Abstract and Applied Analysis

Nonlinear Analysis and Geometric Function Theory

Guest Editors: Ljubomir B. Ćirić, Samuel Krushkal, Qamrul Hasan Ansari, David Kalaj, and Vesna Manojlović

Nonlinear Analysis and Geometric Function Theory

Abstract and Applied Analysis

Nonlinear Analysis and Geometric Function Theory

Guest Editors: Ljubomir B. Ćirić, Samuel Krushkal, Qamrul Hasan Ansari, David Kalaj, and Vesna Manojlović

Editorial Board

Contents

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 656957, 1 page
http://dx.doi.org/10.1155/2014/656957

Editorial

Nonlinear Analysis and Geometric Function Theory

Ljubomir B. Ćirić,[1] Samuel Krushkal,[2] Qamrul Hasan Ansari,[3] David Kalaj,[4] and Vesna Manojlović[5]

[1] *Faculty of Mechanical Engineering, University of Belgrade, Belgrade, Serbia*
[2] *Department of Mathematics, Bar-Ilan University, 5290002 Ramat-Gan, Israel*
[3] *Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India*
[4] *Faculty of Natural Sciences and Mathematics, University of Montenegro, 81000 Podgorica, Montenegro*
[5] *Mathematical Institute SANU and Faculty of Organizational Sciences, University of Belgrade, 11000 Belgrade, Serbia*

Correspondence should be addressed to Ljubomir B. Ćirić; lciric@rcub.bg.ac.rs

Received 30 September 2014; Accepted 30 September 2014; Published 10 November 2014

Sobolev mappings, quasiconformal mappings, or deformations, between subsets of Euclidean space or manifolds and harmonic maps between manifolds, may arise as the solutions to certain optimization problems in the calculus of variations, as stationary points which are the solutions to differential equations (particularly in conformal geometry). The most recent developments in the theory of planar and space quasiconformal mappings are also related to the theory to degenerate elliptic equations, which include p-Laplace equation whose solutions are p-harmonic maps. In particular, Euclidean harmonic (2-harmonic) maps, which are critical points of Dirichlet's integral, are related to the theory of minimal surfaces and are the primary object of the study in classical potential theory. The complex gradient of the p-harmonic operator is quasiregular and it is general feature of nonlinear PDEs.

Harmonic maps have also played a role in compactifications and parametrizations of Teichmuller space, via classical results of Mike Wolf, who gave a compactification in terms of hyperbolic harmonic maps. Teichmuller spaces are central objects in geometry and complex analysis today, with deep connections to quasiconformal mappings (in particular, to extremal problems of quasiconformal mappings), string theory, and hyperbolic 3 manifolds. The study of connection between extremal mappings of finite distortion and harmonic mappings is also initiated. Since its introduction in the early 1980s, quasiconformal surgery has become a major tool in the development of the theory of holomorphic dynamics.

In connection with the general trend of the geometric function theory in Euclidean space to generalize certain aspects of the analytic functions of one complex variable, there is another development related to maps of quasiconformal type, which are solutions of nonlinear second-order elliptic equations or satisfy certain inequality related to Laplacian and gradient.

Variational analysis, fixed point theory, split feasibility problems, and so forth are also some important and vital areas in nonlinear analysis. During the last two decades, several solution methods for these problems have been proposed and analyzed. These areas include application in optimization, medical sciences, image reconstruction, social sciences, engineering, management, and so forth.

In this special issue, thirteen articles were being published. All these articles have given important contributions to different parts of the above-discussed areas. Among them stands out the very important work of Professor M. Mateljević, one of the most prominent scientists in the geometric function theory. In fact, this issue is devoted to his 65th birthday, as many authors emphasized in their articles.

Ljubomir B. Ćirić
Samuel Krushkal
Qamrul Hasan Ansari
David Kalaj
Vesna Manojlović

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 895074, 20 pages
http://dx.doi.org/10.1155/2014/895074

Research Article

Distortion of Quasiregular Mappings and Equivalent Norms on Lipschitz-Type Spaces

Miodrag Mateljević

Faculty of Mathematics, University of Belgrade, Studentski Trg 16, 11000 Belgrade, Serbia

Correspondence should be addressed to Miodrag Mateljević; miodrag@matf.bg.ac.rs

Received 24 January 2014; Accepted 15 May 2014; Published 21 October 2014

Academic Editor: Ljubomir B. Ćirić

We prove a quasiconformal analogue of Koebe's theorem related to the average Jacobian and use a normal family argument here to prove a quasiregular analogue of this result in certain domains in n-dimensional space. As an application, we establish that Lipschitz-type properties are inherited by a quasiregular function from its modulo. We also prove some results of Hardy-Littlewood type for Lipschitz-type spaces in several dimensions, give the characterization of Lipschitz-type spaces for quasiregular mappings by the average Jacobian, and give a short review of the subject.

1. Introduction

The Koebe distortion theorem gives a series of bounds for a univalent function and its derivative. A result of Hardy and Littlewood relates Hölder continuity of analytic functions in the unit disk with a bound on the derivative (we refer to this result shortly as HL-result).

Astala and Gehring [1] observed that for certain distortion property of quasiconformal mappings the function a_f, defined in Section 2, plays analogous role as $|f'|$ when $n = 2$ and f is conformal, and they establish quasiconformal version of the well-known result due to Koebe, cited here as Lemma 4, and Hardy-Littlewood, cited here as Lemma 5.

In Section 2, we give a short proof of Lemma 4, using a version with the average Jacobian $\underline{J}_f$ instead of a_f, and we also characterize bi-Lipschitz mappings with respect to quasihyperbolic metrics by Jacobian and the average Jacobian; see Theorems 8, 9, and 10 and Proposition 13. Gehring and Martio [2] extended HL-result to the class of uniform domains and characterized the domains D with the property that functions which satisfy a local Lipschitz condition in D for some α always satisfy the corresponding global condition there.

The main result of the Nolder paper [3] generalizes a quasiconformal version of a theorem, due to Astala and Gehring [4, Theorems 1.9 and 3.17] (stated here as Lemma 5) to a quasiregular version (Lemma 33) involving a somewhat larger class of moduli of continuity than t^{α}, $0 < \alpha < 1$.

In the paper [5] several properties of a domain which satisfies the Hardy-Littlewood property with the inner length metric are given and also some results on the Hölder continuity are obtained.

The fact that Lipschitz-type properties are sometimes inherited by an analytic function from its modulus was first detected in [6]. Later this property was considered for different classes of functions and we will call shortly results of this type Dyk-type results. Theorem 22 yields a simple approach to Dyk-type result (the part (ii.1); see also [7]) and estimate of the average Jacobian for quasiconformal mappings in space. The characterization of Lipschitz-type spaces for quasiconformal mappings by the average Jacobian is established in Theorem 23 in space case and Theorem 24 yields Dyk-type result for quasiregular mappings in planar case.

In Section 4, we establish quasiregular versions of the well-known result due to Koebe, Theorem 39 here, and use this result to obtain an extension of Dyakonov's theorem for quasiregular mappings in space (without Dyakonov's hypothesis that it is a quasiregular local homeomorphism), Theorem 40. The characterization of Lipschitz-type spaces for quasiregular mappings by the average Jacobian is also established in Theorem 40.

By $\mathbb{R}^n = \{x = (x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}$ denote the real vector space of dimension n. For a domain D in $\mathbb{R}^n$ with nonempty boundary, we define the distance function $d = d(D) = \operatorname{dist}(D)$ by $d(x) = d(x; \partial D) = \operatorname{dist}(D)(x) = \inf\{|x-y| : y \in \partial D\}$, and if f maps D onto $D' \subset \mathbb{R}^n$, in some settings, it is convenient to use short notation $d_* = d_f(x)$ for $d(f(x); \partial D')$. It is clear that $d(x) = \operatorname{dist}(x, D^c)$, where D^c is the complement of D in $\mathbb{R}^n$.

Let G be an open set in $\mathbb{R}^n$. A mapping $f : G \to \mathbb{R}^m$ is differentiable at $x \in G$ if there is a linear mapping $f'(x) : \mathbb{R}^n \to \mathbb{R}^n$, called the derivative of f at x, such that

$$f(x+h) - f(x) = f'(x)h + |h|\varepsilon(x,h), \tag{1}$$

where $\varepsilon(x,h) \to 0$ as $h \to 0$. For a vector-valued function $f : G \to \mathbb{R}^n$, where $G \subset \mathbb{R}^n$ is a domain, we define

$$\left|f'(x)\right| = \max_{|h|=1}\left|f'(x)h\right|, \qquad l\left(f'(x)\right) = \min_{|h|=1}\left|f'(x)h\right|, \tag{2}$$

when f is differentiable at $x \in G$.

In Section 3, we review some results from [7, 8]. For example, in [7] under some conditions concerning a majorant ω, we showed the following.

Let $f \in C^1(D, \mathbb{R}^n)$ and let ω be a continuous majorant such that $\omega_*(t) = \omega(t)/t$ is nonincreasing for $t > 0$.

Assume f satisfies the following property (which we call Hardy-Littlewood (c,ω)-property):

$$\begin{aligned} (\mathrm{HL}(c,\omega)) \quad & d(x)\left|f'(x)\right| \le c\omega(d(x)), \\ x \in D, \quad & \text{where } d(x) = \operatorname{dist}(x, D^c). \end{aligned} \tag{3}$$

Then

$$(\mathrm{loc}\,\Lambda) \quad f \in \mathrm{loc}\,\Lambda_\omega(G). \tag{4}$$

If, in addition, f is harmonic in D or, more generally, $f \in OC^2(D)$, then $(\mathrm{HL}(c,\omega))$ is equivalent to $(\mathrm{loc}\,\Lambda)$. If D is a Λ_ω-extension domain, then $(\mathrm{HL}(c,\omega))$ is equivalent to $f \in \Lambda_\omega(D)$.

In Section 3, we also consider Lipschitz-type spaces of pluriharmonic mappings and extend some results from [9].

In Appendices A and B we discuss briefly distortion of harmonic qc maps, background of the subject, and basic property of qr mappings, respectively. For more details on related qr mappings we refer the interested reader to [10].

2. Quasiconformal Analogue of Koebe's Theorem and Applications

Throughout the paper we denote by Ω, G, and D open subset of $\mathbb{R}^n$, $n \ge 1$.

Let $B^n(x,r) = \{z \in \mathbb{R}^n : |z-x| < r\}$, $S^{n-1}(x,r) = \partial B^n(x,r)$ (abbreviated $S(x,r)$) and let $\mathbb{B}^n$, S^{n-1} stand for the unit ball and the unit sphere in $\mathbb{R}^n$, respectively. Sometimes we write $\mathbb{D}$ and $\mathbb{T}$ instead of $\mathbb{B}^2$ and $\partial\mathbb{D}$, respectively. For a domain $G \subset \mathbb{R}^n$ let $\rho : G \to [0,\infty)$ be a continuous function. We say that ρ is a weight function or a metric density if, for every locally rectifiable curve γ in G, the integral

$$l_\rho(\gamma) = \int_\gamma \rho(x)\,ds \tag{5}$$

exists. In this case we call $l_\rho(\gamma)$ the ρ-length of γ. A metric density defines a metric $d_\rho : G \times G \to [0,\infty)$ as follows. For $a, b \in G$, let

$$d_\rho(a,b) = \inf_\gamma l_\rho(\gamma), \tag{6}$$

where the infimum is taken over all locally rectifiable curves in G joining a and b.

For the modern mapping theory, which also considers dimensions $n \ge 3$, we do not have a Riemann mapping theorem and therefore it is natural to look for counterparts of the hyperbolic metric. So-called hyperbolic type metrics have been the subject of many recent papers. Perhaps the most important metrics of these metrics are the quasihyperbolic metric κ_G and the distance ratio metric j_G of a domain $G \subset \mathbb{R}^n$ (see [11, 12]). The quasihyperbolic metric $\kappa = \kappa_G$ of G is a particular case of the metric d_ρ when $\rho(x) = 1/d(x, \partial G)$ (see [11, 12]).

Given a subset E of $\mathbb{C}^n$ or $\mathbb{R}^n$, a function $f : E \to \mathbb{C}$ (or, more generally, a mapping f from E into $\mathbb{C}^m$ or $\mathbb{R}^m$) is said to belong to the Lipschitz space $\Lambda_\omega(E)$ if there is a constant $L = L(f) = L(f; E) = L(f, \omega; E)$, which we call Lipschitz constant, such that

$$\left|f(x) - f(y)\right| \le L\omega\left(|x-y|\right) \tag{7}$$

for all $x, y \in E$. The norm $\|f\|_{\Lambda_\omega(E)}$ is defined as the smallest L in (7).

There has been much work on Lipschitz-type properties of quasiconformal mappings. This topic was treated, among other places, in [1–5, 7, 13–22].

As in most of those papers, we will currently restrict ourselves to the simplest majorants $\omega_\alpha(t) := t^\alpha$ $(0 < \alpha \le 1)$. The classes $\Lambda_\omega(E)$ with $\omega = \omega_\alpha$ will be denoted by Λ^α (or by $\mathrm{Lip}(\alpha; E) = \mathrm{Lip}(\alpha, L; E)$). $L(f)$ and α are called, respectively, Lipschitz constant and exponent (of f on E). We say that a domain $G \subset R^n$ is uniform if there are constants a and b such that each pair of points $x_1, x_2 \in G$ can be joined by rectifiable arc γ in G for which

$$\begin{aligned} l(\gamma) &\le a\left|x_1 - x_2\right|, \\ \min l\left(\gamma_j\right) &\le bd(x, \partial G) \end{aligned} \tag{8}$$

for each $x \in \gamma$; here $l(\gamma)$ denotes the length of γ and γ_1, γ_2 the components of $\gamma \setminus \{x\}$. We define $C(G) = \max\{a, b\}$. The smallest $C(G)$ for which the previous inequalities hold is called the uniformity constant of G and we denote it by $c^* = c^*(G)$. Following [2, 17], we say that a function f belongs to the local Lipschitz space $\mathrm{loc}\,\Lambda_\omega(G)$ if (7) holds, with a fixed $C > 0$, whenever $x \in G$ and $|y - x| < (1/2)d(x, \partial G)$. We say that G is a Λ_ω-extension domain if $\Lambda_\omega(G) = \mathrm{loc}\,\Lambda_\omega(G)$. In particular if $\omega = \omega_\alpha$, we say that G is a Λ^α-extension domain; this class includes the uniform domains mentioned above.

Suppose that Γ is a curve family in $\overline{\mathbb{R}}^n$. We denote by $\mathscr{F}(\Gamma)$ the family whose elements are nonnegative Borel-measurable functions ρ which satisfy the condition $\int_\gamma \rho(x)|dx| \geq 1$ for every locally rectifiable curve $\gamma \in \Gamma$, where $ds = |dx|$ denotes the arc length element. For $p \geq 1$, with the notation

$$A_p(\rho) = \int_{\mathbb{R}^n} |\rho(x)|^p dV(x), \tag{9}$$

where $dV(x) = dx$ denotes the Euclidean volume element $dx_1 dx_2 \cdots dx_n$, we define the p-modulus of Γ by

$$M_p(\Gamma) = \inf\left\{A_p(\rho) : \rho \in \mathscr{F}(\Gamma)\right\}. \tag{10}$$

We will denote $M_n(\Gamma)$ simply by $M(\Gamma)$ and call it the modulus of Γ.

Suppose that $f : \Omega \to \Omega^*$ is a homeomorphism. Consider a path family Γ in Ω and its image family $\Gamma^* = \{f \circ \gamma : \gamma \in \Gamma\}$. We introduce the quantities

$$K_I(f) = \sup \frac{M(\Gamma^*)}{M(\Gamma)}, \qquad K_O(f) = \sup \frac{M(\Gamma)}{M(\Gamma^*)}, \tag{11}$$

where the suprema are taken over all path families Γ in Ω such that $M(\Gamma)$ and $M(\Gamma^*)$ are not simultaneously 0 or ∞.

Definition 1. Suppose that $f : \Omega \to \Omega^*$ is a homeomorphism; we call $K_I(f)$ the inner dilatation and $K_O(f)$ the outer dilatation of f. The maximal dilatation of f is $K(f) = \max\{K_O(f), K_I(f)\}$. If $K(f) \leq K < \infty$, we say f is K-quasiconformal (abbreviated qc).

Suppose that $f : \Omega \to \Omega^*$ is a homeomorphism and $x \in \Omega$, $x \neq \infty$, and $f(x) \neq \infty$.

For each $r > 0$ such that $S(x; r) \subset \Omega$ we set $L(x, f, r) = \max_{|y-x|=r} |f(y) - f(x)|$, $l(x, f, r) = \min_{|y-x|=r} |f(y) - f(x)|$.

Definition 2. The linear dilatation of f at x is

$$H(x, f) = \limsup_{r \to 0} \frac{L(x, f, r)}{l(x, f, r)}. \tag{12}$$

Theorem 3 (the metric definition of quasiconformality). *A homeomorphism $f : \Omega \to \Omega^*$ is qc if and only if $H(x, f)$ is bounded on Ω.*

Let Ω be a domain in R^n and let $f : \Omega \to R^n$ be continuous. We say that f is quasiregular (abbreviated qr) if

(1) f belongs to Sobolev space $W^n_{1,\mathrm{loc}}(\Omega)$,

(2) there exists K, $1 \leq K < \infty$, such that

$$\left|f'(x)\right|^n \leq K J_f(x) \ \text{ a.e.} \tag{13}$$

The smallest K in (13) is called the outer dilatation $K_O(f)$. A qr mapping is a qc if and only if it is a homeomorphism. First we need Gehring's result on the distortion property of qc (see [23, page 383], [24, page 63]).

Gehring's Theorem. *For every $K \geq 1$ and $n \geq 2$, there exists a function $\theta^n_K : (0,1) \to \mathbb{R}$ with the following properties:*

(1) *θ^n_K is increasing,*

(2) $\lim_{r \to 0} \theta^n_K(r) = 0$,

(3) $\lim_{r \to 1} \theta^n_K(r) = \infty$,

(4) *Let Ω and Ω' be proper subdomains of $\mathbb{R}^n$ and let $f : \Omega \to \Omega'$ be a K-qc. If x and y are points in Ω such that $0 < |y - x| < d(x, \partial\Omega)$, then*

$$\frac{|f(y) - f(x)|}{d(f(x), \partial\Omega')} \leq \theta^n_K\left(\frac{|y - x|}{d(x, \partial\Omega)}\right). \tag{14}$$

Introduce the quantity, mentioned in the introduction,

$$a_f(x) = a_{f,G}(x) := \exp\left(\frac{1}{n|B_x|}\int_{B_x} \log J_f(z)\, dz\right), \quad x \in G, \tag{15}$$

associated with a quasiconformal mapping $f : G \to f(G) \subset \mathbb{R}^n$; here J_f is the Jacobian of f, while $\mathbf{B}_x = \mathbf{B}_{x,G}$ stands for the ball $B(x; d(x, \partial G))$ and $|\mathbf{B}_x|$ for its volume.

Lemma 4 (see [4]). *Suppose that G and G' are domains in R^n: If $f : G \to G'$ is K-quasiconformal, then*

$$\frac{1}{c}\frac{d(f(x), \partial G')}{d(x, \partial G)} \leq a_{f,G}(x) \leq c\frac{d(f(x), \partial G')}{d(x, \partial G)}, \quad x \in G, \tag{16}$$

where c is a constant which depends only on K and n.

Set $\alpha_0 = \alpha^n_K = K^{1/(1-n)}$.

Lemma 5 (see [4]). *Suppose that D is a uniform domain in R^n and that α and m are constants with $0 < \alpha \leq 1$ and $m \geq 0$. If f is K-quasiconformal in D with $f(D) \subset R^n$ and if*

$$a_{f,D}(x) \leq m d(x, \partial D)^{\alpha - 1} \quad \text{for } x \in D, \tag{17}$$

then f has a continuous extension to $\overline{D} \setminus \{\infty\}$ and

$$\left|f(x_1) - f(x_2)\right| \leq C\left(\left|x_1 - x_2\right| + d(x_1, \partial D)\right)^\alpha \tag{18}$$

for $x_1, x_2 \in \overline{D} \setminus \{\infty\}$, where the constant $C = \widehat{c}(D)$ depends only on K, n, a, m and the uniformity constant $c^* = c^*(D)$ for D. In the case $\alpha \leq \alpha_0$, (18) *can be replaced by the stronger conclusion that*

$$\left|f(x_1) - f(x_2)\right| \leq C\left(\left|x_1 - x_2\right|\right)^\alpha \quad \left(x_1, x_2 \in \overline{D} \setminus \{\infty\}\right). \tag{19}$$

Example 6. The mapping $f(x) = |x|^{a-1}x$, $a = K^{1/(1-n)}$ is K-qc with a_f bounded in the unit ball. Hence f satisfies the hypothesis of Lemma 5 with $\alpha = 1$. Since $|f(x) - f(0)| \leq |x - 0|^a = |x|^a$, we see that when $K > 1$, that is, $a < 1 = \alpha$, the conclusion (18) in Lemma 5 cannot be replaced by the stronger assertion $|f(x_1) - f(x_2)| \leq cm|x_1 - x_2|$.

Let $\Omega \in \mathbb{R}^n$ and $\mathbb{R}^+ = [0, \infty)$ and $f, g : \Omega \to \mathbb{R}^+$. If there is a positive constant c such that $f(x) \leq cg(x)$, $x \in \Omega$, we write $f \preceq g$ on Ω. If there is a positive constant c such that

$$\frac{1}{c} g(x) \leq f(x) \leq cg(x), \quad x \in \Omega, \tag{20}$$

we write $f \approx g$ (or $f \asymp g$) on Ω.

Let $G \subset \mathbb{R}^2$ be a domain and let $f : G \to \mathbb{R}^2$, $f = (f_1, f_2)$ be a harmonic mapping. This means that f is a map from G into $\mathbb{R}^2$ and both f_1 and f_2 are harmonic functions, that is, solutions of the two-dimensional Laplace equation

$$\Delta u = 0. \tag{21}$$

The above definition of a harmonic mapping extends in a natural way to the case of vector-valued mappings $f : G \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$, defined on a domain $G \subset \mathbb{R}^n$, $n \geq 2$.

Let h be a harmonic univalent orientation preserving mapping on a domain D, $D' = h(D)$ and $d_h(z) = d(h(z), \partial D')$. If $h = f + \overline{g}$ has the form, where f and g are analytic, we define $\lambda_h(z) = D^-(z) = |f'(z)| - |g'(z)|$, and $\Lambda_h(z) = D^+(z) = |f'(z)| + |g'(z)|$.

2.1. Quasihyperbolic Metrics and the Average Jacobian. For harmonic qc mappings we refer the interested reader to [25–28] and references cited therein.

Proposition 7. *Suppose D and D' are proper domains in $\mathbb{R}^2$. If $h : D \to D'$ is K-qc and harmonic, then it is bi-Lipschitz with respect to quasihyperbolic metrics on D and D'.*

Results of this type have been known to the participants of Belgrade Complex Analysis seminar; see, for example, [29, 30] and Section 2.2 (Proposition 13, Remark 14 and Corollary 16). This version has been proved by Manojlović [31] as an application of Lemma 4. In [8], we refine her approach.

Proof. Let $h = f + \overline{g}$ be local representation on B_z.

Since h is K-qc, then

$$\left(1 - k^2\right) \left|f'\right|^2 \leq J_h \leq K \left|f'\right|^2 \tag{22}$$

on B_z and since $\log |f'(\zeta)|$ is harmonic,

$$\log \left|f'(z)\right| = \frac{1}{2|B_z|} \int_{B_z} \log \left|f'(\zeta)\right|^2 d\xi\, d\eta. \tag{23}$$

Hence,

$$\begin{aligned}\log a_{h,D}(z) &\leq \frac{1}{2} \log K + \frac{1}{2|B_z|} \int_{B_z} \log \left|f'(\zeta)\right|^2 d\xi\, d\eta \\ &= \log \sqrt{K} \left|f'(z)\right|.\end{aligned} \tag{24}$$

Hence, $a_{h,D}(z) \leq \sqrt{K} |f'(z)|$ and $\sqrt{1-k} |f'(z)| \leq a_{h,D}(z)$. Using Astala-Gehring result, we get

$$\Lambda(h, z) \asymp \frac{d\left(hz, \partial D'\right)}{d(z, \partial D)} \asymp \lambda(h, z). \tag{25}$$

This pointwise result, combined with integration along curves, easily gives

$$k_{D'}\left(h(z_1), h(z_2)\right) \asymp k_D(z_1, z_2), \quad z_1, z_2 \in D. \tag{26}$$

Note that we do not use that $\log(1/J_h)$ is a subharmonic function. □

The following follows from the proof of Proposition 7:

(I) $\Lambda(h, z) \approx a_{h,D} \approx \lambda(h, z)$ and $\sqrt[2]{J_f} \approx \underline{J}_f$; see below for the definition of average jacobian $\underline{J}_f$.

When underlining a symbol (we also use other latex-symbols) we want to emphasize that there is a special meaning of it; for example we denote by c a constant and by $\underline{c}, \widehat{c}, \widetilde{c}$ some specific constants.

Our next result concerns the quantity

$$\underline{E}_{f,G}(x) := \frac{1}{|\underline{B}_x|} \int_{\underline{B}_x} J_f(z)\, dz, \quad x \in G, \tag{27}$$

associated with a quasiconformal mapping $f : G \to f(G) \subset \mathbb{R}^n$; here J_f is the Jacobian of f, while $\underline{B}_x = \underline{B}_{x,G}$ stands for the ball $B(x, d(x, \partial G)/2)$ and $|B_x|$ for its volume.

Define

$$\underline{J}_f = \underline{J}_{f,G} = \sqrt[n]{\underline{E}_{f,G}}. \tag{28}$$

Using the distortion property of qc (see [24, page 63]) we give short proof of a quasiconformal analogue of Koebe's theorem (related to Astala and Gehring's results from [4], cited as Lemma 4 here).

Theorem 8. *Suppose that G and G' are domains in $\mathbb{R}^n$: If $f : G \to G'$ is K-quasiconformal, then*

$$\frac{1}{c} \frac{d\left(f(x), \partial G'\right)}{d(x, \partial G)} \leq \underline{J}_{f,G}(x) \leq c \frac{d\left(f(x), \partial G'\right)}{d(x, \partial G)}, \quad x \in G, \tag{29}$$

where c is a constant which depends only on K and n.

Proof. By the distortion property of qc (see [23, page 383], [24, page 63]), there are the constants C_* and c_* which depend on n and K only, such that

$$B\left(f(x), c_* d_*\right) \subset f\left(B_x\right) \subset B\left(f(x), C_* d_*\right), \quad x \in G, \tag{30}$$

where $d_*(x) := d(f(x)) = d(f(x), \partial G')$ and $d(x) := d(x, \partial G)$. Hence

$$\left|B\left(f(x), c_* d_*\right)\right| \leq \int_{B_x} J_f(z)\, dz \leq \left|B\left(f(x), C_* d_*\right)\right| \tag{31}$$

and therefore we prove Theorem 8. □

We only outline proofs in the rest of this subsection.

Suppose that Ω and Ω' are domains in $\mathbb{R}^n$ different from $\mathbb{R}^n$.

Theorem 9. *Suppose $f : \Omega \to \Omega'$ is a C^1 and the following hold.*

(i1) *f is c-Lipschitz with respect to quasihyperbolic metrics on Ω and Ω'; then*

(I1) *$d|f'(x_0)| \leq cd_*(x_0)$ for every $x_0 \in \Omega$.*

(i2) *If f is a qc c-quasihyperbolic-isometry, then*

(I2) *f is a c^2-qc mapping,*

(I3) *$|f'| \approx d_*/d$.*

Proof. Since Ω and Ω' are different from $\mathbb{R}^n$, there are quasihyperbolic metrics on Ω and Ω'. Then for a fixed $x_0 \in \Omega$, we have

$$k(x, x_0) = d(x_0)^{-1}|x - x_0|(1 + o(1)), \quad x \longrightarrow x_0,$$
$$k(f(x), f(x_0)) = \frac{|f(x) - f(x_0)|}{d_*(f(x_0))}(1 + o(1)), \quad \text{when } x \longrightarrow x_0. \tag{32}$$

Hence, (i1) implies (I1). If f is a c-quasihyperbolic-isometry, then $d|f'(x_0)| \leq cd_*$ and $dl(f'(x_0)) \geq d_*/c$. Hence, (I2) and (I3) follow. □

Theorem 10. *Suppose $f : \Omega \to \Omega'$ is a C^1 qc homeomorphism. The following conditions are equivalent:*

(a.1) *f is bi-Lipschitz with respect to quasihyperbolic metrics on Ω and Ω',*

(b.1) *$\sqrt[n]{J_f} \approx d_*/d$,*

(c.1) *$\sqrt[n]{J_f} \approx a_f$,*

(d.1) *$\sqrt[n]{J_f} \approx \underline{l}_f$,*

where $d(x) = d(x, \partial\Omega)$ and $d_*(x) = d(f(x), \partial\Omega')$.

Proof. It follows from Theorem 9 that (a) is equivalent to (b) (see, e.g., [32, 33]).

Theorem 8 states that $\underline{l}_f \approx d_*/d$. By Lemma 4, $a_f \approx d_*/d$ and therefore (b) is equivalent to (c). The rest of the proof is straightforward. □

Lemma 11. *If $f \in C^{1,1}$ is a K-quasiconformal mapping defined in a domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$, then*

$$J_f(x) > 0, \quad x \in \Omega \tag{33}$$

provided that $K < 2^{n-1}$. The constant 2^{n-1} is sharp.

If $\overline{G} \subset \Omega$, then it is bi-Lipschitz with respect to Euclidean and quasihyperbolic metrics on G and $G' = f(G)$.

It is a natural question whether is there an analogy of Theorem 10 if we drop the C^1 hypothesis.

Suppose $f : \Omega \to \Omega'$ is onto qc mapping. We can consider the following conditions:

(a.2) f is bi-Lipschitz with respect to quasihyperbolic metrics on Ω and Ω';

(b.2) $\sqrt[n]{J_f} \approx d_*/d$ a.e. in Ω;

(c.2) $\sqrt[n]{J_f} \approx a_f$ a.e. in Ω;

(d.2) $\sqrt[n]{J_f} \approx \underline{l}_f$ a.e. in Ω.

It seems that the above conditions are equivalent, but we did not check details.

If Ω is a planar domain and f harmonic qc, then we proved that (d.2) holds (see Proposition 18).

2.2. Quasi-Isometry in Planar Case. For a function h, we use notations $\partial h = (1/2)(h'_x - ih'_y)$ and $\overline{\partial} h = (1/2)(h'_x + ih'_y)$; we also use notations Dh and $\overline{D}h$ instead of ∂h and $\overline{\partial} h$, respectively, when it seems convenient. Now we give another proof of Proposition 7, using the following.

Proposition 12 (see [29]). *Let h be an euclidean harmonic orientation preserving univalent mapping of the unit disc $\mathbb{D}$ into $\mathbb{C}$ such that $h(\mathbb{D})$ contains a disc $B_R = B(a; R)$ and $h(0) = a$. Then*

$$|\partial h(0)| \geq \frac{R}{4}. \tag{34}$$

If, in addition, h is K-qc, then

$$\lambda_h(0) \geq \frac{1-k}{4}R. \tag{35}$$

For more details, in connection with material considered in this subsection, see also Appendix A, in particular, Proposition A.7.

Let $h = f + \overline{g}$ be a harmonic univalent orientation preserving mapping on the unit disk $\mathbb{D}$, $\Omega = h(\mathbb{D})$, $d_h(z) = d(h(z), \partial\Omega)$, $\lambda_h(z) = D^-(z) = |f'(z)| - |g'(z)|$, and $\Lambda_h(z) = D^+(z) = |f'(z)| + |g'(z)|$. By the harmonic analogue of the Koebe Theorem, then

$$\frac{1}{16}D^-(0) \leq d_h(0) \tag{36}$$

and therefore

$$\left(1 - |z|^2\right)\frac{1}{16}D^-(z) \leq d_h(z). \tag{37}$$

If, in addition, h is K-qc, then

$$\left(1 - |z|^2\right)D^+(z) \leq 16Kd_h(z). \tag{38}$$

Using Proposition 12, we also prove

$$\left(1 - |z|^2\right)\lambda_h(z) \geq \frac{1-k}{4}d_h(z). \tag{39}$$

If we summarize the above considerations, we have proved the part (a.3) of the following proposition.

Proposition 13 (e-qch, hyperbolic distance version). *(a.3). Let $h = f + \overline{g}$ be a harmonic univalent orientation preserving K-qc mapping on the unit disk $\mathbb{D}$, $\Omega = h(\mathbb{D})$. Then, for $z \in \mathbb{D}$,*

$$\begin{aligned}\left(1-|z|^2\right)\Lambda_h(z) &\leq 16Kd_h(z),\\ \left(1-|z|^2\right)\lambda_h(z) &\geq \frac{1-k}{4}d_h(z).\end{aligned} \tag{40}$$

(b.3) If h is a harmonic univalent orientation preserving K-qc mapping of domain D onto D', then

$$\begin{aligned}d(z)\Lambda_h(z) &\leq 16Kd_h(z),\\ d(z)\lambda_h(z) &\geq \frac{1-k}{4}d_h(z),\end{aligned} \tag{41}$$

and

$$\begin{aligned}\frac{1-k}{4}\kappa_D\left(z,z'\right) &\leq \kappa_{D'}\left(hz,hz'\right)\\ &\leq 16K\kappa_D\left(z,z'\right),\end{aligned} \tag{42}$$

where $z, z' \in D$.

Remark 14. In particular, we have Proposition 7, but here the proof of Proposition 13 is very simple and it it is not based on Lemma 4.

Proof of (b.3). Applying (a.3) to the disk B_z, $z \in D$, we get (41). It is clear that (41) implies (42).

For a planar hyperbolic domain in $\mathbb{C}$, we denote by $\rho = \rho_D$ and $d_{\text{hyp}} = d_{\text{hyp};D}$ the hyperbolic density and metric of D, respectively.

We say that a domain $D \subset \mathbb{C}$ is *strongly hyperbolic* if it is hyperbolic and diameters of boundary components are uniformly bounded from below by a positive constant.

Example 15. The Poincaré metric of the punctured disk $\mathbb{D}' = \{0 < |z| < 1\}$ is obtained by mapping its universal covering, an infinitely-sheeted disk, on the half plane $\Pi^- = \{\operatorname{Re} w < 0\}$ by means of $w = \ln z$ (i.e., $z = e^w$). The metric is

$$\frac{|dw|}{|\operatorname{Re} w|} = \frac{|dz|}{|z|\ln(1/|z|)}. \tag{43}$$

Since a boundary component is the point 0, the punctured disk is not a strongly hyperbolic domain. Note also that ρ/d and $1/\ln(1/d)$ tend to 0 if z tends to 0. Here $d = d(z) = \operatorname{dist}(z, \partial\mathbb{D}')$ and ρ is the hyperbolic density of $\mathbb{D}'$. Therefore one has the following:

(A.1) for $z \in \mathbb{D}'$, $\kappa_{\mathbb{D}'}(z,z') = \ln(1/d(z))d_{\text{hyp},\mathbb{D}'}(z,z')$ when $z' \to z$.

There is no constant c such that

(A.2) $\kappa_{\mathbb{D}'}(z,z') \leq c\kappa_{\Pi^-}(w,w')$ for every $w, w' \in \Pi^-$, where $z = e^w$ and $z' = e^{w'}$.

Since $d_{\text{hyp},\Pi^-} \approx \kappa_{\Pi^-}$, if (A.2) holds, we conclude that $\kappa_{\mathbb{D}'} \approx d_{\text{hyp},\mathbb{D}'}$, which is a contradiction by (A.1).

Let D be a planar hyperbolic domain. Then if D is simply connected,

$$\frac{1}{2d} \leq \rho \leq \frac{2}{d}. \tag{44}$$

For general domain as $z \to \partial D$,

$$\frac{1+o(1)}{d\ln(1/d)} \leq \rho \leq \frac{2}{d}. \tag{45}$$

Hence $d_{\text{hyp}}(z,z') \leq 2\kappa_D(z,z')$, $z, z' \in D$.

If Ω is a strongly hyperbolic domain, then there is a hyperbolic density ρ on the domain Ω, such that $\rho \approx d^{-1}$, where $d(w) = d(w, \partial\Omega)$; see, for example, [34]. Thus $d_{\text{hyp};D} \approx \kappa_D$. Hence, we find the following.

Corollary 16. *Every e-harmonic quasiconformal mapping of the unit disc (more generally of a strongly hyperbolic domain) is a quasi-isometry with respect to hyperbolic distances.*

Remark 17. Let D be a hyperbolic domain, let h be e-harmonic quasiconformal mapping of D onto Ω, and let $\phi : \mathbb{D} \to D$ be a covering and $h^* = h \circ \phi$.

(a.4) Suppose that D is simply connected. Thus ϕ and h^* are one-to-one. Then

$$\left(1-|z|^2\right)\lambda_{h^*}(\zeta) \geq \frac{1-k}{4}d_h(z), \tag{46}$$

where $z = \phi(\zeta)$.

Hence, since $\lambda_{h^*}(\zeta) = l_h(z)|\phi'(\zeta)|$ and $\rho_{\text{hyp};D}(z)|\phi'(\zeta)| = \rho_{\mathbb{D}}(\zeta)$,

$$\lambda_h(z) \geq \frac{1-k}{4}d_h(z)\rho_D(z). \tag{47}$$

Using Hall's sharp result, one can also improve the constant in the second inequality in Propositions 13 and 12 (i.e., the constant $1/4$ can be replaced by τ_0; see below for more details):

$$\left(1-|z|^2\right)\lambda_h(z) \geq (1-k)\tau_0 d_h(z), \tag{48}$$

where $\tau_0 = 3\sqrt{3}/2\pi$.

(b.4) Suppose that D is not a simply connected. Then h^* is not one-to-one and we cannot apply the procedure as in (a.4).

(c.4) It seems natural to consider whether there is an analogue in higher dimensions of Proposition 13.

Proposition 18. *For every e-harmonic quasiconformal mapping f of the unit disc (more generally of a hyperbolic domain D) the following holds:*

(e.2) $\sqrt{J_f} \approx d_*/d$.

In particular, it is a quasi-isometry with respect to quasihyperbolic distances.

Proof. For $z \in D$, by the distortion property,

$$|f(z_1) - f(z)| \le cd(f(z)) \quad \text{for every } z_1 \in \underline{B}_z. \tag{49}$$

Hence, by Schwarz lemma for harmonic maps, $d|f'(z)| \le c_1 d(f(z))$. Proposition 12 yields (e) and an application of Proposition 13 gives the proof. □

Recall $\underline{B}_x$ stands for the ball $B(x; d(x, \partial G)/2)$ and $|\underline{B}_x|$ for its volume. If V is a subset of $\mathbb{R}^n$ and $u : V \to \mathbb{R}^m$, we define

$$\begin{aligned} \operatorname{osc}_V u &:= \sup\{|u(x) - u(y)| : x, y \in V\}, \\ \omega_u(r, x) &:= \sup\{|u(x) - u(y)| : |x - y| = r\}. \end{aligned} \tag{50}$$

Suppose that $G \subset \mathbb{R}^n$ and $\underline{B}_x = B(x, d(x)/2)$. Let $OC^1(G) = OC^1(G; c_1)$ denote the class of $f \in C^1(G)$ such that

$$d(x)\left|f'(x)\right| \le c_1 \operatorname{osc}_{\underline{B}_x} f \tag{51}$$

for every $x \in G$.

Proposition 19. *Suppose $f : \Omega \to \Omega'$ is a C^1. Then $f \in OC^1(\Omega; c)$, if and only if f is c-Lipschitz with respect to quasihyperbolic metrics on B and $f(B)$ for every ball $B \subset \Omega$.*

2.3. Dyk-Type Results. The characterization of Lipschitz-type spaces for quasiconformal mappings in space and planar quasiregular mappings by the average Jacobian are the main results in this subsection. In particular, using the distortion property of qc mappings we give a short proof of a quasiconformal version of a Dyakonov theorem which states:

> Suppose G is a Λ^α-extension domain in $\mathbb{R}^n$ and f is a K-quasiconformal mapping of G onto $f(G) \subset \mathbb{R}^n$. Then $f \in \Lambda^\alpha(G)$ if and only if $|f| \in \Lambda^\alpha(G)$.

This is Theorem B.3, in Appendix B below. It is convenient to refer to this result as Theorem Dy; see also Theorems 23–24 and Proposition A, Appendix B.

First we give some definitions and auxiliary results. Recall, Dyakonov [15] used the quantity

$$\underline{E}_{f,G}(x) : \frac{1}{|\underline{B}_x|} \int_{\underline{B}_x} J_f(z)\, dz, \quad x \in G, \tag{52}$$

associated with a quasiconformal mapping $f : G \to f(G) \subset R^n$; here J_f is the Jacobian of f, while $\underline{B}_x = \underline{B}_{x,G}$ stands for the ball $B(x; d(x, \partial G)/2)$ and $|\underline{B}_x|$ for its volume.

Define $\alpha_0 = \alpha_K^n = K^{1/(1-n)}$, $C_* = \theta_K^n(1/2)$, $\theta_K^n(c_*) = 1/2$, and

$$\underline{J}_{f,G} = \sqrt[n]{\underline{E}_{f,G}}. \tag{53}$$

For a ball $B = B(x, r) \subset \mathbb{R}^n$ and a mapping $f : B \to \mathbb{R}^n$, we define

$$E_f(B; x) := \frac{1}{|B|} \int_B J_f(z)\, dz, \quad x \in G, \tag{54}$$

and $J_f^{av}(B; x) = \sqrt[n]{E_f(B; x)}$; we also use the notation $J_f(B; x)$ and $J_f(x; r)$ instead of $J_f^{av}(B; x)$.

For $x, y \in \mathbb{R}^n$, we define the Euclidean inner product $\langle \cdot, \cdot \rangle$ by

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n. \tag{55}$$

By $\mathbb{E}^n$ we denote $\mathbb{R}^n$ with the Euclidean inner product and call it Euclidean space n-space (space of dimension n). In this paper, for simplicity, we will use also notation $\mathbb{R}^n$ for $\mathbb{E}^n$. Then the Euclidean length of $x \in \mathbb{R}^n$ is defined by

$$|x| = \langle x, x \rangle^{1/2} = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{1/2}. \tag{56}$$

The minimal analytic assumptions necessary for a viable theory appear in the following definition.

Let Ω be a domain in $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^n$ be continuous. We say that f has finite distortion if

(1) f belongs to Sobolev space $W^1_{1,\text{loc}}(\Omega)$;

(2) the Jacobian determinant of f is locally integrable and does not change sign in Ω;

(3) there is a measurable function $K_O = K_O(x) \ge 1$, finite a.e., such that

$$\left|f'(x)\right|^n \le K_O(x)\left|J_f(x)\right| \text{ a.e.} \tag{57}$$

The assumptions (1), (2), and (3) do not imply that $f \in W^n_{1,\text{loc}}(\Omega)$, unless of course K_O is a bounded function.

If K_O is a bounded function, then f is qr. In this setting, the smallest K in (57) is called the outer dilatation $K_O(f)$.

If f is qr, also

$$J_f(x) \le K' l\big(f'(x)\big)^n \text{ a.e.} \tag{58}$$

for some K', $1 \le K' < \infty$, where $l(f'(x)) = \inf\{|f'(x)h| : |h| = 1\}$. The smallest K' in (58) is called the inner dilatation $K_I(f)$ and $K(f) = \max(K_O(f), K_I(f))$ is called the maximal dilatation of f. If $K(f) \le K$, f is called K-quasiregular.

In a highly significant series of papers published in 1966–1969 Reshetnyak proved the fundamental properties of qr mappings and in particular the main theorem concerning topological properties of qr mappings: every nonconstant qr map is discrete and open; cf. [11, 35] and references cited there.

Lemma 20 (see Morrey's Lemma, Lemma 6.7.1, [10, page 170]). *Let f be a function of the Sobolev class $W^{1,p}(\mathbf{B}, \mathbb{E}^n)$ in the ball $\mathbf{B} = 2B = B(x_0, 2R)$, $\underline{B} = B(x_0, R)$, $1 \le p < \infty$, such that*

$$D_p(f, B) = \left(\frac{1}{|B|} \int_B \left|f'\right|^p\right)^{1/p} \le M r^{\gamma - 1}, \tag{59}$$

where $0 < \gamma \le 1$, holds for every ball $B = B(a, r) \subset \mathbf{B}$. Then f is Hölder continuous in $\underline{B}$ with exponent γ, and one has $|f(x) - f(y)| \le C|x - y|^\gamma$, for all $x, y \in \underline{B}$, where $c = 4M\gamma^{-1}2^{-\gamma}$. Here and in some places we omit to write the volume element dx.

We need a quasiregular version of this Lemma.

Lemma 21. *Let $f : \mathbf{B} \to \mathbb{R}^n$ be a K-quasiregular mapping, such that*

$$J_f(a; r) \leq M r^{\gamma-1}, \tag{60}$$

where $0 < \gamma \leq 1$, holds for every ball $B = B(a, r) \subset \mathbf{B}$. Then f is Hölder continuous in $\underline{B}$ with exponent γ, and one has $|f(x) - f(y)| \leq C|x - y|^\gamma$, for all $x, y \in \underline{B}$, where $C = 4MK^{1/n}\gamma^{-1}2^{-\gamma}$.

Proof. By hypothesis, f satisfies (57) and therefore $D_n(f, B) \leq K^{1/n} J_f(a; r)$. An application of Lemma 20 to $p = n$ yields proof. □

(a.0) By $\widehat{B}(G)$ denote a family of ball $B = B(x) = B(x, r)$ such that $\mathbf{B} = B(x, 2r) \subset G$. For $B \in \widehat{B}(G)$, define $d' = d'_{f,\mathbf{B}} = d(f(x), \partial \mathbf{B}')$ and $R(f(x), \mathbf{B}') := c_* d'$, where $\mathbf{B}' = f(\mathbf{B})$.

Define $\omega_{\mathrm{loc}}(f, r) = \omega_{\mathrm{loc},G}(f, r) = \sup \mathrm{osc}_B f$ and $r_0(G) = \sup r(B)$, where supremum is taken over all balls $B = B(a) = B(a, r)$ such that $B_1 = B(a, 2r) \subset G$ and $r(B)$ denotes radius of B.

Theorem 22. *Let $0 < \alpha \leq 1$, $K \geq 1$. Suppose that* (a.5) *G is a domain in $\mathbb{R}^n$ and $f : G \to \mathbb{R}^n$ is K-quasiconformal and $G' = f(G)$. Then one has the following.*

(i.1) *For every $x \in G$, there exists two points $x_1, x_2 \in \underline{B}_x$ such that $|y_2| - |y_1| \geq R(x)$, where $y_k = f(x_k)$, $k = 1, 2$, and $R(x) := c_* d_*(x)$.*

(ii.1) $\omega_{\mathrm{loc},G}(f, r) \leq 2c_0 \omega_{\mathrm{loc},G}(|f|, r)$, $0 \leq r \leq r_0(G)$, *where* $c_0 = 2C_*/c_*$.

(iii.1) *If, in addition, one supposes that* (b.5) $|f| \in \mathrm{loc}\,\Lambda^\alpha(G)$, *then for all balls $B = B(x) = B(x, r)$ such that $B_1 = B(x, 2r) \subset G$, $d_1 \preceq r^\alpha$ and in particular $R(x) := c_* d_*(x) \leq L d(x)^\alpha$, $x \in G$.*

(iv.1) *There is a constant $\underline{c}$ such that*

$$J_f(x; r) \leq \underline{c}\, r^{\alpha-1} \tag{61}$$

for every $x_0 \in G$ and $B = B(x, r) \subset \underline{B}_{x_0}$.

(v.1) *If, in addition, one supposes that* (c.5) *G is a Λ^α-extension domain in $\mathbb{R}^n$, then $f \in \mathrm{Lip}(\alpha, L_2; G)$.*

Proof. By the distortion property (30), we will prove the following.

(vi.1) For a ball $B = B(x) = B(x, r)$ such that $B_1 = B(x, 2r) \subset G$ there exist two points $x_1, x_2 \in B(x)$ such that $|y_2| - |y_1| \geq R(x)$, where $y_k = f(x_k)$, $k = 1, 2$.

Let l be line throughout 0 and $f(x)$ which intersects the $S(f(x), (x))$ at points y_1 and y_2 and $x_k = f^{-1}(y_k)$, $k = 1, 2$. By the left side of (30), $x_1, x_2 \in B(x)$. We consider two cases:

(a) if $0 \notin B(f(x), R(x))$ and $|y_2| \geq |y_1|$, then $|y_2| - |y_1| = 2R(x)$ and $|y_2| - |y_1| = 2R(x)$;

(b) if $0 \in B(f(x), R(x))$, then, for example, $0 \in [y_1, f(x)]$ and if we choose $x_1 = x$, we find $|y_2| - |f(x)| = R(x)$ and this yields (vi.1).

Let $x \in G$. An application of (vi.1) to $B = \underline{B}_x$ yields (i.1). If $|x - x'| \leq r$, then $c_* d' \leq |y_2| - |y_1|$ and therefore we have the following:

(vi.1′) $|f(x) - f(x')| \leq C_* d' \leq c_0(|y_2| - |y_1|)$, where $c_0 = C_*/c_*$.

If $z_1, z_2 \in B(x, r)$, then $|f(z_1) - f(z_2)| \leq |f(z_1) - f(x)| + |f(z_2) - f(x)|$. Hence, using (vi.1′), we find (ii.1). Proof of (iii.1). If the hypothesis (b) holds with multiplicative constant L, and $|x - x'| = r$, then $c_* d' \leq |y_2| - |y_1| \leq 2^\alpha L r^\alpha$ and therefore $|f(x) - f(x')| \leq C_* d' \leq c_1 r^\alpha$, where $c_1 = (C_*/c_*)2^\alpha L$. Hence $d_1 \preceq r^\alpha$ and therefore, in particular, $R(x) := c_* d_*(x) \leq L d(x)^\alpha$, $x \in G$.

It is clear, by the hypothesis (b), that there is a fixed constant L_1 such that (vii.1) $|f|$ belongs to $\mathrm{Lip}(\alpha, L_1; B)$ for every ball $B = B(x, r)$ such that $B_1 = B(x, 2r) \subset G$ and, by (ii.1), we have the following:

(viii.1) $L(f, B) \leq L_2$ for a fixed constant L_2 and $f \in \mathrm{loc}\,\mathrm{Lip}(\alpha, L_2; G)$.

Hence, since, by the hypothesis (c.5), G is a L^α-extension domain, we get (v.1).

Proof of (iv.1). Let $B = B(x) = B(x, r)$ be a ball such that $\mathbf{B} = B(x, 2r) \subset G$. Then $E_g(B; x) \approx d_1^n / r^n$, $J_f(B; x) \approx d_1 / r$, and, by (iii.1), $d_1 \preceq r^\alpha$ on G. Hence

$$J_f(B; x) \preceq r^{\alpha-1}, \tag{62}$$

on G.

Note that one can also combine (iii.1) and Lemma 33 (for details see proof of Theorem 40 below) to obtain (v.1). □

Note that as an immediate corollary of (ii.1) we get a simple proof of Dyakonov results for quasiconformal mappings (without appeal to Lemma 33 or Lemma 21, which is a version of Morrey's Lemma).

We enclose this section by proving Theorems 23 and 24 mentioned in the introduction; in particular, these results give further extensions of Theorem-Dy.

Theorem 23. *Let $0 < \alpha \leq 1$, $K \geq 1$, and $n \geq 2$. Suppose G is a Λ^α-extension domain in $\mathbb{R}^n$ and f is a K-quasiconformal mapping of G onto $f(G) \subset \mathbb{R}^n$. The following are equivalent:*

(i.2) $f \in \Lambda^\alpha(G)$,

(ii.2) $|f| \in \Lambda^\alpha(G)$,

(iii.2) *there is a constant c_3 such that*

$$J_f(x; r) \leq c_3 r^{\alpha-1} \tag{63}$$

for every $x_0 \in G$ and $B = B(x, r) \subset \underline{B}_{x_0}$. If, in addition, G is a uniform domain and if $\alpha \leq \alpha_0 = K^{1/(1-n)}$, then (i.2) and (ii.2) are equivalent to

(iv.2) $|f| \in \Lambda^\alpha(G, \partial G)$.

Proof. Suppose (ii.2) holds, so that $|f|$ is L-Lipschitz in G. Then (iv.1) shows that (iii.2) holds. By Lemma 21, (iii.2) implies (i.2). □

We outline less direct proof that (ii.2) implies (ii.1).

One can show first that (ii.2) implies (18) (or more generally (iv.1); see Theorem 22). Using $A(z) = a + Rz$, we conclude that $c^*(B(a,R)) = c^*(\mathbb{B})$ and $\hat{c}(B) = \hat{c}$ is a fixed constant (which depends only on K, n, c_1), for all balls $B \subset G$. Lemma 5 tells us that (18) holds, with a fixed constant, for all balls $B \subset G$ and all pairs of points $x_1, x_2 \in B$. Further, we pick two points $x, y \in G$ with $|x-y| < d(x, \partial G)$ and apply (18) with $D = B(x; |x-y|)$, letting $x_1 = x$ and $x_2 = y$. The resulting inequality

$$\left|f(x) - f(y)\right| \le \text{const}\left|x-y\right|^{\alpha} \tag{64}$$

shows that $f \in \text{loc}\,\Lambda^{\alpha}(G)$, and since G is a Λ^{α}-extension domain, we conclude that $f \in \Lambda^{\alpha}(G)$.

The implication (ii.4) ⇒ (i.4) is thus established. The converse is clear. For the proof that (iii.4) implies (i.4) for $\alpha \le \alpha_0$, see [15].

For a ball $B = B(x,r) \subset \mathbb{R}^n$ and a mapping $f : B \to \mathbb{R}^n$, we define

$$E_f(B; x) := \frac{1}{|B|}\int_B J_f(z)\,dz, \quad x \in G, \tag{65}$$

and $J_f(B;x) = \sqrt[n]{E_f(B;x)}$.

We use the factorization of planar quasiregular mappings to prove the following.

Theorem 24. *Let $0 < \alpha \le 1$, $K \ge 1$. Suppose D is a Λ^{α}-extension domain in $\mathbb{C}$ and f is a K-quasiregular mapping of D onto $f(D) \subset \mathbb{C}$. The following are equivalent:*

(i.3) $f \in \Lambda^{\alpha}(D)$,

(ii.3) $|f| \in \Lambda^{\alpha}(D)$,

(iii.3) *there is a constant $\check{c}$ such that*

$$J_f(B; z) \le \check{c}r^{\alpha-1}(z), \tag{66}$$

for every $z_0 \in D$ and $B = B(z,r) \subset \underline{B}_{z_0}$.

Proof. Let D and W be domains in C and $f : D \to W$ qr mapping. Then there is a domain G and analytic function ϕ on G such that $f = \phi \circ g$, where g is quasiconformal; see [36, page 247].

Our proof will rely on distortion property of quasiconformal mappings. By the triangle inequality, (i.4) implies (ii.4). Now, we prove that (ii.4) implies (iii.4).

Let $z_0 \in D$, $\zeta_0 = g(z_0)$, $G_0 = g(B_{z_0})$, $W_0 = \phi(G_0)$, $d = d(z_0) = d(z_0, \partial D)$, and $d_1 = d_1(\zeta_0) = d(\zeta_0, \partial G)$. Let $B = B(z,r) \subset B_{z_0}$. As in analytic case there is $\zeta_1 \in g(B)$ such that $||\phi(\zeta_1)| - |\phi(\zeta_0)|| \ge c d_1(\zeta_0)|\phi'(\zeta_0)|$. If $z_1 = g^{-1}(\zeta_1)$, then $||\phi(\zeta_1)| - |\phi(\zeta_0)|| = ||f(z_1)| - |f(z_0)||$. Hence, if $|f|$ is α-Hölder, then $d_1(\zeta)|\phi'(\zeta)| \le c_1 r^{\alpha}$ for $\zeta \in G_0$. Hence, since

$$E_f(B; z) = \frac{1}{|B_z|}\int_{B_z}\left|\phi'(\zeta)\right| J_g(z)\,dx\,dy, \tag{67}$$

we find

$$J_{f,D}(z_0) \le c_2 \frac{r^{\alpha}}{d_1(\zeta_0)} J_{g,G}(z_0) \tag{68}$$

and therefore, using $E_g(B;z) \approx d_1^2/r^2$, we get

$$J_f(B; z) \le c_3 r^{\alpha-1}. \tag{69}$$

Thus we have (iii.3) with $\check{c} = c_3$.

Proof of the implication (iii.4) ⇒ (i.4).

An application of Lemma 21 (see also a version of Astala-Gehring lemma) shows that f is α-Hölder on $\underline{B}_{z_0}$ with a Lipschitz (multiplicative) constant which depends only on α and it gives the result. □

3. Lipschitz-Type Spaces of Harmonic and Pluriharmonic Mappings

3.1. Higher Dimensional Version of Schwarz Lemma. Before giving a proof of the higher dimensional version of the Schwarz lemma we first establish notation.

Suppose that $h : \overline{B}^n(a,r) \to \mathbb{R}^m$ is a continuous vector-valued function, harmonic on $B^n(a,r)$, and let

$$M_a^* = \sup\left\{\left|h(y) - h(a)\right| : y \in S^{n-1}(a,r)\right\}. \tag{70}$$

Let $h = (h^1, h^2, \dots, h^m)$. A modification of the estimate in [37, Equation (2.31)] gives

$$r\left|\nabla h^k(a)\right| \le nM_a^*, \quad k = 1, \dots, m. \tag{71}$$

We next extend this result to the case of vector-valued functions. See also [38] and [39, Theorem 6.16].

Lemma 25. *Suppose that $h : \overline{B}^n(a,r) \to \mathbb{R}^m$ is a continuous mapping, harmonic in $B^n(a,r)$. Then*

$$r\left|h'(a)\right| \le nM_a^*. \tag{72}$$

Proof. Without loss of generality, we may suppose that $a = 0$ and $h(0) = 0$. Let

$$K(x,y) = K_y(x) = \frac{r^2 - |x|^2}{n\omega_n r|x-y|^n}, \tag{73}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Hence, as in [8], for $1 \le j \le n$, we have

$$\frac{\partial}{\partial x_j}K(0,\xi) = \frac{\xi_j}{\omega_n r^{n+1}}. \tag{74}$$

Let $\eta \in S^{n-1}$ be a unit vector and $|\xi| = r$. For given ξ, it is convenient to write $K_{\xi}(x) = K(x,\xi)$ and consider K_{ξ} as function of x.

Then

$$K_{\xi}'(0)\eta = \frac{1}{\omega_n r^{n+1}}(\xi, \eta). \tag{75}$$

Since $|(\xi,\eta)| \le |\xi||\eta| = r$, we see that

$$\left|K'_\xi(0)\eta\right| \le \frac{1}{\omega_n r^n}, \quad \text{and therefore } \left|\nabla K_\xi(0)\right| \le \frac{1}{\omega_n r^n}. \quad (76)$$

This last inequality yields

$$\begin{aligned}\left|h'(0)(\eta)\right| &\le \int_{S^{n-1}(a,r)} \left|\nabla K^y(0)\right| \left|h(y)\right| d\sigma(y) \\ &\le \frac{M_0^* n\omega_n r^{n-1}}{\omega_n r^n} = \frac{M_0^* n}{r},\end{aligned} \quad (77)$$

where $d\sigma(y)$ is the surface element on the sphere, and the proof is complete. □

Let $\mathbb{C}^n = \{z = (z_1,\ldots,z_n) : z_1,\ldots,z_n \in \mathbb{C}\}$ denote the complex vector space of dimension n. For $a = (a_1,\ldots,a_n) \in \mathbb{C}^n$, we define the Euclidean inner product $\langle\cdot,\cdot\rangle$ by

$$\langle z,a\rangle = z_1\overline{a}_1 + \cdots + z_n\overline{a}_n, \quad (78)$$

where $\overline{a}_k$ ($k \in \{1,\ldots,n\}$) denotes the complex conjugate of a_k. Then the Euclidean length of z is defined by

$$|z| = \langle z,z\rangle^{1/2} = \left(|z_1|^2 + \cdots + |z_n|^2\right)^{1/2}. \quad (79)$$

Denote a ball in $\mathbb{C}^n$ with center z' and radius $r > 0$ by

$$\mathbb{B}^n\left(z',r\right) = \left\{z \in \mathbb{C}^n : \left|z - z'\right| < r\right\}. \quad (80)$$

In particular, $\mathbb{B}^n$ denotes the unit ball $\mathbb{B}^n(0,1)$ and $\mathbb{S}^{n-1}$ the sphere $\{z \in \mathbb{C}^n : |z| = 1\}$. Set $\mathbb{D} = \mathbb{B}^1$, the open unit disk in $\mathbb{C}$, and let $T = \mathbb{S}^0$ be the unit circle in $\mathbb{C}$.

A continuous complex-valued function f defined in a domain $\Omega \subset \mathbb{C}^n$ is said to be *pluriharmonic* if, for fixed $z \in \Omega$ and $\theta \in \mathbb{S}^{n-1}$, the function $f(z+\theta\zeta)$ is harmonic in $\{\zeta \in \mathbb{C} : |\theta\zeta - z| < d_\Omega(z)\}$, where $d_\Omega(z)$ denotes the distance from z to the boundary $\partial\Omega$ of Ω. It is easy to verify that the real part of any holomorphic function is pluriharmonic; cf. [40].

Let $\omega : [0,+\infty) \to [0,+\infty)$ with $\omega(0) = 0$ be a continuous function. We say that ω is a *majorant* if

(1) $\omega(t)$ is increasing,

(2) $\omega(t)/t$ is nonincreasing for $t > 0$.

If, in addition, there is a constant $C > 0$ depending only on ω such that

$$\int_0^\delta \frac{\omega(t)}{t}\,dt \le C\omega(\delta), \quad 0 < \delta < \delta_0, \quad (81)$$

$$\delta\int_\delta^\infty \frac{\omega(t)}{t^2}\,dt \le C\omega(\delta), \quad 0 < \delta < \delta_0 \quad (82)$$

for some δ_0, then we say that ω is a *regular majorant*. A majorant is called *fast* (resp., *slow*) if condition (81) (resp., (82)) is fulfilled.

Given a majorant ω, we define $\Lambda_\omega(\Omega)$ (resp., $\Lambda_\omega(\partial\Omega)$) to be the *Lipschitz-type* space consisting of all complex-valued functions f for which there exists a constant C such that, for all z and $w \in \Omega$ (resp., z and $w \in \partial\Omega$),

$$\left|f(z) - f(w)\right| \le C\omega(|z-w|). \quad (83)$$

Using Lemma 25, one can prove the following.

Proposition 26. *Let ω be a regular majorant and let f be harmonic mapping in a simply connected Λ_ω-extension domain $D \subset R^n$. Then $h \in \Lambda_\omega(D)$ if and only if $|\nabla f| \le C(\omega(1-|z|)/(1-|z|))$.*

It is easy to verify that the real part of any holomorphic function is pluriharmonic. It is interesting that the converse is true in simply connected domains.

Lemma 27. *(i) Let u be pluriharmonic in $B_0 = B(a;r)$. Then there is an analytic function f in B such that $f = u + iv$.*

(ii) Let Ω be simply connected and u be pluriharmonic in Ω. Then there is analytic function f in Ω such that $f = u + iv$.

Proof. (i) Let $B_0 = B(a;r) \subset \Omega$, $P_k = -u_{y_k}$, and $Q = u_{x_k}$; define form

$$\omega_k = P_k dx_k + Q_k dy_k = -u_{y_k}dx_k + u_{x_k}dy_k, \quad (84)$$

$\omega = \sum_{k=1}^n \omega_k$ and $v(x,y) = \int_{(x_0,y_0)}^{(x,y)} \omega$. Then (i) holds for $f_0 = u + iv$, which is analytic on B_0.

(ii) If $z \in \Omega$, there is a chain $C = (B_0, B_1, \ldots, B_n)$ in Ω such that z is center of B_n and, by the lemma, there is analytic chain (B_k, f_k), $k = 0,\ldots,n$. We define $f(z) = f_n(z)$. As in in the proof of monodromy theorem in one complex variable, one can show that this definition does not depend of chains C and that $f = u + iv$ in Ω. □

The following three theorems in [9] are a generalization of the corresponding one in [15].

Theorem 28. *Let ω be a fast majorant, and let $f = h + \overline{g}$ be a pluriharmonic mapping in a simply connected Λ_ω-extension domain Ω. Then the following are equivalent:*

(1) $f \in \Lambda_\omega(\Omega)$;

(2) $h \in \Lambda_\omega(\Omega)$ and $g \in \Lambda_\omega(\Omega)$;

(3) $|h| \in \Lambda_\omega(\Omega)$ and $|g| \in \Lambda_\omega(\Omega)$;

(4) $|h| \in \Lambda_\omega(\Omega,\partial\Omega)$ and $|g| \in \Lambda_\omega(\Omega,\partial\Omega)$,

where $\Lambda_\omega(\Omega,\partial\Omega)$ denotes the class of all continuous functions f on $\Omega\cup\partial\Omega$ which satisfy (83) with some positive constant C, whenever $z \in \Omega$ and $w \in \partial\Omega$.

Define $Df = (D_1 f,\ldots,D_n f)$ and $\overline{D}f = (\overline{D}_1 f,\ldots,\overline{D}_n f)$, where $z_k = x_k + iy_k$, $D_k f = (1/2)(\partial_{x_k} f - i\partial_{y_k} f)$ and $\overline{D}_k f = (1/2)(\partial_{x_k} f + i\partial_{y_k} f)$, $k = 1,2,\ldots,n$.

Theorem 29. *Let $\Omega \subset \mathbb{C}^n$ be a domain and f analytic in Ω. Then*

$$d_\Omega(z)\left|\nabla f(z)\right| \le 4\omega_{|f|}\left(d_\Omega(z)\right). \quad (85)$$

If $|f| \in \Lambda_\omega(\Omega)$, then $d_\Omega(z)|\nabla f(z)| \le 4C\omega(d_\Omega(z))$.

If $f = h + \overline{g}$ is a pluriharmonic mapping, where g and h are analytic in Ω, then $|g'| \le |f'|$ and $|h'| \le |f'|$. In particular, $|D_i g|, |D_i h| \le |f'| \le |Dg| + |Dh|$.

Proof. Using a version of Koebe theorem for analytic functions (we can also use Bloch theorem), we outline a proof. Let $z \in \Omega$, $a = (a_1, \dots, a_n) \in \mathbb{C}^n$, $|a| = 1$, $F(\lambda) = F(\lambda, a) = f(z + \lambda a)$, $\lambda \in B(0, d_\Omega(z)/2)$, and $u = |F|$.

By the version of Koebe theorem for analytic functions, for every line L which contains $w = f(z)$, there are points $w_1, w_2 \in L$ such that $d_\Omega(z)|F'(0)| \le 4|w_k - w|$, $k = 1, 2$, and $w \in [w_1, w_2]$. Hence $d_\Omega(z)|F'(0)| \le 4\omega_u(d_\Omega(z))$ and $d_\Omega(z)|F'(0)| \le 4\omega_{|f|}(d_\Omega(z))$.

Define $f'(z) = (f'_{z_1}(z), \dots f'_{z_n}(z))$. Since $F'(0) = \sum_{k=1}^n D_k f(z)\, a_k = (f'(z), \overline{a})$, we find $|\nabla f(z)| = |f'(z)|$, where $\overline{a} = (\overline{a_1}, \dots, \overline{a_n})$. Hence

$$d_\Omega(z)\left|\nabla f(z)\right| \le 4\omega_u\left(d_\Omega(z)\right) \le 4\omega_{|f|}\left(d_\Omega(z)\right). \tag{86}$$

Finally if $|f| \in \Lambda_\omega(\Omega)$, we have $d_\Omega(z)|\nabla f(z)| \le 4C\omega(d_\Omega(z))$. □

For B^n the following result is proved in [9].

Theorem 30. *Let ω be a regular majorant and let Ω be a simply connected Λ_ω-extension domain. A function f pluriharmonic in belongs to $\Lambda_\omega(\Omega)$ if and only if, for each $i \in \{1, 2, \dots, n\}$, $d_\Omega(z)|D_i f(z)| \le \omega(d_\Omega(z))$ and $d_\Omega(z)\, |\overline{D}_i f(z)| \le \omega(d_\Omega(z))$ for some constant C depending only on f, ω, Ω, and n.*

We only outline a proof: let $f = h + \overline{g}$. Note that $D_i f = D_i h$ and $\overline{D}_i f = \overline{D_i g}$.

We can also use Proposition 26.

3.2. Lipschitz-Type Spaces. Let $f : G \to G'$ be a C^2 function and $\underline{B}_x = B(x, d(x)/2)$. We denote by $OC^2(G)$ the class of functions which satisfy the following condition:

$$\sup_{B_x} d^2(x)\left|\Delta f(x)\right| \le c\, \mathrm{osc}_{B_x} f, \tag{87}$$

for every $x \in G$.

It was observed in [8] that $OC^2(G) \subset OC^1(G)$. In [7], we proved the following results.

Theorem 31. *Suppose that*

(a1) *D is a Λ^α-extension domain in $\mathbb{R}^n$, $0 < \alpha \le 1$, and f is continuous on $\overline{D}$ which is a K-quasiconformal mapping of D onto $G = f(D) \subset \mathbb{R}^n$;*

(a2) *∂D is connected;*

(a3) *f is Hölder on ∂D with exponent α;*

(a4) *$f \in OC^2(D)$. Then f is Hölder on $\overline{D}$ with exponent α*

The proof in [7] is based on Lemmas 3 and 8 in the paper of Martio and Nakki [18]. In the setting of Lemma 8, $d|f'(y)| \le \widehat{M} d^\alpha$. In the setting of Lemma 3, using the fact that ∂D is connected, we get similar estimate for d small enough.

Theorem 32. *Suppose that D is a domain in $\mathbb{R}^n$, $0 < \alpha \le 1$, and f is harmonic (more generally $OC^2(D)$) in D. Then one has the following:*

(i.1) *$f \in \mathrm{Lip}(\omega, c, D)$ implies*

(ii.1) *$|f'(x)| \le c d(x)^{\alpha - 1}$, $x \in D$.*

(ii.1) *implies*

(iii.1) *$f \in \mathrm{loc\ Lip}(\alpha, L_1; D)$.*

4. Theorems of Koebe and Bloch Type for Quasiregular Mappings

We assume throughout that $G \subset \mathbb{R}^n$ is an open connected set whose boundary, ∂G, is nonempty. Also $B(x, R)$ is the open ball centered at $x \in G$ with radius R. If $B \subset \mathbb{R}^n$ is a ball, then σB, $\sigma > 0$, denotes the ball with the same center as B and with radius equal to σ times that of B.

The spherical (chordal) distance between two points a, $b \in \overline{\mathbb{R}}^n$ is the number

$$\underline{q}(a, b) = \left|p(a) - p(b)\right|, \tag{88}$$

where $p : \overline{\mathbb{R}}^n \to S(e_{n+1}/2, 1/2)$ is stereographic projection, defined by

$$p(x) = e_{n+1} + \frac{x - e_{n+1}}{\left|x - e_{n+1}\right|^2}. \tag{89}$$

Explicitly, if $a \ne \infty \ne b$,

$$\underline{q}(a, b) = |a - b|\left(1 + |a|^2\right)^{-1/2}\left(1 + |b|^2\right)^{-1/2}. \tag{90}$$

When $f : G \to \mathbb{R}^n$ is differentiable, we denote its Jacobi matrix by Df or f' and the norm of the Jacobi matrix as a linear transformation by $|f'|$. When Df exists a.e. we denote the local Dirichlet integral of f at $x \in G$ by $\underline{D}_f(x) = \underline{D}_{f,G}(x) = (1/|B| \int_{\underline{B}} |f'|^n)^{1/n}$, where $\underline{B} = \underline{B}_x = \underline{B}_{x,G}$. If there is no chance of confusion, we will omit the index G. If $\mathbf{B} = \mathbf{B}_x$, then $\underline{D}_{f,G}(x) = \underline{D}_{f,\mathbf{B}}(x)$ and if $\mathbb{B}$ is the unit ball, we write $\underline{D}(f)$ instead of $\underline{D}_{f,\mathbb{B}}(0)$.

When the measure is omitted from an integral, as here, integration with respect to n-dimensional Lebesgue measure is assumed.

A continuous increasing function $\omega(t) : [0, \infty) \to [0, \infty)$ is a majorant if $\omega(0) = 0$ and if $\omega(t_1 + t_2) \le \omega(t_1) + \omega(t_2)$ for all $t_1, t_2 \ge 0$.

The main result of the paper [3] generalizes Lemma 5 to a quasiregular version involving a somewhat larger class of moduli of continuity than t^α, $0 < \alpha < 1$.

Lemma 33 (see [3]). *Suppose that G is Λ_ω-extension domain in $\mathbb{R}^n$. If f is K-quasiregular in G with $f(G) \subset \mathbb{R}^n$ and if*

$$\underline{D}_f(x) \le C_1 \omega(d(x))\, d(x, \partial D)^{-1} \quad \text{for } x \in G, \tag{91}$$

then f has a continuous extension to $\overline{D} \setminus \{\infty\}$ and

$$\left|f(x_1) - f(x_2)\right| \le C_2 \omega\left(\left|x_1 - x_2\right| + d(x_1, \partial G)\right), \tag{92}$$

for $x_1, x_2 \in \overline{D} \setminus \{\infty\}$, *where the constant* C_2 *depends only on* K, n, ω, C_1, *and* G.

If G *is uniform, the constant* $C_2 = \widehat{c}(G)$ *depends only on* K, n, C_1 *and the uniformity constant* $c^* = c^*(G)$ *for* G.

Conversely if there exists a constant C_2 *such that* (92) *holds for all* $x_1, x_2 \in G$, *then* (91) *holds for all* $x \in G$ *with* C_1 *depending only on* C_2, K, n, ω, *and* G.

Now suppose that $\omega = \omega_\alpha$ and $\alpha \le \alpha_0$.

Also in [3], Nolder, using suitable modification of a theorem of Näkki and Palka [41], shows that (92) can be replaced by the stronger conclusion that

$$|f(x_1) - f(x_2)| \le C(|x_1 - x_2|)^\alpha \quad \left(x_1, x_2 \in \overline{D} \setminus \{\infty\}\right). \tag{93}$$

Remark 34. Simple examples show that the term $d(x_1, \partial G)$ cannot in general be omitted. For example, $f(x) = x|x|^{a-1}$ with $a = K^{1/(1-n)}$ is K-quasiconformal in $B = B(0;1)$. $\underline{D}_f(x)$ is bounded over $x \in B$ yet $f \in \mathrm{Lip}_a(B)$; see Example 6.

If $n = 2$, then $|f'(z)| \approx \rho^{a-1}$, where $z = \rho e^{i\theta}$, and if $B_r = B(0, r)$, then $\int_{B_r} |f'(z)| dx\, dy = \int_0^r \int_0^{2\pi} \rho^{2(a-1)} \rho d\rho d\theta \approx r^{2a}$ and $\underline{D}_{f,B_r}(0) \approx r^{a-1}$.

Note that we will show below that if $|f| \in L^\alpha(G)$, the conclusion (92) in Lemma 33 can be replaced by the stronger assertion $|f(x_1) - f(x_2)| \le cm|x_1 - x_2|^\alpha$.

Lemma 35 (see [3]). *If* $f = (f_1, f_2, \ldots, f_n)$ *is* K*-quasiregular in* G *with* $f(G) \subset R^n$ *and if* B *is a ball with* $\sigma B \subset G$, $\sigma > 1$, *then there exists a constant* C, *depending only on* n, *such that*

$$\left(\frac{1}{|B|}\int_B \left|f'\right|^n\right)^{1/n} \le CK\frac{\sigma}{\sigma - 1}\left(\frac{1}{|\sigma B|}\int_{\sigma B} \left|f_j - a\right|^n\right)^{1/n} \tag{94}$$

for all $a \in \mathbb{R}$ *and all* $j = 1, 2, \ldots, n$. *Here and in some places we omit to write the volume and the surface element.*

Lemma 36 (see[42], second version of Koebe theorem for analytic functions). *Let* $B = B(a;r)$; *let* f *be holomorphic function on* B, $D = f(B)$, $f(a) = b$, *and let the unbounded component* D_∞ *of* D *be not empty, and let* $d_\infty = d_\infty(b) = \mathrm{dist}(b, D_\infty)$. *Then* (a.1) $r|f'(a)| \le 4d_\infty$; (b.1) *if, in addition,* D *is simply connected, then* D *contains the disk* $B(b, \rho)$ *of radius* ρ, *where* $\rho = \rho_f(a) = r|f'(a)|/4$.

The following result can be considered as a version of this lemma for quasiregular mappings in space.

Theorem 37. *Suppose that* f *is a* K*-quasiregular mapping on the unit ball* $\mathbb{B}$, $f(0) = 0$ *and* $|\underline{D}(f)| \ge 1$.

Then, there exists an absolute constant α *such that for every* $x \in \mathbb{S}$ *there exists a point* y *on the half-line* $\Lambda_x = \Lambda(0, x) = \{\rho x : \rho \ge 0\}$, *which belongs to* $f(\mathbb{B})$, *such that* $|y| \ge 2\alpha$.

If f *is a* K*-quasiconformal mapping, then there exists an absolute constant* ρ_1 *such that* $f(\mathbb{B})$ *contains* $\mathbb{B}(0;\rho_1)$.

For the proof of the theorem, we need also the following result, Theorem 18.8.1 [10].

Lemma 38. *For each* $K \ge 1$ *there is* $m = m(n, K)$ *with the following property. Let* $\epsilon > 0$, *let* $G \subset \overline{\mathbb{R}}^n$ *be a domain, and let* $\widehat{\mathrm{F}}$ *denote the family of all qr mappings with* $\max\{K_O(x, f), K_I(x, f)\} \le K$ *and* $f : G \to \mathbb{R}^n \setminus \{a_{1,f}, a_{2,f}, \ldots, a_{m,f}\}$ *where* $a_{i,f}$ *are points in* $\overline{\mathbb{R}}^n$ *such that* $\underline{q}(a_{i,f}, a_{j,f}) > \epsilon$, $i \ne j$. *Then* $\widehat{\mathrm{F}}$ *forms a normal family in* G.

Now we prove Theorem 37.

Proof. If we suppose that this result is not true then there is a sequence of positive numbers a_n, which converges to zero, and a sequence of K-quasiregular functions f_n, such that $f_n(\mathbb{B})$ does not intersect $[a_n, +\infty)$, $n \ge 1$. Next, the functions $g_n = f_n/a_n$ map $\mathbb{B}$ into $G = \mathbb{R}^n \setminus [1, +\infty)$ and hence, by Lemma 38, the sequence g_n is equicontinuous and therefore forms normal family. Thus, there is a subsequence, which we denote again by g_n, which converges uniformly on compact subsets of $\mathbb{B}$ to a quasiregular function g. Since $\underline{D}(g_n)$ converges to $\underline{D}(g)$ and $|\underline{D}(g_n)| = |\underline{D}(f_n)|/a_n$ converges to infinity, we have a contradiction by Lemma 35. □

A path-connected topological space X with a trivial fundamental group $\pi_1(X)$ is said to be simply connected. We say that a domain V in $\mathbb{R}^3$ is spatially simply connected if the fundamental group $\pi_2(V)$ is trivial.

As an application of Theorem 37, we immediately obtain the following result, which we call the Koebe theorem for quasiregular mappings.

Theorem 39 (second version of Koebe theorem for K-quasiregular functions). *Let* $B = B(a;r)$; *let* f *be* K*-quasiregular function on* $B \subset R^n$, $D = f(B)$, $f(a) = b$, *and let the unbounded component* D_∞ *of* D^c *be not empty, and let* $d_\infty = d_\infty(b) = \mathrm{dist}(b, D_\infty)$. *Then there exists an absolute constant c:*

(a.1) $r|\underline{D}_f(a)| \le \tilde{c}d_\infty$;

(a.2) *if, in addition,* $D \subset \mathbb{R}^3$ *and* D *is spatially simply connected, then* D *contains the disk* $B(b, \rho)$ *of radius* ρ, *where* $\rho = \rho_f(a) = r|\underline{D}_f(a)|/\tilde{c}$.

(A.1) Now, using Theorem 39 and Lemma 20, we will establish the characterization of Lipschitz-type spaces for quasiregular mappings by the average Jacobian and in particular an extension of Dyakonov's theorem for quasiregular mappings in space (without Dyakonov's hypothesis that it is a quasiregular local homeomorphism). In particular, our approach is based on the estimate (a.3) below.

Theorem 40 (Theorem-DyMa). *Let* $0 < \alpha \le 1$, $K \ge 1$, *and* $n \ge 2$. *Suppose* G *is a* L^α*-extension domain in* $\mathbb{R}^n$ *and* f *is a* K*-quasiregular mapping of* G *onto* $f(G) \subset \mathbb{R}^n$. *The following are equivalent:*

(i.4) $f \in \Lambda^\alpha(G)$,

(ii.4) $|f| \in \Lambda^\alpha(G)$,

(iii.4) $|\underline{D}_f(r, x)| \le cr^{\alpha-1}$ *for every ball* $B = B(x, r) \subset G$.

Proof. Suppose that f is a K-quasiregular mapping of G onto $f(G) \subset \mathbb{R}^n$. We first establish that for every ball $B = B(x, r) \subset G$, one has the following:

(a.3) $r|\underline{D}_f(r, x)| \le \tilde{c}\omega_{|f|}(r, x)$, where $\underline{D}_f(r, x) = \underline{D}_{f,B}(x)$.

Let l be line throughout 0 and $f(x)$ and denote by $D_{\infty,r}$ the unbounded component of D^c, where $D = f(B(x, r))$. Then, using a similar procedure as in the proof of Theorem 22, the part (iii.1), one can show that there is $z' \in \partial D_{\infty,r} \cap l$ such that $d_\infty(x') \le |z' - f(x)| = ||z'| - |f(x)||$, where $x' = f(x)$.

Take a point $z \in f^{-1}(z')$. Then $z' = f(z)$ and $|z - x| = r$. Since $||f(z)| - |f(x)|| \le \omega_{|f|}(x, r)$. Then using Theorem 39, we find (a.3).

Now we suppose (ii.4), that is, $|f| \in \mathrm{Lip}(\alpha, L; G)$.

Thus we have $\omega_{|f|}(x, r) \le Lr^\alpha$. Hence, we get

(a.4) $r|\underline{D}_f(r, x)| \le cr^\alpha$, where $c = L\tilde{c}$.

It is clear that (iii.4) is denoted here as (a.4). The implication (ii.4) $\Rightarrow$ (iii.4) is thus established.

If we suppose (iii.4), then an application of Lemma 20 shows that $f \in \mathrm{loc}\,\Lambda^\alpha(G)$, and since G is a L^α-extension domain, we conclude that $f \in \Lambda^\alpha(G)$. Thus (iii.4) implies (i.4).

Finally, the implication (i.4) $\Rightarrow$ (ii.4) is a clear corollary of the triangle inequality. □

(A.2) Now we give another outline that (i.4) is equivalent to (ii.4).

Here, we use approach as in [15]. In particular, (a.4) implies that the condition (91) holds.

We consider two cases:

(1) $d(x) \le 2|x - y|$;

(2) $s = |x - y| \le d(x)/2$.

Then we apply Lemma 33 on G and $A = B(x, 2s)$ in Case (1) and Case (2), respectively.

In more detail, if $|f| \in L^\alpha(G)$, then for every ball $B(x, r) \subset G$, by (a.3), $r|\underline{D}_f(x)| \le cr^{\alpha-1}$ and the condition (91) holds. Then Lemma 33 tells us that (92) holds, with a fixed constant, for all balls $B \subset G$ and all pairs of points $x_1, x_2 \in B$. Next, we pick two points $x, \in G$ with $|x - y| < d(x, \partial G)$ and apply (92) with $G = A$, where $A = B(x; |x - y|)$, letting $x_1 = x$ and $x_2 = y$. The resulting inequality

$$|f(x) - f(y)| \le \mathrm{const}|x - y|^\alpha \tag{95}$$

shows that $f \in \mathrm{loc}\,\Lambda^\alpha(G)$, and since G is a L^α-extension domain, we conclude that $f \in \Lambda^\alpha(G)$.

The implication (ii.4) $\Rightarrow$ (i.4) is thus established. The converse being trivially true.

The consideration in (A.1) shows that (i.4) and (ii.4) are equivalent with (a.4).

Appendices

A. Distortion of Harmonic Maps

Recall by $\mathbb{D}$ and $\mathbb{T} = \partial\mathbb{D}$ we denote the unit disc and the unit circle respectively, and we also use notation $z = re^{i\theta}$. For a function h we denote by h'_r, h'_x and h'_y (or sometimes by $\partial_r h$, $\partial_x h$, and $\partial_x h$) partial derivatives with respect to r, x, and y, respectively. Let $h = f + \overline{g}$ be harmonic, where f and g are analytic, and every complex valued harmonic function h on simply connected set D is of this form. Then $\partial h = f'$, $h'_r = f'_r + \overline{g'_r}$, $f'_r = f'(z)e^{i\theta}$, and $J_h = |f'|^2 - |g'|^2$. If h is univalent, then $|g'| < |f'|$ and therefore $|h'_r| \le |f'_r| + |g'_r|$ and $|h'_r| < 2|f'|$.

After writing this paper and discussion with some colleagues (see Remark A.11 below), the author found out that it is useful to add this section. For origins of this section see also [29].

Theorem A.1. *Suppose that*

(a) *h is an euclidean univalent harmonic mapping from an open set D which contains $\overline{\mathbb{D}}$ into $\mathbb{C}$;*

(b) *$h(\overline{\mathbb{D}})$ is a convex set in $\mathbb{C}$;*

(c) *$h(\mathbb{D})$ contains a disc $B(a; R)$, $h(0) = a$, and $h(\mathbb{T})$ belongs to the boundary of $h(\overline{\mathbb{D}})$.*

Then

(d) $|h'_r(e^{i\varphi})| \ge R/2$, $0 \le \varphi \le 2\pi$.

A generalization of this result to several variables has been communicated at Analysis Belgrade Seminar, cf. [32, 33].

Proposition A.2. *Suppose that*

(a$'$) *h is an euclidean harmonic orientation preserving univalent mapping from an open set D which contains $\overline{\mathbb{D}}$ into $\mathbb{C}$;*

(b$'$) *$h(\overline{\mathbb{D}})$ is a convex set in $\mathbb{C}$;*

(c$'$) *$h(\mathbb{D})$ contains a disc $B(a; R)$ and $h(0) = a$.*

Then

(d$'$) $|\partial h(z)| \ge R/4$, $z \in \mathbb{D}$.

By (d), we have

(e) $|f'| \ge R/4$ on $\mathbb{T}$.

Since h is an euclidean univalent harmonic mapping, $f' \ne 0$. Using (e) and applying Maximum Principle to the analytic function $f' = \partial h$, we obtain Proposition A.2.

Proof of Theorem A.1. Without loss of generality we can suppose that $h(0) = 0$. Let $0 \le \varphi \le 2\pi$ be arbitrary. Since $h(\mathbb{D})$ is a bounded convex set in $\mathbb{C}$ there exists $\tau \in [0, 2\pi]$ such that harmonic function u, defined by $u = \mathrm{Re}\, H$, where $H(z) = e^{i\tau}h(z)$, has a maximum on $\overline{\mathbb{D}}$ at $e^{i\varphi}$.

Define $u_0(z) = u(e^{i\varphi}) - u(z)$, $z \in \mathbb{D}$. By the mean value theorem, $(1/2\pi)\int_0^{2\pi} u_0(e^{i\theta})\, d\theta = u(e^{i\varphi}) - u(0) \ge R$.

Since Poisson kernel for $\mathbb{D}$ satisfies

$$P_r(\theta) \ge \frac{1 - r}{1 + r}, \tag{A.1}$$

using Poisson integral representation of the function $u_0(z) = u(e^{i\varphi}) - u(z)$, $z \in \mathbb{D}$, we obtain

$$u\left(e^{i\varphi}\right) - u\left(re^{i\varphi}\right) \geq \frac{1-r}{1+r}\left(u\left(e^{i\varphi}\right) - u(0)\right), \tag{A.2}$$

and hence (d). □

Now we derive a slight generalization of Proposition A.2. More precisely, we show that we can drop the hypothesis (a$'$) and suppose weaker hypothesis (a$''$).

Proposition A.3. *Suppose that* (a'') *h is an euclidean harmonic orientation preserving univalent mapping of the unit disc onto convex domain* Ω. *If* Ω *contains a disc* $B(a;R)$ *and* $h(0) = a$, *then*

$$|\partial h(z)| \geq \frac{R}{4}, \quad z \in \mathbb{D}. \tag{A.3}$$

A proof of the proposition can be based on Proposition A.2 and the hereditary property of convex functions: (i) if an analytic function maps the unit disk univalently onto a convex domain, then it also maps each concentric subdisk onto a convex domain. It seems that we can also use the approach as in the proof of Proposition A.7, but an approximation argument for convex domain G, which we outline here, is interesting in itself:

(ii) approximation of convex domain G with smooth convex domains.

Let ϕ be conformal mapping of $\mathbb{D}$ onto G, $\phi'(0) > 0$, $G_n = \phi(r_n\mathbb{D})$, $r_n = n/(n+1)$, h is univalent mapping of the unit disc onto convex domain Ω and $D_n = h^{-1}(G_n)$.

(iii) Let φ_n be conformal mapping of $\mathbb{U}$ onto D_n, $\varphi_n(0) = 0$, $\varphi_n'(0) > 0$, and $h_n = h \circ \varphi_n$. Since $D_n \subset D_{n+1}$ and $\cup D_n = \mathbb{D}$, we can apply the Carathéodory theorem; φ_n tends to z, uniformly on compacts, whence $\varphi_n'(z) \to 1$ ($n \to \infty$). By the hereditary property G_n is convex.

(iv) Since the boundary of D_n is an analytic Jordan curve, the mapping φ_n can be continued analytically across $\mathbb{T}$, which implies that h_n has a harmonic extension across $\mathbb{T}$.

Thus we have the following.

(v) h_n are harmonic on $\overline{\mathbb{D}}$, $G_n = h_n(\mathbb{D})$ are smooth convex domains, and h_n tends to h, uniformly on compacts subset of $\mathbb{D}$.

Using (v), an application of Proposition A.2 to h_n, gives the proof.

As a corollary of Proposition A.3 we obtain (A.4).

Proposition A.4. *Let h be an euclidean harmonic orientation preserving K-qc mapping of the unit disc* $\mathbb{D}$ *onto convex domain* Ω. *If* Ω *contains a disc* $B(a;R)$ *and* $h(0) = a$, *then*

$$\lambda_h(z) \geq \frac{1-k}{4}R, \tag{A.4}$$

$$\left|h(z_2) - h(z_1)\right| \geq c'\left|z_2 - z_1\right|, \quad z_1, z_2 \in \mathbb{D}. \tag{A.5}$$

(i.0) *In particular,* f^{-1} *is Lipschitz on* Ω.

It is worthy to note that (A.4) holds (i.e., f^{-1} is Lipschitz) under assumption that Ω is convex (without any smoothness hypothesis).

