# Advanced Nonlinear Dynamics of Population Biology and Epidemioloqy 

Guest Editors: Weiming Wang, Yun Kang, Malay Banerjee, and Kaila Wang


# Advanced Nonlinear Dynamics of Population Biology and Epidemiology 

## Abstract and Applied Analysis

## Advanced Nonlinear Dynamics of Population Biology and Epidemiology

Guest Editors: Weiming Wang, Yun Kang, Malay Banerjee, and Kaifa Wang

Copyright © 2014 Hindawi Publishing Corporation. All rights reserved.
This is a special issue published in "Abstract and Applied Analysis." All articles are open access articles distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

## Editorial Board

Ravi P. Agarwal, USA
Bashir Ahmad, Saudi Arabia
M. O. Ahmedou, Germany

Nicholas D. Alikakos, Greece
Debora Amadori, Italy
Pablo Amster, Argentina
Douglas R. Anderson, USA
Jan Andres, Czech Republic
Giovanni Anello, Italy
Stanislav Antontsev, Portugal
Mohamed Kamal Aouf, Egypt
Narcisa C. Apreutesei, Romania
Natig M. Atakishiyev, Mexico
Ferhan M. Atici, USA
Ivan G. Avramidi, USA
Soohyun Bae, Korea
Zhanbing Bai, China
Chuanzhi Bai, China
Dumitru Baleanu, Turkey
Józef Banaś, Poland
Gerassimos Barbatis, Greece
Martino Bardi, Italy
Roberto Barrio, Spain
Feyzi Başar, Turkey
Abdelghani Bellouquid, Morocco
Daniele Bertaccini, Italy
Michiel Bertsch, Italy
Lucio Boccardo, Italy
Igor Boglaev, New Zealand Martin J. Bohner, USA
Julian F. Bonder, Argentina
Geraldo Botelho, Brazil
Elena Braverman, Canada
Romeo Brunetti, Italy
Janusz Brzdek, Poland
Detlev Buchholz, Germany
Sun-Sig Byun, Korea
Fabio M. Camilli, Italy
Jinde Cao, China
Anna Capietto, Italy
Jianqing Chen, China
Wing-Sum Cheung, Hong Kong
Michel Chipot, Switzerland
Changbum Chun, Korea
Soon Y. Chung, Korea

Jaeyoung Chung, Korea
Silvia Cingolani, Italy
Jean M. Combes, France
Monica Conti, Italy
Diego Córdoba, Spain
Juan C. Cortés, Spain
Graziano Crasta, Italy
Bernard Dacorogna, Switzerland
Vladimir Danilov, Russia
Mohammad T. Darvishi, Iran
Luis F. Pinheiro de Castro, Portugal
T. Diagana, USA

Jesús I. Díaz, Spain
Josef Diblík, Czech Republic
Fasma Diele, Italy
Tomas Dominguez, Spain
Alexander I. Domoshnitsky, Israel
Marco Donatelli, Italy
Bo-Qing Dong, China
Ondřej Došlý, Czech Republic
Wei-Shih Du, Taiwan
Luiz Duarte, Brazil
Roman Dwilewicz, USA
Paul W. Eloe, USA
Ahmed El-Sayed, Egypt
Luca Esposito, Italy
Jose A. Ezquerro, Spain
Khalil Ezzinbi, Morocco
Dashan Fan, USA
Angelo Favini, Italy
Marcia Federson, Brazil
Stathis Filippas, Equatorial Guinea
Alberto Fiorenza, Italy
Tore Flåtten, Norway
Ilaria Fragala, Italy
Bruno Franchi, Italy
Xianlong Fu, China
Massimo Furi, Italy
Giovanni P. Galdi, USA
Isaac Garcia, Spain
Jesús García Falset, Spain
José A. García-Rodríguez, Spain
Leszek Gasinski, Poland
György Gát, Hungary
Vladimir Georgiev, Italy

Lorenzo Giacomelli, Italy
Jaume Giné, Spain
Valery Y. Glizer, Israel
Laurent Gosse, Italy
Jean P. Gossez, Belgium
Jose L. Gracia, Spain
Maurizio Grasselli, Italy
Qian Guo, China
Yuxia Guo, China
Chaitan P. Gupta, USA
Uno Hämarik, Estonia
Ferenc Hartung, Hungary
Behnam Hashemi, Iran
Norimichi Hirano, Japan
Jiaxin Hu, China
Chengming Huang, China
Zhongyi Huang, China
Gennaro Infante, Italy
Ivan Ivanov, Bulgaria
Hossein Jafari, Iran
Jaan Janno, Estonia
Aref Jeribi, Tunisia
Un C. Ji, Korea
Zhongxiao Jia, China
L. Jdar, Spain

Jong Soo Jung, Republic of Korea
Henrik Kalisch, Norway
Hamid Reza Karimi, Norway
Satyanad Kichenassamy, France
Tero Kilpelinen, Finland
Sung Guen Kim, Republic of Korea
Ljubisa Kocinac, Serbia
Andrei Korobeinikov, Spain
Pekka Koskela, Finland
Victor Kovtunenko, Austria
Ren-Jieh Kuo, Taiwan
Pavel Kurasov, Sweden
Miroslaw Lachowicz, Poland
Kunquan Lan, Canada
Ruediger Landes, USA
Irena Lasiecka, USA
Matti Lassas, Finland
Chun-Kong Law, Taiwan
Ming-Yi Lee, Taiwan
Gongbao Li, China

Elena Litsyn, Israel
Yansheng Liu, China
Shengqiang Liu, China
Carlos Lizama, Chile
Milton C. Lopes Filho, Brazil
Julian López-Gómez, Spain
Guozhen Lu, USA
Jinhu Lü, China
Grzegorz Lukaszewicz, Poland
Shiwang Ma, China
Wanbiao Ma, China
Eberhard Malkowsky, Turkey
Salvatore A. Marano, Italy
Cristina Marcelli, Italy
Paolo Marcellini, Italy
Jesús Marín-Solano, Spain
Jose M. Martell, Spain
MieczysThlaw MastyThlo, Poland
Ming Mei, Canada
Taras Mel'nyk, Ukraine
Anna Mercaldo, Italy
Changxing Miao, China
Stanislaw Migorski, Poland
Mihai Mihǎilescu, Romania
Feliz Minhós, Portugal
Dumitru Motreanu, France
Roberta Musina, Italy
G. M. N'Guérékata, USA

Maria Grazia Naso, Italy
Sylvia Novo, Spain
Micah Osilike, Nigeria
Mitsuharu Ôtani, Japan
Turgut Öziş, Turkey
Filomena Pacella, Italy
Nikolaos S. Papageorgiou, Greece
Sehie Park, Korea
Alberto Parmeggiani, Italy
Kailash C. Patidar, South Africa
Kevin R. Payne, Italy
Ademir Fernando Pazoto, Brazil
Josip E. Pečarić, Croatia
Shuangjie Peng, China
Sergei V. Pereverzyev, Austria
Maria Eugenia Perez, Spain
Josefina Perles, Spain
Allan Peterson, USA
Andrew Pickering, Spain
Cristina Pignotti, Italy

Somyot Plubtieng, Thailand
Milan Pokorny, Czech Republic
Sergio Polidoro, Italy
Ziemowit Popowicz, Poland
Maria M. Porzio, Italy
Enrico Priola, Italy
Vladimir S. Rabinovich, Mexico
Irena Rachu̇nková, Czech Republic
Maria Alessandra Ragusa, Italy
Simeon Reich, Israel
Abdelaziz Rhandi, Italy
Hassan Riahi, Malaysia
Juan P. Rincón-Zapatero, Spain Luigi Rodino, Italy
Yuriy V. Rogovchenko, Norway
Julio D. Rossi, Argentina
Wolfgang Ruess, Germany
Bernhard Ruf, Italy
Marco Sabatini, Italy
Satit Saejung, Thailand
Stefan G. Samko, Portugal
Martin Schechter, USA
Javier Segura, Spain
Sigmund Selberg, Norway
Valery Serov, Finland
Naseer Shahzad, Saudi Arabia
Andrey Shishkov, Ukraine
Stefan Siegmund, Germany
Abdel-Maksoud A. Soliman, Egypt
Pierpaolo Soravia, Italy
Marco Squassina, Italy
Svatoslav Staněk, Czech Republic
Stevo Stevic, Serbia
Antonio Suárez, Spain
Wenchang Sun, China
Robert Szalai, UK
Sanyi Tang, China
Chun-Lei Tang, China
Youshan Tao, China
Gabriella Tarantello, Italy
Nasser-eddine Tatar, Saudi Arabia
Susanna Terracini, Italy
Gerd Teschke, Germany
Alberto Tesei, Italy
Bevan Thompson, Australia
Sergey Tikhonov, Spain
Claudia Timofte, Romania
Thanh Tran, Australia

