E \end{aligned} \tag{90}$$

for any $\xi_u, \xi_{u_1}, \xi_{u_2} \in \mathscr{B}_R$, $t, t_1, t_2 \in [0,T]$, which imply that the mapping F is 1/2-Hölder continuous. Therefore, $\mathbb{A}_{\exp} = F\{[0,T] \times A\}$ (the image of $[0,T] \times A$) is compact in E.

(iii) Consider the following:

$$\begin{aligned} \dim_f\left\{\mathbb{A}_{\exp}, E\right\} &\le 2\dim_f\{[0,S] \times A, \mathbb{R}^+ \times E\} \\ &\le 2\left(1 + \dim_f\{A, E\}\right) < \infty. \end{aligned} \tag{91}$$

(iv) For any $t \in \mathbb{R}^+$, there exists a $k \in \mathbb{N}^+$, such that $S(t)\mathscr{B}_R \subset T(kT)\mathscr{B}_R$ as $t \in (kT, (k+1)T]$. On account of $A = S(0)A \subset \mathbb{A}_{\exp}$,

$$\begin{aligned} &\operatorname{dist}_E\left\{S(t)\mathscr{B}_R, \mathbb{A}_{\exp}\right\} \le \operatorname{dist}_E\left\{S(kT)\mathscr{B}_R, A\right\} \\ &\quad \le \sup_{\xi_u \in \mathscr{B}_R} \inf_{\xi_v \in A} \left\|S(t)\xi_u - \xi_v\right\|_E^{1/2} \le Cq^{k/2} \le Ce^{-\kappa t/2}. \end{aligned} \tag{92}$$

Therefore, Theorem 8 is proved. □

4. Global and Exponential Attractor in Nonsupercritical Case

Theorem 13. *Let the assumptions of Theorem 1 be in force, with $1 \le p \le \tilde{p} = N/(N-2)^+$. Then problem (14)-(15) admits a unique weak solution u, with $(u, u_t) \in C_b(\mathbb{R}^+, E_1) \equiv L^\infty(\mathbb{R}^+, E_1) \cap C(\mathbb{R}^+, E_1)$, and the solution is Lipschitz continuous in $E_1 = V_1 \times H$; that is,*

$$\left\|(z(t), z_t(t))\right\|_{E_1}^2 \le Ce^{kt}\left\|(z(0), z_t(0))\right\|_{E_1}^2, \quad t \ge 0, \tag{93}$$

for some $C, k > 0$, where $z = u - v$, u and v are, respectively, the weak solutions of (14) corresponding to initial data (u_0, u_1) and (v_0, v_1).

Proof. The existence of the weak solutions can be easily proved by the same way of Theorem 1. So we only prove (93) here. Taking H-inner product by z_t in (36), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\|A^{-1/4}z_t\right\|^2 + \|z\|^2 + \left\|A^{1/4}z\right\|^2\right) + \|z_t\|^2 \\ &\quad = -(g(u) - g(v), z_t) \\ &\quad \le C\left(1 + \|u\|_{2p}^{p-1} + \|v\|_{2p}^{p-1}\right)\|z\|_{2p}\|z_t\| \\ &\quad \le \frac{1}{2}\|z_t\|^2 + C\left\|A^{1/4}z\right\|^2. \end{aligned} \tag{94}$$

Applying the Gronwall inequality to (94) we obtain (93). □

Remark 14. (i) When $1 \le p \le \tilde{p}$, by (93), define the continuous semigroup

$$\begin{aligned} &S(t) : E_1 \longrightarrow E_1, \\ &S(t)\varphi_0 = \varphi_u(t) = (u(t), u_t(t)), \end{aligned} \tag{95}$$

where $\varphi_u = (u, u_t) \in C_b(\mathbb{R}^+, E_1)$ as shown in Theorem 13.

(ii) It follows from Theorem 8 and Remark 14 that the dynamical system $(S(t), E_1)$ is dissipative; that is, it has a bounded absorbing set $\mathscr{B}_R$. Without loss of generality we assume that $\mathscr{B}_R$ is positive invariant; that is, $S(t)\mathscr{B}_R \subset \mathscr{B}_R$ for $t \ge 0$.

Theorem 15. *Let the assumptions of Theorem 13 be in force, especially when $p = \tilde{p}$, $g \in C^2(\mathbb{R})$,*

$$\left|g''(s)\right| \le C\left(1 + |s|^{p-2}\right), \quad s \in \mathbb{R} \text{ with } p \ge 2. \tag{96}$$

Then the following conclusions are valid.

(i) The solution semigroup $S(t)$ possesses in E_1 a compact global attractor $\mathscr{A}$, which has finite fractal dimension.

(ii) Any full trajectory $\nu = \{\varphi_u(t) = (u(t), u_t(t)) \mid t \in \mathbb{R}\} \subset \mathscr{A}$ possesses the property

$$\left(u, u_t, u_{tt}\right) \in L^\infty\left(\mathbb{R}; V_2 \times V_1 \times H\right), \tag{97}$$

and there exists constant $R > 0$ such that

$$\sup_{\nu \subset \mathscr{A}} \sup_{t \in \mathbb{R}} \left(\|u(t)\|_{V_2}^2 + \|u_t(t)\|_{V_1}^2 + \|u_{tt}(t)\|^2\right) \le R^2. \tag{98}$$

(iii) The global attractor $\mathscr{A}$ consists of full trajectory $\nu = \{\varphi_u(t) \mid t \in \mathbb{R}\}$ such that

$$\begin{aligned} &\lim_{t \to -\infty} \operatorname{dist}_{E_1}\{\varphi_u(t), \mathscr{N}\} = 0, \\ &\lim_{t \to +\infty} \operatorname{dist}_{E_1}\{\varphi_u(t), \mathscr{N}\} = 0, \end{aligned} \tag{99}$$

where $\mathscr{N}$ is the set of all fixed points of $T(t)$; that is,

$$\mathscr{N} = \left\{(u, 0) \in E_1 \mid \left(I + A^{1/2}\right)u + g(u) = A^{-1/2} f\right\}. \tag{100}$$

Furthermore, for any $\zeta \in E_1$,

$$\lim_{t \to +\infty} \operatorname{dist}_{E_1}\{S(t)\zeta, \mathscr{N}\} = 0. \tag{101}$$

(iv) The semigroup $S(t)$ has in E_1 an exponential attractor.

Lemma 16. *Let $y : \mathbb{R}^+ \to \mathbb{R}^+$ be an absolutely continuous function satisfying*

$$\frac{d}{dt} y(t) + 2\epsilon y(t) \le h(t)\, y(t) + z(t), \quad t > 0, \tag{102}$$

where $\epsilon > 0$, $z \in L^1_{loc}(\mathbb{R}^+)$, $\int_s^t h(\tau) d\tau \le \epsilon(t-s) + m$ for $t \ge s \ge 0$ and some $m > 0$. Then

$$y(t) \le e^m \left(y(0) e^{-\epsilon t} + \int_0^t |z(\tau)| e^{-\epsilon(t-\tau)} d\tau\right), \quad t > 0. \tag{103}$$

Lemma 17 (quasi-stability). *Let the assumptions of Theorem 13 be valid and let u, v be the solutions of problem (14)-(15) with initial data in $\mathscr{B}_R$. Then $z = u - v$ satisfies the relation*

$$\begin{aligned} \|(z(t), z_t(t))\|_{E_1}^2 &\le C\|(z(0), z_t(0))\|_{E_1}^2 e^{-\kappa t} \\ &\quad + K \sup_{0 \le s \le t} \|z(s)\|^2 \end{aligned} \tag{104}$$

for some constants $C, K > 0$.

Proof. (i) When $1 \le p < \tilde{p}$, taking H-inner product by $z_t + \epsilon z$ in (48), with $\eta = 0$, we get

$$\begin{aligned} &\frac{d}{dt} H_4(z, z_t) + \|z_t\|^2 - \epsilon \left\|A^{-1/4} z_t\right\|^2 \\ &\quad + \epsilon\left(\|z\|^2 + \left\|A^{1/4} z\right\|^2\right) \\ &\quad = -\left(g(u) - g(v), z_t + \epsilon z\right), \end{aligned} \tag{105}$$

where

$$\begin{aligned} H_4(z, z_t) &= \frac{1}{2}\left(\left\|A^{-1/4} z_t\right\|^2 + (1+\epsilon)\|z\|^2 + \left\|A^{1/4} z\right\|^2\right. \\ &\quad \left. + 2\epsilon\left(A^{-1/2} z_t, z\right)\right) \sim \|(z, z_t)\|_{E_1}^2 \end{aligned} \tag{106}$$

for $\epsilon > 0$ suitably small. On account of $p < \tilde{p}$, $V_{1-\delta} \hookrightarrow L^{2p}$ for $\delta : 0 < \delta \ll 1$ and the interpolation theorem we have the control

$$\begin{aligned} &\left|\left(g(u) - g(v), z_t + \epsilon z\right)\right| \le C\left(1 + \|u\|_{2p}^{p-1} + \|v\|_{2p}^{p-1}\right) \\ &\quad \cdot \left(\|z\|_{2p}\|z_t\| + \epsilon\|z\|_{2p}\|z\|\right) \le C\|z\|_{V_{1-\delta}} \\ &\quad \cdot \left(\|z_t\| + \epsilon\|z\|\right) \le \frac{1}{2}\|z_t\|^2 + \frac{\epsilon}{2}\left\|A^{1/4} z\right\|^2 + C\|z\|^2. \end{aligned} \tag{107}$$

Therefore, there exists constant $\kappa > 0$ such that

$$\begin{aligned} &\frac{d}{dt} H_4(z, z_t) + \kappa H_4(z, z_t) \le C\|z\|^2, \\ &\|(z(t), z_t(t))\|_{E_1}^2 \\ &\quad \le C\|(z(0), z_t(0))\|_{E_1}^2 e^{-\kappa t} \\ &\qquad + C\int_0^t e^{-\kappa(t-\tau)}\|z(\tau)\|^2 d\tau \\ &\quad \le C\|(z(0), z_t(0))\|_{E_1}^2 e^{-\kappa t} + K \sup_{0 \le \tau \le t}\|z(\tau)\|^2, \end{aligned} \tag{108}$$

where $K = C/\kappa$.

(ii) When $p = \tilde{p}$, rewrite (105) in the form

$$\begin{aligned} &\frac{d}{dt}\left(H_4(z, z_t) + \frac{1}{2}\left(g(u) - g(v), z\right)\right) + \|z_t\|^2 \\ &\quad - \epsilon\left\|A^{-1/4} z_t\right\|^2 + \epsilon\left(\|z\|^2 + \left\|A^{1/4} z\right\|^2\right) \\ &\quad + \epsilon\left(g(u) - g(v), z\right) = \frac{1}{2}\left(\tilde{g}(u, v), z^2\right), \end{aligned} \tag{109}$$

where

$$\begin{aligned} &\tilde{g}(u, v) \\ &\quad = \int_0^1 g''(\lambda u + (1-\lambda) v)\left(\lambda u_t + (1-\lambda) v_t\right) d\lambda. \end{aligned} \tag{110}$$

Since $V_1 \hookrightarrow L^{2p}$, there exists constant $l > 0$ such that

$$\begin{aligned} &\left|\left(g(u) - g(v), z\right)\right| \\ &\quad \le C\left(1 + \|u\|_{2p}^{p-1} + \|v\|_{2p}^{p-1}\right)\|z\|_{2p}\|z\| \le C\|z\|_{V_1}\|z\| \\ &\quad \le \frac{1}{2}\|z\|_{V_1}^2 + l\|z\|^2, \end{aligned} \tag{111}$$

which means

$$\begin{aligned} &\left(g(u) - g(v), z\right) + l\|z\|^2 \ge -\frac{1}{2}\|z\|_{V_1}^2, \\ &\left|\left(\tilde{g}(u,v), z^2\right)\right| \\ &\quad \le C\left(1 + \|u\|_{2p}^{p-2} + \|v\|_{2p}^{p-2}\right)\left(\|u_t\| + \|v_t\|\right)\|z\|_{2p}^2 \\ &\quad \le C\left(\|u_t\| + \|v_t\|\right)\|z\|_{V_1}^2. \end{aligned} \tag{112}$$

We infer from (109) that

$$\begin{aligned} &\frac{d}{dt}H_5(z, z_t) + \frac{1}{2}\|z_t\|^2 + K_5(z, z_t) \\ &\quad \le C\left(\|u_t\| + \|v_t\|\right)\|z\|_{V_1}^2 + l(z, z_t) \\ &\quad \le C\left(\|u_t\| + \|v_t\|\right)H_5(z, z_t) + \frac{1}{2}\|z_t\|^2 + l^2\|z\|^2, \end{aligned} \tag{113}$$

where

$$\begin{aligned} &H_5(z, z_t) \\ &\quad = H_4(z, z_t) + \frac{1}{2}\left(\left(g(u) - g(v), z\right) + l\|z\|^2\right) \\ &\quad \sim \|z_t\|^2 + \left\|A^{1/4}z\right\|^2, \\ &K_5(z, z_t) \\ &\quad = \left(\frac{1}{2} - \epsilon - \frac{\epsilon}{\sqrt{\lambda_1}}\right)\|z_t\|^2 \\ &\qquad + \epsilon\left(\|z\|^2 + \left\|A^{1/4}z\right\|^2 + \left(g(u) - g(v), z\right)\right) \\ &\quad \ge 2\kappa H_5(z, z_t) - l\epsilon\|z\|^2 \end{aligned} \tag{114}$$

for $\epsilon > 0$ suitably small. Inserting (114) into (113), we get

$$\begin{aligned} &\frac{d}{dt}H_5(z, z_t) + 2\kappa H_5(z, z_t) \\ &\quad \le C\left(\|u_t\| + \|v_t\|\right)H_5(z, z_t) + l^2\|z\|^2. \end{aligned} \tag{115}$$

There exists $m > 0$ such that

$$\begin{aligned} &C\int_s^t \left(\|u_t(\tau)\| + \|v_t(\tau)\|\right)d\tau \\ &\quad \le C\left(\int_s^t \left(\|u_t(\tau)\|^2 + \|v_t(\tau)\|^2\right)d\tau\right)^{1/2}(t-s)^{1/2} \\ &\quad \le \kappa(t-s) + m, \quad \text{for } t \ge s \ge 0. \end{aligned} \tag{116}$$

Applying Lemma 16 to (115), we get

$$\begin{aligned} &\|(z(t), z_t(t))\|_{E_1}^2 \le Ce^m\left(\|(z(0), z_t(0))\|_{E_1}^2 e^{-\kappa t}\right. \\ &\quad \left. + l^2\int_0^t e^{-\kappa(t-\tau)}\|z(\tau)\|^2 d\tau\right) \\ &\quad \le Ce^m\|(z(0), z_t(0))\|_{E_1}^2 e^{-\kappa t} + K\sup_{0\le\tau\le t}\|z(\tau)\|^2, \end{aligned} \tag{117}$$

where $K = Ce^m l^2/\kappa$. Lemma 17 is proved. □

Proof of Theorem 15. The estimates (93) and (104) show that the dissipative system $(T(t), E_1)$ is quasi-stable on the absorbing set $\mathscr{B}_R$, so the conclusions (i) and (ii) follow directly from the standard theory on global attractor (cf. Theorems 7.9.4–7.9.6 and 7.9.8 in [30]).

The energy equality holds and shows that $H(u, u_t)$ is a strictly Lyapunov function on E_1, so the dynamical system $(T(t), E_1)$ is gradient, and by conclusion (ii), it has a compact global attractor. Therefore, the conclusion (iii) of Theorem 4 holds (cf. Theorems 2.28 and 2.31 in [20]).

We see from the conclusion (ii) that the global attractor $\mathscr{A}$ is included and bounded in $E_2 = V_2 \times V_1$. Let $\mathscr{D}$ be the closure of the 1-neighborhood of $\mathscr{A}$ in E_2; that is,

$$\mathscr{D} = \left[\left\{\zeta \in E_2 \mid \operatorname{dist}_{E_2}\{\zeta, \mathscr{A}\} \le 1\right\}\right]_{E_1}. \tag{118}$$

Then $\mathscr{D}$ is bounded in E_2 and closed in E_1, and it is an absorbing set of $T(t)$; without loss of generality we assume that $T(t)\mathscr{D} \subset \mathscr{D}$, $t \ge 0$. By Lemma 17, $T(t)$ is quasi-stable on $\mathscr{D}$. For every $\varphi_0 \in \mathscr{D}$, $\varphi(t) = S(t)\varphi_0 = (u(t), u_t(t)) \in \mathscr{D}$ and by (14), $\|u_{tt}\| \le C(\mathscr{D})$,

$$\begin{aligned} &\|T(t_2)\varphi_0 - T(t_1)\varphi_0\|_{E_1} \le \int_{t_1}^{t_2}\|\varphi'(t)\|_{E_1} dt \\ &\quad \le C(\mathscr{D})|t_2 - t_1|. \end{aligned} \tag{119}$$

So $T(t)$ has in E_1 an exponential attractor (cf. Theorem 7.9.9 in [30]). Theorem 15 is proved. □

Remark 18. Comparing Theorem 8 with Theorem 2.2 in [17] one finds that the critical case $p = \tilde{p}$ is solved in natural energy space E, the restriction $N \le 5$ is removed in the subcritical case $p < \tilde{p}$, the higher regularity of the global attractor is obtained, and the exponential attractor is established in E_1.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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Research Article

Bifurcation Behavior Analysis in a Predator-Prey Model

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A predator-prey model is studied mathematically and numerically. The aim is to explore how some key factors influence dynamic evolutionary mechanism of steady conversion and bifurcation behavior in predator-prey model. The theoretical works have been pursuing the investigation of the existence and stability of the equilibria, as well as the occurrence of bifurcation behaviors (transcritical bifurcation, saddle-node bifurcation, and Hopf bifurcation), which can deduce a standard parameter controlled relationship and in turn provide a theoretical basis for the numerical simulation. Numerical analysis ensures reliability of the theoretical results and illustrates that three stable equilibria will arise simultaneously in the model. It testifies the existence of Bogdanov-Takens bifurcation, too. It should also be stressed that the dynamic evolutionary mechanism of steady conversion and bifurcation behavior mainly depend on a specific key parameter. In a word, all these results are expected to be of use in the study of the dynamic complexity of ecosystems.