Example A.5. $f(z) = (z-1)^2$ is univalent on $\mathbb{D}$. Since $f'(z) = 2(z-1)$ it follows that $f'(z)$ tends 0 if z tends to 1. This example shows that we cannot drop the hypothesis that $f(\mathbb{D})$ is a convex domain in Proposition A.4.

Proof. Let $c' = ((1-k)/4)R$. Since

$$\left|\overline{D}h(z)\right| \leq k\ |Dh(z)|, \tag{A.6}$$

it follows that $\lambda_h(z) \geq c_0 = ((1-k)/4)R$ and therefore $|h(z_2) - h(z_1)| \geq c'|z_2 - z_1|$. □

Hall, see [43, pages 66–68], proved the following.

Lemma A.6 (Hall Lemma). *For all harmonic univalent mappings f of the unit disk onto itself with* $f(0) = 0$,

$$\left|a_1\right|^2 + \left|b_1\right|^2 \geq c_0 = \frac{27}{4\pi^2}, \tag{A.7}$$

where $a_1 = Df(0)$, $b_1 = \overline{D}f(0)$, *and* $c_0 = 27/4\pi^2 = 0.6839\ldots$.

Set $\tau_0 = \sqrt{c_0} = 3\sqrt{3}/2\pi$. Now we derive a slight generalization of Proposition A.3. More precisely, we show that we can drop the hypothesis that the image of the unit disc is convex.

Proposition A.7. *Let h be an euclidean harmonic orientation preserving univalent mapping of the unit disc into* $\mathbb{C}$ *such that* $f(\mathbb{D})$ *contains a disc* $B_R = B(a;R)$ *and* $h(0) = a$.

(i.1) *Then*

$$|\partial h(0)| \geq \frac{R}{4}. \tag{A.8}$$

(i.2) *The constant* $1/4$ *in inequality* (A.8) *can be replaced with sharp constant* $\tau_1 = 3\sqrt{3/2}/2\pi$.

(i.3) *If in addition h is K-qc mapping and* $k = (K-1)/(K+1)$, *then*

$$\lambda_h(0) \geq R\frac{\tau_0(1-k)}{\sqrt{1+k^2}}. \tag{A.9}$$

Proof. Let $0 < r < R$ and $V = V_r = h^{-1}(B_r)$ and let φ be a conformal mapping of the unit disc $\mathbb{D}$ onto V such that $\varphi(0) = 0$ and let $h_r = h \circ \varphi$. By Schwarz lemma

$$\left|\varphi'(0)\right| \leq 1. \tag{A.10}$$

The function h_r has continuous partial derivatives on $\overline{\mathbb{D}}$. Since $\partial h_r(0) = \partial h(0)\varphi'(0)$, by Proposition A.3, we get $|\partial h_r(0)| = |\partial h(0)||\varphi'(0)| \geq r/4$. Hence, using (A.10) we find $|\partial h(0)| \geq r/4$ and if r tends to R, we get (A.8).

(i2) If $r_0 = \max\{|z| : z \in V_R\}$ and $q_0 = 1/r_0$, then

(vi) $q_0|\partial h_r(0)| \leq |\partial h(0)|$.

Hence, by the Hall lemma, $2|a_1|^2 > |a_1|^2 + |b_1|^2 \ge c_0$ and therefore

(vii) $|a_1| \ge \tau_1$, where $\tau_1 = 3\sqrt{3/2}/2\pi$. Combining (vi) and (vii), we prove (i2).

(i3) If $h = f + \overline{g}$, where f and g are analytic, then $\lambda_h(z) = |f'(z)| - |g'(z)|$ and $\lambda_h(z) = |f'(z)| - |g'(z)| \ge (1-k)|f'(z)|$. Set $a_1 = Dh(0)$ and $b_1 = \overline{D}h(0)$. By the Hall sharp form $|a_1|^2 + |b_1|^2 \ge c_0 R^2$ we get $|a_1|^2(1+k^2) \ge c_0 R^2$, then $|a_1| \ge R(\tau_0/\sqrt{1+k^2})$ and therefore (A.9).

□

Also as a corollary of Proposition A.3 we obtain the following.

Proposition A.8 (see [27, 44]). *Let h be an euclidean harmonic diffeomorphism of the unit disc onto convex domain Ω. If Ω contains a disc $B(a;R)$ and $h(0) = a$, then*

$$e(h)(z) \ge \frac{1}{16}R^2, \quad z \in \mathbb{D}, \tag{A.11}$$

where $e(h)(z) = |\partial h(z)|^2 + |\overline{\partial} h(z)|^2$.

The following example shows that Theorem A.1 and Propositions A.2, A.3, A.4, A.7, and A.8 are not true if we omit the condition $h(0) = a$.

Example A.9. The mapping

$$\varphi_b(z) = \frac{z-b}{1-\overline{b}z}, \quad |b| < 1, \tag{A.12}$$

is a conformal automorphism of the unit disc onto itself and

$$\left|\varphi_b'(z)\right| = \frac{1-|b|^2}{\left|1-\overline{b}z\right|^2}, \quad z \in \mathbb{D}. \tag{A.13}$$

In particular $\varphi_b'(0) = 1 - |b|^2$.

Heinz proved (see [45]); that if h is a harmonic diffeomorphism of the unit disc onto itself such that $h(0) = 0$, then

$$e(h)(z) \ge \frac{1}{\pi^2}, \quad z \in \mathbb{D}. \tag{A.14}$$

Using Proposition A.3 we can prove another Heinz theorem.

Theorem A.10 (Heinz). *There exists no euclidean harmonic diffeomorphism from the unit disc $\mathbb{D}$ onto $\mathbb{C}$.*

Note that this result was a key step in his proof of the Bernstein theorem for minimal surfaces in $\mathbb{R}^3$.

Remark A.11. Professor Kalaj turned my attention to the fact that in Proposition 12, the constant $1/4$ can be replaced with sharp constant $\tau_1 = 3\sqrt{3/2}/2\pi$ which is approximately 0.584773.

Thus Hall asserts the sharp form $|a_1|^2 + |b_1|^2 \ge c_0 = 27/4\pi^2$, where $c_0 = 27/4\pi^2 = 0.6839\ldots$ and therefore if $b_1 = 0$, then $|a_1| \ge \tau_0$, where $\tau_0 = \sqrt{c_0} = 3\sqrt{3}/2\pi$.

If we combine Hall's sharp form with the Schwarz lemma for harmonic mappings, we conclude that if a_1 is real, then $3\sqrt{3}/2\pi \le a_1 \le 1$.

Concerning general codomains, the author, using Hall's sharp result, communicated around 1990 at Seminar University of Belgrade a proof of Corollary 16 and a version of Proposition 13; cf. [32, 33].

A.1. Characterization of Harmonic qc Mappings (See [25]). By χ we denote restriction of h on $\mathbb{R}$. If $h \in HQC_0(\mathbb{H})$, it is well-known that $\chi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and $\operatorname{Re} h = P[\chi]$. Now we give characterizations of $h \in HQC_0(\mathbb{H})$ in terms of its boundary value χ.

Suppose that h is an orientation preserving diffeomorphism of $\mathbb{H}$ onto itself, continuous on $\mathbb{H} \cup \mathbb{R}$ such that $h(\infty) = \infty$, and χ the is restriction of h on $\mathbb{R}$. Recall $h \in HQC_0(\mathbb{H})$ if and only if there is analytic function $\phi : \mathbb{H} \to \Pi^+$ such that $\phi(\mathbb{H})$ is relatively compact subset of Π^+ and $\chi'(x) = \operatorname{Re}\phi^*(x)$ a.e.

We give similar characterizations in the case of the unit disk and for smooth domains (see below).

Theorem A.12. *Let ψ be a continuous increasing function on $\mathbb{R}$ such that $\psi(t+2\pi) - \psi(t) = 2\pi$, $\gamma(t) = e^{i\psi(t)}$ and $h = P[\gamma]$. Then h is qc if and only if the following hold:*

(1) ess inf $\psi' > 0$;

(2) *there is analytic function $\phi : \mathbb{D} \to \Pi^+$ such that $\phi(U)$ is relatively compact subset of Π^+ and $\psi'(x) = \operatorname{Re}\phi^*(e^{ix})$ a.e.*

In the setting of this theorem we write $h = h^\phi$. The reader can use the above characterization and functions of the form $\phi(z) = 2 + M(z)$, where M is an inner function, to produce examples of HQC mappings $h = h^\phi$ of the unit disk onto itself so the partial derivatives of h have no continuous extension to certain points on the unit circle. In particular we can take $M(z) = \exp((z+1)/(z-1))$; for the subject of this subsections cf. [25, 28] and references cited therein.

Remark A.13. Because of lack of space in this paper we could not consider some basic concepts related to the subject and in particular further distortion properties of qc maps as Gehring and Osgood inequality [12]. For an application of this inequality, see [25, 28].

B. Quasi-Regular Mappings

(A) The theory of holomorphic functions of one complex variable is the central object of study in complex analysis. It is one of the most beautiful and most useful parts of the whole mathematics.

Holomorphic functions are also sometimes referred to as analytic functions, regular functions, complex differentiable functions or conformal maps.

This theory deals only with maps between two-dimensional spaces (Riemann surfaces).

For a function which has a domain and range in the complex plane and which preserves angles, we call a conformal map. The theory of functions of several complex variables has a different character, mainly because analytic functions of several variables are not conformal.

Conformal maps can be defined between Euclidean spaces of arbitrary dimension, but when the dimension is greater than 2, this class of maps is very small. A theorem of J. Liouville states that it consists of Mobius transformations only; relaxing the smoothness assumptions does not help, as proved by Reshetnyak. This suggests the search of a generalization of the property of conformality which would give a rich and interesting class of maps in higher dimension.

The general trend of the geometric function theory in $\mathbb{R}^n$ is to generalize certain aspects of the analytic functions of one complex variable. The category of mappings that one usually considers in higher dimensions is the mappings with finite distortion, thus, in particular, quasiconformal and quasiregular mappings.

For the dimensions $n = 2$ and $K = 1$, the class of K-quasiregular mappings agrees with that of the complex-analytic functions. Injective quasi-regular mappings in dimensions $n \geq 2$ are called quasiconformal. If G is a domain in $\mathbb{R}^n$, $n \geq 2$, we say that a mapping $f : G \to \mathbb{R}^n$ is discrete if the preimage of a point is discrete in the domain G. Planar quasiregular mappings are discrete and open (a fact usually proved via Stoïlow's Theorem).

Theorem A (Stoïlow's Theorem). *For $n = 2$ and $K \geq 1$, a K-quasi-regular mapping $f : G \to \mathbb{R}^2$ can be represented in the form $f = \phi \circ g$, where $g : G \to G'$ is a K-quasi-conformal homeomorphism and ϕ is an analytic function on G'.*

There is no such representation in dimensions $n \geq 3$ in general, but there is representation of Stoïlow's type for quasiregular mappings f of the Riemann n-sphere $\mathbb{S}^n = \overline{\mathbb{R}}^n$, cf. [46].

Every quasiregular map $f : \mathbb{S}^n \to \mathbb{S}^n$ has a factorization $f = \phi \circ g$, where $g : \mathbb{S}^n \to \mathbb{S}^n$ is quasiconformal and $\phi : \mathbb{S}^n \to \mathbb{S}^n$ is uniformly quasiregular.

Gehring-Lehto Lemma: let f be a complex, continuous, and open mapping of a plane domain Ω which has finite partial derivatives a.e. in Ω. Then f is differentiable a.e. in Ω.

Let U be an open set in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^m$ be a mapping. Set

$$L(x, f) = \limsup \frac{|f(x+h) - f(x)|}{h}. \tag{B.1}$$

If f is differentiable at x, then $L(x, f) = |f'(x)|$. The theorem of Rademacher-Stepanov states that if $L(x, f) < \infty$ a.e., then f is differentiable a.e.

Note that Gehring-Lehto Lemma is used in dimension $n = 2$ and Rademacher-Stepanov theorem to show the following.

For all dimensions, $n \geq 2$, a quasiconformal mapping $f : G \to \mathbb{R}^n$, where G is a domain in $\mathbb{R}^n$, is differentiable a.e. in G.

Therefore the set S_f of those points where it is not differentiable has Lebesgue measure zero. Of course, S_f may be nonempty in general and the behaviour of the mapping may be very interesting at the points of this set. Thus, there is a substantial difference between the two cases $K > 1$ and $K = 1$. This indicates that the higher dimensions theory of quasiregular mappings is essentially different from the theory in the complex plane. There are several reasons for this:

(a) there are neither general representation theorems of Stoïlow's type nor counterparts of power series expansions in higher dimensions;

(b) the usual methods of function theory based on Cauchy's Formula, Morera's Theorem, Residue Theorem, The Residue Calculus and Consequences, Laurent Series, Schwarz's Lemma, Automorphisms of the Unit Disc, Riemann Mapping Theorem, and so forth, are not applicable in the higher-dimensional theory;

(c) in the plane case the class of conformal mappings is very rich, while in higher dimensions it is very small (J. Liouville proved that for $n \geq 3$ and $K = 1$, sufficiently smooth quasiconformal mappings are restrictions of Möbius transformations);

(d) for dimensions $n \geq 3$ the branch set (i.e., the set of those points at which the mapping fails to be a local homeomorphism) is more complicated than in the two-dimensional case; for instance, it does not contain isolated points.

Injective quasiregular maps are called quasiconformal. Using the interaction between different coordinate systems, for example, spherical coordinates $(\rho, \theta, \phi) \in R_+ \times [0, 2\pi) \times [0, \pi]$ and cylindrical coordinates (r, θ, t), one can construct certain qc maps.

Define f by $(\rho, \theta, \phi) \mapsto (\phi, \theta, \ln \rho)$. Then f maps the cone $C(\phi_0) = \{(\rho, \theta, \phi) : 0 \leq \phi < \phi_0\}$ for $0 < \phi_0 \leq \pi$ onto the infinite cylinder $\{(r, \theta, t) : 0 \leq r < \phi_0\}$. We leave it to the reader as an exercise to check that the linear distortion H depends only on ϕ and that $H(\rho, \theta, \phi) = \phi / \sin \phi \leq \phi_0 / \sin \phi_0$.

For $\phi_0 = \pi/2$ we obtain a qc map of the half-space onto the cylinder with the linear distortion bounded by $\pi/2$.

Since the half space and ball are conformally equivalent, we find that there is a qc map of the unit ball onto the infinite cylinder with the linear distortion bounded by $\pi/2$.

A simple example of noninjective quasiregular map f is given in cylindrical coordinates in 3-space by the formula $(r, \theta, z) \mapsto (r, 2\theta, z)$. This map is two-to-one and it is quasiregular on any bounded domain in $\mathbb{R}^3$ whose closure does not intersect the z-axis. The Jacobian $J(f)$ is different from 0 except on the z-axis, and it is smooth everywhere except on the z-axis.

Set $S_+ = \{(x, y, z) : x^2 + y^2 + z^2 = 1,\ z \geq 0\}$ and $\overline{H}^3 = \{(x, y, z) : z \geq 0\}$. The Zorich map $Z : \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$ is a quasiregular analogue of the exponential function. It can be defined as follows.

(1) Choose a bi-Lipschitz map $h : [-\pi/2, \pi/2]^2 \to S_+$.

(2) Define $Z : [-\pi/2, \pi/2]^2 \times R \to H^3$ by $Z(x, y, z) = e^z h(x, y)$.

(3) Extend Z to all of $\mathbb{R}^3$ by repeatedly reflecting in planes.

Quasiregular maps of $\mathbb{R}^n$ which generalize the sine and cosine functions have been constructed by Drasin, by Mayer, and by Bergweiler and Eremenko, see in Fletcher and Nicks paper [47].

(B) There are some new phenomena concerning quasiregular maps which are local homeomorphisms in dimensions $n \geq 3$. A remarkable fact is that all smooth quasiregular maps are local homeomorphisms. Even more remarkable is the following result of Zorich [48].

Theorem B.1. *Every quasiregular local homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$, where $n \geq 3$, is a homeomorphism.*

The result was conjectured by M. A. Lavrentev in 1938. The exponential function exp shows that there is no such result for $n = 2$. Zorich's theorem was generalized by Martio et al., cf. [49] (for a proof see also, e.g., [35, Chapter III, Section 3]).

Theorem B.2. *There is a number $r = r(n, K)$ (which one calls the injectivity radius) such that every K-quasiregular local homeomorphism $g : \mathbb{B} \to \mathbb{R}^n$ of the unit ball $\mathbb{B} \subset R^n$ is actually homeomorphic on $B(0; r)$.*

An immediate corollary of this is Zorich's result.

This explains why in the definition of quasiregular maps it is not reasonable to restrict oneself to smooth maps: all smooth quasiregular maps of $\mathbb{R}^n$ to itself are quasiconformal.

In each dimension $n \geq 3$ there is a positive number ϵ_n such that every nonconstant quasiregular mapping $f : D \to \mathbb{R}^n$ whose distortion function satisfies $K_{\alpha,\beta}(x, f) \leq 1 + \epsilon_n$ for some $1 \leq \alpha, \beta \leq n - 1$ is locally injective.

(C) Despite the differences between the two theories described in parts (A) and (B), many theorems about geometric properties of holomorphic functions of one complex variable have been extended to quasiregular maps. These extensions are usually highly nontrivial.

(e) In a pioneering series of papers, Reshetnyak proved in 1966–1969 that these mappings share the fundamental topological properties of complex-analytic functions: nonconstant quasi-regular mappings are discrete, open, and sense-preserving, cf. [50]. Here we state only the following.

Open Mapping Theorem. If D is open in $\mathbb{R}^n$ and f is a nonconstant qr function from D to $\mathbb{R}^n$, we have that $f(D)$ is open set. (Note that this does not hold for real analytic functions).

An immediate consequence of the open mapping theorem is the maximum modulus principle. It states that if f is qr in a domain D and $|f|$ achieves its maximum on D, then f is constant. This is clear. Namely, if $a \in D$ then the open mapping theorem says that $f(a)$ is an interior point of $f(D)$ and hence there is a point in D with larger modulus.

(f) Reshetnyak also proved important convergence theorems for these mappings and several analytic properties: they preserve sets of zero Lebesgue-measure, are differentiable almost everywhere, and are Hölder continuous. The Reshetnyak theory (which uses the phrase mapping with bounded distortion for "quasiregular mapping") makes use of Sobolev spaces, potential theory, partial differential equations, calculus of variations, and differential geometry. Those mappings solve important first-order systems of PDEs analogous in many respects to the Cauchy-Riemann equation. The solutions of these systems can be viewed as "absolute" minimizers of certain energy functionals.

(g) We have mentioned that all pure topological results about analytic functions (such as the Maximum Modulus Principle and Rouché's theorem) extend to quasiregular maps. Perhaps the most famous result of this sort is the extension of Picard's theorem which is due to Rickman, cf. [35]:

A K-quasiregular map $\mathbb{R}^n \to \mathbb{R}^n$ can omit at most a finite set.

When $n = 2$, this omitted set can contain at most two points (this is a simple extension of Picard's theorem). But when $n > 2$, the omitted set can contain more than two points, and its cardinality can be estimated from above in terms of n and K. There is an integer $q = q(n, K)$ such that every K-qr mapping $f : \mathbb{R}^n \to \overline{\mathbb{R}}^n \setminus \{a_1, a_2, \ldots, a_q\}$, where a_j are disjoint, is constant. It was conjectured for a while that $q(n, K) = 2$. Rickman gave a highly nontrivial example to show that it is not the case: for every positive integer p there exists a nonconstant K-qr mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ omitting p points.

(h) It turns out that these mappings have many properties similar to those of plane quasiconformal mappings. On the other hand, there are also striking differences. Probably the most important of these is that there exists no analogue of the Riemann mapping theorem when $n > 2$. This fact gives rise to the following two problems. Given a domain D in Euclidean n-space, does there exist a quasiconformal homeomorphism f of D onto the n-dimensional unit ball $\mathbb{B}^n$? Next, if such a homeomorphism f exists, how small can the dilatation of f be?

Complete answers to these questions are known when $n = 2$. For a plane domain D can be mapped quasiconformally onto the unit disk B if and only if D is simply connected and has at least two boundary points. The Riemann mapping theorem then shows that if D satisfies these conditions, there exists a conformal homeomorphism f of D onto $\mathbb{D}$. The situation is very much more complicated in higher dimensions, and the Gehring-Väisälä paper [51] is devoted to the study of these two questions in the case where $n = 3$.

(D) We close this subsection with short review of Dyakonov's approach [15].

The main result of Dyakonov's papers [6] (published in Acta Math.) is as follows.

Theorem B. *Lipschitz-type properties are inherited from its modulus by analytic functions.*

A simple proof of this result was also published in Acta Math. by Pavlović, cf. [52].

(D1) In this item we shortly discuss the Lipschitz-type properties for harmonic functions. The set of harmonic functions on a given open set U can be seen as the kernel of the Laplace operator Δ and is therefore a vector space over $\mathbb{R}$: sums, differences, and scalar multiples of harmonic functions are again harmonic. In several ways, the harmonic functions are real analogues to holomorphic functions. All harmonic functions are real analytic; that is, they can be locally expressed as power series, they satisfy the mean value theorem, there is Liouville's type theorem for them, and so forth.

Harmonic quasiregular (briefly, $h$qr) mappings in the plane were studied first by Martio in [53]; for a review of this subject and further results, see [25, 54] and the references cited there. The subject has grown to include study of $h$qr maps in higher dimensions, which can be considered as a natural generalization of analytic function in plane and good candidate for a generalization of Theorem B.

For example, Chen et al. [19] and the author [55] have shown that Lipschitz-type properties are inherited from its modulo for K-quasiregular and harmonic mappings in planar case. These classes include analytic functions in planar case, so this result is a generalization of Theorem A.

(D2) Quasiregular mappings. Dyakonov [15] made further important step and roughly speaking showed that Lipschitz-type properties are inherited from its modulus by qc mappings (see two next results).

Theorem B.3 (Theorem Dy, see [15, Theorem 4]). *Let $0 < \alpha \le 1$, $K \ge 1$, and $n \ge 2$. Suppose G is a Λ^α-extension domain in $\mathbb{R}^n$ and f is a K-quasiconformal mapping of G onto $f(G) \subset \mathbb{R}^n$. Then the following conditions are equivalent:*

(i.5) $f \in \Lambda^\alpha(G)$;

(ii.5) $|f| \in \Lambda^\alpha(G)$.

If, in addition, G is a uniform domain and if $\alpha \le \alpha_0 = \alpha_K^n = K^{1/(1-n)}$, then (i.5) and (ii.5) are equivalent to

(iii.5) $|f| \in \Lambda^\alpha(G, \partial G)$.

An example in Section 2 [15] shows that the assumption $\alpha \le \alpha_0 = \alpha_K^n = K^{1/(1-n)}$ cannot be dropped. For $n \ge 3$, we have the following generalization—but also a consequence of Theorem B.3 dealing with quasiregular mappings that are local homeomorphisms (i.e., have no branching points).

Theorem B.4 (see [15, Proposition 3]). *Let $0 < \alpha \le 1$, $K \ge 1$, and $n \ge 3$. Suppose G is a Λ^α-extension domain in $\mathbb{R}^n$ and f is a K-quasiregular locally injective mapping of G onto $f(G) \subset \mathbb{R}^n$. Then $f \in \Lambda^\alpha(G)$ if and only if $|f| \in \Lambda^\alpha(G)$.*

We remark that a quasiregular mapping f in dimension $n \ge 3$ will be locally injective (or, equivalently, locally homeomorphic) if the dilatation $K(f)$ is sufficiently close to 1. For more sophisticated local injectivity criteria, see the literature cited in [10, 15].

Here we only outline how to reduce the proof of Theorem B.4 to qc case.

It is known that, by Theorem B.2, for given $n \ge 3$ and $K \ge 1$, there is a number $r = r(n, K)$ such that every K-quasiregular local homeomorphism $g : \mathbb{B} \to R^n$ of the unit ball $\mathbb{B} \subset \mathbb{R}^n$ is actually homeomorphic on $B(0; r)$. Applying this to the mappings $g_x(w) = f(x+d(x)w)$, $w \in \mathbb{B}$, where $x \in G$ and $d(x) = d(x, \partial G)$, we see that f is homeomorphic (and hence quasiconformal) on each ball $B_r(x) := B(0; rd(x))$. The constant $r = r(n, K) \in (0, 1)$, coming from the preceding statement, depends on n and $K = K(f)$, but not on x.

Note also the following.

(i0) Define $m_f(x, r) = \min\{|f(x') - f(x)| : |x' - x| = r\}$ and $M_f(x, r) = \max\{|f(x') - f(x)| : |x' - x| = r\}$. We can express the distortion property of qc mappings in the following useful form.

Proposition A. *Suppose that G is a domain in $\mathbb{R}^n$ and $f : G \to \mathbb{R}^n$ is K-quasiconformal and $G' = f(G)$. There is a constant c such that $M_f(x, r) \le c m_f(x, r)$ for $x \in G$ and $r = d(x)/2$. This form shows in an explicit way that the maximal dilatation of a qc mapping is controlled by minimal and it is convenient for some applications. For example, Proposition A also yields a simple proof of Theorem B.3.*

Recall the following. (i1) *Roughly speaking, quasiconformal and quasiregular mappings in $\mathbb{R}^n$, $n \ge 3$, are natural generalizations of conformal and analytic functions of one complex variable.*

(i2) *Dyakonov's proof of Theorem 6.4 in [15] (as we have indicated above) is reduced to quasiconformal case, but it seems likely that he wanted to consider whether the theorem holds more generally for quasiregular mappings.*

(i3) *Quasiregular mappings are much more general than quasiconformal mappings and in particular analytic functions.*

(E) Taking into account the above discussion it is natural to explore the following research problem.

Question A. Is it possible to drop local homeomorphism hypothesis in Theorem B.4?

It seems that using the approach from [15], we cannot solve this problem and that we need new techniques.

(i4) However, we establish the second version of Koebe theorem for K-quasiregular functions, Theorem 39, and the characterization of Lipschitz-type spaces for quasiregular mappings by the average Jacobian, Theorem 40. Using these theorems, we give a positive solution to Question A.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author is grateful to the anonymous referees for a number of helpful suggestions and in particular to the Editor in Chief Professor Cirić for patiently reading through the whole text at several stages of its revision. This research was partially supported by MNTRS, Serbia, Grant no. 174 032.

References

[1] K. Astala and F. W. Gehring, "Injectivity, the BMO norm and the universal Teichmüller space," *Journal d'Analyse Mathématique*, vol. 46, pp. 16–57, 1986.

[2] F. W. Gehring and O. Martio, "Lipschitz classes and quasiconformal mappings," *Annales Academiæ Scientiarum Fennicæ. Series A I. Mathematica*, vol. 10, pp. 203–219, 1985.

[3] C. A. Nolder, "A quasiregular analogue of a theorem of Hardy and Littlewood," *Transactions of the American Mathematical Society*, vol. 331, no. 1, pp. 215–226, 1992.

[4] K. Astala and F. W. Gehring, "Quasiconformal analogues of theorems of Koebe and Hardy-Littlewood," *The Michigan Mathematical Journal*, vol. 32, no. 1, pp. 99–107, 1985.

[5] K. Kim, "Hardy-Littlewood property with the inner length metric," *Korean Mathematical Society*, vol. 19, no. 1, pp. 53–62, 2004.

[6] K. M. Dyakonov, "Equivalent norms on Lipschitz-type spaces of holomorphic functions," *Acta Mathematica*, vol. 178, no. 2, pp. 143–167, 1997.

[7] A. Abaob, M. Arsenović, M. Mateljević, and A. Shkheam, "Moduli of continuity of harmonic quasiregular mappings on bounded domains," *Annales Academiæ Scientiarum Fennicæ*, vol. 38, no. 2, pp. 839–847, 2013.

[8] M. Mateljević and M. Vuorinen, "On harmonic quasiconformal quasi-isometries," *Journal of Inequalities and Applications*, vol. 2010, Article ID 178732, 2010.

[9] J. Qiao and X. Wang, "Lipschitz-type spaces of pluriharmonic mappings," *Filomat*, vol. 27, no. 4, pp. 693–702, 2013.

[10] T. Iwaniec and G. Martin, *Geometric Function Theory and Nonlinear Analysis*, Syraccuse, Aucland, New Zealand, 2000.

[11] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, vol. 1319 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1988.

[12] F. W. Gehring and B. G. Osgood, "Uniform domains and the quasihyperbolic metric," *Journal d'Analyse Mathématique*, vol. 36, pp. 50–74, 1979.

[13] M. Arsenović, V. Manojlović, and M. Mateljević, "Lipschitz-type spaces and harmonic mappings in the space," *Annales Academiæ Scientiarum Fennicæ. Mathematica*, vol. 35, no. 2, pp. 379–387, 2010.

[14] Sh. Chen, M. Mateljević, S. Ponnusamy, and X. Wang, "Lipschitz type spaces and Landau-Bloch type theorems for harmonic functions and solutions to Poisson equations," to appear.

[15] K. M. Dyakonov, "Holomorphic functions and quasiconformal mappings with smooth moduli," *Advances in Mathematics*, vol. 187, no. 1, pp. 146–172, 2004.

[16] A. Hinkkanen, "Modulus of continuity of harmonic functions," *Journal d'Analyse Mathématique*, vol. 51, pp. 1–29, 1988.

[17] V. Lappalainen, "Lip_h-extension domains," *Annales Academiæ Scientiarum Fennicæ. Series A I. Mathematica Dissertationes*, no. 56, p. 52, 1985.

[18] O. Martio and R. Näkki, "Boundary Hölder continuity and quasiconformal mappings," *Journal of the London Mathematical Society. Second Series*, vol. 44, no. 2, pp. 339–350, 1991.

[19] S. Chen, S. Ponnusamy, and X. Wang, "On planar harmonic Lipschitz and planar harmonic Hardy classes," *Annales Academiæ Scientiarum Fennicæ. Mathematica*, vol. 36, no. 2, pp. 567–576, 2011.

[20] Sh. Chen, S. Ponnusamy, and X. Wang, "Equivalent moduli of continuity, Bloch's theorem for pluriharmonic mappings in B^n," *Proceedings of Indian Academy of Sciences—Mathematical Sciences*, vol. 122, no. 4, pp. 583–595, 2012.

[21] Sh. Chen, S. Ponnusamy, M. Vuorinen, and X. Wang, "Lipschitz spaces and bounded mean oscillation of harmonic mappings," *Bulletin of the Australian Mathematical Society*, vol. 88, no. 1, pp. 143–157, 2013.

[22] S. Chen, S. Ponnusamy, and A. Rasila, "Coefficient estimates, Landau's theorem and Lipschitz-type spaces on planar harmonic mappings," *Journal of the Australian Mathematical Society*, vol. 96, no. 2, pp. 198–215, 2014.

[23] F. W. Gehring, "Rings and quasiconformal mappings in space," *Transactions of the American Mathematical Society*, vol. 103, pp. 353–393, 1962.

[24] J. Väisälä, *Lectures on n-Dimensional Quasiconformal Mappings*, Springer, New York, NY, USA, 1971.

[25] M. Mateljević, "Quasiconformality of harmonic mappings between Jordan domains," *Filomat*, vol. 26, no. 3, pp. 479–510, 2012.

[26] D. Kalaj and M. S. Mateljević, "Harmonic quasiconformal self-mappings and Möbius transformations of the unit ball," *Pacific Journal of Mathematics*, vol. 247, no. 2, pp. 389–406, 2010.

[27] D. Kalaj, "On harmonic diffeomorphisms of the unit disc onto a convex domain," *Complex Variables*, vol. 48, no. 2, pp. 175–187, 2003.

[28] M. Mateljević, *Topics in Conformal, Quasiconformal and Harmonic Maps*, Zavod za Udzbenike, Belgrade, Serbia, 2012.

[29] M. Mateljević, "Distortion of harmonic functions and harmonic quasiconformal quasi-isometry," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 51, no. 5-6, pp. 711–722, 2006.

[30] M. Knežević and M. Mateljević, "On the quasi-isometries of harmonic quasiconformal mappings," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 1, pp. 404–413, 2007.

[31] V. Manojlović, "Bi-Lipschicity of quasiconformal harmonic mappings in the plane," *Filomat*, vol. 23, no. 1, pp. 85–89, 2009.

[32] M. Mateljević, *Communications at Analysis Seminar*, University of Belgrade, 2012.

[33] M. Mateljević, "Quasiconformal and Quasiregular harmonic mappings and Applications," *Annales Academiæ Scientiarum Fennicæ Mathematica*, vol. 32, pp. 301–315, 2007.

[34] I. Anić, V. Marković, and M. Mateljević, "Uniformly bounded maximal φ-disks, Bers space and harmonic maps," *Proceedings of the American Mathematical Society*, vol. 128, no. 10, pp. 2947–2956, 2000.

[35] S. Rickman, *Quasiregular Mappings*, vol. 26, Springer, Berlin, Germany, 1993.

[36] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer, New York, NY, USA, 2nd edition, 1973.

[37] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, vol. 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer, Berlin, Germany, 2nd edition, 1983.

[38] B. Burgeth, “A Schwarz lemma for harmonic and hyperbolic-harmonic functions in higher dimensions,” *Manuscripta Mathematica*, vol. 77, no. 2-3, pp. 283–291, 1992.

[39] S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory*, vol. 137 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1992.

[40] W. Rudin, *Function Theory in the Unit Ball of C^n*, Springer, Berlin, Germany, 1980.

[41] R. Näkki and B. Palka, “Lipschitz conditions and quasiconformal mappings,” *Indiana University Mathematics Journal*, vol. 31, no. 3, pp. 377–401, 1982.

[42] M. Mateljević, “Versions of Koebe 1/4 theorem for analytic and quasiregular harmonic functions and applications,” *Institut Mathématique. Publications. Nouvelle Série*, vol. 84, pp. 61–72, 2008.

[43] P. Duren, *Harmonic Mappings in the Plane*, vol. 156 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 2004.

[44] D. Kalaj, *Harmonic functions and quasiconformal mappings [M.S. thesis]*, Harmonijske Funkcije i Kvazikonformna Preslikavanja, Belgrade, Serbia, 1998.

[45] E. Heinz, “On one-to-one harmonic mappings,” *Pacific Journal of Mathematics*, vol. 9, no. 1, pp. 101–105, 1959.

[46] G. Martin and K. Peltonen, “Stoïlow factorization for quasiregular mappings in all dimensions,” *Proceedings of the American Mathematical Society*, vol. 138, no. 1, pp. 147–151, 2010.

[47] A. N. Fletcher and D. A. Nicks, “Iteration of quasiregular tangent functions in three dimensions,” *Conformal Geometry and Dynamics*, vol. 16, pp. 1–21, 2012.

[48] V. A. Zorich, “The global homeomorphism theorem for space quasiconformal mappings, its development and related open problems,” in *Quasiconformal Space Mappings*, vol. 1508 of *Lecture Notes in Mathematics*, pp. 132–148, Springer, Berlin, Germany, 1992.

[49] O. Martio, S. Rickman, and J. Väisälä, “Topological and metric properties of quasiregular mappings,” *Annales Academiæ Scientiarum Fennicæ Mathematica*, vol. 488, pp. 1–31, 1971.

[50] Yu. G. Reshetnyak, “Space mappings with bounded distortion,” *Siberian Mathematical Journal*, vol. 8, no. 3, pp. 466–487, 1967.

[51] F. W. Gehring and J. Väisälä, “The coefficients of quasiconformality of domains in space,” *Acta Mathematica*, vol. 114, pp. 1–70, 1965.

[52] M. Pavlović, “On Dyakonov’s paper ‘Equivalent norms on lipschitz-type spaces of holomorphic functions’,” *Acta Mathematica*, vol. 183, no. 1, pp. 141–143, 1999.

[53] O. Martio, “On harmonic quasiconformal mappings,” *Annales Academiæ Scientiarum Fennicæ Mathematica*, vol. 425, pp. 3–10, 1968.

[54] M. Arsenović, V. Kojić, and M. Mateljević, “On Lipschitz continuity of harmonic quasiregular maps on the unit ball in R^n,” *Annales Academiæ Scientiarum Fennicæ. Mathematica*, vol. 33, no. 1, pp. 315–318, 2008.

[55] M. Mateljević, “A version of Bloch’s theorem for quasiregular harmonic mappings,” *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 47, no. 5-6, pp. 705–707, 2002.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 196751, 9 pages
http://dx.doi.org/10.1155/2014/196751

Research Article

Complete Self-Shrinking Solutions for Lagrangian Mean Curvature Flow in Pseudo-Euclidean Space

Ruiwei Xu and Linfen Cao

Department of Mathematics, Henan Normal University, Xinxiang, Henan 453007, China

Correspondence should be addressed to Linfen Cao; caolf2010@yahoo.com

Received 14 January 2014; Accepted 15 June 2014; Published 6 July 2014

Academic Editor: Ljubomir B. Ćirić

Let $f(x)$ be a smooth strictly convex solution of $\det(\partial^2 f/\partial x_i \partial x_j) = \exp\{(1/2)\sum_{i=1}^n x_i(\partial f/\partial x_i) - f\}$ defined on a domain $\Omega \subset \mathbb{R}^n$; then the graph $M_{\nabla f}$ of ∇f is a space-like self-shrinker of mean curvature flow in Pseudo-Euclidean space $\mathbb{R}_n^{2n}$ with the indefinite metric $\sum dx_i dy_i$. In this paper, we prove a Bernstein theorem for complete self-shrinkers. As a corollary, we obtain if the Lagrangian graph $M_{\nabla f}$ is complete in R_n^{2n} and passes through the origin then it is flat.

1. Introduction

Let M be an n-dimensional submanifold immersed into the Euclidean space $\mathbb{R}^{n+m}$. Mean curvature flow is a one-parameter family $X_t = X(\cdot, t)$ of immersions $X_t : M \to \mathbb{R}^{n+m}$ with corresponding images $M_t = X_t(M)$ such that

$$\frac{d}{dt}X(x,t) = H(x,t), \quad x \in M,$$
$$X(x,0) = X(x) \tag{1}$$

is satisfied, where $H(x,t)$ is the mean curvature vector of M_t at $X(x,t)$ in $\mathbb{R}^{n+m}$. Self-similar solutions to the mean curvature flow play an important role in understanding the behavior of the flow and the types of singularities. They satisfy a system of quasilinear elliptic PDE of the second order as follows:

$$H = -\frac{X^{\perp}}{2}, \tag{2}$$

where $(\cdots)^{\perp}$ stands for the orthogonal projection into the normal bundle NM.

Self-shrinkers in the ambient Euclidean space have been studied by many authors; for example, see [1–6] and so forth. For recent progress and related results, see the introduction in [7]. When the ambient space is a pseudo-Euclidean space, there are many classification works about self-shrinkers; for example, see [8–13] and so forth. But very little is known when self-shrinkers are complete not compact with respect to induced metric from pseudo-Euclidean space. In this paper, we will characterize self-shrinkers for Lagrangian mean curvature flow in the pseudo-Euclidean space from this aspect.

Let $(x_1, \ldots, x_n; y_1, \ldots, y_n)$ be null coordinates in $2n$-dimensional pseudo-Euclidean space $\mathbb{R}_n^{2n}$. Then, the indefinite metric (cf. [14]) is defined by $ds^2 = \sum_{i=1}^n dx_i dy_i$. Suppose $f(x)$ is a smooth strictly convex function defined on domain $\Omega \subset \mathbb{R}^n$. The graph $M_{\nabla f}$ of ∇f can be written as $(x_1, \ldots, x_n; \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$. Then, the induced Riemannian metric on $M_{\nabla f}$ is given by

$$G = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j. \tag{3}$$

In particular, if function f satisfies

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \exp\left\{\frac{1}{2}\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} - f\right\}, \tag{4}$$

then the graph $M_{\nabla f}$ of ∇f is a space-like self-shrinking solution for mean curvature flow in $\mathbb{R}_n^{2n}$.

Huang and Wang [12] and Chau et al. [8] have used different methods to investigate the entire solutions to the above equation and showed that an entire smooth strictly convex solution to (4) in $\mathbb{R}^n$ is the quadratic polynomial under the decay condition on Hessian of f. Later Ding and Xin in [10] improve the previous ones in [8, 12] by removing the additional assumption and prove the following.

Theorem 1. *Any space-like entire graphic self-shrinking solution to Lagrangian mean curvature flow in R^{2n}_n with the indefinite metric $\sum_i dx_i dy_i$ is flat.*

These rigidity results assume that the self-shrinker graphs are entire. Namely, they are Euclidean complete. Here, we will characterize the rigidity of self-shrinker graphs from another completeness and pose the following problem.

If a graphic self-shrinker is complete with respect to induced metric from ambient space R^{2n}_n, then is it flat?

In this paper, we will use *affine technique* (see [15–18]) to prove the following Bernstein theorem. As a corollary, it gives a partial affirmative answer to the above problem.

Theorem 2. *Let $f(x)$ be a C^∞ strictly convex function defined on a convex domain $\Omega \subseteq \mathbb{R}^n$ satisfying the PDE (4). If there is a positive constant α depending only on n such that the hypersurface $M = \{(x, f(x))\}$ in $\mathbb{R}^{n+1}$ is complete with respect to the metric*

$$\widetilde{G} = \exp\left\{\alpha \sum x_i \frac{\partial f}{\partial x_i}\right\} \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j, \tag{5}$$

then f is the quadratic polynomial.

Remark 3. If $f(x)$ is a strictly convex solution to (4), then the graph $\{(x, \nabla f/2n\alpha)\}$ is a minimal manifold in R^{2n}_n endowed with the conformal metric $ds^2 = \exp\{-\alpha x \cdot y\} dx \cdot dy$.

As a direct application of Theorem 2, we have the following.

Corollary 4. *Let f be a strictly convex C^∞-function defined on a convex domain $\Omega \subset \mathbb{R}^n$. If the graph $M_{\nabla f} = \{(x, \nabla f(x))\}$ in $\mathbb{R}^{2n}_n$ is a complete space-like self-shrinker for mean curvature flow and the sum $\sum x_i(\partial f/\partial x_i)$ has a lower bound, then $M_{\nabla f}$ is flat.*

When the shrinker passes through the origin especially, we have the following corollary.

Corollary 5. *If the graph $M_{\nabla f} = \{(x, \nabla f(x))\}$ in $\mathbb{R}^{2n}_n$ is a complete space-like self-shrinker for mean curvature flow and passes through the origin, then $M_{\nabla f}$ is flat.*

2. Preliminaries

Let $f(x_1, \ldots, x_n)$ be a strictly convex C^∞-function defined on a domain $\Omega \subset \mathbb{R}^n$. Consider the graph hypersurface

$$M := \{(x, f(x)) \mid x_{n+1} = f(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in \Omega\}. \tag{6}$$

For M, we choose the canonical relative normalization $Y = (0, 0, \ldots, 1)$. Then, in terms of the language of the relative affine differential geometry, the *Calabi metric*

$$G = \sum f_{ij} dx_i dx_j \tag{7}$$

is the relative metric with respect to the normalization Y. For the position vector $y = (x_1, \ldots, x_n, f(x_1, \ldots, x_n))$, we have

$$y_{,ij} = \sum A^k_{ij} y_k + f_{ij} Y, \tag{8}$$

where "," denotes the covariant derivative with respect to the Calabi metric G. We recall some fundamental formulas for the graph M; for details, see [19]. The Levi-Civita connection with respect to the metric G has the *Christoffel symbols*

$$\Gamma^k_{ij} = \frac{1}{2} \sum f^{kl} f_{ijl}. \tag{9}$$

The *Fubini-Pick tensor* A_{ijk} satisfies

$$A_{ijk} = -\frac{1}{2} f_{ijk}. \tag{10}$$

Consequently, for the *relative Pick invariant*, we have

$$J = \frac{1}{4n(n-1)} \sum f^{il} f^{jm} f^{kn} f_{ijk} f_{lmn}. \tag{11}$$

The *Gauss integrability conditions* and the *Codazzi equations* read

$$R_{ijkl} = \sum f^{mh} \left(A_{jkm} A_{hil} - A_{ikm} A_{hjl}\right), \tag{12}$$

$$A_{ijk,l} = A_{ijl,k}. \tag{13}$$

From (12), we get the Ricci tensor

$$R_{ik} = \sum f^{mh} f^{lj} \left(A_{iml} A_{hjk} - A_{imk} A_{hlj}\right). \tag{14}$$

Introduce the Legendre transformation of f

$$\begin{gathered} \xi_i = \frac{\partial f}{\partial x_i}, \quad i = 1, 2, \ldots, n, \\ u(\xi_1, \ldots, \xi_n) = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} - f(x). \end{gathered} \tag{15}$$

Define the functions

$$\begin{gathered} \rho := \left[\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)} = \left[\det\left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}\right)\right]^{1/(n+2)}, \\ \Phi := \sum f^{ij} (\ln \rho)_i (\ln \rho)_j = \frac{\|\nabla \rho\|^2}{\rho^2}, \end{gathered} \tag{16}$$

here and later the norm $\|\cdot\|$ is defined with respect to the Calabi metric. From the PDE (4), we obtain

$$\frac{\partial \ln \rho}{\partial x_i} = \frac{1}{2(n+2)} (f - u)_i = \frac{1}{2(n+2)} \{f_i - x_k f_{ki}\}. \tag{17}$$

That is,

$$x_i = \sum f^{ik}\left(f_k - 2(n+2)(\ln\rho)_k\right). \tag{18}$$

Using (17) and (18), we can get

$$\rho_{ij} = \frac{\rho_i\rho_j}{\rho} + f^{kl}f_{ijk}\rho_l - \frac{f^{kl}f_{ijk}f_l\rho}{2(n+2)}. \tag{19}$$

Put $\tau := (1/2)\sum f^{ij}(\rho_i/\rho)f_j$. From (19), we have

$$\Delta\rho = -\frac{n}{2}\frac{\|\nabla\rho\|^2}{\rho} + \tau\rho. \tag{20}$$

By (17), we get

$$4(n+2)^2\Phi = \|\nabla u\|^2 + \|\nabla f\|^2 - 2(f+u), \tag{21}$$

and then

$$\|\nabla(f+u)\|^2 = 4(n+2)^2\Phi + 4(f+u). \tag{22}$$

Using (17) yields

$$\Delta(f+u) = 2n + (n+2)^2\Phi. \tag{23}$$

Define a conformal Riemannian metric $\widetilde{G} := \exp\{\alpha(f+u)\}G$, where α is a constant.

Conformal Ricci Curvature. Denote by $\widetilde{R}_{ij}$ the Ricci curvature with respect to the metric $\widetilde{G}$; then

$$\begin{aligned}\widetilde{R}_{ij} = R_{ij} &- \frac{(n-2)\alpha}{2}(f+u)_{,ij} + \frac{(n-2)\alpha^2}{4}(f+u)_{,i}(f+u)_{,j} \\ &- \frac{1}{2}\left(\alpha\Delta(f+u) + \frac{(n-2)\alpha^2}{2}\|\nabla(f+u)\|^2\right)G_{ij},\end{aligned} \tag{24}$$

where "," again denotes the covariant derivation with respect to the Calabi metric.

Using the above formulas, we can get the following crucial estimates.

Proposition 6. *Let $f(x_1,\ldots,x_n)$ be a C^∞ strictly convex function satisfying PDE* (4)*. Then, the following estimate holds:*

$$\begin{aligned}\Delta\Phi \geq{} & A_1\langle\nabla\Phi,\nabla\ln\rho\rangle + \frac{1}{4}\langle\nabla(f+u),\nabla\Phi\rangle - A_2\frac{\|\nabla\Phi\|^2}{\Phi} \\ & + A_3\Phi^2 + \Phi,\end{aligned} \tag{25}$$

where

$$\begin{aligned}A_1 &= \frac{6n^2-n+16}{(n-1)(3n+4)}, \qquad A_2 = \frac{3n^2+32n}{8(n-1)(3n+4)}, \\ A_3 &= \frac{64n^3-72n^2-46n-72}{5n(n-1)(3n+4)}.\end{aligned} \tag{26}$$

Because its calculation is standard as in [16], we will give its proof in the appendix.

For affine hyperspheres, Calabi in [20] calculated the Laplacian of the Pick invariant J. Later, for a general convex function, Li and Xu proved the following lemma in [17].

Lemma 7. *The Laplacian of the relative Pick invariant J satisfies*

$$\begin{aligned}\Delta J \geq{} & \frac{n+2}{n(n-1)}\sum f^{il}f^{jm}f^{kn}A_{ijk}(\ln\rho)_{,lmn} \\ & + \frac{2}{n(n-1)}\|\nabla A\|^2 + 2J^2 - \frac{(n+2)^4}{4}\Phi^2,\end{aligned} \tag{27}$$

where "," denotes the covariant derivative with respect to the Calabi metric.

Using Lemma 7, we get the following corollary. For the proof, see the appendix.

Corollary 8. *Let $f(x_1,\ldots,x_n)$ be a C^∞ strictly convex function satisfying PDE* (4)*; then*

$$\begin{aligned}\Delta J \geq{} & J^2 - 20(n+2)^8\Phi^2 + \frac{1}{4}\langle\nabla J,\nabla(f+u)\rangle \\ & + J - \sqrt{n(n-1)}\,\|\nabla(f+u)\|\,J^{3/2}.\end{aligned} \tag{28}$$

3. Proof of Theorem 2

It is our aim to prove $\Phi \equiv 0$; thus, from definition of ρ,

$$\det\left(f_{ij}\right) = \text{const.} \tag{29}$$

everywhere on M. As in [8], by Euler homogeneous theorem, we get Theorem 2.

Denote by $s(p_0,p)$ the geodesic distance function from $p_0 \in M$ with respect to the metric $\widetilde{G}$. For any positive number a, let $B_a(p_0,\widetilde{G}) := \{p \in M \mid s(p_0,p) \leq a\}$. Denote

$$\begin{aligned}\mathscr{A} &:= \max_{B_a(p_0,\widetilde{G})}\left\{\left(a^2-s^2\right)^2\exp\{-\alpha(f+u)\}\Phi\right\}, \\ \mathscr{B} &:= \max_{B_a(p_0,\widetilde{G})}\left\{\left(a^2-s^2\right)^2\exp\{-\alpha(f+u)\}J\right\}.\end{aligned} \tag{30}$$

Lemma 9. *Let f be a strictly convex C^∞-function satisfying the PDE* (4)*. Then, there exist positive constants α and C, depending only on n, such that*

$$\mathscr{A} \leq C\left(a^2 + a^3\right). \tag{31}$$

Proof. Step 1. We will prove that there exists a constant C depending only on n such that

$$\mathscr{A} \leq C\left(\mathscr{B}^{1/2}a + a^2 + a^3\right). \tag{32}$$

To this end, consider the function

$$F := \left(a^2-s^2\right)^2\exp\{-\alpha(f+u)\}\Phi \tag{33}$$

defined on $B_a(p_0, \widetilde{G})$, where α is a positive constant to be determined later. Obviously, F attains its supremum at some interior point p^*. We may assume that s^2 is a C^2-function in a neighborhood of p^*. Choose an orthonormal frame field on M around p^* with respect to the Calabi metric G. Then, at p^*,

$$\frac{\Phi_{,i}}{\Phi} - \alpha(f+u)_{,i} - \frac{4ss_{,i}}{a^2-s^2} = 0, \tag{34}$$

$$\frac{\Delta\Phi}{\Phi} - \frac{\sum(\Phi_{,i})^2}{\Phi^2} - \alpha\Delta(f+u) - \frac{12a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2} - \frac{4s\Delta s}{a^2-s^2} \le 0, \tag{35}$$

where "," denotes the covariant derivative with respect to the Calabi metric G as before, and we used the fact $\|\nabla s\|_G^2 = \exp\{\alpha(f+u)\}$. Inserting Proposition 6 into (35), we get

$$\begin{aligned} &-(1+A_2)\frac{\sum(\Phi_{,i})^2}{\Phi^2} + A_3\Phi + \frac{1}{4}(f+u)_{,i}\frac{\Phi_{,i}}{\Phi} \\ &+ A_1\frac{\Phi_{,i}}{\Phi}\frac{\rho_{,i}}{\rho} + 1 - \alpha\left(2n+(n+2)^2\Phi\right) \\ &- \frac{12a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2} - \frac{4s\Delta s}{a^2-s^2} \le 0. \end{aligned} \tag{36}$$

Combining (34) with (36) and using the Schwarz inequality, we have

$$\begin{aligned} \frac{1}{4}\sum(f+u)_{,i}\frac{\Phi_{,i}}{\Phi} &\ge \frac{1}{8}\alpha\sum\left[(f+u)_{,i}\right]^2 - \frac{2}{\alpha}\frac{a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2}, \\ A_1\sum\frac{\Phi_{,i}}{\Phi}\frac{\rho_{,i}}{\rho} &\ge -\frac{A_3}{4}\Phi - \frac{A_1^2}{A_3}\frac{\Phi_{,i}^2}{\Phi^2}, \\ \frac{\sum(\Phi_{,i})^2}{\Phi^2} &\le 2\left(\alpha^2\sum\left[(f+u)_{,i}\right]^2 + 16\frac{a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2}\right). \end{aligned} \tag{37}$$

Choose α small enough such that

$$2\left(1+A_2+\frac{A_1^2}{A_3}\right)\alpha \le \frac{1}{16}, \qquad \alpha(n+2)^2 \le \frac{A_3}{4}, \qquad 100n\alpha \le 1. \tag{38}$$

Then, by substituting the three estimates above, we get

$$\frac{A_3}{2}\Phi + \frac{1}{16}\alpha(f+u)_{,i}^2 - \exp\{\alpha(f+u)\}\frac{Ca^2}{(a^2-s^2)^2} - \frac{4s\Delta s}{a^2-s^2} \le 0, \tag{39}$$

here and later C denotes positive constant depending only on n.

Denote $a^* = s(p_0, p^*)$. If $a^* = 0$, from (39), it is easy to complete the proof of the lemma. In the following, we assume that $a^* > 0$. Now, we calculate the term $4s\Delta s/(a^2-s^2)$. Firstly, we will give a lower bound of the Ricci curvature $\mathrm{Ric}(M, \widetilde{G})$. Assume that

$$\begin{aligned} \max_{B_{a^*}(p_0,\widetilde{G})}\{\exp\{-\alpha(f+u)\}\Phi\} &= \exp\{-\alpha(f+u)\}\Phi(\widetilde{p}), \\ \max_{B_{a^*}(p_0,\widetilde{G})}\{\exp\{-\alpha(f+u)\}J\} &= \exp\{-\alpha(f+u)\}J(\widetilde{q}). \end{aligned} \tag{40}$$

For any $p \in B_{a^*}(p_0, \widetilde{G})$, by a coordinate transformation, $f_{ij}(p) = \delta_{ij}$ and $R_{ij}(p) = 0$ hold for $i \ne j$. Then, at p,

$$\begin{aligned} R_{ii} &\ge \frac{1}{4}\left(\sum_m f_{mii}^2 + (n+2)\sum_m f_{mii}\frac{\partial}{\partial x_m}\ln\rho\right) \ge -\frac{(n+2)^2}{16}\Phi, \\ &\frac{(n-2)\alpha}{2}(f+u)_{,ii} \\ &= \frac{(n-2)\alpha}{2}\left(2 - \frac{1}{2}f_{iik}(f+u)_k\right) \\ &\le (n-2)\alpha + \frac{(n-2)\alpha^2}{4}\|\nabla(f+u)\|^2 + CJ. \end{aligned} \tag{41}$$

Then, using the Schwarz inequality and (22)–(24), we know that at the point p

$$\begin{aligned} \mathrm{Ric}(M, \widetilde{G}) &\ge -\exp\{-\alpha(f+u)\} \\ &\quad\times\{C\Phi + CJ + \alpha[3(n-2)\alpha(f+u) + 2(n-1)]\}\widetilde{G}. \end{aligned} \tag{42}$$

If $3(n-2)\alpha(f+u) + 2(n-1) \le 0$, then

$$-\exp\{-\alpha(f+u)\}\alpha[3(n-2)\alpha(f+u) + 2(n-1)] \ge 0. \tag{43}$$

Otherwise,

$$\exp\{-\alpha(f+u)\}\alpha[3(n-2)\alpha(f+u) + 2(n-1)] \le C. \tag{44}$$

Then, the Ricci curvature $\mathrm{Ric}(M, \widetilde{G})$ on $B_{a^*}(p_0, \widetilde{G})$ is bounded from below by

$$\mathrm{Ric}(M, \widetilde{G}) \ge -C\left(\frac{\Phi}{\exp\{\alpha(f+u)\}}(\widetilde{p}) + \frac{J}{\exp\{\alpha(f+u)\}}(\widetilde{q}) + 1\right)\widetilde{G}. \tag{45}$$

By the Laplacian comparison theorem, we get

$$\begin{aligned}
&\frac{s\Delta s}{a^2-s^2}\\
&\quad = \exp\{\alpha(f+u)\}\frac{s\widetilde{\Delta} s}{a^2-s^2} - \frac{(n-2)\alpha}{2}\frac{s(f+u)_{,i}s_{,i}}{a^2-s^2}\\
&\quad \le C_3\frac{\exp\{\alpha(f+u)\}(p^*)}{a^2-s^2}\\
&\qquad \times\left(\sqrt{\exp\{-\alpha(f+u)\}\Phi(\widetilde{p})}\right.\\
&\qquad\qquad \left.+\sqrt{\exp\{-\alpha(f+u)\}J(\widetilde{q})}+1\right)s\\
&\qquad + \frac{\alpha}{16}(f+u)_{,i}^2 + C\frac{a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2},
\end{aligned} \tag{46}$$

where $\widetilde{\Delta}$ denotes the Laplacian with respect to the metric $\widetilde{G}$.

Substituting (46) into (39) yields

$$\begin{aligned}
&\exp\{-\alpha(f+u)\}\Phi\\
&\quad \le aC\left(\sqrt{\frac{\Phi}{\exp\{\alpha(f+u)\}}(\widetilde{p})}\right.\\
&\qquad\qquad \left.+\sqrt{\frac{J}{\exp\{\alpha(f+u)\}}(\widetilde{q})}+1\right)\times\left(a^2-s^2\right)^{-1}\\
&\qquad + \frac{Ca^2}{(a^2-s^2)^2}.
\end{aligned} \tag{47}$$

Note that

$$\begin{aligned}
\mathscr{A} &\ge \left[\left(a^2-s^2\right)^2\exp\{-\alpha(f+u)\}\Phi\right](\widetilde{p})\\
&\ge \left(a^2-s^2\right)^2(p^*)\exp\{-\alpha(f+u)\}(\widetilde{p})\Phi(\widetilde{p}),\\
\mathscr{B} &\ge \left[\left(a^2-s^2\right)^2\exp\{-\alpha(f+u)\}J\right](\widetilde{q})\\
&\ge \left(a^2-s^2\right)^2(p^*)\exp\{-\alpha(f+u)\}(\widetilde{q})J(\widetilde{q}).
\end{aligned} \tag{48}$$

Multiplying by $(a^2-s^2)^2(p^*)$, at both sides of (47), yields

$$\mathscr{A} \le Ca\left(\mathscr{A}^{1/2}+\mathscr{B}^{1/2}\right)+C\left(a^2+a^3\right). \tag{49}$$

Using the Schwarz inequality, we complete Step 1.

Step 2. We will prove that there is a constant C depending only on n such that

$$\mathscr{B} \le C\left(\mathscr{A}+a^2+a^4\right). \tag{50}$$

Consider

$$H = \left(a^2-s^2\right)^2\exp\{-\alpha(f+u)\}J \tag{51}$$

defined on $B_a(p_0,\widetilde{G})$, where α is the constant in (38). Obviously, H attains its supremum at some interior point q^*. Choose an orthonormal frame field on M around q^* with respect to the Calabi metric G. Then, at q^*,

$$\frac{J_{,i}}{J} - \alpha(f+u)_{,i} - \frac{4ss_{,i}}{a^2-s^2} = 0, \tag{52}$$

$$\begin{aligned}
&\frac{\Delta J}{J} - \frac{\sum(J_{,i})^2}{J^2} - \alpha\Delta(f+u) - \frac{12a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2}\\
&\quad - \frac{4s\Delta s}{a^2-s^2} \le 0,
\end{aligned} \tag{53}$$

where "," denotes the covariant derivative with respect to the Calabi metric G as before. Inserting Corollary 8 into (53), we get

$$\begin{aligned}
&J - 20(n+2)^8\frac{\Phi^2}{J} + \frac{1}{4}\sum\frac{J_{,i}}{J}(f+u)_{,i} + 1 - \frac{\sum(J_{,i})^2}{J^2}\\
&\quad - \alpha\left(2n+(n+2)^2\Phi\right) - \sqrt{n(n-1)}\left\|\nabla(f+u)\right\|J^{1/2}\\
&\quad - \frac{12a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2} - \frac{4s\Delta s}{a^2-s^2} \le 0.
\end{aligned} \tag{54}$$

Applying the Schwarz inequality, we have

$$\begin{aligned}
&\frac{1}{4}\sum\frac{J_{,i}}{J}(f+u)_{,i} \ge \frac{\alpha}{8}\sum\left[(f+u)_{,i}\right]^2 - C\frac{a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2},\\
&\sum\frac{(J_{,i})^2}{J^2} \le 2\alpha^2\sum\left[(f+u)_{,i}\right]^2 + \frac{32a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2},\\
&\sqrt{n(n-1)}\left\|\nabla(f+u)\right\|J^{1/2}\\
&\quad \le \frac{J}{4} + 4n(n-1)\left((n+2)^2\Phi+(f+u)\right).
\end{aligned} \tag{55}$$

Inserting these estimates into (54) yields

$$\begin{aligned}
&\frac{3}{4}J - 20(n+2)^8\frac{\Phi^2}{J} - C\Phi + \frac{\alpha}{16}\sum(f+u)_{,i}^2\\
&\quad - C\frac{a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2} - C(f+u) - \frac{4s\Delta s}{a^2-s^2} \le 0,
\end{aligned} \tag{56}$$

here and later C denotes different positive constants depending only on n.

We discuss two subcases.

Case 1. If

$$\frac{J}{\exp\{\alpha(f+u)\}}(q^*) \le \frac{\Phi}{\exp\{\alpha(f+u)\}}(q^*), \tag{57}$$

then $\mathscr{B} \le \mathscr{A}$. In this case, Step 2 is complete.

Case 2. Now, assume that

$$\frac{J}{\exp\{\alpha(f+u)\}}(q^*) > \frac{\Phi}{\exp\{\alpha(f+u)\}}(q^*). \tag{58}$$

Then, $1 > (\Phi/J)(q^*)$. Thus,

$$\frac{3}{4}J - C\Phi + \frac{\alpha}{16}\sum\left[(f+u)_{,i}\right]^2 - C\frac{a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2} - C(f+u) - \frac{4s\Delta s}{a^2-s^2} \le 0. \tag{59}$$

The rest of the estimate is almost the same as in Step 1. The only difference is to deal with the term $(f+u)$. If $(f+u)(q^*) \le 0$, then $-C(f+u)(q^*) \ge 0$. We can drop this term.

Otherwise, $\exp\{-\alpha(f+u)\}(f+u)$ has a uniform upper bound.

Using the same method as in Step 1, we can estimate the term $4s\Delta s/(a^2-s^2)$ and finally get

$$\mathscr{B} \le C\left(\mathscr{A} + a^2 + a^4\right). \tag{60}$$

Then, combining the conclusion of Step 1, we get

$$\mathscr{A} \le C\left(a^2 + a^3\right). \tag{61}$$

This completes the proof of Lemma 9. □

Proof of Theorem 2. For any point $q \in M$, choose sufficient large constant R_0 such that $q \in B_{R_0}(p_0, \widetilde{G})$. Then, for all $a \ge R_0$, $q \in B_a(p_0)$. Using Lemma 9, we know

$$\exp\{-\alpha(f+u)\}\Phi(q) \le \frac{C(n)\left(a^2+a^3\right)}{(a^2-s^2)^2}. \tag{62}$$

Now, let $a \to +\infty$, and we have

$$0 \le \exp\{-\alpha(f+u)\}\Phi(q) \le 0. \tag{63}$$

Consequently,

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(q) = \text{const}. \tag{64}$$

This completes the proof of Theorem 2. □

4. Appendix

Proof of Proposition 6. Let $p \in M$, and we choose a local orthonormal frame field of the metric G around p. Then,

$$\Phi = \frac{\sum(\rho_{,j})^2}{\rho^2}, \qquad \Phi_{,i} = 2\sum\frac{\rho_{,j}\rho_{,ji}}{\rho^2} - 2\rho_{,i}\frac{\sum(\rho_{,j})^2}{\rho^3},$$
$$\Delta\Phi = 2\frac{\sum(\rho_{,ji})^2}{\rho^2} + 2\sum\frac{\rho_{,j}\rho_{,jii}}{\rho^2} - 8\sum\frac{\rho_{,j}\rho_{,i}\rho_{,ji}}{\rho^3} + (n+6)\Phi^2 - 2\tau\Phi, \tag{65}$$

where we used (20). In the case $\Phi(p) = 0$, it is easy to get, at p,

$$\Delta\Phi \ge 2\frac{\sum(\rho_{,ij})^2}{\rho^2}. \tag{66}$$

Now, we assume that $\Phi(p) \ne 0$. Choose a local orthonormal frame field of the metric G around p such that $\rho_{,1}(p) = \|\nabla\rho\|(p) > 0$, $\rho_{,i}(p) = 0$, for all $i > 1$. Then,

$$\Delta\Phi = 2(1-\delta+\delta)\sum\frac{(\rho_{,ij})^2}{\rho^2} + 2\sum\frac{\rho_{,j}\rho_{,jii}}{\rho^2} - 8\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} + (n+6)\Phi^2 - 2\tau\Phi, \tag{67}$$

where $1 > \delta > 0$ is a constant to be determined later. Applying (20), we obtain

$$2\frac{\sum(\rho_{,ij})^2}{\rho^2} \ge \frac{2n}{n-1}\frac{(\rho_{,11})^2}{\rho^2} + 4\frac{\sum_{i>1}(\rho_{,1i})^2}{\rho^2} + \frac{2n}{n-1}\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} + \frac{n^2}{2(n-1)}\Phi^2 - \frac{4}{n-1}\frac{\rho_{,11}}{\rho}\tau + \frac{2}{n-1}\tau^2 - \frac{2n}{n-1}\Phi\tau. \tag{68}$$

An application of the Ricci identity shows that

$$\frac{2}{\rho^2}\sum\rho_{,j}\rho_{,jii} = -2n\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} + n\frac{(\rho_{,1})^4}{\rho^4} + 2R_{11}\frac{(\rho_{,1})^2}{\rho^2} + 2\frac{\rho_{,1}}{\rho^2}(\rho\tau)_{,1}. \tag{69}$$

Substituting (68) and (69) into (67), we obtain

$$\begin{aligned}\Delta\Phi \ge{}& 2\delta\sum\frac{(\rho_{,ij})^2}{\rho^2} + \left(-2n - 8 + \frac{2n(1-\delta)}{n-1}\right) \times \frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} + 2R_{11}\frac{(\rho_{,1})^2}{\rho^2} \\ &+ \left(\frac{n^2(1-\delta)}{2(n-1)} + 2(n+3)\right)\frac{(\rho_{,1})^4}{\rho^4} + 2\frac{\rho_{,1}}{\rho^2}(\rho\tau)_{,1} - \frac{4n-2-2n\delta}{n-1}\Phi\tau + (1-\delta) \\ &\times \left(\frac{2n}{n-1}\frac{(\rho_{,11})^2}{\rho^2} + 4\frac{\sum_{i>1}(\rho_{,1i})^2}{\rho^2} - \frac{4}{n-1}\frac{\rho_{,11}}{\rho}\tau + \frac{2}{n-1}\tau^2\right).\end{aligned} \tag{70}$$

Note that

$$\frac{(\rho_{,11})^2}{\rho^2} = \frac{1}{4}\sum\frac{(\Phi_{,i})^2}{\Phi} - \frac{\sum_{i>1}(\rho_{,1i})^2}{\rho^2} + 2\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} - \frac{(\rho_{,1})^4}{\rho^4}. \tag{71}$$

Then, (70) and (71) together give us

$$\begin{aligned}\Delta\Phi \geq{}& 2\delta\sum\frac{(\rho_{,ij})^2}{\rho^2}+\frac{n(1-\delta)}{2(n-1)}\frac{\sum(\Phi_{,i})^2}{\Phi}\\&+\left(\frac{6n(1-\delta)}{n-1}-2(n+4)\right)\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3}+2R_{11}\frac{(\rho_{,1})^2}{\rho^2}\\&+\left[\frac{(n^2-4n)(1-\delta)}{2(n-1)}+2(n+3)\right]\frac{(\rho_{,1})^4}{\rho^4}\\&+\frac{1-\delta}{n-1}\left(2\tau^2-4\frac{\rho_{,11}}{\rho}\tau\right)-\frac{4n-2-2n\delta}{n-1}\Phi\tau\\&+2\frac{\rho_{,1}}{\rho^2}(\rho\tau)_{,1}.\end{aligned}\tag{72}$$

Using the Schwarz inequality gives

$$2\frac{\rho_{,11}}{\rho}\tau\leq\frac{7}{3}\sum\frac{(\rho_{,ij})^2}{\rho^2}+\frac{3}{7}\tau^2.\tag{73}$$

Using

$$\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3}=\frac{1}{2}\Phi_{,i}\frac{\rho_{,i}}{\rho}+\Phi^2,\tag{74}$$

and choosing $\delta=7/(3n+4)$, we get

$$\begin{aligned}\Delta\Phi \geq{}& \frac{n(1-\delta)}{2(n-1)}\frac{\sum(\Phi_{,i})^2}{\Phi}+\left(\frac{3n(1-\delta)}{n-1}-(n+4)\right)\\&\times\sum\Phi_{,i}\frac{\rho_{,i}}{\rho}+2R_{11}\frac{(\rho_{,1})^2}{\rho^2}\\&+\left[\frac{(n^2+8n)(1-\delta)}{2(n-1)}-2\right]\Phi^2+\frac{8(1-\delta)}{7(n-1)}\tau^2\\&-\frac{4n-2-2n\delta}{n-1}\Phi\tau+2\frac{\rho_{,1}}{\rho^2}(\rho\tau)_{,1}.\end{aligned}\tag{75}$$

In the following, we will calculate the terms $R_{11}((\rho_{,1})^2/\rho^2)$ and $(\rho_{,1}/\rho^2)(\rho\tau)_{,1}$. Note that (17) is invariant under an affine transformation of coordinates that preserved the origin. So, we can choose the coordinates $x_1, x_2, \ldots, x_n$ such that $f_{ij}(p)=\delta_{ij}$ and $\partial\rho/\partial x_1=\|\mathrm{grad}\rho\|(p)>0$, $(\partial\rho/\partial x_i)(p)=0$, for all $i>1$. From (19), we easily obtain

$$\rho_{,ij}=\rho_{ij}+A_{ij1}\rho_{,1}=\frac{\rho_{,i}\rho_{,j}}{\rho}-A_{ij1}\rho_{,1}+\frac{A_{ijk}f_k\rho}{n+2}.\tag{76}$$

Thus, we get

$$\Phi_{,i}=\frac{2\rho_{,1}\rho_{,1i}}{\rho^2}-2\frac{\rho_{,i}(\rho_{,1})^2}{\rho^3}=-2A_{i11}\frac{(\rho_{,1})^2}{\rho^2}+2\frac{\rho_{,1}f_kA_{ki1}}{(n+2)\rho},\tag{77}$$

$$\sum\Phi_{,i}\frac{\rho_{,i}}{\rho}=-2A_{111}\frac{(\rho_{,1})^3}{\rho^3}+2\frac{f_kA_{k11}}{n+2}\frac{(\rho_{,1})^2}{\rho^2}.\tag{78}$$

By the same method, as deriving (69), we have

$$\begin{aligned}\sum(A_{ml1})^2\geq{}&\frac{n}{n-1}\sum(A_{i11})^2-\frac{2}{n-1}A_{111}\sum A_{ii1}\\&+\frac{1}{n-1}\left(\sum A_{ii1}\right)^2.\end{aligned}\tag{79}$$

Note that $\sum A_{ii1}=((n+2)/2)(\rho_1/\rho)$. Therefore, by (14), (77), (78), and (79), we obtain

$$\begin{aligned}2R_{11}\frac{(\rho_{,1})^2}{\rho^2}={}&2\sum(A_{kj1})^2\frac{(\rho_{,1})^2}{\rho^2}-(n+2)A_{111}\frac{(\rho_{,1})^3}{\rho^3}\\\geq{}&\frac{n}{2(n-1)}\frac{\sum(\Phi_{,i}-2(\rho_{,1}f_kA_{ki1}/(n+2)\rho))^2}{\Phi}\\&+\frac{(n+2)(n+1)}{2(n-1)}\sum\Phi_{,i}\frac{\rho_{,i}}{\rho}\\&-\frac{n+1}{n-1}f_kA_{k11}\frac{(\rho_{,1})^2}{\rho^2}+\frac{(n+2)^2}{2(n-1)}\Phi^2.\end{aligned}\tag{80}$$

On the other hand, we have

$$2\frac{\rho_{,1}}{\rho^2}(\rho\tau)_{,1}=2\Phi\tau+\frac{1}{n+2}\sum A_{1ik}f_kf_i\frac{\rho_{,1}}{\rho}+\Phi.\tag{81}$$

Then, inserting (80) and (81) into (75), we get

$$\begin{aligned}\Delta\Phi\geq{}&\frac{2n-n\delta}{2(n-1)}\sum\frac{(\Phi_{,i})^2}{\Phi}-\frac{(n+2)(n-5)+6n\delta}{2(n-1)}\\&\times\sum\Phi_{,i}\frac{\rho_{,i}}{\rho}+\Phi+\frac{2(n+2)^2-(n^2+8n)\delta}{2(n-1)}\Phi^2\\&+\frac{8(1-\delta)}{7(n-1)}\tau^2-\frac{2n(1-\delta)}{n-1}\Phi\tau+\frac{1}{n+2}\\&\times\sum A_{1ik}f_kf_i\frac{\rho_{,1}}{\rho}-\frac{n+1}{n-1}\frac{(\rho_{,1})^2}{\rho^2}f_kA_{k11}\\&-\frac{2n}{(n-1)(n+2)}\frac{\sum\Phi_{,i}f_kA_{ki1}}{\sqrt{\Phi}}+\frac{2n}{(n-1)(n+2)^2}\\&\times\sum(f_kA_{ki1})^2.\end{aligned}\tag{82}$$

Using (77), we have

$$\begin{aligned}&\frac{1}{n+2}\sum A_{1ik}f_kf_i\frac{\rho_{,1}}{\rho}-\frac{n+1}{n-1}\frac{(\rho_{,1})^2}{\rho^2}f_kA_{k11}\\&\quad=\frac{1}{2}f_i\Phi_{,i}-\frac{2}{n-1}\frac{(\rho_{,1})^2}{\rho^2}f_kA_{k11}.\end{aligned}\tag{83}$$

One observes that the Schwarz inequality gives

$$\begin{aligned}
&\frac{2n}{(n-1)(n+2)}\frac{\sum \Phi_{,i} f_k A_{ki1}}{\sqrt{\Phi}} \\
&\quad \le \frac{9n}{8(n-1)}\sum\frac{(\Phi_{,i})^2}{\Phi}+\frac{8n}{9(n-1)(n+2)^2} \\
&\quad\quad \times \sum\left(f_k A_{ki1}\right)^2, \\
&\frac{2}{n-1}\frac{(\rho_{,1})^2}{\rho^2} f_k A_{k11} \\
&\quad \le \frac{9(n+2)^2}{10n(n-1)}\Phi^2+\frac{10n}{9(n-1)(n+2)^2} \\
&\quad\quad \times \sum\left(f_k A_{ki1}\right)^2, \\
&\quad\quad\quad 2n\Phi\tau \le \tau^2+n^2\Phi^2.
\end{aligned} \tag{84}$$

Note that by (17) we have

$$\begin{aligned}
\frac{1}{4}f^{ij}\Phi_j f_i &= \frac{n+2}{2}f^{ij}\Phi_j(\ln\rho)_i+\frac{1}{4}\Phi_j x_j \\
&= \frac{n+2}{2}f^{ij}\Phi_j(\ln\rho)_i+\frac{1}{4}f^{ij}\frac{\partial\Phi}{\partial x_i}\frac{\partial u}{\partial x_j}.
\end{aligned} \tag{85}$$

Then, inserting these estimates into (82) yields Proposition 6. □

Proof of Corollary 8. Now, we will calculate the term $(\ln\rho)_{,ijk}$. In particular, if f satisfies PDE (4), choose the coordinate $(x_1, x_2, \dots, x_n)$ such that $f_{ij}(p) = \delta_{ij}$; then we have

$$\begin{aligned}
(\ln\rho)_{,ijk} &= \frac{1}{n+2}\left(A_{ijk}+A_{ijk,p}f_{,p}\right)-(\ln\rho)_{,l}A_{ijk,l} \\
&\quad + A_{ijl}A_{klp}\left(3(\ln\rho)_{,p}-\frac{2}{n+2}f_{,p}\right).
\end{aligned} \tag{86}$$

Using (17), we have

$$3(\ln\rho)_{,p}-\frac{2}{n+2}f_{,p} = -\frac{1}{n+2}(f+u)_{,p}+(\ln\rho)_{,p}. \tag{87}$$

By the Young inequality and the Schwarz inequality, we have

$$\begin{aligned}
&\frac{n+2}{n(n-1)}\sum A_{ijk}A_{ijl}A_{klh}(\ln\rho)_{,h} \\
&\quad \le \frac{1}{2}J^2+16n^2(n-1)^2(n+2)^4\Phi^2, \\
&\frac{1}{n(n-1)}\sum A_{ijk}A_{ijl,k}f_l \\
&\quad = \frac{1}{2}\sum J_{,l}f_{,l} = \frac{1}{4}\langle\nabla J,\nabla(f+u)\rangle+\frac{n+2}{2}\langle\nabla J,\nabla\ln\rho\rangle, \\
&\frac{n+2}{n(n-1)}\sum A_{ijk}A_{ijl,k}(\ln\rho)_{,l} \\
&\quad = \frac{n+2}{2}\sum J_{,i}(\ln\rho)_{,i} \\
&\quad \le \frac{1}{n(n-1)}\sum\left(A_{ijk,l}\right)^2+\frac{(n+2)^2}{4}J\Phi \\
&\quad \le \frac{1}{n(n-1)}\sum\left(A_{ijk,l}\right)^2+\frac{1}{4}J^2+\frac{(n+2)^4}{16}\Phi^2, \\
&\frac{1}{n(n-1)}\sum A_{ijk}A_{jil}A_{klp}(f+u)_{,p} \\
&\quad \le \sqrt{n(n-1)}\left\|\nabla(f+u)\right\|J^{3/2}.
\end{aligned} \tag{88}$$

Thus, by inserting (88) into Lemma 7, we obtain Corollary 8. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The first author is supported by Grants (nos. 11101129 and 11201318) of NSFC. Part of this work was done when the first author was visiting the University of Washington and he thanks Professor Yu Yuan and the support of China Scholarship Council. The second author is supported by Grants (nos. U1304101 and 11171091) of NSFC and NSF of Henan Province (no. 132300410141).

References

[1] T. H. Colding and I. Minicozzi, "Generic mean curvature flow I: generic singularities," *Annals of Mathematics*, vol. 175, no. 2, pp. 755–833, 2012.

[2] K. Ecker and G. Huisken, "Mean curvature evolution of entire graphs," *Annals of Mathematics*, vol. 130, no. 3, pp. 453–471, 1989.

[3] G. Huisken, "Asymptotic behavior for singularities of the mean curvature flow," *Journal of Differential Geometry*, vol. 31, no. 1, pp. 285–299, 1990.

[4] G. Huisken, "Local and global behaviour of hypersurfaces moving by mean curvature," in *Differential Geometry: Partial Differential Equations on Manifolds*, vol. 54 of *Proceedings of Symposia in Pure Mathematics*, American Mathematical Society, 1993.

[5] K. Smoczyk, "Self-shrinkers of the mean curvature flow in arbitrary codimension," *International Mathematics Research Notices*, no. 48, pp. 2983–3004, 2005.

[6] L. Wang, "A Bernstein type theorem for self-similar shrinkers," *Geometriae Dedicata*, vol. 151, pp. 297–303, 2011.

[7] H. D. Cao and H. Li, "A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension," *Calculus of Variations and Partial Differential Equations*, vol. 46, no. 3-4, pp. 879–889, 2013.

[8] A. Chau, J. Chen, and Y. Yuan, "Rigidity of entire self-shrinking solutions to curvature flows," *Journal für die Reine und Angewandte Mathematik*, vol. 664, pp. 229–239, 2012.

[9] Q. Ding and Z. Z. Wang, *On the self-shrinking systems in arbitrary codimension spaces*, http://arxiv.org/abs/1012.0429.

[10] Q. Ding and Y. L. Xin, "The Rigidity theorems for Lagrangian self-shrinkers," *Journal für die Reine und Angewandte Mathematik*, 2012.

[11] K. Ecker, "Interior estimates and longtime solutions for mean curvature flow of noncompact spacelike hypersurfaces in Minkowski space," *Journal of Differential Geometry*, vol. 46, no. 3, pp. 481–498, 1997.

[12] R. Huang and Z. Wang, "On the entire self-shrinking solutions to Lagrangian mean curvature flow," *Calculus of Variations and Partial Differential Equations*, vol. 41, no. 3-4, pp. 321–339, 2011.

[13] Y. L. Xin, "Mean curvature flow with bounded Gauss image," *Results in Mathematics*, vol. 59, no. 3-4, pp. 415–436, 2011.

[14] Y. L. Xin, *Minimal Submanifolds and Related Topics*, World Scientific Publishing, 2003.

[15] A.-M. Li, R. Xu, U. Simon, and F. Jia, *Affine Bernstein Problems and Monge-Ampère Equations*, World Scientific, Singapore, 2010.

[16] A. M. Li and R. W. Xu, "A rigidity theorem for an affine Kähler-Ricci flat graph," *Results in Mathematics*, vol. 56, no. 1–4, pp. 141–164, 2009.

[17] A.-M. Li and R. Xu, "A cubic form differential inequality with applications to affine Kähler-Ricci flat manifolds," *Results in Mathematics*, vol. 54, no. 3-4, pp. 329–340, 2009.

[18] R. W. Xu and R. L. Huang, "On the rigidity theorems for Lagrangian translating solitons in pseudo-Euclidean space I," *Acta Mathematica Sinica (English Series)*, vol. 29, no. 7, pp. 1369–1380, 2013.

[19] A. V. Pogorelov, *The Minkowski Multidimensional Problem*, John Wiley & Sons, New York, NY, USA, 1978.

[20] E. Calabi, "Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens," *The Michigan Mathematical Journal*, vol. 5, pp. 105–126, 1958.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 985080, 10 pages
http://dx.doi.org/10.1155/2014/985080

Research Article

The Generalized Weaker (α-ϕ-φ)-Contractive Mappings and Related Fixed Point Results in Complete Generalized Metric Spaces

Maryam A. Alghamdi,[1] Chi-Ming Chen,[2] and Erdal Karapınar[3,4]

[1] *Department of Mathematics, Sciences Faculty for Girls, King Abdulaziz University, P.O. Box 4087, Jeddah 21491, Saudi Arabia*
[2] *Department of Applied Mathematics, National Hsinchu University of Education, Taiwan*
[3] *Department of Mathematics, Atilim University, Incek, 06836 Ankara, Turkey*
[4] *Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, Jeddah, Saudi Arabia*

Correspondence should be addressed to Maryam A. Alghamdi; maaalghamdi1@kau.edu.sa

Received 6 February 2014; Accepted 7 May 2014; Published 26 May 2014

Academic Editor: Qamrul Hasan Ansari

We introduce the notion of generalized weaker (α-ϕ-φ)-contractive mappings in the context of generalized metric space. We investigate the existence and uniqueness of fixed point of such mappings. Some consequences on existing fixed point theorems are also derived. The presented results generalize, unify, and improve several results in the literature.

1. Introduction and Preliminaries

In [1], Branciari introduced the notion of generalized metric space by weakening the triangular inequality of metric assumption with quadrilateral inequality. The author [1] characterized and proved the analog of famous Banach fixed point theorem in the setting of generalized metric space. Although the theorem of Branciari [1] is correct, the proofs had gaps [2] since the topology of generalized metric space is not strong enough as the topology of metric space. The disadvantages of generalized metric space can be listed as follows:

($w1$) generalized metric need not be continuous;

($w2$) a convergent sequence in generalized metric space need not be Cauchy;

($w3$) generalized metric space need not be Hausdorff, and hence the uniqueness of limits cannot be guaranteed.

Despite the weakness of the topology of generalized metric space, in [3, 4], the authors suggested some techniques to get a (unique) fixed point in such spaces.

On the other hand, Samet et al. [5] introduced the notion of α-ψ contraction mappings and proved the existence and uniqueness of such mappings in complete metric space. The results of this paper are very impressive since several existing results derived from the main theorem of Samet et al. [5] quiet easily. Later, a number of authors have appreciated these results and have used this technique to get further generalization via α-ψ contraction mappings; see, for example, [6–10].

In this paper, we introduce the generalized weaker α-ψ contraction mappings in the setting of generalized metric spaces. Consequently, we investigate the existence and uniqueness of fixed point by caring the problems ($w1$)–($w3$) mentioned above.

Let us recall basic definitions and notations and interesting results that will be in the sequel.

Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) ψ is nondecreasing;

(ii) there exist $k_0 \in \mathbb{N}$ and $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k, \quad (1)$$

for $k \geq k_0$ and any $t \in \mathbb{R}^+$.

In the literature such functions are called either Bianchini-Grandolfi gauge functions (see, e.g., [11–13]) or (c)-comparison functions (see, e.g., [14]).

Lemma 1 (see, e.g., [14]). *If $\psi \in \Psi$, then the following hold:*

(i) $(\psi^n(t))_{n\in\mathbb{N}}$ *converges to 0 as $n \to \infty$ for all $t \in \mathbb{R}^+$;*

(ii) $\psi(t) < t$, *for any $t \in \mathbb{R}^+$;*

(iii) ψ *is continuous at 0;*

(iv) *the series $\sum_{k=1}^{\infty} \psi^k(t)$ converges for any $t \in \mathbb{R}^+$.*

In the following, we recall the notion of generalized metric spaces.

Definition 2 (see [1]). Let X be a nonempty set and let $d : X \times X \to [0,\infty]$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y:

$$\begin{aligned}
&(\text{GMS1}) \quad d(x,y) = 0 \quad \text{iff } x = y,\\
&(\text{GMS2}) \quad d(x,y) = d(y,x),\\
&(\text{GMS3}) \quad d(x,y) \leq d(x,u) + d(u,v) + d(v,y).
\end{aligned} \quad (2)$$

Then, the map d is called generalized metric. Here, the pair (X,d) is called a generalized metric space and abbreviated as GMS.

In the above definition, if d satisfies only (GMS1) and (GMS2), then it is called semimetric (see, e.g., [15]).

The concepts of convergence, Cauchy sequence, and completeness in a GMS are defined as follows.

Definition 3. (1) A sequence $\{x_n\}$ in a GMS (X,d) is GMS convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(2) A sequence $\{x_n\}$ in a GMS (X,d) is GMS Cauchy if and only if for every $\varepsilon > 0$ there exists positive integer $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.

(3) A GMS (X,d) is called complete if every GMS Cauchy sequence in X is GMS convergent.

The following assumption was suggested by Wilson [15] to replace the triangle inequality with the weakened condition.

(W) For each pair of (distinct) points u, v there is a number $r_{u,v} > 0$ such that, for every $z \in X$,

$$r_{u,v} < d(u,z) + d(z,v). \quad (3)$$

Proposition 4 (see [3]). *In a semimetric space, the assumption (W) is equivalent to the assertion that limits are unique.*

Proposition 5 (see [3]). *Suppose that $\{x_n\}$ is a Cauchy sequence in a GMS (X,d) with $\lim_{n\to\infty} d(x_n, u) = 0$, where $u \in X$. Then $\lim_{n\to\infty} d(x_n, z) = d(u,z)$ for all $z \in X$. In particular, the sequence $\{x_n\}$ does not converge to z if $z \neq u$.*

A function $\phi : [0,\infty) \to [0,\infty)$ is said to be a Meir-Keeler function [16] if, for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0,\infty)$ with $\eta \leq t < \eta + \delta$, we have $\phi(t) < \eta$. Such mapping has been improved and used by several authors [17, 18]. In what follows we recall the notion of weaker Meir-Keeler function.

Definition 6 (see, e.g., [19]). We call $\phi : [0,\infty) \to [0,\infty)$ a weaker Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0,\infty)$ with $\eta \leq t < \eta + \delta$, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(t) < \eta$.

Let Φ be the class of all nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ϕ_1) $\phi : [0,\infty) \to [0,\infty)$ is a weaker Meir-Keeler function;

(ϕ_2) $0 < \phi(t) < t$ for all $t > 0$, $\phi(0) = 0$;

(ϕ_3) for all $t \in (0,\infty)$, $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;

(ϕ_4) if $\lim_{n\to\infty} t_n = \gamma$, then $\lim_{n\to\infty} \phi(t_n) \leq \gamma$.

Let Θ be the class of functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(φ_1) φ is continuous;

(φ_2) $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$.

By using the auxiliary functions, defined above, Chen and Sun [19] proved the following theorem.

Theorem 7. *Let (X,d) be a Hausdorff and complete generalized metric space, and let $f : X \to X$ be a function satisfying*

$$d(fx, fy) \leq \phi(d(x,y)) - \varphi(d(x,y)) \quad (4)$$

for all $x, y \in X$ and $\phi \in \Phi$, $\varphi \in \Theta$. Then f has a periodic point μ in X; that is, there exists $\mu \in X$ such that $\mu = f^p\mu$ for some $p \in \mathbb{N}$.

Another interesting auxiliary function, α-admissible, was defined by Samet et al. [5].

Definition 8 (see [5]). For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be mappings. We say that T is α-admissible if

$$\alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1, \quad (5)$$

for all $x, y \in X$.

Example 9. Let $X = [2,\infty)$ and $T : X \to X$ by $Tx = (x+1)/(x-1)$. Define $\alpha(x,y) : X \times X \to [0,\infty)$ and

$$\alpha(x,y) = \begin{cases} e^{x+1} & \text{if } x \geq y, \\ 0 & \text{if otherwise.} \end{cases} \quad (6)$$

Then T is α-admissible.

Example 10. Let $X = \mathbb{R}$ and $T : X \to X$. Define $\alpha(x, y) : X \times X \to [0, \infty)$ by $Tx = e^{x+1}$ and

$$\alpha(x, y) = \begin{cases} x^2 & \text{if } x \geq y, \\ 0 & \text{if otherwise.} \end{cases} \quad (7)$$

Then T is α-admissible.

Some interesting examples of such mappings were given in [5].

The notion of an α-ψ contractive mapping is defined in the following way.

Definition 11 (see [5]). Let (X, d) be a metric space and let $T : X \to X$ be a given mapping. We say that T is an α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in X. \quad (8)$$

Clearly, any contractive mapping, that is, a mapping satisfying the Banach contraction, is an α-ψ contractive mapping with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$, $k \in (0, 1)$.

Very recently, Karapınar [20] gave the analog of the notion of an α-ψ contractive mapping, in the context of generalized metric spaces as follows.