Juan J. Trujillo, Spain
Gabriel Turinici, France
Milan Tvrdy, Czech Republic
Mehmet Unal, Turkey
Csaba Varga, Romania
Carlos Vazquez, Spain
Gianmaria Verzini, Italy
Jesus Vigo-Aguiar, Spain
Qing-Wen Wang, China
Yushun Wang, China
Shawn X. Wang, Canada
Youyu Wang, China
Jing Ping Wang, UK
Peixuan Weng, China
Noemi Wolanski, Argentina
Ngai-Ching Wong, Taiwan
Patricia J. Y. Wong, Singapore
Zili Wu, China
Yong Hong Wu, Australia
Shanhe Wu, China
Tie-cheng Xia, China
Xu Xian, China
Yanni Xiao, China
Fuding Xie, China
Gonang Xie, China
Naihua Xiu, China
Daoyi Xu, China
Zhenya Yan, China
Xiaodong Yan, USA
Norio Yoshida, Japan
Beong In Yun, Korea
Vjacheslav Yurko, Russia
Agacik Zafer, Turkey
Sergey V. Zelik, UK
Jianming Zhan, China
Meirong Zhang, China
Weinian Zhang, China
Chengjian Zhang, China
Zengqin Zhao, China
Sining Zheng, China
Tianshou Zhou, China
Yong Zhou, China
Qiji J. Zhu, USA
Chun-Gang Zhu, China
Malisa R. Zizovic, Serbia
Wenming Zou, China

## Contents

Advanced Nonlinear Dynamics of Population Biology and Epidemiology, Weiming Wang, Yun Kang, Malay Banerjee, and Kaifa Wang
Volume 2014, Article ID 214514, 3 pages
Nonlinear Dynamic in an Ecological System with Impulsive Effect and Optimal Foraging,
Min Zhao and Chuanjun Dai
Volume 2014, Article ID 169609, 12 pages
Successive Vaccination and Difference in Immunity of a Delay SIR Model with a General Incidence
Rate, Yongzhen Pei, Li Changguo, Qianyong Wu, and Yunfei Lv
Volume 2014, Article ID 678723, 10 pages
Spatial Complexity of a Predator-Prey Model with Holling-Type Response, Lei Zhang and Zhibin Li Volume 2014, Article ID 675378, 15 pages

Dynamical Behavior and Stability Analysis in a Hybrid Epidemiological-Economic Model with
Incubation, Chao Liu, Wenquan Yue, and Peiyong Liu
Volume 2014, Article ID 639405, 22 pages
A Stochastic Predator-Prey System with Stage Structure for Predator, Shufen Zhao and Minghui Song Volume 2014, Article ID 518695, 7 pages

Traveling Waves in a Diffusive Predator-Prey Model Incorporating a Prey Refuge, Xiujuan Wu, Yong Luo, and Yizheng Hu
Volume 2014, Article ID 679131, 13 pages
Modeling Peer-to-Peer Botnet on Scale-Free Network, Liping Feng, Hongbin Wang, Qi Han, Qingshan Zhao, and Lipeng Song
Volume 2014, Article ID 212478, 8 pages
Stability Analysis of a Multigroup SEIR Epidemic Model with General Latency Distributions, Nan Wang, Jingmei Pang, and Jinliang Wang Volume 2014, Article ID 740256, 8 pages

Spatiotemporal Patterns in a Ratio-Dependent Food Chain Model with Reaction-Diffusion, Lei Zhang Volume 2014, Article ID 130851, 9 pages

Bifurcations of Tumor-Immune Competition Systems with Delay, Ping Bi and Heying Xiao Volume 2014, Article ID 723159, 12 pages

Stability and Hopf Bifurcation Analysis of a Gene Expression Model with Diffusion and Time Delay, Yahong Peng and Tonghua Zhang
Volume 2014, Article ID 738682, 9 pages
Traveling Wave Solutions for a Delayed SIRS Infectious Disease Model with Nonlocal Diffusion and Nonlinear Incidence, Xiaohong Tian and Rui Xu
Volume 2014, Article ID 795320, 9 pages
Hopf Bifurcation of a Delayed Epidemic Model with Information Variable and Limited Medical Resources, Caijuan Yan and Jianwen Jia
Volume 2014, Article ID 109372, 11 pages

The Space-Jump Model of the Movement of Tumor Cells and Healthy Cells, Meng-Rong Li, Yu-Ju Lin, and Tzong-Hann Shieh
Volume 2014, Article ID 840891, 7 pages
Stability of Virus Infection Models with Antibodies and Chronically Infected Cells,
Mustafa A. Obaid and A. M. Elaiw
Volume 2014, Article ID 650371, 12 pages
Dynamic Analysis of Nonlinear Impulsive Neutral Nonautonomous Differential Equations with Delays, Jinxian Li
Volume 2014, Article ID 624897, 7 pages
Minimal Wave Speed of Bacterial Colony Model with Saturated Functional Response,
Tianran Zhang and Qingming Gou
Volume 2014, Article ID 510671, 9 pages
Modeling Saturated Diagnosis and Vaccination in Reducing HIV/AIDS Infection,
Can Chen and Yanni Xiao
Volume 2014, Article ID 414383, 12 pages
Comparison of Three Measures to Promote National Fitness in China by Mathematical Modeling,
Pan Tang, Daqing Xu, Qing Dai, and Tingting Huang
Volume 2014, Article ID 685468, 7 pages
Multiple Positive Periodic Solutions for Functional Differential Equations with Impulses and a
Parameter, Zhenguo Luo
Volume 2014, Article ID 812867, 13 pages
Existence of Traveling Wave Solutions for Cholera Model, Tianran Zhang, Qingming Gou, and Xiaoli Wang
Volume 2014, Article ID 201094, 11 pages
Dynamics of a Stochastic Functional System for Wastewater Treatment, Xuehui Ji and Sanling Yuan
Volume 2014, Article ID 831573, 18 pages
Pattern Formation in a Bacterial Colony Model, Xinze Lian, Guichen Lu, and Hailing Wang
Volume 2014, Article ID 149801, 10 pages
Threshold Dynamics of a Huanglongbing Model with Logistic Growth in Periodic Environments, Jianping Wang, Shujing Gao, Yueli Luo, and Dehui Xie
Volume 2014, Article ID 841367, 10 pages
Geometric Analysis of an Integrated Pest Management Model Including Two State Impulses,
Wencai Zhao, Yulin Liu, Tongqian Zhang, and Xinzhu Meng
Volume 2014, Article ID 963072, 18 pages
Analysis of a Patch Model for the Dynamical Transmission of Echinococcosis, Kai Wang,
Xueliang Zhang, Zhidong Teng, Lei Wang, and Liping Zhang
Volume 2014, Article ID 576365, 13 pages