1. Introduction

The dynamical behaviors between different populations and their complex properties have been given close attention by biologists and ecologists. Since the pioneering work of Lotka and Volterra, the research interest in predator-prey dynamics has achieved constant attention. It is well known that these models can directly reflect changes in the size of populations. Considerable improvements are that the relevant theories become more and more complete in this category in recent years [1–5].

The predator-prey models have extensive applicability in the field of biological problems. The biologist can use them to study the relationship between species in different domains [6–10]. Yang and Zhao [11] have established a fish-algae consumption model to explore how to apply the complex dynamics between fish-algae populations to expound the mechanism of algae blooms; these results will be helpful in controlling algae bloom. González-Olivares and Rojas-Palma [12] have established a Gause type predator-prey model with Allee effects and considered three standard functional responses, respectively. They found that different types of functional responses will lead to a model's dynamic behavior change. Their results perfectly explore that the expression of one interaction term has a significant effect on the stability and persistence of the population model, which is meaningful in establishing biological mathematical model. Dhar et al. [13] have proposed a mathematical model to study how instantaneous nutrient recycling affects the dynamic characteristic of aquatic ecosystem. The nutrient supply rate was found to affect the local stability of equilibrium; this work was significant for models involving flowing waters. Luo [14] has considered a mathematical model to study how the periodic environment influences the internal operating characteristics of the aquatic ecosystem. He pointed out that the temperature has a certain time periodicity, the fluctuation of temperature can lead to changes in intrinsic carrying capacity, and the growth rate of prey populations can also be influenced by time periodicity; these results in [14] are more accordant with the actual situation.

It is a common phenomenon in nature that one predator lives on multiple prey species. To address this issue, some researchers have begun to consider alternative prey to depict the dynamic predator mechanism [15–18]. It is easy to find out that alternative prey can variously affect the dynamic capture feature. On one hand, alternative prey can increase the predation quantity for the focal prey, because more prey biomass may result in higher predation rates for both prey items. On the other hand, the alternative prey population can also lower predation on the focal prey because of predator preference for the alternative prey resources [19]. At the same time, many studies show that the alternative population can intensively influence the dynamic behavior of the aquatic ecosystem [20, 21]. Based on this mechanism, Kar and Chattopadhyay [22] have developed a two-species predator-prey model which includes the effect of alternative prey. The model can be depicted as

$$\begin{aligned} \frac{dN}{dT} &= r_1 N\left(1-\frac{N}{k_1}\right)-\frac{a_1 N}{b_1+N}P, \\ \frac{dP}{dT} &= \frac{a_1 e_1 N}{b_1+N}P+s_1 P\left(1-\frac{N}{k_1}\right)-d_1 P, \end{aligned} \tag{1}$$

where N and P represent prey and predator population densities or biomass at time T, respectively. Here, r_1 stands for the intrinsic growth rate of the prey without any environment limitations, k_1 is the environmental carrying capacity of the prey in the absence of the predator, $a_1N/(b_1+N)$ is the Holling type-II functional response [23], which is used to depict the average feeding rate of the predator when the predator spends time seeking prey, where b_1 is the half saturation constant for the Holling type-II, a_1 is the grazing rate of the predator population, e_1 and d_1 are the conversional rate and mortality rate of predator, respectively, and $s_1P(1-N/k_1)$ indicates the portion of biomass of predator increments from the alternative prey, where s_1 represents the growth rate of the predator on account of the alternative prey. From the formula, we can see that when the quantity of focal prey N approaches the environmental carrying capacity k_1, the amount of alternative prey consumed by the predator will tend to be zero [24].

The concept of Allee effect was firstly derived from the research of Allee and Bowen [25]. Since then, the Allee effect received the attention of many researchers [26–29]. Allee effects can be roughly classified into two types: strong and weak [30]. For these two forms, there is a critical value that is referred to as the Allee threshold, respectively. The fist form means that if the population size is below the threshold, the species will become extinct. When the growth rate gradually decreases but remains positive with a low population size, the Allee effect is described as weak. Aulisa and Jang [31] have established a continuous-time predator-prey model to study influences in dynamical behaviors when the prey population possesses Allee effect. They pointed out that both species will become extinct if the prey population size falls below a certain threshold. Pan et al. [32] have considered a reaction-diffusion phytoplankton-zooplankton model with double Allee effects on prey population. They pointed out that the Allee effects can make the dynamical behaviors of a system increasingly complex.

The strong Allee effect can be depicted by the following form:

$$\frac{dN}{dT} = r_1 N\left(1-\frac{N}{k_1}\right)\left(\frac{N}{m_1}-1\right), \tag{2}$$

where m_1 is Allee effect threshold. If population density or size is below the threshold, this population is doomed to extinction. Here $0 < m_1 < k_1$, the other parameters' significance is the same as model (1).

Now we will establish a predator-prey model with strong Allee effects:

$$\begin{aligned} \frac{dN}{dT} &= r_1 N\left(1-\frac{N}{k_1}\right)\left(\frac{N}{m_1}-1\right)-\frac{a_1 N}{b_1+N}P, \\ \frac{dP}{dT} &= \frac{a_1 e_1 N}{b_1+N}P+s_1 P\left(1-\frac{N}{k_1}\right)-d_1 P. \end{aligned} \tag{3}$$

The parameters are all greater than zero and have the same significance as above. For simplicity, we write the above model in dimensionless form as follows: we take the scaling $N = k_1 n$, $P = r_1 p/a_1$, and $T = t/r_1$; then, model (3) can be simplified as

$$\frac{dn}{dt} = n(1-n)(mn-1)-\frac{n}{b+n}p = F_1, \tag{4a}$$

$$\frac{dp}{dt} = \frac{en}{b+n}p+sp(1-n)-dp = F_2, \tag{4b}$$

where $e = a_1e_1/r_1$, $b = b_1/k_1$, $s = s_1/r_1$, $d = d_1/r_1$, $m = k_1/m_1$, and apparently $m > 1$.

In the succedent subsections of this paper, we introduce the problem of determining the number of equilibria. Stability analysis of equilibrium point is also presented. Then, we provide a demonstration of several types of bifurcation and give a set of parameter values to prove the existence of Bogdanov-Takens (BT) bifurcation. In Section 3, numerical simulations are given to illustrate the theoretical analysis results, followed by conclusions in Section 4.

2. Qualitative Analysis

2.1. Equilibria. In this subsection, we mainly concentrate on the existence of positive equilibrium of model (4a) and (4b). Model (4a) and (4b) has three boundary equilibria: $E_0(0,0)$, $E_1(1,0)$, and $E_2(1/m,0)$ and they are always existent. The interior equilibria are the intersection points of the vertical isocline and horizontal isocline in the interior of the first quadrant. The expressions of vertical and horizontal isoclines are as follows:

$$(1-n)(mn-1)-\frac{p}{b+n} = 0, \tag{5a}$$

$$\frac{en}{b+n}+s(1-n)-d = 0. \tag{5b}$$

The horizontal isocline is vertical line and the number of perpendiculars is decided by (5b). We denote (5b) by $H(n) = G(n)/(b+n) = 0$, where $G(n) = -sn^2+(e+s-sb-d)n+b(s-d)$.

$G(n) = 0$ have the same solutions as (5b), and two solutions exist in the equation at most. We denoted them by n_3 and n_4, where $n_4 < n_3$. From the geometric properties of isoclines we can know that model (4a) and (4b) has interior equilibrium points if and only if $1/m < n_3 < 1$ or $1/m < n_4 < 1$. Then, the maximum number of equilibria for model (4a) and (4b) is five. We use $E_3(n_3, p_3)$ and $E_4(n_4, p_4)$ to signify the interior equilibria, the existence and stability of which are shown as follows.

The Jacobian matrix of model (4a) and (4b) at the equilibrium point E_i is

$$J_{(n_i,p_i)} = \begin{pmatrix} A(n_i, p_i) & B(n_i, p_i) \\ C(n_i, p_i) & D(n_i, p_i) \end{pmatrix}, \tag{6}$$

$$\begin{aligned} A(n_i, p_i) &= -3mn_i^2 + (2m+2)n_i - 1 - \frac{bp_i}{(b+n_i)^2}, \\ B(n_i, p_i) &= -\frac{n_i}{b+n_i}, \\ C(n_i, p_i) &= \frac{bep_i}{(b+n_i)^2} - sp_i, \\ D(n_i, p_i) &= \frac{en_i}{b+n_i} + s - d - sn_i. \end{aligned} \tag{7}$$

Theorem 1. *(1) $E_0(0,0)$ is an asymptotically stable node point if $s < d$ and a saddle point for $s > d$; if $s = d$, it is a high order singularity. The Jacobian matrix around E_0 shows that*

$$J_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & s-d \end{pmatrix}. \tag{8}$$

(2) $E_1(1,0)$ is a asymptotically stable node point when $e/(b+1) < d$ while it is a saddle point when $e/(b+1) > d$. The Jacobian matrix around E_1 is

$$J_{(1,0)} = \begin{pmatrix} 1-m & -\dfrac{1}{b+1} \\ 0 & \dfrac{e}{b+1} - d \end{pmatrix}. \tag{9}$$

(3) $E_2(1/m, 0)$ is always unstable. The Jacobian matrix around E_2 is

$$J_{(1/m,0)} = \begin{pmatrix} 1-\dfrac{1}{m} & -\dfrac{1}{bm+1} \\ 0 & \dfrac{e}{bm+1} + s - d - \dfrac{s}{m} \end{pmatrix}. \tag{10}$$

One of the eigenvalues is $1 - 1/m > 0$, so E_2 is unstable.

Theorem 2. *If $e - d - bd > 0$ and $-s/m^2 + (e + s - sb - d)/m + b(s - d) < 0$ hold, model (4a) and (4b) has only one interior equilibrium $E_3(n_3, p_3)$. Furthermore, $E_3(n_3, p_3)$ is always unstable, especially when $d = s + e + sb - 2\sqrt{bse}$; it is a high-order singularity, where*

$$n_3 = \frac{s + e - sb - d + \sqrt{(sb - s + d - e)^2 + 4bs(s-d)}}{2s}, \tag{11a}$$

$$p_3 = (1 - n_3)(mn_3 - 1)(b + n_3). \tag{11b}$$

Proof. According to (6),

$$\begin{aligned} \det\left(J_{(n_3,p_3)}\right) &= -C(n_3, p_3)\, B(n_3, p_3) \\ &= \frac{n_3 p_3}{b+n_3}\left(\frac{be}{(b+n_3)^2} - s\right). \end{aligned} \tag{12}$$

Let $\phi = n_3 - \sqrt{be/s} + b$; substituting (11a) into ϕ, we get

$$\begin{aligned} \phi &= \frac{s + e - d + sb - 2\sqrt{bes} + \sqrt{(s + e - d - sb)^2 + 4bs(s-d)}}{2s} \\ &= \frac{s + e - d + sb - 2\sqrt{bes} + \sqrt{(s + e - d + sb)^2 - 4bse}}{2s} \geq 0. \end{aligned} \tag{13}$$

Then, we know when $n_3 \geq \sqrt{be/s} - b$ and $\det J_{(n_3,p_3)} \leq 0$, E_3 is a saddle point or a high-order singularity. □

Theorem 3. *If $e-d-bd < 0$ and $-s/m^2+(e+s-sb-d)/m+b(s-d) > 0$, model (4a) and (4b) has only one interior equilibrium $E_4(n_4, p_4)$. Furthermore, $E_4(n_4, p_4)$ is locally asymptotically stable when $n_4 \in L$, where $L = \{n \mid A(n) < 0,\ n \in (1/m, 1)\}$ and*

$$n_4 = \frac{s + e - sb - d - \sqrt{(sb - s + d - e)^2 + 4bs(s-d)}}{2s}, \tag{14a}$$

$$p_4 = (1 - n_4)(mn_4 - 1)(b + n_4). \tag{14b}$$

Proof. According to (6),

$$\begin{aligned} \operatorname{trace}\left(J_{(n_4,p_4)}\right) &= A(n_4) \\ &= -3mn_4^2 + (2m+2)n_4 - 1 \\ &\quad - \frac{b(1-n_4)(mn_4-1)}{b+n_4}, \\ \det\left(J_{(n_4,p_4)}\right) &= -C(n_4)\, B(n_4) \\ &= \frac{n_4}{b+n_4}\left(\frac{bep_4}{(b+n_4)^2} - sp_4\right). \end{aligned} \tag{15}$$

Imitating the proof of Theorem 2, we can know

$$n_4 < \sqrt{\frac{be}{s}} - b; \tag{16}$$

then, $\det(J_{(n_4,p_4)}) = -B(n_4)C(n_4) > 0$, which means two eigenvalues have the same sign.

Since

$$A\left(\frac{1}{m}\right) = 1 - \frac{1}{m} > 0, \qquad A(1) = 1 - m < 0. \tag{17}$$

Solving $A(n) = 0$, we can get that there exists a set $L \subset (1/m, 1)$, such that $A(n) < 0$ for all $n \in L$, where $L = ((1 + m - mb + \sqrt{1 - m + mb + m^2(1 + b + b^2)})/3m, 1)$. Then, we know that if $n_4 \in L$, trace $(J_{(n_4,p_4)}) = A(n_4) < 0$ and both eigenvalues of $J_{(n_4,p_4)}$ have negative real parts, and $E_4(n_4, p_4)$ is locally asymptotically stable.

From the above discussion, we find that model (4a) and (4b) may possess two interior equilibria E_3 and E_4 simultaneously if $e - d - bd < 0$, $-s/m^2 + (e + s - sb - s)/m + b(s - d) < 0$, $(sb - s + d - e)^2 + 4bs(s - d) > 0$, and $2s/m < e + s - sb - d < 2s$. The stability condition of E_4 is still $n_4 \in L$.

Considering the existence and stability conditions integrated, we know that there will appear three stable equilibria synchronously if the scope of those parameters satisfies the above conditions. Based on the above demonstration, some cases where the stability of equilibria may occur are cited in Table 1, where the existence of equilibria is classified according to the magnitude of s and d. □

2.2. Local Bifurcation. From Table 1 we know that the variation of parameter value will lead to the number change of interior equilibria. We take d as variable parameter and find that changing the value of d will vary equilibrium's number; when the value of d increases across the threshold $d_{\mathrm{TC1}} = e/(b + 1)$, the interior equilibrium E_4 bifurcates from E_1, and when the value of d is across $d_{\mathrm{TC2}} = e/(bm + 1) + s(1 - 1/m)$, another interior equilibrium E_3 can bifurcate from E_2. Then, there exist two transcritical bifurcations, which are denoted by TC1 and TC2, respectively.

Theorem 4. *(1) Model (4a) and (4b) undergoes transcritical bifurcation at $E_1(1, 0)$ when the value of parameter d equals the transcritical bifurcation threshold $d_{TC1} = e/(b + 1)$.*

(2) Model (4a) and (4b) undergoes another transcritical bifurcation at $E_2(1/m, 0)$ when the value of parameter d equals the transcritical bifurcation threshold $d_{TC2} = e/(bm+1)+s(1-1/m)$.

Proof. It can be easily seen that $E_1(1, 0)$ coincides with $E_4(n_4, p_4)$ when $d = e/(b + 1)$. According to the theorems in [33, 34], it can be found that one interior equilibrium point branches off from E_1 when d passes the threshold $d_{\mathrm{TC1}} = e/(b + 1)$ and also conforms with the transversality condition for transcritical bifurcation. The same notation we followed in this paper is mentioned in [33].

We can calculate the value of Jacobian matrix of model (4a) and (4b) as $\mathrm{Det}\,(J_{(1,0)})|_{d_{\mathrm{TC1}}}$, and then $(1, 0)$ is a nonhyperbolic equilibrium point when $d = d_{\mathrm{TC1}}$. Let the eigenvectors $v = [1, (b + 1)(1 - m)]^T$ and $w = [0, 1]^T$ indicate the eigenvectors corresponding to zero eigenvalues of $J_{((1,0),d_{\mathrm{TC1}})}$ and $[J_{((1,0),d_{\mathrm{TC1}})}]^T$, respectively. Next the transversality conditions for the transcritical bifurcation are satisfied be verified, where $F = (F_1, F_2)^T$:

$$\begin{aligned} &w^T F_d\left((1,0); d_{\mathrm{TC1}}\right) = (0,1)\,(0,0)^T = 0,\\ &w^T DF_d\left((1,0); d_{\mathrm{TC1}}\right) v = (b+1)\,(m-1) \neq 0,\\ &w^T D^2 F\left((1,0); d_{\mathrm{TC1}}\right)(v,v)\\ &\quad = \left(\frac{eb}{(1+b)^2} - s\right)(b+1)\,(1-m). \end{aligned} \tag{18}$$

We notice that $m > 1$ and $w^T D^2 F((1, 0); d_{\mathrm{TC1}})(v, v) < 0$ if $e > s(b + 1)^2/b$ and then the transcritical bifurcation is supercritical, which means that an interior equilibrium point arises through E_1 under this condition. Another aspect is that if $e < s(b+1)^2/b$, the transcritical bifurcation is subcritical and an interior equilibrium vanishes across E_1.