Definition 12. Let (X, d) be a generalized metric space and let $T : X \to X$ be a given mapping. We say that T is an α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in X. \quad (9)$$

Karapınar [20] also stated the following fixed point theorems.

Theorem 13. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be an α-ψ contractive mapping. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*

(iii) *T is continuous.*

Then there exists a $u \in X$ such that $Tu = u$.

Theorem 14. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be an α-ψ contractive mapping. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then there exists a $u \in X$ such that $Tu = u$.

For the uniqueness, Karapınar [20] (see also [21]) added the following additional conditions.

(U) For all $x, y \in \text{Fix}(T)$, we have $\alpha(x, y) \geq 1$, where $\text{Fix}(T)$ denotes the set of fixed points of T.

(H) For all $x, y \in \text{Fix}(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Theorem 15. *Adding condition (U) to the hypotheses of Theorem 13 (resp., Theorem 14), one obtains that u is the unique fixed point of T.*

Theorem 16. *Adding conditions (H) and (W) to the hypotheses of Theorem 13 (resp., Theorem 14), one obtains that u is the unique fixed point of T.*

Corollary 17. *Adding condition (H) to the hypotheses of Theorem 13 (resp., Theorem 14) and assuming that (X, d) is Hausdorff, one obtains that u is the unique fixed point of T.*

In this paper, we define the notion of weaker generalized α-ψ contractive mappings and prove some fixed point results in the setting of generalized metric spaces by using such mappings. We state some examples to illustrate the validity of the main results of this paper.

2. Main Results

In this section, we will state and prove our main results.

We give an extension of the notion of α-ψ contractive mappings, in the context of generalized metric space as follows.

Definition 18. Let (X, d) be a generalized metric space and let $T : X \to X$ be a given mapping. We say that T is a (α-ϕ-φ)-contractive mapping of type I if there exist functions $\alpha : X \times X \to [0, \infty)$, $\varphi \in \Theta$, and $\phi \in \Phi$ such that

$$\alpha(x, y) d(Tx, Ty) \leq \phi(M(x, y)) - \varphi(M(x, y)) \quad (10)$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}. \quad (11)$$

Definition 19. Let (X, d) be a generalized metric space and let $T : X \to X$ be a given mapping. We say that T is a (α-ϕ-φ)-contractive mapping of type II if there exist functions $\alpha : X \times X \to [0, \infty)$, $\varphi \in \Theta$, and $\phi \in \Phi$ such that

$$\alpha(x, y) d(Tx, Ty) \leq \phi(N(x, y)) - \varphi(N(x, y)) \quad (12)$$

for all $x, y \in X$, where

$$N(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}\right\}. \quad (13)$$

Next, we introduce the notion of *triangular α-admissible* as follows.

Definition 20. Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. The mapping T is said to be weak *triangular α-admissible* if for all $x \in X$, one has

$$\alpha(x, Tx) \geq 1, \qquad \alpha(Tx, T^2x) \geq 1 \Longrightarrow \alpha(x, T^2x) \geq 1. \tag{14}$$

Now, we state the first fixed point theorem.

Theorem 21. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a $(\alpha$-ϕ-$\varphi)$-contractive mapping of type I. Suppose that*

(i) *T is weak triangular α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous.*

Then, T has a fixed point $u \in X$; that is $Tu = u$.

Proof. Due to statement (ii) of the theorem, there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$. First, we define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. Notice that if $x_{n_0} = x_{n_0+1}$ for some n_0, then the proof is completed. Indeed, we have $u = x_{n_0} = x_{n_0+1} = Tx_{n_0} = Tu$. Thus, for the rest of the proof, we assume that

$$x_n \neq x_{n+1} \quad \forall n. \tag{15}$$

Owing to the fact that T is α-admissible, we derive that

$$\begin{aligned} \alpha(x_0, x_1) &= \alpha(x_0, Tx_0) \geq 1 \\ &\Longrightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1. \end{aligned} \tag{16}$$

Utilizing the expression above, we find that

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall n = 0, 1, \ldots. \tag{17}$$

Since T is a weak triangular α-admissible mapping, we obtain that

$$\alpha(x_0, x_2) \geq 1. \tag{18}$$

Since $\alpha(x_0, T_0^x) \geq 1$ and $\alpha(Tx_0, T^2x_0) = \alpha(x_0, x_2)$, iteratively, we conclude that

$$\alpha(x_n, x_{n+k}) \geq 1, \quad \forall n, k = 0, 1, \ldots. \tag{19}$$

Taking (10) and (17) into account, we observe that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \alpha(x_n, x_{n-1})\, d(Tx_n, Tx_{n-1}) \\ &\leq \phi(M(x_n, x_{n-1})) - \varphi(M(x_n, x_{n-1})), \end{aligned} \tag{20}$$

for all $n \geq 1$, where

$$\begin{aligned} &M(x_n, x_{n-1}) \\ &\quad = \max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} \\ &\quad = \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &\quad = \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}. \end{aligned} \tag{21}$$

If $M(x_n, x_{n-1}) = d(x_n, x_{n+1})$, then by (15) and property of the function φ, inequality (20) turns into

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \phi(M(x_n, x_{n-1})) - \varphi(M(x_n, x_{n-1})) \\ &= \phi(d(x_n, x_{n+1})) - \varphi(d(x_n, x_{n+1})) \\ &< \phi(d(x_n, x_{n+1})). \end{aligned} \tag{22}$$

Since $\{\phi^n(t)\}$ is decreasing, the inequality above yields a contradiction. Hence, we conclude that $M(x_n, x_{n-1}) = d(x_n, x_{n-1})$ and (20) becomes

$$d(x_{n+1}, x_n) \leq \phi(d(x_n, x_{n-1})), \tag{23}$$

for all $n \geq 1$. Recursively, we derive that

$$d(x_{n+1}, x_n) \leq \phi^n(d(x_1, x_0)), \quad \forall n \geq 1. \tag{24}$$

Owing to the fact that the sequence $\{\phi^n(d(x_0, x_1))\}_{n\in\mathbb{N}}$ is decreasing, it converges to some $\eta \geq 0$. We will show that $\eta = 0$. Suppose, on the contrary, that $\eta > 0$. Taking the definition of weaker Meir-Keeler function ϕ into account, there exists $\delta > 0$ such that for $x_0, x_1 \in X$ with $\eta \leq d(x_0, x_1) < \delta + \eta$, and there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(d(x_0, x_1)) < \eta$. Regarding $\lim_{n\to\infty}\phi^n(d(x_0, x_1)) = \eta$, there exists $p_0 \in \mathbb{N}$ such that $\eta \leq \phi^p(d(x_0, x_1)) < \delta + \eta$, for all $p \geq p_0$. Hence, we deduce that $\phi^{p_0+n_0}(d(x_0, x_1)) < \eta$, which is a contradiction. Thus, $\lim_{n\to\infty}\phi^n(d(x_0, x_1)) = 0$, and hence

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{25}$$

Regarding (10) and (19), we deduce that

$$\begin{aligned} d(x_{n+2}, x_n) &= d(Tx_{n+1}, Tx_{n-1}) \\ &\leq \alpha(x_{n+1}, x_{n-1})\, d(Tx_{n+1}, Tx_{n-1}) \\ &\leq \phi(M(x_{n+1}, x_{n-1})) - \varphi(M(x_{n+1}, x_{n-1})), \end{aligned} \tag{26}$$

for all $n \geq 1$, where

$$\begin{aligned} &M(x_{n+1}, x_{n-1}) \\ &\quad = \max\{d(x_{n+1}, x_{n-1}), d(x_{n+1}, Tx_{n+1}), d(x_{n-1}, Tx_{n-1})\} \\ &\quad = \max\{d(x_{n+1}, x_{n-1}), d(x_{n+1}, x_{n+2}), d(x_{n-1}, x_n)\}. \end{aligned} \tag{27}$$

If $M(x_n, x_{n-1}) = d(x_{n-1}, x_{n+1})$ then inequality (26) turns into

$$\begin{aligned} d(x_{n+2}, x_n) &\leq \phi(d(x_{n+1}, x_{n-1})) - \varphi(d(x_{n+1}, x_{n-1})) \\ &\leq \phi(d(x_{n+1}, x_{n-1})) \end{aligned} \tag{28}$$

for all $n \geq 1$. By repeating the same argument, inequality (15) implies that

$$d(x_{n+2}, x_n) \leq \phi^n(d(x_2, x_0)), \quad \forall n \geq 1. \tag{29}$$

Due to the fact that the sequence $\{\phi^n(d(x_0, x_2))\}_{n\in\mathbb{N}}$ is decreasing, we conclude that

$$\lim_{n\to\infty} d(x_{n+2}, x_n) = 0, \tag{30}$$

by following the lines at the proof of (25).

If either $M(x_n, x_{n-1}) = d(x_{n-1}, x_n)$ or $M(x_{n+1}, x_{n+2}) = d(x_{n+1}, x_{n+2})$, then inequality (26) becomes either

$$\begin{aligned} d(x_{n+2}, x_n) &\le \phi(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, x_n)) \\ &< \phi(d(x_{n-1}, x_n)) \end{aligned} \tag{31}$$

or

$$\begin{aligned} d(x_{n+2}, x_n) &\le \phi(d(x_{n+1}, x_{n+2})) - \varphi(d(x_{n+1}, x_{n+2})) \\ &< \phi(d(x_{n+1}, x_{n+2})), \end{aligned} \tag{32}$$

for all $n \ge 1$. Letting $n \to \infty$ in any of the cases, (31) or (32), together with (25), we have

$$\begin{gathered} \lim_{n\to\infty} d(x_{n+2}, x_n) \le \lim_{n\to\infty} \phi(d(x_{n-1}, x_n)) \le 0 \\ \text{or} \\ \lim_{n\to\infty} d(x_{n+2}, x_n) \le \lim_{n\to\infty} \phi(d(x_{n+1}, x_{n+2})) \le 0. \end{gathered} \tag{33}$$

Let $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \ne n$. Without loss of generality, assume that $m > n$. Thus, $x_m = T^{m-n}(T^n x_0) = T^n x_0 = x_n$. Regarding (15), we consider now

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_n, x_n) &= d(Tx_m, x_m) \\ &= d(Tx_m, Tx_{m-1}) \\ &\le \alpha(x_m, x_{m-1}) d(Tx_m, Tx_{m-1}) \\ &\le \phi(M(x_m, x_{m-1})) - \varphi(M(x_m, x_{m-1})), \end{aligned} \tag{34}$$

where

$$\begin{aligned} &M(x_m, x_{m-1}) \\ &\quad = \max\{d(x_m, x_{m-1}), d(x_m, Tx_m), d(x_{m-1}, Tx_{m-1})\} \\ &\quad = \max\{d(x_m, x_{m-1}), d(x_m, x_{m+1}), d(x_{m-1}, x_m)\} \\ &\quad = \max\{d(x_{m-1}, x_m), d(x_m, x_{m+1})\}. \end{aligned} \tag{35}$$

If $M(x_m, x_{m-1}) = d(x_{m-1}, x_m)$, then from (34) and (23) we get that

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_n, x_n) &= d(Tx_m, x_m) \\ &= d(Tx_m, Tx_{m-1}) \\ &\le \alpha(x_m, x_{m-1}) d(Tx_m, Tx_{m-1}) \\ &\le \phi(d(x_m, x_{m-1})) - \varphi(d(x_m, x_{m-1})) \\ &\le \phi(d(x_m, x_{m-1})) \\ &\le \phi^{m-n}(d(x_{n+1}, x_n)). \end{aligned} \tag{36}$$

If $M(x_m, x_{m-1}) = d(x_m, x_{m+1})$, inequalities (34) and (23) become

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_n, x_n) &= d(Tx_m, x_m) \\ &= d(Tx_m, Tx_{m-1}) \\ &\le \alpha(x_m, x_{m-1}) d(Tx_m, Tx_{m-1}) \\ &\le \phi(d(x_m, x_{m+1})) - \varphi(d(x_m, x_{m+1})) \\ &\le \phi(d(x_m, x_{m+1})) \\ &\le \phi^{m-n+1}(d(x_{n+1}, x_n)). \end{aligned} \tag{37}$$

Due to (ϕ_2), inequalities (36) and (37) yield that

$$\begin{aligned} d(x_{n+1}, x_n) &\le \phi^{m-n}(d(x_{n+1}, x_n)) < d(x_{n+1}, x_n), \\ d(x_{n+1}, x_n) &\le \phi^{m-n+1}(d(x_{n+1}, x_n)) < d(x_{n+1}, x_n), \end{aligned} \tag{38}$$

which is a contradiction. Hence $\{x_n\}$ has no periodic point.

In what follows we will prove that the sequence $\{x_n\}$ is Cauchy by standard technique. Suppose, on the contrary, that there exists $\varepsilon > 0$ such that for any $k \in \mathbb{N}$, there are $m(k), n(k) \in \mathbb{N}$ with $n(k) > m(k) > k$ satisfying

$$d(x_{m(k)}, x_{n(k)}) \ge \varepsilon. \tag{39}$$

Furthermore, corresponding to $m(k)$, one can choose $n(k)$ in a way that it is the smallest integer $n(k) > m(k)$ with $d(x_{m(k)}, x_{n(k)}) \ge \varepsilon$. Consequently, we have $d(x_{m(k)}, x_{n(k)-1}) < \varepsilon$. Consider

$$\begin{aligned} \varepsilon &\le d(x_{n(k)}, x_{m(k)}) \\ &\le d(x_{n(k)}, x_{n(k)-2}) + d(x_{n(k)-2}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \\ &\le d(x_{n(k)}, x_{n(k)-2}) + d(x_{n(k)-2}, x_{n(k)-1}) + \varepsilon. \end{aligned} \tag{40}$$

Letting $k \to \infty$, we get that

$$d(x_{n(k)}, x_{m(k)}) \longrightarrow \varepsilon. \tag{41}$$

On the other hand, again by using the quadrilateral inequality, we find

$$\begin{aligned} &d(x_{n(k)}, x_{m(k)}) \\ &\quad \le d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) \\ &\qquad + d(x_{m(k)-1}, x_{m(k)}), \\ &d(x_{n(k)-1}, x_{m(k)-1}) \\ &\quad \le d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}). \end{aligned} \tag{42}$$

Letting $n \to \infty$, in the inequalities above, we get that

$$d(x_{n(k)-1}, x_{m(k)-1}) \longrightarrow \varepsilon. \tag{43}$$

On account of (10), we have

$$\begin{aligned} & d\left(x_{n(k)}, x_{m(k)}\right) \\ & \quad = d\left(Tx_{n(k)-1}, Tx_{m(k)-1}\right) \\ & \quad \leq \alpha\left(x_{n(k)-1}, x_{m(k)-1}\right) d\left(Tx_{n(k)-1}, Tx_{m(k)-1}\right) \\ & \quad \leq \phi\left(M\left(x_{n(k)-1}, x_{m(k)-1}\right)\right) - \varphi\left(M\left(x_{n(k)-1}, x_{m(k)-1}\right)\right), \end{aligned} \tag{44}$$

where

$$\begin{aligned} & M\left(x_{n(k)-1}, x_{m(k)-1}\right) \\ & \quad = \max\left\{d\left(x_{n(k)-1}, x_{m(k)-1}\right), d\left(x_{n(k)-1}, x_{n(k)}\right),\right. \\ & \qquad \left. d\left(x_{m(k)-1}, x_{m(k)}\right)\right\}. \end{aligned} \tag{45}$$

Letting $n \to \infty$, in (44), and regarding definitions of auxiliary functions ϕ, φ and (45), we conclude that

$$\varepsilon \leq \varepsilon - \varphi(\varepsilon), \tag{46}$$

which yields that $\varphi(\varepsilon) = 0$. By definition of φ, we derive that $\varepsilon = 0$, which is a contradiction. Hence, we conclude that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d\left(x_n, u\right) = 0. \tag{47}$$

Since T is continuous, we obtain from (47) that

$$\lim_{n\to\infty} d\left(x_{n+1}, Tu\right) = \lim_{n\to\infty} d\left(Tx_n, Tu\right) = 0. \tag{48}$$

From (47) and (48) we get immediately that $\lim_{n\to\infty} T^n x_0 = \lim_{n\to\infty} Tx_n = Tu$. Taking Proposition 5 into account, we conclude that $Tu = u$. □

The following result is deduced from the obvious inequality $N(x, y) \leq M(x, y)$.

Theorem 22. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a (α-ϕ-φ)-contractive mapping of type II. Suppose that*

(i) *T is a weak triangular α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous.*

Then there exists a $u \in X$ such that $Tu = u$.

Theorem 23. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a (α-ϕ-φ)-contractive mapping of type I. Suppose that*

(i) *T is a weak triangular α-admissible and ϕ is upper semicontinuous function;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof. Following the proof of Theorem 21, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ converges for some $u \in X$. We will show that $Tu = u$. Suppose, on the contrary, that $Tu \neq u$. From (17) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. By applying the quadrilateral inequality together with (10) and (15), for all k, we get that

$$\begin{aligned} & d(u, Tu) \\ & \quad \leq d\left(u, x_{n(k)+2}\right) + d\left(x_{n(k)+2}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, Tu\right) \\ & \quad \leq d\left(u, x_{n(k)+2}\right) + d\left(x_{n(k)+2}, x_{n(k)+1}\right) + d\left(Tx_{n(k)}, Tu\right) \\ & \quad \leq d\left(u, x_{n(k)+2}\right) + d\left(x_{n(k)+2}, x_{n(k)+1}\right) \\ & \qquad + \alpha\left(x_{n(k)}, u\right) d\left(Tx_{n(k)}, Tu\right) \\ & \quad \leq d\left(u, x_{n(k)+2}\right) + d\left(x_{n(k)+2}, x_{n(k)+1}\right) + \phi\left(M\left(x_{n(k)}, u\right)\right), \end{aligned} \tag{49}$$

where

$$M\left(x_{n(k)}, u\right) = \max\left\{d\left(x_{n(k)}, u\right), d\left(x_{n(k)}, Tx_{n(k)}\right), d(u, Tu)\right\}. \tag{50}$$

Letting $k \to \infty$ in the above equality and regarding that the ϕ is an upper semicontinuous mapping, we find that

$$d(u, Tu) \leq \phi(d(u, Tu)). \tag{51}$$

It implies that from (ϕ_2)

$$d(u, Tu) \leq \phi(d(u, Tu)) < d(u, Tu), \tag{52}$$

which is a contradiction. Hence, we obtain that u is a fixed point of T; that is, $Tu = u$. □

In the following theorem, we remove the semicontinuity of ϕ by weakening the contractive mapping type.

Theorem 24. *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a (α-ϕ-φ)-contractive mapping of type II. Suppose that*

(i) *T is a weak triangular α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof. Following the proof of Theorem 21, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ converges for some $u \in X$. We will show that $Tu = u$. Suppose, on the contrary, that $Tu \neq u$. From (17) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$

for all k. By applying the quadrilateral inequality together with (10) and (15), for all k, we get that

$$\begin{aligned} d(u,Tu) &\le d(u,x_{n(k)+2}) + d(x_{n(k)+2},x_{n(k)+1}) + d(x_{n(k)+1},Tu) \\ &\le d(u,x_{n(k)+2}) + d(x_{n(k)+2},x_{n(k)+1}) + d(Tx_{n(k)},Tu) \\ &\le d(u,x_{n(k)+2}) + d(x_{n(k)+2},x_{n(k)+1}) \\ &\quad + \alpha(x_{n(k)},u)\, d(Tx_{n(k)},Tu) \\ &\le d(u,x_{n(k)+2}) + d(x_{n(k)+2},x_{n(k)+1}) + \phi(N(x_{n(k)},u)), \end{aligned} \tag{53}$$

where

$$N(x_{n(k)},u) = \max\left\{d(x_{n(k)},u), \frac{d(x_{n(k)},Tx_{n(k)}) + d(u,Tu)}{2}\right\}. \tag{54}$$

Letting $k \to \infty$ in the above equality and regarding (ϕ_4), we find that

$$d(u,Tu) \le \phi\left(\frac{d(u,Tu)}{2}\right) \le \frac{d(u,Tu)}{2} \tag{55}$$

which is a contradiction. Hence, we obtain that u is a fixed point of T; that is, $Tu = u$. □

Theorem 25. *Adding condition (U) to the hypotheses of Theorem 21 (resp., Theorem 23), one obtains that u is the unique fixed point of T.*

Proof. In what follows we will show that u is a unique fixed point of T. We will use the *reductio ad absurdum*. Let v be another fixed point of T with $v \neq u$. It is evident that $\alpha(u,v) = \alpha(Tu,Tv)$.

Now, due to (10) and (ϕ_2), we have

$$\begin{aligned} d(u,v) &\le \alpha(u,v)\, d(u,v) \\ &= \alpha(u,v)\, d(Tu,Tv) \\ &\le \phi(M(u,v)) - \varphi(M(u,v)) \\ &= \phi(d(u,v)) - \varphi(d(u,v)) \\ &\le \phi(d(u,v)) \\ &< d(u,v) \end{aligned} \tag{56}$$

which is a contradiction, where

$$M(u,v) = \max\{d(u,v), d(u,Tu), d(v,Tv)\} = d(u,v). \tag{57}$$

Hence, $u = v$. □

Theorem 26. *Adding condition (U) to the hypotheses of Theorem 22 (resp., Theorem 24), one obtains that u is the unique fixed point of T.*

Proof. The proof is analog of the proof of Theorem 25 which will be concluded by using the *reductio ad absurdum*. Suppose, on the contrary, that v is another fixed point of T with $v \neq u$. It is evident that $\alpha(u,v) = \alpha(Tu,Tv)$.

Now, due to (12) and (ϕ_2), we have

$$\begin{aligned} d(u,v) &\le \alpha(u,v)\, d(u,v) \\ &= \alpha(u,v)\, d(Tu,Tv) \\ &\le \phi(N(u,v)) - \varphi(N(u,v)) \\ &= \phi(d(u,v)) - \varphi(d(u,v)) \\ &= \phi(d(u,v)) \\ &< d(u,v) \end{aligned} \tag{58}$$

which is a contradiction, where

$$N(u,v) = \max\left\{d(u,v), \frac{d(u,Tu) + d(v,Tv)}{2}\right\} = d(u,v). \tag{59}$$

Hence, $u = v$. □

For the uniqueness, we can also consider the following condition.

(H^*) For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $\alpha(x,z) \ge 1$ and $\alpha(y,z) \ge 1$. Further, $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$, where $z_1 = z$ and $z_{n+1} = Tz_n$ for $n = 1, 2, 3, \ldots$.

Theorem 27. *Adding conditions (H^*) and (W) to the hypotheses of Theorem 21 (resp., Theorem 23), one obtains that u is the unique fixed point of T.*

Proof. Suppose that v is another fixed point of T. From (H^*), there exists $z \in X$ such that

$$\alpha(u,z) \ge 1, \qquad \alpha(v,z) \ge 1. \tag{60}$$

Since T is α-admissible, from (60), we have

$$\alpha(u,T^n z) \ge 1, \quad \alpha(v,T^n z) \ge 1, \quad \forall n. \tag{61}$$

Define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$ for all $n \ge 0$ and $z_0 = z$. From (61), for all n, we have

$$\begin{aligned} d(u,z_{n+1}) = d(Tu,Tz_n) &\le \alpha(u,z_n)\, d(Tu,Tz_n) \\ &\le \phi(M(u,z_n)) - \varphi(M(u,z_n)), \end{aligned} \tag{62}$$

where

$$\begin{aligned} M(u,z_n) &= \max\{d(u,z_n), d(u,Tu), d(z_n,Tz_n)\} \\ &= \max\{d(u,z_n), d(z_n,Tz_n)\}. \end{aligned} \tag{63}$$

If $M(u,z_n) = d(z_n,Tz_n)$ then by letting $n \to \infty$ in (62) we get that

$$\lim_{n\to\infty} d(z_n,u) = 0, \tag{64}$$

due to the continuity of φ, (ϕ_4) and the fact that $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$. If $M(u, z_n) = d(u, z_n)$ then (62) turns into

$$d(u, z_{n+1}) \leq \phi(d(u, z_n)) - \varphi(d(u, z_n)) \leq \phi(d(u, z_n)). \tag{65}$$

Iteratively, by using inequality (62), we get that

$$d(u, z_{n+1}) \leq \phi^n(d(u, z_0)), \tag{66}$$

for all n. Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} d(z_n, u) = 0. \tag{67}$$

Similarly, one can show that

$$\lim_{n\to\infty} d(z_n, v) = 0. \tag{68}$$

Regarding (W) together with (67) and (68), it follows that $u = v$. Thus we proved that u is the unique fixed point of T. □

Theorem 28. *Adding conditions (H^*) and (W) to the hypotheses of Theorem 22 (resp., Theorem 24), one obtains that u is the unique fixed point of T.*

The proof is the analog of the proof of Theorem 27; hence we omit it.

Corollary 29. *Adding condition (H^*) to the hypotheses of Theorem 21 (resp., Theorems 23, 22, and 24) and assuming that (X, d) is Hausdorff, one obtains that u is the unique fixed point of T.*

The proof is clear, and hence it is omitted. Indeed, Hausdorffness implies the uniqueness of the limit. Thus, the theorem above yields the conclusions.

3. Consequences

Now, we will show that many existing results in the literature can be deduced easily from Theorems 13 and 14.

Definition 30. Let (X, d) be a generalized metric space and let $T : X \to X$ be a given mapping. We say that T is a (α-ϕ-φ)-contractive mapping of type III if there exist functions $\alpha : X \times X \to [0, \infty)$, $\varphi \in \Theta$, and $\phi \in \Phi$ such that

$$\alpha(x, y) d(Tx, Ty) \leq \phi(d(x, y)) - \varphi(d(x, y)) \tag{69}$$

for all $x, y \in X$.

Now, we state the first fixed point theorem.

Theorem 31. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a (α-ϕ-φ)-contractive mapping of type III. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*

(iii) *T is continuous.*

Then, T has a fixed point $u \in X$; that is, $Tu = u$.

We omit the proof of Theorem 31, since it can be derived easily by following the lines in the proof of Theorem 21, analogously.

Theorem 32. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a (α-ϕ-φ)-contractive mapping of type III. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof. Following the proof of Theorem 21 (resp., Theorem 31), we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ converges for some $u \in X$. We will show that $Tu = u$. Suppose, on the contrary, that $Tu \neq u$. From (17) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. By applying the quadrilateral inequality together with (10) and (15), for all k, we get that

$$\begin{aligned} d(u, Tu) &\leq d(u, x_{n(k)+2}) + d(x_{n(k)+2}, x_{n(k)+1}) + d(x_{n(k)+1}, Tu) \\ &\leq d(u, x_{n(k)+2}) + d(x_{n(k)+2}, x_{n(k)+1}) + d(Tx_{n(k)}, Tu) \\ &\leq d(u, x_{n(k)+2}) + d(x_{n(k)+2}, x_{n(k)+1}) \\ &\quad + \alpha(x_{n(k)}, u) d(Tx_{n(k)}, Tu) \\ &\leq d(u, x_{n(k)+2}) + d(x_{n(k)+2}, x_{n(k)+1}) + \phi(d(x_{n(k)}, u)). \end{aligned} \tag{70}$$

Letting $k \to \infty$ in the above equality and regarding (ϕ_4), we find that

$$d(u, Tu) \leq \lim_{n\to\infty} \phi(d(x_{n(k)}, u)) \leq 0, \tag{71}$$

which is a contradiction. Hence, we obtain that u is a fixed point of T; that is, $Tu = u$. □

Theorem 33. *Adding condition (U) to the hypotheses of Theorem 31 (resp., Theorem 32), one obtains that u is the unique fixed point of T.*

Proof. In what follows we will show that u is a unique fixed point of T. We will use the *reductio ad absurdum*. Let v be another fixed point of T with $v \neq u$. It is evident that $\alpha(u, v) = \alpha(Tu, Tv)$.

Now, due to (10) and (ϕ_2), we have

$$\begin{aligned} d(u,v) &\le \alpha(u,v)\,d(u,v) \\ &= \alpha(Tu,Tv)\,d(Tu,Tv) \\ &\le \phi(d(u,v)) - \varphi(d(u,v)) \\ &\le \phi(d(u,v)) \\ &< d(u,v) \end{aligned} \tag{72}$$

which is a contradiction. □

Theorem 34. *Adding conditions (H) and (W) to the hypotheses of Theorem 31 (resp., Theorem 32), one obtains that u is the unique fixed point of T.*

Corollary 35. *Let (X,d) be a complete generalized metric space and let $T : X \to X$ be a given mapping. Suppose that there exist $\phi \in \Phi$ and $\varphi \in \Theta$ such that*

$$d(Tx,Ty) \le \phi(M(x,y)) - \varphi(M(x,y)), \tag{73}$$

for all $x, y \in X$, where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}. \tag{74}$$

Then T has a unique fixed point.

Proof. Let $\alpha : X \times X \to [0,\infty)$ be the mapping defined by $\alpha(x,y) = 1$, for all $x, y \in X$. Then T is a $(\alpha\text{-}\phi\text{-}\varphi)$-contractive mapping of type I. It is evident that all conditions of Theorem 21 are satisfied. Hence, T has a unique fixed point. □

Corollary 36. *Let (X,d) be a complete generalized metric space and let $T : X \to X$ be a given mapping. Suppose that there exist $\phi \in \Phi$ and $\varphi \in \Theta$ such that*

$$d(Tx,Ty) \le \phi(N(x,y)) - \varphi(N(x,y)), \tag{75}$$

for all $x, y \in X$, where

$$N(x,y) = \max\left\{d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}\right\}. \tag{76}$$

Then T has a unique fixed point.

The following Corollary is stronger than the main result of [19]. Notice that we do not need the Hausdorffness condition although it was required in [19].

Corollary 37. *Let (X,d) be a complete generalized metric space and let $T : X \to X$ be a given mapping. Suppose that there exist $\phi \in \Phi$ and $\varphi \in \Theta$ such that*

$$d(Tx,Ty) \le \phi(d(x,y)) - \varphi(d(x,y)), \tag{77}$$

for all $x, y \in X$. Then T has a unique fixed point.

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

References

[1] A. Branciari, "A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces," *Publicationes Mathematicae Debrecen*, vol. 57, no. 1-2, pp. 31–37, 2000.

[2] B. Samet, "Discussion on: a fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces by A. Branciari," *Publicationes Mathematicae Debrecen*, vol. 76, no. 3-4, pp. 493–494, 2010.

[3] W. A. Kirk and N. Shahzad, "Generalized metrics and Caristi's theorem," *Fixed Point Theory and Applications*, vol. 2013, article 129, 2013.

[4] M. Jleli and B. Samet, "The Kannan's fixed point theorem in a cone rectangular metric space," *Journal of Nonlinear Science and its Applications*, vol. 2, no. 3, pp. 161–167, 2009.

[5] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for α-ψ-contractive type mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[6] M. U. Ali and T. Kamran, "On (α^*, ψ)-contractive multi-valued mappings," *Fixed Point Theory and Applications*, vol. 2013, article 137, 2013.

[7] M. Jleli, E. Karapınar, and B. Samet, "Best proximity points for generalized alpha-psi-proximal contractive type mappings," *Journal of Applied Mathematics*, vol. 2013, Article ID 534127, 10 pages, 2013.

[8] M. Jleli, E. Karapınar, and B. Samet, "Fixed point results for α-ψ_λ-contractions on gauge spaces and applications," *Abstract and Applied Analysis*, vol. 2013, Article ID 730825, 7 pages, 2013.

[9] E. Karapınar and B. Samet, "Generalized α-ψ contractive type mappings and related fixed point theorems with applications," *Abstract and Applied Analysis*, vol. 2012, Article ID 793486, 17 pages, 2012.

[10] B. Mohammadi, S. Rezapour, and N. Shahzad, "Some results on fixed points of α-ψ-Ciric generalized multifunctions," *Fixed Point Theory and Applications*, vol. 2013, article 24, 2013.

[11] R. M. Bianchini and M. Grandolfi, "Trasformazioni di tipo contrattivo generalizzato in uno spazio metrico," *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali*, vol. 45, pp. 212–216, 1968.

[12] P. D. Proinov, "A generalization of the Banach contraction principle with high order of convergence of successive approximations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 8, pp. 2361–2369, 2007.

[13] P. D. Proinov, "New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems," *Journal of Complexity*, vol. 26, no. 1, pp. 3–42, 2010.

[14] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca Romania, 2001.

[15] W. A. Wilson, "On semi-metric spaces," *The American Journal of Mathematics*, vol. 53, no. 2, pp. 361–373, 1931.

[16] A. Meir and E. Keeler, "A theorem on contraction mappings," *Journal of Mathematical Analysis and Applications*, vol. 28, pp. 326–329, 1969.

[17] A. A. Eldred and P. Veeramani, "Existence and convergence of best proximity points," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1001–1006, 2006.

[18] M. de la Sen, "Linking contractive self-mappings and cyclic Meir-Keeler contractions with Kannan self-mappings," *Fixed Point Theory and Applications*, vol. 2010, Article ID 572057, 23 pages, 2010.

[19] C. M. Chen and W. Y. Sun, "Periodic points and fixed points for the weaker (ϕ, φ)-contractive mappings in complete generalized metric spaces," *Journal of Applied Mathematics*, vol. 2012, Article ID 856974, 7 pages, 2012.

[20] E. Karapınar, "Discussion on α-ψ contractions on generalized metric spaces," *Abstract and Applied Analysis*, vol. 2014, Article ID 962784, 7 pages, 2014.

[21] H. Aydi, E. Karapınar, and B. Samet, "Fixed points for generalized (alpha, psi)-contractions on generalized metric spaces," *Journal of Inequalities and Applications*. In press.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 568718, 18 pages
http://dx.doi.org/10.1155/2014/568718

Research Article

Coupled and Tripled Coincidence Point Results with Application to Fredholm Integral Equations

Marwan Amin Kutbi,[1] Nawab Hussain,[1] Jamal Rezaei Roshan,[2] and Vahid Parvaneh[3]

[1] *Department of Mathematics, King AbdulAziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[2] *Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran*
[3] *Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran*

Correspondence should be addressed to Vahid Parvaneh; zam.dalahoo@gmail.com

Received 25 January 2014; Accepted 29 March 2014; Published 18 May 2014

Academic Editor: Ljubomir B. Ćirić

The aim of this paper is to define weak α-ψ-φ-contractive mappings and to establish coupled and tripled coincidence point theorems for such mappings defined on G_b-metric spaces using the concept of rectangular G-α-admissibility. As an application, we derive new coupled and tripled coincidence point results for weak ψ-φ-contractive mappings in partially ordered G_b-metric spaces. Our results are generalizations and extensions of some recent results in the literature. We also present an example as well as an application to nonlinear Fredholm integral equations in order to illustrate the effectiveness of our results.

1. Introduction and Mathematical Preliminaries

The concept of generalized metric space, or a G-metric space, was introduced by Mustafa and Sims.

Definition 1 (G-metric space [1]). Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

($G1$) $G(x, y, z) = 0$ if and only if $x = y = z$;

($G2$) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

($G3$) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;

($G4$) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);

($G5$) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Recently, Aghajani et al. in [2] motivated by the concept of b-metric [3] introduced the concept of generalized b-metric spaces (G_b-metric spaces) and then they presented some basic properties of G_b-metric spaces.

The following is their definition of G_b-metric spaces.

Definition 2 (see [2]). Let X be a nonempty set and let $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{R}^+$ satisfies the following:

(G_b1) $G(x, y, z) = 0$ if $x = y = z$,

(G_b2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),

(G_b5) $G(x, y, z) \leq s[G(x, a, a) + G(a, y, z)]$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a generalized b-metric and the pair (X, G) is called a generalized b-metric space or a G_b-metric space.

Each G-metric space is a G_b-metric space with $s = 1$.

Example 3 (see [2]). Let (X, G) be a G-metric space and $G_*(x, y, z) = G(x, y, z)^p$, where $p > 1$ is a real number. Then G_* is a G_b-metric with $s = 2^{p-1}$.

Example 4 (see [4]). Let $X = \mathbb{R}$ and $d(x,y) = |x-y|^2$. We know that (X,d) is a b-metric space with $s = 2$. Let $G(x,y,z) = d(x,y) + d(y,z) + d(z,x)$, it is easy to see that (X,G) is not a G_b-metric space. Indeed, (G_b3) is not true for $x = 0$, $y = 2$, and $z = 1$. However, $G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$ is a G_b-metric on $\mathbb{R}$ with $s = 2$.

Definition 5 (see [2]). A G_b-metric G is said to be symmetric if $G(x,y,y) = G(y,x,x)$, for all $x, y \in X$.

Proposition 6 (see [2]). *Let X be a G_b-metric space. Then for each $x, y, z, a \in X$ it follows that*

(1) *if $G(x,y,z) = 0$, then $x = y = z$,*

(2) $G(x,y,z) \leq s(G(x,x,y) + G(x,x,z))$,

(3) $G(x,y,y) \leq 2sG(y,x,x)$,

(4) $G(x,y,z) \leq s(G(x,a,z) + G(a,y,z))$.

Definition 7 (see [2]). Let X be a G_b-metric space. One defines $d_G(x,y) = G(x,y,y) + G(x,x,y)$, for all $x, y \in X$. It is easy to see that d_G defines a b-metric d on X, which one calls the b-metric associated with G.

Definition 8 (see [2]). Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be

(1) G_b-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$;

(2) G_b-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m, n \geq n_0$, $G(x_n, x_m, x) < \varepsilon$.

Proposition 9 (see [2]). *Let X be a G_b-metric space. Then the following are equivalent:*

(1) *the sequence $\{x_n\}$ is G_b-Cauchy,*

(2) *for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geq n_0$.*

Proposition 10 (see [2]). *Let X be a G_b-metric space. The following are equivalent.*

(1) *$\{x_n\}$ is G_b-convergent to x.*

(2) *$G(x_n, x_n, x) \to 0$, as $n \to +\infty$.*

(3) *$G(x_n, x, x) \to 0$, as $n \to +\infty$.*

Definition 11 (see [2]). A G_b-metric space X is called G_b-complete if every G_b-Cauchy sequence is G_b-convergent in X.

Proposition 12. *Let (X,G) and (X',G') be two G_b-metric spaces. Then a function $f : X \to X'$ is G_b-continuous at a point $x \in X$ if and only if it is G_b-sequentially continuous at x; that is, whenever $\{x_n\}$ is G_b-convergent to x, $\{f(x_n)\}$ is G'_b-convergent to $f(x)$.*

Proposition 13. *Let (X,G) be a G_b-metric space. A mapping $F : X \times X \to X$ is said to be continuous if, for any two G_b-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y, respectively, $\{F(x_n, y_n)\}$ is G_b-convergent to $F(x,y)$.*

In general, a G_b-metric function $G(x,y,z)$ for $s > 1$ is not jointly continuous in all its variables. The following is an example of a discontinuous G_b-metric.

Example 14 (see [4]). Let $X = \mathbb{N} \cup \{\infty\}$ and let $D : X \times X \to \mathbb{R}$ be defined by

$$D(m,n) = \begin{cases} 0, & \text{if } m = n, \\ \left|\dfrac{1}{m} - \dfrac{1}{n}\right|, & \text{if one of } m, n \text{ is even and the} \\ & \text{other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the} \\ & \text{other is odd (and } m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases} \tag{1}$$

Then it is easy to see that, for all $m, n, p \in X$, we have

$$D(m,p) \leq \frac{5}{2}(D(m,n) + D(n,p)). \tag{2}$$

Thus, (X,D) is a b-metric space with $s = 5/2$ (see [5]).

Let $G(x,y,z) = \max\{D(x,y), D(y,z), D(z,x)\}$. It is easy to see that G is a G_b-metric with $s = 5/2$ which is not a continuous function.

We will need the following simple lemma about the G_b-convergent sequences in the proof of our main results.

Lemma 15 (see [4]). *Let (X,G) be a G_b-metric space with $s > 1$ and suppose that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are G_b-convergent to x, y, and z, respectively. Then one has*

$$\begin{aligned} \frac{1}{s^3} G(x,y,z) &\leq \liminf_{n\to\infty} G(x_n, y_n, z_n) \\ &\leq \limsup_{n\to\infty} G(x_n, y_n, z_n) \\ &\leq s^3 G(x,y,z). \end{aligned} \tag{3}$$

In particular, if $x = y = z$, then we have $\lim_{n\to\infty} G(x_n, y_n, z_n) = 0$.

The existence of fixed points, coupled fixed points, and tripled fixed points for contractive type mappings in partially ordered metric spaces has been considered recently by several authors (see [6–28], etc.)

Lakshmikantham and Ćirić [17] introduced the notions of mixed g-monotone mapping and coupled coincidence point and proved some coupled coincidence point and common coupled fixed point theorems in partially ordered complete metric spaces.

Definition 16 (see [17]). Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$ and $g : X \to X$ be two mappings. F has the mixed g-monotone property, if F is monotone

g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument; that is, for all $x_1, x_2 \in X$, $gx_1 \preceq gx_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for any $y \in X$ and for all $y_1, y_2 \in X$, $gy_1 \preceq gy_2$ implies $F(x, y_1) \succeq F(x, y_2)$ for any $x \in X$.

Definition 17 (see [7, 17]). An element $(x, y) \in X \times X$ is called

(1) a coupled fixed point of mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$,

(2) a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$,

(3) a common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Definition 18 (see [17]). Let X be a nonempty set. We say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if $g(F(x, y)) = F(gx, gy)$ for all $x, y \in X$.

Choudhury and Maity [10] have established some coupled fixed point results for mappings with mixed monotone property in partially ordered G-metric spaces. They obtained the following results.

Theorem 19 (see [10, Theorem 3.1]). *Let $(X, \preceq)$ be a partially ordered set and let G be a G-metric on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ such that*

$$G(F(x, y), F(u, v), F(w, z)) \leq \frac{k}{2}[G(x, u, w) + G(y, v, z)], \tag{4}$$

for all $x \preceq u \preceq w$ and $y \succeq v \succeq z$, where either $u \neq w$ or $v \neq z$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point in X; that is, there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Theorem 20 (see [10, Theorem 3.2]). *If, in the above theorem, in place of the continuity of F, one assumes the following conditions, namely,*

(i) *if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*

(ii) *if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n,*

then F has a coupled fixed point.

Definition 21 (see [29]). Let $(X, \preceq)$ be a partially ordered set and let G be a G-metric on X. One says that $(X, G, \preceq)$ is regular if the following conditions hold.

(i) If $\{x_n\}$ is a nondecreasing sequence with $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

(ii) If $\{x_n\}$ is a nonincreasing sequence with $x_n \to x$, then $x_n \succeq x$ for all $n \in \mathbb{N}$.

Definition 22 (see [10]). Let $(\mathcal{X}, G)$ be a generalized b-metric space. Mappings $f : \mathcal{X}^2 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ are called compatible if

$$\begin{gathered} \lim_{n \to \infty} G(gf(x_n, y_n), f(gx_n, gy_n), f(gx_n, gy_n)) = 0, \\ \lim_{n \to \infty} G(gf(y_n, x_n), f(gy_n, gx_n), f(gy_n, gx_n)) = 0 \end{gathered} \tag{5}$$

hold whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathcal{X}$ such that

$$\begin{gathered} \lim_{n \to \infty} f(x_n, y_n) = \lim_{n \to \infty} gx_n, \\ \lim_{n \to \infty} f(y_n, x_n) = \lim_{n \to \infty} gy_n. \end{gathered} \tag{6}$$

On the other hand, Berinde and Borcut [25] introduced the concept of tripled fixed point and obtained some tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. For a survey of tripled fixed point theorems and related topics we refer the reader to [25–28, 30].

Definition 23 (see [25, 26]). Let $(\mathcal{X}, \preceq)$ be a partially ordered set, $f : \mathcal{X}^3 \to \mathcal{X}$, and $g : \mathcal{X} \to \mathcal{X}$.

(1) An element $(x, y, z) \in \mathcal{X}^3$ is called a tripled fixed point of f if $f(x, y, z) = x$, $f(y, x, y) = y$, and $f(z, y, x) = z$.

(2) An element $(x, y, z) \in \mathcal{X}^3$ is called a tripled coincidence point of the mappings f and g if $f(x, y, z) = gx$, $f(y, x, y) = gy$, and $f(z, y, x) = gz$.

(3) An element $(x, y, z) \in \mathcal{X}^3$ is called a tripled common fixed point of f and g if $x = g(x) = f(x, y, z)$, $y = g(y) = f(y, x, y)$, and $z = g(z) = f(z, y, x)$.

(4) One says that f has the mixed g-monotone property if $f(x, y, z)$ is g-nondecreasing in x, g-nonincreasing in y, and g-nondecreasing in z; that is, if, for any $x, y, z \in \mathcal{X}$,

$$\begin{gathered} x_1, x_2 \in \mathcal{X}, \quad gx_1 \preceq gx_2 \Longrightarrow f(x_1, y, z) \preceq f(x_2, y, z), \\ y_1, y_2 \in \mathcal{X}, \quad gy_1 \preceq gy_2 \Longrightarrow f(x, y_1, z) \succeq f(x, y_2, z), \\ z_1, z_2 \in \mathcal{X}, \quad gz_1 \preceq gz_2 \Longrightarrow f(x, y, z_1) \preceq f(x, y, z_2). \end{gathered} \tag{7}$$

Definition 24 (see [28]). Let $\mathcal{X}$ be a nonempty set. One says that the mappings $f : \mathcal{X}^3 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ commute if $g(f(x, y, z)) = f(gx, gy, gz)$, for all $x, y, z \in \mathcal{X}$.

In [26], Borcut obtained the following.

Theorem 25 (see [26, Corollary 1]). *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose there is a metric d on $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Let $f : \mathcal{X}^3 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ be such that f has the g-mixed monotone property. Assume that there exists $k \in [0, 1)$ such that*

$$\begin{aligned} & d(f(x, y, z), f(u, v, w)) \\ & \quad \leq k \max\{d(gx, gu), d(gy, gv), d(gz, gw)\} \end{aligned} \tag{8}$$

for all $x, y, z, u, v, w \in \mathcal{X}$ with $gx \preceq gu$, $gy \succeq gv$, and $gz \preceq gw$. Suppose $f(\mathcal{X}^3) \subseteq g(\mathcal{X})$ and g is continuous and commutes with f and also suppose either

(a) *f is continuous, or*

(b) *$\mathcal{X}$ has the following properties:*

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*

(ii) *if a nonincreasing sequence $y_n \to y$, then $y_n \succeq y$ for all n.*

If there exist $x_0, y_0, z_0 \in \mathcal{X}$ such that $gx_0 \preceq f(x_0, y_0, z_0)$, $gy_0 \succeq f(y_0, x_0, z_0)$, and $gz_0 \preceq f(z_0, y_0, x_0)$, then f and g have a tripled coincidence point.

Definition 26. Let $(\mathcal{X}, G)$ be a generalized b-metric space. Mappings $f : \mathcal{X}^3 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ are called compatible if

$$\begin{gathered}
\lim_{n\to\infty} G\left(gf\left(x_n, y_n, z_n\right), f\left(gx_n, gy_n, gz_n\right), f\left(gx_n, gy_n, gz_n\right)\right) = 0, \\
\lim_{n\to\infty} G\left(gf\left(y_n, x_n, y_n\right), f\left(gy_n, gx_n, gy_n\right), f\left(gy_n, gx_n, gy_n\right)\right) = 0, \\
\lim_{n\to\infty} G\left(gf\left(z_n, y_n, x_n\right), f\left(gz_n, gy_n, gx_n\right), f\left(gz_n, gy_n, gx_n\right)\right) = 0
\end{gathered} \tag{9}$$

hold whenever $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are sequences in $\mathcal{X}$ such that

$$\begin{gathered}
\lim_{n\to\infty} f\left(x_n, y_n, z_n\right) = \lim_{n\to\infty} gx_n, \\
\lim_{n\to\infty} f\left(y_n, x_n, y_n\right) = \lim_{n\to\infty} gy_n, \\
\lim_{n\to\infty} f\left(z_n, y_n, x_n\right) = \lim_{n\to\infty} gz_n.
\end{gathered} \tag{10}$$

Let $\psi : [0, +\infty) \to [0, +\infty)$ satisfies the following:

(i) ψ is continuous and nondecreasing,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

That is, ψ is an altering distance function.

In this paper, we obtain some coupled and tripled coincidence point theorems for nonlinear (ψ, φ) weakly contractive mappings which are G-α-admissible with respect to another function in partially ordered G_b-metric spaces. These results generalize and modify several comparable results in the literature.

2. Main Results

Samet et al. [31] defined the notion of α-admissible mapping as follows.

Definition 27. Let T be a self-mapping on X and let $\alpha : X \times X \to [0, +\infty)$ be a function. One says that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha\left(x, y\right) \geq 1 \Longrightarrow \alpha\left(Tx, Ty\right) \geq 1. \tag{11}$$

Definition 28 (see [32]). Let (X, G) be a G-metric space, let T be a self-mapping on X, and let $\alpha : X^3 \to [0, +\infty)$ be a function. One says that T is an G-α-admissible mapping if

$$x, y, z \in X, \quad \alpha\left(x, y, z\right) \geq 1 \Longrightarrow \alpha\left(Tx, Ty, Tz\right) \geq 1. \tag{12}$$

Following the recent work in [33–35] we present the following definition in the setting of G-metric spaces.

Definition 29. Let (X, G) be a G-metric space and let $f, g : X \to X$ and $\alpha : X^3 \to [0, +\infty)$. One says that f is a rectangular G-α-admissible mapping with respect to g if

(R1) $\alpha(gx, gy, gz) \geq 1$ implies $\alpha(fx, fy, fz) \geq 1$, $x, y, z \in X$,

(R2) $\{\alpha(gx, gy, gy) \geq 1, \alpha(gy, gz, gz) \geq 1\}$ implies $\alpha(gx, gz, gz) \geq 1$, $x, y, z \in X$.

Lemma 30. *Let f be a rectangular G-α-admissible mapping with respect to g such that $f(X) \subseteq g(X)$. Assume that there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0, fx_0) \geq 1$. Define sequence $\{y_n\}$ by $y_n = gx_n = fx_{n-1}$. Then*

$$\alpha\left(y_n, y_m, y_m\right) \geq 1 \quad \forall n, m \in \mathbb{N} \text{ with } n < m. \tag{13}$$

Now, we prove the following coincidence point result.

Theorem 31. *Let (X, G) be a generalized b-metric space and let $f, g : X \to X$ satisfy the following condition:*

$$\begin{aligned}
&\alpha\left(gx, gy, gz\right) \psi\left(sG\left(fx, fy, fz\right)\right) \\
&\quad \leq \psi\left(G\left(gx, gy, gz\right)\right) - \varphi\left(G\left(gx, gy, gz\right)\right)
\end{aligned} \tag{14}$$

for all $x, y, z \in \mathcal{X}$, where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are two altering distance mappings, $\alpha : X^3 \to [0, +\infty)$, and f is a rectangular G-α-admissible mapping with respect to g.

Then, maps f and g have a coincidence point if

(i) *$f(X) \subseteq g(X)$,*

(ii) *there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0, fx_0) \geq 1$,*

(iii) *f and g are continuous and compatible and (X, G) is complete,*

(iii′) *one of $f(X)$ or $g(X)$ is complete and assume that whenever $\{x_n\}$ in X is a sequence such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Proof. Let $x_0 \in X$ be such that $\alpha(gx_0, fx_0, fx_0) \geq 1$. According to (i) one can define the sequence $\{y_n\}$ as $y_{n+1} = gx_{n+1} = fx_n$ for all $n = 0, 1, 2, \ldots$.

As $\alpha(gx_0, gx_1, gx_1) = \alpha(gx_0, fx_0, fx_0) \geq 1$ and since f is an G-α-admissible mapping with respect to g, then

$\alpha(y_1, y_2, y_2) = \alpha(fx_0, fx_1, fx_1) \geq 1$. Continuing this process, we get $\alpha(y_n, y_{n+1}, y_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

If $y_n = y_{n+1}$, then x_n is a coincidence point of f and g.

Now, assume that $y_n \neq y_{n+1}$ for all n; that is,

$$G(y_n, y_{n+1}, y_{n+2}) > 0, \tag{15}$$

for all n. Let $G_n = G(y_n, y_{n+1}, y_{n+2})$. Then, from (14) we obtain that

$$\begin{aligned}
&\psi(sG(y_{n+1}, y_{n+2}, y_{n+3})) \\
&\quad \leq \alpha(y_n, y_{n+1}, y_{n+2}) \psi(sG(y_{n+1}, y_{n+2}, y_{n+3})) \\
&\quad = \alpha(gx_n, gx_{n+1}, gx_{n+2}) \psi(sG(fx_n, fx_{n+1}, fx_{n+2})) \\
&\quad \leq \psi(G(y_n, y_{n+1}, y_{n+2})) - \varphi(G(y_n, y_{n+1}, y_{n+2})).
\end{aligned} \tag{16}$$

We prove that $G_{n+1} \leq G_n$ for each $n \in \mathbb{N}$. If $G_{n+1} > G_n$ for some $n \in \mathbb{N}$, then from (16) we have $\psi(G_{n+1}) \leq \psi(G_{n+1}) - \varphi(G_n)$ which implies that $G_n = 0$, a contradiction to (15).

Hence, we have $0 < G_{n+1} \leq G_n$ for each $n \in \mathbb{N}$. Thus, the sequence $\{G_n\}$ is nonincreasing and so there exists $r \geq 0$ such that $\lim_{n\to\infty} G_n = r \geq 0$.

Suppose that $r > 0$. Then from (16), taking the limit as $n \to \infty$ implies that

$$\psi(r) \leq \psi(r) - \varphi(r) \tag{17}$$

a contradiction. Hence,

$$\lim_{n\to\infty} G_n = \lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+2}) = 0. \tag{18}$$

Since $y_{n+1} \neq y_{n+2}$ for every n, so by property (G_b3) we obtain

$$G(y_n, y_{n+1}, y_{n+1}) \leq G(y_n, y_{n+1}, y_{n+2}). \tag{19}$$

Hence,

$$\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = 0. \tag{20}$$

Also, by part (3) of Proposition 6 we have

$$\lim_{n\to\infty} G(y_n, y_n, y_{n+1}) = 0. \tag{21}$$

Now, we prove that $\{y_n\}$ is a G_b-Cauchy sequence. Assume on contrary that $\{y_n\}$ is not a G_b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that m_k is the smallest index for which $m_k > n_k > k$ and

$$G(y_{n_k}, y_{m_k}, y_{m_k}) \geq \varepsilon. \tag{22}$$

This means that

$$G(x_{n_k}, x_{m_k-1}, x_{m_k-1}) < \varepsilon. \tag{23}$$

Since f is a rectangular G-α-admissible mapping with respect to g, then from Lemma 30 $\alpha(y_{n_k}, y_{m_k-1}, y_{m_k-1}) \geq 1$. Now, from (14) we have

$$\begin{aligned}
&\psi(sG(y_{n_k+1}, y_{m_k}, y_{m_k})) \\
&\leq \alpha(y_{n_k}, y_{m_k-1}, y_{m_k-1}) \psi(sG(y_{n_k+1}, y_{m_k}, y_{m_k})) \\
&= \alpha(gx_{n_k}, gx_{m_k-1}, gx_{m_k-1}) \psi(sG(fx_{n_k}, fx_{m_k-1}, fx_{m_k-1})) \\
&\leq \psi(G(y_{n_k}, y_{m_k-1}, y_{m_k-1})) - \varphi(G(y_{n_k}, y_{m_k-1}, y_{m_k-1})).
\end{aligned} \tag{24}$$

Using (G_b5) we obtain that

$$\begin{aligned}
&G(y_{n_k}, y_{m_k}, y_{m_k}) \\
&\quad \leq sG(y_{n_k}, y_{n_k+1}, y_{n_k+1}) + sG(y_{n_k+1}, y_{m_k}, y_{m_k}).
\end{aligned} \tag{25}$$

Taking the upper limit as $k \to \infty$ and using (20) and (23) we obtain that

$$\limsup_{k\to\infty} G(y_{n_k+1}, y_{m_k}, y_{m_k}) \geq \frac{\varepsilon}{s}. \tag{26}$$

Using (G_b5) we obtain that

$$\begin{aligned}
&G(y_{n_k}, y_{m_k}, y_{m_k}) \\
&\quad \leq sG(y_{n_k}, y_{m_k-1}, y_{m_k-1}) + sG(y_{m_k-1}, y_{m_k}, y_{m_k}).
\end{aligned} \tag{27}$$

Taking the upper limit as $k \to \infty$ and using (20) and (23) we obtain that

$$\liminf_{k\to\infty} G(y_{n_k}, y_{m_k-1}, y_{m_k-1}) \geq \frac{\varepsilon}{s}. \tag{28}$$

Taking the upper limit as $k \to \infty$ in (24) and using (23) and (26) we obtain that

$$\begin{aligned}
\psi(\epsilon) &\leq \psi\left(s \limsup_{k\to\infty} G(y_{n_k+1}, y_{m_k}, y_{m_k})\right) \\
&\leq \psi\left(\limsup_{k\to\infty} G(y_{n_k}, y_{m_k-1}, y_{m_k-1})\right) \\
&\quad - \liminf_{k\to\infty} \varphi(G(y_{n_k}, y_{m_k-1}, y_{m_k-1})) \\
&\leq \psi(\epsilon) - \varphi\left(\liminf_{k\to\infty} G(y_{n_k}, y_{m_k-1}, y_{m_k-1})\right)
\end{aligned} \tag{29}$$

which implies that

$$\varphi\left(\liminf_{k\to\infty} G(y_{n_k}, y_{m_k-1}, y_{m_k-1})\right) = 0, \tag{30}$$

so $\lim_{k\to\infty} \inf G(y_{n_k}, y_{m_k-1}, y_{m_k-1}) = 0$, a contradiction to (28). It follows that $\{y_n\}$ is a G_b-Cauchy sequence in X.

Suppose first that (iii) holds. Then there exists

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in \mathcal{X}. \tag{31}$$

Further, since f and g are continuous and compatible, we get that

$$\lim_{n\to\infty} fgx_n = fz, \qquad \lim_{n\to\infty} gfx_n = gz, \qquad \lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n) = 0. \tag{32}$$

We will show that $fz = gz$. Indeed, we have

$$\begin{aligned} G(fz, gz, gz) &\le s[G(fz, fgx_n, fgx_n) + G(fgx_n, gz, gz)] \\ &\le sG(fz, fgx_n, fgx_n) \\ &\quad + s^2[G(fgx_n, gfx_n, gfx_n) \\ &\quad + G(gfx_n, gz, gz)] \\ &\longrightarrow s\cdot 0 + s^2\cdot 0 + s^2\cdot 0 = 0 \quad \text{as } (n\to\infty), \end{aligned} \tag{33}$$

and it follows that $fz = gz$. It means that f and g have a coincidence point.

In the case (iii′), if we assume that $g(X)$ is G_b-complete, then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu = z \tag{34}$$

for some $u \in \mathcal{X}$. Also, from (iii′) we have $\alpha(gx_n, gu, gu) \ge 1$. Applying (14) with $x = x_n$ and $y = u$, we have

$$\begin{aligned} &\psi(G(fx_n, fu, fu)) \\ &\quad \le \alpha(gx_n, gu, gu)\psi(sG(fx_n, fu, fu)) \\ &\quad \le \psi(G(gx_n, gu, gu)) - \varphi(G(gx_n, gu, gu)). \end{aligned} \tag{35}$$

It follows that $G(fx_n, fu, fu) \to 0$ when $n \to \infty$; that is, $fx_n \to fu$. Uniqueness of the limit yields that $fu = z = gu$. Hence, f and g have a coincidence point $u \in X$. □

Theorem 32. *Let $(X, G, \preceq)$ be an ordered generalized b-metric space and let $f, g : X \to X$ satisfy the following condition:*

$$\begin{aligned} &\psi(sG(fx, fy, fz)) \\ &\quad \le \psi(G(gx, gy, gz)) - \varphi(G(gx, gy, gz)) \end{aligned} \tag{36}$$

for all $x, y, z \in X$, with $gx \preceq gy \preceq gz$, where $\psi, \varphi : [0,\infty) \to [0,\infty)$ are two altering distance functions.

Then, maps f and g have a coincidence point if

(i) *f is g-nondecreasing with respect to $\preceq$ and $f(X) \subseteq g(X)$;*

(ii) *there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$;*

(iii) *f and g are continuous and compatible and (X, G) is complete, or*

(iii′) *$(X, G, \preceq)$ is regular and one of $f(X)$ or $g(X)$ is complete.*

Proof. Define $\alpha : X \times X \times X \to [0, +\infty)$ by

$$\alpha(x, y, z) = \begin{cases} 1, & \text{if } x \preceq y \preceq z, \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

First, we prove that f is a rectangular G-α-admissible mapping with respect to g. Assume that $\alpha(gx, gy, gz) \ge 1$. Therefore, we have $gx \preceq gy \preceq gz$. Since f is g-nondecreasing with respect to $\preceq$, we get $fx \preceq fy \preceq fz$; that is, $\alpha(fx, fy, fz) \ge 1$. Also, let $\alpha(x, z, z) \ge 1$ and $\alpha(z, y, y) \ge 1$, and then $x \preceq z$ and $z \preceq y$. Consequently, we deduce that $x \preceq y \preceq y$; that is, $\alpha(x, y, y) \ge 1$. Thus, f is a rectangular G-α-admissible mapping with respect to g. Since

$$\begin{aligned} &\psi(sG(fx, fy, fz)) \\ &\quad \le \psi(G(gx, gy, gz)) - \varphi(G(gx, gy, gz)) \end{aligned} \tag{38}$$

for all $x, y, z \in X$, with $gx \preceq gy \preceq gz$, then

$$\begin{aligned} &\alpha(gx, gy, gz)\psi(sG(fx, fy, fz)) \\ &\quad \le \psi(G(gx, gy, gz)) - \varphi(G(gx, gy, gz)). \end{aligned} \tag{39}$$

Moreover, from (ii) there exists $x_0 \in X$ such that $gx_0 \preceq fx_0 \preceq fx_0$; that is, $\alpha(gx_0, fx_0, fx_0) \ge 1$. Hence, all the conditions of Theorem 31 are satisfied and therefore f and g have a coincidence point. □

If $\alpha(x, y, z) = 1$ for all $x, y, z \in X$ in Theorem 31, then we obtain the following coincidence point result.

Theorem 33. *Let (X, G) be a generalized b-metric space and let $f, g : X \to X$ satisfy the following condition:*

$$\begin{aligned} &\psi(sG(fx, fy, fz)) \\ &\quad \le \psi(G(gx, gy, gz)) - \varphi(G(gx, gy, gz)) \end{aligned} \tag{40}$$

for all $x, y, z \in \mathcal{X}$, where $\psi, \varphi : [0,\infty) \to [0,\infty)$ are two altering distance functions.

Then, maps f and g have a coincidence point if

(i) *$f(X) \subseteq g(X)$,*

(ii) *f and g are continuous and compatible and (X, G) is complete, or*

(ii′) *one of $f(X)$ or $g(X)$ is complete.*

3. Coupled Fixed Point Results

We will use the following simple lemma in proving our next results. A similar case in the context of b-metric spaces can be found in [24].

Lemma 34. *Let $(\mathcal{X}, G)$ be a generalized b-metric space (with the parameter s) and let $f : \mathcal{X}^2 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$. Suppose that $F : \mathcal{X}^2 \to \mathcal{X}^2$ is given by*

$$FX = (f(x, y), f(y, x)), \quad X = (x, y) \in \mathcal{X}^2 \tag{41}$$

and $H : \mathcal{X}^2 \to \mathcal{X}^2$ is defined by

$$HX = (gx, gy), \quad X = (x, y) \in \mathcal{X}^2. \tag{42}$$

(a) *If a mapping $\Omega_2^m : \mathcal{X}^2 \times \mathcal{X}^2 \times \mathcal{X}^2 \to \mathbb{R}^+$ is given by*

$$\begin{aligned} &\Omega_2^m (X, U, A) = \max \{G (x, u, a), G (y, v, b)\}, \\ &X = (x, y), \quad U = (u, v), \quad A = (a, b) \in \mathcal{X}^2, \end{aligned} \tag{43}$$

then $(\mathcal{X}^2, \Omega_2^m)$ is a generalized b-metric space (with the same parameter s). The space $(\mathcal{X}^2, \Omega_2^m)$ is G_b-complete if and only if $(\mathcal{X}, G)$ is G_b-complete.

(b) *If f is continuous from $(\mathcal{X}^2, \Omega_2^m)$ to $(\mathcal{X}, G)$, then F is continuous in $(\mathcal{X}^2, \Omega_2^m)$.*

(c) *If f and g are compatible, then F and H are compatible.*

(d) *The mapping $f : X^2 \to X$ is G-α-admissible with respect to g; that is,*

$$\begin{aligned} &x, y, u, v, a, b \in X, \\ &\quad \alpha ((gx, gy), (gu, gv), (ga, gb)) \geq 1 \\ &\Longrightarrow \alpha (f (x, y), f (y, x)), \\ &\qquad (f (u, v), f (v, u)), \\ &\qquad ((f (a, b), f (b, a))) \geq 1 \end{aligned} \tag{44}$$

if and only if the mapping $F : \mathcal{X}^2 \to \mathcal{X}^2$ is G-α-admissible with respect to H; that is,

$$\begin{aligned} &X, U, A \in X^2, \\ &\qquad \alpha (HX, HU, HA) \geq 1 \\ &\quad \Longrightarrow \alpha (FX, FU, FA) \geq 1, \end{aligned} \tag{45}$$

where $\alpha : X^2 \times X^2 \times X^2 \to [0, \infty)$ is a function.

(e) *The statement (d) holds if we replace the G-α-admissibility by rectangular G-α-admissibility.*

Let $(\mathcal{X}, G)$ be a generalized b-metric space, $f : \mathcal{X}^2 \to \mathcal{X}$, and $g : \mathcal{X} \to \mathcal{X}$. In the rest of this paper unless otherwise stated, for all $x, y, u, v, z, w \in \mathcal{X}$, let

$$\begin{aligned} &N_f^m (x, y, u, v, z, w) \\ &\quad = \max \{G (f (x, y), f (u, v), f (z, w)), \\ &\qquad\qquad G (f (y, x), f (v, u), f (w, z))\}, \end{aligned} \tag{46}$$

$$\begin{aligned} &N_g^m (x, y, u, v, z, w) \\ &\quad = \max \{G (gx, gu, gz), G (gy, gv, gw)\}. \end{aligned}$$

Now, we have the following coupled coincidence point result.

Theorem 35. *Let $(\mathcal{X}, G)$ be a generalized b-metric space with the parameter s and let $f : \mathcal{X}^2 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$. Assume that*

$$\begin{aligned} &\alpha ((gx, gy), (gu, gv), (gz, gw)) \\ &\quad \times \psi (sN_f^m (x, y, u, v, z, w)) \\ &\quad \leq \psi (N_g^m (x, y, u, v, z, w)) - \varphi (N_g^m (x, y, u, v, z, w)), \end{aligned} \tag{47}$$

for all $x, y, u, v, z, w \in \mathcal{X}$, where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions, $\alpha : (X^2)^3 \to [0, \infty)$, and f is a rectangular G-α-admissible mapping with respect to g.

Assume also that

(1) *$f(\mathcal{X}^2) \subseteq g(\mathcal{X})$;*

(2) *there exist $x_0, y_0 \in \mathcal{X}$ such that*

$$\begin{aligned} &\alpha ((gx_0, gy_0), (f (x_0, y_0), f (y_0, x_0)), \\ &\qquad f (x_0, y_0), f (y_0, x_0)) \geq 1, \\ &\alpha ((gy_0, gx_0), (f (y_0, x_0), f (x_0, y_0)), \\ &\qquad f (y_0, x_0), f (x_0, y_0)) \geq 1. \end{aligned} \tag{48}$$

Also, suppose that either

(a) *f and g are continuous, the pair (f, g) is compatible, and $(\mathcal{X}, G)$ is G_b-complete, or*

(b) *$(g(\mathcal{X}), G)$ is G_b-complete and assume that whenever $\{x_n\}$ and $\{y_n\}$ in X are sequences such that*

$$\begin{aligned} &\alpha ((x_n, y_n), (x_{n+1}, y_{n+1}), (x_{n+1}, y_{n+1})) \geq 1, \\ &\alpha ((y_n, x_n), (y_{n+1}, x_{n+1}), (y_{n+1}, x_{n+1})) \geq 1 \end{aligned} \tag{49}$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$, $y_n \to y$ as $n \to +\infty$, we have

$$\begin{aligned} &\alpha ((x_n, y_n), (x, y), (x, y)) \geq 1, \\ &\alpha ((y_n, x_n), (y, x), (y, x)) \geq 1 \end{aligned} \tag{50}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Then, f and g have a coupled coincidence point in $\mathcal{X}$.

Proof. Let Ω_2^m be the generalized b-metric on $\mathcal{X}^2$ defined in Lemma 34. Also, define the mappings $F, H : \mathcal{X}^2 \to \mathcal{X}^2$ by $FX = (f(x, y), f(y, x))$ and $HX = (gx, gy)$, $X = (x, y)$ as in Lemma 34. Then, $(\mathcal{X}^2, \Omega_2^m)$ is a generalized b-metric space (with the same parameter s as $\mathcal{X}$), such that $F(\mathcal{X}^2) \subseteq H(\mathcal{X}^2)$. Moreover, the contractive condition (47) implies that

$$\begin{aligned} &\alpha (HX, HU, HA) \psi (s\Omega_2^m (FX, FU, FA)) \\ &\quad \leq \psi (\Omega_2^m (HX, HU, HA)) - \varphi (\Omega_2^m (HX, HU, HA)) \end{aligned} \tag{51}$$

holds for all $X, U, A \in \mathcal{X}^2$. Also, one can show that all conditions of Theorem 31 are satisfied for F and H and we have proved in Theorem 31 that, under these conditions, it follows that F and H have a coincidence point $\overline{X} = (\overline{x}, \overline{y}) \in \mathcal{X}^2$ which is obviously a coupled coincidence point of f and g. □

In the following theorem, we give a sufficient condition for the uniqueness of the common coupled fixed point (see also [23]).

Theorem 36. *In addition to the hypotheses of Theorem 35, suppose that f and g are commutative and that, for all (x, y) and $(x^*, y^*) \in \mathcal{X}^2$, there exists $(u, v) \in \mathcal{X}^2$, such that $\alpha((gx, gy), (gu, gv), (gu, gv)) \geq 1$ and $\alpha((gx^*, gy^*), (gu, gv), (gu, gv)) \geq 1$. Then, f and g have a unique common coupled fixed point of the form (a, a).*

Proof. We will use the notations as in the proof of Theorem 35. It was proved in this theorem that the set of coupled coincidence points of f and g; that is, the set of coincidence points of F and H in $\mathcal{X}^2$ is nonempty. We will show that if X and X^* are coincidence points of F and H, that is,

$$HX = FX, \quad HX^* = FX^*, \tag{52}$$

then $HX = HX^*$.

Choose an element $U = (u, v) \in \mathcal{X}^2$ such that $\alpha((gx, gy), (gu, gv), (gu, gv)) \geq 1$ and $\alpha((gx^*, gy^*), (gu, gv), (gu, gv)) \geq 1$. Let $U_0 = U$ and choose $U_1 \in \mathcal{X}^2$ so that $HU_1 = FU_0$. Then, we can inductively define a sequence $\{HU_n\}$ such that $HU_{n+1} = FU_n$.

As $\alpha((gx, gy), (gu, gv), (gu, gv)) \geq 1$ and f is rectangular G-α-admissible with respect to g, then $\alpha((f(x, y), f(y, x)), (f(u, v), f(v, u)), (f(u, v), f(v, u))) \geq 1$; that is, $\alpha(HX, HU_0, HU_0) \geq 1$ yields that

$$\begin{aligned}\alpha(FX, FU, FU) &= \alpha(FX, FU_0, FU_0) \\ &= \alpha(HX, HU_1, HU_1) \\ &\geq 1.\end{aligned} \tag{53}$$

Therefore, by the mathematical induction, we obtain that $\alpha(HX, HU_n, HU_n) \geq 1$, for all $n \geq 0$.

Applying (47), one obtains that

$$\begin{aligned}&\psi(s\Omega_2^m(FX, HU_{n+1}, HU_{n+1})) \\ &\quad\leq \alpha(HX, HU_n, HU_n)\,\psi(s\Omega_2^m(FX, FU_n, FU_n)) \\ &\quad\leq \psi(\Omega_2^m(HX, HU_n, HU_n)) - \varphi(\Omega_2^m(HX, HU_n, HU_n)) \\ &\quad\leq \psi(\Omega_2^m(HX, HU_n, HU_n)).\end{aligned} \tag{54}$$

From the properties of ψ, we deduce that the sequence $\{\Omega_2^m(HX, HU_n, HU_n)\}$ is nonincreasing. Hence, if we proceed as in Theorem 31, we can show that

$$\lim_{n\to\infty} \Omega_2^m(HX, HU_n, HU_n) = 0; \tag{55}$$

that is, $\{HU_n\}$ is G_b-convergent to HX.

Similarly, we can show that $\{HU_n\}$ is G_b-convergent to HX^*. Since the limit is unique, it follows that $HX = HX^*$.

The compatibility of f and g yields that F and H are compatible, and hence F and H are weak compatible. Since $HX = FX$, we have $HHX = HFX = FHX$. Let $HX = A$. Then, $HA = FA$. Thus, A is another coincidence point of F and H. Then, $A = HX = HA$. Therefore, $A = (a, b)$ is a coupled common fixed point of f and g.

To prove the uniqueness, assume that P is another common fixed point of F and H. Then, $P = HP = FP$ and also $HP = HA$. Thus, $P = HP = HA = A$. Hence, the coupled common fixed point is unique. Also, if (a, b) is a common coupled fixed point of f and g, then (b, a) is also a common coupled fixed point of f and g. Uniqueness of the common coupled fixed point yields that $a = b$. □

Let $\Omega_2^a : \mathcal{X}^2 \times \mathcal{X}^2 \times \mathcal{X}^2 \to \mathbb{R}^+$ be given by

$$\begin{gathered}\Omega_2^a(X, U, A) = \frac{G(x, u, a) + G(y, v, b)}{2}, \\ X = (x, y), \quad U = (u, v), \quad A = (a, b) \in \mathcal{X}^2,\end{gathered} \tag{56}$$

and then $(\mathcal{X}^2, \Omega_2^a)$ is a generalized b-metric space (with the same parameter s).

Let $(\mathcal{X}, G)$ be a generalized b-metric space, $f : \mathcal{X}^2 \to \mathcal{X}$, and $g : \mathcal{X} \to \mathcal{X}$. For all $x, y, u, v, z, w \in \mathcal{X}$, let

$$\begin{aligned}&N_f^a(x, y, u, v, z, w) \\ &\quad= \frac{G(f(x, y), f(u, v), f(z, w))}{2} \\ &\qquad+ \frac{G(f(y, x), f(v, u), f(w, z))}{2}, \\ &N_g^a(x, y, u, v, z, w) = \frac{G(gx, gu, gz) + G(gy, gv, gw)}{2}.\end{aligned} \tag{57}$$

Remark 37. The result of Theorems 35 and 36 holds, if we replace Ω_2^m, N_f^m, and N_g^m by Ω_2^a, N_f^a, and N_g^a, respectively.

4. Coupled Fixed Point Results in Partially Ordered Generalized b-Metric Spaces

We will use the following simple lemma in proving our results.

Lemma 38. *Let $(\mathcal{X}, G_b, \preceq)$ be an ordered generalized b-metric space (with the parameter s) and let $f : \mathcal{X}^2 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$. Let $F : \mathcal{X}^2 \to \mathcal{X}^2$ be given by*

$$FX = (f(x, y), f(y, x)), \quad X = (x, y) \in \mathcal{X}^2 \tag{58}$$

and $H : \mathcal{X}^2 \to \mathcal{X}^2$ is defined by

$$HX = (gx, gy), \quad X = (x, y) \in \mathcal{X}^2. \tag{59}$$

(a) If a relation $\sqsubseteq_2$ is defined on $\mathcal{X}^2$ by

$$X \sqsubseteq_2 U \Longleftrightarrow x \preceq u \wedge y \succeq v, \quad X = (x, y),\ U = (u, v) \in \mathcal{X}^2, \tag{60}$$

then $(\mathcal{X}^2, \Omega_2^m, \sqsubseteq_2)$ and $(\mathcal{X}^2, \Omega_2^a, \sqsubseteq_2)$ are ordered generalized b-metric spaces (with the same parameter s).

(b) If the mapping f has the g-mixed monotone property, then the mapping $F : \mathcal{X}^2 \to \mathcal{X}^2$ is G-nondecreasing with respect to $\sqsubseteq_2$; that is,

$$HX \sqsubseteq_2 HU \Longrightarrow FX \sqsubseteq_2 FU. \tag{61}$$

Theorem 39. *Let $(\mathcal{X}, G_b, \preceq)$ be a partially ordered generalized b-metric space with the parameter s and let $f : \mathcal{X}^2 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$. Assume that*

$$\psi\left(sN_f^m(x, y, u, v, z, w)\right) \le \psi\left(N_g^m(x, y, u, v, z, w)\right) - \varphi\left(N_g^m(x, y, u, v, z, w)\right), \tag{62}$$

for all $x, y, u, v, z, w \in \mathcal{X}$ with $gx \preceq gu \preceq gz$ and $gy \succeq gv \succeq gw$, where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions.

Assume also that

(1) f has the mixed g-monotone property and $f(\mathcal{X}^2) \subseteq g(\mathcal{X})$;

(2) there exist $x_0, y_0 \in \mathcal{X}$ such that $gx_0 \preceq f(x_0, y_0)$ and $gy_0 \succeq f(y_0, x_0)$.

Also, suppose that either

(a) f and g are continuous, the pair (f, g) is compatible, and $(\mathcal{X}, G)$ is G_b-complete, or

(b) $(\mathcal{X}, G_b)$ is regular and $(g(\mathcal{X}), G)$ is G_b-complete.

Then, f and g have a coupled coincidence point in $\mathcal{X}$.

Proof. By Lemma 38, $(\mathcal{X}^2, \Omega_2^m, \sqsubseteq_2)$ is an ordered generalized b-metric space (with the same parameter s).

Define $\alpha : X^2 \times X^2 \times X^2 \to [0, +\infty)$ by

$$\alpha((x, y), (u, v), (a, b)) = \begin{cases} 1, & \text{if } (x, y) \sqsubseteq_2 (u, v) \sqsubseteq_2 (a, b), \\ 0, & \text{otherwise.} \end{cases} \tag{63}$$

First, we prove that F is a rectangular G-α-admissible mapping with respect to H. Hence, we assume that $\alpha(HX, HU, HA) \ge 1$, where $X = (x, y)$, $U = (u, v)$, and $A = (a, b)$. Therefore, we have $HX \sqsubseteq_2 HU \sqsubseteq_2 HA$. Since f has the mixed g-monotone property, then from Lemma 38, the mapping $F : \mathcal{X}^2 \to \mathcal{X}^2$ is G-nondecreasing with respect to $\sqsubseteq_2$; that is,

$$FX \sqsubseteq_2 FU \sqsubseteq_2 FA; \tag{64}$$

that is, $\alpha(FX, FU, FA) \ge 1$. Also, let $\alpha(X, A, A) \ge 1$ and $\alpha(A, U, U) \ge 1$; then $X \sqsubseteq_2 A$ and $A \sqsubseteq_2 U$. Consequently, we deduce that $X \sqsubseteq_2 U$; that is, $\alpha(X, U, U) \ge 1$. Thus, F is a rectangular G-α-admissible mapping with respect to H.

From (62) and the definition of α and $\sqsubseteq_2$,

$$\psi\left(s\Omega_2^m(FX, FU, FA)\right) \le \psi\left(\Omega_2^m(HX, HU, HA)\right) - \varphi\left(\Omega_2^m(HX, HU, HA)\right), \tag{65}$$

for all $X, U, A \in X^2$ with $HX \sqsubseteq_2 HU \sqsubseteq_2 HA$. Moreover, from (2) there exists $(x_0, y_0) \in X^2$ such that

$$H(x_0, y_0) = (gx_0, gy_0) \sqsubseteq_2 ((x_0, y_0), f(y_0, x_0)) = F(x_0, y_0). \tag{66}$$

Hence, all the conditions of Theorem 32 are satisfied and so F and H have a coincidence point $\overline{X} = (\overline{x}, \overline{y}) \in \mathcal{X}^2$ which is a coupled coincidence point of f and g. □

In the following theorem, we give a sufficient condition for the uniqueness of the common coupled fixed point (see also [25, 28, 30]).

Theorem 40. *In addition to the hypotheses of Theorem 39, suppose that, for all (x, y) and $(x^*, y^*) \in \mathcal{X}^2$, there exists $(u, v) \in \mathcal{X}^2$, such that (gu, gv) is comparable with (gx, gy) and (gx^*, gy^*). Then, f and g have a unique common coupled fixed point of the form (a, a).*

Proof. It was proved in Theorem 39 that the set of coupled coincidence points of f and g, that is, the set of coincidence points of F and H in $\mathcal{X}^2$, is nonempty. We will show that if X and X^* are coincidence points of F and H, that is,

$$HX = FX, \quad HX^* = FX^*, \tag{67}$$

then $HX = HX^*$. There exists $(u, v) \in \mathcal{X}^2$, such that (gu, gv) is comparable with (gx, gy) and (gx^*, gy^*). Without any loss of generality, we may assume that $(gx, gy) \sqsubseteq_2 (gu, gv)$ and $(gx^*, gy^*) \sqsubseteq_2 (gu, gv)$. According to the definition of α in the above theorem, $\alpha((gx, gy), (gu, gv), (gu, gv)) \ge 1$ and $\alpha((gx^*, gy^*), (gu, gv), (gu, gv)) \ge 1$.

Now, following the proof of Theorem 36, one can obtain that $HX = HX^*$. The remainder part of proof is analogous to the proof of Theorem 36 and so we omit it. □

Remark 41. In Theorem 39, we can replace the contractive condition (62) by the following:

$$\alpha((gx, gy), (gu, gv), (gz, gw)) \times \psi\left(sN_f^a(x, y, u, v, z, w)\right) \le \psi\left(N_g^a(x, y, u, v, z, w)\right) - \varphi\left(N_g^a(x, y, u, v, z, w)\right). \tag{68}$$

Remark 42. Theorem 39 provides conclusions of Theorems 3.1 and 3.2 of [4] for more general pair of compatible maps.

In Theorem 39, if we take $\psi(t) = t$ for all $t \in [0, \infty)$, we obtain the following result.

Corollary 43. *Let $(\mathcal{X}, G, \preceq)$ be a partially ordered G_b-complete generalized b-metric space with the parameter $s \geq 1$ and let $f : \mathcal{X}^2 \to \mathcal{X}$. Assume that*

$$\begin{aligned} &\frac{G(f(x,y), f(u,v), f(z,w))}{2} \\ &\quad + \frac{G(f(y,x), f(v,u), f(w,z))}{2} \\ &\leq \frac{1}{s}\frac{G(gx,gu,gz) + G(gy,gv,gw)}{2} \\ &\quad - \frac{1}{s}\varphi\left(\frac{G(gx,gu,gz) + G(gy,gv,gw)}{2}\right), \end{aligned} \tag{69}$$

for all $x, y, u, v, z, w \in \mathcal{X}$ with $x \preceq u \preceq z$ and $y \succeq v \succeq w$, where $\varphi : [0,\infty) \to [0,\infty)$ is an altering distance function.

Assume also that

(1) *f has the mixed g-monotone property and $f(\mathcal{X}^2) \subseteq g(\mathcal{X})$;*

(2) *there exist $x_0, y_0 \in \mathcal{X}$ such that $gx_0 \preceq f(x_0, y_0)$ and $gy_0 \succeq f(y_0, x_0)$.*

Also, suppose that either

(a) *f is continuous, or*

(b) *$(\mathcal{X}, G)$ is regular.*

Then, f and g have a coupled coincidence point in $\mathcal{X}$.

In addition, suppose that, for all (x, y) and $(x^*, y^*) \in \mathcal{X}^2$, there exists $(u, v) \in \mathcal{X}^2$, such that (u, v) is comparable with (x, y) and (x^*, y^*). Then, f and g have a unique coupled fixed point of the form (a, a).

In Corollary 43, if we take $\varphi(t) = (1-k)t$ for all $t \in [0,\infty)$, where $k \in [0, 1)$, we obtain the following extension of the results by Choudhury and Maity (Theorems 19 and 20).

Corollary 44. *Let $(\mathcal{X}, G, \preceq)$ be a partially ordered G_b-complete generalized b-metric space with the parameter $s \geq 1$ and let $f : \mathcal{X}^2 \to \mathcal{X}$. Assume that*

$$\begin{aligned} &\frac{G(f(x,y), f(u,v), f(z,w))}{2} \\ &\quad + \frac{G(f(y,x), f(v,u), f(w,z))}{2} \\ &\leq \frac{k}{s}\frac{G(x,u,z) + G(y,v,w)}{2}, \end{aligned} \tag{70}$$

for all $x, y, u, v, z, w \in \mathcal{X}$ with $x \preceq u \preceq z$ and $y \succeq v \succeq w$, or $z \preceq u \preceq x$ and $w \succeq v \succeq y$, where $k \in [0, 1)$.

Assume also that

(1) *f has the mixed monotone property;*

(2) *there exist $x_0, y_0 \in \mathcal{X}$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$.*

Also, suppose that either

(a) *f is continuous, or*

(b) *$(\mathcal{X}, G)$ is regular.*

Then, f has a coupled fixed point in $\mathcal{X}$.

In addition, suppose that, for all (x, y) and $(x^*, y^*) \in \mathcal{X}^2$, there exists $(u, v) \in \mathcal{X}^2$, such that (u, v) is comparable with (x, y) and (x^*, y^*). Then, f has a unique coupled fixed point of the form (a, a).

Now, we present an example to illustrate Theorem 39 and Remark 41.

Example 45. Let $X = \mathbb{R}$ be endowed with the usual ordering and let G_b-metric G on X be given by $G(x, y, z) = \max\{|x-y|^2, |y-z|^2, |x-z|^2\}$, where $s = 2$. Define $F : X \times X \to X$ as

$$F(x,y) = \frac{x-y}{9}, \quad (x,y) \in X \times X. \tag{71}$$

We define $\psi, \varphi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \ln(t+1), \qquad \varphi(t) = \ln\left(\frac{t+1}{ct+1}\right), \tag{72}$$

where $c = 8/81$. Also, F has mixed monotone property and satisfies the condition (68). Indeed, for all $x, y, u, v, a, b \in X$ with $x \leq u \leq a$ and $y \geq v \geq b$, we have

$$\begin{aligned} &\psi(2(G(F(x,y), F(u,v), F(a,b)))) \\ &= \ln\left(2\max\left\{\left(\frac{x-y}{9} - \frac{u-v}{9}\right)^2, \left(\frac{u-v}{9} - \frac{a-b}{9}\right)^2,\right.\right. \\ &\qquad \left.\left.\left(\frac{a-b}{9} - \frac{x-y}{9}\right)^2\right\} + 1\right) \\ &= \ln\left(2\max\left\{\left(\frac{x-u+v-y}{9}\right)^2, \left(\frac{u-a+b-v}{9}\right)^2,\right.\right. \\ &\qquad \left.\left.\left(\frac{a-x+y-b}{9}\right)^2\right\} + 1\right) \\ &= \ln\left(\frac{2}{81}\max\left\{(x-u+v-y)^2, (u-a+b-v)^2,\right.\right. \\ &\qquad \left.\left.(a-x+y-b)^2\right\} + 1\right) \\ &\leq \ln\left(\frac{4}{81}\max\left\{(x-u)^2 + (v-y)^2, (u-a)^2\right.\right. \\ &\qquad \left.\left.+(b-v)^2, (a-x)^2 + (y-b)^2\right\} + 1\right) \\ &\leq \ln\left(\frac{4}{81}\left[\max\left\{(x-u)^2, (u-a)^2, (a-x)^2\right\}\right.\right. \\ &\qquad \left.\left.+\max\left\{(v-y)^2, (b-v)^2, (y-b)^2\right\}\right] + 1\right) \end{aligned}$$

$$= \ln\left(\frac{8}{81}\frac{G(x,u,a)+G(y,v,b)}{2}+1\right)$$
$$= \ln\left(c\frac{G(x,u,a)+G(y,v,b)}{2}+1\right)$$
$$= \ln\left(\frac{G(x,u,a)+G(y,v,b)}{2}+1\right)$$
$$-\ln\left(\frac{((G(x,u,a)+G(y,v,b))/2)+1}{c((G(x,u,a)+G(y,v,b))/2)+1}\right)$$
$$= \psi\left(\frac{G(x,u,a)+G(y,v,b)}{2}\right)$$
$$-\varphi\left(\frac{G(x,u,a)+G(y,v,b)}{2}\right). \tag{73}$$

Similarly,

$$\begin{aligned}\psi(2(G(F(y,x),F(v,u)),F(b,a))) &\le \psi\left(\frac{G(x,u,a)+G(y,v,b)}{2}\right) \\ &\quad -\varphi\left(\frac{G(x,u,a)+G(y,v,b)}{2}\right).\end{aligned} \tag{74}$$

So,

$$\begin{aligned}\psi\left(2\left(\frac{G(F(x,y),F(u,v),F(a,b))}{2}\right.\right. &\left.\left.+\frac{G(F(y,x),F(v,u),F(b,a))}{2}\right)\right) \\ &\le \psi\left(\frac{G(x,u,a)+G(y,v,b)}{2}\right) \\ &\quad -\varphi\left(\frac{G(x,u,a)+G(y,v,b)}{2}\right).\end{aligned} \tag{75}$$

Finally, there are obviously $x_0, y_0 \in X$ such that $x_0 \le F(x_0, y_0)$ and $y_0 \ge F(y_0, x_0)$. Thus, we conclude that the mapping F has a coupled fixed point (which is $(0,0)$).

Consider now the same example, but with the G-metric

$$G(x,y,z) = \max\{|x-y|, |y-z|, |x-z|\} \tag{76}$$

on $X = \mathbb{R}$. The respective contractive condition

$$\begin{aligned}\psi(G(F(x,y),F(u,v),F(a,b))) &\le \psi\left(\frac{G(x,u,a)+G(y,v,b)}{2}\right) \\ &\quad -\varphi\left(\frac{G(x,u,a)+G(y,v,b)}{2}\right)\end{aligned} \tag{77}$$

does not hold for all $x, y, u, v, a, b \in X$ such that $x \ge u \ge a$ and $y \le v \le b$. Indeed, for $x = 1$, $y = u = v = a = b = 0$ it reduces to

$$\begin{aligned}&\psi(G(F(1,0),F(0,0),F(0,0))) \\ &\quad = \psi\left(\frac{1}{9}\right) = 0.10536051565 \\ &\quad \nleq 0.4054651081 - 0.35726300629 \\ &\quad = \psi\left(\frac{1}{2}\right) - \varphi\left(\frac{1}{2}\right) \\ &\quad = \psi\left(\frac{G(1,0,0)+G(0,0,0)}{2}\right) \\ &\qquad -\varphi\left(\frac{G(1,0,0)+G(0,0,0)}{2}\right).\end{aligned} \tag{78}$$

We conclude that, using a G_b-metric instead of the standard one, one has more possibilities for choosing a control function in order to get a coupled fixed point result.