Global Behaviors of a Class of Discrete SIRS Epidemic Models with Nonlinear Incidence Rate, Lei Wang, Zhidong Teng, and Long Zhang
Volume 2014, Article ID 249623, 18 pages
Global Property in a Delayed Periodic Predator-Prey Model with Stage-Structure in Prey and Density-Independence in Predator, Xiaolin Fan, Zhidong Teng, and Haijun Jiang
Volume 2014, Article ID 172380, 12 pages
The Stability of SI Epidemic Model in Complex Networks with Stochastic Perturbation, Jinqing Zhao, Maoxing Liu, Wanwan Wang, and Panzu Yang
Volume 2014, Article ID 610959, 14 pages
Global Stability for a Predator-Prey Model with Dispersal among Patches,
Yang Gao and Shengqiang Liu
Volume 2014, Article ID 176493, 6 pages
Global Exponential Stability of Pseudo Almost Periodic Solutions for SICNNs with Time-Varying Leakage Delays, Wentao Wang and Bingwen Liu
Volume 2014, Article ID 967328, 17 pages
The Dynamics of a Nonautonomous Predator-Prey Model with Infertility Control in the Prey, Xiaomei Feng, Zhidong Teng, and Fengqin Zhang
Volume 2014, Article ID 304568, 10 pages
Mathematical Model of Schistosomiasis under Flood in Anhui Province,
Longxing Qi, Jing-an Cui, Tingting Huang, Fengli Ye,
and Longzhi Jiang
Volume 2014, Article ID 972189, 7 pages
Estimation of Hospital Potential Capacity and Basic Reproduction Number, Fei Wang, Linhua Wang, and Peng Wang
Volume 2014, Article ID 875474, 5 pages
Hopf Bifurcation Analysis in a Modified Price Differential Equation Model with Two Delays, Yanhui Zhai, Ying Xiong, and Xiaona Ma
Volume 2014, Article ID 686274, 8 pages
Optimal Control Strategies in an Alcoholism Model, Xun-Yang Wang, Hai-Feng Huo, Qing-Kai Kong, and Wei-Xuan Shi
Volume 2014, Article ID 954069, 18 pages
Dynamics Analysis of a Stochastic SIR Epidemic Model,
Feng Rao
Volume 2014, Article ID 356013, 9 pages
Stability of a Mathematical Model of Malaria Transmission with Relapse,
Hai-Feng Huo and Guang-Ming Qiu
Volume 2014, Article ID 289349, 9 pages

An SIRS Model for Assessing Impact of Media Coverage,
Jing'an Cui and Zhanmin Wu
Volume 2014, Article ID 424610, 6 pages
Antimicrobial Resistance within Host: A Population Dynamics View, Chunji Huang and Aijun Fan
Volume 2014, Article ID 173952, 5 pages
An SIR Epidemic Model with Time Delay and General Nonlinear Incidence Rate,
Mingming Li and Xianning Liu
Volume 2014, Article ID 131257, 7 pages
Dynamics of a Viral Infection Model with General Contact Rate between Susceptible Cells and Virus Particles, Chenxi Dai, Cui Ma, Lijuan Song, and Kaifa Wang
Volume 2014, Article ID 546795, 5 pages
Stochastic Permanence, Stationary Distribution and Extinction of a Single-Species Nonlinear Diffusion System with Random Perturbation, Li Zu, Daqing Jiang, and Donal O’Regan
Volume 2014, Article ID 320460, 14 pages
Bifurcation Analysis of an SIR Epidemic Model with the Contact Transmission Function,
Guihua Li and Gaofeng Li
Volume 2014, Article ID 930541, 7 pages
Modelling the Drugs Therapy for HIV Infection with Discrete-Time Delay,
Xueyong Zhou and Xiangyun Shi
Volume 2014, Article ID 294052, 14 pages
Nonlinear Dynamics of a Nutrient-Plankton Model, Yapei Wang, Min Zhao, Chuanjun Dai, and Xinhong Pan
Volume 2014, Article ID 451757, 10 pages
Multiple Periodic Solutions of a Nonautonomous Plant-Hare Model, Yongfei Gao, P. J. Y. Wong, Y. H. Xia, and Xiaoqing Yuan
Volume 2014, Article ID 130856, 7 pages
Stochastic Analysis of a Hassell-Varley Type Predation Model, Feng Rao, Shunjun Jiang, Yanqiu Li, and Hao Liu
Volume 2013, Article ID 738342, 10 pages
Stochastic Extinction in an SIRS Epidemic Model Incorporating Media Coverage, Liyan Wang, Huilin Huang, Ancha Xu, and Weiming Wang
Volume 2013, Article ID 891765, 8 pages
The Existence of Positive Nonconstant Steady States in a Reaction: Diffusion Epidemic Model,
Yuan Yuan, Hailing Wang, and Weiming Wang
Volume 2013, Article ID 921401, 7 pages

## Editorial

# Advanced Nonlinear Dynamics of Population Biology and Epidemiology 

Weiming Wang, ${ }^{1}$ Yun Kang, ${ }^{2}$ Malay Banerjee, ${ }^{3}$ and Kaifa Wang ${ }^{4}$<br>${ }^{1}$ College of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, China<br>${ }^{2}$ Science and Mathematics Faculty, School of Letters and Sciences, Arizona State University, Mesa, AZ 85212, USA<br>${ }^{3}$ Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur 208016, India<br>${ }^{4}$ School of Biomedical Engineering, Third Military Medical University, Chongqing 400038, China<br>Correspondence should be addressed to Weiming Wang; weimingwang2003@163.com

Received 6 November 2014; Accepted 6 November 2014; Published 22 December 2014
Copyright © 2014 Weiming Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Modern biology and epidemiology have become more and more driven by the need of mathematical models and theory to elucidate general phenomena arising from the complexity of interactions on the numerous spatial, temporal, and hierarchical scales at which biological systems operate and diseases spread. Epidemic modeling and study of disease spread such as gonorrhea, HIV/AIDS, BSE, foot and mouth disease, measles, and rubella have had an impact on public health policy around the world which includes the United Kingdom, The Netherlands, Canada, and the United States. A wide variety of modeling approaches are involved in building up suitable models. Ordinary differential equation models, partial differential equation models, delay differential equation models, stochastic differential equation models, difference equation models, and nonautonomous models are examples of modeling approaches that are useful and capable of providing applicable strategies for the coexistence and conservation of endangered species, to prevent the overexploitation of natural resources, to control disease's outbreak, and to make optimal dosing polices for the drug administration, and so forth.

This special issue is concerned with the nonlinear dynamic modeling and related analysis of interacting populations and important epidemic diseases. All papers submitted to this special issue went through a thorough peer-refereeing process. Based on the reviewer's reports, we collect 50 original research articles by more than 100 active international researchers on differential equations. In the following, we briefly review each of the papers by highlighting the significance of the key contributions.