In the following, we demonstrate that another interior equilibrium point bifurcates from $E_2(1/m, 0)$ through transcritical bifurcation at the threshold $d_{\mathrm{TC2}} = e/(bm + 1) + s(1 - 1/m)$. Similarly, we can get $\mathrm{Det}\,(J_{(1/m,0)})|_{d_{\mathrm{TC2}}} = 0$, $v = [1, (b+1/m)(m-1)]^T$, and $w = [0, 1]^T$ are the eigenvectors corresponding to $J_{((1/m,0),d_{\mathrm{TC2}})}$ and $[J_{((1/m,0),d_{\mathrm{TC2}})}]^T$, respectively. We have

$$\begin{aligned} &w^T F_d\left(\left(\frac{1}{m}, 0\right); d_{\mathrm{TC2}}\right) = (0,1)\,(0,0)^T = 0,\\ &w^T DF_d\left(\left(\frac{1}{m}, 0\right); d_{\mathrm{TC2}}\right) v = (1-m)\left(b + \frac{1}{m}\right) \neq 0,\\ &w^T D^2 F\left(\left(\frac{1}{m}, 0\right); d_{\mathrm{TC2}}\right)(v,v)\\ &\quad = \left(\frac{be}{(b+1/m)^2} - s\right)\left(b + \frac{1}{m}\right)(m-1). \end{aligned} \tag{19}$$

If $e < s(b+1/m)^2/b$ and $w^T D^2 F((1/m, 0); d_{\mathrm{TC2}})(v, v) < 0$, the transcritical bifurcation is supercritical and an interior equilibrium point appears across E_2 under this condition. Another aspect is that if $e > s(b + 1/m)^2/b$, the transcritical bifurcation is subcritical and an interior equilibrium vanishes across E_2. □

Theorem 5. *Model (4a) and (4b) undergoes saddle-node bifurcation when $d = d_{SN}$, where $d_{SN} = s + e + bs - 2\sqrt{bse}$.*

Proof. It is easy to calculate that the discriminant of $G(n) = 0$ is equal to zero when $d = s + e + bs - 2\sqrt{bse}$, which means $G(n) = 0$ has a twofold root. We denote this root by n^*. Model (4a) and (4b) has only one interior equilibrium point $E^*(n^*, p^*)$ correspondingly and the constituents are given by

$$\begin{aligned} n^* &= \sqrt{\frac{eb}{s}} - b,\\ p^* &= (1 - n^*)(mn^* - 1)(b + n^*). \end{aligned} \tag{20}$$

Table 1

		$E_0(0,0)$	$E_1(1,0)$	$E_2(1/m,0)$	$E_3(n_3,p_3)$	$E_4(n_4,p_4)$
$s > d$		Saddle point	Stable node point	Unstable node point	Saddle point	Nonexistent
$s = d$		Unstable (high-order singularity)	Stable node point	Unstable node point	Saddle point	Nonexistent
$s < d$	$\Delta = 0$	Stable node point	Stable node point	Saddle point	E_3 and E_4 coincide and are a saddle-node point	
	$\Delta > 0$	Stable node point	Stable node point	Unstable node point	Saddle point	Nonexistent
		Stable node point	Saddle point	Saddle point	Nonexistent	Stable node point
		Stable node point	Stable node point	Saddle point	Saddle point	Stable node point

The Jacobian matrix evaluate at E^* is given by

$$J_{((n^*,p^*),d_{\mathrm{SN}})} = \begin{pmatrix} A(n^*,p^*,d_{\mathrm{SN}}) & B(n^*,p^*,d_{\mathrm{SN}}) \\ C(n^*,p^*,d_{\mathrm{SN}}) & D(n^*,p^*,d_{\mathrm{SN}}) \end{pmatrix} = \begin{pmatrix} M & \sqrt{\frac{bs}{e}} - 1 \\ 0 & 0 \end{pmatrix}, \tag{21}$$

where $M = A(n^*,p^*,d_{\mathrm{SN}})$. Obviously, matrix $J_{((n^*,p^*),d_{\mathrm{SN}})}$ has a double zero eigenvalues when $\mathrm{Det}(J_{((n^*,p^*),d_{\mathrm{SN}})}) = 0$. We can count out the eigenvectors corresponding to the zero eigenvalue, which are $v = [1, M/(1-\sqrt{bs/e})]^T$ and $w = [0,1]^T$. Utilizing the expressions of vectors v and w, as well as $n^* \in (1/m, 1)$, we can get

$$\begin{aligned} &w^T F_d((n^*,p^*);d_{\mathrm{SN}}) \\ &\quad = -(1-n^*)(mn^*-1)(b+n^*) < 0, \\ &w^T\left[D^2F((n^*,p^*);d_{\mathrm{SN}})\right](v,v) \\ &\quad = \frac{-eb(1-n^*)(mn^*-1)}{(b+n^*)^2}. \end{aligned} \tag{22}$$

Given the value range of n^*, we know that $w^T[D^2F((n^*,p^*);d_{\mathrm{SN}})](v,v)$ cannot be equal to zero. Then, by Sotomayor's theorem we can prove that the model undergoes a saddle-node bifurcation when the parameter d goes via the critical threshold $d = d_{\mathrm{SN}}$. □

Theorem 6. *In the case of $s < d$, the interior equilibrium point E_4 changes its stability through the Hopf-bifurcation at the threshold $b = b_H$, where $b_H = -(3mn_4{}^2 - 2(m+1)n_4 + 1)/(2mn_4 - m - 1)$.*

Proof. Because the interior equilibrium point E_3 is always a saddle, then Hopf bifurcation can only take place at E_4. Parameter b can drive equilibrium E_4 into an unstable state when $b > b_H$, so $b = b_H$ is the critical value where the stability of E_4 changes. Next we will prove the necessary condition for Hopf bifurcation to occur. To testify the transversality condition of the Hopf bifurcation of the model's solution, we take $\lambda = \alpha(b) + \beta(b)i$, where λ is an eigenvalue of the Jacobian matrix $J_{((n_4,p_4),b_H)}$. If the Hopf bifurcation occurs at $b = b_H$, λ is a purely imaginary number, such that $\mathrm{Tr}(J_{((n_4,p_4),b_H)}) = 0$, $\mathrm{Det}(J_{((n_4,p_4),b_H)}) \neq 0$, and $d\mathrm{Tr}(J_{((n_4,p_4),b_H)})/db \neq 0$. Substituting b by $b_H = -(3mn_4{}^2 - 2(m+1)n_4 + 1)/(2mn_4 - m - 1)$, we can get

$$\begin{aligned} &\mathrm{Tr}\left(J_{(n_4,p_4,b_H)}\right) = 0, \\ &\mathrm{Det}\left(J_{(n_4,p_4,b_H)}\right) \\ &\quad = n_4(1-n_4)(mn_4-1)\left(\frac{b_H e}{(b_H+n_4)^2} - s\right), \\ &\frac{d}{db}\mathrm{Tr}\left(J_{(n_4,p_4,b_H)}\right) = -\frac{n_4(1-n_4)(mn_4-1)}{(b+n_4)^2}. \end{aligned} \tag{23}$$

Using (16), we know $\mathrm{Det}(J_{(n_4,p_4,b_H)}) > 0$ and $d\mathrm{Tr}(J_{(n_4,p_4,b_H)})/db \neq 0$. Then, the transversality condition of a Hopf bifurcation is satisfied [35]. □

Theorem 7. *Model (4a) and (4b) undergoes a Bogdanov-Takens (BT) bifurcation of codimension two.*

As we know, $\mathrm{Tr}(J) = 0$ and $\mathrm{Det}(J) = 0$ are the necessary conditions of the occurrence of Hopf and saddle-node, respectively. If both Hopf and saddle-node conditions hold, there will be a new bifurcation called Bogdanov-Takens bifurcation. In this situation, the Jacobian matrix has a double zero eigenvalue [36]. Since the explicitly analytical expressions for thresholds of BT bifurcation are quite difficult to determine, we give a numerical example to confirm the system exhibit BTs bifurcation. We fix $m = 1.5$, $s = 0.1$, and $d = 0.4$. We find that, at $(e_{BT}, b_{BT}) = (0.4930734388, 0.2386077213)$, $\mathrm{Tr}(J_{E^*})|_{(e_{BT},b_{BT})} = 0$ and $\mathrm{Det}(J_{E^*})|_{(e_{BT},b_{BT})} = 0$. Moreover, $(\partial^2F/\partial n^2 - (\partial F/\partial n)(\partial^2F/\partial n\partial p))/(\partial^2F/\partial n\partial p + \partial^2F/\partial n\partial p) = -2.800482957$ and $((1/2)(\partial F/\partial n)(\partial^2F/\partial n^2) - (\partial F/\partial n)^2(\partial^2F/\partial n\partial p))/(\partial F/\partial p + (1/2)(\partial F/\partial p)(\partial^2F/\partial n^2) - (\partial F/\partial n)(\partial^2F/\partial n\partial p)) = -0.0000734288\,78$; these expressions prove the transversality conditions for a BT bifurcation [37].

3. Numerical Results

In order to verify the correctness and feasibility of the theoretical results, a series of numerical simulations will be depicted in detail. Many phase diagrams are given to display the dynamics properties of model (4a) and (4b),

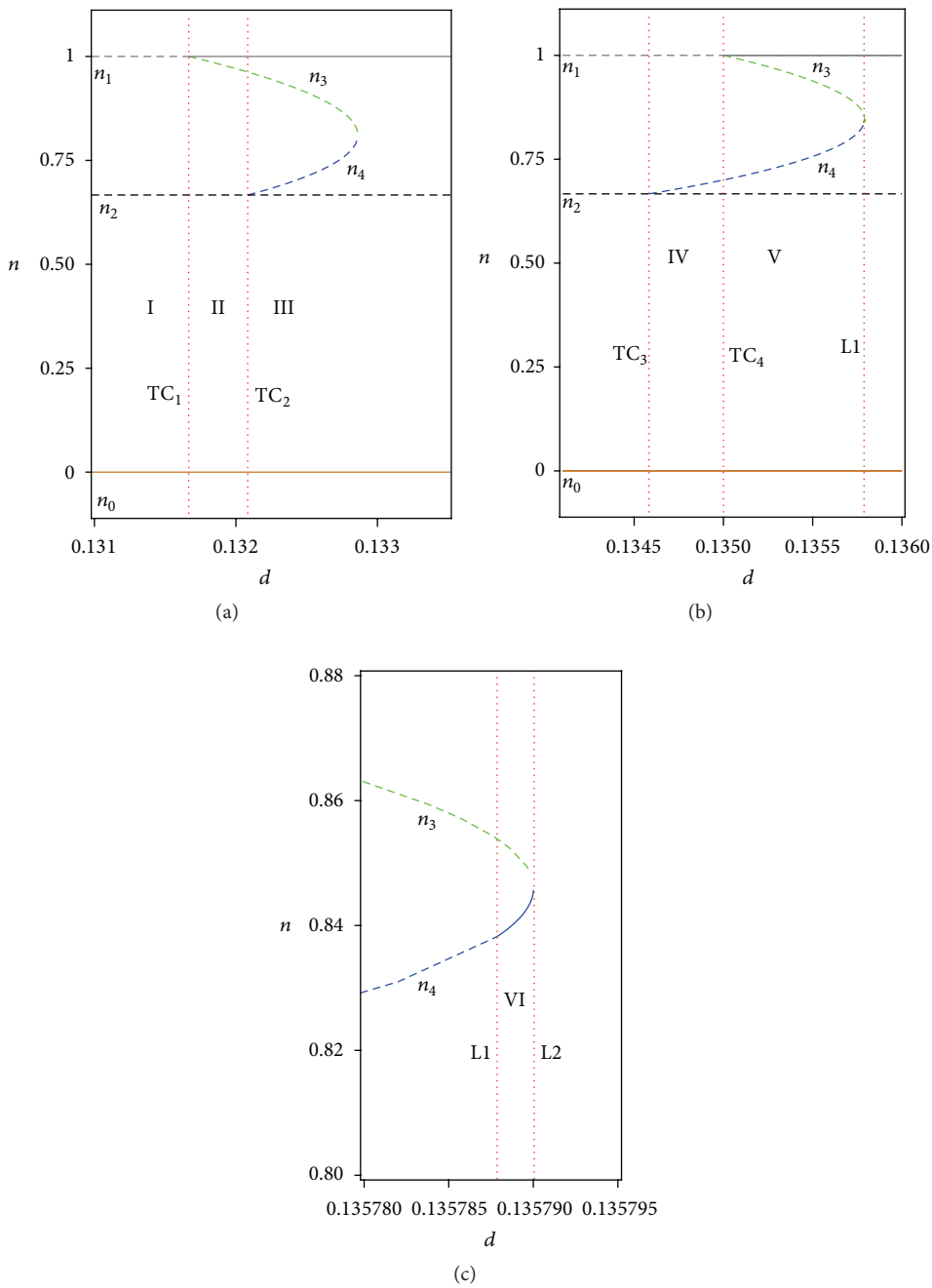

Figure 1: Bifurcation diagram of model (4a) and (4b) in $n - d$ plane constructed for $m = 1.5$, $b = 2.0$, $s = 0.1$, $e = 0.395$ (a), and $e = 0.405$ (b, c). Red perpendicular dashed lines represent the critical value for different bifurcation occurrence. Three horizontal lines stand for three boundary equilibria: E_0 (golden), E_1 (gray), and E_2 (black). Two interior equilibria E_3 and E_4 are presented by blue and green curves, respectively. The solid lines indicate equilibrium in stable state and dotted line indicates unstable state, where, according to the stability theorem, these results are calculated and drawn based on Maple 14 platform.

which are based on pplane8 routines [38] in Matlab 7.1. In fact, according to [39], we can know pplane8 is a powerful tool for studying planar autonomous systems of differential equations, which can rapidly and accurately draw trajectories of each phase plane, count each critical point, and correctly characterize each equilibrium point of the studied systems. It is easy to find from Table 1 that model (4a) and (4b) only has an interior equilibrium point E_3 if $s \geq d$; then, the premise of parametric ranges in Figure 1 is $s > d$. It should be pointed out from Figure 1(a) that two vertical lines passing through points $(e/(b+1), 0)$ and $(e/(bm+1) + s(1 - s/m), 0)$ are the transcritical bifurcation curves, which have been named TC1 and TC2, respectively. Furthermore, if $d < e/(b+1)$ is feasible, that is to say, model (4a) and (4b) does not have any interior equilibrium point when the value of d is positioned within region I of Figure 1(a), three boundary equilibria $E_0(0, 0)$,

$E_1(1,0)$, and $E_2(2/3,0)$ exist. $E_0(0,0)$ is asymptotically stable, while $E_1(1,0)$ and $E_2(2/3,0)$ are unstable, the results of which have been shown in Figure 2(a) ($d = 0.131$). We know those three boundary equilibria $E_0(0,0)$, $E_1(1,0)$, and $E_2(2/3,0)$ always exist no matter what the value of d is. Thus, in the following cases we do not introduce the existence of boundary equilibria separately. However, if the value of d is gradually increasing across the transcritical bifurcation TC1 and finally enters into region II, model (4a) and (4b) has an interior equilibrium point E_3. It is worth emphasizing that the critical value of transcritical bifurcation is $d_{\mathrm{TC1}} = e/(b+1) = 0.13167$ and the interior equilibrium point $E_3(0.97064, 0.039763)$ is a saddle point, boundary equilibria E_0 and E_1 are stable, and E_2 is an unstable node point, which has been shown in Figure 2(b) ($d = 0.132$). As the value of d gradually increases to $d_{\mathrm{TC2}} = e/(bm+1) + s(1-1/m) = 0.132083$, which is a critical value of another transcritical bifurcation and finally enters into region III, model (4a) and (4b) has two interior equilibria $E_3(0.91328, 0.093458)$ and $E_4(0.71172, 0.052831)$, which are unstable; E_0 and E_1 are stable, and E_2 is a saddle point (see Figure 2(c), $d = 0.1325$).

To further fully explore the existence of the equilibria and the occurrence of the bifurcation behavior in model (4a) and (4b), we take another set of parameter values in Figure 1(b) and repeat the steps in Figure 1(a). The bifurcation diagram in Figure 1(b) has been depicted in detail as follows. It is obvious to find that model (4a) and (4b) has an interior equilibrium point E_4 if the value of d exceeds the threshold value $d_{\mathrm{TC3}} = e/(bm+1) + s(1-1/m) = 0.134583$, which is transcritical bifurcation threshold TC3. It is worthwhile to point out that the interior equilibrium point $E_4(0.69128, 0.030678)$ is unstable on the account of $n_4 = 0.69128 \notin L(0.83822, 1)$. The boundary equilibrium $E_0(0,0)$ is stable, and $E_1(1,0)$ and $E_2(2/3,0)$ are saddle points (see Figure 2(d), $d = 0.1349$). We take another set of parameters values as the supplement to display the case that there exists only one interior equilibrium point and it is locally asymptotically stable (see Figure 3(a)), where $E_1(1,0)$ and $E_2(2/3,0)$ are saddle points, $E_0(0,0)$ and $E_4(0.886762, 0.040628)$ are stable, and $L = (0.84652, 1)$. Furthermore, if the value of d increases beyond the threshold value of transcritical bifurcation threshold $d_{\mathrm{TC4}} = e/(b+1) = 0.135$ and finally enters into region V, te model (4a) and (4b) has three boundary equilibria $E_0(0,0)$, $E_1(1,0)$, and $E_2(2/3,0)$ and two interior equilibria $E_3(0.97862, 0.029801)$ and $E_4(0.71938, 0.060341)$ (see Figure 2(e), $d = 0.1352$). It is necessary to underline that $E_0(0,0)$ and $E_1(1,0)$ are stable and $n_4 = 0.71938 \notin L(0.83822, 1)$, so the interior equilibria E_4 and E_3 are all unstable.

In order to clearly explore the steady characteristic of the interior equilibrium point E_4, Figure 1(c) will be given, which is the partially enlarged view of Figure 1(b). It should be stressed that the interior equilibrium E_4 will change the stable state if the value of d increased beyond the line L1 and finally enters into region VI, which suggests that a Hopf bifurcation can lead to the appearance of a limit cycle in the vicinity of the interior equilibrium $E_4(0.83816, 0.11816)$ if $d = 0.1357878256\,59$ (see Figure 2(f)). From Figure 2(f), we can observe that this limit cycle around E_4 is stable as it attracts two neighboring trajectories: the trajectory (bottle green curve) lying inside the limit cycle and the trajectory (blue curve) lying outside; these two trajectories move ectad and entad, respectively, and converge on the limit cycle. But at this time the interior equilibrium point $E_3(0.85397, 0.11709)$ and boundary equilibrium point $E_2(2/3,0)$ are unstable, and $E_0(0,0)$ and $E_1(1,0)$ are stable. When the value of d enters into domain VI, $E_4(0.86298, 0.042889)$ becomes stable, $E_3(0.92702, 0.032122)$ remains unstable, and E_0 and E_1 are all stable in this domain. Thus, it is interesting to know that model (4a) and (4b) will show three stationary phenomenon (see Figure 2(g), $d = 0.13579$). Due to the parameter values, the nature of E_3 and E_4 only can be seen clearly on the enlarged view (see Figure 2(g)). Therefore, we take another group of values to exhibit the three stable states in overall view in Figure 3(b). If the value of d gradually increases and reaches $d_{\mathrm{SN}} = bs + e + 2\sqrt{bse} + s = 0.1357900212$, it is easy to find that $E_3(0.84605, 0.11789)$ and $E_4(0.84605, 0.11789)$ coincide at line L2 and the coincident point is called saddle-node point, which has features of both saddle and node points (see Figure 2(h)). However, it will disappear if the value of d is increased higher than d_{SN}, which signifies that model (4a) and (4b) will undergo a saddle-node bifurcation if the value of d increases across the threshold value $d_{\mathrm{SN}} = bs+e+2\sqrt{bse}+s$. In a word, it is worthy of our summary that model (4a) and (4b) can show a complex dynamic evolutionary process of steady conversion and bifurcation behavior with increase of key parameter d.