Remark 46. Theorems 3.5, 3.6, and 3.7 of [4] are special cases of Theorem 39.

5. Tripled Coincidence Point Results

In this section we prove some tripled coincidence and tripled common fixed point results.

Lemma 47. *Let $(\mathcal{X}, G, \preceq)$ be an ordered generalized b-metric space (with the parameter s) and let $f : \mathcal{X}^3 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$.*

(a) If a relation $\sqsubseteq_3$ is defined on $\mathcal{X}^3$ by

$$\begin{aligned}&X \sqsubseteq_3 U \Longleftrightarrow x \preceq u \wedge y \succeq v \wedge z \preceq w, \\ &X = (x,y,z), \quad U = (u,v,w) \in \mathcal{X}^3,\end{aligned} \tag{79}$$

and a mapping $\Omega_3^m : \mathcal{X}^3 \times \mathcal{X}^3 \times \mathcal{X}^3 \to \mathbb{R}^+$ is given by

$$\Omega_3^m(X,U,A) = \max\{G(x,u,a), G(y,v,b), G(z,w,c)\}, \tag{80}$$

for all $X = (x,y,z)$, $U = (u,v,w)$, and $A = (a,b,c) \in \mathcal{X}^3$, then $(\mathcal{X}^3, \Omega_3^m, \sqsubseteq_3)$ is an ordered generalized b-metric space (with the same parameter s). The space $(\mathcal{X}^3, \Omega_3^m)$ is G_b-complete if and only if $(\mathcal{X}, G)$ is G_b-complete.

(b) If the mapping f has the g-mixed monotone property, then the mapping $F : \mathcal{X}^3 \to \mathcal{X}^3$ given by

$$\begin{aligned}FX = (f(x,y,z), f(y,x,y), f(z,y,x)), \\ X = (x,y,z) \in \mathcal{X}^3\end{aligned} \tag{81}$$

is G-nondecreasing with respect to $\sqsubseteq_3$; that is,

$$HX \sqsubseteq_3 HU \Longrightarrow FX \sqsubseteq_3 FU, \tag{82}$$

where $H : \mathcal{X}^3 \to \mathcal{X}^3$ is defined by

$$HX = (gx, gy, gz), \quad X = (x,y,z) \in \mathcal{X}^3. \tag{83}$$

(c) If f is continuous from $(\mathcal{X}^3, \Omega_3^m)$ to $(\mathcal{X}, G)$, then F is continuous in $(\mathcal{X}^3, \Omega_3^m)$.

(d) If f and g are compatible, then F and H are compatible.

(e) The mapping $f : X^3 \to X$ is G-α-admissible with respect to g; that is,

$$\begin{aligned} & x, y, z, u, v, w, a, b, c \in X, \\ & \qquad \alpha((gx, gy, gz), (gy, gx, gy), (gz, gy, gx)) \geq 1 \\ & \quad \Longrightarrow \alpha((f(x, y, z), f(y, x, y), f(z, y, x)), \\ & \qquad (f(u, v, w), f(v, u, v), f(w, v, u)), \\ & \qquad (f(a, b, c), f(b, a, b), f(c, b, a))) \geq 1 \end{aligned} \tag{84}$$

if and only if the mapping $F : \mathcal{X}^3 \to \mathcal{X}^3$ is G-α-admissible with respect to H; that is,

$$\begin{aligned} & X, U, A \in X^3, \\ & \qquad \alpha(HX, HU, HA) \geq 1 \\ & \quad \Longrightarrow \alpha(FX, FU, FA) \geq 1, \end{aligned} \tag{85}$$

where $\alpha : X^3 \times X^3 \times X^3 \to [0, \infty)$ is a function.

Let $(\mathcal{X}, G, \preceq)$ be an ordered generalized b-metric space, $f : \mathcal{X}^3 \to \mathcal{X}$, and $g : \mathcal{X} \to \mathcal{X}$. For all $x, y, z, u, v, w, a, b, c \in \mathcal{X}$, let

$$\begin{aligned} & M_f^m(x, y, z, u, v, w, a, b, c) \\ & \quad = \max\{G(f(x, y, z), f(u, v, w), f(a, b, c)), \\ & \qquad G(f(y, x, y), f(v, u, v), f(b, a, b)), \\ & \qquad G(f(z, y, x), f(w, v, u), f(c, b, a))\}, \\ & M_g^m(x, y, z, u, v, w, a, b, c) \\ & \quad = \max\{G(gx, gu, ga), G(gy, gv, gb), G(gz, gw, gc)\}. \end{aligned} \tag{86}$$

Theorem 48. *Let $(\mathcal{X}, G)$ be a generalized b-metric space with the parameter s and let $f : \mathcal{X}^3 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$. Assume that*

$$\begin{aligned} & \alpha((gx, gy, gz), (gu, gv, gw), (ga, gb, gc)) \\ & \qquad \times \psi(sM_f^m(x, y, z, u, v, w, a, b, c)) \\ & \quad \leq \psi(M_g^m(x, y, z, u, v, w, a, b, c)) \\ & \qquad - \varphi(M_g^m(x, y, z, u, v, w, a, b, c)), \end{aligned} \tag{87}$$

for all $x, y, z, u, v, w, a, b, c \in \mathcal{X}$ where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions and $\alpha : (X^3)^3 \to [0, \infty)$ is a mapping such that f is a rectangular G-α-admissible mapping with respect to g.

Assume also that

(1) *$f(\mathcal{X}^3) \subseteq g(\mathcal{X})$;*

(2) *there exist $x_0, y_0, z_0 \in \mathcal{X}$ such that*

$$\begin{aligned} & \alpha((gx_0, gy_0, gz_0), \\ & \quad (f(x_0, y_0, z_0), f(y_0, x_0, y_0), f(z_0, y_0, x_0)), \\ & \quad (f(x_0, y_0, z_0), f(y_0, x_0, y_0), f(z_0, y_0, x_0))) \geq 1, \\ & \alpha((gy_0, gx_0, gy_0), \\ & \quad (f(y_0, x_0, y_0), f(x_0, y_0, z_0), f(y_0, x_0, y_0)), \\ & \quad (f(y_0, x_0, y_0), f(x_0, y_0, z_0), f(y_0, x_0, y_0))) \geq 1, \\ & \alpha((gz_0, gy_0, gx_0), \\ & \quad (f(z_0, y_0, x_0), f(y_0, x_0, y_0), f(x_0, y_0, z_0)), \\ & \quad (f(z_0, y_0, x_0), f(y_0, x_0, y_0), f(x_0, y_0, z_0))) \geq 1. \end{aligned} \tag{88}$$

Also, suppose that either

(a) *f and g are continuous, the pair (f, g) is compatible, and $(\mathcal{X}, G)$ is G_b-complete, or*

(b) *$(g(\mathcal{X}), G)$ is G_b-complete and assume that whenever $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ in X are sequences such that*

$$\begin{aligned} & \alpha((x_n, y_n, z_n), (x_{n+1}, y_{n+1}, z_{n+1}), (x_{n+1}, y_{n+1}, z_{n+1})) \geq 1, \\ & \alpha((y_n, x_n, y_n), (y_{n+1}, x_{n+1}, y_{n+1}), (y_{n+1}, x_{n+1}, y_{n+1})) \geq 1, \\ & \alpha((z_n, y_n, x_n), (z_{n+1}, y_{n+1}, x_{n+1}), (z_{n+1}, y_{n+1}, x_{n+1})) \geq 1 \end{aligned} \tag{89}$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$, $y_n \to y$, and $z_n \to z$ as $n \to +\infty$, we have

$$\begin{aligned} & \alpha((x_n, y_n, z_n), (x, y, z), (x, y, z)) \geq 1, \\ & \alpha((y_n, x_n, y_n), (y, x, y), (y, x, y)) \geq 1, \\ & \alpha((z_n, y_n, x_n), (z, y, x), (z, y, x)) \geq 1 \end{aligned} \tag{90}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Then, f and g have a tripled coincidence point in $\mathcal{X}$.

Theorem 49. *In addition to the hypotheses of Theorem 48, suppose that f and g are commutative and that, for all (x, y, z) and $(x^*, y^*, z^*) \in \mathcal{X}^3$, there exists $(u, v, w) \in \mathcal{X}^3$, such that*

$$\begin{aligned} & \alpha((gx, gy, gz), (gu, gv, gw), (gu, gv, gw)) \geq 1, \\ & \alpha((gx^*, gy^*, gz^*), (gu, gv, gw), (gu, gv, gw)) \geq 1. \end{aligned} \tag{91}$$

Then, f and g have a unique common tripled fixed point of the form (a, a, a).

Let

$$\Omega_3^a(X,U,A) = \frac{G(x,u,a)+G(y,v,b)+G(z,w,c)}{3},$$
$$X=(x,y,z),\quad U=(u,v,w),\quad A=(a,b,c)\in\mathcal{X}^3,$$
$$M_f^a(x,y,z,u,v,w,a,b,c) = \frac{G(f(x,y,z),f(u,v,w),f(a,b,c))}{3} + \frac{G(f(y,x,y),f(v,u,v),f(b,a,b))}{3} + \frac{G(f(z,y,x),f(w,v,u),f(c,b,a))}{3},$$
$$M_g^a(x,y,z,u,v,w,a,b,c) = \frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gw,gc)}{3}. \tag{92}$$

Remark 50. In Theorem 48, we can replace the contractive condition (87) by the following:

$$\alpha((gx,gy,gz),(gu,gv,gw),(ga,gb,gc)) \times \psi\left(sM_f^a(x,y,z,u,v,w,a,b,c)\right) \le \psi\left(M_g^a(x,y,z,u,v,w,a,b,c)\right) - \varphi\left(M_g^a(x,y,z,u,v,w,a,b,c)\right). \tag{93}$$

The following tripled fixed point results in ordered metric spaces can be obtained.

Theorem 51. *Let $(\mathcal{X},G,\preceq)$ be a partially ordered generalized b-metric space with the parameter s and let $f:\mathcal{X}^3\to\mathcal{X}$ and $g:\mathcal{X}\to\mathcal{X}$. Assume that*

$$\psi\left(sM_f^m(x,y,z,u,v,w,a,b,c)\right) \le \psi\left(M_g^m(x,y,z,u,v,w,a,b,c)\right) - \varphi\left(M_g^m(x,y,z,u,v,w,a,b,c)\right), \tag{94}$$

for all $x,y,z,u,v,w,a,b,c\in\mathcal{X}$ with $gx\preceq gu\preceq ga$, $gy\succeq gv\succeq gb$, and $gz\preceq gw\preceq gc$, where $\psi,\varphi:[0,\infty)\to[0,\infty)$ are altering distance functions.

Assume also that

(1) *f has the mixed g-monotone property and $f(\mathcal{X}^3)\subseteq g(\mathcal{X})$;*

(2) *there exist $x_0,y_0,z_0\in\mathcal{X}$ such that $gx_0\preceq f(x_0,y_0,z_0)$, $gy_0\succeq f(y_0,x_0,y_0)$, and $gz_0\preceq f(z_0,y_0,x_0)$.*

Also, suppose that either

(a) *f and g are continuous, the pair (f,g) is compatible, and $(\mathcal{X},G)$ is G_b-complete, or*

(b) *$(\mathcal{X},G)$ is regular and $(g(\mathcal{X}),G)$ is G_b-complete.*

Then, f and g have a tripled coincidence point in $\mathcal{X}$.

Theorem 52. *In addition to the hypotheses of Theorem 48, suppose that f and g are commutative and that, for all (x,y,z) and $(x^*,y^*,z^*)\in\mathcal{X}^3$, there exists $(u,v,w)\in\mathcal{X}^3$, such that (gu,gv,gw) is comparable with (gx,gy,gz) and (gx^*,gy^*,gz^*). Then, f and g have a unique common tripled fixed point of the form (a,a,a).*

Remark 53. In Theorem 51, we can replace the contractive condition (94) by the following:

$$\psi\left(sM_f^a(x,y,z,u,v,w,a,b,c)\right) \le \psi\left(M_g^a(x,y,z,u,v,w,a,b,c)\right) - \varphi\left(M_g^a(x,y,z,u,v,w,a,b,c)\right). \tag{95}$$

Remark 54. Theorem 51 extends Theorem 2.1 of [36] to a compatible pair.

Remark 55. Corollaries 2.2, 2.3, 2.6, 2.7, and 2.8 of [36] are special cases of Theorem 51.

Remark 56. Theorem 25 is a special case of Theorems 51.

6. Application to Integral Equations

As an application of the (coupled) fixed point theorems established in Section 4, we study the existence and uniqueness of a solution to a Fredholm nonlinear integral equation.

In order to compare our results to the ones in [37, 38] we will consider the same integral equation; that is,

$$x(t) = \int_a^b (K_1(t,s)+K_2(t,s))(f(s,x(s))+g(s,x(s)))\,ds + h(t), \tag{96}$$

where $t\in I=[a,b]$.

Let Θ denote the set of all functions $\theta:[0,\infty)\to[0,\infty)$ satisfying the following.

(i_θ) θ is nondecreasing and $(\theta(r))^p\le\theta(r^p)$, for all $p\ge 1$.

(ii_θ) There exists $\varphi\in\Phi$ such that $2\theta(r)=(r/2)-\varphi(r/2)$, for all $r\in[0,\infty)$.

Θ is nonempty, as $\theta_1(r)=2kr$ with $0\le 4k<1$ is an element of Θ.

Like in [38], we assume that the functions K_1, K_2, f, and g fulfill the following conditions.

Assumption 57. Consider the following:

(i) $K_1(t,s)\ge 0$ and $K_2(t,s)\le 0$, for all $t,s\in I$;

(ii) there exist two positive numbers λ and μ and $\theta \in \Theta$ such that, for all $x, y \in \mathbb{R}$ with $x \geq y$, the following Lipschitzian type conditions hold:

$$\begin{gathered} 0 \leq f(t, x) - f(t, y) \leq \lambda \theta (x - y), \\ -\mu \theta (x - y) \leq g(t, x) - g(t, y) \leq 0; \end{gathered} \tag{97}$$

(iii)

$$(\max\{\lambda^p, \mu^p\}) \cdot \sup_{t \in I} \left[\left(\int_a^b K_1(t, s)\, ds \right)^p + \left(\int_a^b -K_2(t, s)\, ds \right)^p \right] \leq \frac{1}{2^{3p-3}}. \tag{98}$$

Definition 58 (see [38]). A pair $(\alpha, \beta) \in X^2$ with $X = C(I, \mathbb{R})$ is called a coupled lower-upper solution of (96) if, for all $t \in I$,

$$\begin{aligned} \alpha(t) \leq & \int_a^b K_1(t, s) [f(s, \alpha(s)) + g(s, \beta(s))]\, ds \\ & + \int_a^b K_2(t, s) [f(s, \beta(s)) + g(s, \alpha(s))]\, ds + h(t), \\ \beta(t) \geq & \int_a^b K_1(t, s) [f(s, \beta(s)) + g(s, \alpha(s))]\, ds \\ & + \int_a^b K_2(t, s) [f(s, \alpha(s)) + g(s, \beta(s))]\, ds + h(t). \end{aligned} \tag{99}$$

Theorem 59. *Consider the integral equation* (96) *with*

$$K_1, K_2 \in C(I \times I, \mathbb{R}), \qquad h \in C(I, \mathbb{R}). \tag{100}$$

Suppose that there exists a coupled lower-upper solution (α, β) *of* (96) *with* $\alpha \leq \beta$ *and that Assumption 57 is satisfied. Then the integral equation* (96) *has a unique solution in* $C(I, \mathbb{R})$.

Proof. Consider on $X = C(I, \mathbb{R})$ the natural partial order relation; that is, for $x, y \in C(I, \mathbb{R})$,

$$x \leq y \Longleftrightarrow x(t) \leq y(t), \quad \forall t \in I. \tag{101}$$

It is well known that X is a complete metric space with respect to the sup metric:

$$d(x, y) = \sup_{t \in I} |x(t) - y(t)|, \qquad x, y \in C(I, \mathbb{R}). \tag{102}$$

Now for $p \geq 1$ we define

$$\begin{aligned} \rho(x, y) = d(x, y)^p &= \left(\sup_{t \in I} |x(t) - y(t)| \right)^p \\ &= \sup_{t \in I} |x(t) - y(t)|^p, \quad x, y \in C(I, \mathbb{R}). \end{aligned} \tag{103}$$

Define

$$G(x, y, z) = \max\{\rho(x, y), \rho(y, z), \rho(z, x)\}. \tag{104}$$

It is easy to see that (X, G) is a complete G_b-metric space with $s = 2^{p-1}$ (see Example 3).

Now define on X^2 the following partial order:
for all $(x, y), (u, v) \in X^2$

$$(x, y) \leq (u, v) \Longleftrightarrow x(t) \leq u(t), \quad y(t) \geq v(t) \quad \forall t \in I. \tag{105}$$

Obviously, for any $(x, y), (z, t) \in X^2$, the element $(\max\{F(x, y), F(z, t)\}, \min\{F(y, x), F(t, z)\})$ is comparable with $(F(x, y), F(y, x))$ and $(F(z, t), F(t, z))$.

Define now the mapping $F : X \times X \to X$ by

$$\begin{aligned} F(x, y)(t) = & \int_a^b K_1(t, s) [f(s, x(s)) + g(s, y(s))]\, ds \\ & + \int_a^b K_2(t, s) [f(s, y(s)) + g(s, x(s))]\, ds \\ & + h(t), \quad \forall t \in I. \end{aligned} \tag{106}$$

It is not difficult to prove, like in [38], that F has the mixed monotone property. Now for $x, y, u, v, a, b \in X$ with $x \geq u \geq a$ and $y \leq v \leq b$, we have

$$\rho(F(x, y), F(u, v)) = \sup_{t \in I} |F(x, y)(t) - F(u, v)(t)|^p. \tag{107}$$

Let us first evaluate the expression in the right hand side. According to the computations done by Berinde in [37],

$$\begin{aligned} & F(x, y)(t) - F(u, v)(t) \\ & = \int_a^b K_1(t, s) [f(s, x(s)) + g(s, y(s))]\, ds \\ & \quad + \int_a^b K_2(t, s) [f(s, y(s)) + g(s, x(s))]\, ds \\ & \quad - \int_a^b K_1(t, s) [f(s, u(s)) + g(s, v(s))]\, ds \\ & \quad - \int_a^b K_2(t, s) [f(s, v(s)) + g(s, u(s))]\, ds \\ & = \int_a^b K_1(t, s) [f(s, x(s)) - f(s, u(s)) + g(s, y(s)) \\ & \qquad - g(s, v(s))]\, ds \\ & \quad + \int_a^b K_2(t, s) [f(s, y(s)) - f(s, v(s)) + g(s, x(s)) \\ & \qquad - g(s, u(s))]\, ds \\ & = \int_a^b K_1(t, s) [(f(s, x(s)) - f(s, u(s))) \\ & \qquad - (g(s, v(s)) - g(s, y(s)))]\, ds \end{aligned}$$

$$
\begin{aligned}
&-\int_a^b K_2(t,s)\left[(f(s,y(s))-f(s,v(s)))\right.\\
&\qquad\left.-(g(s,u(s))-g(s,x(s)))\right]ds\\
&\leq\int_a^b K_1(t,s)\left[\lambda\theta(x(s)-u(s))+\mu\theta(v(s)-y(s))\right]ds\\
&\quad-\int_a^b K_2(t,s)\left[\lambda\theta(y(s)-v(s))+\mu\theta(u(s)-x(s))\right]ds.
\end{aligned}
\tag{108}
$$

Since the function θ is nondecreasing and $x \geq u$ and $y \leq v$, we have

$$
\begin{aligned}
\theta(x(s)-u(s)) &\leq \theta\left(\sup_{t\in I}|x(t)-u(t)|\right)=\theta(d(x,u)),\\
\theta(v(s)-y(s)) &\leq \theta\left(\sup_{t\in I}|y(t)-v(t)|\right)=\theta(d(y,v)).
\end{aligned}
\tag{109}
$$

Hence, by (108), in view of the fact that $K_2(t,s)\leq 0$, we obtain that

$$
\begin{aligned}
&|F(x,y)(t)-F(u,v)(t)|\\
&\quad\leq\int_a^b K_1(t,s)\left[\lambda\theta(d(x,u))+\mu\theta(d(y,v))\right]ds\\
&\quad\quad-\int_a^b K_2(t,s)\left[\lambda\theta(d(y,v))+\mu\theta(d(x,u))\right]ds,
\end{aligned}
\tag{110}
$$

as all quantities in the right hand side of (108) are nonnegative.

Now from (108) we have

$$
\begin{aligned}
&|F(x,y)(t)-F(u,v)(t)|^p\\
&\leq\left(\int_a^b K_1(t,s)\left[\lambda\theta(d(x,u))+\mu\theta(d(y,v))\right]ds\right.\\
&\qquad\left.-\int_a^b K_2(t,s)\left[\lambda\theta(d(v,y))+\mu\theta(d(x,u))\right]ds\right)^p\\
&\leq 2^{p-1}\left(\left(\int_a^b K_1(t,s)\,ds\right)^p(\lambda\theta(d(x,u))+\mu\theta(d(v,y)))^p\right.\\
&\qquad+\left(-\int_a^b K_2(t,s)\,ds\right)^p\\
&\qquad\left.\times\ (\lambda\theta(d(v,y))+\mu\theta(d(x,u)))^p\right)\\
&\leq 2^{p-1}\left(\left(\int_a^b K_1(t,s)\,ds\right)^p\right.\\
&\qquad\times 2^{p-1}\left(\lambda^p\theta(d(x,u))^p+\mu^p\theta(d(v,y))^p\right)\\
&\qquad+\left(-\int_a^b K_2(t,s)\,ds\right)^p\\
&\qquad\left.\times 2^{p-1}\left(\lambda^p\theta(d(v,y))^p+\mu^p\theta(d(x,u))^p\right)\right)\\
&\leq 2^{p-1}\left(\left(\int_a^b K_1(t,s)\,ds\right)^p\right.\\
&\qquad\times 2^{p-1}\left(\lambda^p\theta(\rho(x,u))+\mu^p\theta(\rho(v,y))\right)\\
&\qquad+\left(-\int_a^b K_2(t,s)\,ds\right)^p\\
&\qquad\left.\times 2^{p-1}\left(\lambda^p\theta(\rho(v,y))+\mu^p\theta(\rho(x,u))\right)\right)\\
&\leq 2^{2p-2}\left[\left(\lambda^p\left(\int_a^b K_1(t,s)\,ds\right)^p\right.\right.\\
&\qquad\left.+\mu^p\left(-\int_a^b K_2(t,s)\,ds\right)^p\right)\theta\rho(x,u)\\
&\qquad+\left(\mu^p\left(\int_a^b K_1(t,s)\,ds\right)^p\right.\\
&\qquad\left.\left.+\lambda^p\left(-\int_a^b K_2(t,s)\,ds\right)^p\right)\theta\rho(v,y)\right].
\end{aligned}
\tag{111}
$$

So, we have

$$
\begin{aligned}
&|F(x,y)(t)-F(u,v)(t)|^p\\
&\quad\leq 2^{2p-2}\left[\left(\lambda^p\left(\int_a^b K_1(t,s)\,ds\right)^p\right.\right.\\
&\qquad\left.+\mu^p\left(-\int_a^b K_2(t,s)\,ds\right)^p\right)\theta\rho(x,u)\\
&\qquad+\left(\mu^p\left(\int_a^b K_1(t,s)\,ds\right)^p\right.\\
&\qquad\left.\left.+\lambda^p\left(-\int_a^b K_2(t,s)\,ds\right)^p\right)\theta\rho(y,v)\right].
\end{aligned}
\tag{112}
$$

Similarly, one can obtain that

$$
\begin{aligned}
&|F(u,v)(t)-F(z,t)(t)|^p\\
&\quad\leq 2^{2p-2}\left[\left(\lambda^p\left(\int_a^b K_1(t,s)\,ds\right)^p\right.\right.
\end{aligned}
$$

$$+\mu^p\left(-\int_a^b K_2(t,s)\,ds\right)^p\Big)\theta\rho(u,z)$$

$$+\left(\mu^p\left(\int_a^b K_1(t,s)\,ds\right)^p\right.$$

$$\left.+\lambda^p\left(-\int_a^b K_2(t,s)\,ds\right)^p\right)\theta\rho(v,t)\Bigg],$$

$$\begin{aligned}
&\left|F(z,t)(t)-F(x,y)(t)\right|^p\\
&\le 2^{2p-2}\Bigg[\left(\lambda^p\left(\int_a^b K_1(t,s)\,ds\right)^p\right.\\
&\quad\left.+\mu^p\left(-\int_a^b K_2(t,s)\,ds\right)^p\right)\theta\rho(x,z)\\
&\quad+\left(\mu^p\left(\int_a^b K_1(t,s)\,ds\right)^p\right.\\
&\quad\left.+\lambda^p\left(-\int_a^b K_2(t,s)\,ds\right)^p\right)\theta\rho(y,t)\Bigg].
\end{aligned} \tag{113}$$

Taking the supremum with respect to t and using (98) we get

$$\begin{aligned}
&G(F(x,y),F(u,v),F(z,t))\\
&=\max\Big\{\sup_{t\in I}\left|F(x,y)(t)-F(u,v)(t)\right|^p,\\
&\qquad\sup_{t\in I}\left|F(u,v)(t)-F(z,t)(t)\right|^p,\\
&\qquad\sup_{t\in I}\left|F(z,t)(t)-F(x,y)(t)\right|^p\Big\}\\
&\le 2^{2p-2}\left(\max\{\lambda^p,\mu^p\}\right)\\
&\quad\times\sup_{t\in I}\left[\left(\int_a^b K_1(t,s)\,ds\right)^p+\left(\int_a^b -K_2(t,s)\,ds\right)^p\right]\\
&\quad\times\max\{\theta(\rho(x,u))+\theta(\rho(y,v)),\theta(\rho(u,z))\\
&\qquad+\theta(\rho(v,t)),\theta(\rho(x,z))+\theta(\rho(y,t))\}\\
&\le 2^{2p-2}\left(\max\{\lambda^p,\mu^p\}\right)\\
&\quad\times\sup_{t\in I}\left[\left(\int_a^b K_1(t,s)\,ds\right)^p+\left(\int_a^b -K_2(t,s)\,ds\right)^p\right]\\
&\quad\times\theta(G(x,u,z))+\theta(G(y,v,t))\\
&\le\frac{2^{2p-2}}{2^{3p-3}}\left[\theta(G(x,u,z))+\theta(G(y,v,t))\right]\\
&=\frac{1}{2^{p-1}}\left[\theta(G(x,u,z))+\theta(G(y,v,t))\right].
\end{aligned} \tag{114}$$

Now, since θ is nondecreasing, we have

$$\begin{aligned}
\theta(G(x,u,z))&\le\theta(G(x,u,z)+G(y,v,t)),\\
\theta(G(y,v,t))&\le\theta(G(x,u,z)+G(y,v,t))
\end{aligned} \tag{115}$$

and so

$$\begin{aligned}
&\theta(G(x,u,z))+\theta(G(y,v,t))\\
&\quad\le 2\theta(G(x,u,z)+G(y,v,t))\\
&\quad=\frac{G(x,u,z)+G(y,v,t)}{2}\\
&\qquad-\varphi\left(\frac{G(x,u,z)+G(y,v,t)}{2}\right),
\end{aligned} \tag{116}$$

by the definition of θ. Finally, from (114) we get that

$$\begin{aligned}
&G(F(x,y),F(u,v),F(z,t))\\
&\quad\le\frac{1}{2^{p-1}}\frac{G(x,u,z)+G(y,v,t)}{2}\\
&\qquad-\frac{1}{2^{p-1}}\varphi\left(\frac{G(x,u,z)+G(y,v,t)}{2}\right).
\end{aligned} \tag{117}$$

Similarly, we can obtain that

$$\begin{aligned}
&G(F(y,x),F(v,u),F(t,z))\\
&\quad\le\frac{1}{2^{p-1}}\frac{G(x,u,z)+G(y,v,t)}{2}\\
&\qquad-\frac{1}{2^{p-1}}\varphi\left(\frac{G(x,u,z)+G(y,v,t)}{2}\right),
\end{aligned} \tag{118}$$

which is just the contractive condition (69) in Corollary 43.

Now, let $(\alpha,\beta)\in X^2$ be a coupled upper-lower solution of (96). Then we have

$$\alpha(t)\le F(\alpha,\beta)(t),\qquad \beta(t)\ge F(\beta,\alpha)(t), \tag{119}$$

for all $t\in I$, which show that all hypotheses of Corollary 43 are satisfied.

This proves that F has a coupled fixed point (x_*,y_*) in X^2.

Since $\alpha\le\beta$, by Corollary 43 it follows that $x_*=y_*$; that is,

$$x_*=F(x_*,x_*), \tag{120}$$

and therefore $x_*\in C(I,\mathbb{R})$ is the solution of the integral equation (96). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publications of this paper.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

Acknowledgments

This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. Therefore, the first and second authors express their thanks to DSR, KAU for financial support.

References

[1] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.

[2] A. Aghajani, M. Abbas, and J. R. Roshan, "Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces," *Filomat*. In press.

[3] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[4] Z. Mustafa, J. R. Roshan, and V. Parvaneh, "Coupled coincidence point results for (ψ, ϕ)-weakly contractive mappings in partially ordered G_b-metric spaces," *Fixed Point Theory and Applications*, vol. 2013, article 206, 2013.

[5] N. Hussain, D. Dorić, Z. Kadelburg, and S. Radenović, "Suzuki-type fixed point results in metric type spaces," *Fixed Point Theory and Applications*, vol. 2012, article 126, 2012.

[6] H. Aydi, M. Postolache, and W. Shatanawi, "Coupled fixed point results for (ψ, φ)-weakly contractive mappings in ordered G-metric spaces," *Computers & Mathematics with Applications*, vol. 63, no. 1, pp. 298–309, 2012.

[7] T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.

[8] Y. J. Cho, M. H. Shah, and N. Hussain, "Coupled fixed points of weakly F-contractive mappings in topological spaces," *Applied Mathematics Letters*, vol. 24, no. 7, pp. 1185–1190, 2011.

[9] B. S. Choudhury and A. Kundu, "A coupled coincidence point result in partially ordered metric spaces for compatible mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 8, pp. 2524–2531, 2010.

[10] B. S. Choudhury and P. Maity, "Coupled fixed point results in generalized metric spaces," *Mathematical and Computer Modelling*, vol. 54, no. 1-2, pp. 73–79, 2011.

[11] L. Ćirić, B. Damjanović, M. Jleli, and B. Samet, "Coupled fixed point theorems for generalized Mizoguchi-Takahashi contractions with applications," *Fixed Point Theory and Applications*, vol. 2012, article 51, 2012.

[12] H.-S. Ding, L. Li, and S. Radenović, "Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 96, 2012.

[13] D. J. Guo and V. Lakshmikantham, "Coupled fixed points of nonlinear operators with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 11, no. 5, pp. 623–632, 1987.

[14] N. Hussain, M. Abbas, A. Azam, and J. Ahmad, "Coupled coincidence point results for a generalized compatible pair with applications," *Fixed Point Theory and Applications*, vol. 2014, article 62, 2014.

[15] N. Hussain, P. Salimi, and S. al-Mezel, "Coupled fixed point results on quasi-Banach spaces with application to a system of integral equations," *Fixed Point Theory and Applications*, vol. 2013, article 261, 2013.

[16] N. Hussain, A. Latif, and M. H. Shah, "Coupled and tripled coincidence point results without compatibility," *Fixed Point Theory and Applications*, vol. 2012, article 77, 2012.

[17] V. Lakshmikantham and L. Ćirić, "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 12, pp. 4341–4349, 2009.

[18] N. V. Luong and N. X. Thuan, "Coupled fixed points in partially ordered metric spaces and application," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 3, pp. 983–992, 2011.

[19] J. J. Nieto and R. Rodríguez-López, "Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations," *Acta Mathematica Sinica*, vol. 23, no. 12, pp. 2205–2212, 2007.

[20] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.

[21] J. R. Roshan, V. Parvaneh, and I. Altun, "Some coincidence point results in ordered b-metric spaces and applications in a system of integral equations," *Applied Mathematics and Computation*, vol. 226, pp. 725–737, 2014.

[22] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei, and W. Shatanawi, "Common fixed points of almost generalized $(\psi, \varphi)_s$-contractive mappings in ordered b-metric spaces," *Fixed Point Theory and Applications*, vol. 2013, article 159, 2013.

[23] M. Mursaleen, S. A. Mohiuddine, and R. P. Agarwal, "Coupled fixed point theorems for α-ψ-contractive type mappings in partially ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 228, 2012.

[24] V. Parvaneh, J. R. Roshan, and S. Radenović, "Existence of tripled coincidence points in ordered b-metric spaces and an application to a system of integral equations," *Fixed Point Theory and Applications*, vol. 2013, article 130, 2013.

[25] V. Berinde and M. Borcut, "Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 15, pp. 4889–4897, 2011.

[26] M. Borcut, "Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces," *Applied Mathematics and Computation*, vol. 218, no. 14, pp. 7339–7346, 2012.

[27] M. Borcut and V. Berinde, "Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces," *Applied Mathematics and Computation*, vol. 218, no. 10, pp. 5929–5936, 2012.

[28] B. S. Choudhury, E. Karapınar, and A. Kundu, "Tripled coincidence point theorems for nonlinear contractions in partially ordered metric spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, Article ID 329298, 14 pages, 2012.

[29] Y. J. Cho, B. E. Rhoades, R. Saadati, B. Samet, and W. Shatanawi, "Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type," *Fixed Point Theory and Applications*, vol. 2012, article 8, 2012.

[30] H. Aydi, E. Karapınar, and W. Shatanawi, "Tripled coincidence point results for generalized contractions in ordered generalized metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 101, 2012.

[31] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for α-ψ-contractive type mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[32] M. A. Alghamdi and E. Karapınar, "G-β-ψ contractive-type mappings and related fixed point theorems," *Journal of Inequalities and Applications*, vol. 2013, article 70, 2013.

[33] N. Hussain, S. al-Mezel, and P. Salimi, "Fixed points for ψ-graphic contractions with application to integral equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 575869, 11 pages, 2013.

[34] E. Karapınar, P. Kumam, and P. Salimi, "On α-ψ-Meir-Keeler contractive mappings," *Fixed Point Theory and Applications*, vol. 2013, article 94, 2013.

[35] N. Hussain, E. Karapınar, P. Salimi, and P. Vetro, "Fixed point results for G^m-Meir-Keeler contractive and G-(α, ψ)-Meir-Keeler contractive mappings," *Fixed Point Theory and Applications*, vol. 2013, article 34, 2013.

[36] Z. Mustafa, J. R. Roshan, and V. Parvaneh, "Existence of tripled coincidence point in ordered G_b-metric spaces and applications to a system of integral equations," *Journal of Inequalities and Applications*, vol. 2013, article 453, 2013.

[37] V. Berinde, "Coupled fixed point theorems for ϕ-contractive mixed monotone mappings in partially ordered metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 6, pp. 3218–3228, 2012.

[38] N. V. Luong and N. X. Thuan, "Coupled fixed point theorems in partially ordered G-metric spaces," *Mathematical and Computer Modelling*, vol. 55, no. 3-4, pp. 1601–1609, 2012.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 363925, 7 pages
http://dx.doi.org/10.1155/2014/363925

Research Article

Generalized Common Fixed Point Results with Applications

Marwan Amin Kutbi,[1] Muhammad Arshad,[2] Jamshaid Ahmad,[3] and Akbar Azam[3]

[1] *Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia*
[2] *Department of Mathematics & Statistics, International Islamic University H-10, Islamabad, Pakistan*
[3] *Department of Mathematics, COMSATS Institute of Information Technology, Chak Shahzad, Islamabad 44000, Pakistan*

Correspondence should be addressed to Muhammad Arshad; marshad_zia@yahoo.com

Received 10 February 2014; Accepted 23 March 2014; Published 27 April 2014

Academic Editor: Ljubomir B. Ćirić

We obtained some generalized common fixed point results in the context of complex valued metric spaces. Moreover, we proved an existence theorem for the common solution for two Urysohn integral equations. Examples are presented to support our results.

1. Introduction and Preliminaries

Since the appearance of the Banach contraction mapping principle, a number of papers were dedicated to the improvement and generalization of that result. Most of these deal with the generalizations of the contractive condition in metric spaces.

Gähler [1] generalized the idea of metric space and introduced a 2-metric space which was followed by a number of papers dealing with this generalized space. Plenty of material is also available in other generalized metric spaces, such as, rectangular metric spaces, semimetric spaces, pseudometric spaces, probabilistic metric spaces, fuzzy metric spaces, quasimetric spaces, quasisemi metric spaces, D-metric spaces, G-metric space, partial metric space, and cone metric spaces (see [2–14]). Azam et al. [15] improved the Banach contraction principle by generalizing it in complex valued metric space involving rational inequity which could not be handled in cone metric spaces [3, 5, 11, 15] due to limitations regarding product and quotient. Rouzkard and Imdad [16] extended the work of Azam et al. [15]. Sintunavarat and Kumam [17] obtained common fixed point results by replacing constant of contractive condition to control functions. Recently, Klin-eam and Suanoom [12] extend the concept of complex valued metric spaces and generalized the results of Azam et al. [15] and Rouzkard and Imdad [16]. In this paper we continue the study of complex valued metric spaces and established some fixed point results for mappings satisfying a rational inequality. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces and then it will bring wonderful research activities in nonlinear analysis.

In this paper we continue our investigations initiated by Azam et al. [15] and prove a common fixed point result for two mappings and applied it to get the coincidence and common fixed points of three and four mappings.

We begin with listing some notations, definitions, and basic facts on these topics that we will need to convey our theorems. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$\text{iff } z_1 \precsim z_2, \quad \operatorname{Re}(z_1) \leqslant \operatorname{Re}(z_2), \quad \operatorname{Im}(z_1) \leqslant \operatorname{Im}(z_2). \tag{1}$$

It follows that

$$z_1 \precsim z_2 \tag{2}$$

if one of the following conditions is satisfied:

$$\begin{aligned}
&\text{(i) } \operatorname{Re}(z_1) = \operatorname{Re}(z_2), && \operatorname{Im}(z_1) < \operatorname{Im}(z_2),\\
&\text{(ii) } \operatorname{Re}(z_1) < \operatorname{Re}(z_2), && \operatorname{Im}(z_1) = \operatorname{Im}(z_2),\\
&\text{(iii) } \operatorname{Re}(z_1) < \operatorname{Re}(z_2), && \operatorname{Im}(z_1) < \operatorname{Im}(z_2),\\
&\text{(iv) } \operatorname{Re}(z_1) = \operatorname{Re}(z_2), && \operatorname{Im}(z_1) = \operatorname{Im}(z_2).
\end{aligned} \tag{3}$$

In particular, we will write $z_1 \precsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \precsim z_1 \precsim z_2 \Longrightarrow |z_1| < |z_2|, \qquad z_1 \preceq z_2, \quad z_2 \prec z_3 \Longrightarrow z_1 \prec z_3. \tag{4}$$

Definition 1. Let X be a nonempty set. Suppose that the self-mapping $d : X \times X \to \mathbb{C}$ satisfies:

(1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space. A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A. \tag{5}$$

A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$,

$$B(x, r) \cap (A \setminus \{x\}) \neq \phi. \tag{6}$$

A is called open whenever each element of A is an interior point of A. Moreover, a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F = \{B(x, r) : x \in X, 0 \prec r\} \tag{7}$$

is a subbasis for a Hausdorff topology τ on X.

Let x_n be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x, and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space. Let X be a nonempty set and $T, f : X \to X$. The mappings T, f are said to be weakly compatible if they commute at their coincidence point (i.e., $Tfx = fTx$ whenever $Tx = fx$). A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$. We require the following lemmas.

Lemma 2 (see [15]). *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 3 (see [15]). *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

Definition 4 (see [18]). Two families of self-mappings $\{T_i\}_1^m$ and $\{S_i\}_1^n$ are said to be pairwise commuting if:

(1) $T_iT_j = T_jT_i$, $i, j \in \{1, 2, \ldots, m\}$;

(2) $S_kS_l = S_lS_k$, $k, l \in \{1, 2, \ldots, n\}$;

(3) $T_iS_k = S_kT_i$, $i \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, n\}$.

Lemma 5 (see [19]). *Let X be a nonempty set and $f : X \to X$ a function. Then there exists a subset $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one to one.*

Lemma 6 (see [20]). *Let X be a nonempty set and the mappings $S, T, f : X \to X$ have a unique point of coincidence v in X. If (S, f) and (T, f) are weakly compatible, then fv is a unique common fixed point of S, T, f.*

2. Main Results

Theorem 7. *Let (X, d) be a complete complex valued metric space and $0 \le h < 1$. If the self-mappings $S, T : X \to X$ satisfy*

$$d(Sx, Ty) \precsim hL(x, y) \tag{8}$$

for all $x, y \in X$, where

$$L(x, y) \in \left\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Sx)\,d(y, Ty)}{1 + d(x, y)}\right\} \tag{9}$$

then S and T have a unique common fixed point.

Proof. We will first show that fixed point of one map is a fixed point of the other. Suppose that $p = Tp$. Then from (8)

$$d(Sp, p) = d(Sp, Tp) \precsim hL(p, p). \tag{10}$$

Case 1

$$d(Sp, p) \precsim hd(p, p) = 0, \quad p = Sp. \tag{11}$$

Case 2

$$d(Sp, p) \precsim hd(p, Sp), \tag{12}$$

which yields that $p = Sp$.

Case 3

$$d(Sp, p) \precsim hd(p, Tp) = 0, \quad p = Sp. \tag{13}$$

Case 4

$$d(Sp, p) \precsim h\left[\frac{d(p, Sp)\,d(p, Tp)}{1 + d(p, p)}\right], \tag{14}$$

which implies that $d(Sp, p) \precsim 0$, and hence $p = Sp$. In a similar manner it can be shown that any fixed point of S is also the fixed point of T. Let $x_0 \in X$and define

$$x_{2n+1} = Sx_{2n}, \qquad x_{2n+2} = Tx_{2n+1}, \quad n \ge 0. \tag{15}$$

We will assume that $x_n \neq x_{n+1}$ for each n. Otherwise, there exists an n such that $x_{2n} = x_{2n+1}$. Then $x_{2n} = Sx_{2n}$ and x_{2n} is a fixed point of S, hence a fixed point of T. Similarly, if $x_{2n+1} = x_{2n+2}$ for some n, then x_{2n+1} is common fixed point of T and hence of S. From (8)

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}) \leq hL(x_{2n}, x_{2n+1}). \quad (16)$$

Case 1

$$|d(x_{2n+1}, x_{2n+2})| = |d(Sx_{2n}, Tx_{2n+1})| \leq h|d(x_{2n}, x_{2n+1})|. \quad (17)$$

Case 2

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| &= |d(Sx_{2n}, Tx_{2n+1})| \\ &\leq h|d(x_{2n}, Sx_{2n})| \\ &= h|d(x_{2n}, x_{2n+1})|. \end{aligned} \quad (18)$$

Case 3

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| &= |d(Sx_{2n}, Tx_{2n+1})| \\ &\leq h|d(x_{2n+1}, Tx_{2n+1})| \\ &= h|d(x_{2n+1}, x_{2n+2})|, \end{aligned} \quad (19)$$

which implies that

$$x_{2n+1} = x_{2n+2}, \quad (20)$$

a contradiction to our assumption.

Case 4

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| &= |d(Sx_{2n}, Tx_{2n+1})| \\ &\leq h\left|\frac{d(x_{2n}, Sx_{2n})\, d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})}\right| \\ &\leq h\left|\frac{d(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1})}\right| |d(x_{2n+1}, x_{2n+2})| \\ &\leq h|d(x_{2n+1}, x_{2n+2})|. \end{aligned} \quad (21)$$

That is

$$x_{2n+1} = x_{2n+2}, \quad (22)$$

a contradiction to our assumption.

Thus, $|d(x_{2n+1}, x_{2n+2})| \leq h|d(x_{2n}, x_{2n+1})|$. Similarly, one can show that $|d(x_{2n+2}, x_{2n+3})| \leq h|d(x_{2n+1}, x_{2n+2})|$. It follows that, for all n,

$$\begin{aligned} |d(x_n, x_{n+1})| &\leq h|d(x_{n-1}, x_n)| \\ &\leq h^2|d(x_{n-2}, x_{n-1})| \leq \cdots \leq h^n|d(x_0, x_1)|. \end{aligned} \quad (23)$$

Now for any $m > n$,

$$\begin{aligned} |d(x_m, x_n)| &\leq |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| \\ &\quad + \cdots + |d(x_{m-1}, x_m)| \\ &\leq \left[h^n + h^{n+1} + \cdots + h^{m-1}\right] |d(x_0, x_1)| \\ &\leq \left[\frac{h^n}{1-h}\right] |d(x_0, x_1)| \end{aligned} \quad (24)$$

and so

$$|d(x_m, x_n)| \leq \frac{h^n}{1-h}|d(x_0, x_1)| \longrightarrow 0, \quad \text{as } m, n \longrightarrow \infty. \quad (25)$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u$. It follows that $u = Su$; otherwise $d(u, Su) = z \succ 0$ and we would then have

$$\begin{aligned} z &\precsim d(u, x_{2n+2}) + d(x_{2n+2}, Su) \\ &\precsim d(u, x_{2n+2}) + d(Tx_{2n+1}, Su) \\ &\precsim d(u, x_{2n+2}) + hL(u, x_{2n+1}). \end{aligned} \quad (26)$$

Case 1

$$|z| \leq |d(u, x_{2n+2})| + h|d(u, x_{2n+1})|. \quad (27)$$

That is, $|z| \leq 0$, a contradiction and hence $u = Su$.

Case 2

$$|z| \leq |d(u, x_{2n+2})| + h|d(u, Su)|. \quad (28)$$

That is, $|z| \leq 0$, a contradiction and hence $u = Su$.

Case 3

$$\begin{aligned} |z| &\leq |d(u, x_{2n+2})| + h|d(x_{2n+1}, Tx_{2n+1})| \\ &\leq |d(u, x_{2n+2})| + h|d(x_{2n+1}, x_{2n+2})| \\ &\leq |d(u, x_{2n+2})| + h^{2n+2}|d(x_0, x_1)|. \end{aligned} \quad (29)$$

This in turn gives us $|z| \leq 0$, a contradiction and hence $u = Su$.

Case 4

$$\begin{aligned} |z| &\leq |d(u, x_{2n+2})| + h\left|\frac{d(u, Su)\, d(x_{2n+1,}Tx_{2n+1})}{1 + d(u, x_{2n+1})}\right| \\ &\leq d(u, x_{2n+2}) + h\frac{|d(u, Su)|\,|d(x_{2n+1,}x_{2n+2})|}{|1 + d(u, x_{2n+1})|} \\ &\leq |d(u, x_{2n+2})| + h^{2n+2}\frac{|d(u, Su)|\,|d(x_{0,}x_1)|}{|1 + d(u, x_{2n+1})|}. \end{aligned} \quad (30)$$

That is, $|z| \leq 0$ and hence $u = Su$. It follows similarly that $u = Tu$. We now show that S and T have unique common fixed point. For this, assume that u^* in X is another common fixed point of S and T. Then $d(u,u^*) = d(Su,Tu^*) \precsim hL(u,u^*)$.

Case 1

$$d(u,u^*) \precsim hd(u,u^*). \tag{31}$$

Case 2

$$d(u,u^*) \precsim hd(u,Su) \precsim hd(u,u) = 0. \tag{32}$$

This gives us $u = u^*$.

Case 3

$$d(u,u^*) \precsim hd(u^*,Tu^*) = hd(u^*,u^*) = 0. \tag{33}$$

Case 4

$$d(u,u^*) \precsim \frac{hd(u,Su)\,d(u^*,Tu^*)}{1+d(u,u^*)} = 0. \tag{34}$$

Hence, in all cases $u^* = u$. This completes the proof of the theorem. □

Corollary 8 (see [15]). *Let (X,d) be a complete complex valued metric space and let $S,T : X \to X$ and $0 < h < 1$. If the self-mappings S, Tsatisfy*

$$d(Sx,Ty) \precsim hL(x,y) \tag{35}$$

for all $x, y \in X$, where

$$L(x,y) \in \left\{d(x,y), \frac{d(x,Sx)\,d(y,Ty)}{1+d(x,y)}\right\}, \tag{36}$$

then S and T have a unique common fixed point.

Corollary 9. *Let (X,d) be a complete complex valued metric space and $0 \le h < 1$. If the self-mapping $T : X \to X$ satisfies*

$$d(Tx,Ty) \precsim hL(x,y) \tag{37}$$

for all $x, y \in X$, where

$$L(x,y) \in \left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)\,d(y,Ty)}{1+d(x,y)}\right\} \tag{38}$$

then T has a unique fixed point.

Corollary 10 (see [15]). *Let (X,d) be a complete complex valued metric space and let $T : X \to X$ and $0 \le h < 1$. If the self-mapping Tsatisfies*

$$d(Tx,Ty) \precsim hL(x,y) \tag{39}$$

for all $x, y \in X$, where

$$L(x,y) \in \left\{d(x,y), \frac{d(x,Tx)\,d(y,Ty)}{1+d(x,y)}\right\}, \tag{40}$$

then T has a unique fixed point.

As an application of Theorem 7, we prove the following theorem for two finite families of mappings.

Theorem 11. *If $\{T_i\}_1^m$ and $\{S_i\}_1^n$ are two finite pairwise commuting finite families of self-mapping defined on a complete complex valued metric space (X,d) such that the mappings S and T (with $T = T_1,T_2,\ldots,T_m$ and $S = S_1,S_2,\ldots,S_n$) satisfy the contractive condition (8), then the component maps of the two families $\{T_i\}_1^m$ and $\{S_i\}_1^n$ have a unique common fixed point.*

Proof. From theorem we can say that the mappings T and S have a unique common fixed point z; that is, $Tz = Sz = z$. Now our requirement is to show that z is a common fixed point of all the component mappings of both families. In view of pairwise commutativity of the families $\{T_i\}_1^m$ and $\{S_i\}_1^n$, (for every $1 \le k \le m$) we can write $T_kz = T_kTz = TT_kz$ and $T_kz = T_kSz = ST_kz$ which show that T_kz (for every k) is also a common fixed point of T and S. By using the uniqueness of common fixed point, we can write $T_kz = z$ (for every k) which shows that z is a common fixed point of the family $\{T_i\}_1^m$. Using the same argument one can also show that (for every $1 \le k \le n$) $S_kz = z$. Thus component maps of the two families $\{T_i\}_1^m$ and $\{S_i\}_1^n$ have a unique common fixed point. □

By setting $T_1 = T_2 = \cdots = T_m = F$ and $S_1 = S_2 = \cdots = S_n = G$, in Theorem 11, we get the following corollary.

Corollary 12. *If F and G are two commuting self-mappings defined on a complete complex valued metric space (X,d) satisfying the condition*

$$d(F^mx,G^ny) \precsim hL(x,y) \tag{41}$$

for all $x, y \in X$ and $0 \le h < 1$, where

$$L(x,y) \in \left\{d(x,y), d(x,F^mx), d(y,G^ny), \frac{d(x,F^mx)\,d(y,G^ny)}{1+d(x,y)}\right\}, \tag{42}$$

then F and G have a unique common fixed point.

Corollary 13. *Let (X,d) be a complete complex valued metric space and let $T : X \to X$ be a self-mapping satisfying*

$$d(T^nx,T^ny) \precsim hL(x,y) \tag{43}$$

for all $x, y \in X$ and $0 \le h < 1$, where

$$L(x,y) \in \left\{d(x,y), d(x,T^mx), d(y,T^ny), \frac{d(x,T^nx)\,d(y,T^ny)}{1+d(x,y)}\right\}. \tag{44}$$

Then T has a unique fixed point.

Corollary 14 (see [15]). *Let (X,d) be a complete complex valued metric space and $T : X \to X$ and $0 \le h < 1$. The self-mapping T satisfies*

$$d(T^nx,T^ny) \precsim hL(x,y) \tag{45}$$

for all $x, y \in X$, where

$$L(x,y) \in \left\{ d(x,y), \frac{d(x,T^n x)\, d(y,T^n y)}{1+d(x,y)} \right\}. \tag{46}$$

Then T has a unique fixed point.

Our next example exhibits the superiority of Corollary 13 over Corollary 9.

Example 15. Let $X_1 = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1, \operatorname{Im} z = 0\}$ and $X_2 = \{z \in \mathbb{C} : 0 \le \operatorname{Im} z \le 1, \operatorname{Re} z = 0\}$ and let $X = X_1 \cup X_2$. Then with $z = x + iy$, set $S = T$ and define $T : X \to X$ as follows:

$$T(x,y) = \begin{cases} (0,0) & \text{if } x, y \in Q \\ (1,0) & \text{if } x \in Q^c,\ y \in Q \\ (0,1) & \text{if } x \in Q,\ y \in Q^c \\ (1,1) & \text{if } x, y \in Q^c. \end{cases} \tag{47}$$

Consider a complex valued metric $d : X \times X \to \mathbb{C}$ as follows:

$$d(z_1,z_2) = \begin{cases} \frac{2i}{3}\left|x_1 - x_2\right|, & \text{if } z_1, z_2 \in X_1 \\ \frac{i}{3}\left|y_1 - y_2\right|, & \text{if } z_1, z_2 \in X_2 \\ i\left(\frac{2}{3}x_1 + \frac{1}{3}y_2\right), & \text{if } z_1 \in X_1,\ z_2 \in X_2 \\ i\left(\frac{1}{3}y_1 + \frac{2}{3}x_2\right) & \text{if } z_1 \in X_2,\ z_2 \in X_1, \end{cases} \tag{48}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in X$. Then (X, d) is a complete complex valued metric space. By a routine calculation, one can verify that the map T^2 satisfies condition (43) with $\lambda = (1/3)$(say). It is interesting to notice that this example cannot be covered by Corollary 9 as $z_1 = (1,0)$, $z_2 = (1/2, 0) \in X$ implies

$$\frac{2i}{3} = d(Tz_1, Tz_2) \le d(z_1, z_2) = \frac{i}{3} \tag{49}$$

a contradiction for every choice of λ which amounts to say that condition (37) is not satisfied. Notice that the point $0 \in X$ remains fixed under T and T^2 and is indeed unique.

3. Application

By providing the following result, we establish an existence theorem for the common solution for two Urysohn integral equations.

Theorem 16. *Let $X = C([a,b], \mathbb{R}^n)$, $a > 0$, and $d : X \times X \to \mathbb{C}$ is defined as follows:*

$$d(x,y) = \max_{t\in[a,b]} \left\| x(t) - y(t) \right\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}. \tag{50}$$

Consider the Urysohn integral equations

$$x(t) = \int_a^b K_1(t,s,x(s))\, ds + g(t), \tag{α}$$

$$x(t) = \int_a^b K_2(t,s,x(s))\, ds + h(t), \tag{β}$$

where $t \in [a,b] \subset \mathbb{R}$, $x, g, h \in X$.

Suppose that $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $F_x, G_x \in X$ for each $x \in X$, where,

$$\begin{aligned} F_x(t) &= \int_a^b K_1(t,s,x(s))\, ds, \\ G_x(t) &= \int_a^b K_2(t,s,x(s))\, ds, \quad \forall t \in [a,b]. \end{aligned} \tag{51}$$

If there exists $0 \le h < 1$ such that for every $x, y \in X$

$$\begin{aligned} &\left\| F_x(t) - G_y(t) + g(t) - h(t) \right\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a} \\ &\quad \precsim hL(x,y)(t), \end{aligned} \tag{52}$$

where

$$\begin{aligned} &L(x,y)(t) \\ &\quad \in \{A(x,y)(t), B(x,y)(t), C(x,y)(t), D(x,y)(t)\}, \\ &A(x,y)(t) = \|x(t) - y(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}, \\ &B(x,y)(t) = \|F_x(t) + g(t) - x(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}, \\ &C(x,y)(t) = \|G_y(t) + h(t) - y(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}, \\ &D(x,y)(t) \\ &\quad = \frac{\|F_x(t) + g(t) - x(t)\|_\infty \|G_y(t) + h(t) - y(t)\|_\infty}{1 + \max_{t\in[a,b]} A(x,y)(t)} \\ &\quad \times \sqrt{1+a^2} e^{i \tan^{-1} a}, \end{aligned} \tag{53}$$

then the system of integral equations (α) and (β) has a unique common solution.

Proof. Define $S, T : X \to X$ by

$$Sx = F_x + g, \qquad Tx = G_x + h. \tag{54}$$

Then

$$\begin{aligned} d(Sx,Ty) &= \max_{t\in[a,b]}\left\|F_x(t)-G_y(t)+g(t)-h(t)\right\|_\infty \\ &\quad\times\sqrt{1+a^2}e^{i\tan^{-1}a}, \\ d(x,y) &= \max_{t\in[a,b]}A(x,y)(t), \\ d(x,Sx) &= \max_{t\in[a,b]}B(x,y)(t), \\ d(y,Ty) &= \max_{t\in[a,b]}C(x,y)(t), \\ \frac{d(x,Sx)\,d(y,Ty)}{1+d(x,y)} &= \max_{t\in[a,b]}D(x,y)(t). \end{aligned} \tag{55}$$

It is easily seen that $d(Sx,Ty)\precsim hL(x,y)$, where

$$L(x,y) \in \left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Sx)\,d(y,Ty)}{1+d(x,y)}\right\} \tag{56}$$

for every $x,y\in X$. By Theorem 7, the Urysohn integral equations (α) and (β) have a unique common solution. □

Remark 17. Now we will apply techniques of [6] to obtain the common fixed points of three and four mappings by using a common fixed point result for two mappings.

Theorem 18. *Let (X,d) be a complete complex valued metric space and $0\le h<1$. Let $S,T,f:X\to X$ by the self-mappings such that $SX\cup TX\subset fX$. Assume that the following holds:*

$$d(Sx,Ty)\precsim hL(x,y) \tag{57}$$

for all $x,y\in X$, where

$$L(x,y)\in\left\{d(fx,fy), d(fx,Sx), d(fy,Ty), \frac{d(fx,Sx)\,d(fy,Ty)}{1+d(fx,fy)}\right\}. \tag{58}$$

If (S,f) and (T,f) are weakly compatible and fX is closed, then S, T, and f have a unique common fixed point in X.

Proof. By Lemma 5, there exists $E\subset X$ such that $fE=fX$ and $f:E\to X$ is one to one. Now define the self-mappings $g,h:fE\to fE$ by $g(fx)=Sx$ and $h(fx)=Tx$, respectively. Since f is one to one on E, then g, h are well defined. Note that

$$d(g(fx),h(fx))\precsim hL(x,y), \tag{59}$$

where

$$L(x,y)\in\left\{d(fx,fy), d(fx,g(fx)), d(fy,h(fy)), \frac{d(fx,g(fx))\,d(fy,h(fy))}{1+d(fx,fy)}\right\}. \tag{60}$$

By Theorem 7 as fE is complete, we deduce that there exists a unique common fixed point $fz\in fE$ of g and h; that is, $fz=g(fz)=h(fz)$. Thus, z is a coincidence point of S, T, and f. Now we show that S, T, and f have unique point of coincidence. Now let $x\in X$ such that $Sx=Tx=fx$, $fx\neq fz$. Then fx is another common fixed point of g and h, which is a contradiction, which implies that S, T, and f have a unique point of coincidence. Since (S,f) and (T,f) are weakly compatible by Lemma 6, we deduce that fz is a unique common fixed point of S, T, and f. □

Theorem 19. *Let (X,d) be a complete complex valued metric space and $0\le h<1$. Let $S,T,f,g:X\to X$ by the self-mappings such that $SX,TX\subset fX=gX$. Assume that the following holds:*

$$d(Sx,Ty)\precsim hL(x,y) \tag{61}$$

for all $x,y\in X$, where

$$L(x,y)\in\left\{d(fx,gy), d(fx,Sx), d(gy,Ty), \frac{d(fx,Sx)\,d(gy,Ty)}{1+d(fx,gy)}\right\}. \tag{62}$$

If (S,f) and (T,g) are weakly compatible and fX is closed in X, then S, T, f, and g have a unique common fixed point in X.

Proof. By Lemma 5, there exists $E_1,E_2\subset X$ such that $fE_1=fX=gX=gE_2$, $f:E_1\to X$, $g:E_2\to X$ are one to one. Now define the mappings $A,B:fE_1\to fE_1$ by $Af(x)=Sx$ and $Bg(x)=Tx$, respectively. Since f, g are one to one on E_1 and E_2, respectively, then the mappings A, B are well-defined. Now

$$d(Sx,Ty)=d(A(fx),B(gy))\precsim hL(x,y), \tag{63}$$

where

$$L(x,y)\in\left\{d(fx,gy), d(fx,A(fx)), d(gy,B(gy)), \frac{d(fx,A(fx))\,d(gy,B(gy))}{1+d(fx,gy)}\right\} \tag{64}$$

for all $fx,gy\in fE_1$. By Theorem 7, as fE_1 is complete subspace of X, we deduce that there exists a unique common fixed point $fz\in fE_1$ of A and B; that is, $A(fz)=Bf(z)=fz$.

This implies that $Sz = fz$; let $v \in X$ such that $fz = gv$. We have $B(gv) = gv \Rightarrow Tv = gv$. We show that S and f have a unique point of coincidence. If $Sw = fw$ then fw is a fixed point of A. By the proof of Theorem 7 fw is another common fixed point of A and B which is a contradiction. Hence, S and f have a unique point of coincidence. By Lemma 6, it follows that fz is a unique common fixed point of S and f. Similarly, gv is the unique common fixed point for $T$and g. This proves that $fz = gv$ is the unique common fixed point for S, T, f, and g. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

Acknowledgments

This paper is dedicated to Professor Miodrag Mateljević on the occasion of his 65th birthday. The first author gratefully acknowledges the support from the Deanship of Scientific Research (DSR) at King Abdulaziz University (KAU) during this research. The third author gratefully acknowledges the support from the Higher Education Commission of Pakistan.

References

[1] S. Gähler, "2-metrische raume und ihre topologische strukture," *Mathematische Nachrichten*, vol. 26, pp. 115–148, 1963.

[2] M. Abbas, B. Fisher, and T. Nazir, "Well-Posedness and periodic point property of mappings satisfying a rational inequality in an ordered complex valued metric space," *Numerical Functional Analysis and Optimization*, vol. 243, p. 32, 2011.

[3] M. Abbas and B. E. Rhoades, "Fixed and periodic point results in cone metric spaces," *Applied Mathematics Letters*, vol. 22, no. 4, pp. 511–515, 2009.

[4] J. Ahmad, C. Klin-Eam, and A. Azam, "Common fixed points for multivalued mappings in complex valued metric spaces with applications," *Abstract and Applied Analysis*, vol. 2013, Article ID 854965, 12 pages, 2013.

[5] M. Arshad, A. Azam, and P. Vetro, "Some common fixed point results in cone metric spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 493965, 11 pages, 2009.

[6] C. di Bari and P. Vetro, "Common fixed pointsfor three or four mappings via common fixed point for two mappings," http://arxiv.org/abs/1302.3816.

[7] Lj. Cirić, A. Razani, S. Radenović, and J. S. Ume, "Common fixed point theorems for families of weakly compatible maps," *Computers & Mathematics with Applications*, vol. 55, no. 11, pp. 2533–2543, 2008.

[8] Lj. Cirić, "A generalization of Banach's contraction principle," *Proceedings of the American Mathematical Society*, vol. 45, pp. 267–273, 1974.

[9] Lj. Cirić, N. Hussain, and N. Cakić, "Common fixed points for Cirić type f-weak contraction with applications," *Publicationes Mathematicae Debrecen*, vol. 76, no. 1-2, pp. 31–49, 2010.

[10] Lj. Cirić, M. Abbas, R. Saadati, and N. Hussain, "Common fixed points of almost generalized contractive mappings in ordered metric spaces," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5784–5789, 2011.

[11] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[12] C. Klin-eam and C. Suanoom, "Some common fixed-point theorems for generalized-contractive-type mappings on complex-valued metric spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 604215, 6 pages, 2013.

[13] M. A. Kutbi, J. Ahmad, and A. Azam, "On fixed points of α-ψ-contractive multivalued mappings in cone metric spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 313782, 6 pages, 2013.

[14] M. A. Kutbi, A. Azam, J. Ahmad, and C. Di Bari, "Some common coupled fixed point results for generalized contraction in complex-valued metric spaces," *Journal of Applied Mathematics*, vol. 2013, Article ID 352927, 10 pages, 2013.

[15] A. Azam, B. Fisher, and M. Khan, "Common fixed point theorems in complex valued metric spaces," *Numerical Functional Analysis and Optimization*, vol. 32, no. 3, pp. 243–253, 2011.

[16] F. Rouzkard and M. Imdad, "Some common fixed point theorems on complex valued metric spaces," *Computers & Mathematics with Applications*, vol. 64, no. 6, pp. 1866–1874, 2012.

[17] W. Sintunavarat and P. Kumam, "Generalized common fixed point theorems in complex valued metric spaces and applications," *Journal of Inequalities and Applications*, vol. 2012, article 84, 2012.

[18] M. Imdad, J. Ali, and M. Tanveer, "Coincidence and common fixed point theorems for nonlinear contractions in Menger PM spaces," *Chaos, Solitons & Fractals*, vol. 42, no. 5, pp. 3121–3129, 2009.

[19] R. H. Haghi, Sh. Rezapour, and N. Shahzad, "Some fixed point generalizations are not real generalizations," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 74, no. 5, pp. 1799–1803, 2011.

[20] A. Azam and M. Arshad, "Common fixed points of generalized contractive maps in cone metric spaces," *Iranian Mathematical Society. Bulletin*, vol. 35, no. 2, pp. 255–264, 2009.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 213296, 11 pages
http://dx.doi.org/10.1155/2014/213296

Research Article

Convergence of Numerical Solution of Generalized Theodorsen's Nonlinear Integral Equation

Mohamed M. S. Nasser[1,2]

[1] *Department of Mathematics, Faculty of Science, King Khalid University, P.O. Box 9004, Abha, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Ibb University, P.O. Box 70270, Ibb, Yemen*

Correspondence should be addressed to Mohamed M. S. Nasser; mms_nasser@hotmail.com

Received 7 February 2014; Accepted 24 February 2014; Published 2 April 2014

Academic Editor: Samuel Krushkal

We consider a nonlinear integral equation which can be interpreted as a generalization of Theodorsen's nonlinear integral equation. This equation arises in computing the conformal mapping between simply connected regions. We present a numerical method for solving the integral equation and prove the uniform convergence of the numerical solution to the exact solution. Numerical results are given for illustration.

1. Introduction

Numerical methods for conformal mapping from a simply connected region onto another simply connected region are available only when one of the region is a standard region, mostly the unit disk D. Let G and Ω be bounded simply connected regions in the z-plane and w-plane, respectively, such that their boundaries $\Gamma := \partial G$ and $L := \partial\Omega$ are smooth Jordan curves. Then the mapping $\Psi : G \to \Omega$ is calculated as the composition of the maps $G \to D \to \Omega$.

Recently, a numerical method has been proposed in [1] for direct approximation of the mapping $\Psi : G \to \Omega$. Assume that Γ and L are star-like with respect to the origin and defined by polar coordinates

$$\eta(t) = \rho(t)\, e^{\mathrm{i}t}, \qquad \zeta(t) = R(t)\, e^{\mathrm{i}t}, \quad 0 \le t \le 2\pi, \tag{1}$$

respectively, such that both ρ and R are 2π-periodic continuously differentiable positive real functions with nonvanishing derivatives. By the Riemann-mapping theorem, there exists a unique conformal mapping function $\Psi : G \to \Omega$ normalized by $\Psi(0) = 0$, $\Psi'(0) > 0$. The boundary value of the function Ψ is on the boundary L and can be described as

$$\Psi^{+}(\eta(t)) = \zeta(S(t)), \quad 0 \le t \le 2\pi, \tag{2}$$

where $S(t)$ is the *boundary correspondence function* of the mapping function Ψ. The function $S(t)$ is a strictly increasing function so that $S(t) - t$ is a 2π-periodic function.

The function $S(t)$ is the unique solution of a nonlinear integral equation which can be interpreted as a generalization of Theodorsen's nonlinear integral equation [1]. The proof of the existence and the uniqueness of the solution of the nonlinear integral equation was given in [1] for regions Ω of which boundaries $L = \partial\Omega$ satisfy the so-called ϵ-condition; that is,

$$\varepsilon := \max_{0 \le t \le 2\pi} \left| \frac{R'(t)}{R(t)} \right| < 1. \tag{3}$$

In this paper, the nonlinear integral equation is solved by an iterative method. Each iteration of the iterative method requires solving an $n \times n$ linear system which is obtained by discretizing the integrals in the integral equation by the trapezoidal rule. The linear system is solved by a combination of the generalized minimal residual (GMRES) method and the fast multipole method (FMM) in $O(n \ln n)$ operations. The main objective of this paper is to prove the uniform convergence of the numerical solution to the exact solution. We also study the properties of the generalized conjugation operator. Numerical results are presented for illustration.

2. Auxiliary Materials

2.1. The Functions θ and τ. Let $w = f(z)$ be the mapping function from the simply connected region G onto the unit disk D with the normalization $f(0) = 0$ and $f'(0) > 0$. Then the boundary value of the function f is on the unit circle and can be described as

$$f(\eta(t)) = e^{i\theta(t)}, \quad 0 \le t \le 2\pi. \tag{4}$$

The function $\theta(t)$ is the *boundary correspondence function* of the mapping function f where $\theta(t) - t$ is a 2π-periodic function and $\theta'(t) > 0$ for all $t \in [0, 2\pi]$. Let $\tau(t)$ be the inverse of the function $\theta(t)$. Then $\tau(t)$ is the boundary correspondence function of the inverse mapping function $z = f^{-1}(w)$ from D onto G; that is,

$$f^{-1}\left(e^{it}\right) = \eta(\tau(t)), \quad 0 \le t \le 2\pi, \tag{5}$$

where $\tau(t) - t$ is a 2π-periodic function and $\tau'(t) > 0$ for all $t \in [0, 2\pi]$.

2.2. The Norms. Let H be the space of all real Hölder continuous 2π-periodic functions on $[0, 2\pi]$. With the inner product

$$(\gamma, \psi) = \frac{1}{2\pi}\int_0^{2\pi} \gamma(s)\psi(s)\,ds, \tag{6}$$

the space H is a pre-Hilbert space. We define the norm $\|\cdot\|_2$ by

$$\|\gamma\|_2 := (\gamma, \gamma)^{1/2}. \tag{7}$$

Since $s = \tau(t)$and if $t = \theta(s)$, we have

$$\begin{aligned} \left\|\left(\theta'\right)^{1/2}\gamma\right\|_2 &= \int_0^{2\pi} \theta'(s)\gamma(s)^2 ds \\ &= \int_0^{2\pi} \gamma(\tau(t))^2 dt = \|\gamma\circ\tau\|_2. \end{aligned} \tag{8}$$

With the norm $\|\cdot\|_2$, we define a norm $\|\cdot\|_\theta$ by

$$\|\gamma\|_\theta := \left\|\left(\theta'\right)^{1/2}\gamma\right\|_2 = \|\gamma\circ\tau\|_2. \tag{9}$$

We define also the maximum norm $\|\cdot\|_\theta$ by

$$\|\gamma\|_\infty := \max_{0\le t\le 2\pi}|\gamma(t)|. \tag{10}$$

Since $\theta(2\pi) - \theta(0) = 2\pi$, we have

$$\begin{aligned} \|\gamma\|_\theta^2 &= \frac{1}{2\pi}\int_0^{2\pi} \theta'(t)\gamma^2(t)\,dt \\ &\le \|\gamma\|_\infty^2 \frac{1}{2\pi}\int_0^{2\pi}\theta'(t)\,dt = \|\gamma\|_\infty^2, \end{aligned} \tag{11}$$

which implies that

$$\|\gamma\|_\theta \le \|\gamma\|_\infty. \tag{12}$$

Theorem 1. *If $\int_0^{2\pi}\theta'(s)\gamma(s)ds = 0$, then*

$$\|\gamma\|_\infty^2 \le 2\pi\|\gamma\|_\theta\left\|\frac{\gamma'}{\theta'}\right\|_\theta. \tag{13}$$

Proof. Since $s = \tau(t)$ and if $t = \theta(s)$, we have

$$\begin{aligned} \int_0^{2\pi}(\gamma\circ\tau)(t)\,dt &= \int_0^{2\pi}\gamma(\tau(t))\,dt \\ &= \int_0^{2\pi}\theta'(s)\gamma(s)\,ds = 0. \end{aligned} \tag{14}$$

Thus, it follows from [2, page 68] that

$$\|\gamma\circ\tau\|_\infty^2 \le 2\pi\|\gamma\circ\tau\|_2\left\|(\gamma\circ\tau)'\right\|_2. \tag{15}$$

We have also

$$\begin{aligned} \left\|(\gamma\circ\tau)'\right\|_2^2 &= \frac{1}{2\pi}\int_0^{2\pi}\left|(\gamma\circ\tau)'(t)\right|^2 dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\gamma'(\tau(t))^2\tau'(t)^2 dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\gamma'(s)^2\frac{1}{\theta'(s)}ds \\ &= \frac{1}{2\pi}\int_0^{2\pi}\theta'(s)\left(\frac{\gamma'(s)}{\theta'(s)}\right)^2 ds. \end{aligned} \tag{16}$$

Hence,

$$\left\|(\gamma\circ\tau)'\right\|_2^2 = \left\|\frac{\gamma'}{\theta'}\right\|_\theta^2. \tag{17}$$

Since $\tau(\cdot) : [0, 2\pi] \to [0, 2\pi]$ is bijective, we have

$$\begin{aligned} \|\gamma\circ\tau\|_\infty &:= \max_{0\le t\le 2\pi}|\gamma(\tau(t))| \\ &= \max_{0\le\tau\le 2\pi}|\gamma(\tau)| = \|\gamma\|_\infty. \end{aligned} \tag{18}$$

Hence, (15) and (18) imply that

$$\|\gamma\|_\infty \le 2\pi\|\gamma\circ\tau\|_2\left\|(\gamma\circ\tau)'\right\|_2. \tag{19}$$

Then (13) follows from (9), (17), and (19). □

*2.3. The Operators **K** and **J**.* The conjugation operator $\mathbf{K}$ is defined by

$$\mathbf{K}\mu = \int_0^{2\pi}\frac{1}{2\pi}\cot\frac{s-t}{2}\mu(t)\,dt. \tag{20}$$

Let $\mathbf{J}$ be the operator defined by

$$\mathbf{J}\mu = \frac{1}{2\pi}\int_0^{2\pi}\mu(t)\,dt. \tag{21}$$

Hence, the operators $\mathbf{K}$ and $\mathbf{J}$ satisfy [3]

$$\mathbf{JK} = 0, \qquad \mathbf{K}^2 = -\mathbf{I} + \mathbf{J}. \tag{22}$$

3. The Generalized Conjugation Operator

Let A be the complex 2π-periodic continuously differentiable function:

$$A(s) := \eta(s). \tag{23}$$

We define the real kernels M and N as real and imaginary parts:

$$M(s,t) + \mathrm{i}N(s,t) := \frac{1}{\pi}\frac{A(s)}{A(t)}\frac{\eta'(t)}{\eta(t)-\eta(s)}. \tag{24}$$

The kernel $N(s,t)$ is called the generalized Neumann kernel formed with A and η. The kernel $N(s,t)$ is continuous and the kernel M has the representation

$$M(s,t) = -\frac{1}{2\pi}\cot\frac{s-t}{2} + M_1(s,t), \tag{25}$$

with a continuous kernel M_1. See [4] for more details.

We define the Fredholm integral operators $\mathbf{N}$ and $\mathbf{M}_1$ and the singular integral operator $\mathbf{M}$ on H by

$$\begin{aligned}\mathbf{N}\mu &= \int_0^{2\pi} N(s,t)\,\mu(t)\,dt,\\ \mathbf{M}_1\mu &= \int_0^{2\pi} M_1(s,t)\,\mu(t)\,dt,\\ \mathbf{M}\mu &= \int_0^{2\pi} M(s,t)\,\mu(t)\,dt.\end{aligned} \tag{26}$$

We define an operator $\mathbf{E}$ on H by

$$\mathbf{E} = -(\mathbf{I}-\mathbf{N})^{-1}\mathbf{M}. \tag{27}$$

The operator $\mathbf{E}$ is singular but bounded on H [1]. Finally, we define an operator $\mathbf{J}_\theta$ by

$$\mathbf{J}_\theta\mu = \frac{1}{2\pi}\int_0^{2\pi}\theta'(t)\,\mu(t)\,dt. \tag{28}$$

Remark 2. When Γ reduces to the unit, then $\theta'(t) = 1$, the operator $\mathbf{J}_\theta$ reduces to the operator $\mathbf{J}$, and the operator $\mathbf{E}$ reduces to the operator $\mathbf{K}$; that is, the operator $\mathbf{E}$ is a generalization of the well-known conjugation operator $\mathbf{K}$ (see [1] for more details).

The operator $\mathbf{E}$ is related to the operator $\mathbf{K}$ by [1]

$$\mu = \mathbf{E}\gamma \quad \text{iff}\, \mu\circ\tau = \mathbf{K}(\gamma\circ\tau). \tag{29}$$

Since $\mu = (\mu\circ\theta)\circ\tau$ and $\gamma = (\gamma\circ\theta)\circ\tau$, it follows from (29) that

$$\mu = \mathbf{K}\gamma \quad \text{iff}\, \mu\circ\theta = \mathbf{E}(\gamma\circ\theta). \tag{30}$$

Lemma 3 (see [1]). *Let $\gamma,\mu \in H$ be given functions. Then $f(\eta(t)) = \gamma(t) + i\mu(t)$ is the boundary value of an analytic function in G with* $\operatorname{Im} f(0) = 0$ *if and only if*

$$\mu = \mathbf{E}\gamma. \tag{31}$$

Lemma 4 (see [1]). *If $\gamma \in H$ and $\mu = \mathbf{E}\gamma$, then $\gamma = c - \mathbf{E}\mu$ with a real constant $c = f(0)$ where f is the unique analytic function in G with the boundary values $f(\eta(t)) = \gamma(t) + i\mu(t)$ and* $\operatorname{Im} f(0) = 0$.

Lemma 5 (see [1]). *The operator $\mathbf{E}$ has the following properties:*

$$\begin{aligned}&Null(\mathbf{E}) = span\{1\},\\ &\mathbf{E}^3 = -\mathbf{E},\\ &\sigma(\mathbf{E}) = \{0,\pm i\}.\end{aligned} \tag{32}$$

Lemma 6. *The operator $\mathbf{E}$ has the norm*

$$\|\mathbf{E}\|_\theta = 1. \tag{33}$$

Proof. The operator $\mathbf{E}$ has the norm $\|\mathbf{E}\|_\theta \le 1$ [1]. Since $\mathrm{i} \in \sigma(\mathbf{E})$, hence $1 = |\mathrm{i}| \le \|\mathbf{E}\|_\theta \le 1$. Hence, we obtain (33). □

Lemma 7. *The operators $\mathbf{J}_\theta$ and $\mathbf{E}$ satisfy*

$$\mathbf{J}_\theta\mathbf{E} = 0. \tag{34}$$

Proof. For any $\gamma \in H$, let $\mu = \mathbf{E}\gamma$, $\widehat{\mu} = \mu\circ\tau$, and $\widehat{\gamma} = \gamma\circ\tau$. Then, it follows from (29) that $\widehat{\mu} = \mathbf{K}\widehat{\gamma}$. Thus,

$$\begin{aligned}\mathbf{J}_\theta\mathbf{E}\gamma(s) &= \mathbf{J}_\theta\mu(s)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\theta'(t)\,\mu(t)\,dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}\mu(\tau(s))\,ds\\ &= \frac{1}{2\pi}\int_0^{2\pi}\widehat{\mu}(s)\,ds\\ &= \mathbf{J}\widehat{\mu}(s)\\ &= \mathbf{J}\mathbf{K}\widehat{\gamma}(s),\end{aligned} \tag{35}$$

which by (22) implies that

$$\mathbf{J}_\theta\mathbf{E}\gamma(s) = 0. \tag{36}$$

Since (36) holds for all functions $\gamma \in H$, the operator identity (34) follows. □

Lemma 8. *The operator $\mathbf{E}$ satisfies*

$$\mathbf{E}^2 = -\mathbf{I} + \mathbf{J}_\theta. \tag{37}$$

Proof. Let $\gamma \in H$ and $\mu = \mathbf{E}\gamma$. Then, by Lemma 4, $\gamma = c - \mathbf{E}\mu$ with a real constant c. By the definition of the operator $\mathbf{J}_\theta$, we have $\mathbf{J}_\theta c = c$. Since $\mathbf{J}_\theta\mathbf{E} = 0$, we have

$$\mathbf{J}_\theta\gamma = \mathbf{J}_\theta c - \mathbf{J}_\theta\mathbf{E} = c. \tag{38}$$

Hence,

$$\mathbf{E}^2\gamma = \mathbf{E}\mu = -\gamma + c = (-\mathbf{I} + \mathbf{J}_\theta)\gamma \tag{39}$$

holds for all $\gamma \in H$. Thus, the operator identities (37) follow. □

Lemma 9. *For all functions $\gamma \in H$, we have*

$$\|\mathbf{E}\gamma\|_\theta \le \|\gamma\|_\theta, \tag{40}$$

with equality for all γ with $\mathbf{J}_\theta\gamma = 0$.

Proof. For all functions $\gamma \in H$, the inequality (40) follows from (33).

For all functions γ with $\mathbf{J}_\theta\gamma = 0$, we have from (37) that $\gamma = -\mathbf{E}^2\gamma$. Hence

$$\|\gamma\|_\theta = \left\|\mathbf{E}^2\gamma\right\|_\theta \le \|\mathbf{E}\gamma\|_\theta \le \|\gamma\|_\theta, \tag{41}$$

which means that $\|\mathbf{E}\gamma\|_\theta = \|\gamma\|_\theta$. □

Theorem 10. *Let $\gamma \in H$ and $\mu = \mathbf{E}\gamma$. Then*

$$\frac{\mu'}{\theta'} = \mathbf{E}\left(\frac{\gamma'}{\theta'}\right). \tag{42}$$

Proof. For $\gamma \in H$ and $\mu = \mathbf{E}\gamma$, we have from (29) that $\mu \circ \tau = \mathbf{K}(\gamma \circ \tau)$. Then, it follows from [2, page 64] that

$$(\mu \circ \tau)' = \mathbf{K}(\gamma \circ \tau)'. \tag{43}$$

Hence, by (30), we have

$$(\mu \circ \tau)' \circ \theta = \mathbf{E}\left((\gamma \circ \tau)' \circ \theta\right), \tag{44}$$

which implies that

$$\frac{((\mu \circ \tau) \circ \theta)'}{\theta'} = \mathbf{E}\left(\frac{((\gamma \circ \tau) \circ \theta)'}{\theta'}\right). \tag{45}$$

Hence, we obtain (42). □

4. The Generalized Theodorsen Nonlinear Integral Equation

The boundary correspondence function $S(t)$ is the unique solution of the nonlinear integral equation

$$S(t) - t = \mathbf{E}\ln\frac{R(S(\cdot))}{\rho(\cdot)}(t) \tag{46}$$

which is a generalization of the well-known Theodorsen integral equation [1]. Nonlinear integral equation (46) can be solved by the iterative method

$$S_k(t) - t = \mathbf{E}\ln\frac{R(S_{k-1}(\cdot))}{\rho(\cdot)}(t), \quad k = 1, 2, 3, \ldots. \tag{47}$$

Then we have [1]

$$\|S_k - S\|_\theta \le \varepsilon^k \|S_0 - S\|_\theta. \tag{48}$$

Thus, if the curve L satisfies the ε-condition (3), then

$$\|S_k - S\|_\theta \longrightarrow 0. \tag{49}$$

That is, the approximate solutions $S_k(t)$ converge to $S(t)$ with respect to the norm $\|\cdot\|_\theta$ if $\varepsilon < 1$.

In this section, we will prove the uniform convergence of the approximate solutions $S_k(t)$ to the exact solution $S(t)$. We will use the approach used in the proof of Proposition 1.5 in [2, page 69] related to Theodorsen's integral equation. See also [5, 6].

Lemma 11. *Consider*

$$\mathbf{E}[\ln\rho(\cdot)](t) = t - \theta(t). \tag{50}$$

Proof. The function θ is the boundary correspondence function of the conformal mapping f from G onto the unit disk. Hence, the function $\theta(t) - t$ satisfies [1]

$$\theta(t) - t = \mathbf{E}\ln\frac{1}{\rho(\cdot)}(t). \tag{51}$$

Then (50) follows from (51). □

The previous lemma implies that (46) can be rewritten as

$$S(t) - \theta(t) = \mathbf{E}[\ln R(S(\cdot))](t), \tag{52}$$

and (47) can be rewritten as

$$S_k(t) - \theta(t) = \mathbf{E}[\ln R(S_{k-1}(\cdot))](t). \tag{53}$$

Thus

$$S_k(t) - S(t) = \mathbf{E}[\ln R(S_{k-1}(\cdot)) - \ln R(S(\cdot))](t). \tag{54}$$

Lemma 12. *Consider*

$$\|S_k - S\|_\theta \le (\varepsilon + \|S_0 - \theta\|_\theta)\,\varepsilon^k. \tag{55}$$

Proof. Let a be such that

$$\frac{1}{1+\varepsilon} \le \frac{R(t)}{a} \le 1 + \varepsilon, \quad \forall t. \tag{56}$$

Then

$$\left|\ln\frac{R(t)}{a}\right| \le \ln(1+\varepsilon) < \varepsilon, \quad \forall t. \tag{57}$$

Hence,

$$\left\|\ln\frac{R}{a}\right\|_\infty < \varepsilon. \tag{58}$$

Thus

$$\begin{aligned} \left\|\mathbf{E}\left[\ln\left(\frac{R(S(\cdot))}{a}\right)\right]\right\|_\theta &\le \|\mathbf{E}\|_\theta \left\|\ln\left(\frac{R(S)}{a}\right)\right\|_\theta \\ &\le \left\|\ln\left(\frac{R(S)}{a}\right)\right\|_\infty < \varepsilon. \end{aligned} \tag{59}$$

Since

$$\begin{aligned} S(t) - S_0(t) &= S(t) - \theta(t) + \theta(t) - S_0(t) \\ &= \mathbf{E}[\ln R(S(\cdot))](t) + \theta(t) - S_0(t) \end{aligned} \tag{60}$$

and $\mathbf{E}(\ln a) = 0$, we have

$$S(t) - S_0(t) = \mathbf{E}\left[\ln\left(\frac{R(S(\cdot))}{a}\right)\right](t) + \theta(t) - S_0(t), \quad (61)$$

which implies that

$$\begin{aligned}\|S - S_0\|_\theta &\le \left\|\mathbf{E}\left[\ln\left(\frac{R(S(\cdot))}{a}\right)\right]\right\|_\theta \\ &\quad + \|\theta - S_0\|_\theta \le \varepsilon + \|\theta - S_0\|_\theta.\end{aligned} \quad (62)$$

Hence (55) follows from (48). □

Lemma 13. *Consider*

$$\left\|\frac{S_k' - S'}{\theta'}\right\|_\theta \le \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\left(1 + \left\|\frac{S_0'}{\theta'}\right\|_\theta\right). \quad (63)$$

Proof. We have

$$\begin{aligned}\left\|\frac{S_k' - S'}{\theta'}\right\|_\theta &= \left\|\frac{S_k' - \theta' + \theta' - S'}{\theta'}\right\|_\theta \\ &\le \left\|\frac{S_k'}{\theta'} - 1\right\|_\theta + \left\|\frac{S'}{\theta'} - 1\right\|_\theta.\end{aligned} \quad (64)$$

Since

$$\begin{aligned}\left\|\frac{S'}{\theta'} - 1\right\|_\theta^2 &= \frac{1}{2\pi}\int_0^{2\pi}\theta'(t)\left(\frac{S'}{\theta'} - 1\right)^2 dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\theta'(t)\left(\frac{S'}{\theta'}\right)^2 dt - 2\frac{1}{2\pi}\int_0^{2\pi}S'(t)\,dt \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi}\theta'(t)\,dt,\end{aligned} \quad (65)$$

$\int_0^{2\pi} S'(t)dt = 2\pi$, and $\int_0^{2\pi}\theta'(t)dt = 2\pi$, we obtain

$$\left\|\frac{S'}{\theta'} - 1\right\|_\theta^2 = \left\|\frac{S'}{\theta'}\right\|_\theta^2 - 1. \quad (66)$$

Similarly, we have

$$\left\|\frac{S_k'}{\theta'} - 1\right\|_\theta^2 = \left\|\frac{S_k'}{\theta'}\right\|_\theta^2 - 1. \quad (67)$$

In view of Theorem 10, it follows from (52) and (53) that

$$\frac{S'(t)}{\theta'(t)} - 1 = \mathbf{E}\left[\frac{R'(S(\cdot))}{R(S(\cdot))}\frac{S'(\cdot)}{\theta'(\cdot)}\right](t), \quad (68)$$

$$\frac{S_k'(t)}{\theta'(t)} - 1 = \mathbf{E}\left[\frac{R'(S_{k-1}(\cdot))}{R(S_{k-1}(\cdot))}\frac{S_{k-1}'(\cdot)}{\theta'(\cdot)}\right](t). \quad (69)$$

Hence, it follows from (68) that

$$\begin{aligned}\left\|\frac{S'}{\theta'} - 1\right\|_\theta &= \left\|\mathbf{E}\left[\frac{R'(S(\cdot))}{R(S(\cdot))}\frac{S'(\cdot)}{\theta'(\cdot)}\right]\right\|_\theta \\ &\le \|\mathbf{E}\|_\theta\left\|\frac{R'(S(\cdot))}{R(S(\cdot))}\frac{S'(\cdot)}{\theta'(\cdot)}\right\|_\theta \le \varepsilon\left\|\frac{S'}{\theta'}\right\|_\theta.\end{aligned} \quad (70)$$

By (70) and (66), we have

$$\left\|\frac{S'}{\theta'} - 1\right\|_\theta^2 \le \varepsilon^2\left\|\frac{S'}{\theta'}\right\|_\theta^2 = \varepsilon^2 + \varepsilon^2\left\|\frac{S'}{\theta'} - 1\right\|_\theta^2. \quad (71)$$

Hence,

$$\left\|\frac{S'}{\theta'} - 1\right\|_\theta^2 \le \frac{\varepsilon^2}{1-\varepsilon^2}. \quad (72)$$

Similarly, it follows from (69) that

$$\left\|\frac{S_k'}{\theta'} - 1\right\|_\theta \le \varepsilon\left\|\frac{S_{k-1}'}{\theta'}\right\|_\theta, \quad (73)$$

which by (67) implies that

$$\begin{aligned}\left\|\frac{S_k'}{\theta'} - 1\right\|_\theta^2 &\le \varepsilon^2\left\|\frac{S_{k-1}'}{\theta'}\right\|_\theta^2 \\ &= \varepsilon^2 + \varepsilon^2\left\|\frac{S_{k-1}'}{\theta'} - 1\right\|_\theta^2.\end{aligned} \quad (74)$$

Hence,

$$\begin{aligned}\left\|\frac{S_k'}{\theta'} - 1\right\|_\theta^2 &\le \varepsilon^2 + \varepsilon^2\left\|\frac{S_{k-1}'}{\theta'} - 1\right\|_\theta^2 \\ &\le \cdots \le \varepsilon^2 + \varepsilon^4 + \cdots + \varepsilon^{2k}\left\|\frac{S_0'}{\theta'} - 1\right\|_\theta^2,\end{aligned} \quad (75)$$

which, in view of (67), implies that

$$\left\|\frac{S_k'}{\theta'} - 1\right\|_\theta^2 \le \frac{\varepsilon^2}{1-\varepsilon^2}\left(1 + \left\|\frac{S_0'}{\theta'} - 1\right\|_\theta^2\right) = \frac{\varepsilon^2}{1-\varepsilon^2}\left\|\frac{S_0'}{\theta'}\right\|_\theta^2. \quad (76)$$

□

Then (63) follows from (64), (72), and (76).

