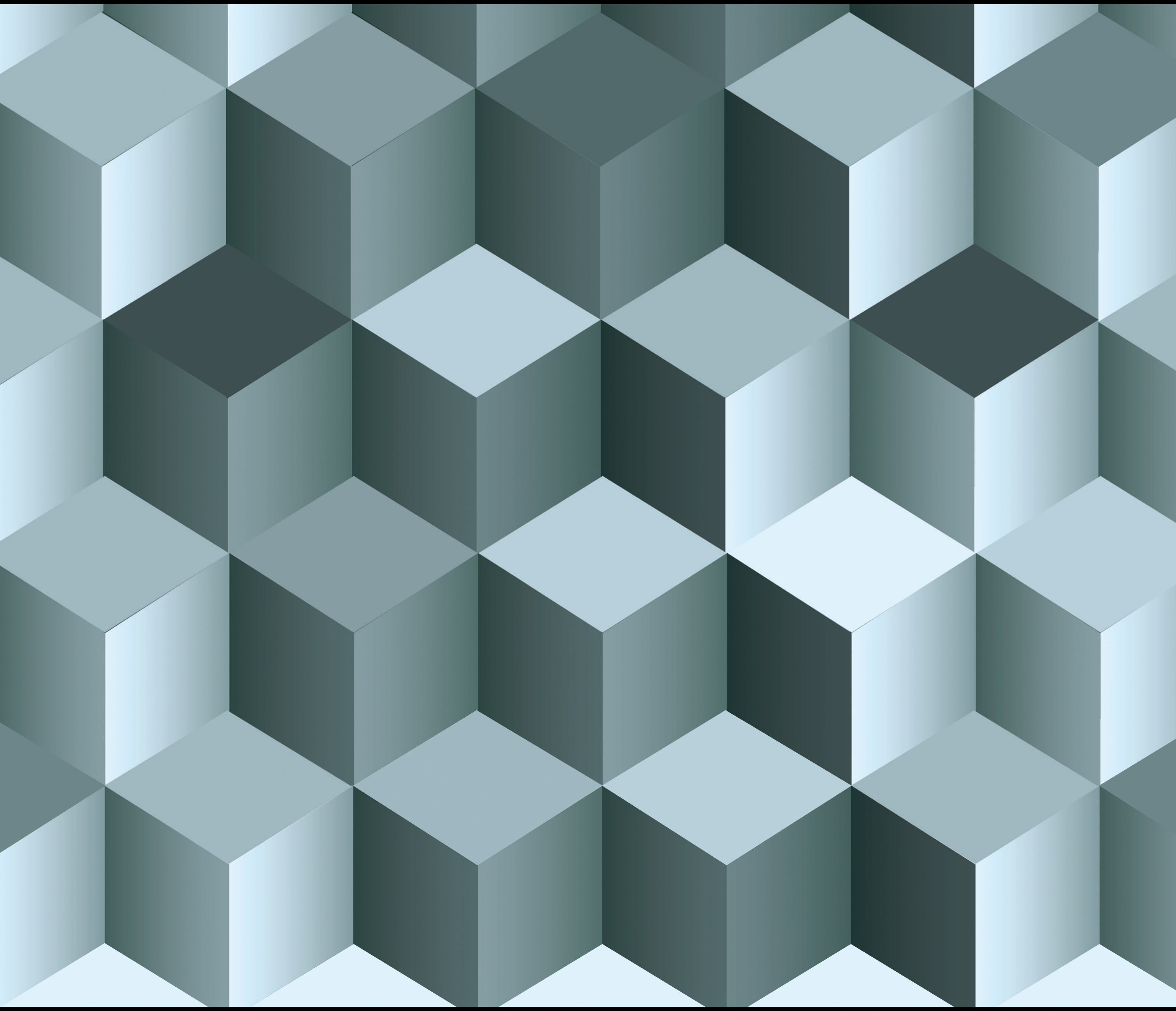
Journal of Function Spaces

Nonlinear and Variational Analysis and their Applications 2021

Lead Guest Editor: Fanglei Wang
Guest Editors: Mustafa Avci, Yuanfang Ru, and Zisen Mao



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Retraction

Retracted: Evaluating Investors' Recognition Abilities for Risk and Profit in Online Loan Markets Using Nonlinear Models and Financial Big Data

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] Q. He, P. Xia, B. Li, and J. Liu, "Evaluating Investors' Recognition Abilities for Risk and Profit in Online Loan Markets Using Nonlinear Models and Financial Big Data," *Journal of Function Spaces*, vol. 2021, Article ID 5178970, 15 pages, 2021.

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Research Article

Two Nonmonotonic Self-Adaptive Strongly Convergent Projection-Type Methods for Solving Pseudomonotone Variational Inequalities

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The primary objective of this study is to introduce two novel extragradient-type iterative schemes for solving variational inequality problems in a real Hilbert space. The proposed iterative schemes extend the well-known subgradient extragradient method and are used to solve variational inequalities involving the pseudomonotone operator in real Hilbert spaces. The proposed iterative methods have the primary advantage of using a simple mathematical formula for step size rule based on operator information rather than the Lipschitz constant or another line search method. Strong convergence results for the suggested iterative algorithms are well-established for mild conditions, such as Lipschitz continuity and mapping monotonicity. Finally, we present many numerical experiments that show the effectiveness and superiority of iterative methods.

1. Introduction

The primary objective of this research is to investigate the iterative methodologies used to estimate the solution of variational inequalities in a real Hilbert space. To establish the convergence analysis theorems, the following conditions need to be satisfied:

Condition 1. The solution set of the problem (VIP) denoted by Ω and it is nonempty.

Condition 2. A mapping $\mathscr{L} : \mathscr{E} \longrightarrow \mathscr{E}$ is said to be pseudo-monotone if

$$\langle \mathscr{L}(p_1), p_2 - p_1 \rangle \geq 0 \Rightarrow \langle \mathscr{L}(p_2), p_1 - p_2 \rangle \leq 0, \quad \forall p_1, p_2 \in \mathscr{A} (\mathrm{PM}). \tag{1}$$

Condition 3. A mapping $\mathscr{L} : \mathscr{E} \longrightarrow \mathscr{E}$ is said to be *Lipschitz continuous* with constant $L > 0$ if

$$\|\mathscr{L}(p_1) - \mathscr{L}(p_2)\| \leq L\|p_1 - p_2\|, \quad \forall p_1, p_2 \in \mathscr{A} (\mathrm{LC}). \tag{2}$$

Condition 4. A mapping $\mathscr{L} : \mathscr{E} \longrightarrow \mathscr{E}$ is said to be *weakly sequentially continuous* if $\{\mathscr{L}(u_n)\}$ converges weakly to $\mathscr{L}(u)$ for each sequence $\{u_n\}$ converges weakly to an element u.

Let $\mathscr{E}$ be any real Hilbert space and $\mathscr{A}$ be any nonempty convex closed subset of a Hilbert space $\mathscr{E}$. Assume that $\mathscr{L} : \mathscr{E} \longrightarrow \mathscr{E}$ be an arbitrary mapping. The *variational inequality problem* for an operator $\mathscr{L}$ on $\mathscr{A}$ is defined in the following manner [1, 2]:

$$\text{Find } q^* \in \mathscr{A} \text{ such that } \langle \mathscr{L}(q^*), y - q^* \rangle \geq 0, \quad \forall y \in \mathscr{A} \ (\mathrm{VIP}). \tag{3}$$

Let Ω stand for the solution set for the problem (VIP). The mathematical model of variational inequalities covers many mathematical problems, such as partial differential equations, optimization, optimal control, mechanics, finance, and mathematical programming (see for details [3–9]) and others in [10–20]. Since it is a fundamental problem in the applied sciences and nonlinear functional analysis, many researchers are investigating not only the stability and existence of solutions to such problems but also iterative methods for solving them numerically. In order to solve variational inequalities numerically, projection iterative methods are a very important tool. Many researchers have provided various projection method extensions and modifications to solve the problem (VIP) (see [21–33]). The extragradient method described below was developed by Korpelevich [25] and Antipin [34]. Their method takes the form of

$$\begin{cases} u_1 \in \mathscr{A}, \\ p_n = P_{\mathscr{A}}[u_n - \delta \mathscr{L}(u_n)], \\ u_{n+1} = P_{\mathscr{A}}[u_n - \delta \mathscr{L}(p_n)], \end{cases} \tag{4}$$

where $0 < \delta < 1/L$. For each iteration of the above iterative scheme, two projections on the feasible set $\mathscr{A}$ are required to be figured out. Of course, if the feasible set $\mathscr{A}$ has a complicated framework, this can affect the method's computational effectiveness. The first one is to follow the subgradient extragradient method designed by Censor et al. [22] to overcome this deficiency. This method is in the form of

$$\begin{cases} u_1 \in \mathscr{A}, \\ p_n = P_{\mathscr{A}}[u_n - \delta \mathscr{L}(u_n)], \\ u_{n+1} = P_{\mathscr{Z}_n}[u_n - \delta \mathscr{L}(p_n)], \end{cases} \tag{5}$$

where $0 < \delta < 1/L$ and

$$\mathscr{Z}_n = \{z \in \mathscr{Z} : \langle u_n - \delta \mathscr{L}(u_n) - p_n, z - p_n \rangle \le 0\}. \tag{6}$$

It is a key point to note that the above-mentioned well-established methods have two major drawbacks. The first is the fixed constant step size, which needs knowledge or approximation of the appropriate operator Lipschitz constant, and also is only weakly convergent in Hilbert spaces. Using a fixed step size can be difficult in terms of computation, affecting the method convergence rate and efficiency.

Hence, a natural question arises:

"Is it possible to propose two new strongly convergent subgradient extragradient algorithms with a nonmonotone self-adaptive step size rule to solve the problem (VIP)?"

The primary objective of this study is to introduce two new strongly convergent subgradient extragradient methods for enhancing the convergence rate of an iterative sequence. The answer to the above question is given in this study, which would be the subgradient extragradient algorithms, which set up a strong convergent iterative sequence by letting a variable nonmonotone step size rule. The suggested methods are employed to solve variational inequality problems involving pseudomonotone and Lipschitz regular operators in real Hilbert space. The proposed methods are based on the projection method [22] as well as the methods proposed in [26, 35]. The established method only needs to compute one projection onto the feasible set and one projection onto the half-space for each iteration. The iterative sequences established by the proposed method strongly converge to some solution of the underlined problem in the framework of some appropriate conditions on control parameters. A number of numerical examples are also added to elaborate on the computational effectiveness of the new methods over some existing methods presented in [36, 37].

The paper is arranged in the following manner: In Section 2, we provide some basic identities and preliminary results that were used in this paper. Section 3 includes the proposed methods and proves their convergence analysis. Finally, Section 4 presents some numerical results to illustrate the convergence and the effectiveness of the proposed methods.

2. Preliminaries

This section contains a number of important identities, as well as useful lemmas and definitions. For all $u, y \in \mathscr{Z}$, we have

$$\|u + y\|^2 = \|u\|^2 + 2\langle u, y \rangle + \|y\|^2. \tag{7}$$

A metric projection $P_{\mathscr{A}}(p_1)$ of an element $p_1 \in \mathscr{Z}$ is evaluated by

$$P_{\mathscr{A}}(p_1) = \operatorname{argmin}\{\|p_1 - p_2\| : p_2 \in \mathscr{A}\}. \tag{8}$$

Next, we list some of the important identities that are used to prove the convergence analysis.

Lemma 1 (see [38]). *Let $P_{\mathscr{A}} : \mathscr{Z} \longrightarrow \mathscr{A}$ be a metric projection on set $\mathscr{A}$. For each $p_1, p_2 \in \mathscr{Z}$ and $\ell \in \mathbb{R}$, then the following inequalities are satisfied:*

(i) $p_3 = P_{\mathscr{A}}(p_1)$ is true if and only if

$$\langle p_1 - p_3, p_2 - p_3 \rangle \le 0, \quad \forall p_2 \in \mathscr{A}, \tag{9}$$

(ii)

$$\|p_1 - P_{\mathscr{A}}(p_2)\|^2 + \|P_{\mathscr{A}}(p_2) - p_2\|^2 \le \|p_1 - p_2\|^2, \quad p_1 \in \mathscr{A}, p_2 \in \mathscr{Z}, \tag{10}$$

(iii)

$$\|p_1 - P_{\mathscr{A}}(p_1)\| \le \|p_1 - p_2\|, \quad p_2 \in \mathscr{A}, p_1 \in \mathscr{X}, \tag{11}$$

(iv)

$$\|\ell p_1 + (1-\ell)p_2\|^2 = \ell\|p_1\|^2 + (1-\ell)\|p_2\|^2 - \ell(1-\ell)\|p_1 - p_2\|^2, \tag{12}$$

(v)

$$\|p_1 + p_2\|^2 \le \|p_1\|^2 + 2\langle p_2, p_1 + p_2\rangle. \tag{13}$$

Lemma 2 (see [39]). *Assuming that* $\{c_n\} \subset [0,+\infty)$, *a sequence meets the following criteria:*

$$c_{n+1} \le (1-d_n)c_n + d_n e_n, \quad \forall n \in \mathbb{N}. \tag{14}$$

Moreover, $\{d_n\} \subset (0,1)$ *and* $\{e_n\} \subset \mathbb{R}$ *are two sequences such that*

$$\lim_{n \longrightarrow +\infty} d_n = 0, \sum_{n=1}^{+\infty} d_n = +\infty and \limsup_{n \longrightarrow +\infty} e_n \le 0. \tag{15}$$

Then, $\lim_{n \longrightarrow +\infty} c_n = 0$.

Lemma 3 (see [40]). *Assume that a sequence* $\{c_n\}$ *of real numbers and there is* $\{n_i\}$ *subsequence of* $\{n\}$ *such that* $c_{n_i} < c_{n_{i+1}}$, $\forall i \in \mathbb{N}$.*(102)*

Thus, there exists a natural nondecreasing sequence $\{m_j\}$ *with* $m_j \longrightarrow +\infty$ *as* $j \longrightarrow +\infty$ *and satisfies the following criteria for* $j \in \mathbb{N}$:

$$c_{m_j} \le c_{m_{j+1}}, c_j \le c_{m_{j+1}}. \tag{16}$$

Indeed, $m_j = \max\{j \le j : c_j \le c_{j+1}\}$.

Lemma 4 (see [41]). *Assume that* $\mathscr{L} : \mathscr{A} \longrightarrow \mathscr{X}$ *is a pseudomonotone and continuous mapping. Then,* q^* *is a solution of the problem (VIP) if and only if* q^* *is a solution of the following problem:*

$$\text{Find } u \in \mathscr{A} \text{ such that } \langle \mathscr{L}(y), y - u\rangle \ge 0, \quad \forall y \in \mathscr{A}. \tag{17}$$

3. Main Results

In this part of the research article, we propose two new methods and the corresponding strong convergence theorems. Both methods are presented in the following manner. The first method is of the following form.

Assume that $g : \mathscr{X} \longrightarrow \mathscr{X}$ is a contraction having constant $\xi \in [0,1)$. The second major contribution of this study work is as follows. The second main algorithm has the following form.

Lemma 5. *A sequence* $\{\delta_n\}$ *generated by (3.1) is convergent to* δ *and satisfies the following inequality:*

$$\min\left\{\frac{\mu}{L}, \delta_1\right\} \le \delta \le \delta_1 + P \quad \text{where } P = \sum_{n=1}^{+\infty} \varphi_n. \tag{18}$$

Proof. Let $\langle \mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle > 0$ such that

$$\begin{aligned} \frac{\mu\left(\|u_n - p_n\|^2 + \|q_n - p_n\|^2\right)}{2\langle \mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle} &\ge \frac{2\mu\|u_n - p_n\|\|q_n - p_n\|}{2\|\mathscr{L}(u_n) - \mathscr{L}(p_n)\|\|q_n - p_n\|} \\ &\ge \frac{2\mu\|u_n - p_n\|\|q_n - p_n\|}{2L\|u_n - p_n\|\|q_n - p_n\|} \ge \frac{\mu}{L}. \end{aligned} \tag{19}$$

By using mathematical induction on the definition of δ_{n+1}, we have

$$\min\left\{\frac{\mu}{L}, \delta_1\right\} \le \delta_n \le \delta_1 + P. \tag{20}$$

Let

$$\begin{aligned} [\delta_{n+1} - \delta_n]^+ &= \max\{0, \delta_{n+1} - \delta_n\}, \\ [\delta_{n+1} - \delta_n]^- &= \max\{0, -(\delta_{n+1} - \delta_n)\}. \end{aligned} \tag{21}$$

Due to expression of $\{\delta_n\}$, we can write

$$\sum_{n=1}^{+\infty} (\delta_{n+1} - \delta_n)^+ = \sum_{n=1}^{+\infty} \max\{0, \delta_{n+1} - \delta_n\} \le P < +\infty. \tag{22}$$

Thus, $\sum_{n=1}^{+\infty}(\delta_{n+1} - \delta_n)^+$ is convergent. Next, we have to prove the convergence of the following series:

$$\sum_{n=1}^{+\infty} (\delta_{n+1} - \delta_n)^-. \tag{23}$$

Let $\sum_{n=1}^{+\infty}(\delta_{n+1} - \delta_n)^- = +\infty$. Thus, we have $\delta_{n+1} - \delta_n = (\delta_{n+1} - \delta_n)^+ - (\delta_{n+1} - \delta_n)^-$. Thus, we have

$$\delta_{k+1} - \delta_1 = \sum_{n=0}^{k} (\delta_{n+1} - \delta_n) = \sum_{n=0}^{k} (\delta_{n+1} - \delta_n)^+ - \sum_{n=0}^{k} (\delta_{n+1} - \delta_n)^-. \tag{24}$$

Letting $k \longrightarrow +\infty$ in (24), we obtain $\delta_k \longrightarrow -\infty$ as $k \longrightarrow \infty$. This is a contradiction. Due to the convergence of the series $\sum_{n=0}^{k}(\delta_{n+1} - \delta_n)^+$ and $\sum_{n=0}^{k}(\delta_{n+1} - \delta_n)^-$ taking $k \longrightarrow +\infty$ in (24), we obtain $\lim_{n \longrightarrow \infty} \delta_n = \delta$. This completes the proof.□□ □

Lemma 6. *Let* $\mathscr{L} : \mathscr{X} \longrightarrow \mathscr{X}$ *be an operator satisfies the criteria Condition 1–Condition 4. For a given* $q^* \in \Omega \ne \emptyset$, *we* *have*

$$\|q_n - q^*\|^2 \le \|u_n - q^*\|^2 - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n - p_n\|^2 - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n - p_n\|^2. \tag{25}$$

Proof. We have to evaluate

$$\begin{aligned}\|q_n - q^*\|^2 &= \left\|P_{\mathscr{Z}_n}[u_n - \delta_n\mathscr{L}(p_n)] - q^*\right\|^2 \\ &= \left\|P_{\mathscr{Z}_n}[u_n - \delta_n\mathscr{L}(p_n)] + [u_n - \delta_n\mathscr{L}(p_n)] \right. \\ &\quad \left. - [u_n - \delta_n\mathscr{L}(p_n)] - q^*\right\|^2 \\ &= \|[u_n - \delta_n\mathscr{L}(p_n)] - q^*\|^2 + \left\|P_{\mathscr{Z}_n}[u_n - \delta_n\mathscr{L}(p_n)] \right. \\ &\quad \left. - [u_n - \delta_n\mathscr{L}(p_n)]\right\|^2 + 2\left\langle P_{\mathscr{Z}_n}[u_n - \delta_n\mathscr{L}(p_n)] \right. \\ &\quad \left. - [u_n - \delta_n\mathscr{L}(p_n)], [u_n - \delta_n\mathscr{L}(p_n)] - q^*\right\rangle.\end{aligned} \tag{26}$$

It is given that $q^* \in \Omega \subset \mathscr{A} \subset \mathscr{Z}_n$ such that

$$\begin{aligned}&\left\|P_{\mathscr{Z}_n}[u_n - \delta_n\mathscr{L}(p_n)] - [u_n - \delta_n\mathscr{L}(p_n)]\right\|^2 \\ &\quad + \left\langle P_{\mathscr{Z}_n}[u_n - \delta_n\mathscr{L}(p_n)] - [u_n - \delta_n\mathscr{L}(p_n)], [u_n - \delta_n\mathscr{L}(p_n)] \right. \\ &\quad \left. - q^*\right\rangle = \left\langle [u_n - \delta_n\mathscr{L}(p_n)] - P_{\mathscr{Z}_n}[u_n - \delta_n\mathscr{L}(p_n)], q^* \right. \\ &\quad \left. - P_{\mathscr{Z}_n}[u_n - \delta_n\mathscr{L}(p_n)]\right\rangle \le 0.\end{aligned} \tag{27}$$

Furthermore, it implies that

$$\begin{aligned}&\left\langle P_{\mathscr{Z}_n}[u_n - \delta_n\mathscr{L}(p_n)] - [u_n - \delta_n\mathscr{L}(p_n)], [u_n - \delta_n\mathscr{L}(p_n)] - q^*\right\rangle \\ &\quad \le -\left\|P_{\mathscr{Z}_n}[u_n - \delta_n\mathscr{L}(p_n)] - [u_n - \delta_n\mathscr{L}(p_n)]\right\|^2.\end{aligned} \tag{28}$$

By using expressions (26) and (28), we obtain

$$\begin{aligned}\|q_n - q^*\|^2 &\le \|u_n - \delta_n\mathscr{L}(p_n) - q^*\|^2 - \left\|P_{\mathscr{Z}_n}[u_n - \delta_n\mathscr{L}(p_n)] \right. \\ &\quad \left. - [u_n - \delta_n\mathscr{L}(p_n)]\right\|^2 \\ &\le \|u_n - q^*\|^2 - \|u_n - q_n\|^2 + 2\delta_n\langle\mathscr{L}(p_n), q^* - q_n\rangle.\end{aligned} \tag{29}$$

Since q^* is the solution of problem (VIP), we have

$$\langle\mathscr{L}(q^*), y - q^*\rangle \ge 0, \quad \forall y \in \mathscr{A}. \tag{30}$$

We have condition on mapping $\mathscr{L}$ with feasible set $\mathscr{A}$, we get

$$\langle\mathscr{L}(y), y - q^*\rangle \ge 0, \quad \forall y \in \mathscr{A}. \tag{31}$$

By substituting $y = p_n \in \mathscr{A}$, we get

$$\langle\mathscr{L}(p_n), p_n - q^*\rangle \ge 0. \tag{32}$$

Therefore, we have

$$\begin{aligned}\langle\mathscr{L}(p_n), q^* - q_n\rangle &= \langle\mathscr{L}(p_n), q^* - p_n\rangle + \langle\mathscr{L}(p_n), p_n - q_n\rangle \\ &\le \langle\mathscr{L}(p_n), p_n - q_n\rangle.\end{aligned} \tag{33}$$

From (29) and (33), we get

$$\begin{aligned}\|q_n - q^*\|^2 &\le \|u_n - q^*\|^2 - \|u_n - q_n\|^2 + 2\delta_n\langle\mathscr{L}(p_n), p_n - q_n\rangle \\ &\le \|u_n - q^*\|^2 - \|u_n - p_n + p_n - q_n\|^2 \\ &\quad + 2\delta_n\langle\mathscr{L}(p_n), p_n - q_n\rangle \\ &\le \|u_n - q^*\|^2 - \|u_n - p_n\|^2 - \|p_n - q_n\|^2 \\ &\quad + 2\langle u_n - \delta_n\mathscr{L}(p_n) - p_n, q_n - p_n\rangle.\end{aligned} \tag{34}$$

Since $q_n = P_{\mathscr{Z}_n}[u_n - \delta_n\mathscr{L}(p_n)]$ and from δ_{n+1}, we have

$$\begin{aligned}&2\langle u_n - \delta_n\mathscr{L}(p_n) - p_n, q_n - p_n\rangle = 2\langle u_n - \delta_n\mathscr{L}(u_n) - p_n, q_n - p_n\rangle \\ &\quad + 2\delta_n\langle\mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle \\ &\le \frac{\delta_n}{\delta_{n+1}} 2\delta_{n+1}\langle\mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle \\ &\le \frac{\mu\delta_n}{\delta_{n+1}}\|u_n - p_n\|^2 + \frac{\mu\delta_n}{\delta_{n+1}}\|q_n - p_n\|^2.\end{aligned} \tag{35}$$

Combining expressions (34) and (35), we obtain

$$\begin{aligned}\|q_n - q^*\|^2 &\le \|u_n - q^*\|^2 - \|u_n - p_n\|^2 - \|p_n - q_n\|^2 \\ &\quad + \frac{\delta_n}{\delta_{n+1}}\left[\mu\|u_n - p_n\|^2 + \mu\|q_n - p_n\|^2\right] \\ &\le \|u_n - q^*\|^2 - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n - p_n\|^2 \\ &\quad - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n - p_n\|^2.\end{aligned} \tag{36}$$

□□ □

Lemma 7. *Let $\mathscr{L} : \mathscr{Z} \longrightarrow \mathscr{Z}$ be a mapping meet the items Condition 1–Condition 4. If there exists a subsequence $\{u_{n_k}\}$ convergent weakly to $\hat{u}$ and $\lim_{k\longrightarrow\infty}\|u_{n_k} - p_{n_k}\| = 0$, then $\hat{u}$ is the solution of (VIP).*

Proof. We need to prove that $\hat{u} \in \Omega$. Indeed, we have

$$p_{n_k} = P_{\mathscr{A}}\left[u_{n_k} - \delta_{n_k}\mathscr{L}\left(u_{n_k}\right)\right], \tag{37}$$

that is equivalent to

$$\left\langle u_{n_k} - \delta_{n_k}\mathscr{L}\left(u_{n_k}\right) - p_{n_k}, y - p_{n_k}\right\rangle \le 0, \quad \forall y \in \mathscr{A}. \tag{38}$$

The above inequality implies that

$$\left\langle u_{n_k} - p_{n_k}, y - p_{n_k} \right\rangle \le \delta_{n_k} \left\langle \mathscr{L}(u_{n_k}), y - p_{n_k} \right\rangle, \quad \forall y \in \mathscr{A}. \tag{39}$$

Thus, we obtain

$$\frac{1}{\delta_{n_k}} \left\langle u_{n_k} - p_{n_k}, y - p_{n_k} \right\rangle + \left\langle \mathscr{L}(u_{n_k}), p_{n_k} - u_{n_k} \right\rangle \le \left\langle \mathscr{L}(u_{n_k}), y - u_{n_k} \right\rangle, \quad \forall y \in \mathscr{A}. \tag{40}$$

Since $\min \{\mu/L, \delta_1\} \le \delta \le \delta_1 + P$ and $\{u_{n_k}\}$ is a bounded sequence. By taking $\lim_{k\longrightarrow\infty} \|u_{n_k} - p_{n_k}\| = 0$ and $k \longrightarrow \infty$ in (18), we obtain

$$\liminf_{k\longrightarrow\infty} \left\langle \mathscr{L}(u_{n_k}), y - u_{n_k} \right\rangle \ge 0, \quad \forall y \in \mathscr{A}. \tag{41}$$

Moreover, we have

$$\begin{aligned} \left\langle \mathscr{L}(p_{n_k}), y - p_{n_k} \right\rangle &= \left\langle \mathscr{L}(p_{n_k}) - \mathscr{L}(u_{n_k}), y - u_{n_k} \right\rangle \\ &\quad + \left\langle \mathscr{L}(u_{n_k}), y - u_{n_k} \right\rangle + \left\langle \mathscr{L}(p_{n_k}), u_{n_k} - p_{n_k} \right\rangle. \end{aligned} \tag{42}$$

Since $\lim_{k\longrightarrow\infty} \|u_{n_k} - p_{n_k}\| = 0$ and $\mathscr{L}$ is L-Lipschitz continuity on $\mathscr{X}$, we have

$$\lim_{k\longrightarrow\infty} \left\| \mathscr{L}(u_{n_k}) - \mathscr{L}(p_{n_k}) \right\| = 0, \tag{43}$$

which together with (42) and (43), we obtain

$$\liminf_{k\longrightarrow\infty} \left\langle \mathscr{L}(p_{n_k}), y - p_{n_k} \right\rangle \ge 0, \quad \forall y \in \mathscr{A}. \tag{44}$$

Next, we have

$$\left\langle \mathscr{L}(u_{n_i}), y - u_{n_i} \right\rangle + \varepsilon_k \ge 0, \quad \forall i \ge m_k. \tag{45}$$

Due to $\{\varepsilon_k\}$ decreasing, this implies that $\{m_k\}$ is increasing.□□ □

Case I. Suppose that a subsequence $u_{n_{m_{k_j}}}$ of $u_{n_{m_k}}$ such as $\mathscr{L}(u_{n_{m_{k_j}}}) = 0$ ($\forall j$). Let $j \longrightarrow \infty$, we get

$$\langle \mathscr{L}(\hat{u}), y - \hat{u} \rangle = \lim_{j\longrightarrow\infty} \left\langle \mathscr{L}\left(u_{n_{m_{k_j}}}\right), y - \hat{u} \right\rangle = 0. \tag{46}$$

Then, $\hat{u} \in \mathscr{A}$ which further implies that $\hat{u} \in \Omega$.

Case II. Suppose that there exists $N_0 \in \mathbb{N}$ such that for all $n_{m_k} \ge N_0$, $\mathscr{L}(u_{n_{m_k}}) \ne 0$. Consider that

$$\Xi_{n_{m_k}} = \frac{\mathscr{L}\left(u_{n_{m_k}}\right)}{\left\|\mathscr{L}\left(u_{n_{m_k}}\right)\right\|^2}, \quad \forall n_{m_k} \ge N_0. \tag{47}$$

Due to the above definition, we obtain

$$\left\langle \mathscr{L}\left(u_{n_{m_k}}\right), \Xi_{n_{m_k}} \right\rangle = 1, \quad \forall n_{m_k} \ge N_0. \tag{48}$$

Moreover, expressions (45) and (48) for all $n_{m_k} \ge N_0$, we have

$$\left\langle \mathscr{L}\left(u_{n_{m_k}}\right), y + \varepsilon_k \Xi_{n_{m_k}} - u_{n_{m_k}} \right\rangle \ge 0. \tag{49}$$

Due to the pseudomonotonicity of $\mathscr{L}$ for $n_{m_k} \ge N_0$, we have

$$\left\langle \mathscr{L}\left(y + \varepsilon_k \Xi_{n_{m_k}}\right), y + \varepsilon_k \Xi_{n_{m_k}} - u_{n_{m_k}} \right\rangle \ge 0. \tag{50}$$

For all $n_{m_k} \ge N_0$, we have

$$\begin{aligned} \left\langle \mathscr{L}(y), y - u_{n_{m_k}} \right\rangle &\ge \left\langle \mathscr{L}(y) - \mathscr{L}\left(y + \varepsilon_k \Xi_{n_{m_k}}\right), y + \varepsilon_k \Xi_{n_{m_k}} - u_{n_{m_k}} \right\rangle \\ &\quad - \varepsilon_k \left\langle \mathscr{L}(y), \Xi_{n_{m_k}} \right\rangle. \end{aligned} \tag{51}$$

Let us consider that $\mathscr{L}(\hat{u}) \ne 0$; we obtain

$$\|\mathscr{L}(\hat{u})\| \le \liminf_{k\longrightarrow\infty} \left\|\mathscr{L}(u_{n_k})\right\|. \tag{52}$$

Thus, we obtain

$$0 \le \lim_{k\longrightarrow\infty} \left\|\varepsilon_k \Xi_{n_{m_k}}\right\| = \lim_{k\longrightarrow\infty} \frac{\varepsilon_k}{\left\|\mathscr{L}\left(u_{n_{m_k}}\right)\right\|} \le \frac{0}{\|\mathscr{L}(\hat{u})\|} = 0. \tag{53}$$

By letting $k \longrightarrow \infty$ in (51), we obtain

$$\langle \mathscr{L}(y), y - \hat{u} \rangle \ge 0, \quad \forall y \in \mathscr{A}. \tag{54}$$

Due to the Minty Lemma in [41], we infer that $\hat{u} \in \Omega$.

Theorem 8. *Let $\mathscr{L} : \mathscr{X} \longrightarrow \mathscr{X}$ be a mapping that satisfies the conditions Condition 1–Condition 4. Then, $\{u_n\}$ sequence generated by Algorithm 1 strongly converges to an element $q^* \in \Omega$.*

Step 0: take $u_1 \in \mathscr{A}, \delta_1 > 0$ and select a nonnegative sequence of real numbers $\{\varphi_n\}$ such that $\sum_{n=1}^{+\infty}\varphi_n < +\infty$. Moreover, $\{\phi_n\} \subset (a, b) \subset (0, 1 - \psi_n)$ and $\{\psi_n\} \subset (0, 1)$ meet the following criteria:

$\lim_{n\longrightarrow+\infty}\psi_n = 0$ and $\sum_{n=1}^{+\infty}\psi_n = +\infty$.

Step 1: evaluate

$p_n = P_{\mathscr{A}}(u_n - \delta_n \mathscr{L}(u_n))$.

If $u_n = p_n$, then STOP. Otherwise, go to Step 2.

Step 2: firstly, construct a half-space

$\mathscr{Z}_n = \{z \in \mathscr{Z} : \langle u_n - \delta_n \mathscr{L}(u_n) - p_n, z - p_n\rangle \le 0\}$,

and compute

$q_n = P_{\mathscr{Z}_n}(u_n - \delta_n \mathscr{L}(p_n))$.

Step 3: evaluate

$u_{n+1} = (1 - \phi_n - \psi_n)u_n + \phi_n q_n$.

Step 4: evaluate

$$\delta_{n+1} = \begin{cases} \min\left\{\delta_n + \varphi_n, \left(\mu\|u_n - p_n\|^2 + \mu\|q_n - p_n\|^2/2[\langle\mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle]\right)\right\} & \text{if}\langle\mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle > 0, \\ \varphi_n + \delta_n, & \text{otherwise.} \end{cases} \quad (3.1)$$

Set $n := n + 1$ and go back to Step 1.

Algorithm 1: Nonmonotonic explicit Mann-type subgradient extragradient method.

Proof. Since $\delta_n \longrightarrow \delta$, there exists a fixed number $\varepsilon \in (0, 1 - \mu)$ such that

$$\lim_{n\longrightarrow\infty}\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right) = 1 - \mu > \varepsilon > 0, \quad \forall n \ge n_0. \tag{55}$$

Thus, expression (36) gives that

$$\|q_n - q^*\|^2 \le \|u_n - q^*\|^2, \quad \forall n \ge n_0. \tag{56}$$

It is given that $q^* \in \Omega$; we obtain

$$\begin{aligned}\|u_{n+1} - q^*\| &= \|(1 - \phi_n - \psi_n)u_n + \phi_n q_n - q^*\| \\ &= \|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*) - \psi_n q^*\| \\ &\le \|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*)\| + \psi_n\|q^*\|.\end{aligned} \tag{57}$$

Next, we have to evaluate the following:

$$\begin{aligned}&\|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*)\|^2 \\ &\quad = (1 - \phi_n - \psi_n)^2\|u_n - q^*\|^2 + \phi_n^2\|q_n - q^*\|^2 \\ &\qquad + 2\langle(1 - \phi_n - \psi_n)(u_n - q^*), \phi_n(q_n - q^*)\rangle \\ &\quad \le (1 - \phi_n - \psi_n)^2\|u_n - q^*\|^2 + \phi_n^2\|q_n - q^*\|^2 \\ &\qquad + 2\phi_n(1 - \phi_n - \psi_n)\|u_n - q^*\|\|q_n - q^*\|,\end{aligned} \tag{58}$$

$$\begin{aligned}&\le (1 - \phi_n - \psi_n)^2\|q_n - q^*\|^2 + \phi_n^2\|q_n - q^*\|^2 \\ &\qquad + \phi_n(1 - \phi_n - \psi_n)\|u_n - q^*\|^2 + \phi_n(1 - \phi_n - \psi_n)\|q_n - q^*\|^2 \\ &\quad \le (1 - \phi_n - \psi_n)(1 - \psi_n)\|u_n - q^*\|^2 + \phi_n(1 - \psi_n)\|q_n - q^*\|^2.\end{aligned} \tag{59}$$

Substituting (56) into (59), we obtain

$$\begin{aligned}&\|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*)\|^2 \\ &\quad \le (1 - \phi_n - \psi_n)(1 - \psi_n)\|u_n - q^*\|^2 + \phi_n(1 - \psi_n)\|u_n - q^*\|^2 \\ &\quad = (1 - \psi_n)^2\|u_n - q^*\|^2.\end{aligned} \tag{60}$$

Next, we have

$$\|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*)\| \le (1 - \psi_n)\|u_n - q^*\|. \tag{61}$$

From expressions (57) and (61), we obtain

$$\begin{aligned}\|u_{n+1} - q^*\| &\le (1 - \psi_n)\|u_n - q^*\| + \psi_n\|q^*\| \le \max\{\|u_n - q^*\|, \|q^*\|\} \\ &\le \max\{\|u_{n_0} - q^*\|, \|q^*\|\}.\end{aligned} \tag{62}$$

Thus, from the above relation, we obtain that $\{u_n\}$ is bounded sequence. From sequence $\{u_{n+1}\}$, we can write

$$\begin{aligned}\|u_{n+1} - q^*\|^2 &= \|(1 - \phi_n - \psi_n)u_n + \phi_n q_n - q^*\|^2 \\ &= \|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*) - \psi_n q^*\|^2 \\ &= \|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*)\|^2 \\ &\quad + \psi_n^2\|q^*\|^2 - 2\langle(1 - \phi_n - \psi_n)(u_n - q^*) \\ &\quad + \phi_n(q_n - q^*), \psi_n q^*\rangle.\end{aligned} \tag{63}$$

By the use of expression (59), we have

$$\begin{aligned}&\|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*)\|^2 \\ &\quad \le (1 - \phi_n - \psi_n)(1 - \psi_n)\|u_n - q^*\|^2 + \phi_n(1 - \psi_n)\|q_n - q^*\|^2.\end{aligned} \tag{64}$$

Combining expressions (63) and (64) (for some $K_2 > 0$), we obtain

$$\begin{aligned}\|u_{n+1}-q^*\|^2 &\le (1-\phi_n-\psi_n)(1-\psi_n)\|u_n-q^*\|^2+\phi_n(1-\psi_n)\|q_n-q^*\|^2+\psi_n K_2\\ &\le (1-\phi_n-\psi_n)(1-\psi_n)\|u_n-q^*\|^2+\psi_n K_2+\phi_n(1-\psi_n)\\ &\quad\cdot\left[\|u_n-q^*\|^2-\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n-p_n\|^2-\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n-p_n\|^2\right]\\ &=(1-\psi_n)^2\|u_n-q^*\|^2+\psi_n K_2-\phi_n(1-\psi_n)\\ &\quad\cdot\left[\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n-p_n\|^2+\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n-p_n\|^2\right]\\ &\le \|u_n-q^*\|^2+\psi_n K_2-\phi_n(1-\psi_n)\\ &\quad\cdot\left[\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n-p_n\|^2+\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n-p_n\|^2\right].\end{aligned}\tag{65}$$

□

The remainder of the proof is now split into two parts:

Case 1. Suppose that there exists a fixed number $n_1 \in \mathbb{N}$ $(n_1 \ge n_0)$ such that

$$\|u_{n+1}-q^*\| \le \|u_n-q^*\|,\quad \forall n \ge n_1.\tag{66}$$

Then, $\lim_{n\longrightarrow\infty}\|u_n-q^*\|$ exists. From (65), we have

$$\begin{aligned}&\phi_n(1-\psi_n)\left[\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n-p_n\|^2+\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n-p_n\|^2\right]\\ &\quad\le \|u_n-q^*\|^2+\psi_n K_2-\|u_{n+1}-q^*\|^2.\end{aligned}\tag{67}$$

Due to the existence of $\lim_{n\longrightarrow+\infty}\|u_n-q^*\|$ and $\psi_n \longrightarrow 0$, we infer that

$$\lim_{n\longrightarrow\infty}\|u_n-p_n\| = \lim_{n\longrightarrow\infty}\|q_n-p_n\| = 0.\tag{68}$$

It follows that

$$\lim_{n\longrightarrow\infty}\|u_n-q_n\| \le \lim_{n\longrightarrow\infty}\|u_n-p_n\| + \lim_{n\longrightarrow\infty}\|p_n-q_n\| = 0.\tag{69}$$

It follows from expression (69) and $\psi_n \longrightarrow 0$ that

$$\begin{aligned}\|u_{n+1}-u_n\| &= \|(1-\phi_n-\psi_n)u_n+\phi_n q_n-u_n\|\\ &= \|u_n-\psi_n u_n+\phi_n q_n-\phi_n u_n-u_n\|\\ &\le \phi_n\|q_n-u_n\|+\psi_n\|u_n\|,\end{aligned}\tag{70}$$

which implies that

$$\|u_{n+1}-u_n\| \longrightarrow 0 \text{ as } n \longrightarrow +\infty.\tag{71}$$

We have $q^* = P_\Omega(0)$ and by using Lemma 1 (i), we can write

$$\langle 0-q^*, y-q^*\rangle \le 0,\quad \forall y \in \Omega.\tag{72}$$

By using expression (72) and Lemma 7, we obtain

$$\limsup_{n\longrightarrow\infty}\langle q^*, q^*-u_n\rangle = \limsup_{k\longrightarrow\infty}\langle q^*, q^*-u_{n_k}\rangle = \langle q^*, q^*-\hat{u}\rangle \le 0.\tag{73}$$

Due to $\lim_{n\longrightarrow\infty}\|u_{n+1}-u_n\| = 0$. It gives that

$$\limsup_{n\longrightarrow\infty}\langle q^*, q^*-u_{n+1}\rangle \le \limsup_{n\longrightarrow\infty}\langle q^*, q^*-u_n\rangle + \limsup_{n\longrightarrow\infty}\langle q^*, u_n-u_{n+1}\rangle \le 0.\tag{74}$$

Next, we assume that

$$t_n = (1-\phi_n)u_n+\phi_n q_n.\tag{75}$$

Thus, we have

$$\begin{aligned}u_{n+1} &= t_n-\psi_n u_n = (1-\psi_n)t_n-\psi_n(u_n-t_n)\\ &= (1-\psi_n)t_n-\psi_n\phi_n(u_n-q_n),\end{aligned}\tag{76}$$

where

$$u_n-t_n = u_n-(1-\phi_n)u_n-\phi_n q_n = \phi_n(u_n-q_n).\tag{77}$$

Thus, we obtain

$$\begin{aligned}\|u_{n+1}-q^*\|^2 &= \|(1-\psi_n)t_n+\phi_n\psi_n(q_n-u_n)-q^*\|^2\\ &= \|(1-\psi_n)(t_n-q^*)+[\phi_n\psi_n(q_n-u_n)-\psi_n q^*]\|^2\\ &\le (1-\psi_n)^2\|t_n-q^*\|^2+2\langle\phi_n\psi_n(q_n-u_n)\\ &\quad-\psi_n q^*, (1-\psi_n)(t_n-q^*)+\phi_n\psi_n(q_n-u_n)-\psi_n q^*\rangle\\ &= (1-\psi_n)^2\|t_n-q^*\|^2+2\langle\phi_n\psi_n(q_n-u_n)\\ &\quad-\psi_n q^*, t_n-\psi_n t_n-\psi_n(u_n-t_n)-q^*\rangle\\ &= (1-\psi_n)\|t_n-q^*\|^2+2\phi_n\psi_n\langle q_n-u_n, u_{n+1}-q^*\rangle\\ &\quad+2\psi_n\langle q^*, q^*-u_{n+1}\rangle\\ &\le (1-\psi_n)\|t_n-q^*\|^2+2\phi_n\psi_n\|q_n-u_n\|\|u_{n+1}\\ &\quad-q^*\|+2\psi_n\langle q^*, q^*-u_{n+1}\rangle.\end{aligned}\tag{78}$$

Next, we have to compute

$$\begin{aligned}\|t_n-q^*\|^2 &= \|(1-\phi_n)u_n+\phi_n q_n-q^*\|^2\\ &= \|(1-\phi_n)(u_n-q^*)+\phi_n(q_n-q^*)\|^2\\ &= (1-\phi_n)^2\|u_n-q^*\|^2+\phi_n^2\|q_n-q^*\|^2\\ &\quad+2\langle(1-\phi_n)(u_n-q^*), \phi_n(q_n-q^*)\rangle\\ &\le (1-\phi_n)^2\|u_n-q^*\|^2+\phi_n^2\|q_n-q^*\|^2\\ &\quad+2\phi_n(1-\phi_n)\|u_n-q^*\|\|q_n-q^*\|\\ &\le (1-\phi_n)^2\|u_n-q^*\|^2+\phi_n^2\|q_n-q^*\|^2\\ &\quad+\phi_n(1-\phi_n)\|u_n-q^*\|^2+\phi_n(1-\phi_n)\|q_n-q^*\|^2\\ &= (1-\phi_n)\|u_n-q^*\|^2+\phi_n\|q_n-q^*\|^2\\ &\le (1-\phi_n)\|u_n-q^*\|^2+\phi_n\|u_n-q^*\|^2 = \|u_n-q^*\|^2.\end{aligned}\tag{79}$$

Step 0: take $u_1 \in \mathscr{A}, \delta_1 > 0$ and select a nonnegative sequence of real number $\{\varphi_n\}$ such that $\sum_{n=1}^{+\infty}\varphi_n < +\infty$. Moreover, $\{\psi_n\} \subset (0,1)$ meet the following criteria:
$\lim_{n\longrightarrow+\infty}\psi_n = 0$ and $\sum_{n=1}^{+\infty}\psi_n = +\infty$.
Step 1: evaluate
$p_n = P_{\mathscr{A}}(u_n - \delta_n \mathscr{L}(u_n))$.
If $u_n = p_n$, then STOP. Otherwise, go to Step 2.
Step 2: evaluate
$\mathscr{Z}_n = \{z \in \mathscr{Z} : \langle u_n - \delta_n \mathscr{L}(u_n) - p_n, z - p_n\rangle \le 0\}$,
and compute
$q_n = P_{\mathscr{Z}_n}(u_n - \delta_n \mathscr{L}(p_n))$.
Step 3: evaluate
$u_{n+1} = \psi_n g(u_n) + (1-\psi_n)q_n$.
Step 4: evaluate

$$\delta_{n+1} = \begin{cases} \min\left\{\delta_n + \varphi_n, \left(\mu\|u_n - p_n\|^2 + \mu\|q_n - p_n\|^2/2[\langle \mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle]\right)\right\} & \text{if } \langle \mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle > 0, \\ \varphi_n + \delta_n, & \text{otherwise.} \end{cases} \quad (3.2)$$

Set $n := n+1$ and go back to Step 1.

Algorithm 2: Nonmonotonic explicit viscosity-type subgradient extragradient method.

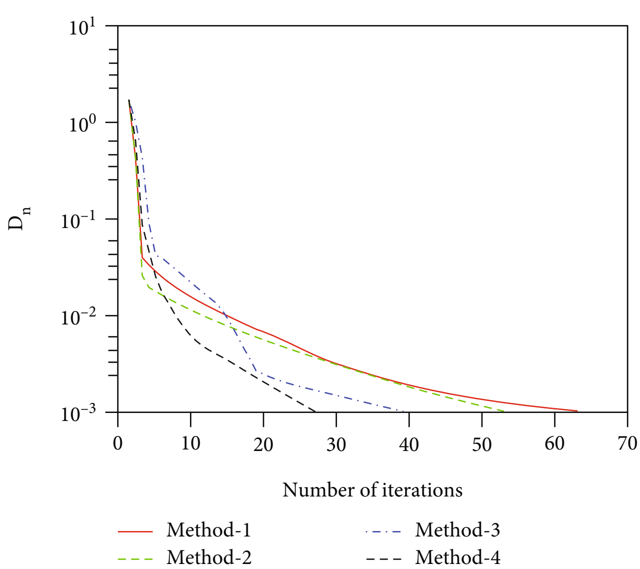

Figure 1: Algorithmic descriptions of Algorithm 1 and Algorithm 2 as well Algorithm 2 in [37] and Algorithm 1 in [36] when $m = 5$.

From expressions (78) and (79), we have

$$\|u_{n+1} - q^*\|^2 \le (1-\psi_n)\|u_n - q^*\|^2 + \psi_n[2\phi_n\|q_n - u_n\|\|u_{n+1} - q^*\| + 2\langle q^*, q^* - u_{n+1}\rangle]. \quad (80)$$

By the use of expressions (74) and (80) and Lemma 2, we can derive that $\|u_n - q^*\| \longrightarrow 0$ as $n \longrightarrow +\infty$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ in order that

$$\left\|u_{n_i} - q^*\right\| \le \left\|u_{n_{i+1}} - q^*\right\|, \quad \forall i \in \mathbb{N}. \quad (81)$$

By Lemma 3, we have

$$\left\|u_{m_k} - q^*\right\| \le \left\|u_{m_{k+1}} - q^*\right\| \quad \text{and} \quad \left\|u_k - q^*\right\| \le \left\|u_{m_{k+1}} - q^*\right\|, \quad \forall k \in \mathbb{N}. \quad (82)$$

By the use of expression (67), we have

$$\phi_{m_k}\left(1-\psi_{m_k}\right)\left[\left(1 - \frac{\mu\delta_{m_k}}{\delta_{m_k+1}}\right)\left\|u_{m_k} - p_{m_k}\right\|^2 + \left(1 - \frac{\mu\delta_{m_k}}{\delta_{m_k+1}}\right)\left\|q_{m_k} - p_{m_k}\right\|^2\right] \le \left\|u_{m_k} - q^*\right\|^2 + \psi_{m_k}K_2 - \left\|u_{m_k+1} - q^*\right\|^2. \quad (83)$$

Due to $\psi_{m_k} \longrightarrow 0$, we can deduce as follows:

$$\lim_{k\longrightarrow\infty}\left\|u_{m_k} - p_{m_k}\right\| = \lim_{k\longrightarrow\infty}\left\|q_{m_k} - p_{m_k}\right\| = 0. \quad (84)$$

It follows that

$$\lim_{k\longrightarrow\infty}\left\|u_{m_k} - q_{m_k}\right\| \le \lim_{k\longrightarrow\infty}\left\|u_{m_k} - p_{m_k}\right\| + \lim_{k\longrightarrow\infty}\left\|p_{m_k} - q_{m_k}\right\| = 0. \quad (85)$$

Further, it implies that

$$\begin{aligned}\left\|u_{m_k+1} - u_{m_k}\right\| &= \left\|\left(1 - \phi_{m_k} - \psi_{m_k}\right)u_{m_k} + \phi_{m_k}q_{m_k} - u_{m_k}\right\| \\ &= \left\|u_{m_k} - \psi_{m_k}u_{m_k} + \phi_{m_k}q_{m_k} - \phi_{m_k}u_{m_k} - u_{m_k}\right\| \\ &\le \phi_{m_k}\left\|q_{m_k} - u_{m_k}\right\| + \psi_{m_k}\left\|u_{m_k}\right\| \longrightarrow 0.\end{aligned} \quad (86)$$

Similar to Case 1, we obtain

$$\limsup_{k\longrightarrow\infty}\left\langle q^*, u_{m_k+1} - q^*\right\rangle \le 0. \quad (87)$$

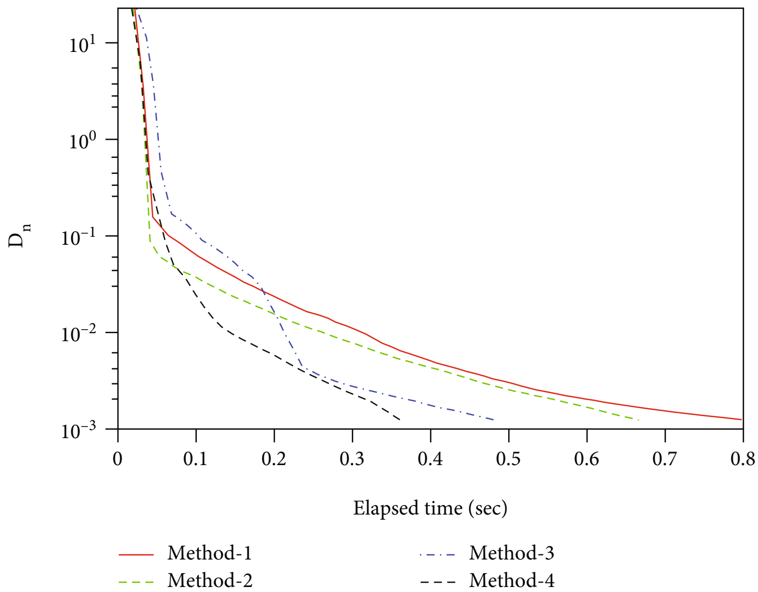

FIGURE 2: Algorithmic descriptions of Algorithm 1 and Algorithm 2 with Algorithm 2 in [37] and Algorithm 1 in [36] when $m = 5$.

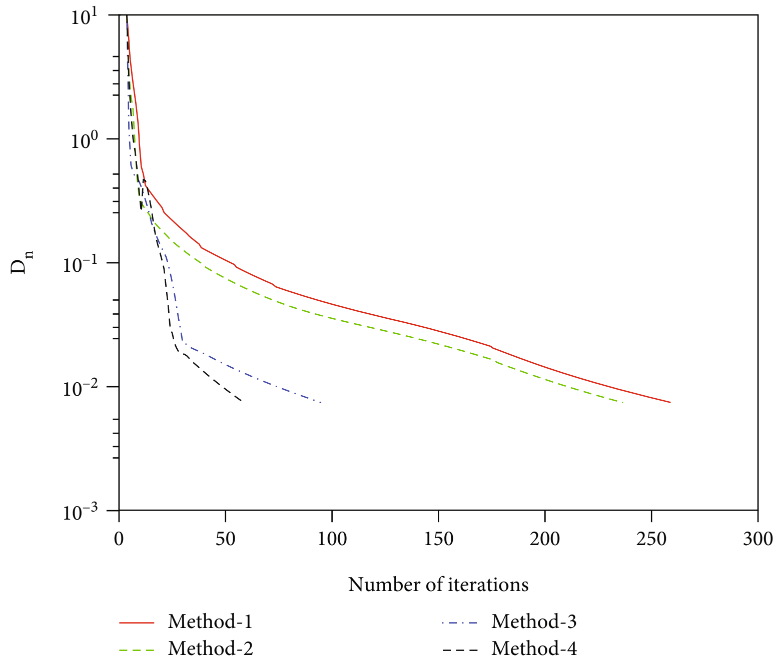

FIGURE 3: Algorithmic descriptions of Algorithm 1 and Algorithm 2 with Algorithm 2 in [37] and Algorithm 1 in [36] when $m = 10$.

By the use of expressions (80) and (82), we have

$$\begin{aligned} \left\| u_{m_k+1} - q^* \right\|^2 \leq{} & \left(1 - \psi_{m_k}\right) \left\| u_{m_k} - q^* \right\|^2 + \psi_{m_k} \\ & \cdot \left[2\phi_{m_k} \left\| q_{m_k} - u_{m_k} \right\| \left\| u_{m_k+1} - q^* \right\| + 2\left\langle q^*, q^* - u_{m_k+1} \right\rangle \right] \\ \leq{} & \left(1 - \psi_{m_k}\right) \left\| u_{m_{k+1}} - q^* \right\|^2 + \psi_{m_k} \\ & \cdot \left[2\phi_{m_k} \left\| q_{m_k} - u_{m_k} \right\| \left\| u_{m_k+1} - q^* \right\| + 2\left\langle q^*, q^* - u_{m_k+1} \right\rangle \right]. \end{aligned} \tag{88}$$

It follows that

$$\left\| u_{m_k+1} - q^* \right\|^2 \leq 2\phi_{m_k} \left\| q_{m_k} - u_{m_k} \right\| \left\| u_{m_k+1} - q^* \right\| + 2\left\langle q^*, q^* - u_{m_k+1} \right\rangle. \tag{89}$$

Since $\psi_{m_k} \longrightarrow 0$ and $\| u_{m_k} - q^* \|$ is a bounded sequence with (87) and (89), we have

$$\left\| u_{m_k+1} - q^* \right\|^2 \longrightarrow 0, \text{as} k \longrightarrow \infty. \tag{90}$$

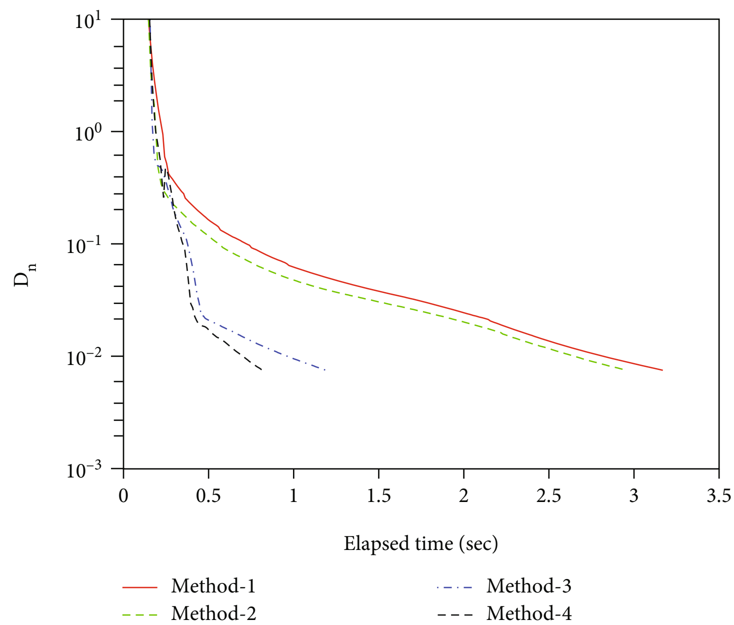

FIGURE 4: Algorithmic descriptions of Algorithm 1 and Algorithm 2 with Algorithm 2 in [37] and Algorithm 1 in [36] when $m = 10$.

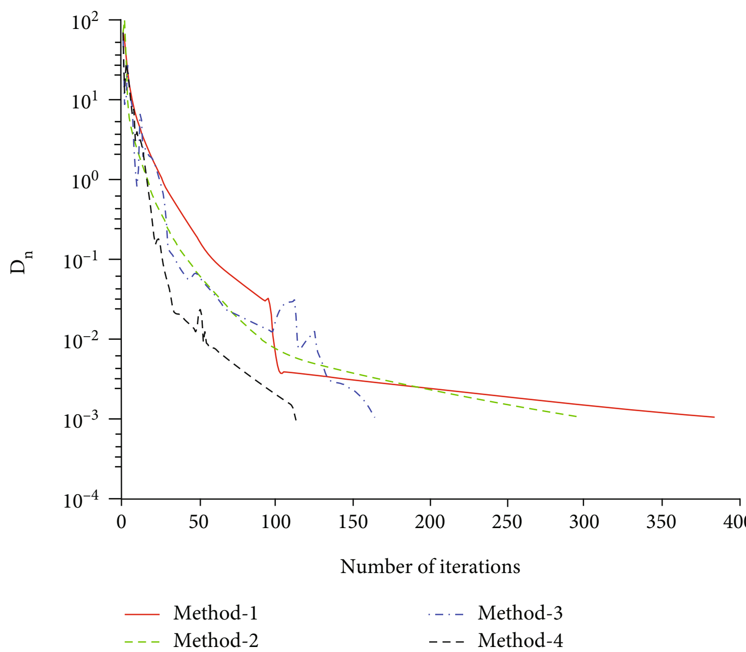

FIGURE 5: Algorithmic descriptions of Algorithm 1 and Algorithm 2 with Algorithm 2 in [37] and Algorithm 1 in [36] when $m = 20$.

The above expression implies that

$$\lim_{k\longrightarrow\infty} \|u_k - q^*\|^2 \le \lim_{k\longrightarrow\infty} \left\|u_{m_k+1} - q^*\right\|^2 \le 0. \tag{91}$$

From the above discussions, we have $u_n \longrightarrow q^*$ as $n \longrightarrow \infty$.

Theorem 9. *Assume that $\mathscr{L} : \mathscr{Z} \longrightarrow \mathscr{Z}$ is a mapping that meets the conditions Condition 1–Condition 4. Then, the $\{u_n\}$ sequence generated by Algorithm 2 strongly converges to an element $q^* \in \Omega$.*

Proof. Since $\delta_n \longrightarrow \delta$, there exists a positive number $\varepsilon \in (0, 1 - \mu)$ such that

$$\lim_{n\longrightarrow\infty} \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right) = 1 - \mu > \varepsilon > 0. \tag{92}$$

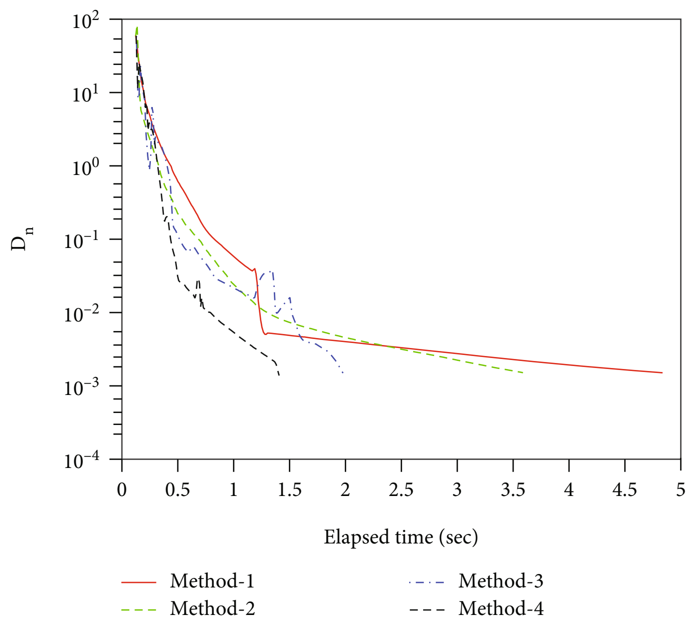

Figure 6: Algorithmic descriptions of Algorithm 1 and Algorithm 2 with Algorithm 2 in [37] and Algorithm 1 in [36] when $m = 20$.

Table 1: Algorithmic study for Example 11.

m	Method-1		Method-2		Method-3		Method-4	
	Iter.	Time	Iter.	Time	Iter.	Time	Iter.	Time
5	68	0.798005200	57	0.665091600	42	0.480156900	29	0.360117900
10	264	3.218157200	241	2.998657400	95	1.121305500	58	0.733156900
20	384	4.884210700	297	3.593486400	164	1.930182900	113	1.339267400
30	649	8.356499700	545	7.850765000	308	3.986678100	196	2.593555200
40	1138	16.96909990	997	15.25866440	357	4.784712300	238	3.410400500
50	2467	37.42977650	17962	24.32523500	558	10.32619485	487	6.132444757

Table 2: Algorithmic study for Example 12.

m	Method-1		Method-2		Method-3		Method-4	
	Iter.	Time	Iter.	Time	Iter.	Time	Iter.	Time
1	482	0.294028700	406	0.211852600	349	0.166720900	308	0.166098200
t^2	268	0.136482920	196	0.112184600	155	0.103728300	112	0.094193700
e^t	371	0.145282000	311	0.123552000	255	0.099748260	211	0.073510000
$\sin(t)$	431	0.212849200	389	0.201682200	279	0.112940100	239	0.102749200

Thus, there exists a number $N_1 \in \mathbb{N}$ such that

$$\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right) > \varepsilon > 0, \quad \forall n \geq N_1. \tag{93}$$

From expression (36), we obtain

$$\|q_n - q^*\|^2 \leq \|u_n - q^*\|^2, \quad \forall n \geq N_1. \tag{94}$$

Next, we have

$$\begin{aligned} \|u_{n+1} - q^*\| &= \|\psi_n g(u_n) + (1-\psi_n)q_n - q^*\| \\ &= \|\psi_n[g(u_n) - q^*] + (1-\psi_n)[q_n - q^*]\| \\ &= \|\psi_n[g(u_n) + g(q^*) - g(q^*) - q^*] + (1-\psi_n)[q_n - q^*]\| \\ &\leq \psi_n\|g(u_n) - g(q^*)\| + \psi_n\|g(q^*) - q^*\| + (1-\psi_n)\|q_n - q^*\| \\ &\leq \psi_n\xi\|u_n - q^*\| + \psi_n\|g(q^*) - q^*\| + (1-\psi_n)\|q_n - q^*\|. \end{aligned} \tag{95}$$

From expressions (94) and (95) and $\psi_n \subset (0,1)$, we

Table 3: Example 13: algorithmic description of Algorithm 2 in [37] and $u_1 = (1, 2, 3, 4)^T$.

Iter. (n)	u_1	u_2	u_3	u_4
1	4.59999999902338	18.5464999999117	18.7004999999153	17.5944999993118
2	-6.26760004899506	16.7378886580863	16.9671141650914	15.8548101409442
3	-5.91418181483705	16.0308795300640	16.2986437617166	15.2919549387799
4	-5.38921479466356	4.78801650028211	4.64461284974750	5.38714304326419
5	1.00580406596249	4.73241274186297	4.62870240188635	5.25179596894124
6	1.23013013951725	4.70677542647729	4.63592929873170	4.97142857141815
7	1.44950907993550	4.70213655930865	4.65912535719577	4.97499999984847
8	1.66717886007189	4.71660015705847	4.69751098404819	4.97777776498426
9	1.88563318471514	4.74871960483109	4.75038581190807	4.97999999900850
10	2.10707812245062	4.79781281512387	4.81759412391964	4.98181818166635
11	2.33359792353143	4.86377467908333	4.89941458277316	4.98333333327664
12	2.56667054007920	4.94667114193612	4.98265426879577	4.98294766642007
13	2.79924540130681	4.96031745289142	4.97036949333673	4.98462062077961
14	3.03033454990915	4.96124045379347	4.97356261422028	4.98546003308369
15	3.26012741095086	4.96358686948872	4.97505259464489	4.98655189412727
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
1086	4.99924843119192	4.99944280475227	4.99963717745358	4.99983154929598
1087	4.99924912189630	4.99944331684910	4.99963751094444	4.99983170418246
1088	4.99924981133227	4.99944382800546	4.99963784382278	4.99983185878434
1089	4.99925049950333	4.99944433822395	4.99963817609027	4.99983201310240
1090	4.99925118641296	4.99944484750715	4.99963850774860	4.99983216713743
1091	4.99925187206462	4.99944535585762	4.99963883879944	4.99983232089020
1092	4.99925255646177	4.99944586327793	4.99963916924446	4.99983247436150
1093	4.99925323960783	4.99944636977061	4.99963949908532	4.99983262755208
1094	4.99925392150626	4.99944687533823	4.99963982832368	4.99983278046273
1095	4.99925460216044	4.99944737998330	4.99964015696118	4.99983293309420
1096	4.99925528157380	4.99944788370835	4.99964048499947	4.99983308544727
1097	4.99925595974971	4.99944838651589	4.99964081244018	4.99983323752269
1098	4.99925663669156	4.99944888840844	4.99964113928495	4.99983338932122
1099	4.99925731240271	4.99944938938848	4.99964146553540	4.99983354084361
1100	4.99925798688652	4.99944988945849	4.99964179119316	4.99983369209063
1101	4.99925866014632	4.99945038862097	4.99964211625983	4.99983384306301
1102	4.99925933218544	4.99945088687837	4.99964244073702	4.99983399376152
1103	4.99926000300721	4.99945138423316	4.99964276462634	4.99983414418688
1104	4.99926067261492	4.99945188068778	4.99964308792938	4.99983429433984
1105	4.99926134101187	4.99945237624468	4.99964341064774	4.99983444422115
1106	4.99926200820134	4.99945287090629	4.99964373278299	4.99983459383154
1107	4.99926267418660	4.99945336467504	4.99964405433671	4.99983474317174
CPU time is seconds	8.339245			

obtain

$$\begin{aligned}\|u_{n+1}-q^*\| &\le \psi_n\xi\|u_n-q^*\| + \psi_n\|g(q^*)-q^*\| + (1-\psi_n)\|u_n-q^*\| \\ &= [1-\psi_n+\xi\psi_n]\|u_n-q^*\| + \psi_n(1-\xi)\frac{\|g(q^*)-q^*\|}{(1-\xi)} \\ &\le \max\left\{\|u_n-q^*\|, \frac{\|g(q^*)-q^*\|}{(1-\xi)}\right\} \le \max\left\{\|u_{N_1}-q^*\|, \frac{\|g(q^*)-q^*\|}{(1-\xi)}\right\}.\end{aligned} \tag{96}$$

Therefore, we infer that the $\{u_n\}$ is a bounded sequence. Now, we are in a position to use the Banach contraction theorem for the existence of a unique fixed point $q^* \in \Omega$ such that

$$q^* = P_\Omega(g(q^*)). \tag{97}$$

Due to the projection mapping, we can write

$$\langle g(q^*) - q^*, y - q^* \rangle \le 0, \quad \forall y \in \Omega. \tag{98}$$

From Lemma 1 and expression (36), we have

$$\begin{aligned}\|u_{n+1} - q^*\|^2 &= \|\psi_n g(u_n) + (1-\psi_n)q_n - q^*\|^2 \\ &= \|\psi_n[g(u_n) - q^*] + (1-\psi_n)[q_n - q^*]\|^2 \\ &= \psi_n\|g(u_n) - q^*\|^2 + (1-\psi_n)\|q_n - q^*\|^2 \\ &\quad - \psi_n(1-\psi_n)\|g(u_n) - q_n\|^2 \\ &\le \psi_n\|g(u_n) - q^*\|^2 + (1-\psi_n) \\ &\quad \cdot \left[\|\|u_n - q^*\|^2 - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n - p_n\|^2 \right.\\ &\quad \left. - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n - p_n\|^2\right] - \psi_n(1-\psi_n)\|g(u_n) - q_n\|^2 \\ &\le \psi_n\|g(u_n) - q^*\|^2 + \|u_n - q^*\|^2 - (1-\psi_n) \\ &\quad \cdot \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n - p_n\|^2 - (1-\psi_n)\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n - p_n\|^2.\end{aligned} \tag{99}$$

The above expression implies that

$$\begin{aligned}&(1-\psi_n)\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n - p_n\|^2 + (1-\psi_n)\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n - p_n\|^2 \\ &\quad \le \psi_n\|g(u_n) - q^*\|^2 + \|u_n - q^*\|^2 - \|u_{n+1} - q^*\|^2.\end{aligned} \tag{100}$$

□ □

Case 1. Suppose that there exists a fixed number $N_2 \in \mathbb{N}$ $(N_2 \ge N_1)$ such that

$$\|u_{n+1} - q^*\| \le \|u_n - q^*\|, \forall n \ge N_2. \tag{101}$$

Then, $\lim_{n\longrightarrow\infty}\|u_n - q^*\|$ exists and let $\lim_{n\longrightarrow\infty}\|u_n - q^*\| = l$. By the use of expression (99), we have

$$\begin{aligned}&(1-\psi_n)\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n - p_n\|^2 + (1-\psi_n)\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n - p_n\|^2 \\ &\quad \le \psi_n\|g(u_n) - q^*\|^2 + \|\|u_n - q^*\|^2 - \|u_{n+1} - q^*\|^2.\end{aligned} \tag{102}$$

By the of existence of a limit of a sequence $\lim_{n\longrightarrow\infty}\|u_n - q^*\|$ and $\psi_n \longrightarrow 0$, we obtain

$$\lim_{n\longrightarrow\infty}\|u_n - p_n\| = \lim_{n\longrightarrow\infty}\|q_n - p_n\| = 0. \tag{103}$$

Furthermore, it implies that

$$\lim_{n\longrightarrow\infty}\|u_n - p_n\| \le \lim_{n\longrightarrow\infty}\|u_n - p_n\| + \lim_{n\longrightarrow\infty}\|q_n - p_n\| = 0. \tag{104}$$

Thus, we have

$$\begin{aligned}\|u_{n+1} - u_n\| &= \|\psi_n g(u_n) + (1-\psi_n)q_n - u_n\| \\ &= \|\psi_n[g(u_n) - u_n] + (1-\psi_n)[q_n - u_n]\| \\ &\le \psi_n\|g(u_n) - u_n\| + (1-\psi_n)\|q_n - u_n\| \longrightarrow 0.\end{aligned} \tag{105}$$

Thus, we have

$$\lim_{n\longrightarrow\infty}\|u_{n+1} - u_n\| = 0. \tag{106}$$

By using expression (97) and Lemma 7, we obtain

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\langle g(q^*) - q^*, u_n - q^*\rangle &= \limsup_{k\longrightarrow\infty}\langle g(q^*) - q^*, u_{n_k} - q^*\rangle \\ &= \langle g(q^*) - q^*, \hat{u} - q^*\rangle \le 0.\end{aligned} \tag{107}$$

By the use of $\lim_{n\longrightarrow\infty}\|u_{n+1} - u_n\| = 0$, we have

$$\begin{aligned}&\limsup_{n\longrightarrow\infty}\langle g(q^*) - q^*, u_{n+1} - q^*\rangle \le \limsup_{n\longrightarrow\infty}\langle g(q^*) - q^*, u_{n+1} - u_n\rangle \\ &\quad + \limsup_{n\longrightarrow\infty}\langle g(q^*) - q^*, u_n - q^*\rangle \le 0.\end{aligned} \tag{108}$$

By using Lemma 1 and expression (94), we have

$$\begin{aligned}\|u_{n+1} - q^*\|^2 &= \|\psi_n g(u_n) + (1-\psi_n)q_n - q^*\|^2 \\ &= \|\psi_n[g(u_n) - q^*] + (1-\psi_n)[q_n - q^*]\|^2 \\ &\le (1-\psi_n)^2\|q_n - q^*\|^2 + 2\psi_n\langle g(u_n) - q^*, (1-\psi_n)[q_n - q^*] \\ &\quad + \psi_n[g(u_n) - q^*]\rangle \\ &= (1-\psi_n)^2\|q_n - q^*\|^2 + 2\psi_n\langle g(u_n) - g(q^*) \\ &\quad + g(q^*) - q^*, u_{n+1} - q^*\rangle \\ &= (1-\psi_n)^2\|q_n - q^*\|^2 + 2\psi_n\langle g(u_n) - g(q^*), u_{n+1} - q^*\rangle \\ &\quad + 2\psi_n\langle g(q^*) - q^*, u_{n+1} - q^*\rangle \\ &\le (1-\psi_n)^2\|q_n - q^*\|^2 + 2\psi_n\xi\|u_n - q^*\|^2\|u_{n+1} - q^*\| \\ &\quad + 2\psi_n\langle g(q^*) - q^*, u_{n+1} - q^*\rangle \\ &\le \left(1 + \psi_n^2 - 2\psi_n\right)\|u_n - q^*\|^2 + 2\psi_n\xi\|u_n - q^*\|^2 \\ &\quad + 2\psi_n\langle g(q^*) - q^*, u_{n+1} - q^*\rangle \\ &= (1 - 2\psi_n)\|u_n - q^*\|^2 + \psi_n^2\|u_n - q^*\|^2 + 2\psi_n\xi\|u_n - q^*\|^2 \\ &\quad + 2\psi_n\langle g(q^*) - q^*, u_{n+1} - q^*\rangle = [1 - 2\psi_n(1-\xi)]\|u_n - q^*\|^2 \\ &\quad + 2\psi_n(1-\xi)\left[\frac{\psi_n\|u_n - q^*\|^2}{2(1-\xi)} + \frac{\langle g(q^*) - q^*, u_{n+1} - q^*\rangle}{1-\xi}\right].\end{aligned} \tag{109}$$

It is clear from expressions (108) and (109) such that

$$\limsup_{n\longrightarrow\infty}\left[\frac{\psi_n\|u_n - q^*\|^2}{2(1-\xi)} + \frac{\langle g(q^*) - q^*, u_{n+1} - q^*\rangle}{1-\xi}\right] \le 0. \tag{110}$$

By choosing $n \ge N_3 \in \mathbb{N}$ $(N_3 \ge N_2)$ large enough such that $2\gamma_n(1-\xi) < 1$. By the use of (109) and (110) and

Table 4: Example 13: algorithmic description of Algorithm 1 in [36] and $u_1 = (1, 2, 3, 4)^T$.

Iter. (n)	u_1	u_2	u_3	u_4
1	4.54999999902338	18.4464999999117	18.5504999999153	17.3944999993118
2	-6.24222797440616	13.5241851137974	13.6213471992304	12.5302619081771
3	-0.659876119446077	4.68238127646957	4.65194330093169	5.00329162462314
4	0.946802477625882	4.61084600770298	4.58119171017774	4.92362497958044
5	1.04934390009719	4.55766490194321	4.52868493067142	4.86118347395020
6	1.14854226167386	4.51824332123436	4.48986816365673	4.81270779537308
7	1.24574590629948	4.48946698164860	4.46164839905812	4.77550686438977
8	1.34146056234125	4.46933050144783	4.44203357836196	4.74729888976619
9	1.43627311838568	4.45647220663972	4.42967080774664	4.72665515468439
10	1.53065237050934	4.44994299224969	4.42361803309317	4.71255603858976
11	1.62498530558409	4.44907178330946	4.42320908822800	4.70425974095071
12	1.71962000624484	4.45338248767389	4.42797158878528	4.70123046557113
13	1.81488364798492	4.46254250802635	4.43757564752582	4.70308628647431
14	1.91109217493918	4.47632857351891	4.45180046474299	4.70956149682268
15	2.00855776273081	4.49460389018728	4.47051224149727	4.72048089550204
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
1081	4.99949078231531	4.99949078231531	4.99949078231531	4.99949078231531
1082	4.99949130039142	4.99949130039142	4.99949130039142	4.99949130039142
1083	4.99949181741442	4.99949181741442	4.99949181741442	4.99949181741442
1084	4.99949233338752	4.99949233338752	4.99949233338752	4.99949233338752
1085	4.99949284831392	4.99949284831392	4.99949284831392	4.99949284831392
1086	4.99949336219679	4.99949336219679	4.99949336219679	4.99949336219679
1087	4.99949387503931	4.99949387503931	4.99949387503931	4.99949387503931
1088	4.99949438684463	4.99949438684463	4.99949438684463	4.99949438684463
1089	4.99949489761590	4.99949489761590	4.99949489761590	4.99949489761590
1090	4.99949540735624	4.99949540735624	4.99949540735624	4.99949540735624
1091	4.99949591606878	4.99949591606878	4.99949591606878	4.99949591606878
1092	4.99949642375661	4.99949642375661	4.99949642375661	4.99949642375661
1093	4.99949693042284	4.99949693042284	4.99949693042284	4.99949693042284
1094	4.99949743607054	4.99949743607054	4.99949743607054	4.99949743607054
1095	4.99949794070278	4.99949794070278	4.99949794070278	4.99949794070278
1096	4.99949844432262	4.99949844432262	4.99949844432262	4.99949844432262
1097	4.99949894693310	4.99949894693310	4.99949894693310	4.99949894693310
1098	4.99949944853725	4.99949944853725	4.99949944853725	4.99949944853725
1099	4.99949994913809	4.99949994913809	4.99949994913809	4.99949994913809
1000	4.99950044873863	4.99950044873863	4.99950044873863	4.99950044873863
1001	4.99950094734186	4.99950094734186	4.99950094734186	4.99950094734186
CPU time is seconds	7.690413			

through Lemma 2, we conclude that $\|u_n - q^*\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Case 2. Consider that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\left\|u_{n_i} - q^*\right\| \le \left\|u_{n_{i+1}} - q^*\right\|, \quad \forall i \in \mathbb{N}. \tag{111}$$

Then, using Lemma 3, there exists a sequence $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \longrightarrow \infty$, such that

$$\left\|u_{m_k} - q^*\right\| \le \left\|u_{m_{k+1}} - q^*\right\|, \left\|u_k - q^*\right\| \le \left\|u_{m_{k+1}} - q^*\right\|, \quad \text{for all} k \in \mathbb{N}. \tag{112}$$

Table 5: Example 13: algorithmic description of Algorithm 1 and $u_1 = (1, 2, 3, 4)^T$.

Iter. (n)	u_1	u_2	u_3	u_4
1	4.76666666563579	19.3990833332401	19.4727499999106	18.2164166659403
2	-6.89912059339238	18.1614280199001	18.2430063869481	16.8357425433078
3	-6.79443152520300	17.8896263780042	17.9698451056013	16.5860356593549
4	-6.48577908884055	4.74876581346600	4.68957409065068	5.74296770759381
5	0.967259458426478	4.73386092054218	4.67600506823831	5.28600264404358
6	1.95405193843723	6.82802561081780	6.99483339988724	5.16669743240466
7	3.19683502766684	5.04658834550239	5.03534741320422	5.05670610709957
8	4.88155078605583	4.96308402800923	4.96500619808367	4.96133883492052
9	1.29027081990860	6.20942957200619	6.32344981434144	6.10583077439888
10	-16.8977269679284	39.2523493068962	39.5416762684644	38.9810641968152
11	-16.8686988161152	39.2288250269955	39.5165446165549	38.9590470564150
12	-17.5936196810597	5.25777948675977	5.05059553613937	5.45334598035476
13	0.953804394536662	5.24340705918321	5.03779221686998	5.23720308394482
14	-2.97340727848293	17.2929482826102	17.1772480445095	17.2896868859967
15	-2.98444635460549	17.2911791507813	17.1759609970061	17.2879313433204
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
718	4.99913721960184	4.99966942735891	4.99966941266818	4.99966942693145
719	4.99913841886068	4.99966988620944	4.99966987154072	4.99966988578262
720	4.99913961479023	4.99967034378793	4.99967032914114	4.99967034336174
721	4.99914080740432	4.99967080009965	4.99967078547474	4.99967079967410
722	4.99914199671672	4.99967125514986	4.99967124054676	4.99967125472495
723	4.99914318274110	4.99967170894380	4.99967169436244	4.99967170851951
724	4.99914436549112	4.99967216148664	4.99967214692697	4.99967216106299
725	4.99914554498034	4.99967261278354	4.99967259824550	4.99967261236052
726	4.99914672122220	4.99967306283966	4.99967304832317	4.99967306241727
727	4.99914789423003	4.99967351166012	4.99967349716513	4.99967351123835
728	4.99914906401720	4.99967395924998	4.99967394477643	4.99967395882883
729	4.99915023059699	4.99967440561429	4.99967439116212	4.99967440519377
730	4.99915139398255	4.99967485075809	4.99967483632724	4.99967485033819
731	4.99915255418697	4.99967529468638	4.99967528027678	4.99967529426709
732	4.99915371122326	4.99967573740412	4.99967572301572	4.99967573698545
733	4.99915486510439	4.99967617891628	4.99967616454901	4.99967617849822
734	4.99915601584329	4.99967661922774	4.99967660488156	4.99967661881030
735	4.99915716345274	4.99967705834342	4.99967704401826	4.99967705792660
736	4.99915830794551	4.99967749626818	4.99967748196397	4.99967749585196
737	4.99915944933426	4.99967793300684	4.99967791872353	4.99967793259123
738	4.99916058763160	4.99967836856424	4.99967835430177	4.99967836814924
739	4.99916172285006	4.99967880294516	4.99967878870346	4.99967880253076
740	4.99916285500216	4.99967923615434	4.99967922193337	4.99967923574055
741	4.99916398410028	4.99967966819653	4.99967965399622	4.99967966778334
742	4.99916511015677	4.99968009907643	4.99968008489673	4.99968009866384
743	4.99916623318388	4.99968052879874	4.99968051463958	4.99968052838674
744	4.99916735319382	4.99968095736811	4.99968094322945	4.99968095695672
745	4.99916847019876	4.99968138478917	4.99968137067095	4.99968138437837
CPU time is seconds	5.182248			

Table 6: Example 13: algorithmic description of Algorithm 2 and $u_1 = (1, 2, 3, 4)^T$.

Iter. (n)	u_1	u_2	u_3	u_4
1	3.72399999925560	14.5721099999327	14.9072699999354	14.2820299994755
2	-2.15456883397507	14.1819518247169	14.5351508101264	13.8743472579798
3	-1.91884564644980	14.7335505679521	15.0699505703768	14.4407029303704
4	-3.75031965270586	8.60066516780067	8.66301230690689	8.54719173870520
5	-0.610354905863696	5.97030538364658	5.97335233684089	5.96773839611341
6	-0.760180052633688	6.39612802072305	6.39915756275061	6.39357570169468
7	0.883426357410548	5.46334299554989	5.45041815549462	5.47423303492034
8	2.24275125763655	5.11576078694906	5.12103138003741	5.11131429071216
9	1.88904098349673	6.05244056925146	6.05529370932879	6.05002755957568
10	2.94173536203857	5.29641543454450	5.29701704593678	5.29590634994195
11	3.95963752720103	5.10693325887312	5.10709889135520	5.10679307211866
12	4.05539113558902	5.22425012478815	5.22438849354865	5.22413300798106
13	4.48640280826840	5.09245571277969	5.09251889856043	5.09240223042630
14	4.63687195165905	5.07427886151574	5.07432105870254	5.07424314405336
15	4.78911903854907	5.03528185258641	5.03530457387877	5.03526262019362
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
616	4.99913861830495	4.99967028584338	4.99967032490328	4.99967025278006
617	4.99914001259341	4.99967081825953	4.99967085725320	4.99967078525225
618	4.99914140237539	4.99967134895898	4.99967138788665	4.99967131600757
619	4.99914278767268	4.99967187795002	4.99967191681191	4.99967184505430
620	4.99914416850695	4.99967240524090	4.99967244403722	4.99967237240068
621	4.99914554489976	4.99967293083979	4.99967296957076	4.99967289805489
622	4.99914691687247	4.99967345475484	4.99967349342067	4.99967342202507
623	4.99914828444637	4.99967397699410	4.99967401559502	4.99967394431929
624	4.99914964764256	4.99967449756562	4.99967453610183	4.99967446494558
625	4.99915100648201	4.99967501647738	4.99967505494909	4.99967498391194
626	4.99915236098562	4.99967553373728	4.99967557214471	4.99967550122626
627	4.99915371117408	4.99967604935322	4.99967608769656	4.99967601689643
628	4.99915505706797	4.99967656333301	4.99967660161249	4.99967653093029
629	4.99915639868777	4.99967707568443	4.99967711390025	4.99967704333559
631	4.99915773605379	4.99967758641521	4.99967762456757	4.99967755412008
632	4.99915906918624	4.99967809553302	4.99967813362214	4.99967806329143
633	4.99916039810519	4.99967860304549	4.99967864107156	4.99967857085726
634	4.99916172283057	4.99967910896021	4.99967914692344	4.99967907682517
635	4.99916304338221	4.99967961328470	4.99967965118530	4.99967958120268
636	4.99916435977982	4.99968011602645	4.99968015386462	4.99968008399728
637	4.99916567204293	4.99968061719291	4.99968065496885	4.99968058521642
638	4.99916698019104	4.99968111679145	4.99968115450536	4.99968108486747
639	4.99916828424343	4.99968161482945	4.99968165248151	4.99968158295780
CPU time is seconds	4.517220			

Thus, we have

$$\left(1-\psi_{m_k}\right)\left(1-\frac{\mu\delta_{m_k}}{\delta_{m_k+1}}\right)\left\|u_{m_k}-p_{m_k}\right\|^2+\left(1-\psi_{m_k}\right)\left(1-\frac{\mu\delta_{m_k}}{\delta_{m_k+1}}\right)\left\|q_{m_k}-p_{m_k}\right\|^2 \le \psi_{m_k}\left\|g\left(u_{m_k}\right)-q^*\right\|^2+\left\|u_{m_k}-q^*\right\|^2-\left\|u_{m_k+1}-q^*\right\|^2. \tag{113}$$

Since $\psi_{m_k} \longrightarrow 0$ implies that

$$\lim_{k\longrightarrow\infty}\left\|u_{m_k}-p_{m_k}\right\| = \lim_{k\longrightarrow\infty}\left\|q_{m_k}-p_{m_k}\right\| = 0. \tag{114}$$

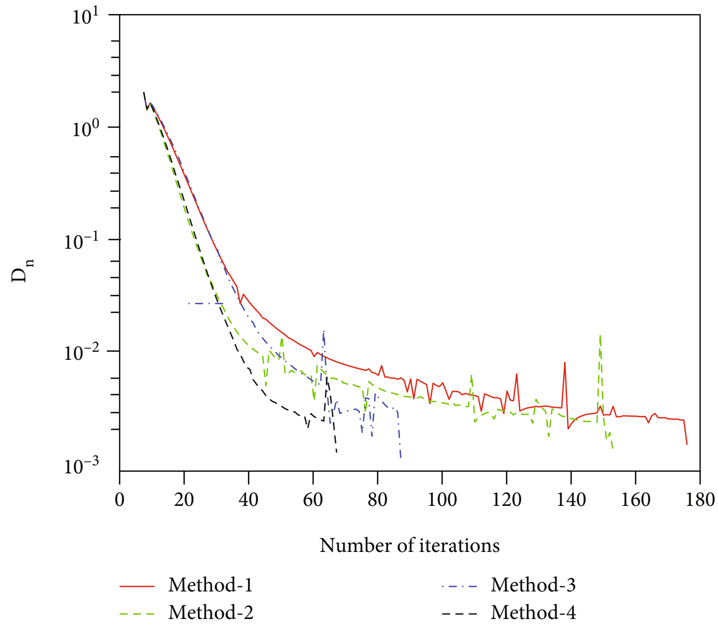

Figure 7: Algorithmic descriptions of Algorithm 1 and Algorithm 2 as well as Algorithm 2 in [37] and Algorithm 1 in [36] when $u_1 = [1.5,1.7]^T$.

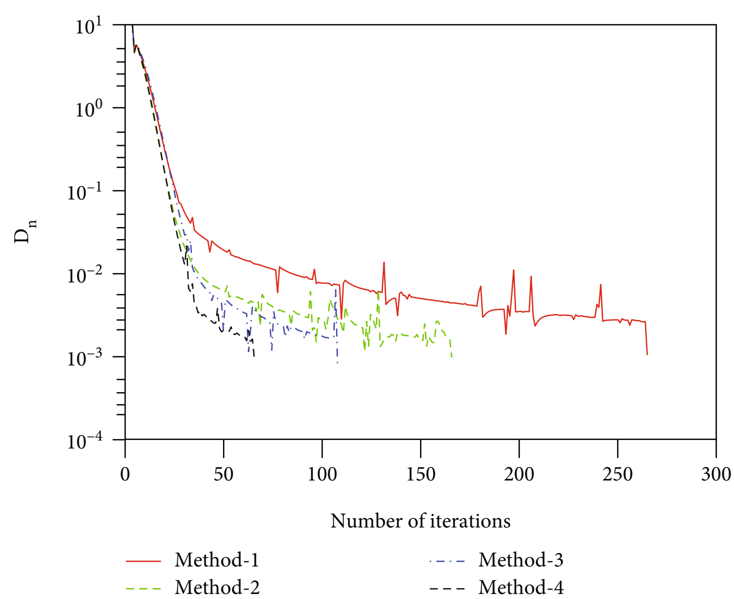

Figure 8: Algorithmic descriptions of Algorithm 1 and Algorithm 2 as well as Algorithm 2 in [37] and Algorithm 1 in [36] when $u_1 = [0,0]^T$.

Next, we obtain

$$\begin{aligned}\left\|u_{m_k+1}-u_{m_k}\right\| &= \left\|\psi_{m_k} g\left(u_{m_k}\right)+\left(1-\psi_{m_k}\right) q_{m_k}-u_{m_k}\right\| \\ &= \left\|\psi_{m_k}\left[g\left(u_{m_k}\right)-u_{m_k}\right]+\left(1-\psi_{m_k}\right)\left[q_{m_k}-u_{m_k}\right]\right\| \\ &\leq \psi_{m_k}\left\|g\left(u_{m_k}\right)-u_{m_k}\right\|+\left(1-\psi_{m_k}\right)\left\|q_{m_k}-u_{m_k}\right\| \longrightarrow 0.\end{aligned} \tag{115}$$

Similar to the Case 1, we can write

$$\limsup_{k\longrightarrow\infty}\left\langle g\left(q^*\right)-q^*, u_{m_k+1}-q^*\right\rangle \leq 0. \tag{116}$$

By using (109) and (112), we have

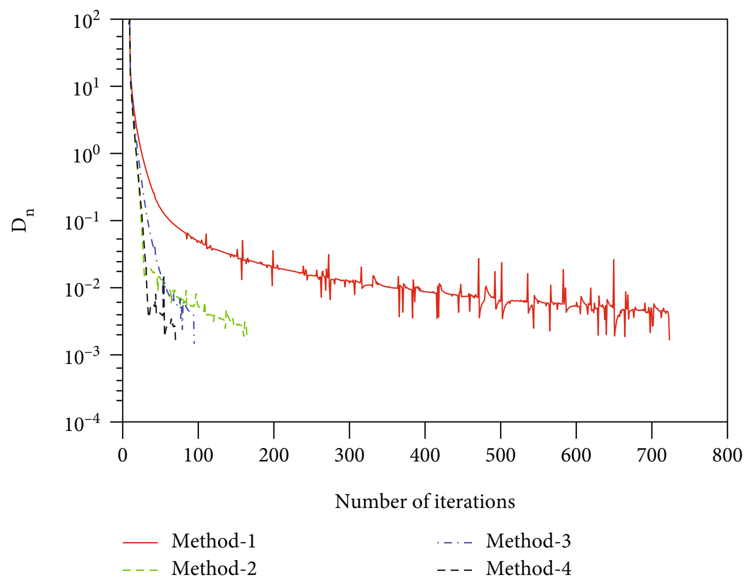

FIGURE 9: Algorithmic descriptions of Algorithm 1 and Algorithm 2 as well as Algorithm 2 in [37] and Algorithm 1 in [36] when $u_1 = [10, 10]^T$.

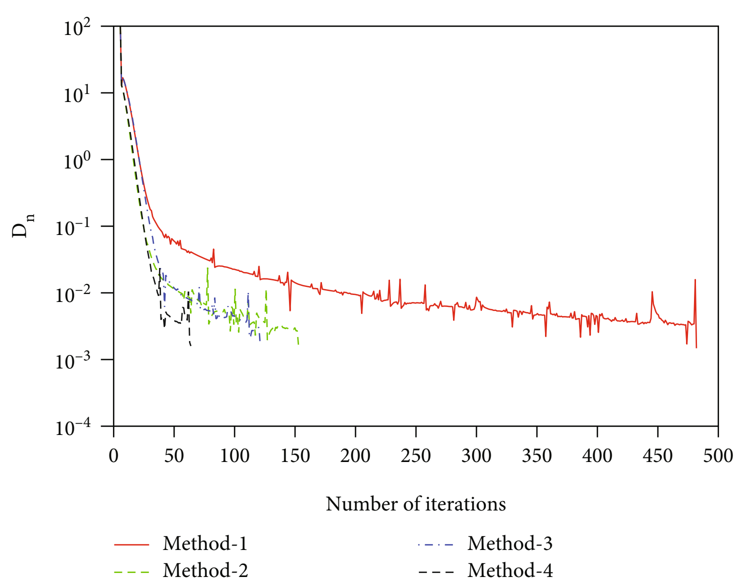

FIGURE 10: Algorithmic descriptions of Algorithm 1 and Algorithm 2 as well as Algorithm 2 in [37] and Algorithm 1 in [36] when $u_1 = [-5,-5]^T$.

TABLE 7: Algorithmic study for Example 14.

u_1	Method-1		Method-2		Method-3		Method-4	
	Iter.	Time	Iter.	Time	Iter.	Time	Iter.	Time
$[1.5,1.7]^T$	170	9.38124300	147	8.166378500	81	5.082575400	61	3.489680300
$[0, 0]^T$	267	14.7065173	166	9.695011100	107	5.894192000	64	3.753339200
$[10, 10]^T$	729	40.4305133	160	8.596874100	89	4.762929600	64	3.520466200
$[-5,-5]^T$	483	28.03565270	150	8.570527900	118	6.676780500	60	3.331014400

$$\begin{aligned}\left\|u_{m_k+1}-q^*\right\|^2 &\le \left[1-2\psi_{m_k}(1-\xi)\right]\left\|u_{m_k}-q^*\right\|^2+2\psi_{m_k}(1-\xi)\\ &\quad\cdot\left[\frac{\psi_{m_k}\left\|u_{m_k}-q^*\right\|^1}{2(1-\xi)}+\frac{\left\langle g(q^*)-q^*, u_{m_k+1}-q^*\right\rangle}{1-\xi}\right]\\ &\le \left[1-2\psi_{m_k}(1-\xi)\right]\left\|u_{m_k+1}-q^*\right\|^2+2\psi_{m_k}(1-\xi)\\ &\quad\cdot\left[\frac{\psi_{m_k}\left\|u_{m_k}-q^*\right\|^2}{2(1-\xi)}+\frac{\left\langle g(q^*)-q^*, u_{m_k+1}-q^*\right\rangle}{1-\xi}\right].\end{aligned} \tag{117}$$

Moreover, we have

$$\left\|u_{m_k+1}-q^*\right\|^2 \le \frac{\psi_{m_k}\|u_{m_k}-q^*\|^2}{2(1-\xi)}+\frac{\left\langle g(q^*)-q^*, u_{m_k+1}-q^*\right\rangle}{1-\xi}. \tag{118}$$

Since $\psi_{m_k} \longrightarrow 0$ and $\|u_{m_k}-q^*\|$ is a bounded sequence. Then, the expressions (116) and (118) imply that

$$\left\|u_{m_k+1}-q^*\right\|^2 \longrightarrow 0, \quad \text{as} k \longrightarrow \infty. \tag{119}$$

The above expression implies that

$$\lim_{k\longrightarrow\infty}\|u_k-q^*\|^2 \le \lim_{k\longrightarrow\infty}\left\|u_{m_k+1}-q^*\right\|^2 \le 0. \tag{120}$$

Consequently, $u_n \longrightarrow q^*$ as $n \longrightarrow \infty$. This completes the proof of the theorem. □

Remark 10.

(i) Two nonmonotonic explicit extragradient-type methods for finding an approximate solution of variational inequalities involving pseudomonotone mapping in a real Hilbert space have been established

(ii) Two strongly convergent results, corresponding to the proposed algorithms have been proven

(iii) It is important to note that these methods use nonmonotonic step size rules that use an operator value rather than the Lipschitz constant of an operator

4. Numerical Illustrations

In this section, computational results of the proposed methods are described and compared to existing related work in the literature. All computations are done in MATLAB R2018b and run on an HP i-5 Core(TM)i5-6200 8.00 GB (7.78 GB usable) RAM laptop.

Example 11. Let a mapping $\mathcal{L}:\mathbb{R}^m \longrightarrow \mathbb{R}^m$ be defined by

$$\mathcal{L}(u)=Mu+q, \tag{121}$$

with $q\in\mathbb{R}^m$ and

$$M=NN^T+B+D. \tag{122}$$

During this experiment, we have chosen $N=\text{rand}(m)$ to be a random matrix and $B=0.5K-0.5K^T$ to be a skew-symmetric matrix with $K=\text{rand}(m)$, and $D=\text{diag}(\text{rand}(m,1))$ is a diagonal matrix. The feasible set $\mathcal{A}$ is defined in the subsequent sense:

$$\mathcal{A}=\{u\in\mathbb{R}^m : Qu\le b\}, \tag{123}$$

where $Q=\text{rand}(100,m)$ and $b=\text{rand}(100{,}1)$. It is obvious that $\mathcal{L}$ is monotone and Lipschitz continuous by $L=\|M\|$. The starting point is $u_1=(2,2,\cdots,2)$ and $D_n=\|u_n-p_n\|\le 1\,0^{-3}$. The numerical results of these methods are shown in Figures 1–6 and Table 1. The control conditions are taken in the following manner: (1) Algorithm 2 in [37] (shortly, Method-1): $\delta_1=0.12, \mu=0.55, \phi_n=1/50(n+2)$; (2) Algorithm 1 in [36] (shortly, Method-2): $\delta_1=0.12, \gamma=0.55, \phi_n=1/2(n+2), f(u)=u/3$; (3) Algorithm 1 (shortly, Method-3): $\delta_1=0.12, \mu=0.55, \psi_n=1/2(n+2), \phi_n=(5/10)(1-\psi_n), \varphi_n=100/(n+1)^2$; and (4) Algorithm 2 (shortly, Method-4): $\delta_1=0.12, \mu=0.55, \psi_n=1/2(n+2), f(u)=u/3$, $\varphi_n=100/(n+1)^2$.

Example 12. Let $\mathcal{Z}=L^2([0,1])$ be a Hilbert space having an inner product

$$\langle u,y\rangle=\int_0^1 u(t)y(t)dt, \quad \forall u,y\in\mathcal{Z}, \tag{124}$$

and the induced norm is determined as follows:

$$\|u\|=\sqrt{\int_0^1|u(t)|^2dt}. \tag{125}$$

Let $\mathcal{A}:=\{u\in L^2([0,1]) : \|u\|\le 1\}$ be a unit ball. A mapping $\mathcal{L}:\mathcal{A}\longrightarrow\mathcal{Z}$ is defined by

$$\mathcal{L}(u)(t)=\int_0^1(u(t)-H(t,s)f(u(s)))ds+g(t), \tag{126}$$

where

$$H(t,s)=\frac{2tse^{(t+s)}}{e\sqrt{e^2-1}}, f(u)=\cos u, g(t)=\frac{2te^t}{e\sqrt{e^2-1}}. \tag{127}$$

It can easily be seen that $\mathcal{L}$ is Lipschitz-continuous with the constant $L=2$ and monotone. The starting point for this experiment is taken differently, and $D_n=\|u_n-p_n\|\le 10^{-3}$. The numerical results of these methods are shown in Table 2. The control conditions are taken as follows: (1) Algorithm 2 in [37] (shortly, Method-1): $\delta_1=0.33, \mu=0.75$, $\phi_n=1/100(n+2)$; (2) Algorithm 1 in [36] (shortly, Method-2): $\delta_1=0.33, \gamma=0.75, \phi_n=1/100(n+2), f(u)=u/2$

; (3) Algorithm 1 (shortly, Method-3): $\delta_1 = 0.33, \mu = 0.75, \psi_n = 1/100(n+2), \phi_n = (7/10)(1-\psi_n), \varphi_n = 100/(n+1)^2$; and (4) Algorithm 2 (shortly, Method-4): $\delta_1 = 0.33, \mu = 0.75, \psi_n = 1/100(n+2), f(u) = u/2, \varphi_n = 100/(n+1)^2$.

Example 13. Let a mapping $\mathscr{L} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be defined by

$$\mathscr{L}(u) = \begin{pmatrix} u_1 + u_2 + u_3 + u_4 - 4u_2u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_3 \end{pmatrix}. \tag{128}$$

Moreover, the constraint set $\mathscr{A}$ is defined by

$$\mathscr{A} = \left\{u \in \mathbb{R}^4 : 1 \le u_i \le 5, i = 1, 2, 3, 4\right\}. \tag{129}$$

It is clear to see that $\mathscr{L}$ is not monotone on the set $\mathscr{A}$. The starting point for this experiment is $u_1 = (1, 2, 3, 4)^T$ and $D_n = \|u_n - p_n\| \le 10^{-3}$. The numerical results of these methods are shown in Tables 3–6. The control conditions are taken as follows: (1) Algorithm 2 in [37] (shortly, Method-1): $\delta_1 = 0.05, \mu = 0.33, \phi_n = 1/20(n+2)$; (2) Algorithm 1 in [36] (shortly, Method-2): $\delta_1 = 0.05, \gamma = 0.33, \phi_n = 1/20(n+2), f(u) = u/4$; (3) Algorithm 1 (shortly, Method-3): $\delta_1 = 0.05, \mu = 0.33, \psi_n = 1/20(n+2), \phi_n = (6/10)(1-\psi_n), \varphi_n = 100/(n+1)^2$; and (4) Algorithm 2 (shortly, Method-4): $\delta_1 = 0.05, \mu = 0.33, \psi_n = 1/20(n+2), f(u) = u/4, \varphi_n = 100/(n+1)^2$.

Example 14. This test problem is taken from [42]. Let a mapping $\mathscr{L} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$\mathscr{L}(u) = \begin{pmatrix} 0.5u_1u_2 - 2u_2 - 10^7 \\ -4u_1 - 0.1u_2^2 - 10^7 \end{pmatrix}, \tag{130}$$

where

$$\mathscr{A} = \left\{u \in \mathbb{R}^2 : (u_1 - 2)^2 + (u_2 - 2)^2 \le 1\right\}. \tag{131}$$

We can easily observe that the mapping $\mathscr{L}$ is not monotone on $\mathscr{A}$ but pseudomonotone and Lipschitz continuous with $L = 5$. This problem has a unique solution that is $u^* = (2.707,2.707)^T$. The starting point for this experiment is taken differently, and $D_n = \|u_n - p_n\| \le 10^{-3}$. The numerical results of these methods are shown in Figures 7–10 and Table 7. The control conditions are taken as follows: (1) Algorithm 2 in [37] (shortly, Method-1): $\delta_1 = 0.333, \mu = 0.90, \phi_n = 1/3(n+2)$; (2) Algorithm 1 in [36] (shortly, Method-2): $\delta_1 = 0.333, \gamma = 0.90, \phi_n = 1/2(n+2), f(u) = u/5$; (3) Algorithm 1 (shortly, Method-3): $\delta_1 = 0.333, \mu = 0.90, \psi_n = 1/2(n+2), \phi_n = (5/10)(1-\psi_n), \varphi_n = 100/(n+1)^2$; and (4) Algorithm 2 (shortly, Method-4): $\delta_1 = 0.333, \mu = 0.90, \psi_n = 1/2(n+2), f(u) = u/5, \varphi_n = 100/(n+1)^2$.

5. Conclusion

Two nonmonotonic explicit extragradient-type methods for finding an approximate solution of variational inequalities involving pseudomonotone mapping in a real Hilbert space have been established. Two strongly convergent results, corresponding to the proposed algorithms have been proven. The numerical results were interpreted to demonstrate that the proposed algorithms worked numerically better than current methods. According to these numerical findings, the nonmonotone variable step size rule improves the efficiency of the iterative sequence in this case.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

No potential conflict of interest was reported by the authors.