In addition, model (4a) and (4b) has three boundary equilibria $E_0(0,0)$, $E_1(1,0)$, and $E_2(2/3,0)$ and an interior equilibrium point E_3, which is unstable if $s > d$ or $s = d$. However, it is worth stressing that the boundary equilibrium point $E_0(0,0)$ is a saddle point if $s > d$ and is an unstable high-order singularity if $s = d$ (see Figures 4(a) and 4(b)). At the same time, it can be known from Theorem 7 that model (4a) and (4b) can undergo a Bogdanov-Takens bifurcation if $\mathrm{Tr}\,(J_{E^*})|_{(e_{\mathrm{BT}}, b_{\mathrm{BT}})} = 0$ and $\mathrm{Det}\,(J_{E^*})|_{(e_{\mathrm{BT}}, b_{\mathrm{BT}})} = 0$, which is shown in Figure 4(c).

Based on the above analysis, the key parameter d can impose influence on dynamic evolutionary mechanism of steady conversion and bifurcation behavior and lead to model (4a) and (4b) having three stationary phenomena and multiple bifurcation behaviors, which can in turn prove that the theoretical results are correct and the complex dynamics of model (4a) and (4b) mainly depend on some key parameters. Moreover, these results show that the method of using mathematical model to study the ecological problems is feasible.

4. Conclusions

On the basis of the theories and methods of ecology, a predator-prey model is studied numerically and analytically in this paper. The aim is to probe how some key factors influence dynamic evolutionary mechanism of steady conversion and bifurcation behavior in predator-prey model. The theoretical works have been promoting the investigation of the existence and stability of the equilibria, as well as some conditions of some bifurcations behaviors, such as transcritical bifurcation, saddle-node bifurcation, and Hopf

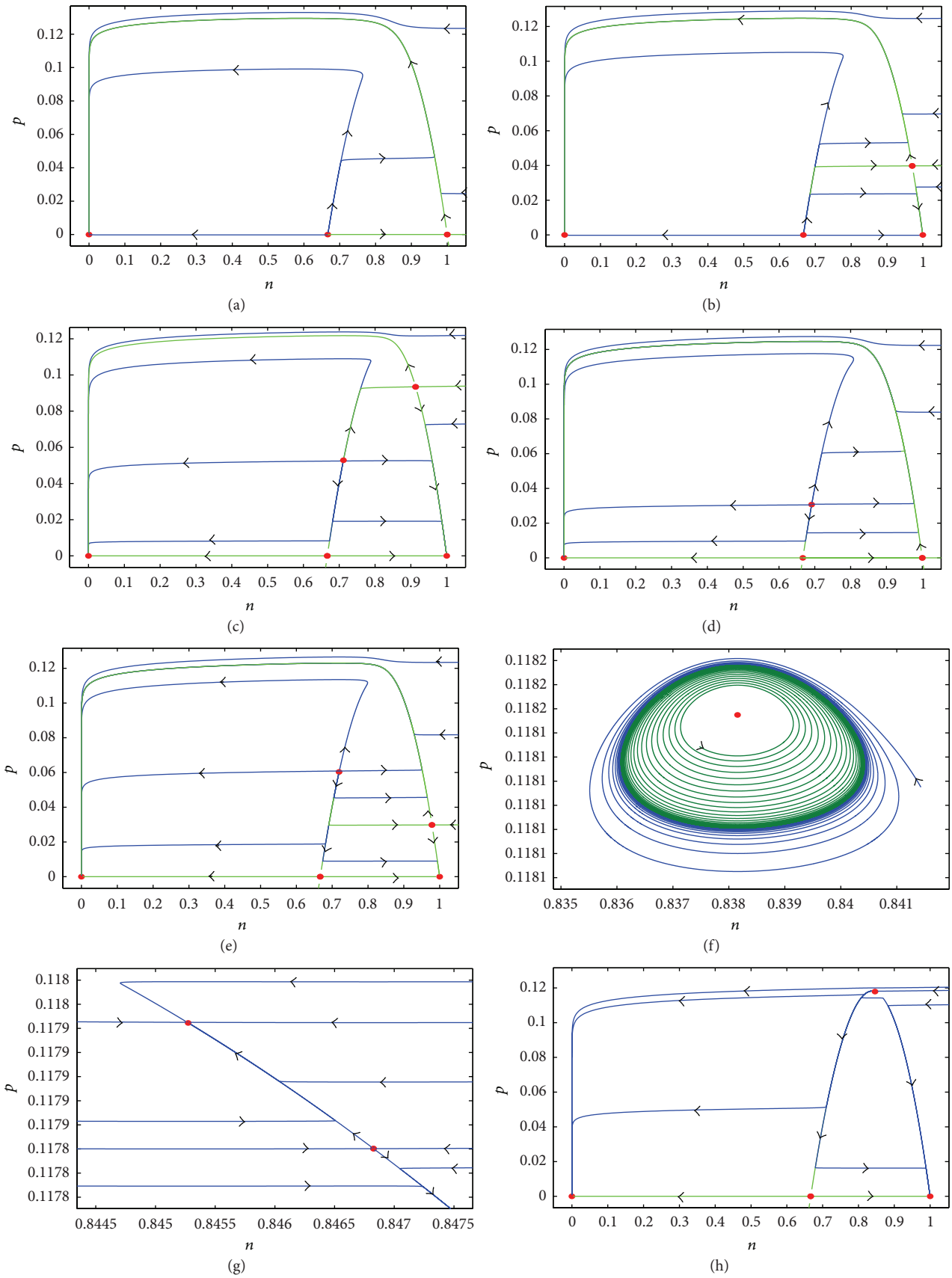

Figure 2: Phase portraits for different concomitant case of the model's equilibria. The horizontal axis is prey population n and the vertical axis is predator population p. The green curves are stable or unstable orbits; the red point is the equilibrium point. We take $m = 1.5$, $b = 2.0$, $s = 0.1$, $e = 0.395$ in (a–c), and $e = 0.405$ in (d–h).

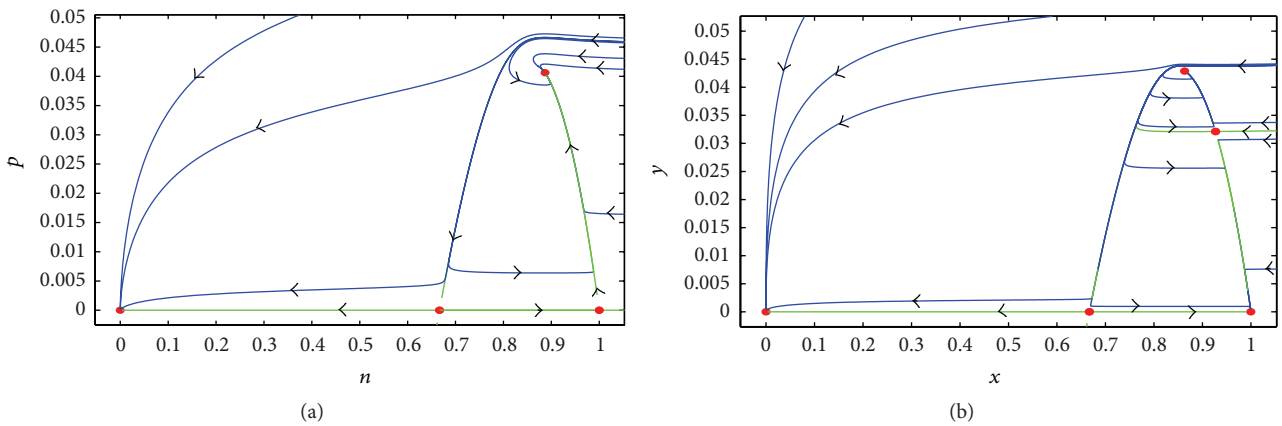

Figure 3: Phase portraits for $m = 1.5$, $s = 0.1$, and $b = 0.2$; (a) $e = 0.844$, $d = 0.7$; (b) $e = 0.599$, $d = 0.5$.

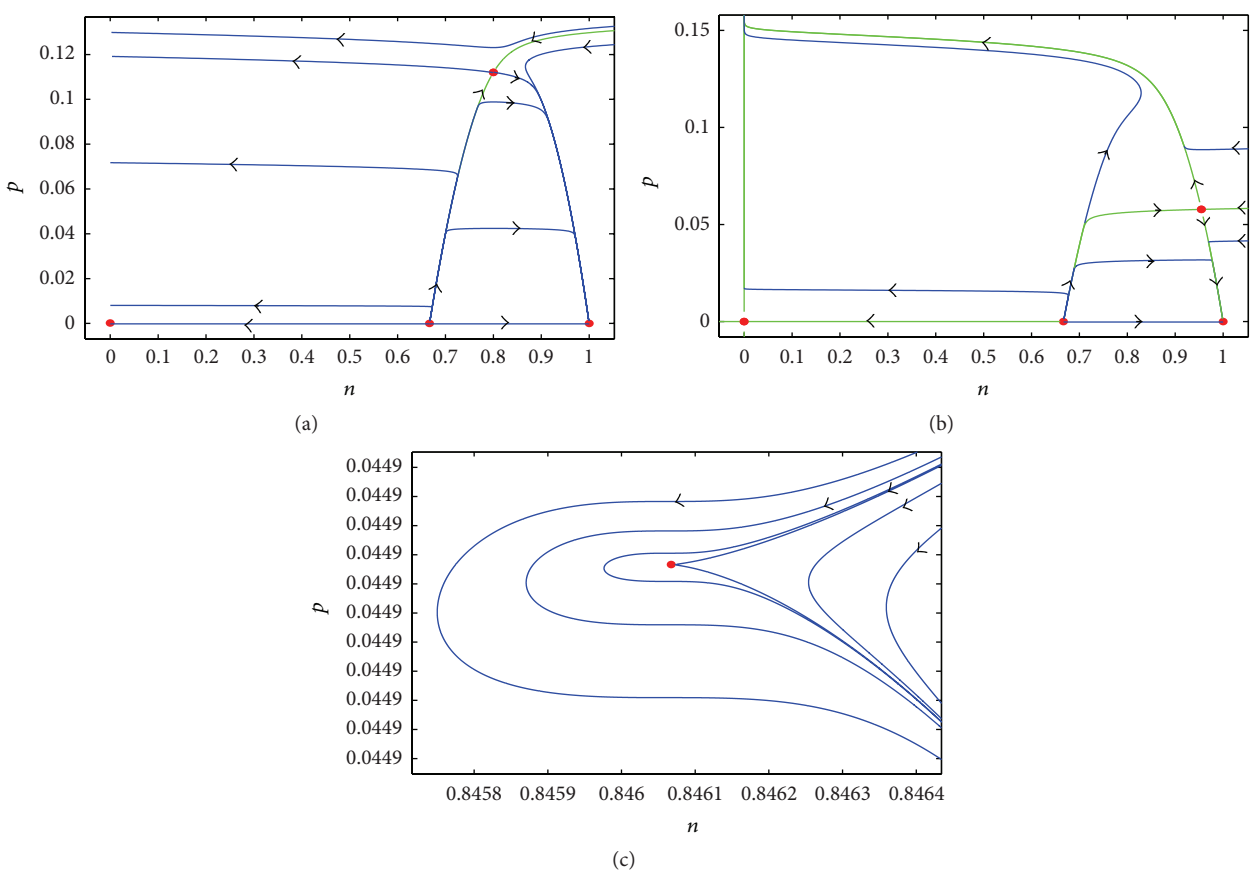

Figure 4: Phase portraits for $m = 1.5$, $s = 0.1$, $e = 0.28$, $b = 2.0$, (a) $d = 0.1$, and (b) $d = 0.095$. The parameter values in image (c) are introduced in Theorem 7.

bifurcation, which can deduce a standard parameter controlled relationship and in turn provide a theoretical basis for the numerical simulation.

Numerical analysis indicates that the dynamic evolutionary mechanism of steady conversion and bifurcation behavior mainly depend on a specific key parameter d. Within this framework, the direct and indirect effects caused by the specific key parameter d are investigated by means of bifurcation analysis and phase diagram. It is obvious to find that the existence and stability of interior equilibria E_3 and E_4 mainly depend on a key parameter d. These results suggest that the key parameter d plays an important role in the prey-predator model. In addition, when the value of the key parameter d is larger than some critical value, the model can possess multiple bifurcation behaviors, such as transcritical bifurcation, saddle-node bifurcation, and Hopf

bifurcation. Thus, it is worthwhile to remark that the key parameter d has a profound effect on the population bifurcation dynamical behaviors. In a word, some key parameters can alter population dynamics and features in prey-predator model. In addition, it is our hope that all these results can be applied in the study of the dynamic complexity of ecosystems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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Research Article

Stochastic Resonance in a Multistable System Driven by Gaussian Noise

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Stochastic resonance (SR) is investigated in a multistable system driven by Gaussian white noise. Using adiabatic elimination theory and three-state theory, the signal-to-noise ratio (SNR) is derived. We find the effects of the noise intensity and the resonance system parameters b, c, and d on the SNR; the results show that SNR is a nonmonotonic function of the noise intensity; therefore, a multistable SR is found in this system, and the value of the peak changes with changing the system parameters.

1. Introduction

Stochastic resonance (SR) is first introduced by Benzi et al. [1] in 1981. In the past decades, SR has received considerable attention in the field of meteorology, and the topic has flourished in physics and neuroscience and weak signal detection [2–6].

There have been many theoretical developments of SR in conventional bistable systems [7–12]. Recently, there have appeared some extensions of SR, such as stochastic resonance in a harmonic oscillator [13], ghost stochastic resonance in the FitzHugh-Nagumo neuron model [14, 15], Transition in a Bistable Duffing System [16], time delay SR [17], trichotomous noise induced SR in a linear system [18], and superthreshold SR [19]. Literature [20–22] proposes a new model of multistable system. However, [7–22] did not study the SNR. In this paper, we use the model of multistable system driven by periodic signal and white noise which can realize the maximum utilization of noise and obtain better detection effects. So it is necessary to discuss the SNR of the multistable system.

In order to describe SR, McNamara and Wiesenfeld [7] introduced the signal-to-noise ratio, which is often used as an indicator of signal processing performance. Numerous studies have been developed to explain SR in continuous time using tools of statistical physics.

Literature [25] studied a solution of Kramers turnover problem for the case of two symmetric deep wells connected through a single shallow well; literature [26] analysed the occurrence of vibrational resonance in a damped quantic oscillator with double-well and triple-well potentials driven by both low-frequency force and high-frequency force; the splitting of the Kramers escape rate in an overdamped system with a triple-well potential was studied in [27].

The paper is organized as follows. In Section 2, we present the model for the multistable system. Then, the expression of the signal-to-noise ratio is derived. In Section 3, the effects of noise intensity and the resonance system parameters b, c, and d on SNR are discussed. A discussion of the effects concludes the paper in Section 4.

2. SNR of Multistable SR

The model of multistable SR is a multistable nonlinear system driven by periodic signal and white noise. The equation can be written as follows:

$$\frac{dx}{dt} = -\frac{dU(x)}{dx} + s(t) + \eta(t), \tag{1}$$

where $s(t) = A\cos(2\pi f t)$ is the input signal, A is the periodic signal amplitude, f is the driving frequency, $\eta(t) = \sqrt{2D}\varepsilon(t)$ in which D is the noise intensity, and $\varepsilon(t)$ represents

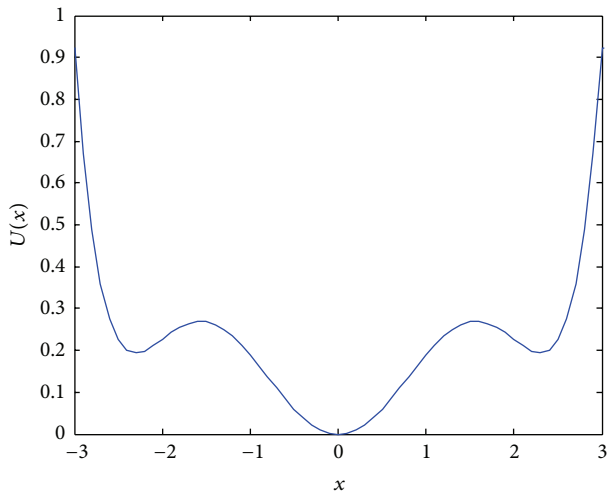

FIGURE 1: The multistable potential function $U(x)$.

a Gaussian white noise with zero mean and unit variance. $x(t)$ is the multistable SR output signal. The potential function for the above multistable system can be denoted as [21, 22]

$$U(x) = \frac{b}{2}x^2 + \frac{c}{4}x^4 + \frac{d}{6}x^6, \tag{2}$$

where b, c, and d are system parameters. As shown in Figure 1, the potential function $U(x)$ is symmetrical and has three stable points ($-x_2$, x_0 and x_2) and two unstable points ($-x_1$, x_1):

$$\begin{aligned} x_0 &= 0, \\ x_1 &= \sqrt{\frac{-1}{2d}\left(c + \sqrt{c^2 - 4bd}\right)}, \\ x_2 &= \sqrt{\frac{-1}{2d}\left(c - \sqrt{c^2 - 4bd}\right)}. \end{aligned} \tag{3}$$

From (1) and (2), the Fokker-Planck equation [26] is given by

$$\begin{aligned} &\frac{\partial \rho(x,t)}{\partial t} \\ &\quad = -\frac{\partial}{\partial x}\left[-bx - cx^3 - dx^5 + A\cos(2\pi f t)\rho(x,t)\right] \\ &\qquad + D\frac{\partial^2}{\partial x^2}\rho(x,t). \end{aligned} \tag{4}$$

Formula (4) contains nonlinear components, so it cannot obtain the steady state solution.