Theorem 14. *If $\varepsilon < 1$, then the approximate solution S_k converges uniformly to the exact solution S with*

$$\begin{aligned}\|S_k - S\|_\infty &\le \sqrt{\frac{2\pi}{\sqrt{1-\varepsilon^2}}} \\ &\quad \times \sqrt{\left(\varepsilon + \|S_0 - \theta\|_\theta\right)\left(1 + \left\|\frac{S_0'}{\theta'}\right\|_\theta\right)}\varepsilon^{k/2+1/2}.\end{aligned} \quad (77)$$

Proof. In view of (54), Lemma 7 implies that

$$\int_0^{2\pi}\theta'(t)\left(S_k(t) - S(t)\right)dt = 0. \quad (78)$$

Thus, we have from (13) that

$$\|S_k - S\|_\infty^2 \le 2\pi\|S_k - S\|_\theta\left\|\frac{S_k' - S'}{\theta'}\right\|_\theta. \quad (79)$$

Hence (77) follows from (55) and (63). □

The following corollary follows from the previous theorem.

Corollary 15. *If*

$$S_0(t) = \theta(t) = t - \mathbf{E}[\ln\rho(\cdot)](t), \tag{80}$$

then

$$\|S_k - S\|_\infty \le 2\sqrt{\frac{\pi}{\sqrt{1-\varepsilon^2}}}\varepsilon^{k/2+1}. \tag{81}$$

Remark 16. When Γ reduces to the unit, then

$$\rho(t) = 1, \qquad \theta(t) = t, \qquad \mathbf{E} = \mathbf{K}. \tag{82}$$

Hence, the results presented in this section reduces to the results presented in [2] for Theodorsen's integral equation.

5. Discretizing (47)

In this paper, we will discretize (47) instead of (46). The numerical method used here is based on strict discretization of the integrals in the operator **E** by the trapezoidal rule which gives accurate results since the integrals are over 2π-periodic. Let n be a given even positive integer. We define n equidistant collocation points s_i in the interval $[0, 2\pi]$ by

$$t_i := (i-1)\frac{2\pi}{n}, \quad i = 1, 2, \ldots, n. \tag{83}$$

Then, for 2π-periodic function $\gamma(t)$, the trapezoidal rule approximates the integral

$$I = \int_0^{2\pi} \gamma(t)\,dt \tag{84}$$

by

$$I_n = \frac{2\pi}{n}\sum_{i=1}^{n}\gamma(t_i). \tag{85}$$

If the function $\gamma(t)$ is continuous, then $|I - I_n| \to 0$. If the integrand $\gamma(t)$ is k times continuously differentiable, then the rate of convergence of the trapezoidal rule is $O(1/n^k)$. For analytic $\gamma(t)$, the rate of convergence is better than $O(1/n^k)$ for any positive integer k [7, page 83]. See also [8].

For $\gamma \in H$, the integral operator **N** will be discretized by the Nyström method as follows:

$$\mathbf{N}_n\gamma(s) = \frac{2\pi}{n}\sum_{j=1}^{n} N(s, t_j)\,\gamma(t_j). \tag{86}$$

Hence, we have

$$\|(\mathbf{N} - \mathbf{N}_n)\gamma\|_\infty = \max_{s\in[0,2\pi]}\left|\int_0^{2\pi} N(s,t)\,\gamma(t)\,dt - \frac{2\pi}{n}\sum_{j=1}^{n} N(s,t_j)\,\gamma(t_j)\right|. \tag{87}$$

Since the kernel $N(s, t)$ is continuous on both variables and since the function $\gamma(t)$ is continuous, we have [9]

$$\|(\mathbf{N} - \mathbf{N}_n)\gamma\|_\infty \longrightarrow 0. \tag{88}$$

The integral operator $\mathbf{M}_1$ will be discretized by the Nyström method as follows:

$$\mathbf{M}_{1,n}\gamma(s) = \frac{2\pi}{n}\sum_{j=1}^{n} M_1(s, t_j)\left[\gamma(t_j) - \gamma(s)\right]. \tag{89}$$

Since the kernel $M_1(s, t)$ is continuous on both variables and since the function $\gamma(t)$ is continuous, we have [9]

$$\|(\mathbf{M}_1 - \mathbf{N}_{1,n})\gamma\|_\infty = \max_{s\in[0,2\pi]}\left|\int_0^{2\pi} M_1(s,t)\,\gamma(t)\,dt - \frac{2\pi}{n}\sum_{j=1}^{n} M_1(s,t_j)\,\gamma(t_j)\right| \longrightarrow 0. \tag{90}$$

To discretize the operator $\mathbf{K}\gamma(s)$, we first approximate the function $\gamma(s)$ by the interpolating trigonometric polynomial of degree $n/2$ which interpolates $\gamma(s)$ at the n points t_j, $j = 1, 2, \ldots, n$. That is,

$$\gamma(s) \approx \sum_{i=0}^{n/2} a_i \cos is + \sum_{i=0}^{n/2-1} b_i \sin is. \tag{91}$$

Then $\mathbf{K}\gamma(s)$ is approximated by

$$\mathbf{K}_n\gamma(s) = \sum_{i=1}^{n/2} a_i \sin is - \sum_{i=0}^{n/2-1} b_i \cos is, \tag{92}$$

where [6]

$$\|(\mathbf{K} - \mathbf{K}_n)\gamma\|_\infty \longrightarrow 0. \tag{93}$$

The integral operator **M** is then discretized by

$$\mathbf{M}_n = \mathbf{M}_{1,n} - \mathbf{K}_n. \tag{94}$$

Then, it follows from (90) and (93) that

$$\|(\mathbf{M} - \mathbf{M}_n)\gamma\|_\infty \longrightarrow 0. \tag{95}$$

The operator $\mathbf{M}_n$ is bounded operator since the operator $\mathbf{M}_{1,n}$ is bounded ($M_1(s, t)$ is continuous) and the operator $\mathbf{K}_n$ is bounded operator (see [6]).

Since the kernel $N(s, t)$ is continuous and $\lambda = 1$ is not an eigenvalue of the kernel $N(s, t)$ [1], the operators $\mathbf{I} - \mathbf{N}_n$ are invertible and $(\mathbf{I} - \mathbf{N}_n)^{-1}$ are uniformly bounded for sufficiently large n [9]. Hence, we discretize the operator **E** by the bounded operator

$$\mathbf{E}_n := -(\mathbf{I} - \mathbf{N}_n)^{-1}\mathbf{M}_n. \tag{96}$$

Lemma 17. *If $\gamma \in H$, then*

$$\left\|\left(\mathbf{E}-\mathbf{E}_n\right)\gamma\right\|_\infty \longrightarrow 0. \tag{97}$$

Proof. Let $\phi := \mathbf{E}\gamma$ and $\phi_n := \mathbf{E}_n\gamma$, then

$$(\mathbf{I}-\mathbf{N})\,\phi = -\mathbf{M}\gamma, \qquad \left(\mathbf{I}-\mathbf{N}_n\right)\phi_n = -\mathbf{M}_n\gamma. \tag{98}$$

Let also $\widehat{\phi}_n$ be the unique solution of the discretized equation

$$\left(\mathbf{I}-\mathbf{N}_n\right)\widehat{\phi}_n = -\mathbf{M}\gamma. \tag{99}$$

Thus, we have

$$\begin{aligned}\left\|\left(\mathbf{E}-\mathbf{E}_n\right)\gamma\right\|_\infty = \left\|\phi-\phi_n\right\|_\infty &\le \left\|\phi-\widehat{\phi}_n\right\|_\infty \\ &+ \left\|\widehat{\phi}_n-\phi_n\right\|_\infty.\end{aligned} \tag{100}$$

Since the kernel N is continuous and $\mathbf{N}_n$ is the discretization of $\mathbf{N}$, then it follows from [9, page 108] that

$$\left\|\phi-\widehat{\phi}_n\right\|_\infty \longrightarrow 0. \tag{101}$$

Since $(\mathbf{I}-\mathbf{N}_n)^{-1}$ is bounded and γ is continuous, then (95) implies that

$$\begin{aligned}&\left\|\widehat{\phi}_n-\phi_n\right\|_\infty \\ &\quad= \left\|\left(\mathbf{I}-\mathbf{N}_n\right)^{-1}\left(\mathbf{M}-\mathbf{M}_n\right)\gamma\right\|_\infty \\ &\quad\le \left\|\left(\mathbf{I}-\mathbf{N}_n\right)^{-1}\right\|_\infty \left\|\left(\mathbf{M}-\mathbf{M}_n\right)\gamma\right\|_\infty \longrightarrow 0,\end{aligned} \tag{102}$$

which with (100) and (101) implies (97). □

To calculate the function S_k in (47) for a given S_{k-1}, we replace the operator $\mathbf{E}$ in (47) by the approximate operator $\mathbf{E}_n$ to obtain

$$S_{k,n}(s) - s = \mathbf{E}_n \ln \frac{R\left(S_{k-1}(\cdot)\right)}{\rho(\cdot)}(s), \tag{103}$$

where $S_{k,n}$ is an approximation to S_k. Substituting $s = t_i$ and $i = 1, 2, \ldots, n$, in (103) we obtain

$$S_{k,n}\left(t_i\right) - t_i = \mathbf{E}_n \ln \frac{R\left(S_{k-1}(\cdot)\right)}{\rho(\cdot)}\left(t_i\right), \quad i = 1, 2 \ldots, n. \tag{104}$$

Equation (104) can be rewritten as

$$\begin{aligned}\left(\mathbf{I}-\mathbf{N}_n\right)\left[S_{k,n}\left(t_i\right)-t_i\right] = -\mathbf{M}_n \ln \frac{R\left(S_{k-1}(\cdot)\right)}{\rho(\cdot)}\left(t_i\right), \\ i = 1, 2, \ldots, m,\end{aligned} \tag{105}$$

which represents an $n \times n$ linear system for the unknown $S_{k,n}(t_1), S_{k,n}(t_2), \ldots, S_{k,n}(t_n)$. By obtaining $S_{k,n}(t_i)$ for $i = 1, 2 \ldots, n$, the function $S_{k,n}(s)$ can be calculated for $s \in [0, 2\pi]$ by the Nyström interpolating formula. In the following lemma, we prove the uniform convergence of the approximate solution $S_{k,n}$ of discretized equation (103) to the solution S_k of (46).

Lemma 18. *Consider*

$$\left\|S_{k,n}-S_k\right\|_\infty \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{106}$$

Proof. Let $\gamma(s) := \ln(R(S_{k-1}(s)))/\rho(s)$. Then, we have

$$S_{k,n} - S_k = \mathbf{E}_n\gamma - \mathbf{E}\gamma = \left(\mathbf{E}_n - \mathbf{E}\right)\gamma. \tag{107}$$

Hence,

$$\left\|S_{k,n}-S_k\right\|_\infty \le \left\|\left(\mathbf{E}-\mathbf{E}_n\right)\gamma_n\right\|_\infty. \tag{108}$$

The lemma is then followed from (97). □

The proof of the uniform convergence of the approximate solution $S_{k,n}$ to the boundary correspondence function S is given in the following theorem.

Theorem 19. *If $\varepsilon < 1$, then*

$$\left\|S_{k,n}-S\right\|_\infty \longrightarrow 0 \quad \text{as } k, n \longrightarrow \infty. \tag{109}$$

Proof. We have

$$\left\|S_{k,n}-S\right\|_\infty \le \left\|S_{k,n}-S_k\right\|_\infty + \left\|S_k-S\right\|_\infty. \tag{110}$$

Since $\varepsilon < 1$, it follows from (77) that $\|S_k - S\|_\infty \to 0$ as $k \to \infty$. The theorem is then followed from (106). □

6. The Algebraic System

Let $\mathbf{t}$ be the $n \times 1$ vector $\mathbf{t} := (t_1, t_2, \ldots, t_n)^T$ where T denotes transposition. Then, for any function $\gamma(t)$ defined on $[0, 2\pi]$, we define $\gamma(\mathbf{t})$ as the $n \times 1$ vector obtained by componentwise evaluation of the function $\gamma(t)$ at the points t_i, $i = 1, 2, \ldots, n$. As in MATLAB, for any two vectors $\mathbf{x}$ and $\mathbf{y}$, we define $\mathbf{x} .* \mathbf{y}$ as the componentwise vector product of $\mathbf{x}$ and $\mathbf{y}$. If $\mathbf{y}_j \ne 0$, for all $j = 1, 2 \ldots, (m+1)n$, we define $\mathbf{x} ./ \mathbf{y}$ as the componentwise vector division of $\mathbf{x}$ by $\mathbf{y}$. For simplicity, we denote $\mathbf{x} .* \mathbf{y}$ by $\mathbf{xy}$ and $\mathbf{x} ./ \mathbf{y}$ by $\mathbf{x}/\mathbf{y}$.

Let $\mathbf{x}_{k-1} = S_{k-1}(\mathbf{t}) - \mathbf{t}$ (given) and $\mathbf{x}_k = S_{k,n}(\mathbf{t}) - \mathbf{t}$ (unknown). Then system (105) can be rewritten as

$$(I-B)\,\mathbf{x}_k = -C \ln \frac{R\left(\mathbf{x}_{k-1}+\mathbf{t}\right)}{\rho(\mathbf{t})}, \quad k = 1, 2, \ldots, \tag{111}$$

where I is the $n \times n$ identity matrix, B is the discretized matrix of the operator $\mathbf{N}$, and C is the discretized matrix of the operator $\mathbf{M}$ [1]. Linear system (111) is uniquely solvable [4, 10, 11].

We start the iteration in (47) with $S_k(t) = t$ and iterate until $\|S_k - S_{k-1}\|_\infty < \text{tol}$ where tol is a given tolerance; that is, we start the iteration in (111) with $\mathbf{x}_0 = \mathbf{0}$ and iterate until $\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_\infty < \text{tol}$. Each iteration in (111) requires solving a linear system for $\mathbf{x}_k$ given $\mathbf{x}_{k-1}$. Linear system (111) is solved in $O(n \ln n)$ operations by the fast method presented in [11, 12] which is based on a combination of the MATLAB function `gmres` and the MATLAB function `zfmm2dpart` in the MATLAB toolbox `FMMLIB2D` [13]. In the numerical results below, for

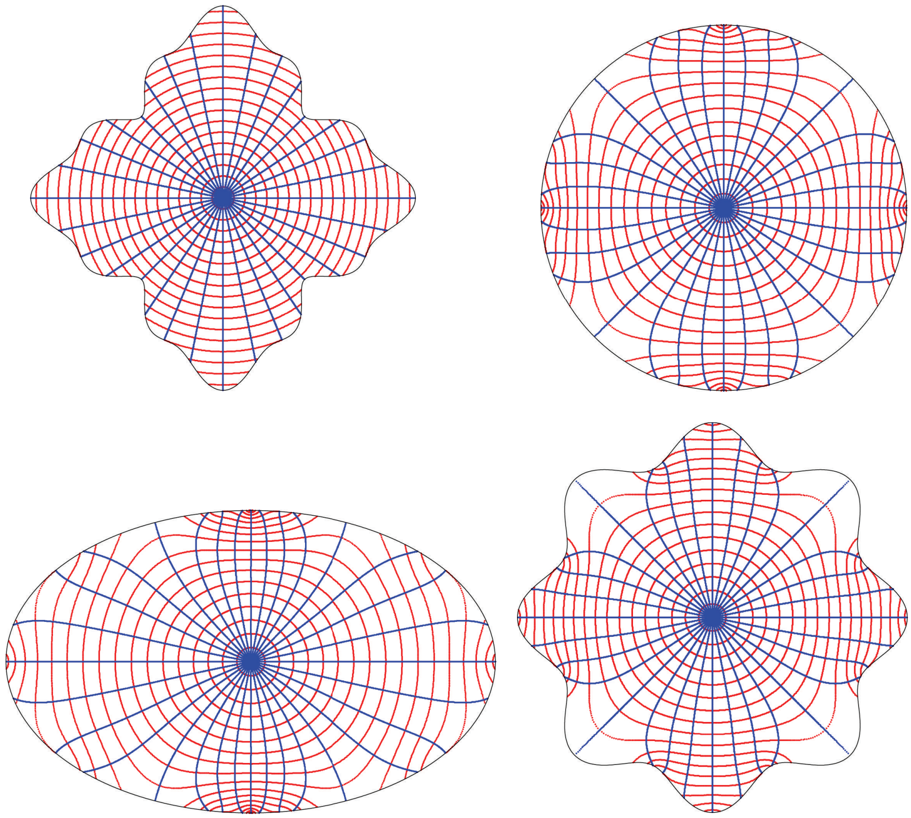

FIGURE 1: The conformal mappings from G_1 onto Ω_1, Ω_2, and Ω_3.

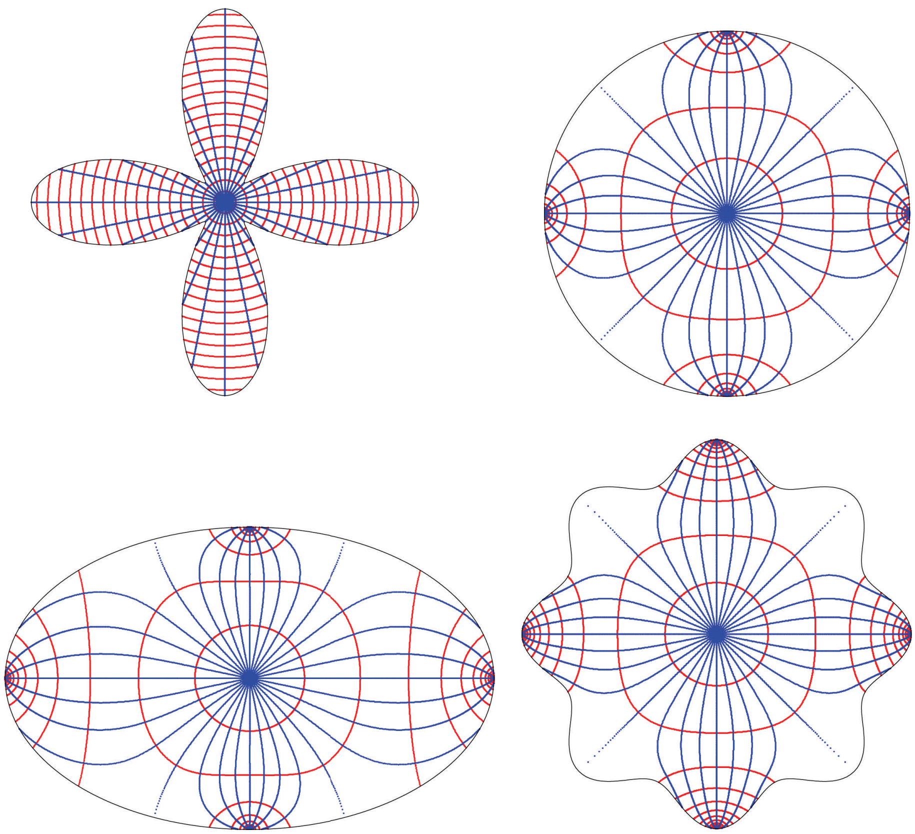

FIGURE 2: The conformal mappings from G_2 onto Ω_1, Ω_2, and Ω_3.

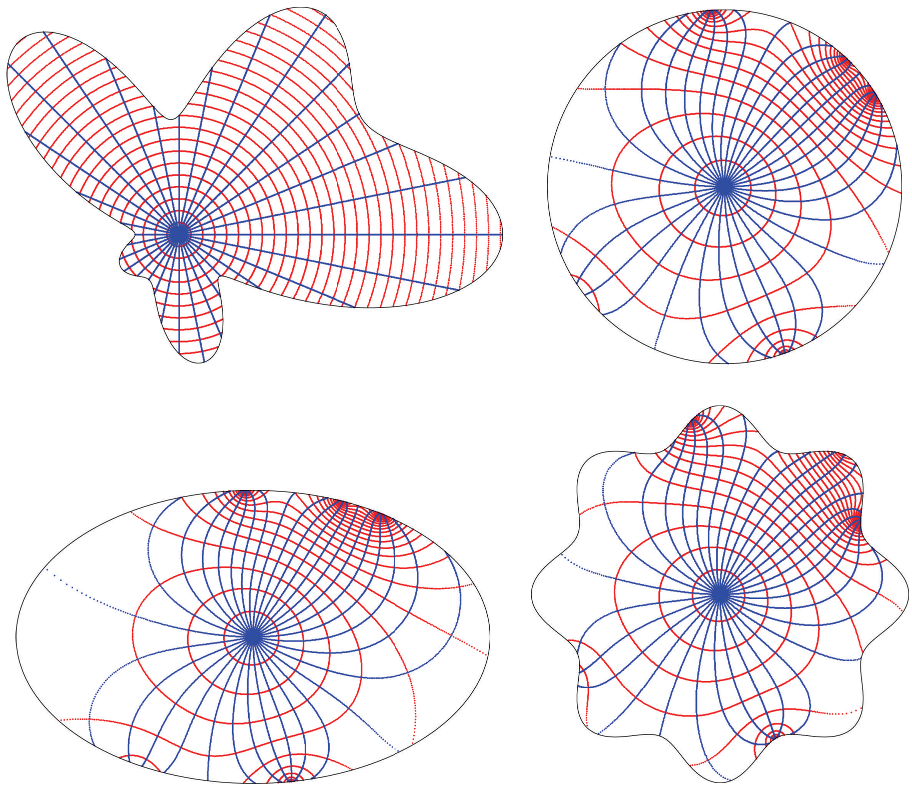

FIGURE 3: The conformal mappings from G_3 onto Ω_1, Ω_2, and Ω_3.

function `zfmm2dpart`, we assume that `iprec` = 4 which means that the tolerance of the FMM is 0.5×10^{-12}. For the function `gmres`, we choose the parameters `restart` = 10, `gmrestol` = 10^{-12}, and `maxit` = 10, which means that the GMRES method is restarted every 10 inner iterations, the tolerance of the GMRES method is 10^{-12}, and the maximum number of outer iterations of GMRES method is 10. See [11, 12] for more details.

By obtaining $\mathbf{x}_k$, we obtain the values $S_{k,n}(t_i)$ for $i = 1, 2, \ldots, n$. Then, the function $S_{k,n}(s) - s$ can be calculated for $s \in [0, 2\pi]$ by the Nyström interpolating formula. The convergence of $S_{k,n}(s)$ to $S(s)$ follows from Theorem 19. Then the values of the mapping function Ψ can be computed from (2). The interior values of the mapping function can be computed by the Cauchy integral formula which can be computed using the fast method presented in [12].

7. Numerical Examples

In this section, we will compute the conformal mapping from three simply connected regions G_1, G_2, and G_3 onto three simply connected regions Ω_1, Ω_2, and Ω_3. The boundaries Γ_1, Γ_2, and Γ_3 of the regions G_1, G_2, and G_3 are parameterized by

$$\eta(t) = \rho(t) e^{\mathrm{i}t}, \quad 0 \le t \le 2\pi, \tag{112}$$

where the function $\rho(t)$ is given by

$$\begin{gathered} \Gamma_1\colon\ \rho(t) = 1 + \frac{1}{4}\cos^3 4t, \\ \Gamma_2\colon\ \rho(t) = 1 + \frac{3}{4}\cos 4t, \\ \Gamma_3\colon\ \rho(t) = e^{\cos t}\cos^2 2t + e^{\sin t}\sin^2 2t. \end{gathered} \tag{113}$$

The boundaries L_1, L_2, and L_3 of the regions Ω_1, Ω_2, and Ω_3 are parameterized by

$$\eta(t) = R(t) e^{\mathrm{i}t}, \quad 0 \le t \le 2\pi, \tag{114}$$

where the function $R(t)$ is given by

$$\begin{gathered} L_1\colon\ R(t) = 1, \\ L_2\colon\ R(t) = \frac{\alpha}{\sqrt{1-(1-\alpha^2)\cos^2 t}}, \\ \alpha = 0.6180339630899485, \\ L_3\colon\ R(t) = 1 + \frac{1}{10}\cos 8t. \end{gathered} \tag{115}$$

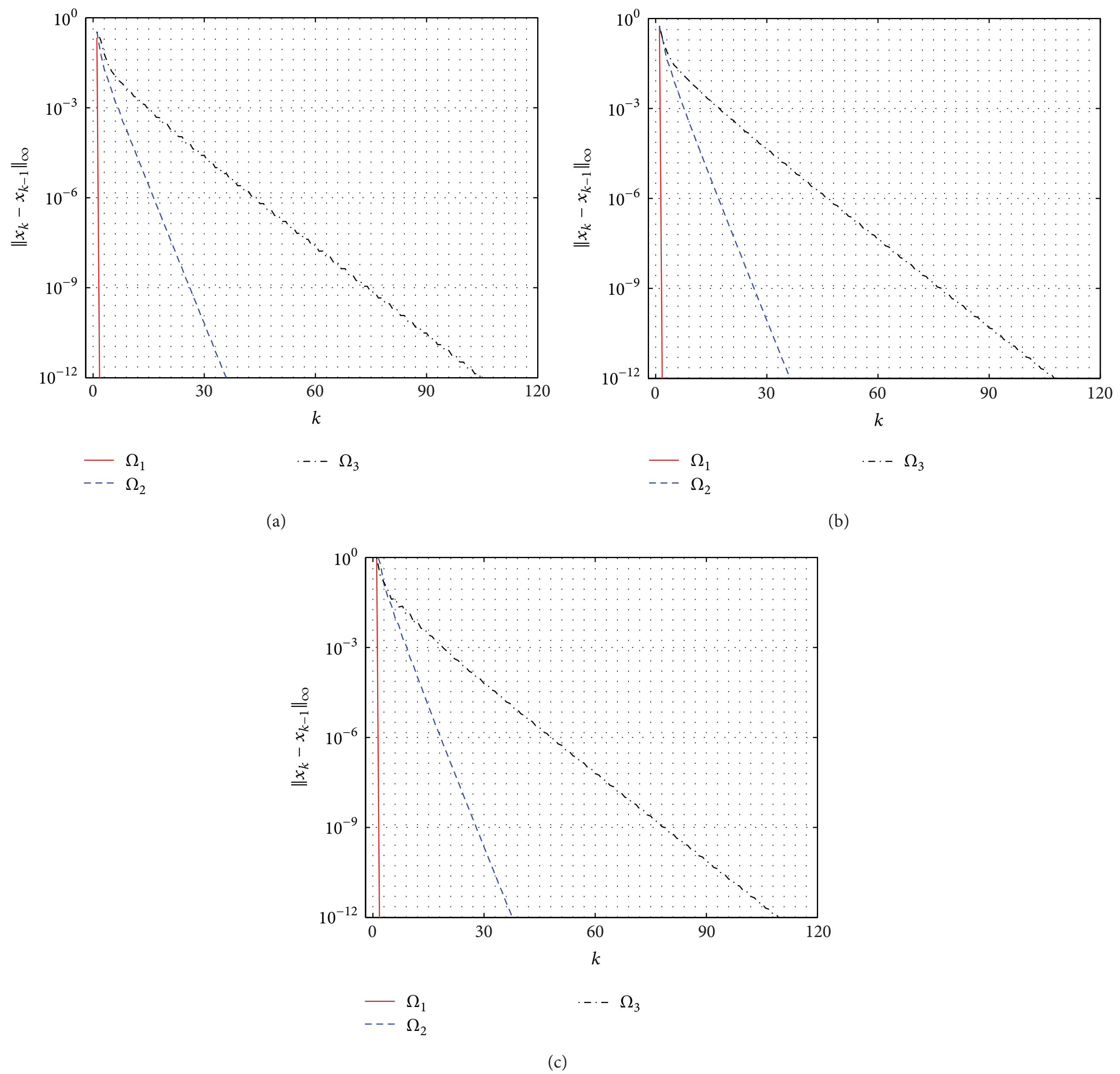

FIGURE 4: The error norm $\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_\infty$ versus the iteration number k for (a) the conformal mapping from G_1 onto Ω_1, Ω_2, and Ω_3, (b) the conformal mapping from G_2 onto Ω_1, Ω_2, and Ω_3, and (c) the conformal mapping from G_3 onto Ω_1, Ω_2, and Ω_3.

The curves L_1, L_2, and L_3 satisfy the ε-condition where

$$\begin{gathered} L_1\colon\ \varepsilon = \left\| \frac{R'(t)}{R(t)} \right\|_\infty = 0, \\ L_2\colon \varepsilon = \left\| \frac{R'(t)}{R(t)} \right\|_\infty = 0.5 < 1, \\ L_3\colon\ \varepsilon = \left\| \frac{R'(t)}{R(t)} \right\|_\infty = 0.80403 < 1. \end{gathered} \tag{116}$$

The numerical results obtained with $n = 4096$ and tol $= 10^{-12}$ are shown in Figures 1, 2, and 3. The error norm $\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_\infty$ versus the iteration number k in (111) is shown in Figure 4. It is clear from Figure 4 that the number of iterations in (111) depends only on the boundary L of the image region. More precisely, it depends on ε. The iterations in (111) converge only if $\varepsilon < 1$. For small ε, a few number of iterations are required for convergence. For values of ε close to 1, a large number of iterations are required for convergence.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] M. M. S. Nasser, "A nonlinear integral equation for numerical conformal mapping," *Advances in Pure and Applied Mathematics*, vol. 1, no. 1, pp. 47–63, 2010.

[2] D. Gaier, *Konstruktive Methoden der konformen Abbildung*, Springer, Berlin, Germany, 1964.

[3] R. Wegmann, "Discretized versions of Newton type iterative methods for conformal mapping," *Journal of Computational and Applied Mathematics*, vol. 29, no. 2, pp. 207–224, 1990.

[4] R. Wegmann, A. H. M. Murid, and M. M. S. Nasser, "The Riemann-Hilbert problem and the generalized Neumann kernel," *Journal of Computational and Applied Mathematics*, vol. 182, no. 2, pp. 388–415, 2005.

[5] D. Gaier, "Numerical methods in conformal mapping," in *Computational Aspects of Complex Analysis*, K. E. Werner, L. Wuytack, and E. Ng, Eds., vol. 102, pp. 51–78, Reidel, Dordrecht, The Netherlands, 1983.

[6] R. Wegmann, "Methods for numerical conformal mapping," in *Handbook of Complex Analysis: Geometric Function Theory*, R. Kuhnau, Ed., vol. 2, pp. 351–477, Elsevier B. V., Amsterdam, The Netherlands, 2005.

[7] A. R. Krommer and C. W. Ueberhuber, *Numerical Integration on Advanced Computer Systems*, vol. 848, Springer, Berlin, Germany, 1994.

[8] P. J. Davis and P. Rabinowitz, *Methods of Numerical Integration*, Academic Press, New York, NY, USA, 2nd edition, 1984.

[9] K. E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, vol. 4, Cambridge University Press, Cambridge, UK, 1997.

[10] M. M. S. Nasser, "A boundary integral equation for conformal mapping of bounded multiply connected regions," *Computational Methods and Function Theory*, vol. 9, no. 1, pp. 127–143, 2009.

[11] M. M. S. Nasser, "Fast solution of boundary integral equations with thegeneralized Neumann kernel," http://arxiv.org/abs/1308.5351.

[12] M. M. S. Nasser and F. A. A. Al-Shihri, "A fast boundary integral equation method for conformal mapping of multiply connected regions," *SIAM Journal on Scientific Computing*, vol. 35, no. 3, pp. A1736–A1760, 2013.

[13] L. Greengard and Z. Gimbutas, "FMMLIB2D: a MATLAB toolbox for fast multipole method in two dimensions," Version 1.2, 2012, http://www.cims.nyu.edu/cmcl/fmm2dlib/fmm2dlib.html.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 683295, 6 pages
http://dx.doi.org/10.1155/2014/683295

Research Article

Implicit Approximation Scheme for the Solution of K-Positive Definite Operator Equation

Naseer Shahzad,[1] Arif Rafiq,[2] and Habtu Zegeye[3]

[1] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21859, Saudi Arabia*
[2] *Department of Mathematics, Lahore Leads University, Lahore 54810, Pakistan*
[3] *Departement of Mathematics, University of Botswana, Private Bag Box 00704, Gaborone, Botswana*

Correspondence should be addressed to Naseer Shahzad; nshahzad@kau.edu.sa

Received 5 December 2013; Accepted 9 February 2014; Published 23 March 2014

Academic Editor: Ljubomir B. Ćirić

We construct an implicit sequence suitable for the approximation of solutions of K-positive definite operator equations in real Banach spaces. Furthermore, implicit error estimate is obtained and the convergence is shown to be faster in comparsion to the explicit error estimate obtained by Osilike and Udomene (2001).

1. Introduction

Let E be a real Banach space and let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \ \|f^*\| = \|x\|\right\}, \quad (1)$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single valued. We will denote the single-valued duality mapping by j.

Let E be a Banach space. The *modulus of smoothness* of E is the function.

$\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : \|x\| \le 1, \|y\| \le t\right\}. \quad (2)$$

The Banach space E is called *uniformly smooth* if

$$\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0. \quad (3)$$

A Banach space E is said to be *strictly convex* if for two elements $x, y \in E$ which are linearly independent we have that $\|x + y\| < \|x\| + \|y\|$.

Let E_1 be a dense subspace of a Banach space E. An operator T with domain $D(T) \supseteq E_1$ is called continuously E_1-invertible if the range of T, $R(T)$, with T in E considered as an operator restricted to E_1, is dense in E and T has a bounded inverse on $R(T)$.

Let E be a Banach space and let A be a linear unbounded operator defined on a dense domain, $D(A)$, in E. An operator A will be called K positive definite (Kpd) [1] if there exist a continuously $D(A)$-invertible closed linear operator K with $D(A) \subset D(K)$ and a constant $c > 0$ such that $j(Kx) \in J(Kx)$,

$$\langle Ax, j(Kx)\rangle \ge c\|Kx\|^2, \quad \forall x \in D(A). \quad (4)$$

Without loss of generality, we assume that $c \in (0, 1)$.

In [1], Chidume and Aneke established the extension of Kpd operators of Martynjuk [2] and Petryshyn [3, 4] from Hilbert spaces to arbitrary real Banach spaces. They proved the following result.

Theorem 1. *Let E be a real separable Banach space with a strictly convex dual E and let A be a Kpd operator with $D(A) = D(K)$. Suppose*

$$\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle, \quad \forall x, y \in D(A). \quad (5)$$

Then, there exists a constant $\alpha > 0$ such that for all $x \in D(A)$

$$\|Ax\| \le \alpha \|Kx\|. \quad (6)$$

Furthermore, the operator A is closed, $R(A) = E$, and the equation $Ax = f$ has a unique solution for any given $f \in E$.

As the special case of Theorem 1 in which $E = L_p\ (l_p)$ spaces, $2 \leq p < \infty$, Chidume and Aneke [1] introduced an iteration process which converges strongly to the unique solution of the equation $Ax = f$, where A and K are commuting. Recently, Chidume and Osilike [5] extended the results of Chidume and Aneke [1] to the more general real separable q-uniformly smooth Banach spaces, $1 < q < \infty$, by removing the commutativity assumption on A and K. Later on, Chuanzhi [6] proved convergence theorems for the iterative approximation of the solution of the Kpd operator equation $Ax = f$ in more general separable uniformly smooth Banach spaces.

In [7], Osilike and Udomene proved the following result.

Theorem 2. *Let E be a real separable Banach space with a strictly convex dual and let $A : D(A) \subseteq E \to E$ be a Kpd operator with $D(A) = D(K)$. Suppose $\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle$ for all $x, y \in D(A)$. Choose any $\epsilon_1 \in (0, c^2/(1 + \alpha(1-c) + \alpha^2))$ and define $T_\epsilon : D(A) \subseteq E \to E$ by*

$$T_\epsilon x = x + \epsilon K^{-1} f - \epsilon K^{-1} Ax. \tag{7}$$

Then the Picard iteration scheme generated from an arbitrary $x_0 \in D(A)$ by

$$x_{n+1} = T_\epsilon x_n = T_\epsilon^n x_0 \tag{8}$$

converges strongly to the solution of the equation $Ax = f$. Moreover, if x^* denotes the solution of the equation $Ax = f$, then

$$\left\| x_{n+1} - x^* \right\| \leq (1 - c\epsilon (1-c))^n \beta^{-1} \left\| Kx_0 - Kx^* \right\|. \tag{9}$$

The most general iterative formula for approximating solutions of nonlinear equation and fixed point of nonlinear mapping is the Mann iterative method [8] which produces a sequence $\{x_n\}$ via the recursive approach $x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n$, for nonlinear mapping $T : C = D(T) \to C$, where the initial guess $x_0 \in C$ is chosen arbitrarily. For convergence results of this scheme and related iterative schemes, see, for example, [9–15].

In [16], Xu and Ori introduced the implicit iteration process $\{x_n\}$, which is the modification of Mann, generated by $x_0 \in C$, $x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n$, for $T_i, i = 1, 2, \ldots, N$, nonexpansive mappings, and $T_n = T_n\ (\mathrm{mod} N)$ and $\{\alpha_n\} \subset (0, 1)$. They proved the weak convergence of this process to a common fixed point of the finite family of nonexpansive mappings in Hilbert spaces. Since then fixed point problems and solving (or approximating) nonlinear equations based on implicit iterative processes have been considered by many authors (see, e.g., [17–21]).

It is our purpose in this paper to introduce implicit scheme which converges strongly to the solution of the Kpd operator equation $Ax = f$ in a separable Banach space. Even though our scheme is implicit, the error estimate obtained indicates that the convergence of the implicit scheme is faster in comparison to the explicit scheme obtained by Osilike and Udomene [7].

2. Main Results

We need the following results.

Lemma 3 (see [10]). *If E^* is uniformly convex then there exists a continuous nondecreasing function $b : [0, \infty) \to [0, \infty)$ such that $b(0) = 0$, $b(\delta t) \leq \delta b(t)$ for all $\delta \geq 1$ and*

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x) \rangle + \max \{\|x\|, 1\} \|y\| b (\|y\|), \tag{R}$$

for all $x, y \in E$.

Lemma 4 (see [22]). *If there exists a positive integer N such that for all $n \geq N$, $n \in \mathbb{N}$ (the set of all positive integers),*

$$\rho_{n+1} \leq (1 - \theta_n) \rho_n + b_n, \tag{10}$$

then

$$\lim_{n \to \infty} \rho_n = 0, \tag{11}$$

where $\theta_n \in [0, 1)$, $\sum_{n=1}^{\infty} \theta_n = \infty$ and $b_n = o(\theta_n)$.

Remark 5 (see [6]). Since K is continuously $D(A)$ invertible, there exists a constant $\beta > 0$ such that

$$\|Kx\| \geq \beta \|x\|, \quad \forall x \in D(K) = D(A). \tag{12}$$

In the continuation $c \in (0, 1)$, α and β are the constants appearing in (4), (6), and (12), respectively. Furthermore, $\epsilon > 0$ is defined by

$$\epsilon = \frac{c - \eta}{\alpha (1 - \eta)}, \quad \eta \in (0, c). \tag{13}$$

With these notations, we now prove our main results.

Theorem 6. *Let E be a real separable Banach space with a strictly convex dual and let $A : D(A) \subseteq E \to E$ be a Kpd operator with $D(A) = D(K)$. Suppose $\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle$ for all $x, y \in D(A)$. Let x^* denote a solution of the equation $Ax = f$. For arbitrary $x_0 \in E$, define the sequence $\{x_n\}_{n=0}^{\infty}$ in E by*

$$x_n = x_{n-1} + \epsilon K^{-1} f - \epsilon K^{-1} A x_n, \quad n \geq 0. \tag{14}$$

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^* with

$$\left\| x_n - x^* \right\| \leq \rho^n \beta^{-1} \left\| Kx_0 - Kx^* \right\|, \tag{15}$$

where $\rho = 1 - ((c - \eta)/(\alpha(1-\eta) + c - \eta))\eta \in (0,1)$. Thus, the choice $\eta = c/2$ yields $\rho = 1 - (c^2/(4\alpha(1 - c/2) + 2c))$. Moreover, x^* is unique.

Proof. The existence of the unique solution to the equation $Ax = f$ comes from Theorem 1. From (4) we have

$$\langle Ax - cKx, j(Kx) \rangle \geq 0, \tag{16}$$

and from Lemma 1.1 of Kato [23], we obtain that

$$\|Kx\| \leq \left\| Kx + \gamma (Ax - cKx) \right\|, \tag{17}$$

for all $x \in D(A)$ and $\gamma > 0$. Now, from (14), linearity of K and the fact that $Ax^* = f$ we obtain that

$$\begin{aligned} Kx_n &= Kx_{n-1} + \epsilon f - \epsilon Ax_n \\ &= Kx_{n-1} + \epsilon Ax^* - \epsilon Ax_n, \end{aligned} \tag{18}$$

which implies that

$$Kx_{n-1} = Kx_n - \epsilon Ax^* + \epsilon Ax_n, \tag{19}$$

so that

$$Kx_{n-1} - Kx^* = Kx_n - Kx^* - \epsilon Ax^* + \epsilon Ax_n. \tag{20}$$

With the help of (14) and Theorem 1, we have the following estimate:

$$\begin{aligned} \|Ax_n - Ax^*\| = & \|A(x_n - x^*)\| \le \alpha \|K(x_n - x^*)\| \\ &= \alpha \|Kx_n - Kx^*\| \\ &= \alpha \|Kx_{n-1} - Kx^* - \epsilon(Ax_n - Ax^*)\| \\ &\le \alpha \|Kx_{n-1} - Kx^*\| + \alpha\epsilon \|Ax_n - Ax^*\|, \end{aligned} \tag{21}$$

which gives

$$\|Ax_n - Ax^*\| \le \frac{\alpha}{1-\alpha\epsilon} \|Kx_{n-1} - Kx^*\|. \tag{22}$$

Furthermore, inequality (20) can be rewritten as

$$\begin{aligned} &Kx_{n-1} - Kx^* \\ &\quad = (1+\epsilon)(Kx_n - Kx^*) \\ &\qquad + \epsilon(Ax_n - Ax^* - c(Kx_n - Kx^*)) \\ &\qquad - \epsilon(1-c)(Kx_n - Kx^*) \\ &\quad = (1+\epsilon)\Big[Kx_n - Kx^* \\ &\qquad\qquad + \frac{\epsilon}{1+\epsilon}(Ax_n - Ax^* - c(Kx_n - Kx^*))\Big] \\ &\qquad - \epsilon(1-c)(Kx_n - Kx^*) \\ &\quad = (1+\epsilon)\Big[Kx_n - Kx^* \\ &\qquad\qquad + \frac{\epsilon}{1+\epsilon}(Ax_n - Ax^* - c(Kx_n - Kx^*))\Big] \\ &\qquad - \epsilon(1-c)(Kx_{n-1} - Kx^*) + \epsilon^2(1-c)(Ax_n - Ax^*). \end{aligned} \tag{23}$$

In addition, from (17) and (22), we get that

$$\begin{aligned} &\|Kx_{n-1} - Kx^*\| \\ &\quad \ge (1+\epsilon)\Big\|Kx_n - Kx^* \\ &\qquad\qquad + \frac{\epsilon}{1+\epsilon}(Ax_n - Ax^* - c(Kx_n - Kx^*))\Big\| \\ &\qquad - \epsilon(1-c)\|Kx_{n-1} - Kx^*\| - \epsilon^2(1-c)\|Ax_n - Ax^*\| \\ &\quad \ge (1+\epsilon)\|Kx_n - Kx^*\| - \epsilon(1-c)\|Kx_{n-1} - Kx^*\| \\ &\qquad - \epsilon^2(1-c)\frac{\alpha}{1-\alpha\epsilon}\|Kx_{n-1} - Kx^*\|, \end{aligned} \tag{24}$$

which implies that

$$\begin{aligned} &\|Kx_n - Kx^{**}\| \\ &\quad \le \frac{1 + \epsilon(1-c) + \epsilon^2(1-c)(\alpha/(1-\alpha\epsilon))}{1+\epsilon}\|Kx_{n-1} - Kx^*\| \\ &\quad = \rho\|Kx_{n-1} - Kx^*\|, \end{aligned} \tag{25}$$

where

$$\begin{aligned} \rho &= \frac{1 + \epsilon(1-c) + \epsilon^2(1-c)(\alpha/(1-\alpha\epsilon))}{1+\epsilon} \\ &= 1 - \frac{\epsilon}{1+\epsilon}\left(c - \epsilon(1-c)\frac{\alpha}{1-\alpha\epsilon}\right) \\ &= 1 - \frac{\epsilon}{1+\epsilon}\eta \\ &= 1 - \frac{c-\eta}{\alpha(1-\eta)+c-\eta}\eta \\ &= 1 - \frac{c^2}{4\alpha(1-c/2)+2c}. \end{aligned} \tag{26}$$

From (25) and (26), we have that

$$\|Kx_n - Kx^*\| \le \rho\|Kx_{n-1} - Kx^*\| \le \cdots \le \rho^n\|K(x_0 - x^*)\|. \tag{27}$$

Hence by Remark 5, we get that

$$\begin{aligned} &\|x_n - x^*\| \\ &\quad \le \beta^{-1}\|Kx_n - Kx^*\| \le \cdots \le \rho^n\beta^{-1}\|Kx_0 - Kx^*\| \longrightarrow 0, \end{aligned} \tag{28}$$

as $n \to \infty$. Thus, $x_n \to x^*$ as $n \to \infty$. □

In [6], Chuanzhi provided the following result.

Theorem 7. *Let E be a real uniformly smooth separable Banach space, and let $A : D(A) \subseteq E \to E$ be a Kpd operator with $D(A) = D(K)$. Suppose $\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle$ for all $x, y \in D(A)$. For arbitrary $f \in E$ and $x_0 \in D(A)$, define the sequence $\{x_n\}_{n=0}^{\infty}$ by*

$$\begin{gathered} x_{n+1} = x_n + t_n y_n, \\ y_n = K^{-1}f - K^{-1}Ax_n, \\ 0 \le t_n \le \frac{1}{2c}, \\ \sum t_n = 0, \qquad \lim_{n\to\infty} t_n = 0, \\ b(\alpha t_n) \le \frac{2c}{B\alpha}, \quad n \ge 0, \end{gathered} \tag{29}$$

where $b(t)$ is as in (R), α is the constant appearing in inequality (6), *c is the constant appearing in inequality* (4), *and*

$$B = \max\{\|K\gamma_0\|, 1\}. \quad (30)$$

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Ax = f$.

However, its implicit version is as follows.

Theorem 8. *Let E be a real uniformly smooth separable Banach space, and let $A : D(A) \subseteq E \to E$ be a Kpd operator with $D(A) = D(K)$. Suppose $\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle$ for all $x, y \in D(A)$. For arbitrary $f \in E$ and $x_0 \in D(A)$, define the sequence $\{x_n\}_{n=0}^{\infty}$ by*

$$x_n = x_{n-1} + t_n\gamma_n, \quad (31)$$

$$\gamma_n = K^{-1}f - K^{-1}Ax_n, \quad (32)$$

$$\sum t_n = \infty, \quad \lim_{n\to\infty} t_n = 0, \quad n \geq 0. \quad (33)$$

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Ax = f$.

Proof. The existence of the unique solution to the equation $Ax = f$ comes from Theorem 1. Using (31) and (32) we obtain

$$K\gamma_n = K\gamma_{n-1} - t_nA\gamma_n. \quad (34)$$

Consider

$$\begin{aligned}\|K\gamma_n\|^2 &= \langle K\gamma_n, j(K\gamma_n)\rangle = \langle K\gamma_{n-1} - t_nA\gamma_n, j(K\gamma_n)\rangle \\ &= \langle K\gamma_{n-1}, j(K\gamma_n)\rangle - t_n\langle A\gamma_n, j(K\gamma_n)\rangle \\ &\leq \|K\gamma_{n-1}\|\,\|K\gamma_n\| - ct_n\|K\gamma_n\|^2,\end{aligned} \quad (35)$$

which implies that

$$\|K\gamma_n\| \leq \|K\gamma_{n-1}\| - ct_n\|K\gamma_n\|. \quad (36)$$

Hence, $\{K\gamma_n\}_{n=0}^{\infty}$ is bounded. Let

$$M_1 = \sup_{n\geq 0}\|K\gamma_n\|. \quad (37)$$

Also from (6) it can be easily seen that $\{A\gamma_n\}_{n=0}^{\infty}$ is also bounded. Let

$$M_2 = \sup_{n\geq 0}\|A\gamma_n\|. \quad (38)$$

Denote $M = M_1 + M_2$; then $M < \infty$.

By using (34) and Lemma 3, we have

$$\begin{aligned}\|K\gamma_n\|^2 &= \|K\gamma_{n-1} - t_nA\gamma_n\|^2 \\ &\leq \|K\gamma_{n-1}\|^2 - 2t_n\langle A\gamma_n, j(K\gamma_{n-1})\rangle \\ &\quad + \max\{\|K\gamma_{n-1}\|, 1\}\|t_nA\gamma_n\|\,b(\|t_nA\gamma_n\|) \\ &= \|K\gamma_{n-1}\|^2 - 2t_n\langle A\gamma_{n-1}, j(K\gamma_{n-1})\rangle \\ &\quad + 2t_n\langle A\gamma_{n-1} - A\gamma_n, j(K\gamma_{n-1})\rangle \\ &\quad + \max\{\|K\gamma_{n-1}\|, 1\}t_n\|A\gamma_n\|\,b(t_n\|A\gamma_n\|) \\ &\leq (1 - 2ct_n)\|K\gamma_{n-1}\|^2 + 2t_n\|A\gamma_{n-1} - A\gamma_n\|\,\|K\gamma_{n-1}\| \\ &\quad + \max\{\|K\gamma_{n-1}\|, 1\}\alpha t_n\|K\gamma_n\|\,b(\alpha t_n\|K\gamma_n\|) \\ &\leq (1 - 2ct_n)\|K\gamma_{n-1}\|^2 + 2Mt_n\eta_n \\ &\quad + \max\{M, 1\}\alpha^2M^2t_nb(t_n),\end{aligned} \quad (39)$$

where

$$\eta_n = \|A\gamma_{n-1} - A\gamma_n\|. \quad (40)$$

By using (6) and (34) we obtain that

$$\begin{aligned}\|A\gamma_{n-1} - A\gamma_n\| &= \|A(\gamma_{n-1} - \gamma_n)\| \leq \alpha\|K(\gamma_{n-1} - \gamma_n)\| \\ &= \alpha t_n\|A\gamma_n\| \leq M\alpha t_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \quad (41)$$

Thus,

$$\eta_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (42)$$

Denote

$$\begin{aligned}\rho_n &= \|x_n - p\|, \\ \theta_n &= 2ct_n, \\ \sigma_n &= 2Mt_n\eta_n + \max\{M, 1\}\alpha^2M^2t_nb(t_n).\end{aligned} \quad (43)$$

Condition (33) assures the existence of a rank $n_0 \in \mathbb{N}$ such that $\theta_n = 2ct_n \leq 1$, for all $n \geq n_0$. Since $b(t)$ is continuous, so $\lim_{n\to\infty} b(t_n) = 0$ (by condition (33)). Now with the help of (33), (42), and Lemma 4, we obtain from (39) that

$$\lim_{n\to\infty}\|K\gamma_n\| = 0. \quad (44)$$

At last by Remark 5, $\gamma_n \to 0$ as $n \to \infty$; that is $Ax_n \to f$ as $n \to \infty$. Because A has bounded inverse, this implies that $x_n \to A^{-1}f$, the unique solution of $Ax_n = f$. This completes the proof. □

Remark 9. (1) According to the estimates (6–8) of Martynjuk [2], we have

$$\begin{aligned}&\|Kx_{n+1} - Kx^*\| \\ &\quad \leq \frac{1 + \epsilon_1(1 - c) + \alpha\epsilon_1^2(1 - c + \alpha)}{1 + \epsilon_1}\|Kx_n - Kx^*\| \\ &\quad = \theta\|Kx_n - Kx^*\|,\end{aligned} \quad (45)$$

where

$$\begin{aligned}\theta &= \frac{1 + \epsilon_1(1 - c) + \alpha\epsilon_1^2(1 - c + \alpha)}{1 + \epsilon_1} \\ &= 1 - \frac{\epsilon_1}{1 + \epsilon_1}(c - \alpha(1 - c + \alpha)\epsilon_1) \\ &= 1 - \frac{\epsilon_1}{1 + \epsilon_1}\eta,\end{aligned} \quad (46)$$

for $\eta = c - \alpha(1 - c + \alpha)\epsilon_1$ or $\epsilon_1 = (c - \eta)/\alpha(1 - c + \alpha)$, $\eta \in (0, c)$. Thus,

$$\begin{aligned}\theta &= 1 - \frac{c - \eta}{\alpha(1 - c + \alpha) + c - \eta}\eta \\ &= 1 - \frac{c^2}{4\alpha(1 - c + \alpha) + 2c}.\end{aligned} \quad (47)$$

Table 1

n	1	2	3	4	5	6	7	8
x_n	0.0922	0.0851	0.07859	0.07528	0.06947	0.06411	0.05916	0.05459

Table 2

n	1	2	3	4	5	6	7	8
x_n	0.0893	0.0798	0.07136	0.06376	0.05698	0.05092	0.04550	0.04066

Table 3

n	1	2	3	4	5	6	7	8
x_n	0.0098	0.0096	0.00949	0.00933	0.00917	0.00901	0.00885	0.00870

Table 4

n	1	2	3	4	5	6	7	8
x_n	0.0098	0.0096	0.00941	0.00923	0.00905	0.00887	0.00869	0.00852

(2) For $\alpha > c/2$, we observe that

$$\rho = 1 - \frac{c^2}{4\alpha(1-c/2)+2c} = \theta - \frac{4\alpha c^2}{(4\alpha(1-c/2)+2c)(4\alpha(1-c+\alpha)+2c)}\left(\alpha - \frac{c}{2}\right). \quad (48)$$

Thus, the relation between Martynjuk [2] and our parameter of convergence, that is, between θ and ρ, respectively, is the following:

$$\rho < \theta. \quad (49)$$

Despite the fact that our scheme is implicit, inequality (49) shows that the results of Osilike and Udomene [7] are improved in the sense that our scheme converges faster.

Example 10. Suppose $E = \mathbb{R}$, $D(A) = \mathbb{R}_+$, $Ax = x$, $Kx = 2I$ ($x^* = 0$ is the solution of $Ax = f$); then for the explicit iterative scheme due to Osilike and Udomene [7] we have

$$Kx_{n+1} = Kx_n - \epsilon_1 Ax_n, \quad (50)$$

which implies that

$$2x_{n+1} = 2x_n - \epsilon_1 x_n, \quad (51)$$

and hence

$$x_{n+1} = \left(1 - \frac{\epsilon_1}{2}\right)x_n. \quad (52)$$

Also for the implicit iterative scheme we have that

$$Kx_n = Kx_{n-1} - \epsilon Ax_n, \quad (53)$$

which implies that

$$x_n = \frac{1}{1+\epsilon/2}x_{n-1}. \quad (54)$$

It can be easily seen that for $c \le 1/2$ and $\alpha \ge 1/2$, (4) and (6) are satisfied. Suppose $c = 1/4$ and $\alpha = 3/5$; then $\eta = 0.125$, $\epsilon = (c-\eta)/\alpha(1-\eta) = 0.23810$, $\epsilon_1 = (c-\eta)/\alpha(1-c+\alpha) = 0.15432$, $\rho = 0.97596$, and $\theta = 0.983288$ and so $\rho < \theta$. Take $x_0 = 0.1$; then from (52) we have Table 1 and for (54) we get Table 2.

Example 11. Let us take $E = \mathbb{R}$, $D(A) = \mathbb{R}_+$, $Ax = (1/4)x$, $Kx = 2x$ ($x^* = 0$ is the solution of $Ax = f$); then for the explicit iterative scheme due to Osilike and Udomene [7] we have

$$Kx_{n+1} = Kx_n - \epsilon_1 Ax_n, \quad (55)$$

which implies that

$$2x_{n+1} = 2x_n - \frac{\epsilon_1}{4}x_n, \quad (56)$$

and hence

$$x_{n+1} = \left(1 - \frac{\epsilon_1}{8}\right)x_n. \quad (57)$$

Also for the implicit iterative scheme we have that

$$Kx_n = Kx_{n-1} - \epsilon Ax_n, \quad (58)$$

which implies that

$$x_n = \frac{1}{1+\epsilon/8}x_{n-1}. \quad (59)$$

It can be easily seen that for $c \le 1/8$ and $\alpha \ge 1/8$, (4) and (6) are satisfied. Suppose $c = 0.0625$ and $\alpha = 0.2$; then $\eta = 0.03125$, $\epsilon = (c-\eta)/\alpha(1-\eta) = 0.16129$, $\epsilon_1 = (c-\eta)/\alpha(1-c+\alpha) = 0.13736$, $\rho = 0.99566$, and $\theta = 0.99623$ and so $\rho < \theta$. Take $x_0 = 0.01$; then from (57) we have Table 3 and for (59) we get Table 4.

Even though our scheme is implicit we observe that it converges strongly to the solution of the Kpd operator equation $Ax = f$ with the error estimate which is faster in comparison to the explicit error estimate obtained by Osilike and Udomene [7].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The first author, therefore, acknowledges his thanks to DSR for the financial support. This paper is dedicated to Professor Miodrag Mateljevi'c on the occasion of his 65th birthday.

References

[1] C. E. Chidume and S. J. Aneke, "Existence, uniqueness and approximation of a solution for a K-positive definite operator equation," *Applicable Analysis*, vol. 50, no. 3-4, pp. 285–294, 1993.

[2] A. E. Martynjuk, "Some new applications of methods of Galerkin type," vol. 49, no. 1, pp. 85–108, 1959.

[3] W. V. Petryshyn, "Direct and iterative methods for the solution of linear operator equations in Hilbert space," *Transactions of the American Mathematical Society*, vol. 105, no. 1, pp. 136–175, 1962.

[4] W. V. Petryshyn, "On a class of K-p.d. and non-K-p.d. operators and operator equations," *Journal of Mathematical Analysis and Applications*, vol. 10, no. 1, pp. 1–24, 1965.

[5] C. E. Chidume and M. O. Osilike, "Approximation of a solution for a K-Positive definite operator equation," *Journal of Mathematical Analysis and Applications*, vol. 210, no. 1, pp. 1–7, 1997.

[6] B. Chuanzhi, "Approximation of a solution for a K-positive definite operator equation in uniformly smooth separable Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 236, no. 2, pp. 236–242, 1999.

[7] M. O. Osilike and A. Udomene, "A note on approximation of solutions of a K-positive definite operator equations," *Bulletin of the Korean Mathematical Society*, vol. 38, no. 2, pp. 231–236, 2001.

[8] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.

[9] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.

[10] S. Reich, "An iterative procedure for constructing zeros of accretive sets in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 2, no. 1, pp. 85–92, 1978.

[11] N. Shahzad and H. Zegeye, "On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 3-4, pp. 838–844, 2009.

[12] C. H. Xiang, Z. Chen, and K. Q. Zhao, "Some convergence theorems for a class of generalized Φ-hemicontractive mappings," *Journal of Concrete and Applicable Mathematics*, vol. 8, no. 4, pp. 638–644, 2010.

[13] Y. Xu, "Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations," *Journal of Mathematical Analysis and Applications*, vol. 224, no. 1, pp. 91–101, 1998.

[14] H. Zegeye and N. Shahzad, "Convergence of Mann's type iteration method for generalized asymptotically nonexpansive mappings," *Computers and Mathematics with Applications*, vol. 62, no. 11, pp. 4007–4014, 2011.

[15] H. Zegeye and N. Shahzad, "Approximation of the common minimum-norm fixed point of a finite family of asymptotically nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2013, no. 1, article 1, 2013.

[16] H.-K. Xu and R. G. Ori, "An implicit iteration process for nonexpansive mappings," *Numerical Functional Analysis and Optimization*, vol. 22, no. 5-6, pp. 767–773, 2001.

[17] N. Shahzad and H. Zegeye, "Strong convergence of an implicit iteration process for a finite family of generalized asymptotically quasi-nonexpansive maps," *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1058–1065, 2007.

[18] N. Shahzad and H. Zegeye, "Strong convergence results for nonself multimaps in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 136, no. 2, pp. 539–548, 2008.

[19] Y. Zhou and S.-S. Chang, "Convergence of implicit iteration process for a finite family of asymptotically nonexpansive mappings in Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 23, no. 7-8, pp. 911–921, 2002.

[20] N. Shahzad and H. Zegeye, "Viscosity approximation methods for nonexpansive multimaps in Banach spaces," *Acta Mathematica Sinica*, vol. 26, no. 6, pp. 1165–1176, 2010.

[21] C. E. Chidume and N. Shahzad, "Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 62, no. 6, pp. 1149–1156, 2005.

[22] X. Weng, "Fixed point iteration for local strictly pseudocontractive mapping," *Proceedings of the American Mathematical Society*, vol. 113, no. 3, pp. 727–731, 1991.

[23] T. Kato, "Nonlinear semigroups and evolution equations," *Journal of the Mathematical Society of Japan*, vol. 19, no. 4, pp. 508–520, 1967.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 303484, 5 pages
http://dx.doi.org/10.1155/2014/303484

Research Article

Caristi Fixed Point Theorem in Metric Spaces with a Graph

M. R. Alfuraidan[1] and M. A. Khamsi[2]

[1] *Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia*
[2] *Department of Mathematical Science, The University of Texas at El Paso, El Paso, TX 79968, USA*

Correspondence should be addressed to M. R. Alfuraidan; monther@kfupm.edu.sa

Received 27 January 2014; Accepted 11 February 2014; Published 13 March 2014

Academic Editor: Qamrul Hasan Ansari

We discuss Caristi's fixed point theorem for mappings defined on a metric space endowed with a graph. This work should be seen as a generalization of the classical Caristi's fixed point theorem. It extends some recent works on the extension of Banach contraction principle to metric spaces with graph.

Dedicated to Rashed Saleh Alfuraidan and Prof. Miodrag Mateljević for his 65th birthday

1. Introduction

This work was motivated by some recent works on the extension of Banach contraction principle to metric spaces with a partial order [1] or a graph [2]. Caristi's fixed point theorem is maybe one of the most beautiful extensions of Banach contraction principle [3, 4]. Recall that this theorem states the fact that any map $T : M \to M$ has a fixed point provided that M is a complete metric space and there exists a lower semicontinuous map $\phi : M \to [0, +\infty)$ such that

$$d(x, Tx) \le \phi(x) - \phi(Tx), \tag{1}$$

for every $x \in M$. Recall that $x \in M$ is called a fixed point of T if $T(x) = x$. This general fixed point theorem has found many applications in nonlinear analysis. It is shown, for example, that this theorem yields essentially all the known inwardness results [5] of geometric fixed point theory in Banach spaces. Recall that inwardness conditions are the ones which assert that, in some sense, points from the domain are mapped toward the domain. Possibly, the weakest of the inwardness conditions, the Leray-Schauder boundary condition, is the assumption that a map points x of ∂M anywhere except to the outward part of the ray originating at some interior point of M and passing through x.

The proofs given to Caristi's result vary and use different techniques (see [3, 6–8]). It is worth to mention that because of Caristi's result of close connection to the Ekeland's [9] variational principle, many authors refer to it as Caristi-Ekeland fixed point result. For more on Ekeland's variational principle and the equivalence between Caristi-Ekeland fixed point result and the completeness of metric spaces, the reader is advised to read [10].

2. Main Results

Maybe one of the most interesting examples of the use of metric fixed point theorems is the proof of the existence of solutions to differential equations. The general approach is to convert such equations to integral equations which describes exactly a fixed point of a mapping. The metric spaces in which such mapping acts are usually a function space. Putting a norm (in the case of a vector space) or a distance gives us a metric structure rich enough to use the Banach contraction principle or other known fixed point theorems. But one structure naturally enjoyed by such function spaces is rarely used. Indeed we have an order on the functions inherited from the order of $\mathbb{R}$. In the classical use of Banach contraction principle, the focus is on the metric behavior of the mapping. The connection with the natural order is usually ignored.

In [1, 11], the authors gave interesting examples where the order is used combined with the metric conditions.

Example 1 (see [1]). Consider the periodic boundary value problem

$$u'(t) = f(t, u(t)), \quad t \in [0, T], \qquad u(0) = u(T), \tag{2}$$

where $T > 0$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Clearly any solution to this problem must be continuously differentiable on $[0, T]$. So the space to be considered for this problem is $C^1([0, T], \mathbb{R})$. The above problem is equivalent to the integral problem

$$u(t) = \int_0^T G(t, s)\left[f(s, u(s)) + \lambda u(s)\right] ds, \tag{3}$$

where $\lambda > 0$ and the Green function is given by

$$G(t, s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} & 0 \le s < t \le T, \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} & 0 \le t < s \le T. \end{cases} \tag{4}$$

Define the mapping $\mathscr{A} : C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R})$ by

$$\mathscr{A}(u)(t) = \int_0^T G(t, s)\left[f(s, u(s)) + \lambda u(s)\right] ds. \tag{5}$$

Note that if $u(t) \in C([0, T], \mathbb{R})$ is a fixed point of $\mathscr{A}$, then $u(t) \in C^1([0, T], \mathbb{R})$ is a solution to the original boundary value problem. Under suitable assumptions, the mapping $\mathscr{A}$ satisfies the following property:

(i) if $u(t) \le v(t)$, then we have $\mathscr{A}(u) \le \mathscr{A}(v)$;

(ii) if $u(t) \le v(t)$, then

$$\|\mathscr{A}(u) - \mathscr{A}(v)\| \le k\|u - v\|, \tag{6}$$

for a constant $k < 1$ independent of u and v.

The contractive condition is only valid for comparable functions. It does not hold on the entire space $C([0, T], \mathbb{R})$. This condition led the authors in [1] to use a weaker version of the Banach contraction principle to prove the existence of the solution to the original boundary value problem, a result which was already known [12] using different techniques.

Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$. We will say that T is monotone increasing if

$$x \preceq y \Longrightarrow T(x) \preceq T(y), \quad \text{for any } x, y \in X. \tag{7}$$

The main result of [1] is the following theorem.

Theorem 2 (see [1]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a distance d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and monotone increasing mapping such that there exists $k < 1$ with*

$$d(T(x), T(y)) \le k d(x, y), \quad \text{whenever } y \preceq x. \tag{8}$$

If there exists $x_0 \in X$, with $x_0 \preceq T(x_0)$, then T has a fixed point.

Clearly, from this theorem, one may see that the contractive nature of the mapping T is restricted to the comparable elements of $(X, \preceq)$ not to the entire set X. The detailed investigation of the example above shows that such mappings may exist which are not contractive on the entire set X. Therefore the classical Banach contraction principle will not work in this situation. The analogue to Caristi's fixed theorem in this setting is the following result.

Theorem 3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a distance d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and monotone increasing mapping. Assume that there exists a lower semicontinuous function $\phi : X \to [0, +\infty)$ such that*

$$d(x, T(x)) \le \phi(x) - \phi(T(x)), \quad \text{whenever } T(x) \preceq x. \tag{9}$$

Then T has a fixed point if and only if there exists $x_0 \in X$, with $T(x_0) \preceq x_0$.

Proof. Clearly, if x_0 is a fixed point of T, that is, $T(x_0) = x_0$, then we have $T(x_0) \preceq x_0$. Assume that there exists $x_0 \in X$ such that $T(x_0) \preceq x_0$. Since T is monotone increasing, we have $T^{n+1}(x_0) \preceq T^n(x_0)$, for any $n \ge 1$. Hence

$$d\left(T^n(x_0), T^{n+1}(x_0)\right) \le \phi(T^n(x_0)) - \phi\left(T^{n+1}(x_0)\right), \quad n = 1, 2, \ldots. \tag{10}$$

Hence $\{\phi(T^n(x_0))\}$ is a decreasing sequence of positive numbers. Let $\phi_0 = \lim_{n\to\infty}\phi(T^n(x_0))$. For any $n, h \ge 1$, we have

$$\begin{aligned} d\left(T^n(x_0), T^{n+h}(x_0)\right) &\le \sum_{k=0}^{h-1} d\left(T^{n+k}(x_0), T^{n+k+1}(x_0)\right) \\ &\le \phi(T^n(x_0)) - \phi\left(T^{n+h}(x_0)\right). \end{aligned} \tag{11}$$

Therefore $\{T^n(x_0)\}$ is a Cauchy sequence in X. Since X is complete, there exists $\overline{x} \in X$ such that $\lim_{n\to\infty}T^n(x_0) = \overline{x}$. Since T is continuous, we conclude that $T(\overline{x}) = \overline{x}$; that is, $\overline{x}$ is a fixed point of T. □

The continuity assumption of T may be relaxed if we assume that X satisfies the property (OSC).

Definition 4. Let $(X, \preceq)$ be a partially ordered set. Let d be a distance defined on X. One says that X satisfies the property (OSC) if and only if for any convergent decreasing sequence $\{x_n\}$ in X, that is, $x_{n+1} \preceq x_n$, for any $n \ge 1$, one has $\lim_{m\to\infty}x_m = \inf\{x_n, n \ge 1\}$.

One has the following improvement to Theorem 3.

Theorem 5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a distance d in X such that (X, d) is a complete metric space. Assume that X satisfies the property (OSC). Let $T : X \to X$ be a monotone increasing mapping. Assume that* *there exists a lower semicontinuous function* $\phi : X \to [0, +\infty)$ *such that*

$$d(x, T(x)) \leq \phi(x) - \phi(T(x)), \quad \text{whenever } T(x) \preceq x. \tag{12}$$

Then T has a fixed point if and only if there exists $x_0 \in X$, *with* $T(x_0) \preceq x_0$.

Proof. We proceed as we did in the proof of Theorem 3. Let $x_0 \in X$ such that $T(x_0) \preceq x_0$. Write $x_n = T^n(x_0)$, $n \geq 1$. Then we have the fact that $\{x_n\}$ is decreasing and $\lim_{n\to\infty} x_n = x_\omega$ exists in X. Since we did not assume T continuous, then x_ω may not be a fixed point of T. The idea is to use transfinite induction to build a transfinite orbit to help catch the fixed point. Note that since X satisfies (OSC), then we have $x_\omega = \inf\{x_n, n \geq 1\}$. Since T is monotone increasing, then we will have $T(x_\omega) \preceq x_\omega$. These basic facts so far will help us seek the transfinite orbit $\{x_\alpha\}_{\alpha\Gamma}$, where Γ is the set of all ordinals. This transfinite orbit must satisfy the following properties:

(1) $T(x_\alpha) = x_{\alpha+1}$, for any $\alpha \in \Gamma$;

(2) $x_\alpha = \inf\{x_\beta, \beta < \alpha\}$, if α is a limit ordinal;

(3) $x_\alpha \preceq x_\beta$, whenever $\beta < \alpha$;

(4) $d(x_\alpha, x_\beta) \leq \phi(x_\beta) - \phi(x_\alpha)$, whenever $\beta < \alpha$.

Clearly the above properties are satisfied for any $\alpha \in \{0, 1, \ldots, \omega\}$. Let α be an ordinal number. Assume that the properties (1)–(4) are satisfied by $\{x_\beta\}_{\beta<\alpha}$. We have two cases as follows.

(i) If $\alpha = \beta + 1$, then set $x_\alpha = T(x_\beta)$.

(ii) Assume that α is a limit ordinal. Set $\phi_0 = \inf\{\phi(x_\beta), \beta < \alpha\}$. Then one can easily find an increasing sequence of ordinals $\{\beta_n\}$, with $\beta_n < \alpha$, such that $\lim_{n\to\infty}\phi(x_{\beta_n}) = \phi_0$. Property (4) will force $\{x_{\beta_n}\}$ to be Cauchy. Since X is complete, then $\lim_{n\to\infty} x_{\beta_n} = \overline{x}$ exists in X. The property (OSC) will then imply $\overline{x} = \inf\{x_{\beta_n}, n \geq 1\}$. Let us show that $\overline{x} = \inf\{x_\beta, \beta < \alpha\}$. Let $\beta < \alpha$. If $\beta_n < \beta$, for all $n \geq 1$, then we have

$$d\left(x_\beta, x_{\beta_n}\right) \leq \phi\left(x_{\beta_n}\right) - \phi\left(x_\beta\right), \quad n = 1, 2, \ldots. \tag{13}$$

But $\phi(x_\beta) \geq \phi_0 = \lim_{n\to\infty}\phi(x_{\beta_n}) \geq \phi(x_\beta)$. Hence $\phi(x_\beta) = \phi_0$ which implies that $\lim_{n\to\infty} x_{\beta_n} = x_\beta$. Hence $x_\beta = \overline{x}$. Assume otherwise that there exists $n_0 \geq 1$ such that $\beta < \beta_{n_0}$. Hence $x_{\beta_{n_0}} \preceq x_\beta$ which implies $\overline{x} \preceq x_\beta$. In any case, we have $\overline{x} \preceq x_\beta$, for any $\beta < \alpha$. Therefore we have

$$\overline{x} = \inf\left\{x_{\beta_n}, n \geq 1\right\} \leq \inf\left\{x_\beta, \beta < \alpha\right\} \leq \inf\left\{x_{\beta_n}, n \geq 1\right\}. \tag{14}$$

Hence $\overline{x} = \inf\{x_\beta, \beta < \alpha\}$. Set $x_\alpha = \overline{x}$. Let us prove that $\{x_\beta, \beta \leq \alpha\}$ satisfies all properties (1)–(4). Clearly (1) and (2) are satisfied. Let us focus on (3) and (4). Let $\beta < \alpha$. We need to show that $x_\alpha \preceq x_\beta$. If α is a limit ordinal, this is obvious. Assume that $\alpha - 1$ exists. We have two cases; if $\alpha - 2$ exists, then we have $x_{\alpha-1} \preceq x_{\alpha-2}$. Since T is monotone increasing, then $T(x_{\alpha-1}) \preceq T(x_{\alpha-2})$); that is, $x_\alpha \preceq x_{\alpha-1}$. Otherwise, if $\alpha - 2$ is an ordinal limit, then $x_{\alpha-2} = \inf\{x_\gamma, \gamma < \alpha - 2\}$. Since T is monotone increasing, then we have

$$x_{\alpha-1} = T\left(x_{\alpha-2}\right) \preceq x_{\gamma+1}, \quad \text{for any } \gamma < \alpha - 2, \tag{15}$$

which implies $x_\alpha \preceq x_{\gamma+2}$, for any $\gamma < \alpha - 2$. Therefore we have $x_\alpha \preceq x_\beta$, which completes the proof of (3). Let us prove (4). Let $\beta < \alpha$. First assume that $\alpha - 1$ exists. Then, in the proof of (3), we saw that $x_\alpha = T(x_{\alpha-1}) \preceq x_{\alpha-1}$. Our assumption on T will then imply

$$d\left(x_{\alpha-1}, x_\alpha\right) \leq \phi\left(x_{\alpha-1}\right) - \phi\left(x_\alpha\right). \tag{16}$$

If $\beta = \alpha - 1$, we are done. Otherwise, if $\beta < \alpha - 1$, then we use the induction assumption to get

$$d\left(x_\beta, x_{\alpha-1}\right) \leq \phi\left(x_\beta\right) - \phi\left(x_{\alpha-1}\right). \tag{17}$$

The triangle inequality will then imply

$$d\left(x_\beta, x_\alpha\right) \leq \phi\left(x_\beta\right) - \phi\left(x_\alpha\right). \tag{18}$$

Next we assume that α is a limit ordinal. Then there exists an increasing sequence of ordinals $\{\beta_n\}$, with $\beta_n < \alpha$, such that $x_\alpha = \lim_{n\to\infty} x_{\beta_n}$. Given $\beta < \alpha$, assume that we have $\beta_n < \beta$, for all $n \geq 1$. In this case, we have seen that $x_\alpha = x_\beta$. Otherwise, let us assume that there exists $n_0 \geq 1$ such that $\beta < \beta_{n_0}$. In this case, from our induction assumption and the triangle inequality, we get

$$d\left(x_\beta, x_{\beta_n}\right) \leq \phi\left(x_\beta\right) - \phi\left(x_{\beta_n}\right), \quad n \geq n_0. \tag{19}$$

Using the lower semicontinuity of ϕ, we conclude that

$$d\left(x_\beta, x_\alpha\right) \leq \phi\left(x_\beta\right) - \phi\left(x_\alpha\right), \tag{20}$$

which completes the proof of (4). By the transfinite induction we conclude that the transfinite orbit $\{x_\alpha\}$ exists which satisfies the properties (1)–(4). Using Proposition A.6 ([13, page 284]), there exists an ordinal β such that $\phi(x_\alpha) = \phi(x_\beta)$, for any $\alpha \geq \beta$. In particular, we have $\phi(x_\alpha) = \phi(x_{\alpha+1})$, for any $\alpha \geq \beta$. Property (4) will then force $x_{\alpha+1} = x_\alpha$; that is, $T(x_\alpha) = x_\alpha$. Therefore T has a fixed point.

One may wonder if Theorem 5 is truly an extension of the main results of [1, 2, 11]. The following example shows that it is the case.

Example 6. Let $X = L^1([0,1], dx)$ be the classical Banach space with the natural pointwise order generated by $\mathbb{R}$. Let $C = \{f \in X, f(t) \geq 0 \ a.e.\}$ be the positive cone of X. Define $T : C \to C$ by

$$T(f)(t) = \begin{cases} f(t) & \text{if } f(t) > \dfrac{1}{2}, \\ 0 & \text{if } f(t) \leq \dfrac{1}{2}. \end{cases} \tag{21}$$

First note that C is a closed subset of X. Hence C is complete for the norm-1 distance. Also it is easy to check that the property (OSC) holds in this case. Note that, for any

$f \in C$, we have $0 \leq T(f) \leq f$. Also we have $T^2(f) = T(T(f)) = T(f))$; that is, $T(f)$ is a fixed point of T for any $f \in C$. Note that for any $f \in C$ we have

$$\begin{aligned} d(f, T(f)) &= \int_0^1 |f(t) - T(f)(t)|\, dt \\ &= \int_0^1 f(t)\, dt - \int_0^1 T(f)(t)\, dt = \|f\| - \|T(f)\|. \end{aligned} \tag{22}$$

Therefore all the assumptions of Theorem 5 are satisfied. But T fails to satisfy the assumptions of [1, 2, 11]. Indeed if we take

$$f(t) = \frac{1}{2}, \quad f_n(t) = \frac{1}{2} + \frac{1}{n}, \quad n \geq 1, \tag{23}$$

then we have $T(f) = 0$ and $T(f_n) = f_n$, for any $n \geq 1$. Therefore T is not continuous since $\{f_n\}$ converges uniformly (and in norm-1 as well) to f. Note also that $f \leq f_n$, for $n \geq 1$. So any Lipschitz condition on the partial order of C will not be satisfied by T in this case. □

3. Caristi's Theorem in Metric Spaces with Graph

It seems that the terminology of graph theory instead of partial ordering sets can give more clear pictures and yield to generalize the theorems above. In this section, we give the graph versions of our two main results.

Throughout this section we assume that (X, d) is a metric space and G is a directed graph (digraph) with set of vertices $V(G) = X$ and set of edges $E(G)$ containing all the loops; that is, $(x, x) \in E(G)$ for any $x \in X$. We also assume that G has no parallel edges (arcs) and so we can identify G with the pair $(V(G), E(G))$. Our graph theory notations and terminology are standard and can be found in all graph theory books, like [14, 15]. A digraph G is called an oriented graph; if whenever $(u, v) \in E(G)$, then $(v, u) \notin E(G)$.

Let $(X, \preceq)$ be a partially ordered set. We define the oriented graph $G_{\preceq}$ on X as follows. The vertices of $G_{\preceq}$ are the elements of X, and two vertices $x, y \in X$ are connected by a directed edge (arc) if $x \preceq y$. Therefore, $G_{\preceq}$ has no parallel arcs as $x \preceq y$ & $y \preceq x \Rightarrow x = y$.

If x, y are vertices of the digraph G, then a directed path from x to y of length N is a sequence $\{x_i\}_{i=0}^N$ of $N+1$ vertices such that

$$\begin{gathered} x_0 = x, \qquad x_N = y, \\ (x_i, x_{i+1}) \in E(G), \quad i = 0, 1, \ldots, N. \end{gathered} \tag{24}$$

A closed directed path of length $N > 1$ from x to y, that is, $x = y$, is called a directed cycle. An acyclic digraph is a digraph that has no directed cycle.

Given an acyclic digraph, G, we can always define a partially order $\preceq_G$ on the set of vertices of G by defining that $x \preceq_G y$ whenever there is a directed path from x to y.

Definition 7. One says that a mapping $T : X \to X$ is G-edge preserving if

$$\forall x, y \in X, \quad (x, y) \in E(G) \Longrightarrow (Tx, Ty) \in E(G). \tag{25}$$

T is said to be a Caristi G-mapping if there exists a lower semicontinuous function $\phi : X \to [0, +\infty)$ such that

$$d(x, Tx) \leq \phi(x) - \phi(Tx), \quad \text{whenever } (T(x), x) \in E(G). \tag{26}$$

One can now give the graph theory versions of our two mean Theorems 3 and 5 as follows.

Theorem 8. *Let G be an oriented graph on the set X with $E(G)$ containing all loops and suppose that there exists a distance d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be continuous, G-edge preserving, and a G-Caristi mapping. Then T has a fixed point if and only if there exists $x_0 \in X$, with $(T(x_0), x_0) \in E(G)$.*

Proof. G has all the loops. In particular, if x_0 is a fixed point of T, that is, $T(x_0) = x_0$, then we have $(T(x_0), x_0) \in E(G)$. Assume that there exists $x_0 \in X$ such that $(T(x_0), x_0) \in E(G)$. Since T is G-edge preserving, we have $(T^{n+1}(x_0), T^n(x_0)) \in E(G)$, for any $n \geq 1$. Hence

$$\begin{aligned} & d\left(T^n(x_0), T^{n+1}(x_0)\right) \\ & \quad \leq \phi\left(T^n(x_0)\right) - \phi\left(T^{n+1}(x_0)\right), \quad n = 1, 2, \ldots. \end{aligned} \tag{27}$$

Hence $\{\phi(T^n(x_0))\}$ is a decreasing sequence of positive numbers. Let $\phi_0 = \lim_{n\to\infty} \phi(T^n(x_0))$. For any $n, h \geq 1$, we have

$$\begin{aligned} d\left(T^n(x_0), T^{n+h}(x_0)\right) &\leq \sum_{k=0}^{h-1} d\left(T^{n+k}(x_0), T^{n+k+1}(x_0)\right) \\ &\leq \phi\left(T^n(x_0)\right) - \phi\left(T^{n+h}(x_0)\right). \end{aligned} \tag{28}$$

Therefore $\{T^n(x_0)\}$ is a Cauchy sequence in X. Since X is complete, there exists $\overline{x} \in X$ such that $\lim_{n\to\infty} T^n(x_0) = \overline{x}$. Since T is continuous, we conclude that $T(\overline{x}) = \overline{x}$; that is, $\overline{x}$ is a fixed point of T. □

The following definition is needed to prove the analogue to Theorem 5.

Definition 9. Let G be an acyclic oriented graph on the set X with $E(G)$ containing all loops. One says that G satisfies the property (OSC) if and only if $(X, \preceq_G)$ satisfies (OSC).

The analogue to Theorem 5 may be stated as follows.

Theorem 10. *Let G be an oriented graph on the set X with $E(G)$ containing all loops and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that G satisfies the property (OSC). Let $T : X \to X$ be a G-edge preserving and a Caristi G-mapping. Then T has a fixed point if and only if there exists $x_0 \in X$, with $(T(x_0), x_0) \in E(G)$.*

The proof of Theorem 10 is similar to the proof of Theorem 5. In fact it is easy to check that these two theorems are equivalent.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.

[2] J. Jachymski, "The contraction principle for mappings on a metric space with a graph," *Proceedings of the American Mathematical Society*, vol. 136, no. 4, pp. 1359–1373, 2008.

[3] A. Brøndsted, "Fixed points and partial orders," *Proceedings of the American Mathematical Society*, vol. 60, pp. 365–366, 1976.

[4] J. Caristi, "Fixed point theory and inwardness conditions," in *Applied Nonlinear Analysis*, pp. 479–483, Academic Press, New York, NY, USA, 1979.

[5] B. R. Halpern and G. M. Bergman, "A fixed-point theorem for inward and outward maps," *Transactions of the American Mathematical Society*, vol. 130, pp. 353–358, 1968.

[6] F. E. Browder, "On a theorem of Caristi and Kirk," in *Proceedings of the Seminar on Fixed Point Theory and Its Applications*, pp. 23–27, Dalhousie University, Academic Press, June 1975.

[7] J. Caristi, "Fixed point theorems for mappings satisfying inwardness conditions," *Transactions of the American Mathematical Society*, vol. 215, pp. 241–251, 1976.

[8] W. A. Kirk and J. Caristi, "Mappings theorems in metric and Banach spaces," *Bulletin de l'Académie Polonaise des Sciences*, vol. 23, no. 8, pp. 891–894, 1975.

[9] I. Ekeland, "Sur les problèmes variationnels," *Comptes Rendus de l'Académie des Sciences*, vol. 275, pp. A1057–A1059, 1972.

[10] F. Sullivan, "A characterization of complete metric spaces," *Proceedings of the American Mathematical Society*, vol. 83, no. 2, pp. 345–346, 1981.

[11] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.

[12] G. S. Ladde, V. Lakshmikantham, and A. S. Vatsala, *Monotone Iterative Techniques for Non-Linear di Erential Equations*, Pitman, Boston, Mass, USA, 1985.

[13] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, Pure and Applied Mathematics, Wiley-Interscience, New York, NY, USA, 2001.

[14] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & Digraphs*, CRC Press, New York, NY, USA, 2011.

[15] J. L. Gross and J. Yellen, *Graph Theory and Its Applications*, CRC Press, New York, NY, USA, 1999.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 634710, 4 pages
http://dx.doi.org/10.1155/2014/634710

Research Article

Applications of Differential Subordination for Argument Estimates of Multivalent Analytic Functions

Meng-Ting Lu, Ting Jia, Xing-Qian Ling, and Jin-Lin Liu

Department of Mathematics, Yangzhou University, Yangzhou 225002, China

Correspondence should be addressed to Jin-Lin Liu; jlliu@yzu.edu.cn

Received 21 November 2013; Accepted 20 January 2014; Published 26 February 2014

Academic Editor: David Kalaj

By using the method of differential subordinations, we derive some properties of multivalent analytic functions. All results presented here are sharp.

This paper is dedicated to Professor Miodrag Mateljević on the occasion of his 65th birthday

1. Introduction

Let $A(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in N = \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic in the open unit disk $D = \{z \in C : |z| < 1\}$. Let $f(z)$ and $g(z)$ be analytic in D. Then, we say that $f(z)$ is subordinate to $g(z)$ in D, written as $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in D, such that $|w(z)| \le |z|$ and $f(z) = g(w(z))$ $(z \in D)$. If $g(z)$ is univalent in D, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(D) \subset g(D)$. Let $p(z) = 1 + p_1 z + \cdots$ be analytic in D. Then, for $-1 \le B < A \le 1$, it is clear that

$$p(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in D) \tag{2}$$

if and only if

$$\left| p(z) - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \quad (-1 < B < A \le 1;\ z \in D), \tag{3}$$

$$\operatorname{Re} p(z) > \frac{1 - A}{2} \quad (B = -1;\ z \in D). \tag{4}$$

Recently, a number of results for argument properties of analytic functions have been obtained by several authors (see, e.g., [1–5]). The objective of the present paper is to derive some further interesting properties of multivalent analytic functions. The basic tool used here is the method of differential subordinations.

To derive our results, we need the following lemmas.

Lemma 1 (see [6, Theorem 1, page 776]). *Let $h(z)$ be analytic and starlike univalent in D with $h(0) = 0$. If $g(z)$ is analytic in D and $zg'(z) \prec h(z)$, then*

$$g(z) \prec g(0) + \int_0^z \frac{h(t)}{t}\, dt. \tag{5}$$

Lemma 2 (see [5, Theorem 1, page 1814]). *Let $0 < \alpha_1 \le 1$, $0 < \alpha_2 \le 1$, $\beta = (\alpha_1 - \alpha_2)/(\alpha_1 + \alpha_2)$, and $c = e^{\beta\pi i}$. Also let*

$$\begin{gathered} \lambda_0 a \ge 0, \quad \lambda(b+2) \ge 0, \quad (b+1) \operatorname{Re} \mu \ge 0, \\ |b+1| \le \frac{2}{\alpha_1 + \alpha_2}, \qquad |a - b - 1| \le \frac{1}{\max\{\alpha_1, \alpha_2\}}. \end{gathered} \tag{6}$$

If $q(z)$ is analytic in D with $q(0) = 1$ and

$$\begin{gathered} \lambda_0 (q(z))^a + \lambda (q(z))^{b+2} + \mu (q(z))^{b+1} \\ + z q'(z) (q(z))^b \prec h(z) \quad (z \in D), \end{gathered} \tag{7}$$

where

$$h(z) = \lambda_0\left(\frac{1+cz}{1-z}\right)^{a((\alpha_1+\alpha_2)/2)} + \left(\frac{1+cz}{1-z}\right)^{(1/2)(b+1)(\alpha_1+\alpha_2)} \times \left(\mu + \lambda\left(\frac{1+cz}{1-z}\right)^{(\alpha_1+\alpha_2)/2} + \frac{\alpha_1+\alpha_2}{2}\left(\frac{z}{1-z}+\frac{cz}{1+cz}\right)\right) \quad (8)$$

is (close-to-convex) univalent in D, then

$$-\frac{\pi}{2}\alpha_2 < \arg\left(q(z)\right) < \frac{\pi}{2}\alpha_1 \quad (z \in D). \quad (9)$$

The bounds α_1 and α_2 in (9) are sharp for the function $q(z)$ defined by

$$q(z) = \left(\frac{1+cz}{1-z}\right)^{(\alpha_1+\alpha_2)/2}. \quad (10)$$

Remark 3 (see [5, Lemma 2, page 1813]). The function $q(z)$ defined by (10) is analytic and univalent convex in D and

$$q(D) = \left\{w : w \in C,\ -\frac{\pi}{2}\alpha_2 < \arg w < \frac{\pi}{2}\alpha_1\right\}. \quad (11)$$

2. Main Results

Our first result is contained in the following.