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Research Article

The Study of Mean-Variance Risky Asset Management with State-Dependent Risk Aversion under Regime Switching Market

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How do investors require a distribution of the wealth among multiple risky assets while facing the risk of the uncontrollable payment for random liabilities? To cope with this problem, firstly, this paper explores the approach of asset-liability management under the state-dependent risk aversion with only risky assets, which has been considered under a continuous-time Markov regime-switching setting. Next, based on this realistic modelling, an extended Hamilton-Jacob-Bellman (HJB) system has been necessarily established for solving the optimization problem of asset-liability management. It has been derived closed-form analytical expressions applied in the time-inconsistent investment with optimal control theory to see that happens to the optimal value of the function. Ultimately, numerical examples presented with comparisons of the analytical results under different market conditions are exposed to analyse numerically the developed mean variance asset liability management strategy. We find that our proposed model can explain the financial phenomena more effectively and accurately.

1. Introduction

Portfolio optimization selection problem, well known as an essential topic in financial markets, has been done in deep researches by many scholars after the first reported by Markowitz [1]. The most frequently used method of optimal asset allocation strategies is HJB equation, i.e., the Hamilton-Jacobi-Bellman equation (see Detemple and Fernando [2], Björk et al. [3]). In the analysis of portfolio optimization, utility function and several system parameters are given to find the optimal values of the control parameters to realise the final utility maximization. Previous researches in this area are classified for the endogenous habit formation [2], the classic constant relative risk aversion (CRRA) by Yu and Yuan [4], the hyperbolic discounting [3], and the utilities like the mean-variance utility proposed by Li et al. [5]. In recent paper by Li et al. [6], the analytical solution portfolio optimization problem involving stochastic short-term interest rates is provided, which can be controlled by the mean-variance utility function with state dependent risk aversion (SDRA). The paper [6] uses the Nash equilibrium for the subgame strategy to concrete analytical expressions of value function and control policy of equilibrium and figure out under the condition of the stochastic short-term interest rates, how do investors with "natural risk aversion" achieve optimal control policies by simplifying financial settings.

Under the framework of mean variance equilibrium asset liability management with SDRA, some extended models have been constructed, such as the mean-variance asset-liability management problem by regime-switching models, as well as mean-variance models with only risky assets (see Bening and Koroley [7]; the asset-only models to asset-liability models have been greatly expanded by Yao et al. [8, 9]). A geometric method raised by Leippold et al. [10] is supposed to apply into the multiperiod mean-variance asset-liability management model by taking the implied mean-variance of liability frontier into consideration. A study by Chiu and Li [11] reported that the influence of the rebalancing frequency is quantified to determine the allocation of optimal initial funds. The work of extension into a continuous-time setting has been developed with the aid of a stochastic linear quadratic control approach. Based on the assumptions used in Leippold et al. [10], analytical results have been derived in a complete market with discussing the impact of liability on the optimal funding ratio. To construct more realistic models, more focus has been put on studying the asset-liability management under the market behavior in the face of many restriction conditions, for example, an uncertain investment horizon (see Li and Ng [12]; Li and Yao [13]), regime-switching to describe phenomena between "Bullish" and "Bearish" markets (see Elliott et al. [14]; Wei et al. [15]; Wu and Li [16]; Wu and Chen [17]; Yu [18]), the choice of optimal portfolio selection for assets with transaction costs without short sales (see Li et al. [19]), portfolio selection under partial information (see Xiong and Zhou [20]), bankruptcy control (see Li and Li [21]), jump-diffusion in financial markets (see Lim [22]; Zeng and Li [23]), and stochastic volatility and stochastic interest rates (see Lim [22]; Lim and Zhou [24]). Also, various studies of assets and liabilities management problem have been carried out in some particular field with application in insurance and pension fund, including Drijver [25] for pension funds Hilli et al. [26] for a Finnish pension company, and Gerstner et al. [27] and Chiu and Wong [28] for life insurance policies.

Among them, regime-switching models have become popular in finance and related fields, which is expected to describe the characteristics of different markets (called "Bullish" versus"Bearish"). A limited number of regimes have been applied to represent the various patterns of the market states. According to diverse financial markets as the change market pattern occurs, indices for instance the interest rate, appreciation rates, and volatilities of stock and liability may be different. Boyle and Draviam's study [29] is an interesting example of regime-switching modelling applied in option pricing achieved by [29], followed by Elliott and Siu [30], who have embedded the regime-switching modelling into the bond valuation, the concept of which has been put forward in the portfolio selection problem by Zhou and Yin [31], Chen et al. [32], and Chen and Yang [33]. The research studied in [32, 33] involves both the asset-liability feature and Markovian regime-switching modelling. As we all know, the models with only risk assets are valuable to be studied. Yao et al. [8] were working on the research of the continuous-time mean-variance model for only risky assets. It is rare for risk-free asset in reality; as a matter of fact, a relatively long investment is considered, corresponding to the stochastic nature real interest rates and the inflation risk (see Viceira [34]). The previous method of a nominal risk-free asset incorporated into the market will simplify the process of selecting portfolio but degenerate the GMV strategy to a bank deposit strategy with zero risks, which is not favourable to investors. Besides, the empirical evidence in the study by DeMiguel et al. [35] shows that the static global minimum-variance (GMV) strategy with only risky assets (derived by Markowitz [1]) tends to be better in performance out-of-sample among all estimated optimal strategies. Then in general, the properties of the time-consistent MV strategies have been shown in a market only with risky assets by Chi [36] on the analysis of Yao et al. [8] and Zhang et al. [37].

In this paper, on the basis of the work of Björk et al. [3], it is determinate to make a further realistic financial model, and it makes sense to select a regime-switching market with only risky assets. Afterwards, the general expansion of the HJB equation will be reached according to the control theory with time inconsistency by Björk and Murgoci [38]. Finally, it proceeds the numerical illustrations to show our extended results and state the relationships with previous researches. The rest of the paper is completed as follows: the setting of the financial market will be explained in Section 2, with the developed structure of mean-variance asset-liability management with state-dependent risk aversion in a regime-switching market with only risky assets. Also, the HJB equation is generalized to the general situation. In Section 3, three different cases with derived solutions will be illustrated in details. More numerical examples are presented in Section 4 with corresponding figures and illustrations, and a conclusion is given in Section 5.

2. Model Formulation

In a given probability space filtered, $(\Omega, \mathbf{P}, \mathscr{F}, \{\mathfrak{I}_t\}_{0\le t\le T})$, let $W(t) = (W_1(t), W_2(t), \cdots, W_m(t))'$ be a standard m-dimensional Brownian motion with definition of $(\Omega, \mathbf{P}, \mathscr{F})$ over the period of $[0, T]$. Since the involution of individual investments has been found, a few number of investors will not make much effect on the whole market. The mode of the market dynamics is described by a Markov chain process $\alpha(t)$. For that sense, the processes of $W(t)$ and $\alpha(t)$ are independent of each other. $\mathfrak{I}_t = \sigma\{W(s), \alpha(s) ; 0 \le s \le t\}$ could be augmented in the case of all the $\mathbf{P}$-null sets in $\mathscr{F}$, where $\mathscr{F} = \mathfrak{I}_T$. Some finite T is used to denote the range of investment time. All random variables taken into consideration, in this paper, are defined within this filtered probability space. Assuming that there are d regimes for the market state, it means that the Markov chain α_t gets one of the values from the set of $\{1, 2, 3, \cdots, d\}$ every time. By assumption, a generator $Q = (q_{kj})_{d\times d}$ in the Markov chain with the stationary transition probabilities such that $p_{kj}(t) = \mathbb{P}(\alpha(s+t) = j \mid \alpha(s) = k)$, where $s, t \ge 0, k, j = 1, 2, \cdots, d$, $q_{kj} = (d/dt)p_{kj}(t)|_{t=0}$ and $\sum_{j=1}^{d} q_{kj} = 0, q_{jj} = -\sum_{j=1}^{d} q_{kj} < 0, q_{kj} > 0$. A financial market with continuous-time under the standard assumptions has been considered. Concretely speaking, the market assumptions in this paper are listed here with permission for continuous trading, no transaction cost or tax in trading, and infinitely divisible assets.

2.1. Financial Market

2.1.1. Assets. Suppose that an investor decides to allocate his wealth among $n+1$ risky assets. The prices of these risky assets meet the following requirements of stochastic differential equations (SDE) driven by the geometric Brownion motion (GBM) (1):

$$\begin{cases} dP_i(t) = P_i(t)\left(b_i(t,\alpha(t))dt + \sum_{h=1}^{m}\sigma_{ih}(t,\alpha(t))dW_h(t)\right), & 0 \le t \le T, \\ P_i(0) = p_i > 0, \quad i = 0, 1, 2, \cdots, n, \end{cases} \tag{1}$$

where $P_i(t)$ means the initial prices of the risky assets, $(p_i, i = 0, 1, 2, \cdots, n)$; $\alpha(t)$ is defined as volatility factor; and $(b_0(t, \alpha(t)), b_1(t, \alpha(t)), \cdots, b_n(t, \alpha(t)))$ and $(\sigma_{ih}(t, \alpha(t)))_{(n+1)\times m}$ refer to the appreciation rate vector and the volatility matrix of these assets, respectively, with assumption of positive continuous bounded deterministic functions of time t. As mentioned above, the GBM vector $\{W(t) = (W_1(t), W_2(t), \cdots, W_m(t))'\}$ is supposed to have a detailed description of all the random factors influencing the prices of risky assets.

2.1.2. Wealth Process. It is assumed that an investor endowed with an initial wealth X_0 at time 0 is intended to invest in the market dynamically through the period of $[0, T]$. Here, $X(t)$ stands for the total wealth at time t for an investor, and $u_i(t)$ denotes the amount investment in asset i and $N_i(t)$ for the total of units of asset i in an investor's portfolio, $i = 0, 1, 2 \cdots, n$. The sum of investment in the 0th asset after the deduction of liability is described as $\{X(t) - \sum_{i=1}^{n} u_i(t)\}$. Therefore, under the conditions above, the wealth held by the investor at time t, $X(t)$ is shown as follows:

$$dX(t) = \sum_{i=0}^{n} N_i(t)dP_i(t) = \sum_{i=0}^{n} u_i(t)\frac{dP_i(t)}{P_i(t)}. \tag{2}$$

To further simplify $dX(t)$ in equation (2), the SDE will be represented as

$$\begin{cases} dX(t) = \left[b_0(t,\alpha(t))X(t) + B'(t,\alpha(t))u(t)\right]dt + \left[X(t)\sigma_0'(t,\alpha(t)) + u'(t)\sigma(t,\alpha(t))\right]dW(t), \\ X(0) = X_0, \\ \alpha(0) = k_0, \end{cases} \tag{3}$$

where

$$\begin{cases} u(t) = (u_1(t), u_2(t), \cdots, u_n(t))'_{n\times 1}, \\ B(t,k) = (b_1(t,k) - b_0(t,k), b_2(t,k) - b_0(t,k) \cdots, b_n(t,k) - b_0(t,k))'_{n\times 1}, \\ \sigma_i(t,k) = (\sigma_{i1}(t,k), \sigma_{i2}(t,k), \cdots, \sigma_{im}(t,k))'_{m\times 1}, \\ \sigma(t,k) = (\sigma_1(t,k) - \sigma_0(t,k), \sigma_2(t,k) - \sigma_0(t,k), \cdots, \sigma_n(t,k) - \sigma_0(t,k))'_{n\times m}. \end{cases} \tag{4}$$

We assume that all the functions are measurable and uniformly bounded in $[0, T]$. Here, $L^2_{\mathcal{F}}(t, T; \mathbb{R}^{n+1})$ is denoted as the set of all $\mathbb{R}^{n+1}$-valued and measurable stochastic processes $f(s, \alpha(s))$ are adjusted to $\{\Im_s\}_{s\ge t}$ on $[0, T]$ such that

$$E\left[\int_t^T |f(s,\alpha(s))|^2 ds\right] < +\infty. \tag{5}$$

2.1.3. Liability Process. In fact, the investor in the financial market is exposed to the uncontrollable liability, with value process by the following SDE:

$$\begin{cases} dL(t) = \mu(t,\alpha(t))dt + \rho'(t,\alpha(t))dW(t), \\ L(0) = l_0, \end{cases} \tag{6}$$

where $L(t)$ is the stochastic liability process and l_0 is defined as the initial value of the liability. Besides, $\mu(t, \alpha(t))$ and $\rho(t, \alpha(t)) = (\rho_1(t, \alpha(t)), \rho_2(t, \alpha(t)), \cdots, \rho_m(t, \alpha(t)))'$ are expressed as the appreciation and volatility in liability, respectively, on the assumption of stochastic functions at time t with Markov process $\alpha(t)$. In addition, generally, the liability is functioned as the real liability excluding the random income of the investor. As a result, it turns out to be negative liabilities; the random income of the investor can be more than the real liability.

Remark 1. It is clearly to see the correlation between liability value and risky assets in the dynamic processes by m-dimensional geometric Brownian motion $W(t)$. Since the investment portfolio has $n + 1$ risky assets and one liability, it leads to $m \ge n + 2$ in the asset liability management model.

2.1.4. Surplus Process. Let $S(t)=X(t)-L(t)$ represent the current surplus of our fortune. By substituting the wealth and liability processes, we have

$$\begin{cases} dS(t)=\left\{b_0(t,\alpha(t))S(t)-\mu(t,\alpha(t))+B'(t,\alpha(t))u(t)\right\}dt+\left\{S(t)\sigma_0'(t,\alpha(t))-\rho'(t,\alpha(t))+u'(t)\sigma(t,\alpha(t))\right\}dW(t), \\ dL(t)=\mu(t,\alpha(t))dt+\rho'(t,\alpha(t))dW(t), \\ S(0)=s_0=x_0-l_0, \\ L(0)=l_0, \alpha(0)=k_0. \end{cases} \tag{7}$$

2.2. Mean-Variance Risky Asset Management (MVRAM). First, $\mathscr{U}[0,T]$ is denoted as the set of all avaliable strategies $\{(u_0(t),u_1(t),\cdots,u_n(t))\}$ over $[0,T]$. Naturally, the MVRAM problem will give emphasis on finding optimal admissible strategy to maximize mean-variance utility at terminal time T. So the objective function of $J(t,s,l,k,u)$ and the equilibrium value function of $V(t,s,l,k)$ are described mathematically as follows:

$$J(t,s,k,u)=E_{t,s,k}[S^{\mathbf{u}}(T)]-\frac{\gamma(s,k)}{2}Var_{t,s,k}[S^{\mathbf{u}}(T)], \tag{8}$$

$$\begin{aligned} V(t,s,k)&=\max_{u(\cdot)\in\mathscr{U}[0,T]}\left\{E_{t,s,k}[S(T)]-\frac{\gamma(s,k)}{2}Var_{t,s,k}[S(T)]\right\} \\ &=J(t,s,k,u^*). \end{aligned} \tag{9}$$

where $E_{t,s,k}[\cdot]=E[\cdot|S^{\mathbf{u}}(t,k)=s,L^{\mathbf{u}}(t,k)=l,\alpha^{\mathbf{u}}(t)=k]$, in which $S^{\mathbf{u}}(t,k),L^{\mathbf{u}}(t,k),\alpha^{\mathbf{u}}(t)$ successively represent the surplus process, liability, and market dynamics obtained by using the control strategy $\mathbf{u}=(u_0(t),u_1(t),\cdots,u_n(t)$, and $r(s,k)$ means the risk aversion coefficient depending on s and k. As a result, $J(t,s,k,u)$ can be defined as

$$J(t,s,k,u)=E_{t,s,k}[F(s,k,S^{\mathbf{u}}(T))]+G(s,k,E_{t,s,k}[S^{\mathbf{u}}(T)]), \tag{10}$$

where

$$F(s,k,y)=y-\frac{\gamma(s,k)}{2}y^2, G(s,k,y)=\frac{\gamma(s,k)}{2}y^2, \tag{11}$$

where in here y represents $E_{t,s,k}[S^{\mathbf{u}}(T)]$.

Second, let$A$be infinitesimal generator, for any fixed $u\in\mathscr{U}$; the controlled infinitesimal generator A^u corresponded to

$$\begin{aligned} A^uW(t,s,k)&=\frac{\partial W(t,s,k)}{\partial t}+\left[b_0s-\mu+B'u\right]\frac{\partial W(t,s,k)}{\partial s} \\ &\quad+\sum_{j=1}^{d}q_{kj}W(t,s,j)+\frac{1}{2}\left\{s\sigma_0'-\rho'+u'\sigma\right\} \\ &\quad\cdot\frac{\partial W(t,s,k)}{\partial s^2}\left\{s\sigma_0-\rho+\sigma'u\right\}. \end{aligned} \tag{12}$$

Based on the analysis of Björk Bjrk and Murgoci [38], with the definition of equilibrium control in equation (9) and the infinitesimal generator A^u in equation (12), the HJB equation will be extended as follow, as well as the verification of theorem.

Theorem 1 (verification theorem). *It is assumed that (V,f,g) is a solution to the following extended HJB system with the supremum of control law $\hat{u}$ in the equation. Then, $\hat{u}$ is subject to an equilibrium control law, and V is supposed to the corresponding value function.*

$$\begin{cases} \sup_{u\in\mathscr{U}}\left\{A^uV(t,s,k)-A^uf(t,s,k,s,k)+A^uf^{s,k}(t,s,k)-A^u(G\circ g)(t,s,k)+H^ug(t,s,k)\right\}=0, \quad 0\le t\le T, \\ A^{u\wedge}f^{m,h}(t,s,k)=0, \quad 0\le t\le T, \\ A^{u\wedge}g(t,s,k)=0, \quad 0\le t\le T, \\ V(T,s,k)=s, \\ f^{m,h}(T,s,k)=s-\dfrac{\gamma(m,h)}{2}s^2, \\ g(T,s,k)=s, \end{cases} \tag{13}$$

with the following conditions and definitions:

$$\begin{cases} f^{m,h}(t,s,k) = f(t,s,k,m,h), \\ (G \circ g)(t,s,k) = G(s,k,g(t,s,k)), \\ H^u g(t,s,k) = G_y(s,k,y) \times A^u g(t,s,k), \\ G_y(s,k,y) = \dfrac{\partial G(s,k,y)}{\partial y}. \end{cases} \quad (14)$$

Moreover, f and g have the following probabilistic representations:

$$\begin{aligned} f(t,s,k,m,h) &= E_{t,s,k}\left[F\left(m,h,S^{u\wedge}(T)\right)\right], \\ g(t,s,k) &= E_{t,s,k}\left[S^{u\wedge}(T)\right], \\ V(t,s,k) &= f(t,s,k) + G(s,k,g(t,s,k)). \end{aligned} \quad (15)$$

Proof. On the basis of the HJB equation in [2] and the objective function [3], it can be derived as

$$V(t,S_t,k_t) = \sup_{u\in U} J(t,S_t,k_t,U) \quad (16)$$

and consequently have

$$J(t,S_t,k_t,U) = E_{t,S_t,k_t}\left[F\left(s,k,S_T^U\right)\right] + G\left(s,k,E_{t,S_t,k_t}\left[S_T^U\right]\right), \quad (17)$$

where the forms of the function F and function G are described in (11).

When $l > t$, we have

$$\begin{aligned} J(l,S_l,k_l,U) &= E_{l,S_l,k_l}\left[F\left(s_l,k_l,S_T^U\right)\right] + G\left(s_l,k_l,E_{l,S_l,k_l}\left[S_T^U\right]\right), \\ E_{l,S_l,k_l}\left[F\left(s_l,k_l,S_T^U\right)\right] &= f^U(l,S_l,k_l,s_l,k_l), \\ E_{l,S_l,k_l}\left[S_T^U\right] &= g^U(l,S_l,k_l). \end{aligned} \quad (18)$$

Hence, equation (18) above can simply be represented as

$$J(l,S_l,k_l,U) = f^U(l,S_l,k_l,s_l,k_l) + G^U\left(s_l,k_l,g^U(l,S_l,k_l)\right). \quad (19)$$

So the expectations of the equation can be shown

$$\begin{aligned} E_{t,S_t,k_t}[J(l,S_l,k_l,U)] = E_{t,S_t,k_t}\Big[&f^U(l,S_l,k_l,s_l,k_l) \\ &+ E_{t,S_t,k_t}\left[G\left(s_l,k_l,g^U(l,S_l,k_l)\right)\right]\Big], \end{aligned} \quad (20)$$

and substituting this result into the definition of (17), we then have

$$\begin{aligned} E_{t,S_t,k_t}[J(l,S_l,k_l,U)] = J(t,S_t,k_t,U) &+ E_{t,S_t,k_t} \\ &\cdot \left[f^U(l,S_l,k_l,s_l,k_l)\right] - E_{t,S_t,k_t}\left[F\left(s,k,S_T^U\right)\right] \\ &+ E_{t,S_t,k_t}\left[G\left(s_l,k_l,g^U(l,S_l,k_l)\right)\right] \\ &- G\left(s,k,E_{t,S_t,k_t}\left[S_T^U\right]\right). \end{aligned} \quad (21)$$

After the process of iteration, we obtain

$$\begin{aligned} E_{t,S_t,k_t}\left[F\left(s_l,k_l,S_T^U\right)\right] &= E_{t,S_t,k_t}\left[E_{l,S_l,k_l}\left[F\left(s_l,k_l,S_T^U\right)\right]\right] \\ &= E_{t,S_t,k_t}\left[f^U(l,S_l,k_l,s_l,k_l)\right], \end{aligned} \quad (22)$$

$$E_{t,S_t,k_t}\left[S_T^U\right] = E_{t,S_t,k_t}\left[E_{l,S_l,k_l} S_T^U\right] = E_{t,S_t,k_t}\left[g^U(l,S_l,k_l)\right]. \quad (23)$$

By substituting the results of (22) and (23) back into the equation of (21), we can get

$$\begin{aligned} E_{t,S_t,k_t}[J(l,S_l,k_l,U)] - E_{t,S_t,k_t}&\left[f^U(l,S_l,k_l,s_l,k_l)\right] \\ &+ E_{t,S_t,k_t}\left[f^U(l,S_l,k_l,s_l,k_l)\right] \\ &- J(t,S_t,k_t,U) \\ &- E_{t,S_t,k_t}\left[G\left(s_l,k_l,g^U(l,S_l,k_l)\right)\right] \\ &+ G\left(s,k,E_{t,S_t,k_t}\left[g^U(l,S_l,k_l)\right]\right) = 0. \end{aligned} \quad (24)$$

Then

$$\begin{aligned} \sup_{u\in U}\Big\{&E_{t,S_t,k_t}[J(l,S_l,k_l,U)] - E_{t,S_t,k_t}\left[f^U(l,S_l,k_l,s_l,k_l)\right] \\ &+ E_{t,S_t,k_t}\left[f^U(l,S_l,k_l,s_l,k_l)\right] - J(t,S_t,k_t,U) \\ &- E_{t,S_t,k_t}\left[G\left(s_l,k_l,g^U(l,S_l,k_l)\right)\right] \\ &+ G\left(s_t,k_t,E_{t,S_t,k_t}\left[g^U(l,S_l,k_l)\right]\right)\Big\} = 0. \end{aligned} \quad (25)$$

Through our proposed problem (16) with the definition of the control law in the classic work, it can be found out that the control U coincides with the equilibrium law $\hat{u}$ in $[l,T]$, and we formulate the following results:

$$\begin{aligned} J(l,S_l,k_l,\hat{u}) &= V(l,S_l,k_l), \\ f^U(l,S_l,k_l,s,k) &= f(l,S_l,k_l,s,k), \\ g^U(l,S_l,k_l) &= g(l,S_l,k_l). \end{aligned} \quad (26)$$

Thus, the optimization problem of (25) can be solved as

$$\begin{aligned}\sup_{u\in U}\{&E_{t,S_t,k_t}[V(l,S_l,k_l)]-E_{t,S_t,k_t}[f(l,S_l,k_l,s_l,k_l)]\\&+E_{t,S_t,k_t}[f(l,S_l,k_l,s_l,k_l)]-V(t,S_t,k_t)\\&-E_{t,S_t,k_t}[G(s_l,k_l,g(l,S_l,k_l))]\\&+G\left(y_t,E_{t,S_t,k_t}\left[g^U(l,S_l,k_l)\right]\right)\}=0.\end{aligned}\tag{27}$$

Here, by using the operator denotations of similarity, we have

$$\begin{aligned}E_{t,S_t,k_t}[V(l,S_l,k_l)]-V(t,S_t,k_t)&=A^uV,\\E_{t,S_t,k_t}[f(l,S_l,k_l,s_l,k_l)]&=A^uf,\\E_{t,S_t,k_t}[f(l,S_l,k_l)]&=A^uf^{s,k},\\E_{t,S_t,k_t}[G(s_l,k_l,g(l,S_l,k_l))]&=A^uG,\\G\left(s_t,k_t,E_{t,S_t,k_t}\left[g^U(l,S_l,k_l)\right]\right)&=H^ug.\end{aligned}\tag{28}$$

The derivation of the extended HJB equation with stochastic volatility will be given as,

$$\begin{aligned}\sup_{u\in U}\{&A^uV(t,s,k)-A^uf(t,s,k,s,k)\\&+A^uf^{s,k}(t,s,k)-A^u(G*g)(t,s,k)\\&+H^ug(t,s,k)\}=0,\ 0\le t\le T.\end{aligned}\tag{29}$$

□

3. Solution Scheme

3.1. The Case with a Generated $\gamma(s,k)$

Theorem 2. *Under the general form of states dependent risk aversion, the optimal control strategy of MVRAM among risky assets is*

$$\widehat{u}(t,s,k)=-\left(\sigma\acute{\sigma}\right)^{-1}\left\{\frac{f_s^{s,k}+\gamma(s,k)gg_s}{f_{ss}^{s,k}+\gamma(s,k)gg_{ss}}B+(s\sigma\sigma_0-\sigma\rho)\right\},\tag{30}$$

where $f_{\cdot}^{s,k}$ and g represent the function $f_{\cdot}^{s,k}(t,s,k)$ and $g(t,s,k)$ by partial equations in (32).

Proof. By using the definition of the infinitesimal generator, we simplify and thus have

$$\begin{aligned}&A^uV(t,s,k)-A^uf(t,s,k,s,k)+A^uf^{s,k}(t,s,k)\\&\quad-A^u(G\circ g)(t,s,k)+H^ug(t,s,k)\\&=A^uf^{s,k}(t,s,k)+H^ug(t,s,k).\end{aligned}\tag{31}$$

The resulting extended HJB equations can be rewritten as

$$\begin{cases}\sup_{u\in\mathcal{U}}\left\{f_t^{s,k}(t,s,k)+\gamma(s,k)g(t,s,k)[g_t(t,s,k)+\mu l\left[f_l^{s,k}(t,s,k)+\gamma(s,k)g(t,s,k)g_l(t,s,k)\right]+\left[b_0s-\mu+B'u\right]\left[f_s^{s,k}(t,s,k)+\gamma(s,k)g(t,s,k)g_s(t,s,k)\right]+\frac{1}{2}\left\{s\sigma_0'-\rho'+u'\sigma\right\}\left[f_{ss}^{s,k}(t,s,k)+\gamma(s,k)g(t,s,k)g_{ss}(t,s,k)\right]\left\{s\sigma_0{}'+\left(\sigma_0{}'-\rho'\right)l+u'\sigma\right\}'+\sum_{j=1}^{d}q_{kj}\left[f^{s,k}(t,s,j)+\gamma(s,k)g(t,s,k)g(t,s,j)\right]\right\}=0,\\ f_t^{m,h}(t,s,k)+\left[b_0s-\mu+B'\widehat{u}\right]f_s^{m,h}(t,s,k)+\frac{1}{2}\left\{s\sigma_0'+-\rho'+u\wedge'\sigma\right\}f_{ss}^{m,h}(t,s,k)\left\{s\sigma_0'-\rho'+u\wedge'\sigma\right\}'+\sum_{j=1}^{d}q_{kj}f^{m,h}(t,s,j)=0,\\ g_t(t,s,k)+\left[b_0s-\mu+B'\widehat{u}\right]g_s(t,s,k)+\frac{1}{2}\left\{s\sigma_0'-\rho'+u\wedge'\sigma\right\}g_{ss}(t,s,k)\left\{s\sigma_0'-\rho'+u\wedge'\sigma\right\}'+\sum_{j=1}^{d}q_{kj}g(t,s,j)=0,\\ f(T,s,k,m,h)=s-\frac{\gamma(m,h)}{2}s^2,\\ g(T,s,k)=s.\end{cases}\tag{32}$$

Adding up all the terms related to u in (32), we have

$$\begin{aligned}&\frac{1}{2}u'\sigma\sigma'u\left[f_{ss}^{s,k}+\gamma(s,k)gg_{ss}\right]\\&\quad+\left\{\left[f_s^{s,k}+\gamma(s,k)gg_s\right]B'u+\left[s\sigma_0'-\rho'\right]'\sigma u\left[f_{ss}^{s,k}+\gamma(s,k)gg_{ss}\right]\right\}\\&=\frac{1}{2}\varepsilon_1\left\{u'\sigma\sigma'u+\frac{2}{\varepsilon_1}\left[\varepsilon_2B'\left(\sigma\acute{\sigma}\right)^{-1}\sigma+\left(s\sigma_0'-\rho'\right)\varepsilon_1\right]\sigma'u\right\}\\&=\frac{1}{2}\varepsilon_1\left\{\sigma'u+\frac{1}{\varepsilon_1}\left[\varepsilon_2\sigma'\left(\sigma\sigma'\right)^{-1}B+\left(s\sigma_0'-\rho'\right)\varepsilon_1\right]\right\}\\&\quad\cdot\left\{\sigma'u+\frac{1}{\varepsilon_1}\left[\varepsilon_2B'\left(\sigma\sigma'\right)^{-1}\sigma+\left(s\sigma_0'-\rho'\right)\varepsilon_1\right]'\right.\\&\quad-\frac{1}{2\varepsilon_1}\left\{\varepsilon_2\sigma'\left(\sigma\sigma'\right)^{-1}B+\left(s\sigma_0'-\rho'\right)\varepsilon_1\right\}\\&\quad\cdot\left\{\varepsilon_2\sigma'\left(\sigma\sigma'\right)^{-1}B+\left(s\sigma_0'-\rho'\right)\varepsilon_1\right\},\end{aligned}\tag{33}$$

where

$$\varepsilon_1=f_{ss}^{s,k}+\gamma(s,k)gg_{ss},\ \varepsilon_2=f_s^{s,k}+\gamma(s,k)gg_s.\tag{34}$$

Therefore, the first-order condition for $u(t)$ corresponding to optimal strategy can be described as

$$\begin{aligned}\hat{u}(t,s,k) &= -\left(\sigma\sigma'\right)^{-1}\left[\frac{\varepsilon_2}{\varepsilon_1}B + (s\sigma\sigma_0 - \sigma\rho)\right]\\ &= -\left(\sigma\sigma'\right)^{-1}\left\{\frac{f_s^{s,k} + \gamma(s,k)gg_s}{f_{ss}^{s,k} + \gamma(s,k)gg_{ss}}B + (s\sigma\sigma_0 - \sigma\rho)\right\}.\end{aligned} \tag{35}$$

□

3.2. The Case with a Natural Choice $\gamma(s,k)$. Here, we have $\gamma(s,k) = \gamma(k)/s$ as a special form of $\gamma(s,k)$. Then, equation (30) has been changed into the following form:

$$\hat{u}(t,s,k) = -\left(\sigma\sigma'\right)^{-1}\left\{\frac{sf_s^{s,k} + \gamma(k)gg_s}{sf_{ss}^{s,k} + \gamma(k)gg_{ss}}B + (s\sigma\sigma_0 - \sigma\rho)\right\}. \tag{36}$$

Then, we make use of the following natural choice of $\gamma(s,k)$:

$$\begin{aligned} g(t,s,k) &= a(t,k)s + n(t,k),\\ f(t,s,k,m,h) &= a(t,k)s + n(t,k) - \frac{\gamma(h)}{2m}\\ &\quad\cdot\left[A(t,k)s^2 + 2D(t,k)s + N(t,k)\right].\end{aligned} \tag{37}$$

By differentiation, we have

$$\begin{aligned} &g_t = \dot{a}s + \dot{n},\ g_s = a,\ g_{ss} = 0,\\ &f_t = \dot{a}s + \dot{n} - \frac{\gamma(h)}{2m}\left[\dot{A}s^2 + 2\dot{D}s + \dot{N}\right],\end{aligned}$$

$$f_s = a - \frac{\gamma(h)}{m}[As + D],\ f_{ss} = -\frac{\gamma(h)}{m}A. \tag{38}$$

Substituting the above expressions into $\hat{u}(t,s,k)$, we have

$$\hat{u} = \mathrm{k}_1 s + k_2, \tag{39}$$

where

$$\begin{aligned} k_1 &= \left(\sigma\acute{\sigma}\right)^{-1}\left(\frac{a - \gamma(k)A + \gamma(k)a^2}{\gamma(k)A}B - \sigma\sigma_0\right),\\ k_2 &= \left(\sigma\acute{\sigma}\right)^{-1}\left[\frac{a\cdot n - D}{A}B + \sigma\rho\right].\end{aligned} \tag{40}$$

Then, we can simplify

$$\begin{aligned} b_0 s - \mu + B'\hat{u} &= \left(b_0 + B'k_1\right)s + \left(B'k_2 - \mu\right) = M_1 s + M_2,\\ s\sigma_0 - \rho + \sigma'\hat{u} &= \left(\sigma_0 + \sigma'k_1\right)s + \left(\sigma'k_2 - \rho\right) = H_1 s + H_2.\end{aligned} \tag{41}$$

By substituting these expressions into (32), we also have the following alternative expressions for $A^{u\wedge}f^{m,h}$ and $A^{u\wedge}g$:

$$\begin{cases} A^{u\wedge}f^{m,h} = -\frac{\gamma(h)}{2m}\left[\dot{A} + 2AM_1 + H_1{'}H_1A + \sum_{j=1}^d q_{kj}A_j\right]s^2 + \left[\dot{a} - \frac{\gamma(h)}{2m}2\dot{D} + M_1\left(a - \frac{\gamma(h)}{m}D\right) + M_2\left(-\frac{\gamma(h)}{m}A\right) - \frac{\gamma(h)}{m}AH_1{'}H_2 + \sum_{j=1}^d q_{kj}\left(a_j - \frac{\gamma(h)}{2m}2D_j\right)\right]s + \left[\dot{n} - \frac{\gamma(h)}{2m}\dot{N} + M_2\left(a - \frac{\gamma(h)}{m}D\right) - \frac{\gamma(h)}{2m}AH_2{'}H_2 + \sum_{j=1}^d q_{kj}\left(n_j - \frac{\gamma(h)}{2m}N_j\right)\right] = 0,\\ A^{u\wedge}g = \left(\dot{a} + M_1 a + \sum_{j=1}^d q_{kj}a_j\right)s + \left(\dot{n} + M_2 a + \sum_{j=1}^d q_{kj}n_j\right) = 0.\end{cases} \tag{42}$$

From equation (42), we can have the following system of ordinary differential equations (ODE):

$$\begin{cases} \dot{A} + 2M_1A + H_1{'}H_1A + \sum_{j=1}^d q_{kj}A_j = 0,\\ \dot{a} - \frac{\gamma(h)}{m}\dot{D} + M_1\left(a - \frac{\gamma(h)}{m}D\right) - \frac{\gamma(h)}{m}M_2A - \frac{\gamma(h)}{m}AH_1{'}H_2 + \sum_{j=1}^d q_{kj}\left(a_j - \frac{\gamma(h)}{m}D_j\right) = 0,\\ \dot{n} - \frac{\gamma(h)}{2m}\dot{N} + M_2\left(a - \frac{\gamma(h)}{m}D\right) - \frac{\gamma(h)}{2m}AH_2{'}H_2 + \sum_{j=1}^d q_{kj}\left(n_j - \frac{\gamma(h)}{2m}N_j\right) = 0,\\ \dot{a} + M_1 a + \sum_{j=1}^d q_{kj}a_j = 0,\\ \dot{n} + M_2 a + \sum_{j=1}^d q_{kj}n_j = 0.\end{cases} \tag{43}$$

By simplifying and substituting the expressions of M_1, M_2, H_1, H_2, and k_1, k_2 into the above equation (43), we have the following system of ODEs:

$$\begin{cases} \dot{A} + [2(b_0 - \delta_2) - \delta_1]A + \frac{(a + \gamma a^2)^2\delta_1}{\gamma^2 A} + \sum_{j=1}^d q_{kj}A_j = 0,\\ \dot{D} + (b_0 - \delta_2 - \delta_1)D + \frac{a^2(1 + \gamma a)n\delta_1}{\gamma A} + (\delta_3 - \mu)A + \sum_{j=1}^d q_{kj}D_j = 0,\\ \dot{N} - \frac{\delta_1}{A}(a^2n^2 - D^2) + 2(\delta_3 - \mu)D + \sum_{j=1}^d q_{kj}N_j = 0,\\ \dot{a} + \left(b_0 + \frac{a - \gamma A + \gamma a^2}{\gamma A}\delta_1 - \delta_2\right)a + \sum_{j=1}^d q_{kj}a_j = 0,\\ \dot{n} + \frac{a^2\delta_1}{A}n - \frac{aD}{A}\delta_1 + (\delta_3 - \mu)a + \sum_{j=1}^d q_{kj}n_j = 0.\end{cases} \tag{44}$$

with the terminal conditions

$$A(T,k)=a(T,k)=1, D(T,k)=N(T,k)=n(T,k)=0, \quad (45)$$

where

$$\delta_1=B'\left(\sigma\sigma'\right)^{-1}B, \delta_2=B'\left(\sigma\sigma'\right)^{-1}\sigma\sigma_0, \delta_3=B'\left(\sigma\sigma'\right)^{-1}\sigma\rho. \quad (46)$$

Following the process of simplification, the equilibrium control can be represented as (39)

$$\widehat{u}(t,s,k)=k_1(t,k)s+k_2, \quad (47)$$

where

$$k_1=\left(\sigma\sigma'\right)^{-1}\left(\frac{a-\gamma(k)A+\gamma(k)a^2}{\gamma(k)A}B-\sigma\sigma_0\right),$$
$$k_2=\left(\sigma\sigma'\right)^{-1}\left[\frac{a\cdot n-D}{A}B+\sigma\rho\right], \quad (48)$$

and the equilibrium value function of correspondence is given as

$$\begin{aligned}V(t,s,k)=&\left[\frac{\gamma(k)}{2}\left(a^2(t,k)-A(t,k)\right)+a(t,k)\right]s\\&+\frac{\gamma(k)}{2s}\left[n^2(t,k)-N(t,k)\right]\\&+[\gamma(k)(a(t,k)n(t,k)-D(t,k))+n(t,k)],\end{aligned} \quad (49)$$

where $A(\cdot), D(\cdot), N(\cdot), a(\cdot)$, and $n(\cdot)$ satisfy the ODE system in (44).

3.3. The Case without Liability. By letting $l(t)=0$, the asset liability problem will be tackled as a portfolio selection problem. We have $S(t)=X(t)$, $V(t,s,k)=J(t,s,k,\widehat{u}(\cdot))$, and then, the portfolio optimal control $\widehat{u}$ turns to be

$$\widehat{u}(t,x,k)=\left(\sigma\sigma'\right)^{-1}\left[\left(\frac{a-\gamma A+\gamma a^2}{\gamma A}B-\sigma\sigma_0\right)x+\frac{B}{A}(a\cdot n-D)\right], \quad (50)$$

which corresponds to equilibrium value function defined as

$$\begin{aligned}V(t,x,k)=&\left[\frac{\gamma}{2}\left(a^2-A\right)+a\right]x+[\gamma(a\cdot n-D)+n]\\&+\frac{\gamma}{2x}\left(n^2-N\right),\end{aligned} \quad (51)$$

where A, a, D, n, N satisfy the following system of ODEs:

$$\begin{cases}\dot{A}+[2(b_0-\delta_2)-\delta_1]A+\dfrac{(a+\gamma a^2)^2\delta_1}{\gamma^2 A}+\sum_{j=1}^d q_{kj}A_j=0,\\ \dot{D}+(b_0-\delta_2-\delta_1)D+\dfrac{a^2(1+\gamma a)n\delta_1}{\gamma A}+\sum_{j=1}^d q_{kj}D_j=0,\\ \dot{N}+\dfrac{a^2n^2-D^2}{A}\delta_1+\sum_{j=1}^d q_{kj}N_j=0,\\ \dot{a}+\left(b_0+\dfrac{a-\gamma A+\gamma a^2}{\gamma A}\delta_1-\delta_2\right)a+\sum_{j=1}^d q_{kj}a_j=0,\\ \dot{n}+\dfrac{a^2\delta_1}{A}n-\dfrac{aD}{A}\delta_1+\sum_{j=1}^d q_{kj}n_j=0,\end{cases} \quad (52)$$

with the terminal conditions

$$A(T,k)=a(T,k)=1, D(T,k)=N(T,k)=n(T,k)=0. \quad (53)$$

The ODEs are still complicated, which requires to make a further restriction such that the first asset is risk-free, namely, $b_0(t,k)=r(t), \sigma_0(t,k)=(0,\cdots,0)'$, and we have $\delta_2=0$, nor does liability. Hence, the portfolio optimal control $\widehat{u}$ is shown as follows:

$$\widehat{u}(t,x,k)=\left(\sigma\sigma'\right)^{-1}\left[\left(\frac{a-\gamma A+\gamma a^2}{\gamma A}Bx+\frac{B}{A}(a\cdot n-D)\right]. \quad (54)$$

Then, it comes to the equilibrium value function as

$$V(t,x,k)=\left[\frac{\gamma}{2}\left(a^2-A\right)+a\right]x+[\gamma(a\cdot n-D)+n]+\frac{\gamma}{2x}\left(n^2-N\right), \quad (55)$$

where A, a, D, n, N satisfy the following system of ODEs:

$$\begin{cases}\dot{A}+[2r-\delta_1]A+\dfrac{(a+\gamma a^2)^2\delta_1}{\gamma^2 A}+\sum_{j=1}^d q_{kj}A_j=0,\\ \dot{D}+(r-\delta_1)D+\dfrac{a^2(1+\gamma a)n\delta_1}{\gamma A}+\sum_{j=1}^d q_{kj}D_j=0,\\ \dot{N}+\dfrac{a^2n^2-D^2}{A}\delta_1+\sum_{j=1}^d q_{kj}N_j=0,\\ \dot{a}+\left(r+\dfrac{a-\gamma A+\gamma a^2}{\gamma A}\delta_1\right)a+\sum_{j=1}^d q_{kj}a_j=0,\\ \dot{n}+\dfrac{a^2\delta_1}{A}n-\dfrac{aD}{A}\delta_1+\sum_{j=1}^d q_{kj}n_j=0,\end{cases} \quad (56)$$

with the terminal conditions

$$A(T,k)=a(T,k)=1, D(T,k)=N(T,k)=n(T,k)=0. \quad (57)$$

4. Numerical Example

According to the research studied by Chen et al. [32] and Wei et al. [15], the market state is either "Bullish" or "Bearish" with the assumption of $d=2$, where Regime 1 refers to "Bullish" and Regime 2 to "Bearish." In this section, it is exposed to illustrate the results numerically obtained in Section 3. We present the optimal control strategy and the equilibrium value in three different situations: the first situation is state-dependent risk aversion in a regime switching market with one bond and one risky asset without liability; the second situation is different from the first with liability, while the last one is in the same situation but with two risky assets.

4.1. The Case with Liability. Similarly, all the related constant parameters in situation two can been seen in Table 1, which are specified for the illustrative purpose.

The corresponding ODE system becomes

$$\begin{cases} \dot{A}_1+[2r_1-\delta_{11}]A_1+\dfrac{(a_1+\gamma_1 a_1^2)^2\delta_{11}}{\gamma_1^2 A_1}+q_{11}A_1+q_{12}A_2=0,\\ \dot{A}_2+[2r_2-\delta_{12}]A_2+\dfrac{(a_2+\gamma_2 a_2^2)^2\delta_{12}}{\gamma_2^2 A_2}+q_{21}A_1+q_{22}A_2=0,\\ \dot{D}_1+(r_1-\delta_{11})D_1+(\delta_{31}-\mu_1)A_1+\dfrac{a_1^2(1+\gamma_1 a_1)n_1\delta_{11}}{\gamma_1 A_1}+q_{11}D_1+q_{12}D_2=0,\\ \dot{D}_2+(r_2-\delta_{12})D_2+(\delta_{32}-\mu_2)A_2+\dfrac{a_2^2(1+\gamma_2 a_2)n_2\delta_{12}}{\gamma_2 A_2}+q_{21}D_1+q_{22}D_2=0,\\ \dot{N}_1+2(\delta_{31}-\mu_1)D_1+\dfrac{a_1^2 n_1^2-D_1^2}{A_1}\delta_{11}+q_{11}N_1+q_{12}N_2=0,\\ \dot{N}_2+2(\delta_{32}-\mu_2)D_2+\dfrac{a_2^2 n_2^2-D_2^2}{A_2}\delta_{12}+q_{21}N_1+q_{22}N_2=0,\\ \dot{a}_1+\left(r_1+\dfrac{a_1-\gamma_1 A_1+\gamma_1 a_1^2}{\gamma_1 A_1}\delta_{11}\right)a_1+q_{11}a_1+q_{12}a_2=0,\\ \dot{a}_2+\left(r_2+\dfrac{a_2-\gamma_2 A_2+\gamma_2 a_2^2}{\gamma_2 A_2}\delta_{12}\right)a_2+q_{21}a_1+q_{22}a_2=0,\\ \dot{n}_1+\dfrac{a_1^2\delta_{11}}{A_1}n_1-\dfrac{a_1 D_1}{A_1}\delta_{11}+(\delta_{31}-\mu_1)a_1+q_{11}n_1+q_{12}n_2=0,\\ \dot{n}_2+\dfrac{a_2^2\delta_{12}}{A_2}n_2-\dfrac{a_2 D_2}{A_2}\delta_{12}+(\delta_{32}-\mu_2)a_2+q_{21}n_1+q_{22}n_2=0, \end{cases} \quad (58)$$

with the terminal conditions

$$\begin{gathered} A_1(T)=A_2(T)=a_1(T)=a_2(T)=1,\\ D_1(T)=D_2(T)=N_1(T)=N_2(T)=n_1(T)=n_2(T)=0. \end{gathered} \quad (59)$$

Table 1: All the related constant parameters in the case with liability.

Parameter (symbol)	Regime 1 ($i=1$)	Regime 2 ($i=2$)
Exit time (T_i)	10	10
Risk-free interest rate (r_i)	0.04	0.04
Risky asset appreciation rate (b_i)	0.2	0.05
Risky asset volatility (σ_i)	0.3	0.07
Liability appreciation rate (μ_i)	0.08	0.04
Liability volatility (ρ_i)	0.3	0.1
Differentiation of transition probabilities (q_i)	0.3	0.7
Risk aversion (γ_i)	0.5	0.9

where

$$\delta_{11}=\frac{(b_1-r_1)^2}{\sigma_1^2}, \delta_{12}=\frac{(b_2-r_2)^2}{\sigma_2^2},$$

$$\delta_{31}=\frac{(b_1-r_1)\rho_1}{\sigma_1}, \delta_{32}=\frac{(b_2-r_2)\rho_2}{\sigma_2}. \quad (60)$$

4.2. The Case with a Natural Choice $\gamma(s,k)$. Again, all the related constant parameters in situation three have been represented in Table 2. All parameters are specified for the illustrative purpose.

The resulting ODE system are shown as follows:

$$\begin{cases} \dot{A}_1+[2(b_{01}-\delta_{21})-\delta_{11}]A_1+\dfrac{(a_1+\gamma_1 a_1^2)^2\delta_{11}}{\gamma_1^2 A_1}+q_{11}A_1+q_{12}A_2=0,\\ \dot{A}_2+[2(b_{02}-\delta_{22})-\delta_{12}]A_2+\dfrac{(a_2+\gamma_2 a_2^2)^2\delta_{12}}{\gamma_2^2 A_2}+q_{21}A_1+q_{22}A_2=0,\\ \dot{D}_1+(b_{01}-\delta_{21}-\delta_{11})D_1+(\delta_{31}-\mu_1)A_1+\dfrac{a_1^2(1+\gamma_1 a_1)n_1\delta_{11}}{\gamma_1 A_1}+q_{11}D_1+q_{12}D_2=0,\\ \dot{D}_2+(b_{02}-\delta_{22}-\delta_{12})D_2+(\delta_{32}-\mu_2)A_2+\dfrac{a_2^2(1+\gamma_2 a_2)n_2\delta_{12}}{\gamma_2 A_2}+q_{21}D_1+q_{22}D_2=0,\\ \dot{N}_1+2(\delta_{31}-\mu_1)D_1+\dfrac{a_1^2 n_1^2-D_1^2}{A_1}\delta_{11}+q_{11}N_1+q_{12}N_2=0,\\ \dot{N}_2+2(\delta_{32}-\mu_2)D_2+\dfrac{a_2^2 n_2^2-D_2^2}{A_2}\delta_{12}+q_{21}N_1+q_{22}N_2=0,\\ \dot{a}_1+\left(b_{01}-\delta_{21}+\dfrac{a_1-\gamma_1 A_1+\gamma_1 a_1^2}{\gamma_1 A_1}\delta_{11}\right)a_1+q_{11}a_1+q_{12}a_2=0,\\ \dot{a}_2+\left(b_{02}-\delta_{22}+\dfrac{a_2-\gamma_2 A_2+\gamma_2 a_2^2}{\gamma_2 A_2}\delta_{12}\right)a_2+q_{21}a_1+q_{22}a_2=0,\\ \dot{n}_1+\dfrac{a_1^2\delta_{11}}{A_1}n_1-\dfrac{a_1 D_1}{A_1}\delta_{11}+(\delta_{31}-\mu_1)a_1+q_{11}n_1+q_{12}n_2=0,\\ \dot{n}_2+\dfrac{a_2^2\delta_{12}}{A_2}n_2-\dfrac{a_2 D_2}{A_2}\delta_{12}+(\delta_{32}-\mu_2)a_2+q_{21}n_1+q_{22}n_2=0, \end{cases} \quad (61)$$

with the terminal conditions

$$\begin{gathered} A_1(T) = A_2(T) = a_1(T) = a_2(T) = 1, \\ D_1(T) = D_2(T) = N_1(T) = N_2(T) = n_1(T) = n_2(T) = 0. \end{gathered} \tag{62}$$

where

$$\begin{aligned} \delta_{11} &= \frac{(b_{11} - b_{01})^2}{(\sigma_{11} - \sigma_{01})^2}, \delta_{12} = \frac{(b_{12} - b_{02})^2}{(\sigma_{12} - \sigma_{02})^2}, \delta_{21} = \frac{(b_{11} - b_{01})\sigma_{01}}{\sigma_{11} - \sigma_{01}}, \\ \delta_{22} &= \frac{(b_{12} - b_{02})\sigma_{02}}{\sigma_{12} - \sigma_{02}}, \delta_{31} = \frac{(b_{11} - b_{01})\rho_1}{\sigma_{11} - \sigma_{01}}, \delta_{32} = \frac{(b_{12} - b_{02})\rho_2}{\sigma_{12} - \sigma_{02}}. \end{aligned} \tag{63}$$

4.3. The Case without Liability. All the related constant parameters in situation one have been displayed in Table 3, which are specified for the illustrative purpose.

The ODE system of (56) can then be simplified in the form of the following expressions:

$$\begin{cases} \dot{A}_1 + [2r_1 - \delta_{11}]A_1 + \dfrac{(a_1 + \gamma_1 a_1^2)^2 \delta_{11}}{\gamma_1^2 A_1} + q_{11}A_1 + q_{12}A_2 = 0, \\ \dot{A}_2 + [2r_2 - \delta_{12}]A_2 + \dfrac{(a_2 + \gamma_2 a_2^2)^2 \delta_{12}}{\gamma_2^2 A_2} + q_{21}A_1 + q_{22}A_2 = 0, \\ \dot{D}_1 + (r_1 - \delta_{11})D_1 + \dfrac{a_1^2(1 + \gamma_1 a_1)n_1\delta_{11}}{\gamma_1 A_1} + q_{11}D_1 + q_{12}D_2 = 0, \\ \dot{D}_2 + (r_2 - \delta_{12})D_2 + \dfrac{a_2^2(1 + \gamma_2 a_2)n_2\delta_{12}}{\gamma_2 A_2} + q_{21}D_1 + q_{22}D_2 = 0, \\ \dot{N}_1 + \dfrac{a_1^2 n_1^2 - D_1^2}{A_1}\delta_{11} + q_{11}N_1 + q_{12}N_2 = 0, \\ \dot{N}_2 + \dfrac{a_2^2 n_2^2 - D_2^2}{A_2}\delta_{12} + q_{21}N_1 + q_{22}N_2 = 0, \\ \dot{a}_1 + \left(r_1 + \dfrac{a_1 - \gamma_1 A_1 + \gamma_1 a_1^2}{\gamma_1 A_1}\delta_{11}\right)a_1 + q_{11}a_1 + q_{12}a_2 = 0, \\ \dot{a}_2 + \left(r_2 + \dfrac{a_2 - \gamma_2 A_2 + \gamma_2 a_2^2}{\gamma_2 A_2}\delta_{12}\right)a_2 + q_{21}a_1 + q_{22}a_2 = 0, \\ \dot{n}_1 + \dfrac{a_1^2\delta_{11}}{A_1}n_1 - \dfrac{a_1 D_1}{A_1}\delta_{11} + q_{11}n_1 + q_{12}n_2 = 0, \\ \dot{n}_2 + \dfrac{a_2^2\delta_{12}}{A_2}n_2 - \dfrac{a_2 D_2}{A_2}\delta_{12} + q_{21}n_1 + q_{22}n_2 = 0, \end{cases} \tag{64}$$

with the terminal conditions

$$\begin{gathered} A_1(T) = A_2(T) = a_1(T) = a_2(T) = 1, \\ D_1(T) = D_2(T) = N_1(T) = N_2(T) = n_1(T) = n_2(T) = 0. \end{gathered} \tag{65}$$

Table 2: All the related constant parameters in the case with a natural choice $\gamma(s, k)$.

Parameter (symbol)	Regime 1 ($i = 1$)	Regime 2 ($i = 2$)
Exit time (T_i)	10	10
The first risky asset appreciation rate (b_{0i})	0.3	0.06
Other risky asset appreciation rate (b_{1i})	0.2	0.05
The first risky asset volatility (σ_{0i})	0.3	0.07
Other risky asset volatility (σ_{1i})	0.2	0.09
Liability appreciation rate (μ_i)	0.08	0.04
Liability volatility (ρ_i)	0.3	0.1
Differentiation of transition probabilities (q_i)	0.3	0.7
Risk aversion (γ_i)	0.5	0.9

Table 3: All the related constant parameters in the case without liability.

Parameter (symbol)	Regime 1 ($i = 1$)	Regime 2 ($i = 2$)
Exit time (T_i)	10	10
Risk-free interest rate (r_i)	0.04	0.04
Risky asset appreciation rate (b_i)	0.2	0.05
Risky asset volatility (σ_i)	0.3	0.07
Liability appreciation rate (μ_i)	0.08	0.04
Liability volatility (ρ_i)	0.3	0.1
Differentiation of transition probabilities (q_i)	0.3	0.7
Risk aversion (γ_i)	0.5	0.9

where

$$\delta_{11} = \frac{(b_1 - r_1)^2}{\sigma_1^2}, \delta_{12} = \frac{(b_2 - r_2)^2}{\sigma_2^2}. \tag{66}$$

From Figures 1–6, in the "Bearish" market, we could see the optimal control strategy u is increasing as time goes by along with the increase of equilibrium value. They capture the optimal strategy of the utility and keep it for the whole time, because the investor pays attention to the utility functions during the entire time. It is reasonable for the time consistent investor to sacrifice parts of the utility or happiness to secure sufficient budget in order to avoid unpredictable deficits. On the other hand, from Figures 1–6 for the "Bullish" market, the optimal control strategy is growing with time while the equilibrium value decreases. At the beginning of the investment in the "Bullish" market, the investor is confident enough to employ the strategy which would lead to a optimize equilibrium value. Over time, the investor will have less investment time to invest to maximize

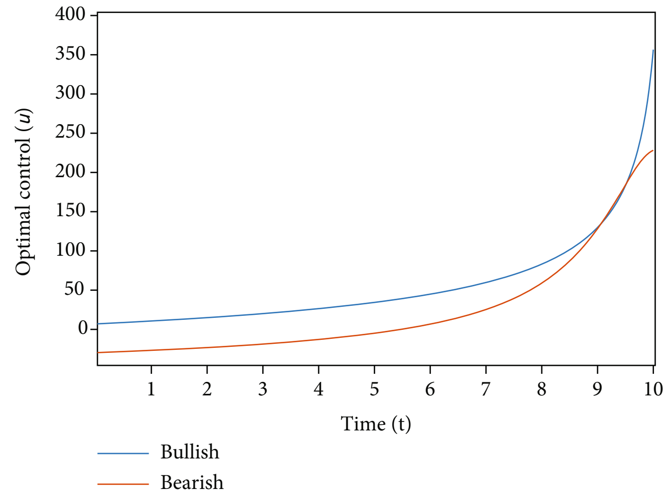

Figure 1: Optimal Control u.

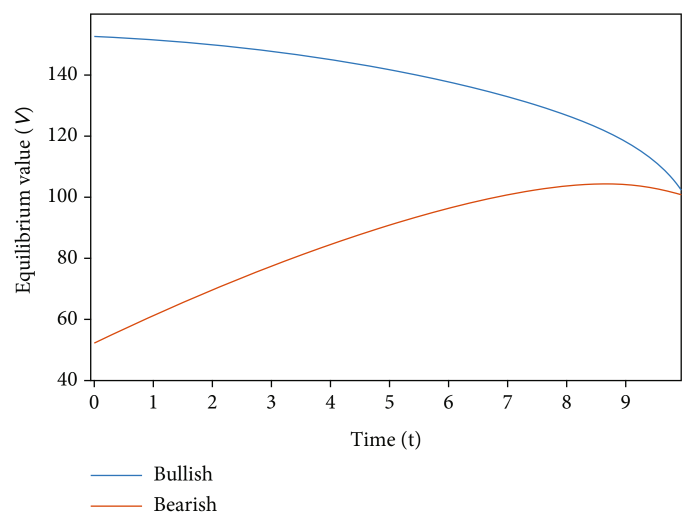

Figure 2: Equilibrium value V.

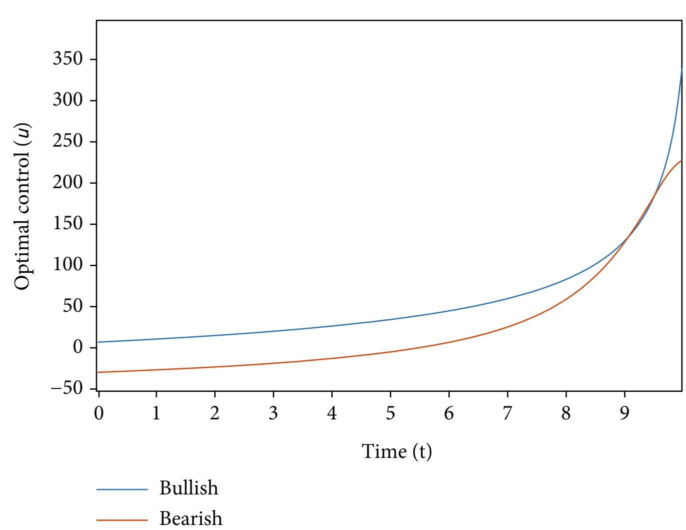

Figure 3: Optimal control u.

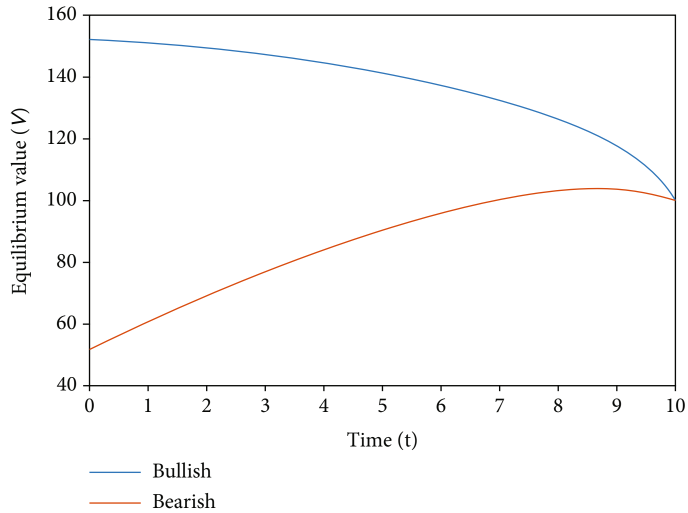

FIGURE 4: Equilibrium value V.

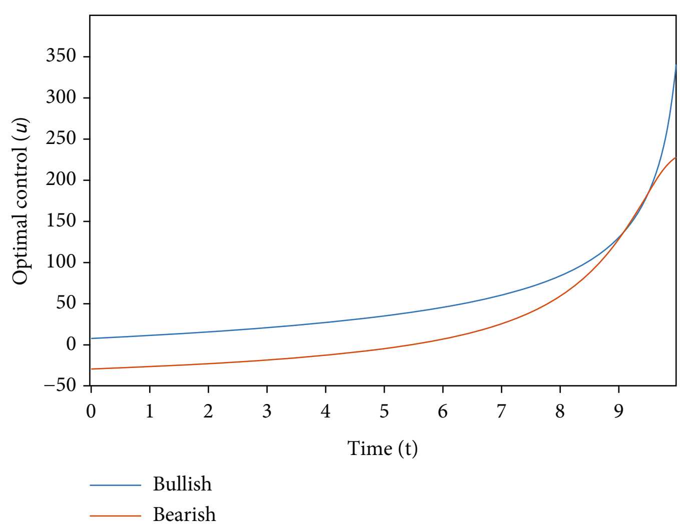

FIGURE 5: Optimal control u.

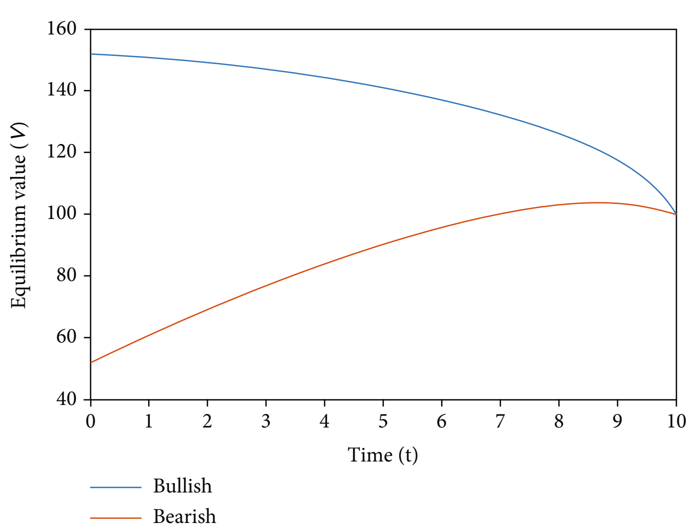

FIGURE 6: Equilibrium value V.

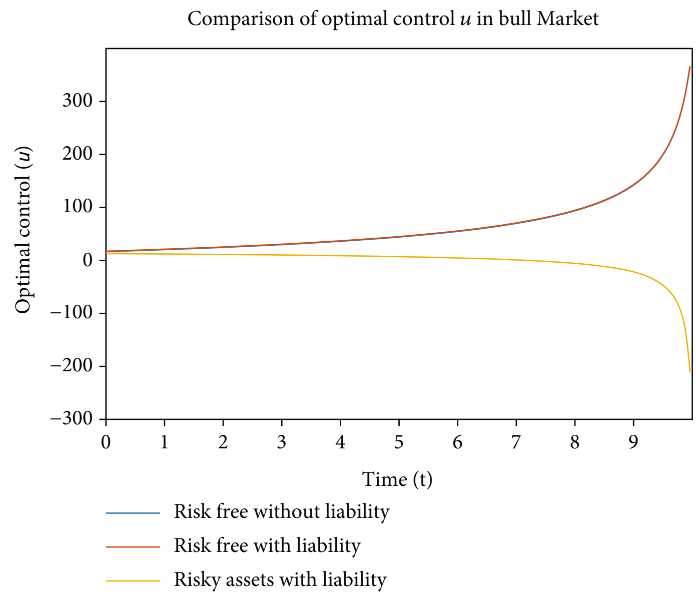

Figure 7: Comparison of optimal control u in Bull market.

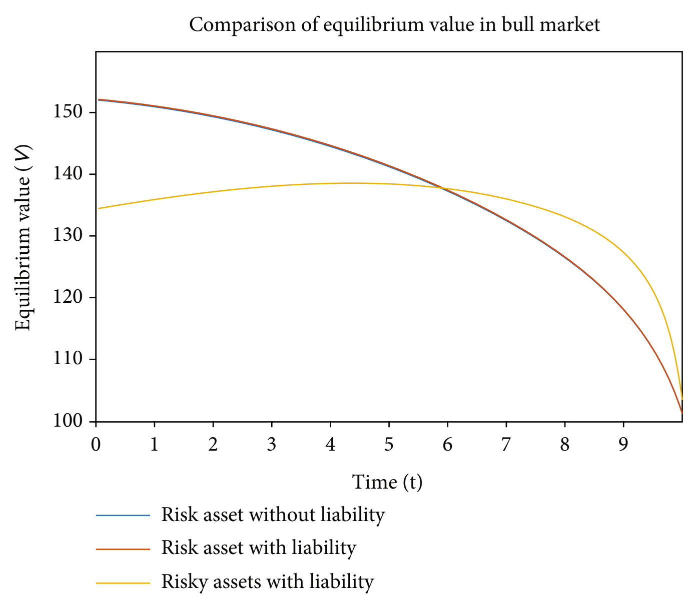

Figure 8: Comparison of equilibrium value in Bull market.

their current utility as well as the benefit from the "Bullish" market because of the state-dependent risk aversion.

In Figures 7 and 8, they compare the results of optimal control strategy and the equilibrium value among three different situations under the "Bullish" market.

In Figure 7, with the comparison of optimal control strategy (u) in Bull market among three different situations, we find that there is no significant difference between the situations with and without liability. When the investors come across the "Bullish" market, they are willing to invest the risky asset instead of bearing the liability. However, in the situation with all risky assets, the optimal control strategy provides a way to invest in one of risky assets, which highly depends on the parameters of the two risky assets. In Figure 8, the comparison analysis on the equilibrium values in Bull market among three different situations has been conducted. The figure shows decreasing evidence along with levelling off, of which the reason has been explained above. For the situation with all risky assets, the equilibrium value has a more stable trend which also depends on the selected two risky assets.

In the last picture, the comparison has been made about the optimal control strategy by varying the risk aversion coefficient. The effect of the risk aversion has been analysed in Figure 9, which presents the optimal control strategy (u) under three different risk aversions. Obviously, as the risk aversion increases, it can be seen that the investor is less likely to invest in risky assets, which is consistent with common sense.

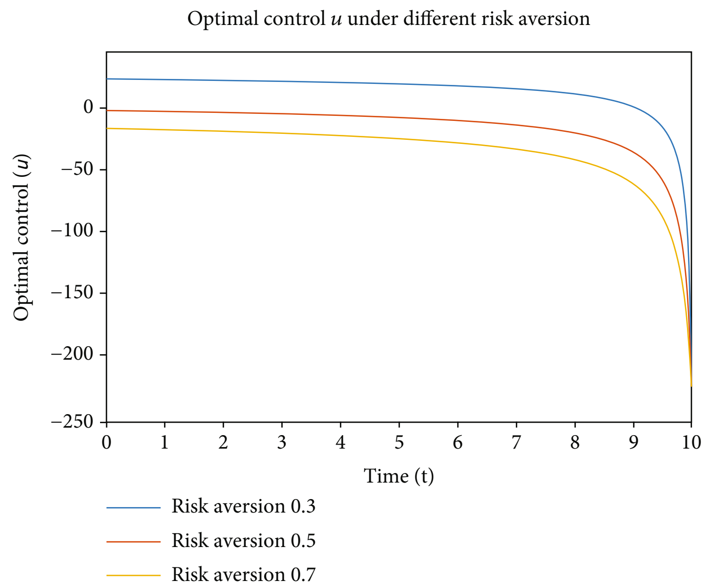

Figure 9: Optimal control u under different risk aversions.

5. Conclusions

In this paper, the mean variance model of asset-liability management has been discussed in the case of state-dependent risk aversion with only risky assets. Based on the continuous-time Markov regime-switching, this paper derives an analytical optimal control expressions theoretically in a more realistic financial market and then makes numerical analysis on a series of special cases. From the numerical results, this paper reveals the feasibility and application of introducing factors, such as regime-switching, liability, and risky asset in a mean-variance optimization framework, and also shows the relationships between a set of risk aversions and the optimal controller.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Lie Symmetry Analysis for the General Classes of Generalized Modified Kuramoto-Sivashinsky Equation

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Lie symmetry analysis of differential equations proves to be a powerful tool to solve or at least reduce the order and nonlinearity of the equation. Symmetries of differential equations is the most significant concept in the study of DE's and other branches of science like physics and chemistry. In this present work, we focus on Lie symmetry analysis to find symmetries of some general classes of KS-type equation. We also compute transformed equivalent equations and some invariant solutions of this equation.

1. Introduction

Symmetry has been a source of inspiration as a powerful tool in the formulation of the laws of the universe. A great number of physical phenomena is transformed into differential equations. Lie symmetry analysis can change the given differential equation into an equivalent form which is easier to solve. In the analysis of differential equations, the symmetry group approach is quite useful. Galois's use of finite groups to solve algebraic equations of degrees two, three, and four, as well as to prove that the general polynomial equation of degrees larger than four could not be solved by radicals, served as the paradigm for this application [1–7]. The symmetry group approach is well-known for its importance in the field of differential equations analysis. Sophus Lie is credited with the invention of group categorization methods and the theoretical basis for the Lie groups.

There are many different methods for computing the symmetries of differential equations. But Lie symmetry analysis is the best because it is a systematic and algorithmic procedure that does not take into account any guesses or approximations. The principal paper on Lie symmetry is [1], in which Lie demonstrated that a linear 2D, 2nd-order PDE admits at most three boundary invariance group. He processed the maximal invariance group of the one-dimensional heat conductivity and used this analysis to compute its explicit solutions. Symmetry reduction is a leading strategy for resolving nonlinear PDEs. Ovisiannikov made a substantial contribution in persisting with these techniques. He presented the strategy of partially invariant solutions [2, 3]. In this work, he gave a methodology that is based on the idea of group called the equivalence group. Gazizov and Ibragimov [8] tracked down the total symmetry analysis of the one-dimensional Black-Scholes model. Shu-Yong and Feng-Xiang, [9] discussed about the connection between the form invariance and Lie symmetry of nonholonomic framework. Buckwar and Luchko [4] initiated the study of symmetry group of scaling transformation for PDEs of fractional order. Yan et al. [6] performed Lie symmetry analysis and fundamental similarity reductions for the coupled Kuramoto-Sivashinsky(KS) equations. Bozhkov and Dimas [10] computed the conversation laws and group classification for generalized 2D KS equation. Nadjafikhah and Ahangari [7] determined the Lie symmetries and reduction for the two-dimensional damped

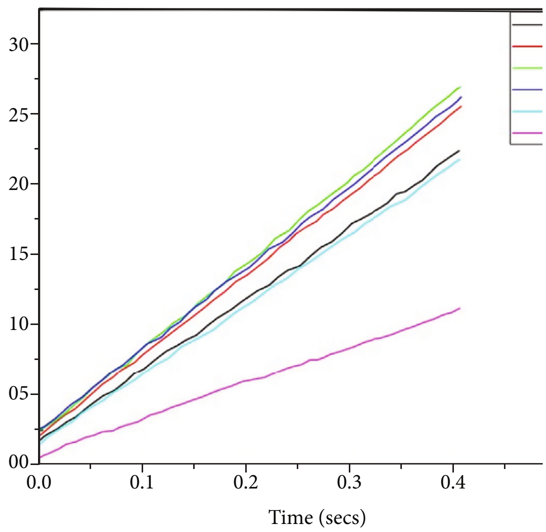

Figure 1: Flame front position.

Kuramoto-Sivashinsky ((2D) DKS) equation. Najafikhah and Ahangari also computed Lie symmetry of 2D generalized Kuramoto-Sivashinsky (KS) equation in [11]. The one-dimensional modified KS-type equation is

$$u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma u_{xx}^2 = 0, \tag{1}$$

Chou [12] determined the solution of the cauchy problem for the MKS equation and also computed the solvability with the help of the blow up theorem.

In the present paper, we deal with the generalized modified one-dimensional Kuramoto-Sivashinsky (GMKS) type equation and determine the symmetry algebra by using Lie symmetry analysis. In particular, we want to find the optimal system and similarity solutions corresponding to some special cases of GMKS equation. The GMKS type equation is given as

$$f(u)u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma u_{xx}^2 = 0. \tag{2}$$

We seek the Lie symmetry algebras for this GMKS equations for $f(u) = u^n$, $f(u) = e^{nu}$, and $f(u) = e^{u^n}$ where λ and σ are arbitrary constant and $\lambda \neq 1$. For $\lambda = 1$, equation (2) is proposed in [13]. Its second derivative satisfies an equation of Cahn-Hilliard type in [14]. This equation has various applications as physical models in biofluids, mechanics, and liquids. In equation (2), u is the velocity function, x is space parameter, and t is time variable. This equation can also be derived from a model in the continuity equation by fitting a suitable function [15]. Actually, the Kuramoto-Sivashinsky equation gives the change of the position of a flame front (Figure 1). It shows the flame front position against time for horizontally propagating methane flame, the movement of a fluid going down a vertical wall, or a spatially uniform oscillating chemical reaction in a homogeneous medium [16]. This equation is also helpful to display solitary pulses in a falling slender film [17]. Figure 2 shows the schematic representation of the flow showing a film flowing vertically down, subjected to an electric field imposed across electrodes separated by a distance d.

2. Lie Symmetry of Generalized Modified Kuramoto-Sivashinsky Equation

In this part, we compute our main results.

Consider one parameter local Lie group of transformation for the independent factors x, t and dependent factor u as follows:

$$\begin{aligned} x^* &= x + \delta\alpha(x,t,u) + O(\delta^2), \\ t^* &= t + \delta\beta(x,t,u) + O(\delta^2), \\ u^* &= u + \delta\gamma(x,t,u) + O(\delta^2), \end{aligned} \tag{3}$$

in which $\delta \in \mathbb{R}$ is the parameter.

Proposition 1. *For all $n \geq 1, n \in N$, the algebra of symmetries of*

$$u^n u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma u_{xx}^2 = 0, \tag{4}$$

is 2-dimensional Abelian Lie algebra.

Proof. The general infinitesimal generator (symmetries) is

$$H = \alpha(x,t,u)\partial_x + \beta(x,t,u)\partial_t + \gamma(x,t,u)\partial_u. \tag{5}$$

The derivation of nth prolongation of H

$$pr^n H = H + \sum_{i=1}^{q}\sum_{P}\gamma_i^P(x,u^n)\left(\frac{\partial}{\partial u_P^i}\right), \tag{6}$$

interprets the relating jet space $Q^n \subset X \times U^n$, where q is a dependent variables, and $P = (P_1, P_2, \cdots P_k)$ with $1 \leq P_k \leq p$, $1 \leq k \leq n$ and

$$\gamma_i^P(x,u^n) = D_P\left(\gamma_i - \sum_{l=1}^{p}\alpha^l u^i{}_l\right) + \sum_{l=1}^{p}\alpha^l u^i{}_{P,l}, \tag{7}$$

where p is an independent variable and $u^i{}_l = \partial u^i/\partial u^l$ and $u^i{}_{P,l} = \partial u^i{}_P/\partial x^l$.

The fourth-order prolongation of H is

$$\begin{aligned} Pr^{(4)}H = H &+ \gamma^x\frac{\partial}{\partial u_x} + \gamma^t\frac{\partial}{\partial u_t} + \gamma^{xx}\frac{\partial}{\partial u_{xx}} + \gamma^{xt}\frac{\partial}{\partial u_{xt}} \\ &+ \gamma^{tt}\frac{\partial}{\partial u_{tt}} + \gamma^{xxx}\frac{\partial}{\partial u_{xxx}} + \gamma^{xxt}\frac{\partial}{\partial u_{xxt}} + \gamma^{xtt}\frac{\partial}{\partial u_{xtt}} \\ &+ \gamma^{ttt}\frac{\partial}{\partial u_{ttt}} + \gamma^{xxxx}\frac{\partial}{\partial u_{xxxx}} + \gamma^{xxxt}\frac{\partial}{\partial u_{xxxt}} \\ &+ \gamma^{xxtt}\frac{\partial}{\partial u_{xxtt}} + \gamma^{xttt}\frac{\partial}{\partial u_{xttt}} + \gamma^{tttt}\frac{\partial}{\partial u_{tttt}}, \end{aligned} \tag{8}$$

$$\begin{gathered} pr^4\left[u^n u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma u_{xx}^2\right] \equiv 0 \mod (2), \\ nu^{n-1}\gamma^t + \gamma^{xx} + \gamma^{xxxx} + 2(\lambda - 1)u_x\gamma^x - 2\sigma u_{xx}\gamma^{xx} \equiv 0 \mod (2). \end{gathered} \tag{9}$$

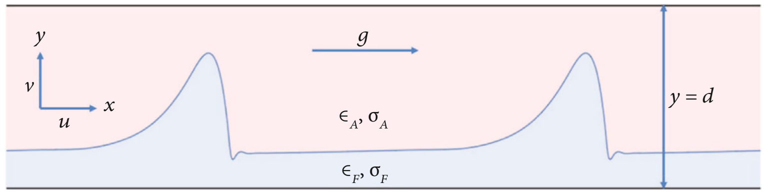

Figure 2: Schematic representation of the flow showing a film flowing vertically down.

Table 1: Commutator table.

[.., ..]	H_1	H_2
H_1	0	0
H_2	0	0

Table 2: Adjoint table.

ad	H_1	H_2
H_1	H_1	H_2
H_2	H_1	H_2

We can calculate γ^t,γ^x, γ^{xx}, and γ^{xxxx} from equation (7) such that

$$\begin{aligned}\gamma^t &= D_t(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xt} + \beta u_{tt},\\ \gamma^x &= D_x(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xx} + \beta u_{xt},\\ \gamma^{xx} &= D_x^2(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xxx} + \beta u_{xxt},\\ \gamma^{xxxx} &= D_x^4(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xxxxx} + \beta u_{xxxxt},\end{aligned} \tag{10}$$

where D_x and D_t are total derivatives.□ □

Putting all the above in equation (9) and eliminating u_t by using the relation $u_t = 1/u^n(\sigma u_{xx}^2 - (\lambda - 1)u_x^2 - u_{xx} - u_{xxxx})$, we get a polynomial equation containing the different differentials of u. Equating the coefficient of u to zero, which are some derivatives of α, β, and γ, it gives the total set of determining equations.

$$\alpha_x = \alpha_t = \alpha_u = 0, \beta_x = \beta_t = \beta_u = 0, \gamma = 0. \tag{11}$$

This gives

$$\alpha(x, t, u) = 0, \beta(x, t, u) = 0, \gamma(x, t, u) = 0. \tag{12}$$

This implies that the Lie group (algebra) of infinitesimal generators of equaution (2) is comprised of two vector fields:

$$\begin{aligned}H_1 &= \partial_x,\\ H_2 &= \partial_t.\end{aligned} \tag{13}$$

The commutator table of the Lie group for equation (2) is given as in Table 1,

The adjoint table of infinitesimal symmetries for equation (2) is given as in Table 2,

In this case, we have only two different basis for a Lie algebra of symmetries.

Hence, this shows that the group of symmetries of equation (2) is two dimensional and tables ensure that it is abelian.

Proposition 2. *For all $n > 1$ and $n \in N$, the group of symmetries of*

$$e^{u^n} u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma u_{xx}^2 = 0, \tag{14}$$

is two-dimensional abelian.

Proof. The infinitesimal generator is

$$X = \alpha(x, t, u)\partial x + \beta(x, t, u)\partial t + \gamma(x, t, u)\partial u. \tag{15}$$

In order to find the symmetry group of equation (14), we have to apply invariance condition that is $X^{(4)}(5) \equiv 0$ mod (6) on equation .(14) where $X^{(4)}$ is the fourth-order prolongation of X given as

$$\begin{aligned}Pr^{(4)}X = X &+ \gamma^x \frac{\partial}{\partial u_x} + \gamma^t \frac{\partial}{\partial u_t} + \gamma^{xx} \frac{\partial}{\partial u_{xx}} + \gamma^{xt} \frac{\partial}{\partial u_{xt}}\\ &+ \gamma^{tt} \frac{\partial}{\partial u_{tt}} + \gamma^{xxx} \frac{\partial}{\partial u_{xxx}} + \gamma^{xxt} \frac{\partial}{\partial u_{xxt}} + \gamma^{xtt} \frac{\partial}{\partial u_{xtt}}\\ &+ \gamma^{ttt} \frac{\partial}{\partial u_{ttt}} + \gamma^{xxxx} \frac{\partial}{\partial u_{xxxx}} + \gamma^{xxxt} \frac{\partial}{\partial u_{xxxt}}\\ &+ \gamma^{xxtt} \frac{\partial}{\partial u_{xxtt}} + \gamma^{xttt} \frac{\partial}{\partial u_{xttt}} + \gamma^{tttt} \frac{\partial}{\partial u_{tttt}}.\end{aligned} \tag{16}$$

After applying an invariance condition on equation (14), we get

$$\begin{aligned}&nu^{n-1}e^{u^n}\gamma^t + \gamma^{xx} + \gamma^{xxxx} + 2(\lambda - 1)u_x\gamma^x\\ &\quad - 2\sigma u_{xx}\gamma^{xx} \equiv 0.\text{mod } (2).\end{aligned} \tag{17}$$

We can calculate γ^t, γ^x, γ^{xx}, and γ^{xxxx} from equation (7) such that

$$\begin{aligned}\gamma^t &= D_t(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xt} + \beta u_{tt},\\ \gamma^x &= D_x(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xx} + \beta u_{xt},\\ \gamma^{xx} &= D_x^2(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xxx} + \beta u_{xxt},\\ \gamma^{xxxx} &= D_x^4(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xxxxx} + \beta u_{xxxxt},\end{aligned} \tag{18}$$

where D_x and D_t are total derivatives.□ □

Table 3: Commutator table.

[.., ..]	X_1	X_2
X_1	0	0
X_2	0	0

Table 4: Adjoint table.

ad	X_1	X_2
X_1	X_1	X_2
X_2	X_1	X_2

After putting all the above in equation (17), and eliminating u_t by using the relation $u_t = 1/e^{u^n}(\sigma u_{xx}^2 - (\lambda-1)u_x^2 - u_{xx} - u_{xxxx})$, we get a polynomial equation containing the different differentials of u. Equating the coefficient of u to zero, which are some derivatives of α, β, and γ, it gives the total set of determining equations, as given by

$\alpha_x = \alpha_t = \alpha_u = 0$, $\beta_x = \beta_t = \beta_u = 0$, $\gamma = 0$,

that gives

$$\alpha(x,t,u) = 0, \beta(x,t,u) = 0, \gamma(x,t,u) = 0. \tag{19}$$

This implies that the Lie group (algebra) of infinitesimal generators of equation (9) comprises two vector fields. Following, Table 3 gives the commutator table as

$$\begin{aligned} X_1 &= \partial_x, \\ X_2 &= \partial_t. \end{aligned} \tag{20}$$

The commutator table of the Lie group for equation (14) is given in Table 3,

The adjoint table of infinitesimal symmetries for equation (14) is given in Table 4,

In this case, we have only two different basis for Lie algebra.

Hence, this shows that the group of symmetries of equation (14) is two-dimensional abelian.

2.1. Symmetry Algebra for $e^{u^n}u_t + u_{xx} + u_{xxxx} + (\lambda-1)u_x^2 - \sigma u_{xx}^2 = 0$ When $n = 1$. For $n = 1$, the equation is

$$\Delta : e^u u_t + u_{xx} + u_{xxxx} + (\lambda-1)u_x^2 - \sigma u_{xx}^2 = 0. \tag{21}$$

The general infinitesimal generator is

$$V = \tau(x,t,u)\partial x + \mu(x,t,u)\partial t + \nu(x,t,u)\partial u. \tag{22}$$

In order to find the symmetry algebra, we have to apply invariance condition that is

$$V^{(n)}(\Delta) \equiv 0 \bmod (\Delta), \tag{23}$$

on equation (21) where $V^{(4)}$ is the fourth-order prolongation of V such that

$$\begin{aligned} Pr^{(4)}V = V &+ \nu^x \frac{\partial}{\partial u_x} + \nu^t \frac{\partial}{\partial u_t} + \nu^{xx} \frac{\partial}{\partial u_{xx}} + \nu^{xt} \frac{\partial}{\partial u_{xt}} \\ &+ \nu^{tt} \frac{\partial}{\partial u_{tt}} + \nu^{xxx} \frac{\partial}{\partial u_{xxx}} + \nu^{xxt} \frac{\partial}{\partial u_{xxt}} + \nu^{xtt} \frac{\partial}{\partial u_{xtt}} \\ &+ \nu^{ttt} \frac{\partial}{\partial u_{ttt}} + \nu^{xxxx} \frac{\partial}{\partial u_{xxxx}} + \nu^{xxxt} \frac{\partial}{\partial u_{xxxt}} \\ &+ \nu^{xxtt} \frac{\partial}{\partial u_{xxtt}} + \nu^{xttt} \frac{\partial}{\partial u_{xttt}} + \nu^{tttt} \frac{\partial}{\partial u_{tttt}}. \end{aligned} \tag{24}$$

After applying invariance condition (23) on equation (21)

$$ue^u \nu^t + \nu^{xx} + \nu^{xxxx} + 2(\lambda-1)u_x \nu^x - 2\sigma u_{xx} \nu^{xx} \equiv 0 \bmod (\Delta), \tag{25}$$

where ν^t,ν^x, ν^{xx}, and ν^{xxxx} from equation (7) such that

$$\begin{aligned} \nu^t &= D_t(\nu - \tau u_x - \mu u_t) + \tau u_{xt} + \mu u_{tt}, \\ \nu^x &= D_x(\nu - \tau u_x - \mu u_t) + \tau u_{xx} + \mu u_{xt}, \\ \nu^{xx} &= D_x^2(\nu - \tau u_x - \mu u_t) + \tau u_{xxx} + \mu u_{xxt}, \\ \nu^{xxxx} &= D_x^4(\nu - \tau u_x - \mu u_t) + \tau u_{xxxxx} + \mu u_{xxxxt}, \end{aligned} \tag{26}$$

where D_x and D_t are total derivative.

Putting all the above in equation (25), we eliminateu_t by using the relation $u_t = 1/e^u(\sigma u_{xx}^2 - (\lambda-1)u_x^2 - u_{xx} - u_{xxxx})$ and get a polynomial equation containing the different differentials of u. Equating the coefficient of u to zero, which are some derivatives of τ, μ, and ν, it gives the total set of determining equations.

$\tau_x = \tau_t = \tau_u = 0$, $\mu_x = \mu_t = \mu_u = 0$, $\nu = 0$,

that gives

$$\tau(x,t,u) = c_3, \mu(x,t,u) = c_1 t + c_2, \nu(x,t,u) = c_1. \tag{27}$$

This implies that the Lie group (algebra) of infinitesimal generators of equation (21) is comprised of three vector fields:

$$V_1 = \partial_u + t\partial_t, \tag{28}$$

$$V_2 = \partial_t, \tag{29}$$

$$V_3 = \partial_x. \tag{30}$$

The commutator table of the Lie group for equation (21) is given in Table 5,

The adjoint table of infinitesimal symmetries for equation (21) is given in Table 6,

In this case, we have three different Lie algebras.

Theorem 3. *The algebra of symmetries of Kuramoto-Sivashinsky type equation is*

$$e^{u^n}u_t + u_{xx} + u_{xxxx} + (\lambda-1)u_x^2 - \sigma u_{xx}^2 = 0, \tag{31}$$

Table 5: Commutator table.

[.., ..]	V_1	V_2	V_3
V_1	0	$-V_2$	0
V_2	V_2	0	0
V_3	0	0	0

Table 6: Adjoint table.

$[ad]$	V_1	V_2	V_3
V_1	V_1	$V_2 e^{\varepsilon}$	V_3
V_2	$V_1 - \varepsilon V_2$	V_2	V_3
V_3	V_1	V_2	V_3

where it is two-dimensional abelian for all $n > 1$, $n \varepsilon N$ and three-dimensional nonabelian for $n = 1$.

Proof. The proof follows easily using Propositions 1 and 2.□ □

Theorem 4. *If $G_s^i(x,t,u)$ be the one parameter group generated by equation (28) then*

$$\begin{aligned} G_s^1(x,t,u) &= (x, e^s t, u), \\ G_s^2(x,t,u) &= (x, t+s, u), \\ G_s^3(x,t,u) &= (x+s, t, u). \end{aligned} \tag{32}$$

There will be a family of solutions to each one parameter subgroups of the full symmetry group of a system called group invariant solutions.

Theorem 5. *If $u = f(x,t)$ is a solution of equation (21), so are the functions*

$$\begin{aligned} \tilde{u}^1 &= f(x, e^{-s} t), \\ \tilde{u}^2 &= f(x, t-s), \\ \tilde{u}^3 &= f(x-s, t), \end{aligned} \tag{33}$$

where $\tilde{u}^i = G_s^i * f(x,t)$, $i = 1,2,3$ and $s << 1$ is any positive number.

Proof. The one parameter Lie group of equation (21) is

$$G_s^1 : (x,t,u) \longrightarrow (x, e^s t, u), \tag{34}$$

with the infinitesimal generator

$$V_1 = t\partial t + \partial u, \tag{35}$$

if $\tilde{u}^1(x,t)$ is any function then it transformed by G_s^1 as

$$\begin{aligned} \tilde{u}^1 &= u, \\ \tilde{u}^1 &= f(x,t), \end{aligned} \tag{36}$$

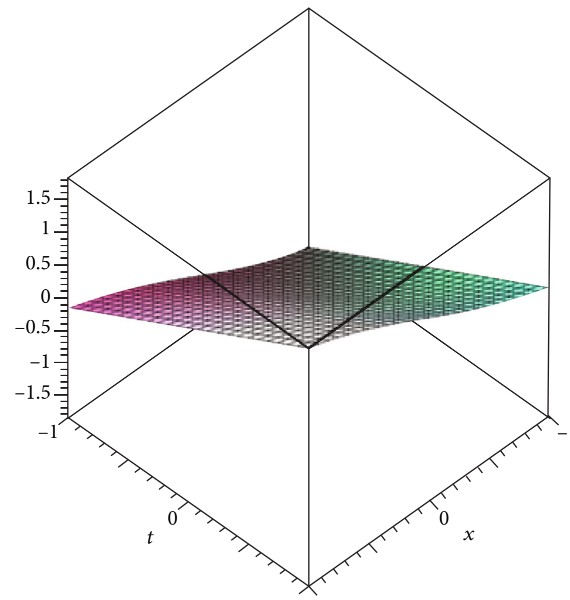

Figure 3: For $u^1(x,t) = \sin(x) + e^{-s}(t)$, $s = 0.00001$.

now

$$(\tilde{x}, \tilde{t}) = (x, e^{-s} t), \tag{37}$$

therefore,

$$\tilde{u}^1 = f(x, e^{-s} t). \tag{38}$$

The graph for $\tilde{u}^1 = f(x, e^{-s} t)$ is given in Figure 3. The one parameter Lie group of equation (21) is

$$G_s^2 : (x,t,u) \longrightarrow (x, t-s, u), \tag{39}$$

with the infinitesimal generator

$$V_2 = t\partial t, \tag{40}$$

if $\tilde{u}^2(x,t)$ is any function then it transformed by G_s^2 as

$$\begin{aligned} \tilde{u}^2 &= u, \\ \tilde{u}^2 &= f(x,t), \end{aligned} \tag{41}$$

now

$$(\tilde{x}, \tilde{t}) = (x, t-s), \tag{42}$$

therefore

$$\tilde{u}^2 = f(x, t-s). \tag{43}$$

The graph for $\tilde{u}^2 = f(x, t-s)$ is given in Figure 4. The one parameter Lie group of equation (21) is

$$G_s^3 : (x,t,u) \longrightarrow (x-s, t, u), \tag{44}$$

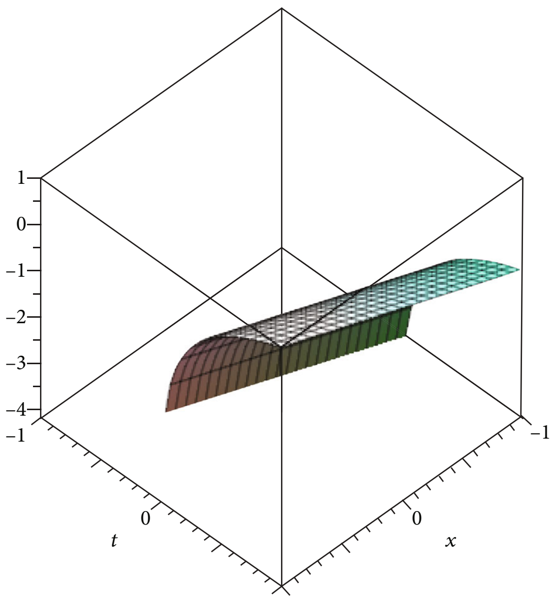

Figure 4: For $u^1(x,t) = \log(x+t) - s, s = 0.00001$.

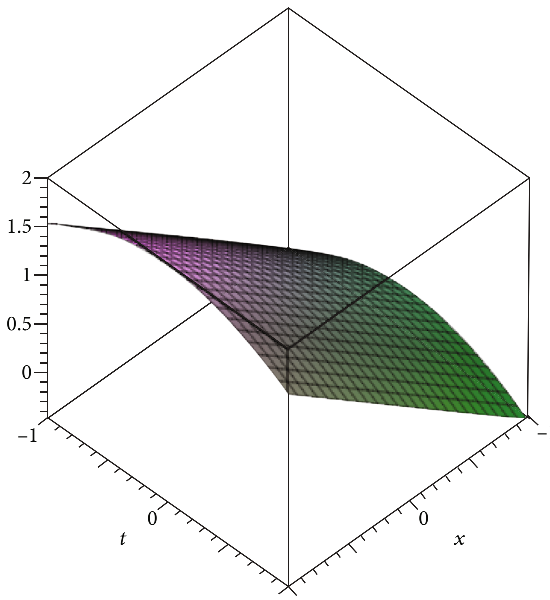

Figure 5: For $u^3(x,t) = (x-s) + \cos(t), s = 0.00001$.

with the infinitesimal generator

$$V_3 = t\partial x, \tag{45}$$

if $\tilde{u}^3(x,t)$ is any function then it transformed by G_s^3 as

$$\begin{aligned} \tilde{u}^3 &= u, \\ \tilde{u}^3 &= f(x,t), \end{aligned} \tag{46}$$

now

$$(\tilde{x}, \tilde{t}) = (x - s, t), \tag{47}$$

therefore

$$\tilde{u}^3 = f(x - s, t). \tag{48}$$

The graph for $\tilde{u}^3 = f(x-s,t)$ as a solution is given in Figure 5.□ □

3. Optimal System of Subalgebras

This is remarkable that the Lie symmetry technique assumes a significant part to determine the solutions of PDEs as well as performing the symmetric reductions. Every combination (should be linear) of infinitesimal symmetries(generators) is a result of another infinitesimal symmetry(generator). As any transformation in the full symmetry groups plot a solution to another, it is sufficient to determine the invariant solution which are not related by transformations in the full symmetry group; this prompted the Optimal system [18, 19].

Theorem 6. *A 1D optimal system of equation (21) is given by those generated by*

$$\begin{aligned} Y_1 &= V_1 \\ Y_2 &= V_2, \\ Y_3 &= V_3, \\ Y_4 &= V_1 + V_3, \\ Y_5 &= V_3 - V_1. \end{aligned} \tag{49}$$

Proof. Since the combination of vector field (infinitesimal generator) is also a vector field. Consider a linear combination V of V_1, V_2, and V_3,

$$V = \sum_{i=1}^{3} = b_i V_i, \tag{50}$$

a nonzero vector field. Here, for proof, we will improve as many of the coefficient $b_i{}'s$ as possible by using adjoint application on V.□ □

Case 1. Firstly assume that $b_3 \neq 0$ then

$$V = b_1 V_1 + b_2 V_2 + V_3, \tag{51}$$

acting on V with $Adj(\exp(b_2/b_1)V_2)$ by using the adjoint table (adjoint Table 3)

$$V' = Adj\left(\exp \frac{b_2}{b_1} V_2\right) V = b_1 V_1 + V_3. \tag{52}$$

When $b_1 > 0$, then we get Y_4.
When $b_1 < 0$, then we get Y_5.
When $b_1 = 0$, then we get Y_3.

Case 2. Let $b_3 = 0$ and $b_1 = 0$,

$$V = b_2 V_2, \quad (53)$$

when $b_1 = 1$ then we get Y_2, Let $b_3 = 0$, $b_2 = 0$ and b_1, then we get Y_1 There is no any more cases for consultation and the proof is complete.

4. Lie Invariants and Similarity Solutions

We can discover that the invariants correlate with the infinitesimal symmetries (28); they can be determined by solving the equations (by using characteristic method). For $V_2 = \partial t$, the characteristic equation is $dx/0 = dt/1 = du/0$ and the corresponding invariants of this system $x = r$ and $u = w$.

We obtain a similar solution of the form $w = w(r)$, and we put it into equation (21) to obtain the form of the function w, and then, we conclude that $w = w(r) = w(x)$ solution of the following differential equation as similarity reduce equation:

$$w_{rr} + w_{rrrr} + (\lambda - 1)w_r^2 - \sigma w_{rr}^2 = 0. \quad (54)$$

For other example, take $V_3 = \partial x$; the characteristic equation for this has the form $dx/1 = dt/0 = du/0$ so the corresponding invariants are $t = r$ and $u = w$.

Taking into account the last invariants, the following similarity solution is obtained $w = w(r) = w(t)$ where the solution satisfied the similarity reduce equation:

$$e^w w_r = 0. \quad (55)$$

5. Conclusions

The present paper addresses Lie symmetries for some general cases of modified one-dimensional Kuramoto-Sivashinky equation (MKS) as well as its similarity solutions using a symmetry operator. In Section 2, we discussed general results for Lie algebras for some general cases of MKS and provide a comparison between them and obtained some general results. In Section 3, we find the optimal system for (MKS). In the last section, we obtained similarity solutions and Lie invariants.

Remarks. It is worthmentioning that $f(u)$ can be any arbitrary function. For other similar functions chosen as $f(u)$, the procedure for symmetry analysis can be very tedious and symmetry algebra can be different.

Data Availability

Data sharing is not applicable to this article as no dataset was generated or analyzed during the current study

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Blow-Up Results for a Class of Quasilinear Parabolic Equation with Power Nonlinearity and Nonlocal Source

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This paper deals with a class of quasilinear parabolic equation with power nonlinearity and nonlocal source under homogeneous Dirichlet boundary condition in a smooth bounded domain; we obtain the blow-up condition and blow-up results under the condition of nonpositive initial energy.

1. Introduction

In this paper, we consider the following quasilinear parabolic equation with power nonlinearity and nonlocal source term:

$$\begin{cases} u_t = \Delta_p u + \mu u^p \int_\Omega u^{p+1}(y,t)dy - k|u|^{p-2}u, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the standard p-Laplace operator with $p > 2$, $\mu, k > 0$, $u_0(x) \in W_0^{1,p}(\Omega) \setminus \{0\}$.

In the past decades, many physical phenomena have been expressed as nonlocal mathematical models (see [1, 2]). It is also suggested that the nonlocal growth term provides a more realistic model for the physical model of compressible reaction gas. Problem (1) appears in the study of fluid flow through porous media with integral source (see [3, 4]) and population dynamics (see [5, 6]). Actually, equations of the above form are mathematical models occurring in studies of the p-Laplace equation ([7–15] and references therein), generalized reaction-diffusion theory [16], non-Newtonian fluid theory [17, 18], non-Newtonian filtration theory [19, 20], and the turbulent flow of a gas in porous medium [7]. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids. When $p \neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p = 2$ seem to be lose or at least difficult to verify.

Blow-up results of parabolic equations with nonlocal sources have been studied as well. For example, the problem of the form

$$\begin{cases} u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \int_\Omega u^q dx, & (x,t) \in \Omega \times (0,T) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2}$$

was studied by Li and Xie [7]. They established global existence of solutions and discussed the blow-up properties of solutions.

The authors in [8] studied the following p-Laplacian equation with power nonlinearity

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{p-2}u = |u|^{p-2}u \ln|u|, & x \in \Omega, t > 0, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3}$$

By using an efficient technique and according to some sufficient conditions, the global existence and decay estimates of solutions under some sufficient conditions are discussed.

The authors in [9] considered the Neumann problem to the following initial parabolic equation with logarithmic source:

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p-2}u \log|u| - \oint_\Omega |u|^{p-2}u \log|u|dx, & x \in \Omega, t > 0, \\ \dfrac{\partial u(x,t)}{\partial \eta} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4}$$

in a bounded domain with smooth boundary, $p > 2$. By using the logarithmic Sobolev inequality and potential wells method, they obtain the decay, blow-up, and nonextinction of solutions under some conditions.

In [10], the following model of a quasilinear diffusion equation with interior logarithmic source has been studied:

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p-2}u \log|u|, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{5}$$

in which $p > 2$, $u_0(x) \in W_0^{1,p}(\Omega) \setminus \{0\}$. By using the potential well method and a logarithmic Sobolev inequality, the authors obtained results of existence or nonexistence of global weak solution. They also provided sufficient conditions for the large time decay of global weak solutions and for the finite time blow-up of weak solutions. Among some other interesting results, they showed that the weak solution $u(x,t)$ of problem (5) blows up at finite time under the condition $J(u_0) \le M$ and $I(u_0) < 0$, where $M > 0$ is a constant; the energy functional $J(u)$ and Nehari functional $I(u)$ are defined as follows:

$$\begin{aligned} J(u) &= \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \log|u|dx + \frac{1}{p^2}\|u\|_p^p, \\ I(u) &= \|\nabla u\|_p^p - \int_\Omega |u|^p \log|u|dx, \end{aligned} \tag{6}$$

in which $\|\cdot\|_p = \left(\int_\Omega |\cdot|^p dx\right)^{1/p}$.

Motivated by the above studies, in this paper, we investigate blow-up results of problem (1). We will give the conditions for the blow-up results and establish the lower bounds for the blow-up rate. Our main results are as follows.

Theorem 1. *Assume that $J_1(u_0) < 0$. Then, the weak solution $u = u(x,t)$ of problem (1) blows up at finite time.*

Theorem 2. *Assume that $u_0 \in H_0^1(\Omega)$ and $J_1(u_0) < 0$, let $u = u(x,t)$ be the nonnegative solution of problem (1), then u blows up in finite time*

$$T := \frac{\|u_0\|_2^2}{-p(4p+4)J_1(u_0)}. \tag{7}$$

Theorem 3. *Assume that $u_0 \in H_0^1(\Omega)$, $J_1(u_0) \le 0$, $\int_0^{t_0} \|u_s(\cdot,s)\|_2^2 ds > 0$ for any $t_0 > 0$, then, the weak solution $u = u(x,t)$ of problem (1) blows up at infinity. Moreover, if $\|u_0\|_2 \le (-J(u_0))^{2/p}$, the lower bound for blow-up rate can be estimated by*

$$\|u\|_2^2 \ge \|u_0\|_2^2. \tag{8}$$

2. Criterions of Blow-Up

2.1. Preliminaries. In this section, we start with the definition of weak solution and blow-up at infinity of (1).

Definition 4 (weak solution). Let $T > 0$. A function $u = u(x,t) \in L^\infty(0,T;X_0)$ with $u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^2(0,T;L^2(\Omega))$ is called a weak solution to problem (1) in $\Omega \times [0,T)$, if $u(x,0) = u_0(x) \in X_0$ and $u(x,t)$ satisfies (1) in the sense of distribution, i.e.,

$$\langle u_t, \omega \rangle + \left\langle |\nabla u|^{p-2}\nabla u, \nabla \omega \right\rangle + k\left\langle |u|^{p-2}u, \omega \right\rangle = \mu\left\langle u^p \int_\Omega u^{p+1}(y,t)dy, \omega \right\rangle, \tag{9}$$

for all $\omega \in W_0^{1,p}(\Omega), t \in (0,T)$, where $X_0 = W_0^{1,p}(\Omega) \setminus \{0\}$, $W^{-1,\tilde{p}}(\Omega)$ to denote the dual space of $W_0^{1,p}(\Omega)$, and $\tilde{p}$ is Holder conjugate exponent of $p > 1$.

Definition 5 (blow-up at infinity). Let $u(x,t)$ be a weak solution of (1), we call $u(x,t)$ blow-up at $+\infty$ if the maximal existence time $T = +\infty$ and

$$\lim_{t \longrightarrow +\infty} \|u(\cdot,t)\|_2 = \infty. \tag{10}$$

To obtain the blow-up results, define the potential energy functional and the Nehari's functional as follows:

$$\begin{aligned} J_1(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{k}{p}\|u\|_p^p - \frac{\mu}{2p+2}\int_\Omega\int_\Omega u^{p+1}(x,t)u^{p+1}(y,t)dxdy, \\ I_1(u) &= \|\nabla u\|_p^p + k\|u\|_p^p - \mu\int_\Omega\int_\Omega u^{p+1}(x,t)u^{p+1}(y,t)dxdy. \end{aligned} \tag{11}$$

To prove the main result, we need the following lemmas.