When the input signal and noise intensity are very small,

$$\begin{aligned} A &\ll 1, \\ D &\ll 1. \end{aligned} \tag{5}$$

The whole x area can be divided into three attraction domains; the first is the attraction domain of the steady-state solution $x = -\sqrt{(-1/2d)(c - \sqrt{c^2 - 4bd})}$, the second is the attraction domain of the steady-state solution $x = 0$, and the last is the attraction domain of the steady-state solution $x = \sqrt{(-1/2d)(c - \sqrt{c^2 - 4bd})}$. In the three attraction domains, the total probability of them contains, respectively [20],

$$\begin{aligned} P_1(t) &= \int_{-\infty}^{-x_1} \rho(x,t)\,dx, \\ P_2(t) &= \int_{-x_1}^{x_1} \rho(x,t)\,dx, \\ P_3(t) &= \int_{x_1}^{+\infty} \rho(x,t)\,dx. \end{aligned} \tag{6}$$

Obviously, $P_1(t) + P_2(t) + P_3(t) = 1$, when the frequency of input signal is very low

$$f \ll 1. \tag{7}$$

In the condition of adiabatic approximation, we can get the master equation for the probability of exchange among the three quantities by simplifying (3):

$$\begin{aligned} P_1'(t) &= -R_1(t)P_1(t) + \frac{1}{2}R_2(t)P_2(t) \\ &= \frac{1}{2}R_2(t) - \left[R_1(t) + R_2(t)\right]P_1(t), \\ P_2'(t) &= -R_2(t)P_2(t) + R_1(t)P_1(t) + R_3(t)P_3(t), \\ P_3'(t) &= -R_3(t)P_3(t) + \frac{1}{2}R_2(t)P_2(t), \end{aligned} \tag{8}$$

where $R_{1,2,3}(t)$ are the escape rate [7]. They are considered as function of a weak periodic signal $A\cos(2\pi f t)$, when $A \ll 1$, under the adiabatic approximation, the escape rate of $R_{1,2,3}(t)$ series expansion, ignoring the higher order terms, you can get the following expression:

$$\begin{aligned} R_1 &= R_3 \\ &= \frac{1}{2}\left(R_0 + R_1\beta\cos(2\pi f t) + R_2\beta^2\cos^2(2\pi f t) + \cdots\right), \\ R_2 &= \frac{1}{2}\left(R_0 - R_1\beta\cos(2\pi f t) + R_2\beta^2\cos^2(2\pi f t) - \cdots\right); \end{aligned} \tag{9}$$

then,

$$R_1 + R_2 = R_0 + R_2\beta^2\cos^2(2\pi f t). \tag{10}$$

Equations (8) can be solved as

$$\begin{aligned} P_1(t) &= \frac{1}{4}\left\{e^{-R_0(t-t_0)}\left[2P_1(t_0) - 1 + m\right] + 1 - n\right\}, \\ P_2(t) &= \frac{1}{2}\left\{e^{-R_0(t-t_0)}\left[2P_2(t_0) - 1 - m\right] + 1 + n\right\}, \\ P_3(t) &= \frac{1}{4}\left\{e^{-R_0(t-t_0)}\left[2P_3(t_0) - 1 + m\right] + 1 - n\right\}, \end{aligned} \tag{11}$$

where

$$m = \frac{R_1\beta\cos(2\pi f t_0 - \theta)}{\left(R_0{}^2 + (2\pi f)^2\right)^{1/2}},$$
$$n = \frac{R_1\beta\cos(2\pi f t - \theta)}{\left(R_0{}^2 + (2\pi f)^2\right)^{1/2}},$$
$$\sin\theta = \frac{2\pi f}{\left(R_0{}^2 + (2\pi f)^2\right)^{1/2}},$$
$$\cos\theta = \frac{R_0}{\left(R_0{}^2 + (2\pi f)^2\right)^{1/2}}. \tag{12}$$

When $t_0 \to -\infty$, $P_{1,2,3}(t)$ approaches $P^s_{1,2,3}(t)$:

$$P^s_1(t) = P^s_3(t) = \frac{1}{4}(1-n),$$
$$P^s_2(t) = \frac{1}{2}(1+n). \tag{13}$$

Let $P_i(t+\tau \mid j,t)$ donate the probability to the system which is in j area at t moment when it is in i area at $t+\tau$ moment $(i, j = 1, 2, 3)$:

$$P_1(t+\tau \mid 1,t) = \frac{1}{4}\left[e^{-R_0\tau}(-1+n)+1-q\right],$$
$$P_1(t+\tau \mid 2,t) = \frac{1}{4}\left[e^{-R_0\tau}(1+n)+1-q\right],$$
$$P_2(t+\tau \mid 2,t) = \frac{1}{2}\left[e^{-R_0\tau}(1-n)+1+q\right],$$
$$P_2(t+\tau \mid 1,t) = \frac{1}{4}\left[e^{-R_0\tau}(-1-n)+1+q\right],$$
$$P_2(t+\tau \mid 3,t) = \frac{1}{4}\left[e^{-R_0\tau}(-1-n)+1+q\right],$$
$$P_3(t+\tau \mid 2,t) = \frac{1}{4}\left[e^{-R_0\tau}(1+n)+1-q\right],$$
$$P_3(t+\tau \mid 3,t) = \frac{1}{4}\left[e^{-R_0\tau}(-1+n)+1-q\right]. \tag{14}$$

In the progressive state, the correlation function of random variable is given by

$$\begin{aligned}
\langle x(t)\,x(t+\tau)\rangle &= \lim_{t_0\to-\infty}\iint xyP_y(t+\tau \mid x,t)\,\rho(x,t)\\
&= (-x_2)(-x_2)P_1(t+\tau \mid 1,t)P^s_1(t)\\
&\quad + (-x_2)x_0P_2(t+\tau \mid 1,t)P^s_1(t)\\
&\quad + x_0x_0P_2(t+\tau \mid 2,t)P^s_2(t)\\
&\quad + x_0(-x_2)P_1(t+\tau \mid 2,t)P^s_2(t)\\
&\quad + x_0x_2P_3(t+\tau \mid 2,t)P^s_2(t)\\
&\quad + x_2x_2P_3(t+\tau \mid 3,t)P^s_3(t)\\
&\quad + x_2x_0P_2(t+\tau \mid 3,t)P^s_3(t)\\
&= \frac{1}{8}x_2^2\left[e^{-R_0|\tau|}\left(-1+2n-n^2\right)+1-q-n+nq\right]\\
&\quad + \frac{1}{4}x_0^2\left[e^{-R_0|\tau|}\left(1-n^2\right)+1+q+n+nq\right].
\end{aligned} \tag{15}$$

The correlation function is not only related with the time interval but also related with the start value of the time. So we take the average value of the correlation function

$$\begin{aligned}
\langle x(t)\,x(t+\tau)\rangle_{\text{Average}} &= f\int_0^{1/f}\langle x(t)\,x(t+\tau)\rangle\,dt\\
&= \left(\frac{-c}{8d} - \frac{1}{4}\sqrt{\frac{b}{d}}\right)\\
&\quad\cdot\left\{e^{-R_0|\tau|}\left[-1-\frac{R_1^2\beta^2}{2\left(R_0{}^2+(2\pi f)^2\right)}\right]+1\right.\\
&\quad\left.+\frac{R_1^2\beta^2\cos(2\pi f\tau)}{2\left(R_0{}^2+(2\pi f)^2\right)}\right\}+\left[\frac{-1}{8d}\left(c+\sqrt{c^2-4bd}\right)\right]\\
&\quad\cdot\left\{e^{-R_0|\tau|}\left[1-\frac{R_1^2\beta^2}{2\left(R_0{}^2+(2\pi f)^2\right)}\right]+1\right.\\
&\quad\left.+\frac{R_1^2\beta^2\cos(2\pi f\tau)}{2\left(R_0{}^2+(2\pi f)^2\right)}\right\}.
\end{aligned} \tag{16}$$

Within the deduction made above, the output power spectral density of a multistable SR system can be obtained:

$$\begin{aligned}
S(w) &= \int_{-\infty}^{+\infty}\langle x(t)\,x(t+\tau)\rangle_{\text{Average}}\,e^{-iw\tau}d\tau\\
&= S_1(w) + S_2(w),
\end{aligned} \tag{17}$$

where $S_1(w)$ and $S_2(w)$ are the power spectral densities of the output signal and the output noise, which are derived

from the periodic input signal and the noise, respectively, as follows:

$$S_1(w) = \left(\frac{-c}{8d} - \frac{1}{4}\sqrt{\frac{b}{d} - \frac{c+\sqrt{c^2-4bd}}{8d}}\right) \cdot \frac{\pi R_1^2 \beta^2}{2\left(R_0{}^2 + (2\pi f)^2\right)}\delta(w - 2\pi f),$$
$$S_2(w) = \left(\frac{c}{8d} + \frac{1}{4}\sqrt{\frac{b}{d} - \frac{c+\sqrt{c^2-4bd}}{8d}}\right) \cdot \frac{2R_0}{R_0{}^2 + (w)^2}. \tag{18}$$

Put $A\cos(2\pi ft)$ as constant processing; we can get the steady state solution of the available equation (4), the potential function of $\Phi(x)$:

$$\Phi(x) = \frac{b}{2}x^2 + \frac{c}{4}x^4 + \frac{d}{6}x^6 - Ax\cos(2\pi ft). \tag{19}$$

The probability transition rate of type 1 can be obtained:

$$R_1(t) = \frac{\left|U''(-x_1)U''(-x_2)\right|^{1/2}}{2\pi} \cdot \exp\left\{-\frac{\Phi(-x_1) - \Phi(-x_2)}{D}\right\}. \tag{20}$$

Make $x_1 = 0$,

$$\begin{aligned}
x_0 &= -\sqrt{\frac{-1}{2d}\left(c + \sqrt{c^2-4bd}\right)},\\
x_2 &= \sqrt{\frac{-1}{2d}\left(c - \sqrt{c^2-4bd}\right)} - \sqrt{\frac{-1}{2d}\left(c + \sqrt{c^2-4bd}\right)},\\
R_0 &= 2\cdot R_1(t)\big|_{A\cos(2\pi ft)=0}\\
&= \frac{\sqrt{-9b^2 - 4bc^2/d + 25b^2/d^2}}{\pi}\\
&\quad \cdot e^{-(-b\sqrt{c^2-4bd}/2d + c^2\sqrt{c^2-4bd}/4d^2 + (-c^2\sqrt{c^2-4bd} - c^3 + 4bcd)/3)/D},\\
\frac{1}{2}R_1 &= -\left.\frac{dR_1(t)}{d(A\cos(2\pi ft))}\right|_{A\cos(2\pi ft)=0},\\
R_1\beta &= \frac{R_0 A(x_2 - x_1)}{D}.
\end{aligned} \tag{21}$$

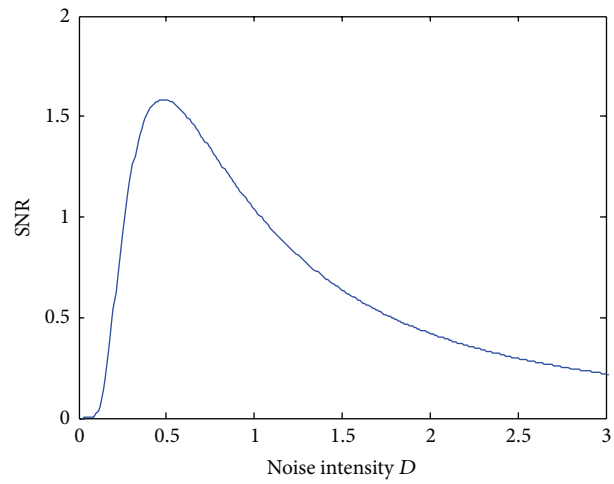

Figure 2: SNR versus noise intensity D with $b = 0.52$, $c = -0.31$, and $d = 0.04$.

To clearly describe the energy distribution of the system output, the SNR of the system output can be calculated as follows:

$$\begin{aligned}
\mathrm{SNR} &= \frac{\int_0^\infty S_1(w)\,dw}{S_2(w = 2\pi f)}\\
&= \frac{-c/8d - (1/4)\sqrt{b/d} - \left(c + \sqrt{c^2-4bd}\right)/8d}{c/8d + (1/4)\sqrt{b/d} - \left(c + \sqrt{c^2-4bd}\right)/8d}\\
&\quad \cdot \frac{\pi R_0 A^2\left(-c/d - 2\sqrt{b/d}\right)}{4D^2}.
\end{aligned} \tag{22}$$

3. The Effects of the Noise Intensity and System Parameters

In this section, we discuss the effect of each parameter on the system SNR.

Figure 2 shows the change trends of the SNR of a multistable SR method with $b = 0.52$, $c = -0.31$, and $d = 0.04$ versus noise intensity D.

It can be seen from Figure 2 that the change curve of the SNR is first increased and then decreased with the variation in noise intensity D; therefore, there exists an optimal noise for the maximum SNR. This typical phenomenon is a signature of multistable SR. Noise plays a role in the SNR within certain range of scale.

The SNR as a function of noise intensity D with different system parameters b is shown in Figure 3. It is seen that the positions of the higher peaks and the lower peaks are both shifting to the left with the increase of b and the SNR is decreasing with the increase of b.

Figure 4 shows the curves of SNR versus noise intensity D with different system parameters c. With the increase of c, the whole curves are shifting to left and SNR is increasing.

Figure 5 shows the curves of SNR versus noise intensity D with different system parameters d. With the increase of d, the whole curves are shifting to the left and the SNR is increasing.

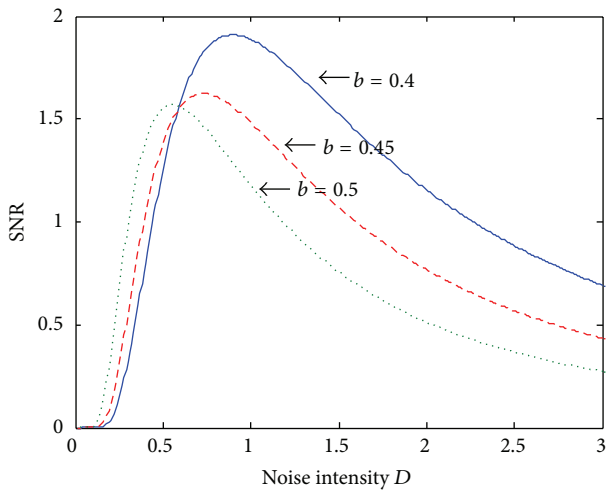

FIGURE 3: SNR versus noise intensity D for different system parameters b: 0.4, 0.45, and 0.5. Other parameters are $c = -0.31$ and $d = 0.04$.

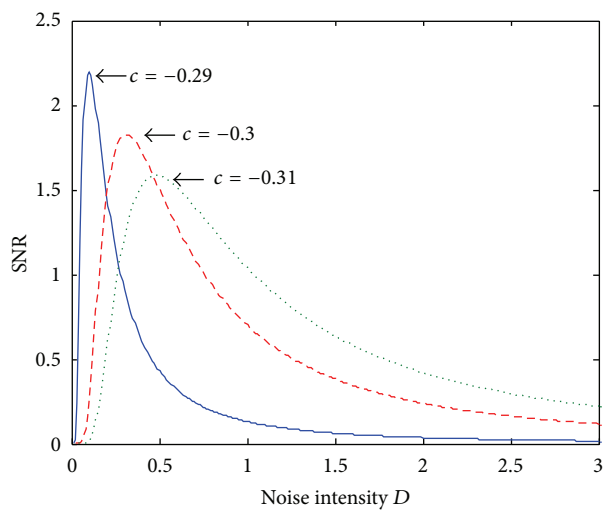

FIGURE 4: SNR versus noise intensity D for different system parameters c: −0.31, −0.3, and −0.29. Other parameters are $b = 0.52$ and $d = 0.04$.

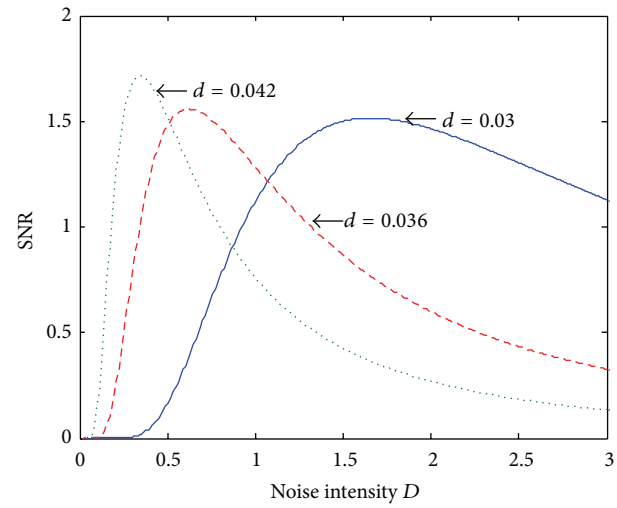

FIGURE 5: SNR versus noise intensity D for different system parameters d: 0.03, 0.036, and 0.042. Other parameters are $b = 0.52$ and $c = -0.31$.

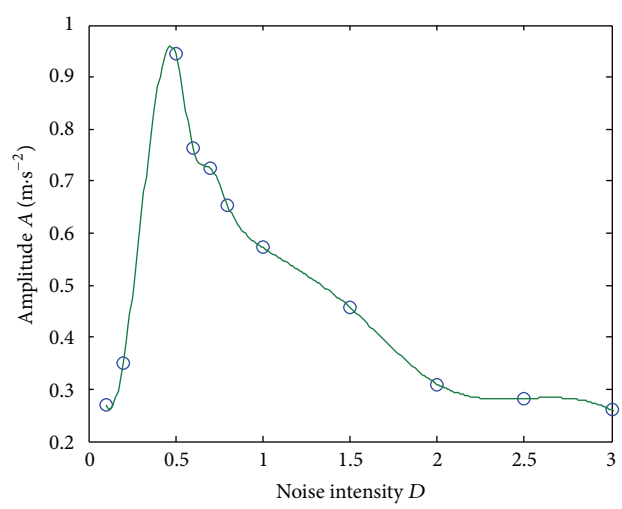

FIGURE 6: The variation curve of the output signal amplitude with the addition of noise D.