Twenty papers are concerned about the disease dynamics of differential equations on time scales. N. Wang et al. study the global stability of a multigroup SEIR epidemic model with general latency distribution and general incidence rate and define the basic reproduction number $R_{0}$ as the role of a threshold. L. Feng et al. present a mathematical model which combines the scale-free trait of Internet with the formation of P2P botnet and demonstrate that the model has a globally stable endemic equilibrium when the infection rate is greater than a critical value. C. Liu et al. establish a hybrid SIR vector disease model with incubation and show that there is a phenomenon of singularity inducing bifurcation as well as local stability switch around interior equilibrium when the economic interest increases forward zero. L. Qi et al. establish a mathematical model of schistosomiasis transmission under flood in Anhui province, China, and show that the diseasefree equilibrium is locally asymptotically stable if the basic reproduction number is less than one, and the stability of the unique endemic equilibrium may be changed under some conditions even if the basic reproduction number is larger than one. J. Wang et al. analyze the impact of seasonal activity of psyllid on the dynamics of Huanglongbing (HLB) infection and establish a new model about HLB transmission and show that if $R_{0}<1$, the disease-free periodic solution is globally asymptotically stable while if $R_{0}>1$ the disease persists. F. Wang et al. present an estimating formula for hospital potential capacity and demonstrate that the formula is useful to estimate the basic reproduction number in epidemiology. The results may contribute to the improvement of decision-making in the allocation of medical resources and
the evaluation of the interventions and control efforts of the infectious disease. C. Yan and J. Jia study the local stability of the disease-free, endemic equilibria and Hopf bifurcation of a delayed SIR epidemic model with information variables and limited medical resources. Y. Pei et al. propose a delay SIR epidemic model with difference in immunity and successive vaccination and obtain that the basic reproduction number governs the dynamic behavior of the system. C. Huang and A. Fan study the relationship between antimicrobial resistance and the concentration of antibiotics with dynamical model of competitive population and indicate that long-term highstrength antibiotic treatment and prevention can induce the extinction of susceptible strain. M. Li and X. Liu investigate the disease dynamics of an SIR epidemic model with nonlinear incidence rate and show that the global properties of the system depend on both properties of these general functions. C. Dai et al. investigate the dynamic behavior of a viral infection model with general contact rate between susceptible host cells and free virus particles and give the local stability of the equilibria. M. A. Obaid and A. M. Elaiw propose and analyze two virus infection models with antibody immune response and chronically infected cells and give the global asymptotic stability of all steady states of the models. G. Li and G. Li consider an SIR endemic model in which the contact transmission function is related to the number of infected population and show that the model exhibits the bistability and undergoes saddle-node bifurcation, the Hopf bifurcation, and the Bogdanov-Takens bifurcation. H.-F. Huo and G.-M. Qiu study the dynamics of a malaria model and show that the disease-free equilibrium is globally asymptotically stable if $R_{0}<1$, and the system is uniformly persistent if $R_{0}>1$. C. Chen and Y. Xiao propose a mathematical model to consider the effects of saturated diagnosis and vaccination on HIV/AIDS infection and find that there exists a backward bifurcation which suggests that the disease cannot be eradicated even if the basic reproduction number is less than unity. When the basic reproduction number is greater than unity, the system is uniformly persistent. The findings suggest that increasing vaccination rate and vaccine efficacy and enhancing interventions like reducing share injectors can greatly reduce the transmission of HIV among IDUs in Yunnan province, China. J. Cui and Z. Wu consider an SIRS model incorporating a general nonlinear contact function and find that when the basic reproduction number $R_{0}<1$, the disease-free equilibrium is locally asymptotically stable, while when $R_{0}>1$, there is a unique endemic equilibrium that is locally asymptotically stable. P. Bi and H. Xiao consider a tumor-immune competition model with delay which consists of two-dimensional nonlinear differential equation and give the general formulas for the direction, period, and stability of the bifurcated periodic solutions are given for codimension-1 and codimension-2 bifurcations, including Hopf bifurcation and BT bifurcation. L. Wang et al. study a class of discrete SIRS epidemic models with nonlinear incidence rate and find that if basic reproduction number $R_{0}<1$, then the disease-free equilibrium is globally asymptotically stable, and if $R_{0}>1$, then the model has a unique endemic equilibrium and when some additional conditions hold, the endemic equilibrium is also globally asymptotically stable.
X. Zhou and X. Shi analyze a discrete-time-delay differential mathematical model that describes HIV infection of $\mathrm{CD} 4^{+} \mathrm{T}$ cells with drugs therapy and give the stability of the two equilibria and the existence of Hopf bifurcation at the positive equilibrium. K. Wang et al. propose a patch model for echinococcosis due to dogs migration and show that the dynamics of the model can be completely determined by $R_{0}$. If $R_{0}<1$, the disease-free equilibrium is globally asymptotically stable. When $R_{0}>1$, the model is permanent and endemic equilibrium is globally asymptotically stable.

Seven papers are developed to discuss the stochastic dynamics of population models. S. Zhao and M. Song consider the global existence and positivity of the solution and give sufficient conditions for the global stability in probability of a stochastic predator-prey system with BeddingtonDeAngelis functional response and stage structure. X. Ji and S. Yuan study the dynamics of a delayed stochastic model simulating wastewater treatment process and give the sufficient conditions for the stochastic stability of its positive equilibrium. F. Rao investigates an SIR epidemic model with stochastic perturbations and gives the existence of global positive solutions, stochastic boundedness, and permanence. L. Wang et al. study the stochastic dynamics of an SIRS epidemic model incorporating media coverage and find that if the intensity of noise is large, then the disease is prone to extinction, which can provide us with some useful control strategies to regulate disease dynamics. J. Zhao et al. investigate a stochastic SI epidemic model in the complex networks and show that the solution will oscillate around the disease-free equilibrium of deterministic system when $R_{0} \leq 1$, while it is persistent when $R_{0}>1$. F. Rao et al. investigate a Hassell-Varley type predator-prey model with stochastic perturbations and find some sufficient conditions for stochastically asymptotical boundedness, permanence, persistence in mean, and extinction of the solution. L. Zu et al. analyze the influence of stochastic perturbations on a singlespecies logistic model with the population's nonlinear diffusion among $n$ patches and give the sufficient conditions for stochastic permanence and persistence in mean, stationary distribution, and extinction.

Four papers focus on the traveling wave solutions. X. Tian and R. Xu investigate a delayed SIRS infectious disease model with nonlocal diffusion and nonlinear incidence and derive the existence of a traveling wave solution connecting the disease-free steady state and the endemic steady state. X. Wu et al. establish the existence of traveling wave solutions and small amplitude traveling wave train solutions of a reactiondiffusion system based on a predator-prey model incorporating a prey refuge and analyze the dynamic behavior of this model in the three-dimensional phase space. T. Zhang and Q . Gou consider the minimal wave speed of bacterial colony model with saturated functional response and give the existence and nonexistence of the traveling wave solutions. T. Zhang et al. investigate the spreading speed of a reactiondiffusion cholera model and find that there exists a traveling wave solution.

Four papers study the impulsive dynamics of population models. M. Zhao and C. Dai investigate the population
dynamics of a three-species ecological system with impulsive effect and give the conditions for the system to be permanent when the number of predators released is less than some critical value. In particular, the authors find that less beneficial prey can support the predator alone when more beneficial prey goes extinct. J. Li considers a class of neural networks described by nonlinear impulsive neutral nonautonomous differential equations with delays and gives the criteria on global exponential stability. M. Zhao et al. investigate the dynamics of a Holling-Tanner predator-prey system with state-dependent impulsive effects and give the existence of periodic solution of the system with statedependent impulsive effects. Z. Luo investigates the existence of multiple positive periodic solutions of a class of impulsive functional differential equations with a parameter.

Five papers consider the stationary patterns in the reaction-diffusion equations. L. Zhang focuses on the pattern formation of a ratio-dependent food chain model and finds that the model dynamics exhibits complex pattern replication. X. Lian et al. investigate the spatiotemporal dynamics of a bacterial colony model and derive the conditions for Hopf and Turing bifurcations. L. Zhang and Z. Li focus on a spatially extended Holling-type IV predator-prey model that contains some important factors, such as noise (random fluctuations), external periodic forcing, and diffusion processes, and find that noise or external periodic forcing can induce instability and enhance the oscillation of the species density, and the cooperation between noise and external periodic forces inherent to the deterministic dynamics of periodically driven models gives rise to the appearance of a rich transport phenomenology. Y. Yuan et al. investigate the disease dynamics of a reaction-diffusion epidemic model and give the conditions of the existence and nonexistence of the positive nonconstant steady states, which guarantees the existence of the stationary patterns. Y. Wang et al. investigate a nonlinear reaction-advection-diffusion model of the interaction between nutrients and plankton and find that if the sinking velocity exceeds a certain critical value, the stable state becomes unstable and the wavelength of phytoplankton increases with the increase of sinking velocity.

Three papers investigate the predation dynamics. Y. Gao and S. Liu investigate a predator-prey model with dispersal for both predator and prey among $n$ patches and derive sufficient conditions under which the positive coexistence equilibrium of this model is unique and globally asymptotically stable if it exists. X. Feng et al. formulate and investigate a nonautonomous predator-prey model with infertility control in the prey and give the conditions for the permanence and extinction of fertility prey and infertility prey. X. Fan et al. study the global property in a delayed periodic predator-prey model with stage-structure in prey and density-independence in predator and give the sufficient conditions of the integrable form for the permanence and extinction.

There are seven new results in the special issue. Y. Peng and T. Zhang investigate stability and Hopf bifurcation analysis of a gene expression model with diffusion and time delay and give the local stability and delay-induced Hopf bifurcation. M. Li et al. establish the interaction model of
two-cell populations following the concept of the randomwalk. After assuming that the cell movement is constrained by space limitation primarily, the authors analyze the model to obtain the behavior of two-cell populations as time is close to initial state and far into the future. P. Tang et al. establish a mathematical model for national fitness in China. The results indicate that national fitness put forward by the Chinese government is reasonable, and, in order to increase the number of people who frequently participate in sport exercise in a short period of time, if only one measure can be chosen, guiding people who never take part in physical exercise will be the best measure. X. Wang et al. presents a deterministic SATQ-type mathematical model for the spread of alcoholism with two control strategies and give some properties of the solutions to the model including positivity, existence, and stability. Y. Gao et al. give the sufficient conditions for the existence of at least two positive periodic solutions for a plant-hare model with toxin-determined functional response (nonmonotone). Y. Zhai et al. investigate the behavior of price differential equation model based on economic theory with two delays and show the linear stability and local Hopf bifurcation. W. Wang and B. Liu consider the shunting inhibitory cellular neural networks with time-varying delays in the leakage (or forgetting) terms and employ a novel argument to establish a criterion on the global exponential stability of pseudo-almost periodic solutions.