Lemma 6. *Assume that $u(x,t)$ is a weak solution of (1). Then, $J_1(u)$ is nonincreasing with respect to t and satisfies the energy inequality*

$$\int_0^t \|u_s(\cdot,s)\|_2^2 ds + J_1(u) \le J_1(u_0), t \in [0, T_0). \tag{12}$$

Proof. Similar to the proof in [8, 10, 11], we can get the result. □

Lemma 7. *[8] $J_1(u)$ is a nonincreasing function, for $t \ge 0$,*

$$J_1'(u) = -\|u_t\|^2 \le 0. \tag{13}$$

Lemma 8 [12]. *Let Φ be a positive, twice differentiable function satisfying the following conditions:*

$$\begin{aligned} &\Phi(\bar{t}) > 0, \\ &\Phi'(\bar{t}) > 0, \end{aligned} \tag{14}$$

for some $\bar{t} \in [0, T)$, and the inequality

$$\Phi(t)\Phi''(t) - \alpha\left(\Phi'(t)\right)^2 \ge 0, \forall t \in [\bar{t}, T], \tag{15}$$

where $\alpha > 1$. Then, we have

$$\Phi(t) \ge \left(\frac{1}{\Phi^{1-\alpha}(\bar{t}) - \tilde{\Phi}(t-\bar{t})}\right)^{1/(\alpha-1)}, t \in [t, T^*], \tag{16}$$

in which $\tilde{\Phi}$ is a positive constant, and

$$T^* = \bar{t} + \frac{\Phi(\bar{t})}{(\alpha-1)\Phi'(\bar{t})}. \tag{17}$$

This implies

$$\lim_{t \longrightarrow T^*} \Phi(t) = +\infty. \tag{18}$$

Lemma 9 [9]. *Suppose that $\theta > 0$, $\alpha > 0$, $\beta > 0$, and $h(t)$ is a nonnegative and absolutely continuous function satisfying $h'(t) + \alpha h^\theta(t) \ge \beta$, then for $0 < t < +\infty$, it holds*

$$h(t) \ge \min\left\{h(0), \left(\frac{\beta}{\alpha}\right)^{1/\theta}\right\}. \tag{19}$$

2.2. Proof of the Main Results. We will consider the finite time blow-up results of problem (1) under the condition of nonpositive initial energy. The theorems are proved as follows.

Proof of Theorem 1. Assume that $u(x,t)$ is the weak solution of problem (1), for any $T > 0$, we define the functional

$$\Gamma(t) = \int_0^t \|u(\cdot,s)\|_2^2 ds + (T-t)\|u_0\|_2^2, t \in [0, T]. \tag{20}$$

It is obvious that $\Gamma(t) > 0$ for all $t \in [0, T]$. Since Γ is continuous, there exists $\rho > 0$ (independent of T) such that $\Gamma(t) > \rho$ for all $t \in [0, T]$.

Then, we have

$$\Gamma'(t) = \|u(\cdot,t)\|_2^2 - \|u_0\|_2^2, \tag{21}$$

$$\Gamma''(t) = 2\int_\Omega u_t u dx = -2I_1(u) \ge -2pJ_1(u). \tag{22}$$

By using (12) in Lemma 7, we have

$$-2pJ_1(u) \ge 2p\int_0^t \|u_s(\cdot,s)\|_2^2 ds - 2pJ_1(u_0). \tag{23}$$

From $J_1(u_0) < 0$, (22) and (23), we get

$$\Gamma''(t) \ge 2p\int_0^t \|u_s(\cdot,s)\|_2^2 ds - 2pJ_1(u_0) > 2p\int_0^t \|u_s(\cdot,s)\|_2^2 ds. \tag{24}$$

Now, multiplying (24) by $\Gamma(t)$, we have

$$\begin{aligned} \Gamma''(t)\Gamma(t) &> 2p\int_0^t \|u_s(\cdot,s)\|_2^2 ds\,\Gamma(t) \\ &= 2p\int_0^t \|u_s(\cdot,s)\|_2^2 ds \int_0^t \|u(\cdot,s)\|_2^2 ds \\ &\quad + 2p(T-t)\|u_0\|_2^2 \int_0^t \|u_s(\cdot,s)\|_2^2 ds. \end{aligned} \tag{25}$$

Noticing that

$$\begin{aligned} \Gamma'(t) &= \|u(\cdot,t)\|_2^2 - \|u_0\|_2^2 \\ &= \int_0^t \frac{d}{ds}\left(\|u(\cdot,s)\|_2^2\right) ds \\ &= 2\int_0^t\int_\Omega u_s u dx ds, \end{aligned} \tag{26}$$

then

$$\left(\Gamma'(t)\right)^2 = 4\left(\int_0^t\int_\Omega u_s u dx ds\right)^2. \tag{27}$$

With the help of Cauchy-Schwarz inequality, we have

$$\left(\Gamma'(t)\right)^2 \le 4\int_0^t \|u_s(\cdot,s)\|_2^2 ds \int_0^t \|u(\cdot,s)\|_2^2 ds. \tag{28}$$

Using (25) and (28), we further get

$$\Gamma''(t)\Gamma(t) - \frac{p}{2}\left(\Gamma'(t)\right)^2 > 2p(T-t)\|u_0\|_2^2\int_0^t \|u_s(\cdot,s)\|_2^2 ds > 0, \tag{29}$$

for all $t \in [0, T]$.

By Lemma 8, there exists $T^* > 0$ such that

$$\lim_{t \longrightarrow T^*} \Gamma(t) = +\infty, \tag{30}$$

which implies

$$\lim_{t \longrightarrow T^*} \int_0^t \|u(\cdot,s)\|_2^2 ds = +\infty. \tag{31}$$

As a consequence, we get

$$\lim_{t \longrightarrow T^*} \|u(\cdot,t)\|_2^2 ds = +\infty, \tag{32}$$

this means $u(x,t)$ blows up at finite time T^*. □

Proof of Theorem 2. Set $G(t) = \|u(\cdot,s)\|_2^2$, then

$$\begin{aligned} G'(t) &= 2\int_\Omega u_t u dx \\ &= 2\left(-\|\nabla u\|_p^p + \mu\int_\Omega\int_\Omega u^{p+1}(x,t)u^{p+1}(y,t)dxdy - k\|u\|_p^p\right) \\ &= -2I_1(u) \\ &\geq F(t), \end{aligned} \tag{33}$$

in which

$$\begin{aligned} F(t) &:= -(4p+4)J_1(u) \\ &= -\frac{4p+4}{p}\|\nabla u\|_p^p - \frac{k(4p+4)}{p}\|u\|_p^p \\ &\quad + 2\mu\int_\Omega\int_\Omega u^{p+1}(x,t)u^{p+1}(y,t)dxdy. \end{aligned} \tag{34}$$

By Lemma 6, we can get

$$F'(t) = -(4p+4)\frac{d}{dt}J_1(u) = (4p+4)\int_\Omega |u_t|^2 dx > 0. \tag{35}$$

From (35) and the condition $J_1(u_0) < 0$, we can get $F(t) > 0$. Then, by Cauchy-Schwarz inequality and (21), the following inequality can be obtained:

$$\begin{aligned} G(t)F'(t) &= (4p+4)\int_\Omega |u|^2 dx \int_\Omega |u_t|^2 dx \\ &\geq (4p+4)\left(\int_\Omega u u_t dx\right)^2 \\ &= (p+1)\left[G'(t)\right]^2 \\ &\geq (p+1)G'(t)F(t), \end{aligned} \tag{36}$$

that is,

$$\frac{F'(t)}{F(t)} \geq (p+1)\frac{G'(t)}{G(t)}. \tag{37}$$

Integrating (37) on $[0, t]$, we can get

$$\ln[F(t)] - \ln[F(0)] \geq (p+1)\ln[G(t)] - (p+1)\ln[G(0)], \tag{38}$$

that is,

$$\frac{F(t)}{[G(t)]^{p+1}} \geq \frac{F(0)}{[G(0)]^{p+1}}. \tag{39}$$

By (33), we have

$$\frac{G'(t)}{[G(t)]^{p+1}} \geq \frac{F(0)}{[G(0)]^{p+1}}. \tag{40}$$

By integrating (40) on $[0, t]$, we can get

$$\frac{1}{[G(t)]^p} \leq \frac{1}{[G(0)]^p} - \frac{pF(0)}{[G(0)]^{p+1}}t = \frac{G(0) - pF(0)t}{[G(0)]^{p+1}}. \tag{41}$$

Take the reciprocal of (41) to get

$$[G(t)]^p \geq \frac{[G(0)]^{p+1}}{G(0) - pF(0)t}. \tag{42}$$

Let

$$T = \frac{G(0)}{pF(0)} = \frac{\|u_0\|_2^2}{-p(4p+4)J_1(u_0)}, \tag{43}$$

when $t \longrightarrow T^-$, we can get $[G(t)]^p \longrightarrow +\infty$; this means that u blows up in a finite time. □

Proof of Theorem 3. Assume that $u(x,t)$ is the weak solution of problem (1). Set $G(t) = \|u(\cdot,s)\|_2^2$, then

$$G'(t) = 2\int_\Omega u_t u dx = -2I_1(u) \geq -2pJ_1(u), \tag{44}$$

By (12) in Lemma 6 and the condition $J_1(u_0) \leq 0$, we can get

$$-2pJ_1(u) \geq 2p\int_0^t \|u_s(\cdot, s)\|_2^2 ds. \tag{45}$$

Then, by (44) and (45), we have

$$G'(t) \geq 2p\int_0^t \|u_s(\cdot, s)\|_2^2 ds > 0. \tag{46}$$

Fix $t_0 > 0$ and let $\kappa = \int_0^t \|u\|_2^2 ds$. By the condition $\int_0^{t_0} \|u_s(\cdot, s)\|_2^2 ds > 0$, we can get κ, which is a positive constant. Integrating (46) over (t_0, t), we can get

$$\begin{aligned} G(t) &\geq G(t_0) + 2p\int_{t_0}^t \int_0^t \|u_s(\cdot, s)\|_2^2 ds d\tau \\ &\geq G(t_0) + 2p\int_{t_0}^t \kappa d\tau \\ &\geq 2p\kappa(t - t_0). \end{aligned} \tag{47}$$

Hence, we have

$$\lim_{t \longrightarrow +\infty} G(t) = \infty. \tag{48}$$

This means that the weak solution $u = u(x, t)$ of problem (1) blows up at infinity.

From (12) and (44), we have

$$\begin{aligned} G'(t) &\geq -2pJ_1(u) \geq -2pJ_1(u_0), \\ G'(t) + G^{p/2}(t) &\geq -2pJ_1(u_0) \geq -J_1(u_0), \end{aligned} \tag{49}$$

for $p > 2$.

By Lemma 9, consider $J_1(u_0) < 0$ and $\|u_0\|_2^2 \leq (-J_1(u_0))^{2/p}$, we have

$$G(t) \geq \min\left\{\|u_0\|_2^2, (-J_1(u_0))^{2/p}\right\} \geq \|u_0\|_2^2, \tag{50}$$

this ends the proof. □

3. Conclusions

In this work, we consider the initial boundary value problem for a class of quasilinear parabolic equation with power nonlinearity and nonlocal source under homogeneous Dirichlet boundary condition in a smooth bounded domain. Some new results of blow-up and blow-up time under the condition of nonpositive initial energy are obtained. The blow-up results of problem (1) with arbitrary initial energy will be the direction of further research.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

Xiaorong Zhang and Zhoujin Cui contributed equally to this work. All authors carried out the proof and conception of the study. All authors read and approved the final manuscript.

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Research Article

On the Multiplicity of Solutions for the Discrete Boundary Problem Involving the Singular ϕ-Laplacian

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In this paper, we consider the multiplicity of solutions for a discrete boundary value problem involving the singular ϕ-Laplacian. In order to apply the critical point theory, we extend the domain of the singular operator to the whole real numbers. Instead, we consider an auxiliary problem associated with the original one. We show that, if the nonlinear term oscillates suitably at the origin, there exists a sequence of pairwise distinct nontrivial solutions with the norms tend to zero. By our strong maximum principle, we show that all these solutions are positive under some assumptions. Moreover, the solutions of the auxiliary problem are solutions of the original one if the solutions are appropriately small. Lastly, we give an example to illustrate our main results.

1. Introduction

Let Z and R denote the sets of integers and real numbers, respectively. For $a, b \in Z$, define $Z(a) = \{a, a+1, \cdots\}$ and $Z(a, b) = \{a, a+1, \cdots, b\}$ when $a \le b$.

In this paper, we consider the following boundary value problem of prescribed mean curvature equations in Minkowski spaces:

$$\begin{cases} \nabla\left(\dfrac{\Delta u_k}{\sqrt{1-(\Delta u_k)^2}}\right) + \lambda f(k, u_k) = 0, & k \in Z(1, T), \\ u_0 = \alpha u_1, \quad u_{T+1} = 0, \end{cases} \tag{1}$$

where T is a given positive integer, α is a constant in $[0, 1]$, ∇ is the backward difference operator defined by $\nabla u_k = u_k - u_{k-1}$, Δ is the forward difference operator defined by $\Delta u_k = u_{k+1} - u_k$, and $f(k, \cdot) \in C(R, R)$ for each $k \in Z(1, T)$.

In 2019, Chen et al. in [1] considered problem (1) in the case where $f(k, x) = \mu_k x^q$ and $\alpha = 1$, that is,

$$\begin{cases} \nabla\left(\dfrac{\Delta u_k}{\sqrt{1-(\Delta u_k)^2}}\right) + \lambda \mu_k (u_k)^q = 0, & k \in Z(1, T), \\ \Delta u_0 = 0 = u_{T+1}. \end{cases} \tag{2}$$

By using upper and lower solutions, the Brouwer degree theory, and Szulkin's critical point theory for convex, lower semicontinuous perturbations of C^1-functions, the authors obtained the intervals of the parameter λ such that problem (2) has zero, one, or two positive solutions. Earlier in 2008, Bereanu and Mawhin in [2] obtained the existence of at least one or two solutions for the boundary value problems of second-order nonlinear differences with singular ϕ-Laplacian by using the Brouwer degree together with fixed point reformulations. For the existence and multiplicity of positive solutions of the associated differential problems to (1), we refer to [3, 4]. And for the boundary value problems of nonsingular differential equations, we refer to [5–9].

Difference equations arise in various research fields. For the existence and multiplicity of solutions of boundary value problems of difference equations, the classical methods are fixed point theory, the method of upper and lower solution techniques, Rabinowitz's global bifurcation theorem, etc. (see [2, 10, 11]). Since 2003, variational methods have been employed to study difference equations [12], by which various results are obtained. See, for example, periodic solutions and subharmonic solutions [13, 14], homoclinic solutions [15–22], heteroclinic solutions [23], and boundary value problems [24–27]. In recent years, boundary value problems of difference equations involving ϕ-Laplacian have aroused extensive attention from scholars; for example, in 2019, Zhou and Ling in [28] considered the following Dirichlet problem of the second-order nonlinear difference equation:

$$\begin{cases} -\Delta(\phi_c(\Delta u_{k-1})) = \lambda f(k, u_k), & k \in Z(1, T), \\ u_0 = u_{T+1} = 0, \end{cases} \tag{3}$$

where ϕ_c is the mean curvature operator defined by $\phi_c(s) = s/\sqrt{1+s^2}$. The authors obtained the existence of infinitely many positive solutions for problem (3). The authors in [29] extended the results of [28] to the following Dirichlet problem:

$$\begin{cases} -\Delta(\phi_c(\Delta u_{k-1})) + q_k\phi_c(u_k) = \lambda f(k, u_k), & k \in Z(1, T), \\ u_0 = u_{T+1} = 0. \end{cases} \tag{4}$$

For the Robin problem of the second-order nonlinear difference equation,

$$\begin{cases} -\Delta\left(\varphi_p(\Delta u_{k-1})\right) + q_k\varphi_p(u_k) = \lambda f(k, u_k), & k \in Z(1, T), \\ \Delta u_0 = u_{T+1} = 0, \end{cases} \tag{5}$$

where φ_p is a special ϕ-Laplacian operator [13] defined by $\varphi_p(s) = p|s|^{p-2}s/2\sqrt{1+|s|^p}$ with $p \geq 2$; we refer to [30].

Up to now, there is less work on the boundary value problems of difference equations involving the singular ϕ-Laplcian; the known results are the existence of at least one or two solutions. The aim of this paper is to obtain the existence of infinitely many solutions for problem (1), and problem (1) contains the Dirichlet problem (when $\alpha = 0$) and Robin problem (when $\alpha = 1$) as special cases. The difficulty lies that the domain of the singular ϕ-Laplcian is a finite open interval $(-a, a)$, not $(-\infty, +\infty)$. The tool is a critical point result in [31]. However, we can not apply the critical point theory to the problem (1) directly, since the domain of the singular ϕ-Laplcian in (1) is $(-1, 1)$; we need to extend the domain of the singular ϕ-Laplcian to $(-\infty, +\infty)$. Instead, we consider the auxiliary problem (20) associated with problem (1) in Section 2. We will show that solutions of problem (20) are solutions of problem (1) if the solutions are appropriately small. For general background on difference equations, we refer the reader to monographs [32, 33].

This paper is organized as follows. In Section 2, an auxiliary problem associated with problem (1) is established, the variational framework associated with this auxiliary problem is established, and the abstract critical point theorem is recalled. In Section 3, our main results are presented. And we also establish a strong maximum principle and obtain the existence of infinitely many positive solutions for (1) according to the oscillating behavior of f at the origin. Finally, in Section 4, an example is given to illustrate our main results.

2. Preliminaries

In this section, we will first introduce a lemma (Theorem 2.5 of [31]).

Let X be a reflexive real Banach space, and let $I_\lambda : X \longrightarrow R$ be a function satisfying the following structure hypothesis:

$(\Lambda) I_\lambda(u) \coloneqq \Phi(u) - \lambda\Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \longrightarrow R$ are two functions of class C^1 on X with Φ coercive, i.e., $\lim_{\|u\|\longrightarrow+\infty}\Phi(u) = +\infty$, and λ is a real positive parameter.

If $\inf_X\Phi < r$, let

$$\mu(r) \coloneqq \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}, \\ \delta \coloneqq \liminf_{r\longrightarrow\left(\inf_X \Phi\right)^+} \mu(r). \tag{6}$$

Obviously, $\delta \geq 0$. When $\delta = 0$, in the sequel, we agree to read $1/\delta$ as $+\infty$.

Lemma 1. *Assume that the condition (Λ) holds, and $\delta < +\infty$; then, for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*

(a_1) there is a global minimum of Φ which is a local minimum of I_λ, or

(a_2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ, with $\lim_{n\longrightarrow+\infty}\Phi(u_n) = \inf_X\Phi$, which weakly converges to the global minimum of Φ.

We will use this lemma to investigate problem (1). Now, we establish the variational framework associated with problem (1). We consider the T-dimensional Banach space.

$$S = \{u : Z(0, T+1) \longrightarrow R : u_0 = \alpha u_1, u_{T+1} = 0\} \tag{7}$$

endowed with the norm

$$\|u\| \coloneqq \left(\sum_{k=1}^{T} (\Delta u_k)^2\right)^{1/2}. \tag{8}$$

We consider another norm in S, that is,

$$\|u\|_\infty := \max\{|u_k|: k \in Z(1,T)\}. \tag{9}$$

For each $u \in S$, there exists a $\tau \in Z(1,T)$, such that

$$\|u\|_\infty = |u_\tau| = \left|\sum_{k=\tau}^{T} \Delta u_k\right| \le \sum_{k=1}^{T} |\Delta u_k| \le T^{1/2}\|u\|, \tag{10}$$

thus,

$$\|u\| \ge \frac{1}{T^{1/2}}\|u\|_\infty. \tag{11}$$

We mention that the equality in (11) holds if we let $u_k = (T+1-k)c, k \in Z(1,T), u_0 = \alpha T c$, where c is a nonzero constant. In fact, in this case, $\|u\| = T^{1/2}|c|$ and $\|u\|_\infty = T|c|$.

We notice that the singular operator $\phi(s) = s/\sqrt{1-s^2}$ in problem (1) only defined for $s \in (-1,1)$. In order to use Lemma 1, we need to extend the domain of the singular operator ϕ to $(-\infty,\infty)$. Take

$$g(s) = \begin{cases} \dfrac{s}{\sqrt{1-s^2}}, & |s| \le \dfrac{\sqrt{3}}{2}, \\ 2s, & |s| > \dfrac{\sqrt{3}}{2}. \end{cases} \tag{12}$$

Then, $g(s)$ is continuous in $(-\infty, +\infty)$, and the primary function of g is given by

$$G(s) = \begin{cases} \dfrac{s}{1+\sqrt{1-s^2}}, & |s| \le \dfrac{\sqrt{3}}{2}, \\ s^2 - \dfrac{1}{4}, & |s| > \dfrac{\sqrt{3}}{2}. \end{cases} \tag{13}$$

We define

$$\begin{aligned} \Phi(u) &= \sum_{k=0}^{T} G(\Delta u_k) + \frac{\alpha G(\Delta u_0)}{1-\alpha}, \\ \Psi(u) &= \sum_{k=1}^{T} F(k, u_k), \end{aligned} \tag{14}$$

for each $u \in S$, where $F(k,u) = \int_0^u f(k,\tau)d\tau$ for every $k \in Z(1,T)$. When $\alpha = 1$, we read $(\alpha G(\Delta u_0))/(1-\alpha)$ as 0 in (14), since

$$\lim_{\alpha \longrightarrow 1} \frac{\alpha G(\Delta u_0)}{1-\alpha} = \lim_{\alpha \longrightarrow 1} \frac{\alpha(1-\alpha)^2 u_1^2}{(1-\alpha)(1+\sqrt{1-(1-\alpha)^2 u_1^2})} = 0. \tag{15}$$

Put

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u), \tag{16}$$

for $u \in S$. Then, Φ and Ψ are two functionals of class $C^1(S,R)$ whose Gâteaux derivatives at the point $u \in S$ are given by

$$\begin{aligned} \Phi'(u)(v) &= \sum_{k=0}^{T} g(\Delta u_k)\Delta v_k + \frac{\alpha g(\Delta u_0)\Delta v_0}{1-\alpha}, \\ \Psi'(u)(v) &= \sum_{k=1}^{T} f(k,u_k)v_k, \end{aligned} \tag{17}$$

for all $u, v \in S$. It is clear that

$$\begin{aligned} \sum_{k=0}^{T} g(\Delta u_k)\Delta v_k &= \sum_{k=0}^{T} g(\Delta u_k)v_{k+1} - \sum_{k=0}^{T} g(\Delta u_k)v_k \\ &= \sum_{k=1}^{T} g(\Delta u_{k-1})v_k - \sum_{k=1}^{T} g(\Delta u_k)v_k - g(\Delta u_0)v_0 \\ &= -\sum_{k=1}^{T} \nabla(g(\Delta u_k))v_k - \alpha g(\Delta u_0)v_1 \\ &= -\sum_{k=1}^{T} \nabla(g(\Delta u_k))v_k - \frac{\alpha g(\Delta u_0)\Delta v_0}{1-\alpha}, \end{aligned} \tag{18}$$

then,

$$\left[\Phi'(u) - \lambda\Psi'(u)\right](v) = -\sum_{k=1}^{T} [\nabla(g(\Delta u_k)) + \lambda f(k,u_k)]v_k. \tag{19}$$

Consequently, the critical points of I_λ in S are exactly the solutions of the following boundary value problem:

$$\begin{cases} \nabla(g(\Delta u_k)) + \lambda f(k,u_k) = 0, & k \in Z(1,T), \\ u_0 = \alpha u_1, u_{T+1} = 0. \end{cases} \tag{20}$$

Remark 2. If $u \in S$ is a solution of problem (20) with $|\Delta u_k| \le \sqrt{3}/2$ for $k \in Z(0,T)$, then, u is a solution of problem (1).

3. Main Results

First, we consider the existence of nontrivial solutions for problem (1). Let

$$B := \limsup_{t \longrightarrow 0^+} \frac{\sum_{k=1}^{T} F(k,t)}{t^2}. \tag{21}$$

We have the following:

Theorem 3. *Assume that there exist two real sequences $\{u_n\}$ and $\{v_n\}$, with $v_n > 0$ and $\lim_{n \longrightarrow +\infty} v_n = 0$, such that*

$$\frac{u_n^2}{1+\sqrt{1-u_n^2}} + \frac{(1-\alpha)u_n^2}{1+\sqrt{1-(1-\alpha)^2 u_n^2}} < \frac{v_n^2}{2T}, \quad n \in Z(1), \tag{22}$$

$$\gamma := \liminf_{n\longrightarrow\infty} \frac{\sum_{k=1}^{T} \max_{|t|\le v_n} F(k,t) - \sum_{k=1}^{T} F(k,u_n)}{v_n^2/2T - \left(u_n^2/\left(1+\sqrt{1-u_n^2}\right)\right) - \left(((1-\alpha)u_n^2)/\left(1+\sqrt{1-(1-\alpha)^2 u_n^2}\right)\right)} < \frac{2B}{2-\alpha}. \tag{23}$$

Then, for each $\lambda \in ((2-\alpha)/2B, 1/\gamma)$, problem (1) admits a sequence of nontrivial solutions which converges to zero.

Remark 4. A sequence $\{u^{(n)}\}$ in S is said to converge to zero if $\|u^{(n)}\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. Take Φ and Ψ as defined by (14); we will prove Theorem 3 by using Lemma 1. Since $\lim_{|s|\longrightarrow+\infty} G(s) = \lim_{|s|\longrightarrow+\infty} s^2 - (1/4) = +\infty$, it is easy to see that $\lim_{\|u\|\longrightarrow+\infty} \Phi(u) = +\infty$, and (Λ) is satisfied. Put

$$r_n = \frac{v_n^2}{2T}. \tag{24}$$

Since $\lim_{n\longrightarrow\infty} v_n = 0$, there is no harm in assuming that $v_n \le \sqrt{T}$; then, $r_n \le 1/2$. If $u \in S$ and

$$\Phi(u) = \sum_{k=0}^{T} G(\Delta u_k) + \frac{\alpha G(\Delta u_0)}{1-\alpha} < r_n, \tag{25}$$

then, $G(\Delta u_k) < (1/2)$ for $k \in Z(0,T)$. Noting that $G(s) \ge 1/2$ for $|s| \ge \sqrt{3}/2$, we see that $|\Delta u_k| < \sqrt{3}/2$ for $k \in Z(0,T)$ and $\Phi(u)$ takes the form

$$\Phi(u) = \sum_{k=0}^{T} \frac{(\Delta u_k)^2}{1+\sqrt{1-(\Delta u_k)^2}} + \frac{\alpha(1-\alpha)u_1^2}{1+\sqrt{1-(1-\alpha)^2 u_1^2}}. \tag{26}$$

Therefore,

$$\frac{1}{2}\sum_{k=1}^{T} (\Delta u_k)^2 \le \Phi(u) < r_n, \tag{27}$$

which implies that $\|u\| < \sqrt{2r_n}$. By (11), we have

$$\|u\|_\infty < \sqrt{2Tr_n} = v_n. \tag{28}$$

According to the definition of μ, we have

$$\mu(r_n) \le \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\sum_{k=1}^{T} \max_{|t|\le v_n} F(k,t) - \sum_{k=1}^{T} F(k,u_k)}{(v_n^2/2T) - \Phi(u)}. \tag{29}$$

For each $n \in Z(1)$, let $w_n \in S$ be defined by $(w_n)_k = u_n$ for every $k \in Z(1,T)$ and $(w_n)_0 = \alpha(w_n)_1$, $(w_n)_{T+1} = 0$. Then,

$$\Phi(w_n) = \frac{u_n^2}{1+\sqrt{1-u_n^2}} + \frac{(1-\alpha)u_n^2}{1+\sqrt{1-(1-\alpha)^2 u_n^2}} < \frac{v_n^2}{2T} = r_n, \tag{30}$$

by using (22). Thus,

$$\mu(r_n) \le \frac{\sum_{k=1}^{T} \max_{|t|\le v_n} F(k,t) - \sum_{k=1}^{T} F\left(k,(w_n)_k\right)}{(v_n^2/2T) - \Phi(w_n)} = \frac{\sum_{k=1}^{T} \max_{|t|\le v_n} F(k,t) - \sum_{k=1}^{T} F(k,u_n)}{(v_n^2/2T) - \left(u_n^2/\left(1+\sqrt{1-u_n^2}\right)\right) - \left(((1-\alpha)u_n^2)/\left(1+\sqrt{1-(1-\alpha)^2 u_n^2}\right)\right)}. \tag{31}$$

Therefore, by (23), we know that $\delta \le \liminf_{n\longrightarrow+\infty} \mu(r_n) \le \gamma < +\infty$.

To get our results, we need to show that conclusion (a_2) of Lemma 1 holds. Therefore, we want to show that the global minimum $u \equiv 0$ of Φ is not a local minimum of I_λ. To prove this, we consider two cases: $B = +\infty$ and $B < +\infty$. In the case where $B = +\infty$, let $\{t_n\}$ be a sequence of positive numbers, with $t_n \in (0, \sqrt{3}/2)$ and $\lim_{n\longrightarrow+\infty} t_n = 0$, such that

$$\sum_{k=1}^{T} F(k, t_n) \geq \frac{3t_n^2}{\lambda}, \quad n \in Z(1). \tag{32}$$

Defining a sequence $\{\theta_n\}$ in S by $(\theta_n)_k = t_n$ for $k \in Z(1, T)$ and $(\theta_n)_0 = \alpha(\theta_n)_1, (\theta_n)_{T+1} = 0$, we have

$$\begin{aligned} I_\lambda(\theta_n) &= \frac{t_n^2}{1+\sqrt{1-t_n^2}} + \frac{(1-\alpha)t_n^2}{1+\sqrt{1-(1-\alpha)^2 t_n^2}} - \lambda \sum_{k=1}^{T} F(k, t_n) \\ &= -(1+\alpha)t_n^2 < 0. \end{aligned} \tag{33}$$

In the case where $B < +\infty$, since $\lambda > (2-\alpha)/2B$, we can choose a small number $\varepsilon \in (0, \sqrt{3}/2)$ such that

$$\frac{1}{1+\sqrt{1-\varepsilon^2}} + \frac{(1-\alpha)}{1+\sqrt{1-(1-\alpha)^2\varepsilon^2}} < \lambda(B-\varepsilon). \tag{34}$$

By the definition of B, we can find a sequence of real numbers $\{\tau_n\}$ with $\tau_n \in (0, \varepsilon)$ such that $\lim_{n\longrightarrow+\infty}\tau_n = 0$ and

$$\sum_{k=1}^{T} F(k, \tau_n) \geq (B-\varepsilon)\tau_n^2. \tag{35}$$

Defining a sequence $\{\xi_n\}$ in S by $(\xi_n)_k = \tau_n$ for $k \in Z(1, T)$ and $(\xi_n)_0 = \alpha(\xi_n)_1, (\xi_n)_{T+1} = 0$, we have

$$\begin{aligned} I_\lambda(\xi_n) &= \frac{\tau_n^2}{1+\sqrt{1-\tau_n^2}} + \frac{(1-\alpha)\tau_n^2}{1+\sqrt{1-(1-\alpha)^2\tau_n^2}} - \lambda \sum_{k=1}^{T} F(k, \tau_n) \\ &\leq \left(\frac{1}{1+\sqrt{1-\varepsilon^2}} + \frac{(1-\alpha)}{1+\sqrt{1-(1-\alpha)^2\varepsilon^2}} - \lambda(B-\varepsilon)\right)\tau_n^2 \\ &< 0. \end{aligned} \tag{36}$$

Noticing that $I_\lambda(0) = 0$, we see that $u \equiv 0$ is not a local minimum of I_λ by combining the above two cases. Therefore, by Lemma 1 and Remark 2, we know the conclusion of Theorem 3 holds.

Now, let

$$B_* = \liminf_{t\longrightarrow 0^+} \frac{\sum_{k=1}^{T} \max_{|s|\leq t} F(k, s)}{t^2}. \tag{37}$$

Then, there exists a sequence $\{v_n\}$ of positive numbers with $\lim_{n\longrightarrow+\infty} v_n = 0$ such that

$$\lim_{n\longrightarrow\infty} \frac{\sum_{k=1}^{T} \max_{|t|\leq v_n} F(k, t)}{v_n^2} = B_*. \tag{38}$$

Taking $u_n = 0$ for all $n \in Z(1)$, by Theorem 3, we get the following corollary.

Corollary 5. *If*

$$TB_* < \frac{B}{2-\alpha}. \tag{39}$$

Then, for each $\lambda \in ((2-\alpha)/2B, 1/2TB_*)$, problem (1) admits a sequence of nontrivial solutions which converges to zero.

To obtain the positive solutions of problem (20), we need the following strong maximum principle.

Theorem 6. *Assume $u \in S$ such that either*

$$u_k > 0 \text{ or} \nabla(g(\Delta u_k)) \leq 0, \tag{40}$$

for all $k \in Z(1, T)$. Then, either $u_k > 0$ for all $k \in Z(1, T)$ or $u \equiv 0$.

Proof. There exists $\tau \in Z(1, T)$ such that

$$u_\tau = \min\{u_k : k \in N(1, T)\}. \tag{41}$$

If $u_\tau > 0$, then $u_k > 0$ for all $k \in Z(1, T)$ and the proof is complete.

If $u_\tau \leq 0$, then $u_\tau = \min\{u_k : k \in N(0, T+1)\}$. Because $\Delta u_{\tau-1} = u_\tau - u_{\tau-1} \leq 0$ and $\Delta u_\tau = u_{\tau+1} - u_\tau \geq 0$, $g(s)$ is increasing in s, and $g(0) = 0$, we have

$$g(\Delta u_\tau) \geq 0 \geq g(\Delta u_{\tau-1}). \tag{42}$$

On the other hand, let $k = \tau$; (40) implies

$$g(\Delta u_\tau) \leq g(\Delta u_{\tau-1}). \tag{43}$$

By combining (42) with (43), we get $g(\Delta u_\tau) = 0 = g(\Delta u_{\tau-1})$. That is $u_{\tau+1} = u_{\tau-1} = u_\tau$. If $\tau + 1 = T + 1$, we have $u_\tau = 0$. Otherwise, $\tau + 1 \in N(1, T)$. Replacing τ by $\tau + 1$, we get $u_{\tau+2} = u_{\tau+1}$. Continuing this process $T + 1 - \tau$ times, we have $u_{\tau+j} = 0$ for $j \in Z(0, T+1-\tau)$. Similarly, we have $u_j = 0$ for $j \in Z(0, \tau)$. Thus, $u \equiv 0$ and the proof is complete.

Now, we are ready to establish the existence of positive solutions for problem (1); we have

Corollary 7. *If $f(k, 0) \geq 0$ for all $k \in Z(1, T)$,*

$$A_* := \liminf_{t\longrightarrow 0^+} \frac{2T\sum_{k=1}^{T} \max_{0\leq s\leq t} \int_0^s f(k, x)dx}{t^2} < \frac{2B}{2-\alpha}. \tag{44}$$

Then, for each $\lambda \in ((2-\alpha)/2B, 1/A_*)$, problem (1) admits a sequence of positive solutions which converges to zero.

Proof. Put

$$f^*(k,x)=\begin{cases} f(k,x), & \text{if } x>0,\\ f(k,0), & \text{if } x\le 0.\end{cases} \tag{45}$$

Noticing that $f(k,0)\ge 0$, we see that

$$\max_{0\le|s|\le t}\int_0^s f^*(k,x)dx=\max_{0\le s\le t}\int_0^s f(k,x)dx, \tag{46}$$

for all $t\ge 0$. By Corollary 5, we know that problem (1) with f replaced by f^* admits a sequence of nontrivial solutions which converges to zero for each $\lambda\in((2-\alpha)/2B,1/A_*)$. And by Theorem 6, we know that all these solutions are positive.

4. An Example

In this section, we give an example to illustrate our main results.

Example 8. Consider the boundary value problem (1) with

$$f(k,x)=f(x)=\begin{cases} x(2+2\varepsilon+2\sin(\varepsilon\ln|x|)+\varepsilon\cos(\varepsilon\ln|x|)), & x\ne 0,\\ 0, & x=0,\end{cases} \tag{47}$$

for $k\in Z(1,T)$. Then,

$$F(k,x)=F(x)=\int_0^x f(s)ds=x^2(1+\varepsilon+\sin(\varepsilon\ln x)),\quad \text{for } x>0. \tag{48}$$

Since $f(x)\ge 0$ for $x\ge 0$, we see that $F(x)$ is increasing in $x\in[0,+\infty)$. Thus,

$$\begin{aligned} A_* &:= \liminf_{t\longrightarrow 0^+}\frac{2T\sum_{k=1}^{T}\max_{0\le s\le t}\int_0^s f(k,x)dx}{t^2}\\ &= \liminf_{t\longrightarrow 0^+} 2T^2(1+\varepsilon+\sin(\varepsilon\ln t))=2T^2\varepsilon,\\ B &:= \limsup_{t\longrightarrow 0^+}\frac{\sum_{k=1}^{T}F(k,t)}{t^2}\\ &= \limsup_{t\longrightarrow 0^+} T(1+\varepsilon+\sin(\varepsilon\ln t))\\ &= T(2+\varepsilon). \end{aligned} \tag{49}$$

Let $\varepsilon\in(0,2/((2T-\alpha T-1)))$; then (44) holds. By Corollary 7, for each $\lambda\in((2-\alpha)/(2T(2+\varepsilon)),1/2T^2\varepsilon)$, problem (1) admits a sequence of positive solutions which converges to zero.

5. Conclusions

In this paper, we consider a discrete boundary value problem involving the singular ϕ-Laplacian. The problem contains the Dirichlet problem (when $\alpha=0$) and Robin problem (when $\alpha=1$) as special cases. Since the domain of the singular operator ϕ is $(-1,1)$, we can not apply the critical point theory to this problem directly. Therefore, we extend the domain of the singular operator to the whole real numbers and consider an auxiliary problem associated with the original one. The conditions for the multiplicity of positive solutions of the discrete boundary problem are found, and an illustrative example is given. The method in this paper provides a new way to discuss the boundary value problems containing a singular Laplacian.

Data Availability

No data were used to support the study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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Research Article

Φ-Haar Wavelet Operational Matrix Method for Fractional Relaxation-Oscillation Equations Containing Φ-Caputo Fractional Derivative

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This paper proposes a numerical method for solving fractional relaxation-oscillation equations. A relaxation oscillator is a type of oscillator that is based on how a physical system returns to equilibrium after being disrupted. The primary equation of relaxation and oscillation processes is the relaxation-oscillation equation. The fractional derivatives in the relaxation-oscillation equations under consideration are defined in the Φ-Caputo sense. The numerical method relies on a novel type of operational matrix method, namely, the Φ-Haar wavelet operational matrix method. The operational matrix approach has a lower computational complexity. The proposed scheme simplifies the main problem to a set of linear algebraic equations. Numerical examples demonstrate the validity and applicability of the proposed technique.

1. Introduction

The history of fractional or noninteger order differential and integral operators can be traced back to the origins of integer order calculus [1]. In recent years, fractional differential equations have attracted a lot of attention. In fields such as damping laws, diffusion processes, and other physical phenomena, fractional differential equations have proven to be adequate models. Since the majority of fractional differential equations do not have analytical solutions, we must use an approximate method. Many studies have analyzed solution techniques for fractional differential equation such as the collocation method, Adomian decomposition method, variational iteration method, tau method, and operational matrix method [2–12].

A relaxation oscillator is a type of oscillator that is based on a physical system's ability to return to equilibrium after being disrupted. The main equation of relaxation and oscillation processes is the relaxation-oscillation (R-O) equation. A relaxation equation in its standard form is given by

$$u'(\chi) + \mu u(\chi) = g(\chi), \tag{1}$$

where μ is a real number and g is a given function. Equation (1) can be used to represent a variety of physical processes, such as the Maxwell model, which uses a spring and a dashpot in succession to explain the characteristics of a viscoelastic material. A simple physical process with a regulated phase shift is described by the standard oscillation equation. The equation that defines the oscillation u of a system corresponding to an external force g has the simplest linear form as

$$u''(\chi) + \mu u(\chi) = g(\chi), \tag{2}$$

where μ is the oscillator's natural frequency. In the relaxation and oscillation models given in Equations (1) and (2), fractional derivatives are used to depict slow relaxation and damped oscillation (see [13, 14]). The fractional relaxation-oscillation differential equation (FRODE) is given by

$$\mathbb{D}^{\alpha} u(\chi) + \mu u(\chi) = g(\chi), \chi > 0, 0 < \alpha < 2 \quad \text{where } \alpha \neq 1, \tag{3}$$

having the following initial conditions: $u(0) = u_0$, if $0 < \alpha < 1$, and $u'(0) = u_1$, if $1 < \alpha < 2$, where $\mathbb{D}^{\alpha}$ denotes a fractional differential operator of order α.

The numerical investigation of FRODEs has received a lot of interest recently. The numerical solution of problem (3) (with $g(\chi) = 0$) was investigated in [15] by taking into account the positive fractional and fractal derivatives. The authors of [16] employed a Taylor matrix method to find the numerical solution of problem (3) by taking into account the Caputo fractional derivative. This approach is based on a fractional version of Taylor's formula, which was first proposed in [17]. The numerical solution of problem (3) is achieved by the optimal homotopy asymptotic approach in [18], where the fractional derivative is given in the Caputo sense. In [19], a trapezoidal approximation of the fractional integral is used to get the numerical solution of problem (3) with Caputo fractional derivative. To solve problem (3), [20] proposes a generalized wavelet collocation operational matrix approach based on the Haar wavelet (HW), where the fractional derivative is represented in the Caputo sense. Inspired by the above-mentioned studies, this paper focuses on a numerical solution of the fractional differential equation of the form

$$\mathbb{D}^{\alpha,\Phi} u(\chi) + \mu u(\chi) = g(\chi), \quad \chi \in [a, b], \tag{4}$$

with initial conditions $(d_\Phi)^{\mathfrak{n}} u(a) = u_{\mathfrak{n}}, n = 0, 1, 2, \cdots, \{\mathfrak{m} - 1$, where $\max\{\mathfrak{m} - 1, 1/2\} < \alpha < \mathfrak{m}$, and $\mathfrak{m}$ is a natural number. $\mathbb{D}^{\alpha,\Phi}$ is the Φ-Caputo fractional derivative of order α, and

$$(d_\Phi)^{\mathfrak{n}} u(\chi) = \begin{cases} u(\chi), & \text{if } \mathfrak{n} = 0, \\ \left(\dfrac{1}{\Phi'}\dfrac{d}{d\chi}\right)^{\mathfrak{n}} u(\chi), & \text{if } \mathfrak{n} = 1, 2, 3, \cdots, \mathfrak{m} - 1. \end{cases} \tag{5}$$

These problems are studied in [21] by using an operational matrix of Φ-shifted Legendre polynomials.

As far as we know, there is no open literature article dealing with the numerical treatment of FRODEs involving the Φ-Caputo fractional derivative employing HW. Therefore, the basic aim of this paper is to provide a numerical technique for solving Φ-FRODEs that arise in physics. Our method is based on a new type of operational matrix of fractional integration called the Φ-HW operational matrix. We provide a rigorous verification of convergence for the suggested method. Furthermore, numerical experiments are presented to demonstrate the convergence of the procedure by comparing the exact values to numerical approximation.

Different types of orthogonal polynomials, including Chebyshev polynomials [22], Legendre polynomials [23], and Laguerre and Hermite polynomials [24], have been utilized with the operational matrix of integer order integration. Many authors then expanded it to the fractional case, as seen in [25–31] and the references therein. Only Caputo or Riemann-Liouville fractional derivatives were examined in all of the above listed papers.

The following is a description of this paper's structure.

In Section 2, we go through the basic fundamentals of fractional calculus. We introduce HW and function approximation using HW in Section 3. In Section 4, we construct an explicit formula for the Φ-fractional integration of the HW and the Φ-Haar wavelet operational matrix. Section 5 discusses the numerical scheme as well as the method's convergence.

2. Preliminaries

We will go over some definitions of Φ-fractional integral and differential operators in this section.

Let the function $g : [a_1, a_2] \longrightarrow \mathbb{R}$ be integrable, α is a positive real number, n is a natural number, $\Phi \in C^1([a_1, a_2])$, and $\Phi'(\chi) \neq 0 \forall \chi \in [a_1, a_2]$, where Φ is increasing.

Definition 1 (see [32–34]). The Φ-Riemann-Liouvile (Φ-RL) fractional integral operator of order α is defined by

$$\mathcal{J}_{a_1}^{\alpha,\Phi} g(\chi) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{\chi} \Phi'(s)(\Phi(\chi) - \Phi(s))^{\alpha-1} g(s) ds. \tag{6}$$

The Φ-RL differential operator of fractional α is defined by

$$\begin{aligned} \mathbb{D}_{a_1}^{\alpha,\Phi} g(\chi) &= \left(\frac{1}{\Phi'(\chi)}\frac{d}{d\chi}\right)^{\mathfrak{n}} \mathcal{J}_{a_1}^{\mathfrak{n}-\alpha,\Phi} g(\chi) \\ &= \frac{1}{\Gamma(\mathfrak{n} - \alpha)} \left(\frac{1}{\Phi'(\chi)}\frac{d}{d\chi}\right)^{\mathfrak{n}} \int_{a_1}^{\chi} \Phi'(s)(\Phi(\chi) \\ &\quad - \Phi(s))^{\mathfrak{n}-\alpha-1} g(s) ds, \quad \text{where } \mathfrak{n} = \lfloor \alpha \rfloor + 1. \end{aligned} \tag{7}$$

Definition 2 (see [21, 35, 36]). Let α be a positive real number, $\mathfrak{n}$ a natural number, $g, \Phi \in C^{\mathfrak{n}}([a_1, a_2])$, and $\Phi'(\chi) \neq 0 \forall \chi \in [a_1, a_2]$, where Φ is increasing. The Φ-Caputo differential operator of fractional-order α is defined by

$${}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi} g(\chi) = \frac{1}{\Gamma(\mathfrak{n} - \alpha)} \int_{a_1}^{\chi} \Phi'(s)(\Phi(\chi) - \Phi(s))^{\mathfrak{n}-\alpha-1} \mathbb{D}^{\mathfrak{n},\Phi} g(s) ds, \tag{8}$$

where $\mathbb{D}^{\mathfrak{n},\Phi}g(\chi)=((1/\Phi'(\chi))(d/d\chi))^{\mathfrak{n}}g(\chi)$, $\mathfrak{n}=\lfloor\alpha\rfloor+1$ for $\alpha\notin\mathbb{N}$ and $\mathfrak{n}=\alpha$ when $\alpha\in\mathbb{N}$.

2.1. Function Approximation by Haar Wavelet. HW are the simplest wavelets with a compact support among the various wavelet families. These wavelets have been shown to be an effective method for numerical function approximation. The Haar functions, which are orthogonal, contain only one wavelet during some subinterval of time and remain zero elsewhere.

The ith HW, $h_i(\chi)$, where $\chi\in[a_1,a_2]$ in the HW family is defined as [37]

$$h_i(\chi)=\begin{cases}1, & \text{when } \chi\in\left[a_1+(a_2-a_1)\dfrac{\kappa}{\mathfrak{m}},a_1+(a_2-a_1)\dfrac{2\kappa+1}{2\mathfrak{m}}\right),\\ -1, & \text{when } \chi\in\left[a_1+(a_2-a_1)\dfrac{2\kappa+1}{2\mathfrak{m}},a_1+(a_2-a_1)\dfrac{\kappa+1}{\mathfrak{m}}\right),\\ 0, & \text{otherwise,}\end{cases} \tag{9}$$

where $\mathfrak{m}=2^j, j=0,1,2,3,\cdots,J$, is the dilation parameter and $\kappa=0,1,2,3,\cdots,\mathfrak{m}-1$ is the translation parameter. J is the max resolution level of HW. The parameters i, j, and κ are related by the equation $i=2^j+\kappa+1$; i is called the wavelet number. Equation (9) holds true for $i\geq 3$.

For $i=1$ and $i=2$, the scaling functions for HW family are defined, respectively, by

$$\begin{aligned}h_1(\chi)&=\begin{cases}1, & \text{when } \chi\in[a,b),\\ 0, & \text{otherwise,}\end{cases}\\ h_2(\chi)&=\begin{cases}1, & \text{when } \chi\in\left[a,\dfrac{a+b}{2}\right),\\ -1, & \text{when } \chi\in\left[\dfrac{a+b}{2},b\right),\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{10}$$

A square integrable function on (a,b), $u(\chi)$ can be approximated by HW in the following way:

$$u(\chi)=\sum_{i=0}^{\infty}c_ih_i(\chi), \tag{11}$$

where c_i is defined by the inner product of $u(\chi)$ and $h_i(\chi)$ and $\langle . \rangle$ represents the inner product. The first $\mathfrak{m}$ terms are employed for function approximation, that is,

$$u(\chi)\cong u_{\mathfrak{m}}(\chi)=\sum_{i=0}^{\mathfrak{m}-1}c_ih_i(\chi), \tag{12}$$

which can be written in the matrix notation as

$$u(\chi)\cong u_{\mathfrak{m}}(\chi)=C_{\mathfrak{m}}^TH_{\mathfrak{m}}(\chi), \tag{13}$$

where $C=[c_0,c_1,c_2,\cdots,c_{\mathfrak{m}-1}]^T$ is the coefficient matrix determined by $c_i=\langle u(\chi),h_i(\chi)\rangle$ and $H=[h_0(\chi),h_1(\chi),h_2(\chi),\cdots,h_{\mathfrak{m}-1}(\chi)]^T$ is the vector of Haar functions.

3. Φ-HW Operational Matrix

The fractional-ordered Φ-RL integral of the HW is defined by

$$\begin{aligned}\mathcal{J}^{\alpha,\Phi}h_1(\chi)&=\frac{1}{\Gamma(\alpha+1)}[\Phi(\chi)-\Phi(a_1)]^\alpha,\\ P_l^{\alpha,\Phi}(\chi)=\mathcal{J}^{\alpha,\Phi}h_l(\chi)&=\frac{1}{\Gamma(\alpha)}\int_{a_1}^{\chi}\Phi'(s)(\Phi(\chi)-\Phi(s))^{\alpha-1}h_i(s)ds\\ &=\frac{1}{\Gamma(\alpha+1)}\begin{cases}0, & \text{if } \chi<\zeta_1(l),\\ (\Phi(\chi)-\Phi(\zeta_1(l)))^\alpha, & \text{if } \chi\in[\zeta_1(l),\zeta_2(l)),\\ (\Phi(\chi)-\Phi(\zeta_1(l)))^\alpha-2(\Phi(\chi)-\Phi(\zeta_2(l)))^\alpha, & \text{if } \chi\in(\zeta_2(l),\zeta_3(l)],\\ (\Phi(\chi)-\Phi(\zeta_1(l)))^\alpha-2(\Phi(\chi)-\Phi(\zeta_2(l)))^\alpha+(\Phi(\chi)-\Phi(\zeta_3(l)))^\alpha, & \text{if } \chi>\zeta_3(l),\end{cases}\end{aligned} \tag{14}$$

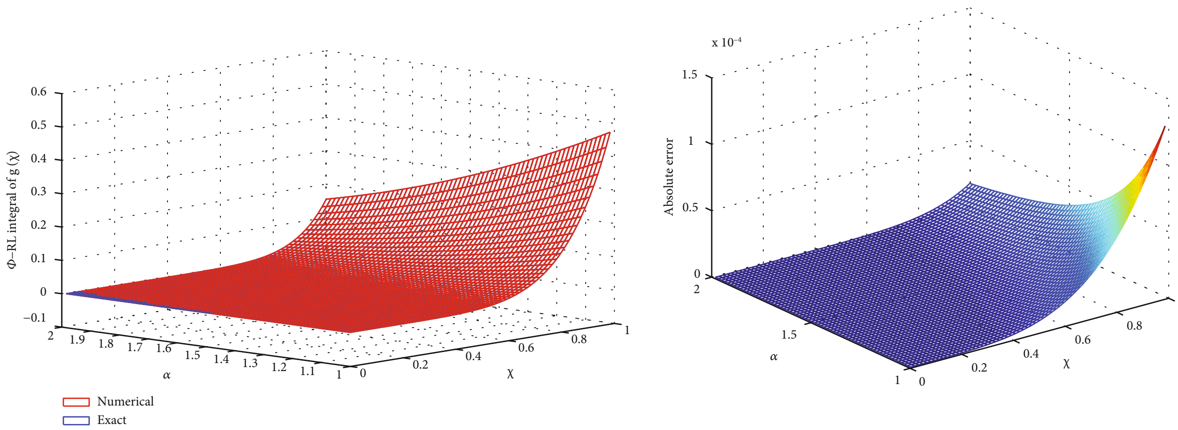


FIGURE 1: The exact and numerical Φ-RL integral of $g(\chi)=\Phi(\chi)$ for $J=5$ and $1<\alpha\leq 2$ and their absolute error.

where $\zeta_1(l)=a_1+(a_2-a_1)(\kappa/\mathfrak{m}), \zeta_2(l)=a_1+(a_2-a_1)((2\kappa+1)/2\mathfrak{m}), \zeta_3(l)=a_1+(a_2-a_1)((\kappa+1)/\mathfrak{m})$.

The Φ-HW operational matrix $P^{\alpha,\Phi}$ is computed in the interval $[0, 1]$ for $\Phi(\chi)=\chi^3$ and $\alpha=0.75$. The numerical and exact Φ-RL integration of the function $\Phi(\chi)=\chi^3$ for $J=6$ and different values of α is plotted in Figure 1.

$$P^{\alpha,\Phi}=\begin{bmatrix} 0.3331 & -0.2641 & -0.0553 & -0.2141 & -0.0115 & -0.0449 & -0.0850 & -0.1295 \\ -0.1258 & 0.1948 & -0.0553 & 0.2584 & -0.0115 & -0.0449 & 0.1201 & 0.1433 \\ -0.0662 & 0.0395 & 0.0403 & -0.0179 & -0.0115 & 0.0537 & -0.0146 & -0.0053 \\ 0.0386 & -0.0386 & 0 & 0.1455 & 0 & 0 & -0.1026 & 0.1988 \\ -0.0130 & -0.0007 & 0.0080 & -0.0016 & 0.0078 & -0.0035 & -0.0010 & -0.0005 \\ -0.0175 & 0.0251 & -0.0076 & -0.0088 & 0 & 0.0310 & -0.0081 & -0.0022 \\ -0.0051 & 0.0051 & 0 & 0.0680 & 0 & 0 & 0.0623 & -0.0135 \\ 0.0285 & -0.0285 & 0 & -0.0571 & 0 & 0 & 0 & 0.0999 \end{bmatrix}. \tag{15}$$

4. Error Analysis

Caputo-type FDEs have recently been investigated in context of error analysis in [38]. In addition, the convergence analysis of solution of nonlinear Fredholm integral equations by HW is given in [39].

Using the Φ-Caputo fractional differential operator, we estimated the max error, demonstrating the efficiency of the Φ-HW approach for Φ-FDEs.

Theorem 3. *Let $\mathbb{D}^{\ell}y$ be continuous on $[a_1, a_2]$ and suppose that $M>0$ so that $|\mathbb{D}^{\ell,\Phi}u(\chi)|\leq M\forall\chi\in[a_1,a_2]$, where $a_1, a_2\in\mathbb{R}^+$, $\mathbb{D}^{\ell,\Phi}u(\chi)=((1/\Phi'(\chi))(d/d\chi))^{\ell}u(\chi)$ and ${}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi)$ is approximated by ${}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u_{\ell}(\chi)$, then we have*

$$\begin{aligned}&\left\|{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi)-{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u_{\ell}(\chi)\right\|_E\\ &\quad\leq\frac{(a_2-a_1)M\left(\Phi'(a_2)\right)^{\ell-\alpha}}{\Gamma(\ell-\alpha+1)}\frac{1}{\kappa^{(\ell-\alpha)}}\frac{1}{\left[1-2^{2(\alpha-\ell)}\right]^{1/2}}.\end{aligned} \tag{16}$$

Proof. ${}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}y$ can be approximated by HW as

$${}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi)=\sum_{i=a}^{\infty}c_i h_i(\chi). \tag{17}$$

Here, c_i is given by

$$c_i=\left\langle{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi),h_i(\chi)\right\rangle=\int_{a_1}^{b}\left({}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi)\right)h_i(\chi)d\chi. \tag{18}$$

Let the approximation of ${}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u$ be ${}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u_\ell$ which is defined by

$$ {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u_\ell(\chi) = \sum_{i=0}^{\ell-1} c_i h_i(\chi), \tag{19} $$

in which $\ell = 2^{\beta+1}$, $\beta = 1, 2, 3, \cdots$.

Therefore,

$$ {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi) - {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u_\ell(\chi) = \sum_{i=m}^{\infty} c_i h_i(\chi) = \sum_{i=2^{\beta+1}}^{\infty} c_i h_i(\chi). \tag{20} $$

This gives

$$ \begin{aligned} &\left\| {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi) {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u_\ell(\chi) \right\|_E^2 \\ &= \int_{a_1}^{\chi} \left({}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi) - {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u_\ell(\chi) \right)^2 d\chi \\ &= \sum_{i=2^{\beta+1}}^{\infty} \sum_{i'=2^{\beta+1}}^{\infty} c_i c_{i'} \int_{a_1}^{\chi} h_i(\chi) h_{i'}(\chi) d\chi. \end{aligned} \tag{21} $$

The sequence $\{h_{\mathfrak{m}}(\chi)\}$ being orthogonal, we get $\int_{a_1}^{a_2} h_{\mathfrak{m}}(\chi) h_{\mathfrak{m}}(\chi) d\chi = I_{\mathfrak{m}}$, $I_{\mathfrak{m}}$ which represents the identity matrix of order $\mathfrak{m}$.□ □

Therefore, from Equation (21) we have

$$ \left\| {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi) - {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u_\ell(\chi) \right\|_E^2 = \sum_{i'=2^{\beta+1}}^{\infty} c_i^2. \tag{22} $$

Equation (18) gives

$$ \begin{aligned} c_i &= \int_{a_1}^{b} \left({}^{CC}\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi) \right) h_i(\chi) d\chi \\ &= 2^{\frac{j}{2}} \left\{ \int_{a_1+(a_2-a_1)\kappa 2^{-j}}^{a_1+(a_2-a_1)\left(\kappa+\frac{1}{2}\right)2^{-j}} {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi) d\chi \right. \\ &\quad \left. - \int_{a_1+(a_2-a_1)\left(\kappa+\frac{1}{2}\right)2^{-j}}^{a_1+(a_2-a_1)(\kappa+1)2^{-j}} {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi) d\chi \right\}. \end{aligned} \tag{23} $$

Employing mean value theorem of integration $\exists \chi_1, \chi_2 \in (a_1, a_2)$ where

$$ \begin{aligned} &a_1 + (a_2 - a_1)\kappa 2^{-j} < \chi_1 < a_1 + (a_2 - a_1)\left(\kappa + \frac{1}{2}\right)2^{-j}, \\ &a_1 + (a_2 - a_1)\left(\kappa + \frac{1}{2}\right)2^{-j} < \chi_2 < a_1 + (a_2 - a_1)(\kappa + 1)2^{-j}, \end{aligned} \tag{24} $$

so that we arrive at

$$ \begin{aligned} c_i &= 2^{j/2}(a_2 - a_1)\left\{ \left(a_1 + \left(\kappa + \frac{1}{2}\right)2^{-j} - \left(a_1 + \kappa 2^{-j}\right) \right) {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi_1) \right. \\ &\quad \left. - \left(a_1 + (\kappa+1)2^{-j} - \left(a_1 + \left(\kappa + \frac{1}{2}\right)2^{-j}\right) {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi_2) \right\} \\ &= 2^{j/2}(a_2 - a_1)\left\{ 2^{-j-1}\left({}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi_1) - {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi_2) \right) \right\}. \end{aligned} \tag{25} $$

Therefore,

$$ c_i^2 = 2^{-j-2}(a_2 - a_1)^2 \left({}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi_1) - {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi_2) \right)^2. \tag{26} $$

Applying the Φ-Caputo fractional differential operator along with the facts that Φ is increasing and $|\mathbb{D}^{\ell,\Phi}u(\chi)| \leq M$, we get

$$ \begin{aligned} &\left| {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi_1) - {}^C\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi_2) \right| \\ &= \frac{1}{\Gamma(\ell-\alpha)} \left| \int_{a_1}^{\chi_1} \Phi'(\chi)(\Phi(\chi_1) - \Phi(\chi))^{\ell-\alpha-1} \mathbb{D}^{\ell,\Phi}u(\chi) d\chi \right. \\ &\quad \left. - \int_{a_1}^{\chi_2} \Phi'(\chi)(\Phi(\chi_2) - \Phi(\chi))^{\ell-(\alpha+1)} \mathbb{D}^{\ell,\Phi}u(\chi) d\chi \right| \\ &= \frac{1}{\Gamma(\ell-\alpha)} \left| \int_{a_1}^{\chi_1} \Phi'(\chi)(\Phi(\chi_1) - \Phi(\chi))^{\ell-\alpha-1} \mathbb{D}^{\ell,\Phi}u(\chi) d\chi \right. \\ &\quad - \int_{a_1}^{\chi_1} \Phi'(\chi)(\Phi(\chi_2) - \Phi(\chi))^{\ell-(\alpha+1)} \mathbb{D}^{\ell,\Phi}u(\chi) d\chi \\ &\quad \left. - \int_{\chi_1}^{\chi_2} \Phi'(\chi)(\Phi(\chi_2) - \Phi(\chi))^{\ell-(\alpha+1)} \mathbb{D}^{\ell,\Phi}u(\chi) d\chi \right| \\ &\leq \frac{1}{\Gamma(\ell-\alpha)} \left| \int_{a_1}^{\chi_1} \Phi'(\chi)(\Phi(\chi_1) - \Phi(\chi))^{\ell-(\alpha+1)} \mathbb{D}^{\ell,\Phi}u(\chi) d\chi \right. \\ &\quad \left. - \int_{a_1}^{\chi_1} \Phi'(\chi)(\Phi(\chi_2) - \Phi(\chi))^{\ell-(\alpha+1)} \mathbb{D}^{\ell,\Phi}u(\chi) d\chi \right| \\ &\quad + \left| \int_{\chi_1}^{\chi_2} \Phi'(\chi)(\Phi(\chi_2) - \Phi(\chi))^{\ell-(\alpha+1)} \mathbb{D}^{\ell,\Phi}u(\chi) d\chi \right| \\ &= \frac{1}{\Gamma(\ell-\alpha)} \left(\left| \int_{a_1}^{\chi_1} \Phi'(\chi) \left[(\Phi(\chi_1) - \Phi(\chi))^{\ell-(\alpha+1)} \right. \right. \right. \\ &\quad \left. \left. - (\Phi(\chi_2) - \Phi(\chi))^{\ell-(\alpha+1)} \right] \mathbb{D}^{\ell,\Phi}u(\chi) d\chi \right| \\ &\quad \left. + \left| \int_{\chi_1}^{\chi_2} \Phi'(\chi)(\Phi(\chi_2) - \Phi(\chi))^{\ell-(\alpha+1)} \mathbb{D}^{\ell,\Phi}u(\chi) d\chi \right| \right) \end{aligned} $$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\ell-\alpha)}\left(\int_{a_1}^{\chi_1}\left|\Phi'(\chi)\left[(\Phi(\chi_1)-\Phi(\chi))^{\ell-(\alpha+1)}\right.\right.\right.\\
&\quad\left.-(\Phi(\chi_2)-\Phi(\chi))^{\ell-(\alpha+1)}\right]\mathbb{D}^{\ell,\Phi}u(\chi)\Big|d\chi\\
&\quad\left.+\int_{\chi_1}^{\chi_2}\left|\Phi'(\chi)(\Phi(\chi_2)-\Phi(\chi))^{\ell-(\alpha+1)}\mathbb{D}^{\ell,\Phi}u(\chi)\right|d\chi\right)\\
&\leq \frac{1}{\Gamma(\ell-\alpha)}\left(\int_{a_1}^{\chi_1}\Phi'(\chi)\left[(\Phi(\chi_1)-\Phi(\chi))^{\ell-(\alpha+1)}\right.\right.\\
&\quad\left.-(\Phi(\chi_2)-\Phi(\chi))^{\ell-(\alpha+1)}\right]\left|\mathbb{D}^{\ell,\Phi}u(\chi)\right|d\chi\\
&\quad\left.+\int_{\chi_1}^{\chi_2}\Phi'(\chi)(\Phi(\chi_2)-\Phi(\chi))^{\ell-(\alpha+1)}\left|\mathbb{D}^{\ell,\Phi}u(\chi)\right|d\chi\right),\\
&\quad\text{where } \ell>1+\alpha \leq \frac{M}{\Gamma(\ell-\alpha)}\left(\int_{a_1}^{\chi_1}\Phi'(\chi)\left[(\Phi(\chi_1)\right.\right.\\
&\quad\left.-\Phi(\chi))^{\ell-(\alpha+1)}-(\Phi(\chi_2)-\Phi(\chi))^{\ell-(\alpha+1)}\right]d\chi\\
&\quad\left.+\int_{\chi_1}^{\chi_2}\Phi'(\chi)(\Phi(\chi_2)-\Phi(\chi))^{\ell-(\alpha+1)}d\chi\right)\\
&=\frac{M}{\Gamma(\ell-\alpha)}\frac{1}{(\ell-\alpha)}\Big((\Phi(\chi_1)-\Phi(a_1))^{\ell-\alpha}+(\Phi(\chi_2)\\
&\quad-\Phi(\chi_1))^{\ell-\alpha}-(\Phi(\chi_2)-\Phi(a_1))^{\ell-\alpha}\\
&\quad+(\Phi(\chi_2)-\Phi(\chi_1))^{\ell-\alpha}\Big)=\frac{M}{\Gamma(\ell-\alpha+1)}\Big((\Phi(\chi_1)\\
&\quad-\Phi(a_1))^{\ell-\alpha}-(\Phi(\chi_2)-\Phi(a_1))^{\ell-\alpha}\\
&\quad+2(\Phi(\chi_2)-\Phi(\chi_1))^{\ell-\alpha}\Big).
\end{aligned} \tag{27}$$

As $\chi_1, \chi_2 > a$ and $\chi_1 < \chi_2$ also $\Phi(\chi)$ are increasing, so

$$(\Phi(\chi_1)-\Phi(a_1))^{\ell-\alpha}-(\Phi(\chi_2)-\Phi(a_1))^{\ell-\alpha}<0. \tag{28}$$

Therefore,

$$\begin{aligned}
&\left|{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}y(\chi_1)-{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}y(\chi_2)\right|\\
&\quad\leq\frac{2M}{\Gamma(\ell-\alpha+1)}(\Phi(\chi_2)-\Phi(\chi_1))^{\ell-\alpha}.
\end{aligned} \tag{29}$$

According to the mean value theorem, $\exists\zeta\in[\chi_1,\chi_2]\subseteq[a_1,a_2]$ such that $\Phi(\chi_2)-\Phi(\chi_1)\leq(\chi_2-\chi_1)\Phi'(\zeta)$, we get

$$\begin{aligned}
&\left|{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}y(\chi_1)-{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}y(\chi_2)\right|\\
&\quad\leq\frac{2M}{\Gamma(\ell-\alpha+1)}\left((\chi_2-\chi_1)\Phi'(\zeta)\right)^{\ell-\alpha}\\
&\quad\leq\frac{2M}{\Gamma(\ell-\alpha+1)2^{j(\ell-\alpha)}}\left(\Phi'(a_2)\right)^{\ell-\alpha},
\end{aligned} \tag{30}$$

which implies that

$$\begin{aligned}
&\left({}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}y(\chi_1)-{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}y(\chi_2)\right)^2\\
&\quad\leq\frac{4M^2}{\Gamma^2(\ell-\alpha+1)2^{2j(\ell-\alpha)}}\left(\Phi'(a_2)\right)^{2(\ell-\alpha)}.
\end{aligned} \tag{31}$$

Putting (31) in (26), we get

$$c_i^2\leq 2^{-j-2}(a_2-a_1)^2\frac{4M^2}{\Gamma^2(\ell-\alpha+1)2^{2j(\ell-\alpha)}}\left(\Phi'(a_2)\right)^{2(\ell-\alpha)}. \tag{32}$$

Putting together equations (22) and (32), we have

$$\begin{aligned}
&\left\|{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi)-{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u_\ell(\chi)\right\|_E^2\\
&\quad=\sum_{i=2^{\beta+1}}^{\infty}c_i^2=\sum_{j=\beta+1}^{\infty}\left(\sum_{i=2^j}^{2^{j+1}-1}c_i^2\right)\\
&\quad\leq\sum_{j=\beta+1}^{\infty}(a_2-a_1)^2\frac{M^2}{\Gamma^2(\ell-\alpha+1)2^{2j(\ell-\alpha)+j}}\\
&\qquad\left(\Phi'(a_2)\right)^{2(\ell-\alpha)}\left(2^{j+1}-1-2^j+1\right)\\
&\quad=\frac{(a_2-a_1)^2M^2\left(\Phi'(a_2)\right)^{2(\ell-\alpha)}}{\Gamma^2(\ell-\alpha+1)}\sum_{j=\beta+1}^{\infty}\frac{1}{2^{2j(\ell-\alpha)}}\\
&\quad=\frac{(a_2-a_1)^2M^2\left(\Phi'(a_2)\right)^{2(\ell-\alpha)}}{\Gamma^2(\ell-\alpha+1)}\frac{1}{2^{2(\beta+1)(\ell-\alpha)}}\frac{1}{1-2^{2(\alpha-\ell)}},
\end{aligned} \tag{33}$$

which implies that

$$\begin{aligned}
&\left\|{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi)-{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u_\ell(\chi)\right\|_E\\
&\quad\leq\frac{(a_2-a_1)M\left(\Phi'(a_2)\right)^{\ell-\alpha}}{\Gamma(\ell-\alpha+1)}\frac{1}{2^{(\beta+1)(\ell-\alpha)}}\frac{1}{\left[1-2^{2(\alpha-\ell)}\right]^{1/2}}.
\end{aligned} \tag{34}$$

Let $\kappa=2^{\beta+1}$, (34) can also be written as

$$\begin{aligned}
&\left\|{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u(\chi)-{}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u_\ell(\chi)\right\|_E\\
&\quad\leq\frac{(a_2-a_1)M\left(\Phi'(a_2)\right)^{\ell-\alpha}}{\Gamma(\ell-\alpha+1)}\frac{1}{\kappa^{(\ell-\alpha)}}\frac{1}{\left[1-2^{2(\alpha-\ell)}\right]^{1/2}}.
\end{aligned} \tag{35}$$

To compute the error bound, we need the value of M. So we will estimate M first. Since $\mathbb{D}^{\ell}u(\chi)$ is continuous and bounded on $[a_1,a_2]$, so is $\mathbb{D}^{\ell,\Phi}u(\chi)$ and it is approximated by

$$\mathbb{D}^{\ell,\Phi}u(\chi) \cong \sum_{i=0}^{r-1} c_i h_i(\chi) = C_r^T H_r(\chi), \tag{36}$$

where $C_r = [c_0, c_1, c_2, \cdots, c_{r-1}]^T$ and $H_r(\chi) = [h_0(\chi), h_1(\chi), h_2(\chi), \cdots, h_{r-1}(\chi)]^T$.

Integration of (36) gives

$$\begin{aligned}\mathbb{D}^{\ell-1,\Phi}u(\chi) &= \int_{a_1}^{\chi} \mathbb{D}^{\ell,\Phi}u(\chi)d\chi + \mathbb{D}^{\ell-1,\Phi}u(a_1)\\ &= \int_{a_1}^{\chi} \mathbb{D}^{\ell,\Phi}u(\chi)d\chi \cong C_r^T P^{1,\Phi} H_\ell(\chi).\end{aligned} \tag{37}$$

Similarly,

$$\begin{aligned}\mathbb{D}^{\ell-2,\Phi}u(\chi) &= \int_{a_1}^{\chi} \mathbb{D}^{\ell-1,\Phi}u(\chi)d\chi + \mathbb{D}^{\ell-2,\Phi}y(a_1)\\ &= \int_{a_1}^{\chi} \mathbb{D}^{\ell-1,\Phi}u(\chi)d\chi \cong C_\ell^T P^{2,\Phi} H_\ell(\chi).\end{aligned} \tag{38}$$

Continuing in the same manner, we arrive at

$$\mathbb{D}^{\Phi}u(\chi) \cong C_\ell^T P^{\ell,\Phi} H_\ell(\chi). \tag{39}$$

Taking $\chi_j = (j - 1/2)/\ell, j = 0, 1, 2, \cdots, m$, and putting it in (39), we have

$$\mathbb{D}^{\Phi}u\left(\chi_j\right) \cong C_\ell^T P^{\ell,\Phi} H_\ell\left(\chi_j\right). \tag{40}$$

The matrix form of (40) is as

$$\begin{aligned}\mathbb{D}^{\Phi}U^T \cong C_\ell^T P^{\ell,\Phi} H_\ell\left(\chi_j\right) \quad &\text{where } \mathbb{D}^{\Phi}U^T\\ = \left[\mathbb{D}^{\Phi}u(\chi_1), \mathbb{D}^{\Phi}u(\chi_2), \mathbb{D}^{\Phi}u(\chi_3), \cdots, \mathbb{D}^{\Phi}u(\chi_\ell)\right]^T&.\end{aligned} \tag{41}$$

The linear system in (41) determines the value of the vector C_ℓ^T; by putting this value in (36), $\mathbb{D}^{\ell,\Phi}(\chi)$ can be obtained $\forall \chi \in [a_1, a_2]$.

Let $\tau_i \in [a_1, a_2]$, then $\mathbb{D}^{\ell,\Phi}u(t_i)$ can be computed for the equidistant points $i = 1, 2, 3, \cdots, \ell$, then $\varepsilon + \max |\mathbb{D}^{\ell}u(t_i)|$ is the approximation of M.

Theorem 4. *Assume that ${}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u_\ell$, computed from Φ-HW is estimated by ${}^{C}\mathbb{D}_{a_1}^{\alpha,\Phi}u$, then we have*

$$\|u(\chi) - u_\ell(\chi)\|_E \le \frac{MN}{\Gamma(\alpha+1)\Gamma(\ell-\alpha+1)} \frac{1}{\kappa^{(\ell-\alpha)}} \frac{1}{\left[1 - 2^{2(\alpha-\ell)}\right]^{1/2}}, \tag{42}$$

where $N = \max |(a_2 - a_1)(\Phi(a_2))^{\ell-\alpha}(\Phi(\chi) - \Phi(0))^{\alpha}|$.

Theorem 4 can be proven easily by following the procedure of Theorem 3. From Equation (42), we noted that $\|u(\chi) - u_\ell(\chi)\|_E \longrightarrow 0$ as $\ell \longrightarrow \infty$. Thus, the convergence of the Φ-HW method is inferred.

5. Numerical Examples

We present several examples of how to get numerical solutions of Φ-FRODEs using the Φ-HW operational matrix approach.

Example 5. Consider the Φ-FRODE

$$\mathbb{D}^{\alpha,\Phi}u(\chi) + au(\chi) = g(\chi), 0 < \alpha \le 1, \chi \in [0, 1], u(0) = 0. \tag{43}$$

For $a = 2/\Gamma(3-\alpha)$ and $g(\chi) = (2/\Gamma(3-\alpha))(\Phi(\chi))^{2-\alpha} + (\Phi(\chi)^2)$, the actual solution of Equation (43) is $u(\chi) = (\Phi(\chi))^2$. We use the Φ-HW technique to solve problem (43).

Let

$${}^{C}D^{\alpha,\Phi}u(\chi) = C_{\mathbf{m}}^T H_{\mathbf{m}}(\chi). \tag{44}$$

Integrating Equation (44) with respect to $\mathcal{J}_a^{\alpha,\Phi}$ and using the initial conditions, we have

$$u(\chi) = \mathcal{J}^{\alpha,\Phi} C_{\mathbf{m}}^T H_{\mathbf{m}}(\chi) = C_{\mathbf{m}}^T P_{\mathbf{m}\times\mathbf{m}}^{\alpha,\Phi} H_{\mathbf{m}}(\chi). \tag{45}$$

Substituting (44) and (45) into (43), we have

$$C_{\mathbf{m}}^T\left(H_{\mathbf{m}}(\chi) + aP_{\mathbf{m}\times\mathbf{m}}^{\alpha,\Phi} H_{\mathbf{m}}(\chi)\right) = g(\chi). \tag{46}$$

Equation (46) has the following matrix form:

$$C_{\mathbf{m}}^T\left(H_{\mathbf{m}}(\chi) + aP_{\mathbf{m}\times\mathbf{m}}^{\alpha,\Phi} H_{\mathbf{m}}(\chi)\right) = G, \tag{47}$$

where G is the matrix representation of g at the collocation points.

Solving the algebraic system given by Equation (47) for C_m^T and substituting this value into Equation (45), we will have the required numerical solution. In Table 1, the max absolute error is given for $J = 6$ and $\varphi(\zeta) = \zeta^3$. Approximate solutions for $J = 6$ and $\alpha = 0.5$ and different choices of the function Φ are plotted in Figure 2. Also, actual and approximate results and the absolute error are given for $J = 6$, $\alpha = 0.8$, and $\Phi(\chi) = 1/3(\chi^3 - \chi^2 - \chi)$ in Figure 2.

Example 6. Consider the composite Φ-FRODE

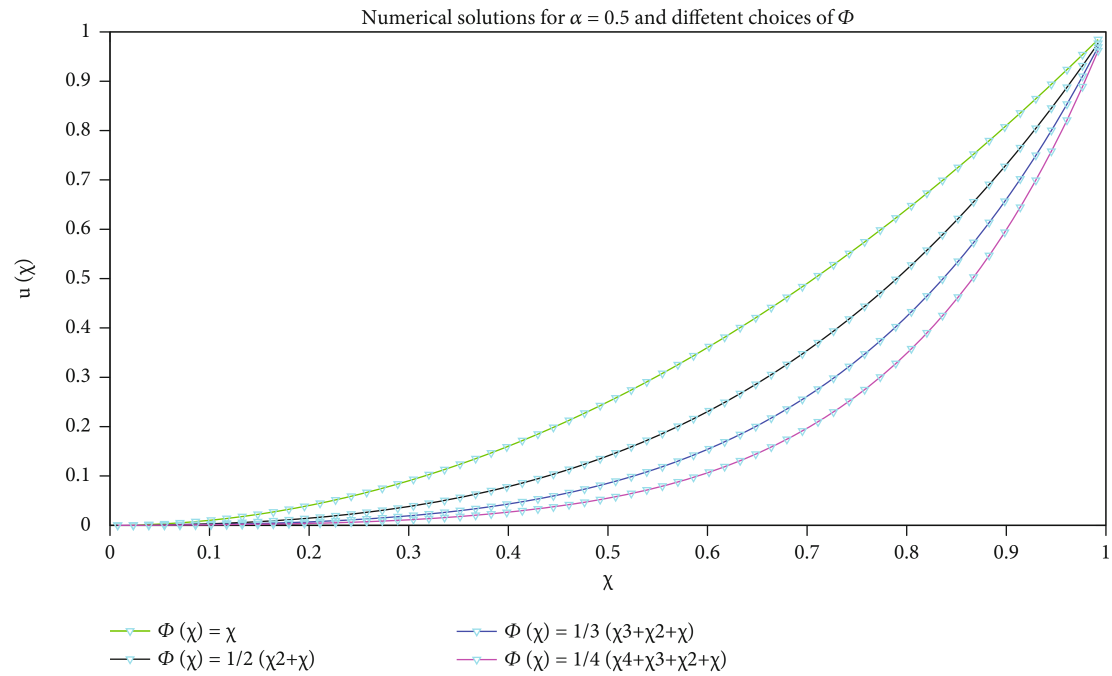
$$\begin{aligned}\mathbb{D}^{\alpha,\Phi}u(\chi) + bu(\chi) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(\Phi(\chi))^{\alpha}[1 + (\Phi(\chi))^{\alpha}], 0 < \alpha\\ \le 1, x \in [0, 1],\end{aligned} \tag{48}$$

$$u(0) = 0. \tag{49}$$

For $b = \Gamma(2\alpha+1)/\Gamma(\alpha+1)$.

Table 1: Max absolute error for various choices of α and J.

α	$J = 5$	$J = 6$	$J = 7$	$J = 8$	$J = 9$
0.6	1.0141×10^{-4}	3.2169×10^{-5}	1.0296×10^{-5}	3.3179×10^{-6}	1.0751×10^{-6}
0.7	9.2421×10^{-5}	2.7180×10^{-5}	8.0595×10^{-6}	2.4056×10^{-6}	7.2213×10^{-7}
0.8	7.8481×10^{-5}	2.1470×10^{-5}	5.9091×10^{-6}	1.6349×10^{-6}	4.5452×10^{-7}
0.9	6.3749×10^{-5}	1.6437×10^{-5}	4.2464×10^{-6}	1.0994×10^{-6}	2.8527×10^{-7}
1.0	5.3010×10^{-5}	1.3250×10^{-5}	3.3127×10^{-6}	8.2817×10^{-7}	2.0704×10^{-7}



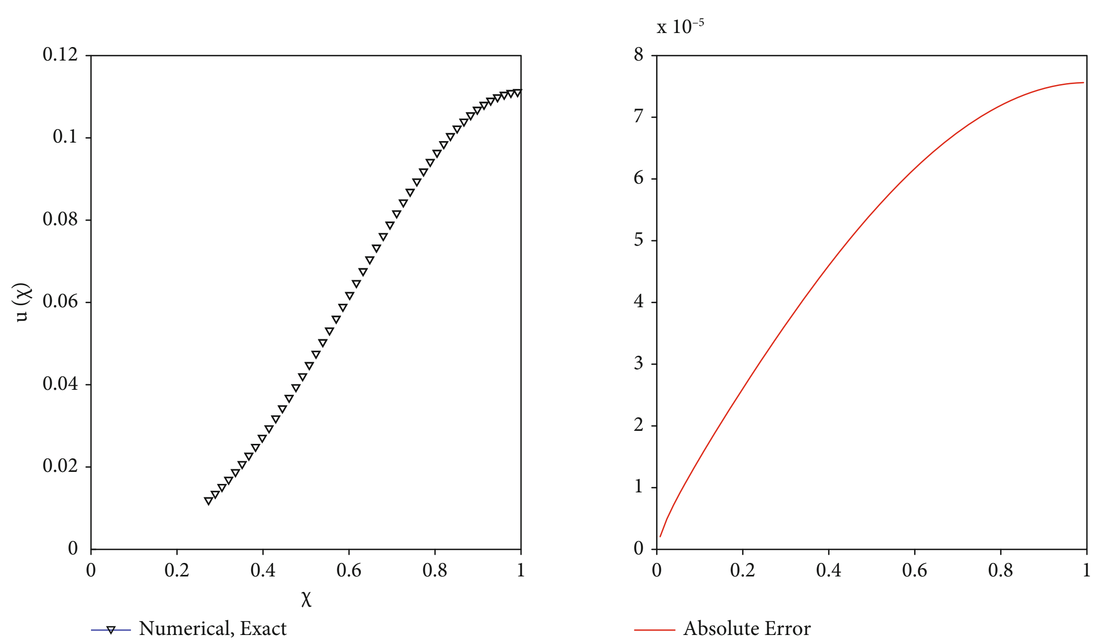


Figure 2: Approximate and exact results of Equation (43) and their absolute error.

Table 2: Max absolute error for various choices of α and J.

α	$J=5$	$J=6$	$J=7$	$J=8$	$J=9$
0.6	1.2082×10^{-4}	3.9804×10^{-5}	1.3061×10^{-5}	4.2822×10^{-6}	1.4045×10^{-6}
0.7	9.1203×10^{-5}	2.8002×10^{-5}	8.5466×10^{-6}	2.6042×10^{-6}	7.9372×10^{-7}
0.8	6.7053×10^{-5}	1.9253×10^{-5}	5.4799×10^{-6}	1.5545×10^{-6}	4.4065×10^{-7}
0.9	4.8650×10^{-5}	1.3151×10^{-5}	3.5128×10^{-6}	9.3312×10^{-7}	2.4729×10^{-7}
1.0	3.5205×10^{-5}	9.0544×10^{-6}	2.2954×10^{-6}	5.7785×10^{-7}	1.4496×10^{-7}

The exact solution for the problem (48) is $u(\chi)=(\Phi(\chi))^{2\alpha}$.

For the numerical solution, we employ the Φ-HW technique.

Let

$$^{C}D^{\alpha,\Phi}u(\chi)=C_{\mathrm{m}}^{T}H_{\mathrm{m}}(\chi). \tag{50}$$

Integrating Equation (50) with respect to $\mathcal{J}_a^{\alpha,\Phi}$ and utilizing the initial condition, we have

$$u(\chi)=\mathcal{J}^{\alpha,\Phi}C_{\mathrm{m}}^{T}H_{\mathrm{m}}(x)=C_{\mathrm{m}}^{T}P_{\mathrm{m\times m}}^{\alpha,\Phi}H_{\mathrm{m}}(\chi). \tag{51}$$

Substituting Equations (50) and (51) in Equation (48), we get

$$C_{\mathrm{m}}^{T}\left(H_{\mathrm{m}}(\chi)+P_{\mathrm{m\times m}}^{\alpha,\Phi}H_{\mathrm{m}}(\chi)\right)=g(\chi), \tag{52}$$

where $(\Gamma(2\alpha+1)/\Gamma(\alpha+1))(\Phi(\chi))^{\alpha}[1+(\Phi(\chi))^{\alpha}]$. Equation (52) in matrix form is given as

$$C_{\mathrm{m}}^{T}\left(H_{\mathrm{m}}(\chi)+P_{\mathrm{m\times m}}^{\alpha,\Phi}H_{\mathrm{m}}(\chi)\right)=G. \tag{53}$$

where G is the matrix representation of $g(\chi)$.

Required approximate solutions can be obtained by using the value of C_m^T from Equation (53) into Equation (51).

The max absolute error is tabulated for $\Phi(\chi)=(\chi)^3/5$ and various choices of J and α in Table 2, which shows that the Maximum Absolute Error decreases by increasing the values of J. Figure 3 represents approximate solutions for different choices of α. Also, comparison of actual and approximate results and their absolute error is displayed in Figure 3.

Example 7. Consider the Φ-FRODE

$$\mathbb{D}^{\alpha,\Phi}u(\chi)+u(\chi)=1-4\Phi(\chi)+5(\Phi(\chi))^2-\frac{4}{\Gamma(2-\alpha)}(\Phi(\chi))^{1-\alpha}+\frac{10}{\Gamma(3-\alpha)}(\Phi(\chi))^{2-\alpha}, \tag{54}$$

where $0<\alpha\le1$, $x\in[0,1]$, and $u(0)=1$. It is easy to verify that $u(\chi)=1-4\Phi(\chi)+5(\Phi(\chi))^2$ is the actual solution of Equation (54). For numerical approximation, we employ the Φ-HW technique.

Let

$$^{C}D^{\alpha,\Phi}u(\chi)=C_{\mathrm{m}}^{T}H_{\mathrm{m}}(\chi). \tag{55}$$

Integrating Equation (55) with respect to $\mathcal{J}_a^{\alpha,\Phi}$ and using the initial conditions, we have

$$u(\chi)=\mathcal{J}^{\alpha,\Phi}C_{\mathrm{m}}^{T}H_{\mathrm{m}}(x)+u(0)=C_{\mathrm{m}}^{T}P_{\mathrm{m\times m}}^{\alpha,\Phi}H_{\mathrm{m}}(\chi)+1. \tag{56}$$

Substituting (55) and (56) into (54), we have

$$C_{\mathrm{m}}^{T}\left(H_{\mathrm{m}}(\chi)+P_{\mathrm{m\times m}}^{\alpha,\Phi}H_{\mathrm{m}}(\chi)\right)=g(\chi), \tag{57}$$

where $g(\chi)=-4\Phi(\chi)+5(\Phi(\chi))^2-(4/\Gamma(2-\alpha))(\Phi(\chi))^{1-\alpha}+(10/\Gamma(3-\alpha))(\Phi(\chi))^{2-\alpha}$.

The matrix form of Equation (57) is

$$C_{\mathrm{m}}^{T}\left(H_{\mathrm{m}}(\chi)+P_{\mathrm{m\times m}}^{\alpha,\Phi}H_{\mathrm{m}}(\chi)\right)=G, \tag{58}$$

where G is the matrix representation of $g(\chi)$.

Required approximate solutions can be obtained by using the value of C_m^T from Equation (58) in Equation (56). Table 3 shows that the Maximum Absolute Error decreases by increasing the values of J. Approximate solutions are displayed in Figure 4 for various values of Φ. Also, Figure 4 represents approximate and exact solutions and their max absolute error for $\alpha=0.75$, $J=6$, and $\Phi(\chi)=(\chi)^3/15$.

Example 8. Consider the Φ-FRODE

$$\mathbb{D}^{\alpha,\Phi}u(\chi)+\mu u(\chi)=g(\chi), 0<\alpha\le1, x\in[0,1]\ u(0)=0. \tag{59}$$

For $\mu=1$ and $g(\chi)=(\Gamma(2\alpha+1)/\Gamma(1+\alpha))(\Phi(\chi))^{\alpha}+(\Gamma(2)/\Gamma(2-\alpha))(\Phi(\chi))^{1+\alpha}+(\Phi(\chi))^{2\alpha}+\Phi(\chi)$, the exact solution of Equation (59) is $(\Phi(\chi))^{2\alpha}+\Phi(\chi)$.

For approximate solutions, we use the Φ-HW technique.

Let

$$^{C}D^{\alpha,\Phi}u(\chi)=C_{\mathrm{m}}^{T}H_{\mathrm{m}}(\chi). \tag{60}$$

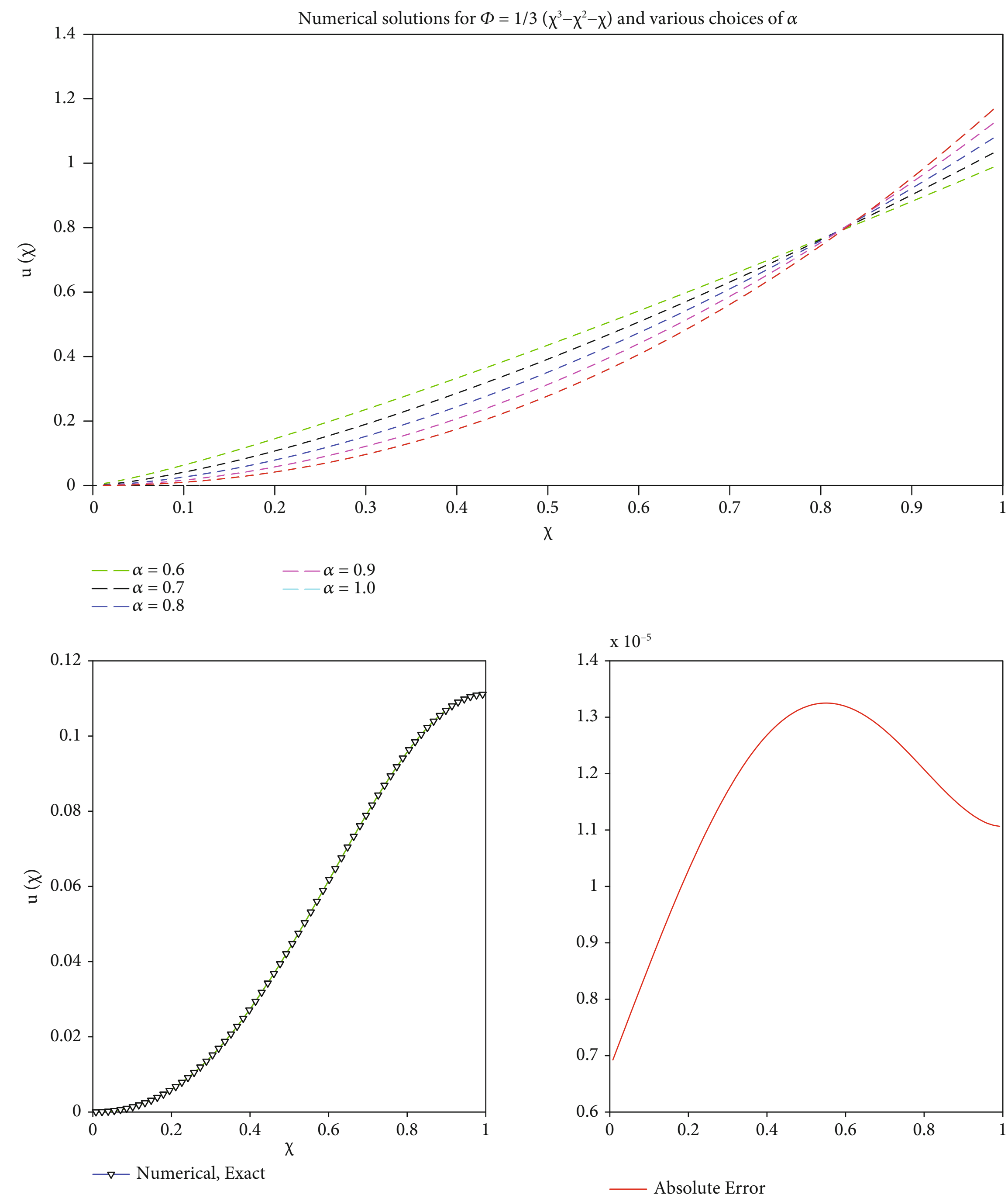


Figure 3: Approximate results of Equation (48) for various choices of α, actual, approximate results, and the absolute error.

Table 3: Max absolute error for $\Phi(\chi) = \chi^3/15$ and various choices of J and α.

α	$J = 5$	$J = 6$	$J = 7$	$J = 8$	$J = 9$
0.6	9.0149×10^{-5}	2.9995×10^{-5}	9.9647×10^{-6}	3.3058×10^{-6}	1.0954×10^{-6}
0.7	4.3216×10^{-5}	1.3568×10^{-5}	4.2468×10^{-6}	1.3254×10^{-6}	4.1271×10^{-7}
0.8	1.5201×10^{-5}	4.5357×10^{-6}	1.3466×10^{-6}	3.9805×10^{-7}	1.1721×10^{-7}
0.9	2.2599×10^{-5}	6.0704×10^{-6}	1.5981×10^{-6}	4.1680×10^{-7}	1.0825×10^{-7}
1.0	2.2302×10^{-5}	5.7525×10^{-6}	1.4605×10^{-6}	3.6795×10^{-7}	9.2342×10^{-8}

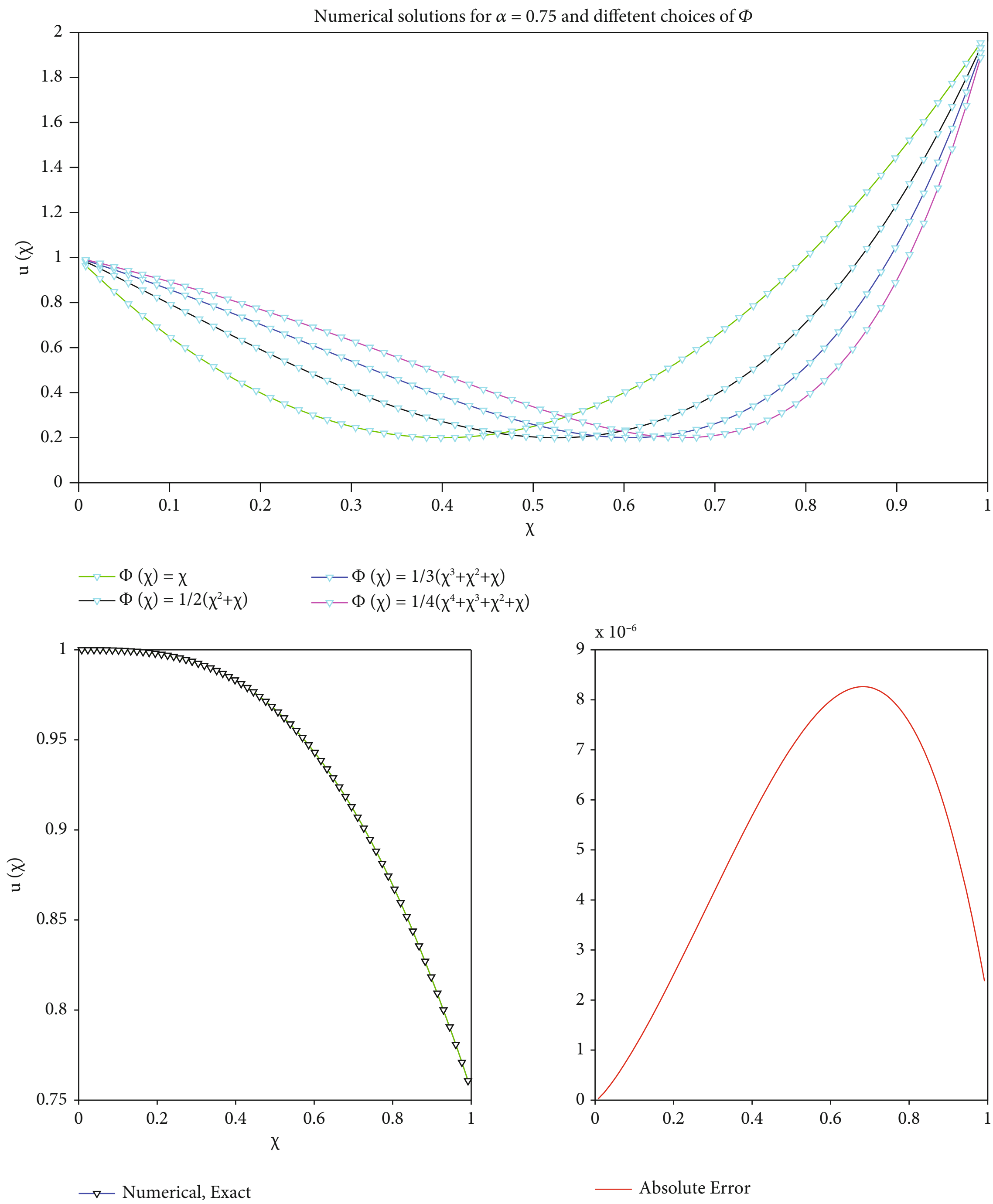


Figure 4: Approximate solutions for different choices of α and functions $\Phi(x)$.

Integrating Equation (60) with respect to $\mathcal{J}_0^{\alpha,\Phi}$ and using the initial conditions, we have

$$u(\chi) = \mathcal{J}_0^{\alpha,\Phi} C_{\mathbf{m}}^T H_{\mathbf{m}}(x) = C_{\mathbf{m}}^T P_{\mathbf{m}\times\mathbf{m}}^{\alpha,\Phi} H_{\mathbf{m}}(\chi). \tag{61}$$

Substituting (60) and (61) into (59), we have

$$C_{\mathbf{m}}^T \left(H_{\mathbf{m}}(\chi) + \mu P_{\mathbf{m}\times\mathbf{m}}^{\alpha,\Phi} H_{\mathbf{m}}(\chi) \right) = g(\chi). \tag{62}$$

The matrix representation of Equation (62) is

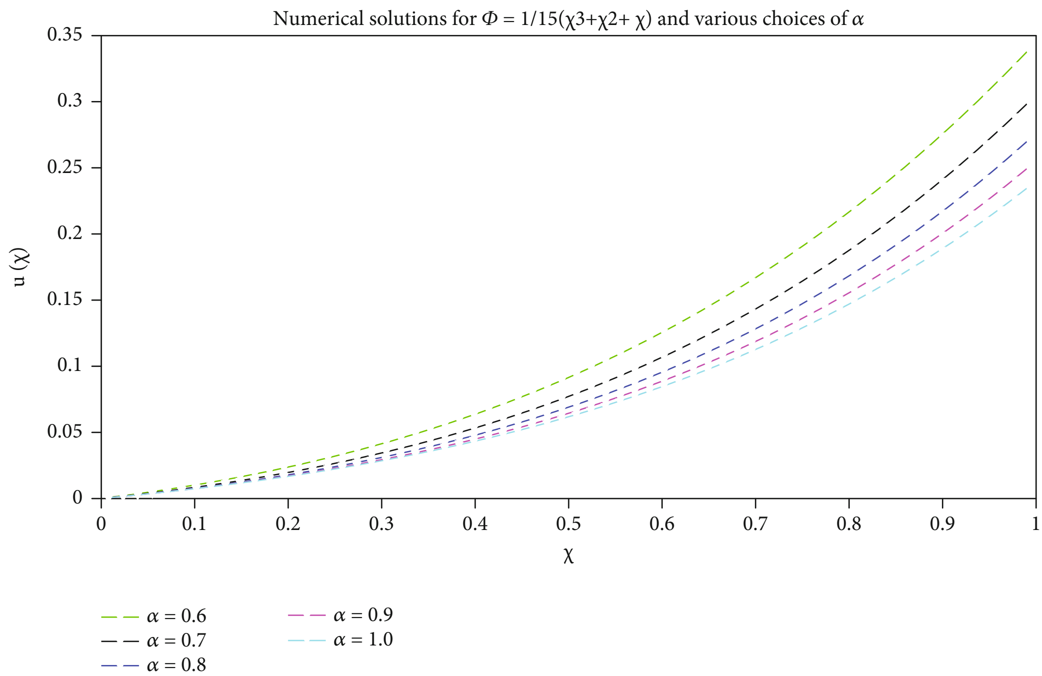
$$C_{\mathbf{m}}^T \left(H_{\mathbf{m}}(\chi) + \mu P_{\mathbf{m}\times\mathbf{m}}^{\alpha,\Phi} H_{\mathbf{m}}(\chi) \right) = G, \tag{63}$$

where G is the matrix representation of $g(\chi)$ at the collocation points.

Required approximate solutions can be obtained by using the value of C_m^T from Equation (58) in Equation (61). Table 4 shows that the Maximum Absolute Error decreases

Table 4: Max absolute error for $\Phi(\chi) = \chi^2/15$ and various choices of J and α.

α	$J = 5$	$J = 6$	$J = 7$	$J = 8$	$J = 9$
0.6	3.8255×10^{-5}	1.2688×10^{-5}	4.1987×10^{-6}	1.3876×10^{-6}	4.5829×10^{-7}
0.7	2.0382×10^{-5}	6.3431×10^{-6}	1.9679×10^{-6}	6.0937×10^{-7}	1.8843×10^{-7}
0.8	9.0481×10^{-6}	2.6383×10^{-6}	7.6857×10^{-7}	2.2372×10^{-7}	6.5069×10^{-8}
0.9	3.0670×10^{-6}	8.0025×10^{-7}	2.1048×10^{-7}	5.5848×10^{-8}	1.4938×10^{-8}
1.0	2.0360×10^{-6}	5.1692×10^{-7}	1.3022×10^{-7}	3.2678×10^{-8}	8.1852×10^{-9}



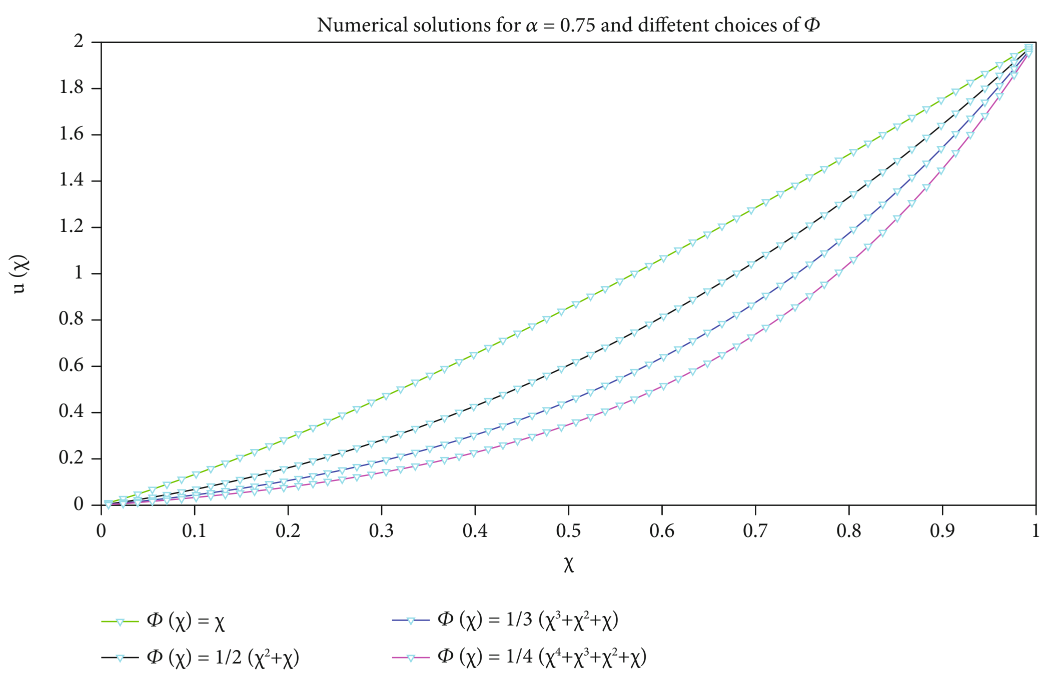


Figure 5: Numerical solutions for $\alpha = 1$, $J = 8$ and for different functions $\Phi(\chi)$.

by increasing the values of J. Also the approximate solutions are displayed in Figure 5 for various values of α.