4. The Simulation

Take the same parameters as in Figure 2 to detect the weak signal with the multistable stochastic resonance and then let D take different values; and the amplitude of the corresponding characteristic frequency is recorded; finally, the curve of amplitude versus the noise is made. It can be seen that the simulation result in Figure 6 is consistent with the analysis in Figure 2.

Take the same parameters as in Figure 3 to detect the weak signal with the multistable stochastic resonance. First, take b equal to 0.4 and let D take N different values; then, the amplitude of the corresponding characteristic frequency is recorded and the curve of amplitude versus the noise is finally made. Second, take b equal to 0.45 and 0.5 and repeat the above operation, respectively. It can be seen that the simulation result in Figure 7 is consistent with the analysis in Figure 3.

Take the same parameters as in Figure 4 to detect the weak signal with the multistable stochastic resonance. First, take c equal to −0.31 and let D take N different values; then, the amplitude of the corresponding characteristic frequency is recorded and the curve of amplitude versus the noise is finally made. Second, take c equal to −0.3 and −0.29 and repeat the above operation, respectively. It can be seen that the simulation result in Figure 8 is consistent with the analysis in Figure 4.

Take the same parameters as in Figure 5 to detect the weak signal with the multistable stochastic resonance. First, take d equal to 0.03 and let D take N different values; then, the amplitude of the corresponding characteristic frequency is recorded and the curve of amplitude versus the noise is

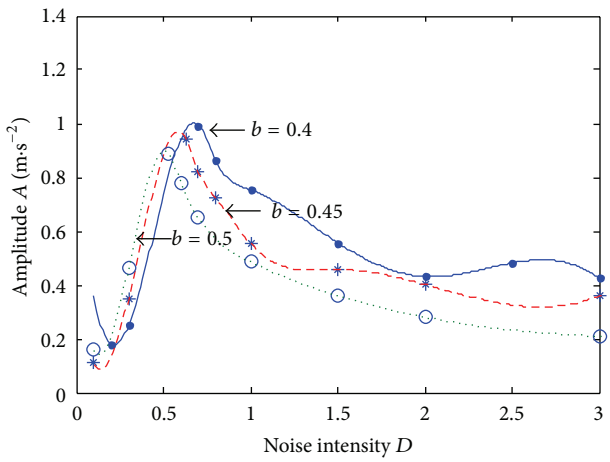

Figure 7: The variation curve of the output signal amplitude with the addition of noise D under different b value.

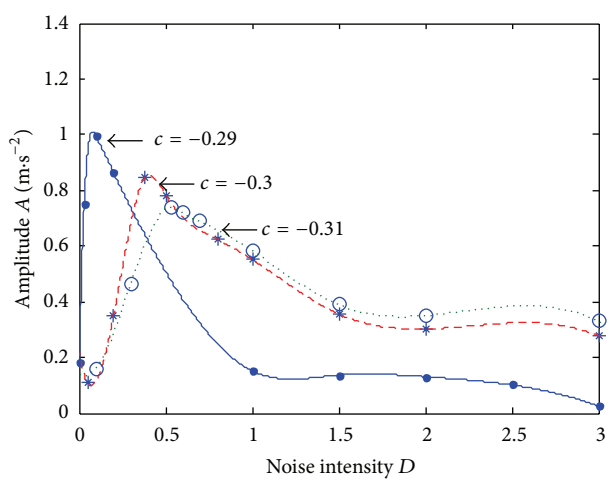

Figure 8: The variation curve of the output signal amplitude with the addition of noise D under different c value.

finally made. Second, take d equal to 0.036 and 0.042 and repeat the above operation, respectively. It can be seen that the simulation result in Figure 9 is consistent with the analysis in Figure 5.

5. Conclusion

In the paper, we first derive the expression of the multistable system SNR. Through the research about the effects of Gauss noise and system parameters on the multistable system SNR, we can draw the following conclusions: (1) the SNR expression is applicable to arbitrary signal amplitude; (2) the curve of the SNR versus noise intensity is nonmonotonic, which is a typical phenomenon of multistable SR; (3) the SNR peak is increasing gradually with the increase of system parameters c and d, but it is decreasing with the increase of system parameters b. The SNR as a function of system parameters b, c, and d will not be described in this paper.

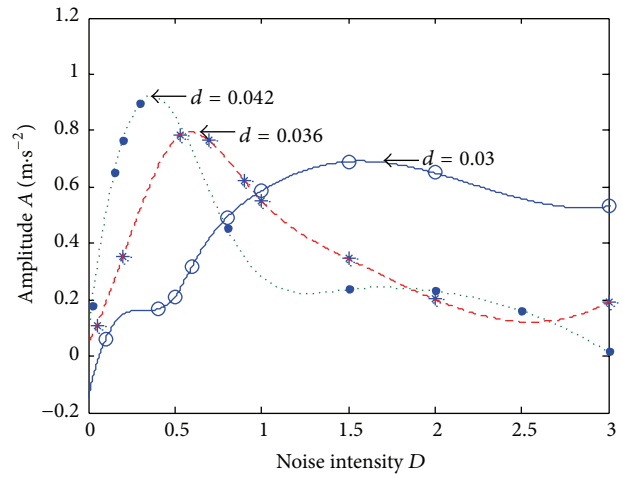

Figure 9: The variation curve of the output signal amplitude with the addition of noise D under different d value.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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Research Article

The Contrast Structures for a Class of Singularly Perturbed Systems with Heteroclinic Orbits

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Singularly perturbed problems are often used as the models of ecology and epidemiology. In this paper, a class of semilinear singularly perturbed systems with contrast structures are discussed. Firstly, we verify the existence of heteroclinic orbits connecting two equilibrium points about the associated systems for contrast structures in the corresponding phase space. Secondly, the asymptotic solutions of the contrast structures by the method of boundary layer functions and smooth connection are constructed. Finally, the uniform validity of the asymptotic expansion is defined and the existence of the smooth solutions is proved.

1. Introduction

Contrast structures in singularly perturbed problems can be classified as step-type contrast structures or spike-type contrast structures [1–3]. The existence of contrast structures is relevant to the existence of homoclinic orbits and heteroclinic orbits of their associated systems in [4–6]. Recently, contrast structures in singularly perturbed problems are attached great importance. Ni and Wang study four-dimensional contrast structures in singularly perturbed problems with fast variables in [7]. In [8], Wang considers a kind of step-type contrast structure for singularly perturbed problems with slow and fast variables and proves the existence of a heteroclinic orbit of its associated system in the corresponding phase space.

Since contrast structures can express the instantaneous transformation more accurately, we often use them in singularly perturbed problems as the models of the collision of cars and the transfer law of neurons. In [9], a kind of epidemical model with spike-type contrast structures is proposed. Chattoadhyay and Bairagi build an ecoepidemiological model in [10]:

$$\begin{aligned}
\frac{du}{dt} &= d\Delta u + f_1(u,v), \quad (t,x) \in (0,+\infty)\times\Omega,\\
\frac{dv}{dt} &= d\Delta v + f_2(u,v), \quad (t,x) \in (0,+\infty)\times\Omega,\\
\frac{dw}{dt} &= d\Delta w + f_3(u,v), \quad (t,x) \in (0,+\infty)\times\Omega,\\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, \quad (t,x) \in (0,+\infty)\times\partial\Omega,\\
&u(x,0) \ge 0,\ v(x,0) \ge 0,\ w(x,0) \ge 0,\ x \in \Omega,
\end{aligned} \tag{1}$$

where d is a diffusion coefficient, Ω is a population habitat, u is susceptible prey, v is infected prey, w is density of predators, and n is a unit outer normal vector. By geometric methods and functional skills, Chattoadhyay and Bairagi study the existence of $u(x)$, which is a stable solution.

Considering the complexity of the ecoepidemiological model and the small parameter d, we propose the following semilinear singularly perturbed system with contrast structures:

$$\begin{aligned}
\frac{dz}{dt} &= g(u,v,w,z,t),\\
\mu\frac{du}{dt} &= f_1(u,v,w,z,t),\\
\mu\frac{dv}{dt} &= f_2(u,v,w,z,t),\\
\mu\frac{dw}{dt} &= f_3(u,v,w,z,t),
\end{aligned}$$

$$u(0,\mu) = u^0,$$
$$v(0,\mu) = v^0,$$
$$w(1,\mu) = w^1,$$
$$z(0,\mu) = z^0, \tag{2}$$

where $0 < \mu \ll 1, 0 \le t \le 1$, the functions f_1, f_2, f_3, and g are sufficiently smooth on the domain $\mathbf{G} = \{(u,v,w,z,t) \mid \|u\| \le l_1, \|v\| \le l_2, \|w\| \le l_3, \|z\| \le l_4, 0 \le t \le 1\}$, and l_i $(i = 1,2,3,4)$ are given positive real numbers. Assuming that $\mathbf{y} = (u,v,w)^T$ and $\mathbf{f} = (f_1, f_2, f_3)^T$, system (2) is equivalent to the following system:

$$\begin{aligned} \frac{dz}{dt} &= g(\mathbf{y}, z, t), \\ \mu\frac{d\mathbf{y}}{dt} &= \mathbf{f}(\mathbf{y}, z, t), \\ u(0,\mu) &= u^0, \\ v(0,\mu) &= v^0, \\ w(1,\mu) &= w^1, \\ z(0,\mu) &= z^0. \end{aligned} \tag{3}$$

The degenerate equations of (3) are

$$\begin{aligned} \mathbf{f}(\bar{\mathbf{y}}, \bar{z}, t) &= 0, \\ \frac{d\bar{z}}{dt} &= g(\bar{\mathbf{y}}, \bar{z}, t). \end{aligned} \tag{4}$$

In the following, we let

(A1) the degenerate equation $\mathbf{f}(\bar{\mathbf{y}}, \bar{z}, t) = 0$ have two isolated smooth solutions $\bar{\mathbf{y}} = \alpha(\bar{z}, t)$ and $\bar{\mathbf{y}} = \beta(\bar{z}, t)$ on $D = \{(\bar{z}, t) \mid |\bar{z}| \le l_5, 0 \le t \le 1\}$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$, $\beta = (\beta_1, \beta_2, \beta_3)^T$, and l_5 is a given positive real number.

According to (A1), the initial value problem

$$\frac{d\bar{z}^{(-)}}{dt} = g\left(\alpha\left(\bar{z}^{(-)}, t\right), \bar{z}^{(-)}, t\right), \quad \bar{z}^{(-)}(0) = z^0, \tag{5}$$

has the unique solution $\bar{z}^{(-)}(t)$ in $[0,1]$, $0 < t_0 < 1$, and

$$\frac{d\bar{z}^{(+)}}{dt} = g\left(\beta\left(\bar{z}^{(+)}, t\right), \bar{z}^{(+)}, t\right), \quad \bar{z}^{(+)}(t_0) = \bar{z}^{(-)}(t_0), \tag{6}$$

has the unique solution $\bar{z}^{(+)}(t)$ in [0,1]. The associated system of (3) is

$$\frac{d\tilde{\mathbf{y}}}{d\tau} = \mathbf{f}(\tilde{\mathbf{y}}, \bar{z}, \bar{t}), \quad \tau = \frac{\mu}{t} \ge 0, \tag{7}$$

where $\tilde{\mathbf{y}}(\tau) = (\tilde{u}(\tau), \tilde{v}(\tau), \tilde{w}(\tau))^T$ and $\bar{t}$ and t are parameters. Obviously, system (7) exhibits two families of equilibrium points $M_1(\alpha(\bar{z}^{(-)}, \bar{t}), \bar{z}^{(-)}, \bar{t})$ and $M_2(\beta(\bar{z}^{(+)}, \bar{t}), \bar{z}^{(+)}, \bar{t})$. Assuming that $B_l(t) = D_{\tilde{\mathbf{y}}}\mathbf{f}(\tilde{\mathbf{y}}, z, t)|_{M_l}$, $l = 1,2$, we can classify sixteen cases (can be seen in [11, 12]) on relations between M_1 and M_2 by the symbols of eigenvalues. There might exist interior layers satisfying one of the following cases:

(1) $M_1[-,-,+]$, $M_2[-,-,+]$; (2) $M_1[-,+,+]$, $M_2[-,+,+]$. We will discuss case (1); case (2) can be debated similarly.

(A2) Assume that $B_l(t)$ has three real-valued eigenvalues and satisfies the inequality $\lambda_{l1} < \lambda_{l2} < 0 < \lambda_{l3}$.

By (A2), system (7) may exhibit a heteroclinic orbit connecting M_1 to M_2. To give the necessary conditions about the existence of a heteroclinic orbit, we introduce the following hypothesis:

(A3) The associated system (7) has two manifolds expressed by $\beta_l(\tilde{\mathbf{y}}, \tilde{z}, \bar{t}) = C_l$, $l = 1,2$.

The manifold crossing through M_1 is

$$\Phi_l(\tilde{\mathbf{y}}, \tilde{z}, \bar{t}) = \Phi_l^{(-)}\left(\alpha\left(\bar{z}^{(-)}, \bar{t}\right), \bar{z}^{(-)}, \bar{t}\right). \tag{8}$$

The manifold crossing through M_2 is

$$\Phi_l(\tilde{\mathbf{y}}, \tilde{z}, \bar{t}) = \Phi_l^{(+)}\left(\beta\left(\bar{z}^{(+)}, \bar{t}\right), \bar{z}^{(+)}, \bar{t}\right). \tag{9}$$

According to (8) and (9), the necessary conditions about the existence of heteroclinic orbits can be obtained by

$$\Phi_l^{(-)}\left(\alpha\left(\bar{z}^{(-)}, \bar{t}\right), \bar{z}^{(-)}, \bar{t}\right) = \Phi_l^{(+)}\left(\beta\left(\bar{z}^{(+)}, \bar{t}\right), \bar{z}^{(+)}, \bar{t}\right). \tag{10}$$

By (A3), (8) and (9) can be expressed by

$$\begin{aligned} \tilde{v}^{(-)}(\tau) &= \Psi_2^{(-)}\left(\tilde{u}^{(-)}, \bar{z}^{(-)}, \bar{t}\right), \\ \tilde{w}^{(-)}(\tau) &= \Psi_3^{(-)}\left(\tilde{u}^{(-)}, \bar{z}^{(-)}, \bar{t}\right), \\ \tilde{v}^{(+)}(\tau) &= \Psi_2^{(+)}\left(\tilde{u}^{(+)}, \bar{z}^{(+)}, \bar{t}\right), \\ \tilde{w}^{(+)}(\tau) &= \Psi_3^{(+)}\left(\tilde{u}^{(+)}, \bar{z}^{(+)}, \bar{t}\right). \end{aligned} \tag{11}$$

Supposing that

$$H_h(\bar{t}) = \Psi_h^{(-)}\left(\tilde{u}^{(-)}, \bar{z}^{(-)}, \bar{t}\right) - \Psi_h^{(+)}\left(\tilde{u}^{(+)}, \bar{z}^{(+)}, \bar{t}\right), \quad h = 2,3; \tag{12}$$

we can give the following hypothesis:

(A4) Assume that (12) has a solution of $\bar{t} = t_0$, and $(\partial H_2/\partial \bar{t})|_{\bar{t}=t_0}$, $(\partial H_3/\partial \bar{t})|_{\bar{t}=t_0}$ are not simultaneously equal to zero.

In accordance with (A4), we can realize that $H_h(t_0) = 0$, namely, there exists a heteroclinic orbit connecting M_1 and M_2.

Lemma 1. *Under conditions (A1)–(A4), the associated system (7) has a heteroclinic orbit connecting M_1 and M_2, which is expressed by (10) and (12); therefore, system (3) has the solution with interior layer.*

2. The Construction of Asymptotic Expansion

In accordance with (A1)–(A4), system (3) has a step-like solution from $\alpha(t)$ to $\beta(t)$. Hence, we can suppose that

$$t^* = t_0 + \mu t_1 + \cdots + \mu^k t_k + \cdots,$$
$$\mathbf{x}(t^*, \mu) = \mathbf{x}_0^* + \mu \mathbf{x}_1^* + \cdots + \mu^k \mathbf{x}_k^* + \cdots, \tag{13}$$

where t^* is the transit point from $\alpha(t)$ to $\beta(t)$ and t_i and $\mathbf{x}_i^*$, $i = 1, 2, \ldots$, are undetermined coefficients. The interior-layer solution of system (3) can be divided into two parts. The left problem is $(0 \le t \le t^*)$

$$\begin{aligned}
\frac{dz^{(-)}}{dt} &= g\left(\mathbf{y}^{(-)}, z^{(-)}, t\right),\\
\mu\frac{d\mathbf{y}^{(-)}}{dt} &= \mathbf{f}\left(\mathbf{y}^{(-)}, z^{(-)}, t\right),\\
u^{(-)}(0, \mu) &= u^0,\\
v^{(-)}(0, \mu) &= v^0,\\
w^{(-)}(t^*, \mu) &= w^*,\\
z^{(-)}(0, \mu) &= z^0.
\end{aligned} \tag{14}$$

The right problem is $(t^* \le t \le 1)$

$$\begin{aligned}
\frac{dz^{(+)}}{dt} &= g\left(\mathbf{y}^{(+)}, z^{(+)}, t\right),\\
\mu\frac{d\mathbf{y}^{(+)}}{dt} &= \mathbf{f}\left(\mathbf{y}^{(+)}, z^{(+)}, t\right),\\
u^{(+)}(t^*, \mu) &= u^*,\\
v^{(+)}(t^*, \mu) &= v^*,\\
w^{(+)}(1, \mu) &= w^1,\\
z^{(+)}(t^*, \mu) &= z^{(-)}(t^*, \mu).
\end{aligned} \tag{15}$$

The step-like contrast structure of (3) can be regarded as the smooth connection at the point of t^* by two solutions of (14) and (15). Let $\mathbf{x} = (u, v, w, z)^T$, and by the method of boundary layer functions [8–10], the asymptotic expansion of (14) can be constructed as follows:

$$\mathbf{x}^{(-)}(t, \mu) = \sum_{k=0}^{\infty} \mu^k \left[\overline{\mathbf{x}}_k^{(-)}(t) + L_k \mathbf{x}(\tau_0) + Q_k^{(-)} \mathbf{x}(\tau)\right],$$
$$\tau_0 = \frac{t}{\mu}, \quad \tau = \frac{t - t^*}{\mu}. \tag{16}$$

Also, the asymptotic expansion of (15) can be constructed as follows:

$$\mathbf{x}^{(+)}(t, \mu) = \sum_{k=0}^{\infty} \mu^k \left[\overline{\mathbf{x}}_k^{(+)}(t) + Q_k^{(+)} \mathbf{x}(\tau) + R_k \mathbf{x}(\tau_1)\right],$$
$$\tau_1 = \frac{t - 1}{\mu}, \tag{17}$$

where $\overline{\mathbf{x}}_k^{(\pm)}(t)$ are the coefficients of regular series, $L_k\mathbf{x}(\tau_0)$ are the coefficients of left boundary layer series, $R_k\mathbf{x}(\tau_1)$ are the coefficients of right boundary layer series, and $Q_k^{(\pm)}\mathbf{x}(\tau)$ are the coefficients of interior-layer series.