Theorem 4. *Let $\alpha \in (0,1/2]$ and $\beta \in (0,1)$. If $f(z) \in A(p)$ satisfies $f(z) \neq 0$ $(0 < |z| < 1)$ and*

$$\left|\frac{z^p}{f(z)}\left(\frac{zf'(z)}{f(z)} - p\right)\right| < \delta \quad (z \in D), \quad (12)$$

where δ is the smallest positive root of the equation

$$\alpha \sin\left(\frac{\pi\beta}{2}\right)x^2 - x + (1-\alpha)\sin\left(\frac{\pi\beta}{2}\right) = 0, \quad (13)$$

then

$$\left|\arg\left(\frac{f(z)}{z^p} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad (z \in D). \quad (14)$$

The bound β is sharp for each $\alpha \in (0,1/2]$.

Proof. Let

$$g(x) = \alpha \sin\left(\frac{\pi\beta}{2}\right)x^2 - x + (1-\alpha)\sin\left(\frac{\pi\beta}{2}\right). \quad (15)$$

We can see easily that (13) has two positive roots. Since $g(0) > 0$ and $g(1) < 0$, we have

$$0 < \frac{\alpha}{1-\alpha}\delta \leq \delta < 1. \quad (16)$$

Put

$$\frac{f(z)}{z^p} = \alpha + (1-\alpha)\,p(z). \quad (17)$$

Then, from the assumption of the theorem, we can see that $p(z)$ is analytic in D with $p(0) = 1$ and $\alpha + (1-\alpha)p(z) \neq 0$ for all $z \in D$. Taking the logarithmic differentiations in both sides of (17), we get

$$\frac{zf'(z)}{f(z)} - p = \frac{(1-\alpha)\,zp'(z)}{\alpha + (1-\alpha)\,p(z)}, \quad (18)$$

$$\frac{z^p}{f(z)}\left(\frac{zf'(z)}{f(z)} - p\right) = \frac{(1-\alpha)\,zp'(z)}{\left(\alpha + (1-\alpha)\,p(z)\right)^2} \quad (19)$$

for all $z \in D$. Thus, inequality (12) is equivalent to

$$\frac{(1-\alpha)\,zp'(z)}{\left(\alpha + (1-\alpha)\,p(z)\right)^2} \prec \delta z. \quad (20)$$

By using Lemma 1, (20) leads to

$$\int_0^z \frac{(1-\alpha)\,p'(t)}{\left(\alpha + (1-\alpha)\,p(t)\right)^2}dt \prec \delta z \quad (21)$$

or to

$$1 - \frac{1}{\alpha + (1-\alpha)\,p(z)} \prec \delta z. \quad (22)$$

According to (16), (22) can be written as

$$p(z) \prec \frac{1 + (\alpha/(1-\alpha))\,\delta z}{1-\delta z}. \quad (23)$$

Now, by taking $A = (\alpha/(1-\alpha))\delta$ and $B = -\delta$ in (2) and (3), we have

$$\left|\arg\left(\frac{f(z)}{z^p} - \alpha\right)\right| = \left|\arg p(z)\right| < \arcsin\left(\frac{\delta}{1-\alpha+\alpha\delta^2}\right) = \frac{\pi}{2}\beta \quad (24)$$

for all $z \in D$ because of $g(\delta) = 0$. This proves (14).

Next, we consider the function $f(z)$ defined by

$$f(z) = \frac{z^p}{1-\delta z} \quad (25)$$

for all $z \in D$. It is easy to see that

$$\left|\frac{z^p}{f(z)}\left(\frac{zf'(z)}{f(z)} - p\right)\right| = |\delta z| < \delta \quad (26)$$

for all $z \in D$. Since

$$\frac{f(z)}{z^p} - \alpha = (1-\alpha)\,\frac{1 + (\alpha/(1-\alpha))\,\delta z}{1-\delta z}, \quad (27)$$

it follows from (3) that

$$\sup_{z\in U}\left|\arg\left(\frac{f(z)}{z^p} - \alpha\right)\right| = \arcsin\left(\frac{\delta}{1-\alpha+\alpha\delta^2}\right) = \frac{\pi}{2}\beta. \quad (28)$$

Hence, we conclude that the bound β is the best possible for each $\alpha \in (0,1/2]$. □

Next, we derive the following.

Theorem 5. *If $f(z) \in A(p)$ satisfies $f(z) \neq 0$ $(0 < |z| < 1)$ and*

$$\operatorname{Re}\left\{\frac{z^p}{f(z)}\left(\frac{zf'(z)}{f(z)} - p\right)\right\} < \gamma \quad (z \in D), \tag{29}$$

where

$$0 < \gamma < \frac{1}{2\log 2}, \tag{30}$$

then

$$\operatorname{Re}\frac{z^p}{f(z)} > 1 - 2\gamma\log 2 \quad (z \in D). \tag{31}$$

The bound in (31) *is sharp.*

Proof. Let

$$p(z) = \frac{f(z)}{z^p}. \tag{32}$$

Then, from the assumption of the theorem we can see that $p(z)$ is analytic in D with $p(0) = 1$ and $p(z) \neq 0$ for all $z \in D$. According to (32) and (29), we have immediately

$$1 - \frac{zp'(z)}{\gamma p^2(z)} \prec \frac{1+z}{1-z}; \tag{33}$$

that is,

$$z\left(\frac{1}{p(z)}\right)' \prec \frac{2\gamma z}{1-z}. \tag{34}$$

Now, by using Lemma 1, we obtain

$$\frac{1}{p(z)} \prec 1 - 2\gamma\log(1-z). \tag{35}$$

Since the function $1 - 2\gamma\log(1-z)$ is convex univalent in D and

$$\operatorname{Re}\left(1 - 2\gamma\log(1-z)\right) > 1 - 2\gamma\log 2 \quad (z \in D), \tag{36}$$

from (35), we get inequality (31).

To show that the bound in (31) cannot be increased, we consider

$$f(z) = \frac{z^p}{1 - 2\gamma\log(1-z)} \quad (z \in D). \tag{37}$$

It is easy to verify that the function $f(z)$ satisfies inequality (29). On the other hand, we have

$$\operatorname{Re}\frac{z^p}{f(z)} \longrightarrow 1 - 2\gamma\log 2 \tag{38}$$

as $z \to -1$. Now, the proof of the theorem is complete. □

Finally, we discuss the following theorem.

Theorem 6. *Let $\alpha, \gamma \in (0,1)$. If $f(z) \in A(p)$ satisfies $f(z) \neq 0$ $(0 < |z| < 1)$ and*

$$\left|\arg\left\{\frac{f(z)}{z^p}\left(\gamma\left(\frac{zf'(z)}{f(z)} - p\right) + (1-\gamma)\frac{f(z)}{z^p}\right) - (1-\gamma)\alpha^2\right\}\right| < \pi\delta \tag{39}$$

for all $z \in D$, where

$$\delta = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{\sqrt{\gamma\left(2(1-\alpha)(1-\gamma) + \gamma\right)}}{2\alpha(1-\gamma)}\right), \tag{40}$$

then

$$\operatorname{Re}\frac{f(z)}{z^p} > \alpha \quad (z \in D). \tag{41}$$

The bound δ in (39) *is sharp.*

Proof. Define the function $p(z)$ by (17). For $\alpha, \gamma \in (0,1)$, it follows from (17) and (18) that

$$\begin{aligned}\frac{1}{\gamma(1-\alpha)}&\left\{\frac{f(z)}{z^p}\left(\gamma\left(\frac{zf'(z)}{f(z)} - p\right) + (1-\gamma)\frac{f(z)}{z^p}\right) - (1-\gamma)\alpha^2\right\}\\ &= \frac{(1-\alpha)(1-\gamma)}{\gamma}p^2(z) + \frac{2\alpha(1-\gamma)}{\gamma}p(z) + zp'(z)\end{aligned} \tag{42}$$

for all $z \in D$. Putting

$$a = b = \lambda_0 = 0, \qquad \alpha_1 = \alpha_2 = 1, \qquad \lambda = \frac{(1-\alpha)(1-\gamma)}{\gamma}, \qquad \mu = \frac{2\alpha(1-\gamma)}{\gamma} \tag{43}$$

in Lemma 2 and using (42), we see that if

$$\frac{1}{\gamma(1-\alpha)}\left\{\frac{f(z)}{z^p}\left(\gamma\left(\frac{zf'(z)}{f(z)} - p\right) + (1-\gamma)\frac{f(z)}{z^p}\right) - (1-\gamma)\alpha^2\right\} \prec h(z), \tag{44}$$

where

$$h(z) = \left(\frac{1+z}{1-z}\right)\left(\frac{(1-\alpha)(1-\gamma)}{\gamma}\left(\frac{1+z}{1-z}\right) + \frac{2\alpha(1-\gamma)}{\gamma} + \frac{2z}{1-z^2}\right), \tag{45}$$

then (41) holds true.

Letting $0 < \theta < \pi$ and $x = \cot(\theta/2)$, we deduce that

$$\begin{aligned} \arg h\left(e^{i\theta}\right) &= \frac{\pi}{2} + \arg\left\{\frac{(1-\alpha)(1-\gamma)}{\gamma} x e^{\pi i/2} + \frac{2\alpha(1-\gamma)}{\gamma} + \frac{i}{2}\left(x + \frac{1}{x}\right)\right\} \\ &= \frac{\pi}{2} + \tan^{-1}\left(\frac{(2(1-\alpha)(1-\gamma)+\gamma)x^2 + \gamma}{4\alpha(1-\gamma)x}\right). \end{aligned} \tag{46}$$

Making use of (46), we obtain that

$$\begin{aligned} \inf_{|z|=1 (z \neq \pm 1)} \left|\arg h(z)\right| &= \min_{0<\theta<\pi} \arg h\left(e^{i\theta}\right) \\ &= \frac{\pi}{2} + \min_{x>0} \tan^{-1}\left(\frac{(2(1-\alpha)(1-\gamma)+\gamma)x^2 + \gamma}{4\alpha(1-\gamma)x}\right) \\ &= \frac{\pi}{2} + \tan^{-1}\left(\frac{\sqrt{\gamma(2(1-\alpha)(1-\gamma)+\gamma)}}{2\alpha(1-\gamma)}\right) \\ &= \pi\delta. \end{aligned} \tag{47}$$

Therefore, if $f(z) \in A(p)$ satisfies (39), then the subordination (44) holds, and, thus, we obtain (41).

For the function

$$\frac{f(z)}{z^p} = \frac{1 + (1-2\alpha)z}{1-z}, \tag{48}$$

we find that

$$\frac{1}{\gamma(1-\alpha)}\left\{\frac{f(z)}{z^p}\left(\gamma\left(\frac{zf'(z)}{f(z)} - p\right) + (1-\gamma)\frac{f(z)}{z^p}\right) - (1-\gamma)\alpha^2\right\} = h(z), \tag{49}$$

where $h(z)$ is defined by (45). In view of (46) and (49), we conclude that the bound δ in (39) is the largest number such that (41) holds true. This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors would like to express their sincere thanks to the referees for careful reading and suggestions which helped them improve the paper.

References

[1] A. Gangadharan, V. Ravichandran, and T. N. Shanmugam, "Radii of convexity and strong starlikeness for some classes of analytic functions," *Journal of Mathematical Analysis and Applications*, vol. 211, no. 1, pp. 301–313, 1997.

[2] J.-L. Liu, "The Noor integral and strongly starlike functions," *Journal of Mathematical Analysis and Applications*, vol. 261, no. 2, pp. 441–447, 2001.

[3] M. Nunokawa, S. Owa, E. Y. Duman, and M. Aydoğan, "Some properties of analytic functions relating to the Miller and Mocanu result," *Computers & Mathematics with Applications*, vol. 61, no. 5, pp. 1291–1295, 2011.

[4] N.-E. Xu, D.-G. Yang, and S. Owa, "On strongly starlike multivalent functions of order β and type α," *Mathematische Nachrichten*, vol. 283, no. 8, pp. 1207–1218, 2010.

[5] D.-G. Yang and J.-L. Liu, "Argument inequalities for certain analytic functions," *Mathematical and Computer Modelling*, vol. 52, no. 9-10, pp. 1812–1821, 2010.

[6] T. J. Suffridge, "Some remarks on convex maps of the unit disk," *Duke Mathematical Journal*, vol. 37, no. 4, pp. 775–777, 1970.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 478486, 5 pages
http://dx.doi.org/10.1155/2014/478486

Research Article

Existence Theorems of ε-Cone Saddle Points for Vector-Valued Mappings

Tao Chen

College of Culture and Tourism, Yunnan Open University, Kunming 650223, China

Correspondence should be addressed to Tao Chen; chentao1cq@126.com

Received 19 November 2013; Accepted 22 January 2014; Published 25 February 2014

Academic Editor: Qamrul Hasan Ansari

A new existence result of ε-vector equilibrium problem is first obtained. Then, by using the existence theorem of ε-vector equilibrium problem, a weakly ε-cone saddle point theorem is also obtained for vector-valued mappings.

1. Introduction

Saddle point problems are important in the areas of optimization theory and game theory. As for optimization theory, the main motivation of studying saddle point has been their connection with characterized solutions to minimax dual problems. Also, as for game theory, the main motivation has been the determination of two-person zero-sum games based on the minimax principle.

In recent years, based on the development of vector optimization, a great deal of papers have been devoted to the study of cone saddle points problems for vector-valued mappings and set-valued mappings, such as [1–8]. Nieuwenhuis [5] introduced the notion of cone saddle points for vector-valued functions in finite-dimensional spaces and obtained a cone saddle point theorem for general vector-valued mappings. Gong [2] established a strong cone saddle point theorem of vector-valued functions. Li et al. [4] obtained an existence theorem of lexicographic saddle point for vector-valued mappings. Bigi et al. [1] obtained a cone saddle point theorem by using an existence theorem of a vector equilibrium problem. Zhang et al. [9] established a general cone loose saddle point for set-valued mappings. Zhang et al. [8] obtained a minimax theorem and an existence theorem of cone saddle points for set-valued mappings by using Fan-Browder fixed point theorem. Some other types of existence results can be found in [3, 10–18].

On the other hand, in some situations, it may not be possible to find an exact solution for an optimization problem, or such an exact solution simply does not exist, for example, if the feasible set is not compact. Thus, it is meaningful to look for an approximate solution instead. There are also many papers to investigate the approximate solution problem, such as [19–21]. Kimura et al. [20] obtained several existence results for ε-vector equilibrium problem and the lower semicontinuity of the solution mapping of ε-vector equilibrium problem. Anh and Khanh [19] have considered two kinds of solution sets to parametric generalized ε-vector quasiequilibrium problems and established the sufficient conditions for the Hausdorff semicontinuity (or Berge semicontinuity) of these solution mappings. X. B. Li and S. J. Li [21] established some semicontinuity results on ε-vector equilibrium problem.

The aim of this paper is to characterize the ε-cone saddle point of vector-valued mappings. For this purpose, we first establish an existence theorem for ε-vector equilibrium problem. Then, by this existence result, we obtain an existence theorem for ε-cone saddle point of vector-valued mappings.

2. Preliminaries

Let X be a real Hausdorff topological vector space and let V be a real local convex Hausdorff topological vector space. Assume that S is a pointed closed convex cone in V with nonempty interior $\operatorname{int} S \neq \emptyset$. Let V^* be the topological dual space of V. Denote the dual cone of S by S^*:

$$S^* = \{s^* \in V^* : s^*(s) \geq 0, \forall s \in S\}. \tag{1}$$

Note that from Lemma 3.21 in [22] we have

$$z \in S \Longleftrightarrow \{\langle z^*, z\rangle \geq 0, \forall z^* \in S^*\},$$
$$z \in \operatorname{int} S \Longleftrightarrow \{\langle z^*, z\rangle > 0, \forall z^* \in S^* \setminus \{0\}\}. \tag{2}$$

Definition 1 (*see [7, 23]*). Let $f : X \to V$ be a vector-valued mapping. f is said to be S-upper semicontinuous on X if and only if, for each $x \in X$ and any $s \in \operatorname{int} S$, there exists an open neighborhood U_x of x such that

$$f(u) \in f(x) + s - \operatorname{int} S, \quad \forall u \in U_x. \tag{3}$$

f is said to be S-lower semicontinuous on X if and only if $-f$ is S-upper semicontinuous on X.

Lemma 2 (see [17]). *Let $f : X \times X \to V$ be a vector-valued mapping and $s^* \in S^* \setminus \{0\}$. If f is S-lower semicontinuous, then $s^* \circ f$ is lower semicontinuous.*

Definition 3 (*see [24]*). Let A and B be nonempty subsets of X and $f : A \times B \to V$ be a vector-valued mapping.

(i) f is said to be S-concavelike in its first variable on A if and only if, for all $x_1, x_2 \in A$ and $l \in [0, 1]$, there exists $\overline{x} \in A$ such that

$$f(\overline{x}, y) \in lf(x_1, y) + (1 - l) f(x_2, y) + S, \quad \forall y \in B. \tag{4}$$

(ii) f is said to be S-convexlike in its second variable on B if and only if, for all $y_1, y_2 \in B$ and $l \in [0, 1]$, there exists $\overline{y} \in B$ such that

$$f(x, \overline{y}) \in lf(x, y_1) + (1 - l) f(x, y_2) - S, \quad \forall x \in A. \tag{5}$$

(iii) f is said to be S-concavelike-convexlike on $A \times B$ if and only if f is S-concavelike in its first variable and S-convexlike in its second variable.

Definition 4. Let $A \subset V$ be a nonempty subset and $\varepsilon \in \operatorname{int} S$.

(i) A point $z \in A$ is said to be a weak ε-minimal point of A if and only if $A \cap (z - \varepsilon - \operatorname{int} S) = \emptyset$ and $\operatorname{Min}_\varepsilon A$ denotes the set of all weak ε-minimal points of A.

(ii) A point $z \in A$ is said to be a weak ε-maximal point of A if and only if $A \cap (z + \varepsilon + \operatorname{int} S) = \emptyset$ and $\operatorname{Max}_\varepsilon A$ denotes the set of all weak ε-maximal points of A.

Definition 5. Let $f : A \times B \to V$ be a vector-valued mapping and $\varepsilon \in \operatorname{int} S$. A point $(a, b) \in A \times B$ is said to be a weak ε-S-saddle point of f on $A \times B$ if

$$f(a, b) \in \operatorname{Max}_\varepsilon f(A, b) \bigcap \operatorname{Min}_\varepsilon f(a, B). \tag{6}$$

3. Existence of ε-Vector Equilibrium Problem

In this section, we deal with the following ε-vector equilibrium problem (for short VAEP). Find $\overline{x} \in E$ such that

$$f(x, y) + \varepsilon \notin -\operatorname{int} S, \quad \forall y \in E, \tag{7}$$

where $f : X \times X \to V$ is a vector-valued mapping, E is a nonempty subset of X, and $\varepsilon \in \operatorname{int} S$.

If $f(x, y) = g(y) - g(x)$, $x, y \in E$, and if $\overline{x} \in E$ is a solution of VAEP, then $\overline{x} \in E$ is a solution of ε-vector optimization of g, where g is a vector-valued mapping.

Denote the ε-solution set of (VAEP) by

$$S(\varepsilon) := \{\overline{x} \in E : f(x, y) + \varepsilon \notin -\operatorname{int} S, \forall y \in E\}. \tag{8}$$

Lemma 6 (see [20]). *Let E be a nonempty subset of X. Suppose that $f : X \times X \to V$ is a vector-valued mapping and the following conditions are satisfied:*

(i) $\operatorname{cl} E$ *is a compact set;*

(ii) $\{x \in \operatorname{cl} E : f(x, y) \notin -\operatorname{int} S, \forall y \in \operatorname{cl} E\} \neq \emptyset$*;*

(iii) f *is S-lower semicontinuous on* $\operatorname{cl} E \times \operatorname{cl} E$.

Then, for each $\varepsilon \in \operatorname{int} S$, $S(\varepsilon) \neq \emptyset$.

Next, we give a sufficient condition for the condition (ii) in Lemma 6.

Lemma 7. *Let E be a nonempty subset of X. Suppose that $f : X \times X \to V$ is a vector-valued mapping with $f(x, x) = 0$ for all $x \in X$ and the following conditions are satisfied:*

(i) $\operatorname{cl} E$ *is a compact set;*

(ii) f *is S-concavelike-convexlike on* $\operatorname{cl} E \times \operatorname{cl} E$*;*

(iii) *for each $x \in \operatorname{cl} E$, $f(x, \cdot)$ is S-lower semicontinuous on* $\operatorname{cl} E$.

Then, there exists $\overline{x} \in \operatorname{cl} E$ such that

$$f(\overline{x}, y) \notin -\operatorname{int} S, \quad \forall y \in \operatorname{cl} E. \tag{9}$$

Proof. For any $t < 0$ and $s^* \in S^* \setminus \{0\}$, we define a multifunction $G : \operatorname{cl} E \to 2^{\operatorname{cl} E}$ by

$$G(x) = \{y \in \operatorname{cl} E : s^*(f(x, y)) \leq t\}, \quad \forall x \in \operatorname{cl} E. \tag{10}$$

First, by assumptions, we must have

$$\bigcap_{x \in \operatorname{cl} E} G(x) = \emptyset. \tag{11}$$

In fact, if there exists $\overline{y} \in \operatorname{cl} E$ such that $\overline{y} \in G(x)$, for all $x \in \operatorname{cl} E$, then

$$s^*(f(x, \overline{y})) \leq t, \quad \forall x \in \operatorname{cl} E. \tag{12}$$

Particularly, taking $x = \overline{y}$, we have $0 = s^*(f(\overline{y}, \overline{y})) \leq t$, which contradicts the assumption about t.

Then, by Lemma 2, $G(x)$ is a closed set, for each $x \in \operatorname{cl} E$. By (11), for any $y \in \operatorname{cl} E$, we have

$$y \in V \setminus \bigcap_{x \in \operatorname{cl} E} G(x) = \bigcup_{x \in \operatorname{cl} E} V \setminus G(x). \tag{13}$$

Since $\operatorname{cl} E$ is compact, there exists a finite point set $\{x_1, x_2, \ldots, x_n\}$ in $\operatorname{cl} E$ such that

$$\operatorname{cl} E \subset \bigcup_{1 \leq i \leq n} V \setminus G(x_i). \tag{14}$$

Namely, for each $y \in \operatorname{cl} E$, there exists $i \in \{1, 2, \ldots, n\}$ such that

$$s^*\left(f\left(x_i, y\right)\right) > t. \tag{15}$$

Now, we consider the set

$$\begin{aligned} M := \Big\{&\left(z_1, z_2, \ldots, z_n, r\right) \in R^{n+1} \mid \exists y \in \operatorname{cl} E, \\ &s^*\left(f\left(x_i, y\right)\right) \le r + z_i, \forall i = 1, 2, \ldots, n\Big\}. \end{aligned} \tag{16}$$

Obviously, by the condition (ii), M is a convex set. By (15), we have the fact that $(0_{R^n}, t) \notin M$.

By the separation theorem of convex sets, there exists $(\lambda_1, \lambda_2, \ldots, \lambda_n, \bar{r}) \neq 0_{R^n}$ such that

$$\sum_{i=1}^{n} \lambda_i z_i + \bar{r} r \ge \bar{r} t, \quad \forall \left(z_1, z_2, \ldots, z_n, r\right) \in M. \tag{17}$$

Since $M + R^{n+1} \subset M$, we can get $\lambda_i \ge 0$ and $\bar{r} \ge 0$, for all $i = 1, 2, \ldots, n$. By the definition of M, for each $y \in \operatorname{cl} E$,

$$\left(0_{R^n}, 1 + \max_{1 \le i \le n} s^*\left(f\left(x_i, y\right)\right)\right) \in \operatorname{int} M, \tag{18}$$

$$\begin{aligned} &\left(s^*\left(f\left(x_1, y\right)\right) - r,\right. \\ &\quad \left. s^*\left(f\left(x_2, y\right)\right) - r, \ldots, s^*\left(f\left(x_n, y\right)\right) - r, r\right) \in M. \end{aligned} \tag{19}$$

By (18), $\bar{r} > 0$. Then, by (17) and (19),

$$\sum_{i=1}^{n} \frac{\lambda_i}{\bar{r}} s^*\left(f\left(x_i, y\right)\right) + r\left(1 - \sum_{i=1}^{n} \frac{\lambda_i}{\bar{r}}\right) \ge t. \tag{20}$$

By (20), $\sum_{i=1}^{n}(\lambda_i/\bar{r}) = 1$. Thus, by the condition (ii), for each $y \in \operatorname{cl} E$, there exists $\bar{x} \in \operatorname{cl} E$ such that

$$s^*\left(f\left(\bar{x}, y\right)\right) \ge \sum_{i=1}^{n} \frac{\lambda_i}{\bar{r}} s^*\left(f\left(x_i, y\right)\right) \ge t. \tag{21}$$

By the assumption about t and s^*, there exists $\bar{x} \in \operatorname{cl} E$ such that

$$f\left(\bar{x}, y\right) \notin -\operatorname{int} S, \quad \forall y \in \operatorname{cl} E. \tag{22}$$

This completes the proof. □

By Lemmas 6 and 7, we can get the following result.

Theorem 8. *Let E be a nonempty subset of X. Suppose that $f : X \times X \to V$ is a vector-valued mapping with $f(x, x) = 0$ for all $x \in X$ and the following conditions are satisfied:*

(i) $\operatorname{cl} E$ *is a compact set;*

(ii) f *is S-concavelike-convexlike on* $\operatorname{cl} E \times \operatorname{cl} E$*;*

(iii) f *is S-lower semicontinuous on* $\operatorname{cl} E \times \operatorname{cl} E$.

Then, for each $\varepsilon \in \operatorname{int} S$, $S(\varepsilon) \neq \emptyset$.

Remark 9. Note that the condition (i) does not require the fact that $\operatorname{cl} E$ is a convex set. So Theorem 8 is different from Theorem 3.2 in [20]. The following example explains this case.

Example 10. Let $X = R$, $V = R^2$, and $E = [0, 1/3] \cup [2/3, 1]$,

$$\begin{aligned} f\left(x, y\right) &= \left\{\left(xy, xz\right) \in R^2 \mid z = 1 - y^2\right\}, \quad x, y \in X, \\ S &= \left\{\left(x, y\right) \in R^2 \mid x \ge 0, y \ge 0\right\}. \end{aligned} \tag{23}$$

Obviously, $\operatorname{cl} E$ is a compact set. However, $\operatorname{cl} E$ is not a convex set. So, Theorem 3.2 in [20] is not applicable. By the definition of f, f is S-concavelike-convexlike on $\operatorname{cl} E \times \operatorname{cl} E$ and S-lower semicontinuous on $\operatorname{cl} E \times \operatorname{cl} E$. Thus, all conditions of Theorem 8 hold. Indeed, for each $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \operatorname{int} S$,

$$f\left(0, y\right) + \varepsilon = \left(\varepsilon_1, \varepsilon_2\right) \notin -\operatorname{int} S, \quad \forall y \in E. \tag{24}$$

Namely, $0 \in S(\varepsilon)$.

4. Existence of ε-Cone Saddle Points

Lemma 11. *Let E be a nonempty subset of X and $E = A \times B$. Let $\varepsilon \in \operatorname{int} S$ and let $f : X \times X \to V$ be a vector-valued mapping with $f(x, y) = g(a, v) - g(u, b)$, where $x = (a, b)$, $y = (u, v)$, $a, u \in A$, and $v, b \in B$. If there exists $\bar{x} = (\bar{a}, \bar{b}) \in E$ such that*

$$f\left(\bar{x}, y\right) + \varepsilon \notin -\operatorname{int} S, \quad \forall y \in E, \tag{25}$$

then $(\bar{a}, \bar{b}) \in A \times B$ is a weak ε-S-saddle point of g on $A \times B$.

Proof. By assumptions, we have

$$f\left(\bar{x}, y\right) + \varepsilon \notin -\operatorname{int} S, \quad \forall y \in E. \tag{26}$$

Then,

$$g\left(\bar{a}, v\right) - g\left(u, \bar{b}\right) + \varepsilon \notin -\operatorname{int} S, \quad \forall \left(u, v\right) \in A \times B. \tag{27}$$

By (27), taking $u = \bar{a}$,

$$g\left(\bar{a}, v\right) - g\left(\bar{a}, \bar{b}\right) + \varepsilon \notin -\operatorname{int} S, \quad \forall v \in B, \tag{28}$$

which implies $g(\bar{a}, \bar{b}) \in \operatorname{Min}_\varepsilon g(\bar{a}, B)$. Then, by (27), taking $v = \bar{b}$,

$$g\left(\bar{a}, \bar{b}\right) - g\left(u, \bar{b}\right) + \varepsilon \notin -\operatorname{int} S, \quad \forall u \in A, \tag{29}$$

which implies $g(\bar{a}, \bar{b}) \in \operatorname{Max}_\varepsilon g(A, \bar{b})$. Thus, $(\bar{a}, \bar{b}) \in A \times B$ is a weak ε-S-saddle point of g on $A \times B$. This completes the proof. □

Theorem 12. *Let A and B be nonempty sets and $\varepsilon \in \operatorname{int} S$. Suppose that g is a vector-valued mapping and the following conditions are satisfied:*

(i) $\operatorname{cl} A$ *and* $\operatorname{cl} B$ *are compact sets;*

(ii) g *is S-concavelike-convexlike on* $\operatorname{cl} A \times \operatorname{cl} B$*;*

(iii) g *is S-upper semicontinuous on* $\operatorname{cl} A \times \operatorname{cl} B$*;*

(iv) g *is S-lower semicontinuous on* $\operatorname{cl} A \times \operatorname{cl} B$.

Then, g has a weak ε-S-saddle point on $A \times B$.

Proof. Let $A \times B = E$ and $f : \mathrm{cl}\, E \times \mathrm{cl}\, E \to V$ be a vector-valued mappings by

$$f(x, y) = g(a, v) - g(u, b), \quad \forall\ x = (a, b) \in \mathrm{cl}\, E, \quad y = (u, v) \in \mathrm{cl}\, E. \tag{30}$$

Next, we show that all assumptions of Theorem 8 are satisfied by g.

Clearly, by the condition (i), $\mathrm{cl}\, E$ is compact. Then, by the condition (ii), we have the fact that, for each $a_1, a_2 \in \mathrm{cl}\, A$ and $l \in [0, 1]$, there exists $a_3 \in \mathrm{cl}\, A$ such that

$$g(a_3, b) \in lg(a_1, b) + (1 - l)\, g(a_2, b) + S, \quad \forall b \in \mathrm{cl}\, B \tag{31}$$

and, for each $b_1, b_2 \in \mathrm{cl}\, B$ and $l \in [0, 1]$, there exists $b_3 \in \mathrm{cl}\, B$

$$g(a, b_3) \in lg(a, b_1) + (1 - l)\, g(a, b_2) - S, \quad \forall a \in \mathrm{cl}\, A. \tag{32}$$

By (31) and (32), for each $(a_1, b_1), (a_2, b_2) \in \mathrm{cl}\, E$ and $l \in [0, 1]$, there exists $(a_3, b_3) \in \mathrm{cl}\, E$ such that

$$\begin{aligned} g(a_3, b) - g(a, b_3) &\in l\left(g(a_1, b) - g(a, b_1)\right) \\ &\quad + (1 - l)\left(g(a_2, b) - g(a, b_2)\right) + S, \\ &\quad \forall (a, b) \in \mathrm{cl}\, E, \\ g(a, b_3) - g(a_3, b) &\in l\left(g(a, b_1) - g(a_1, b)\right) \\ &\quad + (1 - l)\left(g(a, b_2) - g(a_2, b)\right) - S, \\ &\quad \forall (a, b) \in \mathrm{cl}\, E. \end{aligned} \tag{33}$$

Namely, f is S-concavelike-convexlike on $\mathrm{cl}\, E \times \mathrm{cl}\, E$.

Now, we show that f is S-lower semicontinuous on $\mathrm{cl}\, E \times \mathrm{cl}\, E$. By the condition (iii), for each $(a, v) \in \mathrm{cl}\, A \times \mathrm{cl}\, B$ and $s \in \mathrm{int}\, S$, there exists an open neighborhood U_a of a and U_v of v such that

$$g(u_a, u_v) \in g(a, v) - \frac{s}{2} + \mathrm{int}\, S, \quad \forall u_a \in U_a,\ u_v \in U_v, \tag{34}$$

and, for each $(u, b) \in \mathrm{cl}\, A \times \mathrm{cl}\, B$ and $s \in \mathrm{int}\, S$, there exists an open neighborhood U_u of u and U_b of b such that

$$g(u_u, u_b) \in g(u, b) + \frac{s}{2} - \mathrm{int}\, S, \quad \forall u_u \in U_u,\ u_b \in U_b. \tag{35}$$

By (34) and (35), we have the fact that, for any $((a, b), (u, v)) \in \mathrm{cl}\, E \times \mathrm{cl}\, E$,

$$g(u_a, u_v) - g(u_u, u_b) \in g(a, v) - g(u, b) - s + \mathrm{int}\, S, \quad \forall\left((u_a, u_b), (u_u, u_v)\right) \in U_a \times U_b \times U_u \times U_v. \tag{36}$$

Namely, f is S-lower semicontinuous on $\mathrm{cl}\, E \times \mathrm{cl}\, E$. Therefore, by Lemma 11, g has a weak ε-S-saddle point on $A \times B$. This completes the proof. □

Remark 13. The conditions (iii) and (iv) of Theorem 12 do not imply that g is continuous (see [23]).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author would like to thank the anonymous referees for their valuable comments and suggestions, which helped to improve the paper. This paper is dedicated to Professor Miodrag Mateljevi'c on the occasion of his 65th birthday.

References

[1] G. Bigi, A. Capătă, and G. Kassay, "Existence results for strong vector equilibrium problems and their applications," *Optimization*, vol. 61, no. 5, pp. 567–583, 2012.

[2] X.-H. Gong, "The strong minimax theorem and strong saddle points of vector-valued functions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 8, pp. 2228–2241, 2008.

[3] X. Gong, "Strong vector equilibrium problems," *Journal of Global Optimization*, vol. 36, no. 3, pp. 339–349, 2006.

[4] X. B. Li, S. J. Li, and Z. M. Fang, "A minimax theorem for vector-valued functions in lexicographic order," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 73, no. 4, pp. 1101–1108, 2010.

[5] J. W. Nieuwenhuis, "Some minimax theorems in vector-valued functions," *Journal of Optimization Theory and Applications*, vol. 40, no. 3, pp. 463–475, 1983.

[6] T. Tanaka, "A characterization of generalized saddle points for vector-valued functions via scalarization," *Nihonkai Mathematical Journal*, vol. 1, no. 2, pp. 209–227, 1990.

[7] T. Tanaka, "Generalized semicontinuity and existence theorems for cone saddle points," *Applied Mathematics and Optimization*, vol. 36, no. 3, pp. 313–322, 1997.

[8] Y. Zhang, S. J. Li, and S. K. Zhu, "Minimax problems for set-valued mappings," *Numerical Functional Analysis and Optimization*, vol. 33, no. 2, pp. 239–253, 2012.

[9] Q.-b. Zhang, M.-j. Liu, and C.-z. Cheng, "Generalized saddle points theorems for set-valued mappings in locally generalized convex spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 1-2, pp. 212–218, 2009.

[10] Q. H. Ansari, I. V. Konnov, and J. C. Yao, "Existence of a solution and variational principles for vector equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 110, no. 3, pp. 481–492, 2001.

[11] Q. H. Ansari, I. V. Konnov, and J. C. Yao, "Characterizations of solutions for vector equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 113, no. 3, pp. 435–447, 2002.

[12] Q. H. Ansari, X. Q. Yang, and J.-C. Yao, "Existence and duality of implicit vector variational problems," *Numerical Functional Analysis and Optimization*, vol. 22, no. 7-8, pp. 815–829, 2001.

[13] J. W. Chen, Z. Wan, and Y. J. Cho, "The existence of solutions and well-posedness for bilevel mixed equilibrium problems in Banach spaces," *Taiwanese Journal of Mathematics*, vol. 17, no. 2, pp. 725–748, 2013.

[14] Y. J. Cho, S. S. Chang, J. S. Jung, S. M. Kang, and X. Wu, "Minimax theorems in probabilistic metric spaces," *Bulletin of the Australian Mathematical Society*, vol. 51, no. 1, pp. 103–119, 1995.

[15] Y. J. Cho, M. R. Delavar, S. A. Mohammadzadeh, and M. Roohi, "Coincidence theorems and minimax inequalities in abstract convex spaces," *Journal of Inequalities and Applications*, vol. 2011, article 126, 2011.

[16] Y.-P. Fang and N.-J. Huang, "Vector equilibrium type problems with $(S)_+$-conditions," *Optimization*, vol. 53, no. 3, pp. 269–279, 2004.

[17] B. T. Kien, N. C. Wong, and J.-C. Yao, "Generalized vector variational inequalities with star-pseudomonotone and discontinuous operators," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 9, pp. 2859–2871, 2008.

[18] X. B. Li and S. J. Li, "Existence of solutions for generalized vector quasi-equilibrium problems," *Optimization Letters*, vol. 4, no. 1, pp. 17–28, 2010.

[19] L. Q. Anh and P. Q. Khanh, "Semicontinuity of the approximate solution sets of multivalued quasiequilibrium problems," *Numerical Functional Analysis and Optimization*, vol. 29, no. 1-2, pp. 24–42, 2008.

[20] K. Kimura, Y. C. Liou, Yao, and J. C. :, "Semicontinuty of the solution mapping of ε-vector equilibrium problem," in *Globalization Challenge and Management Transformation*, J. L. Zhang, W. Zhang, X. P. Xia, and J. Q. Liao, Eds., pp. 103–113, Science Press, Beijing, China, 2007.

[21] X. B. Li and S. J. Li, "Continuity of approximate solution mappings for parametric equilibrium problems," *Journal of Global Optimization*, vol. 51, no. 3, pp. 541–548, 2011.

[22] J. Jahn, *Vector Optimization: Theory, Applications, and Extensions*, Springer, Berlin, Germany, 2004.

[23] D. H. Luc, *Theory of Vector Optimization*, vol. 319 of *Lecture Notes in Economics and Mathematical Systems*, Springer, Berlin, Germany, 1989.

[24] Y. Zhang and S. J. Li, "Minimax theorems for scalar set-valued mappings with nonconvex domains and applications," *Journal of Global Optimization*, vol. 57, no. 4, pp. 1359–1373, 2013.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 280817, 11 pages
http://dx.doi.org/10.1155/2014/280817

Research Article

Fixed Point Theory in α-Complete Metric Spaces with Applications

N. Hussain,[1] M. A. Kutbi,[1] and P. Salimi[2]

[1] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, 21589 Jeddah, Saudi Arabia*
[2] *Young Researchers and Elite Club, Rasht Branch, Islamic Azad University, Rasht, Iran*

Correspondence should be addressed to P. Salimi; salimipeyman@gmail.com

Received 28 November 2013; Accepted 14 December 2013; Published 6 February 2014

Academic Editor: Ljubomir B. Ćirić

The aim of this paper is to introduce new concepts of α-η-complete metric space and α-η-continuous function and establish fixed point results for modified α-η-ψ-rational contraction mappings in α-η-complete metric spaces. As an application, we derive some Suzuki type fixed point theorems and new fixed point theorems for ψ-graphic-rational contractions. Moreover, some examples and an application to integral equations are given here to illustrate the usability of the obtained results.

This paper is dedicated to Professor Miodrag Mateljević on the occasion of his 65th birthday

1. Preliminaries

We know by the Banach contraction principle [1], which is a classical and powerful tool in nonlinear analysis, that a self-mapping f on a complete metric space (X, d) such that $d(fx, fy) \leq c\ d(x, y)$ for all $x, y \in X$, where $c \in [0, 1)$, has a unique fixed point. Since then, the Banach contraction principle has been generalized in several directions (see [2–26] and references cited therein).

In 2008, Suzuki [21] proved the following result that is an interesting generalization of the Banach contraction principle which also characterizes the metric completeness.

Theorem 1. *Let (X, d) be a complete metric space and let T be a self-mapping on X. Define a nonincreasing function $\theta : [0, 1) \to (1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1, & if\ 0 \leq r \leq \dfrac{(\sqrt{5}-1)}{2}, \\ (1-r)r^{-2}, & if\ \dfrac{(\sqrt{5}-1)}{2} < r < 2^{-1/2}, \\ (1+r)^{-1}, & if\ 2^{-1/2} \leq r < 1. \end{cases} \tag{1}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r)\, d(x, Tx) \leq d(x, y) \quad implies\ d(Tx, Ty) \leq rd(x, y) \tag{2}$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover, $\lim_{n\to+\infty} T^n x = z$ for all $x \in X$.

In 2012, Samet et al. [19] introduced the concepts of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings defined on complete metric spaces. Afterwards Salimi et al. [16] and Hussain et al. [7] modified the notions of α-ψ-contractive and α-admissible mappings and established fixed point theorems which are proper generalizations of the recent results in [12, 19].

Definition 2 (see [19]). Let T be a self-mapping on X and let $\alpha : X \times X \to [0, +\infty)$ be a function. One says that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{3}$$

Definition 3 (see [16]). Let T be a self-mapping on X and let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. One says that T is an α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \Longrightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty). \tag{4}$$

Note that if we take $\eta(x, y) = 1$, then this definition reduces to Definition 2. Also, if we take $\alpha(x, y) = 1$, then we say that T is an η-subadmissible mapping.

Here we introduce the notions of α-η-complete metric space and α-η-continuous function and establish fixed point results for modified α-η-ψ-rational contractions in α-η-complete metric spaces which are not necessarily complete. As an application, we derive some Suzuki type fixed point theorems and new fixed point theorems for ψ-graphic-rational contractions. Moreover, some examples and an application to integral equations are given here to illustrate the usability of the obtained results.

2. Main Results

First, we introduce the notions of α-η-complete metric space and α-η-continuous function.

Definition 4. Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, +\infty)$. The metric space X is said to be α-η-complete if and only if every Cauchy sequence $\{x_n\}$ with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ converges in X. One says X is an α-complete metric space when $\eta(x, y) = 1$ for all $x, y \in X$ and one says (X, d) is an η-complete metric space when $\alpha(x, y) = 1$ for all $x, y \in X$.

Example 5. Let $X = (0, \infty)$ and $d(x, y) = |x - y|$ be a metric function on X. Let A be a closed subset of X. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} (x + y)^2, & \text{if } x, y \in A, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(x, y) = 2xy. \tag{5}$$

Clearly, (X, d) is not a complete metric space, but (X, d) is an α-η-complete metric space. Indeed, if $\{x_n\}$ is a Cauchy sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $x_n \in A$ for all $n \in \mathbb{N}$. Now, since (A, d) is a complete metric space, then there exists $x^* \in A$ such that $x_n \to x^*$ as $n \to \infty$.

Remark 6. Let $T : X \to X$ be a self-mapping on metric space X and let X be an orbitally T-complete. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 3, & \text{if } x, y \in O(w), \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(x, y) = 1, \tag{6}$$

where $O(w)$ is an orbit of a point $w \in X$. Then (X, d) is an α-η-complete metric space. Indeed, if $\{x_n\}$ be a Cauchy sequence, where $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq O(w)$. Now, since X is an orbitally T-complete metric space, then $\{x_n\}$ converges in X. That is, (X, d) is an α-η-complete metric space. Also, suppose that $\alpha(x, y) \geq \eta(x, y)$; then $x, y \in O(w)$. Hence, $Tx, Ty \in O(w)$. That is, $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. Thus, T is an α-admissible mapping with respect to η.

Definition 7. Let (X, d) be a metric space. Let $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. One says T is an α-η-continuous mapping on (X, d), if for given $x \in X$ and sequence $\{x_n\}$ with

$$x_n \longrightarrow x, \quad \text{as } n \longrightarrow \infty,$$
$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \quad \forall n \in \mathbb{N} \Longrightarrow Tx_n \longrightarrow Tx. \tag{7}$$

Example 8. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ be a metric on X. Assume that $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be defined by

$$Tx = \begin{cases} x^5, & \text{if } x \in [0, 1], \\ \sin \pi x + 2, & \text{if } (1, \infty), \end{cases}$$
$$\alpha(x, y) = \begin{cases} x^2 + y^2 + 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$
$$\eta(x, y) = x^2.$$

Clearly, T is not continuous, but T is α-η-continuous on (X, d). Indeed, if $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$, then $x_n \in [0, 1]$ and so $\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_n^5 = x^5 = Tx$.

Remark 9. Define (X, d) and $\alpha, \eta : X \times X \to [0, +\infty)$ as in Remark 6. Let $T : X \to X$ be a an orbitally continuous map on (X, d). Then T is α-η-continuous on (X, d). Indeed if $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, so $x_n \in O(w)$ for all $n \in \mathbb{N}$, then there exists sequence $(k_i)_{i \in \mathbb{N}}$ of positive integer such that $x_n = T^{k_i} w \to x$ as $i \to \infty$. Now since T is an orbitally continuous map on (X, d), then $Tx_n = T(T^{k_i} w) \to Tx$ as $i \to \infty$ as required.

A function $\psi : [0, \infty) \to [0, \infty)$ is called Bianchini-Grandolfi gauge function [13, 14, 27] if the following conditions hold:

(i) ψ is nondecreasing;

(ii) there exist $k_0 \in \mathbb{N}$ and $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \leq a \psi^k(t) + v_k, \tag{9}$$

for $k \geq k_0$ and any $t \in \mathbb{R}^+$.

In some sources, Bianchini-Grandolfi gauge function is known as (c)—comparison function (see e.g., [2]). We denote by Ψ the family of Bianchini-Grandolfi gauge functions. The following lemma illustrates the properties of these functions.

Lemma 10 (see [2]). *If $\psi \in \Psi$, then the following hold:*

(i) $(\psi^n(t))_{n\in\mathbb{N}}$ *converges to 0 as $n \to \infty$ for all $t \in \mathbb{R}^+$;*

(ii) $\psi(t) < t$, *for any $t \in (0,\infty)$;*

(iii) ψ *is continuous at 0;*

(iv) *the series $\sum_{k=1}^{\infty}\psi^k(t)$ converges for any $t \in \mathbb{R}^+$.*

Definition 11. Let (X,d) be a metric space and let T be a self-mapping on X. Let

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)}{1+d(x,Tx)}, \frac{d(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}. \tag{10}$$

Then,

(a) we say T is a modified α-η-ψ-rational contraction mapping if

$$x, y \in X, \quad \eta(x,Tx) \le \alpha(x,y) \Longrightarrow d(Tx,Ty) \le \psi(M(x,y)), \tag{11}$$

where $\psi \in \Psi$;

(b) we say T is a modified α-ψ-rational contraction mapping if

$$x, y \in X, \quad \alpha(x,y) \ge 1 \Longrightarrow d(Tx,Ty) \le \psi(M(x,y)), \tag{12}$$

where $\psi \in \Psi$.

The following is our first main result of this section.

Theorem 12. *Let (X,d) be a metric space and let T be a self-mapping on X. Also, suppose that $\alpha, \eta : X \times X \to [0,\infty)$ are two functions and $\psi \in \Psi$. Assume that the following assertions hold true:*

(i) *(X,d) is an α-η-complete metric space;*

(ii) *T is an α-admissible mapping with respect to η;*

(iii) *T is modified α-η-ψ-rational contraction mapping on X;*

(iv) *T is an α-η-continuous mapping on X;*

(v) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge \eta(x_0,Tx_0)$.*

Then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0,Tx_0) \ge \eta(x_0,Tx_0)$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for T and the result is proved. Hence, we suppose that $x_{n+1} \ne x_n$ for all $n \in \mathbb{N}$. Since T is α-admissible mapping with respect to η and $\alpha(x_0,Tx_0) \ge \eta(x_0,Tx_0)$, we deduce that $\alpha(x_1,x_2) = \alpha(Tx_0,T^2x_0) \ge \eta(Tx_0,T^2x_0) = \eta(x_1,x_2)$. Continuing this process, we get

$$\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1}) = \eta(x_n,Tx_n) \tag{13}$$

for all $n \in \mathbb{N}\cup\{0\}$. Now, by (a) we get

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \le \psi(M(x_{n-1},x_n)), \tag{14}$$

where

$$\begin{aligned} M(x_{n-1},x_n) &= \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},Tx_{n-1})}{1+d(x_{n-1},Tx_{n-1})}, \frac{d(x_n,Tx_n)}{1+d(x_n,Tx_n)}, \frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2}\right\} \\ &= \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)}{1+d(x_{n-1},x_n)}, \frac{d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})}, \frac{d(x_{n-1},x_{n+1})}{2}\right\} \\ &\le \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right\} \\ &= \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} \end{aligned} \tag{15}$$

and so, $M(x_{n-1},x_n) \le \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}$. Now since ψ is nondecreasing, so from (14), we have

$$d(x_n,x_{n+1}) \le \psi(\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}). \tag{16}$$

Now, if $\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$ for some $n \in \mathbb{N}$, then

$$\begin{aligned} d(x_n,x_{n+1}) &\le \psi(\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}) \\ &= \psi(d(x_n,x_{n+1})) < d(x_n,x_{n+1}) \end{aligned} \tag{17}$$

which is a contradiction. Hence, for all $n \in \mathbb{N}$ we have

$$d(x_n,x_{n+1}) \le \psi(d(x_{n-1},x_n)). \tag{18}$$

By induction, we have

$$d(x_n,x_{n+1}) \le \psi^n(d(x_0,x_1)). \tag{19}$$

Fix $\epsilon > 0$; there exists $N \in \mathbb{N}$ such that

$$\sum_{n\ge N}\psi^n(d(x_0,x_1)) < \epsilon. \tag{20}$$

Let $m, n \in \mathbb{N}$ with $m > n \ge N$. Then by triangular inequality we get

$$d(x_n,x_m) \le \sum_{k=n}^{m-1} d(x_k,x_{k+1}) \le \sum_{n\ge N}\psi^n(d(x_0,x_1)) < \epsilon. \tag{21}$$

Consequently $\lim_{m,n,\to+\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. On the other hand from (13) we know that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Now since X is an α-η-complete metric space, there is $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also, since T is an α-η-continuous mapping, so $x_{n+1} = Tx_n \to Tz$ as $n \to \infty$. That is, $z = Tz$ as required. □

Example 13. Let $X = (-\infty, -2) \cup [-1, 1] \cup (2, +\infty)$. We endow X with the metric

$$d(x, y) = \begin{cases} \max\{|x|, |y|\}, & \text{if } x \neq y, \\ 0, & x = y. \end{cases} \quad (22)$$

Define $T : X \to X$, $\alpha, \eta : X \times X \to [0, \infty)$, and $\psi : [0, \infty) \to [0, \infty)$ by

$$Tx = \begin{cases} \sqrt{2x^2 - 1}, & \text{if } x \in (-\infty, -3], \\ x^3 - 1, & \text{if } x \in (-3, -2), \\ \frac{1}{4}x^2, & \text{if } x \in [-1, 0], \\ \frac{1}{4}x, & \text{if } x \in (0, 1], \\ 5 + \sin \pi x, & \text{if } x \in (2, 4), \\ 3x^3 + \ln x + 1, & \text{if } x \in [4, \infty), \end{cases} \quad (23)$$

$$\alpha(x, y) = \begin{cases} x^2 + y^2 + 1, & \text{if } x, y \in [-1, 1], \\ x^2, & \text{otherwise}, \end{cases}$$

$$\eta(x, y) = x^2 + y^2,$$

$$\psi(t) = \frac{1}{2}t.$$

Clearly, (X, d) is not a complete metric space. However, it is an α-η-complete metric space. In fact, if $\{x_n\}$ is a Cauchy sequence such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [-1, 1]$ for all $n \in \mathbb{N}$. Now, since $([-1, 1], d)$ is a complete metric space, then the sequence $\{x_n\}$ converges in $[-1, 1] \subseteq X$. Let $\alpha(x, y) \geq \eta(x, y)$; then $x, y \in [-1, 1]$. On the other hand, $Tw \in [-1, 1]$ for all $w \in [-1, 1]$. Then, $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. That is, T is an α-admissible mapping with respect to η. Let $\{x_n\}$ be a sequence, such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n+1}, x_n) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Then, $\{x_n\} \subseteq [-1, 1]$ for all $n \in \mathbb{N}$. So, $\{Tx_n\} \subseteq [-1, 1]$ (since $Tw \in [-1, 1]$ for all $w \in [-1, 1]$). Now, since T is continuous on $[-1, 1]$. Then, $Tx_n \to Tx$ as $n \to \infty$. That is, T is an α-η-continuous mapping. Clearly, $\alpha(0, T0) \geq \eta(0, T0)$. Let $\alpha(x, y) \geq \eta(x, Tx)$. Now, if $x \notin [-1, 1]$ or $y \notin [-1, 1]$, then $x^2 \geq x^2 + y^2 + 1$ which implies $y^2 + 1 \leq 0$ which is a contradiction. Then, $x, y \in [-1, 1]$. Now we consider the following cases:

(i) let $x, y \in [-1, 0)$ with $x \neq y$; then,

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{4} \max\{x^2, y^2\} \\ &\leq \frac{1}{2} \max\{|x|, |y|\} = \psi(d(x, y)) \leq \psi(M(x, y)); \end{aligned} \quad (24)$$

(ii) let $x, y \in (0, 1]$ with $x \neq y$; then

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{4} \max\{|x|, |y|\} \\ &\leq \frac{1}{2} \max\{|x|, |y|\} = \psi(d(x, y)) \leq \psi(M(x, y)); \end{aligned} \quad (25)$$

(iii) let $x \in (-1, 0)$ and $y \in (0, 1)$; then

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{4} \max\{x^2, y\} \\ &\leq \frac{1}{2} \max\{|x|, |y|\} = \psi(d(x, y)) \leq \psi(M(x, y)) \end{aligned} \quad (26)$$

(iv) let $x = y \in [-1, 0)$, $x = y \in (0, 1]$ or let $x = -1$, $y = 1$; then, $Tx = Ty$. That is,

$$d(Tx, Ty) = 0 \leq \psi(M(x, y)). \quad (27)$$

Thus T is a modified α-η-ψ-rational contraction mapping. Hence all conditions of Theorem 12 are satisfied and T has a fixed point. Here, $x = 0$ is fixed point of T.

By taking $\eta(x, y) = 1$ for all $x, y \in X$ in Theorem 12, we obtain the following corollary.

Corollary 14. *Let (X, d) be a metric space and let T be a self-mapping on X. Also, suppose that $\alpha : X \times X \to [0, \infty)$ is a function and $\psi \in \Psi$. Assume that the following assertions hold true:*

(i) *(X, d) is an α-complete metric space;*

(ii) *T is an α-admissible mapping;*

(iii) *T is a modified α-ψ-rational contraction on X;*

(iv) *T is an α-continuous mapping on X;*

(v) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.*

Then T has a fixed point.

Theorem 15. *Let (X, d) be a metric space and let T be a self-mapping on X. Also, suppose that $\alpha, \eta : X \times X \to [0, \infty)$ are two functions and $\psi \in \Psi$. Assume that the following assertions hold true:*

(i) *(X, d) is an α-η-complete metric space;*

(ii) *T is an α-admissible mapping with respect to η;*

(iii) *T is a modified α-η-ψ-rational contraction on X;*

(iv) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(v) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *with* $x_n \to x$ *as* $n \to \infty$, *then either*

$$\eta\left(Tx_n, T^2x_n\right) \leq \alpha\left(Tx_n, x\right) \quad \text{or } \eta\left(T^2x_n, T^3x_n\right) \leq \alpha\left(T^2x_n, x\right) \tag{28}$$

holds for all $n \in \mathbb{N}$.

Then T *has a fixed point.*

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Now as in the proof of Theorem 12 we have $\alpha(x_{n+1}, x_n) \geq \eta(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$ and there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Let $d(z, Tz) \neq 0$. From (v) either

$$\eta\left(Tx_{n-1}, T^2x_{n-1}\right) \leq \alpha\left(Tx_{n-1}, z\right) \quad \text{or } \eta\left(T^2x_{n-1}, T^3x_{n-1}\right) \leq \alpha\left(T^2x_{n-1}, z\right) \tag{29}$$

holds for all $n \in \mathbb{N}$. Then,

$$\eta\left(x_n, x_{n+1}\right) \leq \alpha\left(x_n, z\right) \quad \text{or} \quad \eta\left(x_{n+1}, x_{n+2}\right) \leq \alpha\left(x_{n+1}, z\right) \tag{30}$$

holds for all $n \in \mathbb{N}$. Let $\eta(x_n, x_{n+1}) \leq \alpha(x_n, z)$ hold for all $n \in \mathbb{N}$. Now from (a) we get

$$\begin{aligned}
&d\left(x_{n_k+1}, Tz\right) \\
&= d\left(Tx_{n_k}, Tz\right) \\
&\leq \psi\left(\max\left\{d\left(x_{n_k}, z\right), \frac{d\left(x_{n_k}, Tx_{n_k}\right)}{1 + d\left(x_{n_k}, Tx_{n_k}\right)}, \frac{d(z, Tz)}{1 + d(z, Tz)}, \frac{d\left(x_{n_k}, Tz\right) + d\left(z, Tx_{n_k}\right)}{2}\right\}\right) \\
&= \psi\left(\max\left\{d\left(x_{n_k}, z\right), \frac{d\left(x_{n_k}, x_{n_k+1}\right)}{1 + d\left(x_{n_k}, x_{n_k+1}\right)}, \frac{d(z, Tz)}{1 + d(z, Tz)}, \frac{d\left(x_{n_k}, Tz\right) + d\left(z, x_{n_k+1}\right)}{2}\right\}\right) \\
&< \max\left\{d\left(x_{n_k}, z\right), \frac{d\left(x_{n_k}, x_{n_k+1}\right)}{1 + d\left(x_{n_k}, x_{n_k+1}\right)}, \frac{d(z, Tz)}{1 + d(z, Tz)}, \frac{d\left(x_{n_k}, Tz\right) + d\left(z, x_{n_k+1}\right)}{2}\right\}.
\end{aligned} \tag{31}$$

By taking limit as $k \to \infty$ in the above inequality we get

$$d(z, Tz) \leq \max\left\{\frac{d(z, Tz)}{1 + d(z, Tz)}, \frac{d(z, Tz)}{2}\right\} < d(z, Tz) \tag{32}$$

which is a contradiction. Hence, $d(z, Tz) = 0$ implies $z = Tz$. By the similar method we can show that $z = Tz$ if $\eta(x_{n+1}, x_{n+2}) \leq \alpha(x_{n+1}, z)$ holds for all $n \in \mathbb{N}$. □

Example 16. Let $X = (0, +\infty)$. We endow X with usual metric. Define $T : X \to X$, $\alpha, \eta : X \times X \to [0, \infty)$, and $\psi : [0, \infty) \to [0, \infty)$ by

$$\begin{gathered}
Tx = \begin{cases} \dfrac{\sqrt{x^2+1}}{\sin x + \cos x + 3}, & \text{if } x \in (0,1), \\ \dfrac{1}{16}x^2 + 1, & \text{if } x \in [1,2], \\ \dfrac{x^3+1}{\sqrt{x^2+1}}, & \text{if } x \in (2,\infty), \end{cases} \\
\alpha(x, y) = \begin{cases} \dfrac{1}{2}, & \text{if } x, y \in [1,2], \\ 0, & \text{otherwise}, \end{cases} \\
\eta(x, y) = \frac{1}{4}, \qquad \psi(t) = \frac{1}{4}t.
\end{gathered} \tag{33}$$

Note that (X, d) is not a complete metric space. But it is an α-η-complete metric space. Indeed, if $\{x_n\}$ is a Cauchy sequence such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [1,2]$ for all $n \in \mathbb{N}$. Now, since $([1,2], d)$ is a complete metric space, then the sequence $\{x_n\}$ converges in $[1,2] \subseteq X$. Let $\alpha(x, y) \geq \eta(x, y)$; then $x, y \in [1,2]$. On the other hand, $Tw \in [1,2]$ for all $w \in [1,2]$. Then, $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. That is, T is an α-admissible mapping with respect to η. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$. Then, $Tx_n, T^2x_n, T^3x_n \in [1,2]$ for all $n \in \mathbb{N}$. That is,

$$\eta\left(Tx_n, T^2x_n\right) \leq \alpha\left(Tx_n, x\right), \quad \eta\left(T^2x_n, T^3x_n\right) \leq \alpha\left(T^2x_n, x\right), \tag{34}$$

holds for all $n \in \mathbb{N}$. Clearly, $\alpha(0, T0) \geq \eta(0, T0)$. Let, $\alpha(x, y) \geq \eta(x, Tx)$. Now, if $x \notin [1,2]$ or $y \notin [1,2]$, then $0 \geq 1/4$, which is a contradiction. So, $x, y \in [1,2]$. Therefore,

$$\begin{aligned}
d(Tx, Ty) &= \frac{1}{16}\left|x^2 - y^2\right| \\
&= \frac{1}{16}|x - y||x + y| \leq \frac{1}{4}|x - y| \\
&= \frac{1}{4}d(x, y) \leq \frac{1}{4}M(x, y) = \psi(M(x, y)).
\end{aligned} \tag{35}$$

Therefore T is a modified α-η-ψ-rational contraction mapping. Hence all conditions of Theorem 15 hold and T has a fixed point. Here, $x = 8 - 2\sqrt{14}$ is a fixed point of T.

If in Theorem 15 we take $\eta(x, y) = 1$ for all $x, y \in X$, then we obtain the following result.

Corollary 17. *Let* (X, d) *be a metric space and let* T *be a self-mapping on* X. *Also, suppose that* $\alpha : X \times X \to [0, \infty)$ *is a function and* $\psi \in \Psi$. *Assume that the following assertions hold true:*

(i) (X, d) *is a* α*-complete metric space;*

(ii) T *is an* α*-admissible mapping;*

(iii) T *is a modified* α*-*ψ*-rational contraction mapping on* X*;*

(iv) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*

(v) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *with* $x_n \to x$ *as* $n \to \infty$*, then either*

$$\alpha\left(Tx_n, x\right) \geq 1 \quad \text{or } \alpha\left(T^2 x_n, x\right) \geq 1 \tag{36}$$

holds for all $n \in \mathbb{N}$*.*

Then T *has a fixed point.*

Corollary 18. *Let* (X, d) *be a complete metric space and let* T *be a continuous self-mapping on* X*. Assume that* T *is a modified rational contraction mapping, that is,*

$$\forall x, y \in X, \quad d\left(Tx, Ty\right) \leq \psi\left(M\left(x, y\right)\right), \tag{37}$$

where $\psi \in \Psi$*. Then* T *has a fixed point.*

Corollary 19. *Let* (X, d) *be a complete metric space and let* T *be a continuous self-mapping on* X*. Assume that* T *satisfies the following rational inequality:*

$$\forall x, y \in X, \quad d\left(Tx, Ty\right) \leq rM\left(x, y\right), \tag{38}$$

where $0 \leq r < 1$ *and*

$$M\left(x, y\right) = \max\left\{d\left(x, y\right), \frac{d\left(x, Tx\right)}{1 + d\left(x, Tx\right)}, \frac{d\left(y, Ty\right)}{1 + d\left(y, Ty\right)}, \frac{d\left(x, Ty\right) + d\left(y, Tx\right)}{2}\right\}. \tag{39}$$

Then T *has a fixed point.*

3. Consequences

3.1. Suzuki Type Fixed Point Results. From Theorem 12 we deduce the following Suzuki type fixed point result.

Theorem 20. *Let* (X, d) *be a complete metric space and let* T *be a continuous self-mapping on* X*. Assume that there exists* $r \in [0, 1)$ *such that*

$$d\left(x, Tx\right) \leq d\left(x, y\right) \quad \text{implies } d\left(Tx, Ty\right) \leq rM\left(x, y\right) \tag{40}$$

for all $x, y \in X$*, where*

$$M\left(x, y\right) = \max\left\{d\left(x, y\right), \frac{d\left(x, Tx\right)}{1 + d\left(x, Tx\right)}, \frac{d\left(y, Ty\right)}{1 + d\left(y, Ty\right)}, \frac{d\left(x, Ty\right) + d\left(y, Tx\right)}{2}\right\}. \tag{41}$$

Then T *has a unique fixed point.*

Proof. Define $\alpha, \eta : X \times X \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$ by

$$\alpha\left(x, y\right) = d\left(x, y\right), \qquad \eta\left(x, y\right) = d\left(x, y\right), \tag{42}$$

for all $x, y \in X$ and $\psi(t) = rt$, where $0 \leq r < 1$. Clearly, $\eta(x, y) \leq \alpha(x, y)$ for all $x, y \in X$. That is, conditions (i)–(v) of Theorem 12 hold true. Let $\eta(x, Tx) \leq \alpha(x, y)$. Then, $d(x, Tx) \leq d(x, y)$. Now from (40) we have $d(Tx, Ty) \leq rM(x, y) = \psi(M(x, y))$. That is, T is a modified α-η-ψ-rational contraction mapping on X. Then all conditions of Theorem 12 hold and T has a fixed point. The uniqueness of the fixed point follows easily from (40). □

Corollary 21. *Let* (X, d) *be a complete metric space and let* T *be a continuous self-mapping on* X*. Assume that there exists* $r \in [0, 1)$ *such that*

$$d\left(x, Tx\right) \leq d\left(x, y\right) \quad \text{implies } d\left(Tx, Ty\right) \leq rd\left(x, y\right) \tag{43}$$

for all $x, y \in X$*. Then* T *has a unique fixed point.*

Now, we prove the following Suzuki type fixed point theorem without continuity of T.

Theorem 22. *Let* (X, d) *be a complete metric space and let* T *be a self-mapping on* X*. Define a nonincreasing function* $\rho : [0, 1) \to (1/2, 1]$ *by*

$$\rho\left(r\right) = \frac{1}{1 + r}. \tag{44}$$

Assume that there exists $r \in [0, 1)$ *such that*

$$\rho\left(r\right) d\left(x, Tx\right) \leq d\left(x, y\right) \quad \text{implies } d\left(Tx, Ty\right) \leq rd\left(x, y\right) \tag{45}$$

for all $x, y \in X$*. Then* T *has a unique fixed point.*

Proof. Define $\alpha, \eta : X \times X \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$ by

$$\alpha\left(x, y\right) = d\left(x, y\right), \qquad \eta\left(x, y\right) = \rho\left(r\right) d\left(x, y\right) \tag{46}$$

for all $x, y \in X$ and $\psi(t) = rt$, where $0 \leq r < 1$. Now, since $\rho(r)d(x, y) \leq d(x, y)$ for all $x, y \in X$, $\eta(x, y) \leq \alpha(x, y)$ for all $x, y \in X$. That is, conditions (i)–(iv) of Theorem 15 hold true. Let $\{x_n\}$ be a sequence with $x_n \to x$ as $n \to \infty$. Since $\rho(r)d(Tx_n, T^2 x_n) \leq d(Tx_n, T^2 x_n)$ for all $n \in \mathbb{N}$, then from (45) we get

$$d\left(T^2 x_n, T^3 x_n\right) \leq rd\left(Tx_n, T^2 x_n\right) \tag{47}$$

for all $n \in \mathbb{N}$.

Assume there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \eta\left(Tx_{n_0}, T^2 x_{n_0}\right) &> \alpha\left(Tx_{n_0}, x\right), \\ \eta\left(T^2 x_{n_0}, T^3 x_{n_0}\right) &> \alpha\left(T^2 x_{n_0}, x\right); \end{aligned} \tag{48}$$

then,

$$\rho(r)\,d\left(Tx_{n_0},T^2x_{n_0}\right) > d\left(Tx_{n_0},x\right),$$
$$\rho(r)\,d\left(T^2x_{n_0},T^3x_{n_0}\right) > d\left(T^2x_{n_0},x\right), \tag{49}$$

and so by (47) we have

$$\begin{aligned} & d\left(Tx_{n_0},T^2x_{n_0}\right) \\ & \leq d\left(Tx_{n_0},x\right)+d\left(T^2x_{n_0},x\right) \\ & < \rho(r)\,d\left(Tx_{n_0},T^2x_{n_0}\right)+\rho(r)\,d\left(T^2x_{n_0},T^3x_{n_0}\right) \\ & \leq \rho(r)\,d\left(Tx_{n_0},T^2x_{n_0}\right)+r\rho(r)\,d\left(Tx_{n_0},T^2x_{n_0}\right) \\ & = \rho(r)(1+r)\,d\left(Tx_{n_0},T^2x_{n_0}\right)=d\left(Tx_{n_0},T^2x_{n_0}\right) \end{aligned} \tag{50}$$

which is a contradiction. Hence, either

$$\eta\left(Tx_n,T^2x_n\right) \leq \alpha\left(Tx_n,x\right)$$
$$\text{or } \eta\left(T^2x_n,T^3x_n\right) \leq \alpha\left(T^2x_n,x\right) \tag{51}$$

holds for all $n \in \mathbb{N}$. That is condition (v) of Theorem 15 holds.

Let, $\eta(x,Tx) \leq \alpha(x,y)$. So, $\rho(r)d(x,Tx) \leq d(x,y)$. Then from (45) we get $d(Tx,Ty) \leq rd(x,y) \leq rM(x,y) = \psi(M(x,y))$. Hence, all conditions of Theorem 15 hold and T has a fixed point. The uniqueness of the fixed point follows easily from (45). □

3.2. Fixed Point Results in Orbitally T-Complete Metric Spaces

Theorem 23. *Let (X,d) be a metric space and let $T: X \to X$ be a self-mapping on X. Suppose the following assertions hold:*

(i) *(X,d) is an orbitally T-complete metric space;*

(ii) *there exists $\psi \in \Psi$ such that*

$$d(Tx,Ty) \leq \psi(M(x,y)) \tag{52}$$

holds for all $x,y \in O(w)$ for some $w \in X$, where

$$\begin{aligned} & M(x,y) \\ & = \max\left\{d(x,y),\frac{d(x,Tx)}{1+d(x,Tx)},\right. \\ & \left.\qquad \frac{d(y,Ty)}{1+d(y,Ty)},\frac{d(x,Ty)+d(y,Tx)}{2}\right\}; \end{aligned} \tag{53}$$

(iii) *if $\{x_n\}$ is a sequence such that $\{x_n\} \subseteq O(w)$ with $x_n \to x$ as $n \to \infty$, then $x \in O(w)$.*

Then T has a fixed point.

Proof. Define $\alpha: X \times X \to [0,+\infty)$ as in Remark 6. From Remark 6 we know that (X,d) is an α-complete metric space and T is an α-admissible mapping. Let $\alpha(x,y) \geq 1$; then $x,y \in O(w)$. Then from (ii) we have

$$d(Tx,Ty) \leq \psi(M(x,y)). \tag{54}$$

That is, T is a modified α-ψ-rational contraction mapping. Let $\{x_n\}$ be a sequence such that $\alpha(x_n,x_{n+1}) \geq 1$ with $x_n \to x$ as $n \to \infty$. So, $\{x_n\} \subseteq O(w)$. From (iii) we have $x \in O(w)$. That is, $\alpha(x_n,x) \geq 1$. Hence, all conditions of Corollary 17 hold and T has a fixed point. □

Corollary 24. *Let (X,d) be a metric space and let $T: X \to X$ be a self-mapping on X. Suppose the following assertions hold:*

(i) *(X,d) is an orbitally T-complete metric space;*

(ii) *there exists $r \in [0,1)$ such that*

$$d(Tx,Ty) \leq rM(x,y) \tag{55}$$

holds for all $x,y \in O(w)$ for some $w \in X$, where

$$\begin{aligned} M(x,y) = \max & \left\{d(x,y),\frac{d(x,Tx)}{1+d(x,Tx)},\right. \\ & \left.\frac{d(y,Ty)}{1+d(y,Ty)},\frac{d(x,Ty)+d(y,Tx)}{2}\right\}; \end{aligned} \tag{56}$$

(iii) *if $\{x_n\}$ is a sequence such that $\{x_n\} \subseteq O(w)$ with $x_n \to x$ as $n \to \infty$, then $x \in O(w)$.*

Then T has a fixed point.

3.3. Fixed Point Results for Graphic Contractions. Consistent with Jachymski [11], let (X,d) be a metric space and let Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G),E(G))$. Moreover, we may treat G as a weighted graph (see [11]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length $N(N \in \mathbb{N})$ is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1},x_n) \in E(G)$ for $i = 1,\ldots,N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected (see for details [3, 6, 10, 11]).

Recently, some results have appeared providing sufficient conditions for a mapping to be a Picard operator if (X,d) is endowed with a graph. The first result in this direction was given by Jachymski [11].

Definition 25 (see [11]). We say that a mapping $T: X \to X$ is a Banach G-contraction or simply G-contraction if T preserves edges of G; that is,

$$\forall x,y \in X \quad ((x,y) \in E(G) \Longrightarrow (T(x),T(y)) \in E(G)) \tag{57}$$

and T decreases weights of edges of G in the following way:

$$\exists\,\alpha \in (0,1), \quad \forall x,y \in X$$
$$((x,y) \in E(G) \Longrightarrow d(T(x),T(y)) \leq \alpha d(x,y)). \tag{58}$$

Definition 26 (see [11]). A mapping $T : X \to X$ is called G-continuous, if given $x \in X$ and sequence $\{x_n\}$

$$\begin{gathered} x_n \longrightarrow x, \quad \text{as } n \longrightarrow \infty, \\ (x_n, x_{n+1}) \in E(G), \quad \forall n \in \mathbb{N} \text{ implying } Tx_n \longrightarrow Tx. \end{gathered} \tag{59}$$

Theorem 27. *Let (X, d) be a metric space endowed with a graph G and let T be a self-mapping on X. Suppose that the following assertions hold:*

(i) *for all $x, y \in X$, $(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)$;*

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*

(iii) *there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le \psi(M(x, y)) \tag{60}$$

for all $(x, y) \in E(G)$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}; \tag{61}$$

(iv) *T is G-continuous;*

(v) *if $\{x_n\}$ is a Cauchy sequence in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is convergent in X.*

Then T has a fixed point.

Proof. Define $\alpha : X^2 \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G), \\ 0, & \text{otherwise.} \end{cases} \tag{62}$$

At first we prove that T is an α-admissible mapping. Let $\alpha(x, y) \ge 1$; then $(x, y) \in E(G)$. From (i), we have $(Tx, Ty) \in E(G)$. That is, $\alpha(Tx, Ty) \ge 1$. Thus T is an α-admissible mapping. Let T be G-continuous on (X, d). Then,

$$\begin{gathered} x_n \longrightarrow x, \quad \text{as } n \longrightarrow \infty, \\ (x_n, x_{n+1}) \in E(G), \quad \forall n \in \mathbb{N} \text{ implying } Tx_n \longrightarrow Tx. \end{gathered} \tag{63}$$

That is,

$$\begin{gathered} x_n \longrightarrow x, \quad \text{as } n \longrightarrow \infty, \\ \alpha(x_n, x_{n+1}) \ge 1, \quad \forall n \in \mathbb{N} \text{ implying } Tx_n \longrightarrow Tx \end{gathered} \tag{64}$$

which implies that T is α-continuous on (X, d). From (ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. That is, $\alpha(x_0, Tx_0) \ge 1$. Let $\alpha(x, y) \ge 1$; then $(x, y) \in E(G)$. Now, from (iii) we have $d(Tx, Ty) \le \psi(M(x, y))$. That is,

$$\alpha(x, y) \ge 1 \Longrightarrow d(Tx, Ty) \le \psi(M(x, y)). \tag{65}$$

Condition (v) implies that (X, d) is an α-complete metric space. Hence, all conditions of Corollary 14 are satisfied and T has a fixed point. □

Theorem 28. *Let (X, d) be a complete metric space endowed with a graph G and let T be a self-mapping on X. Suppose that the following assertions hold:*

(i) *for all $x, y \in X$, $(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)$;*

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*

(iii) *there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le \psi(M(x, y)) \tag{66}$$

for all $(x, y) \in E(G)$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}; \tag{67}$$

(iv) *T is G-continuous.*

Then T has a fixed point.

As an application of Corollary 17, we obtain.

Theorem 29. *Let (X, d) be a metric space endowed with a graph G and let T be a self-mapping on X. Suppose that the following assertions hold:*

(i) *for all $x, y \in X$, $(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)$;*

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*

(iii) *there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le \psi(M(x, y)) \tag{68}$$

for all $(x, y) \in E(G)$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}; \tag{69}$$

(iv) *if $\{x_n\}$ is a sequence such that $(x_n, x_{n+1}) \in E(G)$ with $x_n \to x$ as $n \to \infty$, then either*

$$(Tx_n, x) \in E(G) \quad \text{or} \quad (T^2 x_n, x) \in E(G) \tag{70}$$

holds for all $n \in \mathbb{N}$;

(v) *if $\{x_n\}$ is a Cauchy sequence in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then either $\{x_n\}$ is convergent in X or (X, d) is a complete metric space.*

Then T has a fixed point.

Let $(X, d, \preceq)$ be a partially ordered metric space. Define the graph G by

$$E(G) := \{(x, y) \in X \times X : x \preceq y\}. \tag{71}$$

For this graph, condition (i) in Theorem 27 means that T is nondecreasing with respect to this order [5]. From Theorems 27–29 we derive the following important results in partially ordered metric spaces.

Theorem 30. *Let $(X, d, \preceq)$ be a partially ordered metric space and let T be a self-mapping on X. Suppose that the following assertions hold:*

(i) *T is nondecreasing map;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(iii) *there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le \psi(M(x, y)) \tag{72}$$

for all $x \preceq y$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}; \tag{73}$$

(iv) *either for a given $x \in X$ and sequence $\{x_n\}$*

$$x_n \longrightarrow x, \quad \text{as } n \longrightarrow \infty, \qquad x_n \preceq x_{n+1}, \quad \forall n \in \mathbb{N}, \text{ one has } Tx_n \longrightarrow Tx \tag{74}$$

or T is continuous;

(v) *if $\{x_n\}$ is a Cauchy sequence in X with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, then either $\{x_n\}$ is convergent in X or (X, d) is a complete metric space.*

Then T has a fixed point.

Corollary 31 (Ran and Reurings [15]). *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $T : X \to X$ be a continuous nondecreasing self-mapping such that $x_0 \preceq Tx_0$ for some $x_0 \in X$. Assume that*

$$d(Tx, Ty) \le r d(x, y) \tag{75}$$

holds for all $x, y \in X$ with $x \preceq y$, where $0 \le r < 1$. Then T has a fixed point.

Theorem 32. *Let $(X, d, \preceq)$ be a partially ordered metric space and let T be a self-mapping on X. Suppose that the following assertions hold:*

(i) *T is nondecreasing map;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(iii) *there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le \psi(M(x, y)) \tag{76}$$

for all $x \preceq y$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}; \tag{77}$$

(iv) *if $\{x_n\}$ is a sequence such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$, then either*

$$Tx_n \preceq x \quad \text{or } T^2 x_n \preceq x \tag{78}$$

holds for all $n \in \mathbb{N}$;

(v) *if $\{x_n\}$ is a Cauchy sequence in X with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, then either $\{x_n\}$ is convergent in X or (X, d) is a complete metric space.*

Then T has a fixed point.

4. Application to Existence of Solutions of Integral Equations

Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [28–30] and references therein). In this section, we apply our result to the existence of a solution of an integral equation. Let $X = C([0, T], \mathbb{R})$ be the set of real continuous functions defined on $[0, T]$ and let $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x, y) = \|x - y\|_\infty \tag{79}$$

for all $x, y \in X$. Then (X, d) is a complete metric space. Also, assume this metric space endowed with a graph G.

Consider the integral equation as follows:

$$x(t) = p(t) + \int_0^T S(t, s) f(s, x(s)) \, ds \tag{80}$$

and let $F : X \to X$ be defined by

$$F(x)(t) = p(t) + \int_0^T S(t, s) f(s, x(s)) \, ds. \tag{81}$$

We assume that

(A) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous;

(B) $p : [0, T] \to \mathbb{R}$ is continuous;

(C) $S : [0, T] \times \mathbb{R} \to [0, +\infty)$ is continuous;

(D) there exists a $\psi \in \Psi$ such that for all $s \in [0,T]$

$$\forall x, y \in X \quad (x,y) \in E(G) \Longrightarrow (F(x), F(y)) \in E(G),$$

$$\begin{aligned}\forall x, y \in X \quad (x,y) \in E(G) &\Longrightarrow 0 \\ &\le f(s, x(s)) - f(s, y(s)) \\ &\le \psi\Bigg(\max\Bigg\{\frac{|x(s)-y(s)|}{1+|x(s)-y(s)|}, \\ &\qquad \frac{|x(s)-F(x(s))|}{1+|x(s)-F(x(s))|}, |y(s)-F(y(s))|, \\ &\qquad \frac{1}{2}\left[|x(s)-F(y(s))| + |y(s)-F(x(s))|\right]\Bigg\}\Bigg);\end{aligned} \tag{82}$$

(E) there exists $x_0 \in X$ such that $(x_0, F(x_0)) \in E(G)$;

(F) if $\{x_n\}$ is a sequence such that $(x_n, x_{n+1}) \in E(G)$ with $x_n \to x$ as $n \to \infty$, then either

$$(Fx_n, x) \in E(G) \quad \text{or} \quad (F^2 x_n, x) \in E(G) \tag{83}$$

holds for all $n \in \mathbb{N}$;

(G) $\int_0^T S(t,s)ds \le 1$ for all t.

Theorem 33. *Under assumptions (A)–(G), the integral equation* (80) *has a solution in* $X = C([0,T], \mathbb{R})$.

Proof. Consider the mapping $F : X \to X$ defined by (81). Let $(x,y) \in E(G)$. Then from (D) we deduce

$$\begin{aligned}&|F(x)(t) - F(y)(t)| \\ &= \left|\int_0^T S(t,s)\left[f(s,x(s)) - f(s,y(s))\right] ds\right| \\ &\le \int_0^T S(t,s)\left|f(s,x(s)) - f(s,y(s))\right| ds \\ &\le \int_0^T S(t,s)\,\psi \\ &\quad \times \Bigg(\max\Bigg\{|x(s)-y(s)|, \\ &\qquad \frac{|x(s)-F(x(s))|}{1+|x(s)-F(x(s))|}, \frac{|y(s)-F(y(s))|}{1+|y(s)-F(y(s))|}, \\ &\qquad \frac{1}{2}\left[|x(s)-F(y(s))| + |y(s)-F(x(s))|\right]\Bigg\}\Bigg) ds \\ &\le \left(\int_0^T S(t,s)\,ds\right)\psi \\ &\quad \times \Bigg(\max\Bigg\{\|x(s)-y(s)\|, \\ &\qquad \frac{\|x(s)-F(x(s))\|}{1+\|x(s)-F(x(s))\|}, \frac{\|y(s)-F(y(s))\|}{1+\|y(s)-F(y(s))\|}, \\ &\qquad \frac{1}{2}\left[\|x(s)-F(y(s))\| + \|y(s)-F(x(s))\|\right]\Bigg\}\Bigg).\end{aligned} \tag{84}$$

Then

$$\begin{aligned}&\|Fx - Fy\|_\infty \\ &\le \psi\Bigg(\max\Bigg\{\|x(s)-y(s)\|, \\ &\qquad \frac{\|x(s)-F(x(s))\|}{1+\|x(s)-F(x(s))\|}, \frac{\|y(s)-F(y(s))\|}{1+\|y(s)-F(y(s))\|}, \\ &\qquad \frac{1}{2}\left[\|x(s)-F(y(s))\| + \|y(s)-F(x(s))\|\right]\Bigg\}\Bigg).\end{aligned} \tag{85}$$

That is, $(x,y) \in E(G)$ implies

$$\begin{aligned}&\|Fx - Fy\|_\infty \\ &\le \psi\Bigg(\max\Bigg\{\|x-y\|_\infty, \\ &\qquad \frac{\|x-F(x)\|_\infty}{1+\|x-F(x)\|_\infty}, \frac{\|y-F(y)\|_\infty}{1+\|y-F(y)\|_\infty}, \\ &\qquad \frac{1}{2}\left[\|x-F(y)\|_\infty + \|y-F(x)\|_\infty\right]\Bigg\}\Bigg).\end{aligned} \tag{86}$$

It easily shows that all the hypotheses of Theorem 29 are satisfied and hence the mapping F has a fixed point that is a solution in $X = C([0,T], \mathbb{R})$ of the integral equation (80). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. Therefore, the authors acknowledge with thanks DSR, KAU, for the financial support.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] V. Berinde, *Iterative Approximation of Fixed Points*, Springer, Berlin, Germany, 2007.

[3] F. Bojor, "Fixed point theorems for Reich type contractions on metric spaces with a graph," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 75, no. 9, pp. 3895–3901, 2012.

[4] L. Ćirić, N. Hussain, and N. Cakić, "Common fixed points for Ćirić type f-weak contraction with applications," *Publicationes Mathematicae Debrecen*, vol. 76, no. 1-2, pp. 31–49, 2010.

[5] L. Ćirić, M. Abbas, R. Saadati, and N. Hussain, "Common fixed points of almost generalized contractive mappings in ordered metric spaces," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5784–5789, 2011.

[6] R. Espínola and W. A. Kirk, "Fixed point theorems in R-trees with applications to graph theory," *Topology and its Applications*, vol. 153, no. 7, pp. 1046–1055, 2006.

[7] N. Hussain, P. Salimi, and A. Latif, "Fixed point results for single and set-valued α-η-ψ-contractive mappings," *Fixed Point Theory and Applications*, vol. 2013, article 212, 2013.

[8] N. Hussain, E. Karapınar, P. Salimi, and F. Akbar, "α-admissible mappings and related fixed point theorems," *Journal of Inequalities and Applications*, vol. 2013, article 114, 11 pages, 2013.

[9] N. Hussain, M. A. Kutbi, and P. Salimi, "Best proximity point results for modified α-ψ-proximal rational contractions," *Abstract and Applied Analysis*, vol. 2013, Article ID 927457, 14 pages, 2013.

[10] N. Hussain, S. Al-Mezel, and P. Salimi, "Fixed points for ψ-graphic contractions with application to integral equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 575869, 11 pages, 2013.

[11] J. Jachymski, "The contraction principle for mappings on a metric space with a graph," *Proceedings of the American Mathematical Society*, vol. 136, no. 4, pp. 1359–1373, 2008.

[12] E. Karapınar and B. Samet, "Generalized α-ψ contractive type mappings and related fixed point theorems with applications," *Abstract and Applied Analysis*, vol. 2012, Article ID 793486, 17 pages, 2012.

[13] P. D. Proinov, "A generalization of the Banach contraction principle with high order of convergence of successive approximations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 67, no. 8, pp. 2361–2369, 2007.

[14] P. D. Proinov, "New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems," *Journal of Complexity*, vol. 26, no. 1, pp. 3–42, 2010.

[15] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.

[16] P. Salimi, A. Latif, and N. Hussain, "Modified α-ψ-contractive mappings with applications," *Fixed Point Theory and Applications*, vol. 2013, article 151, 2013.

[17] P. Salimi, C. Vetro, and P. Vetro, "Fixed point theorems for twisted (α, β)-ψ-contractive type mappings and applications," *Filomat*, vol. 27, no. 4, pp. 605–615, 2013.

[18] P. Salimi, C. Vetro, and P. Vetro, "Some new fixed point results in non-Archimedean fuzzy metric spaces," *Nonlinear Analysis. Modelling and Control*, vol. 18, no. 3, pp. 344–358, 2013.

[19] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for α-ψ-contractive type mappings," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[20] W. Sintunavarat, S. Plubtieng, and P. Katchang, "Fixed point result and applications on a b-metric space endowed with an arbitrary binary relation," *Fixed Point Theory and Applications*, vol. 2013, article 296, 2013.

[21] T. Suzuki, "A generalized Banach contraction principle that characterizes metric completeness," *Proceedings of the American Mathematical Society*, vol. 136, no. 5, pp. 1861–1869, 2008.

[22] N. Shobkolaei, S. Sedghi, J. R. Roshan, and N. Hussain, "Suzuki-type fixed point results in metric-like spaces," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 143686, 9 pages, 2013.

[23] N. Hussain, D. Dorić, Z. Kadelburg, and S. Radenović, "Suzuki-type fixed point results in metric type spaces," *Fixed Point Theory and Applications*, vol. 2012, article 126, 2012.

[24] D. Dorić, Z. Kadelburg, and S. Radenović, "Edelstein-Suzuki-type fixed point results in metric and abstract metric spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 75, no. 4, pp. 1927–1932, 2012.

[25] Z. Kadelburg and S. Radenović, "Generalized quasicontractions in orbitally complete abstract metric spaces," *Fixed Point Theory*, vol. 13, no. 2, pp. 527–536, 2012.

[26] D. Dorić, Z. Kadelburg, S. Radenović, and P. Kumam, "A note on fixed point results without monotone property in partially ordered metric spaces," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales*.

[27] R. M. Bianchini and M. Grandolfi, "Trasformazioni di tipo contrattivo generalizzato in uno spazio metrico," *Atti della Accademia Nazionale dei Lincei. Rendiconti*, vol. 45, pp. 212–216, 1968.

[28] R. P. Agarwal, N. Hussain, and M.-A. Taoudi, "Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations," *Abstract and Applied Analysis*, vol. 2012, Article ID 245872, 15 pages, 2012.

[29] N. Hussain, A. R. Khan, and R. P. Agarwal, "Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces," *Journal of Nonlinear and Convex Analysis*, vol. 11, no. 3, pp. 475–489, 2010.

[30] N. Hussain and M. A. Taoudi, "Krasnosel'skii-type fixed point theorems with applications to Volterra integral equations," *Fixed Point Theory and Applications*, vol. 2013, article 196, 2013.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 132053, 25 pages
http://dx.doi.org/10.1155/2014/132053

Research Article

Algorithms of Common Solutions for Generalized Mixed Equilibria, Variational Inclusions, and Constrained Convex Minimization

Lu-Chuan Ceng[1] and Suliman Al-Homidan[2]

[1] *Department of Mathematics, Shanghai Normal University and Scientific Computing Key Laboratory of Shanghai Universities, Shanghai 200234, China*
[2] *Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia*

Correspondence should be addressed to Lu-Chuan Ceng; zenglc@hotmail.com

Received 3 November 2013; Accepted 12 November 2013; Published 23 January 2014

Academic Editor: Qamrul Hasan Ansari

We introduce new implicit and explicit iterative algorithms for finding a common element of the set of solutions of the minimization problem for a convex and continuously Fréchet differentiable functional, the set of solutions of a finite family of generalized mixed equilibrium problems, and the set of solutions of a finite family of variational inclusions in a real Hilbert space. Under suitable control conditions, we prove that the sequences generated by the proposed algorithms converge strongly to a common element of three sets, which is the unique solution of a variational inequality defined over the intersection of three sets.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. Let $S : C \to C$ be a self-mapping on C. We denote by Fix (S) the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $A : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Ax - Ay\| \leq L \|x - y\|, \quad \forall x, y \in C. \tag{1}$$

In particular, if $L = 1$, then A is called a nonexpansive mapping [1]; if $L \in [0, 1)$, then A is called a contraction.

A mapping V is called strongly positive on H if there exists a constant $\mu > 0$ such that

$$\langle Vx, x\rangle \geq \mu \|x\|^2, \quad \forall x \in H. \tag{2}$$

Let $A : C \to H$ be a nonlinear mapping on C. We consider the following variational inequality problem (VIP): find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{3}$$

The solution set of VIP (3) is denoted by VI (C, A).