6. Conclusion

This study introduces a numerical approach for solving a class of fractional differential equations with a Φ-Caputo fractional derivative based on a novel type of operational matrix of fractional integration, namely, the Φ-HW operational matrix. The convergence of the proposed method is demonstrated, and the numerical tests reported in Section 5 corroborate the efficacy of our approach.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Evaluating Investors' Recognition Abilities for Risk and Profit in Online Loan Markets Using Nonlinear Models and Financial Big Data

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Evaluating Investors' Recognition Abilities for Risk and Profit in Online Loan Markets Using Nonlinear Models and Financial Big Data

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Financial big data are obtained by web crawler, and investors' recognition abilities for risk and profit in online loan markets are researched using heteroskedastic Probit models. The conclusions are obtained as follows: First, the preference for the item is reflected directly in the time and indirectly in the number of participants for being full, and the larger the preference, the shorter the time and the fewer the participants. Second, investors can discriminate the default risk not reflected by the interest rate, and the bigger the default risk, the longer the time and the more participants being full. Third, investors can discriminate the pure return rate deducted from the maturity term and credit risk, and the higher the return, the shorter the time and the fewer the participants being full. Fourth, default risks are reflected well by online loan platform interest rates, and inventors do not choose the item blindly according to the interest rate but consider comprehensively the profit and the risk. In the future, interest rate liberalization should be deepened, the choosing function of interest rates should be played better, and the information disclosure, investor education, and investor effective usage of other information should be strengthened.

1. Background

With the rapid development of electronic information, internet finance, which has low transaction costs, low participation threshold, and convenient features, gets development in full swing. But there are also many internet finance platforms that have gone bankrupt in recent years. In this context, the government pays more attention to internet finance, and the focus also changes from healthy and standardized development to preventing accumulated risks and strengthening supervision. Thus, it is necessary to research issues of yield and risk of internet finance and to ascertain how investors can participate in internet finance platforms and whether they can discriminate risk and yield.

Traditional debit and credit usually take bank as the intermedium, and depositors and lenders are passive receivers of the interest rate, and thus, characteristics of rate marketization cannot be reflected completely. Although the online-loan-platform interest rate is also one kind of non-fully market-oriented interest rate, the full bid rate, the result of depositors and lenders weighing each other, reflects well the characteristics of interest rate marketization. Traditional financial institutions such as banks do not announce the information of depositors and lenders, but online-loan-platforms announce information of borrowers to potential lenders to promote a deal. Online loan platforms have more characteristics of Financial Big Data than traditional financial institutions and thus can provide rich data resources for researching how online investors weight risks and rewards of online debt items and whether they can.

The online loan platform, without the participation of traditional banks and other financial institutions, can reduce costs and improve the efficiency of capital allocation through direct financing between borrowers and lenders and thus affords a new path to solve the problems of difficult and expensive financing for Small and Medium-Sized

Enterprises (SMEs). The online loan platform, as a new financial medium, broadens investing and financing channels of grassroot debit and credit and makes the common people have a chance to get a higher return. As to whether the online loan platforms have a price discovery function, whether investors are rational and can recognize risks and returns, and whether investors can effectively identify different default risks behind the same interest rate and the net yield difference after deducting maturity period and credit risk, all those need research deeply, and there are great theoretical and practical meanings for understanding and regulating behaviors of Chinese online loan investors and promoting online loan developing healthily. The online loan is an important aspect of internet finances, and the failure probability is very high in recent years, but there are many fake internet finances which are excluded by the paper. The paper researches the recognition ability of risk and return and expects to afford suggestions for formal online loan's healthy and sustainable development based on the Renrendai Online Loan Platform.

2. Literature Reviews

Big financial data accumulated by online loan provide material for deeply researching references and behavior of borrowers and investors. Existing research mainly focuses on researching the full rate, the default rate, and investor's rational consciousness and behavior, and these issues mix up and can be roughly divided into the following categories by evolving process.

Factors influencing the full rate and lender's judgment have been researched by many literatures, and the full rate is a fundamental problem about online loan. Klafft [1] has tested the factors influencing the success of online loans based on the America Prosper platform, and the results show that credit rating, individual character, etc. are important to the success rate of the item. Li et al. [2] analyze the basic statistical characteristics of the online loan item using the data of ppdai and find that basic information of the borrower and item has an important effect on the loan success rate. Li et al. [3] find that descriptive information having a positive impact on the full rate and the more positive information are beneficial to successful fundraising. Liu et al. [4] research lenders' decision-making characteristics and found that friendship has an important effect in the online loan market, and there is a herding effect in the market, namely, lenders following their friends' lending decision. Wan et al. [5] find that the initial trust and consciousness on yield are the main factors affecting lenders' lending in the online loan market.

Some researchers are concerned about borrowers' final activities, namely, factors influencing the default rate, and some are focused on whether principal and interest of online loan can be paid on schedule. Iyer et al. [6] test empirically the role of credit score to online loan's default rate based on data of Prosper, and the results showed that credit level has significant influence on the default rate. Liao et al. [7] test empirically the relationship between interest rates and default rates using data from the Renrendai website. Serrano-Cinca et al. [8] research empirically factors influencing online loan default rates based on data from the Lending Club which is the biggest P2P company in the USA. Emekter et al. [9] research the characteristic of the P2P online loan using data from the Lending Club website, and the empirical test showed that indicators such as credit rating have a significant effect on default rates of items and that high interest rate corresponding to high risk cannot compensate the higher loan default rate. Ge et al. [10] test empirically the influence of social medium information shown by the borrower itself to the online loan default rate. Liu et al. [11] test empirically the forecasting effect of lender's information on default rates of items using data from the Renrendai website.

There are also many literatures that combined the former two questions and researched comprehensively success rates and default rates of fundraising. Using data from the Prosper company, Ravina [12] finds that borrower's individual characteristics such as ethnicity, looking credible, beauty, and body weight have an important effect on financing success rate, but those characteristics except beauty have no influence on the late performance of the item, and although beautiful borrowers more easily get a loan from the online market, they are more likely to delay repayments. Freedman and Jin [13] test empirically the information discrimination of online loans and thought that there are three issues: adverse selection, lender misjudgment, and high interest rate corresponding to high risk in the online loan market, and the former two issues are unique to the online loan market and can be relieved by announcing more borrowers' information and lenders' studies, and the last issue is also existing in traditional markets, and online loans will eventually compete with traditional banks directly. Guo [14] researched roles of internet nicknames and real names on fundraising success rate and default rate, and the empirical test shows that real names cannot increase the success rate and decrease the default rate. Yue et al. [15] test empirically market information's role on investor behavior and forecasting item's default rate. Guo [16] researched the role of marriage in online loans and found that marriage benefits for both increasing fundraising success rate and decreasing default rate. Zhang and Cai [17] research "title bias," namely, the difference in the role of the title to the full rate and default rate. Xu and Chau [18] research the role of communication between lender and borrower on the full rate and default rate and showed that information communication plays a significance effect on the full rate but a nonsignificant effect on the default rate. Caldieraro et al. [19] research the role of nonverified information offered by the borrower to fundraising and the item's late performance and found that verified and nonverified information both have an important effect on online loans. Hu et al. [20] research the performance, financing difficulty, and financing cost of the peasantry and low-income people on the Renrendai platform by combining inclusive finance and online loans. Babaei and Bamdad [21] have evaluated the return and risk of the P2P item using Artificial Neural Network and Logistic function, respectively.

With the deepening of research, some scholars have done deep and detailed research on investors' rational consciousness and choosing behavior. Freedman and Jin [22] show that studying benefits clearing information asymmetry among market participants, and online loan lenders step away from high-risk items and subprime borrowers are excluded from the Prosper website.

Liao et al. [7] research choosing behaviors of online loan investors on the background of China's non-completely-market-oriented interest rate and tested whether information in addition to interest rate has an indicative effect, and investors can discriminate the different default risk implied in the same interest rate based on testing the relationship between interest and default rates, but they have not considered the time value of the interest rate and excluded the credit risk from the interest rate. Gao et al. [23] test "gender effect" using data from the Renrendai website and showed that men operate more frequently based on self-confidence and have a lower yield because of exchanging costs than women in the online loan market. Dorfleitner et al. [24] have researched the role of credit risk and social impact on interest-free P2P lending using Logistic and Tobit models.

Hu and Song [25] test well the investor's rational consciousness from two angles which are the default rate and the full rate and showed that Chinese online investors have rational consciousness of preferring yield and avoiding risk, but they have not tested the relationship between investor number and default rate at full circumstance, and they discriminated yield and default risk by choosing different kinds of items. Although it is rational relatively, there are further chances in the interaction of interest rate and default rate. In the meantime, Liao et al. [7] and Hu and Song [25] have not considered the heteroskedasticity effect of the model. The paper tests the investors' discriminating ability of yield and risk after considering time value of interest rate, relationship between interest rate and credit risk, and heteroskedasticity effect.

Existing research focuses mainly on the full rate and the default rate, and some discuss deep-seated questions such as investors' rational consciousness, but they are not deep enough, and methods used are relatively simple. In the meantime, information asymmetry and adverse selection are more serious in online loan markets than traditional offline markets: on the one hand, online loan lenders have difficulties to get complete credit notes of borrowers; on the other hand, many online loan borrowers are fundraisers who have difficulties in getting credit debts offline [22]. Unlike American market-oriented completely interest rate, the Chinese online loan interest rate is incompletely market-oriented and set up initially by borrowers according to self-conditions within ranges specified by the government and then bidden by lenders according to items' interest rates and information. It is to be called a full bid if the investment fund reaches the amount the borrower is planning to get, and the corresponding interest rate is effective; otherwise, it is to be called a flow bid, and the corresponding interest rate is noneffective. Compared with traditional bank credit debts, lenders in the online loan market are at information disadvantages, and they can only decide based on the items and borrowers' information published by the website and their own experience and then whether the decision is rational, but can the investor discriminate the default rate difference behind the same interest rate? Can the investor discriminate the yield difference behind the same credit risk? There are great theoretical and real meanings for sorting out these questions in the background of constant advancement and coming to an end of the market-oriented Chinese interest rate.

The work of the paper is mainly exhibited: First, the rational consciousness of the online loan investor has been researched from two angles which are the interest rate and the default risk. Interest rate and default risk are influencing each other, but most of the exiting research have not eliminated the mutual influence when testing the impact of interest rate and default rate on investor behavior. And then on the one hand, there may be implied the impact of default risk when testing the impact of interest rate on investor behavior; on the other hand, there may be implied the impact of interest compensation when testing the impact of default risk on investor behavior. (Liao et al. [7] research the impact of default risks not reflected by interest rates on investors' behaviors, and they deduct the implied effect of interest rate on default risk, but they do not deduct the implied effect of default rate on interest rate, namely, the influence of interest rate on investor's behavior maybe is caused by the default risk corresponding to the interest rate. And they have not considered the term structure of the interest rate and heteroskedasticity effect.) The paper not only excludes the role of interest rate from default risk but also excludes the role of default risk from interest rate and tests impacts of default risk not reflected by interest rate and pure yield excluded time term structure and default risk on investor behavior. Second, the paper is based on microdata, and macro and micro are combined. The paper crawled more than 0.3 million data from the Renrendai website using web crawling technology, and the data reflect in detail the microcharacteristics of online loans, and thus, effectiveness, credibility, and reality microfoundation of empirical tests are ensured. Term structure theory of interest rates in the macrofield is applied to the microfield, and the influence of different maturities on the interest rate is considered. Third, at the research perspective, both borrowers' characteristics and lenders' subjective initiatives are considered, and lenders' identification ability of yield and default risk is researched, and at the econometric model, heteroskedasticity is considered, characteristics of Chinese online loans are analyzed, and suggestions on how to develop healthy online loans are offered based on situation analysis, theory combing, and empirical tests.

3. Theoretical Analysis and Research Hypothesis

3.1. Theoretical Analysis. Whether traditional offline or online financial market, investors mainly think of two factors: yield and risk, and for online loan platforms, investors' concern is mainly on the item's interest rate and default risk. Because we research the default risk (credit risk) directly, investors avoiding risk is in keeping with facts. Investors chasing yield and avoiding default risk has become a consistent conclusion. Liao et al. [7] showed that investors in the Renrendai Market are disgusted with risk and thus chase the minimized risk at the equal yield. Hu and Song [25] test the phenomenon that investors of online loans prefer the item with lower default risk at the same yield or higher yield with the same default risk. Next, we should research further how to measure the default risk difference behind the same interest rate, the yield difference behind the same default risk, and whether investors can determine the difference and how to determine it. Akerlof [26] discusses commodity quality uncertainty, information asymmetry, and market structure. In some markets, consumers evaluate utilities of potential purchases according to market statics information, and this will produce the profit difference between the whole and the individual seller. Benefits for all parties can be enhanced by government regulation. Unwritten promises are preconditions for many products and trades that proceed well, but adverse choices caused by information asymmetry may make prices disorderly and the configuration efficiency low. Borrowers in online markets are at an information advantage, and investors can only judge whether borrowers can pay back capital and interest timely in the future according to items and borrowers' information published by borrowers in online platforms and investors' experience. In an imperfect market, different dealers hold different information, some hold specific information and some do not hold specific information only aimed at communication, and some may judge erroneously public information, and generally, dealers who hold advantageous information will gain [27]. People's attention is limited, and mutual interference will appear when attention increase cannot satisfy increased needs [28], and online market platforms publish much information on items and borrowers, and the information mingles with each other, and investors should pay consistent attention to this information and give correct judgments. Peng and Xiong [29] analyze investors' classification learning abilities and their roles in asset dynamic pricing; investors prefer applying limited attention to classification learning and are adept in using markets and industry information but are weak on company-specific information, and thus, investors in online platforms also prefer classification recognition and learning according to information published by the platform and then give investment decision.

In a prefect nonarbitrage market with transparent information, the yield and the risk correspond to each other. But as said in the former, online market information is asymmetric, and risk cannot be indicated by yield and yield cannot be indicated by risk. Default risks behind the two items with the same interest rate have heterogeneity; whether investors can and how to discriminate the heterogeneity need to be discussed further. With the gradual deepening of interest rate marketization, Chinese interest rate marketization has been to the final stage. Online loan interest rate is marketized incompletely, and borrowers decide interest rate levels themselves in the range specified by the government, and lenders decide whether to invest and how much to invest according to information such as borrowers' credit level published in the online website. There are many participants in the online loan website, and information communicates rapidly, and thus, there are generally multiple participants involved in the full item finished in the specified time, and this embodies fully the strength of the market. Investors make decisions mainly considering from the two angles of yield and risk.

In short, theories and existing empirical research show that rational investors in online loan markets chase default risk minimizing at the same yield or yield maximizing at the same risk. The default risk behind the same interest rate and interest rate corresponding to the same default risk may be different, and then, whether Chinese online loan investors can effectively discriminate these differences, and how to behave if they can, all these need to be tested by empirical data.

3.2. Research Hypothesis. The former analyses show that default risk behind the same interest rate in online loan markets may be different, and there may exist a default risk not reflected by the interest rate, which is later called the excess default risk. The interest rate level corresponding to the same default risk may be also different, and there may exist an interest rate that does not correspond to the default risk, which is later called the short excess yield.

Investors who can discriminate the kinds of excess default risk and excess yield will show different preferences to the corresponding item. Generally, the more investors prefer the item, the more investors will make investing decisions rapidly and the investing amount, and thus, items preferred by investors will be full in shorter time and need fewer people, and thus, the preference of investors for the item can be reflected by the full time and full participant number. Rational investors are prone to avoid items with extra default risk, and this will be expressed in two aspects: one is less investors choosing the item and the other is the investor of the item investing with a smaller amount for prudent goal, and the two behaviors will make the full time take longer. Based on the above analysis, we conclude hypothesis 1.

Hypothesis 1 (extra default risk and full time have a positive relationship). At a given yield, the greater the risk uncompensated by the interest rate, the longer the time needed for full.

Not only is the risk not reflected by the interest rate but also the net yield influences investor behavior, and the higher the net yield after being deducted the credit risk and time value, the more investors prefer the item and are willing to invest in the item. Rational investors prefer to choose items with extra yield, and we can get hypothesis 2 similarly to the above analysis.

Hypothesis 2 (extra yield and full time have a reverse relationship). At a given default rate, the greater the net yield after being deducted the credit risk and time value, the shorter the time needed for full.

The preference an investor pays to the item exhibits not only directly in the full time but also indirectly in the participant number for full.

When investors prefer to avoid the item with extra default risk, participant number will appear in two situations: one is that the participant number willing to invest in the item decreases, which makes the item fail, and the other is that at full circumstance, the amount the investor is willing to invest decreases, which leads to the number needed for full to increase. The two situations correspond to two opinions on participant number: one is that the more the public prefers the item, the participant number is more and leads to it easily being full [25], and the other is at full circumstance, the risk not reflected by the interest rate is bigger, the investor is more careful and thus full need the more participant number. And thus, the impact of the investor choosing behavior on the participant number has not coincided, and it is necessary to test empirically further for determining the influencing mechanic and effect. Thus, the impact of investor choosing behavior on participant number has not coincided, and further empirical test is needed to determine the influencing mechanism and result. Because we research the situation of full, it is expected to be the second situation in the paper. Namely, risk premium not reflected by the interest rate has a positive relationship with time and participant number for full. The bigger the risk not predicted by the interest rate exhibits directly in the longer time and indirectly, the more the participant number needed for being full. And thus, we get hypothesis 3.

Hypothesis 3 (extra default risk and participant number for being full have a positive relationship). At a given yield, the bigger the risk not compensated by the interest rate, the more the participant number for being full.

Not only is the risk not reflected by the interest rate but also the net yield has an influence on investor's behavior. The higher the net yield, which is deducted the credit risk and time value, the investor prefers the item more and thus is more willing to invest in it. The higher net yield means fewer participants can complete the full bid, and thus, the smaller the participant number needed for being full. Rational investors prefer to choose items with extra yield, and we can get hypothesis 4 similarly to the above analysis.

Hypothesis 4 (extra yield and participant number for being full have a reverse relationship). At a given default rate, the greater the net yield after being deducted the credit risk and time value, the fewer participants for being full.

4. Research Design, Variable Selection, and Sample Characteristics

4.1. Research Design. The interest rate is one of the most important factors in P2P Lending [24]. Generally, the item's interest rate level corresponds to its risk level, and they interact with each other. If empirical tests directly use the interest rate and default rate of the market, investors' discriminating abilities on risk premium not being reflected by yield and pure yield after the term structure and credit risk have been deducted cannot be measured effectively, because the interacting effect of yield and credit risk is not discriminated. And thus, we conduct, respectively, the default rate not reflected by yield and the pure yield after the term structure and credit risk are deducted based on the relationships among the default rate, interest rate, and credit level.

$$P(d_i = 1|r_i) = f(r_i) + e_{1,i}, \tag{1}$$

$$P(d_i = 1 \mid r_i, x_i) = f(r_t, x_i) + e_{2,i}, \tag{2}$$

$$\Delta P_{d_i} = P(d_i = 1 \mid r_i, x_i) - P(d_i = 1 \mid r_i). \tag{3}$$

d_i expresses whether default, r_i expresses interest rate, and P_{d_i} expresses default probity, equation (1) expresses default probity calculated only based on interest rate, equation (2) expresses default probity calculated based on interest rate and other factors which may influence the default rate, and equation (3) measures the default rate which is not reflected by interest rate, measuring the default difference behind the same interest rate, namely, extra-default rate. The smart investor can discriminate the difference, and it is reflected in whether to bid and the bid amount, and in addition directly reflected the time and indirectly participant number needed for being full. To discriminate different maturity influences, term structure theory of interest rate is applied to online loan interest rate, and the interest rates are converted to continuous compound interest. After discriminating impacts of maturity, interest rates are influenced mainly by credit risk, and each item's credit risk corresponds to a level of interest rate, and the gap between it and the factual interest rate measures the yield no credit risk is corresponding to, namely, the extra yield.

$$\Delta r = \frac{\mathrm{Ln}(1 + t * r_t)}{t} - f\left(\frac{\mathrm{Ln}(1 + t * r_t)/t}{\mathrm{cl, hb, ho}}\right). \tag{4}$$

r_t is the interest rate of term t, $(\mathrm{Ln}(1 + t * r_t))/t$ eliminates the influence of the maturity term and translates r_t to the continuous compound interest, $f((\mathrm{Ln}(1 + t * r_t)/t)/\mathrm{cl, hb, ho})$ is the continuous compound interest corresponding to and predicted by the known credit level, and Δr measures the pure yield that being eliminated the differences of maturity term and credit level, namely, the extra yield.

Extra risk and extra yield can be obtained by formulas (3) and (4), and in addition, equations can be made up by taking the time for being full or the participant number for being full as the explained variable. And specific influencing effects of extra risk and extra yield on the time and the participant number for being full can be tested empirically.

4.2. Variable Selection and Sample Characteristics. Different from the completely marketized mechanism of the online loan interest rate decided by the relationship between borrowers and lenders in USA and England, etc., the Chinese online loan interest rate is determined mainly by borrowers according to their situations and is verified by the platform, and then, potential lenders decide whether to invest and the investment amounts according to information published by the platform. The item fails if the amount investors are willing to invest cannot reach the amount planned to be raised in the given term, which is called out of bid, and the item succeeds if the amount investors are willing to invest reaches the amount planned to be raised in the given term, which is called the full bid. Investors cannot decide the item's interest rate level directly, but they can impact indirectly the final actual interest rate by the model of "vote with feet," and this is an incompletely marketized mechanism of interest rate. Our primary purpose is to research whether investors can discriminate the interest rate and risk of an online loan item, especially the different default risk behind the same interest rate and the different yield behind the same credit risk.

According to the former theoretical analysis, research hypothesis, existing research practices, and data available, we will take the following variables: (1) Whether default (d): it will be assigned 0 if the full item does not default at maturity and 1 if it defaults. (2) Interest rate (r): the promised interest rate of the item. (3) Time for being full (ft): the duration needed for being full, and the unit is day. (4) Number for being full (n): participant number when being full. (5) Total amount (ta): planning to raise the total loan amount, and the unit is yuan. (6) Deadline for repayment (dd): the planned deadline for paying back the raised fund of the item, and the unit is month. (7) Credit level (cl): borrower's credit level, and Renrendai platform offers a comprehensive credit valuing index according to the borrower's various indicators, and seven grades from high to low according to credit level are given—AA, A, B, C, D, E, and HR—and are expressed, respectively, by 7, 6, 5, 4, 3, 2, and 1, and thus, the bigger cl means the higher credit level. (8) Historical borrowing times (hb): the times when the borrower issued a financing project in the Renrendai platform. (9) Historical overdue times (ho): the times when the borrower borrowed successfully and overdue in the Renrendai platform. (10) Year (y): the age of the borrower, and it is required to be from 22 to 70 in the Renrendai platform. (11) Education (e): the borrower's educational background, and it is divided into four grades—high school (or below), college, undergraduate, and postgraduate (above)—is and expressed, respectively, by 0, 1, 2, and 3, and thus, the bigger e means the higher the educational background. (12) Marriage (m): The borrower's marriage situation—1 if married and 0 if not married. (13) Housing (h): the borrower's housing situation—1 if having house and 0 if else. (14) Car (c): the borrower's car situation—1 if having a car and 0 if else. (15) Income (i): the total borrower family income, and the unit is yuan.

In the above, whether default (d) and interest rate (r) are the most concerned indexes by investors, and investors chase high yield and low credit risk, in which generally, the two cannot be both obtained. Time for being full (ft) and number for being full (n) mainly measure investors' preferences to items. Total amount (ta) and deadline for repayment (dd) reflect the basic information of items, credit level (cl), and historical borrowing times (hb), and historical overdue times (ho) reflect borrowers' credit situations. Year (y), education (e), marriage (m), housing (h), car (c), and income (i) reflect borrowers' individual and family situations.

The data are grabbed using web crawl technology and Python3.6 after registering in the Renrendai website which is an online loan platform offering information on loan items. Default or not need not only have been full but also have been finished, and generally, the longest deadline for repayment in the Renrendai platform is 3 years. The sample term we choose is from 2010.10.11 to 2015.01.04, and thus, our entire sample does not conclude the item in repayment. We first grab more than 300,000 sample data and then further remove samples which are not full or are incompatible to the age requirement of the Renrendai website or samples having other obviously abnormal features or missing information, and finally, there are 99,492 samples that can be used.

According to Table 1 at full circumstance, the means of default rates and interest rates are about 5.62% and 12.68%, and the average time and number for being full are about 0.6 hours and 43 people. The average borrowing amount and deadline for repayment are about 56,000 yuan and 2 years. The average credit level is relatively high, historical borrowing times are over 2, and historical overdue times are less than 1. The average age of borrowers is about 39 years, and most borrowers' education gradations are not high. Most borrowers have been married, half of families have a house, and 27.16% of families have a car. Intuitively, the default rate of the Renrendai online loan is relatively low, the interest rate is much higher than the bank interest rate but also in the government-given range, the average time for being full is relatively short and less than 1 hour, and fundraisers who have a historical borrowing record and less overdue are easy to raise funds successfully and be full.

5. Empirical Results

5.1. Measuring Default Rates Not Reflected by Interest Rates. The most fundamental purpose of lenders investing in items on the online platform is to gain yield, and they are concerned mostly with the default rate and interest rate, and they want to gain a stable high yield and simultaneously fear borrowers defaulting and not repaying the principal and interest on time. In circumstances of nonarbitrage and information transparency, the default rate and interest rate have a linear correspondence relationship, and thus, default levels can be reflected by only using interest rates. We construct the single-factor heteroskedasticity Probit model by taking the default rate as an explained variable and the interest rate as an explanatory variable, and the heteroskedasticity is in the form of [30]

$$P_{d_i} = P(d_i = 1 \mid r) = \Phi(r_i) + \varepsilon_i, \varepsilon_i \sim N\left(0, {\sigma_i}^2\right), \quad {\sigma_i}^2 = e^{2f(r_i)}. \tag{5}$$

Parameters can be estimated by the maximum likelihood function [30]:

$$\log L = \frac{n}{2} \log 2\pi - \frac{1}{2}\sum_{i=1}^{n} 2f(r_i) - \frac{1}{2}\sum_{i=1}^{n} e^{-2f(r_i)} \left(P_{d_i} - \Phi(r_i)\right)^2. \tag{6}$$

Table 1: Samples of statistical characteristics.

Variables	Sample number	Mean	Standard deviation	Minimum	Maximum
d	99492	0.0562	0.2304	0.0000	1.0000
r	99492	12.6786	1.3382	3.0000	24.4000
ft	99492	0.6076	2.0330	0.0000	36.9083
n	99492	42.6097	59.8358	1.0000	1840.0000
ta	99492	56242.8200	57495.49	3000.0000	3000000.0000
dd	99492	25.1656	10.8038	1.0000	36.0000
cl	99492	5.4630	1.4584	1.0000	7.0000
hb	99492	2.1502	6.9564	1.0000	148.0000
ho	99492	0.5926	2.8495	0.0000	54.0000
y	99492	39.2862	8.2369	22.0000	60.0000
e	99492	0.9819	0.7374	0.0000	3.0000
m	99492	0.7193	0.4493	0.0000	1.0000
h	99492	0.4789	0.4996	0.0000	1.0000
c	99492	0.2716	0.4448	0.0000	1.0000
i	99492	15677.0400	14727.5000	1000.0000	50000.0000

The statistics for heteroskedasticity obey the chi-square distribution [30]:

$$n \log \sum_{i=1}^{n} \frac{\left(P_{d_i} - \Phi(r_i)\right)^2}{n} - \sum_{i=1}^{n} 2\hat{f}(r_i) \sim \chi^2(1). \tag{7}$$

According to Table 2, first, there is the heteroskedasticity effect. Heteroskedasticity testing shows that the null hypothesis there does not exist and heteroskedasticity is denied, and this illustrates that the traditional Probit model cannot reflect effectively the heteroskedasticity effect that existed in the model, and the heteroskedasticity Probit model needs to be used. Second, interest rates have a significant positive role on default rates. The higher interest rate corresponds to the higher default rate, and this indicates that online loan investors cannot purely choose the item with the high interest rate but must balance default rates and interest rates. Third, the default rate level reflected by the interest rate can be calculated based on formula (3) and the parameters given in Table 2.

If in a complete market-oriented interest rate circumstance as mentioned above, then the interest rate can reflect completely the default risk. In fact, the Chinese online loan interest rate is incomplete market-oriented, it is decided by the borrower unilaterally, and the investor can only take the passive decision mode of "voting with feet." Thus, we add other variables into the default rate estimating model and construct the following multiple-factor heteroskedasticity Probit model, and the heteroskedasticity takes the form of [30]

$$P_{d_i} = P(d_i = 1 \mid r_i, x_i) = \Phi(r_i, x_i) + \varepsilon_i, \varepsilon_i \sim N\left(0, \sigma_i^{\,2}\right), \\ \sigma_i^{\,2} = e^{2f(r_i, x_i)}. \tag{8}$$

Parameters can be estimated by the maximum likelihood function [30]:

$$\log L = \frac{n}{2} \log 2\pi - \frac{1}{2}\sum_{i=1}^{n} 2f(r_i, x_i) - \frac{1}{2}\sum_{i=1}^{n} e^{-2f(r_i, x_i)}\left(P_{d_i} - \Phi(r_i, x_i)\right)^2. \tag{9}$$

The statistics for heteroskedasticity obey the chi-square distribution [30]:

$$n \log \sum_{i=1}^{n} \frac{\left(P_{d_i} - \Phi({}_i r, x_i)\right)^2}{n} - \sum_{i=1}^{n} 2\hat{f}(r_i, x_i) \sim \chi^2(m). \tag{10}$$

According to Table 3, first, there is a heteroskedasticity effect in the driving factor model of the default rate. The heteroskedasticity existing test shows that the null hypothesis that heteroskedasticity does not exist is denied, and this illustrates that the traditional Probit model cannot reflect effectively the heteroskedasticity effect existing in the model, and thus, the heteroskedasticity Probit model needs to be used. Second, total loan amount, deadline, age, and historical overdue times have a significant positive relationship with default rates. The bigger the loan amounts, the bigger the stress for repaying principal and interest in a timely manner, and thus, the bigger the corresponding default rate. The longer the deadline, the larger the uncertainty factor and the bigger the risk for lenders regaining their principal and interest at maturity and thus the corresponding bigger default rate. The older borrowers correspond to the higher default rate possibly because older borrowers in the online loan market have been given the restricted employment chance and reduced ability to repay principal and interest in a timely manner, and thus a corresponding higher default rate. The bigger historical overdue times indicate the

Table 2: Single factor estimated results of heteroskedasticity Probit model of default.

Variables	Explained variable: whether default	
	dy/dx	z-statistics and p value
r	0.0145	28.0700 (≤0.001)***
Testing whether existing heteroskedasticity		
Null hypothesis	Lr-statistics and p value	Conclusions
Homoskedasticity	104.48 (≤0.001)***	Rejecting null hypothesis, namely, existing heteroskedasticity
Variance equation		
Variables	Coefficient estimated values	z-statistics and p value
r	-0.0367	-11.51 (≤0.001)***

Note: $***$ means that the corresponding coefficient is significant at 1% significance level.

borrowers having a relatively poor historical performance record, and the significant positive effect of historical overdue times on default rates indicates that the borrowers' repaying behaviors have inertial characteristics and the borrowers' past performance record has a significant forecasting effect on his future repaying behavior. Total loan amount, deadline, age, and historical overdue times that have a significant positive effect on default rates is not only in accordance with theories but also actual situations. Third, credit level, education, historical borrowing times, income, housing, and car have significant negative effects on default rates. Credit level is a comprehensive valuing index given by the online loan platform according to borrowers' various indexes, and the borrower's higher credit level means the borrower's better credibility and the stronger ability and willingness for borrowers to repay principal and interest in a timely manner. Generally, the higher the education degree means the better the skill, employment, and development opportunities, and the stronger the ability to repay the principal and interest in a timely manner, and thus, the higher borrower's education degree corresponds to the lower default rate. Historical borrowing times represent the success times of the fundraiser in the online loan market in the past, and the more times mean the fundraiser's more successful experience and have a positive feedback on the borrower repaying the principal and interest in a timely manner and thus corresponds to the lower default rate. The higher the fundraiser's income, the better his ability to repay the principal and interest in a timely manner in the future, and thus, the income level has a reverse relationship with the default rate. Housing and car are one kind of capital; on the one hand, owners have had some capital accumulation, and on the other hand, owners' future rigid expenditures on housing and car are lower than those of nonholders, and thus, both housing and car have a significant negative influence on the default rate. Credit level, education, historical borrowing times, income, housing, and car have significant negative influences on default rates which coincide not only on theory requirements but also in actual situations.

In addition, by integrating Tables 2 and 3, seeing from the size and significance of r's coefficient, introducing other variables reduces greatly r's role, and not only does other information have a relatively strong role on the default rate but also the role of the interest rate on the default rate can be embodied by other information, namely, the role of the interest rate on the default rate is partly through indirect credit level indexes. Default rates which synthesize various information, and which utilize only the interest rate, can be deduced by the models corresponding to Tables 2 and 3. The difference between the two is the default rate, which is not reflected by the interest rate, namely, extra default risk, and it has been excluded in the interest rates' role, and this kind of default rate does not have an interest rate return and can be discriminated and avoided by rational investors.

5.2. Measuring Interest Rate Eliminating Time Value and Default Factor. We have measured different risks behind the same interest rate and obtained the default rate not fully reflected by the interest rate. Next, we apply the term structure theory of interest rate to the online loan interest rate and measure the interest rate that excluded the default risk according to the relationship between interest rates and default risk, and it is the pure yield not corresponding to the default risk, and then, we further test whether online loan investors can discriminate the kind of pure yield and make rational choices.

Interest rates published in the online loan website have different maturities, and interest rates with different maturities are different according to the term structure theory of the interest rate. In order to strengthen the comparability and analyze more specifically the investors' yield discriminating ability, we transfer the interest rate into a continuous compound form according to the formula $[\mathrm{Ln}(1 + t * r_t)]/t$, and thus, heterogeneity effect of different maturities is eliminated.

Since the default rate is ex post variable and cannot be determined completely beforehand, namely, the default rate cannot be measured directly, thus, we apply indirect methods: method 1 only using credit factor and method 2 using various factors. Generally, the default risk is mainly decided by borrowers' credit risk level, and thus, borrowers enact the item's interest rate level according to their own credit risk level, and investors decide whether to accept the interest rate level published by borrowers according to the borrowers' credit level. The Renrendai website publishes a credit level (cl) index according comprehensively to borrowers' various information, and the two indexes are directly

Table 3: Multifactor estimated results of heteroskedasticity Probit model of default.

(a)

Variables	Explained variable: whether default dy/dx	z-statistics and p value
r	0.0002	1.56 (0.119)
cl	-0.0019	-8.29 (≤0.001)***
lnta	0.0021	5.41 (≤0.001)***
dd	0.0002***	6.61 (≤0.001)***
y	0.0001	2.58 (0.010)***
e	-0.0008	-2.32 (0.020)**
m	0.0002	0.27 (0.785)
hb	-0.0030	-18.22 (≤0.001)***
ho	0.0177	26.60 (≤0.001)***
lni	-0.0014	-3.82 (≤0.001)***
Housing (h)	-0.0016	-2.81 (0.005)***
c	-0.0013	-1.99 (0.047)**
	Testing whether existing heteroskedasticity	
Null hypothesis	Lr-statistics and p value	Conclusions
Homoskedasticity	8971.15 (≤0.001)	Rejecting null hypothesis, namely, existing heteroskedasticity

(b)

Variables	Variance equation Coefficient estimated values	z-statistics and p value
r	0.0323	4.1500 (≤0.001)***
cl	0.0326	2.3100 (0.0210)**
lnta	-0.0805	-3.6600 (≤0.001)***
dd	-0.0335	-14.3000 (≤0.001)***
y	-0.0071	-2.5300 (0.0110)**
e	0.0109	0.5000 (0.6180)
m	-0.0046	-0.1100 (0.9130)
hb	0.0413	10.8300 (≤0.001)***
ho	0.5852	30.8900 (≤0.001)***
lni	0.0908	4.0400 (≤0.001)***
h	0.1730	4.4800 (≤0.001)***
c	0.0603	1.4900 (0.1350)

Note: *** means that the corresponding coefficient is significant at 1% significance level, and ** means that the corresponding coefficient is significant at 5% significance level.

related borrowers' credit level: historical borrowing times (hb) and historical overdue times (ho). Historical borrowing times (hb) reflect the times which borrowers have issued items and succeeded to be full in the Renrendai website. The more historical borrowing times mean borrowers having better historical records and being accepted by online loan investors in the past, and this shows indirectly borrowers having a higher credit level and more probability to repay the principal and interest rate in a timely manner. Historical overdue times (ho) reflect the times borrowers raise funds successfully in Renrendai platform but cannot repay the principal or interest timely in the end; the more historical overdue times indicate borrowers having the worse records previously, and this indirectly shows that the borrower's credit level is not high, and there is relatively bigger probability to being overdue. To reflecting the impact of the credit risk on the interest rate fully, besides the credit level, historical borrowing times and historical overdue times are also introduced as supplementary variables.

According to Table 4, credit factors have important decisive effects on interest rates, and the credit level and interest rate level are in a significant negative relationship. This is in

Table 4: The influencing results of credit factors on continuous compound interest.

Variables	Explained variable: continuous compound interest Coefficient estimated values	z-statistics and p value
cl	-0.2894	-103.58 (≤0.001)***
hb	-0.0007	-1.29 (0.197)
ho	0.0559	37.78 (≤0.001)***
Constant term	12.7960	795.30 (≤0.001)***

Note: *** means that the corresponding coefficient is significant at 1% significance level.

Table 5: Influencing factors of continuous compound interests.

Variables	Explained variable: continuous compound interest Coefficient estimated values	t statistics and p value
cl	-0.2203	-63.87(≤0.001)***
lnta	-0.2729	-44.83 (≤0.001)***
dd	0.0022	5.03 (≤0.001)***
y	0.0050	11.37 (≤0.001)***
e	-0.0599	-12.83 (≤0.001)***
m	-0.0265	-3.34 (0.001)***
hb	-0.0088	-15.63 (≤0.001)***
ho	0.0588	40.19 (≤0.001)***
lni	0.0058	1.35 (0.175)
h	-0.0304	-3.91 (≤0.001)***
c	0.1021	12.40 (≤0.001)***
Constant term	15.1065	268.56 (≤0.001)***

Note: *** means that the corresponding coefficient is significant at 1% significance level.

accordance with economic theories, and the higher the credit level indicates the lower the borrower's credit risk and thus corresponds necessarily to the smaller interest rate level. Investors select the interest rate according to not only the credit level but also other credit factors, and the significant positive effect of historical overdue times on the interest rate indicates that maybe the higher historical overdue times correspond to the higher default rate and need the higher yield compensation, and thus items with higher historical overdue times generally correspond to a higher interest rate. This indirectly indicates that market investors are clever and can discriminate credit risk and have the willingness to take part in items with higher risk only when are given higher yield compensation.

The influence of the item credit risk on interest rate is mainly indirectly measured using factors which have direct relationship with credit in Table 4. Although default risks behave mostly as credit risks, other factors also have certain influence on default rate. To measure fully the impact of default risks on interest rates, we further consider more factors and indirectly measure the impact using various factors, and the econometric results are seen in Table 5.

Table 5 reflects the driving effects of borrower's information published by the online loan platform on the item's interest rate, and these factors indirectly embody the borrower's default risk. Testing results in Table 5 are also in accordance with economic theories and actual situations: credit level, education, historical borrowing times, and housing have significant negative effects on interest rates, and this goes along with empirical results of Table 3, and these factors also have significant negative effects on default risks, and all these embody low default risks which correspond to low interest rates. Deadline for repayment, year, and historical overdue times have significant positive effects on the interest rate, and this also goes along with the empirical results of Table 3, and these factors also have significant positive effects on default risks, and all these embody high default risks which correspond to high interest rates.

Term structure theory of the interest rate is applied to decide the interest rate level, and the continuous compound interest is calculated. To further eliminate the credit level's impact, the interest rates, corresponding to a certain default risk, are calculated according to the models corresponding to Tables 4 and 5, respectively. The differences between continuous compound interests and the two former interest rates represent pure yields deducted by default risks and are expressed by $\Delta r1$ and $\Delta r2$, and thus, extra yields after the deduced term structure and default risk factor are obtained.

5.3. Research on Participant Numbers and Time for Being Full. Although online loan interest rates are decided voluntarily by the fundraiser at the government-given range, the investors can also determine the final bargaining interest rate level by the mode of "vote with feet." After the interest rate level is decided by the fundraiser, the investor will make the decision whether to invest comprehensively according to the deadline and risk level, and generally, investors make the decision from the two angles of yield and risk, and they chase the minimal risk at the same yield or the maximal yield at the same risk.

As to the default risk, Liao et al. [7] get the default risk which has no interest rate return by excluding the default risk reflected by the interest rate factor, and thus, the impact of the interest rate can be eliminated when measuring the impact of the default risk on the investor's behavior. But as to the interest rate factor, they have not considered the term structure and default risk level, and thus, the item's interest rate may contain the corresponding term and default risk factors. We first apply the term structure theory of the interest rate in the macroeconomic field to online loan interest rates and get the continuous compound interest to eliminate the influences of different deadlines and then compare it with the continuous compound interest corresponding to the default factors and further get the pure yield deducted the impacts of different maturities and default risk. We measure investors' weighing behaviors, respectively, based on the default risk not reflected by the interest rate and pure yield after deducting the impacts of maturity and default risk.

Table 6: Regression results to time for being full.

Explanatory variables	Explained variable: unit financing's time for being full Equation (1) Estimated values and p value	Equation (2) Estimated values and p value	Equation (3) Estimated values and p value	Equation (4) Estimated values and p value	Equation (5) Estimated values and p value
Mean equation					
Constant term	0.1037 (≤0.001)***	0.1006 (≤0.001)***	0.1189 (≤0.001)***	0.0864 (≤0.001)***	0.0865 (≤0.001)***
Δr_1	-0.0610 (≤0.001)***			-0.0077 (≤0.001)***	
Δr_2		-0.0856 (≤0.001)***			-0.0106 (≤0.001)***
ΔP_d			1.0305 (≤0.001)***	0.0375 (≤0.001)***	0.0346 (≤0.001)***
ar(1)	0.8505 (≤0.001)***	0.8588 (≤0.001)***		0.9682 (≤0.001)***	0.9688 (≤0.001)***
ma(1)	-0.6445 (≤0.001)***	-0.6537 (≤0.001)***	-0.0014 (≤0.001)***	-0.7945 (≤0.001)***	-0.7952 (≤0.001)***
Variance equation					
Constant term	0.0161 (≤0.001)***	0.0161 (≤0.001)***		0.0001 (≤0.001)***	0.0001 (≤0.001)***
RESID(-1)^2	0.0779 (≤0.001)***	0.0824 (≤0.001***)	0.2176 (≤0.001)***	0.1010 (≤0.001)***	0.1000 (≤0.001)***
GARCH(-1)	0.8912 (≤0.001)***	0.8886 (≤0.001)***	0.7824 (≤0.001)***	0.8974 (≤0.001)***	0.8984 (≤0.001)***
Residual test					
F-statistics of heteroskedasticity test	0.0549 (0.8147)	0.0357 (0.8501)	0.1607 (0.6885)	0.6629 (0.4155)	0.7008 (0.4025)
DW value of autocorrelation test	2.1004	2.0994	1.7299	1.9312	1.9290
Root inverse of ar and ma	Within unit circle	Within unit circle	Within unit circle	Within unit circle	Within unit circle
ADF statistics of stationarity test	-36.7565 (≤0.001)***	-36.3303 (≤0.001)***	-30.7849 (≤0.001)***	-31.2582 (≤0.001)***	-31.2583 (≤0.001)***

Note: *** means that the corresponding coefficient is significant at 1% significance level.

As to the explained variables time and participant number for being full, Liao et al. [7] apply directly the value corresponding to each item and take the raising amount as the explaining variable.

Considering that time and participant number for being full will certainly be affected by the amount being raised, and to measure better the investors' risk and yield discrimination abilities, we use time and participant number corresponding to unit financing, namely, time and participant number of each item divided by the amount being raised by each item. Empirical results are seen in Tables 6 and 7, and they measure, respectively, online loan investors' weighing behaviors between risk and yield through time and participant number for being full.

According to Table 6, first, it is necessary to consider the model heteroskedasticity. On the one hand, coefficients of AR and GARCH terms in the variance equation are significant, and this illustrates that it is necessary to consider heteroskedasticity; on the other hand, residual testing results show that there no longer exists heteroskedasticity in models after considering the first-order GARCH effect, and at the same time, models' residuals are stationary, and all these explain that the models are effective. Second, the default rate not reflected by the interest rate has a significant positive relationship with time for being full, and the bigger default risk corresponds to the longer time for being full, and thus, hypothesis 1 is verified. The pure yield after deducting the effects of different terms and credit risk levels has a significant negative relationship with time for being full, and the higher yield corresponds to the shorter time, and thus, hypothesis 2 is verified. The front empirical tests are about the time for being full, but the preference of the investor for the item appears in not only time but also participant number for being full. Next, we research how the default risk not reflected by the interest rate and the pure yield after deducting the effects of different terms and credit risk levels influence the participant number for being full.

According to Table 7, the model heteroskedasticity effect can be measured effectively by the GARCH(1,1) model. On the one hand, the equation coefficients are significant, and this indicates that it is necessary to use the GARCH(1,1) model; on the other hand, it is not necessary to use the higher-order GARCH model by model residual tests, and in addition, model residuals also pass the stationarity test, and all these indicate the models' effectiveness.

Results in Table 7 show that the default rate not reflected by the interest rate has a significant positive relationship with the participant number, and this indicates that the investor can discriminate the default risk not compensated by the yield and make prudent measures and generally reduce the item's investing amount, and thus, more participants are needed to complete the full bid, and hypothesis 3 is fulfilled. The pure yield after deducting the term and credit risk has a significant negative relationship with the participant number for being full, and this indicates that the investor prefers more the item with the higher pure yield, and

Table 7: Regression results to participant number for being full.

Explanatory variables	Explained variable: unit financing's participant number for being full				
	Equation (6) Estimated values and p value	Equation (7) Estimated values and p value	Equation (8) Estimated values and p value	Equation (9) Estimated values and p value	Equation (10) Estimated values and p value
Mean equation					
Constant term	0.0872 (≤0.001)***	0.0873 (≤0.001)***	0.0864 (≤0.001)***	0.0864 (≤0.001)***	0.0865 (≤0.001)***
Δr_1	-0.0096 (≤0.001)***			-0.0077 (≤0.001)***	
Δr_2		-0.0121 (≤0.001)***			-0.0106 (≤0.001)***
ΔP_d			0.0465 (≤0.001)***	0.0375 (≤0.001)***	0.0346 (≤0.001)***
ar(1)	0.9696 (≤0.001)***	0.9700 (≤0.001)***	0.9665 (≤0.001)***	0.9682 (≤0.001)***	0.9688 (≤0.001)***
ma(1)	-0.7920 (≤0.001)***	-0.7928 (≤0.001)***	-0.7919 (≤0.001)***	-0.7945 (≤0.001)***	-0.7952 (≤0.001)***
Variance equation					
Constant term	0.0001 (≤0.001)***	0.0001 (≤0.001)***	0.0001 (≤0.001)***	0.0001 (≤0.001)***	0.0001 (≤0.001)***
RESID(-1)^2	0.0990 (≤0.001)***	0.0981 (≤0.001)***	0.1023 (≤0.001)***	0.1010 (≤0.001)***	0.1000 (≤0.001)***
GARCH(-1)	0.8978 (≤0.001)***	0.8988 (≤0.001)***	0.8957 (≤0.001)***	0.8974 (≤0.001)***	0.8984 (≤0.001)***
Residual test					
F-statistics of heteroskedasticity test	0.7385 (0.3901)	0.7781 (0.3777)	0.6376 (0.4246)	0.6629 (0.4155)	0.7008 (0.4025)
DW value of autocorrelation test	1.9312	1.9289	1.9374	1.9312	1.9290
Root inverse of ar and ma	Within unit circle	Within unit circle	Within unit circle	Within unit circle	Within unit circle
ADF statistics of stationarity test	-31.2087 (≤0.001)***	-31.2277 (≤0.001)***	-31.7400 (≤0.001)***	-31.2582 (≤0.001)***	-31.2583 (≤0.001)***

Note: *** means that the corresponding coefficient is significant at 1% significance level.

thus, less participants are needed to complete the full bid and hypothesis 4 is fulfilled.

The front hypotheses are verified by empirical tests, and the following results can be obtained: First, interest rates and risks mingle with each other, online platform interest rates embody default risks, investors can give trade-offs, and incomplete market-oriented online interest rates reflect the matching result well between the investor and the fundraiser. Second, it is necessary to consider model heteroskedasticity in econometric methods and the blending effect of the interest rate and the default risk in index measurements the impacts of different terms and credit risk levels on interest rates, and the impacts that have been reflected by the interest rate as well as the default rate. Third, the investor's preference for the item embodies directly the time and indirectly the participant number for being full, and generally, if the investor prefers the item more, the shorter the time and the less the participant number needed to be full. Fourth, the investor can discriminate not only default risk not reflected by the interest rate but also pure yield after deducting impacts of different terms and credit risks.

Empirical tests have fulfilled the four hypotheses in the paper's front and verified whether the investor who prefers the item directly embodies the time and specifically shows that the more the preference for the item, the shorter the time for being full, and indirectly embodies the participant number and specifically shows that the more preference for the item, the less the participant number for being full. Compared with the results of Liao et al. [7], both the influencing direction and the significance of different default risks behind the same interest rate on time and participant number for being full are consistent, but either the influencing direction or the significance of the interest rate on time and participant number for being full is inconsistent. It may be due that Liao et al. [7] directly use interest rates without deducting default risks, and thus, the empirical tests show that interest rates have no significant impact on time for being full. On the one hand, our empirical results' credibility is verified; on the other hand, not only should default risks deduct the parts reflected by interest rates but also interest rates should deduct the parts used to compensate the default risk.

5.4. Robustness Test. In the front, we only consider the two factors, the interest rate and the default rate, when researching the time and participant number for being full, and to test robustly, we further consider family income and test adding other variables whether they have some impacts on the conclusions.

According to Table 8, after introducing the income factor, although the coefficient's sizes change slightly, significance and signs of coefficients are the same with the former without considering the income factor, and effectiveness and robustness of our empirical testing results are verified. Additionally, the income level has a significant negative effect on time and participant number for being

Table 8: Regression results to time and participant number for being full.

Explanatory variables	Explained variable: time for being full		Explained variable: participant number for being full	
	Equation (11) Estimated values and p value	Equation (12) Estimated values and p value	Equation (13) Estimated values and p value	Equation (14) Estimated values and p value
Mean equation				
Constant term	0.6105 (≤0.001)***	0.4612 (≤0.001)***	0.1266 (≤0.001)***	0.1228 (≤0.001)***
Δr_1	-0.0472 (≤0.001)***		-0.0080 (≤0.001)***	
Δr_2		-0.0361 (≤0.001)***		-0.0104 (≤0.001)***
ΔP_d	1.1429 (≤0.001)***	1.3552 (≤0.001)***	0.0364 (≤0.001)***	0.0340 (≤0.001)***
lni	-0.0538 (≤0.001)***	-0.0401 (≤0.001)***	-0.0043 (≤0.001)***	-0.0039 (≤0.001)***
ar(1)	0.0233 (≤0.001)***	0.9949 (≤0.001)***	0.9686 (≤0.001)***	0.9691 (≤0.001)***
ma(1)		-0.9917 (≤0.001)***	-0.7951 (≤0.001)***	-0.7958 (≤0.001)***
ar(2)	0.8968 (≤0.001)***			
ma(2)	-0.8721 (≤0.001)***			
Variance equation				
Constant term			0.0001 (≤0.001)***	0.0001 (≤0.001)***
RESID(-1)^2	0.0615 (≤0.001)***	0.2222 (≤0.001)***	0.0993 (≤0.001)***	0.0984 (≤0.001)***
GARCH(-1)	-0.0614 (≤0.001)***	-0.2219 (≤0.001)***	0.8983 (≤0.001)***	0.8992 (≤0.001)***
RESID(-2)^2	1.7409 (≤0.001)***	1.4562 (≤0.001)***		
GARCH(-2)	-0.7410 (≤0.001)***	-0.4565 (≤0.001)***		
Residual test				
F-statistics of heteroskedasticity test	0.0718 (0.7888)	0.1073 (0.7432)	0.6906 (0.406)	0.7312 (0.3925)
DW value of autocorrelation test	1.8636	1.8412	1.9292	1.9271
Root inverse of ar and ma	Within unit circle	Within unit circle	Within unit circle	Within unit circle
ADF statistics of stationarity test	-34.3592 (≤0.001)***	-42.9152 (≤0.001)***	-31.3891 (≤0.001)***	-31.4052 (≤0.001)***

Note: *** means that the corresponding coefficient is significant at 1% significance level.

full, and the investor prefers the item corresponding to the higher income level; thus, the full bid needs a relatively shorter time and fewer participants. Our previous basic hypotheses are verified further, namely, investors' preferences for items directly embody the time and express a shorter time needed and indirectly embody the participant number and are expressed as a less participant number needed. Term structure of the interest rate is applied to the online loan interest rate, and the impact of different deadlines is eliminated using continuous compound interest. Further, the annual compound interest is applied (to save space, the testing results are omitted), and the results indicate that signs and significance of coefficients are the same with continuous compound interest except slight changes of coefficient sizes, and effectiveness and robustness of empirical testing results are ensured.

6. Conclusions

Risk levels not reflected by interest rates and pure yields after deducting the term and risk factors are applied to measure investors' discrimination abilities on risks and yields, and online loan investors' discrimination abilities and matching consciousness on risks and yields are researched from the angles of the time and participant number for being full, and the main conclusions and suggestions are obtained as follows.

First, investors have relatively strong matching consciousness, and investors have not blindly selected the item with the high interest rate. Although the higher interest rate corresponds to the higher default risk, the interest rate cannot completely reflect the default risk, and other information are also helpful for reflecting default risk, and investors as passive receivers of online loan rate cannot directly decide the interest rate level, but they can take part in the final interest rate decision by the mode of "vote with feet."

Second, investors' preferences to items embody directly the time and indirectly the participant number for being full. Generally, if the investor prefers the item more, the shorter the time and the less participant number needed for being full. Yield, default rate, and borrower's income are all in accordance with the characteristics.

Third, investors have relatively strong risk discrimination abilities and can identify default risks not reflected by interest rates. If interest rates have been completely market-oriented, then default risks can be reflected entirely by interest rates, and Chinese interest rates are incompletely market-oriented and the same interest rate may correspond to the different default risks, and investors identify this kind of default risk using other information besides interest rates. Investors' discrimination consciousness on this kind of default risk embody directly the time and indirectly the participant number, and items with the bigger kind of default risk need longer time and more participant number for being full.

Fourth, investors have relatively strong yield discrimination abilities and can identify pure yields after deducting deadline and credit risk factors. Investors' discrimination on pure yields embody directly the time and indirectly the participant number, and items with the bigger pure yields need the shorter time and the less participant number for being full.

Fifth, the online loan platform has a relatively strong self-purification function, and historical information (borrowing times and overdue times) have significant forecasting effect on the default risk. The self-purification function of the online loan platform guarantees its sustainable development, but there is further subdivided space in credit valuing, and it is necessary to strengthen the discrimination function of credit valuing to the default risk.

In short, although the online loan interest rate is incompletely market-oriented, it embodies demands of borrowers and lenders and is the weighing result of borrowers and lenders. Investors can identify the different default risks behind the same interest rate and the different yield level behind the same credit risk. Our empirical results show the following: the completely market-oriented online loan interest rate has both actual basis and necessity. Investors having relatively strong rational consciousness can identify not only the different credit risk behind the same interest rate but also the pure yield after deducting the different deadline and credit risk's effects, and these offer feasible actual basis for marketization of online loan interest rates. Additionally, when the default risk has not been compensated by the interest rate or the pure yield, after deducting deadline and credit risk effects, is too low, the item being full bid needs more participants and a longer time. If investors' number is limited in the online loan market, many items may be un-full bid, and the effective funds needed may be suppressed. Complete market-oriented interest rates can make the item be full bid by use of shortening time and reducing participant number for being full, thus satisfying better borrowers' financing needs. ‚Next interest rate marketization reform should be strengthened further, and interest rate marketization and competing mechanism should be loosened and introduced in online loan markets, and thus, the price decision mechanism of the interest rate is played better, and investors' behavior is guided better by the interest rate.

In addition, the online loan platform as a shared information platform can exert the roles of information intermediary, transaction cost reduction, and fund allocation effectiveness improvement, but compared with the traditional financial market, the network information asymmetry is serious. And thus, in the next, information disclosure especially borrowers' historical information should be perfected and strengthened further, and online loan investors' information collection ability and yield and risk identification abilities using existed information should be enhanced.

The paper has researched investors' recognition abilities on default and yield using big financial data but the mathematics deduction has not been offered, and in the future, we will give a rigorous mathematical proof.

Data Availability

The raw data can be obtained through the crawler method after registering a personal account in the Renrendai platform. And the data are also available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Infinitely Many Solutions for Discrete Boundary Value Problems with the (p, q)-Laplacian Operator

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In this paper, we consider the existence and multiplicity of solutions for a discrete Dirichlet boundary value problem involving the (p, q)-Laplacian. By using the critical point theory, we obtain the existence of infinitely many solutions under some suitable assumptions on the nonlinear term. Also, by our strong maximum principle, we can obtain the existence of infinitely many positive solutions.

1. Introduction

Let N be a positive integer and denote with $[1, N]$ the discrete set $\{1, \cdots, N\}$. In this paper, we consider the existence of infinitely many solutions for the following discrete Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u(j-1) - \Delta_q u(j-1) + \alpha(j)\phi_p(u(j)) + \beta(j)\phi_q(u(j)) = \lambda g(j, u(j)), \forall j \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases} \tag{1}$$

where $\Delta_r u(j) \coloneqq \Delta(\phi_r(\Delta u(j)))$ is the discrete r-Laplacian, $\phi_r(u) = |u|^{r-2}u$ with $u \in \mathbb{R}$, $\Delta u(j) = u(j+1) - u(j)$ is the forward difference operator, $g(j, \cdot)$: $\mathbb{R} \longrightarrow \mathbb{R}$ is continuous for each $j \in [1, N]$, $1 < q \le p < +\infty$, λ is a positive parameter, and $\alpha(j), \beta(j) \ge 0$ for all $j \in [1, N]$.

In the past decades, there has been tremendous interest in the study of difference equations, with the development of engineering, physics, economy, and so on (see [1–4]). Most results about the boundary value problems of difference equations are obtained by using the method of upper and lower solutions and fixed point methods (see [5–7]). In 2003, Guo and Yu [8] first applied the critical point theory to study the existence of periodic and subharmonic solutions for a second-order difference equation. Since then, the critical point theory has been employed to study difference equations, and many meaningful results have been obtained, concerning periodic solutions [9, 10], homoclinic solutions [11–13], heteroclinic solutions [14], and especially in boundary value problems [15–20]. For example, Candito and Giovannelli [21] established the existence of multiple solutions of the following problem

$$\begin{cases} -\Delta_p u(j-1) = \lambda f(j, u(j)), j \in [1, N], \\ u(0) = u(N+1) = 0. \end{cases} \tag{2}$$

Later, Bonanno and Candito [22] established the existence of infinitely many solutions of the following problem

$$\begin{cases} -\Delta_p u(j-1) + q(k)\phi_p(u(j)) = \lambda f(j, u(j)), j \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases} \tag{3}$$

where $q(j) \ge 0$ for all $j \in [1, N]$. Obviously, (2) is a special case $(q(j) = 0)$ of (3). After that, under different conditions, D'Aguì et al. [23] established the existence of at least two positive solutions of (3).

In [24], Li and Zhou considered the following discrete mixed boundary value problem

$$\begin{cases} -\Delta_p u(j-1) + s(j)\phi_q(u(j)) = \lambda f(k, u(k)), j \in [1, N], \\ u(0) = \Delta u(N) = 0, \end{cases} \tag{4}$$

where $s(j) \geq 0$ for all $j \in [1, N]$. By using the critical point theory, the authors obtained the existence of at least two positive solutions for (4).

The boundary value problems involving the sum of a p-Laplacian operator and of a q-Laplacian operator is more common, because this arises in the study of stationary solutions of reaction-diffusion systems (see [25]). For example, Mugnai and Papageorgiou [26] and Marano et al. [27] investigated the following Dirichlet problem

$$\begin{cases} -\Delta_p u - \mu\Delta_q u = f(x, u), & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{5}$$

where $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Carathéodory's conditions, and they obtained the existence of multiple solutions of (5).

In [28], Nastasi et al. proved the existence of at least two positive solutions for problem (1). Compared with the discrete boundary value problem involving p-Laplacian operator, there are few results on the discrete boundary value problem with (p, q)-Laplacian operator except [28]. Inspired by the above results, we want to investigate the multiplicity of solutions for problem (1).

In this paper, under suitable assumptions, we use the critical point theory obtained in [29] to establish the existence of infinitely many solutions for discrete (p, q)-Laplacian equations with Dirichlet type boundary conditions. Moreover, by our strong maximum principle, we can obtain the existence of infinitely many positive solutions of (1).

The rest of this paper is organized as follows. In Section 2, we recall the critical point theory and show some basic lemmas. In Section 3, our main results and proofs are presented. After that, we have two examples to explain our main results. We conclude our results in the last section.

2. Preliminaries

Let X be a reflexive real Banach space and let $I_\lambda : X \longrightarrow \mathbb{R}$ be a function satisfying the following structure hypothesis:

(H) $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \longrightarrow \mathbb{R}$ are two functions of class C^1 on X with Φ coercive, i.e., $\lim_{\|u\| \longrightarrow \infty} \Phi(u) = +\infty$, and λ is a real positive parameter

Provided that $\inf_X \Phi < r$, put

$$\varphi(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\left(\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v) \right) - \Psi(u)}{r - \Phi(u)}, \tag{6}$$

and

$$\gamma = \liminf_{r \longrightarrow +\infty} \varphi(r), \delta = \liminf_{r \longrightarrow \left(\inf_X \Phi\right)^+} \varphi(r). \tag{7}$$

There is no doubt that $\gamma \geq 0$ and $\delta \geq 0$. When $\gamma = 0$ (or $\delta = 0$), in the sequel, we agree to regard $1/\gamma$ (or $1/\delta$) as $+\infty$.

Now, we recall Theorem 2.1 of [29], which is our main tool for investigating problem (1).

Lemma 1. *Assume that the condition (H) holds. We have*

(*a*) *For every $r > \inf_X \Phi$ and every $\lambda \in]0, 1/\varphi(r)[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(*b*) *If $\gamma < +\infty$ then, for each $\lambda \in]0, 1/\gamma[$, the following alternative holds: either*

(b_1)*I_λ possesses a global minimum, or*

(b_2) *There is a sequence $\{u_n\}$ of critical points (local minimum) of I_λ such that $\lim_{n \longrightarrow +\infty} \Phi(u_n) = +\infty$*

(*c*) *If $\delta < +\infty$ then, for each $\lambda \in]0, 1/\delta[$, the following alternative holds: either*

(c_1) *There is a global minimum of Φ which is a local minimum of I_λ, or*

(c_2) *There is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ, with $\lim_{n \longrightarrow +\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ*

Here, we consider the N-dimensional Banach space

$$X_d = \{u : [0, N+1] \longrightarrow \mathbb{R} \quad \text{such} \quad \text{that} \quad u(0) = u(N+1) = 0\}, \tag{8}$$

and define the norm

$$\|u\|_{r,h} := \left(\sum_{j=0}^{N} |\Delta u(j)|^r + \sum_{j=1}^{N} h(j)|u(j)|^r \right)^{1/r}, \tag{9}$$

where $h : [1, N] \longrightarrow \mathbb{R}$, with $h(j) \geq 0$ for all $j \in [1, N]$, and $r \in]1, +\infty[$. Then, let X_d be endowed with the norm $\|u\| = \|u\|_{p,\alpha} + \|u\|_{q,\beta}$. We denote the usual sup-norm by $\|u\|_\infty = \max_{j \in [1,N]} |u(j)|$, and then we consider the inequality (see ([30], Lemma 2.2)):

$$\|u\|_\infty \leq \frac{(N+1)^{(r-1)/r}}{2} \|u\|_{r,h} \quad \text{for} \quad \text{all} \quad u \in X_d. \tag{10}$$

Lemma 2. *Let $h = \sum_{j=1}^{N} h(j)$. The following inequalities hold*

$$\frac{2}{(N+1)^{(r-1)/r}} \|u\|_\infty \leq \|u\|_{r,h} \leq (2^r N + h)^{1/r} \|u\|_\infty. \tag{11}$$

Proof. The left-hand side of (11) follows by [30]. Consider the right-hand inequality,

$$\begin{aligned}\|u\|_{r,h}^r &= \sum_{j=0}^{N} |\Delta u(j)|^r + \sum_{j=1}^{N} h(j)|u(j)|^r \\ &= |\Delta u(0)|^r + |\Delta u(N)|^r + \sum_{j=1}^{N-1} |\Delta u(j)|^r + \sum_{j=1}^{N} h(j)|u(j)|^r \\ &\le 2\|u\|_\infty^r + \sum_{j=1}^{N-1} (2\|u\|_\infty)^r + \sum_{j=1}^{N} h(j)\|u\|_\infty^r \\ &\le (2^r N + h)\|u\|_\infty^r. \end{aligned} \tag{12}$$

□

Put

$$\begin{aligned} &A_1(u) = \frac{1}{p}\|u\|_{p,\alpha}^p, A_2(u) = \frac{1}{q}\|u\|_{q,\beta}^q \quad \text{and} \\ &\Psi(u) = \sum_{j=1}^{N} G(j, u(j)), \quad \text{for} \quad \text{all} \quad u \in X_d, \end{aligned} \tag{13}$$

where the function $G : [1, N] \times \mathbb{R} \longrightarrow \mathbb{R}$ is given by $G(j, t) = \int_0^t g(j, s)ds$, for all $t \in \mathbb{R}$, $j \in [1, N]$.

Clearly, $A_1, A_2, \Psi \in C^1(X_d, \mathbb{R})$ and we have the following Gâteaux derivatives at the point $u \in X_d$:

$$\left\langle A_1'(u), v\right\rangle = \sum_{j=0}^{N} \phi_p(\Delta u(j))\Delta v(j) + \sum_{j=1}^{N} \alpha(j)\phi_p(u(j))v(j), \tag{14}$$

$$\left\langle A_2'(u), v\right\rangle = \sum_{j=0}^{N} \phi_q(\Delta u(j))\Delta v(j) + \sum_{j=1}^{N} \beta(j)\phi_q(u(j))v(j), \tag{15}$$

$$\left\langle \Psi'(u), v\right\rangle = \sum_{j=1}^{N} g(j, u(j))v(j), \tag{16}$$

for all $v \in X_d$. Now, for $r \in]1,+\infty[$,

$$\begin{aligned} &\sum_{j=0}^{N} \phi_r(\Delta u(j))\Delta v(j) \\ &\quad = \sum_{j=0}^{N} [\phi_r(\Delta u(j))v(j+1) - \phi_r(\Delta u(j))v(j)] \\ &\quad = \sum_{j=1}^{N} \phi_r(\Delta u(j-1))v(j) - \sum_{j=1}^{N} \phi_r(\Delta u(j))v(j) \\ &\quad = -\sum_{j=1}^{N} \Delta\phi_r(\Delta u(j-1))v(j). \end{aligned} \tag{17}$$

If we plug this result back into the calculation of Gâteaux derivatives above, then

$$\left\langle A_1'(u), v\right\rangle = \sum_{j=1}^{N} \left[-\Delta\phi_p(\Delta u(j-1)) + \alpha(j)\phi_p(u(j))\right]v(j), \tag{18}$$

$$\left\langle A_2'(u), v\right\rangle = \sum_{j=1}^{N} \left[-\Delta\phi_q(\Delta u(j-1)) + \beta(j)\phi_q(u(j))\right]v(j), \tag{19}$$

for all $u, v \in X_d$. Let

$$\Phi(u) = A_1(u) + A_2(u). \tag{20}$$

Consider the functional $I_\lambda : X_d \longrightarrow \mathbb{R}$ given as

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u), \quad \text{for} \quad \text{all} \quad u \in X_d. \tag{21}$$

We have

$$\begin{aligned} &\left\langle I_\lambda'(u), v\right\rangle \\ &\quad = \sum_{j=1}^{N} \left[-\Delta_p u(j-1) - \Delta_q u(j-1) + \alpha(j)\phi_p(u(j)) + \beta(j)\phi_q(u(j)) - \lambda g(j, u(j))\right]v(j), \end{aligned} \tag{22}$$

for all $u, v \in X_d$. Thus, $u \in X_d$ is a solution of problem (1) if and only if u is a critical point of I_λ.

Lemma 3. *Fix $u \in X_d$ such that either*

$$\begin{aligned} u(j) > 0 or &- \Delta_p u(j-1) - \Delta_q u(j-1) \\ &+ \alpha(j)\phi_p(u(j)) + \beta(j)\phi_q(u(j)) \ge 0, \end{aligned} \tag{23}$$

for all $j \in [1, N]$. Then, either $u > 0$ in $[1, N]$ or $u \equiv 0$.

Proof. Fix $u \in X_d \setminus \{0\}$ and $Z = \{j \in [1, N]: u(j) \le 0\}$. If $Z = \varnothing$, then, $u > 0$. Now, if $\min Z = 1$, we can get

$$-\Delta_p u(0) - \Delta_q u(0) + \alpha(1)\phi_p(u(1)) + \beta(1)\phi_q(u(1)) \ge 0, \tag{24}$$

which implies that

$$\begin{aligned} &\Delta\Big(\phi_p(\Delta u(0))\Big) + \Delta\Big(\phi_q(\Delta u(0))\Big) \\ &\quad \le \alpha(1)\phi_p(u(1)) + \beta(1)\phi_q(u(1)) \le 0. \end{aligned} \tag{25}$$

Thus,

$$\phi_p(\Delta u(1)) + \phi_q(\Delta u(1)) \le \phi_p(\Delta u(0)) + \phi_q(\Delta u(0)). \tag{26}$$

Since ϕ_p and ϕ_q are both strictly increasing, we have $\Delta u(1) \le \Delta u(0)$, which implies $u(2) - u(1) \le u(1) - 0 \le 0$. It follows that $u(2) \le 0$, then $\Delta(\phi_p(\Delta u(1))) + \Delta(\phi_q(\Delta u(1))) \le$

$\alpha(2)\phi_p(u(2)) + \beta(2)\phi_q(u(2)) \leq 0$. An easy induction gives

$$0 = u(N+1) \leq u(N) \leq \cdots \leq u(1) \leq 0. \quad (27)$$

That is $u \equiv 0$, and this is absurd. Next, we assume that $\min Z = z \in [2, N]$,

$$\begin{aligned}\Delta\left(\phi_p(\Delta u(z-1))\right) + \Delta\left(\phi_q(\Delta u(z-1))\right) \\ \leq \alpha(z)\phi_p(u(z)) + \beta(z)\phi_q(u(z)) \leq 0.\end{aligned} \quad (28)$$

Due to the monotonicity of ϕ_p and ϕ_q, $\Delta u(z) \leq \Delta u(z-1)$, which means $u(z+1) - u(z) \leq u(z) - u(z-1)$. Because $u(z-1) > 0$, we have $u(z+1) < u(z) \leq 0$. By repeating this argument, it is easy to see

$$0 = u(N+1) < u(N) < \cdots < u(z) \leq 0, \quad (29)$$

which leads to a contradiction. □

Now, consider the function $G^+ : [1, N] \times \mathbb{R} \longrightarrow \mathbb{R}$ given as

$$G^+(j,t) = \int_0^t g(j, s^+)ds, \quad \text{for all} \quad t \in \mathbb{R}, j \in [1, N], \quad (30)$$

where $s^+ = \max\{s, 0\}$. Now, we define $I_\lambda^+(u) = \Phi(u) - \lambda\Psi^+(u)$, for all $u \in X_d$, where $\Psi^+(u) = \sum_{j=1}^N G^+(j, u(j))$. Similarly, the critical points of I_λ^+ are the solutions of the following problem

$$\begin{cases} -\Delta_p u(j-1) - \Delta_q u(j-1) + \alpha(j)\phi_p(u(j)) + \beta(j)\phi_q(u(j)) = \lambda g(j, u^+(j)), \forall j \in [1, N], \\ u(0) = u(N+1) = 0. \end{cases} \quad (31)$$

Lemma 4. *If $g(j, 0) \geq 0$ for all $j \in [1, N]$, then each nonzero critical point of I_λ^+ is a positive solution of (1).*

Proof. We note that each positive solution $u \in X_d$ of (31) is a positive solution of (1). By an application of Lemma 3, we conclude that $u > 0$. It follows that the nonzero solutions of (31) are positive and hence are positive solutions of (1). □

3. Main Results

Let

$$\alpha = \sum_{j=1}^N \alpha(j), \beta = \sum_{j=1}^N \beta(j), L_\infty(j) = \liminf_{t\longrightarrow+\infty} \frac{G(j,t)}{t^p} \text{and} L_\infty = \min_{j\in[1,N]} L_\infty(j). \quad (32)$$

The main results are as follows.

Theorem 5. *Assume that $L_\infty > 0$, and there are two real sequences $\{a_n\}$ and $\{b_n\}$, with $\lim_{n\longrightarrow+\infty} a_n = +\infty$, such that*

$$|b_n| < \min\left\{\frac{2a_n}{(\alpha+2)^{1/p}(N+1)^{(p-1)/p}}, \frac{2a_n}{(\beta+2)^{1/q}(N+1)^{(q-1)/q}}\right\}, \quad \text{for every} \quad n \in \mathbb{N}, \quad (33)$$

$$A_\infty := \liminf_{n\longrightarrow+\infty} \frac{\sum_{j=1}^N \max_{|t|\leq a_n} G(j,t) - \sum_{j=1}^N G(j, b_n)}{\left((2a_n)^p/p(N+1)^{p-1}\right) + \left((2a_n)^q/q(N+1)^{q-1}\right) - [(2+\alpha)/p]|b_n|^p - [(2+\beta)/q]|b_n|^q} < \frac{qL_\infty}{(2^p+2^q)N + \alpha + \beta}. \quad (34)$$

Then for each $\lambda \in][(2^p + 2^q)N + \alpha + \beta]/qL_\infty, 1/A_\infty[$, problem (1) admits an unbounded sequence of solutions.

Proof. Fixλin$][(2^p + 2^q)N + \alpha + \beta]/qL_\infty, 1/A_\infty[$, then, we can take the real Banach space X_d as defined in Section 2, and the definitions of Φ, Ψ, I_λ are the same as before. We will prove Theorem 5 by applying Lemma 1 part (b) to function I_λ. Since (H) is trivial to prove, it suffices to prove $\gamma < +\infty$ and I_λ turns out to be unbounded from below. To this end, let

$$\rho_n := \frac{(2a_n)^p}{p(N+1)^{p-1}} \quad \text{and} \quad \sigma_n := \frac{(2a_n)^q}{q(N+1)^{q-1}}, \quad \text{for every} \quad n \in \mathbb{N}. \quad (35)$$

Since, owing to (10), if $\|u\|_{p,\alpha} \leq (p\rho_n)^{1/p}$ then $\|u\|_\infty \leq a_n$, and if $\|u\|_{q,\beta} \leq (q\sigma_n)^{1/q}$ then $\|u\|_\infty \leq a_n$. So, let $r_n = \rho_n + \sigma_n$. From $\Phi(u) \leq r_n$, we have $\|u\|_\infty \leq a_n$.

We obtain

$$\varphi(r_n) \leq \inf_{\Phi(u)\leq r_n} \frac{\sum_{j=1}^N \max_{|t|\leq a_n} G(j,t) - \sum_{j=1}^N G(j, u(j))}{r_n - \Phi(u)}. \quad (36)$$

Then, we define $w(j)$ such that $w_n(j) = b_n$ for every $j \in [1, N]$, $w_n(0) = w_n(N+1) = 0$. Clearly $w_n(j) \in X_d$ and $\Phi(w_n) < r_n$ owing to (33). One has

$$\varphi(r_n) \leq \frac{\sum_{j=1}^N \max_{|t|\leq a_n} G(j,t) - \sum_{j=1}^N G(j,c)}{\left((2a_n)^p/p(N+1)^{p-1}\right) + \left((2a_n)^q/q(N+1)^{q-1}\right) - [(2+\alpha)/p]|b_n|^p - [(2+\beta)/q]|b_n|^q}. \quad (37)$$

Therefore, $\gamma \leq \liminf_{n\longrightarrow+\infty} \varphi(r_n) \leq A_\infty < +\infty$. It remains to show that I_λ is unbounded from below.

Let $\{u_n\} \subset X_d$ be a sequence with $u_n(j) \geq 1$ for $j \in [1,N]$ such that $\lim_{n\longrightarrow\infty} \|u_n\| = +\infty$. Because $L_\infty > 0$, fix L such that $L_\infty > L > [(2^p + 2^q)N + \alpha + \beta]/q\lambda$, and we deduce that there is $\delta_j > 0$ such that $G(j,t) > Lt^p$ for all $t > \delta_j$. Moreover, since $G(j,t)$ is a continuous function, there exists a constant $C(j) \geq 0$ such that $G(j,t) \geq Lt^p - C(j)$ for all $t \in [0,\delta_j]$. Thus, $G(j,t) \geq Lt^p - C(j)$ for all $t \geq 0$ and $j \in [1,N]$. It follows that

$$\begin{aligned}\Psi(u_n) &= \sum_{j=1}^N G(j,u_n(j)) \geq \sum_{j=1}^N \left[L(u_n(j))^p - C(j)\right] \\ &\geq L\|u_n\|_\infty^p - C, \quad \text{for all} \quad n \in \mathbb{N},\end{aligned} \quad (38)$$

where $C = \sum_{j=1}^N C(j)$. Since $\|u_n\|_\infty \geq 1$, one has

$$\begin{aligned}I_\lambda(u_n) &= \frac{\|u_n\|_{p,\alpha}^p}{p} + \frac{\|u_n\|_{q,\beta}^q}{q} - \lambda\sum_{j=1}^N G(j,u_n(j)) \\ &\leq \frac{2^pN+\alpha}{p}\|u_n\|_\infty^p + \frac{2^qN+\beta}{q}\|u_n\|_\infty^q - \lambda L\|u_n\|_\infty^p + \lambda C \\ &\leq \left[\frac{(2^p+2^q)N+\alpha+\beta}{q} - \lambda L\right]\|u_n\|_\infty^p + \lambda C.\end{aligned} \quad (39)$$

As $[(2^p + 2^q)N + \alpha + \beta]/q - \lambda L < 0$, it is obvious that $\lim_{n\longrightarrow+\infty} I_\lambda(u_n) = -\infty$. Hence, I_λ is unbounded from below and the proof is complete. □

Let

$$B^\infty = \limsup_{t\longrightarrow+\infty} \frac{\sum_{j=1}^N G(j,t)}{t^p}. \quad (40)$$

The following theorem can be obtained if we change some of the conditions.

Theorem 6. *Assume that there are two real sequences $\{a_n\}$ and $\{b_n\}$, with $\lim_{n\longrightarrow+\infty} a_n = +\infty$, such that (33) holds and*

$$\begin{aligned}A_\infty &:= \liminf_{n\longrightarrow+\infty} \frac{\sum_{j=1}^N \max_{|t|\leq a_n} G(j,t) - \sum_{j=1}^N G(j,b_n)}{\left((2a_n)^p/p(N+1)^{p-1}\right) + \left((2a_n)^q/q(N+1)^{q-1}\right) - [(2+\alpha)/p]|b_n|^p - [(2+\beta)/q]|b_n|^q} \\ &< \frac{B^\infty}{4+\alpha+\beta}.\end{aligned} \quad (41)$$

Then, for each $\lambda \in](4+\alpha+\beta)/qB^\infty, 1/A_\infty[$, problem (1) admits an unbounded sequence of solutions.

Proof. The first half of the argument is analogous to that in Theorem 5, and put $\Phi, \Psi, I_\lambda, r_n$ as above. So, we have $\gamma \leq \liminf_{n\longrightarrow+\infty} \varphi(r_n) \leq A_\infty < +\infty$.

Our task now is to verify that I_λ is unbounded from below. First, we assume that $B^\infty = +\infty$. Fix M such that $B^\infty > M > (4+\alpha+\beta)/q\lambda$, and let $\{t_n\}$ be a sequence with $t_n \geq 1$ and $\lim_{n\longrightarrow+\infty} t_n = +\infty$, such that

$$\sum_{j=1}^N G(j,t_n) > Mt_n^p, \quad \text{for all} \quad n \in \mathbb{N}. \quad (42)$$

Taking the sequence x_n in X_d defined by $x_n(j) = t_n$ for every $j \in [1,N]$, $x_n(0) = x_n(N+1) = 0$, we have

$$\begin{aligned}I_\lambda(x_n) &= \frac{\|x_n\|_{p,\alpha}^p}{p} + \frac{\|x_n\|_{q,\beta}^q}{q} - \lambda\sum_{j=1}^N G(j,x_n(j)) \\ &= \frac{2+\alpha}{p}t_n^p + \frac{2+\beta}{q}t_n^q - \lambda\sum_{j=1}^N G(j,t_n) \\ &< \frac{2+\alpha}{p}t_n^p + \frac{2+\beta}{q}t_n^q - \lambda Mt_n^p \\ &< \left(\frac{\alpha+\beta+4}{q} - \lambda M\right)t_n^p.\end{aligned} \quad (43)$$

It is easy to see $\lim_{n\longrightarrow+\infty} I_\lambda(x_n) = -\infty$.

Then, we assume that $B^\infty < +\infty$ and fix $\varepsilon > 0$ such that $\varepsilon < B^\infty - (4+\alpha+\beta)/q\lambda$. Let $\{t_n\}$ be a sequence with $t_n \geq 1$, such that $\lim_{n\longrightarrow+\infty} t_n = +\infty$ and

$$(B^\infty + \varepsilon)t_n^p > \sum_{j=1}^N G(j,t_n) > (B^\infty - \varepsilon)t_n^p, \forall n \in \mathbb{N}. \quad (44)$$

Let the sequence $\{x_n\}$ in X_d be the same as the case where $B^\infty = +\infty$, such that

$$I_\lambda(x_n) < \left[\frac{4+\alpha+\beta}{q} - \lambda(B^\infty - \varepsilon)\right]b_n^p, \quad (45)$$

which implies that $\lim_{n\longrightarrow+\infty} I_\lambda(x_n) = -\infty$.

So, in both cases, I_λ is unbounded from below, which completes the proof of Theorem 6.

Let

$$B^0 := \limsup_{t\longrightarrow0^+} \frac{\sum_{j=1}^N G(j,t)}{t^q}. \quad (46)$$

Applying part (c) of Lemma 1, we get the following theorem. □

Theorem 7. *Assume that there exist two real sequences* $\{c_n\}$ *and* $\{d_n\}$, *with* $\lim_{n\longrightarrow+\infty} d_n = 0$, *such that*

$$|c_n| < \min\left\{\frac{2d_n}{(\alpha+2)^{1/p}(N+1)^{(p-1)/p}}, \frac{2d_n}{(\beta+2)^{1/q}(N+1)^{(q-1)/q}}\right\}, \quad \textit{for every} \quad n\in\mathbb{N}, \tag{47}$$

$$A_0 := \liminf_{n\longrightarrow+\infty} \frac{\sum_{j=1}^N \max_{|t|\le d_n} G(j,t) - \sum_{j=1}^N G(j,c_n)}{\left((2d_n)^p/p(N+1)^{p-1}\right) + \left((2d_n)^q/q(N+1)^{q-1}\right) - [(2+\alpha)/p]|c_n|^p - [(2+\beta)/q]|c_n|^q} < \frac{B^0}{4+\alpha+\beta}. \tag{48}$$

Then, for each $\lambda \in](4+\alpha+\beta)/qB^0, 1/A_0[$, *problem (1) admits a sequence of nonzero solutions which converges to zero.*

Proof. Fix λ in $](4+\alpha+\beta)/qB^0, 1/A_0[$, and we can take the real Banach space X_d and functional Φ, Ψ, I_λ as defined in Section 2. Our aim is to apply Lemma 1 part (c) to function I_λ. To this end, let

$$\rho_n := \frac{(2d_n)^p}{p(N+1)^{p-1}} \quad \text{and} \quad \sigma_n := \frac{(2d_n)^q}{q(N+1)^{q-1}}, \quad \text{for every} \quad n\in\mathbb{N}. \tag{49}$$

Owing to (10), if $\|u\|_{p,\alpha} \le (p\rho_n)^{1/p}$ then $\|u\|_\infty \le d_n$, and if $\|u\|_{q,\beta} \le (q\sigma_n)^{1/q}$ then $\|u\|_\infty \le d_n$. So, let $r_n = \rho_n + \sigma_n$. It follows that if $\Phi(u) \le r_n$, then $\|u\|_\infty \le d_n$. We obtain

$$\varphi(r_n) \le \inf_{\Phi(u)\le r_n} \frac{\sum_{j=1}^N \max_{|t|\le d_n} G(j,t) - \sum_{j=1}^N G(j,u(j))}{r_n - \|u\|_{p,\alpha}^p/p - \|u\|_{q,\beta}^q/q}. \tag{50}$$

Now, for each $n\in\mathbb{N}$, let $v_n(j)$ be defined by $v_n(j) = c_n$ for every $j\in[1,N]$, $v_n(0) = v_n(N+1) = 0$. Clearly $v_n(j) \in X_d$, and $\Phi(v_n) \le r_n$ from (47). We have

$$\varphi(r_n) \le \frac{\sum_{j=1}^N \max_{|t|\le d_n} G(j,t) - \sum_{j=1}^N G(j,c_n)}{\left((2d_n)^p/p(N+1)^{p-1}\right) + \left((2d_n)^q/q(N+1)^{q-1}\right) - [(2+\alpha)/p]|c_n|^p - [(2+\beta)/q]|c_n|^q}. \tag{51}$$

Hence, $\delta \le \liminf_{n\longrightarrow+\infty} \varphi(r_n) \le A_0 < +\infty$ follows.

In fact, $\inf_{X_d} \Phi = 0$, so our task now is to verify that the 0 is not a local minimum of I_λ. First, assume that $B^0 = +\infty$. Fix M such that $B^0 > M > (4+\alpha+\beta)/q\lambda$, and let $\{s_n\}$ be a sequence of positive numbers, with $s_n \le 1$ and $\lim_{n\longrightarrow+\infty} s_n = 0$, such that

$$\sum_{j=1}^N G(j,s_n) > M s_n^q, \quad \text{for all} \quad n\in\mathbb{N}. \tag{52}$$

Thus, taking the sequence $\{y_n\}$ in X_d, let $y_n(j) = s_n$ for every $j\in[1,N]$, $y_n(0) = y_n(N+1) = 0$. Some tedious manipulation yields

$$I_\lambda(y_n) < \left(\frac{4+\alpha+\beta}{q} - \lambda M\right) s_n^q, \tag{53}$$

which implies that $I_\lambda(y_n) < 0$.

Then, we assume that $B^0 < +\infty$ and fix $\varepsilon > 0$ such that $\varepsilon < B^0 - (4+\alpha+\beta)/q\lambda$. Let $\{s_n\}$ be a sequence of positive numbers, with $s_n \le 1$, such that $\lim_{n\longrightarrow+\infty} s_n = 0$ and

$$\left(B^0+\varepsilon\right)s_n^q > \sum_{j=1}^N G(j,s_n) > \left(B^0-\varepsilon\right)s_n^q, \forall n\in\mathbb{N}. \tag{54}$$

Choosing the same $\{y_n\}$ in X_d as the case $B^0 = +\infty$, one has

$$I_\lambda(y_n) < \left[\frac{4+\alpha+\beta}{q} - \lambda\left(B^0-\varepsilon\right)\right] s_n^q. \tag{55}$$

That is $I_\lambda(y_n) < 0$. Since 0 is the global minimum of Φ, in both cases, $u = 0$ is not a local minimum of I_λ and the proof is complete. □

By setting

$$A_* := \liminf_{n\longrightarrow+\infty} \frac{\sum_{j=1}^N \max_{|t|\le a_n} G(j,t)}{\left((2a_n)^p/p(N+1)^{p-1}\right) + \left((2a_n)^q/q(N+1)^{q-1}\right)}, \bar{A}_\infty := \liminf_{t\longrightarrow+\infty} \frac{\sum_{j=1}^N \max_{|\xi|\le t} G(j,\xi)}{t^q + t^p}, \tag{56}$$

we get the following consequences.

Corollary 8. *Assume that*

$$\bar{A}_\infty < \frac{2^q}{p(N+1)^{p-1}(4+\alpha+\beta)} B^\infty. \tag{57}$$

Then, for each $\lambda \in](4+\alpha+\beta)/qB^\infty, 2^q/p(N+1)^{p-1}\bar{A}_\infty[$, *problem (1) admits an unbounded sequence of solutions.*

Proof. Let $\{a_n\}$ be a sequence of positive numbers with $\lim_{n\longrightarrow\infty} a_n = +\infty$, such that

$$\bar{A}_\infty = \liminf_{n\longrightarrow+\infty} \frac{\sum_{j=1}^N \max_{|\xi|\le a_n} G(j,\xi)}{a_n^q + a_n^p}. \tag{58}$$

After simple scaling and calculation, we have

$$A_* \le \frac{p(N+1)^{p-1}}{2^q} \bar{A}_\infty. \tag{59}$$

Taking $b_n = 0$ for each $n\in\mathbb{N}$, from Theorem 6, the conclusion follows. □

If $g(j,0)$ satisfies the nonnegative condition, we have the following conclusion.

Corollary 9. *Assume that $g(j,0) \geq 0$ for all $j \in [1,N]$, and*

$$\bar{A}_\infty < \frac{2^q}{p(N+1)^{p-1}(4+\alpha+\beta)} B^\infty. \quad (60)$$

Then, for each $\lambda \in](4+\alpha+\beta)/qB^\infty, 2^q/p(N+1)^{p-1}\bar{A}_\infty[$, problem (1) admits an unbounded sequence of positive solutions.

Proof. Let

$$g^+(j,t) = \begin{cases} g(j,t), & \text{if } t > 0, \\ g(j,0), & \text{if } t \leq 0. \end{cases} \quad (61)$$

Since $g(j,0) \geq 0$,

$$\max_{0\leq s\leq t} \int_0^s g^+(j,\xi)d\xi = \max_{0\leq s\leq t} \int_0^s g(j,\xi)d\xi, \quad (62)$$

for all $t \geq 0$. From Corollary 8, we know that problem (1) with g replaced by g^+ admits an unbounded sequence of solutions for each $\lambda \in](4+\alpha+\beta)/qB^\infty, 2^q/p(N+1)^{p-1}\bar{A}_\infty[$. Then, all these solutions are positive solutions of problem (1) by Lemma 4.

Let

$$\bar{A}_0 := \liminf_{t\longrightarrow 0^+} \frac{\sum_{j=1}^N \max_{|\xi|\leq t} G(j,\xi)}{t^q + t^p}. \quad (63)$$

Arguing as in the proof of Corollary 8 and taking $c_n = 0$ for each $n \in [1,N]$, by Theorem 7, we have the following corollary. □

Corollary 10. *Assume that*

$$\bar{A}_0 < \frac{2^q}{p(N+1)^{p-1}(4+\alpha+\beta)} B^0. \quad (64)$$

Then, for each $\lambda \in](4+\alpha+\beta)/qB^0, 2^q/p(N+1)^{p-1}\bar{A}_0[$, problem (1) admits a sequence of nonzero solutions which converges to zero.

Arguing as in Corollary 9, we have the following result.

Corollary 11. *Assume that $g(j,0) \geq 0$ for all $j \in [1,N]$, and*

$$\bar{A}_0 < \frac{2^q}{p(N+1)^{p-1}(4+\alpha+\beta)} B^0. \quad (65)$$

Then, for each $\lambda \in](4+\alpha+\beta)/qB^0, 2^q/p(N+1)^{p-1}\bar{A}_0[$, problem (1) admits a sequence of positive solutions which converges to zero.

Finally, we give two easy examples to illustrate our results.

Example 1. Let $\alpha = \beta = 0$, $q = 2$, $p = 3$,

$$g(j,x) = g(x) = \begin{cases} 3x^2 \sin\left(\frac{1}{2}\ln|x|\right) + \frac{1}{2}x^2\cos\left(\frac{1}{2}\ln|x|\right) + \frac{25}{8}x^2, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \quad (66)$$

for each $j \in [1,N]$. Then,

$$\liminf_{t\longrightarrow+\infty} \frac{\max_{|\xi|\leq t}\int_0^\xi \left[3x^2\sin(\ln x/2) + x^2\cos(\ln x/2)/2 + 25x^2/8\right]dx}{t^2+t^3} = \liminf_{t\longrightarrow+\infty} \frac{t^3\sin(\ln t/2) + 25t^3/24}{t^2+t^3} = \frac{1}{24}, \quad (67)$$

and

$$\limsup_{t\longrightarrow+\infty} \frac{\int_0^t \left[3x^2\sin(\ln x/2) + x^2\cos(\ln x/2)/2 + 25x^2/8\right]dx}{t^3} = \limsup_{t\longrightarrow+\infty} \frac{t^3\sin(\ln t/2) + 25t^3/24}{t^3} = \frac{49}{24}. \quad (68)$$

By choosing $N = 3$, we have

$$\frac{2^q}{p(N+1)^{p-1}(4+\alpha+\beta)} = \frac{1}{48}. \quad (69)$$

From the above calculation, we obtain

$$\bar{A}_\infty = \liminf_{t\longrightarrow+\infty} \frac{\sum_{j=1}^3 \max_{|\xi|\leq t}\int_0^\xi x^2[3\sin(\ln x/2) + \cos(\ln x/2)/2 + 25/8]dx}{t^2+t^3} = \frac{1}{8}, \quad (70)$$

$$B^\infty = \limsup_{t\longrightarrow+\infty} \frac{\sum_{j=1}^3 \int_0^t x^2[3\sin(\ln x/2) + \cos(\ln x/2)/2 + 25/8]dx}{t^3} = \frac{49}{8}. \quad (71)$$

It is clear that $\bar{A}_\infty < 2^q B^\infty/p(N+1)^{p-1}(4+\alpha+\beta)$, by Corollary 9, the problem

$$\begin{cases} -(|\Delta u(j)|+1)\Delta u(j) + (|\Delta u(j-1)|+1)\Delta u(j-1) = \frac{1}{2}g(u(j)), \forall j \in [1,3], \\ u(0) = u(4) = 0, \end{cases} \quad (72)$$

admits an unbounded sequence of positive solutions.

Example 2. Let $q = 2, p > 2$ and

$$g(j, x) = g(x) = \begin{cases} x(2 + 2\varepsilon + 2\cos(\varepsilon \ln |x|) - \varepsilon \sin(\varepsilon \ln |x|)), & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0, \end{cases} \tag{73}$$

for each $j \in [1, N]$. Then,

$$G(j, x) = G(x) = \int_0^x g(s) ds = x^2 [1 + \varepsilon + \cos(\varepsilon \ln x)], \tag{74}$$

for $x > 0$. Since $g(x) \geq 0$ for $x \geq 0$, $G(x)$ is increasing. We have

$$\bar{A}_0 = \liminf_{t \longrightarrow 0^+} \frac{\sum_{j=1}^N \max_{0 \leq \xi \leq t} G(j, \xi)}{t^q + t^p} = N \liminf_{t \longrightarrow 0^+} \frac{t^2 [1 + \varepsilon + \cos(\varepsilon \ln t)]}{t^2 + t^p} = N\varepsilon, \tag{75}$$

$$B^0 = \limsup_{t \longrightarrow 0^+} \frac{\sum_{j=1}^N G(j, t)}{t^q} = N \limsup_{t \longrightarrow 0^+} \frac{t^2 [1 + \varepsilon + \cos(\varepsilon \ln t)]}{t^2} = N(2 + \varepsilon). \tag{76}$$

Let ε be a sufficiently small constant, such that

$$N\varepsilon < \frac{2^q}{p(N+1)^{p-1}(4 + \alpha + \beta)} N(2 + \varepsilon). \tag{77}$$

Then, by Corollary 11, for each $\lambda \in](4 + \alpha + \beta)/qB^0, 2^q/p(N+1)^{p-1}\bar{A}_0[$, problem (1) admits a sequence of positive solutions which converges to zero.

4. Conclusions

In this paper, we consider a discrete Dirichlet boundary value problem involving the (p, q)-Laplacian. Unlike the existing result in [28], which is the existence of at least two positive solutions, we consider the existence of infinitely many solutions for problem (1) for the first time. In fact, by using Theorem 2.1 of [29], we show that problem (1) admits a sequence of pairwise distinct solutions under some appropriate assumptions on the nonlinear term near at infinity and at the origin. Moreover, we prove the existence of infinitely many positive solutions through our strong maximum principle. It seems that we can use the method in this paper to study other similar problems, such as the existence and multiplicity of solutions for difference equations with different boundary value conditions. This will be left as our future work.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

The Correction of Multiscale Stochastic Volatility to American Put Option: An Asymptotic Approximation and Finite Difference Approach

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It has been found that the surface of implied volatility has appeared in financial market embrace volatility "Smile" and volatility "Smirk" through the long-term observation. Compared to the conventional Black-Scholes option pricing models, it has been proved to provide more accurate results by stochastic volatility model in terms of the implied volatility, while the classic stochastic volatility model fails to capture the term structure phenomenon of volatility "Smirk." More attempts have been made to correct for American put option price with incorporating a fast-scale stochastic volatility and a slow-scale stochastic volatility in this paper. Given that the combination in the process of multiscale volatility may lead to a high-dimensional differential equation, an asymptotic approximation method is employed to reduce the dimension in this paper. The numerical results of finite difference show that the multiscale volatility model can offer accurate explanations of the behavior of American put option price.

1. Introduction

Compared to the European option, the biggest difference is that American option can be exercised any time before its maturity date. Due to the early exercise feature, the pricing of American option has long been the most challenging research topic in finance (see Karatzas [1], Rogers [2], and Haugh and Kogan [3]). The American option can be valued using an analytic approximation approach called the Barone-Adesi and Whaley method (BAW method) when the underlying asset is driven by a stochastic process with constant volatility [4]. However, the significant leptokurtic feature of the underlying asset process found from lots of empirical evidences indicates that the volatility should be randomly distributed rather than a constant, and thus, the BAW model can hardly be applied if a stochastic process drives the volatility. To solve this problem, many researchers currently resort for the stochastic volatility models. In particular, the multiscale stochastic volatility model has been proposed to deal with the fast data and slow data frequency of the underlying asset.

1.1. Multiscale Stochastic Volatility Modelling. Most of the early studies on the American option always assume that a stochastic process with constant volatility drives the underlying asset. This assumption can simplify the problem but fails to capture the real market feature of the volatility (Giesecke et al. [5]). Numerous extensions have been made for relaxing the overstrict assumption, namely, the stochastic volatility models. The most popular stochastic volatility models are Heston's model (see [6]) and Stein and Stein's model (see [7]), which assume that the volatility is driven by a single-factor stochastic process. However, currently, empirical study shows that the volatility can be rescaled according to the frequency of the observed data. Besides, multifactor stochastic models can be used to improve the accuracy of the option pricing (see Fouque and Zhou [8] and Christoffersen et al. [9]).

The studies by Clarke and Parrott [10] and Muzy et al. [11] defined the time scale as the frequency of the observed data in different time periods. Further study of the time scale divided into fast scale and slow scale has been carried out by Chacko and Viceira [12]. The fast scale describes the short period fluctuation with high frequency, while the slow scale describes the long-term variations with low frequency. A comprehensive empirical analysis was presented by Fouque and Zhou [8]. They proposed the slow-scale volatility model to price the European option and investigated the long run time correction effect on the option pricing. A multiscale volatility model was developed to price option by Fouque et al. [13], from which the authors combined both the fast-scale volatility and slow-scale volatility into the volatility processes. In Fouque et al.'s research, the analysis of the fast-scale volatility is connected with the singular perturbation theory, while the study of the slow-scale volatility is associated with the regular perturbation theory. Therefore, the asymptotic approach can be applied to approximate the volatility correction terms.

Despite the popularity of the classic European option pricing, the multiscale model has also been widely used to price other more complicated financial derivatives, such as the Asian option and the VIX future (see Fouque and Han [14] and Fouque and Saporito [15]). The multiscale volatility model is studied in Liu et al. [16], from which the authors incorporated the jump-diffusion terms in pricing European option and the variance swap and applied the finite element method to approximate the generated high-dimension partial integral differential equation (PIDE). Recently, a perpetual American option is investigated under the stochastic elasticity of variance (CEV) model, and the fast-scale correction of the variance elasticities derived by the multiscale asymptotic method [17–19]. Apart from the derivative pricing, the multi-scale model can also be applied in studying portfolio selection (see Fouque and Hu [20, 21]).

1.2. Pricing Methodologies for American Option. It is well known that American-type option featured by early exercise can be formulated as a free boundary PDE problem, whereas the analytical solution is not available. Hence, numerous studies have been conducted to find an approximation of the American option price including the semianalytical approach, the numerical approach, and the Monte Carlo simulations (see Fouque et al. [22], Longstaff and Schwartz [23], and Stentoft [24]).

The semianalytical approach can be divided into three groups, including an analytical method of lines, the integral equation approach, and capped option approximation approach. A semianalytic valuation method for American option is developed by randomizing the maturity date, from which the maturity date is viewed as several jumps driven by standard Poisson processes [25]. The integral equation approach is built by valuing and hedging American options based on a recursive integration of the early exercise boundary [26]. Instead of approximating the early exercise boundary, the capped method is inclined to impose the lower and upper bounds on American option value [27]. Although the semianalytical methods are supposed to be more efficient, it is hardly applicable if more factors are included. Regarding the simulation approximation, various methods are applied to undertake an analysis of the field. The simplest one is the binomial tree method introduced by Cox et al. [28]. LSM method has been introduced by Longstaff and Schwartz [23], who claimed that the simulation method is favourable for multifactor models. The LSM method approximates and simulates the conditionally expected pay-off function from the cross-sectional information using the least square method. However, the method is proved to be less efficient compared to other numerical and semianalytic approaches.

According to Feynman-Kac theorem, American option pricing problems essentially described by a nonlinear PDE can be solved numerically by the finite difference method and the finite element method [29]. For the time-dependent American option, a free and moving boundary is considered due to the early exercise feature. Two approaches, namely, the fixed-point approach and the penalty approach, are commonly used to deal with the moving boundary. Merton et al. [29] applied the front-fixing transformation to incorporate the unknown boundary into the PDE and solved it numerically as a fixed boundary nonlinear PDE problem. The front-fixing method is highly efficient because it does not have to embed iteration at each time step of evolution. A detailed comparison between the front-fixing method and the penalty method has been carried out by Nielsen et al. [30]. Different from the front-fixing method, the penalty method removes the free boundary by adding a small and continuous penalty term which accurately captures the boundary properties. Nielsen et al. [31] derived a penalty method for solving the American multiasset option pricing problem, and they have proved that the semi-implicit FDM method performs better than the implicit method by avoiding the nonlinear term. However, even though the numerical method is beneficial to dealing with the PDE problem with free boundaries, the method is not perfect for a high-dimensional problem concerning the issue of computation efficiency.

This paper is aimed at expanding current research on the subject of American option pricing problem with stochastic volatilities. In our research, a multiscale stochastic volatility model is incorporated to investigate the value of American option. We compare the fast-scale and slow-scale effects of the volatility, which helps to explain the investors' different behaviors when facing the short-run and long-run volatility risks. Since more factors have been taken into account in the model, it will result in a high-dimensional problem subject to moving boundaries, which is hard to solve analytically. Even though many of the numerical methods are useful in solving the higher-dimensional PDE, such as the finite element method in Liu et al. [16], it is highly time-consuming. In our research, we propose an efficient approach to deal with the problem by combining the asymptotic method with the front-fixing method. Besides, as the valuation of the Greeks is closely related to hedging and the risk exposure of the portfolio selection, in this paper, we study the Greeks numerically to test the sensitivities of model parameters. Last but not least, we apply the real financial market data to calibrate the correction terms, which makes our research more reliable and realistic. Instead of using SPX 500 option quotes to calculate the likelihood of the underlying distribution, we follow Fouque et al.'s calibration framework as an alternative approach to analyse the linear relationship between the implied volatility and the Log-Moneyness in Fouque et al. [13]. The idea of the calibration is straightforward and more efficient with fewer calibrated parameters.

The rest of the paper is organized as follows. Model Setup describes the modelling of the underlying asset price, where its volatility is driven by multiscale stochastic processes. Pricing Approximation presents the asymptotic approximation algorithm which is applied to reduce the dimension of the resulting PDE for option pricing. Numerical Analysis is the numerical approach of finite difference for the solution of the PDE problem with free boundary. Numerical illustrations are presented in Empirical Results and followed by a conclusion with future work in light of the empirical findings.

2. Model Setup

We assume that the price of the underlying asset S is driven by the following stochastic process in the form of stochastic differential equation,

$$dS = \mu S dt + f(y, z) S dW_t^{(0)}, \tag{1}$$

where μ is the constant drift term, $W_t^{(0)}$ is a standard Brownian motion, and $f(y, z)$ is a function of two factors y and z, which represent "fast-" and "slow-"scale volatility, respectively. The fast- and slow-scale volatilities y and z are driven by the following two stochastic processes:

$$\begin{aligned} dy &= \left(\frac{1}{\xi}\alpha(y) - \frac{1}{\sqrt{\xi}}\Gamma_1(1)\right)dt + \frac{1}{\sqrt{\xi}}\beta(y)dW_t^{(1)}, \\ z &= \left(\sigma c(z) - \sqrt{\sigma}\Gamma_2(z)\right)dt + \sqrt{\sigma}g(z)dW_t^{(2)}, \end{aligned} \tag{2}$$

where $1/\xi$ and σ denote, respectively, the fast-scale rate and slow-scale rate of volatility; $\Gamma_1(y)$ and $\Gamma_2(z)$ denote the risk premium of the volatility risk under the risk-neutral assumption; and the functions $\alpha(y), \beta(y), c(z), g(z)$ are smooth and at most linearly grow as $y \longrightarrow \infty$ and $z \longrightarrow \infty$.