Putting (16) and (17) into (14) and (15), and separating equations by scales t, τ_0, τ, and τ_1, then $\overline{\mathbf{x}}_0^{(\pm)}(t)$ satisfies

$$\begin{aligned}
&\mathbf{f}\left(\overline{\mathbf{y}}_0^{(-)}, \overline{z}_0^{(-)}, t\right) = 0,\\
&\left(\overline{z}_0^{(-)}\right)' = g\left(\overline{\mathbf{y}}_0^{(-)}, \overline{z}_0^{(-)}, t\right),\\
&\overline{z}_0^{(-)}(0) = z^0,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\mathbf{f}\left(\overline{\mathbf{y}}_0^{(+)}, \overline{z}_0^{(+)}, t\right) = 0,\\
&\left(\overline{z}_0^{(+)}\right)' = g\left(\overline{\mathbf{y}}_0^{(+)}, \overline{z}_0^{(+)}, t\right),\\
&\overline{z}_0^{(+)}(t_0) = \overline{z}_0^{(-)}(t_0).
\end{aligned} \tag{19}$$

By (A2), we can obtain that $\overline{z}_0^{(-)} = \overline{z}^{(-)}(t)$, $\overline{\mathbf{y}}_0^{(-)} = \alpha(\overline{z}^{(-)}(t), t) = \alpha(t, t_0)$, $\overline{z}_0^{(+)} = \overline{z}^{(+)}(t)$, and $\overline{\mathbf{y}}_0^{(+)} = \beta(\overline{z}^{(+)}(t), t) = \beta(t, t_0)$, where $z^{(\pm)}(t)$ are functions about t and t_0 and t_0 is an undetermined constant.

For $Q_0^{(-)}(\tau)$, we have

$$\begin{aligned}
&\frac{dQ_0^{(-)}z(\tau)}{d\tau} = 0,\\
&\frac{dQ_0^{(-)}\mathbf{y}(\tau)}{d\tau}\\
&\quad = \mathbf{f}\left(\alpha(t_0) + Q_0^{(-)}\mathbf{y}(\tau), \overline{z}_0^{(-)} + Q_0^{(-)}z(\tau), t_0\right),\\
&\alpha_1(t_0) + Q_0^{(-)}u(0) = u^*,\\
&Q_0^{(-)}\mathbf{x}(-\infty) = 0.
\end{aligned} \tag{20}$$

By (20), we can solve $Q_0^{(-)}z(\tau) \equiv 0$. Assuming that $\widetilde{\mathbf{y}}^l = \alpha(t_0) + Q_0^{(-)}\mathbf{y}(\tau)$, (20) can be rewritten as

$$\begin{aligned}
\frac{d\widetilde{\mathbf{y}}^l}{d\tau} &= \mathbf{f}\left(\widetilde{\mathbf{y}}^l, \overline{z}_0^{(-)}(t_0), t_0\right),\\
\widetilde{u}^l(0) &= u^*,\\
\widetilde{\mathbf{y}}^l(-\infty) &= \alpha(t_0).
\end{aligned} \tag{21}$$

For $Q_0^{(+)}(\tau)$, we have

$$\begin{aligned}
&\frac{dQ_0^{(+)}z(\tau)}{d\tau} = 0,\\
&\frac{dQ_0^{(+)}\mathbf{y}(\tau)}{d\tau}\\
&\quad = \mathbf{f}\left(\beta(t_0) + Q_0^{(+)}\mathbf{y}(\tau), \overline{z}_0^{(+)} + Q_0^{(+)}z(\tau), t_0\right),\\
&\beta_1(t_0) + Q_0^{(+)}u(0) = u^*,\\
&Q_0^{(+)}\mathbf{x}(+\infty) = 0.
\end{aligned} \tag{22}$$

By (22), we can solve $Q_0^{(+)}z(\tau) \equiv 0$. Supposing that $\tilde{\mathbf{y}}^r = \beta(t_0) + Q_0^{(+)}\mathbf{y}(\tau)$, (22) can be rewritten as

$$\begin{aligned} \frac{d\tilde{\mathbf{y}}^r}{d\tau} &= \mathbf{f}\left(\tilde{\mathbf{y}}^r, \overline{z}_0^{(+)}(t_0), t_0\right), \\ \tilde{u}^r(0) &= u^*, \\ \tilde{\mathbf{y}}^r(+\infty) &= \beta(t_0). \end{aligned} \tag{23}$$

Obviously, systems (21) and (23) coincide with the associated system (7), so we can consider their combined system

$$\begin{aligned} \frac{d\tilde{\mathbf{y}}}{d\tau} &= \mathbf{f}\left(\tilde{\mathbf{y}}, \overline{z}_0(t_0), t_0\right), \\ \tilde{u}(0) &= u^*, \\ \tilde{\mathbf{y}}(+\infty) &= \beta(t_0), \\ \tilde{\mathbf{y}}(-\infty) &= \alpha(t_0). \end{aligned} \tag{24}$$

By (A1), (A2), and (A3), system (24) has a solution, which is a heteroclinic orbit connecting $M_1(\alpha(t_0), \overline{z}_0^{(-)}(t_0), t_0)$ with $M_2(\beta(t_0), \overline{z}_0^{(+)}(t_0), t_0)$. On the basis of (A4) and (10), the value of t_0 can be confirmed, so $Q_0^{(\pm)}\mathbf{y}(\tau)$ is determined completely.

For $L_0\mathbf{x}(\tau_0)$, we have

$$\begin{aligned} &\frac{dL_0 z(\tau_0)}{d\tau_0} = 0, \\ &\frac{dL_0\mathbf{y}(\tau_0)}{d\tau_0} \\ &\quad = \mathbf{f}\left(\alpha(0) + L_0\mathbf{y}(\tau_0), \overline{z}_0^{(-)}(0) + L_0 z(\tau_0), 0\right), \\ &L_0 u(0) = u^* - \alpha_1(0), \\ &L_0 v(0) = v^0 - \alpha_2(0), \\ &L_0\mathbf{x}(+\infty) = 0. \end{aligned} \tag{25}$$

By (25), we can solve $L_0 z(\tau_0) \equiv 0$. Assuming that $\tilde{\tilde{\mathbf{y}}}^l = \alpha(0) + L_0\mathbf{y}(\tau_0)$, (25) can be rewritten as

$$\begin{aligned} \frac{d\tilde{\tilde{\mathbf{y}}}^l}{d\tau_0} &= \mathbf{f}\left(\tilde{\tilde{\mathbf{y}}}^l, \overline{z}_0^{(-)}(0), 0\right), \\ \tilde{\tilde{u}}^l(0) &= u^0, \\ \tilde{\tilde{v}}^l(0) &= v^0, \\ \tilde{\tilde{\mathbf{y}}}^l(+\infty) &= \alpha(0). \end{aligned} \tag{26}$$

For $R_0\mathbf{x}(\tau_1)$, we have

$$\begin{aligned} &\frac{dR_0 z(\tau_1)}{d\tau_1} = 0, \\ &\frac{dR_0\mathbf{y}(\tau_1)}{d\tau_1} \\ &\quad = \mathbf{f}\left(\beta(1) + R_0\mathbf{y}(\tau_1), \overline{z}_0^{(+)}(1) + R_0 z(\tau_1), 1\right), \\ &R_0 w(0) = w^1 - \beta_3(1), \\ &R_0\mathbf{x}(-\infty) = 0. \end{aligned} \tag{27}$$

According to (27), we can solve $R_0 z(\tau_1) \equiv 0$. Supposing that $\tilde{\tilde{\mathbf{y}}}^r = \beta(1) + R_0\mathbf{y}(\tau_1)$, (27) can be rewritten as

$$\begin{aligned} \frac{d\tilde{\tilde{\mathbf{y}}}^r}{d\tau_1} &= \mathbf{f}\left(\tilde{\tilde{\mathbf{y}}}^r, \overline{z}_0^{(+)}(1), 1\right), \\ \tilde{\tilde{w}}^r(0) &= w^1, \\ \tilde{\tilde{\mathbf{y}}}^r(-\infty) &= \beta(1). \end{aligned} \tag{28}$$

Equations (26) and (28) coincide with the associated system (7). By (A1), systems (26) and (28) have the equilibrium points $\widetilde{M}_1(\varphi(0), \overline{z}_0^{(-)}(0), 0)$ and $\widetilde{M}_2(\psi(1), \overline{z}_0^{(+)}(1), 1)$, respectively. We will give the following hypothesis to obtain the solution of systems (26) and (28):

(A5) The initial values $\tilde{\tilde{u}}^l(0) = u^0$ and $\tilde{\tilde{v}}^l(0) = v^0$ are intersected with the one-dimensional stable manifold $W^s(\widetilde{M}_1(0))$ near the equilibrium point $\widetilde{M}_1$, and the initial value $\tilde{\tilde{w}}^r(0) = w^1$ is intersected with the one-dimensional unstable manifold $W^u(\widetilde{M}_2(1))$.

By (A1)–(A4) and (20)–(24), $Q_0^{(\pm)}\mathbf{x}(\tau)$ is solved, which decays exponentially as $\tau \to \pm\infty$. Then, by (A5) and (25)–(28), we can solve $L_0\mathbf{x}(\tau_0)$, which decays exponentially as $\tau \to +\infty$, and $R_0 z(\tau_1)$, which decays exponentially as $\tau \to -\infty$. So, the following conclusion is obtained.

Lemma 2. *Under conditions (A1)–(A5) and (20)–(28), there exist the interior-layer functions $Q_0^{(\pm)}\mathbf{x}(\tau)$ and the boundary layer functions $L_0\mathbf{x}(\tau_0)$, $R_0\mathbf{x}(\tau_1)$, which satisfy the following inequality:*

$$\begin{aligned} \left\|Q_0^{(-)}\mathbf{x}(\tau)\right\| &\le C_0 e^{k_0\tau}, \\ \left\|Q_0^{(+)}\mathbf{x}(\tau)\right\| &\le C_1 e^{-k_1\tau}, \\ \left\|L_0\mathbf{x}(\tau_0)\right\| &\le C_2 e^{-k_2\tau_0}, \\ \left\|R_0\mathbf{x}(\tau_1)\right\| &\le C_3 e^{k_3\tau_1}, \end{aligned} \tag{29}$$

where C_l and k_l $(l = 0, 1, 2, 3)$ are all positive constants.

Now, the coefficients of zero-order terms for (16) and (17) are completely determined. To determine functions $\overline{\mathbf{y}}_i^{(\pm)}(t)$ and $\overline{z}_i^{(\pm)}(t)$, we need the following hypothesis:

(A6) The determinant of $\mathbf{f}_{\mathbf{y}}(\overline{\mathbf{y}}_0^{(\pm)}(t), \overline{z}_0^{(\pm)}, t)$ is not equal to zero all the time.

For $\overline{\mathbf{x}}_i^{(\pm)}(t)$, we can obtain

$$\begin{aligned}
&\left(\overline{\mathbf{y}}_{i-1}^{(\pm)}(t)\right)' = \mathbf{f}_y \overline{\mathbf{y}}_i^{(\pm)}(t) + \mathbf{f}_z \overline{z}_i^{(\pm)}(t) + h_i(t),\\
&\left(\overline{z}_i^{(\pm)}(t)\right)' = g_y \overline{\mathbf{y}}_i^{(\pm)}(t) + g_z \overline{z}_i^{(\pm)}(t) + g_i(t),\\
&\overline{z}_i^{(-)}(0) = -\Pi_i z(0),\\
&\overline{z}_i^{(-)}(t_0) + Q_i^{(-)} z(0) + \sum_{k=1}^{i} \left[\overline{z}_{i-k}^{(-)}(t_0)\right]^{(k)} t_k\\
&\quad = \overline{z}_i^{(+)}(t_0) + Q_i^{(+)} z(0) + \sum_{k=1}^{i} \left[\overline{z}_{i-k}^{(+)}(t_0)\right]^{(k)} t_k,
\end{aligned} \tag{30}$$

where functions $\mathbf{f}_y, \mathbf{f}_z, g_y$, and g_z take value at $(\overline{\mathbf{y}}_0^{(\pm)}(t), \overline{z}_0^{(\pm)}, t)$, while $h_i(t)$ and $g_i(t)$ are known functions about $\overline{\mathbf{y}}_m(t)$, $\overline{z}_m(t)$ $(m = 0, 1, 2, \ldots, i-1)$. On the basis of (A6) and the first equation of (30), we can ascertain

$$\overline{\mathbf{y}}_i^{(\pm)}(t) = \mathbf{f}_y^{-1}\left[\left(\overline{\mathbf{y}}_{i-1}^{(\pm)}(t)\right)' - \mathbf{f}_z \overline{z}_i^{(\pm)}(t) - h_i(t)\right]. \tag{31}$$

Inserting $\overline{\mathbf{y}}_i^{(\pm)}(t)$ into the second equations of (30), $\overline{z}_i^{(\pm)}(t, c)$ is solved, where c is an undetermined coefficient.

For $L_i\mathbf{x}(\tau_0)$ $(i = 1, 2, \ldots)$, we can obtain

$$\begin{aligned}
&\frac{dL_i\mathbf{y}}{d\tau_0} = \widetilde{\mathbf{f}}_y L_i\mathbf{y} + \widetilde{\mathbf{f}}_z L_i z + G_i(\tau_0),\\
&\frac{dL_i z}{d\tau_0} = \widetilde{g}_y L_{i-1}\mathbf{y} + \widetilde{g}_z L_{i-1} z + M_{i-1}(\tau_0).\\
&L_i u(0) = -\overline{u}_i(0),\\
&L_i\mathbf{x}(+\infty) = 0,
\end{aligned} \tag{32}$$

where $\widetilde{\mathbf{f}}_y, \widetilde{\mathbf{f}}_z, \widetilde{g}_y$, and $\widetilde{g}_z$ take value at $(\alpha(0) + L_0\mathbf{y}(\tau_0), \overline{z}_0(0) + L_0 z(\tau_0), 0)$, $G_i(\tau)$ is a known vector function about $\overline{\mathbf{x}}_{i-1}(t)$ and $L_{i-1}\mathbf{x}(\tau_0)$, and $M_{i-1}(\tau_0)$ is a known vector function about $\overline{\mathbf{x}}_{i-2}(t)$ and $L_{i-2}\mathbf{x}(\tau_0)$. By the second equation of (32), we can solve $L_i z(\tau_0) = \int_{+\infty}^{\tau_0} L_{i-1} g(s)ds$, where $L_{i-1} g(\tau_0) = \widetilde{g}_y L_{i-1}\mathbf{y} + \widetilde{g}_z L_{i-1} z + M_{i-1}(\tau_0)$, so $L_i z(0) = \int_{+\infty}^{0} L_{i-1} g(s)ds$. By (30), we can know

$$\overline{z}_i^{(-)}(t) = -L_i z(0) = \int_0^{+\infty} L_{i-1} g(s)\, ds. \tag{33}$$

By the above condition and (28), $\overline{\mathbf{x}}_i^{(\pm)}(t)$ is confirmed; similarly, we can solve (32).

For $Q_i^{(\pm)}\mathbf{x}(\tau)$ $(i = 1, 2, \ldots)$, we can obtain

$$\begin{aligned}
&\frac{dQ_i^{(\pm)}\mathbf{y}}{d\tau} = \widetilde{\mathbf{f}}_y^{(\pm)} Q_i^{(\pm)}\mathbf{y} + \widetilde{\mathbf{f}}_z^{(\pm)} Q_i^{(\pm)} z + H_i^{(\pm)}(\tau),\\
&\frac{dQ_i^{(\pm)} z}{d\tau} = \widetilde{g}_y^{(\pm)} Q_{i-1}\mathbf{y} + \widetilde{g}_z^{(\pm)} Q_{i-1} z + N_{i-1}^{(\pm)}(\tau),\\
&Q_i^{(\pm)} u(0) = u_i^* - \sum_{k=0}^{i} \left[\overline{u}_{i-k}^{(\pm)}(t_0)\right]^{(k)} t_k,\\
&Q_i^{(-)}\mathbf{x}(-\infty) = Q_i^{(+)}\mathbf{x}(+\infty) = 0,
\end{aligned} \tag{34}$$

where $\widetilde{\mathbf{f}}_y^{(-)}$, $\widetilde{\mathbf{f}}_z^{(-)}$, and $\widetilde{g}_y^{(\pm)}$ take value at $(\alpha(t_0) + Q_0^{(-)}\mathbf{y}(\tau), \overline{z}_0^{(-)}(t_0) + Q_0^{(-)} z(\tau), t_0)$ and $\widetilde{\mathbf{f}}_y^{(+)}$, $\widetilde{\mathbf{f}}_z^{(+)}$, and $\widetilde{g}_z$ take value at $(\beta(t_0) + Q_0^{(+)}\mathbf{y}(\tau), \overline{z}_0^{(+)}(t_0) + Q_0^{(+)} z(\tau), t_0)$. $H_i^{(\pm)}(\tau)$ is a known vector function about $\overline{\mathbf{x}}_{i-n}^{(\pm)}(t)$, $Q_{i-n}^{(\pm)}\mathbf{x}(\tau)$, and t_{i-n} $(0 \le n \le i-1)$. By the condition $Q_i^{(-)} z(-\infty) = 0$ and the second equation of (34), we can solve $Q_i^{(-)} z(\tau) = \int_{-\infty}^{\tau} Q_{i-1}^{(-)} g(s)ds$, where $Q_{i-1}^{(\pm)} g(\tau) = \widetilde{g}_y^{(\pm)} Q_{i-1}\mathbf{y} + \widetilde{g}_z^{(\pm)} Q_{i-1} z + N_{i-1}^{(\pm)}(\tau)$. We can know that

$$Q_i^{(-)} z(0) = \int_{-\infty}^{0} Q_{i-1}^{(-)} g(s)\, ds. \tag{35}$$

By (35) and the first equation of (34), $Q_i^{(-)}\mathbf{x}(\tau)$ is solved, which decays exponentially as $\tau \to -\infty$. Then, we can solve $Q_i^{(+)}\mathbf{x}(\tau)$ which decays exponentially as $\tau \to +\infty$. Inserting (35) into the second condition of (30), t_i is solved.