The VIP (3) was first discussed by Lions [2]. There are many applications of VIP (3) in various fields; see, for example, [3–6]. It is well known that if A is a strongly monotone and Lipschitz continuous mapping on C, then VIP (3) has a unique solution. In 1976, Korpelevič [7] proposed an iterative algorithm for solving the VIP (3) in Euclidean space $\mathbf{R}^n$:

$$\begin{aligned} y_n &= P_C (x_n - \tau A x_n), \\ x_{n+1} &= P_C (x_n - \tau A y_n), \quad \forall n \geq 0, \end{aligned} \tag{4}$$

with $\tau > 0$ a given number, which is known as the extragradient method (see also [8]). The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see, for example, [9–24] and references therein, to name but a few.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function, $A : H \to H$ a nonlinear mapping, and $\Theta : C \times C \to \mathbf{R}$ a bifunction. In 2008, Peng and Yao [12] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{5}$$

We denote the set of solutions of GMEP (5) by GMEP (Θ, φ, A). The GMEP (5) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, and Nash equilibrium problems in noncooperative games. The GMEP is further considered and studied; see, for example, [11, 14, 23, 25–28]. If $\varphi = 0$ and $A = 0$, then GMEP (5) reduces to the equilibrium problem (EP) which is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{6}$$

It is considered and studied in [29]. The set of solutions of EP is denoted by EP (Θ). It is worth mentioning that the EP is a unified model of several problems, namely, variational inequality problems, optimization problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems, and so forth.

Throughout this paper, it is assumed as in [12] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)–(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone; that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;

(A3) Θ is upper-hemicontinuous; that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(tz + (1-t)x, y) \leq \Theta(x, y); \tag{7}$$

(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that, for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{8}$$

(B2) C is a bounded set.

Next we list some elementary results for the MEP.

Proposition 1 (see [26]). *Assume that $\Theta : C \times C \to \mathbf{R}$ satisfies (A1)–(A4) and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$\begin{aligned} T_r^{(\Theta,\varphi)}(x) = \Big\{ z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) \\ + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \Big\}, \end{aligned} \tag{9}$$

for all $x \in H$. Then the following hold:

(i) *for each $x \in H$, $T_r^{(\Theta,\varphi)}(x)$ is nonempty and single-valued;*

(ii) *$T_r^{(\Theta,\varphi)}$ is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\left\| T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y \right\|^2 \leq \left\langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\right\rangle; \tag{10}$$

(iii) Fix $(T_r^{(\Theta,\varphi)}) = MEP(\Theta, \varphi)$;

(iv) *$MEP(\Theta, \varphi)$ is closed and convex;*

(v) *$\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\|^2 \leq ((s-t)/s)\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\rangle$ for all $s, t > 0$ and $x \in H$.*

Let $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N} \in (0, 1]$, $n \geq 1$. Given the nonexpansive mappings $S_1, S_2, \ldots, S_N$ on H, for each $n \geq 1$, the mappings $U_{n,1}, U_{n,2}, \ldots, U_{n,N}$ are defined by

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}S_1 + (1 - \lambda_{n,1}) I, \\ U_{n,2} &= \lambda_{n,2}S_n U_{n,1} + (1 - \lambda_{n,2}) I, \\ U_{n,n-1} &= \lambda_{n-1}T_{n-1}U_{n,n} + (1 - \lambda_{n-1}) I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}S_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1}) I, \\ W_n := U_{n,N} &= \lambda_{n,N} S_N U_{n,N-1} + (1 - \lambda_{n,N}) I. \end{aligned} \tag{11}$$

The W_n is called the W-mapping generated by $S_1, \ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Note that the nonexpansivity of S_i implies the nonexpansivity of W_n.

In 2012, combining the hybrid steepest-descent method in [30] and hybrid viscosity approximation method in [31], Ceng et al. [27] proposed and analyzed the following hybrid iterative method for finding a common element of the set of solutions of GMEP (5) and the set of fixed points of a finite family of nonexpansive mappings $\{S_i\}_{i=1}^N$.

Theorem CGY (see [27, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying assumptions (A1)–(A4) and $\varphi : C \to \mathbf{R}$ a lower semicontinuous and convex function with restriction (B1) or (B2). Let the mapping $A : H \to H$ be δ-inverse-strongly monotone and $\{S_i\}_{i=1}^N$ a finite family of nonexpansive mappings on H such that $\cap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP(\Theta, \varphi, A) \neq \emptyset$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$ and $V : H \to H$ a ρ-Lipschitzian mapping with constant $\rho \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$, $\{\gamma_n\}$ is a sequence in $(0, 2\delta]$, and $\{\lambda_{n,i}\}_{i=1}^N$ is a sequence in $[a, b]$ with $0 < a \leq b < 1$. For every $n \geq 1$, let W_n be the W-mapping generated by $S_1, \ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$.*

Given $x_1 \in H$ arbitrarily, suppose that the sequences $\{x_n\}$ and $\{u_n\}$ are generated iteratively by

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$
$$x_{n+1} = \alpha_n \gamma V x_n + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F) W_n u_n, \quad \forall n \geq 1, \tag{12}$$

where the sequences $\{\alpha_n\}, \{\beta_n\}, \{r_n\}$ and the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^{N}$ satisfy the conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\delta$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$;

(iv) $\lim_{n\to\infty}(\lambda_{n+1,i} - \lambda_{n,i}) = 0$ *for all* $i = 1, 2, \ldots, N$.

Then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in \cap_{i=1}^{N}\mathrm{Fix}(S_i)\cap GMEP(\Theta,\varphi,A)$, where $x^* = P_{\cap_{i=1}^{N}\mathrm{Fix}(S_i)\cap GMEP(\Theta,\varphi,A)}(I - \mu F + \gamma f)x^*$ is a unique solution of the variational inequality problem (VIP):

$$\langle(\mu F - \gamma V)x^*, x^* - x\rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^{N}\mathrm{Fix}(S_i) \cap GMEP(\Theta,\varphi,A). \tag{13}$$

Let B be a single-valued mapping of C into H and R a multivalued mapping with $D(R) = C$. Consider the following variational inclusion: find a point $x \in C$ such that

$$0 \in Bx + Rx. \tag{14}$$

We denote by $I(B,R)$ the solution set of the variational inclusion (14). In particular, if $B = R = 0$, then $I(B,R) = C$. If $B = 0$, then problem (14) becomes the inclusion problem introduced by Rockafellar [32]. It is known that problem (14) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics, and equilibria and game theory.

In 1998, Huang [33] studied problem (14) in the case where R is maximal monotone and B is strongly monotone and Lipschitz continuous with $D(R) = C = H$. Subsequently, Zeng et al. [34] further studied problem (14) in the case which is more general than Huang's one [33]. Moreover, the authors [34] obtained the same strong convergence conclusion as in Huang's result [33]. In addition, the authors also gave the geometric convergence rate estimate for approximate solutions. Also, various types of iterative algorithms for solving variational inclusions have been further studied and developed; for more details, refer to [35–39] and the references therein.

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=0}^{\infty}$ a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a self-mapping W_n on C as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&\vdots\\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{15}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Whenever $C = H$ a real Hilbert space, Yao et al. [11] very recently introduced and analyzed an iterative algorithm for finding a common element of the set of solutions of GMEP (5), the set of solutions of the variational inclusion (14), and the set of fixed points of an infinite family of nonexpansive mappings.

Theorem YCL (see [11, Theorem 3.2]). *Let $\varphi : H \to \mathbf{R}$ be a lower semicontinuous and convex function and $\Theta : H \times H \to \mathbf{R}$ a bifunction satisfying conditions (A1)–(A4) and (B1). Let V be a strongly positive bounded linear operator with coefficient $\mu > 0$ and $R : H \to 2^H$ a maximal monotone mapping. Let the mappings $A, B : H \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $f : H \to H$ be a ρ-contraction. Let $r > 0$, $\gamma > 0$, and $\lambda > 0$ be three constants such that $r < 2\alpha$, $\lambda < 2\beta$, and $0 < \gamma < \mu/\rho$. Let $\{\lambda\}_{n=1}^{\infty}$ be a sequence of positive numbers in $(0,b]$ for some $b \in (0,1)$ and $\{T_n\}_{n=1}^{\infty}$ an infinite family of nonexpansive self-mappings on H such that $\Omega := \cap_{n=1}^{\infty}\mathrm{Fix}(T_n)\cap GMEP(\Theta,\varphi,A)\cap I(B,R) \neq \emptyset$. For arbitrarily given $x_1 \in H$, let the sequence $\{x_n\}$ be generated by*

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle y - u_n, Ax_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H,$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n V] W_n J_{R,\lambda}(u_n - \lambda B u_n), \quad \forall n \geq 1, \tag{16}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real sequences in $[0,1]$ and W_n is the W-mapping defined by (15) *(with $X = H$ and $C = H$). Assume that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

Then the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *where* $x^* = P_\Omega(\gamma f(x^*) + (I - V)x^*)$ *is a unique solution of the VIP:*

$$\left\langle (\gamma f - V)\, x^*, y - x^* \right\rangle \le 0, \quad \forall y \in \Omega. \tag{17}$$

Let $f : C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional. Consider the convex minimization problem (CMP) of minimizing f over the constraint set C

$$\min_{x\in C} f(x) \tag{18}$$

(assuming the existence of minimizers). We denote by Γ the set of minimizers of CMP (18). It is well known that the gradient-projection algorithm (GPA) generates a sequence $\{x_n\}$ determined by the gradient ∇f and the metric projection P_C:

$$x_{n+1} := P_C\left(x_n - \lambda\nabla f\left(x_n\right)\right), \quad \forall n \ge 0, \tag{19}$$

or more generally,

$$x_{n+1} := P_C\left(x_n - \lambda_n\nabla f\left(x_n\right)\right), \quad \forall n \ge 0, \tag{20}$$

where, in both (19) and (20), the initial guess x_0 is taken from C arbitrarilyn and the parameters λ or λ_n are positive real numbers. The convergence of algorithms (19) and (20) depends on the behavior of the gradient ∇f. As a matter of fact, it is known that if ∇f is α-strongly monotone and L-Lipschitz continuous, then, for $0 < \lambda < 2\alpha/L^2$, the operator $P_C(I - \lambda\nabla f)$ is a contraction; hence, the sequence $\{x_n\}$ defined by the GPA (19) converges in norm to the unique solution of CMP (18). More generally, if the sequence $\{\lambda_n\}$ is chosen to satisfy the property

$$0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < \frac{2\alpha}{L^2}, \tag{21}$$

then the sequence $\{x_n\}$ defined by the GPA (20) converges in norm to the unique minimizer of CMP (18). If the gradient ∇f is only assumed to be Lipschitz continuous, then $\{x_n\}$ can only be weakly convergent if H is infinite-dimensional (a counterexample is given in Section 5 of Xu [40]).

Since the Lipschitz continuity of the gradient ∇f implies that it is actually $(1/L)$-inverse-strongly monotone (ism) [41], its complement can be an averaged mapping (i.e., it can be expressed as a proper convex combination of the identity mapping and a nonexpansive mapping). Consequently, the GPA can be rewritten as the composite of a projection and an averaged mapping, which is again an averaged mapping. This shows that averaged mappings play an important role in the GPA. Recently, Xu [40] used averaged mappings to study the convergence analysis of the GPA, which is hence an operator-oriented approach.

Motivated and inspired by the above facts, we in this paper introduce new implicit and explicit iterative algorithms for finding a common element of the set of solutions of the CMP (18) for a convex functional $f : C \to \mathbf{R}$ with L-Lipschitz continuous gradient ∇f, the set of solutions of a finite family of GMEPs, and the set of solutions of a finite family of variational inclusions for maximal monotone and inverse-strong monotone mappings in a real Hilbert space. Under mild control conditions, we prove that the sequences generated by the proposed algorithms converge strongly to a common element of three sets, which is the unique solution of a variational inequality defined over the intersection of three sets. Our iterative algorithms are based on Korpelevich's extragradient method, hybrid steepest-descent method in [30], viscosity approximation method, and averaged mapping approach to the GPA in [40]. The results obtained in this paper improve and extend the corresponding results announced by many others.

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; that is,

$$\omega_w\left(x_n\right) := \left\{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \left\{x_{n_i}\right\} \text{ of } \left\{x_n\right\}\right\}. \tag{22}$$

Recall that a mapping $A : C \to H$ is called

(i) monotone if

$$\left\langle Ax - Ay, x - y\right\rangle \ge 0, \quad \forall x, y \in C; \tag{23}$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\left\langle Ax - Ay, x - y\right\rangle \ge \eta\left\|x - y\right\|^2, \quad \forall x, y \in C; \tag{24}$$

(iii) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\left\langle Ax - Ay, x - y\right\rangle \ge \alpha\left\|Ax - Ay\right\|^2, \quad \forall x, y \in C. \tag{25}$$

It is obvious that if A is α-inverse-strongly monotone, then A is monotone and $(1/\alpha)$-Lipschitz continuous.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\left\|x - P_C x\right\| = \inf_{y\in C}\left\|x - y\right\| =: d\left(x, C\right). \tag{26}$$

Some important properties of projections are gathered in the following proposition.

Proposition 2. *For given $x \in H$ and $z \in C$,*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0$, *for all* $y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, *for all* $y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$, *for all* $y \in H$.

Consequently, P_C is nonexpansive and monotone.

If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $(1/\alpha)$-Lipschitz continuous. We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I - \lambda A) u - (I - \lambda A) v\|^2 &= \|(u - v) - \lambda (Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda \langle Au - Av, u - v\rangle \\ &\quad + \lambda^2 \|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda (\lambda - 2\alpha) \\ &\quad \times \|Au - Av\|^2.\end{aligned} \tag{27}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

Definition 3. A mapping $T : H \rightarrow H$ is said to be

(a) nonexpansive [1] if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H; \tag{28}$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive or, equivalently, if T is 1-inverse-strongly monotone (1-ism),

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H; \tag{29}$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S), \tag{30}$$

where $S : H \rightarrow H$ is nonexpansive; projections are firmly nonexpansive.

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_C is 1-ism. Inverse-strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

Definition 4. A mapping $T : H \rightarrow H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping; that is,

$$T \equiv (1 - \alpha) I + \alpha S, \tag{31}$$

where $\alpha \in (0, 1)$ and $S : H \rightarrow H$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus, firmly nonexpansive mappings (in particular, projections) are $(1/2)$-averaged mappings.

Proposition 5 (see [42]). *Let $T : H \rightarrow H$ be a given mapping.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $(1/2)$-ism.*

(ii) *If T is ν-ism, then, for $\gamma > 0$, γT is (ν/γ)-ism.*

(iii) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $(1/2\alpha)$-ism.*

Proposition 6 (see [42, 43]). *Let $S, T, V : H \rightarrow H$ be given operators.*

(i) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite $T_1 T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1 \alpha_2$.*

(v) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \cdots T_N). \tag{32}$$

The notation $\mathrm{Fix}(T)$ *denotes the set of all fixed points of the mapping T; that is,* $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 7. *Let X be a real inner product space. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X. \tag{33}$$

Lemma 8. *Let $A : C \rightarrow H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see Proposition 2(i)) implies*

$$u \in \mathrm{VI}(C, A) \Longleftrightarrow u = P_C(u - \lambda Au), \quad \textit{for some } \lambda > 0. \tag{34}$$

Lemma 9 (see [44, Demiclosedness principle]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C with* $\mathrm{Fix}(T) \neq \emptyset$. *Then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here I is the identity operator of H.*

Lemma 10 (see [45]). *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the conditions*

$$s_{n+1} \le (1-\alpha_n) s_n + \alpha_n \beta_n, \quad \forall n \ge 1, \tag{35}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *or, equivalently,*

$$\prod_{n=1}^{\infty} (1-\alpha_n) := \lim_{n\to\infty} \prod_{k=1}^{n} (1-\alpha_k) = 0; \tag{36}$$

(ii) $\limsup_{n\to\infty} \beta_n \le 0$, *or* $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 11 (see [46]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1. \tag{37}$$

Suppose that $x_{n+1} = (1-\beta_n) z_n + \beta_n x_n$ for each $n \ge 1$ and

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{38}$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

The following lemma can be easily proven and, therefore, we omit the proof.

Lemma 12. *Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \ge 0$, and let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Then, for $0 \le \gamma l < \mu\eta$,*

$$\langle (\mu F - \gamma V) x - (\mu F - \gamma V) y, x - y \rangle \ge (\mu\eta - \gamma l) \|x - y\|^2, \quad \forall x, y \in H. \tag{39}$$

That is, $\mu F - \gamma V$ is strongly monotone with constant $\mu\eta - \gamma l$.

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to H$, we define the mapping $T^\lambda : C \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C, \tag{40}$$

where $F : H \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on H; that is, F satisfies the conditions:

$$\|Fx - Fy\| \le \kappa \|x - y\|, \qquad \langle Fx - Fy, x - y \rangle \ge \eta \|x - y\|^2, \tag{41}$$

for all $x, y \in H$.

Lemma 13 (see [45, Lemma 3.1]). *T^λ is a contraction provided $0 < \mu < 2\eta/\kappa^2$; that is,*

$$\|T^\lambda x - T^\lambda y\| \le (1 - \lambda\tau) \|x - y\|, \quad \forall x, y \in C, \tag{42}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1]$.

Recall that a set-valued mapping $R : D(R) \subset H \to 2^H$ is called monotone if for all $x, y \in D(R)$, $f \in R(x)$ and $g \in R(y)$ imply

$$\langle f - g, x - y \rangle \ge 0. \tag{43}$$

A set-valued mapping R is called maximal monotone if R is monotone and $(I + \lambda R) D(R) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(R)$ the graph of R. It is known that a monotone mapping R is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y \rangle \ge 0$ for every $(y, g) \in G(R)$ implies $f \in R(x)$.

Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$; that is,

$$N_C v = \{ w \in H : \langle v - u, w \rangle \ge 0, \ \forall u \in C \}. \tag{44}$$

Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{45}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [32].

Assume that $R : D(R) \subset H \to 2^H$ is a maximal monotone mapping. Then, for $\lambda > 0$, associated with R, the resolvent operator $J_{R,\lambda}$ can be defined as

$$J_{R,\lambda} x = (I + \lambda R)^{-1} x, \quad \forall x \in H. \tag{46}$$

In terms of Huang [33] (see also [34]), the following property holds for the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$.

Lemma 14. *$J_{R,\lambda}$ is single-valued and firmly nonexpansive; that is,*

$$\langle J_{R,\lambda} x - J_{R,\lambda} y, x - y \rangle \ge \|J_{R,\lambda} x - J_{R,\lambda} y\|^2, \quad \forall x, y \in H. \tag{47}$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 15 (see [39]). *Let R be a maximal monotone mapping with $D(R) = C$. Then, for any given $\lambda > 0$, $u \in C$ is a solution of problem* (14) *if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda} (u - \lambda B u). \tag{48}$$

Lemma 16 (see [34]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous, and single-valued mapping. Then, for each $z \in H$, the equation $z \in (B + \lambda R) x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 17 (see [39]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a monotone, continuous, and single-valued mapping. Then $(I + \lambda(R + B)) C = H$ for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

3. Implicit Iterative Algorithm and Its Convergence Criteria

We now state and prove the first main result of this paper.

Theorem 18. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let M, N be two integers. Let Θ_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1, 2, \ldots, M\}$. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A_k : H \to H$ and $B_i : C \to H$ be μ_k-inverse-strongly monotone and η_i-inverse-strongly monotone, respectively, where $k \in \{1, 2, \ldots, M\}$, $i \in \{1, 2, \ldots, N\}$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := \cap_{k=1}^{M} GMEP(\Theta_k, \varphi_k, A_k) \cap \cap_{i=1}^{N} I(B_i, R_i) \cap \Gamma \neq \emptyset$ and that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
u_n &= T_{r_{M,n}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n} A_M\right) T_{r_{M-1,n}}^{(\Theta_{M-1},\varphi_{M-1})} \\
&\quad \times \left(I - r_{M-1,n} A_{M-1}\right) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)} \left(I - r_{1,n} A_1\right) x_n, \\
v_n &= J_{R_N,\lambda_{N,n}} \left(I - \lambda_{N,n} B_N\right) J_{R_{N-1},\lambda_{N-1,n}} \\
&\quad \times \left(I - \lambda_{N-1,n} B_{N-1}\right) \cdots J_{R_1,\lambda_{1,n}} \left(I - \lambda_{1,n} B_1\right) u_n, \\
x_n &= s_n \gamma V x_n + \left(I - s_n \mu F\right) T_n v_n, \\
&\qquad \forall n \geq 1,
\end{aligned} \tag{49}$$

where $P_C(I - \lambda_n \nabla f) = s_n I + (1 - s_n) T_n$ (here T_n is nonexpansive and $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$). Assume that the following conditions hold:

(i) *$s_n \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$, $\lim_{n\to\infty} s_n = 0$ ($\Leftrightarrow$ $\lim_{n\to\infty} \lambda_n = 2/L$);*

(ii) *$\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, for all $i \in \{1, 2, \ldots, N\}$;*

(iii) *$\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, for all $k \in \{1, 2, \ldots, M\}$.*

Then $\{x_n\}$ converges strongly as $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) to a point $q \in \Omega$, which is a unique solution of the VIP:

$$\left\langle \left(\mu F - \gamma V\right) q, p - q \right\rangle \geq 0, \quad \forall p \in \Omega. \tag{50}$$

Equivalently, $q = P_\Omega(I - \mu F + \gamma V) q$.

Proof. First of all, let us show that the sequence $\{x_n\}$ is well defined. Indeed, since ∇f is L-Lipschitzian, it follows that ∇f is $1/L$-ism; see [41]. By Proposition 5(ii) we know that, for $\lambda > 0$, $\lambda \nabla f$ is $(1/\lambda L)$-ism. So by Proposition 5(iii) we deduce that $I - \lambda \nabla f$ is $(\lambda L/2)$-averaged. Now since the projection P_C is $(1/2)$-averaged, it is easy to see from Proposition 6(iv) that the composite $P_C(I - \lambda \nabla f)$ is $(2 + \lambda L)/4$-averaged for $\lambda \in (0, 2/L)$. Hence, we obtain that, for each $n \geq 1$, $P_C(I - \lambda_n \nabla f)$ is $((2 + \lambda_n L)/4)$-averaged for each $\lambda_n \in (0, 2/L)$. Therefore, we can write

$$P_C\left(I - \lambda_n \nabla f\right) = \frac{2 - \lambda_n L}{4} I + \frac{2 + \lambda_n L}{4} T_n = s_n I + \left(1 - s_n\right) T_n, \tag{51}$$

where T_n is nonexpansive and $s_n := s_n(\lambda_n) = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$. It is clear that

$$\lambda_n \longrightarrow \frac{2}{L} \Longleftrightarrow s_n \longrightarrow 0. \tag{52}$$

Put

$$\begin{aligned}
\Delta_n^k &= T_{r_{k,n}}^{(\Theta_k,\varphi_k)} \left(I - r_{k,n} A_k\right) T_{r_{k-1,n}}^{(\Theta_{k-1},\varphi_{k-1})} \\
&\quad \times \left(I - r_{k-1,n} A_{k-1}\right) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)} \left(I - r_{1,n} A_1\right) x_n,
\end{aligned} \tag{53}$$

for all $k \in \{1, 2, \ldots, M\}$ and $n \geq 1$,

$$\begin{aligned}
\Lambda_n^i &= J_{R_i,\lambda_{i,n}} \left(I - \lambda_{i,n} B_i\right) J_{R_{i-1},\lambda_{i-1,n}} \\
&\quad \times \left(I - \lambda_{i-1,n} B_{i-1}\right) \cdots J_{R_1,\lambda_{1,n}} \left(I - \lambda_{1,n} B_1\right),
\end{aligned} \tag{54}$$

for all $i \in \{1, 2, \ldots, N\}$ and $n \geq 1$, and $\Delta_n^0 = \Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Delta_n^M x_n$ and $v_n = \Lambda_n^N u_n$.

Consider the following mapping G_n on H defined by

$$G_n x = s_n \gamma V x + \left(I - s_n \mu F\right) T_n \Lambda_n^N \Delta_n^M x, \quad \forall x \in H,\ n \geq 1, \tag{55}$$

where $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$. By Proposition 1(ii) and Lemma 13 we obtain from (27) that for all $x, y \in H$

$$\begin{aligned}
&\left\| G_n x - G_n y \right\| \\
&\leq s_n \gamma \left\| V x - V y \right\| + \left\| \left(I - s_n \mu F\right) T_n \Lambda_n^N \Delta_n^M x \right. \\
&\qquad \left. - \left(I - s_n \mu F\right) T_n \Lambda_n^N \Delta_n^M y \right\| \\
&\leq s_n \gamma l \left\| x - y \right\| + \left(1 - s_n \tau\right) \left\| \Lambda_n^N \Delta_n^M x - \Lambda_n^N \Delta_n^M y \right\| \\
&\leq s_n \gamma l \left\| x - y \right\| + \left(1 - s_n \tau\right) \left\| \left(I - \lambda_{N,n} B_N\right) \Lambda_n^{N-1} \Delta_n^M x \right. \\
&\qquad \left. - \left(I - \lambda_{N,n} B_N\right) \Lambda_n^{N-1} \Delta_n^M y \right\| \\
&\leq s_n \gamma l \left\| x - y \right\| + \left(1 - s_n \tau\right) \left\| \Lambda_n^{N-1} \Delta_n^M x - \Lambda_n^{N-1} \Delta_n^M y \right\| \\
&\ \ \vdots \\
&\leq s_n \gamma l \left\| x - y \right\| + \left(1 - s_n \tau\right) \left\| \Lambda_n^0 \Delta_n^M x - \Lambda_n^0 \Delta_n^M y \right\|
\end{aligned}$$

$$\begin{aligned}
&= s_n\gamma l \|x-y\| + (1-s_n\tau)\left\|\Delta_n^M x - \Delta_n^M y\right\| \\
&\le s_n\gamma l \|x-y\| + (1-s_n\tau)\left\|(I-r_{M,n}A_M)\Delta_n^{M-1}x \right. \\
&\qquad \left. - (I-r_{M,n}A_M)\Delta_n^{M-1}y\right\| \\
&\le s_n\gamma l \|x-y\| + (1-s_n\tau)\left\|\Delta_n^{M-1}x - \Delta_n^{M-1}y\right\| \\
&\ \ \vdots \\
&\le s_n\gamma l \|x-y\| + (1-s_n\tau)\left\|\Delta_n^{0}x - \Delta_n^{0}y\right\| \\
&= s_n\gamma l \|x-y\| + (1-s_n\tau)\|x-y\| \\
&= (1-s_n(\tau-\gamma l))\|x-y\|.
\end{aligned} \tag{56}$$

Since $0 < 1 - s_n(\tau - \gamma l) < 1$, $G_n : H \to H$ is a contraction. Therefore, by the Banach contraction principle, G_n has a unique fixed point $x_n \in H$, which uniquely solves the fixed point equation:

$$x_n = s_n\gamma V x_n + (I - s_n\mu F)T_n\Lambda_n^N\Delta_n^M x_n. \tag{57}$$

This shows that the sequence $\{x_n\}$ is defined well.

Note that $0 \le \gamma l < \tau$ and $\mu\eta \ge \tau \Leftrightarrow \kappa \ge \eta$. Hence, by Lemma 12 we know that

$$\langle(\mu F-\gamma V)x-(\mu F-\gamma V)y, x-y\rangle \ge (\mu\eta-\gamma l)\|x-y\|^2, \quad \forall x, y \in H. \tag{58}$$

That is, $\mu F - \gamma V$ is strongly monotone for $0 \le \gamma l < \tau \le \mu\eta$. Moreover, it is clear that $\mu F - \gamma V$ is Lipschitz continuous. So the VIP (50) has only one solution. Below we use $q \in \Omega$ to denote the unique solution of the VIP (50).

Now, let us show that $\{x_n\}$ is bounded. In fact, take $p \in \Omega$ arbitrarily. Then from (27) and Proposition 1(ii) we have

$$\begin{aligned}
\|u_n - p\| &= \left\|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}A_M)\Delta_n^{M-1}x_n \right. \\
&\qquad \left. - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}A_M)\Delta_n^{M-1}p\right\| \\
&\le \left\|(I-r_{M,n}A_M)\Delta_n^{M-1}x_n \right. \\
&\qquad \left. - (I-r_{M,n}A_M)\Delta_n^{M-1}p\right\| \\
&\le \left\|\Delta_n^{M-1}x_n - \Delta_n^{M-1}p\right\| \\
&\ \ \vdots \\
&\le \left\|\Delta_n^{0}x_n - \Delta_n^{0}p\right\| \\
&= \|x_n - p\|.
\end{aligned} \tag{59}$$

Similarly, we have

$$\begin{aligned}
\|v_n - p\| &= \left\|J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n \right. \\
&\qquad \left. - J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}p\right\| \\
&\le \left\|(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n - (I-\lambda_{N,n}B_N)\Lambda_n^{N-1}p\right\| \\
&\le \left\|\Lambda_n^{N-1}u_n - \Lambda_n^{N-1}p\right\| \\
&\ \ \vdots \\
&\le \left\|\Lambda_n^{0}u_n - \Lambda_n^{0}p\right\| \\
&= \|u_n - p\|.
\end{aligned} \tag{60}$$

Combining (59) and (60), we have

$$\|v_n - p\| \le \|x_n - p\|. \tag{61}$$

Since

$$p = P_C(I-\lambda_n\nabla f)p = s_np + (1-s_n)T_np, \quad \forall\lambda_n \in \left(0, \frac{2}{L}\right), \tag{62}$$

where $s_n := s_n(\lambda_n) = (2-\lambda_nL)/4 \in (0, 1/2)$, it is clear that $T_np = p$ for each $\lambda_n \in (0, 2/L)$. Thus, utilizing Lemma 13 and the nonexpansivity of T_n, we obtain from (61) that

$$\begin{aligned}
\|x_n - p\| &= \|s_n(\gamma Vx_n - \mu Fp) + (I-s_n\mu F)T_nv_n \\
&\qquad - (I-s_n\mu F)T_np\| \\
&\le \|(I-s_n\mu F)T_nv_n - (I-s_n\mu F)T_np\| \\
&\quad + s_n\|\gamma Vx_n - \mu Fp\| \\
&\le (1-s_n\tau)\|v_n - p\| \\
&\quad + s_n(\gamma\|Vx_n - Vp\| + \|\gamma Vp - \mu Fp\|) \\
&\le (1-s_n\tau)\|x_n - p\| \\
&\quad + s_n(\gamma l\|x_n - p\| + \|\gamma Vp - \mu Fp\|) \\
&= (1-s_n(\tau-\gamma l))\|x_n - p\| + s_n\|\gamma Vp - \mu Fp\|.
\end{aligned} \tag{63}$$

This implies that $\|x_n - p\| \le \|\gamma Vp - \mu Fp\|/(\tau - \gamma l)$. Hence, $\{x_n\}$ is bounded. So, according to (59) and (61) we know that $\{u_n\}$, $\{v_n\}$, $\{T_nv_n\}$, $\{Vx_n\}$, and $\{FT_nv_n\}$ are bounded.

Next let us show that $\|u_n - x_n\| \to 0$, $\|v_n - u_n\| \to 0$, and $\|x_n - T_nx_n\| \to 0$ as $n \to \infty$.

Indeed, from (27) it follows that for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$

$$\begin{aligned}
\|v_n - p\|^2 &= \left\|\Lambda_n^N u_n - p\right\|^2 \\
&\le \left\|\Lambda_n^i u_n - p\right\|^2 \\
&= \Big\|J_{R_i,\lambda_{i,n}}\left(I - \lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n \\
&\qquad - J_{R_i,\lambda_{i,n}}\left(I - \lambda_{i,n}B_i\right)p\Big\|^2 \\
&\le \left\|\left(I - \lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n - \left(I - \lambda_{i,n}B_i\right)p\right\|^2 \\
&\le \left\|\Lambda_n^{i-1}u_n - p\right\|^2 + \lambda_{i,n}\left(\lambda_{i,n} - 2\eta_i\right) \\
&\qquad \times \left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2 \\
&\le \|u_n - p\|^2 + \lambda_{i,n}\left(\lambda_{i,n} - 2\eta_i\right) \\
&\qquad \times \left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2 \\
&\le \|x_n - p\|^2 + \lambda_{i,n}\left(\lambda_{i,n} - 2\eta_i\right) \\
&\qquad \times \left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2, \\
\|u_n - p\|^2 &= \left\|\Delta_n^M x_n - p\right\|^2 \\
&\le \left\|\Delta_n^k x_n - p\right\|^2 \\
&= \Big\|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}\left(I - r_{k,n}A_k\right)\Delta_n^{k-1}x_n \\
&\qquad - T_{r_{k,n}}^{(\Theta_k,\varphi_k)}\left(I - r_{k,n}A_k\right)p\Big\|^2 \\
&\le \left\|\left(I - r_{k,n}A_k\right)\Delta_n^{k-1}x_n - \left(I - r_{k,n}A_k\right)p\right\|^2 \\
&\le \left\|\Delta_n^{k-1}x_n - p\right\|^2 + r_{k,n}\left(r_{k,n} - 2\mu_k\right) \\
&\qquad \times \left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|^2 \\
&\le \|x_n - p\|^2 + r_{k,n}\left(r_{k,n} - 2\mu_k\right) \\
&\qquad \times \left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|^2.
\end{aligned} \tag{64}$$

Thus, utilizing Lemma 7, from (49) and (64) we have

$$\begin{aligned}
&\|x_n - p\|^2 \\
&\quad = \|s_n(\gamma V x_n - \mu F p) + (I - s_n\mu F)T_n v_n \\
&\qquad - (I - s_n\mu F)T_n p\|^2 \\
&\quad \le \|(I - s_n\mu F)T_n v_n - (I - s_n\mu F)T_n p\|^2 \\
&\qquad + 2s_n\langle \gamma V x_n - \mu F p, x_n - p\rangle \\
&\quad \le (1 - s_n\tau)^2\|v_n - p\|^2 + 2s_n\langle \gamma V x_n - \mu F p, x_n - p\rangle \\
&\quad = (1 - s_n\tau)^2\|v_n - p\|^2 + 2s_n\gamma\langle V x_n - V p, x_n - p\rangle \\
&\qquad + 2s_n\langle \gamma V p - \mu F p, x_n - p\rangle \\
&\quad \le (1 - s_n\tau)^2\|v_n - p\|^2 + 2s_n\gamma l\|x_n - p\|^2 \\
&\qquad + 2s_n\|\gamma V p - \mu F p\|\|x_n - p\| \\
&\quad \le (1 - s_n\tau)^2\Big[\|u_n - p\|^2 + \lambda_{i,n}\left(\lambda_{i,n} - 2\eta_i\right) \\
&\qquad\qquad \times \left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2\Big] \\
&\qquad + 2s_n\gamma l\|x_n - p\|^2 + 2s_n\|\gamma V p - \mu F p\|\|x_n - p\| \\
&\quad \le (1 - s_n\tau)^2\Big[\|x_n - p\|^2 + r_{k,n}\left(r_{k,n} - 2\mu_k\right) \\
&\qquad\qquad \times \left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|^2 \\
&\qquad\qquad + \lambda_{i,n}\left(\lambda_{i,n} - 2\eta_i\right)\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2\Big] \\
&\qquad + 2s_n\gamma l\|x_n - p\|^2 + 2s_n\|\gamma V p - \mu F p\|\|x_n - p\| \\
&\quad = \left[1 - 2s_n(\tau - \gamma l) + s_n^2\tau^2\right]\|x_n - p\|^2 \\
&\qquad - (1 - s_n\tau)^2\Big[r_{k,n}\left(2\mu_k - r_{k,n}\right)\left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|^2 \\
&\qquad\qquad + \lambda_{i,n}\left(2\eta_i - \lambda_{i,n}\right)\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2\Big] \\
&\qquad + 2s_n\|\gamma V p - \mu F p\|\|x_n - p\| \\
&\quad \le \|x_n - p\|^2 + s_n^2\tau^2\|x_n - p\|^2 - (1 - s_n\tau)^2 \\
&\qquad \times \Big[r_{k,n}\left(2\mu_k - r_{k,n}\right)\left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|^2 \\
&\qquad\qquad + \lambda_{i,n}\left(2\eta_i - \lambda_{i,n}\right)\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2\Big] \\
&\qquad + 2s_n\|\gamma V p - \mu F p\|\|x_n - p\|,
\end{aligned} \tag{65}$$

which implies that

$$\begin{aligned}
&(1 - s_n\tau)^2\Big[r_{k,n}\left(2\mu_k - r_{k,n}\right)\left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|^2 \\
&\qquad + \lambda_{i,n}\left(2\eta_i - \lambda_{i,n}\right)\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2\Big] \\
&\quad \le s_n^2\tau^2\|x_n - p\|^2 + 2s_n\|\gamma V p - \mu F p\|\|x_n - p\|.
\end{aligned} \tag{66}$$

Since $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$, from $s_n \to 0$ we conclude immediately that

$$\begin{gathered}
\lim_{n\to\infty}\left\|A_k\Delta_n^{k-1}x_n - A_k p\right\| = 0, \\
\lim_{n\to\infty}\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\| = 0,
\end{gathered} \tag{67}$$

for all $k \in \{1, 2, \ldots, M\}$ and $i \in \{1, 2, \ldots, N\}$.

Furthermore, by Proposition 1(ii) we obtain that for each $k \in \{1, 2, \ldots, M\}$

$$\begin{aligned}
&\left\|\Delta_n^k x_n - p\right\|^2 \\
&\quad = \left\|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}\left(I - r_{k,n}A_k\right)\Delta_n^{k-1}x_n \right.\\
&\qquad \left. - T_{r_{k,n}}^{(\Theta_k,\varphi_k)}\left(I - r_{k,n}A_k\right)p\right\|^2 \\
&\quad \leq \left\langle \left(I - r_{k,n}A_k\right)\Delta_n^{k-1}x_n - \left(I - r_{k,n}A_k\right)p, \Delta_n^k x_n - p\right\rangle \\
&\quad = \frac{1}{2}\left(\left\|\left(I - r_{k,n}A_k\right)\Delta_n^{k-1}x_n - \left(I - r_{k,n}A_k\right)p\right\|^2\right.\\
&\qquad + \left\|\Delta_n^k x_n - p\right\|^2 - \left\|\left(I - r_{k,n}A_k\right)\Delta_n^{k-1}x_n\right.\\
&\qquad - \left(I - r_{k,n}A_k\right)p\\
&\qquad \left.\left. - \left(\Delta_n^k x_n - p\right)\right\|^2\right) \\
&\quad \leq \frac{1}{2}\left(\left\|\Delta_n^{k-1}x_n - p\right\|^2 + \left\|\Delta_n^k x_n - p\right\|^2\right.\\
&\qquad \left. - \left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n - r_{k,n}\left(A_k\Delta_n^{k-1}x_n - A_k p\right)\right\|^2\right),
\end{aligned} \tag{68}$$

which implies that

$$\begin{aligned}
&\left\|\Delta_n^k x_n - p\right\|^2 \\
&\quad \leq \left\|\Delta_n^{k-1}x_n - p\right\|^2 \\
&\qquad - \left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n - r_{k,n}\left(A_k\Delta_n^{k-1}x_n - A_k p\right)\right\|^2 \\
&\quad = \left\|\Delta_n^{k-1}x_n - p\right\|^2 - \left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|^2 \\
&\qquad - r_{k,n}^2\left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|^2 \\
&\qquad + 2r_{k,n}\left\langle \Delta_n^{k-1}x_n - \Delta_n^k x_n, A_k\Delta_n^{k-1}x_n - A_k p\right\rangle \\
&\quad \leq \left\|\Delta_n^{k-1}x_n - p\right\|^2 - \left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|^2 \\
&\qquad + 2r_{k,n}\left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|\left\|A_k\Delta_n^{k-1}x_n - A_k p\right\| \\
&\quad \leq \left\|x_n - p\right\|^2 - \left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|^2 \\
&\qquad + 2r_{k,n}\left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\| \\
&\qquad \times \left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|.
\end{aligned} \tag{69}$$

Also, by Lemma 14, we obtain that for each $i \in \{1, 2, \ldots, N\}$

$$\begin{aligned}
&\left\|\Lambda_n^i u_n - p\right\|^2 \\
&\quad = \left\|J_{R_i,\lambda_{i,n}}\left(I - \lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n - J_{R_i,\lambda_{i,n}}\left(I - \lambda_{i,n}B_i\right)p\right\|^2 \\
&\quad \leq \left\langle \left(I - \lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n - \left(I - \lambda_{i,n}B_i\right)p, \Lambda_n^i u_n - p\right\rangle \\
&\quad = \frac{1}{2}\left(\left\|\left(I - \lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n - \left(I - \lambda_{i,n}B_i\right)p\right\|^2\right.\\
&\qquad + \left\|\Lambda_n^i u_n - p\right\|^2 - \left\|\left(I - \lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n\right.\\
&\qquad - \left(I - \lambda_{i,n}B_i\right)p\\
&\qquad \left.\left. - \left(\Lambda_n^i u_n - p\right)\right\|^2\right) \\
&\quad \leq \frac{1}{2}\left(\left\|\Lambda_n^{i-1}u_n - p\right\|^2 + \left\|\Lambda_n^i u_n - p\right\|^2\right.\\
&\qquad \left. - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}\left(B_i\Lambda_n^{i-1}u_n - B_i p\right)\right\|^2\right) \\
&\quad \leq \frac{1}{2}\left(\left\|u_n - p\right\|^2 + \left\|\Lambda_n^i u_n - p\right\|^2\right.\\
&\qquad \left. - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}\left(B_i\Lambda_n^{i-1}u_n - B_i p\right)\right\|^2\right),
\end{aligned} \tag{70}$$

which implies

$$\begin{aligned}
\left\|\Lambda_n^i u_n - p\right\|^2 &\leq \left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right.\\
&\qquad \left. - \lambda_{i,n}\left(B_i\Lambda_n^{i-1}u_n - B_i p\right)\right\|^2 \\
&= \left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2 \\
&\quad - \lambda_{i,n}^2\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2 \\
&\quad + 2\lambda_{i,n}\left\langle \Lambda_n^{i-1}u_n - \Lambda_n^i u_n,\right.\\
&\qquad \left. B_i\Lambda_n^{i-1}u_n - B_i p\right\rangle \\
&\leq \left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2 \\
&\quad + 2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| \\
&\quad \times \left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|.
\end{aligned} \tag{71}$$

Thus, utilizing Lemma 7, from (49), (69), and (71) we have

$$\begin{aligned}
&\left\|x_n - p\right\|^2 \\
&\quad = \left\|s_n\left(\gamma V x_n - \mu F p\right) + \left(I - s_n\mu F\right)T_n v_n\right.\\
&\qquad \left. - \left(I - s_n\mu F\right)T_n p\right\|^2 \\
&\quad \leq \left\|\left(I - s_n\mu F\right)T_n v_n - \left(I - s_n\mu F\right)T_n p\right\|^2 \\
&\qquad + 2s_n\left\langle \gamma V x_n - \mu F p, x_n - p\right\rangle \\
&\quad \leq \left(1 - s_n\tau\right)^2\left\|v_n - p\right\|^2 + 2s_n\gamma l\left\|x_n - p\right\|^2 \\
&\qquad + 2s_n\left\|\gamma V p - \mu F p\right\|\left\|x_n - p\right\|
\end{aligned}$$

$$\begin{aligned}
&= (1 - s_n\tau)^2 \left\|\Lambda_n^N u_n - p\right\|^2 + 2s_n\gamma l \|x_n - p\|^2 \\
&\quad + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&\le (1 - s_n\tau)^2 \left\|\Lambda_n^i u_n - p\right\|^2 + 2s_n\gamma l \|x_n - p\|^2 \\
&\quad + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&\le (1 - s_n\tau)^2 \Big[\|u_n - p\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \Big] \\
&\quad + 2s_n\gamma l \|x_n - p\|^2 + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&= (1 - s_n\tau)^2 \Big[\left\|\Delta_n^M x_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \Big] \\
&\quad + 2s_n\gamma l \|x_n - p\|^2 + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&\le (1 - s_n\tau)^2 \Big[\left\|\Delta_n^k x_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \Big] \\
&\quad + 2s_n\gamma l \|x_n - p\|^2 + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&\le (1 - s_n\tau)^2 \Big[\|x_n - p\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\qquad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \\
&\qquad \times \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| \\
&\qquad - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \Big] \\
&\quad + 2s_n\gamma l \|x_n - p\|^2 \\
&\quad + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&\le \left(1 - 2s_n (\tau - \gamma l) + s_n^2\tau^2\right) \|x_n - p\|^2 \\
&\quad - (1 - s_n\tau)^2 \Big(\left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\qquad + \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \Big) \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| \\
&\quad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \\
&\quad + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&\le \|x_n - p\|^2 + s_n^2\tau^2 \|x_n - p\|^2 - (1 - s_n\tau)^2 \\
&\quad \times \left(\left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 + \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \right) \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| \\
&\quad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \\
&\quad + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| .
\end{aligned} \tag{72}$$

It immediately follows that

$$\begin{aligned}
&(1 - s_n\tau)^2 \left(\left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 + \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \right) \\
&\quad \le s_n^2\tau^2 \|x_n - p\|^2 + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \\
&\qquad \times \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| .
\end{aligned} \tag{73}$$

Since $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$, from (67) and $s_n \to 0$ we deduce that

$$\lim_{n\to\infty} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| = 0, \qquad \lim_{n\to\infty} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| = 0, \tag{74}$$

for all $k \in \{1, 2, \ldots, M\}$ and $i \in \{1, 2, \ldots, N\}$. Hence, we get

$$\begin{aligned}
\|x_n - u_n\| &= \left\|\Delta_n^0 x_n - \Delta_n^M x_n\right\| \\
&\le \left\|\Delta_n^0 x_n - \Delta_n^1 x_n\right\| + \left\|\Delta_n^1 x_n - \Delta_n^2 x_n\right\| \\
&\quad + \cdots + \left\|\Delta_n^{M-1} x_n - \Delta_n^M x_n\right\| \longrightarrow 0 \\
&\qquad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{75}$$

$$\begin{aligned}
\|u_n - v_n\| &= \left\|\Lambda_n^0 u_n - \Lambda_n^N u_n\right\| \\
&\le \left\|\Lambda_n^0 u_n - \Lambda_n^1 u_n\right\| + \left\|\Lambda_n^1 u_n - \Lambda_n^2 u_n\right\| \\
&\quad + \cdots + \left\|\Lambda_n^{N-1} u_n - \Lambda_n^N u_n\right\| \longrightarrow 0 \\
&\qquad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{76}$$

So, taking into account that $\|x_n - v_n\| \le \|x_n - u_n\| + \|u_n - v_n\|$, we have

$$\lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{77}$$

Thus, from (77) and $s_n \to 0$ we have

$$\begin{aligned}
\|v_n - T_n v_n\| &\le \|v_n - x_n\| + \|x_n - T_n v_n\| \\
&= \|v_n - x_n\| + s_n \|\gamma V x_n - \mu F T_n v_n\| \longrightarrow 0 \\
&\qquad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{78}$$

Now we show that $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$. In fact, from the nonexpansivity of T_n, we have

$$\begin{aligned}\|x_n - T_n x_n\| &\le \|x_n - v_n\| + \|v_n - T_n v_n\| \\ &\quad + \|T_n v_n - T_n x_n\| \\ &\le 2\|x_n - v_n\| + \|v_n - T_n v_n\|.\end{aligned} \tag{79}$$

By (77) and (78), we get

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{80}$$

From (78) it is easy to see that

$$\lim_{n\to\infty} \|x_n - T_n v_n\| = 0. \tag{81}$$

Observe that

$$\begin{aligned}&\|P_C(I - \lambda_n \nabla f) v_n - v_n\| \\ &\quad = \|s_n v_n + (1 - s_n) T_n v_n - v_n\| \\ &\quad = (1 - s_n)\|T_n v_n - v_n\| \\ &\quad \le \|T_n v_n - v_n\|,\end{aligned} \tag{82}$$

where $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$. Hence, we have

$$\begin{aligned}&\left\|P_C\left(I - \frac{2}{L}\nabla f\right) v_n - v_n\right\| \\ &\quad \le \left\|P_C\left(I - \frac{2}{L}\nabla f\right) v_n - P_C(I - \lambda_n \nabla f) v_n\right\| \\ &\qquad + \|P_C(I - \lambda_n \nabla f) v_n - v_n\| \\ &\quad \le \left\|\left(I - \frac{2}{L}\nabla f\right) v_n - (I - \lambda_n \nabla f) v_n\right\| \\ &\qquad + \|P_C(I - \lambda_n \nabla f) v_n - v_n\| \\ &\quad \le \left(\frac{2}{L} - \lambda_n\right)\|\nabla f(v_n)\| + \|T_n v_n - v_n\|.\end{aligned} \tag{83}$$

From the boundedness of $\{v_n\}$, $s_n \to 0$ ($\Leftrightarrow \lambda_n \to 2/L$) and $\|T_n v_n - v_n\| \to 0$ (due to (78)), it follows that

$$\lim_{n\to\infty} \left\|v_n - P_C\left(I - \frac{2}{L}\nabla f\right) v_n\right\| = 0. \tag{84}$$

Further, we show that $\omega_w(x_n) \subset \Omega$. Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to some w. Note that $\lim_{n\to\infty}\|x_n - u_n\| = 0$ (due to (75)). Hence, $x_{n_i} \rightharpoonup w$. Since C is closed and convex, C is weakly closed. So, we have $w \in C$. From (74) and (75), we have that $\Delta^k_{n_i} x_{n_i} \rightharpoonup w$, $\Lambda^m_{n_i} u_{n_i} \rightharpoonup w$, $u_{n_i} \rightharpoonup w$, $v_{n_i} \rightharpoonup w$, where $k \in \{1, 2, \dots, M\}$, $m \in \{1, 2, \dots, N\}$. First, we prove that $w \in \cap_{m=1}^N I(B_m, R_m)$. As a matter of fact, since B_m is η_m-inverse-strongly monotone, B_m is a monotone and Lipschitz continuous mapping. It follows from Lemma 17 that $R_m + B_m$ is maximal monotone. Let $(v, g) \in G(R_m + B_m)$; that is, $g - B_m v \in R_m v$. Again, since $\Lambda^m_n u_n = J_{R_m, \lambda_{m,n}}(I - \lambda_{m,n} B_m)\Lambda^{m-1}_n u_n$, $n \ge 1$, $m \in \{1, 2, \dots, N\}$, we have

$$\Lambda^{m-1}_n u_n - \lambda_{m,n} B_m \Lambda^{m-1}_n u_n \in (I + \lambda_{m,n} R_m)\Lambda^m_n u_n; \tag{85}$$

that is,

$$\frac{1}{\lambda_{m,n}}\left(\Lambda^{m-1}_n u_n - \Lambda^m_n u_n - \lambda_{m,n} B_m \Lambda^{m-1}_n u_n\right) \in R_m \Lambda^m_n u_n. \tag{86}$$

In terms of the monotonicity of R_m, we get

$$\begin{aligned}&\left\langle v - \Lambda^m_n u_n, g - B_m v - \frac{1}{\lambda_{m,n}}\right. \\ &\quad \left. \times \left(\Lambda^{m-1}_n u_n - \Lambda^m_n u_n - \lambda_{m,n} B_m \Lambda^{m-1}_n u_n\right)\right\rangle \ge 0\end{aligned} \tag{87}$$

and hence

$$\begin{aligned}&\langle v - \Lambda^m_n u_n, g\rangle \\ &\quad \ge \left\langle v - \Lambda^m_n u_n, B_m v + \frac{1}{\lambda_{m,n}}\right. \\ &\qquad \left. \times \left(\Lambda^{m-1}_n u_n - \Lambda^m_n u_n - \lambda_{m,n} B_m \Lambda^{m-1}_n u_n\right)\right\rangle \\ &\quad = \left\langle v - \Lambda^m_n u_n, B_m v - B_m \Lambda^m_n u_n + B_m \Lambda^m_n u_n\right. \\ &\qquad \left. - B_m \Lambda^{m-1}_n u_n + \frac{1}{\lambda_{m,n}}\left(\Lambda^{m-1}_n u_n - \Lambda^m_n u_n\right)\right\rangle \\ &\quad \ge \left\langle v - \Lambda^m_n u_n, B_m \Lambda^m_n u_n - B_m \Lambda^{m-1}_n u_n\right\rangle \\ &\qquad + \left\langle v - \Lambda^m_n u_n, \frac{1}{\lambda_{m,n}}\left(\Lambda^{m-1}_n u_n - \Lambda^m_n u_n\right)\right\rangle.\end{aligned} \tag{88}$$

In particular,

$$\begin{aligned}&\left\langle v - \Lambda^m_{n_i} u_{n_i}, g\right\rangle \\ &\quad \ge \left\langle v - \Lambda^m_{n_i} u_{n_i}, B_m \Lambda^m_{n_i} u_{n_i} - B_m \Lambda^{m-1}_{n_i} u_{n_i}\right\rangle \\ &\qquad + \left\langle v - \Lambda^m_{n_i} u_{n_i}, \frac{1}{\lambda_{m,n_i}}\left(\Lambda^{m-1}_{n_i} u_{n_i} - \Lambda^m_{n_i} u_{n_i}\right)\right\rangle.\end{aligned} \tag{89}$$

Since $\|\Lambda^m_n u_n - \Lambda^{m-1}_n u_n\| \to 0$ (due to (74)) and $\|B_m \Lambda^m_n u_n - B_m \Lambda^{m-1}_n u_n\| \to 0$ (due to the Lipschitz continuity of B_m), we conclude from $\Lambda^m_{n_i} u_{n_i} \rightharpoonup w$ and condition (ii) that

$$\lim_{i\to\infty}\left\langle v - \Lambda^m_{n_i} u_{n_i}, g\right\rangle = \langle v - w, g\rangle \ge 0. \tag{90}$$

It follows from the maximal monotonicity of $B_m + R_m$ that $0 \in (R_m + B_m)w$; that is, $w \in I(B_m, R_m)$. Therefore, $w \in \cap_{m=1}^N I(B_m, R_m)$. Next we prove that

$w \in \cap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k)$. Since $\Delta_n^k x_n = T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}A_k)\Delta_n^{k-1}x_n$, $n \geq 1$, $k \in \{1, 2, \ldots, M\}$, we have

$$\begin{aligned}\Theta_k\left(\Delta_n^k x_n, y\right) &+ \varphi_k(y) - \varphi_k\left(\Delta_n^k x_n\right)\\ &+ \left\langle A_k \Delta_n^{k-1} x_n, y - \Delta_n^k x_n\right\rangle\\ &+ \frac{1}{r_{k,n}}\left\langle y - \Delta_n^k x_n, \Delta_n^k x_n - \Delta_n^{k-1} x_n\right\rangle \geq 0.\end{aligned} \tag{91}$$

By (A2), we have

$$\begin{aligned}\varphi_k(y) &- \varphi_k\left(\Delta_n^k x_n\right) + \left\langle A_k \Delta_n^{k-1} x_n, y - \Delta_n^k x_n\right\rangle\\ &+ \frac{1}{r_{k,n}}\left\langle y - \Delta_n^k x_n, \Delta_n^k x_n - \Delta_n^{k-1} x_n\right\rangle \geq \Theta_k\left(y, \Delta_n^k x_n\right).\end{aligned} \tag{92}$$

Let $z_t = ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in C$. This implies that $z_t \in C$. Then, we have

$$\begin{aligned}&\left\langle z_t - \Delta_n^k x_n, A_k z_t\right\rangle\\ &\quad\geq \varphi_k\left(\Delta_n^k x_n\right) - \varphi_k(z_t) + \left\langle z_t - \Delta_n^k x_n, A_k z_t\right\rangle\\ &\qquad - \left\langle z_t - \Delta_n^k x_n, A_k \Delta_n^{k-1} x_n\right\rangle\\ &\qquad - \left\langle z_t - \Delta_n^k x_n, \frac{\Delta_n^k x_n - \Delta_n^{k-1} x_n}{r_{k,n}}\right\rangle + \Theta_k\left(z_t, \Delta_n^k x_n\right)\\ &\quad= \varphi_k\left(\Delta_n^k x_n\right) - \varphi_k(z_t)\\ &\qquad + \left\langle z_t - \Delta_n^k x_n, A_k z_t - A_k \Delta_n^k x_n\right\rangle\\ &\qquad + \left\langle z_t - \Delta_n^k x_n, A_k \Delta_n^k x_n - A_k \Delta_n^{k-1} x_n\right\rangle\\ &\qquad - \left\langle z_t - \Delta_n^k x_n, \frac{\Delta_n^k x_n - \Delta_n^{k-1} x_n}{r_{k,n}}\right\rangle + \Theta_k\left(z_t, \Delta_n^k x_n\right).\end{aligned} \tag{93}$$

By (74), we have $\|A_k \Delta_n^k x_n - A_k \Delta_n^{k-1} x_n\| \to 0$ (due to the Lipschitz continuity of A_k). Furthermore, by the monotonicity of A_k, we obtain $\langle z_t - \Delta_n^k x_n, A_k z_t - A_k \Delta_n^k x_n\rangle \geq 0$. Then, by (A4) we obtain

$$\left\langle z_t - w, A_k z_t\right\rangle \geq \varphi_k(w) - \varphi_k(z_t) + \Theta_k(z_t, w). \tag{94}$$

Utilizing (A1), (A4), and (94), we obtain

$$\begin{aligned}0 &= \Theta_k(z_t, z_t) + \varphi_k(z_t) - \varphi_k(z_t)\\ &\leq t\Theta_k(z_t, y) + (1-t)\Theta_k(z_t, w) + t\varphi_k(y)\\ &\quad + (1-t)\varphi_k(w) - \varphi_k(z_t)\\ &\leq t\left[\Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)\right]\\ &\quad + (1-t)\left\langle z_t - w, A_k z_t\right\rangle\\ &= t\left[\Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)\right]\\ &\quad + (1-t)t\left\langle y - w, A_k z_t\right\rangle,\end{aligned} \tag{95}$$

and hence

$$0 \leq \Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t) + (1-t)\left\langle y - w, A_k z_t\right\rangle. \tag{96}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq \Theta_k(w, y) + \varphi_k(y) - \varphi_k(w) + \left\langle y - w, A_k w\right\rangle. \tag{97}$$

This implies that $w \in \mathrm{GMEP}(\Theta_k, \varphi_k, A_k)$ and hence $w \in \cap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k)$. Further, let us show that $w \in \Gamma$. As a matter of fact, from (84), $v_{n_i} \rightharpoonup w$, and Lemma 9, we conclude that

$$w = P_C\left(I - \frac{2}{L}\nabla f\right)w. \tag{98}$$

So, $w \in \mathrm{VI}(C, \nabla f) = \Gamma$. Therefore, $w \in \cap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k) \cap \cap_{i=1}^{N} I(B_i, R_i) \cap \Gamma =: \Omega$. This shows that $\omega_w(x_n) \subset \Omega$.

Finally, let us show that $x_n \to q$ as $n \to \infty$ where q is the unique solution of the VIP (50). Indeed, we note that, for $w \in \Omega$ with $x_{n_i} \rightharpoonup w$,

$$\begin{aligned}x_n - w = s_n(\gamma V x_n - \mu F w) &+ (I - s_n\mu F)T_n v_n\\ &- (I - s_n\mu F)w.\end{aligned} \tag{99}$$

By (61) and Lemma 13, we obtain that

$$\begin{aligned}\|x_n - w\|^2 &= s_n\langle\gamma V x_n - \mu F w, x_n - w\rangle\\ &\quad + \langle(I - s_n\mu F)T_n v_n - (I - s_n\mu F)w, x_n - w\rangle\\ &= s_n\langle\gamma V x_n - \mu F w, x_n - w\rangle\\ &\quad + \|(I - s_n\mu F)T_n v_n - (I - s_n\mu F)w\|\,\|x_n - w\|\\ &\leq s_n\langle\gamma V x_n - \mu F w, x_n - w\rangle\\ &\quad + (1 - s_n\tau)\|v_n - w\|\,\|x_n - w\|\\ &\leq s_n\langle\gamma V x_n - \mu F w, x_n - w\rangle + (1 - s_n\tau)\|x_n - w\|^2.\end{aligned} \tag{100}$$

Hence, it follows that

$$\begin{aligned}\|x_n - w\|^2 &\leq \frac{1}{\tau}\langle\gamma V x_n - \mu F w, x_n - w\rangle\\ &= \frac{1}{\tau}\left(\gamma\langle V x_n - V w, x_n - w\rangle\right.\\ &\qquad \left.+ \langle\gamma V w - \mu F w, x_n - w\rangle\right)\\ &\leq \frac{1}{\tau}\left(\gamma l\|x_n - w\|^2 + \langle\gamma V w - \mu F w, x_n - w\rangle\right),\end{aligned} \tag{101}$$

which hence leads to

$$\|x_n - w\|^2 \leq \frac{\langle\gamma V w - \mu F w, x_n - w\rangle}{\tau - \gamma l}. \tag{102}$$

In particular, we have

$$\left\|x_{n_i} - w\right\|^2 \leq \frac{\left\langle \gamma Vw - \mu Fw, x_{n_i} - w\right\rangle}{\tau - \gamma l}. \quad (103)$$

Since $x_{n_i} \rightharpoonup w$, it follows from (103) that $x_{n_i} \to w$ as $i \to \infty$.

Now we show that w solves the VIP (50). Since $x_n = s_n\gamma Vx_n + (I - s_n\mu F)T_n v_n$, we have

$$(\mu F - \gamma V)\, x_n = -\frac{1}{s_n}\left((I - s_n\mu F)\, x_n - (I - s_n\mu F)\, T_n v_n\right). \quad (104)$$

It follows that, for each $p \in \Omega$,

$$\begin{aligned}
&\langle(\mu F - \gamma V)\, x_n, x_n - p\rangle \\
&= -\frac{1}{s_n}\langle(I - s_n\mu F)\, x_n - (I - s_n\mu F)\, T_n v_n, x_n - p\rangle \\
&= -\frac{1}{s_n}\langle(I - s_n\mu F)\, x_n - (I - s_n\mu F)\, T_n v_n, x_n - p\rangle \\
&= -\frac{1}{s_n}\left\langle(I - s_n\mu F)\, x_n - (I - s_n\mu F)\, T_n\Lambda_n^N\Delta_n^M x_n, x_n - p\right\rangle \\
&= -\frac{1}{s_n}\left\langle\left(I - T_n\Lambda_n^N\Delta_n^M\right) x_n - \left(I - T_n\Lambda_n^N\Delta_n^M\right) p, x_n - p\right\rangle \\
&\quad + \left\langle\mu Fx_n - \mu FT_n\Lambda_n^N\Delta_n^M x_n, x_n - p\right\rangle \\
&\leq \left\langle\mu Fx_n - \mu FT_n\Lambda_n^N\Delta_n^M x_n, x_n - p\right\rangle,
\end{aligned} \quad (105)$$

since $I - T_n\Lambda_n^N\Delta_n^M$ is monotone (i.e., $\langle(I - T_n\Lambda_n^N\Delta_n^M)x - (I - T_n\Lambda_n^N\Delta_n^M)y, x - y\rangle \geq 0$ for all $x, y \in H$. This is due to the nonexpansivity of $T_n\Lambda_n^N\Delta_n^M$). Since $\|x_n - T_n v_n\| = \|(I - T_n\Lambda_n^N\Delta_n^M)x_n\| \to 0$ as $n \to \infty$, by replacing n in (105) with n_i and letting $i \to \infty$, we get

$$\begin{aligned}
&\langle(\mu F - \gamma V)\, w, w - p\rangle \\
&\quad = \lim_{i\to\infty}\left\langle(\mu F - \gamma V)\, x_{n_i}, x_{n_i} - p\right\rangle \\
&\quad \leq \lim_{i\to\infty}\left\langle\mu Fx_{n_i} - \mu FT_{n_i}v_{n_i}, x_{n_i} - p\right\rangle = 0.
\end{aligned} \quad (106)$$

That is, $w \in \Omega$ is a solution of VIP (50).

Finally, in terms of the uniqueness of solutions of VIP (50), we deduce that $w = q$ and $x_{n_i} \to q$ as $n \to \infty$. So, every weak convergence subsequence of $\{x_n\}$ converges strongly to the unique solution q of VIP (50). Therefore, $\{x_n\}$ converges strongly to the unique solution q of VIP (50). In addition, the VIP (50) can be rewritten as

$$\left\langle(I - \mu F + \gamma V)\, q - q, q - p\right\rangle \geq 0, \quad \forall p \in \Omega. \quad (107)$$

By Proposition 2(i), this is equivalent to the fixed point equation:

$$P_\Omega\,(I - \mu F + \gamma V)\, q = q. \quad (108)$$

This completes the proof. □

Corollary 19. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B_i : C \to H$ be ζ-inverse-strongly monotone and η_i-inverse-strongly monotone, respectively, for $i = 1, 2$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := GMEP(\Theta, \varphi, A) \cap I(B_1, R_1) \cap I(B_2, R_2) \cap \Gamma \neq \emptyset$ and that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&\Theta\,(u_n, y) + \varphi\,(y) - \varphi\,(u_n) + \langle Ax_n, y - u_n\rangle \\
&\qquad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
&v_n = J_{R_2,\lambda_{2,n}}\,(I - \lambda_{2,n}B_2)\, J_{R_1,\lambda_{1,n}}\,(I - \lambda_{1,n}B_1)\, u_n, \\
&x_n = s_n\gamma Vx_n + (I - s_n\mu F)\, T_n v_n, \quad \forall n \geq 1,
\end{aligned} \quad (109)$$

where $P_C(I - \lambda_n\nabla f) = s_n I + (1 - s_n)T_n$ (here T_n is nonexpansive and $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$). Assume that the following conditions hold:

(i) *$s_n \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$, $\lim_{n\to\infty}s_n = 0$ ($\Leftrightarrow$ $\lim_{n\to\infty}\lambda_n = 2/L$);*

(ii) *$\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ for $i = 1, 2$;*

(iii) *$\{r_n\} \subset [e, f] \subset (0, 2\zeta)$.*

Then $\{x_n\}$ converges strongly as $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) to a point $q \in \Omega$, which is a unique solution of the VIP:

$$\left\langle(\mu F - \gamma V)\, q, p - q\right\rangle \geq 0, \quad \forall p \in \Omega. \quad (110)$$

Corollary 20. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $R : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B : C \to H$ be ζ-inverse-strongly monotone and ξ-inverse-strongly monotone, respectively. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := GMEP(\Theta, \varphi, A) \cap I(B, R) \cap \Gamma \neq \emptyset$ and that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&\Theta\,(u_n, y) + \varphi\,(y) - \varphi\,(u_n) + \langle Ax_n, y - u_n\rangle \\
&\qquad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
&x_n = s_n\gamma Vx_n + (I - s_n\mu F)\, T_n J_{R,\rho_n}\,(u_n - \rho_n Bu_n), \\
&\qquad\qquad \forall n \geq 1,
\end{aligned} \quad (111)$$

where $P_C(I-\lambda_n\nabla f) = s_nI+(1-s_n)T_n$ *(here* T_n *is nonexpansive and* $s_n = (2-\lambda_nL)/4 \in (0,1/2)$ *for each* $\lambda_n \in (0,2/L)$*). Assume that the following conditions hold:*

(i) $s_n \in (0,1/2)$ *for each* $\lambda_n \in (0,2/L)$, $\lim_{n\to\infty}s_n = 0$ $(\Leftrightarrow \lim_{n\to\infty}\lambda_n = 2/L)$;

(ii) $\{\rho_n\} \subset [a,b] \subset (0,2\xi)$;

(iii) $\{r_n\} \subset [e,f] \subset (0,2\zeta)$.

Then $\{x_n\}$ *converges strongly as* $\lambda_n \to 2/L$ $(\Leftrightarrow s_n \to 0)$ *to a point* $q \in \Omega$*, which is a unique solution of the VIP:*

$$\langle(\mu F-\gamma V)q, p-q\rangle \geq 0, \quad \forall p \in \Omega. \tag{112}$$

4. Explicit Iterative Algorithm and Its Convergence Criteria

We next state and prove the second main result of this paper.

Theorem 21. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $f : C \to \mathbf{R}$ *be a convex functional with L-Lipschitz continuous gradient* ∇f*. Let* M, N *be two integers. Let* Θ_k *be a bifunction from* $C\times C$ *to* $\mathbf{R}$ *satisfying (A1)–(A4) and let* $\varphi_k : C \to \mathbf{R}\cup\{+\infty\}$ *be a proper lower semicontinuous and convex function, where* $k \in \{1,2,\ldots,M\}$*. Let* $R_i : C \to 2^H$ *be a maximal monotone mapping and let* $A_k : H \to H$ *and* $B_i : C \to H$ *be* μ_k*-inverse-strongly monotone and* η_i*-inverse-strongly monotone, respectively, where* $k \in \{1,2,\ldots,M\}$, $i \in \{1,2,\ldots,N\}$*. Let* $F : H \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with positive constants* $\kappa,\eta > 0$*. Let* $V : H \to H$ *be an* l*-Lipschitzian mapping with constant* $l \geq 0$*. Let* $0 < \mu < 2\eta/\kappa^2$ *and* $0 \leq \gamma l < \tau$*, where* $\tau = 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}$*. Assume that* $\Omega := \cap_{k=1}^M GMEP(\Theta_k,\varphi_k,A_k) \cap \cap_{i=1}^N I(B_i,R_i) \cap \Gamma \neq \emptyset$ *and that either (B1) or (B2) holds. For arbitrarily given* $x_1 \in H$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} u_n &= T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}A_M)T_{r_{M-1,n}}^{(\Theta_{M-1},\varphi_{M-1})} \\ &\quad\times(I-r_{M-1,n}A_{M-1})\cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I-r_{1,n}A_1)x_n, \\ v_n &= J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)J_{R_{N-1},\lambda_{N-1,n}} \\ &\quad\times(I-\lambda_{N-1,n}A_{N-1})\cdots J_{R_1,\lambda_{1,n}}(I-\lambda_{1,n}B_1)u_n, \\ x_{n+1} &= s_n\gamma Vx_n+\beta_nx_n+((1-\beta_n)I-s_n\mu F)T_nv_n, \\ &\qquad \forall n\geq 1, \end{aligned} \tag{113}$$

where $P_C(I-\lambda_n\nabla f) = s_nI+(1-s_n)T_n$ *(here* T_n *is nonexpansive and* $s_n = (2-\lambda_nL)/4 \in (0,1/2)$ *for each* $\lambda_n \in (0,2/L)$*). Assume that the following conditions hold:*

(i) $s_n \in (0,1/2)$ *for each* $\lambda_n \in (0,2/L)$*, and* $\lim_{n\to\infty}s_n = 0$ $(\Leftrightarrow \lim_{n\to\infty}\lambda_n = 2/L)$;

(ii) $\{\beta_n\} \subset (0,1)$ *and* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\{\lambda_{i,n}\} \subset [a_i,b_i] \subset (0,2\eta_i)$ *and* $\lim_{n\to\infty}|\lambda_{i,n+1}-\lambda_{i,n}| = 0$ *for all* $i \in \{1,2,\ldots,N\}$;

(iv) $\{r_{k,n}\} \subset [e_k,f_k] \subset (0,2\mu_k)$ *and* $\lim_{n\to\infty}|r_{k,n+1}-r_{k,n}| = 0$ *for all* $k \in \{1,2,\ldots,M\}$.

Then $\{x_n\}$ *converges strongly as* $\lambda_n \to 2/L$ $(\Leftrightarrow s_n \to 0)$ *to a point* $q \in \Omega$*, which is a unique solution of VIP* (50).

Proof. First of all, repeating the same arguments as in Theorem 18, we can write

$$P_C(I-\lambda_n\nabla f) = \frac{2-\lambda_nL}{4}I+\frac{2+\lambda_nL}{4}T_n = s_nI+(1-s_n)T_n, \tag{114}$$

where T_n is nonexpansive and $s_n := s_n(\lambda_n) = (2-\lambda_nL)/4 \in (0,1/2)$ for each $\lambda_n \in (0,2/L)$. It is clear that

$$\lambda_n \longrightarrow \frac{2}{L} \Longleftrightarrow s_n \longrightarrow 0. \tag{115}$$

Put

$$\begin{aligned} \Delta_n^k &= T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}A_k)T_{r_{k-1,n}}^{(\Theta_{k-1},\varphi_{k-1})} \\ &\quad\times(I-r_{k-1,n}A_{k-1})\cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I-r_{1,n}A_1)x_n \end{aligned} \tag{116}$$

for all $k \in \{1,2,\ldots,M\}$ and $n \geq 1$,

$$\begin{aligned} \Lambda_n^i &= J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)J_{R_{i-1},\lambda_{i-1,n}} \\ &\quad\times(I-\lambda_{i-1,n}B_{i-1})\cdots J_{R_1,\lambda_{1,n}}(I-\lambda_{1,n}B_1) \end{aligned} \tag{117}$$

for all $i \in \{1,2,\ldots,N\}$ and $n \geq 1$, and $\Delta_n^0 = \Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Delta_n^M x_n$ and $v_n = \Lambda_n^N u_n$. In addition, taking into consideration conditions (i) and (ii), we may assume, without loss of generality, that $s_n \leq 1-\beta_n$ for all $n \geq 1$.

We divide the remainder of the proof into several steps.

Step 1. Let us show that $\|x_n-p\| \leq \max\{\|x_1-p\|, \|\gamma Vp-\mu Fp\|/(\tau-\gamma l)\}$ for all $n \geq 1$ and $p \in \Omega$. Indeed, take $p \in \Omega$ arbitrarily. Repeating the same arguments as those of (59)–(61) in the proof of Theorem 18, we obtain

$$\begin{aligned} \|u_n-p\| &\leq \|x_n-p\|, \\ \|v_n-p\| &\leq \|u_n-p\|, \\ \|v_n-p\| &\leq \|x_n-p\|. \end{aligned} \tag{118}$$

Then from (118), $T_np = p$, and Lemma 13, we have

$$\begin{aligned} &\|x_{n+1}-p\| \\ &\quad= \|s_n(\gamma Vx_n-\mu Fp)+\beta_n(x_n-p) \\ &\qquad+((1-\beta_n)I-s_n\mu F)T_nv_n \\ &\qquad-((1-\beta_n)I-s_n\mu F)T_np\| \end{aligned}$$

$$\begin{aligned}
&\le s_n \|\gamma V x_n - \mu F p\| + \beta_n \|x_n - p\| \\
&\quad + (1-\beta_n) \left\| \left(I - \frac{s_n}{1-\beta_n}\mu F\right) T_n v_n \right. \\
&\qquad \left. - \left(I - \frac{s_n}{1-\beta_n}\mu F\right) T_n p \right\| \\
&\le s_n (\|\gamma V x_n - \gamma V p\| + \|\gamma V p - \mu F p\|) \\
&\quad + \beta_n \|x_n - p\| + (1-\beta_n)\left(1 - \frac{s_n \tau}{1-\beta_n}\right) \|v_n - p\| \\
&\le s_n (\|\gamma V x_n - \gamma V p\| + \|\gamma V p - \mu F p\|) \\
&\quad + \beta_n \|x_n - p\| + (1 - \beta_n - s_n\tau) \|x_n - p\| \\
&\le s_n \gamma l \|x_n - p\| + s_n \|\gamma V p - \mu F p\| \\
&\quad + (1 - s_n \tau) \|x_n - p\| \\
&= (1 - s_n (\tau - \gamma l)) \|x_n - p\| \\
&\quad + s_n (\tau - \gamma l) \frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l} \\
&\le \max\left\{ \|x_n - p\|, \frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l} \right\}.
\end{aligned} \tag{119}$$

By induction, we have

$$\|x_n - p\| \le \max\left\{ \|x_1 - p\|, \frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l} \right\}, \quad \forall n \ge 1. \tag{120}$$

Hence, $\{x_n\}$ is bounded. According to (118), $\{u_n\}$, $\{v_n\}$, $\{T_n v_n\}$, $\{V x_n\}$, and $\{F T_n v_n\}$ are also bounded.

Step 2. Let us show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. To this end, define

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n, \quad \forall n \ge 1. \tag{121}$$

Observe that, from the definition of z_n,

$$\begin{aligned}
&z_{n+1} - z_n \\
&\quad = \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&\quad = \frac{s_{n+1}\gamma V x_{n+1} + ((1-\beta_{n+1}) I - s_{n+1}\mu F) T_{n+1} v_{n+1}}{1-\beta_{n+1}} \\
&\qquad - \frac{s_n \gamma V x_n + ((1-\beta_n) I - s_n \mu F) T_n v_n}{1-\beta_n} \\
&\quad = \frac{s_{n+1}}{1-\beta_{n+1}}\gamma V x_{n+1} - \frac{s_n}{1-\beta_n}\gamma V x_n + T_{n+1} v_{n+1} \\
&\qquad - T_n v_n + \frac{s_n}{1-\beta_n}\mu F T_n v_n - \frac{s_{n+1}}{1-\beta_{n+1}}\mu F T_{n+1} v_{n+1} \\
&\quad = \frac{s_{n+1}}{1-\beta_{n+1}} (\gamma V x_{n+1} - \mu F T_{n+1} v_{n+1}) \\
&\qquad + \frac{s_n}{1-\beta_n} (\mu F T_n v_n - \gamma V x_n) + T_{n+1} v_{n+1} - T_n v_n.
\end{aligned} \tag{122}$$

Thus, it follows that

$$\begin{aligned}
\|z_{n+1} - z_n\| &\le \frac{s_{n+1}}{1-\beta_{n+1}} (\gamma \|V x_{n+1}\| + \mu \|F T_{n+1} v_{n+1}\|) \\
&\quad + \frac{s_n}{1-\beta_n} (\mu \|F T_n v_n\| + \gamma \|V x_n\|) \\
&\quad + \|T_{n+1} v_{n+1} - T_n v_n\|.
\end{aligned} \tag{123}$$

On the other hand, since ∇f is $(1/L)$-ism, $P_C(I - \lambda_n \nabla f)$ is nonexpansive for $\lambda_n \in (0, 2/L)$. So, it follows that, for any given $p \in \Omega$,

$$\begin{aligned}
\|P_C (I - \lambda_{n+1}\nabla f) v_n\| &\le \|P_C (I - \lambda_{n+1}\nabla f) v_n - p\| + \|p\| \\
&= \|P_C (I - \lambda_{n+1}\nabla f) v_n \\
&\quad - P_C (I - \lambda_{n+1}\nabla f) p\| + \|p\| \\
&\le \|v_n - p\| + \|p\| \\
&\le \|v_n\| + 2\|p\|.
\end{aligned} \tag{124}$$

This together with the boundedness of $\{v_n\}$ implies that $\{P_C(I - \lambda_{n+1}\nabla f) v_n\}$ is bounded. Also, observe that

$$\begin{aligned}
&\|T_{n+1} v_n - T_n v_n\| \\
&\quad = \left\| \frac{4P_C (I - \lambda_{n+1}\nabla f) - (2 - \lambda_{n+1} L) I}{2 + \lambda_{n+1} L} v_n \right. \\
&\qquad \left. - \frac{4P_C (I - \lambda_n \nabla f) - (2 - \lambda_n L) I}{2 + \lambda_n L} v_n \right\| \\
&\quad \le \left\| \frac{4P_C (I - \lambda_{n+1}\nabla f)}{2 + \lambda_{n+1} L} v_n - \frac{4P_C (I - \lambda_n \nabla f)}{2 + \lambda_n L} v_n \right\| \\
&\qquad + \left\| \frac{2 - \lambda_n L}{2 + \lambda_n L} v_n - \frac{2 - \lambda_{n+1} L}{2 + \lambda_{n+1} L} v_n \right\| \\
&\quad = \| (4 (2 + \lambda_n L) P_C (I - \lambda_{n+1}\nabla f) v_n \\
&\qquad - 4 (2 + \lambda_{n+1} L) P_C (I - \lambda_n \nabla f) v_n) \\
&\qquad \times ((2 + \lambda_{n+1} L)(2 + \lambda_n L))^{-1} \| \\
&\qquad + \frac{4L |\lambda_{n+1} - \lambda_n|}{(2 + \lambda_{n+1} L)(2 + \lambda_n L)} \|v_n\|
\end{aligned}$$

$$
\begin{aligned}
&= \| (4L(\lambda_n - \lambda_{n+1}) P_C (I - \lambda_{n+1}\nabla f) v_n + 4(2 + \lambda_{n+1}L) \\
&\quad \times (P_C (I - \lambda_{n+1}\nabla f) v_n - P_C (I - \lambda_n \nabla f) v_n)) \\
&\quad \times ((2 + \lambda_{n+1}L)(2 + \lambda_n L))^{-1} \| \\
&\quad + \frac{4L|\lambda_{n+1} - \lambda_n|}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} \|v_n\| \\
&\le \frac{4L|\lambda_n - \lambda_{n+1}| \|P_C (I - \lambda_{n+1}\nabla f) v_n\|}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} \\
&\quad + (4(2 + \lambda_{n+1}L) \|P_C (I - \lambda_{n+1}\nabla f) v_n \\
&\qquad - P_C (I - \lambda_n \nabla f) v_n\|) \\
&\quad \times ((2 + \lambda_{n+1}L)(2 + \lambda_n L))^{-1} \\
&\quad + \frac{4L|\lambda_{n+1} - \lambda_n|}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} \|v_n\| \\
&\le |\lambda_{n+1} - \lambda_n| [L \|P_C (I - \lambda_{n+1}\nabla f) v_n\| \\
&\qquad + 4\|\nabla f(v_n)\| + L\|v_n\|] \\
&\le \widetilde{M} |\lambda_{n+1} - \lambda_n|,
\end{aligned} \tag{125}
$$

where $\sup_{n\ge1}\{L\|P_C(I - \lambda_{n+1}\nabla f)v_n\| + 4\|\nabla f(v_n)\| + L\|v_n\|\} \le \widetilde{M}$ for some $\widetilde{M} > 0$. So, by (125), we have that

$$
\begin{aligned}
\|T_{n+1}v_{n+1} - T_n v_n\| &\le \|T_{n+1}v_{n+1} - T_{n+1}v_n\| \\
&\quad + \|T_{n+1}v_n - T_n v_n\| \\
&\le \|v_{n+1} - v_n\| + \widetilde{M} |\lambda_{n+1} - \lambda_n| \\
&\le \|v_{n+1} - v_n\| + \frac{4\widetilde{M}}{L} (s_{n+1} + s_n).
\end{aligned} \tag{126}
$$

Note that

$$
\begin{aligned}
\|v_{n+1} - v_n\| &= \|\Lambda_{n+1}^N u_{n+1} - \Lambda_n^N u_n\| \\
&= \|J_{R_N,\lambda_{N,n+1}} (I - \lambda_{N,n+1}B_N) \Lambda_{n+1}^{N-1} u_{n+1} \\
&\qquad - J_{R_N,\lambda_{N,n}} (I - \lambda_{N,n}B_N) \Lambda_n^{N-1} u_n\| \\
&\le \|J_{R_N,\lambda_{N,n+1}} (I - \lambda_{N,n+1}B_N) \Lambda_{n+1}^{N-1} u_{n+1} \\
&\qquad - J_{R_N,\lambda_{N,n}} (I - \lambda_{N,n}B_N) \Lambda_{n+1}^{N-1} u_{n+1}\| \\
&\quad + \|J_{R_N,\lambda_{N,n}} (I - \lambda_{N,n}B_N) \Lambda_{n+1}^{N-1} u_{n+1} \\
&\qquad - J_{R_N,\lambda_{N,n}} (I - \lambda_{N,n}B_N) \Lambda_n^{N-1} u_n\| \\
&\le \|(I - \lambda_{N,n+1}B_N) \Lambda_{n+1}^{N-1} u_{n+1} \\
&\qquad - (I - \lambda_{N,n}B_N) \Lambda_{n+1}^{N-1} u_{n+1}\| \\
&\quad + \|(I - \lambda_{N,n}B_N) \Lambda_{n+1}^{N-1} u_{n+1} \\
&\qquad - (I - \lambda_{N,n}B_N) \Lambda_n^{N-1} u_n\| \\
&\le |\lambda_{N,n+1} - \lambda_{N,n}| \|B_N \Lambda_{n+1}^{N-1} u_{n+1}\| \\
&\quad + \|\Lambda_{n+1}^{N-1} u_{n+1} - \Lambda_n^{N-1} u_n\| \\
&\le |\lambda_{N,n+1} - \lambda_{N,n}| \|B_N \Lambda_{n+1}^{N-1} u_{n+1}\| \\
&\quad + |\lambda_{N-1,n+1} - \lambda_{N-1,n}| \|B_{N-1} \Lambda_{n+1}^{N-2} u_{n+1}\| \\
&\quad + \|\Lambda_{n+1}^{N-2} u_{n+1} - \Lambda_n^{N-2} u_n\| \\
&\ \ \vdots \\
&\le |\lambda_{N,n+1} - \lambda_{N,n}| \|B_N \Lambda_{n+1}^{N-1} u_{n+1}\| \\
&\quad + |\lambda_{N-1,n+1} - \lambda_{N-1,n}| \|B_{N-1} \Lambda_{n+1}^{N-2} u_{n+1}\| \\
&\quad + \cdots + |\lambda_{1,n+1} - \lambda_{1,n}| \|B_1 \Lambda_{n+1}^{0} u_{n+1}\| \\
&\quad + \|\Lambda_{n+1}^{0} u_{n+1} - \Lambda_n^{0} u_n\| \\
&\le \widetilde{M}_0 \sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \|u_{n+1} - u_n\|,
\end{aligned} \tag{127}
$$

where $\sup_{n\ge1}\{\sum_{i=1}^{N} \|B_i \Lambda_{n+1}^{i-1} u_{n+1}\|\} \le \widetilde{M}_0$ for some $\widetilde{M}_0 > 0$. Also, utilizing Proposition 1(v) we deduce that

$$
\begin{aligned}
&\|u_{n+1} - u_n\| \\
&\quad = \|\Delta_{n+1}^M x_{n+1} - \Delta_n^M x_n\| \\
&\quad = \|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)} (I - r_{M,n+1}A_M) \Delta_{n+1}^{M-1} x_{n+1} \\
&\qquad - T_{r_{M,n}}^{(\Theta_M,\varphi_M)} (I - r_{M,n}A_M) \Delta_n^{M-1} x_n\| \\
&\quad \le \|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)} (I - r_{M,n+1}A_M) \Delta_{n+1}^{M-1} x_{n+1} \\
&\qquad - T_{r_{M,n}}^{(\Theta_M,\varphi_M)} (I - r_{M,n}A_M) \Delta_{n+1}^{M-1} x_{n+1}\| \\
&\qquad + \|T_{r_{M,n}}^{(\Theta_M,\varphi_M)} (I - r_{M,n}A_M) \Delta_{n+1}^{M-1} x_{n+1} \\
&\qquad\quad - T_{r_{M,n}}^{(\Theta_M,\varphi_M)} (I - r_{M,n}A_M) \Delta_n^{M-1} x_n\| \\
&\quad \le \|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)} (I - r_{M,n+1}A_M) \Delta_{n+1}^{M-1} x_{n+1} \\
&\qquad - T_{r_{M,n}}^{(\Theta_M,\varphi_M)} (I - r_{M,n+1}A_M) \Delta_{n+1}^{M-1} x_{n+1}\|
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right.\\
&\qquad\left.-T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\|\\
&+\left\|\left(I-r_{M,n}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}-\left(I-r_{M,n}A_M\right)\Delta_{n}^{M-1}x_{n}\right\|\\
&\le\frac{\left|r_{M,n+1}-r_{M,n}\right|}{r_{M,n+1}}\left\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right.\\
&\qquad\left.-\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\|\\
&+\left|r_{M,n+1}-r_{M,n}\right|\left\|A_M\Delta_{n+1}^{M-1}x_{n+1}\right\|\\
&+\left\|\Delta_{n+1}^{M-1}x_{n+1}-\Delta_{n}^{M-1}x_{n}\right\|\\
&=\left|r_{M,n+1}-r_{M,n}\right|\\
&\quad\times\left[\left\|A_M\Delta_{n+1}^{M-1}x_{n+1}\right\|\right.\\
&\qquad+\frac{1}{r_{M,n+1}}\left\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right.\\
&\qquad\left.\left.-\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\|\right]\\
&+\left\|\Delta_{n+1}^{M-1}x_{n+1}-\Delta_{n}^{M-1}x_{n}\right\|\\
&\vdots\\
&\le\left|r_{M,n+1}-r_{M,n}\right|\\
&\quad\times\left[\left\|A_M\Delta_{n+1}^{M-1}x_{n+1}\right\|\right.\\
&\qquad+\frac{1}{r_{M,n+1}}\left\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right.\\
&\qquad\left.\left.-\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\|\right]\\
&+\cdots+\left|r_{1,n+1}-r_{1,n}\right|\\
&\quad\times\left[\left\|A_1\Delta_{n+1}^{0}x_{n+1}\right\|\right.\\
&\qquad+\frac{1}{r_{1,n+1}}\left\|T_{r_{1,n+1}}^{(\Theta_1,\varphi_1)}\left(I-r_{1,n+1}A_1\right)\Delta_{n+1}^{0}x_{n+1}\right.\\
&\qquad\left.\left.-\left(I-r_{1,n+1}A_1\right)\Delta_{n+1}^{0}x_{n+1}\right\|\right]\\
&+\left\|\Delta_{n+1}^{0}x_{n+1}-\Delta_{n}^{0}x_{n}\right\|\\
&\le\widetilde{M}_1\sum_{k=1}^{M}\left|r_{k,n+1}-r_{k,n}\right|+\left\|x_{n+1}-x_n\right\|,
\end{aligned}
\tag{128}
$$

where $\widetilde{M}_1>0$ is a constant such that for each $n\ge 1$

$$
\begin{aligned}
\sum_{k=1}^{M}&\left[\left\|A_k\Delta_{n+1}^{k-1}x_{n+1}\right\|+\frac{1}{r_{k,n+1}}\right.\\
&\times\left\|T_{r_{k,n+1}}^{(\Theta_k,\varphi_k)}\left(I-r_{k,n+1}A_k\right)\Delta_{n+1}^{k-1}x_{n+1}\right.\\
&\left.\left.-\left(I-r_{k,n+1}A_k\right)\Delta_{n+1}^{k-1}x_{n+1}\right\|\right]\le\widetilde{M}_1.
\end{aligned}
\tag{129}
$$

Combining (123)–(128), we get

$$
\begin{aligned}
&\left\|z_{n+1}-z_n\right\|-\left\|x_{n+1}-x_n\right\|\\
&\le\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&\quad+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)\\
&\quad+\left\|T_{n+1}v_{n+1}-T_nv_n\right\|-\left\|x_{n+1}-x_n\right\|\\
&\le\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&\quad+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)\\
&\quad+\left\|v_{n+1}-v_n\right\|+\frac{4\widetilde{M}}{L}\left(s_{n+1}+s_n\right)-\left\|x_{n+1}-x_n\right\|\\
&\le\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&\quad+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)\\
&\quad+\widetilde{M}_0\sum_{i=1}^{N}\left|\lambda_{i,n+1}-\lambda_{i,n}\right|+\left\|u_{n+1}-u_n\right\|\\
&\quad+\frac{4\widetilde{M}}{L}\left(s_{n+1}+s_n\right)-\left\|x_{n+1}-x_n\right\|\\
&\le\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&\quad+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)\\
&\quad+\widetilde{M}_0\sum_{i=1}^{N}\left|\lambda_{i,n+1}-\lambda_{i,n}\right|+\widetilde{M}_1\sum_{k=1}^{M}\left|r_{k,n+1}-r_{k,n}\right|\\
&\quad+\left\|x_{n+1}-x_n\right\|+\frac{4\widetilde{M}}{L}\left(s_{n+1}+s_n\right)-\left\|x_{n+1}-x_n\right\|\\
&=\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&\quad+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)
\end{aligned}
$$

$$+\widetilde{M}_0\sum_{i=1}^{N}\left|\lambda_{i,n+1}-\lambda_{i,n}\right|+\widetilde{M}_1\sum_{k=1}^{M}\left|r_{k,n+1}-r_{k,n}\right| + \frac{4\widetilde{M}}{L}\left(s_{n+1}+s_n\right). \tag{130}$$

Thus, it follows from (130) and conditions (i)–(iv) that

$$\limsup_{n\to\infty}\left(\left\|z_{n+1}-z_n\right\|-\left\|x_{n+1}-x_n\right\|\right)\le 0. \tag{131}$$

Hence, by Lemma 11 we have

$$\lim_{n\to\infty}\left\|z_n-x_n\right\|=0. \tag{132}$$

Consequently,

$$\lim_{n\to\infty}\left\|x_{n+1}-x_n\right\|=\lim_{n\to\infty}\left(1-\beta_n\right)\left\|z_n-x_n\right\|=0, \tag{133}$$

and, by (126)–(128),

$$\begin{gathered}\lim_{n\to\infty}\left\|u_{n+1}-u_n\right\|=0,\\ \lim_{n\to\infty}\left\|v_{n+1}-v_n\right\|=0,\\ \lim_{n\to\infty}\left\|T_{n+1}v_{n+1}-T_nv_n\right\|=0.\end{gathered} \tag{134}$$

Step 3. Let us show that $\|A_k\Delta_n^{k-1}x_n-A_kp\|\to 0$ and $\|B_i\Lambda_n^{i-1}u_n-B_ip\|\to 0$ for all $k\in\{1,2,\ldots,M\}$ and $i\in\{1,2,\ldots,N\}$.