Here, we assume that the Brownian motion $(W_t^{(0)}, W_t^{(1)}, W_t^{(2)})$ is correlated with the following correlations: $\mathrm{Cov}(W_t^{(0)}, W_t^{(1)}) = \rho_1$, $\mathrm{Cov}(W_t^{(0)}, W_t^{(2)}) = \rho_2$, and $\mathrm{Cov}(W_t^{(1)}, W_t^{(2)}) = \rho_{12}$.

The American option provides the contract holder with the right of early exercise before the expiration date T, which is the key difference from European options. Hence, there exists an optimal exercise boundary $\{B(\tau), t \le T\}$, which divides the area into the holding region and the exercise region. It is optimal to exercise the American put option with strike price K when $S \le B(\tau)$ subject to the following boundary conditions:

$$\begin{aligned} \frac{\partial P(B(t), y, z, t)}{\partial S} &= -1, \\ P(B(t), y, z, t) &= K - B(t). \end{aligned} \tag{3}$$

On the other hand, if $S > B(\tau)$, it is the holding region, and the investors choose to hold the contract until the maturity time. The resulting boundary condition in this region becomes

$$P(S, y, z, t) = (K - S)^+, \tag{4}$$

and the terminal condition of the free boundary at expiration date is defined as

$$B(y, z, T) = K. \tag{5}$$

3. Pricing Approximation

The asymptotic approximation method for American option pricing can refer to the perturbation theorem, which aims at correcting solutions of a PDE with respect to the small variation of the coefficient. We generate the perturbation theorem by introducing fast-scale correction and low-scale correction at the same time into the process of approximation. The perturbation theorem can be grouped into the singular perturbation and the regular perturbation. The PDE degenerates at the singularity point for the singular perturbation, whereas the regular perturbation will not change the nature of the PDE. In order to approximate the altering of the solution under perturbation theorem, the asymptotic approximation is adapted in the series expansion. In our research, the correction of fast-scale stochastic volatility is regarded as a singular perturbation, while the correction of slow-scale stochastic volatility is considered to be a regular perturbation similar to Fouque et al. [13].

Therefore, American option price $P^{\xi,\sigma}$ can be decomposed into the leading term and two correction terms, i.e.,

$$P^{\xi,\sigma} = P_{0,0} + P_{1,0}^{\xi} + P_{1,0}^{\sigma} + \mathrm{Error}, \tag{6}$$

where $P_{0,0}$ denotes the option price without the volatility correction, $P_{1,0}^{\xi} = \sqrt{\xi}P_{1,0}$ the fast-scale volatility correction, and $P_{0,1}^{\sigma} = \sqrt{\sigma}P_{1,0}$ the slow-scale volatility correction. The error term Error is $o(\xi\sqrt{\xi} + \sqrt{\sigma} + \sqrt{\xi\sigma})$ with reference to Fouque et al. [32].

Under the proposed fast-scale and slow-scale stochastic volatility model, the price of an American put option $P(S, y, z, t)$ satisfies the following PDE problem with free boundary conditions:

$$\begin{cases} \mathscr{L}^{\xi,\sigma}P^{\xi,\sigma}=0, \\ P^{\xi,\sigma}(S,y,z,T)=(K-S)^{+}, \\ B^{\xi,\sigma}(y,z,t)<S<+\infty \\ P^{\xi,\sigma}\left(B^{\xi,\sigma}(y,z,t),y,z,t\right)=K-B^{\xi,\sigma}(y,z,t), \\ P_S^{\xi,\sigma}\left(B^{\xi,\sigma}(y,z,t),y,z,t\right)=-1, \\ P_y^{\xi,\sigma}\left(B^{\xi,\sigma}(y,z,t),y,z,t\right)=0, \\ P_z^{\xi,\sigma}\left(B^{\xi,\sigma}(y,z,t),y,z,t\right)=0, \\ P^{\xi,\sigma}(+\infty,y,z,t)=0, \end{cases} \tag{7}$$

where $t\in[0,T]$. The operator $\mathscr{L}^{\xi,\sigma}$ is given by

$$\mathscr{L}^{\xi,\sigma}=\mathscr{L}_0+\frac{1}{\sqrt{\xi}}\mathscr{L}_1+\frac{1}{\xi}\mathscr{L}_2+\sqrt{\sigma}\mathscr{M}_1+\sigma\mathscr{M}_2+\frac{\sqrt{\sigma}}{\sqrt{\xi}}\mathscr{M}_{12}=0, \tag{8}$$

with

$$\begin{cases} \mathscr{L}_0=\dfrac{\partial}{\partial t}+\dfrac{1}{2}f^2(y,z)S^2\dfrac{\partial^2}{\partial S^2}+rS\dfrac{\partial}{\partial S}-r, \\ \mathscr{L}_1=\rho_1\beta(y)f(y,z)S\dfrac{\partial^2}{\partial S\partial y}-\Gamma_1\beta(y)\dfrac{\partial}{\partial y}, \\ \mathscr{L}_2=\dfrac{1}{2}\beta^2(y)\dfrac{\partial^2}{\partial y^2}+\alpha(y)\dfrac{\partial}{\partial y}, \\ \mathscr{M}_1=\rho_2g(z)f(y,z)S\dfrac{\partial^2}{\partial S\partial z}-\Gamma_2g(z)\dfrac{\partial}{\partial z}, \\ \mathscr{M}_2=\dfrac{1}{2}g^2(z)\dfrac{\partial^2}{\partial z^2}+c(z)\dfrac{\partial}{\partial z}, \\ \mathscr{M}_{12}=\rho_{12}\beta(y)g(z)\dfrac{\partial^2}{\partial y\partial z}. \end{cases} \tag{9}$$

To construct a singular perturbation expansion, the asymptotic price $P^{\xi,\sigma}$ is expanded in the form of

$$P^{\xi,\sigma}=\sum_i^n\xi^{i/2}P_i^{\sigma}(S,y,z,t). \tag{10}$$

To construct a regular perturbation expansion, the asymptotic price P^{σ} is expanded in the form of

$$P^{\sigma}=\sum_j^n\sigma^{j/2}P_{i,j}(S,y,z,t). \tag{11}$$

Similarly, the asymptotic price of free boundary $B(y,z,t)$ is of the form

$$B^{\xi,\sigma}=\sum_i^n\xi^{i/2}B_i^{\sigma}(y,z,t), \tag{12}$$

$$B^{\sigma}=\sum_i^n\sigma^{i/2}B_{i,j}(y,z,t). \tag{13}$$

We substitute (10) and (11) into $\mathscr{L}^{\xi,\sigma}P^{\xi,\sigma}=0$ from (7) and collect the similar terms up to order1/2; the results are shown as below:

$$\begin{aligned} &\mathscr{O}\left(\frac{1}{\xi}\right): && \mathscr{L}_2P_{0,0}=0, \\ &\mathscr{O}\left(\frac{1}{\sqrt{\xi}}\right): && \mathscr{L}_2P_{1,0}+\mathscr{L}_1P_{0,0}=0, \\ &\mathscr{O}\left(\frac{\sqrt{\sigma}}{\xi}\right): && \mathscr{L}_2P_{0,1}=0, \\ &\mathscr{O}\left(\frac{\sqrt{\sigma}}{\sqrt{\xi}}\right): && \mathscr{L}_2P_{1,1}+\mathscr{L}_1P_{0,1}+\mathscr{M}_{12}P_{0,0}=0, \end{aligned} \tag{14}$$

$$\begin{aligned} &\mathscr{O}(1): && \mathscr{L}_2P_{2,0}+\mathscr{L}_1P_{1,0}+\mathscr{L}_0P_{0,0}=0, \\ &\mathscr{O}\left(\sqrt{\xi}\right): && \mathscr{L}_2P_{3,0}+\mathscr{L}_1P_{2,0}+\mathscr{L}_0P_{1,0}=0, \\ &\mathscr{O}\left(\sqrt{\sigma}\right): && \mathscr{L}_2P_{2,1}+\mathscr{L}_1P_{1,1}+\mathscr{L}_0P_{0,1}+\mathscr{M}_1P_{0,0}+\mathscr{M}_{12}P_{1,0}=0. \end{aligned} \tag{15}$$

From (9), since the operators $\mathscr{L}_0$ and $\mathscr{L}_1$ only contain the derivatives with respect to y, it can be concluded that $P_{0,0}$, $P_{1,0}$, $P_{0,1}$, and $P_{1,1}$ are independent of y according to (14). As a result of the fact that $\mathscr{L}_1P_{1,0}=0$, equation (15) is a Poisson equation of the form

$$\mathscr{L}_2P+\langle\mathscr{G}\rangle=0, \tag{16}$$

with $\langle\mathscr{G}\rangle\coloneqq\int g(y)\Pi(dy)=0$ according to the centring resolvability of Poisson equation, where Π is an invariant distribution to y. Thus, we obtain

$$\langle\mathscr{L}_0\rangle P_{0,0}=0, \tag{17}$$

$$\left\langle\mathscr{L}_1P_{2,0}\right\rangle+\langle\mathscr{L}_0\rangle P_{1,0}=0, \tag{18}$$

$$\langle\mathscr{M}_1\rangle P_{0,0}+\langle\mathscr{L}_0\rangle P_{0,1}=0. \tag{19}$$

Correspondingly, the free boundary conditions are necessary to be expanded in terms of $\sqrt{\xi}$ and $\sqrt{\sigma}$. By combining (12) and (13) into (7) and applying Taylor expansion, we obtain

$$\begin{aligned} P_{0,0}(B_{0,0},z,t)&+\sqrt{\xi}\left(\frac{\partial P_{0,0}(B_{0,0},z,t)}{\partial S}B_{1,0}+P_{1,0}(B_{0,0},z,t)+B_{1,0}\right) \\ &+\sqrt{\sigma}\left(\frac{\partial P_{0,0}(B_{0,0},z,t)}{\partial S}B_{0,1}+P_{0,1}(B_{0,0},z,t)+B_{0,1}\right)=K-B_{0,0}, \end{aligned} \tag{20}$$

$$\frac{\partial P_{0,0}(B_{0,0},z,t)}{\partial S}+\sqrt{\xi}\left(\frac{\partial P_{0,0}^2(B_{0,0},z,t)}{\partial S^2}B_{1,0}+\frac{\partial P_{0,0}(B_{0,0},z,t)}{\partial S}\right)+\sqrt{\sigma}\left(\frac{\partial P_{0,0}^2(B_{0,0},z,t)}{\partial S^2}B_{0,1}+\frac{\partial P_{0,0}(B_{0,0},z,t)}{\partial S}\right)=-1, \tag{21}$$

$$\frac{\partial P_{0,0}(B_{0,0},z,t)}{\partial z}+\sqrt{\xi}\left(\frac{\partial P_{0,0}^2(B_{0,0},z,t)}{\partial S\partial z}B_{1,0}+\frac{\partial P_{0,0}(B_{0,0},z,t)}{\partial z}\right)+\sqrt{\sigma}\left(\frac{\partial P_{0,0}^2(B_{0,0},z,t)}{\partial S\partial z}B_{0,1}+\frac{\partial P_{0,0}(B_{0,0},z,t)}{\partial z}\right)=0. \tag{22}$$

The boundary conditions can be derived from (20), (21), and (22). The terminal condition can be rewritten as follows:

$$\begin{gathered}P_{0,0}(S,z,T)=(K-S)^+,\\ P_{1,0}(S,z,T)=P_{0,1}(S,z,T)=0.\end{gathered} \tag{23}$$

Here, we propose a series of theorems correspondingly based on the assumptions above to assist for the pricing of American option.

Theorem 1. *Assume $p=P_{0,0}$ to be the leading term without the volatility correction; p can be determined by solving PDE (24):*

$$\begin{gathered}\langle\mathscr{L}_0\rangle p=0,\\ \langle\mathscr{L}_0\rangle=\frac{\partial}{\partial t}+\frac{1}{2}\bar{\delta}^2S^2\frac{\partial^2}{\partial S^2}+rS\frac{\partial}{\partial S}-r,\end{gathered} \tag{24}$$

where $\bar{\delta}^2=\langle f^2(y,z)\rangle:=\int f^2(y,z)\prod(dy)=0$ denotes the mean historical volatility of stock and is subject to the terminal and boundary conditions:

$$\begin{gathered}p(S,T)=(K-S)^+,\\ p(S_{\min},t)=(K-S_{\min})e^{-r(T-t)},\\ p(S_{\max},t)=0,\\ p(B_0,t)=K-B_0,\\ \frac{\partial p(B_0,t)}{\partial x}=-1,\end{gathered} \tag{25}$$

where $B_0=B_{0,0}$ denotes the moving boundary of the leading term.

Proof. The boundary conditions are easily derived from (20).□ □

From Theorem 1, the leading term p can be obtained by solving a one-dimensional PDE with free boundary conditions. The analytical solution is impossible to obtain; thus, we resort to the numerical solution presented in the next section. Apart from the leading term, the fast-scale volatility and slow-scale volatility terms can be determined by Theorem 2 and Theorem 3.

Theorem 2. *Let $P^{\xi}=P_{1,0}$ denote the fast-scale volatility correction term which can be solved by*

$$\langle\mathscr{L}_0\rangle P^{\xi}=V_1^{\xi}S^3\frac{\partial^3 p}{\partial S^3}+V_0^{\xi}S^2\frac{\partial^2 p}{\partial S^2}. \tag{26}$$

$\langle\cdot\rangle$ denotes the integral with respect to y, subject to the terminal and boundary conditions

$$P^{\xi}(S,T)=0,P^{\xi}(B_0,t)=0, \tag{27}$$

where $V_1^{\xi}=\rho_1\cdot\bar{\delta}\cdot\langle\beta(y)\rangle\cdot\langle\Phi_y\rangle$, $V_0^{\xi}=-\Gamma_1\cdot\langle\beta(y)\rangle$.

Proof. From (9) and (17), we obtain

$$\langle\mathscr{L}_0\rangle-\mathscr{L}_0=\frac{1}{2}\left(\bar{\delta}^2-f^2(y,z)\right)S^2\frac{\partial^2}{\partial S^2}. \tag{28}$$

Subtracting the first equation of (15) by (17) and setting

$$\mathscr{L}_2\phi=-\frac{1}{2}\left(\bar{\delta}^2-f^2(y,z)\right), \tag{29}$$

we obtain

$$\langle\mathscr{L}_1P_{2,0}\rangle=\langle\mathscr{L}_1\mathscr{L}_2^{-1}(\langle\mathscr{L}_0\rangle-\mathscr{L}_0)p\rangle=\left\langle\mathscr{L}_1\phi S^2\frac{\partial^2 p}{\partial S^2}\right\rangle, \tag{30}$$

where $\phi=\phi(t,S,y,z)$.

According to the chain rule and PDE (18), we get

$$\langle\mathscr{L}_0\rangle P_{1,0}=-\langle\mathscr{L}_1P_{2,0}\rangle=V_1^{\xi}S^3\frac{\partial^3 p}{\partial S^3}+V_0^{\xi}S^2\frac{\partial^2 p}{\partial S^2}, \tag{31}$$

where $V_1^{\xi}=\rho_1\bar{\delta}\langle\beta(y)\rangle\langle\Phi_y\rangle$ and $V_0^{\xi}=-\Gamma_1\langle\beta(y)\rangle$.

From Theorem 2, the fast-scale volatility term can be determined by solving the PDE problem subject to the fixed boundary condition, or it can be approximated by

$$P_{1,0}=-(T-t)\mathscr{V}p, \tag{32}$$

where $\mathscr{V}=V_0S^3(\partial^3/\partial S^3)+2V_0S^2(\partial^2/\partial S^2)$.

Substituting (32) into (26), we obtain

$$\langle\mathscr{L}_0\rangle P_{1,0}=-\mathscr{V}p+(T-t)\mathscr{V}\langle\mathscr{L}_0\rangle p=-\mathscr{V}p. \tag{33}$$

□ □

Theorem 3. *Let $P^{\sigma}=P_{0,1}$ denote the slow correction term which can be solved from the PIDE:*

$$\langle\mathscr{L}_0\rangle P^{\sigma}=V_1^{\sigma}S\frac{\partial^2 p}{\partial S\partial z}+V_0^{\sigma}\frac{\partial p}{\partial z} \tag{34}$$

subject to the terminal and boundary conditions

$$\begin{aligned} P^{\sigma}(S,T) &= 0, \\ P^{\sigma}(B_0,t) &= 0, \end{aligned} \tag{35}$$

where $V_0^{\sigma} = \Gamma_2 g(z)$ *and* $V_1^{\sigma} = -\rho_2 g(z)\langle f(y,z)\rangle$.

Proof. According to (19), we obtain

$$\langle \mathscr{L}_0 \rangle P_{0,1} = -\langle \mathscr{M}_1 \rangle P_{0,0}, \tag{36}$$

with

$$\langle \mathscr{M}_1 \rangle - V_1^{\sigma} S^2 \frac{\partial^2}{\partial S \partial z} - V_0^{\sigma} S \frac{\partial}{\partial z}. \tag{37}$$

Similarly,

$$P_{0,1} = -(T-t)\langle \mathscr{M}_1 \rangle P_{0,0} = -(T-t)\left(V_1^{\sigma} S \frac{\partial^2 p}{\partial S \partial z} + V_0^{\sigma} \frac{\partial p}{\partial z} \right). \tag{38}$$

where $V_0^{\sigma} = \Gamma_2 g(z)$, and $V_1^{\Sigma} = \rho_2 g(z)\langle f(y,z)\rangle$.□ □

4. Numerical Analysis

For the convenience of numerical computation, both fast-scale stochastic volatility and slow-scale stochastic volatility processes are assumed to be mean-reverting Ornstein-Uhlenbeck (OU) processes. Thus, the stochastic volatility models under the risk-neutral adjustment are

$$\begin{aligned} dS &= rSdt + f(y,z)SdW_t^{*(0)}, \\ dy &= \frac{1}{\xi}(m^* - y)dt + \frac{1}{\sqrt{\xi}} v_1 \sqrt{y} dW_t^{*(1)}, \\ dz &= \sigma(k^* - z)dt + \sqrt{\sigma} v_2 \sqrt{z} dW_t^{*(2)}, \end{aligned} \tag{39}$$

where r is the risk-free interest rate; m^* and k^* are mean reversion level; y and z are the fast-scale stochastic volatility and slow-scale stochastic volatility, respectively; v_1 and v_2 are the volatility of volatility, i.e., vol of vol; and $W_t^{*(i)}$, $i = 0, 1, 2$, are the standard Brownian motions under the risk neutral measure.

As described in the previous section, the option price could be approximated by the summation of the leading term, the fast-scale correction, and the slow-scale correction terms. In this section, the leading term is approximated numerically by applying the front fixing method (FDM) in the following work.

4.1. The Sensitivity Study of the Leading Term. Generally, to obtain the leading term of the original American put option $p = P_{0,0}(S,t)$, we first have to solve the following problem numerically:

$$\frac{\partial p}{\partial t} + \frac{1}{2} S^2 \bar{\delta}^2 \frac{\partial^2 p}{\partial S^2} + rS \frac{\partial p}{\partial S} - rp = 0. \tag{40}$$

By applying the front fixing method (FDM) and letting $x = S/B(t)$, equation (40) becomes

$$\frac{\partial p}{\partial t} + \frac{1}{2} \bar{\delta}^2 x^2 \frac{\partial^2 p}{\partial x^2} + \left(r - \frac{B'(t)}{B(t)} \right) x \frac{\partial p}{\partial x} - rp = 0, \quad x \geq 1, \tag{41}$$

subject to the terminal and the boundary conditions:

$$\begin{aligned} p(x,T) &= 0, \quad x \geq 0, \\ B(T) &= K, \\ p(1,t) &= K - B(t), \\ \frac{\partial p(1,t)}{\partial x} &= -B(t), \\ p(x_{\max}, t) &= 0. \end{aligned} \tag{42}$$

Discretizing (41) gives

$$\begin{aligned} &\frac{p_i^{n+1} - p_i^n}{dt} + \frac{1}{2} \bar{\delta}^2 x_i^2 \frac{p_{i+1}^n - 2p_i^n + p_{i-1}^n}{dx^2} \\ &\quad + \left(r - \frac{B^{n+1} - B^n}{B^n dt} \right) x_i \frac{p_{i+1}^n + p_{i-1}^n}{2dx} - rp_i^n = 0, \end{aligned} \tag{43}$$

which can be simplified to

$$p_i^{n+1} = a_i p_{i+1}^n + b_i p_i^n + c_i p_{i-1}^n, \quad i = 2, \cdots, M, \tag{44}$$

where for simplification, let $l = dt/2dx$ and $k = dt/dx^2$:

$$\begin{cases} a_i = -\dfrac{1}{2} \bar{\delta}^2 k x_i^2 - x_i l \left(r - \dfrac{B^{n+1} - B^n}{B^{n+1} dt} \right), \\ b_i = 1 + \bar{\delta}^2 k x_i^2 + r dt, \\ c_i = -\dfrac{1}{2} \bar{\delta}^2 k x_i^2 + x_i l \left(r - \dfrac{B^{n+1} - B^n}{B^{n+1} dt} \right). \end{cases} \tag{45}$$

The terminal and the boundary conditions (42) have been reduced to

$$\begin{aligned} p_0^n &= K - B^n, \\ p_1^n &= K - B^n(1 + dx), \\ p_M^n &= 0, \\ B^N &= K. \end{aligned} \tag{46}$$

Now, let

$$F(B^n) = a_1(B^n) p_2^n + b_1(K - B^n) + c_1(B^n)(K - B^n(1 + dx)), \tag{47}$$

then the moving boundary can be approximated by applying Newton's method:

$$B^{n,r+1} = B^{n,r} - \frac{F(B^{n,r})}{F'(B^{n,r})}, \tag{48}$$

where r denotes the rth iteration at time n, and the iteration stops once the solution converges.

Option Greeks are of great importance because Greeks measure the evolution of the option price along with the change of the model parameters, such as volatility and stock price. Also, Greeks are useful tools for hedging purpose and estimation of the risk exposure in portfolio selection. Here, we study the Greeks including Delta, Gamma, and Vega of the leading term, and the derived results will be used later in the approximation of the first-order correction terms.

4.2. The Study of Delta. Let $D = \partial p/\partial S$ denote the Delta, which measures the rate of change of the option value for the stock price. It then can be determined by solving the following PDE:

$$\frac{\partial D}{\partial t} + \frac{1}{2}S^2\bar{\delta}^2\frac{\partial^2 D}{\partial S^2} + rS\frac{\partial D}{\partial S} - rD = -S\bar{\delta}^2\frac{\partial^2 p}{\partial S^2} - r\frac{\partial p}{\partial S} \tag{49}$$

subject to the terminal and the boundary conditions:

$$\begin{aligned} D(S,T) &= \begin{cases} -1, & \text{if } K \le S, \\ 0, & \text{if } K > S, \end{cases} \\ D(S_{\min},t) &= -e^{-r(T-t)}, \\ D(S_{\max},t) &= 0, \\ D(B_0,t) &= 0, \\ \frac{\partial D(B_0,t)}{\partial S} &= 0. \end{aligned} \tag{50}$$

Using a similar approach as described in (41) and letting $x = S/B(t)$, equation (49) subject to the terminal and the boundary conditions (50) becomes

$$\frac{\partial D}{\partial t} + \frac{1}{2}\bar{\delta}^2 x^2\frac{\partial^2 D}{\partial x^2} + \left(r - \frac{B'(t)}{B(t)}\right)x\frac{\partial D}{\partial x} - rD = -\frac{1}{2}\bar{\delta}^2\frac{x}{B}\frac{\partial^2 p}{\partial x^2} - \frac{r}{B}\frac{\partial p}{\partial x}, \quad x \ge 1. \tag{51}$$

The terminal and the boundary conditions have been rewritten as follows:

$$\begin{aligned} D(x,T) &= 0, \\ B(T) &= K, \\ (1,t) &= -1, \\ \frac{\partial D(1,t)}{\partial x} &= 0, \\ D(x_{\max},t) &= 0. \end{aligned} \tag{52}$$

Discretizing (51) gives

$$\begin{aligned} &\frac{D_i^{n+1} - D_i^n}{dt} + \frac{1}{2}\bar{\delta}^2 x_i^2\frac{D_{i+1}^n - 2D_i^n + D_{i-1}^n}{dx^2} + \left(r - \frac{B^{n+1} - B^n}{B^n dt}\right)x_i\frac{D_{i+1}^n - D_{i-1}^n}{2dx} - rD_i^n \\ &\quad = \left(-\frac{1}{2}\frac{\bar{\delta}^2 x_i}{B^n dx^2} + \frac{1}{2}\frac{r}{B^n dx}\right)p_{i-1}^n + \frac{\bar{\delta}^2 x_i p_i^n}{B^n dx^2} \\ &\quad + \left(-\frac{1}{2}\frac{\bar{\delta}^2 x_i}{B^n dx^2} - \frac{1}{2}\frac{r}{B^n dx}\right)p_{i+1}^n, \end{aligned} \tag{53}$$

which can be simplified to

$$D_i^{n+1} = a_i D_{i+1}^n + b_i D_i^n + c_i D_{i-1}^n + f_i^n, \quad i = 2, \cdots, M, \tag{54}$$

where a_i, b_i, and c_i are the same as described in (45) and the force term is

$$f_i^n = \left(-\frac{1}{2}\delta^2 x_i k + rl\right)p_{i-1}^n + \delta^2 x_i k p_i^n - \left(\frac{1}{2}\delta^2 x_i k + rl\right)p_{i+1}^n, \quad i = 2, \cdots, M. \tag{55}$$

Formula (54) is an implicit scheme given the terminal condition. It has been approved that the implicit scheme is unconditionally stable, and for a parabolic type of PDE, the convergence of the FDM requires that $dt < < dx$ is satisfied.

4.3. The Study of Gamma. Let $G = \partial^2 p/\partial x^2 = \partial D/\partial x$ denote the Gamma, which measures the rate of change of Delta, and it can be approximated numerically by solving the following PDE:

$$\frac{\partial G}{\partial t} + \frac{1}{2}S^2\bar{\delta}^2\frac{\partial^2 G}{\partial S^2} + \left(r + 2\bar{\delta}^2\right)S\frac{\partial G}{\partial S} - rG = -\left(\bar{\delta}^2 + r\right)\frac{1}{B}\frac{\partial D}{\partial x} \tag{56}$$

subject to the terminal and the boundary conditions:

$$\begin{aligned} G(S,T) &= 0, \\ G(S_{\min},t) &= 0, \\ G(S_{\max},t) &= 0, \\ G(B_0,t) &= 0, \\ \frac{\partial G(B_0,t)}{\partial S} &= 0. \end{aligned} \tag{57}$$

By using the similar approach as in (41), we let $x = S/B(t)$, and then, equation (56) subject to the terminal and the boundary conditions (57) becomes

$$\frac{\partial G}{\partial t} + \frac{1}{2}\bar{\delta}^2 x^2\frac{\partial^2 G}{\partial x^2} + \left(r - \frac{B'(t)}{B(t)}\right)x\frac{\partial G}{\partial x} - rG = -\left(\bar{\delta}^2 + r\right)\frac{1}{B}\frac{\partial D}{\partial x}, \quad x \ge 1, \tag{58}$$

subject to the terminal and the boundary conditions:

$$\begin{aligned} &G(1,t)=0,\\ &\frac{\partial G(1,t)}{\partial x}=0,\\ &G(x,T)=0,\\ &G(x_{\max},t)=0,\\ &G(T)=K. \end{aligned} \tag{59}$$

Discretizing (58) gives

$$\begin{aligned} &\frac{G_i^{n+1}-G_i^n}{dt}+\frac{1}{2}\bar{\delta}^2 x_i^2\frac{G_{i+1}^n-2G_i^n+G_{i-1}^n}{dx^2}\\ &\quad+\left(r-\frac{B^{n+1}-B^n}{B^n dt}\right)x_i\frac{G_{i+1}^n-G_{i-1}^n}{2dx}-rG_i^n\\ &\quad=-\left(\bar{\delta}^2+r\right)\frac{1}{B^n}\frac{D_{i+1}^n-D_{i-1}^n}{2dx}, \end{aligned} \tag{60}$$

which can be simplified to

$$G_i^{n+1}=a_iG_{i+1}^n+b_iG_i^n+c_iG_{i-1}^n+f_i^n,\quad i=2,\cdots,M, \tag{61}$$

where a_i,b_i, and c_i are the same as defined in (45) and the force term is

$$f_i^n=-\left(\bar{\delta}^2+r\right)\frac{1}{B^n}l(D_{i+1}^n-D_{i-1}^n),\quad i=2,\cdots,M. \tag{62}$$

4.4. The Study of Vega. The Vega measures the sensitivity of the option value for the volatility, which can be obtained by solving the following PDE:

$$\frac{\partial V}{\partial t}+\frac{1}{2}S^2\bar{\delta}^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=-\bar{\delta}S^2\frac{\partial^2 p}{\partial S^2} \tag{63}$$

subject to the terminal and the boundary conditions:

$$\begin{aligned} &V(S,T)=0,\\ &V(S_{\min},t)=0,\\ &V(S_{\max},t)=0,\\ &V(B_0,t)=0,\\ &\frac{\partial V(B_0,t)}{\partial S}=0, \end{aligned} \tag{64}$$

where $V=\partial p/\partial\bar{\delta}$ denotes the Vega. Applying the approach as used in (41), letting $x=S/B(t)$, equation (63) subject to the terminal and the boundary conditions (64) becomes

$$\frac{\partial V}{\partial t}+\frac{1}{2}\bar{\delta}^2x^2\frac{\partial^2 V}{\partial x^2}+\left(r-\frac{B'(t)}{B(t)}\right)x\frac{\partial V}{\partial x}-rV=-\bar{\delta}x^2\frac{\partial^2 p}{\partial x^2},\quad x\geq 1, \tag{65}$$

subject to the terminal and the boundary conditions:

$$\begin{aligned} &V(x,T)=0,\\ &B(T)=K,\\ &V(1,t)=0,\\ &\frac{\partial V(1,t)}{\partial x}=0,\\ &V(x_{\max},t)=0. \end{aligned} \tag{66}$$

Discretizing (65) gives

$$\begin{aligned} &\frac{V_i^{n+1}-V_i^n}{dt}+\frac{1}{2}\bar{\delta}^2 x_i^2\frac{V_{i+1}^n-2V_i^n+V_{i-1}^n}{dx^2}\\ &\quad+\left(r-\frac{B^{n+1}-B^n}{B^n dt}\right)x_i\frac{V_{i+1}^n-V_{i-1}^n}{2dx}-rV_i^n\\ &\quad=-\frac{\bar{\delta}(p_{i+1}^n-2p_i^n+p_{i-1}^n)}{dx^2}, \end{aligned} \tag{67}$$

which can be simplified to

$$V_i^{n+1}=a_iV_{i+1}^n+b_iV_i^n+c_iV_{i-1}^n+f_i^n,\quad i=2,\cdots,M, \tag{68}$$

where a_i,b_i, and c_i are the same as defined in (45) and the force term is

$$f_i^n=-\bar{\delta}k(p_{i+1}^n-2p_i^n+p_{i-1}^n),\quad i=2,\cdots,M. \tag{69}$$

5. Empirical Results

In order to study our model effectively and efficiently, we firstly calibrate the proposed model and fit it to the financial market data. The source of the calibrated data in this section is from the Chicago Board Options Exchange (see website [33]), and the method we applied here is the least square method.

5.1. Model Calibration. Instead of estimating the model parameters $(m^*,v_1,k^*,v_2,\rho_1,\rho_2)$ in (39), Fouque et al. [32] suggested that $(V_0^\xi,V_1^\xi,V_0^\sigma,V_1^\sigma)$ could be expressed as

$$\begin{aligned} &V_0^\xi=-\bar{\delta}^3\left[\bar{\delta}-d\right]-a\bar{\delta}\left(r-\frac{\bar{\delta}^2}{2}\right),\\ &V_1^\xi=-\bar{\delta}^3*a,\\ &V_0^\sigma=-\bar{\delta}^3c-b\bar{\delta}\left(r-\frac{\bar{\delta}^2}{2}\right),\\ &V_1^\sigma=-\bar{\delta}^3*b, \end{aligned} \tag{70}$$

where$a,b,c,$ and$d$ are calibrated by the following linear relationship:

$$I\approx d+a*\mathrm{LMMR}+b*\mathrm{LM}+c*\tau, \tag{71}$$

where I denotes the implied volatility, $\mathrm{LM} = \log(X/K)$ denotes the Log-Moneyness, $\mathrm{LMMR} = (\log(X/K))/\tau$ denotes the Log-Moneyness to maturity ratio, and $\tau = T - t$ is the time to maturity.

Table 1 is calculated by fitting the SPX500 implied volatility as a linear function of LMMR, LM, and τ. Tables 1–3 present the calibrated results of three different days: 13/02/2017, 14/03/2017, and 26/05/2017. The calibrated results will be used later in the approximation of the first-order correction terms.

5.2. Numerical Analysis of the Leading Term. Figure 1(a) is the profile of the American option value without volatility correction with different interest rates. Figure 1(b) is the profile of the moving boundary with different interest rates. The region to the left of the free boundary is the holding region, at which the investor will continue to hold the American put option instead of executing the option. The region to the right of the free boundary is the exercising region. The investor will exercise the American put option as soon as the stock price reaches the optimal exercising boundary.

Interest rates play a key role in determining the optimal exercising boundary. The opportunity cost of holding the stock increases with the growth of the interest rate. When r increases, holding stock becomes less attractive, and the investors intend to hold the put option longer. As a result, the put option price intends to decline. The leading term $p = P_{0,0}$, which can be solved numerically from (40), is shown in Figures 2(a) and 2(b) with different interest rates.

To ensure the convergence of FDM scheme, we let the time step size $dt = 10^{-4}$ and the space step size $dx = 0.128$. The trajectories of moving boundaries are compared in Figures 3(a) and 3(b). As shown in Figure 3(a), the value of American put option is increasing along with the volatility, which means that the put option has more chance to be profitable as the volatility grows.

5.3. Sensitivity Analysis. The approximation of Greeks of the leading term is studied in this subsection. Also, the option valuation is intimately related to hedging and the risk exposure of the portfolio selection, and it is interesting to see the evolution of the most critical sensitivities. Here, Delta, Gamma, and Vega will be studied numerically by using the front-fixing method to be applied later in the formation of the first-order correction terms.

Figures 4(a) and 4(b) show the evolution of Delta governed by (49). Delta, the sensitivity of option price to the change of stock price, ranges from −1 to 0. However, the results of Delta in Figures 4(a) and 4(b) are very steep due to the discontinuity of the moving boundary.

A similar result can be viewed from Figures 5(a) and 5(b). The Gamma described by (56) is growing steeply before reaching the peak; Figures 6(a) and 6(b) show the Vega solved by (63), which represents the sensitivity of volatility. The Greeks will be applied subsequently to solve the first-order correction terms.

5.4. Numerical Analysis of Correction Terms. In this subsection, more details on the numerical analysis of the correction terms are provided. According to Theorem 2, the fast-scale correction term can be obtained by solving the following.

PDE:

$$\frac{\partial P^{\xi}}{\partial t} + \frac{1}{2} S^2 \bar{\delta}^2 \frac{\partial^2 P^{\xi}}{\partial S^2} + rS \frac{\partial P^{\xi}}{\partial S} - rP^{\xi} = V_0^{\xi} S^3 \frac{\partial^2 D}{\partial S^2} + V_0^{\xi} S^2 \frac{\partial D}{\partial S} \tag{72}$$

subject to the initial and the boundary conditions:

$$\begin{aligned} P^{\xi}(S, T) &= 0, \\ B_0 &= K, \\ P^{\xi}(B_0, t) &= 0, \\ P^{\xi}(S_{\max}, t) &= 0, \end{aligned} \tag{73}$$

where $V_0 = \rho_1 \bar{\sigma} \langle \beta(y) \rangle \langle \Phi_y \rangle$.

According to Theorem 3, the slow-scale correction term can be solved from the following PDE:

$$\frac{\partial P^{\sigma}}{\partial t} + \frac{1}{2} S^2 \bar{\delta}^2 \frac{\partial^2 P^{\sigma}}{\partial S^2} + rS \frac{\partial P^{\sigma}}{\partial S} - rP^{\sigma} = V_1^{\sigma} S \left(\frac{\partial V}{\partial S} \right)^2 + V_0^{\sigma} V \tag{74}$$

subject to the initial and the boundary conditions

Table 1: Calibrated parameters of 13/02/2017.

a	b	c	d
-0.1955046	-7.857835	-0.1233350	-0.0249121
V_0^{ξ}	V_1^{ξ}	V_0^{σ}	V_1^{σ}
-0.069767	0.003055	-0.010911	0.122779

Table 2: Calibrated parameters of 14/03/2017.

a	b	c	d
-0.1291801	-11.2623374	-0.1975740	-0.0361232
V_0^{ξ}	V_1^{ξ}	V_0^{σ}	V_1^{σ}
-0.072217	0.002018	-0.010438	0.175974

Table 3: Calibrated parameters of 26/05/2017.

a	b	c	d
-0.2610624	-4.2253650	-0.0694265	-0.0119128
V_0^{ξ}	V_1^{ξ}	V_0^{σ}	V_1^{σ}
-0.066865	0.004079	-0.005091	0.066021

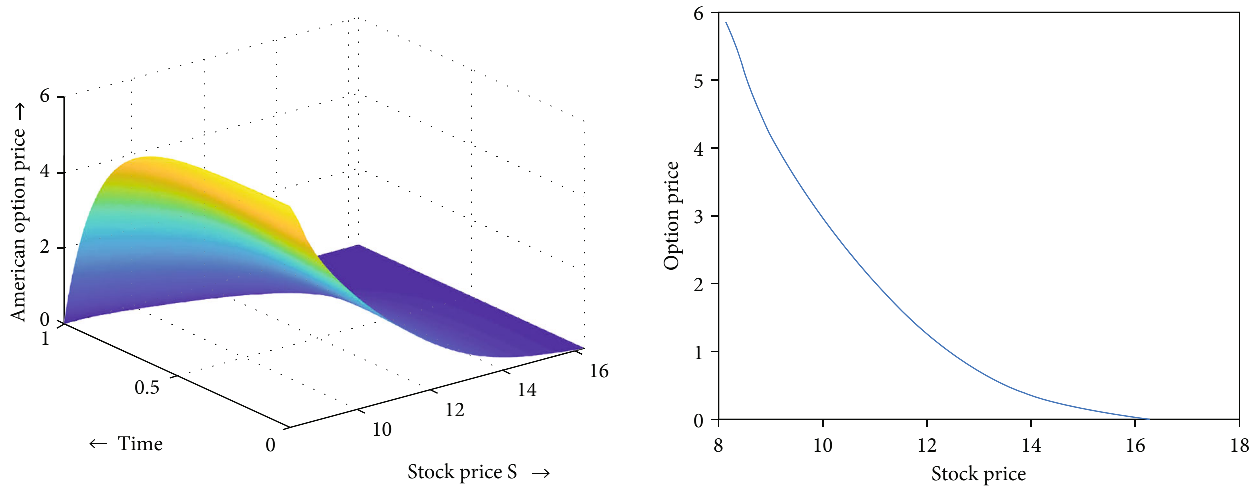

Figure 1: Option value of the leading term.

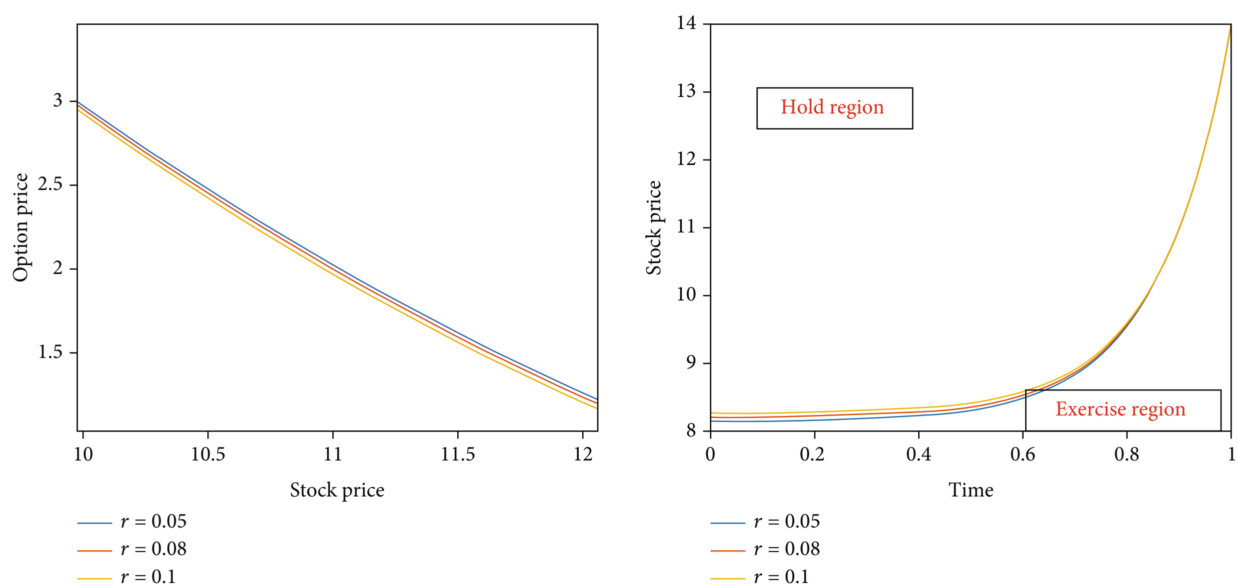

Figure 2: Comparison of American put value and moving boundary for fixed $\bar{\delta} = 0.25$ with different interest rates.

$$\begin{aligned} P^{\sigma}(S, T) &= 0, \\ B_0 &= K, \\ P^{\sigma}(B_0, t) &= 0, \\ P^{\sigma}(S_{\max}, t) &= 0. \end{aligned} \tag{75}$$

The discretizing scheme is the same as (49) with different force terms. Similarly, the front-fixing method is applied to solve the fast-scale volatility and the slow-scale volatility terms P^{ξ} and P^{δ}. From Figures 7(a) and 7(b), they conclude that the inclusion of the fast-scale volatility will result in the increasing of the option price. Figures 8(a) and 8(b) show that the inclusion of the slow-scale volatility also corrects the leading term of American option price positively.

Moreover, it can be summarized from Figures 7(a) and 7(b) that volatility correction terms grow rapidly when the underlying asset price is close to the strike price, and start to decrease quickly when it is close to maturity. Table 4 provides the comparison of the American option price with and without scale correction at the point where the underlying asset price is close to the strike price (at the money). Assuming $S = 13.0216$ when the strike price K equals to 14 and $\epsilon = \sigma = 0.1$, we can conclude that the short-term fast-scale volatility has more significant effects on American option pricing than the long-term slow-scale volatility. The possible explanation behind this phenomenon is that the investors are normally sensitive to the price change in the short run than in the long run. In the long run, investors care more about the quality of the investment rather than the short-term economic indicators.

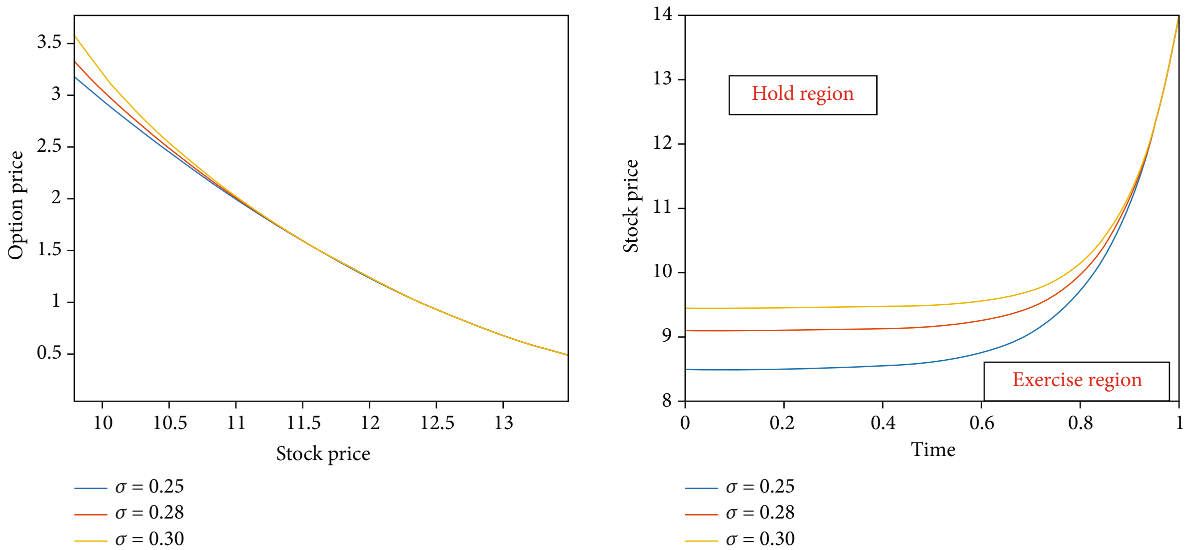

FIGURE 3: Comparison of American option value and moving boundary for fixed $r = 0.005$ with different $\bar{\delta}$.

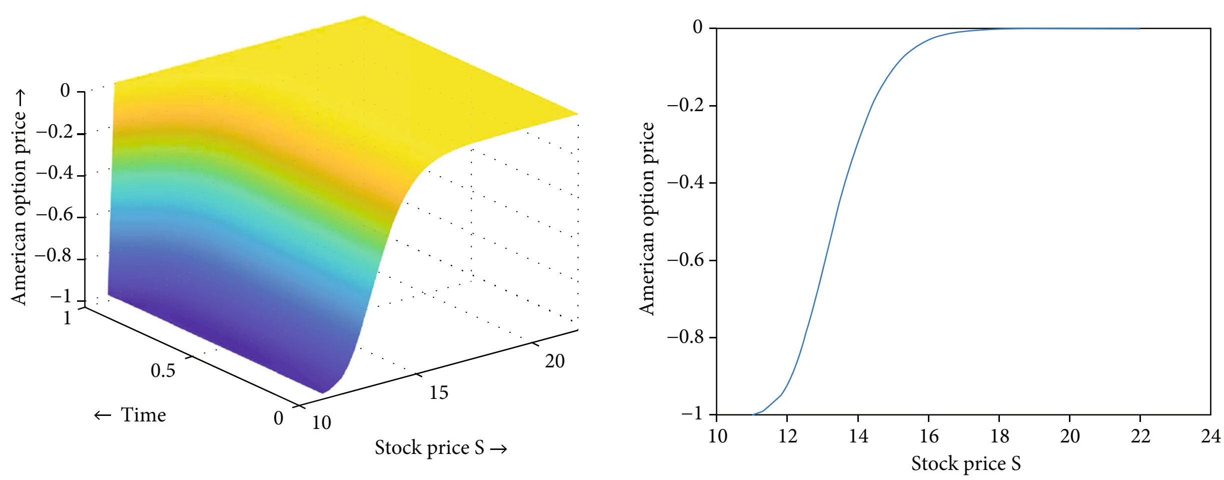

FIGURE 4: Delta of the leading term.

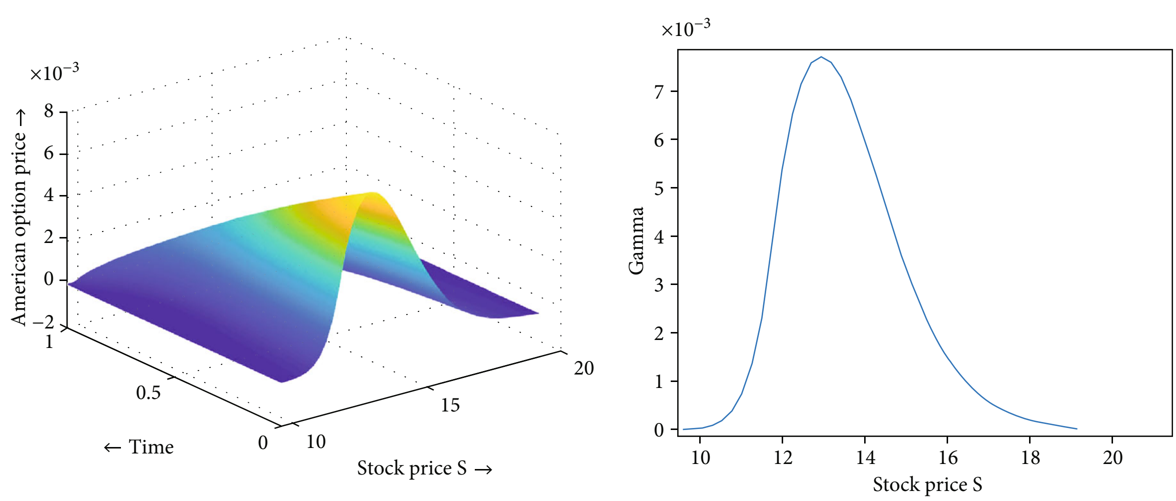

FIGURE 5: Gamma of the leading term.

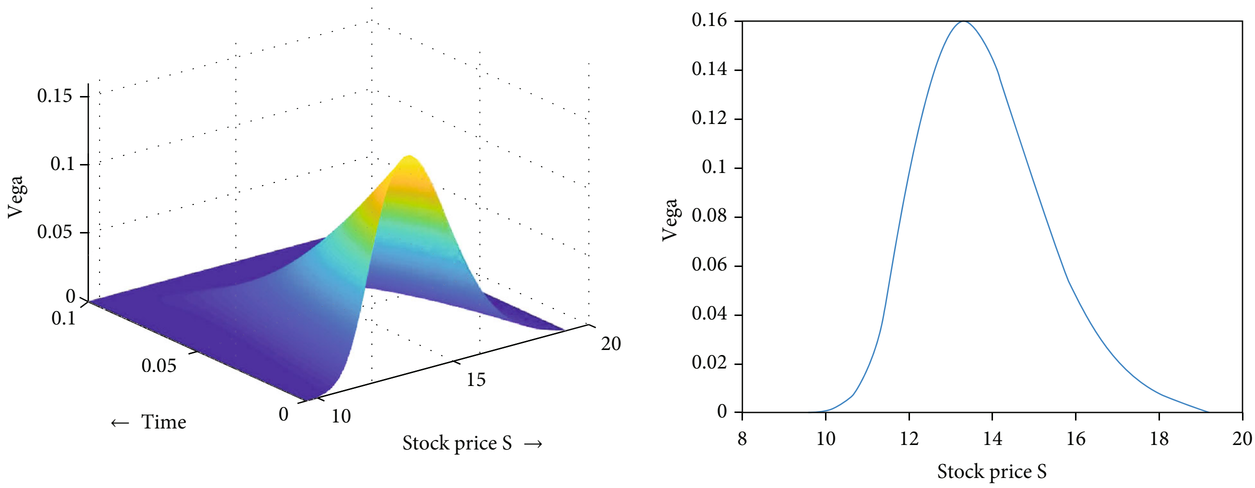

Figure 6: Vega of the leading term.

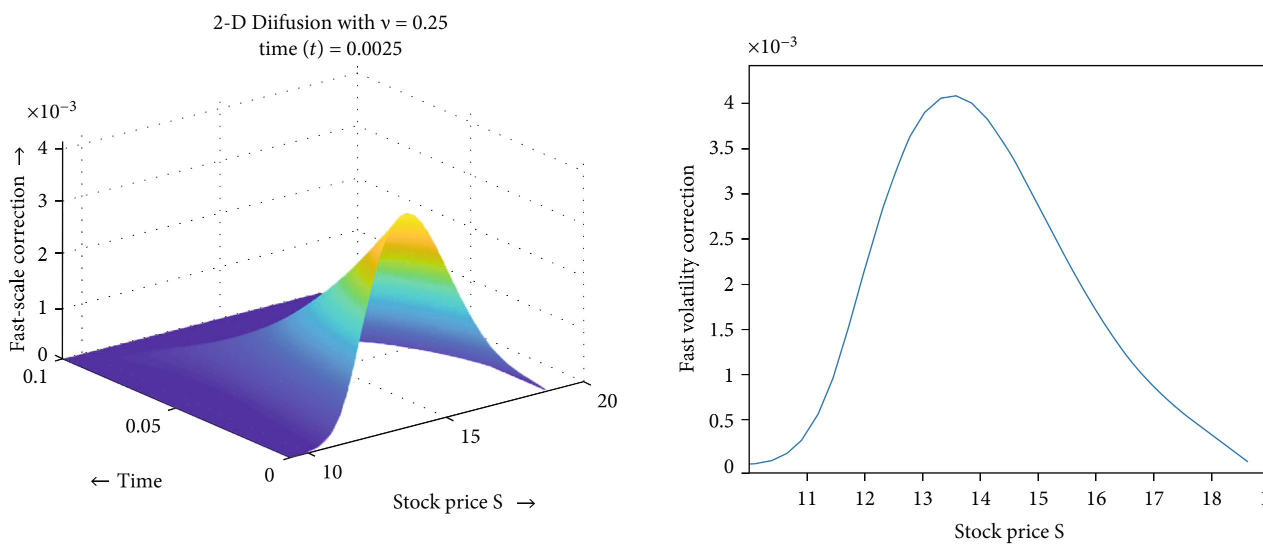

Figure 7: Numerical result of fast-scale correction term.

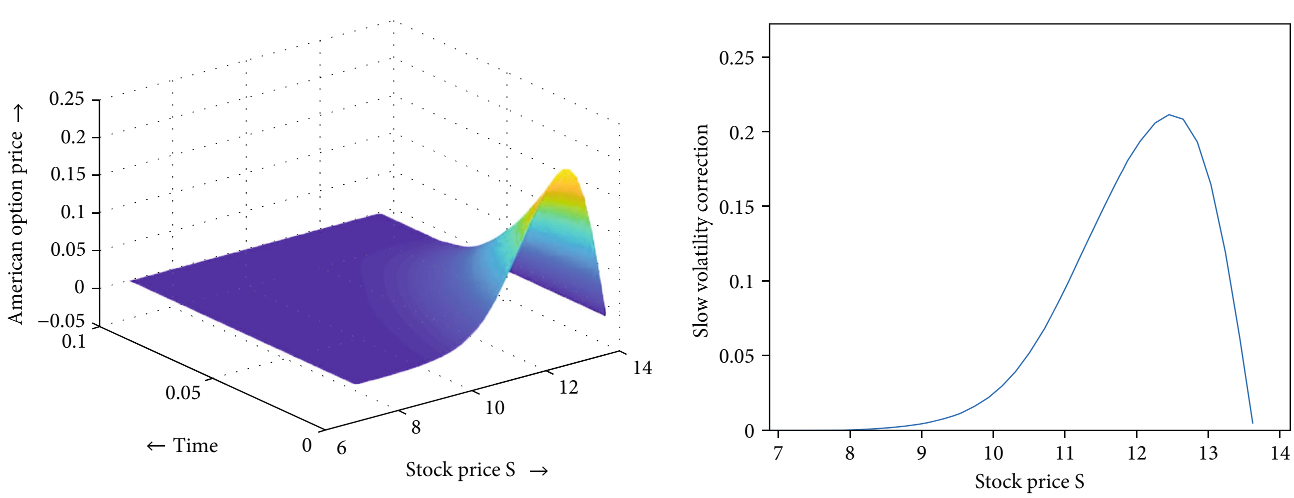

Figure 8: Numerical result of slow-scale correction term.

Table 4: Comparison of the American option pricing with and without scale correction.

Date	Without correction	Fast-scale correction	Slow-scale correction
13/02/2017	0.6957	0.039	0.00494
14/03/2017	0.6957	0.039	0.00462
26/05/2017	0.6957	0.038	0.00224

6. Conclusions

This paper has investigated the correction of multiscale stochastic volatility to American put option pricing. The application of the fast-scale volatility and slow-scale volatility is more practical to capture the behavior of the volatility in a different time periods. The fast-scale volatility is modelled to describe the highly oscillated fluctuation of the return process in the short run. In contrast, the slow-scale volatility makes a description of the slow-varied oscillation fluctuation of long-run return process. The empirical study conducted by Christoffersen et al. [9] offers a better explanation of the leptokurtosis feature of implied volatility surface. In fact, the effects of the multiscale volatility and the early exercising feature of American option can be transferred to a high-dimensional nonlinear partial differential equation subject to a moving boundary.

However, this model is impossible to solve analytically. In this paper, we reduce the high-dimensional PDE to three one-dimensional PDEs with moving boundary by applying the asymptotic approximation method. The asymptotic approximation can be applied because the scales of the fast-scale volatility and slow-scale volatility can be viewed as a singular perturbation and a regular perturbation, respectively. The resulted moving boundary problem has been solved numerically by using the front-fixing method. Moreover, the coefficients in the numerical study are calibrated by the implied volatility derived from S&P 500 options. The numerical results of finite difference show that multiscale volatilities have a significant influence on the American option pricing. The incorporation of the fast-scale volatility and the slow-scale volatility will increase the value of American option pricing, and the effects are more evident in the long run than in the short run. In addition, Greeks are also studied in this paper to show the sensitivity of the model parameters. The first-order correction terms are proved to be related to the sharpness of the leading term. Even though we have successfully applied the multiscale model in studying the pricing problem of the American option, the application of the multiscale volatility model on other path-dependent financial derivatives is also worth to be evaluated in the future research.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Fixed Point Approximation of Monotone Nonexpansive Mappings in Hyperbolic Spaces

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Fixed points of monotone α-nonexpansive and generalized β-nonexpansive mappings have been approximated in Banach space. Our purpose is to approximate the fixed points for the above mappings in hyperbolic space. We prove the existence and convergence results using some iteration processes.

1. Introduction

In 1965, Browder [1], Göhde [2], and Kirk [3] started working in the approximation of fixed point for nonexpansive mappings. Firstly, Browder obtained fixed point theorem for nonexpansive mapping on a subset of a Hilbert space that is closed bounded and convex. Soon after, Browder [1] and Göhde [2] generalized the same result from a Hilbert space to a uniformly convex Banach space. Kirk [3] utilized normal structure property in a reflexive Banach space to sum up the similar results. Recently, Dehici and Najeh [4] and Tan and Cho [5] approximated fixed point result for nonexpansive mappings in Banach space and Hilbert space.

Fixed point theory in partially ordered metric spaces has been initiated by Ran and Reurings [6] for finding application to matrix equation. Nieto and Lopez [7] extended their result for nondecreasing mapping and presented an application to differential equations. Recently, Song et al. [8] extended the notion of α-nonexpansive mapping to monotone α-nonexpansive mapping in order Banach spaces and obtained some existence and convergence theorem for the Mann iteration (see also [9] and the reference therein). Motivated by the work of Suzuki [10], Aoyama and Kohsaka [11], Dehaish and Khamsi [9], and Song et al. [8], Pant and Shukla obtained existence results in ordered Banach space for a wider class of nonexpansive mappings [12, 13]. There are many mathematicians who worked on weak and strong convergence of nonexpansive mappings and its generalizations by using one step, two step, and multistep iteration process ([8, 14, 15]). We obtain existence results in partial ordered hyperbolic space for monotone generalized α-nonexpansive and monotone generalized β-nonexpansive map. Particularly, in Section 3, some auxiliary results and existence theorems for monotone α-nonexpansive mappings in ordered hyperbolic spaces are presented. In Section 4, we presented numerical examples and graphical representation. In Section 5, we obtained some existence results for monotone generalized β-nonexpansive mappings in ordered hyperbolic spaces.

2. Preliminaries

In 1976, the concept of Δ *-convergence* was given by Lim [14].

Lim [14] initiated the idea that in a metric space, Δ -convergence is possible. This concept is adapted for CAT(0) spaces by Kirk and Panyanak [16], and they have indicated that in numerous Banach space, outcomes comprising weak convergence were having exactly accurate analogs in this manner.

Definition 1. A self map $\mathcal{T}$ on W is known as Lipschitz, if there exists $k \geq 0$ such that

$$\sigma(\mathcal{T}\xi, \mathcal{T}\eta) \leq k\sigma(\xi, \eta), \text{ for all } \xi, \eta \in W. \tag{1}$$

$\mathcal{T}$ is called to be contractive if $k \in (0, 1]$, and $\mathcal{T}$ is called nonexpansive mapping, if $k = 1$, that is,

$$\sigma(\mathcal{T}\xi, \mathcal{T}\eta) \leq \sigma(\xi, \eta), \text{ for all } \xi, \eta \in W. \tag{2}$$

Suzuki [10] introduced an interesting generalized nonexpansive mapping as follows.

Definition 2. A self map $\mathcal{T}$ on W is said to satisfy condition (3), if for all $\xi, \eta \in W$,

$$\frac{1}{2}\sigma(\xi, \mathcal{T}\xi) \leq \sigma(\xi, \eta) \Rightarrow \sigma(\mathcal{T}\xi, \mathcal{T}\eta) \leq \sigma(\xi, \eta). \tag{3}$$

Suzuki type generalized nonexpansive mapping is another name of self map $\mathcal{T}$ holding condition (3).

Many generalizations of nonexpansive mapping have been introduced in the literature (see [17–19]). Aoyama and Kohsaka [11] defined a new type of nonexpansive mapping that satisfies the condition (3) known as α-nonexpansive mapping as follows.

Definition 3. Let W be a nonempty subset of a Banach space H. A self map $\mathcal{T}$ on W can be referred α -nonexpansive mapping, if for all $\xi, \eta \in W$ and $\alpha < 1$,

$$\|\mathcal{T}\xi - \mathcal{T}\eta\|^2 \leq \alpha\|\mathcal{T}\xi - \eta\|^2 + \alpha\|\xi - \mathcal{T}\eta\|^2 + (1 - 2\alpha)\|\xi - \eta\|^2. \tag{4}$$

The concept of monotone nonexpansive mappings was introduced in 2015 by Bachar and Khamsi [20], and they studied common approximate fixed point of monotone nonexpansive semigroup. To determine some order fixed points, Dehaish and Khamsi [9] proposed weak convergence theorems of the Mann iteration process for monotone nonexpansive mappings in uniformly convex ordered Banach spaces.

Definition 4. Let T be a self map on W; then, T is said to be

(1) monotone [20] if $T\xi \preceq T\eta$ for all $\xi, \eta \in W$ with $\xi \preceq \eta$;

(2) monotone nonexpansive [20] if T is monotone and

$$\sigma(T\xi, T\eta) \leq \sigma(\xi, \eta), \tag{5}$$

for all $\xi, \eta \in W$ with $\xi \preceq \eta$;

(3) monotone quasi-nonexpansive [8] if T is monotone and

$$\sigma(T\xi, \eta) \leq \sigma(\xi, \eta), \tag{6}$$

where $\xi \in W$ and $\eta \in F(T)$, and $F(T)$ is the set of fixed points.

Definition 5 [21]. (H, σ, S) is called a hyperbolic space, if (H, σ) is a metric space and $S : H \times H \times [0, 1] \longrightarrow H$ is a function holding

(i) $\sigma(\zeta, S(\xi, \eta, \lambda)) \leq (1 - \lambda)\sigma(\zeta, \xi) + \lambda\sigma(\zeta, \eta)$;

(ii) $\sigma(S(\xi, \eta, \lambda_1), S(\xi, \eta, \lambda_2)) \leq |\lambda_1 - \lambda_2|\sigma(\xi, \eta)$;

(iii) $S(\xi, \eta, \lambda) = S(\eta, \xi, 1 - \lambda)$;

(iv) $\sigma(S(\xi, \zeta, \lambda), S(\eta, \nu, \lambda)) \leq (1 - \lambda)\sigma(\xi, \eta) + \lambda\sigma(\zeta, \nu)$,

for all $\xi, \eta, \zeta, \nu \in H$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$.

Here, (i) defines the convexity in metric space (H, σ) that was first considered by Takahashi [22], (ii) provides a unique geodesic between any two elements ξ and η of H by convexity map S that is the space of hyperbolic type in the sense of Goebel and Kirk [23], (iii) provides symmetry along the direction of geodesic, and (iv) defines negative curvature or hyperbolicity of metric space was first considered by Itoh [24].

Theorem 6 [11]. *Let W be a nonempty closed convex subset of a uniformly convex Banach space H and $\mathcal{T} : W \longrightarrow W$ be an α -nonexpansive mapping. Then, $F(\mathcal{T})$ is nonempty iff there exists $\xi \in W$ such that $\{T^n(\xi)\}$ is bounded.*

Since the hyperbolic spaces contain all normed linear spaces and their convex subsets, so uniformly convex Banach space is contained in hyperbolic metric space so it is natural to generalize the above result to hyperbolic metric space.

Definition 7 [16]. A bounded sequence (ξ_n) in H is known as Δ-converge to an element $\xi \in H$, if ξ is *a* unique asymptotic centre of each subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$.

In this section, following definitions and lemma are stated in [9].

Definition 8. Let W be a nonempty set of a hyperbolic metric space (H, σ). A map $\tau : W \longrightarrow [0,\infty)$ is said to be a type function, if there exists a bounded sequence $\{u_n\}$ in H such that

$$\tau(u) = \limsup_{n \longrightarrow \infty} \sigma(u_n, u), \text{ for all } u \in W. \tag{7}$$

It is known that each bounded sequence generates a unique type function.

Lemma 9. *Let $(H, \sigma, \preceq)$ be a uniformly convex hyperbolic metric space and W a nonempty closed convex subset of H. Let $\tau : W \longrightarrow [0,\infty)$ be a type function. Then, τ is continuous. Furthermore, there exists a unique minimum point $\xi \in W$*

such that

$$\tau(\xi) = \inf \{\tau(u)\colon u \in W\}. \tag{8}$$

Definition 10. A hyperbolic space (H, σ) known as uniformly convex, if for every $r > 0$ and $\varepsilon > 0$,

$$\delta(r, \varepsilon) = \inf \left\{1 - \frac{1}{r}\sigma\left(\frac{1}{2}\xi \oplus \frac{1}{2}\eta, a\right); \sigma(\xi, a) \le r, \sigma(\eta, a) \le r, \sigma(\xi, \eta) \ge r\varepsilon\right\} > 0, \tag{9}$$

for any $a \in$ H.

Definition 11. Let W be a nonempty subset of a hyperbolic space H and $\{\xi_v\}$ be a bounded sequence in H. Then, for every $\xi \in H$, define

(i) Asymptotic radius of $\{\xi_v\}$ at ξ by

$$r(\xi, \{\xi_v\}) = \limsup_{v \longrightarrow \infty} \sigma(\xi_v, \xi). \tag{10}$$

(ii) Asymptotic radius of the sequence $\{\xi_v\}$ relative to the above supposed set W by

$$r(\xi, \{\xi_v\}) = \inf \{r(\xi, \xi_v)\colon \xi \in W\}. \tag{11}$$

(iii) Asymptotic centre of the sequence $\{\xi_v\}$ relative to the above supposed set W by

$$A(W, \{\xi_v\}) = \{r(\xi, \{\xi_v\}) = r(W, \{\xi_v\})\colon \xi \in W\}. \tag{12}$$

Note that $A(W, \{\xi_v\}) \ne \varnothing$. Further, $A(W, \{\xi_v\})$ has exactly one point if H is uniformly convex.

From now to onward, we will suppose that the ordered intervals are convex and closed, and they are also contained in ordered hyperbolic space $(H, \preceq, F)$; these are described as follows:

$$[a, \longrightarrow) \coloneqq \{\xi \in H : a \preceq \xi\} \text{ and } (\longleftarrow, b] \coloneqq \{\xi \in H : \xi \preceq b\} \text{ for any } a, b \in H. \tag{13}$$

3. Monotone α-Nonexpansive Mappings

In this section, we will use the following iteration introduced by Kalsoom et al. [25].

$$\begin{cases} \xi \in W, \\ \xi_{v+1} = T((1-\alpha_v)T\xi_v + \alpha_v T\eta_v), \\ \eta_v = (1-\beta_v)z_v + \beta_v T z_v, \\ z_v = (1-\gamma_v)\xi_v + \gamma_v T\xi_v, \quad v \in \mathbb{N}, \end{cases} \tag{14}$$

where $\{\alpha_v\}$, $\{\beta_v\}$ and $\{\gamma_v\}$ are in $(0, 1)$.

Now we define monotone α-nonexpansive mappings in partially ordered hyperbolic metric space $(H, \sigma, \preceq)$ as follows.

Definition 12. Let W be a nonempty closed convex subset of an ordered hyperbolic metric space H. A self map $\mathcal{T}$ on W is monotone α-nonexpansive mapping, if $\mathcal{T}$ is monotone and for some $\alpha < 1$,

$$\sigma^2(\mathcal{T}\xi, \mathcal{T}\eta) \le \alpha\sigma^2(\mathcal{T}\xi, \eta) + \alpha\sigma^2(\xi, \mathcal{T}\eta) + (1-2\alpha)\sigma^2(\xi, \eta), \tag{15}$$

for all $\xi, \eta \in W$ with $\xi \preceq \eta$.

Lemma 13. *Let W be a nonempty closed convex subset of an ordered hyperbolic metric space H. A self map $\mathcal{T}$ on W is monotone α-nonexpansive mapping; then,*

(i) $\mathcal{T}$ is monotone quasi.

(ii) For all $\xi, \eta \in W$ with $\xi \preceq \eta$

$$\sigma^2(\mathcal{T}\xi, \mathcal{T}\eta) \le \sigma^2(\xi, \eta) + \frac{2\alpha}{1-\alpha}\sigma^2(\mathcal{T}\xi, \xi) + \frac{2\alpha}{1-\alpha}\sigma(\mathcal{T}\xi, \xi)[\sigma(\xi, \eta) + \sigma(\mathcal{T}\xi, \mathcal{T}\eta)]. \tag{16}$$

Proof. To prove (i), it is followed by definition of monotone α-nonexpansive mappings that

$$\sigma^2(\mathcal{T}\xi, \eta) \le \alpha\sigma^2(\mathcal{T}\xi, \eta) + \alpha\sigma^2(\xi, \eta) + (1-2\alpha)\sigma^2(\xi, \eta), \tag{17}$$

implies $\sigma(\mathcal{T}\xi, \eta) \le \sigma(\xi, \eta)$.

Hence, $\mathcal{T}$ is monotone quasi for $\xi \in W$ and $\eta \in F(\mathcal{T})$.

Now we will prove (ii), and if $0 < \alpha < 1$ then we have

$$\begin{aligned} \sigma^2(\mathcal{T}\xi, \mathcal{T}\eta) &\le \alpha\sigma^2(\mathcal{T}\xi, \eta) + \alpha\sigma^2(\xi, \mathcal{T}\eta) + (1-2\alpha)\sigma^2(\xi, \eta) \\ &\le \alpha\sigma^2(\mathcal{T}\xi, \xi) + 2\alpha\sigma(\mathcal{T}\xi, \xi)\sigma(\xi, \eta) + \alpha\sigma^2(\xi, \eta) \\ &\quad + \alpha\sigma^2(\mathcal{T}\xi, \mathcal{T}\eta) + 2\alpha\sigma(\mathcal{T}\xi, \mathcal{T}\eta)\sigma(\mathcal{T}\xi, \xi) \\ &\quad + \alpha\sigma^2(\mathcal{T}\xi, \xi) + (1-2\alpha)\sigma^2(\xi, \eta), \end{aligned}$$

$$\sigma^2(\mathcal{T}\xi,\mathcal{T}\eta) \le \sigma^2(\xi,\eta) + \frac{2\alpha}{1-\alpha}\sigma^2(\mathcal{T}\xi,\xi) + \frac{2\alpha}{1-\alpha}\sigma(\mathcal{T}\xi,\xi)[\sigma(\xi,\eta)+\sigma(\mathcal{T}\xi,\mathcal{T}\eta)]. \tag{18}$$

This completes the proof. ☐ ☐

Definition 14. An ordered hyperbolic metric space $(H,\sigma,\preceq)$ is said to be uniformly convex, if for an arbitrary $h \in H, r > 0$ and $\varepsilon > 0$,

$$\delta(r,\varepsilon) = \inf\left[1 - \frac{1}{2}\sigma\left(\frac{1}{2}\xi \oplus \frac{1}{2}\eta, h\right); \sigma(\xi,h) \le r; \sigma(\eta,h) \le r; \sigma(\xi,\eta) \ge r\varepsilon\right] > 0. \tag{19}$$

Now, we utilize iteration processes for monotone α -nonexpansive mappings.

Lemma 15. *Let $(H,\sigma,\preceq)$ be a uniformly convex partially ordered hyperbolic metric space (in short, UCPOHMS) and W a nonempty closed convex subset of H. Let $\mathcal{T}: W \longrightarrow W$ be a monotone mapping and $\xi_1 \in W$ be such that $\xi_1 \preceq \mathcal{T}\xi_1$ [or $T\xi_1 \preceq \xi_1$]. Then, for sequence $\{\xi_v\}$ defined by (14), we have*

(a) $\xi_v \preceq \mathcal{T}\xi_v \preceq \xi_{v+1}$ (or $\xi_{v+1} \preceq \mathcal{T}\xi_v \preceq \xi_v$);

(b) $\xi_v \preceq p (or p \preceq \xi_v)$ provided $\{\xi_v\}\Delta$-converges to a point $p \in W, \forall v \in \mathbb{N}$.

Theorem 16. *Let W be a nonempty closed convex subset of a UCPOHMS $(H,\sigma,\preceq)$ and $\mathcal{T}: W \longrightarrow W$ be a monotone α -nonexpansive mapping. Assume that there exists $\xi_1 \in W$ such that $\xi_1 \preceq \mathcal{T}\xi_1$ and $\{\xi_v\}$ defined in (14) is a bounded sequence with $\xi_v \le \eta$ for all $\eta \in W$ such that*

$$\liminf_{v\longrightarrow\infty} \sigma(\xi_v,\mathcal{T}\xi_v) = 0. \tag{20}$$

Then, $F(\mathcal{T}) \ne \emptyset$.

Proof. Let $\{\xi_v\}$ defined by (14) be a bounded sequence such that

$$\liminf_{v\longrightarrow\infty} \sigma(\xi_v,\mathcal{T}\xi_v) = 0. \tag{21}$$

Then, there exists a subsequence $\{\xi_{v_q}\}$ such that

$$\lim_{k\longrightarrow\infty} \sigma\left(\xi_{v_q},\mathcal{T}\xi_{v_q}\right) = 0. \tag{22}$$

By Lemma 15, we have

$$\xi_1 \preceq \xi_{v_q} \preceq \xi_{v_{q+1}}. \tag{23}$$

Define

$$W_q = \{u \in W : \xi_1 \preceq u\}, \tag{24}$$

for all $q \in \mathbb{N}$. Clearly, for every $q \in \mathbb{N}$, W_q is closed convex. As $u \in W_q$, it shows that $W_q \ne \emptyset$. Define

$$W_\infty = \bigcap_{q=1}^{\infty} W_q \ne \emptyset. \tag{25}$$

Then, W_∞ is a closed convex subset of W. Let $u \in W_\infty$. Then,

$$\xi_{v_q} \preceq u, \forall q \in \mathbb{N}. \tag{26}$$

As we know, $\mathcal{T}$ is a mapping which is monotone; then, for $q \in \mathbb{N}$,

$$\xi_{v_q} \preceq \mathcal{T}\xi_{v_q} \preceq \mathcal{T}u, \tag{27}$$

which implies that $\mathcal{T}(W_\infty) \subset W_\infty$. Let a type function $\tau : W_\infty \longrightarrow [0,\infty)$ generated by $\{\xi_{v_q}\}$ such that

$$\tau(u) = \limsup_{q\longrightarrow\infty} \sigma\left(\xi_{v_q}, u\right). \tag{28}$$

Then, there exists a unique point $z \in W_\infty$ such that

$$\tau(z) = \inf\{\tau(u) : u \in W_\infty\}. \tag{29}$$

By definition of type function,

$$\tau(\mathcal{T}z) = \limsup_{q\longrightarrow\infty} \sigma\left(\xi_{v_q}, \mathcal{T}z\right). \tag{30}$$

By using Lemma 13, we get

$$\sigma^2\left(\mathcal{T}\xi_{v_q},\mathcal{T}z\right) \le \sigma^2\left(\xi_{v_q}, z\right) + \frac{2\alpha}{1-\alpha}\sigma^2\left(\mathcal{T}\xi_{v_q},\xi_{v_q}\right) + \frac{2\alpha}{1-\alpha}\sigma\left(\mathcal{T}\xi_{v_q},\xi_{v_q}\right)\left[\sigma\left(\xi_{v_q}, z\right) + \sigma\left(\mathcal{T}\xi_{v_q},\mathcal{T}z\right)\right]. \tag{31}$$

From the boundedness of the sequence $\{\xi_{v_q}\}$ and $\lim_{q\longrightarrow\infty}\sigma(\xi_{v_q},\mathcal{T}\xi_{v_q}) = 0$, we have

$$\limsup_{q\longrightarrow\infty} \sigma^2\left(\mathcal{T}\xi_{v_q},\mathcal{T}z\right) \le \limsup_{q\longrightarrow\infty} \sigma^2\left(\xi_{v_q}, z\right), \tag{32}$$

implies $\tau(\mathrm{T}z) = \tau(z)$. It shows that $\mathcal{T}z = z$, and hence, $F(\mathcal{T}) \ne \emptyset$. ☐ ☐

Theorem 17. *Let W be a nonempty closed convex subset of a UCPOHMS $(H,\sigma,\preceq)$ and $\mathcal{T}: W \longrightarrow W$ a monotone α*

-nonexpansive mapping. Assume that there exists $\xi_1 \in W$ such that $\mathcal{T}\xi_1 \preceq \xi_1$ and $\{\xi_v\}$ defined by (14) is a bounded sequence with $\xi_v \leq \eta$ for all $\eta \in W$ such that

$$\liminf_{v \longrightarrow \infty} \sigma(\xi_v, \mathcal{T}\xi_v) = 0. \quad (33)$$

Then, $F(\mathcal{T}) \neq \varnothing$.

Theorem 18. *Let $(H, \sigma, \preceq)$ be a UCPOHMS, C a closed convex cone, and $\mathcal{T} : C \longrightarrow C$ a monotone α-nonexpansive mapping. Assume that $\xi_1 = 0$ and $\{\xi_v\}$ defined by (14) is a bounded sequence with $\xi_v \leq \eta$ for all $\eta \in W$ such that*

$$\liminf_{v \longrightarrow \infty} \sigma(\xi_v, \mathcal{T}\xi_v) = 0. \quad (34)$$

Then, $F(\mathcal{T}) \neq \varnothing$.

Proof. With the help of definition of partial order $\preceq$, we know that $\xi_1 = 0 \preceq \mathcal{T}0 = \mathcal{T}\xi_1$; then, the proof is directly from Theorem 16. □ □

We now prove some convergence results for monotone α-nonexpansive mappings.

Lemma 19 [9]. *Suppose (H, σ) be a UC hyperbolic space and also monotone modulus of uniform convexity δ, and suppose $z \in H$ and a sequence $\{\alpha_v\}$ such that $0 < a \leq \alpha_v \leq b < 1$. If the sequences $\{u_v\}$ and $\{v_v\}$ are in H, in such a way that $\limsup_{v \longrightarrow \infty} \sigma(u_v, z) \leq r$, $\limsup_{v \longrightarrow \infty} \sigma(v_v, z) \leq r$, and $\lim_{v \longrightarrow \infty} \sigma(\alpha_v u_v \oplus (1 - \alpha_v) v_v, z) = r$, then $\lim_{v \longrightarrow \infty} \sigma(u_v, v_v) = 0$.*

Theorem 20. *Let W be a nonempty closed convex subset of a UCPOHMS $(H, \sigma, \preceq)$ and $\mathcal{T} : W \longrightarrow W$ a monotone α-nonexpansive mapping. Suppose there exists a sequence $\{\xi_v\}$ defined by (14) with $\xi_1 \preceq \mathcal{T}\xi_1$ (or $\mathcal{T}\xi_1 \preceq \xi_1$) and $F(\mathcal{T}) \neq \varnothing$. Then,*

(1) $\{\xi_v\}$ is bounded

(2) $\sigma(\xi_{v+1}, z) \leq \sigma(\xi_v, z)$ and $\lim_{v \longrightarrow \infty} \sigma(\xi_v, z)$ exists for all $z \in F(\mathcal{T})$

(3) $\lim_{v \longrightarrow \infty} \sigma(\mathcal{T}\xi_v, \xi_v) = 0$.

Proof. Suppose that $\xi_1 \preceq z$ and $z \in F(\mathcal{T})$, and as we know that $\mathcal{T}$ is monotone, then $\mathcal{T}\xi_1 \preceq \mathcal{T}z$, and so

$$\xi_1 \preceq \mathcal{T}\xi_1 \preceq z. \quad (35)$$

It follows from Lemma 15.

$$\xi_v \preceq \xi_{v+1} \preceq \mathcal{T}\xi_v \preceq z, \quad (36)$$

which gives that

$$\xi_{v+1} \preceq z, \forall v \in \mathbb{N}. \quad (37)$$

It shows that $\{\xi_v\}$ is bounded. On the other hand, by using Lemma 13, we get

$$\sigma(\xi_{v+1}, z) \leq \sigma(\xi_v, z), \quad (38)$$

and so

$$\begin{aligned} \sigma(\xi_{v+1}, z) &\leq \beta_v \sigma(\xi_v, z) + (1 - \beta_v)\sigma(\mathcal{T}\xi_v, z) \\ &\leq \beta_v \sigma(\xi_v, z) + (1 - \beta_v)\sigma(\xi_v, z) \\ &= \sigma(\xi_v, z) \\ &\;\;\vdots \\ &\leq \sigma(\xi_1, z). \end{aligned} \quad (39)$$

Then, the sequence $\{\sigma(\xi_v, z)\}$ is nonincreasing and bounded sequence; hence, (i) and (ii) proved. So $\lim_{v \longrightarrow \infty} \sigma(\xi_v, z)$ exists for all $z \in F(\mathcal{T})$ and $v \in \mathbb{N}$. Suppose that

$$\lim_{v \longrightarrow \infty} \sigma(\xi_v, z) \leq s. \quad (40)$$

As $\mathcal{T}$ is monotone quasi,

$$\limsup_{v \longrightarrow \infty} \sigma(\mathcal{T}\xi_v, z) \leq s, \quad (41)$$

and hence,

$$\begin{aligned} &\limsup_{v \longrightarrow \infty} \sigma(\xi_{v+1}, z) \leq s, \\ s &= \limsup_{v \longrightarrow \infty} \{\beta_v \sigma(\xi_v, z) + (1 - \beta_v)\sigma(\mathcal{T}\xi_v, z)\} \\ &= \limsup_{v \longrightarrow \infty} \sigma(\xi_v, z). \end{aligned} \quad (42)$$

It concludes from Lemma 19 that

$$\limsup_{v \longrightarrow \infty} \sigma(\xi_v, z) = 0. \quad (43)$$

□ □

Theorem 21. *Let W be a nonempty closed convex subset of a UCPOHMS $(H, \sigma, \preceq)$ and $\mathcal{T} : W \longrightarrow W$ a monotone α-nonexpansive mapping. Suppose there exists a sequence $\{\xi_v\}$ defined by (14) with $\xi_1 \preceq \mathcal{T}\xi_1$ (or $\mathcal{T}\xi_1 \preceq \xi_1$) and $F(\mathcal{T}) \neq \varnothing$. Then, $\{\xi_v\}$ Δ-converges to a fixed point of $\mathcal{T}$.*

Proof. By the Theorem 18, we have $\{\xi_v\}$ is bounded. So, there exists a subsequence $\{\xi_{v_q}\}$ of $\{\xi_v\}$ Δ-converges to some $p \in W$ such that

$$\xi_1 \preceq \xi_{v_q} \preceq p, \forall q \in \mathbb{N}. \quad (44)$$

In the next step, we prove there exists a unique Δ-limit in $F(\mathcal{T})$ corresponding to each Δ-convergent subsequence of $\{\xi_v\}$. Consider $\{\xi_v\}$ has two subsequences $\{\xi_{v_q}\}$ and $\{\xi_{v_r}\}$ which are Δ-convergent to l and m, respectively. Then, $\{\xi_v\}$ is bounded and

$$\lim_{q\longrightarrow\infty}\sigma\left(\mathscr{T}\xi_{v_q},\xi_{v_q}\right)=0, \tag{45}$$

which concludes that $l\in F(\mathscr{T})$. Let $\tau : W\longrightarrow[0,\infty)$ is a type function which is generated by $\{\xi_{v_q}\}$. Then,

$$\begin{aligned}\tau(l)&=\limsup_{v\longrightarrow\infty}\sigma\left(\xi_{v_q},l\right),\\ \tau(\mathscr{T}l)&=\limsup_{v\longrightarrow\infty}\sigma\left(\xi_{v_q},\mathscr{T}l\right).\end{aligned} \tag{46}$$

By Lemma 13, we infer

$$\begin{aligned}\sigma^2\left(\mathscr{T}\xi_{v_q},\mathscr{T}l\right)\le{}&\sigma^2\left(\xi_{v_q},l\right)+\frac{2\alpha}{1-\alpha}\sigma^2\left(\mathscr{T}\xi_{v_q},\xi_{v_q}\right)\\&+\frac{2\alpha}{1-\alpha}\sigma\left(\mathscr{T}\xi_{v_q},\xi_{v_q}\right)\left[\sigma\left(\xi_{v_q},l\right)\right.\\&\left.+\sigma\left(\mathscr{T}\xi_{v_q},\mathscr{T}l\right)\right],\end{aligned} \tag{47}$$

as

$$\begin{aligned}&\lim_{q\longrightarrow\infty}\sigma\left(\mathscr{T}\xi_{v_q},\xi_{v_q}\right)=0,\\&\limsup_{q\longrightarrow\infty}\sigma\left(\mathscr{T}\xi_{v_q},\mathscr{T}l\right)\le\limsup_{q\longrightarrow\infty}\sigma\left(\xi_{v_q},l\right).\end{aligned} \tag{48}$$

By uniqueness of element l and definition of Δ-convergence, we conclude that

$$\mathscr{T}l=l. \tag{49}$$

Similarly, one can easily show that

$$\mathscr{T}m=m. \tag{50}$$

By continuity of τ and definition of Δ-convergence, we get

$$\begin{aligned}\limsup_{v\longrightarrow\infty}\sigma(\xi_v,l)&\le\limsup_{q\longrightarrow\infty}\sigma\left(\xi_{v_q},l\right)\le\limsup_{q\longrightarrow\infty}\sigma\left(\xi_{v_q},m\right)\\&=\limsup_{v\longrightarrow\infty}\sigma(\xi_v,m)=\limsup_{r\longrightarrow\infty}\sigma\left(\xi_{v_r},m\right)\\&\le\limsup_{r\longrightarrow\infty}\sigma\left(\xi_{v_r},l\right),\end{aligned} \tag{51}$$

which shows that $l=m$. □ □

Theorem 22. *Let W be a nonempty closed convex subset of a UCPOHMS $(H,\sigma,\preceq)$ and $\mathscr{T} : W\longrightarrow W$ a monotone α-nonexpansive mapping. Suppose there exists a sequence $\{\xi_v\}$ defined by (14) with $\xi_1\preceq\mathscr{T}\xi_1$(or $\mathscr{T}\xi_1\preceq\xi_1$) and*

$$\limsup_{v\longrightarrow\infty}\beta_v(1-\beta_v)>0. \tag{52}$$

Then, $\{\xi_v\}$ converges strongly to a fixed point of $\mathscr{T}$.

Proof. By Theorem 18, there exists a subsequence $\{\xi_{v_q}\}$ of $\{\xi_v\}$ which converges strongly to a point $\eta\in W$. From Lemma 15, we get

$$\xi_1\preceq\xi_{v_q}\preceq\eta,\forall q\in\mathbb{N}. \tag{53}$$

By Theorem 17, $F(\mathscr{T})\neq\emptyset$ and $\{\xi_v\}$ is bounded, and

$$\liminf_{v\longrightarrow\infty}\sigma(\mathscr{T}\xi_v,\xi_v)=0. \tag{54}$$

Without loss of generality, we get

$$\lim_{q\longrightarrow\infty}\sigma\left(\mathscr{T}\xi_{v_q},\xi_{v_q}\right)=0. \tag{55}$$

On the other hand, by Lemma 13, we derive

$$\begin{aligned}\sigma^2\left(\mathscr{T}\xi_{v_q},\mathscr{T}\eta\right)\le{}&\sigma^2\left(\xi_{v_q},\eta\right)+\frac{2\alpha}{1-\alpha}\sigma^2\left(\mathscr{T}\xi_{v_q},\xi_{v_q}\right)\\&+\frac{2\alpha}{1-\alpha}\sigma\left(\mathscr{T}\xi_{v_q},\xi_{v_q}\right)\left[\sigma\left(\xi_{v_q},\eta\right)\right.\\&\left.+\sigma\left(\mathscr{T}\xi_{v_q},\mathscr{T}\eta\right)\right],\end{aligned} \tag{56}$$

as

$$\begin{aligned}&\lim_{q\longrightarrow\infty}\sigma\left(\mathscr{T}\xi_{v_q},\xi_{v_q}\right)=0,\\&\limsup_{q\longrightarrow\infty}\sigma\left(\mathscr{T}\xi_{v_q},\mathscr{T}\eta\right)\le\limsup_{q\longrightarrow\infty}\sigma\left(\xi_{v_q},\eta\right).\end{aligned} \tag{57}$$

By boundedness of $\{\xi_{v_q}\}$, we have

$$\begin{aligned}&\limsup_{q\longrightarrow\infty}\sigma\left(\xi_{v_q},\eta\right)=0,\\&\limsup_{q\longrightarrow\infty}\sigma\left(\mathscr{T}\xi_{v_q},\mathscr{T}\eta\right)=0,\end{aligned} \tag{58}$$

and hence,

$$\lim_{q\longrightarrow\infty}\sigma\left(\mathscr{T}\xi_{v_q},\mathscr{T}\eta\right)=0. \tag{59}$$

Therefore,

$$\lim_{q\longrightarrow\infty}\sigma\left(\xi_{v_q},\mathscr{T}\eta\right)=\lim_{q\longrightarrow\infty}\sigma\left(\xi_{v_q},\mathscr{T}\xi_{v_q}\right)+\lim_{q\longrightarrow\infty}\sigma\left(\mathscr{T}\xi_{v_q},\mathscr{T}\eta\right)=0, \tag{60}$$

which shows that $\eta\in F(\mathscr{T})$. By Theorem 18, $\lim_{v\longrightarrow\infty}\sigma(\xi_v,\eta)$ exists, so

$$\lim_{v\longrightarrow\infty}\sigma(\xi_v,\eta)=0. \tag{61}$$

This completes the proof. □ □

Example 1. Let $W = [1, 4]$ be a nonempty closed convex subset of a UCPOHMS $(\mathbb{R}, \sigma, \leq)$ and $T : W \longrightarrow W$ defined by $Tx = (2x+4)^{1/3}$ be a monotone α-nonexpansive mapping. Then, for the sequences $\alpha_n = 3n/(2n+1)^4$, $\beta_n = 2n + 1/(7n+3)^2$, and $\gamma_n = 5n + 2/(3n+2)^3$, there exists a sequence $\{\xi_v\}$ for which all the conditions of Theorem 22 are satisfied by T, and hence, $x = 2$ is the required fixed point.

4. Comparison of Iteration Processes

In this section, we are presenting some iterations [25–31] which we will be used in the numerical example.

Mann iteration process

In 1953, Mann proposed an iteration, namely, *Mann iteration*, for calculation of a fixed point for a nonexpansive mapping $\mathcal{T}$, defined as

$$\xi_{v+1} = \beta_v \xi_v + (1 - \beta_v)\mathcal{T}\xi_v, \tag{62}$$

for each $v \geq 1$ and $\{\beta_v\} \subset (0, 1)$.

Ishikawa iteration process

In 1974, Ishikawa proposed the two-step iteration process as follows:

$$\begin{cases} \xi_1 = \xi \in W, \\ \xi_{v+1} = (1 - \alpha_v)\xi_v + \alpha_v \mathcal{T}\eta_v, \\ \eta_v = (1 - \beta_v)\xi_v + \beta_v \mathcal{T}\xi_v, \quad v \in \mathbb{N}, \end{cases} \tag{63}$$

where $\{\alpha_v\}$ and $\{\beta_v\}$ are in $(0, 1)$.

Noor iteration process

In 2000, Noor proposed the three-step iteration as follows:

$$\begin{cases} \xi_1 = \xi \in W, \\ \xi_{v+1} = (1 - \alpha_v)\xi_v + \alpha_v \mathcal{T}\eta_v, \\ \eta_v = (1 - \beta_v)\xi_v + \beta_v \mathcal{T}z_v, \\ z_v = (1 - \gamma_v)\xi_v + \gamma_v \mathcal{T}\xi_v, \quad v \in \mathbb{N}, \end{cases} \tag{64}$$

where $\{\alpha_v\}$, $\{\beta_v\}$, and $\{\gamma_v\}$ are in $(0, 1)$.

Agarwal iteration process

In 2007, Agarwal et al. introduced the three-step iteration as follows:

$$\begin{cases} \xi_1 = \xi \in W, \\ \xi_{v+1} = (1 - \alpha_v)\mathcal{T}\xi_v + \alpha_v \mathcal{T}\eta_v, \\ \eta_v = (1 - \beta_v)\xi_v + \beta_v \mathcal{T}\xi_v, \quad v \in \mathbb{N}, \end{cases} \tag{65}$$

where $\{\alpha_v\}$ and $\{\beta_v\}$ are in $(0, 1)$.

Abbas and Nazir iteration process

Table 1: Convergence behavior of Mann, Ishikawa, Noor, Agarwal, Abbas, Thakur, and Kalsoom et al. iterations towards fixed point.

Initial points	0.5	0.7	1.3	1.8
Mann	3978	3980	3981	3981
Ishikawa	2656	2657	2659	2600
Noor	2279	2280	2282	2283
Agarwal	1590	1590	1590	1576
Abbas	1000	1000	1000	1001
Thakur	883	883	883	882
Kalsoom	795	795	795	786

In 2014, Abbas and Nazir introduced the three-step iteration as follows:

$$\begin{cases} \xi_1 = \xi \in W, \\ \xi_{v+1} = (1 - \alpha_v)\mathcal{T}z_v + \alpha_v \mathcal{T}\eta_v, \\ \eta_v = (1 - \beta_v)\mathcal{T}\xi_v + \beta_v \mathcal{T}z_v, \\ z_v = (1 - \gamma_v)\xi_v + \gamma_v \mathcal{T}\xi_v, \quad v \in \mathbb{N}, \end{cases} \tag{66}$$

where $\{\alpha_v\}$, $\{\beta_v\}$, and $\{\gamma_v\}$ are in $(0, 1)$.

Thakur iteration process

In 2016, Thakur et al. proposed the three-step iteration as follows:

$$\begin{cases} \xi_1 = \xi \in W, \\ \xi_{v+1} = \mathcal{T}\eta_v, \\ \eta_v = \mathcal{T}((1 - \alpha_v)\xi_v + \alpha_v z_v), \\ z_v = (1 - \beta_v)\xi_v + \beta_v \mathcal{T}\xi_v, \quad v \in \mathbb{N}, \end{cases} \tag{67}$$

where $\{\alpha_v\}$ and $\{\beta_v\}$ are in $(0, 1)$.

Two qualities fastness and stability play a vital role in iteration process to be performed. In [32], Rhoades mentioned that for the increasing functions, Ishikawa [27] iteration process is faster than Mann iteration process [26] but in the case of decreasing function, condition is reverse. In [29], Agarwal et al. proved that their iteration process was more stable than the previous ones. In [31], Thakur iteration process was considered faster convergent than all the abovementioned iteration processes.

Recently, Kalsoom et al. [25] introduced a new iteration process and proved it to be the fastest convergent than all. The following example is given to support this claim.

Example 2. Define

$$\mathcal{T}\xi = \sin \xi - 1 + \cos \xi, \tag{68}$$

for $\xi \in [0, 2)$. Then, $\mathcal{T}$ is monotone nonexpansive mapping as well as $\mathcal{T}$ is monotone α-nonexpansive mapping.

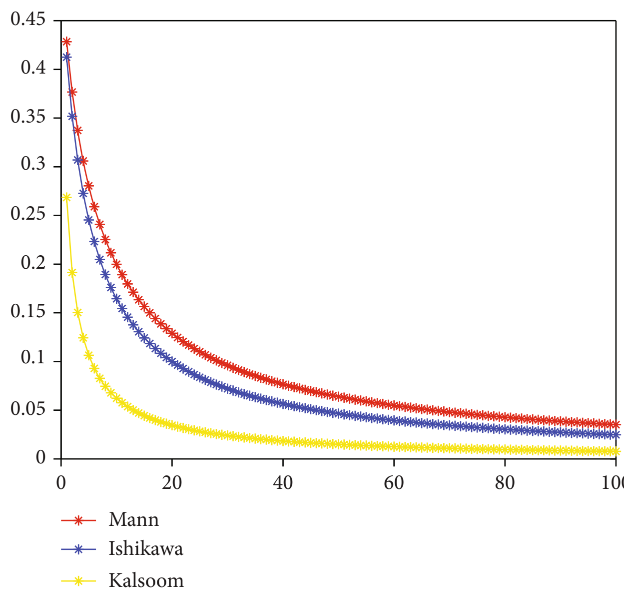

FIGURE 1: Convergence behavior of Mann, Ishikawa, and Kalsoom et al. iterations towards fixed point.

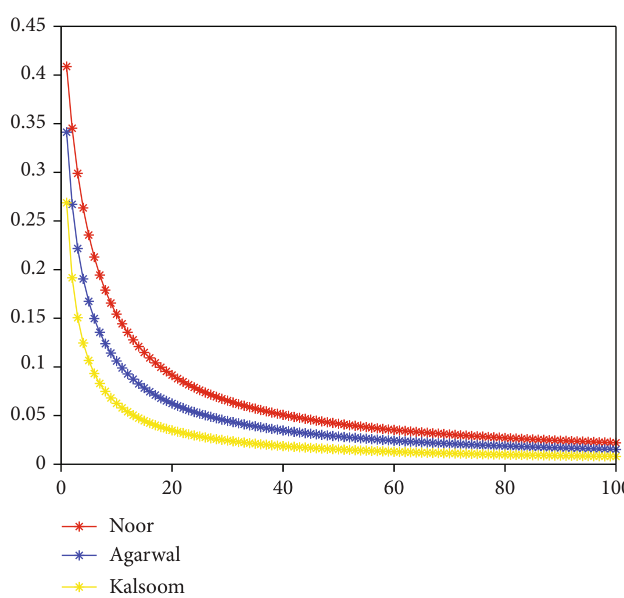

FIGURE 2: Convergence behavior of Noor, Agarwal, and Kalsoom et al. iterations towards fixed point.

Now we will compare abovementioned iteration processes to check the convergence of mapping $\mathcal{T}$. By using MATLAB, we present graphs and table.

In Table 1, we discussed the convergence behavior of some iteration processes. It is clear that all iterations approach to 0 which is the fixed point of mapping $\mathcal{T}$. In this case, Figures 1–3 show that Kalsoom et al. iteration process converges faster to the fixed point as compared the other iterations.

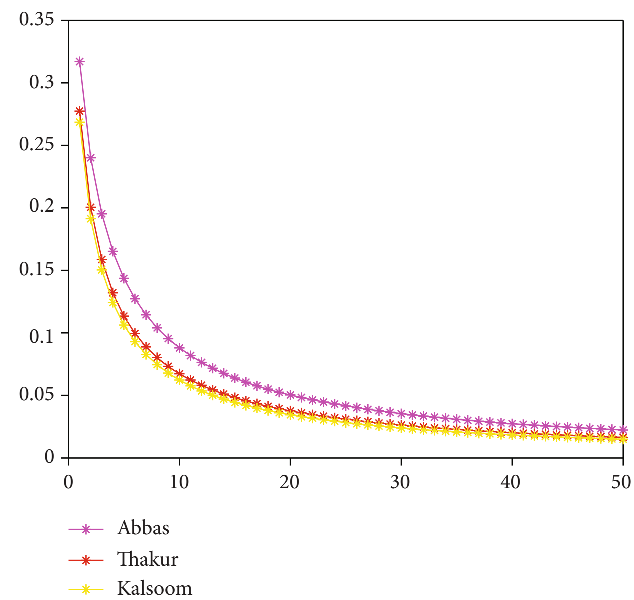

FIGURE 3: Convergence behavior of Abbas, Thakur, and Kalsoom et al. iterations towards fixed point.

5. Monotone Generalized β-Nonexpansive Mappings

In this section, we define monotone generalized β-nonexpansive mapping which generalizes the results of Pandey and Shukla [13] in hyperbolic spaces.

Now we will define monotone generalized β-nonexpansive mappings in hyperbolic space with nontrivial example.

Definition 23. Let W be a nonempty subset of an ordered hyperbolic metric space $(H, \sigma, \preceq)$. A mapping $\mathcal{T} : W \longrightarrow W$ is said to be monotone generalized β-nonexpansive mapping, if $\mathcal{T}$ is monotone and there exists $\beta \in [0, 1)$ such that

$$\begin{aligned}\frac{1}{2}\sigma(\xi, \mathcal{T}\xi) \le \sigma(\xi, \eta) \Rightarrow \sigma(\mathcal{T}\xi, \mathcal{T}\eta) \\ \le \sigma(\xi, \eta) + \beta\{\sigma(\xi, \eta) - \sigma(\mathcal{T}\xi, \mathcal{T}\eta)\},\end{aligned} \tag{69}$$

for all $\xi, \eta \in W$ with $\xi \preceq \eta$.

Proposition 24. *Every monotone nonexpansive mapping is monotone generalized β-nonexpansive mapping, but converse is not true.*

Proof. By putting $\beta = 0$ in (69), we have

$$\frac{1}{2}\sigma(\xi, \mathcal{T}\xi) \le \sigma(\xi, \eta) \Rightarrow \sigma(\mathcal{T}\xi, \mathcal{T}\eta) \le \sigma(\xi, \eta), \tag{70}$$

which shows that monotone generalized β-nonexpansive mapping reduces to monotone nonexpansive mapping satisfying the condition (3). The following example will prove the converse statement does not hold. □ □

Example 3. Let $W = [0, 4]$ be a subset of H endowed with usual order. Define $\mathcal{T} : W \longrightarrow W$ by

$$\mathcal{T}\xi = \begin{cases} 0, & \xi \neq 4, \\ 2, & \xi = 4. \end{cases} \tag{71}$$

Then, for $\xi \in (2, 8/3]$ and $\eta = 4$,

$$\frac{1}{2}\sigma(\xi, \mathcal{T}\xi) \leq \sigma(\xi, \eta), \tag{72}$$

implies

$$\sigma(\mathcal{T}\xi, \mathcal{T}\eta) = 2 > \sigma(\xi, \eta), \tag{73}$$

and so $\mathcal{T}$ does not hold condition (3). Again, $\xi \in (2, 3]$ and $\eta = 4$,

$$\frac{1}{2}\sigma(\eta, \mathcal{T}\eta) \leq \sigma(\xi, \eta) \Rightarrow \sigma(\mathcal{T}\xi, \mathcal{T}\eta) > \sigma(\xi, \eta), \tag{74}$$

and $\mathcal{T}$ does not satisfy condition (3). Nevertheless, $\mathcal{T}$ is generalized β-nonexpansive mapping with $\beta \geq 1/3$.