## Acknowledgments

We would also like to thank the editorial board members of this journal, for their support and help throughout the preparation of this special issue.

Weiming Wang
Yun Kang
Malay Banerjee
Kaifa Wang

## Research Article

# Nonlinear Dynamic in an Ecological System with Impulsive Effect and Optimal Foraging 

Min Zhao ${ }^{1,2}$ and Chuanjun Dai ${ }^{1,2}$<br>${ }^{1}$ School of Life and Environmental Science, Wenzhou University, Wenzhou, Zhejiang 325035, China<br>${ }^{2}$ Zhejiang Provincial Key Laboratory for Water Environment and Marine Biological Resources Protection, Wenzhou University, Wenzhou, Zhejiang 325035, China

Correspondence should be addressed to Min Zhao; zmen@tom.com
Received 13 January 2014; Accepted 15 May 2014; Published 16 June 2014
Academic Editor: Weiming Wang
Copyright © 2014 M. Zhao and C. Dai. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

The population dynamics of a three-species ecological system with impulsive effect are investigated. Using the theories of impulsive equations and small-amplitude perturbation scales, the conditions for the system to be permanent when the number of predators released is less than some critical value can be obtained. Furthermore, because the predator in the system follows the predictions of optimal foraging theory, it follows that optimal foraging promotes species coexistence. In particular, the less beneficial prey can support the predator alone when the more beneficial prey goes extinct. Moreover, the influences of the impulsive effect and optimal foraging on inherent oscillations are studied using simulation, which reveals rich dynamic behaviors such as period-halving bifurcations, a chaotic band, a periodic window, and chaotic crises. In addition, the largest Lyapunov exponent and the power spectra of the strange attractor, which can help analyze the chaotic dynamic behavior of the model, are investigated. This information will be useful for studying the dynamic complexity of ecosystems.


## 1. Introduction

In recent years, interest in studying nonlinear dynamic systems has exploded. In the 1970s, since the pioneering work of May on the relationship between food-web complexity and stability and the chaotic phenomenon [1-3], more and more researchers have become interested in dynamic behavior involving ecological mechanisms that promote species diversity [4-20]. More recently, dynamic systems' studies have benefited from an infusion of interest and new techniques in ecology.

It is known that when a predator is shared by two noncompeting species, predator-mediated apparent competition often leads to competitive exclusion of one prey population [21]. This phenomenon is related to optimal foraging and adaptive foraging. A two-prey-one-predator population model with optimal predator foraging behavior has been studied in a fine-grained environment [22-24]. On this basis, Křivan and Eisner considered a system composed of two prey
types and an optimally foraging predator [25] in a system described by the following model:

$$
\begin{align*}
& \dot{x}(t)=x(t)( r_{1}(x(t)) \\
&\left.-\frac{\lambda_{1} z(t)}{1+h_{1} \lambda_{1} x(t)+u h_{2} \lambda_{2} y(t)}\right) \\
& \dot{y}(t)=y(t)\left(r_{2}(y(t))\right.  \tag{1}\\
&\left.-\frac{u \lambda_{2} z(t)}{1+h_{1} \lambda_{1} x(t)+u h_{2} \lambda_{2} y(t)}\right) \\
& \dot{z}(t)=z(t)\left(\frac{e_{1} \lambda_{1} x(t)+u e_{2} \lambda_{2} y(t)}{1+h_{1} \lambda_{1} x(t)+u h_{2} \lambda_{2} y(t)}-m\right) .
\end{align*}
$$

This paper considers an impulsive differential-equation model based on model (1), which assumes that predators
forage according to optimal foraging theory [23, 24]. This system can be expressed by the following equations:

$$
\begin{gather*}
\dot{x}(t)=r_{1} x(t)\left(\frac{k_{0}-x(t)}{k_{1}-x(t)}\right)-b_{1} x^{2}(t) \\
-\frac{\lambda_{1} x(t) z(t)}{1+h_{1} \lambda_{1} x(t)+u h_{2} \lambda_{2} y(t)} \\
\dot{y}(t)=r_{2} y(t)\left(\frac{k_{2}-y(t)}{k_{3}-y(t)}\right)-b_{2} y^{2}(t) \\
-\frac{u \lambda_{2} y(t) z(t)}{1+h_{1} \lambda_{1} x(t)+u h_{2} \lambda_{2} y(t)} \\
\dot{z}(t)=z(t)\left(\frac{e_{1} \lambda_{1} x(t)+u e_{2} \lambda_{2} y(t)}{1+h_{1} \lambda_{1} x(t)+u h_{2} \lambda_{2} y(t)}-m\right)  \tag{2}\\
t \neq n T \\
x\left(t^{+}\right)=x(t) \\
y\left(t^{+}\right)=y(t) \\
z\left(t^{+}\right)=z(t)+p \\
t=n T
\end{gather*}
$$

where $x(t), y(t), z(t)$ are, respectively, the densities of two prey types and one predator at time $t, r_{i}(i=1,2)$ is the per capita prey intrinsic growth rate, $r_{1} \cdot k_{0}\left(0 \leq k_{0} / k_{1} \leq\right.$ $1), r_{2} \cdot k_{2}\left(0 \leq k_{2} / k_{3} \leq 1\right)$ are the respective carrying capacities of the prey, and $k_{1}, k_{3}$ are the corresponding values of available resources or in the ideal case (i.e., where no resources are wasted) the carrying capacity. However, the ideal case is impossible in reality. The ratios $k_{0} / k_{1}, k_{2} / k_{3}$ express the relative efficiency of nutrient utilization in species $x, y$. At any time, if $x<k_{0}<k_{1}, y<k_{2}<k_{3}$, the efficiency is high as long as $k_{0} / k_{1}, k_{2} / k_{3}$ are close to one; when the values are lower, this indicates that resource limitations are restricting the population increase [23]. $b_{i}(i=1,2)$ are the rate of intraspecific competition of the prey, $\lambda_{i}(i=1,2)$ is the cropping rate of a predator feeding on the $i$ th prey type, $e_{i}(i=1,2)$ is the conversion factor relating predator reproduction to prey consumption, and $h_{i}(i=1,2)$ is the per capita mortality rate for the forager. In this paper, it is assumed that prey type $x$ is more beneficial than the other and hence $e_{1} / h_{1}>e_{2} / h_{2}[26,27]$. To study optimal foraging, a control parameter $u(0 \leq u \leq 1)$ is introduced [25], which represents the probability that the alternative second prey type is included in the predator's diet. $T$ is the period of the impulsive effect, $n \in N$, and $p>0$ is the number of predators released at $t=n T$. To achieve a set of conditions which can guarantee that the system will be permanent and that the numbers of the two prey types are not so large that they go extinct because of exceeding the carrying capacity of the environment, the model will release a certain number of predators only at $t=n T$ because the predator is assumed to be a versatile and advanced predator.

The rest of this paper is organized as follows. Section 2 will review the effect of impulsive perturbations, establish
conditions for extinction, and obtain the conditions for permanence of System (2) using the Floquet theory of impulsive equations at small-amplitude perturbation scales. In Section 3, the results of computer-based numerical analysis are shown and discussed briefly. In addition, the largest Lyapunov exponent, which also indicates the chaotic dynamic behavior of the model, is computed, and the Fourier spectra, which illustrate the qualitative nature of strange attractors, are plotted. Finally, conclusions and remarks are stated.