Indeed, since

$$x_{n+1}=s_n\gamma Vx_n+\beta_nx_n+\left(\left(1-\beta_n\right)I-s_n\mu F\right)T_nv_n, \tag{135}$$

we have

$$\begin{aligned}\left\|x_n-T_nv_n\right\|&\le\left\|x_n-x_{n+1}\right\|+\left\|x_{n+1}-T_nv_n\right\|\\&\le\left\|x_n-x_{n+1}\right\|+s_n\left\|\gamma Vx_n-\mu FT_nv_n\right\|\\&\quad+\beta_n\left\|x_n-T_nv_n\right\|;\end{aligned} \tag{136}$$

that is,

$$\begin{aligned}\left\|x_n-T_nv_n\right\|\le{}&\frac{1}{1-\beta_n}\left\|x_n-x_{n+1}\right\|\\&+\frac{s_n}{1-\beta_n}\left(\gamma\left\|Vx_n\right\|+\mu\left\|FT_nv_n\right\|\right).\end{aligned} \tag{137}$$

So, from $s_n\to 0$, $\|x_{n+1}-x_n\|\to 0$, and condition (ii), it follows that

$$\lim_{n\to\infty}\left\|x_n-T_nv_n\right\|=0. \tag{138}$$

Also, from (27) it follows that for all $i\in\{1,2,\ldots,N\}$ and $k\in\{1,2,\ldots,M\}$

$$\begin{aligned}\left\|v_n-p\right\|^2&=\left\|\Lambda_n^Nu_n-p\right\|^2\\&\le\left\|\Lambda_n^iu_n-p\right\|^2\\&=\left\|J_{R_i,\lambda_{i,n}}\left(I-\lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n\right.\\&\qquad\left.-J_{R_i,\lambda_{i,n}}\left(I-\lambda_{i,n}B_i\right)p\right\|^2\\&\le\left\|\left(I-\lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n\right.\\&\qquad\left.-\left(I-\lambda_{i,n}B_i\right)p\right\|^2\\&\le\left\|\Lambda_n^{i-1}u_n-p\right\|^2+\lambda_{i,n}\left(\lambda_{i,n}-2\eta_i\right)\\&\quad\times\left\|B_i\Lambda_n^{i-1}u_n-B_ip\right\|^2\\&\le\left\|u_n-p\right\|^2+\lambda_{i,n}\left(\lambda_{i,n}-2\eta_i\right)\\&\quad\times\left\|B_i\Lambda_n^{i-1}u_n-B_ip\right\|^2\\&\le\left\|x_n-p\right\|^2+\lambda_{i,n}\left(\lambda_{i,n}-2\eta_i\right)\\&\quad\times\left\|B_i\Lambda_n^{i-1}u_n-B_ip\right\|^2,\\ \left\|u_n-p\right\|^2&=\left\|\Delta_n^Mx_n-p\right\|^2\\&\le\left\|\Delta_n^kx_n-p\right\|^2\\&=\left\|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}\left(I-r_{k,n}A_k\right)\Delta_n^{k-1}x_n\right.\\&\qquad\left.-T_{r_{k,n}}^{(\Theta_k,\varphi_k)}\left(I-r_{k,n}A_k\right)p\right\|^2\\&\le\left\|\left(I-r_{k,n}A_k\right)\Delta_n^{k-1}x_n-\left(I-r_{k,n}A_k\right)p\right\|^2\\&\le\left\|\Delta_n^{k-1}x_n-p\right\|^2+r_{k,n}\left(r_{k,n}-2\mu_k\right)\\&\quad\times\left\|A_k\Delta_n^{k-1}x_n-A_kp\right\|^2\\&\le\left\|x_n-p\right\|^2+r_{k,n}\left(r_{k,n}-2\mu_k\right)\\&\quad\times\left\|A_k\Delta_n^{k-1}x_n-A_kp\right\|^2.\end{aligned} \tag{139}$$

Furthermore, utilizing Lemma 7, we deduce from (113) that

$$\begin{aligned}&\left\|x_{n+1}-p\right\|^2\\&\quad=\left\|s_n\left(\gamma Vx_n-\mu Fp\right)+\beta_n\left(x_n-T_nv_n\right)\right.\\&\qquad\left.+\left(I-s_n\mu F\right)T_nv_n-\left(I-s_n\mu F\right)p\right\|^2\\&\quad\le\left\|\beta_n\left(x_n-T_nv_n\right)+\left(I-s_n\mu F\right)T_nv_n-\left(I-s_n\mu F\right)p\right\|^2\\&\qquad+2s_n\left\langle\gamma Vx_n-\mu Fp,x_{n+1}-p\right\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq \left[\beta_n \|x_n - T_n v_n\| + \|(I - s_n\mu F) T_n v_n - (I - s_n\mu F) T_n p\|\right]^2 \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\leq \left[\beta_n \|x_n - T_n v_n\| + (1 - s_n\tau) \|v_n - p\|\right]^2 \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&= (1 - s_n\tau)^2 \|v_n - p\|^2 + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau) \beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\|.
\end{aligned}
\tag{140}
$$

From (139)–(140), it follows that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|v_n - p\|^2 + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau) \beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\leq \|u_n - p\|^2 + \lambda_{i,n} (\lambda_{i,n} - 2\eta_i) \\
&\quad \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\|^2 + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau) \beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\leq \|x_n - p\|^2 + r_{k,n} (r_{k,n} - 2\mu_k) \\
&\quad \times \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\|^2 \\
&\quad + \lambda_{i,n} (\lambda_{i,n} - 2\eta_i) \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\|^2 \\
&\quad + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau) \beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\|,
\end{aligned}
\tag{141}
$$

and so

$$
\begin{aligned}
&r_{k,n} (2\mu_k - r_{k,n}) \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\|^2 \\
&\quad + \lambda_{i,n} (2\eta_i - \lambda_{i,n}) \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\|^2 \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau) \beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|) \\
&\quad \times \|x_n - x_{n+1}\| + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau) \beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\|.
\end{aligned}
\tag{142}
$$

Since $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$, by (133), (138), and (142) we conclude immediately that

$$
\begin{aligned}
\lim_{n\to\infty} \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| &= 0, \\
\lim_{n\to\infty} \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| &= 0,
\end{aligned}
\tag{143}
$$

for all $k \in \{1, 2, \ldots, M\}$ and $i \in \{1, 2, \ldots, N\}$.

Step 4. Let us show that $\|x_n - u_n\| \to 0$, $\|u_n - v_n\| \to 0$, and $\|v_n - T_n v_n\| \to 0$ as $n \to \infty$.

Indeed, by Proposition 1(iii) we obtain that for each $k \in \{1, 2, \ldots, M\}$

$$
\begin{aligned}
&\left\|\Delta_n^k x_n - p\right\|^2 \\
&\quad = \left\|T_{r_{k,n}}^{(\Theta_k, \varphi_k)} (I - r_{k,n} A_k) \Delta_n^{k-1} x_n - T_{r_{k,n}}^{(\Theta_k, \varphi_k)} (I - r_{k,n} A_k) p\right\|^2 \\
&\quad \leq \left\langle (I - r_{k,n} A_k) \Delta_n^{k-1} x_n - (I - r_{k,n} A_k) p, \Delta_n^k x_n - p \right\rangle \\
&\quad = \frac{1}{2} \left(\left\|(I - r_{k,n} A_k) \Delta_n^{k-1} x_n - (I - r_{k,n} A_k) p\right\|^2 \right. \\
&\qquad + \left\|\Delta_n^k x_n - p\right\|^2 - \left\|(I - r_{k,n} A_k) \Delta_n^{k-1} x_n \right. \\
&\qquad\qquad - (I - r_{k,n} A_k) p \\
&\qquad\qquad \left. \left. - \left(\Delta_n^k x_n - p\right)\right\|^2 \right) \\
&\quad \leq \frac{1}{2} \left(\left\|\Delta_n^{k-1} x_n - p\right\|^2 + \left\|\Delta_n^k x_n - p\right\|^2 \right. \\
&\qquad \left. - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n - r_{k,n} \left(A_k \Delta_n^{k-1} x_n - A_k p\right)\right\|^2 \right),
\end{aligned}
\tag{144}
$$

which implies that

$$
\begin{aligned}
\left\|\Delta_n^k x_n - p\right\|^2 &\leq \left\|\Delta_n^{k-1} x_n - p\right\|^2 \\
&\quad - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n \right. \\
&\qquad \left. - r_{k,n} \left(A_k \Delta_n^{k-1} x_n - A_k p\right)\right\|^2 \\
&= \left\|\Delta_n^{k-1} x_n - p\right\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\quad - r_{k,n}^2 \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\|^2 \\
&\quad + 2r_{k,n} \left\langle \Delta_n^{k-1} x_n - \Delta_n^k x_n, A_k \Delta_n^{k-1} x_n - A_k p \right\rangle \\
&\leq \left\|\Delta_n^{k-1} x_n - p\right\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| \\
&\leq \|x_n - p\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\|.
\end{aligned}
\tag{145}
$$

Also, by Lemma 14, we obtain that for each $i \in \{1, 2, \ldots, N\}$

$$\begin{aligned}
&\left\|\Lambda_n^i u_n - p\right\|^2 \\
&\quad = \left\|J_{R_i,\lambda_{i,n}}\left(I - \lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n - J_{R_i,\lambda_{i,n}}\left(I - \lambda_{i,n}B_i\right)p\right\|^2 \\
&\quad \leq \left\langle \left(I - \lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n - \left(I - \lambda_{i,n}B_i\right)p, \Lambda_n^i u_n - p\right\rangle \\
&\quad = \frac{1}{2}\left(\left\|\left(I - \lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n - \left(I - \lambda_{i,n}B_i\right)p\right\|^2\right. \\
&\qquad + \left\|\Lambda_n^i u_n - p\right\|^2 - \left\|\left(I - \lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n\right. \\
&\qquad - \left(I - \lambda_{i,n}B_i\right)p \\
&\qquad \left.\left. - \left(\Lambda_n^i u_n - p\right)\right\|^2\right) \\
&\quad \leq \frac{1}{2}\left(\left\|\Lambda_n^{i-1}u_n - p\right\|^2 + \left\|\Lambda_n^i u_n - p\right\|^2\right. \\
&\qquad \left. - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}\left(B_i\Lambda_n^{i-1}u_n - B_i p\right)\right\|^2\right) \\
&\quad \leq \frac{1}{2}\left(\left\|u_n - p\right\|^2 + \left\|\Lambda_n^i u_n - p\right\|^2\right. \\
&\qquad \left. - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}\left(B_i\Lambda_n^{i-1}u_n - B_i p\right)\right\|^2\right),
\end{aligned} \tag{146}$$

which implies

$$\begin{aligned}
\left\|\Lambda_n^i u_n - p\right\|^2 &\leq \left\|u_n - p\right\|^2 \\
&\quad - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}\left(B_i\Lambda_n^{i-1}u_n - B_i p\right)\right\|^2 \\
&= \left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2 \\
&\quad - \lambda_{i,n}^2\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2 \\
&\quad + 2\lambda_{i,n}\left\langle \Lambda_n^{i-1}u_n - \Lambda_n^i u_n, B_i\Lambda_n^{i-1}u_n - B_i p\right\rangle \\
&\leq \left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2 \\
&\quad + 2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|.
\end{aligned} \tag{147}$$

Thus, from (140), (145), and (147), we have

$$\begin{aligned}
&\left\|x_{n+1} - p\right\|^2 \\
&\quad \leq \left(1 - s_n\tau\right)^2\left\|v_n - p\right\|^2 + \beta_n^2\left\|x_n - T_n v_n\right\|^2 \\
&\qquad + 2\left(1 - s_n\tau\right)\beta_n\left\|v_n - p\right\|\left\|x_n - T_n v_n\right\| \\
&\qquad + 2s_n\left\|\gamma V x_n - \mu F p\right\|\left\|x_{n+1} - p\right\| \\
&\quad = \left(1 - s_n\tau\right)^2\left\|\Lambda_n^N u_n - p\right\|^2 + \beta_n^2\left\|x_n - T_n v_n\right\|^2 \\
&\qquad + 2\left(1 - s_n\tau\right)\beta_n\left\|v_n - p\right\|\left\|x_n - T_n v_n\right\| \\
&\qquad + 2s_n\left\|\gamma V x_n - \mu F p\right\|\left\|x_{n+1} - p\right\| \\
&\quad \leq \left(1 - s_n\tau\right)^2\left\|\Lambda_n^i u_n - p\right\|^2 + \beta_n^2\left\|x_n - T_n v_n\right\|^2 \\
&\qquad + 2\left(1 - s_n\tau\right)\beta_n\left\|v_n - p\right\|\left\|x_n - T_n v_n\right\| \\
&\qquad + 2s_n\left\|\gamma V x_n - \mu F p\right\|\left\|x_{n+1} - p\right\| \\
&\quad \leq \left(1 - s_n\tau\right)^2\left[\left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2\right. \\
&\qquad + 2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| \\
&\qquad \left.\times\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|\right] \\
&\qquad + \beta_n^2\left\|x_n - T_n v_n\right\|^2 + 2\left(1 - s_n\tau\right)\beta_n\left\|v_n - p\right\| \\
&\qquad \times\left\|x_n - T_n v_n\right\| + 2s_n\left\|\gamma V x_n - \mu F p\right\|\left\|x_{n+1} - p\right\| \\
&\quad = \left(1 - s_n\tau\right)^2\left[\left\|\Delta_n^M x_n - p\right\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2\right. \\
&\qquad + 2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| \\
&\qquad \left.\times\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|\right] \\
&\qquad + \beta_n^2\left\|x_n - T_n v_n\right\|^2 + 2\left(1 - s_n\tau\right)\beta_n\left\|v_n - p\right\| \\
&\qquad \times\left\|x_n - T_n v_n\right\| + 2s_n\left\|\gamma V x_n - \mu F p\right\|\left\|x_{n+1} - p\right\| \\
&\quad \leq \left(1 - s_n\tau\right)^2\left[\left\|\Delta_n^k x_n - p\right\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2\right. \\
&\qquad + 2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| \\
&\qquad \left.\times\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|\right] \\
&\qquad + \beta_n^2\left\|x_n - T_n v_n\right\|^2 + 2\left(1 - s_n\tau\right)\beta_n\left\|v_n - p\right\| \\
&\qquad \times\left\|x_n - T_n v_n\right\| + 2s_n\left\|\gamma V x_n - \mu F p\right\|\left\|x_{n+1} - p\right\| \\
&\quad \leq \left(1 - s_n\tau\right)^2\left[\left\|x_n - p\right\|^2 - \left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|^2\right. \\
&\qquad + 2r_{k,n}\left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\| \\
&\qquad \times\left\|A_k\Delta_n^{k-1}x_n - A_k p\right\| \\
&\qquad - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| \\
&\qquad \left.\times\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|\right] \\
&\qquad + \beta_n^2\left\|x_n - T_n v_n\right\|^2 \\
&\qquad + 2\left(1 - s_n\tau\right)\beta_n\left\|v_n - p\right\|\left\|x_n - T_n v_n\right\| \\
&\qquad + 2s_n\left\|\gamma V x_n - \mu F p\right\|\left\|x_{n+1} - p\right\| \\
&\quad \leq \left\|x_n - p\right\|^2 - \left(1 - s_n\tau\right)^2\left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|^2 \\
&\qquad + 2r_{k,n}\left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|\left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|
\end{aligned}$$

$$
\begin{aligned}
&- (1 - s_n\tau)^2 \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&+ 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \\
&+ \beta_n^2 \left\|x_n - T_n v_n\right\|^2 + 2(1 - s_n\tau)\beta_n \left\|v_n - p\right\| \\
&\times \left\|x_n - T_n v_n\right\| + 2s_n \left\|\gamma V x_n - \mu F p\right\| \left\|x_{n+1} - p\right\|;
\end{aligned} \tag{148}
$$

that is,

$$
\begin{aligned}
&(1 - s_n\tau)^2 \left[\left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 + \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2\right] \\
&\le \left\|x_n - p\right\|^2 - \left\|x_{n+1} - p\right\|^2 \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| \\
&\quad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \\
&\quad + \beta_n^2 \left\|x_n - T_n v_n\right\|^2 \\
&\quad + 2(1 - s_n\tau)\beta_n \left\|v_n - p\right\| \left\|x_n - T_n v_n\right\| \\
&\quad + 2s_n \left\|\gamma V x_n - \mu F p\right\| \left\|x_{n+1} - p\right\| \\
&\le \left(\left\|x_n - p\right\| + \left\|x_{n+1} - p\right\|\right) \left\|x_n - x_{n+1}\right\| \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| \\
&\quad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \\
&\quad + \beta_n^2 \left\|x_n - T_n v_n\right\|^2 + 2(1 - s_n\tau)\beta_n \left\|v_n - p\right\| \\
&\quad \times \left\|x_n - T_n v_n\right\| + 2s_n \left\|\gamma V x_n - \mu F p\right\| \left\|x_{n+1} - p\right\|.
\end{aligned} \tag{149}
$$

So, from $s_n \to 0$, (133), (138), and (143), we immediately get

$$
\lim_{n\to\infty} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| = 0, \qquad \lim_{n\to\infty} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| = 0, \tag{150}
$$

for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$. Note that

$$
\begin{aligned}
\left\|x_n - u_n\right\| &= \left\|\Delta_n^0 x_n - \Delta_n^M x_n\right\| \\
&\le \left\|\Delta_n^0 x_n - \Delta_n^1 x_n\right\| + \left\|\Delta_n^1 x_n - \Delta_n^2 x_n\right\| \\
&\quad + \cdots + \left\|\Delta_n^{M-1} x_n - \Delta_n^M x_n\right\|, \\
\left\|u_n - v_n\right\| &= \left\|\Lambda_n^0 u_n - \Lambda_n^N u_n\right\| \\
&\le \left\|\Lambda_n^0 u_n - \Lambda_n^1 u_n\right\| + \left\|\Lambda_n^1 u_n - \Lambda_n^2 u_n\right\| \\
&\quad + \cdots + \left\|\Lambda_n^{N-1} u_n - \Lambda_n^N u_n\right\|.
\end{aligned} \tag{151}
$$

Thus, from (150) we have

$$
\lim_{n\to\infty} \left\|x_n - u_n\right\| = 0, \qquad \lim_{n\to\infty} \left\|u_n - v_n\right\| = 0. \tag{152}
$$

It is easy to see that as $n \to \infty$

$$
\left\|x_n - v_n\right\| \le \left\|x_n - u_n\right\| + \left\|u_n - v_n\right\| \longrightarrow 0. \tag{153}
$$

Also, observe that

$$
\left\|T_n v_n - v_n\right\| \le \left\|T_n v_n - x_n\right\| + \left\|x_n - v_n\right\|. \tag{154}
$$

Hence, we have from (138)

$$
\lim_{n\to\infty} \left\|T_n v_n - v_n\right\| = 0. \tag{155}
$$

Step 5. Let us show that $\limsup_{n\to\infty} \langle(\mu F - \gamma V)q, q - x_n\rangle \le 0$, where $q \in \Omega$ is the same as in Theorem 18; that is, $q \in \Omega$ is a unique solution of VIP (50). To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$
\begin{aligned}
&\limsup_{n\to\infty} \langle(\mu F - \gamma V) q, q - x_n\rangle \\
&\quad = \lim_{i\to\infty} \left\langle(\mu F - \gamma V) q, q - x_{n_i}\right\rangle.
\end{aligned} \tag{156}
$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup w$. From Step 4, we have that $\Delta_{n_i}^k x_{n_i} \rightharpoonup w$, $\Lambda_{n_i}^m u_{n_i} \rightharpoonup w$, $u_{n_i} \rightharpoonup w$, and $v_{n_i} \rightharpoonup w$, where $k \in \{1, 2, \ldots, M\}$, $m \in \{1, 2, \ldots, N\}$. Since $v_n - T_n v_n \to 0$ by Step 4, by the same arguments as in the proof of Theorem 18, we get $w \in \Omega$. Since $q = P_\Omega(I - \mu F + \gamma V)q$, it follows that

$$
\begin{aligned}
\limsup_{n\to\infty} \langle(\mu F - \gamma V) q, q - x_n\rangle &= \lim_{i\to\infty} \left\langle(\mu F - \gamma V) q, q - x_{n_i}\right\rangle \\
&= \langle(\mu F - \gamma V) q, q - w\rangle \le 0.
\end{aligned} \tag{157}
$$

Step 6. Let us show that $\lim_{n\to\infty}\|x_n - q\| = 0$, where $q \in \Omega$ is the same as in Theorem 18; that is, $q \in \Omega$ is a unique solution of VIP (50). From (113), we know that

$$
\begin{aligned}
x_{n+1} - q = {} & s_n(\gamma V x_n - \mu F q) + \beta_n(T_n v_n - q) \\
& + ((1 - \beta_n) I - s_n\mu F) T_n v_n \\
& - ((1 - \beta_n) I - s_n\mu F) q.
\end{aligned} \tag{158}
$$

Applying Lemmas 7 and 13 and noticing $T_n q = q$ and $\|v_n - q\| \le \|x_n - q\|$ for all $n \ge 1$, we have

$$
\begin{aligned}
&\left\|x_{n+1} - q\right\|^2 \\
&\quad \le \left\|\beta_n(T_n v_n - q) + ((1 - \beta_n) I - s_n\mu F) T_n v_n\right. \\
&\qquad \left. - ((1 - \beta_n) I - s_n\mu F) q\right\|^2 \\
&\qquad + 2s_n \langle\gamma V x_n - \mu F q, x_{n+1} - q\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq [\beta_n \|T_n v_n - q\| + \|((1-\beta_n) I - s_n\mu F) T_n v_n \\
&\qquad - ((1-\beta_n) I - s_n \mu F) q\|]^2 \\
&\quad + 2s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q\rangle \\
&\quad + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\
&= \Big[\beta_n \|T_n v_n - q\| + (1-\beta_n) \\
&\qquad \times \Big\| \Big(I - \frac{s_n}{1-\beta_n}\mu F\Big) T_n v_n - \Big(I - \frac{s_n}{1-\beta_n}\mu F\Big) T_n q \Big\|\Big]^2 \\
&\quad + 2s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q\rangle \\
&\quad + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\
&\leq \Big[\beta_n \|v_n - q\| + (1-\beta_n)\Big(1 - \frac{s_n\tau}{1-\beta_n}\Big)\|v_n - q\|^2\Big]^2 \\
&\quad + 2s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q\rangle \\
&\quad + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\
&= [\beta_n \|v_n - q\| + (1-\beta_n - s_n\tau)\|v_n - q\|]^2 \\
&\quad + 2s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q\rangle \\
&\quad + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\
&\leq [\beta_n \|x_n - q\| + (1-\beta_n - s_n\tau)\|x_n - q\|]^2 \\
&\quad + 2s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q\rangle \\
&\quad + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\
&\leq (1 - s_n\tau)^2 \|x_n - q\|^2 + 2 s_n \gamma l \|x_n - q\| \\
&\quad \times \|x_{n+1} - q\| + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\
&\leq (1 - s_n\tau)^2 \|x_n - q\|^2 + s_n \gamma l \left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\right) \\
&\quad + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle .
\end{aligned}
\tag{159}
$$

This implies that

$$
\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \leq \frac{1 - 2\tau s_n + \tau^2 s_n^2 + s_n\gamma l}{1 - s_n\gamma l}\|x_n - q\|^2 \\
&\qquad + \frac{2s_n}{1 - s_n\gamma l}\langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\
&\quad = \left(1 - \frac{2(\tau - \gamma l) s_n}{1 - s_n\gamma l}\right)\|x_n - q\|^2 + \frac{\tau^2 s_n^2}{1 - s_n\gamma l}\|x_n - q\|^2 \\
&\qquad + \frac{2s_n}{1 - s_n\gamma l}\langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\
&\quad \leq \left(1 - \frac{2(\tau - \gamma l)}{1 - s_n\gamma l} s_n\right)\|x_n - q\|^2 + \frac{2(\tau - \gamma l) s_n}{1 - s_n\gamma l} \\
&\qquad \times \left(\frac{\tau^2 s_n}{2(\tau - \gamma l)}\widetilde{M}_2 \right.\\
&\qquad\qquad \left. + \frac{1}{\tau - \gamma l}\langle \mu F q - \gamma V q, q - x_{n+1}\rangle\right) \\
&\quad = (1 - \sigma_n)\|x_n - q\|^2 + \sigma_n \delta_n,
\end{aligned}
\tag{160}
$$

where $\widetilde{M}_2 = \sup_{n\geq 1}\|x_n - q\|^2$, $\sigma_n = (2(\tau - \gamma l)/(1 - s_n\gamma l)) s_n$, and

$$
\delta_n = \frac{\tau^2 s_n}{2(\tau - \gamma l)}\widetilde{M}_2 + \frac{1}{\tau - \gamma l}\langle \mu F q - \gamma V q, q - x_{n+1}\rangle . \tag{161}
$$

From condition (i) and Step 5, it is easy to see that $\sigma_n \to 0$, $\sum_{n=0}^{\infty}\sigma_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$. Hence, by Lemma 10, we conclude that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 22. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B_i : C \to H$ be ζ-inverse-strongly monotone and η_i-inverse-strongly monotone, respectively, for $i = 1, 2$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := GMEP(\Theta, \varphi, A) \cap I(B_1, R_1) \cap I(B_2, R_2) \cap \Gamma \neq \emptyset$ and that either (B1) or (B2) holds. For arbitrarily given $x_1 \in H$, let $\{x_n\}$ be a sequence generated by*

$$
\begin{gathered}
\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle A x_n, y - u_n\rangle \\
+ \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
v_n = J_{R_2,\lambda_{2,n}}(I - \lambda_{2,n}B_2) J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1) u_n, \\
x_{n+1} = s_n\gamma V x_n + \beta_n x_n + ((1-\beta_n) I - s_n\mu F) T_n v_n, \\
\forall n \geq 1,
\end{gathered}
\tag{162}
$$

where $P_C(I - \lambda_n\nabla f) = s_n I + (1 - s_n) T_n$ (here T_n is nonexpansive and $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$). Assume that the following conditions hold:

(i) *$s_n \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$, and $\lim_{n\to\infty} s_n = 0$ ($\Leftrightarrow \lim_{n\to\infty}\lambda_n = 2/L$);*

(ii) *$\{\beta_n\} \subset (0, 1)$ and $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;*

(iii) *$\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\lim_{n\to\infty}|\lambda_{i,n+1} - \lambda_{i,n}| = 0$ for $i = 1, 2$;*

(iv) *$\{r_n\} \subset [e, f] \subset (0, 2\zeta)$ and $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.*

Then $\{x_n\}$ *converges strongly as* $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) *to a point* $q \in \Omega$*, which is a unique solution of VIP* (110).

Corollary 23. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $f : C \to \mathbf{R}$ *be a convex functional with L-Lipschitz continuous gradient* ∇f*. Let* Θ *be a bifunction from* $C\times C$ *to* $\mathbf{R}$ *satisfying (A1)–(A4) and let* $\varphi : C \to \mathbf{R}\cup\{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* $R : C \to 2^H$ *be a maximal monotone mapping and let* $A : H \to H$ *and* $B : C \to H$ *be* ζ*-inverse-strongly monotone and* ξ*-inverse-strongly monotone, respectively. Let* $F : H \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with positive constants* $\kappa, \eta > 0$*. Let* $V : H \to H$ *be an* l*-Lipschitzian mapping with constant* $l \geq 0$*. Let* $0 < \mu < 2\eta/\kappa^2$ *and* $0 \leq \gamma l < \tau$*, where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$*. Assume that* $\Omega := GMEP(\Theta, \varphi, A) \cap I(B, R) \cap \Gamma \neq \emptyset$ *and that either (B1) or (B2) holds. For arbitrarily given* $x_1 \in H$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} &\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle \\ &\qquad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ &x_{n+1} = s_n\gamma Vx_n + \beta_n x_n \\ &\qquad +((1 - \beta_n) I - s_n\mu F)T_n J_{R,\rho_n}(u_n - \rho_n Bu_n), \quad \forall n \geq 1, \end{aligned} \tag{163}$$

where $P_C(I - \lambda_n\nabla f) = s_n I + (1 - s_n)T_n$ *(here* T_n *is nonexpansive and* $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ *for each* $\lambda_n \in (0, 2/L)$*). Assume that the following conditions hold:*

(i) $s_n \in (0, 1/2)$ *for each* $\lambda_n \in (0, 2/L)$*, and* $\lim_{n\to\infty} s_n = 0$ ($\Leftrightarrow \lim_{n\to\infty}\lambda_n = 2/L$)*;*

(ii) $\{\beta_n\} \subset (0, 1)$ *and* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$*;*

(iii) $\{\rho_n\} \subset [a, b] \subset (0, 2\xi)$ *and* $\lim_{n\to\infty}|\rho_{n+1} - \rho_n| = 0$*;*

(iv) $\{r_n\} \subset [e, f] \subset (0, 2\zeta)$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ *converges strongly as* $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) *to a point* $q \in \Omega$*, which is a unique solution of VIP* (112).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

In this research, the first author was supported by King Fahd University of Petroleum and Minerals Project no. IN111015; he also is grateful to the university for providing excellent research facilities. This research was partially supported by the National Science Foundation of China (11071169), Innovation Program of Shanghai Municipal Education Commission (09ZZ133), and Ph.D. Program Foundation of Ministry of Education of China (20123127110002).

References

[1] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[2] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Lineaires*, Dunod, Paris, France, 1969.

[3] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer, New York, NY, USA, 1984.

[4] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.

[5] J. T. Oden, *Quantitative Methods on Nonlinear Mechanics*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1986.

[6] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, Springer, New York, NY, USA, 1985.

[7] G. M. Korpelevič, "An extragradient method for finding saddle points and for other problems," *Èkonomika i Matematicheskie Metody*, vol. 12, no. 4, pp. 747–756, 1976.

[8] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, vol. 1-2, Springer, New York, NY, USA, 2003.

[9] N. Nadezhkina and W. Takahashi, "Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings," *Journal of Optimization Theory and Applications*, vol. 128, no. 1, pp. 191–201, 2006.

[10] L.-C. Zeng and J.-C. Yao, "Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems," *Taiwanese Journal of Mathematics*, vol. 10, no. 5, pp. 1293–1303, 2006.

[11] Y. Yao, Y. J. Cho, and Y.-C. Liou, "Algorithms of common solutions for variational inclusions, mixed equilibrium problems and fixed point problems," *European Journal of Operational Research*, vol. 212, no. 2, pp. 242–250, 2011.

[12] J.-W. Peng and J.-C. Yao, "A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems," *Taiwanese Journal of Mathematics*, vol. 12, no. 6, pp. 1401–1432, 2008.

[13] L.-C. Ceng, C.-Y. Wang, and J.-C. Yao, "Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities," *Mathematical Methods of Operations Research*, vol. 67, no. 3, pp. 375–390, 2008.

[14] L.-C. Ceng, Q. H. Ansari, and S. Schaible, "Hybrid extragradient-like methods for generalized mixed equilibrium problems, systems of generalized equilibrium problems and optimization problems," *Journal of Global Optimization*, vol. 53, no. 1, pp. 69–96, 2012.

[15] L.-C. Ceng, Q. H. Ansari, M. M. Wong, and J.-C. Yao, "Mann type hybrid extragradient method for variational inequalities, variational inclusions and fixed point problems," *Fixed Point Theory*, vol. 13, no. 2, pp. 403–422, 2012.

[16] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "An extragradient method for solving split feasibility and fixed point problems," *Computers & Mathematics with Applications*, vol. 64, no. 4, pp. 633–642, 2012.

[17] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Relaxed extragradient methods for finding minimum-norm solutions of the split feasibility problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 4, pp. 2116–2125, 2012.

[18] L.-C. Ceng, N. Hadjisavvas, and N.-C. Wong, "Strong convergence theorem by a hybrid extragradient-like approximation

method for variational inequalities and fixed point problems," *Journal of Global Optimization*, vol. 46, no. 4, pp. 635–646, 2010.

[19] L. C. Ceng, M. Teboulle, and J. C. Yao, "Weak convergence of an iterative method for pseudomonotone variational inequalities and fixed-point problems," *Journal of Optimization Theory and Applications*, vol. 146, no. 1, pp. 19–31, 2010.

[20] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Relaxed extragradient iterative methods for variational inequalities," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 1112–1123, 2011.

[21] L.-C. Ceng, Q. H. Ansari, N.-C. Wong, and J.-C. Yao, "An extragradient-like approximation method for variational inequalities and fixed point problems," *Fixed Point Theory and Applications*, vol. 2011, article 22s, 18 pages, 2011.

[22] L.-C. Ceng, S.-M. Guu, and J.-C. Yao, "Finding common solutions of a variational inequality, a general system of variational inequalities, and a fixed-point problem via a hybrid extragradient method," *Fixed Point Theory and Applications*, Article ID 626159, 22 pages, 2011.

[23] L.-C. Ceng and A. Petruşel, "Relaxed extragradient-like method for general system of generalized mixed equilibria and fixed point problem," *Taiwanese Journal of Mathematics*, vol. 16, no. 2, pp. 445–478, 2012.

[24] Y. Yao, Y.-C. Liou, and S. M. Kang, "Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3472–3480, 2010.

[25] L. C. Ceng, H.-Y. Hu, and M. M. Wong, "Strong and weak convergence theorems for generalized mixed equilibrium problem with perturbation and fixed pointed problem of infinitely many nonexpansive mappings," *Taiwanese Journal of Mathematics*, vol. 15, no. 3, pp. 1341–1367, 2011.

[26] L.-C. Ceng and J.-C. Yao, "A hybrid iterative scheme for mixed equilibrium problems and fixed point problems," *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 186–201, 2008.

[27] L.-C. Ceng, S.-M. Guu, and J.-C. Yao, "Hybrid iterative method for finding common solutions of generalized mixed equilibrium and fixed point problems," *Fixed Point Theory and Applications*, vol. 2012, article 92, 19 pages, 2012.

[28] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 3, pp. 1025–1033, 2008.

[29] L. C. Ceng, A. Petruşel, and J. C. Yao, "Iterative approaches to solving equilibrium problems and fixed point problems of infinitely many nonexpansive mappings," *Journal of Optimization Theory and Applications*, vol. 143, no. 1, pp. 37–58, 2009.

[30] I. Yamada, "The hybrid steepest-descent method for the variational inequality problems over the intersection of the fixed-point sets of nonexpansive mappings," in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, D. Batnariu, Y. Censor, and S. Reich, Eds., pp. 473–504, North-Holland, Amsterdam, The Netherlands, 2001.

[31] V. Colao, G. Marino, and H.-K. Xu, "An iterative method for finding common solutions of equilibrium and fixed point problems," *Journal of Mathematical Analysis and Applications*, vol. 344, no. 1, pp. 340–352, 2008.

[32] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976.

[33] N.-J. Huang, "A new completely general class of variational inclusions with noncompact valued mappings," *Computers & Mathematics with Applications*, vol. 35, no. 10, pp. 9–14, 1998.

[34] L.-C. Zeng, S.-M. Guu, and J.-C. Yao, "Characterization of H-monotone operators with applications to variational inclusions," *Computers & Mathematics with Applications*, vol. 50, no. 3-4, pp. 329–337, 2005.

[35] L. C. Ceng, S. M. Guu, and J. C. Yao, "Weak convergence theorem by a modified extragradient method for variational inclusions, variational inequalities and fixed point problems," *Journal of Nonlinear and Convex Analysis*, vol. 14, pp. 21–31, 2013.

[36] Y.-P. Fang and N.-J. Huang, "H-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces," *Applied Mathematics Letters*, vol. 17, no. 6, pp. 647–653, 2004.

[37] S.-S. Zhang, J. H. W. Lee, and C. K. Chan, "Algorithms of common solutions to quasi variational inclusion and fixed point problems," *Applied Mathematics and Mechanics (English Edition)*, vol. 29, no. 5, pp. 571–581, 2008.

[38] J.-W. Peng, Y. Wang, D. S. Shyu, and J.-C. Yao, "Common solutions of an iterative scheme for variational inclusions, equilibrium problems, and fixed point problems," *Journal of Inequalities and Applications*, vol. 2008, Article ID 720371, 15 pages, 2008.

[39] L.-C. Ceng, Q. H. Ansari, M. M. Wong, and J.-C. Yao, "Mann type hybrid extragradient method for variational inequalities, variational inclusions and fixed point problems," *Fixed Point Theory*, vol. 13, no. 2, pp. 403–422, 2012.

[40] H.-K. Xu, "Averaged mappings and the gradient-projection algorithm," *Journal of Optimization Theory and Applications*, vol. 150, no. 2, pp. 360–378, 2011.

[41] J.-B. Baillon and G. Haddad, "Quelques propriétés des opérateurs angle-bornés et n-cycliquement monotones," *Israel Journal of Mathematics*, vol. 26, no. 2, pp. 137–150, 1977.

[42] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.

[43] P. L. Combettes, "Solving monotone inclusions via compositions of nonexpansive averaged operators," *Optimization*, vol. 53, no. 5-6, pp. 475–504, 2004.

[44] K. Geobel and W. A. Kirk, *Topics on Metric Fixed-Point Theory*, Cambridge University Press, Cambridge, UK, 1990.

[45] H. K. Xu and T. H. Kim, "Convergence of hybrid steepest-descent methods for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 119, no. 1, pp. 185–201, 2003.

[46] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 178053, 9 pages
http://dx.doi.org/10.1155/2013/178053

Research Article

A New Iterative Method for Equilibrium Problems and Fixed Point Problems

Abdul Latif[1] and Mohammad Eslamian[2]

[1] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[2] *Department of Mathematics, Mazandaran University of Science and Technology, Behshahr, Iran*

Correspondence should be addressed to Abdul Latif; alatif@kau.edu.sa

Received 31 October 2013; Accepted 15 December 2013

Academic Editor: Ljubomir B. Ćirić

Introducing a new iterative method, we study the existence of a common element of the set of solutions of equilibrium problems for a family of monotone, Lipschitz-type continuous mappings and the sets of fixed points of two nonexpansive semigroups in a real Hilbert space. We establish strong convergence theorems of the new iterative method for the solution of the variational inequality problem which is the optimality condition for the minimization problem. Our results improve and generalize the corresponding recent results of Anh (2012), Cianciaruso et al. (2010), and many others.

"Dedicated to Professor Miodrag Mateljevi'c on the occasion of his 65th birthday"

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H, and let Proj_C be a nearest point projection of H into C; that is, for $x \in H$, $\mathrm{Proj}_C x$ is the unique point in C with the property $\|x - \mathrm{Proj}_C x\| := \inf\{\|x - y\| : y \in C\}$. It is well known that $y = \mathrm{Proj}_C x$ iff $\langle x - y, y - z\rangle \geq 0$ for all $z \in C$.

Let f be a bifunction from $C \times C$ into $\mathbb{R}$, such that $f(x, x) = 0$ for all $x \in C$. Consider the Fan inequality [1]: find a point $x^\star \in C$ such that

$$f(x^\star, y) \geq 0, \quad \forall y \in C, \tag{1}$$

where $f(x,\cdot)$ is convex and subdifferentiable on C for every $x \in C$. The set of solutions of this problem is denoted by $\mathrm{Sol}(f, C)$. In fact, the Fan inequality can be formulated as an equilibrium problem. Such problems arise frequently in mathematics, physics, engineering, game theory, transportation, economics, and network. Due to importance of the solutions of such problems, many researchers are working in this area and studying the existence of the solutions of such problems; for example, see, [2–4]. Further, if $f(x, y) = \langle Fx, y - x\rangle$ for every $x, y \in C$, where F is a mapping from C into H, then the Fan inequality problem (equilibrium problem) becomes the classical variational inequality problem which is formulated as finding a point $x^\star \in C$ such that

$$\langle Fx^\star, y - x^\star\rangle \geq \quad \forall y \in C. \tag{2}$$

Variational inequalities were introduced and studied by Stampacchia [5]. It is well known that this area covers many branches of mathematics, such as partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance; see [6–11].

Here we recall some useful notions.

A mapping $T : C \to H$ is said to be *L-Lipschitz* on C if there exists a constant $L \geq 0$ such that for each $x, y \in C$,

$$\|Tx - Ty\| \leq L\|x - y\|. \tag{3}$$

In particular, if $L \in [0, 1[$, then T is called a *contraction* on C; if $L = 1$, then T is called a *nonexpansive* mapping on C. The set of fixed points of T is denoted by $F(T)$.

A family $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ of mappings on a closed convex subset C of a Hilbert space H is called a *nonexpansive semigroup* if it satisfies the following: (i) $T(0)x = x$ for all $x \in C$; (ii) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$; (iii) for all $x \in C$,

$s \to T(s)x$ is continuous; (iv) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$.

We use $F(\mathcal{T})$ to denote the common fixed point set of the semigroup $\mathcal{T}$; that is, $F(\mathcal{T}) = \{x \in C : T(s)x = x, \forall s \geq 0\}$. It is well known that $F(\mathcal{T})$ is closed and convex [12]. A nonexpansive semigroup $\mathcal{T}$ on C is said to be uniformly asymptotically regular (in short, u.a.r.) on C if for all $h \geq 0$ and any bounded subset B of C,

$$\lim_{t \to \infty} \sup_{x \in B} \|T(h)(T(t)x) - T(t)x\| = 0. \tag{4}$$

For each $h \geq 0$, define $\sigma_t(x) = (1/t)\int_0^t T(s)x\,ds$. Then

$$\lim_{t \to \infty} \sup_{x \in B} \|T(h)(\sigma_t(x)) - \sigma_t(x)\| = 0, \tag{5}$$

Provided that B is closed bounded convex subset of C. It is known that the set $\{\sigma_t(x) : t > 0\}$ is a u.a.r. nonexpansive semigroup; see [13]. The other examples of u.a.r. operator semigroup can be found in [14].

A bifunction $f : C \times C \to \mathbb{R}$ is said to be (i) *strongly monotone* on C with $\alpha > 0$ if $f(x, y) + f(y, x) \leq -\alpha\|x - y\|^2$, $\forall x, y \in C$; (ii) *monotone* on C if $f(x, y) + f(y, x) \leq 0, \forall x, y \in C$; (iii) *pseudomonotone* on C if $f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0$, $\forall x, y \in C$; (iv) *Lipschitz-type continuous* on C with constants $c_1 > 0$ and $c_2 > 0$ if $f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \forall x, y, z \in C$.

Note that if T is L-Lipschitz on C, then for each $x, y \in C$, the function $f(x, y) = \langle Fx, y - x\rangle$ is a Lipschitz-type continuous with constants $c_1 = c_2 = L/2$.

An operator A on H is called *strongly positive* if there is a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{6}$$

Recently, iterative methods for nonexpansive mappings have been applied to solve convex minimization problems. In [15], Xu defined an iterative sequence $\{x_n\}$ in C which converges strongly to the unique solution of the minimization problem under some suitable conditions. A well-known typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping T on a real Hilbert space H:

$$\min_{x \in F(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{7}$$

where b is a given point in H and A is strongly positive operator.

For solving the variational inequality problem, Marino and Xu [16] introduced the following general iterative process for nonexpansive mapping T based on the viscosity approximation method (see [17]):

$$x_{n+1} = a_n\gamma h(x_n) + (I - a_n A)Tx_n, \quad \forall n \geq 0, \tag{8}$$

where A is strongly positive bounded linear operator on H, h is contraction on H, and $\{a_n\} \subset]0, 1[$. They proved that, under some appropriate conditions on the parameters, the sequence $\{x_n\}$ generated by (8) converges strongly to the unique solution $x^\star \in F(T)$ of the variational inequality

$$\langle(A - \gamma h)x^\star, x - x^\star\rangle \geq 0, \quad \forall x \in F(T), \tag{9}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \frac{1}{2}\langle Ax, x\rangle - g(x), \tag{10}$$

where g is a potential function for γh (i.e., $g'(x) = \gamma h(x)$, $\forall x \in H$).

Iterative process for approximating common fixed points of a nonexpansive semigroup has been investigated by various authors (see [13, 14, 18–21]). Recently, Li et al. [19] introduced the following iterative procedure for the approximation of common fixed points of a nonexpansive semigroup $\mathcal{T} = \{T(s) : 0 \leq s < \infty\}$ on a closed convex subset C of a Hilbert space H:

$$x_n = \alpha_n\gamma h(x_n) + (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n\,ds, \quad n \geq 1, \tag{11}$$

where A is a strongly positive bounded linear operator on H and h is a contraction on C. Imposing some appropriate conditions on the parameters, they proved that the iterative sequence $\{x_n\}$ generated by (11) converges strongly to the unique solution $x^\star \in F(\mathcal{T})$ of the variational inequality $\langle(\gamma h - A)x^\star, z - x^\star\rangle \leq 0, \forall z \in F(\mathcal{T})$.

For obtaining a common element of $\mathrm{Sol}(f, C)$ and the set of fixed points of a nonexpansive mapping T, S. Takahashi and W. Takahashi [9] first introduced an iterative scheme by the viscosity approximation method. They proved that under certain conditions the iterative sequences converge strongly to $z = \mathrm{Proj}_{F(T)\cap\mathrm{Sol}(f,C)}(h(z))$.

During last few years, iterative algorithms for finding a common element of the set of solutions of Fan inequality and the set of fixed points of nonexpansive mappings in a real Hilbert space have been studied by many authors (see, e.g., [2, 4, 22–28]). Recently, Anh [22] studied the existence of a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of Fan inequality for monotone and Lipschitz-type continuous bifunctions. He introduced the following new iterative process:

$$\begin{aligned} w_n &= \operatorname{argmin}\left\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : w \in C\right\}, \\ z_n &= \operatorname{argmin}\left\{\lambda_n f(w_n, z) + \frac{1}{2}\|z - x_n\|^2 : z \in C\right\}, \\ x_{n+1} &= \alpha_n h(x_n) + \beta_n x_n + \gamma_n(\mu S(x_n) + (1 - \mu)z_n), \\ &\qquad \forall n \geq 0, \end{aligned} \tag{12}$$

where $\mu \in]0, 1[$, C is nonempty, closed convex subset of a real Hilbert space H, f is monotone, continuous, and Lipschitz-type continuous bifunction, h is self-contraction on C with constant $k \in]0, 1[$, and S is self nonexpansive mapping on C. He proved that, under some appropriate conditions over positive sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$, the sequences $\{x_n\}$, $\{w_n\}$, and $\{z_n\}$ converge strongly to $q \in F(S) \cap \mathrm{Sol}(f, C)$ which is a solution of the variational inequality $\langle (I-h)q, x-q\rangle \geq 0, \forall x \in F(S) \cap \mathrm{Sol}(f, C)$.

In this paper, we introduce a new iterative scheme based on the viscosity method and study the existence of a common element of the set of solutions of equilibrium problems for a family of monotone, Lipschitz-type continuous mappings and the sets of fixed points of two nonexpansive semigroups in a real Hilbert space. We establish strong convergence theorems of the new iterative scheme for the solution of the variational inequality problem which is the optimality condition for the minimization problem. Our results improve and generalize the corresponding recent results of Anh [22], Cianciaruso et al. [18], and many others.

2. Preliminaries

In this section we collect some lemmas which are crucial for the proofs of our results.

Let $\{x_n\}$ be a sequence in H and $x \in H$. In the sequel, $x_n \rightharpoonup x$ denotes that $\{x_n\}$ weakly converges to x and $x_n \to x$ denotes that $\{x_n\}$ weakly converges to x.

Lemma 1. *Let H be a real Hilbert space. Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H. \tag{13}$$

Lemma 2 (see [15]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\eta_n) a_n + \eta_n \delta_n, \quad n \geq 0, \tag{14}$$

where $\{\eta_n\}$ is a sequence in $]0,1[$ and δ_n is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \eta_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\eta_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 3 (see [16]). *Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\,\overline{\gamma}$.*

Lemma 4 (see [29]). *Let H be a Hilbert space and $x_i \in H$, $(1 \leq i \leq m)$. Then for any given $\{\lambda_i\}_{i=1}^m \subset]0,1[$ with $\sum_{i=1}^m \lambda_i = 1$ and for any positive integer k, j with $1 \leq k < j \leq m$,*

$$\left\|\sum_{i=1}^{m} \lambda_i x_i\right\|^2 \leq \sum_{i=1}^{m} \lambda_i \|x_i\|^2 - \lambda_k \lambda_j \|x_k - x_j\|^2. \tag{15}$$

Lemma 5 (see [30]). *Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{\tau(n)\} \subset \mathbb{N}$ such that $\tau(n) \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbb{N}$:*

$$t_{\tau(n)} \leq t_{\tau(n)+1}, \qquad t_n \leq t_{\tau(n)+1}. \tag{16}$$

In fact

$$\tau(n) = \max\{k \leq n : t_k < t_{k+1}\}. \tag{17}$$

Lemma 6 (see [2]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $f : C \times C \to \mathbb{R}$ be a pseudomonotone and Lipschitz-type continuous bifunction with constants $c_1, c_2 \geq 0$. For each $x \in C$, let $f(x, \cdot)$ be convex and subdifferentiable on C. Let $x_0 \in C$ and let $\{x_n\}$, $\{z_n\}$, and $\{w_n\}$ be sequences generated by*

$$\begin{aligned} w_n &= \operatorname{argmin}\left\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : w \in C\right\}, \\ z_n &= \operatorname{argmin}\left\{\lambda_n f(w_n, z) + \frac{1}{2}\|z - x_n\|^2 : z \in C\right\}. \end{aligned} \tag{18}$$

Then for each $x^\star \in Sol(f, C)$,

$$\begin{aligned} \|z_n - x^\star\|^2 \leq{}& \|x_n - x^\star\|^2 - (1 - 2\lambda_n c_1)\|x_n - w_n\|^2 \\ &- (1 - 2\lambda_n c_2)\|w_n - z_n\|^2, \quad \forall n \geq 0. \end{aligned} \tag{19}$$

3. Main Results

In this section, we prove the main strong convergence result which solves the problem of finding a common element of three sets $F(\mathcal{T})$, $F(\mathcal{S})$, and $\mathrm{Sol}(f_i, C)$ for finite family of monotone, continuous, and Lipschitz-type continuous bifunctions f_i in a real Hilbert space H.

Theorem 7. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $f_i : C \times C \to \mathbb{R}$ $(i = 1, 2, \ldots, m)$ be a finite family of monotone, continuous, and Lipschitz-type continuous bifunctions with constants $c_{1,i}$ and $c_{2,i}$. Let $\mathcal{T} = \{T(u) : u \geq 0\}$ and $\mathcal{S} = \{S(u) : u \geq 0\}$ be two u.a.r. nonexpansive self-mapping semigroups on C such that $\Omega = \bigcap_{i=1}^m Sol(f_i, C) \bigcap F(\mathcal{T}) \bigcap F(\mathcal{S}) \neq \emptyset$. Assume that h is a k-contraction self-mapping of C and A is a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} < 1$ and $0 < \gamma < \overline{\gamma}/k$. Let $x_0 \in C$ and let $\{x_n\}$, $\{w_{n,i}\}$, and $\{z_{n,i}\}$ be sequences generated by*

$$\begin{aligned} w_{n,i} &= \operatorname{argmin}\left\{\lambda_{n,i} f_i(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : w \in C\right\}, \\ &\qquad i \in \{1, 2, \ldots, m\}, \\ z_{n,i} &= \operatorname{argmin}\left\{\lambda_{n,i} f_i(w_{n,i}, z) + \frac{1}{2}\|z - x_n\|^2 : z \in C\right\}, \\ &\qquad i \in \{1, 2, \ldots, m\}, \\ y_n &= \alpha_n x_n + \sum_{i=1}^{m} \beta_{n,i} z_{n,i} + \gamma_n T(t_n) x_n + \eta_n S(t_n) x_n, \\ x_{n+1} &= \theta_n \gamma h(x_n) + (I - \theta_n A) y_n, \quad \forall n \geq 0, \end{aligned} \tag{20}$$

where $\alpha_n + \sum_{i=1}^{m} \beta_{n,i} + \gamma_n + \eta_n = 1$ and $\{t_n\}$, $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $\{\gamma_n\}$, $\{\eta_n\}$, $\{\lambda_{n,i}\}$, and $\{\theta_n\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} t_n = \infty$,

(ii) $\{\theta_n\} \subset]0,1[$, $\lim_{n\to\infty}\theta_n = 0$, *and* $\sum_{n=1}^{\infty} \theta_n = \infty$,

(iii) $\{\lambda_{n,i}\} \subset [a,b] \subset]0,1/L[$, *where* $L = \max\{2c_{1,i}, 2c_{2,i}, 1 \le i \le m\}$,

(iv) $\{\alpha_n\}, \{\beta_{n,i}\}, \{\gamma_n\}, \{\eta_n\} \subset]0,1[$, $\liminf_n \alpha_n\beta_{n,i} > 0$, $\liminf_n \alpha_n\gamma_n > 0$, $\liminf_n \alpha_n\eta_n > 0$, *and* $\liminf_n \beta_{n,i}(1-2\lambda_{n,i}\, c_{1,i}) > 0$ *for each* $1 \le i \le m$.

Then, the sequence $\{x_n\}$ converges strongly to $x^\star \in \bigcap_{i=1}^{m} Sol(f_i, C) \bigcap F(\mathcal{T}) \bigcap F(\mathcal{S})$ which solves the variational inequality:

$$\langle (A-\gamma h)\, x^\star, x - x^\star \rangle \ge 0, \quad \forall x \in \Omega. \tag{21}$$

Proof. Since $F(\mathcal{T})$, $F(\mathcal{S})$, and $\mathrm{Sol}(f_i, C)$ are closed and convex, Proj_Ω is well-defined. We claim that $\mathrm{Proj}_\Omega(I-A+\gamma h)$ is a contraction from C into itself. Indeed, for each $x, y \in C$, we have

$$\begin{aligned}
&\left\|\mathrm{Proj}_\Omega (I-A+\gamma h)(x) - \mathrm{Proj}_\Omega (I-A+\gamma h)(y)\right\| \\
&\quad \le \left\|(I-A+\gamma h)(x) - (I-A+\gamma h)(y)\right\| \\
&\quad \le \left\|(I-A)x - (I-A)y\right\| + \gamma\left\|hx - hy\right\| \\
&\quad \le (1-\overline{\gamma})\left\|x-y\right\| + \gamma k\left\|x-y\right\| \\
&\quad \le (1-(\overline{\gamma}-\gamma k))\left\|x-y\right\|.
\end{aligned} \tag{22}$$

Therefore, by the Banach contraction principle, there exists a unique element $x^\star \in C$ such that $x^\star = \mathrm{Proj}_\Omega(I-A+\gamma h)x^\star$. We show that $\{x_n\}$ is bounded. Since $\lim_{n\to\infty}\theta_n = 0$, we can assume, with no loss of generality, that $\theta_n \in (0, \|A\|^{-1})$, for all $n \ge 0$. By Lemma 6, for each $1 \le i \le m$, we have

$$\left\|z_{n,i} - x^\star\right\| \le \left\|x_n - x^\star\right\|. \tag{23}$$

This implies that

$$\begin{aligned}
&\left\|y_n - x^\star\right\| \\
&\quad \le \left\|\alpha_n x_n + \sum_{i=1}^{m}\beta_{n,i} z_{n,i} + \gamma_n T(t_n)x_n + \eta_n S(t_n)x_n - x^\star\right\| \\
&\quad \le \alpha_n\left\|x_n - x^\star\right\| + \sum_{i=1}^{m}\beta_{n,i}\left\|z_{n,i} - x^\star\right\| \\
&\qquad + \gamma_n\left\|T(t_n)x_n - x^\star\right\| + \eta_n\left\|S(t_n)x_n - x^\star\right\| \\
&\quad \le \alpha_n\left\|x_n - x^\star\right\| + \sum_{i=1}^{m}\beta_{n,i}\left\|x_n - x^\star\right\| \\
&\qquad + \gamma_n\left\|x_n - x^\star\right\| + \eta_n\left\|x_n - x^\star\right\| \\
&\quad = \left\|x_n - x^\star\right\|.
\end{aligned} \tag{24}$$

It follows from Lemma 3 that

$$\begin{aligned}
&\left\|x_{n+1} - x^\star\right\| \\
&\quad = \left\|\theta_n(\gamma h(x_n) - Ax^\star) + (I-\theta_n A)(y_n - x^\star)\right\| \\
&\quad \le \theta_n\left\|\gamma h(x_n) - Ax^\star\right\| + \left\|I-\theta_n A\right\|\left\|y_n - x^\star\right\| \\
&\quad \le \theta_n\left\|\gamma h(x_n) - Ax^\star\right\| + (1-\theta_n\overline{\gamma})\left\|x_n - x^\star\right\| \\
&\quad \le \theta_n\gamma\left\|h(x_n) - h(x^\star)\right\| + \theta_n\left\|\gamma h(x^\star) - Ax^\star\right\| \\
&\qquad + (1-\theta_n\overline{\gamma})\left\|x_n - x^\star\right\| \\
&\quad \le \theta_n\gamma k\left\|x_n - x^\star\right\| + \theta_n\left\|\gamma h(x^\star) - Ax^\star\right\| \\
&\qquad + (1-\theta_n\overline{\gamma})\left\|x_n - x^\star\right\| \\
&\quad \le (1-\theta_n(\overline{\gamma}-\gamma k))\left\|x_n - x^\star\right\| + \theta_n\left\|\gamma h(x^\star) - Ax^\star\right\| \\
&\quad = (1-\theta_n(\overline{\gamma}-\gamma k))\left\|x_n - x^\star\right\| \\
&\qquad + \theta_n(\overline{\gamma}-\gamma k)\frac{\left\|\gamma h(x^\star) - Ax^\star\right\|}{\overline{\gamma}-\gamma k} \\
&\quad \le \max\left\{\left\|x_n - x^\star\right\|, \frac{\left\|\gamma h x^\star - \mathscr{A}x^\star\right\|}{\overline{\gamma}-\gamma k}\right\}.
\end{aligned} \tag{25}$$

This implies that $\{x_n\}$ is bounded and so are $\{z_{n,i}\}$, $\{h(x_n)\}$, $\{T(t_n)x_n\}$, and $\{S(t_n)x_n\}$. Next, we show that $\lim_{n\to\infty}\|x_n - T(t_n)x_n\| = \lim_{n\to\infty}\|x_n - S(t_n)x_n\| = 0$. Indeed, by Lemma 6, for each $1 \le i \le m$, we have

$$\begin{aligned}
\left\|z_{n,i} - x^\star\right\|^2 \le{}& \left\|x_n - x^\star\right\|^2 - (1-2\lambda_{n,i}c_{1,i})\left\|x_n - w_{n,i}\right\|^2 \\
&- (1-2\lambda_{n,i}c_{2,i})\left\|w_{n,i} - z_{n,i}\right\|^2.
\end{aligned} \tag{26}$$

Applying Lemma 4 and inequality (26) for $k \in \{1,2,\ldots,m\}$ we have that

$$\begin{aligned}
&\left\|y_n - x^\star\right\|^2 \\
&\quad = \left\|\alpha_n x_n + \sum_{i=1}^{m}\beta_{n,i} z_{n,i} + \gamma_n T(t_n)x_n + \eta_n S(t_n)x_n - x^\star\right\|^2 \\
&\quad \le \alpha_n\left\|x_n - x^\star\right\|^2 + \sum_{i=1}^{m}\beta_{n,i}\left\|z_{n,i} - x^\star\right\|^2 \\
&\qquad + \gamma_n\left\|T(t_n)x_n - x^\star\right\|^2 + \eta_n\left\|S(t_n)x_n - x^\star\right\|^2 \\
&\qquad - \alpha_n\beta_{n,k}\left\|x_n - z_{n,k}\right\|^2 - \alpha_n\gamma_n\left\|x_n - T(t_n)x_n\right\|^2 \\
&\qquad - \alpha_n\eta_n\left\|x_n - S(t_n)x_n\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n \|x_n - x^\star\|^2 + \sum_{i=1}^{m} \beta_{n,i} \|x_n - x^\star\|^2 + \gamma_n \|x_n - x^\star\|^2 \\
&\quad + \eta_n \|x_n - x^\star\|^2 - \alpha_n \beta_{n,k} \|x_n - z_{n,k}\|^2 \\
&\quad - \beta_{n,k} \left(1 - 2\lambda_{n,k} c_{1,k}\right) \|x_n - w_{n,k}\|^2 \\
&\quad - \alpha_n \gamma_n \|x_n - T(t_n) x_n\|^2 - \alpha_n \eta_n \|x_n - S(t_n) x_n\|^2 \\
&= \|x_n - x^\star\|^2 - \beta_{n,k} \left(1 - 2\lambda_{n,k} c_{1,k}\right) \|x_n - w_{n,k}\|^2 \\
&\quad - \alpha_n \beta_{n,k} \|x_n - z_{n,k}\|^2 - \alpha_n \gamma_n \|x_n - T(t_n) x_n\|^2 \\
&\quad - \alpha_n \eta_n \|x_n - S(t_n) x_n\|^2 .
\end{aligned}
\tag{27}
$$

We now compute

$$
\begin{aligned}
&\|x_{n+1} - x^\star\|^2 \\
&\quad = \|\theta_n (\gamma h(x_n) - A x^\star) + (I - \theta_n A)(y_n - x^\star)\|^2 \\
&\quad \leq \theta_n^2 \|\gamma h(x_n) - A x^\star\|^2 + (1 - \theta_n \overline{\gamma})^2 \|y_n - x^\star\|^2 \\
&\qquad + 2\theta_n (1 - \theta_n \overline{\gamma}) \|\gamma h(x_n) - A x^\star\| \|y_n - x^\star\| \\
&\quad \leq \theta_n^2 \|\gamma h(x_n) - A x^\star\|^2 + (1 - \theta_n \overline{\gamma})^2 \|x_n - x^\star\|^2 \\
&\qquad + 2\theta_n (1 - \theta_n \overline{\gamma}) \|\gamma h(x_n) - A x^\star\| \|x_n - x^\star\| \\
&\qquad - (1 - \theta_n \overline{\gamma})^2 \beta_{n,k} \left(1 - 2\lambda_{n,k} c_{1,k}\right) \|x_n - w_{n,k}\|^2 \\
&\qquad - (1 - \theta_n \overline{\gamma})^2 \alpha_n \beta_{n,k} \|x_n - z_{n,k}\|^2 \\
&\qquad - (1 - \theta_n \overline{\gamma})^2 \alpha_n \gamma_n \|x_n - T(t_n) x_n\|^2 \\
&\qquad - (1 - \theta_n \overline{\gamma})^2 \alpha_n \eta_n \|x_n - S(t_n) x_n\|^2 .
\end{aligned}
\tag{28}
$$

Therefore,

$$
\begin{aligned}
&(1 - \theta_n \overline{\gamma})^2 \alpha_n \beta_{n,k} \|x_n - z_{n,k}\|^2 \\
&\quad \leq \|x_n - x^\star\|^2 - \|x_{n+1} - x^\star\|^2 + 2\theta_n (1 - \theta_n \overline{\gamma}) \\
&\qquad \times \|\gamma h(x_n) - A x^\star\| \|x_n - x^\star\| + \theta_n^2 \|\gamma h(x_n) - A x^\star\|^2 .
\end{aligned}
\tag{29}
$$

In order to prove that $x_n \to x^\star$ as $n \to \infty$, we consider the following two cases. □

Case 1. Assume that $\{\|x_n - x^\star\|\}$ is a monotone sequence. In other words, for large enough n_0, $\{\|x_n - x^\star\|\}_{n \geq n_0}$ is either nondecreasing or nonincreasing. Since $\{\|x_n - x^\star\|\}$ is bounded, it is convergent. Since $\lim_{n\to\infty} \theta_n = 0$ and $\{h(x_n)\}$ and $\{x_n\}$ are bounded, from (29), we have

$$
\lim_{n\to\infty} (1 - \theta_n \overline{\gamma})^2 \alpha_n \beta_{n,k} \|x_n - z_{n,k}\|^2 = 0,
\tag{30}
$$

and by assumption we get

$$
\lim_{n\to\infty} \|x_n - z_{n,k}\| = 0, \quad 1 \leq k \leq m.
\tag{31}
$$

By similar argument we can obtain that

$$
\begin{aligned}
\lim_{n\to\infty} \|x_n - w_{n,k}\| &= \lim_{n\to\infty} \|x_n - T(t_n) x_n\| \\
&= \lim_{n\to\infty} \|x_n - S(t_n) x_n\| = 0.
\end{aligned}
\tag{32}
$$

Further, for all $h \geq 0$ and $n \geq 0$, we see that

$$
\begin{aligned}
&\|x_n - T(h) x_n\| \\
&\quad \leq \|x_n - T(t_n) x_n\| + \|T(t_n) x_n - T(h) T(t_n) x_n\| \\
&\qquad + \|T(h) T(t_n) x_n - T(h) x_n\| \\
&\quad \leq 2 \|x_n - T(t_n) x_n\| \\
&\qquad + \sup_{x \in \{x_n\}} \|T(t_n) x_n - T(h) T(t_n) x_n\| .
\end{aligned}
\tag{33}
$$

Since $\{T(t)\}$ is u.a.r. nonexpansive semigroup and $\lim_{n\to\infty} t_n = \infty$, we have

$$
\lim_{n\to\infty} \|x_n - T(h) x_n\| = 0.
\tag{34}
$$

Similarly, for all $h \geq 0$ and $n \geq 0$, we obtain that

$$
\lim_{n\to\infty} \|x_n - S(h) x_n\| = 0.
\tag{35}
$$

Next, we show that

$$
\limsup_{n\to\infty} \left\langle (A - \gamma h) x^\star, x^\star - x_n \right\rangle \leq 0.
\tag{36}
$$

To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$
\begin{aligned}
&\lim_{i\to\infty} \left\langle (A - \gamma h) x^\star, x^\star - x_{n_i} \right\rangle \\
&\quad = \limsup_{n\to\infty} \left\langle (A - \gamma h) x^\star, x^\star - x_n \right\rangle .
\end{aligned}
\tag{37}
$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $\widetilde{x}$. Without loss of generality, we can assume that $x_{n_j} \rightharpoonup \widetilde{x}$. Consider

$$
\begin{aligned}
\left\| x_{n_j} - T(t) \widetilde{x} \right\| &\leq \left\| x_{n_j} - T(t) x_{n_j} \right\| + \left\| T(t) x_{n_j} - T(t) \widetilde{x} \right\| \\
&\leq \left\| x_{n_j} - T(t) x_{n_j} \right\| + \left\| x_{n_j} - \widetilde{x} \right\| .
\end{aligned}
\tag{38}
$$

Thus, we have

$$
\limsup_{n\to\infty} \left\| x_{n_i} - T(t) \widetilde{x} \right\| \leq \limsup_{n\to\infty} \left\| x_{n_i} - \widetilde{x} \right\| .
\tag{39}
$$

By the Opial property of the Hilbert space H we obtain $T(t)\widetilde{x} = \widetilde{x}$ for all $t \geq 0$. Similarly we have that $S(t)\widetilde{x} = \widetilde{x}$ for all $t \geq 0$. This implies that $\widetilde{x} \in F(\mathcal{T}) \bigcap F(\mathcal{S})$. Now we show that $\widetilde{x} \in \mathrm{Sol}(f_i, C)$. For each $1 \leq i \leq m$, since $f_i(x, \cdot)$ is convex on C for each $x \in C$, we see that

$$
w_{n,i} = \operatorname{argmin} \left\{ \lambda_{n,i}\, f_i(x_n, y) + \frac{1}{2} \|y - x_n\|^2 : y \in C \right\}
\tag{40}
$$

if and only if

$$0 \in \partial_2 \left(f_i\left(x_n, y\right) + \frac{1}{2}\|y - x_n\|^2 \right)\left(w_{n,i}\right) + N_C\left(w_{n,i}\right), \quad (41)$$

where $N_C(x)$ is the (outward) normal cone of C at $x \in C$. This follows that

$$0 = \lambda_{n,i} v + w_{n,i} - x_n + u_n, \quad (42)$$

where $v \in \partial_2 f_i(x_n, w_{n,i})$ and $u_n \in N_C(w_{n,i})$. By the definition of the normal cone N_C we have

$$\langle w_{n,i} - x_n, y - w_{n,i} \rangle \geq \lambda_{n,i} \langle v, w_{n,i} - y \rangle, \quad \forall y \in C. \quad (43)$$

Since $f_i(x_n, \cdot)$ is subdifferentiable on C, by the well-known Moreau-Rockafellar theorem [31] (also see [6]), for $v \in \partial_2 f_i(x_n, w_{n,i})$, we have

$$f_i\left(x_n, y\right) - f_i\left(x_n, w_{n,i}\right) \geq \langle v, y - w_{n,i} \rangle, \quad \forall y \in C. \quad (44)$$

Combining this with (43), we have

$$\lambda_{n,i}\left(f_i\left(x_n, y\right) - f_i\left(x_n, w_{n,i}\right)\right) \geq \langle w_{n,i} - x_n, w_{n,i} - y \rangle, \quad \forall y \in C. \quad (45)$$

In particular, we have

$$\lambda_{n_j,i}\left(f_i\left(x_{n_j}, y\right) - f_i\left(x_{n_j}, w_{n,i}\right)\right) \geq \left\langle w_{n_j,i} - x_{n_j}, w_{n_j,i} - y \right\rangle, \quad \forall y \in C. \quad (46)$$

Since $x_{n_j} \rightharpoonup \tilde{x}$, it follows from (32) that $w_{n_j,i} \rightharpoonup \tilde{x}$. And thus we have

$$f_i\left(\tilde{x}, y\right) \geq 0, \quad \forall y \in C. \quad (47)$$

This implies that $\tilde{x} \in \mathrm{Sol}(f_i, C)$ and hence $\tilde{x} \in \Omega$. Since $x^\star = \mathrm{Proj}_\Omega(I - A + \gamma h)x^\star$ and $\tilde{x} \in \Omega$, we have

$$\begin{aligned}
\limsup_{n\to\infty} &\langle (A - \gamma h) x^\star, x^\star - x_n \rangle \\
&= \lim_{i\to\infty} \left\langle (A - \gamma h) x^\star, x^\star - x_{n_i} \right\rangle \\
&= \langle (A - \gamma h) x^\star, x^\star - \tilde{x} \rangle \leq 0.
\end{aligned} \quad (48)$$

From Lemma 1, it follows that

$$\begin{aligned}
&\|x_{n+1} - x^\star\|^2 \\
&\leq \|(I - \theta_n A)(y_n - x^\star)\|^2 \\
&\quad + 2\theta_n \langle \gamma h(x_n) - Ax^\star, x_{n+1} - x^\star \rangle \\
&\leq (1 - \theta_n\overline{\gamma})^2 \|x_n - x^\star\|^2 \\
&\quad + 2\theta_n\gamma \langle h(x_n) - h(x^\star), x_{n+1} - x^\star \rangle \\
&\quad + 2\theta_n \langle \gamma h(x^\star) - Ax^\star, x_{n+1} - x^\star \rangle \\
&\leq (1 - \theta_n\overline{\gamma})^2 \|x_n - x^\star\|^2 + 2\theta_n k\gamma \|x_n - x^\star\| \|x_{n+1} - x^\star\| \\
&\quad + 2\theta_n \langle \gamma h(x^\star) - Ax^\star, x_{n+1} - x^\star \rangle \\
&\leq (1 - \theta_n\overline{\gamma})^2 \|x_n - x^\star\|^2 \\
&\quad + \theta_n k\gamma \left(\|x_n - x^\star\|^2 + \|x_{n+1} - x^\star\|^2\right) \\
&\quad + 2\theta_n \langle \gamma h(x^\star) - Ax^\star, x_{n+1} - x^\star \rangle \\
&\leq \left((1 - \theta_n\overline{\gamma})^2 + \theta_n k\gamma\right) \|x_n - x^\star\|^2 \\
&\quad + \theta_n\gamma k\|x_{n+1} - x^\star\|^2 + 2\theta_n \langle \gamma h(x^\star) - Ax^\star, x_{n+1} - x^\star \rangle.
\end{aligned} \quad (49)$$

This implies that

$$\begin{aligned}
&\|x_{n+1} - x^\star\|^2 \\
&\leq \frac{1 - 2\theta_n\overline{\gamma} + (\theta_n\overline{\gamma})^2 + \theta_n\gamma k}{1 - \theta_n\gamma k} \|x_n - x^\star\|^2 \\
&\quad + \frac{2\theta_n}{1 - \theta_n\gamma k} \langle \gamma h(x^\star) - Ax^\star, x_{n+1} - x^\star \rangle \\
&= \left(1 - \frac{2(\overline{\gamma} - \gamma k)\theta_n}{1 - \theta_n\gamma k}\right) \|x_n - x^\star\|^2 + \frac{(\theta_n\overline{\gamma})^2}{1 - \theta_n\gamma k} \|x_n - x^\star\|^2 \\
&\quad + \frac{2\theta_n}{1 - \theta_n\gamma k} \langle \gamma h(x^\star) - Ax^\star, x_{n+1} - x^\star \rangle \\
&\leq \left(1 - \frac{2(\overline{\gamma} - \gamma k)\theta_n}{1 - \theta_n\gamma k}\right) \|x_n - x^\star\|^2 + \frac{2(\overline{\gamma} - \gamma k)\theta_n}{1 - \theta_n\gamma k} \\
&\quad \times \left(\frac{(\theta_n\overline{\gamma}^2) M}{2(\overline{\gamma} - \gamma k)} + \frac{1}{\overline{\gamma} - \gamma k} \langle \gamma h(x^\star) - Ax^\star, x_{n+1} - x^\star \rangle\right) \\
&= (1 - \eta_n) \|x_n - x^\star\|^2 + \eta_n\delta_n,
\end{aligned} \quad (50)$$

where

$$M = \sup\left\{\|x_n - x^\star\|^2 : n \geq 0\right\}, \qquad \eta_n = \frac{2(\overline{\gamma} - \gamma k)\theta_n}{1 - \theta_n \gamma k},$$
$$\delta_n = \frac{(\theta_n \overline{\gamma}^2) M}{2(\overline{\gamma} - \gamma k)} + \frac{1}{\overline{\gamma} - \gamma k}\langle \gamma h x^\star - A x^\star, x_{n+1} - x^\star \rangle. \tag{51}$$

It is easy to see that $\eta_n \to 0$, $\sum_{n=1}^{\infty} \eta_n = \infty$, and $\limsup_{n\to\infty}\delta_n \leq 0$. Hence, by Lemma 2 the sequence $\{x_n\}$ converges strongly to $x^\star$. From (32) we have that $\{w_{n,i}\}$ and $\{z_{n,i}\}$ converge strongly to $x^\star$.

Case 2. Assume that $\{\|x_n - x^\star\|\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \geq n_0$ (for some large enough n_0) by

$$\tau(n) := \max\left\{k \in \mathbb{N}, k \leq n : \|x_k - x^\star\| < \|x_{k+1} - x^\star\|\right\}. \tag{52}$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$, and for all $n \geq n_0$,

$$\|x_{\tau(n)} - x^\star\| < \|x_{\tau(n)+1} - x^\star\|. \tag{53}$$

From (33) we obtain that

$$\lim_{n\to\infty}\|x_{\tau(n)} - z_{\tau(n),i}\| = \lim_{n\to\infty}\|x_{\tau(n)} - w_{\tau(n),i}\| = \lim_{n\to\infty}\|x_{\tau(n)} - T(t_{\tau_n}) x_{\tau(n)}\| = 0. \tag{54}$$

Following an argument similar to that in Case 1 we have

$$\|x_{\tau(n)+1} - x^\star\|^2 \leq (1 - \eta_{\tau(n)})\|x_{\tau(n)} - x^\star\|^2 + \eta_{\tau(n)}\delta_{\tau(n)}, \tag{55}$$

where $\eta_{\tau(n)} \to 0$, $\sum_{n=1}^{\infty}\eta_{\tau(n)} = \infty$, and $\limsup_{n\to\infty}\delta_{\tau(n)} \leq 0$. Hence, by Lemma 2, we obtain $\lim_{n\to\infty}\|x_{\tau(n)} - x^\star\| = 0$ and $\lim_{n\to\infty}\|x_{\tau(n)+1} - x^\star\| = 0$. Now Lemma 5 implies that

$$\begin{aligned} 0 &\leq \|x_n - x^\star\| \leq \max\left\{\|x_{\tau(n)} - x^\star\|, \|x_n - x^\star\|\right\} \\ &\leq \|x_{\tau(n)+1} - x^\star\|. \end{aligned} \tag{56}$$

Therefore, $\{x_n\}$ converges strongly to $x^\star = \mathrm{Proj}_\Omega(I - A + \gamma h)x^\star$. This completes the proof.

4. Application

In this section, we consider a particular Fan inequality corresponding to the function f defined by the following: for every $x, y \in C$

$$f(x, y) = \langle F(x), y - x\rangle, \tag{57}$$

where $F : C \to H$. Then, we obtain the classical variational inequality as follows.

$$\text{Find } z \in C \quad \text{such that } \langle F(z), y - z\rangle \geq 0, \ \forall y \in C. \tag{58}$$

The set of solutions of this problem is denoted by $VI(F, C)$. In that particular case, the solution y_n of the minimization problem

$$\operatorname{argmin}\left\{\lambda_n\, f(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\} \tag{59}$$

can be expressed as

$$y_n = P_C(x_n - \lambda_n F(x_n)). \tag{60}$$

Let F be L-Lipschitz continuous on C. Then

$$f(x, y) + f(y, z) - f(x, z) = \langle F(x) - F(y), y - z\rangle, \quad x, y, z \in C. \tag{61}$$

Therefore,

$$\begin{aligned} |\langle F(x) - F(y), y - z\rangle| &\leq L\|x - y\|\|y - z\| \\ &\leq \frac{L}{2}\left(\|x - y\|^2 + \|y - z\|^2\right) \end{aligned} \tag{62}$$

hence, f satisfies Lipschitz-type continuous condition with $c_1 = c_2 = L/2$.

Using Theorem 7 we obtain the following convergence theorem.

Theorem 8. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F_i : C \to H$ ($i = 1, 2, \ldots, m$) be functions such that, for each $1 \leq i \leq m$, F_i is monotone and L-Lipschitz continuous on C. Let $\mathcal{T} = \{T(u) : u \geq 0\}$ and $\mathcal{S} = \{S(u) : u \geq 0\}$ be two u.a.r. nonexpansive self-mapping semigroups on C such that $\Omega = \bigcap_{i=1}^{m} VI(F_i, C) \bigcap F(\mathcal{T}) \bigcap F(\mathcal{S}) \neq \emptyset$. Assume that h is a k-contraction of C into itself and A is a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} < 1$ and $0 < \gamma < \overline{\gamma}/k$. Let $x_0 \in C$ and let $\{x_n\}$, $\{w_{n,i}\}$, and $\{z_{n,i}\}$ be sequences generated by*

$$\begin{aligned} w_{n,i} &= P_C(x_n - \lambda_{n,i} F_i(x_n)), \quad i \in \{1, 2, \ldots, m\}, \\ z_{n,i} &= P_C(x_n - \lambda_{n,i} F_i(w_{n,i})), \quad i \in \{1, 2, \ldots, m\}, \\ y_n &= \alpha_n x_n + \sum_{i=1}^{m}\beta_{n,i} z_{n,i} + \gamma_n T(t_n) x_n + \eta_n S(t_n) x_n, \\ x_{n+1} &= \theta_n \gamma h(x_n) + (I - \theta_n A) y_n, \quad \forall n \geq 0, \end{aligned} \tag{63}$$

where $\alpha_n + \sum_{i=1}^{m}\beta_{n,i} + \gamma_n + \eta_n = 1$ and $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $\{\gamma_n\}$, $\{\eta_n\}$, $\{\lambda_{n,i}\}$, and $\{\theta_n\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} t_n = \infty$,

(ii) $\{\theta_n\} \subset]0, 1[$, $\lim_{n\to\infty}\theta_n = 0$, *and* $\sum_{n=1}^{\infty}\theta_n = \infty$,

(iii) $\{\lambda_{n,i}\} \subset [a, b] \subset]0, 1/L[$,

(iv) $\{\alpha_n\}, \{\beta_{n,i}\}, \{\gamma_n\}, \{\eta_n\} \subset]0, 1[$, $\liminf_n \alpha_n\beta_{n,i} > 0$, $\liminf_n \alpha_n\gamma_n > 0$, $\liminf_n \alpha_n\eta_n > 0$, *and* $\liminf_n \beta_{n,i}(1 - 2\lambda_{n,i}c_{1,i}) > 0$ *for each* $1 \leq i \leq m$.

Then, the sequence $\{x_n\}$ converges strongly to $x^\star \in \bigcap_{i=1}^{m} VI(F_i, C) \bigcap F(\mathcal{T}) \bigcap F(\mathcal{S})$ which solves the variational inequality

$$\langle (A - \gamma h) x^\star, x - x^\star\rangle \geq 0, \quad \forall x \in \Omega. \tag{64}$$

In [32], Baillon proved a strong mean convergence theorem for nonexpansive mappings, and it was generalized in [33]. It follows from the above proof that Theorems 7 is valid for nonexpansive mappings. Thus, we have the following mean ergodic theorems for nonexpansive mappings in a Hilbert space.

Theorem 9. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F_i : C \to H$ $(i = 1, 2, \ldots, m)$ be functions such that for each $1 \leq i \leq m$, F_i is monotone and L-Lipschitz continuous on C. Let T and S be two nonexpansive mappings on C such that $\Omega = \bigcap_{i=1}^{m} VI(F_i, C) \bigcap F(T) \bigcap F(S) \neq \emptyset$. Assume that h is a k-contraction of C into itself and A is a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} < 1$ and $0 < \gamma < \overline{\gamma}/k$. Let $\{x_n\}$, $\{w_{n,i}\}$, and $\{z_{n,i}\}$ be sequences generated by $x_0 \in C$ and by*

$$\begin{aligned} w_{n,i} &= P_C\left(x_n - \lambda_{n,i} F_i\left(x_n\right)\right), \quad i \in \{1, 2, \ldots, m\}, \\ z_{n,i} &= P_C\left(x_n - \lambda_{n,i} F_i\left(w_{n,i}\right)\right), \quad i \in \{1, 2, \ldots, m\}, \\ y_n &= \alpha_n x_n + \sum_{i=1}^{m} \beta_{n,i} z_{n,i} + \gamma_n T^n x_n + \eta_n S^n x_n, \\ x_{n+1} &= \theta_n \gamma h\left(x_n\right) + \left(I - \theta_n A\right) y_n, \quad \forall n \geq 0, \end{aligned} \tag{65}$$

where $\alpha_n + \sum_{i=1}^{m} \beta_{n,i} + \gamma_n + \eta_n = 1$ and $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $\{\gamma_n\}$, $\{\eta_n\}$, $\{\lambda_{n,i}\}$, and $\{\theta_n\}$ satisfy the following conditions:

(i) $\{\theta_n\} \subset]0, 1[$, $\lim_{n\to\infty} \theta_n = 0$, *and* $\sum_{n=1}^{\infty} \theta_n = \infty$,

(ii) $\{\lambda_{n,i}\} \subset [a, b] \subset]0, 1/L[$, *where* $L = \max\{2c_{1,i}, 2c_{2,i}, 1 \leq i \leq m\}$,

(iii) $\{\alpha_n\}, \{\beta_{n,i}\}, \{\gamma_n\}, \{\eta_n\} \subset]0, 1[$, $\liminf_n \alpha_n \beta_{n,i} > 0$, $\liminf_n \alpha_n \gamma_n > 0$, $\liminf_n \alpha_n \eta_n > 0$, *and* $\liminf_n \beta_{n,i}(1 - 2\lambda_{n,i}\ c_{1,i}) > 0$ *for each* $1 \leq i \leq m$.

Then, the sequence $\{x_n\}$ converges strongly to $x^\star \in \bigcap_{i=1}^{m} VI(F_i, C) \bigcap F(\mathcal{T}) \bigcap F(\mathcal{S})$ which solves the variational inequality

$$\left\langle \left(A - \gamma h\right) x^\star, x - x^\star \right\rangle \geq 0, \quad \forall x \in \Omega. \tag{66}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. Therefore, the authors acknowledge with thanks DSR, for technical and financial support.

References

[1] K. Fan, "A minimax inequality and applications," in *Inequality III*, pp. 103–113, Academic Press, New York, NY, USA, 1972.

[2] P. N. Anh, "A hybrid extragradient method extended to fixed point problems and equilibrium problems," *Optimization*, vol. 62, no. 2, pp. 271–283, 2013.

[3] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

[4] L.-C. Ceng, N. Hadjisavvas, and N.-C. Wong, "Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems," *Journal of Global Optimization*, vol. 46, no. 4, pp. 635–646, 2010.

[5] G. Stampacchia, "Formes bilinéaires coercitives sur les ensembles convexes," *Comptes rendus de l'Académie des Sciences*, vol. 258, pp. 4413–4416, 1964.

[6] P. Daniele, F. Giannessi, and A. Maugeri, *Equilibrium Problems and Variational Models*, vol. 68, Kluwer Academic, Norwell, Mass, USA, 2003.

[7] J.-W. Peng and J.-C. Yao, "Some new extragradient-like methods for generalized equilibrium problems, fixed point problems and variational inequality problems," *Optimization Methods & Software*, vol. 25, no. 4–6, pp. 677–698, 2010.

[8] D. R. Sahu, N.-C. Wong, and J.-C. Yao, "Strong convergence theorems for semigroups of asymptotically nonexpansive mappings in Banach spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 202095, 8 pages, 2013.

[9] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[10] S. Wang and B. Guo, "New iterative scheme with nonexpansive mappings for equilibrium problems and variational inequality problems in Hilbert spaces," *Journal of Computational and Applied Mathematics*, vol. 233, no. 10, pp. 2620–2630, 2010.

[11] Y. Yao, Y.-C. Liou, and Y.-J. Wu, "An extragradient method for mixed equilibrium problems and fixed point problems," *Fixed Point Theory and Applications*, vol. 2009, Article ID 632819, 15 pages, 2009.

[12] F. E. Browder, "Nonexpansive nonlinear operators in a Banach space," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 54, pp. 1041–1044, 1965.

[13] R. Chen and Y. Song, "Convergence to common fixed point of nonexpansive semigroups," *Journal of Computational and Applied Mathematics*, vol. 200, no. 2, pp. 566–575, 2007.

[14] A. Aleyner and Y. Censor, "Best approximation to common fixed points of a semigroup of nonexpansive operators," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 137–151, 2005.

[15] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.

[16] G. Marino and H.-K. Xu, "A general iterative method for nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 43–52, 2006.

[17] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.

[18] F. Cianciaruso, G. Marino, and L. Muglia, "Iterative methods for equilibrium and fixed point problems for nonexpansive semigroups in Hilbert spaces," *Journal of Optimization Theory and Applications*, vol. 146, no. 2, pp. 491–509, 2010.

[19] S. Li, L. Li, and Y. Su, "General iterative methods for a one-parameter nonexpansive semigroup in Hilbert space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 9, pp. 3065–3071, 2009.

[20] D. R. Sahu, N. C. Wong, and J. C. Yao, "A unified hybrid iterative method for solving variational inequalities involving generalized pseudocontractive mappings," *SIAM Journal on Control and Optimization*, vol. 50, no. 4, pp. 2335–2354, 2012.

[21] N. Shioji and W. Takahashi, "Strong convergence theorems for asymptotically nonexpansive semigroups in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 34, no. 1, pp. 87–99, 1998.

[22] P. N. Anh, "Strong convergence theorems for nonexpansive mappings and Ky Fan inequalities," *Journal of Optimization Theory and Applications*, vol. 154, no. 1, pp. 303–320, 2012.

[23] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Hybrid proximal-type and hybrid shrinking projection algorithms for equilibrium problems, maximal monotone operators, and relatively nonexpansive mappings," *Numerical Functional Analysis and Optimization*, vol. 31, no. 7-9, pp. 763–797, 2010.

[24] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Viscosity approximation methods for generalized equilibrium problems and fixed point problems," *Journal of Global Optimization*, vol. 43, no. 4, pp. 487–502, 2009.

[25] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Relaxed extragradient iterative methods for variational inequalities," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 1112–1123, 2011.

[26] L.-C. Ceng, S.-M. Guu, and J.-C. Yao, "Hybrid iterative method for finding common solutions of generalized mixed equilibrium and fixed point problems," *Fixed Point Theory and Applications*, vol. 2012, article 92, 2012.

[27] M. Eslamian, "Convergence theorems for nonspreading mappings and nonexpansive multivalued mappings and equilibrium problems," *Optimization Letters*, vol. 7, no. 3, pp. 547–557, 2013.

[28] M. Eslamian, "Hybrid method for equilibrium problems and fixed point problems of finite families of nonexpansive semigroups," *RACSAM*, vol. 107, pp. 299–307, 2013.

[29] M. Eslamian and A. Abkar, "One-step iterative process for a finite family of multivalued mappings," *Mathematical and Computer Modelling*, vol. 54, no. 1-2, pp. 105–111, 2011.

[30] P.-E. Maingé, "Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization," *Set-Valued Analysis*, vol. 16, no. 7-8, pp. 899–912, 2008.

[31] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976.

[32] J.-B. Baillon, "Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert," *Comptes Rendus de l'Académie des Sciences*, vol. 280, no. 22, pp. A1511–A1514, 1975.

[33] W. Kaczor, T. Kuczumow, and S. Reich, "A mean ergodic theorem for nonlinear semigroups which are asymptotically nonexpansive in the intermediate sense," *Journal of Mathematical Analysis and Applications*, vol. 246, no. 1, pp. 1–27, 2000.