Proposition 25. *Let W be a nonempty subset of an ordered hyperbolic metric space $(H, \sigma, \preceq)$ and $\mathcal{T} : W \longrightarrow W$ a monotone generalized β-nonexpansive mapping which has a fixed point $\eta \in W$ with $\xi \preceq \eta$. Then, $\mathcal{T}$ can be referred as a monotone quasi-nonexpansive mapping.*

Proof. Let $\eta \in F(\mathcal{T})$ and $\xi \in W$; then, by (69),

$$\begin{aligned} \frac{1}{2}\sigma(\xi, \mathcal{T}\xi) \leq \sigma(\xi, \eta) &\Rightarrow \sigma(\mathcal{T}\xi, \mathcal{T}\eta) \\ &\leq \sigma(\xi, \eta) + \beta\{\sigma(\xi, \eta) - \sigma(\mathcal{T}\xi, \mathcal{T}\eta)\}, \end{aligned}$$

$$\sigma(\mathcal{T}\xi, \eta) \leq \sigma(\xi, \eta) + \beta\sigma(\xi, \eta) - \beta\sigma(\mathcal{T}\xi, \eta),$$

$$(1 + \beta)\sigma(\mathcal{T}\xi, \eta) \leq (1 + \beta)\sigma(\xi, \eta), \tag{75}$$

where $(1 + \beta) > 0$ as $\beta \in [0, 1)$ which shows that $\mathcal{T}$ is monotone quasi-nonexpansive mapping. □ □

Proposition 26. *Let W be a nonempty subset of an ordered hyperbolic metric space $(H, \sigma, \preceq)$. If $\mathcal{T} : W \longrightarrow W$ is monotone generalized β-nonexpansive mapping, then $F(\mathcal{T})$ is closed. Furthermore, if H is strictly convex, then W is convex and $F(\mathcal{T})$ is also convex.*

Proof. Let $\{z_\upsilon\}$ be a sequence in $F(\mathcal{T})$ which converges to $z \in W$. Since

$$\frac{1}{2}\sigma(z_\upsilon, \mathcal{T}z_\upsilon) \leq \sigma(z_\upsilon, z), \tag{76}$$

with the help of continuity of metric, we have

$$\begin{gathered} \lim_{\upsilon\longrightarrow\infty} \sigma(\mathcal{T}z_\upsilon, \mathcal{T}z) \leq \lim_{\upsilon\longrightarrow\infty} \sigma(z_\upsilon, \mathcal{T}z), \\ \lim_{\upsilon\longrightarrow\infty} \sigma(z_\upsilon, \mathcal{T}z) \leq \lim_{\upsilon\longrightarrow\infty} [\sigma(z_\upsilon, z) + \beta\{\sigma(z_\upsilon, z) - \sigma(\mathcal{T}z_\upsilon, \mathcal{T}z)\}], \\ \lim_{\upsilon\longrightarrow\infty} \sigma(z_\upsilon, \mathcal{T}z) \leq \lim_{\upsilon\longrightarrow\infty} \sigma(z_\upsilon, z) + \beta \lim_{\upsilon\longrightarrow\infty} \sigma(z_\upsilon, z) - \beta \lim_{\upsilon\longrightarrow\infty} \sigma(z_\upsilon, \mathcal{T}z), \\ (1 + \beta) \lim_{\upsilon\longrightarrow\infty} \sigma(z_\upsilon, \mathcal{T}z) \leq (1 + \beta) \lim_{\upsilon\longrightarrow\infty} \sigma(z_\upsilon, z), \\ (1 + \beta) > 0 \quad \text{as} \quad \beta \in [0, 1), \\ \lim_{\upsilon\longrightarrow\infty} \sigma(z_\upsilon, \mathcal{T}z) \leq \lim_{\upsilon\longrightarrow\infty} \sigma(z_\upsilon, z), \mathcal{T}z = z. \end{gathered} \tag{77}$$

Hence, $F(\mathcal{T})$ is closed. Now, we assume that H is strictly convex and W is convex. Let $\lambda \in [0, 1)$, $\xi, \eta \in F(\mathcal{T})$ with $\xi \neq \eta$; then, put $z = \lambda\xi \oplus (1 - \lambda)\eta \in W$.

$$\begin{gathered} \frac{1}{2}\sigma(\xi, \mathcal{T}\xi) = 0 < \sigma(\xi, z), \\ \sigma(\mathcal{T}\xi, \mathcal{T}z) \leq \sigma(\xi, z) + \beta\{\sigma(\xi, z) - \sigma(\mathcal{T}\xi, \mathcal{T}z)\}, \\ (1 + \beta)\sigma(\mathcal{T}\xi, \mathcal{T}z) \leq (1 + \beta)\sigma(\xi, z), \\ (1 + \beta) > 0 \text{ as } \beta \in [0, 1), \\ \Rightarrow \sigma(\mathcal{T}\xi, \mathcal{T}z) \leq \sigma(\xi, z). \end{gathered} \tag{78}$$

By similar argument, we get

$$\begin{gathered} \frac{1}{2}\sigma(\eta, \mathcal{T}\eta) = 0 < \sigma(\eta, z), \\ \sigma(\mathcal{T}\eta, \mathcal{T}z) \leq \sigma(\eta, z) + \beta\{\sigma(\eta, z) - \sigma(\mathcal{T}\eta, \mathcal{T}z)\}, \\ (1 + \beta)\sigma(\mathcal{T}\eta, \mathcal{T}z) \leq (1 + \beta)\sigma(\eta, z), \\ (1 + \beta) > 0 \text{ as } \beta \in [0, 1), \\ \Rightarrow \sigma(\mathcal{T}\eta, \mathcal{T}z) \leq \sigma(\eta, z). \end{gathered} \tag{79}$$

Therefore,

$$\begin{aligned} \sigma(\xi, \eta) &\leq \sigma(\xi, \mathcal{T}z) + \sigma(\mathcal{T}z, \eta) \\ &= \sigma(\mathcal{T}\xi, \mathcal{T}z) + \sigma(\mathcal{T}z, \mathcal{T}\eta) \\ &\leq \sigma(\xi, z) + \sigma(z, \eta) \leq \sigma(\xi, \eta). \end{aligned} \tag{80}$$

From strict convexity of H, there exists $\mu \in [0, 1)$ such that

$$\begin{gathered} \mathcal{T}z = \mu\xi \oplus (1 - \mu), \\ \sigma(\xi, \mathcal{T}\xi) \leq \sigma(\xi, z), \\ \sigma(\xi, \mu\xi \oplus (1 - \mu)\eta) \leq \sigma(\xi, z), \\ (1 - \mu)\sigma(\xi, \eta) \leq \sigma(\xi, z)\eta. \end{gathered} \tag{81}$$

By using value of ξ, it gives

$$(1 - \mu)\sigma(\xi, \eta) \leq (1 - \lambda)\sigma(\xi, \eta), \tag{82}$$

and similarly,

$$\sigma(\eta, \mathcal{T} z) \leq \sigma(\eta, z). \tag{83}$$

By putting values of z and $\mathcal{T} z$, we infer

$$\mu\sigma(\xi, \eta) \leq \lambda\sigma(\xi, \eta). \tag{84}$$

From the above two inequalities, it is concluded that

$$\begin{aligned} &(1-\mu) \leq (1-\lambda), \mu \leq \lambda, \\ &\mu = \lambda, \\ &z = \mathcal{T} z \Rightarrow z \in F(\mathcal{T}). \end{aligned} \tag{85}$$

Hence, $F(\mathcal{T})$ is convex. □ □

Lemma 27. *Let W be a nonempty subset of an ordered hyperbolic metric space $(H, \sigma, \preceq)$ and $\mathcal{T} : W \longrightarrow W$ a monotone generalized β-nonexpansive mapping. Then, for each $\xi, \eta \in W$ with $\xi \preceq \eta$,*

(a) $\sigma(T\xi, T^2\xi) \leq \sigma(\xi, T\xi)$.

(b) Either $1/2\sigma(\xi, T\xi) \leq \sigma(\xi, \eta)$ or $1/2\sigma(T\xi, T^2\xi) \leq \sigma(T\xi, \eta)$.

(c) Either $\sigma(T\xi, T\eta) \leq \sigma(\xi, \eta) + \beta\{\sigma(\xi, \eta)\sigma(T\xi, T\eta)\}$ or $\sigma(T^2\xi, T^2\eta) \leq \sigma(T\xi, \eta) + \beta\{-\sigma(T\xi, \eta) - \sigma(T^2\xi, T^2\eta)\}$.

Proof. Since

$$\frac{1}{2}\sigma(\xi, \mathcal{T}\xi) \leq \sigma(\xi, \mathcal{T}\xi), \tag{86}$$

implies

$$\begin{aligned} &\sigma\left(\mathcal{T}\xi, \mathcal{T}^2\xi\right) \leq \sigma(\xi, \mathcal{T}\xi) + \beta\left\{\sigma(\xi, \mathcal{T}\xi) - \sigma\left(\mathcal{T}\xi, \mathcal{T}^2\xi\right)\right\}, \\ &(1+\beta)\sigma\left(\mathcal{T}\xi, \mathcal{T}^2\xi\right) \leq (1+\beta)\sigma(\xi, \mathcal{T}\xi), \\ &(1+\beta) > 0,\ \beta \in [0, 1), \\ &\sigma\left(\mathcal{T}\xi, \mathcal{T}^2\xi\right) \leq \sigma(\xi, \mathcal{T}\xi). \end{aligned} \tag{87}$$

Hence, part (a) is satisfied. Now, we will prove part (b); we argue with contradiction, and suppose

$$\begin{aligned} &\frac{1}{2}\sigma(\xi, \mathcal{T}\xi) > \sigma(\xi, \eta), \\ &\frac{1}{2}\sigma\left(\mathcal{T}\xi, \mathcal{T}^2\xi\right) > \sigma(\mathcal{T}\xi, \eta). \end{aligned} \tag{88}$$

By (a) and triangular inequality,

$$\begin{aligned} \sigma(\xi, \mathcal{T}\xi) &\leq \sigma(\xi, \eta) + \sigma(\mathcal{T}\xi, \eta) < \frac{1}{2}\sigma(\xi, \mathcal{T}\xi) + \frac{1}{2}\sigma\left(\mathcal{T}\xi, \mathcal{T}^2\xi\right) \\ &\leq \frac{1}{2}\sigma(\xi, \mathcal{T}\xi) + \frac{1}{2}\sigma(\xi, \mathcal{T}\xi) \leq \sigma(\xi, \mathcal{T}\xi), \end{aligned} \tag{89}$$

which is contradiction to our supposition, hence proved. The proof of (c) is in a similar way, so we omit that. □ □

Lemma 28. *Let W be a nonempty subset of an ordered hyperbolic metric space $(H, \sigma, \preceq)$ and $\mathcal{T} : W \longrightarrow W$ a monotone generalized β-nonexpansive mapping. Then, for each $\xi, \eta \in W$ with $\xi \preceq \eta$,*

$$\sigma(\xi, \mathcal{T}\eta) \leq \left(\frac{2+\beta}{1-\beta}\right)\sigma(\xi, \mathcal{T}\xi) + \left(\frac{1+\beta}{1-\beta}\right)\sigma(\xi, \eta). \tag{90}$$

Proof. By the help of Lemma 27, we infer either

$$\sigma(\mathcal{T}\xi, \mathcal{T}\eta) \leq \sigma(\xi, \eta) + \beta\{\sigma(\xi, \eta) - \sigma(\mathcal{T}\xi, \mathcal{T}\eta)\}, \tag{91}$$

or

$$\sigma\left(\mathcal{T}^2\xi, \mathcal{T}^2\eta\right) \leq \sigma(\mathcal{T}\xi, \eta) + \beta\left\{\sigma(\mathcal{T}\xi, \eta) - \sigma\left(\mathcal{T}^2\xi, \mathcal{T}^2\eta\right)\right\}. \tag{92}$$

In the first case, we have

$$\begin{aligned} \sigma(\xi, \mathcal{T}\eta) &\leq \sigma(\xi, \mathcal{T}\xi) + \sigma(\xi, \eta) + \beta\{\sigma(\xi, \eta) - \sigma(\mathcal{T}\xi, \mathcal{T}\eta)\} \\ &\leq \sigma(\xi, \mathcal{T}\xi) + \sigma(\xi, \eta) + \beta\{\sigma(\xi, \mathcal{T}\xi) + \sigma(\mathcal{T}\xi, \eta) - \sigma(\mathcal{T}\xi, \mathcal{T}\eta)\} \\ &\leq \sigma(\xi, \mathcal{T}\xi) + \sigma(\xi, \eta) + \beta\{\sigma(\xi, \mathcal{T}\xi) + \sigma(\xi, \eta) - \sigma(\xi, \mathcal{T}\eta)\} \\ &\leq \sigma(\xi, \mathcal{T}\xi) + \sigma(\xi, \eta) + \beta\{\sigma(\xi, \mathcal{T}\xi) + \sigma(\eta, \mathcal{T}\eta)\} \\ &\leq \sigma(\xi, \mathcal{T}\xi) + \sigma(\xi, \eta) + \beta\sigma(\xi, \mathcal{T}\xi) + \beta\sigma(\xi, \eta) + \beta\sigma(\xi, \mathcal{T}\eta), \end{aligned} \tag{93}$$

and so

$$\sigma(\xi, \mathcal{T}\eta) \leq \left(\frac{1+\beta}{1-\beta}\right)\sigma(\xi, \mathcal{T}\xi) + \left(\frac{1+\beta}{1-\beta}\right)\sigma(\xi, \eta). \tag{94}$$

In second case,

$$\begin{aligned} \sigma(\xi, \mathcal{T}\eta) &\leq \sigma(\xi, \mathcal{T}\xi) + \sigma\left(\mathcal{T}\xi, \mathcal{T}^2\xi\right) + \sigma\left(\mathcal{T}^2\xi, \mathcal{T}\eta\right) \\ &\leq \sigma(\xi, \mathcal{T}\xi) + \sigma(\xi, \mathcal{T}\xi) + \sigma\left(\mathcal{T}^2\xi, \mathcal{T}\eta\right) \\ &\leq 2\sigma(\xi, \mathcal{T}\xi) + \sigma(\mathcal{T}\xi, \eta) + \beta\left\{\sigma(\mathcal{T}\xi, \eta) + \sigma\left(\mathcal{T}^2\xi, \mathcal{T}\eta\right)\right\} \\ &\leq 2\sigma(\xi, \mathcal{T}\xi) + \sigma(\mathcal{T}\xi, \eta) + \beta\{\sigma(\xi, \mathcal{T}\xi) + \sigma(\eta, \mathcal{T}\eta)\} \\ &\leq 2\sigma(\xi, \mathcal{T}\xi) + \sigma(\xi, \eta) + \beta\sigma(\xi, \mathcal{T}\xi) + \beta\sigma(\xi, \eta) + \beta\sigma(\xi, \mathcal{T}\eta), \end{aligned} \tag{95}$$

Table 2: Convergence behavior of Mann, Ishikawa, Noor, Agarwal, Abbas, Thakur and Kalsoom et al. iterations towards fixed point.

Steps	Mann	Ishikawa	Noor	Agarwal	Abbas	Thakur	Kalsoom
1	4.807669	4.457118	4.440406	4.264788	4.183261	4.098255	4.080140
2	4.653029	4.210612	4.195196	4.072351	4.034941	4.010133	4.006740
3	4.528465	4.097407	4.086770	4.019942	4.006713	4.001050	4.000569
4	4.427974	4.045132	4.038623	4.005510	4.001291	4.000108	4.000048
5	4.346800	4.020928	4.017202	4.001523	4.000248	4.000011	4.000000
6	4.281160	4.009708	4.007663	4.000421	4.000047	4.000001	4.000000
7	4.228036	4.004504	4.003414	4.000116	4.000009	4.000000	4.000000
8	4.185011	4.002090	4.001521	4.000032	4.000001	4.000000	4
9	4.150144	4.000970	4.000677	4.000008	4.000000	4.000000	4
10	4.121875	4.000450	4.000302	4.000002	4.000000	4.000000	4
11	4.098946	4.000208	4.000134	4.000000	4.000000	4.000000	4
12	4.080342	4.000096	4.000059	4.000000	4.000000	4	4
13	4.065244	4.000044	4.000026	4.000000	4	4	4
14	4.052988	4.000020	4.000011	4.000000	4	4	4
15	4.043038	4.000009	4.000005	4.000000	4	4	4
16	4.034959	4.000004	4.000002	4.000000	4	4	4
17	4.028397	4.000002	4.000001	4	4	4	4
18	4.023069	4.000000	4.000000	4	4	4	4
19	4.018740	4.000000	4.000000	4	4	4	4
20	4.015225	4.000000	4.000000	4	4	4	4
21	4.012369	4	4	4	4	4	4
22	4.010049	4	4	4	4	4	4
23	4.008164	4	4	4	4	4	4
24	4.006633	4	4	4	4	4	4
25	4.006633	4	4	4	4	4	4

and hence,

$$\sigma(\xi, \mathcal{T}\eta) \leq \left(\frac{2+\beta}{1-\beta}\right)\sigma(\xi, \mathcal{T}\xi) + \left(\frac{1+\beta}{1-\beta}\right)\sigma(\xi, \eta). \tag{96}$$

Hence, we get the desired result. □ □

Theorem 29. *Let W be a nonempty convex and closed subset of an ordered hyperbolic metric space $(H, \sigma, \preceq)$ and $\mathcal{T} : W \longrightarrow W$ a monotone generalized β-nonexpansive mapping. Then, $F(\mathcal{T}) \neq \varnothing$ iff $\{T^{\upsilon}\xi\}$ is a sequence which is also bounded for some $\xi \in W$ provides that $\mathcal{T}^{\upsilon}\xi \preceq z$ for some $z \in W$ and $\xi \preceq \mathcal{T}\xi$.*

Proof. Let $\{\mathcal{T}^{\upsilon}\xi\}$ be a bounded sequence for some $\xi \in W$. As we know that $\mathcal{T}$ is monotone and $\xi \preceq \mathcal{T}\xi$, so we get

$$\mathcal{T}\xi \preceq \mathcal{T}^2\xi. \tag{97}$$

In the same manner, we get

$$\mathcal{T}^2\xi \preceq \mathcal{T}^3\xi \preceq \mathcal{T}^4\xi \preceq \cdots \preceq \mathcal{T}^{\upsilon}\xi \preceq \mathcal{T}^{\upsilon+1}\xi \preceq \cdots. \tag{98}$$

Define $\xi_{\upsilon} = \mathcal{T}\xi_{\upsilon-1} = \mathcal{T}^{\upsilon}\xi$, $\forall \upsilon \in \mathbb{N}$. Then, the asymptotic centre of $\{\xi_{\upsilon}\}$ w.r.t W is $A(W, \{\xi_{\upsilon}\}) = \{z\}$ where z is unique and $\xi_{\upsilon} \preceq z$ for all $\upsilon \in \mathbb{N}$. Now, we claim that $\{\sigma(\xi_{\upsilon+1}, \xi_{\upsilon+2})\}$ is a nonincreasing sequence, that is,

$$\sigma(\xi_{\upsilon+1}, \xi_{\upsilon+2}) \leq \sigma(\xi_{\upsilon}, \xi_{\upsilon+1}). \tag{99}$$

Since

$$\frac{1}{2}\sigma(\xi_{\upsilon}, \mathcal{T}\xi_{\upsilon}) \leq \sigma(\xi_{\upsilon}, \xi_{\upsilon+1}), \tag{100}$$

which gives that

$$\begin{aligned}
\sigma(\xi_{\upsilon+1}, \xi_{\upsilon+2}) &= \sigma(\mathcal{T}\xi_{\upsilon}, \mathcal{T}\xi_{\upsilon+1}) \\
&\leq \sigma(\xi_{\upsilon}, \xi_{\upsilon+1}) + \beta\{\sigma(\xi_{\upsilon}, \xi_{\upsilon+1}) - \sigma(\mathcal{T}\xi_{\upsilon}, \mathcal{T}\xi_{\upsilon+1})\} \\
&= \sigma(\xi_{\upsilon}, \xi_{\upsilon+1}) + \beta\{\sigma(\xi_{\upsilon}, \xi_{\upsilon+1}) - \sigma(\xi_{\upsilon+1}, \xi_{\upsilon+2})\} \\
&\Longrightarrow (1+\beta)\sigma(\xi_{\upsilon+1}, \xi_{\upsilon+2}) \leq (1+\beta)\sigma(\xi_{\upsilon}, \xi_{\upsilon+1}) \\
&\Longrightarrow \sigma(\xi_{\upsilon+1}, \xi_{\upsilon+2}) \leq \sigma(\xi_{\upsilon}, \xi_{\upsilon+1}).
\end{aligned} \tag{101}$$

Now, we claim that

$$\begin{gathered}
\sigma(\xi_{\upsilon}, \xi_{\upsilon+1}) \leq 2\sigma(\xi_{\upsilon}, z) \text{ or} \\
\sigma(\xi_{\upsilon+1}, \xi_{\upsilon+2}) \leq 2\sigma(\xi_{\upsilon+1}, z).
\end{gathered} \tag{102}$$

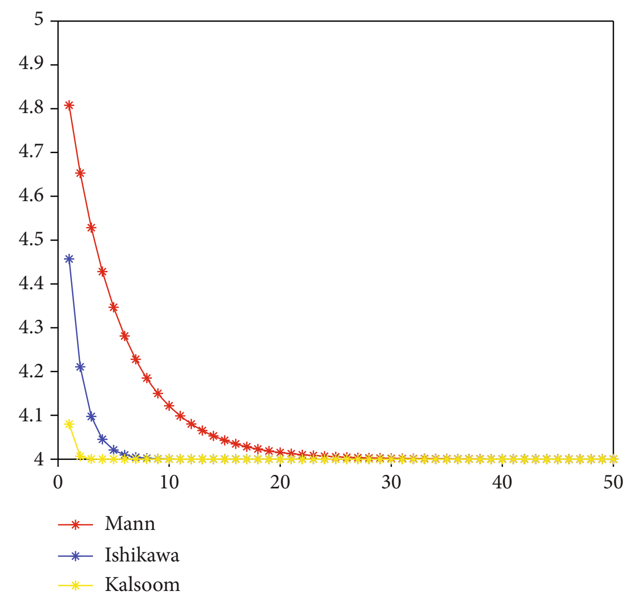

Figure 4: Convergence behavior of Mann, Ishikawa, and Kalsoom et al. iterations towards fixed point.

To prove this, we consider the contradiction

$$\begin{aligned} &2\sigma(\xi_v, z) < \sigma(\xi_v, \xi_{v+1}) \text{ and} \\ &2\sigma(\xi_{v+1}, z) < \sigma(\xi_{v+1}, \xi_{v+2}). \end{aligned} \tag{103}$$

By using triangular inequality,

$$\begin{aligned} \sigma(\xi_v, \xi_{v+1}) &\le \sigma(\xi_v, z) + \sigma(\xi_{v+1}, z) \\ &< \frac{1}{2}\sigma(\xi_v, \xi_{v+1}) + \frac{1}{2}\sigma(\xi_{v+1}, \xi_{v+2}) \\ &< \sigma(\xi_v, \xi_{v+1}), \end{aligned} \tag{104}$$

which is not possible, so (102) is satisfied.

In the first case of (102),

$$\begin{gathered} \frac{1}{2}\sigma(\xi_v, \xi_{v+1}) \le \sigma(\xi_v, z) \\ \frac{1}{2}\sigma(\xi_v, \mathcal{T}\xi_v) \le \sigma(\xi_v, z) \\ \sigma(\mathcal{T}\xi_v, \mathcal{T}z) \le \sigma(\xi_v, z) + \beta\{\sigma(\xi_v, z) - \sigma(\mathcal{T}\xi_v, \mathcal{T}z)\} \\ (1+\beta)\sigma(\mathcal{T}\xi_v, \mathcal{T}z) \le (1+\beta)\sigma(\xi_v, z) \\ \sigma(\mathcal{T}\xi_v, \mathcal{T}z) \le \sigma(\xi_v, z). \end{gathered} \tag{105}$$

Putting lim sup on both sides,

$$\begin{gathered} \limsup_{v\longrightarrow\infty} \sigma(\mathcal{T}\xi_v, \mathcal{T}z) \le \limsup_{v\longrightarrow\infty} \sigma(\xi_v, z), \\ \mathcal{T}z = z. \end{gathered} \tag{106}$$

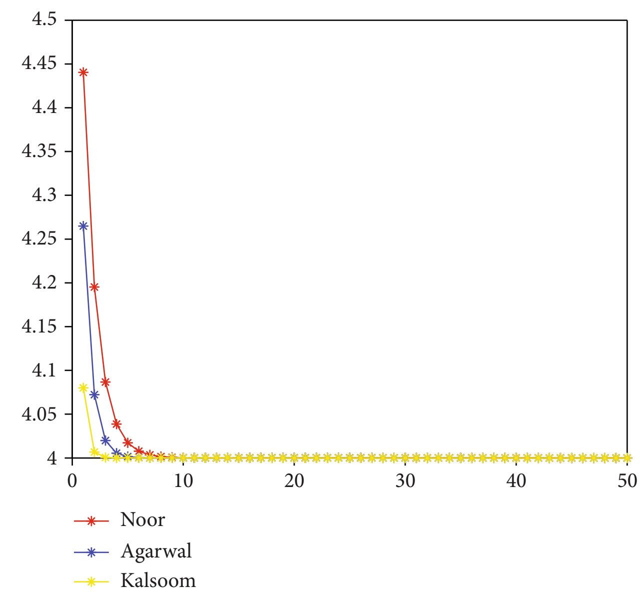

Figure 5: Convergence behavior of Noor, Agarwal, and Kalsoom et al. iterations towards fixed point.

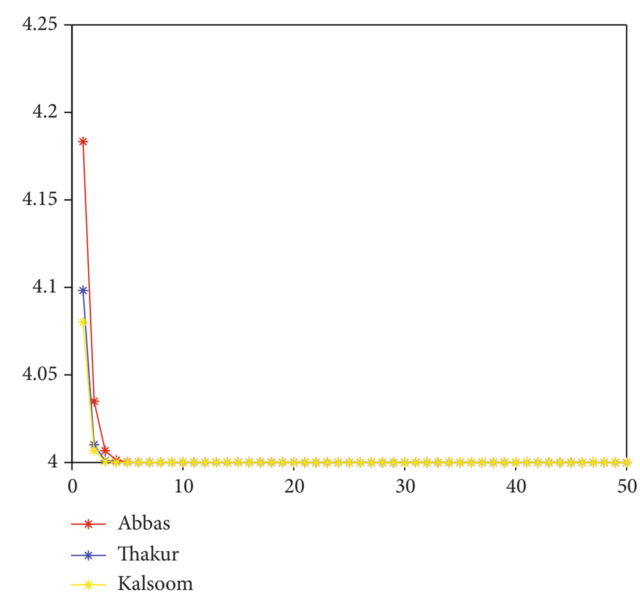

Figure 6: Convergence behavior of Abbas, Thakur, and Kalsoom et al. iterations towards fixed point.

Similarly, in the second case,

$$\begin{gathered} \frac{1}{2}\sigma(\xi_{v+1}, \xi_{v+2}) \le \sigma(\xi_{v+1}, z) \\ \frac{1}{2}\sigma(\xi_{v+1}, \mathcal{T}\xi_{v+1}) \le \sigma(\xi_{v+1}, z) \\ \sigma(\mathcal{T}\xi_{v+1}, \mathcal{T}z) \le \sigma(\xi_{v+1}, z) + \beta\{\sigma(\xi_{v+1}, z) - \sigma(\mathcal{T}\xi_{v+1}, \mathcal{T}z)\} \\ (1+\beta)\sigma(\mathcal{T}\xi_{v+1}, \mathcal{T}z) \le (1+\beta)\sigma(\xi_{v+1}, z) \\ \sigma(\mathcal{T}\xi_{v+1}, \mathcal{T}z) \le \sigma(\xi_{v+1}, z). \end{gathered} \tag{107}$$

Putting lim sup on both sides,

$$\lim_{v\longrightarrow\infty}\sup\sigma(\mathcal{T}\xi_{v+1},\mathcal{T}z)\le\lim_{v\longrightarrow\infty}\sup\sigma(\xi_{v+1},z),\qquad \mathcal{T}z=z. \tag{108}$$

Conversely, $F(\mathcal{T})\neq\varnothing$; then, there exists some $w\in F(\mathcal{T})$ and $\mathcal{T}^v(w)=w\forall v\in\mathbb{N}$; then, $\{T^v(w)\}$ is a constant sequence, and hence, it is bounded and this completes the proof. □ □

Example 4. Let $\mathcal{T}:W\longrightarrow W$ where $W=[3,6]$; then, $\mathcal{T}$ is defined as

$$\mathcal{T}\xi=\sqrt{3\xi+4}, \tag{109}$$

for any $\xi\in W$; take $\alpha_v=0.7$, $\beta_v=0.6$, and $\gamma_v=0.5$. The fixed point of $\mathcal{T}$ is 4, and take initial point as $\xi_0=5$. Then, $\mathcal{T}$ is monotone generalized β-nonexpansive mapping.

In Table 2, we discussed the convergence behavior of some iteration processes. It is clear that all iterations approach to 4 which is the fixed point of $\mathcal{T}$. In this case, Figures 4–6 show that Kalsoom et al. iteration process converges faster to the fixed point as compared the other iterations.

6. Conclusion

It concludes that we have approximated fixed point results of monotone α and generalized β-nonexpansive mappings in hyperbolic spaces. Moreover, we proved some numerical applications and presented the graphical representations by using different iteration processes.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare to have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Some Results on Fractional m-Point Boundary Value Problems

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In this paper, we will apply some fixed-point theorems to discuss the existence of solutions for fractional m-point boundary value problems $D_{0^+}^q(u''(t)) = h(t)f(u(t)), t \in [0,1], 1 < q \le 2, u'(0) = u''(0) = u(1) = 0, u''(1) - \sum_{i=1}^{m-2} \alpha_i u'''(\xi_i) = 0$. In addition, we also present Lyapunov's inequality and Ulam-Hyers stability results for the given m-point boundary value problems.

1. Introduction

Mathematical models due to fractional differential equations can describe the natural phenomenon in physics, population dynamics, chemical technology, biotechnology, aerodynamics, electrodynamics of complex medium, polymer rheology, and control of dynamical systems (see [1–4]). Due to the nonlocal characteristics and the rapid development of the theory of fractional operators, some authors have investigated different aspects of fractional differential equations including existence of solutions, Lyapunov's inequality, and Hyers-Ulam stability for fractional differential equations by different mathematical techniques. For example, first, many authors have discussed the existence of nontrivial solutions of fractional differential equations in nonsingular case as well as singular case. Usually, the proof is based on either the method of upper and lower solutions, fixed-point theorems, alternative principle of Leray-Schauder, topological degree theory, or critical point theory. We refer the readers to [5–20]. Second, Lyapunov, during his study of general theory of stability of motion in 1892, introduced the stability criterion for second-order differential equations, which yielded a counter inequality be called Lyapunov inequality (see [21, 22]). Since then, we can find considerable modifications of Lyapunov-type inequality of differential equations, such as linear differential-algebraic equations, fractional differential equations, extreme Pucci equations, and dynamic equations, which are applied to study the stability and disconjugacy or oscillatory criterion for the mentioned problems, and we refer the readers to [23–32]. Finally, the stability of functional equations was originally raised by Hyers in 1941 (see [33, 34]). Thereafter, the stability properties of all kinds of equations have attracted the attention of many mathematicians. To see more details on the Ulam-Hyers stability and Ulam-Hyers-Rassias of differential equations, we refer the readers to [35–38].

Inspired by the references, this paper is mainly concerned with the existence, Lyapunov's inequality, and Ulam-Hyers stability results for the m-point boundary value problems.

$$\begin{cases} D_{0^+}^q\left(u''(t)\right) = h(t)f(u(t)), t \in [0,1], 1 < q \le 2, \\ u'(0) = u''(0) = u(1) = 0, u''(1) - \sum_{i=1}^{m-2} \alpha_i u'''(\xi_i) = 0, \end{cases} \quad (1)$$

where α_i, ξ_i, h, and f satisfy the following assumptions:

(H1) $\sum_{i=1}^{m-2} \alpha_i \xi_i^{q-2} > 1/q-1$ and $\sum_{i=2}^{m-2} \alpha_i \le (1-\xi_{m-2})^{q-1}/(q-1)\xi_{m-2}^{q-2}((1-\xi_{m-2})^{q-1}/(q-1)\xi_{m-2}^{q-2})$

(H2) $h : [0,1] \longrightarrow \mathbb{R}$ is Lebesgue integral

(H3) $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous

For these goals, we first convert problem (1) into an integral equation via Green function. Furthermore, we study the properties and estimates of the Green function. Then, on the basis of these properties, we apply some fixed-point theorems to establish some existence results of problem (1) under some suitable conditions. In addition, the Lyapunov inequality and Hyers-Ulam stability of the proposed problem are also considered.

2. Preliminaries

Before beginning the main results, we state some classic and modified definitions and lemmas from fractional calculus.

Definition 1 [4]. The fractional integral of order $q > 0$ of a function $u : (0,+\infty) \longrightarrow R$ is given by

$$I_{0^+}^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) ds, \quad (2)$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2 [4]. The fractional derivative of order $q > 0$ of a continuous function $u : (0,+\infty) \longrightarrow R$ is given by

$$D_{0^+}^q u(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{q-n-1}} ds, \quad (3)$$

where $n = [q] + 1$, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 3 [21]. Assume that $q > 0$, then

$$I_{0^+}^q D_{0^+}^q u(t) = u(t) + \sum_{i=1}^n C_i t^{q-i}, \quad (4)$$

for some $C_i \in R$, $i = 1, 2, \cdots, n$, where n is the smallest integer greater than or equal to q.

Lemma 4. *Assume that (H1) holds. Then, for any $y(t) \in L^1[0,1]$, the boundary value problem*

$$\begin{cases} D_{0^+}^q \left(u''(t)\right) = y(t), t \in [0,1], 1 < q \leq 2, \\ u'(0) = u''(0) = u(1) = 0, u''(1) - \displaystyle\sum_{i=1}^{m-2} \alpha_i u'''(\xi_i) = 0, \end{cases} \quad (5)$$

has a unique solution $u(t) = \int_0^1 G(t,s) y(s) ds$. Let $p = 1 - (q-1)\sum_{i=1}^{m-2} \alpha_i \xi_i^{q-2} < 0$,, and we have

(i) for $s \leq t$, $s \leq \xi_1$

$$G(t,s) = \frac{1}{\Gamma(q+2)} \left[(t-s)^{q+1} - (1-s)^{q+1} + 1 - t^{q+1} - \frac{\left(1 - t^{q+1}\right)\left[1 - (1-s)^{q-1}\right]}{p} \right] \quad (6)$$

(ii) for $t \leq s \leq \xi_1$

$$G(t,s) = \frac{1}{\Gamma(q+2)} \left[-(1-s)^{q+1} + 1 - t^{q+1} - \frac{\left(1 - t^{q+1}\right)\left[1 - (1-s)^{q-1}\right]}{p} \right] \quad (7)$$

(iii) for $s \leq t$, $\xi_j \leq s \leq \xi_{j+1}$, $j = 1, 2, \cdots, m-3$

$$G(t,s) = \frac{1}{\Gamma(q+2)} \left[\frac{1 - t^{q+1}}{p} \left[(1-s)^{q-1} - (q-1) \cdot \sum_{i=j+1}^{m-2} \alpha_i (\xi_i - s)^{q-2} \right] + (t-s)^{q+1} - (1-s)^{q+1} \right] \quad (8)$$

(iv) for $t \leq s$, $\xi_j \leq s \leq \xi_{j+1}$, $j = 1, 2, \cdots, m-3$

$$G(t,s) = \frac{1}{\Gamma(q+2)} \left[\frac{1 + t^{q+1}}{p} \left[(1-s)^{q-1} - (q-1) \cdot \sum_{i=j+1}^{m-2} \alpha_i (\xi_i - s)^{q-2} \right] - (1-s)^{q+1} \right] \quad (9)$$

(v) for $\xi_{m-2} \leq s \leq t$

$$G(t,s) = \frac{1}{\Gamma(q+2)} \left[(t-s)^{q+1} - (1-s)^{q+1} + \frac{1 - t^{q+1}}{p} (1-s)^{q-1} \right] \quad (10)$$

(vi) for $\xi_{m-2} \leq s$, $t \leq s$

$$G(t,s) = \frac{1}{\Gamma(q+2)} \left[-(1-s)^{q+1} + \frac{1 - t^{q+1}}{p} (1-s)^{q-1} \right] \quad (11)$$

Proof. From Definition 3, it follows that

$$u''(t) = I_{0^+}^q y(t) - C_1 t^{q-1} - C_2 t^{q-2}. \quad (12)$$

Since $u''(0)=0$, it is clear that $C_2=0$. Then,

$$u''(t)=I_{0^+}^{q}y(t)-C_1t^{q-1}=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}y(s)ds-C_1t^{q-1}. \tag{13}$$

On one hand, taking the derivative of $u''(t)$, we can get

$$u'''(t)=\frac{q-1}{\Gamma(q)}\int_0^t(t-s)^{q-2}y(s)ds-(q-1)C_1t^{q-2}. \tag{14}$$

On the other hand, combining the boundary conditions $u(1)=u'(0)=0$, we have

$$\begin{aligned}u'(t)&=\frac{1}{\Gamma(q)}\int_0^t\int_0^\tau(\tau-s)^{q-1}y(s)d\tau-\frac{C_1}{q}t^q\\&=\frac{1}{\Gamma(q+1)}\int_0^t(t-s)^qy(s)ds-\frac{C_1}{q}t^q.\end{aligned} \tag{15}$$

Furthermore, we have

$$\begin{aligned}u(t)&=-\frac{1}{\Gamma(q+1)}\int_t^1\int_0^\tau(\tau-s)^qy(s)dsd\tau+\frac{C_1}{q}\int_t^1\tau^qd\tau\\&=-\frac{1}{\Gamma(q+2)}\left[\int_0^1(1-s)^{q+1}y(s)ds-\int_0^t(t-s)^{q+1}y(s)ds\right]\\&\quad+\frac{C_1\left(1-t^{q+1}\right)}{q(q+1)}.\end{aligned} \tag{16}$$

According to these above expressions, we have

$$\begin{aligned}u''(1)&=\frac{1}{\Gamma(q)}\int_0^1(1-s)^{q-1}y(s)ds-C_1,\\\sum_{i=1}^{m-2}\alpha_iu'''(\xi_i)&=\frac{q-1}{\Gamma(q)}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}(\xi_i-s)^{q-2}y(s)ds\\&\quad-(q-1)C_1\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-2}.\end{aligned} \tag{17}$$

Then, from $u''(1)-\sum_{i=1}^{m-2}\alpha_iu'''(\xi_1)=0$, it follows that

$$\begin{aligned}C_1&=\frac{1}{p\Gamma(q)}\left[\int_0^1(1-s)^{q-1}y(s)ds\right.\\&\quad\left.-(q-1)\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}(\xi_i-s)^{q-2}y(s)ds\right],\end{aligned} \tag{18}$$

which yields

$$\begin{aligned}u(t)&=\frac{1}{\Gamma(q+2)}\left[\int_0^t(t-s)^{q+1}y(s)ds-\int_0^1(1-s)^{q+1}y(s)ds\right.\\&\quad+\frac{1-t^{q+1}}{p}\left(\int_0^1(1-s)^{q-1}y(s)ds\right.\\&\quad\left.\left.-(q-1)\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}(\xi_i-s)^{q-2}y(s)ds\right)\right].\end{aligned} \tag{19}$$

If $s\le t$, $s\le\xi_1$, we have

$$\begin{aligned}G(t,s)&=\frac{1}{\Gamma(q+2)}\left[(t-s)^{q+1}-(1-s)^{q+1}+\frac{1-t^{q+1}}{p}\right.\\&\quad\left.\cdot\left((1-s)^{q-1}-(q-1)\sum_{i=1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right]\right.\\&=\frac{1}{\Gamma(q+2)}\left[(t-s)^{q+1}-(1-s)^{q+1}+1\right.\\&\quad\left.-t^{q+1}-\frac{1-t^{q+1}}{p}\left(1-(1-s)^{q-1}\right)\right].\end{aligned} \tag{20}$$

□

In the similar way, we also can get the expression of $G(t,s)$ on other intervals.

Lemma 5. *Assume that (H1) holds. Then, $G(t,s)$ satisfies the following properties:*

(I) Sign of $G(t,s)$

(i) $G(t,s)\ge0$, for $0\le s\le\xi_1$

(ii) $G(t,s)\le0$, for $\xi_1<s\le1$

(II) The range of $G(t,s)$

(1) For $0\le s\le\xi_1$

$$0\le G(t,s)<\frac{1}{\Gamma(q+2)}\left[1-(1-s)^{q+1}-\frac{1-(1-s)^{q-1}}{p}\right] \tag{21}$$

(2) For $\xi_j\le s\le\xi_{j+1}$, $j=1,2,\cdots,m-3$

$$\begin{aligned}&\frac{1}{\Gamma(q+2)}\left[\frac{1}{p}\left((1-s)^{q-1}-(q-1)\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right)\right.\\&\quad\left.-(1-s)^{q+1}\right]\le G(t,s)\le0\end{aligned} \tag{22}$$

(3) For $\xi_{m-2} \le s \le 1$

$$\frac{1}{\Gamma(q+2)}\left[\frac{(1-s)^{q-1}}{p}-(1-s)^{q+1}\right] \le G(t,s) \le 0 \tag{23}$$

Proof. For $0 \le s \le \xi_1$, by the definition of $G(t,s)$, it is clear that $G(t,s)$ is continuous and derivativable with respect to t at $[0,1]$. On one hand, if $s \le t \le 1$, we have

$$\frac{\partial G}{\partial t} = \frac{1}{\Gamma(q+1)}\left((t-s)^q - t^q + \frac{t^q\left[1-(1-s)^{q-1}\right]}{p}\right) \le 0. \tag{24}$$

On the other hand, if $0 \le t < s$, we have

$$\frac{\partial G}{\partial t} = \frac{t^q}{\Gamma(q+1)}\left(-1+\frac{1-(1-s)^{q-1}}{p}\right) \le 0. \tag{25}$$

Then, $G(t,s)$ is nonincreasing on t, which yields that

$$\begin{aligned}
&\min\{G(t,s)\colon t\in[0,1]\} = G(1,s) = 0,\\
&\max\{G(t,s)\colon t\in[0,1]\}\\
&\quad = G(0,s) = \frac{1}{\Gamma(q+2)}\left[1-(1-s)^{q+1}-\frac{1-(1-s)^{q-1}}{p}\right].
\end{aligned} \tag{26}$$

So for $0 \le s < \xi_1$, $0 \le t \le 1$, it concludes that

$$0 \le G(t,s) < \frac{1}{\Gamma(q+2)}\left[1-(1-s)^{q+1}-\frac{1-(1-s)^{q-1}}{p}\right]. \tag{27}$$

For $\xi_j \le s \le \xi_{j+1} (j = 1,2,\cdots,m-3)$, we have

$$\begin{aligned}
&\min\{G(t,s)\colon t\in[0,1]\}\\
&\quad = G(0,s) = \frac{1}{\Gamma(q+2)}\left[\frac{1}{p}\left((1-s)^{q-1}\right.\right.\\
&\qquad \left.\left. -(q-1)\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right)-(1-s)^{q+1}\right],\\
&\max\{G(t,s)\colon t\in[0,1]\} = G(1,s) = 0.
\end{aligned} \tag{28}$$

For $\xi_{m-2} \le s \le 1$, we have

$$\begin{aligned}
&\min\{G(t,s)\colon t\in[0,1]\}\\
&\quad = G(0,s) = \frac{1}{\Gamma(q+2)}\left[\frac{(1-s)^{q-1}}{p}-(1-s)^{q+1}\right],\\
&\quad \max\{G(t,s)\colon t\in[0,1]\} = G(1,s) = 0.
\end{aligned} \tag{29}$$

□

Let

$$\begin{aligned}
G^1 &= \max_{s\in[0,\xi_1]}\left\{\frac{1}{\Gamma(q+2)}\left[1-(1-s)^{q+1}-\frac{1-(1-s)^{q-1}}{p}\right]\right\}\\
&= \frac{1}{\Gamma(q+2)}\left[1-(1-\xi_1)^{q+1}-\frac{1-(1-\xi_1)^{q-1}}{p}\right],\\
G^2 &= \max_{s\in[\xi_{m-2},1]}\left\{\frac{1}{\Gamma(q+2)}\left|\frac{(1-s)^{q-1}}{p}-(1-s)^{q+1}\right|\right\}\\
&= \frac{1}{\Gamma(q+2)}\left[(1-\xi_{m-2})^{q+1}-\frac{(1-\xi_{m-2})^{q-1}}{p}\right],\\
&\qquad G^3 = \max_{1\le j\le m-3}\left\{G_j^3\right\},
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
G_j^3 = \max_{s\in[\xi_j,\xi_{j+1}]}&\left\{\frac{1}{\Gamma(q+2)}\left|\frac{1}{p}\left[(1-s)^{q-1}\right.\right.\right.\\
&\left.\left.\left. -(q-1)\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right]-(1-s)^{q+1}\right|\right\}.
\end{aligned} \tag{31}$$

From Lemma 5, it is clear that $|G(t,s)| \le \bar{G}$, where $\bar{G} = \max\{G^1, G^2, G^3\}$.

Lemma 6. *Assume that (H1) holds and $\xi_1 > 1-(1/2)^{1/q-1}$. Then, $\bar{G} = G^1$.*

Proof. Let

$$\begin{aligned}
G_j(s) = \frac{1}{\Gamma(q+2)}&\left|\frac{1}{p}\left[(1-s)^{q-1}-(q-1)\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right]\right.\\
&\left. -(1-s)^{q+1}\right|, s\in[\xi_j,\xi_{j+1}], j=1,2,\cdots,m-3.
\end{aligned} \tag{32}$$

From (H1) and $\xi_1 > 1-(1/2)^{1/q-1}$, we can verify that

$$\begin{aligned}
&G^1 - |G_j(s)|\\
&\quad = \frac{1}{\Gamma(q+2)}\left[1-(1-\xi_1)^{q+1}-\frac{1}{p}\left(1-(1-\xi_1)^{q-1}\right)\right.\\
&\qquad -(1-s)^{q+1}+\frac{1}{p}\left((1-s)^{q-1}\right.\\
&\qquad \left.\left. -(q-1)\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{\Gamma(q+2)}\left[1-(1-\xi_1)^{q+1}-(1-\xi_j)^{q+1}\right.\\
&\quad -\frac{1}{p}\left(1-(1-\xi_1)^{q-1}-(1-\xi_1)^{q-1}\right)\\
&\quad \left.-\frac{q-1}{p}\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-\xi_j)^{q-2}\right]\\
&\geq \frac{1}{\Gamma(q+2)}\left[1-2(1-\xi_1)^{q+1}-\frac{1}{p}\left(1-2(1-\xi_1)^{q-1}\right)\right.\\
&\quad \left.-\frac{q-1}{p}\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-\xi_j)^{q-2}\right]\\
&\geq \frac{1}{\Gamma(q+2)}\left[1-\frac{1}{2^{2/q-1}}-\frac{q-1}{p}\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-\xi_j)\right]>0.
\end{aligned} \tag{33}$$

Also, we can verify that

$$\begin{aligned}
G^1-G^2 &= \frac{1}{\Gamma(q+2)}\left[1-(1-\xi_1)^{q+1}-\frac{1}{p}\left(1-(1-\xi_1)^{q-1}\right)\right.\\
&\quad \left.-(1-\xi_{m-2})^{q+1}+\frac{1}{p}(1-\xi_{m-2})^{q-1}\right]\\
&\geq \frac{1}{\Gamma(q+2)}\left[1-2(1-\xi_1)^{q+1}-\frac{1}{p}\left(1-2(1-\xi_1)^{q-1}\right)\right]\\
&\geq \frac{1}{\Gamma(q+2)}\left(1-\frac{1}{2^{2/q-1}}\right)>0.
\end{aligned} \tag{34}$$

So, it concludes that $G^1 > G^2$, $G^1 > G^3$, namely, $\bar{G} = G^1$. ☐

3. Main Results

3.1. Existence Results

Theorem 7. *Assume that (H1)-(H3) hold. In addition, there exists a positive constant $L > 0$ such that*

$$|f(u)-f(v)| \leq L|u-v|, \forall u, v \in \mathbb{R}. \tag{35}$$

Then, problem (1) has a unique solution if $L\bar{G}|h|_{L^1} < 1$.

Proof. Let $C[0,1] = \{x(t): x(t) \text{ is continuous on } [0,1]\}$ is a Banach space with the norm $\|x\| = \max_{0\leq t\leq 1}|x(t)|$. From Lemma 4, it is clear that solutions of (1) can be rewritten as fixed points of operator T, which is defined by

$$Tu(t) = \int_0^1 G(t,s)h(s)f(u(s))ds. \tag{36}$$

Now, we show that $T: B_r \longrightarrow B_r$ and T is a contraction map, where $B_r = \{u \in E: \|u\| < r\}$ with

$$r > \frac{\bar{G}|h|_{L^1}|f(0)|}{1-\bar{G}L|h|_{L^1}}. \tag{37}$$

On one hand, for any $u \in B_r$, we have

$$\begin{aligned}
\|T(u)(t)\| &= \left\|\int_0^1 G(t,s)h(s)f(u(s))ds\right\|\\
&\leq \bar{G}\int_0^1 |h(s)f(u(s))|ds\\
&\leq \bar{G}\int_0^1 |h(s)|[|f(u)-f(0)|+|f(0)|]ds\\
&\leq \bar{G}\int_0^1 |h(s)|[Lr+|f(0)|]ds\\
&\leq \bar{G}|h|_{L^1}[Lr+|f(0)|] \leq r,
\end{aligned} \tag{38}$$

which implies that $T(B_r) \subset B_r$.

On the other hand, for any $u, v \in E$, we have

$$\begin{aligned}
\|T(u)-T(v)\| &= \left\|\int_0^1 G(t,s)h(s)(f(u(s)))ds\right\|\\
&\leq \bar{G}\int_0^1 |h(s)||f(u(s))-f(v(s))|ds\\
&\leq \bar{G}L\int_0^1 |h(s)||u(s)-v(s)|ds\\
&\leq L\bar{G}|h|_{L^1}\|u(t)-v(t)\|,
\end{aligned} \tag{39}$$

which implies that T is a contraction map. ☐

Therefore, by the Banach contraction mapping principle, it follows that the operator T has a unique fixed point, which is the unique solution for problem (1).

Theorem 8. *Assume that (H1)-(H3) hold. In addition, there exists a positive constant K such that $|f(u)| \leq K$ for $u \in R$. Then, problem (1) has at least one solution.*

The proof is based on the following fixed-point theorem.

Lemma 9 [39]. *Let E be a Banach space, E_1 is a closed, convex subset of E, Ω an open subset of E_1, and $0 \in \Omega$. Suppose that $T: \Omega \longrightarrow E_1$ is completely continuous. Then, either*

(i) T has a fixed point in Ω, or

(ii) there are $u \in \partial\Omega$ (the boundary of Ω in E_1) and $\rho \in (0,1)$ with $u = \rho T u$

Proof of Theorem 8. First, we show that the operator T is uniformly bounded.

For any $u \in \bar{\Omega}_\delta = \{u \in C[0,1]: \|u\| \le \delta\}$, we have

$$\|T(u)(t)\| = \left\| \int_0^1 G(t,s)h(s)f(u(s))ds \right\| \le \bar{G}K|h|_{L^1}, \quad (40)$$

which implies that $T(\Omega_\delta)$ is uniformly bounded.

Second, for $0 \le s \le \xi_1$, from Lemma 4, we have

(i) if $s \le t \le 1$

$$\begin{aligned}\left|\frac{\partial G(t,s)}{\partial t}\right| &= \frac{1}{\Gamma(q+1)}\left|(t-s)^q - t^q + \frac{t^q\left[1-(1-s)^{q-1}\right]}{p}\right| \\ &\le \frac{1}{\Gamma(q+1)}\left((t-s)^q + t^q - \frac{t^q\left[1-(1-s)^{q-1}\right]}{p}\right) \\ &\le \frac{1}{\Gamma(q+1)}\left(2-\frac{1}{p}\right).\end{aligned} \quad (41)$$

(ii) if $0 \le t < s$

$$\begin{aligned}\left|\frac{\partial G(t,s)}{\partial t}\right| &= \frac{1}{\Gamma(q+1)}\left|-t^q + \frac{t^q\left[1-(1-s)^{q-1}\right]}{p}\right| \\ &\le \frac{1}{\Gamma(q+1)}\left(t^q - \frac{t^q\left[1-(1-s)^{q-1}\right]}{p}\right) \\ &\le \frac{1}{\Gamma(q+1)}\left(1-\frac{1}{p}\right),\end{aligned} \quad (42)$$

which implies that $|\partial G(t,s)/\partial t|$ is bounded for $0 \le s < \xi_1$, $0 \le t \le 1$. In the similar way, we know that there exists a $S > 0$ such that $|\partial G(t,s)/\partial t| \le S$ for $0 \le s, t \le 1$.

Furthermore, for $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}&|Tu(t_2) - Tu(t_1)| \\ &= \left|\int_0^1 G(t_2,s)h(s)f(u(s))ds - \int_0^1 G(t_1,s)h(s)f(u(s))ds\right| \\ &= \left|\int_0^1 [G(t_2,s) - G(t_1,s)]h(s)f(u(s))ds\right| \\ &\le SK|h|_{L^1}|t_2 - t_1|.\end{aligned} \quad (43)$$

Therefore, applying the Arzela-Ascoli theorem [39], we can find that $T(\Omega_\delta)$ is relatively compact.

Third, we claim that $T: \bar{\Omega}_\delta \longrightarrow \mathbb{R}$ is continuous. Assume that $\{u_n\}_{n=1}^\infty \subset \bar{\Omega}_\delta$, which converges to u_0 uniformly on [0,1]. Since $\{(Tu_n)(t)\}_{n=1}^\infty$ is uniformly bounded and equicontinuous on [0,1], from the Arzela-Ascoli theorem, it follows that there exists a uniformly convergent subsequence in $\{(Tu_n)(t)\}_{n=1}^\infty$. Let $\{(Tu_{n(m)})(t)\}_{m=1}^\infty$ be a subsequence which converges to $v(t)$ uniformly on [0,1]. Observe that

$$Tu_{n(m)}(t) = \int_0^1 G(t,s)h(s)f\left(u_{n(m)}(s)\right)ds. \quad (44)$$

Furthermore, by Lebesgue's dominated convergence theorem and letting $m \longrightarrow \infty$, we have

$$v(t) = \int_0^1 G(t,s)h(s)f(u_0(s)))ds, \quad (45)$$

namely, $v(t) = Tu_0(t)$. This shows that each subsequence of $\{(Tu_n)(t)\}_{n=1}^\infty$ uniformly converges to $v(t)$. Therefore, the sequence $\{(Tu_n)(t)\}_{n=1}^\infty$ uniformly converges to $Tu_0(t)$. This means that T is continuous at $u_0 \in \Omega_\delta$. So, T is continuous on $\bar{\Omega}_\delta$. Thus, T is completely continuous.

Finally, let $\Omega_\delta = \{u \in C[0,1]: \|u\| < \delta\}$ with $\delta = \bar{G}K|h|_{L^1} + 1$. If u is a solution of problem (1), then, for $\rho \in (0,1)$, $u \in \partial\Omega_\delta$, we have

$$\begin{aligned}\|u\| &= \rho\|Tu(t)\| = \rho\left\|\int_0^1 G(t,s)h(s)f(u(s))ds\right\| \\ &\le \rho\bar{G}\int_0^1 |h(s)f(u(s))|ds \le \bar{G}K|h|_{L^1},\end{aligned} \quad (46)$$

which yields a contradiction. Therefore, by Lemma 9, the operator T has a fixed point in Ω_δ.

Theorem 10. *Assume that (H1)-(H3) hold. In addition, f satisfies the following assumptions:*

(H4) There exists a nondecreasing function $\psi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$|f(u)| \le \psi(\|u\|), \forall u \in \mathbb{R}. \quad (47)$$

(H5) There exists a constant $R > 0$ such that $R/\bar{G}|h|_{L^1}\psi(R) > 1$. Then, problem (1) has at least one solution.

Proof. Now we show that (ii) of Lemma 9 does not hold. If u is a solution of problem (1), then for $\rho \in (0,1)$, we obtain

$$\begin{aligned}\|u\| &= \rho\|T(u(t))\| \le \rho\int_0^1 |G(t,s)h(s)f(u(s))|ds \\ &\le \rho\bar{G}\int_0^1 |h(s)f(u(s))|ds \le \bar{G}|h|_{L^1}\psi(\|u\|).\end{aligned} \quad (48)$$

Let $B_R = \{u \in C[0,1]: \|u\| < R\}$. From the above inequality and (H5), it yields a contradiction. Therefore, by Lemma 9, the operator T has a fixed point in B_R. □

3.2. Lyapunov's Inequality

Theorem 11. *Assume that (H1)-(H3) hold. In addition, $f(u)$ is a concave function on $\mathbb{R}$. Then, for any nontrivial solution of problem (1), we have*

$$\int_0^1 |h(t)|dt > \frac{\|u\|}{\bar{G}\max_{u\in[u_*,u^*]}|f(u)|}, \quad (49)$$

where

$$u_* = \min_{t\in[0,1]} u(t),\ u^* = \max_{t\in[0,1]} u(t). \tag{50}$$

Proof. If $u(t)$ is a nontrivial solution of problem (1), then by Lemma 4, we have

$$u(t) = \int_0^1 G(t,s)h(s)f(u(s))ds. \tag{51}$$

Furthermore, by Lemma 6, we have

$$|u(t)| \le \int_0^1 |G(t,s)h(s)f(u(s))|ds. \tag{52}$$

Since f is continuous and concave, then from Jensen's inequality, it follows that

$$\begin{aligned}\|u(t)\| &\le \max_{t\in[0,1]}\int_0^1 |G(t,s)||h(s)f(u(s))|ds\\ &\le \int_0^1 \left[\max_{t\in[0,1]}|G(t,s)|\right]|h(s)f(u(s))|ds\\ &\le \bar{G}|h(t)|_{L^1}\int_0^1 \frac{|h(s)|}{|A(t)|_{L^1}}|f(u(s))|ds\\ &\le \bar{G}\max_{u\in[u_*,u^*]}|f(u)||h|_{L^1},\end{aligned} \tag{53}$$

namely,

$$\int_0^1 |h(t)dt| > \frac{\|u\|}{\bar{G}\max_{u\in[u_*,u^*]}|f(u)|}. \tag{54}$$

□

3.3. Stability Analysis

Definition 12 [34]. Equation (1) is said to be Ulam-Hyers-Rassias stability with respect to $\Psi \in C[0,1]$ if there exists a nonzero positive real number μ such that for every $\varepsilon > 0$ and each solution $v \in C[0,1]$ of the inequality

$$\left|D_{0^+}^{q} v''(t) - h(t)f(v(t))\right| \le \varepsilon\Psi(t),\ t\in[0,1], \tag{55}$$

there exists a solution $u \in C[0,1]$ of problem (1) such that $|u(t) - v(t)| \le \mu\varepsilon\Psi(t)$, $t \in [0,1]$.

Theorem 13. *Assume that (H1)-(H3) hold. In addition, there exists a positive constant $L > 0$ such that*

$$|f(u) - f(v)| \le L|u - v|, \forall u, v \in C[0,1]. \tag{56}$$

Then, problem (1) is Ulam-Hyers-Rassias stability if $LG|h|_{L^1} < 1$.

Proof. Let $v \in C[0,1]$ be the solution of the inequality (55); then,

$$\left|D_{0^+}^{1} v''(t) - h(t)f(v(t))\right| \le \varepsilon\Psi(t),\ t\in[0,1]. \tag{57}$$

Thus, for $\varepsilon > 0$, we get

$$\left|v(t) - \int_0^1 G(t,s)h(s)f(v(s))ds\right| \le \varepsilon\Psi(t),\ t\in[0,1]. \tag{58}$$

By Theorem 7, problem (1) has a solution $u(t)$ satisfies

$$u(t) = \int_0^1 G(t,s)h(s)f(u(s))ds. \tag{59}$$

Then, for $t \in [0,1]$, we have

$$\begin{aligned}|v(t) - u(t)| &= \left|v(t) - \int_0^1 G(t,s)h(s)f(u(s))ds\right|\\ &\le \left|v(t) - \int_0^1 G(t,s)h(s)f(v(s))ds\right|\\ &\quad + \left|\int_0^1 G(t,s)h(s)(f(u(s)) - f(v(s)))ds\right|\\ &\le \varepsilon\Psi(t) + L\int_0^1 |G(t,s)h(s)(u(s) - v(s))|ds\\ &\le \varepsilon\Psi(t) + L\bar{G}\int_0^1 |h(s)||u(s) - v(s)|ds\\ &\le \varepsilon\Psi(t) + L\bar{G}|h|_{L^1}|u(t) - v(t)|,\end{aligned} \tag{60}$$

which yields

$$|v(t) - u(t)| \le \frac{\varepsilon\Psi(t)}{1 - L\bar{G}|h|_{L^1}} = \mu\varepsilon\Psi(t),\ t\varepsilon[0,1]. \tag{61}$$

Therefore, problem (1) is Ulam-Hyers-Rassias stability. □

4. Examples

Now we give some examples to illustrate our main results.

Example 1. We consider the following problem:

$$\begin{cases} D_{0^+}^{3/2} u''(t) = 6t\ \arctan u,\ t\in[0,1],\\ u'(0) = u''(0) = u(1) = 0,\ u''(1) - 2u'''\left(\frac{4}{5}\right) - \frac{1}{8}u'''\left(\frac{6}{7}\right) = 0,\end{cases} \tag{62}$$

where $h(t) = 12t$ and $f(u) = 1/2 \arctan u$. It is obvious that (H1)-(H3) hold. Via some computations, we have

$$\begin{aligned}
\xi_1 &= \frac{4}{5} > 1 - \frac{1}{2^{1/q-1}} = \frac{3}{4},\\
p &= 1 - \frac{\sqrt{5}}{2} - \frac{\sqrt{7}}{16\sqrt{6}},\\
\bar{G} &= \frac{8}{105\sqrt{\pi}}\left[1 - \frac{1}{\sqrt{5^5}} - \frac{\sqrt{5}-1}{\sqrt{5 - 5/2 - 35/16\sqrt{6}}}\right] \approx 0.170285.
\end{aligned} \tag{63}$$

Since

$$f'(u) = \left(\frac{1}{2}\arctan u\right)' = \frac{1}{2(1+u^2)} \le \frac{1}{2} = L, \tag{64}$$

the function f satisfies the condition

$$|f(u) - f(v)| \le L|u - v|, \forall u, v \in C[0, 1]. \tag{65}$$

Furthermore, we can verify that $GL|h|_{L^1} \approx 0.510855 < 1$. Therefore, by Theorem 7 and Theorem 13, problem (62) has a solution $u(t)$, which is Ulam-Hyers-Rassias stability.

Example 2. Let us consider the following problem:

$$\begin{cases} D_{0^+}^{3/2} u''(t) = 6t\left(2\arctan u^{1/2} + \sin u\right), t \in [0, 1],\\ u'(0) = u''(0) = u(1) = 0, u''(1) - 2u'''\left(\frac{4}{5}\right) - \frac{1}{8}u'''\left(\frac{6}{7}\right) = 0, \end{cases} \tag{66}$$

where $h(t) = 6t$ and

$$|f(u)| = \left|2\arctan u^{1/2} + \sin u\right| \le 2\|u\|^{1/2} + \|u\| = \psi(\|u\|). \tag{67}$$

It is obvious that (H1)-(H4) hold. By computations of Example 2, we have

$$\bar{G} \approx 0.170285. \tag{68}$$

Furthermore, for $R > 4.362954$, the inequality $R/\bar{G}|h|_{L^1}\psi(R) > 1$ holds, Therefore, by Theorem 10, problem (66) has at least one solution.

Data Availability

All data generated or analyzed during this study are included in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Numerical Analysis of the Klein-Gordon Equations by Using the New Iteration Transform Method

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This paper presents an analysis based on a mixture of the Laplace transform and the new iteration method to obtain new approximate results of the fractional-order Klein-Gordon equations in the Caputo-Fabrizio sense. So, a general system to investigate the approximate results of the fractional-order Klein-Gordon equations is obtained. This technique's effectiveness is demonstrated by comparing the actual results of the fractional-order equations suggested with the results achieved.

1. Introduction

Fractional partial differential equations (FPDEs) are critical tools for analyzing and simulating numerous narrative models in physics and mathematical models, such as electrical circuits, fluid dynamics, damping, induction, mathematical biology, ad relaxation, (Klimek, 2005; Baleanu et al., 2009; Kilbas et al., 2010; Jumarie, 2009; Mainardi, 2010; Ortigueira, 2010). Fractional derivatives provide more precise representations of real-world problems than integer-order derivatives; they are regarded as an effective technique for describing such physical problems. The subject of fractional calculus is an important and valuable branch of mathematics that plays a critical and severe role in explaining complex dynamic behavior in a wide range of application areas, helps to understand the essence of the matter as well as simplify the control design without any lack of inherited behavior, and describes even more complex structures [1, 2].

The Klein-Gordon equations (KGEs) play an important role in physics, nonlinear optics, quantum field theory and solid state physics, plasma physics, kinematics, mathematical biology, and the recurrence of the initial state. The modeling of many phenomena, including the behavior of elementary particles and dislocation of crystals propagation, is the important applications of KGEs. To study solitons [3], examining nonlinear wave equations [4] and condensed matter physics equations gained the attention of scholars. In the previous few years, mathematicians have made many considerable efforts to find the solutions to these equations. There are many methods introduced to find the solution of these equations such as the radial basis functions [5], B-spline collocation method [5], auxiliary approach [6], and exponential-type potential, and there are some more methods mentioned in [7–11] for the solution of these equations. To solve the KGEs of a nonlinear type got tremendous attention of scholar, and a verity of methods were developed as mentioned in [12–14]. Some other methods are the stationary solution [15], the Homotopy perturbation technique [16], the tanh technique [17], the variation iteration technique [18], the traveling wave solutions, and so on.

In the recent paper, we are applying new iterative transform method to KGEs of both linear and nonlinear orders of the following form:

$$\frac{\partial^{\varrho}\mu}{\partial\tau^{\varrho}} - \frac{\partial^2\mu}{\partial\zeta^2} + \Psi = g(\zeta), 1 < \rho \leq 2, \tag{1}$$

with the boundary conditions

$$\mu(\zeta, 0) = 0\, \mu_\tau(\zeta, 0) = k(\zeta). \tag{2}$$

Daftardar-Gejji and Jafari developed a new iterative approach for solving nonlinear equations in 2006 [19, 20]. Jafari et al. first applied the Laplace transformation in the iterative technique. They proposed a new straightforward technique called the iterative Laplace transform method (ILTM) [21] to look for the numerical solution of the FPDE system. The iterative Laplace transform method was used to solve linear and nonlinear partial differential equations such as the time-fractional Fokker-Planck equation [22], Zakharov-Kuznetsov equation [23], and Fornberg-Whitham equation [24]. The Elzaki transform was used to modify the iterative technique, known as the new iterative transform method.

The new iterative transform method is implemented to investigate the fractional-order of the Klein-Gordon equations. The solution of the fractional-order problems and integral-order models is calculated applying the current techniques. The proposed approach is also helpful for dealing with other fractional-orders of linear and nonlinear PDEs.

2. Fractional Calculus

This section provides some fundamental concepts of fractional calculus.

Definition 1. The Liouville-Caputo operator (C) is given as [25]

$$D_{\mathfrak{F}}^{\varrho} u(\zeta, \mathfrak{F}) = \frac{1}{\Gamma(n-\beta)} \int_0^{\mathfrak{F}} (\mathfrak{F} - \theta)^{n-1} u^n(\zeta, \theta) d\theta,\ n-1 < \rho < n, \tag{3}$$

where $u^n(\zeta, \theta)$ is the derivative of integer nth order of $u(\zeta, \mathfrak{F})$, $n = 1, 2, \cdots \in N$ and $n - 1 < \varrho \le n$. If $0 < \varrho \le 1$; then, we defined the Laplace transformation for the Caputo fractional derivative as follows:

$$\mathscr{L}\left[D_{\mathfrak{F}}^{\varrho} u(\zeta, \mathfrak{F})\right](s) = s^{\varrho} \mathscr{L}[u(\zeta, \mathfrak{F})](s) - s^{\varrho-1}[u(\zeta, 0)]. \tag{4}$$

Definition 2. The Caputo-Fabrizio operator (CF) is define as given [25]:

$$D_{\mathfrak{F}}^{\varrho} u(\zeta, \mathfrak{F}) = \frac{(2-\varrho) M(\varrho)}{2(n-\varrho)} \int_0^{\mathfrak{F}} \exp\left(-\varrho \frac{(\mathfrak{F} - \theta)}{n - \varrho}\right) u^{(n)}(\zeta, \theta) d\theta,\ n < \varrho \le n+1. \tag{5}$$

$M(\varrho)$ is a normalization form, and $M(0) = M(1) = 1$. The exponential law is used as the nonsingular kernel in this fractional operator.

If $0 < \rho \le 1$, then we define the Caputo-Fabrizio of the Laplace transformation for the fractional derivative is given as

$$\mathscr{L}\left[D_{\mathfrak{F}}^{\varrho} u(\zeta, \mathfrak{F})\right](s) = \left(\frac{s\mathscr{L}[u(\zeta, \mathfrak{F})](s) - u(\zeta, 0)}{s + \varrho(1-s)}\right). \tag{6}$$

3. The Iterative Transform Method Basic Procedure

Consider a particular type of a FPDE.

$$D_\tau^{\varrho} \upsilon(\zeta, \tau) + M\upsilon(\zeta, \tau) + N\upsilon(\zeta, \tau) = h(\zeta, \tau),\ n \in N,\ n-1 < \varrho \le n, \tag{7}$$

where the functions of linear and nonlinear are M and N, respectively.

With the initial condition

$$\upsilon^k(\zeta, 0) = g_k(\zeta),\ k = 0, 1, 2 \cdots n-1, \tag{8}$$

implementing the Laplace transformation of Equation (7), we have

$$L[D_\tau^{\varrho} \upsilon(\zeta, \tau)] + L[M\upsilon(\zeta, \tau) + N\upsilon(\zeta, \tau)] = L[h(\zeta, \tau)]. \tag{9}$$

Applying the Laplace differentiation is given to

$$\begin{aligned} L[\upsilon(\zeta, \tau)] = {} & \frac{1}{s} \upsilon(\zeta, 0) + \frac{s + \varrho(1-s)}{s^2} L[h(\zeta, \tau)] \\ & - \frac{s + \varrho(1-s)}{s^2} L[M\upsilon(\zeta, \tau) + N\upsilon(\zeta, \tau)], \end{aligned} \tag{10}$$

using the inverse Laplace transformation of Equation (10) into

$$\begin{aligned} \upsilon(\zeta, \tau) = {} & L^{-1}\left[\left(\frac{1}{s} \upsilon(\zeta, 0) + \frac{s + \varrho(1-s)}{s^2} L[h(\zeta, \tau)]\right)\right] \\ & - L^{-1}\left[\frac{s + \varrho(1-s)}{s^2} L[M\upsilon(\zeta, \tau) + N\upsilon(\zeta, \tau)]\right]. \end{aligned} \tag{11}$$

As through the iterative technique, we have

$$\upsilon(\zeta, \tau) = \sum_{m=0}^{\infty} \upsilon_m(\zeta, \tau). \tag{12}$$

Further, the operator M is linear; therefore

$$M\left(\sum_{m=0}^{\infty} \upsilon_m(\zeta, \tau)\right) = \sum_{m=0}^{\infty} M[\upsilon_m(\zeta, \tau)], \tag{13}$$

and the operator N is nonlinear; we have the following

$$\begin{aligned} N\left(\sum_{m=0}^{\infty} \upsilon_m(\zeta, \tau)\right) = {} & \upsilon_0(\zeta, \tau) + M\left(\sum_{k=0}^{m} \upsilon_k(\zeta, \tau)\right) \\ & - N\left(\sum_{k=0}^{m} \upsilon_k(\zeta, \tau)\right). \end{aligned} \tag{14}$$

Putting Equations (12)–(14) in Equation (11), we obtain

$$\sum_{m=0}^{\infty} \upsilon_m(\zeta,\tau) = L^{-1}\left[\left(\frac{1}{s}\upsilon(\zeta,0) + \frac{s+\varrho(1-s)}{s^2}L[h(\zeta,\tau)]\right)\right] - L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}E\left[M\left(\sum_{k=0}^{m}\upsilon_k(\zeta,\tau)\right) - N\left(\sum_{k=0}^{m}\upsilon_k(\zeta,\tau)\right)\right]\right]. \tag{15}$$

The new iterative transform method is defined as

$$\begin{aligned}
\upsilon_0(\zeta,\tau) &= L^{-1}\left[\left(\frac{1}{s}\upsilon(\zeta,0) + \frac{s+\varrho(1-s)}{s^2}L(g(\zeta,\tau))\right)\right],\\
\upsilon_1(\zeta,\tau) &= -L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L[M[\upsilon_0(\zeta,\tau)] + N[\upsilon_0(\zeta,\tau)]]\right],\\
\upsilon_{m+1}(\zeta,\tau) &= -L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left[-M\left(\sum_{k=0}^{m}\upsilon_k(\zeta,\tau)\right) - N\left(\sum_{k=0}^{m}\upsilon_k(\zeta,\tau)\right)\right]\right], m \geq 1.
\end{aligned} \tag{16}$$

Finally, Equations (7) and (8) provide the m-terms solution in a series form given as

$$\upsilon(\zeta,\tau) \cong \upsilon_0(\zeta,\tau) + \upsilon_1(\zeta,\tau) + \upsilon_2(\zeta,\tau) + \cdots + \upsilon_m(\zeta,\tau), m = 1, 2, \cdots. \tag{17}$$

4. Applications of the Proposed Method

4.1. Example. Consider the fractional-order Klein-Gordon equation [18]

$$\frac{{}^{CF}\partial^{\varrho+1}\mu(\zeta,\tau)}{\partial\tau^{\varrho+1}} - \frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} + \mu(\zeta,\tau) = 0 \quad 0<\zeta \quad \tau<0 \quad 0<\varrho\leq 1, \tag{18}$$

with the initial conditions

$$\mu(\zeta,0) = 0, \mu_\tau(\zeta,0) = \zeta. \tag{19}$$

Applying the Laplace transform to Equation (18), we have

$$s^{\varrho}L[\mu(\zeta,\tau)] = \mu_{(0)}(\zeta,0)s^{-1} + \mu_{(\tau)}(\zeta,0)s^{-2} + L\left[\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} - \mu(\zeta,\tau)\right], \tag{20}$$

$$L[\mu(\zeta,\tau)] = s^{-1}\mu(\zeta,0) + s^{-2}\mu_{(\tau)}(\zeta,0) + \frac{s+\varrho(1-s)}{s^2}L\left[\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} - \mu(\zeta,\tau)\right]. \tag{21}$$

Applying the inverse Laplace transform of Equation (21), we have

$$\mu(\zeta,\tau) = L^{-1}\left[s^{-1}\mu(\zeta,0) + s^{-2}\mu_{(\tau)}(\zeta,0)\right] + L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} - \mu(\zeta,\tau)\right)\right]. \tag{22}$$

Now, by using the suggested analytical method, we get

$$\begin{aligned}
\mu_0(\zeta,\tau) &= \zeta\tau,\\
\mu_1(\zeta,\tau) &= L^{-1}\left[\frac{s+\rho(1-s)}{s^2}L\left(\frac{\partial^2\mu_0(\zeta,\tau)}{\partial\zeta^2} - \mu_0(\zeta,\tau)\right)\right]\\
&= -\frac{\zeta\tau^2}{6}(\tau\rho + 3 - 3\rho),\\
\mu_2(\zeta,\tau) &= L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_1(\zeta,\tau)}{\partial\zeta^2} - \mu_1(\zeta,\tau)\right)\right]\\
&= \frac{\zeta\tau^3}{120}\left(10\tau\varrho - 10\tau\varrho^2 - 40\varrho + 20 + 20\varrho^2 + \tau^2\varrho^2\right),\\
\mu_3(\zeta,\tau) &= L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_2(\zeta,\tau)}{\partial\zeta^2} - \mu_2(\zeta,\tau)\right)\right],\\
\mu_3(\zeta,\tau) &= -\frac{\zeta\tau^4}{5040}\left(630\varrho^2 - 630\varrho + 210 - 210\varrho^3 + \tau^3\varrho^3\right.\\
&\quad \left. + 21\tau^2\varrho^2 - 21\tau^2\varrho^3 - 252\tau\varrho^2 + 126\tau\varrho + 126\tau\varrho^3\right),\\
\mu_4(\zeta,\tau) &= L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_3(\zeta,\tau)}{\partial\zeta^2} - \mu_3(\zeta,\tau)\right)\right],\\
\mu_4(\zeta,\tau) &= \frac{\zeta\tau^5}{362880}(\tau\varrho + 6 - 6\varrho) * \left(\tau^3\varrho^3 - 30\tau^2\varrho^3 + 30\tau^2\varrho^2\right.\\
&\quad + 252\tau\varrho^3 - 504\tau\varrho^2 + 252\tau\varrho, -504\varrho^3\\
&\quad \left. + 1512\varrho^2 - 1512\varrho + 504\right),\\
&\vdots\\
\mu_n + 1(\zeta,\tau) &= L^{-1}\left[\frac{(s+\varrho(1-s))^n}{s^{2n+2}}L\left(\frac{\partial^2\mu_n(\zeta,\tau)}{\partial\zeta^2} - \mu_n(\zeta,\tau)\right)\right].
\end{aligned} \tag{23}$$

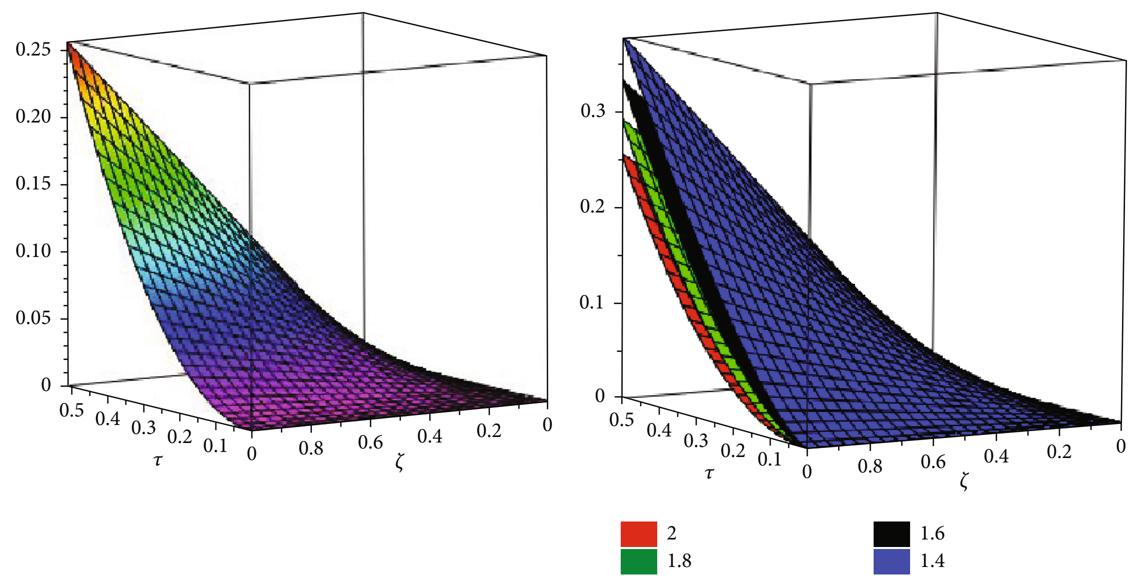
The series form result is

$$\begin{aligned}
\mu(\zeta,\tau) &= \mu_0(\zeta,\tau) + \mu_1(\zeta,\tau) + \mu_2(\zeta,\tau) + \mu_3(\zeta,\tau) + \cdots \mu_n(\zeta,\tau),\\
\mu(\zeta,\tau) &= \zeta\tau - \frac{\zeta\tau^2}{6}(\tau\varrho + 3 - 3\varrho) + \frac{\zeta\tau^3}{120}\left(10\tau\varrho - 10\tau\varrho^2 - 40\varrho\right.\\
&\quad \left. + 20 + 20\varrho^2 + \tau^2\varrho^2\right) - \frac{\zeta\tau^4}{5040}\left(630\varrho^2 - 630\varrho + 210\right.\\
&\quad - 210\varrho^3 + \tau^3\varrho^3 + 21\tau^2\varrho^2 - 21\tau^2\varrho^3 - 252\tau\varrho^2\\
&\quad \left. + 126\tau\varrho + 126\tau\varrho^3\right) + \frac{\zeta\tau^5}{362880}(\tau\varrho + 6 - 6\varrho)\\
&\quad \cdot \left(\tau^3\varrho^3 - 30\tau^2\varrho^3 + 30\tau^2\varrho^2 + 252\tau\varrho^3 - 504\tau\varrho^2\right.\\
&\quad \left. + 252\tau\varrho - 504\varrho^3 + 1512\varrho^2 - 1512\varrho + 504\right) - \cdots.
\end{aligned} \tag{24}$$

The problem has the exact solution at $\varrho = 1$:

$$\mu(\zeta,\tau) = \zeta \sin(\tau). \tag{25}$$

In Figure 1, the exact and the approximate solutions of



(a) (b)

Figure 1: (a) Exact and an approximate graph of problem 1 and (b) different fractional-order graphs of problem 1.

example 1 at $\varrho = 1$ are shown, and the second graph shows the 3D graph of different fractional-order ρ, respectively. From the given graphs, it can be shown that both the approximate and exact solutions are in close relation with each other. Also, in Figure 2, the 2D figure of the approximate solutions of problem 1 is analysis at different fractional-order ρ for ζ and τ. It is demonstrated that the outcomes of time-fractional problems converge to an integer-order effect as the time-fractional evaluation to integer-order.

4.2. Example. Consider the fractional-order Klein-Gordon equation [18]:

$$\frac{{}^{CF}\partial^{\varrho+1}\mu(\zeta,\tau)}{\partial\tau^{\varrho+1}} - \frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} + \mu(\zeta,\tau) = 2\sin(\zeta) \quad 0<\zeta \quad \tau<0 \quad 0<\varrho\le 1, \tag{26}$$

with the initial conditions

$$\mu(\zeta,0) = \sin(\zeta), \mu_\tau(\zeta,0) = 1. \tag{27}$$

We apply the Laplace transformation to Equation (26), and we get

$$\begin{aligned}\frac{s^2}{s+\varrho(1-s)}L[\mu(\zeta,\tau)] &= \mu_{(0)}(\zeta,0)s^{-1} + \mu_{(\tau)}(\zeta,0)s^{-2} \\ &+ L\left[\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} - \mu(\zeta,\tau) + 2\sin(\zeta)\right],\end{aligned} \tag{28}$$

$$\begin{aligned}L[\mu(\zeta,\tau)] &= s^{-1}\mu(\zeta,0) + s^{-2}\mu_{(\tau)}(\zeta,0) + \frac{s+\varrho(1-s)}{s^2}L \\ &\cdot \left[\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} - \mu(\zeta,\tau) + 2\sin(\zeta)\right].\end{aligned} \tag{29}$$

Now, using the inverse Laplace transformation of Equation (29), we have

$$\begin{aligned}\mu(\zeta,\tau) &= L^{-1}\left[s^{-1}\mu(\zeta,0) + s^{-2}\mu_{(\tau)}(\zeta,0)\right] \\ &+ L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} - \mu(\zeta,\tau) + 2\sin(\zeta)\right)\right].\end{aligned} \tag{30}$$

Now, by using the suggested analytical method, we get

$$\mu_0(\zeta,\tau) = \sin(\zeta) + \tau,$$

$$\begin{aligned}\mu_1(\zeta,\tau) &= L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_0(\zeta,\tau)}{\partial\zeta^2} - \mu_0(\zeta,\tau) + 2\sin(\zeta)\right)\right] \\ &= -\frac{\tau^2}{6}(\tau\varrho + 3 - 3\varrho),\end{aligned}$$

$$\mu_2(\zeta,\tau) = L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_1(\zeta,\tau)}{\partial\zeta^2} - \mu_1(\zeta,\tau) + 2\sin(\zeta)\right)\right],$$

$$\begin{aligned}\mu_2(\zeta,\tau) = \frac{\tau^2}{120}\big(&20\tau - 60\tau\varrho + 20\tau\varrho^2 + \tau^3\varrho^2 - 60 \\ &+ 60\varrho + 10\tau^2\varrho - 10\tau^2\varrho^2\big),\end{aligned}$$

$$\mu_3(\zeta,\tau) = L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_2(\zeta,\tau)}{\partial\zeta^2} - \mu_2(\zeta,\tau) + 2\sin(\zeta)\right)\right],$$

$$\begin{aligned}\mu_3(\zeta,\tau) = -\frac{\tau^2}{5040}\big(&4200\tau\varrho - 1680\tau - 1680\tau\varrho^2 - 336\tau^3\varrho^2 \\ &+ 126\varrho\tau^3 + 126\tau^3\varrho^3 + \varrho^3\tau^5 + 2520 - 2520\varrho \\ &+ 1470\tau^2\varrho^2 - 1470\tau^2\varrho + 210\tau^2 - 210\tau^2\varrho^3 \\ &+ 21\varrho^2\tau^4 - 21\tau^4\varrho^3\big),\end{aligned}$$

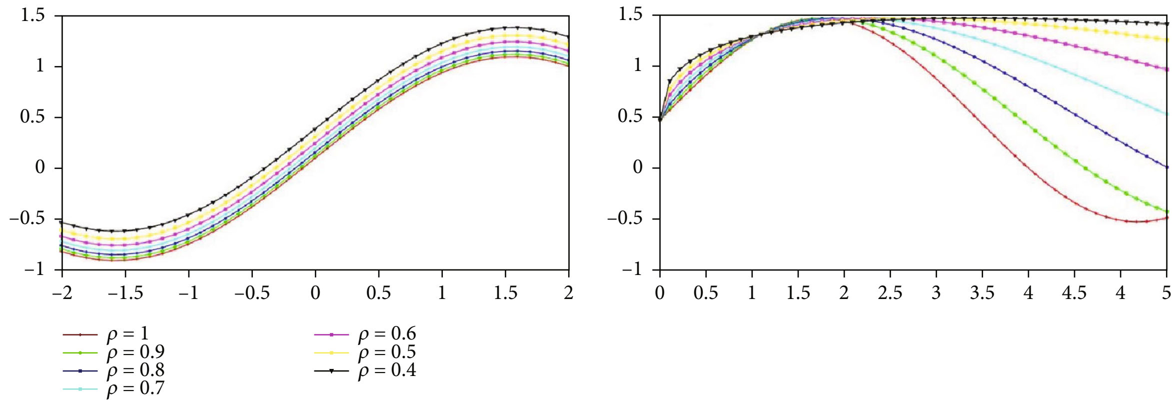


FIGURE 2: The different fractional-orders with respect to ζ and τ of problem 1.

$$\mu_4(\zeta,\tau) = L^{-1}\left[\frac{s+\varrho(1-s)}{s^2} L\left(\frac{\partial^2\mu_3(\zeta,\tau)}{\partial\zeta^2} - \mu_3(\zeta,\tau) + 2\sin(\zeta)\right)\right],$$

$$\begin{aligned}\mu_4(\zeta,\tau) = \frac{\tau^2}{362880}&\left(181440\tau\varrho^2 + 181440\tau - 423360\tau\varrho\right.\\ &+ 81648\tau^3\varrho^2 - 39312\tau^3\varrho^3 - 39312\varrho\tau^3 + 3024\tau^3\\ &+ 3024\tau^3\varrho^4 + 36\varrho^3\tau^6 - 36\varrho^4\tau^6 - 1080\varrho^3\tau^5\\ &+ 432\varrho^2\tau^5 + 432\tau^5\varrho^4 + \tau^7\varrho^4 - 181440 + 11440\varrho\\ &+ 226800\tau^2\varrho + 45360\tau^2\varrho^3 - 226800\tau^2\varrho^2\\ &- 45360\tau^2 - 10584\varrho^2\tau^4 + 10584\tau^4\varrho^3\\ &\left.- 2016\tau^4\varrho^4 + 2016\varrho\tau^4\right),\end{aligned}$$

$$\vdots$$

$$\mu_n(\zeta,\tau) = L^{-1}\left[\frac{(s+\varrho(1-s))^n}{s^{2n+2}} L\left(\frac{\partial^2\mu_n(\zeta,\tau)}{\partial\zeta^2} - \mu_n(\zeta,\tau) + 2\sin(\zeta)\right)\right]. \tag{31}$$

The series form result is

$$\begin{aligned}&\mu(\zeta,\tau) = \mu_0(\zeta,\tau) + \mu_1(\zeta,\tau) + \mu_2(\zeta,\tau) + \mu_3(\zeta,\tau) + \cdots \mu_n(\zeta,\tau),\\ &\mu(\zeta,\tau) == \sin(\zeta) + \tau - \frac{\tau^2}{6}(\tau\varrho + 3 - 3\varrho)\\ &\quad + \frac{\tau^2}{120}\left(20\tau - 60\tau\varrho + 20\tau\varrho^2 + \tau^3\varrho^2 - 60\right.\\ &\quad \left.+ 60\varrho + 10\tau^2\varrho - 10\tau^2\varrho^2\right) + - \frac{\tau^2}{5040}\\ &\quad \cdot \left(4200\tau\varrho - 1680\tau - 1680\tau\varrho^2 - 336\tau^3\varrho^2\right.\\ &\quad + 126\varrho\tau^3 + 126\tau^3\varrho^3 + \varrho^3\tau^5 + 2520 - 2520\varrho\\ &\quad + 1470\tau^2\varrho^2 - 1470\tau^2\varrho + 210\tau^2 - 210\tau^2\varrho^3\\ &\quad \left.+ 21\varrho^2\tau^4 - 21\tau^4\varrho^3\right) + \frac{\tau^2}{362880}\\ &\quad \cdot \left(181440\tau\varrho^2 + 181440\tau - 423360\tau\varrho + 81648\tau^3\varrho^2\right.\\ &\quad - 39312\tau^3\varrho^3 - 39312\varrho\tau^3 + 3024\tau^3 + 3024\tau^3\varrho^4\\ &\quad + 36\varrho^3\tau^6 - 36\varrho^4\tau^6 - 1080\varrho^3\tau^5 + 432\varrho^2\tau^5 + 432\tau^5\varrho^4\\ &\quad + \tau^7\varrho^4 - 181440 + 11440\rho + 226800\tau^2\varrho + 45360\tau^2\varrho^3\\ &\quad - 226800\tau^2\varrho^2 - 45360\tau^2 - 10584\varrho^2\tau^4 + 10584\tau^4\varrho^3\\ &\quad \left.- 2016\tau^4\varrho^4 + 2016\varrho\tau^4\right) + \cdots.\end{aligned} \tag{32}$$

The problem has the exact solution at $\varrho = 1$:

$$\mu(\zeta,\tau) = \sin(\zeta) + \sin(\tau). \tag{33}$$

In Figure 3, the exact and the approximate solutions of example 2 at $\varrho = 1$ are shown. From the given figures, it can be seen that both the approximate and exact solutions are in close contact with each other. Also, in Figure 4, the 2D graph of the approximate results of problem 2 is investigated at different fractional-order ρ for ζ and τ. It is demonstrated that the outcomes of time-fractional problems converge to an integer-order effect as the time-fractional evaluation to integer-order.

4.3. Example. Consider the fractional-order nonlinear Klein-Gordon equation [18]:

$$\frac{{}^{CF}\partial^{\varrho+1}\mu(\zeta,\tau)}{\partial\tau^{\varrho+1}} - \frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} + \mu^2(\zeta,\tau) = \zeta^2\tau^2 \quad 0<\zeta \quad \tau<0 \quad 0<\varrho\le 1, \tag{34}$$

with the initial conditions

$$\mu(\zeta,0) = 0, \mu_\tau(\zeta,0) = \zeta. \tag{35}$$

Using the Laplace transform to Equation (34), we get

$$\begin{aligned}\frac{s^2}{s+\varrho(1-s)} L[\mu(\zeta,\tau)] &= \mu_{(0)}(\zeta,0)s^{-1} + \mu_{(\tau)}(\zeta,0)s^{-2}\\ &+ L\left[\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} - \mu^2(\zeta,\tau) + \zeta^2\tau^2\right],\end{aligned} \tag{36}$$

$$\begin{aligned}L[\mu(\zeta,\tau)] = s^{-1}\mu(\zeta,0) + s^{-2}\mu_{(\tau)}(\zeta,0) + \frac{s+\varrho(1-s)}{s^2} L\\ \cdot \left[\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} - \mu^2(\zeta,\tau) + \zeta^2\tau^2\right].\end{aligned} \tag{37}$$

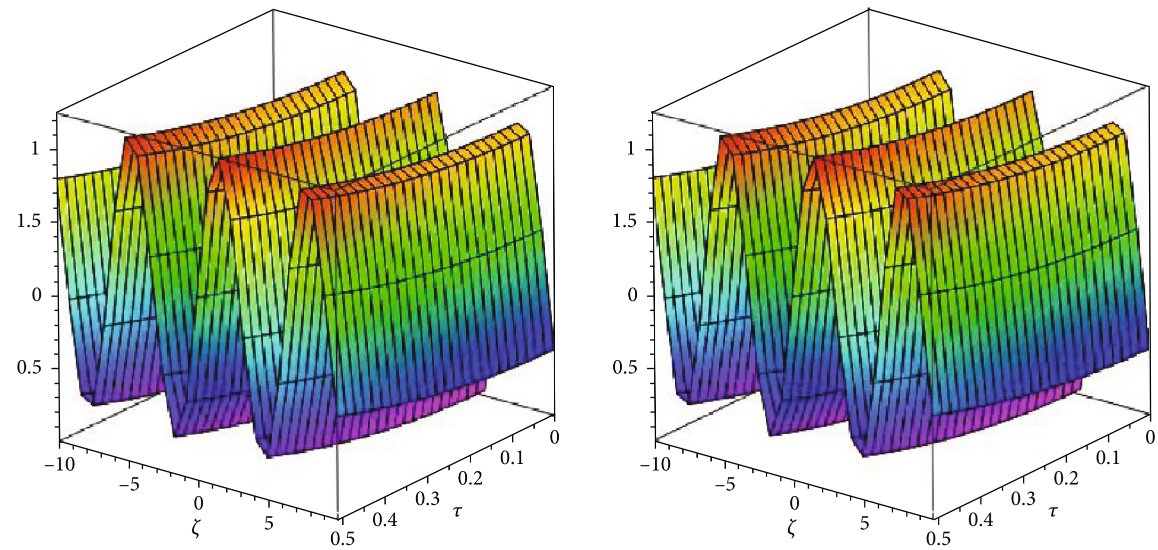


FIGURE 3: The exact and an approximate graph of problem 2.

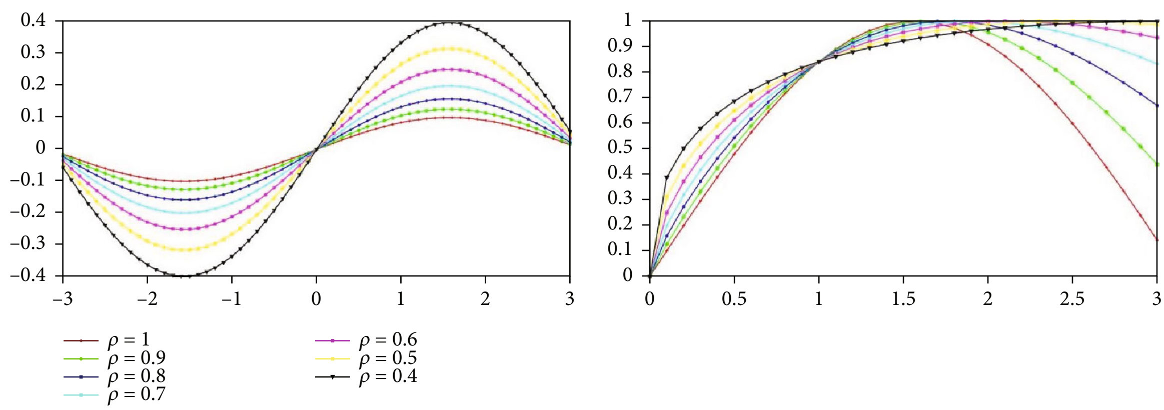


FIGURE 4: The different fractional-orders with respect to ζ and τ of problem 2.

Applying the inverse Laplace transform of Equation (37), we have

$$\mu(\zeta,\tau)=L^{-1}\left[s^{-1}\mu(\zeta,0)+s^{-2}\mu_{(\tau)}(\zeta,0)\right]+L^{-1}\cdot\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu^2(\zeta,\tau)}{\partial\zeta^2}-\mu^2(\zeta,\tau)+\zeta^2\tau^2\right)\right]. \tag{38}$$

Now, by using the suggested an approximate method, we get

$$\begin{aligned}\mu_0(\zeta,\tau)&=\zeta\tau,\\ \mu_1(\zeta,\tau)&=L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_0(\zeta,\tau)}{\partial\zeta^2}-\mu_0^2(\zeta,\tau)+\zeta^2\tau^2\right)\right]=0,\\ &\vdots\\ \mu_n(\zeta,\tau)&=L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_n(\zeta,\tau)}{\partial\zeta^2}-\mu_n^2(\zeta,\tau)+\zeta^2\tau^2\right)\right].\end{aligned} \tag{39}$$

The series form result is

$$\begin{aligned}\mu(\zeta,\tau)&=\mu_0(\zeta,\tau)+\mu_1(\zeta,\tau)+\mu_2(\zeta,\tau)+\mu_3(\zeta,\tau)+\cdots\mu_n(\zeta,\tau),\\ \mu(\zeta,\tau)&=\zeta\tau+0+\cdots.\end{aligned} \tag{40}$$

The problem has the exact solution at $\varrho=2$:

$$\mu(\zeta,\tau)=\zeta\tau. \tag{41}$$

Figure 5 compares the exact solution and approximate solution of example 3 for the nonlinear fractional-order Klein-Gordon equation at $\varrho=1$. The figure shows the close relationship between the exact and an approximate solution.

4.4. Example. Consider the fractional-order nonlinear Klein-Gordon equation [18]:

$$\frac{{}^{CF}\partial^{\varrho+1}\mu(\zeta,\tau)}{\partial\tau^{\varrho+1}}-\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2}+\mu^2(\zeta,\tau)=2\zeta^2-2\tau^2+\zeta^4\tau^4\quad 0<\zeta\quad \tau<0\quad 0<\varrho\le 1, \tag{42}$$

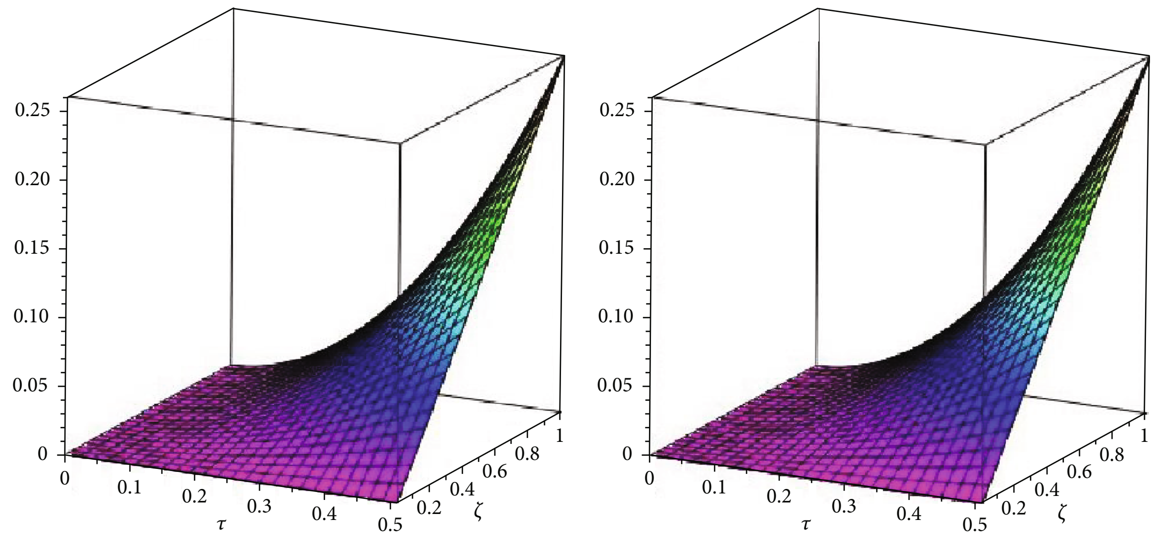


FIGURE 5: The exact and an approximate graph of problem 3.

with the initial conditions

$$\mu(\zeta,0)=\mu_\tau(\zeta,0)=0. \tag{43}$$

Using the Laplace transform to Equation (42), we get

$$\frac{s^2}{s+\varrho(1-s)}L[\mu(\zeta,\tau)]=\mu_{(0)}(\zeta,0)s^{-1}+\mu_{(\tau)}(\zeta,0)s^{-2}+L\left[\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2}-\mu^2(\zeta,\tau)+2\zeta^2-2\tau^2+\zeta^4\tau^4\right], \tag{44}$$

$$L[\mu(\zeta,\tau)]=s^{-1}\mu(\zeta,0)+s^{-2}\mu_{(\tau)}(\zeta,0)+\frac{s+\varrho(1-s)}{s^2}L\cdot\left[\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2}-\mu^2(\zeta,\tau)+2\zeta^2-2\tau^2+\zeta^4\tau^4\right]. \tag{45}$$

Applying the inverse Laplace transform of Equation (45), we have

$$\mu(\zeta,\tau)=L^{-1}\left[s^{-1}\mu(\zeta,0)+s^{-2}\mu_{(\tau)}(\zeta,0)\right]+L^{-1}\cdot\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu^2(\zeta,\tau)}{\partial\zeta^2}-\mu^2(\zeta,\tau)+2\zeta^2-2\tau^2+\zeta^4\tau^4\right)\right]. \tag{46}$$

Now, by using the suggested analytical method, we get

$$\mu_0(\zeta,\tau)=0,$$

$$\mu_1(\zeta,\tau)=L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_0(\zeta,\tau)}{\partial\zeta^2}-\mu_0^2(\zeta,\tau)+2\zeta^2-2\tau^2+\zeta^4\tau^4\right)\right],$$

$$\mu_1(\zeta,\tau)=\frac{\tau}{30}\left(-5\varrho\tau^3+20\tau^2\varrho\zeta^4\tau^5\varrho+60\zeta^2-60\varrho\zeta^2+30\zeta^2\tau\varrho+6\tau^4\zeta^4-6\tau^4\zeta^4\varrho\right),$$

$$\mu_2(\zeta,\tau)=L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_1(\zeta,\tau)}{\partial\zeta^2}-\mu_1^2(\zeta,\tau)+2\zeta^2-2\tau^2+\zeta^4\tau^4\right)\right],$$

$$\begin{aligned}\mu_2(\zeta,\tau)=-\frac{\tau}{162116200}\big(&-32432400\tau\varrho^2-2702700\tau^3\varrho^2\\&-32432400\tau^2\varrho+21621600\tau^2\varrho^2-228228\tau^9\varrho^3\zeta^4\\&+456456\tau^9\varrho^2\zeta^4-228228\tau^9\varrho\zeta^4+12012\tau^9\zeta^6\varrho^3\\&-32760\varrho^2\tau^{10}\zeta^4+32760\varrho^3\tau^10\zeta^4-1365\tau^{11}\varrho^3\zeta^4\\&+22932\tau^{11}\zeta^8\varrho^3+22932\tau^{11}\zeta^8\varrho+5005\tau^9\varrho^3\\&+100100\varrho^2\tau^8-100100\varrho^3\tau^8-1158300\tau^7\varrho^2\\&+579150\tau^7\varrho-480480\tau^8\zeta^4+58968\tau^{10}\zeta^8\\&+1853280\tau^6\zeta^6+3088800\tau^6\varrho^2-3088800\tau^6\varrho\\&+21621600\tau^2\zeta^4+579150\varrho^3\tau^7-8648640\tau^4\zeta^2\\&-6486480\zeta^2\tau^5+1029600\tau^6+240240\varrho^2\tau^8\zeta^6\\&-240240\varrho^3\tau^8\zeta^6+1312740\tau^7\zeta^6\varrho^3-2625480\tau^7\zeta^6\varrho^2\\&-96525\tau^7\varrho^3\zeta^2+1312740\tau^7\zeta^6\varrho-115830\tau^7\varrho^2\zeta^2\\&-6486480\varrho^3\tau^4\zeta^4+6486480\varrho^2\tau^4\zeta^4-43243200\tau^3\varrho^2\zeta^4\\&+21621600\tau^3\varrho^3\zeta^4+21621600\tau^3\varrho\zeta^4+99\zeta^8\varrho^3\tau^{13}\\&-1853280\tau^6\varrho\zeta^2+308880\tau^6\varrho^2\zeta^2+1544400\tau^6\varrho^3\zeta^2\\&+540540\tau^5\varrho^3\zeta^4-6846840\tau^5\varrho^3\zeta^2+6126120\tau^5\varrho\zeta^2\\&+7207200\tau^5\varrho^2\zeta^2+2772\varrho^2\tau^{12}\zeta^8-2772\varrho^3\tau^{12}\zeta^8\\&+480480\tau^8\varrho^3\zeta^4-1441440\tau^8\varrho^2\zeta^4+1441440\tau^8\varrho\zeta^4\\&-58968\tau^{10}\zeta^8\varrho^3-176904\tau^{10}\zeta^8\varrho-1853280\tau^6\zeta^6\varrho^3\\&+5559840\tau^6\zeta^6\varrho^2-5559840\tau^6\zeta^6\varrho+64864800\tau^2\varrho^2\zeta^4\\&-21621600\tau^2\varrho^3\zeta^4-64864800\tau^2\varrho\zeta^4+8648640\tau^4\varrho^3\zeta^2\\&-25945920\tau^4\varrho^2\zeta^2+176904\tau^{10}\zeta^8\varrho^2+25945920\zeta^2\varrho\tau^4\\&-45864\tau^{11}\zeta^8\varrho^2+2702700\varrho\tau^3-32432400\zeta^2\\&-540540\zeta^4\tau^5\varrho-16216200\zeta^2\tau\varrho+3243240\tau^4\zeta^4\rho\\&-1029600\rho^3\tau^6-3243240\tau^4\zeta^4-32432400\tau\\&+32432400\rho\zeta^2+64864800\tau\varrho+10810800\tau^2\big),\ \vdots\end{aligned}$$

$$\mu_n(\zeta,\tau)=L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_n(\zeta,\tau)}{\partial\zeta^2}-\mu_n^2(\zeta,\tau)+2\zeta^2-2\tau^2+\zeta^4\tau^4\right)\right]. \tag{47}$$

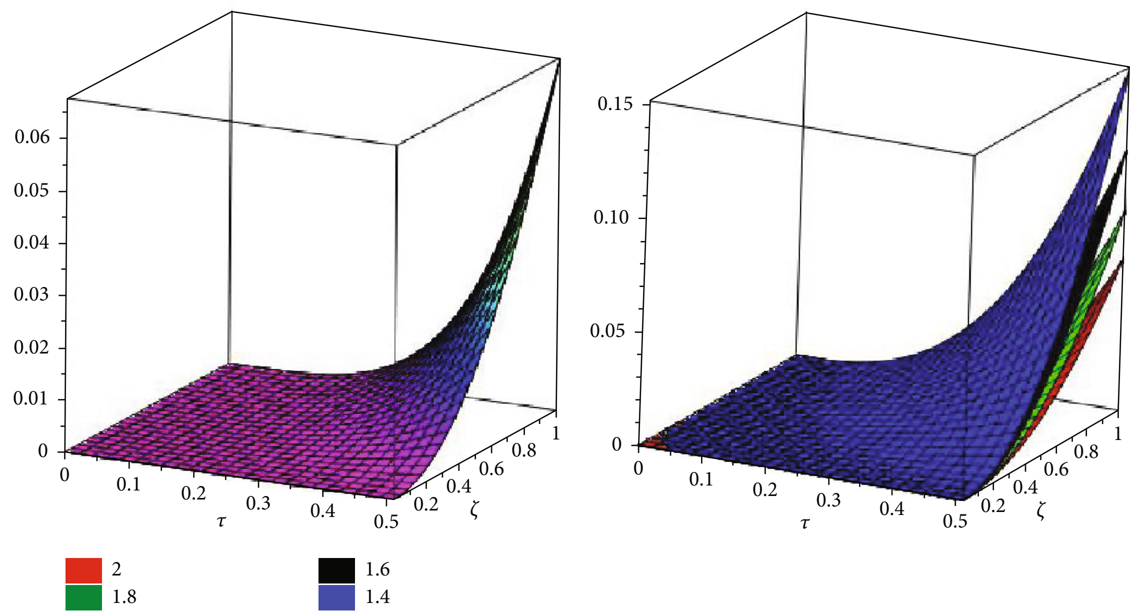


FIGURE 6: The exact and an approximate graph and different fractional-order graphs of problem 4.