## 2. Analysis of the System

Let $R_{+}=[0, \infty), R_{+}=\left\{X \in R^{3}: X \geq 0, X=(x, y, z)\right\}$, $\Omega=\operatorname{int} R_{+}^{3}$, and let $N$ be the set of all nonnegative integers. The map $g=\left(g_{1}, g_{2}, g_{3}\right)^{T}$ is defined by the right-hand side of the first three equations of System (2).

Let $V: R_{+} \times R_{+}^{3} \rightarrow R_{+}$; then $V$ is said to belong to class $V_{0}$ if
(1) $V$ is continuous in $(n T,(n+1) T] \times R_{+}^{3}$, and for each $x \in R_{+}^{3}, n \in N, \lim _{(t, y) \rightarrow\left(n T^{+}, x\right)} V(t, y)=V\left(n T^{+}, x\right)$ exists;
(2) $V$ is locally Lipschitzian in $X$.

Definition 1. Let $V \in V_{0}$; then, for $(t, x) \in(n T,(n+1) T] \times$ $R_{+}^{3}$, the upper right derivative of $V(t, x)$ with respect to the impulsive differential System (2) can be defined as

$$
\begin{equation*}
D^{+} V(t, x)=\lim _{h \rightarrow 0} \sup \frac{1}{h}[V(t+h, x+h g(t, x))-V(t, x)] \tag{3}
\end{equation*}
$$

The solution of System (2) is a piecewise continuous function $X: R_{+} \rightarrow R_{+}^{3}$, where $X(t)$ is continuous on ( $n T$, $(n+$ 1) $T], n \in N$, and $X\left(n T^{+}\right)=\lim _{t \rightarrow n T} X(t)$ exists. Obviously, the smoothness property of $g$ guarantees the global existence and uniqueness of a solution of System (2) (for details, see [28-30]).

Definition 2. System (2) is said to be permanent if there exists a compact region $\Omega=\operatorname{int} R_{+}^{3}$ such that every solution $(x(t), y(t), z(t))$ of System (2) will eventually enter and remain in the region $\Omega$.

The following lemma will now be presented.
Lemma 3. Let $X(t)$ be a solution of System (2) with $X\left(0^{+}\right) \geq 0$; then $X(t) \geq 0$ for all $t \geq 0$, and furthermore $X(t)>0, t \geq 0$ if $X\left(0^{+}\right)>0$.

An important comparison theorem will now be used on the impulsive differential equation.

Lemma 4 (see [28-30]). Let $V \in V_{0}$ and assume that

$$
\begin{array}{cl}
D^{+} V(t, X) \leq f(t, V(t, X)) & t \neq n T \\
V\left(t, X\left(t^{+}\right)\right) \leq \varphi_{n}(V(t, X)) & t=n T \tag{4}
\end{array}
$$

where $f: R_{+} \times R_{+} \rightarrow R$ is continuous in $(n T,(n+1) T] \times R_{+}$and for $\mu \in R_{+}, n \in N, \lim _{(t, y) \rightarrow\left(n T^{+}, \mu\right)} f(t, y)=f\left(n T^{+}, \mu\right)$ exists
and $\varphi_{n}: R_{+} \rightarrow R_{+}$is nondecreasing. Let $r(t)$ be the maximal solution to the scalar impulsive differential equation

$$
\begin{gather*}
\dot{u}(t)=f(t, u(t)) \quad t \neq n T \\
u\left(t^{+}\right)=\varphi_{n}(u(t)) \quad t=n T  \tag{5}\\
u\left(0^{+}\right)=u_{0}
\end{gather*}
$$

existing on $[0, \infty)$. Then $V\left(0^{+}, X_{0}\right) \leq u_{0}$ implies that $V(t, X(t)) \leq r(t), t \geq 0$, where $X(t)$ is any solution to System (2).

Theorem 5. There exists a constant $M$ such that $x(t) \leq$ $M, y(t) \leq M$, and $z(t) \leq M$ for each solution $X(t)=$ $(x(t), y(t), z(t))$ of System (2) for all t large enough.

Proof. Define $V(t, X(t))$ such that

$$
\begin{equation*}
V(t, X(t))=e_{1} x(t)+e_{2} y(t)+z(t) \tag{6}
\end{equation*}
$$

where $V \in V_{0}$. Since $\dot{x}(t) \leq r_{1} x(t)-b_{1} x^{2}(t)$ and $\dot{y}(t) \leq$ $r_{2} y(t)-b_{2} y^{2}(t)$, then $x(t) \leq r_{1} / b_{1}, y(t) \leq r_{2} / b_{2}$, and the upper right derivative of $V(t, X(t))$ can be calculated along a solution of System (2), yielding the following impulsive differential equation:

$$
\begin{align*}
D^{+} V(t)+L V(t)= & L e_{1} x(t)+r_{1} e_{1} x(t)\left(\frac{k_{0}-x(t)}{k_{1}-x(t)}\right) \\
& -b_{1} e_{1} x^{2}(t)+L e_{2} y(t) \\
& +r_{2} e_{2} y(t)\left(\frac{k_{2}-y(t)}{k_{3}-y(t)}\right)-b_{2} e_{2} y^{2}(t) \\
& +(L-m) z(t) \quad t \neq n T \\
V\left(t^{+}\right) & =V(t)+p \quad t=n T . \tag{7}
\end{align*}
$$

Obviously,

$$
\begin{align*}
D^{+} V(t)+L V(t) \leq & \left(L e_{1}+r_{1} e_{1}\right) x(t)-b_{1} e_{1} x^{2}(t) \\
& +\left(L e_{2}+r_{2} e_{2}\right) y(t)-b_{2} e_{2} y^{2}(t)  \tag{8}\\
& +(L-m) z(t) .
\end{align*}
$$

Let $0<L<m$; then $D^{+} V(t)+L V(t)$ is bounded. Select $L_{1}, L_{2}$ such that

$$
\begin{gather*}
D^{+} V(t) \leq-L_{1} V(t)+L_{2} \quad t \neq n T  \tag{9}\\
V\left(t^{+}\right)=V(t)+p \quad t=n T,
\end{gather*}
$$

where $L_{1}, L_{2}$ are two positive constants.

According to Lemma 4,

$$
\begin{align*}
V(t) \leq & \left(V\left(0^{+}\right)-\frac{L_{2}}{L_{1}}\right) \exp \left(-L_{1} t\right) \\
& +\frac{p \exp \left(1-\exp \left(-n L_{1} t\right)\right)}{\exp \left(L_{1} T\right)-1} \exp \left(L_{1} t\right)  \tag{10}\\
& \times \exp \left(-L_{1}(t-n t)\right)+\frac{L_{2}}{L_{1}}
\end{align*}
$$

where $t \in(n T,(n+1) T]$. Hence,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} V(t) \leq \frac{L_{2}}{L_{1}}+\frac{p \exp \left(L_{1} T\right)}{\exp \left(L_{1} T\right)-1} \tag{11}
\end{equation*}
$$

Therefore, $V(t, X(t))$ is ultimately bounded, and it follows that each positive solution of System (2) is uniformly ultimately bounded. This completes the proof.

Next, some basic properties of the following subsystem of System (2), in which the two prey types are absent, will be defined:

$$
\begin{gather*}
\dot{z}(t)=-m z(t) \quad t \neq n T \\
z\left(t^{+}\right)=z(t)+p \quad t=n T  \tag{12}\\
z\left(0^{+}\right)=z_{0} .
\end{gather*}
$$

Clearly, $z^{*}(t)=p \exp (-m(t-n T)) /(1-\exp (-m T)), t \in$ $(n T,(n+1) T], n \in N, z^{*}\left(0^{+}\right)=p /(1-\exp (-m T))$ is a positive periodic solution of System (12). Hence,

$$
\begin{equation*}
z(t)=\left(z\left(0^{+}\right)-\frac{p}{1-\exp (-m T)}\right) \exp (-m T)+z^{*}(t) \tag{13}
\end{equation*}
$$

is a solution of System (12) with initial value $z_{0} \geq 0$, where $t \in(n T,(n+1) T], n \in N$.

Lemma 6. For a positive periodic solution $z^{*}(t)$ of System (12) and every solution $z(t)$ of System (12) with $z_{0} \geq 0, \mid z(t)-$ $z^{*}(t) \mid \rightarrow 0, t \rightarrow \infty$.