The boundary value function $R_i\mathbf{x}(\tau_1)$ satisfies the following equations:

$$\begin{aligned}
&\frac{dR_i\mathbf{y}}{d\tau_1} = \widetilde{\mathbf{f}}_y R_i\mathbf{y} + \widetilde{\mathbf{f}}_z R_i z + G_i(\tau_1),\\
&\frac{dR_i z}{d\tau_1} = \widetilde{g}_y R_{i-1}\mathbf{y} + \widetilde{g}_z R_{i-1} z + M_{i-1}(\tau_1).\\
&R_i w(0) = w^1 - \overline{w}_i(1),\\
&R_i\mathbf{x}(-\infty) = 0.
\end{aligned} \tag{36}$$

The solution of (36) is similar to the solution of (32). From (36), we can verify that there exists the right boundary value function $R_i\mathbf{x}(\tau_1)$, which decays exponentially as $\tau \to -\infty$, and the following conclusion is obtained.

Lemma 3. *Systems (32)–(36) have the solutions $L_i\mathbf{x}(\tau_0)$, $Q_i^{(\pm)}\mathbf{x}(\tau)$, and $R_i\mathbf{x}(\tau_1)$, respectively, satisfying the following inequality:*

$$\begin{aligned}
&\left\|L_i\mathbf{x}(\tau_0)\right\| \le \overline{C}_0 e^{-\overline{k}_0\tau_0},\\
&\left\|Q_i^{(-)}\mathbf{x}(\tau)\right\| \le \overline{C}_1 e^{\overline{k}_1\tau},\\
&\left\|Q_i^{(+)}\mathbf{x}(\tau)\right\| \le \overline{C}_2 e^{-\overline{k}_2\tau},\\
&\left\|R_i\mathbf{x}(\tau_1)\right\| \le \overline{C}_3 e^{\overline{k}_3\tau_1},
\end{aligned} \tag{37}$$

where $\overline{C}_l$ and $\overline{k}_l$ $(l = 0, 1, 2, 3)$ are all positive constants.

3. The Existence of Asymptotic Expansion

There are many methods to prove the existence of the step-like contrast structure for system (3) and we will prove the existence with implicit functions theorems [11]. The solutions of left and right problems can be expressed by (15) and (16) and we will prove that (15) and (16) are connected smoothly at the point of t^*, which is on the neighborhood of t_0. At the point of t^*, (15) and (16) can be expressed by

$$\begin{aligned} \mathbf{x}^{(-)}(t^*,\mu) &= \bar{\mathbf{x}}_0^{(-)}(t^*) + Q_0^{(-)}\mathbf{x}(0) + O(\mu), \\ \mathbf{x}^{(+)}(t^*,\mu) &= \bar{\mathbf{x}}_0^{(+)}(t^*) + Q_0^{(+)}\mathbf{x}(0) + O(\mu). \end{aligned} \tag{38}$$

As $L_0\mathbf{x}(\tau_0)$ and $R_0\mathbf{x}(\tau_1)$ decay exponentially as $t = t^*$, they can be neglected. Because the first two components of the solution for the left and right problems are equal correspondingly at the point of t^*, we can solve t^* by the third component $w^{(\pm)}$. Supposing that $U(t^*,\mu) = u^{(-)}(t^*,\mu) - u^{(+)}(t^*,\mu)$, we have

$$\begin{aligned} U(t^*,\mu) &= \left[\alpha_3(t^*) + Q_0^{(-)}w(0)\right] \\ &\quad - \left[\beta_3(t^*) + Q_0^{(+)}w(0)\right] + O(\mu) \\ &= H_3(t^*) + O(\mu). \end{aligned} \tag{39}$$

According to (A4), $(\partial H_2/\partial\bar{t})|_{\bar{t}=t_0}$ and $(\partial H_3/\partial\bar{t})|_{\bar{t}=t_0}$ are not simultaneously equal to zero. If $(\partial H_2/\partial\bar{t})|_{\bar{t}=t_0} = 0$, we can ascertain that $(\partial H_3/\partial\bar{t})|_{\bar{t}=t_0} \neq 0$. On the basis of the implicit functions theorem, there exists $t^*(\mu) = t_0 + O(\mu)$ causing $U(t^*,\mu) = 0$. So there exists a step-like contrast structure at the point of t^*. If $(\partial H_2/\partial\bar{t})|_{\bar{t}=t_0} \neq 0$, $(\partial H_3/\partial\bar{t})|_{\bar{t}=t_0} = 0$ and t^* can be confirmed by the second component v. By the above discussion, the following theorem is obtained.

Theorem 4. *By (A1)–(A6) and Lemmas 1–3, there exists the step-like solution of (3) as follows:*

$$\mathbf{x}(t,\mu) = \begin{cases} \bar{\mathbf{x}}_0^{(-)}(t) + L_0\mathbf{x}(\tau_0) + Q_0^{(-)}\mathbf{x}(\tau) + O(\mu), & 0 \le t \le t^*, \\ \bar{\mathbf{x}}_0^{(+)}(t) + Q_0^{(+)}\mathbf{x}(\tau) + R_0\mathbf{x}(\tau_1) + O(\mu), & t^* \le t \le 1. \end{cases} \tag{40}$$

4. Example

The equations are given by

$$\begin{aligned} \mu\frac{du}{dt} &= v, \\ \mu\frac{dv}{dt} &= \left(u - t + \frac{1}{2}\right)\left(u^2 - \frac{9}{4}\right), \\ \mu\frac{dw}{dt} &= \frac{3}{\sqrt{2}}(u - w), \\ \frac{dz}{dt} &= u + z, \end{aligned} \tag{41}$$

satisfying the boundary value conditions as follows:

$$\begin{aligned} u(0,\mu) &= u^0, \\ v(0,\mu) &= v^0, \\ w(1,\mu) &= w^1, \\ z(0,\mu) &= z^0. \end{aligned} \tag{42}$$

Assuming that $\mu = 0$, we can solve three groups of isolated solutions:

$$\begin{aligned} \bar{\mathbf{y}} &= \alpha(t) = \left(-\frac{3}{2}, 0, -\frac{3}{2}\right)^T, \\ \bar{\mathbf{y}} &= \chi(t) = \left(t - \frac{1}{2}, 0, t - \frac{1}{2}\right)^T, \\ \bar{\mathbf{y}} &= \beta(t) = \left(\frac{3}{2}, 0, \frac{3}{2}\right)^T. \end{aligned} \tag{43}$$

According to (43), the solution of $dz/dt = u + z$ exists. Considering the associated system,

$$\begin{aligned} \frac{d\tilde{u}}{d\tau} &= \tilde{v}, \\ \frac{d\tilde{v}}{d\tau} &= \left(\tilde{u} - \bar{t} + \frac{1}{2}\right)\left(\tilde{u}^2 - \frac{9}{4}\right), \\ \frac{d\tilde{w}}{d\tau} &= \frac{3}{\sqrt{2}}(\tilde{u} - \tilde{w}). \end{aligned} \tag{44}$$

The corresponding characteristic equation is given by

$$(\lambda + 1)\left[\lambda^2 - F_{y_1}(M_{1,2})\right] = 0. \tag{45}$$

By (45), we can solve the following characteristic roots:

$$\begin{aligned} \lambda_{1,2} &= \pm\sqrt{F_{y_1(M_{1,2})}}, \\ \lambda_3 &= -1, \end{aligned} \tag{46}$$

where $F_{y_1}(M_1) = 3\bar{t} + 3 > 0$ and $F_{y_1}(M_2) = 3(2 - \bar{t}) > 0$. So the two equilibrium points $M_1(-3/2, 0, -3/2)$ and $M_2(3/2, 0, 3/2)$ are all hyperbolic saddle points of (43). To determine t_0, we can discuss the equations $Q_0^{(\pm)}\mathbf{y}(\tau)$ as follows:

$$\begin{aligned} \frac{dQ_0^{(-)}u}{d\tau} &= Q_0^{(-)}v, \\ \frac{dQ_0^{(-)}v}{d\tau} &= \left(-\frac{3}{2} + Q_0^{(-)}u - t_0 + \frac{1}{2}\right)\left[\left(-\frac{3}{2} + Q_0^{(-)}u\right)^2 - \frac{9}{4}\right], \\ \frac{dQ_0^{(-)}w}{d\tau} &= \frac{3}{\sqrt{2}}\left(Q_0^{(-)}u - Q_0^{(-)}w\right), \end{aligned}$$

$$
\begin{aligned}
&-\frac{3}{2}+Q_0^{(-)}u=t_0-\frac{1}{2},\\
&Q_0^{(-)}\mathbf{y}(-\infty)=0,
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
&\frac{dQ_0^{(+)}u}{d\tau}=Q_0^{(+)}v,\\
&\frac{dQ_0^{(+)}v}{d\tau}\\
&\quad=\left(\frac{3}{2}+Q_0^{(+)}u-t_0+\frac{1}{2}\right)\left[\left(\frac{3}{2}+Q_0^{(+)}u\right)^2-\frac{9}{4}\right],\\
&\frac{dQ_0^{(+)}w}{d\tau}=\frac{3}{\sqrt{2}}\left(Q_0^{(+)}u-Q_0^{(+)}w\right),\\
&\frac{3}{2}+Q_0^{(+)}u=t_0-\frac{1}{2},\\
&Q_0^{(+)}\mathbf{y}(+\infty)=0.
\end{aligned}
\tag{48}
$$

Assuming that $\widetilde{u}^{(\pm)}=\pm 3/2+Q_0^{(\pm)}u$, $\widetilde{v}^{(\pm)}=Q_0^{(\pm)}v$, and $\widetilde{w}^{(\pm)}=\pm 3/2+Q_0^{(\pm)}w$, (47) and (48) can be expressed by

$$
\begin{aligned}
&\frac{d\widetilde{u}^{(\pm)}}{d\tau}=\widetilde{v}^{(\pm)},\\
&\frac{d\widetilde{v}^{(\pm)}}{d\tau}=\left(\widetilde{u}^{(\pm)}-t_0+\frac{1}{2}\right)\left[\left(\widetilde{u}^{(\pm)}\right)^2-\frac{9}{4}\right],\\
&\frac{d\widetilde{w}^{(\pm)}}{d\tau}=\frac{3}{\sqrt{2}}\left(\widetilde{u}^{(\pm)}-\widetilde{w}^{(\pm)}\right),\\
&\widetilde{u}^{(\pm)}(0)=t_0-\frac{1}{2}.
\end{aligned}
\tag{49}
$$

The first integral of (49) passing through M_1 and M_2 is

$$
\begin{aligned}
\left[\widetilde{v}^{(\pm)}(\tau)\right]^2&=2\int_{-(3/2)(3/2)}^{\widetilde{u}^{(\mp)}}\left(s-t_0+\frac{1}{2}\right)\left(s^2-\frac{9}{4}\right)ds\\
&=0.
\end{aligned}
\tag{50}
$$

Since the solutions of (41) are connected smoothly at the point of t_0, we have $\widetilde{u}^{(-)}(0)=\widetilde{u}^{(+)}(0)$. Assuming that

$$
H(t_0)=\widetilde{u}^{(-)}(0)-\widetilde{u}^{(+)}(0)=0,
\tag{51}
$$

substituting (50) into (51), we can obtain

$$
\int_{-3/2}^{3/2}\left(s-t_0+\frac{1}{2}\right)\left(s^2-\frac{9}{4}\right)ds=0.
\tag{52}
$$

On the basis of (52), we can solve $t_0=1/2$ and $\partial H(t_0)/\partial t_0=21/4\neq 0$. So the solutions of (41) transfer at the point of $t_0=1/2$.

Substituting $t_0=1/2$ into (49), as $\tau\leq 0$, we have

$$
\begin{aligned}
&\widetilde{u}^{(-)}(\tau)=\frac{3Ce^{(3/\sqrt{2})\tau}+3}{2Ce^{(3/\sqrt{2})\tau}-2},\\
&\widetilde{v}^{(-)}(\tau)=\frac{9\sqrt{2}Ce^{-3/\sqrt{2}}\tau}{4\left(1-Ce^{(3/\sqrt{2})\tau}\right)^2},\\
&\widetilde{w}^{(-)}(\tau)=\frac{3}{2}+\frac{2\ln\left|Ce^{(3/\sqrt{2})\tau}-1\right|}{Ce^{(3/\sqrt{2})\tau}}.
\end{aligned}
\tag{53}
$$

As $\tau\geq 0$, we can solve

$$
\begin{aligned}
&\widetilde{u}^{(+)}(\tau)=\frac{3C+3e^{-(3/\sqrt{2})\tau}}{2C-2e^{-(3/\sqrt{2})\tau}},\\
&\widetilde{v}^{(+)}(\tau)=\frac{9\sqrt{2}Ce^{-3/\sqrt{2}}\tau}{4\left(1-Ce^{-(3/\sqrt{2})\tau}\right)^2},\\
&\widetilde{w}^{(+)}(\tau)=\frac{3}{2}+\frac{2\ln\left|Ce^{-(3/\sqrt{2})\tau}-1\right|}{Ce^{-(3/\sqrt{2})\tau}}.
\end{aligned}
\tag{54}
$$

So we can solve the interior-layer functions as follows:

$$
\begin{aligned}
&Q_0^{(-)}u(\tau)=\frac{3Ce^{(3/\sqrt{2})\tau}}{Ce^{(3/\sqrt{2})\tau}-1},\\
&Q_0^{(-)}v(\tau)=\frac{9\sqrt{2}Ce^{-3/\sqrt{2}}\tau}{4\left(1-Ce^{(3/\sqrt{2})\tau}\right)^2},\\
&Q_0^{(-)}w(\tau)=3+\frac{2\ln\left|Ce^{(3/\sqrt{2})\tau}-1\right|}{Ce^{(3/\sqrt{2})\tau}},\\
&Q_0^{(+)}u(\tau)=\frac{3e^{-(3/\sqrt{2})\tau}}{C-e^{-(3/\sqrt{2})\tau}},\\
&Q_0^{(+)}v(\tau)=\frac{9\sqrt{2}Ce^{-(3/\sqrt{2})}\tau}{4\left(1-Ce^{-(3/\sqrt{2})\tau}\right)^2},\\
&Q_0^{(+)}w(\tau)=\frac{2\ln\left|Ce^{-(3/\sqrt{2})\tau}-1\right|}{Ce^{-(3/\sqrt{2})\tau}}.
\end{aligned}
\tag{55}
$$

Similarly, the boundary layer function $L_0\mathbf{y}(\tau_0)$ can be expressed by

$$
\begin{aligned}
&L_0u(\tau_0)=\frac{3e^{-(3/\sqrt{2})\tau_0}}{e^{-(3/\sqrt{2})\tau_0}-A},\\
&L_0v(\tau_0)=\frac{9\sqrt{2}Ce^{-(3/\sqrt{2})}\tau_0}{4\left(1-Ae^{-(3/\sqrt{2})\tau_0}\right)^2},\\
&L_0w(\tau_0)=\frac{2\ln\left|Ae^{-(3/\sqrt{2})\tau_0}-1\right|}{Ae^{-(3/\sqrt{2})\tau_0}}.
\end{aligned}
\tag{56}
$$

The boundary layer function $R_0\mathbf{y}(\tau_1)$ can be expressed by

$$R_0 u(\tau_1) = \frac{3e^{(3/\sqrt{2})\tau_1}}{B - e^{(3/\sqrt{2})\tau_1}},$$
$$R_0 v(\tau_1) = \frac{9\sqrt{2}Ce^{3/\sqrt{2}}\tau_1}{4\left(1 - Be_1^{(3/\sqrt{2})\tau}\right)^2}, \tag{57}$$
$$R_0 w(\tau_1) = -3 + \frac{2\ln\left|Ae^{(3/\sqrt{2})\tau_1} - 1\right|}{Ae^{(3/\sqrt{2})\tau_1}},$$

so we can construct a zero-order asymptotic solution of (41) and (42) as follows:

$$\mathbf{x}(t,\mu) = \begin{cases} \overline{\mathbf{x}}_0^{(-)}(t) + L_0\mathbf{x}(\tau) + Q_0^{(-)}\mathbf{x}(\tau) + O(\mu), & 0 \le t \le t^*, \\ \overline{\mathbf{x}}_0^{(+)}(t) + Q_0^{(+)}\mathbf{x}(\tau) + R_0\mathbf{x}(\tau_1) + O(\mu), & t^* \le t \le 1, \end{cases} \tag{58}$$

where $t^* = 1/2 + O(\mu)$, $\overline{\mathbf{x}}_0^{(-)}(t) = (-3/2, 0, -3/2)^T$, and $\overline{\mathbf{x}}_0^{(+)}(t) = (3/2, 0, 3/2)^T$.

5. Conclusive Remarks

By the boundary layer function method and smooth connection, we study the contrast structure for a class of semilinear singularly perturbed systems. Under some assumptions, the existence of a step-like contrast structure of system (3) and a heteroclinic orbit connecting two equilibrium points of the corresponding associated systems is determined. Then, we obtain the asymptotic solution of system (3). In comparison with [8, 9], the system we study is more general.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

Han Xu completed the main part of this paper, and Yinlai Jin corrected the main theorems.

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