The series form result is

$$
\begin{aligned}
\mu(\zeta,\tau) &= \mu_0(\zeta,\tau) + \mu_1(\zeta,\tau) + \mu_2(\zeta,\tau) + \mu_3(\zeta,\tau) + \cdots \mu_n(\zeta,\tau),\\
\mu(\zeta,\tau) &= \frac{\tau}{30}\Big(-5\varrho\tau^3 + 20\tau^2\varrho\zeta^4\tau^5\varrho + 60\zeta^2 - 60\varrho\zeta^2\\
&\quad + 30\zeta^2\tau\varrho + 6\tau^4\zeta^4 - 6\tau^4\zeta^4\varrho\Big) - \frac{\tau}{162116200}\\
&\quad \cdot \big(-32432400\tau\varrho^2 - 2702700\tau^3\varrho^2 - 32432400\tau^2\varrho\\
&\quad + 21621600\tau^2\varrho^2 - 228228\tau^9\varrho^3\zeta^4 + 456456\tau^9\varrho^2\zeta^4\\
&\quad - 228228\tau^9\varrho\zeta^4 + 12012\tau^9\zeta^6\varrho^3 - 32760\varrho^2\tau^{10}\zeta^4\\
&\quad + 32760\varrho^3\tau^1 0\zeta^4 - 1365\tau^{11}\varrho^3\zeta^4 + 22932\tau^{11}\zeta^8\varrho^3\\
&\quad + 22932\tau^{11}\zeta^8\varrho + 5005\tau^9\varrho^3 + 100100\varrho^2\tau^8\\
&\quad - 100100\varrho^3\tau^8 - 1158300\tau^7\varrho^2 + 579150\tau^7\varrho\\
&\quad - 480480\tau^8\zeta^4 + 58968\tau^{10}\zeta^8 + 1853280\tau^6\zeta^6\\
&\quad + 3088800\tau^6\varrho^2 - 3088800\tau^6\varrho + 21621600\tau^2\zeta^4\\
&\quad + 579150\varrho^3\tau^7 - 8648640\tau^4\zeta^2 - 6486480\zeta^2\tau^5\\
&\quad + 1029600\tau^6 + 240240\varrho^2\tau^8\zeta^6 - 240240\varrho^3\tau^8\zeta^6\\
&\quad + 1312740\tau^7\zeta^6\varrho^3 - 2625480\tau^7\zeta^6\varrho^2 - 96525\tau^7\varrho^3\zeta^2\\
&\quad + 1312740\tau^7\zeta^6\varrho - 115830\tau^7\varrho^2\zeta^2 - 6486480\varrho^3\tau^4\zeta^4\\
&\quad + 6486480\varrho^2\tau^4\zeta^4 - 43243200\tau^3\varrho^2\zeta^4 + 21621600\tau^3\varrho^3\zeta^4\\
&\quad + 21621600\tau^3\varrho\zeta^4 + 99\zeta^8\varrho^3\tau^{13} - 1853280\tau^6\varrho\zeta^2\\
&\quad + 308880\tau^6\varrho^2\zeta^2 + 1544400\tau^6\varrho^3\zeta^2 + 540540\tau^5\varrho^3\zeta^4\\
&\quad - 6846840\tau^5\varrho^3\zeta^2 + 6126120\tau^5\varrho\zeta^2 + 7207200\tau^5\varrho^2\zeta^2\\
&\quad + 2772\varrho^2\tau^{12}\zeta^8 - 2772\varrho^3\tau^{12}\zeta^8 + 480480\tau^8\varrho^3\zeta^4\\
&\quad - 1441440\tau^8\varrho^2\zeta^4 + 1441440\tau^8\varrho\zeta^4 - 58968\tau^{10}\zeta^8\varrho^3\\
&\quad - 176904\tau^{10}\zeta^8\varrho - 1853280\tau^6\zeta^6\varrho^3 + 5559840\tau^6\zeta^6\varrho^2\\
&\quad - 5559840\tau^6\zeta^6\varrho + 64864800\tau^2\varrho^2\zeta^4 - 21621600\tau^2\varrho^3\zeta^4\\
&\quad - 64864800\tau^2\varrho\zeta^4 + 8648640\tau^4\varrho^3\zeta^2 - 25945920\tau^4\varrho^2\zeta^2\\
&\quad + 176904\tau^{10}\zeta^8\varrho^2 + 25945920\zeta^2\varrho\tau^4 - 45864\tau^{11}\zeta^8\varrho^2\\
&\quad + 2702700\varrho\tau^3 - 32432400\zeta^2 - 540540\zeta^4\tau^5\varrho\\
&\quad - 16216200\zeta^2\tau\varrho + 3243240\tau^4\zeta^4\varrho - 1029600\varrho^3\tau^6\\
&\quad - 3243240\tau^4\zeta^4 - 32432400\tau + 32432400\varrho\zeta^2\\
&\quad + 64864800\tau\varrho + 10810800\tau^2\big) + \cdots.
\end{aligned}
\tag{48}
$$

The problem has the exact solution at $\varrho = 1$:

$$\mu(\zeta,\tau) = \zeta^2\tau^2. \tag{49}$$

In Figure 6, the exact and the approximate solutions of example 4 at $\varrho = 1$ are shown, and the second graph shows the 3D graph of different fractional-order ρ, respectively. From the given figures, it can be seen that both the approximate and exact solutions are in close contact with each other. It is demonstrated that the outcomes of time-fractional problems converge to an integer-order effect as the time-fractional evaluation to integer-order.

5. Conclusion

In this paper, the iterative transformation method is implemented to achieve approximate analytical results of the fractional-order Klein-Gordon equations, which is widely applied in problems for spatial effects in applied sciences. In physical models, the technique yields series form results that converge very quickly. The obtained results in this article are expected to be important for further analysis of the sophisticated nonlinear models. The calculations of this method are very simple and straightforward. As a result, we conclude that this technique can be used to solve a variety of nonlinear fractional-order partial differential equation systems.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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Research Article

A Study of the Anisotropic Static Elasticity System in Thin Domain

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We study the asymptotic behavior of solutions of the anisotropic heterogeneous linearized elasticity system in thin domain of $\mathbb{R}^3$ which has a fixed cross-section in the $\mathbb{R}^2$ plane with Tresca friction condition. The novelty here is that stress tensor has given by the most general form of Hooke's law for anisotropic materials. We prove the convergence theorems for the transition 3D-2D when one dimension of the domain tends to zero. The necessary mathematical framework and (2D) equation model with a specific weak form of the Reynolds equation are determined. Finally, the properties of solution of the limit problem are given, in which it is confirmed that the limit problem is well defined.

1. Introduction

In this paper, we are interested of the asymptotic behavior of the linear elasticity system in a domain of $\mathbb{R}^3$ with a Tresca friction condition where the boundary of this domain has a fixed cross-section in dimension 2 and a small thickness. One of the objectives of this study is to obtain two-dimensional equation that allows a reasonable description of the phenomenon occurring in the three-dimensional domain by passing the limit to 0 on the small thickness of the domain (3D). Let us mention for example [1–8] in which the authors worked on the asymptotic behavior for the linearized elasticity system with different boundary conditions. Some problems of Newtonian or non-Newtonian fluids are considered in [9–11] where the authors proved a limit problem that gives a distribution of velocity and pressure through the weak form of the Reynolds equation. In [6, 7], the authors demonstrate the transition 3D-1D in anisotropic heterogeneous linearized elasticity; so, we mention here that this phenomenon has been studied only about strong solutions, without friction law. Benseridi in [2] investigated the asymptotic analysis of a dynamical problem of linear elasticity with Tresca's friction. The static case with a nonlinear term for linear elastic materials has been considered in [3]. See another situation in [4] where the paper concerns asymptotic derivation of frictionless contact models for elastic rods on a foundation with normal compliance. Recently, the authors in [5, 12] have proved the asymptotic behavior of a frictionless contact problem between two elastic bodies, when the vertical dimension of the two domain reaches zero. However, all these papers have been only restricted in a homogeneous and isotropic case of elastic materials.

The present work is a follow-up of [2, 3, 5] to study the heterogeneous and anisotropic situation with Tresca's friction. Here, the stress tensor with its components is given by the generalized Hooke's law (see [13]): $\sigma^\varepsilon = A^\varepsilon e(u^\varepsilon)$, where u^ε denotes the displacement vector, $e(u^\varepsilon)$ is the linearized strain tensor, and A^ε is the fourth order tensor which describes the elastic properties of the material. Many materials that follow the linear elastic model, although they are well made, are not subject to the assumptions of isotropy, for example, wood, reinforced concrete, composite materials, and many biological materials, where the mechanical properties of these materials differ according to the directions of space; in that case, the elasticity operator depends on the location of the point (see [14, 15]). Necas in [7] and Sofonea in [16] established the existence of a weak solution for the static frictional contact problem involving linearly elastic and viscoelastic materials, by using a results of convex optimization [17], and numerical approximation of this problem was studied in [18]. For the variational analysis of various contact problems, we mention excellent references in [14, 15]. Mathematically, the asymptotic analysis is more difficult since in general, the limit problem involves an equation that takes into account the anisotropy of the medium, and it is thus important to identify the elastic components of A^ε that appear in the (2D) equation model.

The paper is organized as follows; in section 2, the strong and weak formulation of the problem is given in terms of u^ε and also the related existence and uniqueness of the weak solution. In section 3, we introduce a scaling, and we find some estimates on the displacement which are independent of the parameter ε. In section 4, we state the main results concerning the existence of a weak limit u^* of u^ε, the (2D) equation model with a specific weak form of the Reynolds equation is proved, the limit form of the Tresca boundary conditions is formulated, and finally, the uniqueness of u^* is given.

2. Mathematical Formulation

Let ω be an open set in $\mathbb{R}$ with Lipschitz boundary, and we consider a smooth function $h : \omega \longrightarrow \mathbb{R}$ be a class C^1 such that $0 < h_{\min} \le h(x) \le h_{\max}$, for all $x \in \omega$, where $h_{\min}$ and $h_{\max}$ are constants. We define the smooth bounded domain Ω whose boundary has a flat part ω,

$$\Omega = \left\{ (x, z) \in \mathbb{R}^3, x \in \omega,\ 0 < z < h(x) \right\}. \tag{1}$$

We denote by Γ_1 is the upper boundary of the equation $z = h(x)$, and Γ_L is the lateral boundary.

Let $\varepsilon > 0$ be a small parameter, and we define Ω^ε be the change of scale $z = x_3/\varepsilon$ and the points of Ω,

$$\Omega^\varepsilon = \left\{ (x, x_3) \in \mathbb{R}^3, x \in \omega,\ 0 < x_3 < \varepsilon h(x) \right\}. \tag{2}$$

We have $\Gamma^\varepsilon = \bar{\omega} \cup \overline{\Gamma}_1^\varepsilon \cup \overline{\Gamma}_L^\varepsilon$ which its boundary of Ω^ε and where Γ_1^ε is the upper surface defined by $x_3 = \varepsilon h(x)$, and Γ_L^ε is the lateral boundary. The unit outward normal to Γ^ε is denoted by ν. It follows that there is correspondence between the functions $\phi : \Omega^\varepsilon \longrightarrow \mathbb{R}^n$ and $\widehat{\phi} : \Omega \longrightarrow \mathbb{R}^n (n = 1, 2, 3)$ given by $\widehat{\phi}(x, z) = \phi(x, x_3)$.

Let $H^{1/2}(\Gamma)^3$ be the space of traces of functions on Γ of functions from $H^1(\Omega)^3$, and we use the vector function $g \in H^{1/2}(\Gamma)^3$ such that

$$\int_\Gamma g.\nu ds = 0. \tag{3}$$

We denote by $\mathbb{S}_n$ the space of symmetric tensors on $\mathbb{R}^n$ and $|.|$ the Euclidean norm on $\mathbb{R}^n$ and $\mathbb{S}_n$. Here and below, the indices i, j, k, l run between 1 and 3, and the summation convention overrepeated indices is adopted.

The basic equations of frictionless contact problem for the anisotropic heterogeneous elastic body occupy the domain Ω^ε as follows:

The equations of equilibrium are as follows:

$$\frac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} + f_i^\varepsilon = 0 \text{in} \Omega^\varepsilon, \tag{4}$$

$$\sigma_{ij}^\varepsilon(u^\varepsilon) = A_{ijkl}^\varepsilon\, e_{kl}(u^\varepsilon) \text{in} \Omega^\varepsilon, \tag{5}$$

where the vector $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon)$ represents the forces of density, $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ is the displacement field, the elements A_{ijkl}^ε denote the components of elasticity tensor A^ε, and $e_{ij}(u^\varepsilon)$ is the rate of deformation operator,

$$e_{ij}(u^\varepsilon) = (e(u^\varepsilon))_{ij} = \frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i} \right). \tag{6}$$

On Γ_L^ε, the displacement is known:

$$u^\varepsilon = g^\varepsilon \text{on} \Gamma_L^\varepsilon. \tag{7}$$

On Γ_1^ε, we assume that the elastic body is held fixed:

$$u^\varepsilon = g^\varepsilon = 0 \text{on} \Gamma_1^\varepsilon. \tag{8}$$

On the surface ω, we assume that the contact is bilateral:

$$u^\varepsilon.\nu = g^\varepsilon.\nu = 0 \tag{9}$$

and satisfies the Tresca boundary condition [7] with friction function k^ε;

$$\begin{cases} |\sigma_\tau^\varepsilon| < k^\varepsilon u_\tau^\varepsilon = s, \\ |\sigma_\tau^\varepsilon| = k^\varepsilon \exists \lambda \ge 0 \text{suchthat} u_\tau^\varepsilon = s - \lambda \sigma_\tau^\varepsilon, \end{cases} \tag{10}$$

where $s = g^\varepsilon$ on ω. u_τ^ε, σ_τ^ε, and σ_ν^ε are the tangential displacement, the tangential, and the normal stress tensor, respectively, with

$$u_{\tau_i}^\varepsilon = u_i^\varepsilon - u_j^\varepsilon \nu_j \nu_i,\ \sigma_{\tau_i}^\varepsilon = \sigma_{ij}^\varepsilon.\nu_j - (\sigma_\nu^\varepsilon).\nu_i,\ \sigma_\nu^\varepsilon = (\sigma^\varepsilon.\nu).\nu. \tag{11}$$

Consider now the following closed convex subset of $H^1(\Omega^\varepsilon)^3$ given by

$$K^\varepsilon = \left\{ \phi \in H^1(\Omega^\varepsilon)^3 : \phi = g^\varepsilon \text{on} \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon,\ \phi.\nu = 0 \text{on} \omega \right\}. \tag{12}$$

Let us introduce the form $a : K^\varepsilon \times K^\varepsilon \longrightarrow \mathbb{R}$ and the functional $J^\varepsilon : K^\varepsilon \longrightarrow \mathbb{R}^+$ defined by

$$\begin{aligned} a(u^\varepsilon, \phi) &= \int_{\Omega^\varepsilon} A_{ijkl}^\varepsilon e_{kl}(u^\varepsilon) e_{ij}(\phi) dx dx_3, \\ J^\varepsilon(\phi) &= \int_\omega k^\varepsilon |\phi - s|\, dx. \end{aligned} \tag{13}$$

In the study of the mechanical problem (3)–(10), we assume that all components A^{ε}_{ijkl} belong to $L^{\infty}(\Omega^{\varepsilon})$ and satisfy the usual properties of symmetry and ellipticity [19], i.e.,

$$A^{\varepsilon}_{ijkl} = A^{\varepsilon}_{jikl} = A^{\varepsilon}_{klij} \in L^{\infty}(\Omega^{\varepsilon}), \tag{14}$$

and there exists a constant $\mu > 0$ such that

$$A^{\varepsilon}_{ijkl}(y)\xi_{kl}\xi_{ij} \geq \mu|\xi|^2 \forall \xi \in \mathbb{S}_3, \text{a.e.} y \in \Omega^{\varepsilon}. \tag{15}$$

Remark 1. It follows from previous properties and by Korn' s inequality (see [16], pp. 79), that the bilinear form a is coercive and continuous, i.e.,

$$a(\phi,\phi) \geq \mu C_K \|\nabla\phi\|^2_{L^2(\Omega^{\varepsilon})} \forall \phi \in K^{\varepsilon}, \tag{16}$$

$$|a(\varphi,\psi)| \leq M\|\nabla\phi\|_{L^2(\Omega^{\varepsilon})}\|\nabla\psi\|_{L^2(\Omega^{\varepsilon})} \forall \phi,\psi \in K^{\varepsilon}, \tag{17}$$

where $M = \max_{1\leq i,j,k,l\leq 3} \|A^{\varepsilon}_{ijkl}\|_{L^{\infty}(\Omega^{\varepsilon})}$ and C_K denoting a positive constant depends on Ω^{ε},Γ^{ε}_1, and Γ^{ε}_L.

Lemma 2. *Assuming that $f^{\varepsilon} \in L^2(\Omega^{\varepsilon})^3$ and $k^{\varepsilon} \in L^{\infty}(\omega)$, the variational formulation of problem (3)–(10) is equivalent to*

Find $u^{\varepsilon} \in K^{\varepsilon}$ satisfying

$$a(u^{\varepsilon},\phi-u^{\varepsilon}) + J^{\varepsilon}(\phi) - J^{\varepsilon}(u^{\varepsilon}) \geq \int_{\Omega^{\varepsilon}} f^{\varepsilon}.(\phi-u^{\varepsilon})dx dx_3 \forall \phi \in K^{\varepsilon}, \tag{18}$$

for every ε small fixed.

Moreover, if the assumptions of (14) and (15) hold, then the variational inequality (18) has a unique solution $u^{\varepsilon} \in K^{\varepsilon}$.

Remark 3. A problem of the form (18) is called an elliptic variational inequality of the second kind ([17]). The following theorem (see [19], Theorem 6) allows us to replace the variational inequality (18) by a minimization problem. Thus, we will not repeat the proof, but our goal is to study the asymptotic behavior.

3. Some Estimates in Fixed Domain

To be able to study the asymptotic behavior of the solutions of (18), we use the change of variable $z = x_3/\varepsilon$, to return to the fixed domain Ω, and then we define the following functions in Ω:

$$\widehat{u}^{\varepsilon}_i(x,z) = u^{\varepsilon}_i(x,x_3) \quad \text{for} i = 1,2,3. \tag{19}$$

For the data $\widehat{A}_{ijkl}$, $\widehat{f}_i$, and $\widehat{k}$, we have the following relations:

$$\widehat{A}_{ijkl}(x,z) = A^{\varepsilon}_{ijkl}(x,x_3), \quad \widehat{f}_i(x,z) = \varepsilon^2 f^{\varepsilon}_i(x,x_3) \text{ and } \widehat{k} = \varepsilon k^{\varepsilon}, \tag{20}$$

(for $1 \leq i,j,k,l \leq 3$).

Let

$$\begin{aligned} K &= \left\{v \in H^1(\Omega)^3 : v = g \text{on} \Gamma_L \cup \Gamma_1, v.\nu = 0 \text{on} \omega\right\}, \\ V_z &= \left\{ v = (v_1,v_2,v_3) \in L^2(\Omega)^3 : \frac{\partial v_i}{\partial z} \right. \\ &\quad \left. \in L^2(\Omega), i = 1,2,3 \,; v = 0 \text{on} \Gamma_1 \right\}. \end{aligned} \tag{21}$$

V_z is a Banach space for the following norm:

$$\|v\|_{V_z} = \left[\sum_{i=1}^{3}\left(\|v_i\|^2_{L^2(\Omega)} + \left\|\frac{\partial v_i}{\partial z}\right\|^2_{L^2(\Omega)}\right)\right]^{1/2}. \tag{22}$$

Everywhere in the sequel, the indexes α,β,γ and δ run from 1 to 2, and summation over repeated indices is implied. Follow the same steps as in [6, 12], passing to the fixed domain Ω, and using the symmetry of $\sigma^{\varepsilon}_{ij}$ and A^{ε}_{ijkl}, after multiplication by ε, we have (18) that is equivalent to

Find $\widehat{u}^{\varepsilon} \in K$, such that

$$\begin{aligned} &\widehat{a}\left(\widehat{u}^{\varepsilon},\widehat{\phi}-\widehat{u}^{\varepsilon}\right) + \int_{\omega}\widehat{k}\left|\widehat{\phi}-s\right|dx - \int_{\omega}\widehat{k}|\widehat{u}^{\varepsilon}-s|dx \\ &\quad \geq \sum_{i=1}^{3}\int_{\Omega}\widehat{f}_i\left(\widehat{\phi}_i-\widehat{u}^{\varepsilon}_i\right)dx dz, \forall\widehat{\phi} \in K, \end{aligned} \tag{23}$$

where

$$\begin{aligned} \widehat{a}\left(\widehat{u}^{\varepsilon},\widehat{\phi}\right) &= \varepsilon^2\int_{\Omega}\widehat{A}_{\alpha\beta\gamma\theta}\widehat{e}_{\gamma\theta}(\widehat{u}^{\varepsilon})\frac{\partial\widehat{\phi}_{\alpha}}{\partial x_{\beta}}dx dz \\ &\quad + 2\varepsilon\int_{\Omega}\widehat{A}_{\alpha 3\gamma\theta}\widehat{e}_{\gamma\theta}(\widehat{u}^{\varepsilon})\frac{\partial\widehat{\phi}_{\alpha}}{\partial z}dx dz \\ &\quad + 2\varepsilon^2\int_{\Omega}\widehat{A}_{\alpha\beta\gamma 3}\widehat{e}_{\gamma 3}(\widehat{u}^{\varepsilon})\frac{\partial\widehat{\phi}_{\alpha}}{\partial x_{\beta}}dx dz \\ &\quad + 4\varepsilon\int_{\Omega}\widehat{A}_{\alpha 3\gamma 3}\widehat{e}_{\gamma 3}(\widehat{u}^{\varepsilon})\frac{\partial\widehat{\phi}_{\alpha}}{\partial z}dx dz \\ &\quad + \varepsilon^2\int_{\Omega}\widehat{A}_{\alpha\beta 33}\widehat{e}_{33}(\widehat{u}^{\varepsilon})\frac{\partial\widehat{\phi}_{\alpha}}{\partial x_{\beta}}dx dz \\ &\quad + \varepsilon\int_{\Omega}\widehat{A}_{33\alpha\beta}\widehat{e}_{\alpha\beta}(\widehat{u}^{\varepsilon})\frac{\partial\widehat{\phi}_3}{\partial z}dx dz \\ &\quad + 2\varepsilon\int_{\Omega}\widehat{A}_{\alpha 333}\widehat{e}_{33}(\widehat{u}^{\varepsilon})\frac{\partial\widehat{\phi}_{\alpha}}{\partial z}dx dz \\ &\quad + 2\varepsilon\int_{\Omega}\widehat{A}_{33\alpha 3}\widehat{e}_{\alpha 3}(\widehat{u}^{\varepsilon})\frac{\partial\widehat{\phi}_3}{\partial z}dx dz \\ &\quad + \varepsilon\int_{\Omega}\widehat{A}_{3333}\widehat{e}_{33}(\widehat{u}^{\varepsilon})\frac{\partial\widehat{\phi}_3}{\partial z}dx dz, \end{aligned} \tag{24}$$

and $\widehat{e}(\widehat{u}^\varepsilon) = (\widehat{e}_{ij}(\widehat{u}^\varepsilon))_{ij}$ is given by the relations

$$\begin{aligned}
\widehat{e}_{ij}(\widehat{u}^\varepsilon) &= \frac{1}{2}\left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i}\right), \qquad i,j = 1,2,\\
\widehat{e}_{i3}(\widehat{u}^\varepsilon) = \widehat{e}_{3i}(\widehat{u}^\varepsilon) &= \frac{1}{2}\left(\frac{1}{\varepsilon}\frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i}\right), \qquad i = 1,2.\\
\widehat{e}_{33}(\widehat{u}^\varepsilon) &= \frac{1}{\varepsilon}\frac{\partial \widehat{u}_3^\varepsilon}{\partial z},
\end{aligned} \tag{25}$$

Lemma 4. *Under the assumptions of Lemma 2, there exists a constant $C > 0$ independent of ε, such that*

$$\sum_{i=1}^{3}\left\|\frac{\partial \widehat{u}_i^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2 + \varepsilon^2 \sum_{i,j=1}^{2}\left\|\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j}\right\|_{L^2(\Omega)}^2 + \varepsilon^2 \sum_{j=1}^{2}\left\|\frac{\partial \widehat{u}_3^\varepsilon}{\partial x_j}\right\|_{L^2(\Omega)}^2 \le C. \tag{26}$$

Proof. Assume that u^ε is a solution of (2.12). As $J^\varepsilon(u^\varepsilon) \ge 0$, then

$$\begin{aligned}
a(u^\varepsilon, u^\varepsilon) \le a(u^\varepsilon, \phi) + J^\varepsilon(\phi) &+ \int_{\Omega^\varepsilon} f^\varepsilon . u^\varepsilon dx dx_3\\
&- \int_{\Omega^\varepsilon} f^\varepsilon . \phi dx dx_3, \quad \forall \phi \in K^\varepsilon.
\end{aligned} \tag{27}$$

Using the Young's inequality

$$ab \le \eta^2 \frac{a^2}{2} + \eta^{-2}\frac{b^2}{2} \tag{28}$$

in (17) for $\eta = \sqrt{\mu C_K/2}$, we find

$$|a(u^\varepsilon, \phi)| \le \frac{\mu C_K}{4}\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \frac{M}{\mu C_K}\|\nabla \phi\|_{L^2(\Omega^\varepsilon)}^2. \tag{29}$$

Also, by the Cauchy-Schwarz and Poincaré's inequalities, we get

$$\left|\int_{\Omega^\varepsilon} f^\varepsilon . \phi dx dx_3\right| \le \varepsilon h_{\max}\|f^\varepsilon\|_{L^2(\Omega^\varepsilon)}\|\nabla \phi\|_{L^2(\Omega^\varepsilon)}, \tag{30}$$

then using Young's inequality for $\eta = \sqrt{1/\mu C_K}$ to obtain

$$\left|\int_{\Omega^\varepsilon} f^\varepsilon . u^\varepsilon dx dx_3\right| \le \frac{(\varepsilon h_{\max})^2}{2\mu C_K}\|f^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \frac{\mu C_K}{2}\|\nabla \phi\|_{L^2(\Omega^\varepsilon)}^2. \tag{31}$$

Using (16), (29), and (31) in (27), we get

$$\begin{aligned}
\frac{\mu C_K}{4}\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 &\le \frac{M}{\mu C_K}\|\nabla \phi\|_{L^2(\Omega^\varepsilon)}^2 + \frac{(\varepsilon h_{\max})^2}{\mu C_K}\|f^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2\\
&\quad + \frac{\mu C_K}{2}\|\nabla \phi\|_{L^2(\Omega^\varepsilon)}^2 + \int_\omega k^\varepsilon |\phi - s| dx.
\end{aligned} \tag{32}$$

Taking into account the g function introduced in (5) and using [20] (lemma 2 pp.24), there exists a function $\tilde{g} \in H^1(\Omega)^3$ such that

$$\tilde{g} = g \text{on} \Gamma_L \text{and} \quad \tilde{g}.\nu = 0 \text{on} \omega \cup \Gamma_1. \tag{33}$$

Thus, choosing $\widehat{\phi} = \tilde{g}$ in (3.6), then multiplying product inequality by ε, and the fact that $g = s$ on ω, we obtain

$$\begin{aligned}
\frac{\mu C_K}{4}\varepsilon\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 &\le \frac{(h_{\max})^2}{\mu C_K}\left\|\widehat{f}\right\|_{L^2(\Omega)}^2\\
&\quad + \left(\frac{M}{\mu C_K} + \frac{\mu C_K}{2}\right)\|\nabla \tilde{g}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{34}$$

From [6], we can see the constant Korn C_K contained in Remark 1 does not depend on ε and ϕ, for $\varepsilon \in]0,1]$; moreover, by changing the data of A^ε, remark that μ and M are independent of ε. Therefore, passing to the fixed domain Ω, we get

$$\begin{aligned}
\varepsilon\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 &= \sum_{i=1}^{3}\left\|\frac{\partial \widehat{u}_i^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2 + \varepsilon^2\left(\sum_{i,j=1}^{2}\left\|\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j}\right\|_{L^2(\Omega)}^2\right.\\
&\quad \left. + \sum_{j=1}^{2}\left\|\frac{\partial \widehat{u}_3^\varepsilon}{\partial x_j}\right\|_{L^2(\Omega)}^2\right) \le C,
\end{aligned} \tag{35}$$

with

$$C = \frac{4}{\mu C_K}\left[\frac{(h_{\max})^2}{\mu C_K}\left\|\widehat{f}\right\|_{L^2(\Omega)}^2 + \left(\frac{M}{\mu C_K} + \frac{\mu C_K}{2}\right)\|\nabla \tilde{g}\|_{L^2(\Omega)}^2\right]. \tag{36}$$

□

Lemma 5. *Under the assumptions of Lemma 4, there exists $u^* = (u_1^*, u_2^*, u_3^*) \in V_z$ such that*

$$\widehat{u}^\varepsilon \rightharpoonup u^* weakly in V_z, \tag{37}$$

$$\varepsilon\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_\alpha} \rightharpoonup 0 \, weakly in L^2(\Omega) \, (i = 1,2,3 and \alpha = 1,2), \tag{38}$$

$$\varepsilon\widehat{e}_{\alpha\beta}(\widehat{u}^\varepsilon) \rightharpoonup 0, weakly in L^2(\Omega) \, (\alpha, \beta = 1,2), \tag{39}$$

$$\varepsilon\widehat{e}_{\gamma 3}(\widehat{u}^\varepsilon) \rightharpoonup \frac{1}{2}\frac{\partial u_\gamma^*}{\partial z} \, weakly in L^2(\Omega) \, (\gamma = 1,2), \tag{40}$$

$$\varepsilon\widehat{e}_{33}(\widehat{u}^\varepsilon) \rightharpoonup \frac{\partial u_3^*}{\partial z} \, weakly in L^2(\Omega). \tag{41}$$

Proof. From (26), there exists a fixed constant $C > 0$ such that

$$\left\|\frac{\partial \widehat{u}_i^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2 \le C, \quad \text{for} i = 1,2,3. \tag{42}$$

Using Poincaré's inequality in the domain Ω

$$\|\widehat{u}_i^\varepsilon\|_{L^2(\Omega)} \le h_{\max} \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}, \quad \text{for} i = 1, 2, 3, \tag{43}$$

we deduce that $\widehat{u}^\varepsilon$ is bounded in V_z. From the last two estimates, there exists $u^* = (u_1^*, u_2^*, u_3^*) \in V_z$ and satisfies (37). From (26), we can extract a subsequence such that $\varepsilon(\partial \widehat{u}_i^\varepsilon / \partial x_\alpha) \rightharpoonup \eta$ in $L^2(\Omega)$; on the other hand, from (37), we deduce (38). Also, (39)–(41) follow from (37) and (38). ☐

4. Limit Problem and Main Result

At the limit $\varepsilon = 0$, we give the satisfactory equations of u^* and the properties of solution of the limit problem for the system (3)-(10).

Theorem 6. *With the same assumptions as Lemma 5, u^* satisfies*

$$\begin{aligned} &\left\langle A^* \frac{\partial u^*}{\partial z}, \frac{\partial}{\partial z}\left(\widehat{\phi} - u^*\right) \right\rangle_{L^2(\Omega)} + \int_\omega \widehat{k}\left(\left|\widehat{\phi} - s\right| - |u^* - s|\right) dx \\ &\quad \ge \int_\Omega \widehat{f} \cdot \left(\widehat{\phi} - u^*\right) dx dz, \quad \forall \widehat{\phi} \in K, \end{aligned} \tag{44}$$

where the symmetric matrix A^* is given by

$$A^* = \begin{pmatrix} 4\widehat{A}_{1313} & 4\widehat{A}_{1323} & 2\widehat{A}_{1333} \\ 4\widehat{A}_{2313} & 4\widehat{A}_{2323} & 2\widehat{A}_{2333} \\ 2\widehat{A}_{3313} & 2\widehat{A}_{3323} & \widehat{A}_{3333} \end{pmatrix}. \tag{45}$$

Moreover, we have

$$\left.\begin{aligned} &-\frac{\partial}{\partial z}\left\{4\widehat{A}_{\alpha 3\gamma 3} \frac{\partial u_\gamma^*}{\partial z} + 2\widehat{A}_{\alpha 333} \frac{\partial u_3^*}{\partial z}\right\} = \widehat{f}_\alpha, (\alpha = 1, 2) \\ &-\frac{\partial}{\partial z}\left\{2\widehat{A}_{33\alpha 3} \frac{\partial u_\alpha^*}{\partial z} + \widehat{A}_{3333} \frac{\partial u_3^*}{\partial z}\right\} = \widehat{f}_3 \end{aligned}\right\} in L^2(\Omega). \tag{46}$$

Proof. As (23) can be written,

$$\begin{aligned} &\widehat{a}\left(\widehat{u}^\varepsilon, \widehat{\phi}\right) + \int_\omega \widehat{k}\left|\widehat{\phi} - s\right| dx - \sum_{i=1}^3 \int_\Omega \widehat{f}_i\left(\widehat{\phi}_i - \widehat{u}_i^\varepsilon\right) dx dz \\ &\quad \ge \widehat{a}(\widehat{u}^\varepsilon, \widehat{u}^\varepsilon) + \int_\omega \widehat{k}|\widehat{u}^\varepsilon - s| dx. \end{aligned} \tag{47}$$

Since the form $\widehat{a}(.,.)$ is a symmetry and K-elliptic, and the fact that $\widehat{\phi} \longrightarrow \int_\omega \widehat{k}|\widehat{\phi} - s| dx$ is convex and lower semicontinuous, we deduce

$$\begin{aligned} &\liminf_{\varepsilon \longrightarrow 0} \left[\widehat{a}(\widehat{u}^\varepsilon, \widehat{u}^\varepsilon) + \int_\omega \widehat{k}|\widehat{u}^\varepsilon - s| dx\right] \\ &\quad \ge \widehat{a}(u^*, u^*) + \int_\omega \widehat{k}|u^* - s| dx. \end{aligned} \tag{48}$$

Using Lemma 5, we let ε tend to 0 in (47), to obtain

$$\begin{aligned} &4\int_\Omega \widehat{A}_{\alpha 3\gamma 3} \frac{\partial u_\gamma^*}{\partial z} \frac{\partial}{\partial z}\left(\widehat{\phi}_\alpha - u_\alpha^*\right) dx dz \\ &\quad + 2\int_\Omega \widehat{A}_{\alpha 333} \frac{\partial u_3^*}{\partial z} \frac{\partial}{\partial z}\left(\widehat{\phi}_\alpha - u_\alpha^*\right) dx dz \\ &\quad + 2\int_\Omega \widehat{A}_{33\alpha 3} \frac{\partial u_\alpha^*}{\partial z} \frac{\partial}{\partial z}\left(\widehat{\phi}_3 - u_3^*\right) dx dz \\ &\quad + \int_\Omega \widehat{A}_{3333} \frac{\partial u_3^*}{\partial z} \frac{\partial}{\partial z}\left(\widehat{\phi}_3 - u_3^*\right) dx dz \\ &\quad + \int_\omega \widehat{k}\left(\left|\widehat{\phi} - s\right| - |u^* - s|\right) dx \\ &\ge \sum_{i=1}^3 \int_\Omega \widehat{f}_i\left(\widehat{\phi}_i - u_i^*\right) dx dz. \end{aligned} \tag{49}$$

This completes the proof of (44) if we cross (49) in the matrix form A^*. We choose in the variational inequation (49) $\widehat{\phi}_i = u_i^* \pm \psi_i$, where $\psi_i \in H_0^1(\Omega)$ (for $i = 1, 2, 3$), and using Green's formula, we find

$$\begin{aligned} &-\int_\Omega \frac{\partial}{\partial z}\left\{4\widehat{A}_{\alpha 3\gamma 3} \frac{\partial u_\gamma^*}{\partial z} + 2\widehat{A}_{\alpha 333} \frac{\partial u_3^*}{\partial z}\right\} \psi_\alpha dx dz \\ &\quad - \int_\Omega \frac{\partial}{\partial z}\left\{2\widehat{A}_{33\alpha 3} \frac{\partial u_\alpha^*}{\partial z} + \widehat{A}_{3333} \frac{\partial u_3^*}{\partial z}\right\} \psi_3 dx dz \\ &= \sum_{i=1}^3 \int_\Omega \widehat{f}_i \psi_i dx dz, \end{aligned} \tag{50}$$

choosing $\psi_3 = 0$ and $\psi_\alpha \in H_0^1(\Omega)$; then, $\psi_\alpha = 0$ and $\psi_3 \in H_0^1(\Omega)$, we get (59). ☐

Theorem 7. *Under the assumptions of Theorem 6 then, the solution of the limit problem (44)–(46) is unique in V_z.*

Proof. Suppose that there exists two solutions u^* and v^* of the variational inequality (44), and taking $\widehat{\phi} = v^*$ in (44), then $\widehat{\phi} = u^*$ in the inequality relating to v^*. By subtracting the two obtained inequalities, we have

$$\left\langle A^* \cdot \frac{\partial}{\partial z}(v^* - u^*), \frac{\partial}{\partial z}(v^* - u^*) \right\rangle_{L^2(\Omega)} \le 0. \tag{51}$$

We must now check that A^* is ellipticity. So, we return to the properties of $\widehat{A}$ mentioned in (14) and (15); in particular, we choose symmetric tensors ζ that are given by $\xi_{\alpha\beta} = 0$ (for $\alpha, \beta = 1, 2$); otherwise, the rest of the components (ξ_{i3}) let it be whatever. Putting $\eta_i = \xi_{i3}$, for $i = 1, 2, 3$, we will get

$$\widehat{A}_{ijkl}\xi_{kl}\xi_{ij} = 4\widehat{A}_{\alpha 3\beta 3}\eta_\beta\eta_\alpha + 2\widehat{A}_{\alpha 333}\eta_3\eta_\alpha + 2\widehat{A}_{33\alpha 3}\eta_\alpha\eta_3 + \widehat{A}_{3333}\eta_3\eta_3 = A^*_{ij}\eta_j\eta_i. \quad (52)$$

Consequently, and as $|\xi|^2 \geq |\eta|^2$, there exists a positive constant μ, and for all vectors η in $\mathbb{R}^3$, we have

$$A^*_{ij}\eta_j\eta_i \geq \mu|\eta|^2. \quad (53)$$

So, A^* is ellipticity. Thus, the relation (44) implies that

$$\mu\left\|\frac{\partial}{\partial z}(v^* - u^*)\right\|^2_{L^2(\Omega)} \leq 0. \quad (54)$$

Using Poincaré's inequality, we obtain

$$\|v^* - u^*\|^2_{L^2(\Omega)} \leq (h_{\max})^2\left\|\frac{\partial}{\partial z}(v^* - u^*)\right\|^2_{L^2(\Omega)} = 0, \quad (55)$$

and the proof of uniqueness of u^* is complete. □

Theorem 8. *Under the assumptions of Theorem 7, the traces (s^*, π^*) with $s^* = (s^*_i)_{1\leq i\leq 3}$ and $\pi^* = (\pi^*_i)_{1\leq i\leq 3}$ defined by*

$$\begin{aligned} s^*_i(x) &= u^*_i(x, 0), \\ \pi^*_\alpha(x) &= \left[4\widehat{A}_{\alpha 3\gamma 3}\frac{\partial u^*_\gamma}{\partial z} + 2\widehat{A}_{\alpha 333}\frac{\partial u^*_3}{\partial z}\right](x, 0)\,;\, \pi^*_3(x) \\ &= \left[2\widehat{A}_{33\gamma 3}\frac{\partial u^*_\gamma}{\partial z} + \widehat{A}_{3333}\frac{\partial u^*_3}{\partial z}\right](x, 0), \end{aligned} \quad (56)$$

satisfy the following limit form of the Tresca boundary conditions:

$$\int_\omega \widehat{k}|\psi + s^* - s| - |s^* - s|dx - \int_\omega \pi^*.\psi dx \geq 0\, \forall\psi \in L^2(\omega)^3, \quad (57)$$

$$\left.\begin{array}{l} |\pi^*| < \widehat{k} \Rightarrow s^* = s \\ |\pi^*| = \widehat{k} \Rightarrow \exists\lambda > 0 \text{ such that } s^* = s + \lambda\pi^* \end{array}\right\} \text{a.e. in } \omega. \quad (58)$$

Moreover, if the coefficients $\widehat{A}_{i3j3}$ for $1 \leq i, j \leq 3$, depending only on the variable x, we have the following weak form of the Reynolds equation:

$$\int_\omega \left(\tilde{F} - \frac{h}{2}s^* + \int_0^h u^*(x, z)dz\right).\nabla\psi(x)dx = 0,\ \forall\psi \in H^1(\omega), \quad (59)$$

where $InvA^*(x)$ denotes the inverse of $A^*(x)$ and

$$\tilde{F}(x) = \int_0^h \tilde{F}(x, \rho)d\rho - \frac{h}{2}\tilde{F}(x, h), \quad (60)$$

$$\tilde{F}(x, \rho) = InvA^*(x).\int_0^\rho\int_0^\theta \widehat{f}(x, y)dyd\theta. \quad (61)$$

Proof. We now choose in the variational inequality (49) $\widehat{\phi}_i = u^*_i + \psi_i$, where $\psi_i \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega)$ for $i = 1, 2, 3$, and then using Green's formula, we obtain

$$\begin{aligned} &-\int_\Omega \frac{\partial}{\partial z}\left\{4\widehat{A}_{\alpha 3\gamma 3}\frac{\partial u^*_\gamma}{\partial z} + 2\widehat{A}_{\alpha 333}\frac{\partial u^*_3}{\partial z}\right\}\psi_\alpha dxdz \\ &- \int_\omega\left(4\widehat{A}_{\alpha 3\gamma 3}\frac{\partial u^*_\gamma}{\partial z} + 2\widehat{A}_{\alpha 333}\frac{\partial u^*_3}{\partial z}\right)\psi_\alpha(x, 0)dx \\ &- \int_\Omega \frac{\partial}{\partial z}\left\{2\widehat{A}_{33\gamma 3}\frac{\partial u^*_\gamma}{\partial z} + \widehat{A}_{3333}\frac{\partial u^*_3}{\partial z}\right\}\psi_3 dxdz \\ &- \int_\omega\left(2\widehat{A}_{33\gamma 3}\frac{\partial u^*_\gamma}{\partial z} + \widehat{A}_{3333}\frac{\partial u^*_3}{\partial z}\right)\psi_3(x, 0)dx \\ &+ \int_\omega \widehat{k}(|\psi + s^* - s| - |u^* - s|)dx \\ &\geq \sum_{i=1}^3\int_\Omega \widehat{f}_i\psi_i dxdz. \end{aligned} \quad (62)$$

On the other hand, from (46), we have

$$\int_\omega \widehat{k}(|\psi + s^* - s| - |s^* - s|)dx - \int_\omega\left(\sum_\alpha \pi^*_\alpha\psi_\alpha + \pi^*_3\psi_3\right)dx \geq 0. \quad (63)$$

By density theorems, we find (57). For (58), we use the analogue of [10].

To prove (59), we use those similar steps as in [2, 5, 9–11], by integrating (46) from 0 to z, and taking into account $\widehat{A}_{i3j3}$ depending only on x, we obtain

$$\begin{cases} -\left[4\widehat{A}_{\alpha 3\gamma 3}(x)\dfrac{\partial u^*_\gamma}{\partial z}(x, z) + 2\widehat{A}_{\alpha 333}(x)\dfrac{\partial u^*_3}{\partial z}(x, z)\right] + \pi^*_\alpha(x) = \displaystyle\int_0^z \widehat{f}_\alpha(x, y)dy, \\ -\left[2\widehat{A}_{33\gamma 3}(x)\dfrac{\partial u^*_\gamma}{\partial z}(x, z) + \widehat{A}_{3333}(x)\dfrac{\partial u^*_3}{\partial z}(x, z)\right] + \pi^*_3(x) = \displaystyle\int_0^z \widehat{f}_3(x, y)dy, \end{cases} \quad (64)$$

$$-A^*(x).\frac{\partial u^*}{\partial z} + \pi^*(x) = \int_0^z \widehat{f}(x, y)dy. \quad (65)$$

It follows from (51) that it is a invertible matrix $A^*(x)$, for almost every $x \in \omega$. Therefore,

$$-\frac{\partial u^*}{\partial z} + \mathrm{Inv}A^*(x).\pi^*(x) = \mathrm{Inv}A^*(x).\int_0^z \widehat{f}(x,y)dy. \quad (66)$$

By integrating between 0 and z, we obtain

$$-u^*(x,z) + s^*(x) + z\mathrm{Inv}A^*(x).\pi^*(x) = \tilde{F}(x,z). \quad (67)$$

As $u_i^*(x,h(x)) = 0$, we have

$$s^*(x) + h(x)\mathrm{Inv}A^*(x).\pi^*(x) = \int_0^{h(x)} \tilde{F}(x,y)dy. \quad (68)$$

We integrate (67) from 0 to $h(x)$, and we obtain

$$\begin{aligned} -\int_0^{h(x)} u^*(x,z)dz + hs^*(x) + \frac{h(x)^2}{2}\mathrm{Inv}A^*(x).\pi^*(x) \\ = \int_0^{h(x)} \tilde{F}(x,y)dy, \end{aligned} \quad (69)$$

and by (68), we deduce that

$$-\int_0^{h(x)} u^*(x,z)dz + \frac{h(x)}{2}s^*(x) - \tilde{F}(x) = 0, \quad (70)$$

such that $\tilde{F}$ is already defined in (61), and let us finally get the weak form (59) after multiplying (70) by $\nabla\psi(x)$ and integrate it in ω. □

5. Conclusions

We were able to find a framework to conclude that solving our original problem leads to solving a well-defined problem as in (44),(46) and (57)–(59) for the "small" parameter ε.

The key of the problem lies in the relation between the matrices A^ε and A^*. Note that they have the same properties despite the difference in dimensions, therefore it played a key role in the transition from u^ε to u^*.

Indeed, the special case

$$A^\varepsilon_{ijkl} = \mu^\varepsilon\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \lambda^\varepsilon\delta_{ij}\delta_{kl} \quad (71)$$

where λ^ε, $\mu^\varepsilon > 0$ are the Lamé coefficients (see [13] pp. 102-103) corresponds to the homogeneous and isotropic case of elastic materials, and has been studied in [2, 3, 5]. Thus also, the Stokes flow in [11] can be recovered when λ^ε tends to 0.

Data Availability

No data were used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

A Note on the Generalized Nonlinear Vector Variational-Like Inequality Problem

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In this paper, we discuss two variants of the generalized nonlinear vector variational-like inequality problem. We provide their solutions by adopting topological approach. Topological properties such as compactness, closedness, and net theory are used in the proof. The admissibility of the function space topology and KKM-Theorem have played important role in proving the results.

1. Introduction

Variational inequalities have appeared as a working and important tool to investigate various fields of mathematics as well as of sciences including elasticity, vector equilibrium problems, and optimization problems [1–4]. In mid-sixties, Browder [5] formulated and proved the basic existence results for the solutions to a class of nonlinear variational inequality problems. He used a reflexive Banach space X and a monotone nonlinear map T from the space X to its dual space X^*, to set up the nonlinear variational inequality problem. Browder used the property of hemi-continuity and monotonicity of mapping T along with the lower semicontinuity of f, for providing the existence of the solution of nonlinear variational inequality problem. After that, this problem has been generalized and extended in various directions under different set-ups using different techniques. Liu et al. [6], Zhao et al. [7], and Ahmad and Irfan [8] are a few, who extended Browder's results to more generalized nonlinear variational inequalities. In 2009, Farajzadeh et al. [9] considered new kinds of generalized variational-like inequality problems under the frame work of topological vector spaces.

In the subsequent period, generalized quasi-variational inequalities were studied by Hung and others [10–12]. In 2017, Irfan et al. introduced a new generalized variational-like inclusion problem involving relaxed monotone operators [13]. A class of η-generalized operator variational-like inequalities were introduced by Kim et al. in 2018 [14]. In the same year, Tavakoli et al. studied the C-pseudomonotone property for the set-valued mappings in order to solve a generalized variational inequality problems [15]. On the other hand, vector equilibrium problems for the set-valued mappings were studied by Farajzadeh et al. and Chen et al. during this period [16, 17]. This wide range of literature is a clear indication of the importance that variational inequality problems have gained in the recent years. In this paper, we further add to this literature by providing solutions to a generalized nonlinear vector variational-like inequality problem, using topological methods.

Variational-like inequalities have number of applications which make it an interesting discipline for research. Vector variational inequality on flow equilibrium problem on a network has been discussed in [18]. Application of variational-like inequality in fuzzy optimization problem is discussed in [19]. More such studies are available in the literature [20–22].

Motivated by these studies, here, we investigate a generalized nonlinear variational like inequality problem, which was proposed by Farajzadeh et al. [9] as follows:

Generalized nonlinear variational-like inequality problem: let $\langle X, X^* \rangle$ be a dual system of Hausdorff topological vector spaces and K be a nonempty convex subset of X. Given the mappings $f, g, p : X^* \longrightarrow X^*$ and $\eta : K \times K \longrightarrow X$, set-valued map $M, S, T : K \longrightarrow X^*$, and a map $h : K \times K \longrightarrow \mathbb{R}$, consider the following *generalized nonlinear variational-like inequality problem (GNVLIP)*

$$\begin{cases} \text{Find } x \in K \text{ such that for each } y \in K, \\ \exists u \in M(x), v \in S(x),\ w \in T(x) \text{ satisfying} \\ \langle p(u) - (f(v) - g(w)), \eta(y, x) \rangle \geq h(x, y) \end{cases} \tag{1}$$

In this paper, we consider two variants of nonlinear vector variational-like inequality problems in a more general setup as follows:

Let X and Y be two topological vector spaces, and let K be a nonempty, closed, and convex subset of X and $CL(X, Y)$ be the space of all continuous linear mappings from the space X to the space Y. Clearly, $CL(X, Y)$ is nonempty as the zero mapping, that is, $k : X \longrightarrow Y$ defined as $k(x) = \mathbf{0}$ for all $x \in X$ is always linear and continuous, hence belongs to $CL(X, Y)$.

Further, let $M, S, T : K \longrightarrow CL(X, Y)$ be set-valued mappings and $f, g, p : CL(X, Y) \longrightarrow CL(X, Y)$ be single-valued mappings. Suppose the maps $\eta : K \times K \longrightarrow X$ and $h : K \times K \longrightarrow Y$ are two bifunctions.

Problem 1. Suppose $C \subseteq Y$ is a closed, convex, pointed cone with $\text{int}\, C \neq \emptyset$. Then, the generalized nonlinear vector variational-like inequality problem (I) (GNVVLIP (I)) is to find $x_0 \in K$, such that for each $y \in K$, there exist $u \in M(x_0)$, $v \in S(x_0)$, and $w \in T(x_0)$ such that

$$((p(u) - (f(v) - g(w))))(\eta(y, x_0)) - h(x_0, y) \notin -\text{int}\, C. \tag{2}$$

Problem 2. Suppose $C : K \longrightarrow Y$ is a set-valued map such that for every $x \in K$, $C(x)$ is a proper, closed, convex, pointed cone with nonempty interior. Then, the generalized nonlinear vector variational-like inequality problem (II) (GNVVLI (II)) is to find $x_0 \in K$ such that for each $y \in K$, $\exists u \in M(x_0)$, $v \in S(x_0)$, and $w \in T(x_0)$ such that

$$((p(u) - (f(v) - g(w))))(\eta(y, x_0)) - h(x_0, y) \notin -\text{int}\, C(x_0). \tag{3}$$

Here, we are trying to provide solutions to the above stated generalized nonlinear vector variational-like inequality problems (I and II) from a topological point of view. We consider X and Y to be any topological vector spaces and use the concept of admissibility of the function space topology along with net theory to prove the existence of solutions of these generalized nonlinear vector variational-like inequality problems.

2. Preliminaries

Below, we provide some definitions and results related mainly to set-valued maps between topological spaces.

Definition 3 [23]. Suppose (X, τ_1) and (Y, τ_2) are two topological spaces and $F : X \longrightarrow Y$ is a set-valued map.

(i) F is called *upper semicontinuous* (in short, u.s.c.) at a point $x \in X$, if for every open set V in Y such that $F(x) \subseteq V$, there exists an open set U in X with $x \in U$ such that $F(U) \subseteq V$;

(ii) F is called *lower semicontinuous* (in short, l.s.c.) at a point $x \in X$, if for every open set V in Y such that $F(x) \cap V \neq \emptyset$, there exists an open set U in X with $x \in U$ such that for each $u \in U$, $F(u) \cap V \neq \emptyset$;

(iii) F is said to be *continuous* at $x \in X$ if it is both upper semicontinuous and lower semicontinuous at x;

(iv) F is said to be *continuous* (resp. u.s.c. and l.s.c.) if it is so at each point of X.

Lemma 4 [24]. *Suppose (X, τ_1) and (Y, τ_2) are topological spaces and $F : X \longrightarrow Y$ is a set-valued map. Then,*

(i) if F is upper semicontinuous at $x \in X$ and $F(x)$ is compact then for every net $\{x_\alpha\}_{\alpha \in D}$ and $y_\alpha \in F(x_\alpha)$ with $x_\alpha \longrightarrow x$ and $y_\alpha \longrightarrow y$, we have $y \in F(x)$;

(ii) F is lower semicontinuous at $x \in X$ if and only if for every $y \in F(x)$ and every net $\{x_\alpha\}_{\alpha \in D}$ with $x_\alpha \longrightarrow x$, there exists a subnet $\{x_\beta\}_{\beta \in \Omega}$, where Ω is a directed subset of D and a net $\{y_\beta\}_{\beta \in \Omega}$ such that $y_\beta \in F(x_\beta)$ with $y_\beta \longrightarrow y$.

Theorem 5 [23]. *Let (X, τ) and (Y, μ) be two topological spaces. Let $F : X \longrightarrow Y$ be a set-valued map. Then, F is lower semicontinuous at $x \in X$ if for any net $\{x_n\}_{n \in \Delta}$ in X converging to $x \in X$, the image net $\{F(x_n)\}_{n \in \Delta}$ converges to $F(x)$.*

Lemma 6 [25]. *Suppose X and Y are topological spaces and $F : X \longrightarrow Y$ is a set-valued upper semicontinuous function. If $F(x)$ is compact for each $x \in X$, then image of every compact subset of X under F is compact.*

Definition 7 [23, 26]. Let (Y, μ_1) and (Z, μ_2) be two topological spaces. Let $\mathscr{C}(Y, Z)$ be the space of all continuous mappings from Y to Z. A topology τ on $\mathscr{C}(Y, Z)$ is called admissible, if the evaluation map $e : \mathscr{C}(Y, Z) \times Y \longrightarrow Z$, defined by $e(f, y) = f(y)$, is continuous.

Definition 8 [26]. Let $\{f_n\}_{n \in \Delta}$ be a net in $\mathscr{C}(Y, Z)$. Then, $\{f_n\}_{n \in \Delta}$ is said to continuously converge to f if for each net $\{y_m\}_{m \in \sigma}$ in Y converging to y, $\{f_n(y_m)\}_{(n,m) \in \Delta \times \sigma}$ converges to $f(y)$ in Z.

Theorem 9 [23]. *Let (Y,τ) and (Z,μ) be two topological spaces. A topology $\mathfrak{T}$ on $\mathscr{C}(Y,Z)$, the family of continuous mappings from Y to Z, is admissible if and only if for any net $\{f_n\}_{n\in\Delta}$ in $\mathscr{C}(Y,Z)$, $\{f_n\}_{n\in\Delta}$ converges to f in $\mathfrak{T}$ implies continuous convergence of $\{f_n\}_{n\in\Delta}$ to f.*

Definition 10 [27]. Suppose $F: X \longrightarrow Y$ is a set-valued map from X to Y. The graph of F, denoted by $\mathscr{G}(F)$, is

$$\mathscr{G}(F) = \{(x,y)\in X\times Y \mid x\in X, y\in F(x)\}. \tag{4}$$

Definition 11 [28]. Suppose S is a nonempty subset of some topological vector space X. A set-valued map $F: S\longrightarrow X$ is called a KKM-mapping if for every nonempty finite set $\{x_1, x_2,\cdots,x_n\}$ of S, we have

$$\operatorname{conv}\{x_1,x_2,\cdots,x_n\}\subseteq\bigcup_{j=1}^{n}F(x_j). \tag{5}$$

The following result is taken from [28].

Lemma 12 (KKM-Theorem). *Suppose S is a nonempty subset of some topological vector space X and $F{:}S\longrightarrow X$ is a KKM-mapping such that for every $x\in S$, $F(x)$ is a closed subset of X. If there exists a point $x_0\in S$ such that $F(x_0)$ is compact, then $\bigcap_{x\in S}F(x)\neq\varnothing$.*

3. Main Results

Theorem 13. *Suppose X and Y are two topological vector spaces and $CL(X,Y)$ is the space of all continuous linear mappings from the space X to the space Y equipped with an admissible topology. Let K be a nonempty, compact, closed, and convex subset of X. Suppose $C\subseteq Y$ is a closed, convex, pointed cone with* int $C\neq\varnothing$. *Further, let $M,S,T: K\longrightarrow CL(X,Y)$ be set-valued lower semicontinuous mappings and $f,g,p: CL(X,Y)\longrightarrow CL(X,Y)$ be continuous mappings. Suppose the maps $\eta: K\times K\longrightarrow X$ and $h: K\times K\longrightarrow Y$ are affine mappings such that η is continuous in the second argument and h is continuous in the first argument, respectively, with $\eta(x,x)=h(x,x)=0$ for all $x\in K$. Then, the generalized nonlinear vector variational-like inequality problem (I) has a solution. That is, there exists $x_0\in K$ such that for each $y\in K$, there exist $u\in M(x_0)$, $v\in S(x_0)$, and $w\in T(x_0)$ such that*

$$((p(u)-(f(v)-g(w))))(\eta(y,x_0))-h(x_0,y)\notin -\text{int } C. \tag{6}$$

Proof. We define a set-valued map $F: K\longrightarrow K$ by

$$F(y)=\{x\in K \mid \exists\, u\in M(x), v\in S(x), w\in T(x) \text{ such that } \cdot((p(u)-(f(v)-g(w))))(\eta(y,x))-h(x,y)\notin -intC\}. \tag{7}$$

Clearly, $F(y)$ is nonempty as $y\in F(y)$. As $y\in K$, we have $\eta(y,y)=h(y,y)=0$. Thus, for each $u\in M(y)$, $v\in S(y)$, and $w\in T(x)$, we have $(p(u)-(f(v)-g(w)))(\eta(y,y))-h(y,y) = (p(u)-(f(v)-g(w)))(0)-0=\mathbf{0}$. Since C is a closed convex and pointed cone, thus $\mathbf{0}\notin -\text{int } C$.

The proof of the theorem is divided into two parts:

(i) F is a KKM-mapping on K:

Let $A=\{x_1,x_2,\cdots,x_n\}\subseteq K$ be any finite subset of K.

We show that $\operatorname{conv}\{x_1,x_2,\cdots,x_n\}\subseteq\bigcup_{i=1}^{n}F(x_i)$. Let, if possible, $x'\notin\bigcup_{i=1}^{n}F(x_i)$ for some $x'\in\operatorname{conv}\{x_1,x_2,\cdots,x_n\}$. Then, we have $x'=\sum_{i=1}^{n}\mu_i x_i$ for some $\mu_i\geq 0$ and $\sum_{i=1}^{n}\mu_i=1$. Also, as $x'\notin F(x_i)$, for all $u\in M(x')$, $v\in S(x')$, and $w\in T(x')$, we have $((p(u)-(f(v)-g(w))))(\eta(x_i,x'))-h(x',x_i)\in -\text{int } C$, for each $i=1,2,\cdots,n$. Since $-\text{int } C$ is convex and $\mu_i\geq 0$ with $\sum_{i=1}^{n}\mu_i=1$, therefore $\sum_{i=1}^{n}\mu_i[(p(u)-(f(v)-g(w)))(\eta(x_i,x'))-h(x',x_i)]\in -\text{int } C$. As $p(u)$, $f(v)$, and $g(w)$ belong to $CL(X,Y)$, they are linear. Therefore, we have $\sum_{i=1}^{n}\mu_i[(p(u)-(f(v)-g(w)))(\eta(x_i,x'))-h(x',x_i)]=(p(u)-(f(v)-g(w)))(\sum_{i=1}^{n}\mu_i\eta(x_i,x'))-\sum_{i=1}^{n}\mu_i h(x',x_i)$. Again, η and h are affine; hence, $\sum_{i=1}^{n}\mu_i(\eta(x_i,x'))-\sum_{i=1}^{n}\mu_i(h(x',x_i)) =\eta(\sum_{i=1}^{n}\mu_i x_i,\sum_{i=1}^{n}\mu_i x')-h(\sum_{i=1}^{n}\mu_i x',\sum_{i=1}^{n}\mu_i x_i)=\eta(x',x')-h(x',x')=\mathbf{0}$ as $\eta(x',x')=\mathbf{0}=h(x',x')$ by the given hypothesis. Therefore, $\sum_{i=1}^{n}\mu_i[p(u)-(f(v)-g(w))(\eta(x_i,x'))-h(x',x_i)]=\mathbf{0}$. Thus, we have $\mathbf{0}\in -\text{int } C$, where $\mathbf{0}$ is the zero vector in Y. Thus, $\mathbf{0}=-\mathbf{0}\in \text{int } C$, which is a contradiction. Therefore, we have $\operatorname{conv}\{x_1,x_2,\cdots,x_n\}\subseteq\bigcup_{i=1}^{n}F(x_i)$. Hence, F is a KKM-mapping on K.

(ii) $F(y)$ is closed for each $y\in K$:

Let $\{x_\alpha\}_{\alpha\in D}$ be a net in $F(y)$, converging to some z_0 in X. As K is closed, $z_0\in K$. We have to show that $z_0\in F(y)$, that is, there exist $u_0\in M(z_0)$, $v_0\in S(z_0)$, and $w_0\in T(z_0)$ such that $(p(u_0)-(f(v_0)-g(w_0)))(\eta(y,z_0))-h(z_0,y)\notin -\text{int } C$. Since $x_\alpha\in F(y)$, therefore there exist some $u_\alpha\in M(x_\alpha)$, $v_\alpha\in S(x_\alpha)$, and $w_\alpha\in T(x_\alpha)$ such that $(p(u_\alpha)-(f(v_\alpha)-g(w_\alpha)))(\eta(y,x_\alpha))-h(x_\alpha,y)\notin -\text{int } C$. Now, the maps M, S, and T are set-valued lower semicontinuous functions; therefore, for each $u_0\in M(z_0)$ and $\{x_\alpha\}$ converging to z_0, there exists a subnet $\{x_{\alpha_k}\}$ of $\{x_\alpha\}$ with $u_{\alpha_k}\in M(x_{\alpha_k})$ such that u_{α_k} converges to u_0, in view of Lemma 4. Now, $\{x_{\alpha_k}\}$ is a net in itself converging to z_0 and $w_0\in T(z_0)$; therefore, there exists a subnet $\{x_{\alpha_{k_l}}\}$ of $\{x_{\alpha_k}\}$ with $v_{\alpha_{k_l}}\in S(x_{\alpha_{k_l}})$. Similarly, we have a subnet $\{x_{\alpha_{k_{l_m}}}\}$ of $\{x_{\alpha_{k_l}}\}$ with $w_{\alpha_{k_{l_m}}}\in T(x_{\alpha_{k_{l_m}}})$ such that the subnets $\{u_{\alpha_k}\}$, $\{v_{\alpha_{k_l}}\}$, and $\{w_{\alpha_{k_{l_m}}}\}$ converge to u_0, v_0, and w_0, respectively, in view of Lemma 4. As these nets are subnets of u_{α_k}, v_{α_k}, and w_{α_k}, respectively, thus without loss of generality, we denote the subnets $\{u_{\alpha_k}\}$, $\{v_{\alpha_{k_l}}\}$, and $\{w_{\alpha_{k_{l_m}}}\}$ by $\{u_{\alpha_k}\}$, $\{v_{\alpha_k}\}$, and $\{w_{\alpha_k}\}$, respectively, which converge to u_0, v_0, and w_0, respectively.

Since the single-valued map p, f, and g are continuous, therefore we have $p(u_{\alpha_k})$, $f(v_{\alpha_k})$, and $g(w_{\alpha_k})$ converge to $p(u_0)$, $f(v_0)$, and $g(w_0)$, respectively.

By the given hypothesis, that is, η is continuous in the second argument, we have $\eta(y,x_{\alpha_k})$ converges to $\eta(y,z_0)$. Since the space $CL(X,Y)$ is given to be admissible, thus we have $(p(u_{\alpha_k})-(f(v_{\alpha_k})-g(w_{\alpha_k})))(\eta(y,x_{\alpha_k}))$ converges to $(p(u_0)-(f(v_0)-g(w_0)))(\eta(y,z_0))$, by Theorem 9. As the map h is continuous in the first component, therefore $h(x_{\alpha_k},y)$ converges to $h(z_0,y)$. Hence, $((p(u_{\alpha_k})-(f(v_{\alpha_k})-g(w_{\alpha_k})))(\eta(y,x_{\alpha_k}))-h(x_{\alpha_k},y))$ converges to $(p(u_0)-(f(v_0)-g(w_0)))(\eta(y,z_0))-h(z_0,y)$, in view of the fact that $CL(X,Y)$ is admissible.

Now, we will show that $(p(u_0)-(f(v_0)-g(w_0)))(\eta(y,z_0))-h(z_0,y)\notin -\text{int } C$.

Let, if possible, $(p(u_0)-(f(v_0)-g(w_0)))(\eta(y,z_0))-h(z_0,y)\in -\text{int } C$. Then, by the convergence of net, we have $(p(u_{\alpha_k})-(f(v_{\alpha_k})-g(w_{\alpha_k})))(\eta(y,x_{\alpha_k}))-h(x_{\alpha_k},y)\in -\text{int } C$ eventually, which leads to contradiction. Hence, $(p(u_0)-(f(v_0)-g(w_0)))(\eta(y,z_0))-h(z_0,y)\notin -\text{int } C$. Thus, we have $z_0\in F(y)$.

Now $F(y)$ is closed, and K is compact. This implies that $F(y)$ is a compact subset of K. Therefore, by KKM-Theorem, $\bigcap_{y\in K}F(y)\neq\varnothing$. Hence, there exists some $x_0\in K$ such that $x_0\in\bigcap_{y\in K}F(y)$. That is, for each $y\in K$, there exist $u_0\in M(x_0)$, $v_0\in S(x_0)$, and $w_0\in T(x_0)$ such that $(p(u_0)-(f(v_0)-g(w_0)))(\eta(y,x_0))-h(x_0,y)\notin -\text{int } C$, hence the result. ☐

In the above theorem, we have proved that, along with other conditions, lower semicontinuity of M, S, T ensures existence of solutions for GNVVLIP (I).

In the next theorem, we are providing another set of conditions for the existence of solutions for these class of problems.

Theorem 14. *Suppose X and Y are two topological vector spaces and $CL(X,Y)$ is the space of all continuous linear mappings from the space X to the space Y equipped with an admissible topology. Let K be a nonempty, closed, compact, and convex subset of X. Suppose $C\subseteq Y$ is a closed, convex, pointed cone with* int $C\neq\varnothing$. *Further, let $M,S,T:K\longrightarrow CL(X,Y)$ be set-valued upper semicontinuous functions with nonempty compact values, that is, $M(x)$, $S(x)$, and $T(x)$ are compact for every $x\in K$. Let $f,g,p:CL(X,Y)\longrightarrow CL(X,Y)$ be continuous mappings. Suppose the maps $\eta:K\times K\longrightarrow X$ and $h:K\times K\longrightarrow Y$ are affine mappings such that η is continuous in the second argument and h is continuous in the first argument, respectively, with $\eta(x,x)=h(x,x)=0$ for all $x\in K$. Then, there exists a solution to the generalized nonlinear vector variational inequality problem. That is, there exists $x_0\in K$ such that for each $y\in K$, there exist $u\in M(x_0)$, $v\in S(x_0)$, and $w\in T(x_0)$ such that*

$$((p(u)-(f(v)-g(w))))(\eta(y,x_0))-h(x_0,y)\notin -\text{int } C. \quad (8)$$

Proof. Consider a set-valued map $F:K\longrightarrow K$ defined as

$$F(y)=\{x\in K\,|\,\exists\, u\in M(x), v\in S(x), w\in T(x) \text{ such that } \cdot((p(u)-(f(v)-g(w))))(\eta(y,x))-h(x,y)\notin -intC\}. \quad (9)$$

The proof of the theorem is divided into two parts:

(i) F is a KKM-mapping on K;

(ii) $F(y)$ is closed for each $y\in K$.

Proof of part (i) is similar to that of Theorem 13. Therefore, we are providing the proof of part (ii) only.

Let $\{x_\alpha\}_{\alpha\in D}$ be a net in $F(y)$, converging to some $z_0\in X$. As K is closed, $z_0\in K$. We have to show that $z_0\in F(y)$, that is, there exist $u_0\in M(z_0)$, $v_0\in S(z_0)$, and $w_0\in T(z_0)$ such that $(p(u_0)-(f(v_0)-g(w_0)))(\eta(y,z_0))-h(z_0,y)\notin -\text{int } C$. Since $x_\alpha\in F(y)$, therefore there exist some $u_\alpha\in M(x_\alpha)$, $w_\alpha\in T(x_\alpha)$, and $v_\alpha\in S(x_\alpha)$ such that $(p(u_\alpha)-(f(v_\alpha)-g(w_\alpha)))(\eta(y,x_\alpha))-h(x_\alpha,y)\notin -\text{int } C$. Now, K is a compact subset of X, and M, S, and T are set-valued upper semicontinuous functions such that $M(x)$, $S(x)$, and $T(x)$ are compact. Therefore, $M(K)$, $S(K)$, and $T(K)$ are also compact by Lemma 6. As $\{u_\alpha\}$, $\{v_\alpha\}$, and $\{w_\alpha\}$ are nets in $M(K)$, $S(K)$, and $T(K)$, respectively, there exist subnets $\{u_{\alpha_k}\}_{\alpha_k\in D_1}$, $\{v_{\alpha_l}\}_{\alpha_l\in D_2}$, and $\{w_{\alpha_m}\}_{\alpha_m\in D_3}$ such that $\{u_{\alpha_k}\}$, $\{v_{\alpha_l}\}$, and $\{w_{\alpha_m}\}$ converge to some $u_0\in M(K)$, $v_0\in S(K)$, and $w_0\in T(K)$, respectively.

Now, we construct a directed set $D_4\subset D$, defined in the following way:

By the order property of the directed set D, for each triplet $\alpha_k,\alpha_l,\alpha_m\in D$, there exists some $\alpha_\delta\in D$ such that $\alpha_\delta\geq\alpha_k$, $\alpha_\delta\geq\alpha_l$, and $\alpha_\delta\geq\alpha_m$. We denote the collection of such α_δ's by D_4. It can be easily verified that D_4 is a directed set under the induced ordering of D.

Thus, we have subnets $\{u_\delta\}_{\delta\in D_4}$, $\{v_\delta\}_{\delta\in D_4}$, and $\{w_\delta\}_{\delta\in D_4}$ of $\{u_{\alpha_k}\}$, $\{v_{\alpha_l}\}$, and $\{w_{\alpha_m}\}$, respectively, such that $\{u_\delta\}$, $\{v_\delta\}$, and $\{w_\delta\}$ converge to $u_0\in M(K)$, $v_0\in S(K)$, and $w_0\in T(K)$, respectively.

Then, proceeding as in Theorem 13, we have $z_0\in F(y)$. Thus, $F(y)$ is closed, and K is compact. This implies that $F(y)$ is a compact subset of K. Therefore, by KKM-Theorem, there exists some $x_0\in F(y)$ for each $y\in K$. That is, for each $y\in K$, there exist $u_0\in M(x_0)$, $v_0\in S(x_0)$, and $w_0\in T(x_0)$ such that $(p(u_0)-(f(v_0)-g(w_0)))(\eta(y,x_0))-h(x_0,y)\notin -\text{int } C$, hence the result. ☐

In the next theorem, we investigate the properties of solution sets of GNVVLIP (I).

Theorem 15. *Let $S\subseteq K$ be the set of all solutions of a generalized nonlinear vector variational-like inequality problem as obtained in Theorem 13 (respectively, Theorem 14). Then, S is closed and compact in K.*

Proof. Suppose $S\subseteq K$ is the solution set of the generalized nonlinear vector variational-like inequality problem. Then, by Theorem 13, we have $S=\bigcap_{y\in K}F(y)$, where

$$F(y)=\{x\in K\,|\,\exists\, u\in M(x), v\in S(x), w\in T(x) \text{ such that } \cdot((p(u)-(f(v)-g(w))))(\eta(y,x))-h(x,y)\notin -intC\}. \quad (10)$$

It has been proved in Theorem 13 that each $F(y)$ is closed. Therefore, S is a closed set and hence a closed subset of K. Since K is compact, S is compact as well. ☐

In the next set of theorems, we consider the other variant of the generalized nonlinear vector variational-like inequality problem and provide the conditions for the solution.

Theorem 16. *Suppose X and Y are two topological vector spaces and $CL(X, Y)$ is the space of all continuous linear mappings from the space X to the space Y equipped with an admissible topology. Let K be a nonempty, closed, compact, and convex subset of X. Suppose $C : K \longrightarrow Y$ is a set-valued map such that for every $x \in K$, $C(x)$ is a proper closed, convex, pointed cone with $\operatorname{int} C(x) \neq \emptyset$. Further, suppose that $G : K \longrightarrow Y$ is also a set-valued map defined by $G(x) = Y \setminus (-\operatorname{int} C(x))$ such that the graph of G, $\mathscr{G}(G)$, is a closed set in $X \times Y$. Further, let $M, S, T : K \longrightarrow CL(X, Y)$ be set-valued lower semicontinuous mappings and $f, g, p : CL(X, Y) \longrightarrow CL(X, Y)$ be continuous mappings. Suppose the maps $\eta : K \times K \longrightarrow X$ and $h : K \times K \longrightarrow Y$ are affine mappings such that η is continuous in the second argument and h is continuous in the first argument, respectively, with $\eta(x, x) = h(x, x) = 0$ for all $x \in K$. Then, the generalized nonlinear vector variational-like inequality problem (II) has a solution. That is, there exists $x_0 \in K$ such that for each $y \in K$, $\exists u \in M(x_0)$, $v \in S(x_0)$, and $w \in T(x_0)$ such that*

$$((p(u) - (f(v) - g(w))))(\eta(y, x_0)) - h(x_0, y) \notin -\operatorname{int} C(x_0). \tag{11}$$

Proof. Consider a set-valued map $F : K \longrightarrow K$ defined as

$$F(y) = \{x \in K \mid \exists u \in M(x), v \in S(x), w \in T(x) \text{ such that } \cdot ((p(u) - (f(v) - g(w))))(\eta(y, x)) - h(x, y) \notin -int C(x_0)\}. \tag{12}$$

Likewise in Theorem 13, the proof of this theorem is also divided into two parts:

(i) F is a KKM-mapping on K;

(ii) $F(y)$ is closed for each $y \in K$.

Proof of part (i) is similar to that of Theorem 13. Therefore, we are providing the proof of part (ii) only.

As in Theorem 13, it follows that the net $\{(p(u_{\alpha_k}) - (f(v_{\alpha_k}) - g(w_{\alpha_k})))(\eta(y, x_{\alpha_k})) - h(x_{\alpha_k}, y)\}$ converges to $(p(u_0) - (f(v_0) - g(w_0)))(\eta(y, z_0)) - h(z_0, y)$, in view of admissibility of $CL(X, Y)$, lower semicontinuity of T, and the other given conditions of the hypothesis.

Now, we will show that $(p(u_0) - (f(v_0) - g(w_0)))(\eta(y, z_0)) - h(z_0, y) \notin -\operatorname{int} C(z_0)$. As in Theorem 13, we have the net $\{x_\alpha, (p(u_{\alpha_k}) - (f(v_{\alpha_k}) - g(w_{\alpha_k})))(\eta(y, x_{\alpha_k})) - h(x_{\alpha_k}, y)\}$ is convergent and converges to $\{z_0, (p(u_0) - (f(v_0) - g(w_0)))(\eta(y, z_0)) - h(z_0, y)\}$. Also, the net is contained in $\mathscr{G}(G)$, and the graph $\mathscr{G}(G)$ is closed; therefore, $(z_0, (p(u_0) - (f(v_0) - g(w_0)))(\eta(y, z_0)) - h(z_0, y)) \in \mathscr{G}(G)$. Hence, we have $p(u_0) - (f(v_0) - g(w_0)))(\eta(y, z_0)) - h(z_0, y) \notin -\operatorname{int} C(z_0)$. Thus, we have $z_0 \in F(y)$. Therefore, $F(y)$ is closed.

Now $F(y)$ is closed, and K is compact. This implies that $F(y)$ is a compact subset of K. Therefore, by KKM-Theorem, $\bigcap_{y \in K} F(y) \neq \emptyset$. Hence, there exists some $x_0 \in K$ such that $x_0 \in \bigcap_{y \in K} F(y)$. That is, for each $y \in K$, there exist $u_0 \in M(x_0)$, $v_0 \in S(x_0)$, and $w_0 \in T(x_0)$ such that $(p(u_0) - (f(v_0) - g(w_0)))(\eta(y, x_0)) - h(x_0, y) \notin -\operatorname{int} C(x_0)$, hence the result. ☐

Another solution of the generalized nonlinear vector variational-like inequality problem (II) is provided below.

Theorem 17. *Suppose X and Y are two topological vector spaces and $CL(X, Y)$ is the space of all continuous linear mappings from the space X to the space Y equipped with an admissible topology. Let K be a nonempty, closed, compact, and convex subset of X. Suppose $C : K \longrightarrow Y$ is a set-valued map such that for every $x \in K$, $C(x)$ is a proper, closed, convex, pointed cone with $\operatorname{int} C(x) \neq \emptyset$. Further, suppose that $G : K \longrightarrow Y$ is also a set-valued map defined by $G(x) = Y \setminus (-\operatorname{int} C(x))$ such that graph of G, $\mathscr{G}(G)$, is a closed set in $X \times Y$. Further, let $M, S, T : K \longrightarrow CL(X, Y)$ be set-valued upper semicontinuous functions with nonempty compact values, that is, $M(x)$, $S(x)$, and $T(x)$ are compact for every $x \in K$ and let $f, g, p : CL(X, Y) \longrightarrow CL(X, Y)$ be continuous mappings. Suppose the maps $\eta : K \times K \longrightarrow X$ and $h : K \times K \longrightarrow Y$ are affine mappings such that η is continuous in the second argument and h is continuous in the first argument with $\eta(x, x) = h(x, x) = 0$, respectively, for all $x \in K$. Then, the generalized nonlinear vector variational-like inequality problem (II) has a solution. That is, there exists $x_0 \in K$ such that for each $y \in K$, $\exists u \in M(x_0)$, $v \in S(x_0)$, and $w \in T(x_0)$ such that*

$$((p(u) - (f(v) - g(w))))(\eta(y, x_0)) - h(x_0, y) \notin -\operatorname{int} C(x_0). \tag{13}$$

Proof. The result can be proved on the similar lines as that of Theorem 14 and Theorem 16. ☐

We can draw the following conclusions from the results obtained so far:

(i) Suppose $h : K \times K \longrightarrow Y$ is defined as the constant map with $h(x, y) = 0$ for all $(x, y) \in K \times K$ and $\eta : K \times K \longrightarrow Y$ is defined as $\eta(x, y) = x - y$, and M and S are taken as zero functions. If $f, p : CL(X, Y) \longrightarrow CL(X, Y)$ are also defined as zero functions, that if $f(x) = p(x) = 0$ for all $x \in CL(X, Y)$ and $g : CL(X, Y) \longrightarrow CL(X, Y)$ is taken as an identity map, that is, $g(x) = x$ for all $x \in CL(X, Y)$. Then, Theorem 16 ensures solution to the generalized vector variational inequality problem discussed in [29]. Also, we have

(a) Theorem 16 reduces to Theorem 3.6 of [29];

(b) Theorem 17 reduces to Theorem 3.1 of [29].

Further, if T is single-valued, then Theorem 13 reduces to Theorem 3.1 and Theorem 17 reduces to Theorem 3.2 of [30], respectively.

4. Conclusion

In this paper, we have provided solutions to two variants of the generalized vector variational-like inequality problem. Our approach and the result obtained here differ significantly from those of the existing literature. To be precise,

(i) The spaces X and Y considered in this paper are topological vector spaces. In [15], the space Y is taken to be $\mathbb{R}$, whereas [13] deals with real Hilbert spaces;

(ii) We have used conditions of upper semicontinuity as well as lower semicontinuity on the set-valued mappings to obtain our results. In [15], the concepts of transfer closed and intersectionally closed are used along with an assumption milder than C-pseudomonotonicity on the set-valued mappings. On the other hand, relaxed monotonicity and relaxed Lipschitz's continuity are used in [13] and generalized C-quasi-convexity is used in [17] for the set-valued mappings.

The approach adopted in our paper is topological and varies significantly from the rest literature. Net theory is extensively used in all the main results. We have used admissibility of the function space topology to obtain our results. The authors are not aware of any such results in the literature of variational inequality which are proved adopting similar techniques.

Data Availability

No data is used.

Conflicts of Interest

The authors declare there is no conflict of interest.

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Research Article

On the JK Iterative Process in Banach Spaces

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In the recent progress, different iterative procedures have been constructed in order to find the fixed point for a given self-map in an effective way. Among the other things, an effective iterative procedure called the JK iterative scheme was recently constructed and its strong and weak convergence was established for the class of Suzuki mappings in the setting of Banach spaces. The first purpose of this research is to obtain the strong and weak convergence of this scheme in the wider setting of generalized α-nonexpansive mappings. Secondly, by constructing an example of generalized α-nonexpansive maps which is not a Suzuki map, we show that the JK iterative scheme converges faster as compared the other iterative schemes. The presented results of this paper properly extend and improve the corresponding results of the literature.

1. Introduction

A mapping $\mathcal{S}$ on a subset U of a Banach space is called contraction provided that for all $z, z' \in U$ follows that

$$\left\| \mathcal{S}z - \mathcal{S}z' \right\| \leq \delta \left\| z - z' \right\|, \tag{1}$$

where $\delta \in [0, 1)$ is fixed. A point v_0 is called a fixed point for $\mathcal{S}$ if $v_0 = \mathcal{S}v_0$. Normally, we denote the set of all fixed points of $\mathcal{S}$ by $F_{\mathcal{S}}$, that is, $F_{\mathcal{S}} = \{v_0 \in U : \mathcal{S}v_0 = v_0\}$. The Banach–Caccioppoli fixed point theorem (BCFPT) [1, 2] provides the existence of a unique fixed point for every self-contraction of a complete metric space.

We say that a self-map $\mathcal{S} : U \longrightarrow U$ is nonexpansive on the set U provided that

$$\left\| \mathcal{S}z - \mathcal{S}z' \right\| \leq \left\| z - z' \right\|, \quad \text{for all } z, z' \in U. \tag{2}$$

We may observe that every contraction of a subset U of a Banach space is nonexpansive but the converse may not hold in general. Unlike contractions, every self-nonexpansive mapping of a complete metric space does not admit a fixed point. After many years of BCFPT, Browder [3], Gohde [4] and Kirk [5] independently obtained that a self-nonexpansive mapping of a closed bounded convex subset of a uniformly convex Banach space (UCBS, for short) always has a fixed point.

In 2008, Suzuki [6] provided a new type of generalization of nonexpansive mappings and proved some related fixed point results for this class of mappings in Banach spaces. Notice that a self-map $\mathcal{S} : U \longrightarrow U$ is mapping with the (C) property (also called Suzuki mapping) if any $z, z' \in U$ follows that

$$\frac{1}{2} \| z - \mathcal{S}z \| \leq \left\| z - z' \right\| \Rightarrow \left\| \mathcal{S}z - \mathcal{S}z' \right\| \leq \left\| z - z' \right\|. \tag{3}$$

In 2011, Aoyama and Kohsaka [7] provided the idea of α-nonexpansive mappings. A self-map $\mathcal{S} : U \longrightarrow U$ is called α-nonexpansive if any $z, z' \in U$ follows that

$$\begin{aligned} \left\|\mathcal{S}z - \mathcal{S}z'\right\|^2 \le \alpha\left\|z - \mathcal{S}z'\right\|^2 + \alpha\left\|z' - \mathcal{S}z\right\|^2 \\ + (1-2\alpha)\left\|z - z'\right\|^2, \end{aligned} \tag{4}$$

where $\alpha \in [0, 1)$.

In 2017, Pant and Shukla [8] defined a very general class of nonexpansive mappings which properly contains the class of Suzuki mappings and partially extends the class of α-nonexpansive mappings. A self-map $\mathcal{S} : U \longrightarrow U$ is called generalized α-nonexpansive if any $z, z' \in U$ follows that

$$\begin{aligned} \frac{1}{2}\|z - \mathcal{S}z\| \le \left\|z - z'\right\| \Rightarrow \left\|\mathcal{S}z - \mathcal{S}z'\right\| \\ \le \alpha\left\|z - \mathcal{S}z'\right\| + \alpha\left\|z' - \mathcal{S}z\right\| \\ + (1-2\alpha)\left\|z - z'\right\|, \end{aligned} \tag{5}$$

where $\alpha \in [0, 1)$.

Fixed point approximation for nonexpansive mappings under a suitable iterative method is a very active field of research and provides many interesting and important applications in applied sciences (cf. [9–12] and others). Finding the fixed points for nonexpansive and generalized nonexpansive under Picard iteration is not possible in general. A simple situation of such a case which is the rotation of the unit disk about the origin in a plane is a best example of a nonexpansive mapping which has a unique fixed point but Picard iteration does not converge to this point. In order to find fixed points of nonexpansive and hence generalized nonexpansive mappings and secondly to obtain relatively high accuracy, some authors introduced different types of iterative procedures (cf. Mann [13], Ishikawa [14], Noor [15], Agarwal et al. [16], Abbas and Nazir [17], Thakur et al. [18] and references therein). Suppose that U is a closed nonempty convex subset of a given Banach space, and assume further that ξ_k, $\eta_k, \mu_k \in (0, 1)$, $k \in \mathbb{N}$, and $\mathcal{S}$ is a self-map of U.

The Mann [13] iteration process is stated as follows:

$$\begin{aligned} p_1 &= p \in U, \\ p_{k+1} &= (1-\xi_k)p_k + \xi_k\mathcal{S}p_k. \end{aligned} \tag{6}$$

The Ishikawa [14] iterative process may be viewed as a two-step Mann iteration, which is given by

$$\begin{aligned} p_1 &= p \in U, \\ q_k &= (1-\eta_k)p_k + \eta_k\mathcal{S}p_k, \\ p_{k+1} &= (1-\xi_k)p_k + \xi_k\mathcal{S}q_k. \end{aligned} \tag{7}$$

In 2000, Noor [15] suggested a three-step iterative process which is more general than the Mann and Ishikawa iteration processes as follows:

$$\begin{aligned} p_1 &= p \in U, \\ r_k &= (1-\mu_k)p_k + \mu_k\mathcal{S}p_k, \\ q_k &= (1-\eta_k)p_k + \eta_k\mathcal{S}r_k, \\ p_{k+1} &= (1-\xi_k)p_k + \xi_k\mathcal{S}q_k. \end{aligned} \tag{8}$$

In 2007, Agarwal et al. [16] suggested a new iteration process, which converges faster than the Mann iteration for contraction mappings in Banach spaces:

$$\begin{aligned} p_1 &= p \in U, \\ q_k &= (1-\eta_k)p_k + \eta_k\mathcal{S}p_k, \\ p_{k+1} &= (1-\xi_k)\mathcal{S}p_k + \xi_k\mathcal{S}q_k. \end{aligned} \tag{9}$$

In 2014, Abbas and Nazir [17] proposed a new three-step iteration which converges faster than all of the Picard, Mann, Ishikawa, and Agarwal iterative processes for nonexpansive mappings, as follows:

$$\begin{aligned} p_1 &= p \in U, \\ r_k &= (1-\mu_k)p_k + \mu_k\mathcal{S}p_k, \\ q_k &= (1-\eta_k)\mathcal{S}p_k + \eta_k\mathcal{S}r_k, \\ p_{k+1} &= (1-\xi_k)\mathcal{S}q_k + \xi_k\mathcal{S}r_k. \end{aligned} \tag{10}$$

In 2016, Thakur et al. [18] suggested the following iteration process, which converges faster than all of the above iterative processes for Suzuki mappings:

$$\begin{aligned} p_1 &= p \in U, \\ r_k &= (1-\eta_k)p_k + \eta_k\mathcal{S}p_k, \\ q_k &= \mathcal{S}((1-\xi_k)p_k + \xi_k r_k), \\ p_{k+1} &= \mathcal{S}q_k. \end{aligned} \tag{11}$$

Very recently, Ahmad et al. [19] introduced a new iterative process named JK iteration, as follows:

$$\begin{aligned} p_1 &= p \in U, \\ r_k &= (1-\eta_k)p_k + \eta_k\mathcal{S}p_k, \\ q_k &= \mathcal{S}r_k, \\ p_{k+1} &= \mathcal{S}((1-\xi_k)\mathcal{S}r_k + \xi_k\mathcal{S}q_k). \end{aligned} \tag{12}$$

They observed that JK iteration (12) can be used for fixed points of Suzuki mappings. Moreover, they proved by providing a novel example of Suzuki mappings that the JK iteration process converges faster than all of the above iterative processes including the leading Thakur iteration (11). In this paper, firstly, we improve and extend the main results of Ahmad et al. [19] from the context of Suzuki mappings to the more general framework of generalized α-nonexpansive mappings. We then provide a novel example of generalized α-nonexpansive mappings and show that its JK iterative process is better than the mentioned iterative processes. Our results can be used for finding the solutions of split feasibility problems, solutions of differential and integral equations provided that the operator is generalized α-nonexpansive.

2. Preliminaries

We now provide some definitions.

Definition 1 [20]. A Banach space W is said to be endowed with Opial's property if every weakly convergence sequence $\{p_k\} \subseteq W$ having a weak limit $v_0 \in W$ follows that

$$\limsup_{k\longrightarrow\infty} \|p_k - v_0\| < \limsup_{k\longrightarrow\infty} \|p_k - u_0\|, \quad \text{for every choice of } u_0 \neq v_0. \tag{13}$$

Definition 2 [21]. A self-mapping $\mathcal{S}$ of a subset U of a Banach space is said to be endowed with condition I if one has a nondecreasing function $\eta : [0,\infty) \longrightarrow [0,\infty)$ having $\eta(0) = 0$ and $\eta(v) > 0$ for every $v \in (0,\infty)$ and $\|z - \mathcal{S}z\| \geq \eta(d(z, F_{\mathcal{S}}))$ for every $z \in U$; here, $d(z, F_{\mathcal{S}})$ stands for the distance of z from $F_{\mathcal{S}}$.

Definition 3. Suppose that W is any given Banach space and $\{p_k\} \subseteq W$ is bounded. Assume that $\emptyset \neq U \subseteq W$ is closed and convex. Then, the asymptotic radius of the sequence $\{p_k\}$ relative to the set U is given by $r(U, \{p_k\}) = \inf \{\limsup_{k\longrightarrow\infty} \|p_k - z\| : z \in U\}$. Moreover, the asymptotic center of $\{p_k\}$ with respect to U is given by $A(U, p_k\}) = \{z \in U : \limsup_{k\longrightarrow\infty} \|p_k - z\| = r(U, p_k)\}$.

Remark 4. The most well-known fact about the set $A(U, \{p_k\})$ is that it is always singleton whenever W is UCBS [22]. The fact that the set $A(U, \{p_k\})$ is convex and nonempty also known in the case when U is weakly compact and convex [23, 24].

Now, we combine some elementary properties of generalized α-nonexpansive mappings, which can be found in [8].

Proposition 5. *Suppose that U is any nonempty subset of a Banach space W and $\mathcal{S} : U \longrightarrow U$.*

(a) If $\mathcal{S}$ is Suzuki mapping, then, $\mathcal{S}$ is generalized α-nonexpansive

(b) If $\mathcal{S}$ is generalized α-nonexpansive having nonempty fixed point set, then, for any $v_0 \in F_{\mathcal{S}}$, $\|\mathcal{S}z - \mathcal{S}v_0\| \leq \|z - v_0\|$ for all $z \in U$

(c) If $\mathcal{S}$ is generalized α-nonexpansive, then, the set $F_{\mathcal{S}}$ is closed in U. Also, $F_{\mathcal{S}}$ is convex in the case when W is strictly convex and U is convex

(d) If $\mathcal{S}$ is a generalized α-nonexpansive mapping, then, for every choice of $z, z' \in U$, the following holds:

$$\|z - \mathcal{S}z'\| \leq \frac{(3+\alpha)}{(1-\alpha)} \|z - \mathcal{S}z\| + \|z - z'\| \tag{14}$$

(e) Suppose that $\mathcal{S}$ is generalized α-nonexpansive and W is endowed with the Opial property. If $\{p_k\}$ is weakly convergent to l_0 and $\lim_{k\longrightarrow\infty} \|p_k - \mathcal{S}p_k\| = 0$, it follows that $l_0 \in F_{\mathcal{S}}$

The following useful lemma can be found in [25].

Lemma 6. *Suppose that $0 < i \leq y_k \leq j < 1$ for each $k \in \mathbb{N}$ and $\lambda \geq 0$. If $\{p_k\}$ and $\{q_k\}$are any sequences in a UCBS W endowed with $\limsup_{k\longrightarrow\infty} \|p_k\| \leq \lambda$, $\limsup_{k\longrightarrow\infty} \|q_k\| \leq \lambda$, and $\lim_{k\longrightarrow\infty} \|y_k p_k + (1 - y_k) q_k\| = \lambda$, then $\lim_{k\longrightarrow\infty} \|p_k - q_k\| = 0$.*

3. Main Results

The aim of this section is at giving some important weak and strong convergence of JK (12) for the class of generalized α-nonexpansive mappings. We start the section with a key lemma.

Lemma 7. *Suppose that W is UCBS, $\emptyset \neq U \subseteq W$ is closed convex, and $\mathcal{S} : U \longrightarrow U$ is a generalized α-nonexpansive having $F_{\mathcal{S}} \neq \emptyset$. If $\{p_k\}$ is a JK iteration sequence as provided in (12). Then, $\lim_{k\longrightarrow\infty} \|p_k - v_0\|$ exists for each $v_0 \in F_{\mathcal{S}}$.*

Proof. If we choose $v_0 \in F_{\mathcal{S}}$, then, using (12) along with Proposition 5 (b), we have

$$\begin{aligned} \|r_k - v_0\| &= \|(1-\eta_k) p_k + \eta_k \mathcal{S} p_k - v_0\| \\ &\leq (1-\eta_k)\|p_k - v_0\| + \eta_k \|\mathcal{S}p_k - v_0\| \\ &\leq (1-\eta_k)\|p_k - v_0\| + \eta_k \|p_k - v_0\| \\ &\leq \|p_k - v_0\|. \end{aligned} \tag{15}$$

Hence,

$$\begin{aligned} \|p_{k+1} - v_0\| &= \|\mathcal{S}((1-\xi_k)\mathcal{S}r_k + \xi_k \mathcal{S}q_k) - v_0\| \\ &\leq \|(1-\xi_k)\mathcal{S}r_k + \xi_k \mathcal{S}q_k - v_0\| \\ &\leq (1-\xi_k)\|\mathcal{S}r_k - v_0\| + \xi_k \|\mathcal{S}q_k - v_0\| \\ &\leq (1-\xi_k)\|r_k - v_0\| + \xi_k \|q_k - v_0\| \\ &= (1-\xi_k)\|r_k - v_0\| + \xi_k \|\mathcal{S}r_k - v_0\| \\ &\leq (1-\xi_k)\|r_k - v_0\| + \xi_k \|r_k - v_0\| \\ &= \|r_k - v_0\| \leq \|p_k - v_0\|. \end{aligned} \tag{16}$$

Consequently, we conclude that $\{\|p_k - v_0\|\}$ is nonincreasing and bounded; accordingly, we must have that $\lim_{k\longrightarrow\infty} \|p_k - v_0\|$ exists for every element v_0 of $F_{\mathcal{S}}$. □

Theorem 8. *Suppose that W is UCBS, $\emptyset \neq U \subseteq W$ is closed convex, and $\mathcal{S} : U \longrightarrow U$ is a generalized α-nonexpansive.*

If $\{p_k\}$ is a JK iteration sequence as provided in (12). Then, $F_{\mathcal{S}} \neq \emptyset$ if and only if $\{p_k\}$ is bounded and $\lim_{k\longrightarrow\infty}\|p_k - \mathcal{S}p_k\| = 0$.

Proof. Firstly, we may take $F_{\mathcal{S}} \neq \emptyset$. According to Lemma 7, one concludes that $\{p_k\}$ is bounded and $\lim_{k\longrightarrow\infty}\|p_k - v_0\|$ exists for every element v_0 of $F_{\mathcal{S}}$. We now suppose

$$\lim_{k\longrightarrow\infty}\|p_k - v_0\| = \lambda. \tag{17}$$

We need to obtain that $\lim_{k\longrightarrow\infty}\|p_k - \mathcal{S}p_k\| = 0$. Then, using Lemma 7, we have

$$\begin{aligned}\|r_k - v_0\| \le \|p_k - v_0\| \Rightarrow \limsup_{k\longrightarrow\infty}\|r_k - v_0\| \\ \le \limsup_{k\longrightarrow\infty}\|p_k - v_0\| = \lambda.\end{aligned} \tag{18}$$

Since $v_0 \in F_{\mathcal{S}}$, so by Proposition 5 (b), we infer

$$\begin{aligned}\|\mathcal{S}p_k - v_0\| \le \|p_k - v_0\| \Rightarrow \limsup_{k\longrightarrow\infty}\|\mathcal{S}p_k - v_0\| \\ \le \limsup_{k\longrightarrow\infty}\|p_k - v_0\| = \lambda.\end{aligned} \tag{19}$$

Now from (16), we get

$$\|p_{k+1} - v_0\| \le \|r_k - v_0\|. \tag{20}$$

Using this together with (17), we obtain

$$\lambda \le \liminf_{k\longrightarrow\infty}\|r_k - v_0\|. \tag{21}$$

From (18) and (21), we deduce

$$\lambda = \lim_{k\longrightarrow\infty}\|r_k - v_0\|. \tag{22}$$

Using (22), we get

$$\begin{aligned}\lambda &= \lim_{k\longrightarrow\infty}\|r_k - v_0\| = \lim_{k\longrightarrow\infty}\|(1-\eta_k)p_k + \eta_k\mathcal{S}p_k - v_0\| \\ &= \lim_{k\longrightarrow\infty}\|(1-\eta_k)(p_k - v_0) + \eta_k(\mathcal{S}p_k - v_0)\|.\end{aligned} \tag{23}$$

Thus,

$$\lambda = \lim_{k\longrightarrow\infty}\|(1-\eta_k)(p_k - v_0) + \eta_k(\mathcal{S}p_k - v_0)\|. \tag{24}$$

Using (17), (19), and (24) and keeping Lemma 6 in mind, one concludes that

$$\lim_{k\longrightarrow\infty}\|p_k - \mathcal{S}p_k\| = 0. \tag{25}$$

Conversely, we may suppose that $\{p_k\}$ is bounded in U such that $\lim_{k\longrightarrow\infty}\|p_k - \mathcal{S}p_k\| = 0$. The aim is to prove that $F_{\mathcal{S}} \neq \emptyset$. If we take any $v_0 \in A(U, \{p_k\})$, then, using Proposition 5 (d), it follows that

$$\begin{aligned}A(\mathcal{S}v_0, \{p_k\}) &= \limsup_{k\longrightarrow\infty}\|p_k - \mathcal{S}v_0\| \le \frac{(3+\alpha)}{(1-\alpha)}\limsup_{k\longrightarrow\infty}\|p_k - \mathcal{S}p_k\| \\ &\quad + \limsup_{k\longrightarrow\infty}\|p_k - v_0\| = \limsup_{k\longrightarrow\infty}\|p_k - v_0\| \\ &= A(v_0, \{p_k\}).\end{aligned} \tag{26}$$

It follows that $\mathcal{S}v_0 \in A(U, \{p_k\}$. Since W is UCBS, $A(U, \{p_k\}$ contains only one element, that is, we must have $\mathcal{S}v_0 = v_0$. Hence, $v_0 \in F_{\mathcal{S}}$, that is, the fixed point $F_{\mathcal{S}}$ is nonempty. □

Now, we are in the position to prove our weak convergence result.

Theorem 9. *Suppose that W is UCBS, $\emptyset \neq U \subseteq W$ is closed convex, and $\mathcal{S}: U \longrightarrow U$ is a generalized α-nonexpansive having $F_{\mathcal{S}} \neq \emptyset$. If $\{p_k\}$ is a JK iteration sequence as provided in (12) and W has Opial's property, then, $\{p_k\}$ converges weakly to a point of $F_{\mathcal{S}}$.*

Proof. By Theorem 8, $\{p_k\}$ is bounded. The uniform convexity of W follows reflexivity of W, that is, $\{p_k\}$ has a weakly convergent subsequence $\{p_{k_t}\}$ with a weak limit, namely, l_0. According to Theorem 8, $\lim_{m\longrightarrow\infty}\|p_{k_m} - \mathcal{S}p_{k_m}\| = 0$. Hence, using Proposition 5 (e), we get $l_0 \in F_{\mathcal{S}}$. We claim that l_0 is the weak limit of $\{p_k\}$. We may suppose on the contrary that l_0 is not the weak limit of $\{p_k\}$, that is, $\{p_k\}$ has another weakly convergent subsequence $\{p_{k_s}\}$ with a weak limit, namely, $l_0' \neq l_0$. According to Theorem 8, $\lim_{s\longrightarrow\infty}\|p_{k_s} - \mathcal{S}p_{k_s}\| = 0$. Hence, using Proposition 5 (e), we get $l_0' \in F_{\mathcal{S}}$. Now using Lemma 7 and Opial's property, we have

$$\begin{aligned}\limsup_{k\longrightarrow\infty}\|p_k - l_0\| &= \limsup_{m\longrightarrow\infty}\left\|p_{k_m} - l_0\right\| < \limsup_{m\longrightarrow\infty}\left\|p_{k_m} - l_0'\right\| \\ &= \limsup_{k\longrightarrow\infty}\|p_k - l_0'\| = \limsup_{s\longrightarrow\infty}\left\|p_{k_s} - l_0'\right\| \\ &< \limsup_{s\longrightarrow\infty}\left\|p_{k_s} - l_0\right\| = \limsup_{k\longrightarrow\infty}\|p_k - l_0\|.\end{aligned} \tag{27}$$

Consequently, we obtained $\limsup_{k\longrightarrow\infty}\|p_k - l_0\| < \limsup_{k\longrightarrow\infty}\|p_k - l_0\|$, which suggests a contradiction. Therefore, we conclude that l_0 is the weak limit of the sequence $\{p_k\}$. □

Now, we prove the following strong convergence result.

Theorem 10. *Suppose that W is UCBS, $\emptyset \neq U \subseteq W$ is compact convex, and $\mathcal{S}: U \longrightarrow U$ is a generalized α-nonexpansive* *having $F_{\mathcal{S}} \neq \varnothing$. If $\{p_k\}$ is a JK iteration sequence as provided in (12), then, $\{p_k\}$ converges strongly to a point of $F_{\mathcal{S}}$.*

Proof. Since $\{p_k\} \subseteq U$ and U is compact, so we can find a subsequence, namely, $\{p_{k_m}\}$ of $\{p_k\}$ such that $\lim_{m\longrightarrow\infty}\|p_{k_m} - u_0\| = 0$ for some element $u_0 \in U$. Moreover, since $F_{\mathcal{S}} \neq \varnothing$, so according to the Theorem 8, $\lim_{m\longrightarrow\infty}\|p_{k_m} - \mathcal{S}p_{k_m}\| = 0$. Applying Proposition 5 (d), we get

$$\left\|p_{k_m} - \mathcal{S}u_0\right\| \leq \frac{(3+\alpha)}{(1-\alpha)}\left\|p_{k_m} - \mathcal{S}p_{k_m}\right\| + \left\|p_{k_m} - u_0\right\|. \quad (28)$$

Consequently, $p_{k_m} \longrightarrow \mathcal{S}u_0$ provided that $m \longrightarrow \infty$. But W is a Banach space, and so, the limit of a convergent sequence is always unique. Thus, $\mathcal{S}u_0 = u_0$. Lemma 7 provides us that $\lim_{k\longrightarrow\infty}\|p_k - u_0\|$ exists. Hence, u_0 is the strong limit of $\{p_k\}$. □

We now state and then prove another strong convergence theorem as follows.