Hence, when only the predator is present, it is possible to obtain the complete expression for the periodic solution $\left(0,0, z^{*}(t)\right)$ of System (2).

Based on these discussions, the following theorems can be proved.

Theorem 7. Let $(x(t), y(t), z(t))$ be any solution of System (2). Then
(1) $\left(0,0, z^{*}(t)\right)$ is said to be locally asymptotically stable if $p>\max \left(r_{1} k_{0} \operatorname{Tm} / \lambda_{1} k_{1}, r_{2} k_{2} \operatorname{Tm} / u \lambda_{2} k_{3}\right)$;
(2) $\left(0,0, z^{*}(t)\right)$ is said to be globally asymptotically stable if $p>\max \left(r_{1} k_{0} T m / \lambda_{1} k_{1}, r_{2} k_{2} T m / u \lambda_{2} k_{3}\right)$ and

$$
\begin{align*}
\frac{p \exp (-m T)}{1-\exp (-m T)}>\max ( & \frac{r_{1}\left(1+h_{1} \lambda_{1} M+u h_{2} \lambda_{2} M\right)}{\lambda_{1}} \\
& \left.\frac{r_{2}\left(1+h_{1} \lambda_{1} M+u h_{2} \lambda_{2} M\right)}{u \lambda_{2}}\right) . \tag{14}
\end{align*}
$$

Proof. The local stability of the periodic solution $\left(0,0, z^{*}(t)\right)$ may be determined by considering the behavior of smallamplitude perturbations of the solution. Define

$$
\begin{equation*}
x(t)=u(t), \quad y(t)=v(t), \quad z(t)=w(t)+z^{*}(t) . \tag{15}
\end{equation*}
$$

Then substitute (15) into System (2). The linearization of the system becomes

$$
\begin{gather*}
\dot{u}(t)=\left(r_{1} \frac{k_{0}}{k_{1}}-\lambda_{1} z^{*}\right) u(t) \\
\dot{v}(t)=\left(r_{2} \frac{k_{2}}{k_{3}}-u \lambda_{2} z^{*}\right) v(t) \\
\dot{w}(t)=z^{*} e_{1} \lambda_{1} u(t)+z^{*} u e_{2} \lambda_{2} v(t)-m w(t) \\
t \neq n T  \tag{16}\\
u\left(t^{+}\right)=u(t) \\
v\left(t^{+}\right)=v(t) \\
w\left(t^{+}\right)=w(t) \\
t=n T .
\end{gather*}
$$

Therefore,

$$
\left(\begin{array}{c}
u(t)  \tag{17}\\
v(t) \\
w(t)
\end{array}\right)=\Phi(t)\left(\begin{array}{c}
u(0) \\
v(0) \\
w(0)
\end{array}\right) \quad 0 \leq t \leq T,
$$

where $\Phi(t)$ satisfies

$$
\frac{d \Phi}{d t}=\left(\begin{array}{ccc}
r_{1} \frac{k_{0}}{k_{1}}-\lambda_{1} z^{*} & 0 & 0  \tag{18}\\
0 & r_{2} \frac{k_{2}}{k_{3}}-u \lambda_{2} z^{*} & 0 \\
z^{*} e_{1} \lambda_{1} & z^{*} u e_{2} \lambda_{2} & -m
\end{array}\right) \Phi(t)
$$

and $\Phi(0)=I$, the identity matrix, and

$$
\left(\begin{array}{c}
u\left(n T^{+}\right)  \tag{19}\\
v\left(n T^{+}\right) \\
w\left(n T^{+}\right)
\end{array}\right)=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{c}
u(n T) \\
v(n T) \\
w(n T)
\end{array}\right) .
$$

The stability of the periodic solution $\left(0,0, z^{*}(t)\right)$ is determined by the eigenvalues of

$$
\Theta=\left(\begin{array}{lll}
1 & 0 & 0  \tag{20}\\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \Phi(T)
$$

which are

$$
\begin{aligned}
& \mu_{1}=\exp \left(\int_{0}^{T} r_{1} \frac{k_{0}}{k_{1}}-\lambda_{1} z^{*} d t\right) \\
& \mu_{2}=\exp \left(\int_{0}^{T} r_{2} \frac{k_{2}}{k_{3}}-u \lambda_{2} z^{*} d t\right) \\
& \mu_{3}=\exp (-m T)<1
\end{aligned}
$$

According to Floquet theory, $\left(0,0, z^{*}(t)\right)$ is locally asymptotically stable if $\left|\mu_{1}\right|<1$ and $\left|\mu_{2}\right|<1$; that is, $p>$ $\max \left(r_{1} k_{0} T m / \lambda_{1} k_{1}, r_{2} k_{2} T m / u \lambda_{2} k_{3}\right)$.

If $\left(0,0, z^{*}(t)\right)$ is locally asymptotically stable and a global attractor, then $\left(0,0, z^{*}(t)\right)$ is globally asymptotically stable. In the following, global attractiveness will be demonstrated.

Let $V(t)=x(t)+y(t)$; then

$$
\begin{align*}
\left.\dot{V}(t)\right|_{(2)}= & r_{1} x(t)\left(\frac{k_{0}-x(t)}{k_{1}-x(t)}\right)-b_{1} x^{2}(t) \\
& -\frac{\lambda_{1} x(t) z(t)}{1+h_{1} \lambda_{1} x(t)+u h_{2} \lambda_{2} y(t)} \\
& +r_{2} y(t)\left(\frac{k_{2}-y(t)}{k_{3}-y(t)}\right)-b_{2} y^{2}(t) \\
& -\frac{u \lambda_{2} y(t) z(t)}{1+h_{1} \lambda_{1} x(t)+u h_{2} \lambda_{2} y(t)}  \tag{22}\\
\left.\dot{V}(t)\right|_{(2)} \leq & \left(r_{1}-\frac{\lambda_{1} z(t)}{1+h_{1} \lambda_{1} x(t)+u h_{2} \lambda_{2} y(t)}\right) x(t) \\
& -b_{1} x^{2}(t) \\
& +\left(r_{2}-\frac{u \lambda_{2} z(t)}{1+h_{1} \lambda_{1} x(t)+u h_{2} \lambda_{2} y(t)}\right) y(t) \\
& -b_{2} y^{2}(t) .
\end{align*}
$$

By Theorem 5, there exists a constant $M>0$ such that $x(t) \leq M, y(t) \leq M$, and $z(t) \leq M$ for each solution $X(t)=(x(t), y(t), z(t))$ of System (2) with all $t$ large enough. Therefore,

$$
\begin{gather*}
\dot{z}(t)=z(t)\left(\frac{e_{1} \lambda_{1} x(t)+u e_{2} \lambda_{2} y(t)}{1+h_{1} \lambda_{1} x(t)+u h_{2} \lambda_{2} y(t)}-m\right) \\
\geq-m z(t) \quad t \neq n T .  \tag{23}\\
z\left(t^{+}\right)=z(t)+p \quad t=n T
\end{gather*}
$$

By Lemmas 4 and 6, it is known that there exists $t_{1}>0$ and it is possible to select $\varepsilon>0$ small enough so that $z(t) \geq$ $z^{*}(t)-\varepsilon$. Therefore, for all $t \geq t_{1}$,

$$
\begin{align*}
z(t) & \geq z^{*}(t)-\varepsilon \\
& =\frac{p \exp (-m(t-n T))}{1-\exp (-m T)}-\varepsilon  \tag{24}\\
& \geq \frac{p \exp (-m T)}{1-\exp (-m T)}-\varepsilon .
\end{align*}
$$

Define

$$
\begin{equation*}
\gamma=\frac{p \exp (-m T)}{1-\exp (-m T)}-\varepsilon \tag{25}
\end{equation*}
$$

Then $r_{1}-\left(\lambda_{1} \gamma /\left(1+h_{1} \lambda_{1} M+u h_{2} \lambda_{2} M\right)\right)<0$ and $r_{1}-$ $\left(u \lambda_{2} \gamma /\left(1+h_{1} \lambda_{1} M+u h_{2} \lambda_{2} M\right)\right)<0$. Thus, for $t \geq t_{1}$,

$$
\begin{align*}
\left.\dot{V}(t)\right|_{(2)} \leq & \left(r_{1}-\frac{\lambda_{1} \gamma}{1+h_{1} \lambda_{1} M+u h_{2} \lambda_{2} M}\right) x(t)-b_{1} x^{2}(t) \\
& +\left(r_{2}-\frac{u \lambda_{2} \gamma}{1+h_{1} \lambda_{1} M+u h_{2} \lambda_{2} M}\right) y(t) \\
& -b_{2} y^{2}(t)<0 \tag{26}
\end{align*}
$$

So $V(t) \rightarrow 0, x(t) \rightarrow 0, y(t) \rightarrow 0$ as $t \rightarrow \infty$. Note that the limiting case of System (2) is exactly System (12) together with Lemma 6. It follows that the periodic solution $\left(0,0, z^{*}(t)\right)$ is a global attractor. This completes the proof.