Theorem 11. *Suppose that W is UCBS, $\varnothing \neq U \subseteq W$ is closed convex, and $\mathcal{S} : U \longrightarrow U$ is a generalized α-nonexpansive having $F_{\mathcal{S}} \neq \varnothing$. If $\{p_k\}$ is a JK iteration sequence as provided in (12), then, $\{p_k\}$ converges strongly to a point $F_{\mathcal{S}}$ whenever* $\liminf_{k\longrightarrow\infty} d(p_k, F_{\mathcal{S}}) = 0$

Proof. According to Lemma 7, $\lim_{k\longrightarrow\infty}\|p_k - v_0\|$ exists, for every choice of fixed point v_0 of $\mathcal{S}$. It follows that $\lim_{k\longrightarrow\infty} d(p_k, F_{\mathcal{S}})$ exists. Accordingly, we have

$$\lim_{k\longrightarrow\infty} d(p_k, F_{\mathcal{S}}) = 0. \quad (29)$$

The above strong limit suggests the existence of two subsequences $\{p_{k_s}\}$, $\{v_s\}$ in $\{p_k\}$ and $F_{\mathcal{S}}$, respectively, with the property $\|p_{k_s} - v_s\| \leq (1/2^s)$ for every natural constant s. According to the proof of Lemma 7, the iterative sequence $\{p_k\}$ is nonincreasing. Accordingly, we have

$$\left\|p_{k_{s+1}} - v_s\right\| \leq \left\|p_{k_s} - v_s\right\| \leq \frac{1}{2^s}. \quad (30)$$

Using the above and triangle inequality, one has

$$\begin{aligned}\|v_{s+1} - v_s\| &\leq \left\|v_{s+1} - p_{k_{s+1}}\right\| + \left\|p_{k_{s+1}} - v_s\right\| \leq \frac{1}{2^{s+1}} + \frac{1}{2^s} \\ &\leq \frac{1}{2^{s-1}} \longrightarrow 0, \quad \text{provided that } s \longrightarrow \infty.\end{aligned} \quad (31)$$

Accordingly, we obtained $\lim_{s\longrightarrow\infty}\|v_{s+1} - v_s\| = 0$, that is, $\{v_s\}$ form the Cauchy sequence in the closed set $F_{\mathcal{S}} \subseteq U$. It follows that $\lim_{s\longrightarrow\infty} v_s = u_0$ for some $u_0 \in F_{\mathcal{S}}$. Cosequently, $u_0 \in F_{\mathcal{S}}$. By Lemma 7, $\lim_{k\longrightarrow\infty}\|p_k - v_0\|$ exists, that is, u_0 is also the strong limit of $\{p_k\}$. □

Table 1: Strong convergence comparison of JK (12), Thakur (11), Abbas (10), Agarwal (9), Noor (8), Ishikawa (7), and Mann (6) iterates for the self-map $\mathcal{S}$ in Example 13.

k	JK	Thakur	Abbas	Agarwal	Noor	Ishikawa	Mann
1	6.5000	6.5000	6.5000	6.5000	6.5000	6.5000	6.5000
2	5.1645	5.2897	5.3983	5.5794	5.7361	5.8044	5.9750
3	5.0180	5.0559	5.1057	5.2238	5.3613	5.4313	5.6338
4	5.0020	5.0108	5.0281	5.0864	5.1773	5.2313	5.4119
5	5.0002	5.0021	5.0075	5.0334	5.0870	5.1240	5.2678
6	*5.0000*	5.0004	5.0020	5.0129	5.0427	5.0665	5.1740
7	5.0000	5.0001	5.0005	5.0050	5.0210	5.0357	5.1131
8	5.0000	*5.0000*	5.0001	5.0019	5.0103	5.0191	5.0735
9	5.0000	5.0000	*5.0000*	5.0007	5.0050	5.0103	5.0478
10	5.0000	5.0000	5.0000	5.0003	5.0025	5.0055	5.0311
11	5.0000	5.0000	5.0000	5.0001	5.0012	5.0029	5.0202
12	5.0000	5.0000	5.0000	*5.0000*	5.0006	5.0016	5.0131
13	5.0000	5.0000	5.0000	5.0000	5.0003	5.0008	5.0085
14	5.0000	5.0000	5.0000	5.0000	5.0001	5.0005	5.0055
15	5.0000	5.0000	5.0000	5.0000	5.0001	5.0002	5.0036
16	5.0000	5.0000	5.0000	5.0000	*5.0000*	5.0001	5.0023
17	5.0000	5.0000	5.0000	5.0000	5.0000	5.0001	5.0015
18	5.0000	5.0000	5.0000	5.0000	5.0000	*5.0000*	5.0010
19	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0006
20	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0004
21	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0003
22	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0002
23	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0001
24	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0001
25	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	*5.0000*

We finish this section with a strong convergence theorem under the condition I.

Theorem 12. *Suppose that W is UCBS, $\varnothing \neq U \subseteq W$ is closed convex, and $\mathcal{S} : U \longrightarrow U$ is a generalized α-nonexpansive having $F_{\mathcal{S}} \neq \varnothing$. If $\{p_k\}$ is a JK iteration sequence as provided in (12). Then, $\{p_k\}$ converges strongly to a point of $F_{\mathcal{S}}$ whenever $\mathcal{S}$ is endowed with condition I.*

Proof. According to Theorem 8, one can conclude that $\liminf_{k\longrightarrow\infty}\|p_k - \mathcal{S}p_k\| = 0$. Applying the condition I of $\mathcal{S}$, one obtain $\liminf_{k\longrightarrow\infty} d(p_k, F_{\mathcal{S}}) = 0$. It now follows from Theorem 11 that $\{p_k\}$ is strongly convergent in the set $F_{\mathcal{S}}$. □

4. Numerical Example

The aim of this section is to provide a new example of generalized α-nonexpansive mappings that exceeds the class of Suzuki mappings. We connect the mentioned iterative schemes with this example to show the effectiveness of our obtained results.

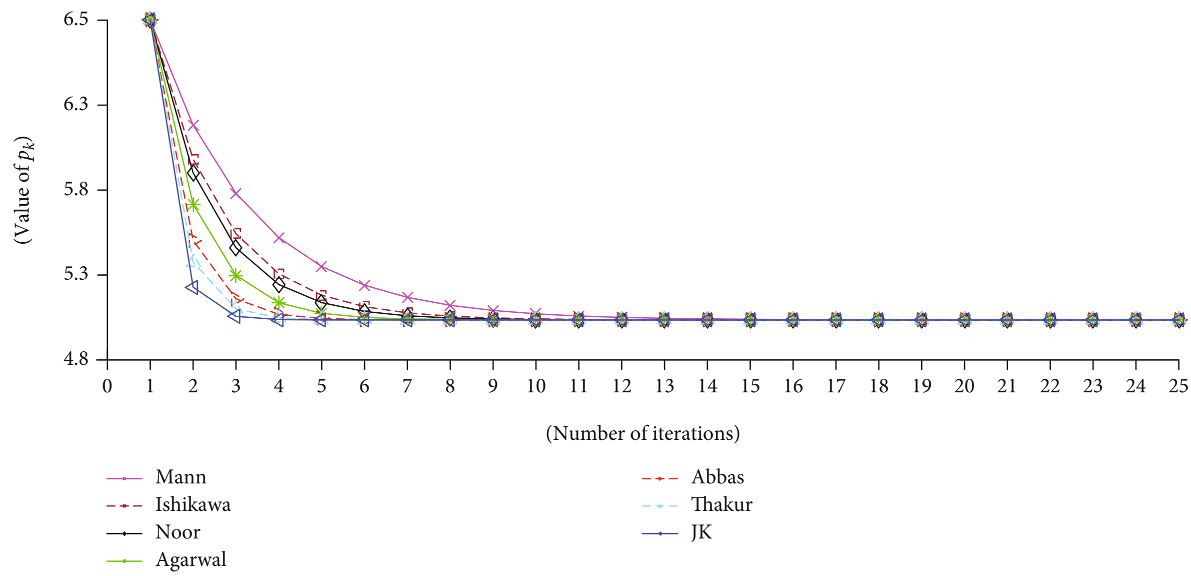

FIGURE 1: Convergence behavior of the sequences developed by remarkable iterative processes.

Example 13. We take a set $U = [5, 10]$ and set a self-map on U by the following rule:

$$\mathcal{S}z = \begin{cases} \dfrac{z+5}{2}, & \text{if } z < 10, \\ 5, & \text{if } z = 10. \end{cases} \tag{32}$$

We show that $\mathcal{S}$ is generalized α-nonexpansive having $\alpha = (1/2)$, but not Suzuki mapping. This example thus exceeds the class of Suzuki mappings.

Case 1. When $z, z' \in \{10\}$, we have

$$\begin{aligned} &\frac{1}{2}\left|z - \mathcal{S}z'\right| + \frac{1}{2}\left|z' - \mathcal{S}z\right| \\ &\quad + \left(1 - 2\left(\frac{1}{2}\right)\right)\left|z - z'\right| \geq 0 = \left|\mathcal{S}z - \mathcal{S}z'\right|. \end{aligned} \tag{33}$$

Case 2. When $z, z' \in [5, 10)$, we have

$$\begin{aligned} &\frac{1}{2}\left|z - \mathcal{S}z'\right| + \frac{1}{2}\left|z' - \mathcal{S}z\right| + \left(1 - 2\left(\frac{1}{2}\right)\right)\left|z - z'\right| \\ &\quad = \frac{1}{2}\left|z' - \left(\frac{z+5}{2}\right)\right| + \frac{1}{2}\left|z - \left(\frac{z'+5}{2}\right)\right| \\ &\quad \geq \frac{1}{2}\left|\left(z' - \left(\frac{z+5}{2}\right)\right) - \left(z - \left(\frac{z'+5}{2}\right)\right)\right| \\ &\quad = \frac{1}{2}\left|\frac{2z' - z - 5 - 2z + z' + 5}{2}\right| = \frac{1}{2}\left|\frac{3z' - 3z}{2}\right| \\ &\quad = \frac{3}{4}\left|z - z'\right| \geq \frac{1}{2}\left|z - z'\right| = \left|\mathcal{S}z - \mathcal{S}z'\right|. \end{aligned} \tag{34}$$

Case 3. When $z' \in \{10\}$ and $z \in [5, 10)$, we have

$$\begin{aligned} &\frac{1}{2}\left|z - \mathcal{S}z'\right| + \frac{1}{2}\left|z' - \mathcal{S}z\right| + \left(1 - 2\left(\frac{1}{2}\right)\right)\left|z - z'\right| \\ &\quad = \frac{1}{2}\left|z - 5\right| + \frac{1}{2}\left|z' - \left(\frac{z+5}{2}\right)\right| \geq \frac{1}{2}\left|z - 5\right| \\ &\quad = \left|\frac{z-5}{2}\right| = \left|\mathcal{S}z - \mathcal{S}z'\right|. \end{aligned} \tag{35}$$

The above cases clearly suggest that $\mathcal{S}$ is generalized 1/2 nonexpansive mapping having $F_{\mathcal{S}} = \{5\}$. Choose $z = 8.8$ and $z' = 10$; then, $|z - z'| = 1.2$, $|\mathcal{S}z - \mathcal{S}z'| = 1.9$, and $(1/2)|z - \mathcal{S}z| = 0.95$. Thus, it is seen that, $(1/2)|z - \mathcal{S}z| < |z - z'|$ but $|\mathcal{S}z - \mathcal{S}z'| > |z - z'|$. Thus, $\mathcal{S}$ exceeds the class of Suzuki mappings.

Now, we by choosing $\xi_k = 0.70$, $\eta_k = 0.65$, and $\mu_k = 0.80$, we may observe in Table 1 and Figure 1 that JK (12) iterative process converges faster to $5 \in F_{\mathcal{S}}$ as compared the other processes.

Now we show the further effectiveness of the JK iteration (12) in the class of generalized α-nonexpansive mappings. Using $\mathcal{S}$ defined in Example 13, we suggest some different values for the parameters and p_1. To do this, we set the stopping criteria $\|p_k - 5\| < 10^{-15}$. The obtained results are provided in Table 2. The bold numbers show that JK iteration (12) requires less iteration numbers as compared to the leading three-step Thakur (11) and leading two-step Agarwal (9).

Remark 14. The main outcome of this paper extended the corresponding results of Ahmad et al. [19] from the class of Suzuki maps to the setting of generalized α-nonexpansive maps. We have observed in Tables 1 and 2 as well as in Figure 1 that the JK iterative scheme (12) is still more effective than the other iterative schemes even in the general setting of generalized α-nonexpansive maps.

Table 2: Influence of parameters: comparison of various iterative schemes.

Iterations	Initial points 5.1	6.0	6.9	7.8	8.9	9.9
	For $\xi_k = (1/k+5)$, $\eta_k = (k/k+3)$					
Agarwal	29	31	31	32	32	32
Thakur	18	20	20	20	20	21
JK	*14*	*15*	*15*	*15*	*16*	*16*
	For $\xi_k = 1/k$, $\eta_k = \left(1/(k+23)^{1/4}\right)$					
Agarwal	43	47	48	48	49	49
Thakur	22	24	24	24	25	25
JK	*18*	*20*	*20*	*20*	*20*	*21*
	For $\xi_k = 1-(1/(4k+6))^{1/5}$, $\eta_k = 1/k^4$					
Agarwal	44	48	49	49	50	50
Thakur	22	24	25	25	25	25
JK	*18*	*20*	*20*	*20*	*20*	*21*
	For $\xi_k = ((k+1)/(2k+1))^{1/2}$, $\eta_k = 1/(3k+7)^{5/30}$					
Agarwal	45	48	49	50	50	50
Thakur	24	25	25	25	25	25
JK	*17*	*18*	*19*	*19*	*19*	*19*
	For $\xi_k = 1-(1/(4k+3))^{1/4}$, $\eta_k = 1/k^7$					
Agarwal	44	48	49	49	50	50
Thakur	22	24	25	25	25	25
JK	*18*	*19*	*20*	*20*	*20*	*20*

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

No competing interests are associated with this paper.

Authors' Contributions

This research work is completed by the equal contribution of each of the author listed in this paper.

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Research Article

Global Existence and General Decay of Solutions for a Quasilinear System with Degenerate Damping Terms

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In this work, we consider a quasilinear system of viscoelastic equations with degenerate damping, dispersion, and source terms under Dirichlet boundary condition. Under some restrictions on the initial datum and standard conditions on relaxation functions, we study global existence and general decay of solutions. The results obtained here are generalization of the previous recent work.

1. Introduction

Let Ω be a bounded domain with a sufficiently smooth boundary in $R^n (n \geq 1)$. We investigate a quasilinear system of two viscoelastic equations in the presence of degenerate damping, dispersion, and source terms, namely,

$$\begin{cases} |u_t|^\eta u_{tt} - \Delta u + \int_0^t h_1(t-s)\Delta u(s)ds - \Delta u_{tt} + \left(|u|^k + |v|^l\right)|u_t|^{j-1}u_t = f_1(u,v), (x,t) \in \Omega \times (0,T), \\ |v_t|^\eta v_{tt} - \Delta v + {}_0^t h_2(t-s)\Delta v(s)ds - \Delta v_{tt} + \left(|v|^\theta + |u|^\varrho\right)|v_t|^{s-1}v_t = f_2(u,v), (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), x \in \Omega, \end{cases} \tag{1}$$

where, $s \geq 1, \eta > 0, k, l, \theta, \varrho \geq 0$, and $h_i(.): R^+ \longrightarrow R^+ (i = 1, 2)$ are positive relaxation functions which will be specified later. $(|(.)|^a + |(.)|^b)|(.)_t|^{\tau-1}(.)_t$ and $-\Delta(.)_{tt}$ are the degenerate damping term and the dispersion term, respectively.

By taking

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(\kappa+1)}(u+v) + b|u|^\kappa u|v|^{\kappa+2}, \\ f_2(u,v) &= a|u+v|^{2(\kappa+1)}(u+v) + b|v|^\kappa v|u|^{\kappa+2}, \end{aligned} \tag{2}$$

in which $a > 0, b > 0$, and

$$1 < \kappa < +\infty \text{if } n = 1, 2 \text{ and } 1 < \kappa \leq \frac{3-n}{n-2} \text{ if } n \geq 3. \quad (3)$$

It is simple to show that

$$uf_1(u,v) + vf_2(u,v) = 2(\kappa+2)F(u,v), \forall (u,v) \in R^2, \quad (4)$$

where

$$F(u,v) = \frac{1}{2(\kappa+2)}\left[a|u+v|^{2(\kappa+2)} + 2b|uv|^{\kappa+2}\right]. \quad (5)$$

To motivate our problem (1), it can trace back to the initial boundary value problem for the single viscoelastic equation of the form

$$|u_t|^\eta u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)ds - \Delta u_{tt} + g(u,u_t) = f(u). \quad (6)$$

This type problem appears a variety of mathematical models in applied science. For instance, in the theory of viscoelasticity, physics, and material science, problem (5) has been studied by various authors, and several results concerning blow-up and energy decay have been studied case $(\eta \geq 0)$. For example, Liu [1] studied a general decay of solutions case $(g(u,u_t) = 0)$. Messaoudi and Tatar [2] applied the potential well method to indicate the global existence and uniform decay of solutions ($g(u,u_t) = 0$ instead of Δu_t). Furthermore, the authors obtained a blow-up result for positive initial energy. Wu [3] studied a general decay of solution case $(g(u,u_t) = |u_t|^m u_t)$. Later, Wu [4] studied the same problem case $(g(u,u_t) = u_t)$ and discussed the decay rate of solution energy. Recently, Yang et al. [5] proved the existence of global solution and asymptotic stability result without restrictive conditions on the relaxation function at infinity case $(f(u) = \sigma(x,t)W_t(t,x))$.

In case $g(u,u_t) = 0$ and without dispersion term, problem (5) has been investigated by Song [6], and the blow-up result for positive initial energy has been proved.

For a coupled system, He [7] investigated the following problem

$$\begin{cases} |u_t|^\eta u_{tt} - \Delta u + \int_0^t h_1(t-s)\Delta u(s)ds - \Delta u_{tt} + |u_t|^{j-2}u_t = f_1(u,v), \\ |v_t|^\eta v_{tt} - \Delta v + \int_0^t h_2(t-s)\Delta v(s)ds - \Delta v_{tt} + |v_t|^{s-2}v_t = f_2(u,v), \end{cases} \quad (7)$$

where $\eta > 0, j, s \geq 2$. The author proved general and optimal decay of solutions. Then, in [8], the author investigated the same problem without damping term and established a general decay of solutions. Furthermore, the author obtained a blow-up of solutions for negative initial energy. In addition, problem (1) with in case $\eta = 0$ and without dispersion term, Wu [9] proved a general decay of solutions. Later, Pişkin and Ekinci [10] studied a general decay and blow-up of solutions with nonpositive initial energy for problem (1) case (Kirchhoff-type instead of Δu and without dispersion term). In recent years, some other authors investigate the hyperbolic type system with degenerate damping term (see [11–14]).

The rest of the paper is arranged as follows: in Section 2, as preliminaries, we give necessary assumptions and lemmas that will be used later and local existence theorem without proof. In Section 3, we prove the global existence of solution. In the last section, we studied the general decay of solutions.

2. Preliminaries

We begin this section with some assumptions, notations, lemmas, and theorems. Denote the standart $L^2(\Omega)$ norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^p(\Omega)$ norm by $\|.\|_p = \|.\|_{L^p(\Omega)}$.

To state and prove our result, we need some assumptions:

(A1) Regarding $h_i : [0,\infty) \longrightarrow (0,\infty), (i = 1,2)$ is C^1 functions and satisfies

$$h_i(\alpha) > 0, h_i'(\alpha) \leq 0, 1 - {}_0^\infty h_i(\alpha)d\alpha = l_i > 0, \alpha \geq 0 \quad (8)$$

and nonincreasing differentiable positive C^1 functions ς_1 and ς_2 such that

$$h_i'(t) \leq -\varsigma_i(t)h_i(t), t \geq 0, i = 1, 2. \quad (9)$$

(A2) For the nonlinearity, we assume that

$$\begin{cases} 1 \leq j, s \text{ if } n = 1, 2, \\ 1 \leq j, s \leq \dfrac{n+2}{n-2} \text{ if } n \geq 3. \end{cases} \quad (10)$$

(A3) Assume that η satisfies

$$\begin{cases} 0 < \eta \text{ if } n = 1, 2, \\ 0 < \eta \leq \dfrac{2}{n-2} \text{ if } n \geq 3. \end{cases} \quad (11)$$

In addition, we present some notations:

$$(\alpha \diamond \nabla w)(t) = \int_0^t \alpha(t-s)\|\nabla w(t) - \nabla w(s)\|^2 ds, \quad l = \min\{l_1, l_2\}. \quad (12)$$

Lemma 1 (Sobolev-Poincare inequality) [15]. *Let q be a number with $2 \leq q < \infty (n = 1,2)$ or $2 \leq q \leq 2n/(n-2) (n \geq 3)$, and then there is a constant $C_* = C_*(\Omega, q)$ such that*

$$\|u\|_q \leq C_* \|\nabla u\| \text{ for } u \in H_0^1(\Omega). \quad (13)$$

Now, we state the local existence theorem that can be established by combining arguments of [7, 10].

Theorem 2. *Assume that (A1)-(A3) and (2) hold. Let $u_0, v_0 \in H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$ be given. Then, for some $T > 0$,* *problem (1) has a unique local weak solution in the following class:*

$$\begin{aligned} & u, v \in C\left([0, T] ; H^2(\Omega) \cap H_0^1(\Omega)\right), \\ & u_t \in C\left([0, T] ; H_0^1(\Omega)\right) \cap L^{j+1}(\Omega), \\ & v_t \in C\left([0, T] ; H_0^1(\Omega)\right) \cap L^{s+1}(\Omega). \end{aligned} \tag{14}$$

We define the energy function as follows:

$$\begin{aligned} E(t) = & \frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2}\right)+\frac{1}{2}\left[(h_1 \diamond \nabla u)(t)\right. \\ & \left.+(h_2 \diamond \nabla v)(t)+\|\nabla u_t\|^2+\|\nabla v_t\|^2\right] \\ & +\frac{1}{2}\left[\left(1-\int_0^t h_1(s) d s\right)\|\nabla u(t)\|^2\right. \\ & \left.+\left(1-\int_0^t h_2(s) d s\right)\|\nabla v(t)\|^2\right]-\int_{\Omega} F(u, v) d x. \end{aligned} \tag{15}$$

Also, we define

$$\begin{aligned} I(t)= & \|\nabla u_t\|^2+\|\nabla v_t\|^2+\left(1-\int_0^t h_1(s) d s\right)\|\nabla u(t)\|^2 \\ & +\left(1-\int_0^t h_2(s) d s\right)\|\nabla v(t)\|^2+(h_1 \diamond \nabla u)(t) \\ & +(h_2 \diamond \nabla v)(t)-2(\kappa+2) \int_{\Omega} F(u, v) d x, \\ J(t)= & \frac{1}{2}\left[\left(1-\int_0^t h_1(s) d s\right)\|\nabla u(t)\|^2+\left(1 \int_0^t h_2(s) d s\right)\right. \\ & \left.\cdot\|\nabla v(t)\|^2\right]+\frac{1}{2}\left[(h_1 \diamond \nabla u)(t)+(h_2 \diamond \nabla v)(t)\right. \\ & \left.+\|\nabla u_t\|^2+\|\nabla v_t\|^2\right]-\int_{\Omega} F(u, v) d x. \end{aligned} \tag{16}$$

By computation, we get

$$\begin{aligned} \frac{d}{d t} E(t) \leq & \frac{1}{2}\left[\left(h_1' \diamond \nabla u\right)(t)+\left(h_2' \diamond \nabla v\right)(t)\right] \\ & -\frac{1}{2}\left(h_1(t)\|\nabla u\|^2+h_2(t)\|\nabla v\|^2\right) \\ & -\int_{\Omega}\left(|u|^k+|v|^l\right)|u_t|^{j+1} d x \\ & -\int_{\Omega}\left(|v|^\theta+|u|^\varrho\right)|v_t|^{s+1} d x \leq 0. \end{aligned} \tag{17}$$

3. Global Existence

In this part, in order to state and prove the global existence of solution (1), we firstly give two lemmas.

Lemma 3 [16]. *Assume that (4) holds. Then, there exist $\rho>0$ such that for the solution (u, v),*

$$\|u+v\|_{2(\kappa+2)}^{2(\kappa+2)}+2\|u v\|_{\kappa+2}^{\kappa+2} \leq \rho\left(l_1\|\nabla u\|^2+l_2\|\nabla v\|^2\right)^{\kappa+2}. \tag{18}$$

Lemma 4. *Let $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$. Suppose that (A1)-(A3) hold. If*

$$I(0)>0 \text{ and } \beta=\rho\left(\frac{2(\kappa+2)}{\kappa+1} E(0)\right)^{\kappa+1}<1, \tag{19}$$

then

$$I(t)>0, \forall t>0. \tag{20}$$

Proof. We have $I(0)>0$ and by continuity of $I(t)$ about t, there exist a maximal time $t_m>0$ such that

$$I(t) \geq 0, \text{ on } t \in [0, t_m]. \tag{21}$$

Let t_0 be as follows:

$$\{I(t_0)=0 \text{ and } I(t)>0, \text{ for all } 0 \leq t<t_0\}. \tag{22}$$

By using (8), (9), and (A1), we get

$$\begin{aligned} J(t)= & \frac{\kappa+1}{2(\kappa+2)}\left\{\left(1-\int_0^t h_1(s) d s\right)\|\nabla u(t)\|^2\right. \\ & +\left(1-\int_0^t h_2(s) d s\right)\|\nabla v(t)\|^2+(h_1 \diamond \nabla u)(t) \\ & \left.+(h_2 \diamond \nabla v)(t)+\|\nabla u_t\|^2+\|\nabla v_t\|^2\right\}+\frac{1}{2(\kappa+2)} I(t) \\ \geq & \frac{\kappa+1}{2(\kappa+2)}\left\{l_1\|\nabla u(t)\|^2+l_2\|\nabla v(t)\|^2+(h_1 \diamond \nabla u)(t)\right. \\ & \left.+(h_2 \diamond \nabla v)(t)+\|\nabla u_t\|^2+\|\nabla v_t\|^2\right\}. \end{aligned} \tag{23}$$

From (7) and (10), we have

$$\begin{aligned} & l_1\|\nabla u(t)\|^2+l_2\|\nabla v(t)\|^2 \\ & \quad \leq \frac{2(\kappa+2)}{\kappa+1} J(t) \leq \frac{2(\kappa+2)}{\kappa+1} E(t) \\ & \quad \leq \frac{2(\kappa+2)}{\kappa+1} E(0), \forall t \in [0, t_0]. \end{aligned} \tag{24}$$

By (11) and (12), we infer that

$$\begin{aligned} & 2(\kappa+2) \int_{\Omega} F(u(t_0), v(t_0)) d x \\ & \quad \leq \rho\left(l_1\|\nabla u(t_0)\|^2+l_2\|\nabla v(t_0)\|^2\right)^{\kappa+2} \\ & \quad \leq \rho\left(\frac{2(\kappa+2)}{\kappa+1} E(0)\right)^{\kappa+1}\left(l_1\|\nabla u(t_0)\|^2+l_2\|\nabla v(t_0)\|^2\right) \\ & \quad \leq \rho\left(l_1\|\nabla u(t_0)\|^2+l_2\|\nabla v(t_0)\|^2\right) \\ & \quad \leq\left(1-\int_0^t h_1(s) d s\right)\|\nabla u(t)\|^2+\left(1-\int_0^t h_2(s) d s\right) \\ & \quad \cdot\|\nabla v(t)\|^2. \end{aligned} \tag{25}$$

Thus, from (8), we obtain

$$I(t_0) > 0, \tag{26}$$

which contradicts to (13). Thus, $I(t) > 0$ on $[0, T]$.

Theorem 5. *Suppose that the conditions of Lemma 4 hold, then the solution (1) is bounded and global in time.*

Proof. We have

$$\begin{aligned} E(0) \geq E(t) &= J(t) + \frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) \\ &\geq \frac{\kappa+1}{2(\kappa+2)}\left(l_1\|\nabla u(t)\|^2 + l_2\|\nabla v(t)\|^2\right. \\ &\quad + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 + (h_1 \diamond \nabla u)(t) \\ &\quad \left.+ (h_2 \diamond \nabla v)(t)\right) + \frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right). \end{aligned} \tag{27}$$

Thus,

$$\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \leq CE(0), \tag{28}$$

where positive constant C depends only on κ, l_1, l_2. This implies that the solution of problem (1) is global in time.

4. General Decay of Solutions

This section is devoted to show the decay of solution (1). Set

$$\Gamma(t) := M\mathrm{E}(t) + \varepsilon\Phi(t) + F(t), \tag{29}$$

where M and ε are positive constants and

$$\begin{aligned} \Phi(t) &= \delta_1(t)\left[\frac{1}{\eta+1}\int_\Omega |u_t|^\eta u_t u dx + \int_\Omega \nabla u_t \nabla u dx\right] \\ &\quad + \delta_2(t)\left[\frac{1}{\eta+1}\int_\Omega |v_t|^\eta v_t v dx + \int_\Omega \nabla v_t \nabla v dx\right], \end{aligned}$$

$$\begin{aligned} F(t) &= \delta_1(t)\left[\int_\Omega\left(\Delta u_t - \frac{|u_t|^\eta u_t}{\eta+1}\right)\int_0^t h_1(t-s)\right. \\ &\quad \left.\cdot (u(t)-u(s))dsdx\right] + \delta_2(t)\left[\int_\Omega\left(\Delta v_t - \frac{|v_t|^\eta v_t}{\eta+1}\right)\right. \\ &\quad \left.\cdot \int_0^t h_2(t-s)(v(t)-v(s))dsdx\right]. \end{aligned} \tag{30}$$

Lemma 6. *For ε which is small enough while M is large enough, the relation*

$$\alpha_1\Gamma(t) \leq E(t) \leq \alpha_2\Gamma(t), \forall t \geq 0 \tag{31}$$

holds for two positive constants α_1 and α_2.

Proof. As references [1, 10], it can be show easily that $\Gamma(t)$ and $E(t)$ are equivalent in the sense that α_1 and α_2 are positive constants, depending on ε and M.

Lemma 7 [3]. *Assume that (12) holds. Let (u, v) be the solution of problem (1). Then, for $\sigma \geq 0$, we get*

$$\begin{cases} \displaystyle\int_\Omega\left(\int_0^t h_1(t-s)(u(t)-u(s))ds\right)^{\sigma+2} dx \leq (1-l_1)^{\sigma+1} c_*^{\sigma+2}\left(\frac{2(\kappa+2)E(0)}{l_1(\kappa+1)}\right)^{\sigma/2}(h_1 \diamond \nabla u)(t), \\ \displaystyle\int_\Omega\left(\int_0^t h_2(t-s)(v(t)-v(s))ds\right)^{\sigma+2} dx \leq (1-l_1)^{\sigma+1} c_*^{\sigma+2}\left(\frac{2(\kappa+2)E(0)}{l_2(\kappa+1)}\right)^{\sigma/2}(h_2 \diamond \nabla v)(t). \end{cases} \tag{32}$$

Lemma 8 [16]. *Let (A1)-(A3) hold. Assume that $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$, be given and satisfying (12). Then, throughout the solution (u, v) of (1), there exist two positive constants β_1 and β_2 such that for any $\delta > 0$ and for all $t \geq 0$,*

$$\begin{aligned} &\int_\Omega f_1(u, v)\int_0^t h_1(t-s)(u(t)-u(s))dsdx \\ &\quad \leq \beta_1\delta\left(l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2\right) + \frac{(1-l_1)c_*^2(h_1 \diamond \nabla u)(t)}{4\delta}, \\ &\int_\Omega f_2(u, v)\int_0^t h_2(t-s)(v(t)-v(s))dsdx \\ &\quad \leq \beta_2\delta\left(l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2\right) + \frac{(1-l_2)c_*^2(h_2 \diamond \nabla u)(t)}{4\delta}. \end{aligned} \tag{33}$$

Lemma 9. *Let $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$, be given and satisfying (12). Suppose that (A1)-(A3) hold. Then, for each $t_0 > 0$, the functional $\Gamma(t)$ verifies, throughout the solution of (1)*

$$\Gamma'(t) \leq -\xi_1 E(t) + \xi_2[(h_1 \diamond \nabla u)(t) + (h_2 \diamond \nabla v)(t)], t \geq t_0, \tag{34}$$

where $\xi_i > 0, (i = 1, 2)$.

Proof. By applying (18) and Eq.(1) and getting $\delta_i \equiv 1 (i = 1, 2)$ in (18), we have

$$\Phi'(t) \leq \frac{1}{\eta+1}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) - \|\nabla u\|^2 - \|\nabla v\|^2$$

$$
\begin{aligned}
&+\|\nabla u_t\|^2+\|\nabla v_t\|^2\int_\Omega \nabla u(t)\int_0^t h_1(t-s)\nabla u(s)dsdx\\
&+\int_\Omega \nabla v(t)\int_0^t h_2(t-s)\nabla v(s)dsdx+2(\kappa+2)\\
&\cdot\int_\Omega F(u,v)dx-\int_\Omega u\left(|u|^k+|v|^l\right)u_t|u_t|^{j-1}dx\\
&-\int_\Omega v\left(|v|^\theta+|u|^\rho\right)v_t|v_t|^{s-1}dx.
\end{aligned}
\tag{35}
$$

For estimating the seventh term in the right side of (22) as follows (see [17]):

$$
\begin{aligned}
&\int_\Omega \nabla u(t)\int_0^t h_1(t-s)\nabla u(s)dsdx\\
&\quad\le \frac{1}{2}\|\nabla u\|^2+\frac{1}{2}\int_\Omega\left(\int_0^t h_1(t-s)(|\nabla u(s)\right.\\
&\quad\left.-\nabla u(t)|+|\nabla u(t)|)ds\right)^2dx.
\end{aligned}
\tag{36}
$$

By exploiting Young's inequality and the assumption that $\int_0^t h_1(s)ds\le\int_0^\infty h_1(s)ds\le 1-l_1$, for $\gamma_1>0$,

$$
\begin{aligned}
&\int_\Omega \nabla u(t)\int_0^t h_1(t-s)\nabla u(s)dsdx\\
&\quad\le \frac{1}{2}\|\nabla u\|^2+\frac{1}{2}(1+\gamma_1)\int_\Omega\left(\int_0^t h_1(t-s)|\nabla u(s)|ds\right)^2dx\\
&\quad+\frac{1}{2}\left(1+\frac{1}{\gamma_1}\right)\int_\Omega\left(\int_0^t h_1(t-s)|\nabla u(s)-\nabla u(t)|ds\right)^2dx\\
&\quad\le \frac{1}{2}\|\nabla u\|^2+\frac{1}{2}(1+\gamma_1)\left(\int_0^t h_1(s)ds\right)^2\|\nabla u\|^2\\
&\quad+\frac{1}{2}\left(1+\frac{1}{\gamma_1}\right)\left(\int_0^t h_1(s)ds\right)(h_1\diamond\nabla u)(t)\\
&\quad\le \frac{1+(1+\gamma_1)(1-l_1)^2}{2}\|\nabla u\|^2\\
&\quad+\frac{(1+(1/\gamma_1))(1-l_1)}{2}(h_1\diamond\nabla u)(t).
\end{aligned}
\tag{37}
$$

Similarly with $\gamma_2>0$,

$$
\begin{aligned}
&\int_\Omega \nabla v(t)\int_0^t h_2(t-s)\nabla v(s)dsdx\\
&\quad\le \frac{1+(1+\gamma_2)(1-l_2)^2}{2}\|\nabla v\|^2\\
&\quad+\frac{(1+(1/\gamma_2))(1-l_2)}{2}(h_2\diamond\nabla v)(t).
\end{aligned}
\tag{38}
$$

By estimating the following terms in (22), we have

$$
\begin{aligned}
&\int_\Omega u\left(|u|^k+|v|^l\right)u_t|u_t|^{j-1}dx\\
&\quad\le {}_\Omega|u|^{k+1}|u_t|^jdx+\int_\Omega |u||v|^l|u_t|^jdx.
\end{aligned}
\tag{39}
$$

Exploiting Young's inequality, Hölder's inequality, Sobolev-Poincare inequality, (A3), and (15) for $\beta_1>0$, one has

$$
\begin{aligned}
&\int_\Omega |u|^{k+1}|u_t|^jdx\\
&\le\left(\int_\Omega |u|^k|u_t|^{j+1}dx\right)^{j/j+1}\left(\int_\Omega |u|^{k+j+1}dx\right)^{j/j+1}\\
&\le\frac{j\beta_1^{-j+1/j}}{j+1}\int_\Omega |u|^k|u_t|^{j+1}dx+\frac{\beta_1^{j+1}C_*^{k+j+1}}{j+1}\|\nabla u\|^{k+j+1}\\
&\le\frac{j\beta_1^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx\\
&+\frac{\beta_1^{j+1}C_*^{k+j+1}\chi_1^{k+j-1/2}}{j+1}\|\nabla u\|^2,\\
&\left|\int_\Omega |u||v|^l|u_t|^jdx\right|\\
&\le\left(\int_\Omega |v|^l|u_t|^{j+1}dx\right)^{j/j+1}\left(\int_\Omega |v|^l|u|^{j+1}dx\right)^{j/j+1}\\
&\le\frac{j\beta_1^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx\\
&+\frac{\beta_1^{j+1}}{2(j+1)}\left(\|v\|_{2l}^{2l}+\|u\|_{2(j+1)}^{2(j+1)}\right)\\
&\le\frac{j\beta_1^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx+\frac{\beta_1^{j+1}C_*^{2l}\chi_2^{l-1}}{2(j+1)}\\
&\cdot\left(\|\nabla v\|^2\right)+\frac{\beta_1^{j+1}C_*^{2j+2}\chi_1^j}{2(j+1)}\left(\|\nabla u\|^2\right),
\end{aligned}
\tag{40}
$$

where

$$
\chi_1=\frac{2(\kappa+2)E(0)}{l_1(\kappa+1)}\ \text{and}\ \chi_2=\frac{2(\kappa+2)E(0)}{l_2(\kappa+1)}.
\tag{41}
$$

By inserting (27) and (28) into (26), we have

$$
\begin{aligned}
&\int_\Omega u\left(|u|^k+|v|^l\right)u_t|u_t|^{j-1}dx\\
&\quad\le\frac{2j\beta_1^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx\\
&\quad+\frac{\beta_1^{j+1}C_*^{2l}\chi_2^{l-1}}{2(j+1)}\left(\|\nabla v\|^2\right)+\frac{\beta_1^{j+1}}{j+1}\\
&\quad\cdot\left(C_*^{k+j+1}\chi_1^{(k+j-1)/2}+\frac{C_*^{2j+2}\chi_1^j}{2}\right)\|\nabla u\|^2.
\end{aligned}
\tag{42}
$$

Similarly, for $\beta_2 > 0$, we have

$$\begin{aligned}
&\int_\Omega v\left(|v|^\theta + |u|^\rho\right) v_t |v_t|^{s-1} dx \\
&\quad \le \frac{2s\beta_2^{-s+1/s}}{s+1} \int_\Omega \left(|v|^\theta + |u|^\rho\right) |v_t|^{s+1} dx \\
&\quad + \frac{\beta_2^{s+1} C_*^{2\rho} \chi_1^{\rho-1}}{2(s+1)} \left(\|\nabla u\|^2\right) + \frac{\beta_2^{s+1}}{s+1} \\
&\quad \cdot \left(C_*^{\theta+s+1} \chi_2^{(\theta+s-1)/2} + \frac{C_*^{2s+2}\chi_2^s}{2}\right) \|\nabla v\|^2.
\end{aligned} \tag{43}$$

Thus, inserting (24) and (25) and (29) and (30) into (22), we obtain

$$\begin{aligned}
\Phi'(t) \le{}& \frac{1}{\eta+1}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) - a_1\|\nabla u\|^2 \\
&- a_2\|\nabla v\|^2 + 2(\kappa+2)\int_\Omega F(u,v)dx + \|\nabla u_t\|^2 \\
&+ \|\nabla v_t\|^2 + \frac{(1+(1/\gamma_1))(1-l_1)}{2}(h_1 \diamond \nabla u)(t) \\
&+ \frac{(1+(1/\gamma_2))(1-l_2)}{2}(h_2 \diamond \nabla v)(t) \\
&+ \frac{2j\beta_1^{-j+1/j}}{j+1}\int_\Omega \left(|u|^k + |v|^l\right)|u_t|^{j+1} dx \\
&+ \frac{2s\beta_2^{-s+1/s}}{s+1}\int_\Omega \left(|v|^\theta + |u|^\rho\right)|v_t|^{s+1} dx,
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
a_1 ={}& \frac{1-(1+\gamma_1)(1-l_1)^2}{2} - \frac{\beta_1^{j+1}}{j+1} \\
&\cdot \left(C_*^{k+j+1}\chi_1^{(k+j-1)/2} + \frac{C_*^{2j+2}\chi_1^j}{2}\right) - \frac{\beta_2^{s+1} C_*^{2\rho}\chi_1^{\rho-1}}{2(s+1)}, \\
a_2 ={}& \frac{1-(1+\gamma_2)(1-l_2)^2}{2} - \frac{\beta_2^{s+1}}{s+1} \\
&\cdot \left(C_*^{\theta+s+1}\chi_2^{(\theta+s-1)/2} + \frac{C_*^{2s+2}\chi_2^s}{2}\right) - \frac{\beta_1^{j+1} C_*^{2l}\chi_2^{l-1}}{2(j+1)}.
\end{aligned} \tag{45}$$

At this moment, choosing $\gamma_1 = l_1/1 - l_1, \gamma_2 = l_2/1 - l_2$, and picking β_1 and β_2 small enough such that

$$\begin{aligned}
&\frac{\beta_1^{j+1}}{j+1}\left(C_*^{k+j+1}\chi_1^{(k+j-1)/2} + \frac{C_*^{2j+2}\chi_1^j}{2}\right) + \frac{\beta_2^{s+1}C_*^{2\rho}\chi_1^{\rho-1}}{2(s+1)} \le \frac{l_1}{4}, \\
&\frac{\beta_2^{s+1}}{s+1}\left(C_*^{\theta+s+1}\chi_2^{(\theta+s-1)/2} + \frac{C_*^{2s+2}\chi_2^s}{2}\right) + \frac{\beta_1^{j+1}C_*^{2l}\chi_2^{l-1}}{2(j+1)} \le \frac{l_2}{4}.
\end{aligned} \tag{46}$$

Consequently, (31) yields

$$\begin{aligned}
\Phi'(t) \le{}& \frac{1}{\eta+1}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) - \frac{l_1}{4}\|\nabla u\|^2 \\
&- \frac{l_2}{4}\|\nabla v\|^2 + 2(\kappa+2)\int_\Omega F(u,v)dx \\
&+ \frac{1-l_1}{2l_1}(h_1\diamond\nabla u)(t) + \frac{1-l_2}{2l_2}(h_2\diamond\nabla v)(t) + \|\nabla u_t\|^2 \\
&+ \|\nabla v_t\|^2 + \frac{2j\beta_1^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k + |v|^l\right)|u_t|^{j+1}dx \\
&+ \frac{2s\beta_2^{-s+1/s}}{s+1}\int_\Omega\left(|v|^\theta + |u|^\rho\right)|v_t|^{s+1}dx.
\end{aligned} \tag{47}$$

In order to estimate the $F'(t)$, we set

$$F_1(t) = \int_\Omega \left(\Delta u_t - \frac{1}{\eta+1}|u_t|^\eta u_t\right)\int_0^t h_1(t-s)(u(t)-u(s))dsdx, \tag{48}$$

Then, by using equations (1), we have

$$\begin{aligned}
F_1'(t) ={}& -\int_\Omega\left(\int_0^t h_1(t-s)\nabla u(s)ds\right) \\
&\cdot\left(\int_0^t h_1(t-s)(\nabla u(t) - \nabla u(s))ds\right)dx \\
&+ \int_\Omega\left(|u|^k + |v|^l\right)|u_t|^{j-1}u_t\int_0^t h_1(t-s) \\
&\cdot(u(t)-u(s))dsdx + \int_\Omega \nabla u(t)\int_0^t h_1(t-s) \\
&\cdot(\nabla u(t) - \nabla u(s))dsdx - \int_\Omega \nabla u_t(t)\int_0^t h_1'(t-s) \\
&\cdot(\nabla u(t) - \nabla u(s))dsdx - \frac{1}{\eta+1}\int_\Omega |u_t|^\eta u_t \\
&\cdot\int_0^t h_1'(t-s)(u(t)-u(s))dsdx \\
&- \int_\Omega f_1(u,v)\int_0^t h_1(t-s)(u(t)-u(s))dsdx \\
&- \frac{1}{\eta+1}\left(\int_0^t h_1(s)\right)\|u_t\|_{\eta+2}^{\eta+2} - \left(\int_0^t h_1(s)\right)\|\nabla u_t\|^2.
\end{aligned} \tag{49}$$

For the first term of (33), by applying (A1), Hölder's inequality, and Young's inequality, we deduce

$$\begin{aligned}&\left|-\int_\Omega\left(\int_0^t h_1(t-s)\nabla u(s)ds\right)\left(\int_0^t h_1(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx\right|\\&\quad\le\delta\int_\Omega\left(\int_0^t h_1(t-s)\nabla u(s)ds\right)^2dx\\&\qquad+\frac{1}{4\delta}\int_\Omega\left(\int_0^t h_1(t-s)(\nabla u(t)-\nabla u(s))ds\right)^2dx\\&\quad\le\delta\int_\Omega\left[\int_0^t h_1(t-s)(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|)ds\right]^2dx\\&\qquad+\frac{1}{4\delta}\left(\int_0^t h_1(s)ds\right)(h_1\diamond\nabla u)(t)\le 2\delta(1-l_1)^2\|\nabla u\|^2\\&\qquad+\left(2\delta+\frac{1}{4\delta}\right)(1-l_1)(h_1\diamond\nabla u)(t),\forall\delta>0.\end{aligned}\tag{50}$$

Then, in order to estimate the following term, we seperate such that

$$\begin{aligned}&\left|\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j-1}u_t\int_0^t h_1(t-s)(u(t)-u(s))dsdx\right|\\&\quad=I_1+I_2,\end{aligned}\tag{51}$$

where

$$\begin{aligned}I_1&=\int_\Omega|u|^k|u_t|^{j-1}u_t\int_0^t h_1(t-s)(u(t)-u(s))dsdx,\\I_2&=\int_\Omega|v|^l|u_t|^{j-1}u_t\int_0^t h_1(t-s)(u(t)-u(s))dsdx.\end{aligned}\tag{52}$$

By Hölder's inequality, Young's inequality, (15), and (21), we get

$$\begin{aligned}|I_1|&\le\left(\int_\Omega|u|^k|u_t|^{j+1}dx\right)^{j/j+1}\left(\int_\Omega|u|^k\left(\int_0^t h_1(t-s)\right.\right.\\&\qquad\left.\left.\cdot(u(t)-u(s))ds\right)^{j+1}dx\right)^{1/j+1}\\&\le\frac{j\delta^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx+\frac{\delta^{j+1}}{j+1}\left(\frac{1}{2}\|u\|_{2k}^{2k}\right.\\&\qquad\left.+\frac{1}{2}\int_\Omega\left(\int_0^t h_1(t-s)(u(t)-u(s))ds\right)^{2(j+1)}dx\right)\\&\le\frac{j\delta^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx+\frac{\delta^{j+1}}{j+1}\left(\frac{c_*^{2k}}{2}\chi_1^{k-1}\|\nabla u\|^2\right.\\&\qquad\left.+\frac{c_*^{2j+2}(2\chi_1)^j(1-l_1)^{2j+1}}{2}(h_1\diamond\nabla u)(t)\right)\\|I_2|&\le\frac{j\delta^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx+\frac{\delta^{j+1}}{j+1}\left(\frac{c_*^{2l}}{2}\chi_2^{l-1}\|\nabla v\|^2\right.\\&\qquad\left.+\frac{c_*^{2j+2}(2\chi_1)^j(1-l_1)^{2j+1}}{2}(h_1\diamond\nabla u)(t)\right).\end{aligned}\tag{53}$$

From (A1) assumption, Hölder's inequality, and Young's inequality, we get

$$\begin{aligned}&\left|\int_\Omega\nabla u(t)\int_0^t h_1(t-s)(\nabla u(t)-\nabla u(s))dsdx\right|\\&\quad\le\delta\|\nabla u\|^2+\frac{1-l_1}{4\delta}(h_1\diamond\nabla u)(t).\end{aligned}\tag{54}$$

In order to estimate the forth term, we use Young's inequality, Sobolev-Poincare inequality, Hölder's inequality, and (A1) assumption

$$\begin{aligned}&\left|-\int_\Omega\nabla u_t(t)\int_0^t h_1'(t-s)(\nabla u(t)-\nabla u(s))dsdx\right|\\&\quad\le\delta\|\nabla u_t\|^2-\frac{h_1(0)}{4\delta}\left(h_1'\diamond\nabla u\right)(t).\\&\left|-\frac{1}{\eta+1}\int_\Omega|u_t|^\eta u_t\int_0^t h_1'(t-s)(u(t)-u(s))dsdx\right|\\&\quad\le\frac{1}{\eta+1}\left[\delta\|u_t\|_{2(\eta+1)}^{2(\eta+1)}+\frac{1}{4\delta}\int_\Omega\left(\int_0^t h_1'(t-s)(u(t)-u(s))ds\right)^2dx\right]\\&\quad\le\frac{1}{\eta+1}\left[\delta\|u_t\|_{2(\eta+1)}^{2(\eta+1)}-\frac{h_1(0)c_*^2}{4\delta}\int_\Omega\int_0^t h_1'(t-s)|\nabla u(t)-\nabla u(s)|^2dsdx\right]\\&\quad\le\frac{\delta c_*^{2(\eta+1)}}{\eta+1}\left(\frac{2(\kappa+2)E(0)}{\kappa+1}\right)^\eta\|\nabla u_t\|^2-\frac{h_1(0)c_*^2}{4\delta(\eta+1)}\left(h_1'\diamond\nabla u\right)(t).\end{aligned}\tag{55}$$

Combining these estimates (34)-(40) and (33) becomes

$$\begin{aligned}F_1'(t)&\le\left[\left(2\delta+\frac{1}{2\delta}\right)(1-l_1)+\frac{\delta^{j+1}c_*^{2j+2}(2\chi_1)^j(1-l_1)^{2j+1}}{j+1}\right]\\&\quad\cdot(h_1\diamond\nabla u)(t)+\left(\delta+2\delta(1-l_1)^2+\frac{\delta^{j+1}}{j+1}\frac{c_*^{2k}}{2}\chi_1^{k-1}\right)\\&\quad\cdot\|\nabla u\|^2+\frac{\delta^{j+1}}{j+1}\frac{c_*^{2l}}{2}\chi_2^{l-1}\|\nabla v\|^2-\frac{h_1(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\\&\quad\cdot\left(h_1'\diamond\nabla u\right)(t)+\left(\delta+\frac{\delta c_*^{2(\eta+1)}}{\eta+1}\left(\frac{2(\kappa+2)E(0)}{\kappa+1}\right)^\eta\right.\\&\quad\left.-\int_0^t h_1(s)ds\right)\|\nabla u_t\|^2+\frac{2j\delta^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)\\&\quad\cdot|u_t|^{j+1}dx-\frac{1}{\eta+1}\left(\int_0^t h_1(s)ds\right)\|u_t\|_{\eta+2}^{\eta+2}\\&\quad-\int_\Omega f_1(u,v)\int_0^t h_1(t-s)(u(t)-u(s))dsdx.\end{aligned}\tag{56}$$

Similarly, let

$$F_2(t)={}_\Omega\left(\Delta v_t-\frac{1}{\eta+1}|v_t|^\eta v_t\right)_0^t h_2(t-s)(v(t)-v(s))dsdx,\tag{57}$$

then

$$F_2'(t) \le \left[\left(2\delta+\frac{1}{2\delta}\right)(1-l_2)+\frac{\delta^{s+1}c_*^{2s+2}(2\chi_2)^s(1-l_2)^{2s+1}}{s+1}\right] \cdot (h_2\diamond\nabla v)(t)+\left(\delta+2\delta(1-l_2)^2+\frac{\delta^{s+1}}{s+1}\frac{c_*^{2\theta}}{2}\delta_2^{\theta-1}\right) \cdot \|\nabla v\|^2+\frac{\delta^{s+1}}{s+1}\frac{c_*^{2\varrho}}{2}\chi_1^{\varrho-1}\|\nabla u\|^2-\frac{h_2(0)}{4\delta}\left(1+\frac{c_*^2}{\eta+1}\right) \cdot \left(h_2'\diamond\nabla v\right)(t)+\left(\delta+\frac{\delta c_*^{2(\eta+1)}}{\eta+1}\left(\frac{2(\kappa+2)E(0)}{\kappa+1}\right)^\eta - \int_0^t h_2(s)ds\right)\|\nabla v_t\|^2+\frac{2s\delta^{-s+1/s}}{s+1}\int_\Omega\left(|v|^\theta+|u|^\varrho\right) \cdot |v_t|^{s+1}dx-\int_\Omega f_2(u,v)\int_0^t h_2(t-s)(v(t)-v(s))dsdx - \frac{1}{\eta+1}\left(\int_0^t h_2(s)ds\right)\|v_t\|_{\eta+2}^{\eta+2}. \tag{58}$$

Since the function h_i $(i=1,2)$ is positive, then for any $t_0 > 0$,

$$\int_0^t h_i(s)ds \ge \int_0^{t_0} h_i(s)ds = h_i(0) \ge h_3 > 0, \forall t \ge t_0. \tag{59}$$

Hence, we conclude from (17), (10), (32), (41), and (42) that

$$\begin{aligned}\Gamma'(t) \le{}& -\frac{1}{\eta+1}(h_3-\varepsilon)\left(\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2}\right)-\left(h_3-\varepsilon-\delta - \frac{\delta c_*^{2(\eta+1)}}{\eta+1}\left(\frac{2(\kappa+2)E(0)}{\kappa+1}\right)^\eta\right)\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\ &-\left(\frac{\varepsilon l_1}{4}-c_2\right)\|\nabla u\|^2-\left(\frac{\varepsilon l_2}{4}-c_3\right)\|\nabla v\|^2 +\left(\frac{\varepsilon(1-l_1)}{2l_1}+c_4\right)(h_1\diamond\nabla u)(t)+\left(\frac{\varepsilon(1-l_2)}{2l_2}+c_5\right) \cdot (h_2\diamond\nabla v)(t)+2\varepsilon(\kappa+2)\int_\Omega F(u,v)dx\\ &+\left[\frac{M}{2}-\frac{h_1(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\right]\left(h_1'\diamond\nabla u\right)(t) +\left[\frac{M}{2}-\frac{h_2(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\right]\left(h_2'\diamond\nabla v\right)(t)\\ &+\left(\frac{2j\beta_1^{-j+1/j}}{j+1}(\varepsilon+1)-M\right)\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx +\left(\frac{2s\beta_2^{-s+1/s}}{s+1}(\varepsilon+1)-M\right)\int_\Omega\left(|v|^\theta+|u|^\rho\right)|v_t|^{s+1}dx\\ &-\int_\Omega f_1(u,v)\int_0^t h_1(t-s)(u(t)-u(s))dsdx -\int_\Omega f_2(u,v)\int_0^t h_2(t-s)(v(t)-v(s))dsdx,\end{aligned} \tag{60}$$

where

$$\begin{aligned}c_2&=2\delta(1-l_1)^2+\frac{\delta^{j+1}c_*^{2k}\chi_1^{k-1}}{2(j+1)}+\frac{\delta^{s+1}c_*^{2\rho}\chi_1^{\rho-1}}{2(s+1)},\\ c_3&=2\delta(1-l_2)^2+\frac{\delta^{s+1}c_*^{2\theta}\chi_2^{\theta-1}}{2(s+1)}+\frac{\delta^{j+1}c_*^{2l}\chi_2^{l-1}}{2(j+1)},\\ c_4&=\left(2\delta+\frac{1}{2\delta}\right)(1-l_1)+\frac{\delta^{j+1}c_*^{2j+2}(2\chi_1)^j(1-l_1)^{2j+1}}{j+1},\\ c_5&=\left(2\delta+\frac{1}{2\delta}\right)(1-l_2)+\frac{\delta^{s+1}c_*^{2s+2}(2\chi_2)^s(1-l_2)^{2s+1}}{s+1}.\end{aligned} \tag{61}$$

By using Lemma 8 and (15) for the last two terms of (43), we obtain

$$\begin{aligned}\Gamma'(t) \le{}& -\frac{1}{\eta+1}(\varepsilon-h_3)\left(\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2}\right) -\left(\varepsilon+\delta+\frac{\delta c_*^{2(\eta+1)}}{\eta+1}\left(\frac{2(\kappa+2)E(0)}{\kappa+1}\right)^\eta-h_3\right) \cdot \left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\ &-\left(\frac{\varepsilon l_1}{4}-(c_2+(\beta_1+\beta_2)\delta l_1)\right) \cdot \|\nabla u\|^2-\left(\frac{\varepsilon l_2}{4}-\varepsilon(c_3+(\beta_1+\beta_2)\delta l_2)\right)\|\nabla v\|^2\\ &+\left[\left(\frac{\varepsilon}{2l_1}+\frac{c_*^2}{4\delta}\right)(1-l_1)+c_4\right](h_1\diamond\nabla u)(t) +\left[\left(\frac{\varepsilon}{2l_2}+\frac{c_*^2}{4\delta}\right)(1-l_2)+c_5\right](h_2\diamond\nabla v)(t)\\ &+\left[\frac{M}{2}-\frac{h_1(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\right]\left(h_1'\diamond\nabla u\right)(t) +\left[\frac{M}{2}-\frac{h_2(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\right]\left(h_2'\diamond\nabla u\right)(t)\\ &+2\varepsilon(\kappa+2)\int_\Omega F(u,v)dx+\left(\frac{2j\beta_1^{-j+1/j}}{j+1}(\varepsilon+1)-M\right) \cdot \int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx+\left(\frac{2s\beta_2^{-s+1/s}}{s+1}(\varepsilon+1)-M\right) \cdot \int_\Omega\left(|v|^\theta+|u|^\rho\right)|v_t|^{s+1}dx.\end{aligned} \tag{62}$$

At this point, we choose ε and δ which are small enough, and we have

$$\begin{aligned}&\frac{1}{\eta+1}(h_3-\varepsilon)>0,\\ &h_3-\varepsilon-\delta-\frac{\delta c_*^{2(\eta+1)}}{\eta+1}\left(\frac{2(\kappa+2)E(0)}{\kappa+1}\right)^\eta>0,\\ &\frac{\varepsilon l_1}{4}-(c_2+(\beta_1+\beta_2)\delta l_1)>0,\\ &\frac{\varepsilon l_2}{4}-(c_2+(\beta_1+\beta_2)\delta l_2)>0.\end{aligned} \tag{63}$$

Further, we pick ε so small and

$$\frac{2j\beta_1^{-j+1/j}}{j+1}(\varepsilon+1)-M<0, \frac{2s\beta_2^{-s+1/s}}{s+1}(\varepsilon+1)-M<0. \tag{64}$$

Once δ is fixed, we choose M that is sufficently large so that

$$\frac{M}{2}-\frac{h_1(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\geq 0, \frac{M}{2}-\frac{h_2(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\geq 0. \tag{65}$$

Consequently, for all $t\geq t_0$, we reach at

$$\Gamma'(t)\leq -\xi_1 E(t)+\xi_2((h_1\diamond\nabla u)(t)+(h_2\diamond\nabla v)(t)), \tag{66}$$

where there are positive constants ξ_i, $i=1,2$.

Now, we are ready to state our stability result.

Theorem 10. *Suppose that (4) and (A1)-(A3) hold, and that* $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ *and* $(v_0,v_1)\in H_0^1(\Omega)\times L^2(\Omega)$ *satisfy* $E(0)<E_1$ *and*

$$\left(l_1\|\nabla u_0\|^2+l_2\|\nabla v_0\|^2\right)^{1/2}<\alpha_*. \tag{67}$$

Then for each, the energy of (1) satisfies

$$E(t)\leq Ke^{-k\int_{t_0}^t\delta(s)ds}, t\geq t_0, \tag{68}$$

where $\delta(t):=\min\{\delta_1(t),\delta_2(t)\}$ *and* K *and* k *are positive constants.*

Proof. Multiplying (46) by $\delta(t)$, we get

$$\delta(t)\Gamma'(t)\leq -\xi_1\delta(t)E(t)+\xi_2\delta(t)[(h_1\diamond\nabla u)(t)+(h_2\diamond\nabla v)(t)]. \tag{69}$$

Applying (A2) and $\delta(t):=\min\{\delta_1(t),\delta_2(t)\}$ and since $-[(h_1'\diamond\nabla u)(t)+(h_2'\diamond\nabla v)(t)]\leq -2E'(t)$ by (10), we obtain

$$\begin{aligned}\delta(t)\Gamma'(t)&\leq -\xi_1\delta(t)E(t)-\xi_2\delta(t)\left[\left(h_1'\diamond\nabla u\right)(t)+\left(h_2'\diamond\nabla v\right)(t)\right]\\&\leq -\xi_1\delta(t)E(t)-2\xi_2E'(t),\forall t\geq t_0.\end{aligned} \tag{70}$$

That is

$$G'(t)\leq -c_*\delta(t)E(t)\leq -k\delta(t)G(t),\forall t\geq t_0. \tag{71}$$

And here, $G(t)=\delta(t)\Gamma(t)+CE(t)$ is equivalent to $E(t)$ due to (20), and k is a positive constant. A simple integration of (50) leads to

$$G(t)\leq G(t_0)e^{-k\int_{t_0}^t\delta(s)ds},\forall t\geq t_0. \tag{72}$$

This completes the proof.

5. Conclusion

As far as we know, there have not been any global existences and general decay results in the literature known for quasilinear viscoelastic equations with degenerate damping terms. Our work extends the works for some quasilinear viscoelastic equations treated in the literature to the quasilinear viscoelastic equation with degenerate damping terms.

Data Availability

No data were used to support the study.

Disclosure

Title for this paper "Global existence and general decay of solutions for a quasilinear system with degenerate damping terms" has been submitted in "Conference Proceeding of 9th International Eurasian Conference on Mathematical Sciences and Applications."

Conflicts of Interest

The authors declare that they have no competing interests.

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Research Article

Iterative Solutions for the Differential Equation with p-Laplacian on Infinite Interval

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This paper systematically investigates a class of fourth-order differential equation with p-Laplacian on infinite interval in Banach space. By means of the monotone iterative technique, we establish not only the existence of positive solutions but also iterative schemes under the suitable conditions. At last, we give an example to demonstrate the application of the main result.

1. Introduction

The partial differential equation with the p-Laplacian operator

$$-\text{div}\left(|\nabla_u(x)|^{p-2}\nabla u(x)\right) + b(x)\varphi(u(x)) = 0, \tag{1}$$

which is often used to describe, for example, diffusion process [1], with a spatial symmetric potential b, can be reduced to $(r(t)\varphi_p(y'(t)))' + c(t)\varphi_p(y(t)) = 0$, where $\varphi_p(x) = |x|^{p-2}x$, $p > 1$. This fact leads us to study the following p-Laplacian boundary value problem (BVP):

$$\begin{cases} \left(\varphi_p\left(u''(t)\right)\right)'' - k^2\varphi_p\left(u''(t)\right) = a(t)f(t, u(t)), \in (0, +\infty), \\ u(0) = \displaystyle\int_0^\infty g(t)u(t)dt, \lim_{t\longrightarrow+\infty} u'(t) = \xi, u''(0) = \lim_{t\longrightarrow+\infty} u''(t) = 0, \end{cases} \tag{2}$$

where $\varphi_p(x) = |x|^{p-2}x$, $p > 1$, $\varphi_q = \varphi_p^{-1}$, $(1/p) + (1/q) = 1$, $k > 0$, $\xi \geq 0$ are real constants, $g : [0, +\infty) \longrightarrow [0, +\infty)$ is a Lebesgue integrable function with $\int_0^\infty g(t)dt < 1$, $\int_0^\infty tg(t)dt < +\infty$, $a : (0, +\infty) \longrightarrow [0, +\infty)$ is continuous and may be singular at $t = 0$, and $f : [0, +\infty) \times [0, +\infty) \longrightarrow [0, +\infty)$ is a continuous function.

The p-Laplacian equation arises quite naturally in the modeling of different physical and natural phenomena. For instance, in fluid mechanics, the shear stress $\vec{\tau}$ and the velocity gradient $\nabla_p u$ of certain fluids obey a relation of the form $\vec{\tau} = a(x)\nabla_p u$, where $\nabla_p u = |\nabla_u|^{p-2}\nabla u$. Here, the real number $p > 1$ and $p = 2$ (respectively, $p < 2$, $p > 2$) designate a Newtonian (respectively, pseudoplastic, dilatant) fluid. Given a is a constant, the resulting equations of motion then involve $\text{div}\,(a\nabla_p u)$, which reduces to $a\Delta_p u = a\ \text{div}\ \nabla_p u$. Over the last couple of decades, many important results including integral and fractional equations with p-Laplacian on certain boundary value conditions had been obtained. We refer the reader to [2–17] and the references cited therein. Liang and Zhang [18] considered the m-point BVP with a p-Laplacian operator

$$\begin{cases} \left(\varphi_p\left(u'(t)\right)\right)' + h(t)f(u(t)) = 0, t \in (0, +\infty), \\ u(0) = \displaystyle\sum_{i=1}^{m-2} \alpha_i u(\xi_i), \lim_{t\longrightarrow+\infty} u'(t) = 0, \end{cases} \tag{3}$$

where $\xi_i \in (0, +\infty)$, with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, and α_i satisfies $\alpha_i \in [0, +\infty)$, $0 \leq \sum_{i=1}^{m-2} \alpha_i < 1$, $h : [0, +\infty) \longrightarrow [0, +\infty)$ and has countably many singularities in $[1, +\infty)$, $f : [$

$0, +\infty) \longrightarrow [0, +\infty)$ is a continuous function. The existence of positive solutions is obtained by applying the fixed-point theorem of Leggett-Williams.

Using the fixed point index theory, Xu and Yang in [19] researched the existence of positive solutions for the fourth order p-Laplacian BVP

$$\begin{cases} \left(\left|u''\right|^{p-1}u''\right)'' = f(t,u), t \in (0,1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, i = 0, 1, \end{cases} \tag{4}$$

where $p > 0$, $f : [0,1] \times [0,+\infty) \longrightarrow [0,+\infty)$ is a continuous function.

The motivation for the present work stems from both practical and theoretical aspects. In fact, many mathematical problems in science and engineering are set in unbounded domains, such as unsteady flow of gas through a semi-infinite porous media, the theory of drain flows, plasma physics, in determining the electrical potential in an isolated neutral atom. In all these applications, it is frequent that only positive solutions are useful. In this paper, we study the differential equation with p-Laplacian operator as BVP (2); when $p = 2$, BVP (2) becomes the ordinary fourth-order differential equation. The results for the existence of the maximal and minimal solutions to the BVP (2) are established. In addition, we establish iterative schemes for approximating the solutions, which start from the known simple linear functions. However, to the best knowledge of the authors, there are few works in the literature dealing with the existence of positive solutions to boundary value problems of differential equation on infinite intervals with p-Laplacian operator by using iterative technique up to now. The goal of the present paper is to fill the gap in this area, so it is interesting and important to study the existence of positive solutions for BVP (2).

2. Preliminaries and Lemmas

The basic space used in this paper is E, where E is denoted by

$$E = \left\{ u \in C[0,+\infty): \sup_{t\in[0,+\infty)} \frac{|u(t)|}{1+t} < +\infty \right\}. \tag{5}$$

Then, E is a Banach space equipped with the norm $\|u\| = \sup_{t\in[0,+\infty)}(|u(t)|/(1+t))$. Define a cone K in the Banach space E by $K = \{u \in E : u(t) \geq 0, t \in [0,+\infty)\}$.

Lemma 1 (see [20]). *Let $y \in C[0,+\infty) \cap L^1(0,+\infty)$, then x is a solution of*

$$\begin{cases} x''(t) - k^2 x(t) + y(t) = 0, t \in (0,+\infty), \\ x(0) = \lim_{t\longrightarrow+\infty} x(t) = 0, \end{cases} \tag{6}$$

if and only if x is a solution of

$$x(t) = \int_0^\infty G(t,s)y(s)ds, \tag{7}$$

where

$$G(t,s) = \frac{1}{2k} \begin{cases} e^{-ks}\left(e^{kt} - e^{-kt}\right), 0 \leq t \leq s < +\infty, \\ e^{-kt}\left(e^{ks} - e^{-ks}\right), 0 \leq s \leq t < +\infty. \end{cases} \tag{8}$$

Lemma 2. *Green's function $G(t,s)$ satisfies*

(1) $G(t,s) \geq 0$, for any $t, s \in [0,+\infty)$

(2) $G(t,s) \leq G(s,s) \leq 1/2k$, for any $t, s \in [0,+\infty)$

In what follows, we list some conditions for convenience.

(H_0) $g : [0,+\infty) \longrightarrow [0,+\infty)$ is a Lebesgue integrable function with $\int_0^\infty g(t)dt < 1$, $\int_0^\infty tg(t)dt < +\infty$.

(H_1) $f : [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty)$ is a continuous function, $f(t,0) \equiv 0$ on $[0,+\infty)$ and $f(t,(1+t)x)$ is bounded, for $t \in [0,+\infty)$, $x \in D$, $D \subset [0,+\infty)$ is a closed subinterval.

(H_2) $a : (0,+\infty) \longrightarrow [0,+\infty)$ is continuous, $a(t) \equiv 0$ on $[0,+\infty)$ and

$$\begin{aligned} 0 &< \int_0^\infty G(s,s)a(s)ds \\ &< +\infty, \int_0^\infty s\varphi_q\left(\int_0^\infty G(s,\varsigma)a(\varsigma)d\varsigma\right)ds < +\infty. \end{aligned} \tag{9}$$

By routine discussion, Lemma 3 is valid.

Lemma 3. *Assume that (H_0) holds and $\varphi_q(x) \in C[0,+\infty) \cap L^1(0,+\infty)$; then, u is a solution of*

$$\begin{cases} u''(t) + \varphi_q(x(t)) = 0, t \in (0,+\infty), \\ u(0) = \int_0^\infty g(t)u(t)dt, \lim_{t\longrightarrow+\infty} u'(t) = \xi, \end{cases} \tag{10}$$

if and only if u is a solution of

$$\begin{aligned} u(t) = {} & \frac{1}{1 - \int_0^\infty g(t)dt}\left(\xi\int_0^\infty tg(t)dt \right. \\ & \left. + \int_0^\infty g(t)\int_0^t\int_s^\infty \varphi_q(x(\tau))d\tau ds dt\right) \\ & + \int_0^t\int_s^\infty \varphi_q(x(\tau))d\tau ds + \xi t. \end{aligned} \tag{11}$$

Let $x(t) = -\varphi_p(u''(t))$, then BVP (2) is divided into the following two parts:

$$\begin{cases} x''(t) - k^2x(t) + a(t)f(t, u(t)) = 0, t \in (0, +\infty), \\ x(0) = \lim_{t\longrightarrow+\infty} x(t) = 0, \end{cases}$$
$$\begin{cases} u''(t) + \varphi_q(x(t)) = 0, t \in (0, +\infty), \\ u(0) = \int_0^\infty g(t)u(t)dt, \lim_{t\longrightarrow+\infty} u'(t) = \xi. \end{cases} \tag{12}$$

From Lemmas 1 and 3, under the above assumptions $(H_0) - (H_2)$, denote the operator $A : K \longrightarrow E$ as follows:

$$\begin{aligned} Au(t) = {} & \frac{1}{1-\int_0^\infty g(t)dt}\left(\xi\int_0^\infty tg(t)dt + \int_0^\infty g(t)\int_0^t\int_s^\infty \varphi_q\right. \\ & \left.\cdot\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right)d\tau ds dt\right) \\ & + \int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right)d\tau ds \\ & + \xi t, t\in[0,+\infty). \end{aligned} \tag{13}$$

Now, we claim that Au is well defined for $u \in K$. In fact, for any $u \in K$, there exists $r > 0$, such that $|(u(t))/(1+t)| \le r, t \in [0, +\infty)$. From (H_1) and the definition of $\|\cdot\|$, we have

$$S_f^r \coloneqq \sup\{f(t,(1+t)x) : (t,x) \in [0,+\infty)\times[0,r]\} < +\infty. \tag{14}$$

Thus, by $(H_1)(H_2)$, we know

$$\begin{aligned} & \int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right)d\tau ds \\ & \quad \le \varphi_q\left(S_f^r\right)\int_0^\infty s\varphi_q\left(\int_0^\infty G(s,\varsigma)a(\varsigma)d\varsigma\right)ds < +\infty. \end{aligned} \tag{15}$$

Since

$$\int_0^\infty g(t)dt < 1, \int_0^\infty tg(t)dt < +\infty. \tag{16}$$

Together with (15), for any $t \in [0, +\infty)$, we can see that

$$\begin{aligned} \frac{|Au(t)|}{1+t} \le {} & \frac{1}{1-\int_0^\infty g(t)dt}\left(\xi\int_0^\infty tg(t)dt + \int_0^\infty g(t)\int_0^t\int_s^\infty \varphi_q\right. \\ & \left.\cdot\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right)d\tau ds dt\right) \\ & + \int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right)d\tau ds + \xi \\ \le {} & \frac{1}{1-\int_0^\infty g(t)dt}\left(\xi\int_0^\infty tg(t)dt + \varphi_q\left(S_f^r\right)\int_0^\infty s\varphi_q\right. \\ & \left.\cdot\left(\int_0^\infty G(s,\varsigma)a(\varsigma)d\varsigma\right)ds\int_0^\infty g(t)dt\right) \\ & + \varphi_q\left(S_f^r\right)\int_0^\infty s\varphi_q\left(\int_0^\infty G(s,\varsigma)a(\varsigma)d\varsigma\right)ds + \xi. \end{aligned} \tag{17}$$

So, by (15) and (17), we obtain

$$\sup_{t\in[0,+\infty)} \frac{|Au(t)|}{1+t} < +\infty. \tag{18}$$

On the other hand, for any $t_1, t_2 \in [0, +\infty)$, by (13), we have

$$\begin{aligned} & |Au(t_1) - Au(t_2)| \\ & \quad \le \left|\int_{t_1}^{t_2}\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right)d\tau ds\right| \\ & \qquad + \xi|t_1 - t_2| \longrightarrow 0, as\, t_1 \longrightarrow t_2. \end{aligned} \tag{19}$$

Therefore, $Au \in C[0, +\infty)$, for any $u \in K$. Hence, $A : K \longrightarrow E$ is well defined. Obviously, u is a positive solution of BVP (2) if and only if u is a fixed point of A in K.

The Arzela-Ascoli theorem fails to work in the Banach space E due to the fact that the infinite interval $[0, +\infty)$ is noncompact. The following compactness criterion will help us to resolve this problem.

Lemma 4 (see [21, 22]). *Let E be defined as (5) and M be any bounded subset of E. Then, M is relatively compact in E if $\{x(t)/(1+t) : x \in M\}$ is equicontinuous on any finite subinterval of $[0, +\infty)$, and for any given $\varepsilon > 0$, there exists $N > 0$, such that $|(x(t_1)/(1+t_1)) - (x(t_2)/(1+t_2))| < \varepsilon$ uniformly with respect to $x \in M$, as $t_1, t_2 > N$.*

3. Main Results

Lemma 5. *Assume that $(H_0) - (H_2)$ hold. Then, $A : K \longrightarrow K$ is completely continuous.*

Proof. It is clear that $Au(t) \ge 0$ for any $u \in K$, $t \in [0, +\infty)$. Thus, $A(K) \subseteq K$. Now, we prove that A is continuous and compact, respectively. Let $M \subset K$ be a bounded subset. Then, there exists $R > 0$, such that $\|x\| < R$, for any $x \in M$. So, for any $x \in M$, we have

$$\|Au\| = \sup_{t\in[0,+\infty)} \frac{1}{1+t} \left| \frac{1}{1-\int_0^\infty g(t)dt} \left(\xi \int_0^\infty tg(t)dt + \int_0^\infty g(t) \cdot \int_0^t \int_s^\infty \varphi_q \left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma \right) d\tau ds dt \right) + \int_0^t \int_s^\infty \varphi_q \left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma \right) d\tau ds + \xi t \right| \le \frac{1}{1-\int_0^\infty g(t)dt} \left(\xi \int_0^\infty tg(t)dt + \varphi_q\left(S_f^R\right) \int_0^\infty s\varphi_q \cdot \left(\int_0^\infty G(s,\varsigma)a(\varsigma)d\varsigma \right) ds \int_0^\infty g(t)dt \right) + \varphi_q\left(S_f^R\right) \int_0^\infty s\varphi_q \left(\int_0^\infty G(s,\varsigma)a(\varsigma)d\varsigma \right) ds + \xi < +\infty, \tag{20}$$

where

$$S_f^R := \sup\{f(t,(1+t)x) : (t,x) \in [0,+\infty)\times[0,R]\} < +\infty. \tag{21}$$

So, AM is bounded in E. Moreover, given $T \in (0,+\infty)$, for any $x \in M$ and $t_1, t_2 \in [0,T]$, without loss of generality, we may assume that $t_1 < t_2$. In fact,

$$\left| \frac{Au(t_1)}{1+t_1} - \frac{Au(t_2)}{1+t_2} \right| \le \left| \frac{1}{1+t_1} - \frac{1}{1+t_2} \right| \frac{1}{1-\int_0^\infty g(t)dt} \left(\xi \int_0^\infty tg(t)dt + \int_0^\infty g(t) \cdot \int_0^t \int_s^\infty \varphi_q \left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma \right) d\tau ds dt \right) + \left| \frac{1}{1+t_1} \int_0^{t_1} \int_s^\infty \varphi_q \left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma \right) d\tau ds - \frac{1}{1+t_2} \int_0^{t_2} \int_s^\infty \varphi_q \left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma \right) d\tau ds \right| + \xi \left| \frac{t_1}{1+t_1} - \frac{t_2}{1+t_2} \right| \le \left| \frac{1}{1+t_1} - \frac{1}{1+t_2} \right| \frac{1}{1-\int_0^\infty g(t)dt} \cdot \left(\xi \int_0^\infty tg(t)dt + \varphi_q\left(S_f^R\right) \int_0^\infty g(t)dt \int_0^\infty s\varphi_q \cdot \left(\int_0^\infty G(s,\varsigma)a(\varsigma)d\varsigma \right) ds \right) + \left| \frac{1}{1+t_1} - \frac{1}{1+t_2} \right| \varphi_q\left(S_f^R\right) \cdot \int_0^{t_1} \int_s^\infty \varphi_q \left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)d\varsigma \right) d\tau ds + \varphi_q\left(S_f^R\right) \int_{t_1}^{t_2} \int_s^\infty \varphi_q \left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)d\varsigma \right) d\tau ds + \xi \left| \frac{t_1}{1+t_1} - \frac{t_2}{1+t_2} \right|. \tag{22}$$

So, for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any t_1, $t_2 \in [0,T]$ with $|t_1 - t_2| < \delta$, and for any $x \in M$, we have

$$\left| \frac{(Au)(t_1)}{1+t_1} - \frac{(Au)(t_2)}{1+t_2} < \right| \delta. \tag{23}$$

Hence, $\{(Au)(t)/(1+t) : u \in M\}$ is equicontinuous on $[0,T]$. Since $T > 0$ is arbitrary, $\{(Au)(t)/(1+t) : u \in M\}$ is locally equicontinuous on $[0,+\infty)$.

Next, we prove that we show that $A : K \longrightarrow K$ is equiconvergent at $+\infty$. For any $u \in M$, we have

$$\lim_{t\to+\infty} \left| \frac{Au(t)}{1+t} \right| = \lim_{t\to+\infty} \frac{1}{1+t} \left| \frac{1}{1-\int_0^\infty g(t)dt} \left(\xi \int_0^\infty tg(t)dt + \int_0^\infty g(t) \cdot \int_0^t \int_s^\infty \varphi_q \left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma \right) d\tau ds dt \right) + \int_0^t \int_s^\infty \varphi_q \left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma \right) d\tau ds + \xi t \right| = \lim_{t\to+\infty} \int_t^\infty \varphi_q \left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma \right) d\tau + \xi = \xi. \tag{24}$$

So, for any $u \in M$, we have

$$\left| \frac{Au(t)}{1+t} - \xi \right| = \left| \frac{1}{1+t} \left(\frac{1}{1-\int_0^\infty g(t)dt} \left(\xi \int_0^\infty tg(t)dt + \int_0^\infty g(t) \cdot \int_0^t \int_s^\infty \varphi_q \left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma \right) d\tau ds dt \right) + \int_0^t \int_s^\infty \varphi_q \left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma \right) d\tau ds + \xi t \right) - \xi \right| \le \left| \frac{1}{1+t} \left(\frac{1}{1-\int_0^\infty g(t)dt} \left(\xi \int_0^\infty tg(t)dt + \varphi_q\left(S_f^R\right) \cdot \int_0^\infty s\varphi_q \left(\int_0^\infty G(s,\varsigma)a(\varsigma)d\varsigma \right) ds \int_0^\infty g(t)dt \right) + \varphi_q\left(S_f^R\right) \int_0^\infty s\varphi_q \left(\int_0^\infty G(s,\varsigma)a(\varsigma)d\varsigma \right) ds \right) - \frac{\xi}{1+t} \right| \longrightarrow 0,\ t \longrightarrow +\infty. \tag{25}$$

Thus, for any $\varepsilon > 0$, there exists $N > 0$, for any $t > N$ and for any $x \in M$, such that

$$\left| \frac{Au(t)}{1+t} - \xi \right| < \frac{\varepsilon}{2}. \tag{26}$$

Consequently, for any $t_1, t_2 > N$ and for any $x \in M$, we have

$$\begin{gathered}\left|\frac{Au(t_1)}{1+t_1} - \xi\right| < \frac{\varepsilon}{2}, \\ \left|\frac{Au(t_2)}{1+t_2} - \xi\right| < \frac{\varepsilon}{2}, \\ t_1, t_2 > N.\end{gathered} \tag{27}$$

Therefore, for any $t_1, t_2 > N$ and for any $x \in M$, we have

$$\begin{aligned}\left|\frac{Au(t_1)}{1+t_1} - \frac{Au(t_2)}{1+t_2}\right| &= \left|\left(\frac{Au(t_1)}{1+t_1} - \xi\right) - \left(\frac{Au(t_2)}{1+t_2} - \xi\right)\right| \\ &\le \left|\frac{Au(t_1)}{1+t_1} - \xi\right| + \left|\frac{Au(t_2)}{1+t_2} - \xi\right| \\ &< \varepsilon, \quad t_1, t_2 > N.\end{aligned} \tag{28}$$

This implies that AM is equiconvergent at $+\infty$.

Let $u_n \longrightarrow u$ as $n \longrightarrow +\infty$; then, there exists r_0 such that $\max_{n\in\mathbb{N}\setminus\{0\}}\{\|u_n\|, \|u\|\} < r_0$, $\mathbb{N}$ is a natural number set. By (H_2), we have

$$\begin{aligned}&\left|\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u_n(\varsigma))d\varsigma - \int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right| \\ &\quad\le \int_0^\infty G(\tau,\varsigma)a(\varsigma)|f(\varsigma,u_n(\varsigma)) - f(\varsigma,u(\varsigma))|d\varsigma \\ &\quad\le 2S_f^{r_0}\int_0^\infty G(\tau,\varsigma)a(\varsigma)d\varsigma < +\infty,\end{aligned} \tag{29}$$

where

$$S_f^{r_0} \coloneqq \sup\{f(t,(1+t)x) : (t,x) \in [0,+\infty) \times [0,r_0]\} < +\infty. \tag{30}$$

So, for any $\varepsilon > 0$, we can find a sufficiently large $H_0 > 0$, such that

$$S_f^{r_0}\int_{H_0}^\infty G(\tau,\varsigma)a(\varsigma)d\varsigma < \frac{\varepsilon}{4}. \tag{31}$$

It follows from the Lebesgue dominated convergence theorem and continuity of f, we can get

$$\begin{aligned}&\left|\int_0^{H_0} G(\tau,\varsigma)a(\varsigma)f(\varsigma,u_n(\varsigma))d\varsigma - \int_0^{H_0} G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right| \\ &\quad\le \int_0^{H_0} G(\tau,\varsigma)a(\varsigma)|f(\varsigma,u_n(\varsigma)) - f(\varsigma,u(\varsigma))|d\varsigma \\ &\quad\longrightarrow 0, n \longrightarrow +\infty, \tau \in [0,+\infty).\end{aligned} \tag{32}$$

So, for the above $\varepsilon > 0$, there exists $N > 0$, when $n > N_0$, we have

$$\int_0^{H_0} G(\tau,\varsigma)a(\varsigma)|f(\varsigma,u_n(\varsigma)) - f(\varsigma,u(\varsigma))|d\varsigma < \frac{\varepsilon}{2}. \tag{33}$$

Therefore, when $n > N_0$,

$$\begin{aligned}&\left|\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u_n(\varsigma))d\varsigma - \int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right| \\ &\quad\le \int_0^\infty G(\tau,\varsigma)a(\varsigma)|f(\varsigma,u_n(\varsigma) - f\varsigma,u_n(\varsigma))|d\varsigma \\ &\quad\le \int_0^{H_0} G(\tau,\varsigma)a(\varsigma)|f(\varsigma,u_n(\varsigma) - f\varsigma,u_n(\varsigma))|d\varsigma \\ &\qquad+ \int_{H_0}^\infty G(\tau,\varsigma)a(\varsigma)|f(\varsigma,u_n(\varsigma) - f\varsigma,u_n(\varsigma))|d\varsigma \\ &\quad\le \frac{\varepsilon}{2} + 2S_f^{r_0}\int_{H_0}^\infty G(\tau,\varsigma)a(\varsigma)d\varsigma < \varepsilon,\end{aligned} \tag{34}$$

which implied that

$$\begin{aligned}&\left|\tau\varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u_n(\varsigma))d\varsigma\right)\right. \\ &\qquad\left.- \tau\varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right)\right| \longrightarrow 0,\end{aligned} \tag{35}$$

as $n \longrightarrow +\infty, \tau \in [0,+\infty)$. Since

$$\begin{aligned}&\left|\int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u_n(\varsigma))d\varsigma\right)d\tau ds\right. \\ &\qquad\left.- \int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right)d\tau ds\right| \\ &\quad\le 2\varphi_q\left(S_f^{r_0}\right)\int_0^\infty s\varphi_q\left(\int_0^\infty G(s,\varsigma)a(\varsigma)d\varsigma\right)ds < +\infty.\end{aligned} \tag{36}$$

Then, by the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}&\|Au_n - Au\| \\ &\quad= \sup_{t\longrightarrow+\infty}\frac{1}{1+t}\left|\frac{1}{1-\int_0^\infty g(t)dt}\int_0^\infty g(t)\right. \\ &\qquad\cdot\int_0^t\int_s^\infty\left(\varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u_n(\varsigma))d\varsigma\right)\right. \\ &\qquad\left.- \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right)\right)d\tau ds dt \\ &\qquad+ \int_0^t\int_s^\infty\left(\varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u_n(\varsigma))d\varsigma\right)\right. \\ &\qquad\left.\left.- \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right)\right)d\tau ds\right|\end{aligned}$$

$$\leq \frac{1}{1-\int_0^\infty g(t)dt}\int_0^\infty g(t)\int_0^t\int_s^\infty \cdot\left|\varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u_n(\varsigma))d\varsigma\right) - \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right)\right| d\tau ds dt + \int_0^t\int_s^\infty \left|\varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u_n(\varsigma))d\varsigma\right) - \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right)\right| d\tau ds$$
$$\leq \left(\frac{\int_0^\infty g(t)dt}{1-\int_0^\infty g(t)dt}+1\right)\int_0^\infty s \cdot\left|\varphi_q\left(\int_0^\infty G(s,\varsigma)a(\varsigma)f(\varsigma,u_n(\varsigma))d\varsigma\right) - \varphi_q\left(\int_0^\infty G(s,\varsigma)a(\varsigma)f(\varsigma,u(\varsigma))d\varsigma\right)\right| ds \longrightarrow 0, n\longrightarrow +\infty. \tag{37}$$

Therefore, $A : K \longrightarrow K$ is continuous. In conclusion, by Lemma 4, we know that $A : K \longrightarrow K$ is completely continuous. The proof is completed. □

Theorem 6. *Assume that $(H_0)-(H_2)$ hold and there exists $d > 2m$ which satisfies the following condition:*

$(H_3) f(t,x_1) \leq f(t,x_2)$, $t\in[0,+\infty)$, $0\leq x_1\leq x_2$.

$(H_4) f(t,(1+t)x) \leq \varphi_p(d/2l)$, $(t,x)\in[0,+\infty)\times[0,d]$, where

$$m = \xi\left(1+\frac{\int_0^\infty tg(t)dt}{1-\int_0^\infty g(t)dt}\right),\quad l = \frac{1}{1-\int_0^\infty g(t)dt}\int_0^\infty s\varphi_q\left(\int_0^\infty G(s,\varsigma)a(\varsigma)d\varsigma\right)ds. \tag{38}$$

Then, BVP (2) has the maximal and minimal positive solutions μ^* and $\bar{\mu}^*$ on $[0,+\infty)$, such that

$$0 < \sup_{t\in[0,+\infty)}\frac{|\mu^*(t)|}{1+t} \leq d,\quad 0 < \sup_{t\in[0,+\infty)}\frac{|\bar{\mu}^*(t)|}{1+t} \leq d. \tag{39}$$

Moreover, for initial values

$$\mu_0(t) = \frac{d}{2}+\xi\left(t+\frac{\int_0^\infty tg(t)dt}{1-\int_0^\infty g(t)dt}\right),\quad \bar{\mu}_0(t) = 0,\quad t\in[0,+\infty), \tag{40}$$

define the iterative sequences $\{\mu_n\}$ and $\{\bar{\mu}_n\}$ by

$$\mu_n = \frac{1}{1-\int_0^\infty g(t)dt}\left(\xi\int_0^\infty tg(t)dt + \int_0^\infty g(t) \cdot\int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,\mu_{n-1}(\varsigma))d\varsigma\right)d\tau ds dt\right) + \int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,\mu_{n-1}(\varsigma))d\varsigma\right)d\tau ds + \xi t,$$
$$\bar{\mu}_n = \frac{1}{1-\int_0^\infty g(t)dt}\left(\xi\int_0^\infty tg(t)dt + \int_0^\infty g(t) \cdot\int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,\bar{\mu}_{n-1}(\varsigma))d\varsigma\right)d\tau ds dt\right) + \int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,\bar{\mu}_{n-1}(\varsigma))d\varsigma\right)d\tau ds + \xi t. \tag{41}$$

Then,

$$\lim_{n\longrightarrow+\infty}\sup_{t\in[0,+\infty)}\frac{|\mu_n(t)-\mu^*(t)|}{1+t} = 0,\quad \lim_{n\longrightarrow+\infty}\sup_{t\in[0,+\infty)}\frac{|\bar{\mu}_n(t)-\bar{\mu}^*(t)|}{1+t} = 0. \tag{42}$$

Proof. From Lemma 5, we know that $A : K \longrightarrow K$ is completely continuous. For any $u_1, u_2 \in K$ with $u_1 \leq u_2$, from the definition of A and (H_3), we know that $Au_1 \leq Au_2$. Let $K_d = \{u\in K : \|u\|\leq d\}$. In what follows, we firstly prove $A : K_d \longrightarrow K_d$. In fact, for any $u\in K_d$, we have $0\leq u(t)/(1+t) \leq d$, $t\in[0,+\infty)$. By (H_4), we know that $f(t,x)\leq\varphi_p(d/2l)$, $(t,x)\in[0,+\infty)\times[0,d]$. Also, by (H_4), we have

$$\mu_{n+1}(t) \leq \mu_n(t),\quad t\in[0,+\infty),\ n=0,1,2,\cdots. \tag{43}$$

Thus, we have proved that $A : K_d \longrightarrow K_d$. Let $\mu_0(t) = (d/2)+\xi(t+((\int_0^\infty tg(t)dt)/(1-\int_0^\infty g(t)dt)))$, $t\in[0,+\infty)$, and then, $\mu_0(t)\in K_d$. Let $\mu_1 = A\mu_0$, $\mu_2 = A\mu_1 = A^2\mu_0$; by Theorem 6, we have $\mu_1,\mu_2\in K_d$. Denote $\mu_{n+1} = A\mu_n = A^n \mu_0$, $n=1,2,\cdots$. Since $A : K_d \longrightarrow K_d$, we have $\mu_n \in A(K_d) \subset K_d$. It follows from the complete continuity of A that $\{\mu_n\}_{n=1}^\infty$ is a sequentially compact set. By (H_4), we have

$$\mu_1(t) = A\mu_0(t) = \frac{1}{1-\int_0^\infty g(t)dt}\left(\xi\int_0^\infty tg(t)dt + \int_0^\infty g(t) \cdot\int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,\mu_0(\varsigma))d\varsigma\right)d\tau ds dt\right) + \int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f(\varsigma,\mu_0(\varsigma))dd\varsigma\right)d\tau ds + \xi t$$

$$\leq \frac{1}{1-\int_0^\infty g(t)dt}\int_0^\infty g(t)dt\int_0^\infty s\varphi_q \cdot \left(\int_0^\infty G(s,\varsigma)a(\varsigma)f(\varsigma,\mu_0(\varsigma))d\varsigma\right)ds + \int_0^\infty s\varphi_q\left(\int_0^\infty G(s,\varsigma)a(\varsigma)f(\varsigma,\mu_0(\varsigma))dd\varsigma\right)ds + \xi\left(\frac{\int_0^\infty tg(t)dt}{1-\int_0^\infty g(t)dt}\right) + \xi t \leq \frac{d}{2l\left(1-\int_0^\infty g(t)dt\right)} \cdot \int_0^\infty s\varphi_q\left(\int_0^\infty G(s,\varsigma)a(\varsigma)d\varsigma\right)ds + \xi\left(t+\frac{\int_0^\infty tg(t)dt}{1-\int_0^\infty g(t)dt}\right) \leq \frac{d}{2}+\xi\left(t+\frac{\int_0^\infty tg(t)dt}{1-\int_0^\infty g(t)dt}\right) = \mu_0(t). \tag{44}$$

Together with (H_4), we also have $\mu_2(t) = A\mu_1(t) \leq A\mu_0(t) = \mu_1(t)$, $t \in [0,+\infty)$. By induction, we obtain

$$\mu_{n+1}(t) \leq \mu_n(t), \quad t \in [0,+\infty),\ n = 0,1,2,\cdots. \tag{45}$$

Therefore, there exists $\mu^* \in K_d$ such that $\mu_n \longrightarrow \mu^*$ as $n \longrightarrow +\infty$. Applying the continuity of A and $\mu_{n+1} = A\mu_n$, we get that $A\mu^* = \mu^*$.

On the other hand, let $\bar{\mu}_0(t) = 0$, $t \in [0,+\infty)$, and then $\bar{\mu}_0(t) \in K_d$. Let $\bar{\mu}_1 = A\bar{\mu}_0$, $\bar{\mu}_2 = A\bar{\mu}_1 = A^2\bar{\mu}_0$; then, by Theorem 6, we have $\bar{\mu}_1, \bar{\mu}_2 \in K_d$. Denote $\bar{\mu}_{n+1} = A\bar{\mu}_n = A^n\bar{\mu}_0$, $= 1,2,\cdots$. Since $A : K_d \longrightarrow K_d$, we have $\bar{\mu}_n \in A(K_d) \subset K_d$. It follows from the complete continuity of A that $\{\bar{\mu}_n\}_{n=1}^{\infty}$ is a sequentially compact set. Since $\bar{\mu}_1 = A\bar{\mu}_0 \in K_d$, we have

$$\bar{\mu}_2(t) = A\bar{\mu}_1(t) = A0(t) \geq 0, t \in [0,+\infty). \tag{46}$$

By induction, we get

$$\bar{\mu}_{n+1}(t) \geq \bar{\mu}_n(t), \in [0,+\infty), \quad n = 1,2,\cdots. \tag{47}$$

Thus, there exists $\bar{\mu}^* \in K$ such that $\bar{\mu}_n \longrightarrow \bar{\mu}^*$ as $n \longrightarrow +\infty$. Applying the continuity of A and $\bar{\mu}_{n+1} = A\bar{\mu}_n$, we get that $A\bar{\mu}^* = \bar{\mu}^*$.

Now, we are in a position to show that μ^* and $\bar{\mu}^*$ are the maximal and minimal positive solutions of BVP (2) in $(0,(d/2)+\xi(t+((\int_0^\infty tg(t)dt)/(1-\int_0^\infty g(t)dt)))]$. Let $u \in [0,(d/2)+\xi(t+((\int_0^\infty tg(t)dt)/(1-\int_0^\infty g(t)dt)))]$ be any solution of BVP (2), that is, $Au = u$. Noting that A is nondecreasing and $\bar{\mu}_0(t) = 0 \leq u(t) \leq (d/2)+\xi(t+((\int_0^\infty tg(t)dt)/(1-\int_0^\infty g(t)dt))) = \mu_0(t)$, then we have $\bar{\mu}_1(t) = A\bar{\mu}_0(t) \leq u(t) \leq A\mu_0(t) = \mu_1(t)$, for all $t \in [0,+\infty)$. By induction, we have

$$\bar{\mu}_n(t) \leq u(t) \leq \mu_n(t), \quad n = 1,2,3,\cdots. \tag{48}$$

Since $\mu^* = \lim_{n\longrightarrow+\infty}\mu_n$, $\bar{\mu}^* = \lim_{n\longrightarrow+\infty}\bar{\mu}_n$, it follows from (42)–(45) that

$$\bar{\mu}_0 \leq \bar{\mu}_1 \leq \cdots \bar{\mu}_n \leq \cdots \leq \bar{\mu}^* \leq u \leq \mu^* \leq \cdots \leq \mu_n \leq \cdots \leq \mu_1 \leq \mu_0. \tag{49}$$

In virtue of $f(t,0)$, $t \in [0,+\infty)$, then the zero function is not the solution of BVP (2). Therefore, by (46), we know that μ^* and $\bar{\mu}^*$ are the maximal and minimal positive solutions of BVP (2) in $(0,(d/2)+\xi(t+((\int_0^\infty tg(t)dt)/(1-\int_0^\infty g(t)dt)))]$, which can be obtained by the corresponding iterative sequences $\mu_n = A\mu_{n-1}$, $\bar{\mu}_n = A\bar{\mu}_{n-1}$. The proof is completed. □

Remark 1. The iterative schemes in Theorem 6 start with a known simple linear function and the zero function, respectively. This is very convenient in application. So Theorem 6 is very interesting and importance. Similarly, we can obtain Theorem 7.

Theorem 7. *Assume that $(H_0)-(H_3)$ hold and there exist $d_n > d_{n-1} > \cdots > d_2 > d_1 > 2m$ satisfying the following condition:*

$(H'_4)f(t,(1+t)x) \leq \varphi_p(d_j/2l)$, $(t,x) \in [0,+\infty)\times[0,d_j]$, $j = 1,2,\cdots,n.m,l$ are defined as Theorem 6. Then, BVP (2) has the maximal and minimal positive solutions μ_j^* and $\bar{\mu}_j^*$, on $[0,+\infty)$, such that

$$0 < \sup_{t\in[0,+\infty)}\frac{|\mu_j^*(t)|}{1+t} \leq d, \quad 0 < \sup_{t\in[0,+\infty)}\frac{|\bar{\mu}_j^*(t)|}{1+t} \leq d. \tag{50}$$

Moreover, for initial values

$$\mu_{j0}(t) = \frac{d_j}{2}+\xi\left(t+\frac{\int_0^\infty tg(t)dt}{1-\int_0^\infty g(t)dt}\right), \quad \bar{\mu}_{j0}(t) = 0, \quad t \in [0,+\infty), \tag{51}$$

define the iterative sequences $\{\mu_{jn}\}$ and $\{\bar{\mu}_{jn}\}$ by

$$\mu_{jn} = \frac{1}{1-\int_0^\infty g(t)dt}\left(\xi\int_0^\infty tg(t)dt + \int_0^\infty g(t) \cdot \int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f\left(\varsigma,\mu_{j(n-1)}(\varsigma)\right)d\varsigma\right)d\tau ds dt\right) + \int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f\left(\varsigma,\mu_{j(n-1)}(\varsigma)\right)d\varsigma\right) \cdot d\tau ds + \xi t,$$

$$\begin{aligned}\bar{\mu}_{jn} = {} & \frac{1}{1-\int_0^\infty g(t)dt}\left(\xi\int_0^\infty tg(t)dt + \int_0^\infty g(t)\right.\\ & \cdot \int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f\left(\varsigma,\mu_{j(n-1)}(\varsigma)\right)d\varsigma\right)d\tau dsdt\Bigg)\\ & + \int_0^t\int_s^\infty \varphi_q\left(\int_0^\infty G(\tau,\varsigma)a(\varsigma)f\left(\varsigma,\mu_{j(n-1)}(\varsigma)\right)d\varsigma\right)\\ & \cdot d\tau ds + \xi t.\end{aligned} \tag{52}$$

Then,

$$\begin{aligned}&\lim_{n\longrightarrow+\infty}\sup_{t\in[0,+\infty)}\frac{|\mu_{jn}(t)-\mu_j^*(t)|}{1+t}=0,\\ &\lim_{n\longrightarrow+\infty}\sup_{t\in[0,+\infty)}\frac{|\bar{\mu}_{jn}(t)-\bar{\mu}_j^*(t)|}{1+t}=0.\end{aligned} \tag{53}$$

Remark 2. We note that μ^* and $\bar{\mu}^*$ in Theorem 6 may coincide; then, BVP (2) has only one solution in K_d. Similarly, positive solutions μ_j^* and $\bar{\mu}_j^*$ in Theorem 7 may also coincide.

4. Example

Consider the following BVP

$$\begin{cases}\left(\left|u''(t)\right|^{-1/2}u''(t)\right)'' - \left|u''(t)\right|^{-1/2}u''(t) = a(t)f(t,u(t)), t\in(0,+\infty),\\ u(0)=\int_0^\infty \frac{1}{(1+t)^3}u(t)dt, \lim_{t\longrightarrow+\infty}u'(t)=\frac{\sqrt{3}}{4}, ''(0)=\lim_{t\longrightarrow+\infty}u''(t)=0.\end{cases} \tag{54}$$

Obviously, we know that $p=3/2$, $k=1$, $\xi=\sqrt{3}/4$, $g(t)=1/(1+t)^3$, so we can get

$$\begin{aligned}&\int_0^\infty g(t)dt = \int_0^\infty \frac{1}{(1+t)^3}dt = \frac{1}{2} < 1, \int_0^\infty tg(t)dt\\ &\quad = \int_0^\infty \frac{t}{(1+t)^3}dt = \frac{1}{2} < +\infty,\\ &G(t,s) = \frac{1}{2}\begin{cases} e^{-s}\left(e^t - e^{-t}\right), 0\le t\le s<+\infty,\\ e^{-t}\left(e^s - e^{-s}\right), 0\le s\le t<+\infty.\end{cases}\end{aligned} \tag{55}$$

Let

$$\begin{aligned}a(t) &= e^{-2t}, f(t,x)\\ &= \begin{cases} |\sin(55t+21)| + \dfrac{1}{144}\left(\dfrac{x}{1+t}\right)^3, & x\le 2,\\ |\sin(55t+21)| + \dfrac{1}{144}\left(\dfrac{2}{1+t}\right)^3, & x\ge 2.\end{cases}\end{aligned} \tag{56}$$

By computation, we get

$$\begin{aligned}&\int_0^\infty G(s,s)a(s)ds \le \frac{1}{2}\int_0^\infty e^{-2s}ds = \frac{1}{4} < +\infty,\\ &\int_0^\infty s\varphi_q\left(\int_0^\infty G(s,\varsigma)a(\varsigma)d\varsigma\right)ds\\ &\quad = \int_0^\infty s\varphi_q\left(\int_0^s G(s,\varsigma)a(\varsigma)d\varsigma\right.\\ &\quad\quad + \left.\int_s^\infty G(s,\varsigma)a(\varsigma)d\varsigma\right)ds = \frac{13}{432} < +\infty.\end{aligned} \tag{57}$$

So, the conditions $(H_0)-(H_3)$ hold. For $m=\sqrt{3}/2$, $l=13/216$. Take $d=10$, it follows that

$$\begin{aligned}f_1(t,(1+t)x) &\le \frac{10}{9} < \varphi_p\left(\frac{d}{2l}\right) = \varphi_p\left(\frac{10}{2\cdot(13/216)}\right)\\ &= \sqrt{\frac{1080}{13}}, \quad (t,x)\in[0,+\infty)\times[0,10].\end{aligned} \tag{58}$$

Hence, condition (H_4) holds, that is, all conditions of Theorem 6 are satisfied. Therefore, BVP (51) has the minimal and maximal positive solutions in $(0, 5+(\sqrt{3}/4)(t+1)]$, which can be obtained by two explicit monotone iterative sequences.

Data Availability

No data was generated or used during the study; this paper was not about data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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