Theorem 8. System (2) is permanent if

$$
\begin{gather*}
p<\min \left(\frac{r_{1} k_{0} T m}{\lambda_{1} k_{1}}, \frac{r_{2} k_{2} T m}{u \lambda_{2} k_{3}}\right) \\
\frac{p \exp (-m T)}{1-\exp (-m T)}<\min \left(\frac{r_{1}\left(1+h_{1} \lambda_{1} M+u h_{2} \lambda_{2} M\right)}{\lambda_{1}},\right. \\
\left.\frac{r_{2}\left(1+h_{1} \lambda_{1} M+u h_{2} \lambda_{2} M\right)}{u \lambda_{2}}\right) . \tag{27}
\end{gather*}
$$

Proof. Let $X(t)=(x(t), y(t), z(t))$ be any solution of System (2) with $X(0)>0$. From Theorem 5, assume that $x(t) \leq$ $M, y(t) \leq M, z(t) \leq M$.

From System (2), $\dot{x}(t) \leq\left(r_{1}-b_{1} x(t)\right) x(t)$.
Consider the following equation:

$$
\begin{align*}
\dot{w}(t) & =w(t)\left(r_{1}-b_{1} w(t)\right) \\
w(0) & =x(0) . \tag{28}
\end{align*}
$$

It is possible to obtain $x(t) \leq w(t)$ and $w(t) \rightarrow r_{1} / b_{1}$ as $t \rightarrow \infty$. Hence, for any $\varepsilon_{1}>0, x(t)<r_{1} / b_{1}+\varepsilon_{1}$ for all $t$ sufficiently large. For simplification, it may be assumed that $x(t)<r_{1} / b_{1}+\varepsilon_{1}$ holds for all $t>0$. The same arguments can be made for any $\varepsilon_{2}>0, y(t)<r_{2} / b_{2}+\varepsilon_{2}$ for all $t>0$. Let $m_{3}=(p \exp (-m T)) /(1-\exp (-m T))-\varepsilon(\varepsilon>0)$. Note that $\dot{z}(t) \geq-m z(t)$, and consider the following equation:

$$
\begin{align*}
\dot{w}(t) & =-b_{3} w(t) \quad t \neq n T \\
w\left(t^{+}\right) & =w(t)+p \quad t=n T  \tag{29}\\
w\left(0^{+}\right) & =z\left(0^{+}\right)>0 .
\end{align*}
$$

From Lemmas 4 and $6, z(t) \geq w(t)>z^{*}(t)-\varepsilon>0$, and hence $z(t)>m_{3}$ for all $t$ sufficiently large. Therefore, it is necessary to find $l_{1}>0$ and $l_{2}>0$ such that $x(t) \geq l_{1}$, $y(t) \geq l_{2}$ for all $t$ large enough. This can be done as shown in the following two steps.

First, choose $0<l_{1}<m / 2 e_{1} \lambda_{1}, 0<l_{2}<m / 2 u e_{2} \lambda_{2}$ such that $e_{1} \lambda_{1} l_{1}+u e_{2} \lambda_{2} l_{2}<m$. Then there exist $t_{1} \in(0, \infty)$ and $t_{2} \in(0, \infty)$ such that $x\left(t_{1}\right) \geq l_{1}, y\left(t_{2}\right) \geq l_{2}$. Otherwise,
(1) there exists a $t_{2}>0$ such that $y\left(t_{2}\right) \geq l_{2}$, but $x\left(t_{1}\right)<l_{1}$ for all $t>0$;
(2) there exists a $t_{1}>0$ such that $x\left(t_{1}\right) \geq l_{1}$, but $y\left(t_{2}\right)<l_{2}$ for all $t>0$;
(3) there are $x\left(t_{1}\right)<l_{1}$ and $y\left(t_{2}\right)<l_{2}$ for all $t>0$.

For case (1), according to Theorem 8, select $\eta_{1}>0$ small enough so that

$$
\begin{equation*}
\omega_{1}=\int_{n T}^{(n+1) T}\left(r_{1}-b_{1} l_{1}-\lambda_{1}\left(v^{*}+\eta_{1}\right)\right) d t>0 \tag{30}
\end{equation*}
$$

From case (1),

$$
\begin{align*}
& \dot{z}(t) \leq\left(e_{1} \lambda_{1} x(t)+u e_{2} \lambda_{2} y(t)-m\right) z(t) \\
& \leq\left(-m+e_{1} \lambda_{1} l_{1}+u e_{2} \lambda_{2}\left(\frac{r_{2}}{b_{2}}+\varepsilon_{1}\right)\right) z(t)  \tag{31}\\
&=A z(t) \quad t \neq n T \\
& \quad z\left(t^{+}\right)=z(t)+p \quad t=n T,
\end{align*}
$$

where $A=-m+e_{1} \lambda_{1} l_{1}+u e_{2} \lambda_{2}\left(r_{2} / b_{2}+\varepsilon_{1}\right)$. Therefore, $z(t) \leq$ $v(t)$ and $v(t) \rightarrow v^{*}(t)$ as $t \rightarrow \infty$, where $v(t)$ is the solution of the following equation:

$$
\begin{gather*}
\dot{v}(t)=A v(t) \quad t \neq n T \\
v\left(t^{+}\right)=v(t)+p \quad t=n T  \tag{32}\\
v\left(0^{+}\right)=z\left(0^{+}\right) \geq 0
\end{gather*}
$$

and $v^{*}(t)=p \exp (A(t-n T)) /(1-\exp (A T))$. Therefore, there exists a $T_{1}>0$, when $t>T_{1}$,

$$
\begin{gather*}
z(t) \leq v(t)<v^{*}(t)+\eta_{1},  \tag{33}\\
\dot{x}(t) \geq\left(r_{1}-b_{1} l_{1}-\lambda_{1}\left(v^{*}(t)+\eta_{1}\right)\right) x(t) \quad t \neq n T \\
x\left(t^{+}\right)=x(t) \quad t=n T . \tag{34}
\end{gather*}
$$

Let $N_{1} \in N$ and $N_{1} T \geq T_{1}$. Integrating (34) on ( $n T$, $(n+$ 1)T], $n \geq N_{1}$, the following result can be obtained:

$$
\begin{align*}
& x((n+1) T) \\
& \quad \geq x(n T) \exp \left(\int_{n T}^{(n+1) T}\left(r_{1}-b_{1} l_{1}-\lambda_{1}\left(v^{*}+\eta_{1}\right)\right) d t\right) \\
& \quad=x(n T) \exp \left(\omega_{1}\right) . \tag{35}
\end{align*}
$$

Then $x\left(\left(N_{1}+k\right) T\right) \geq x\left(N_{1} T\right) \exp \left(k \omega_{1}\right) \rightarrow \infty$ as $k \rightarrow \infty$, which is a contradiction.

For case (2), the same arguments can be used.
Now consider case (3). Choose $\eta_{2}>0$ small enough so that

$$
\begin{equation*}
\omega_{2}=\int_{n T}^{(n+1) T}\left(r_{1}-b_{1} l_{1}-\lambda_{1}\left(u^{*}+\eta_{2}\right)\right) d t>0 \tag{36}
\end{equation*}
$$

From case (3),

$$
\begin{align*}
\dot{z}(t) \leq & \left(e_{1} \lambda_{1} x(t)+u e_{2} \lambda_{2} y(t)-m\right) z(t) \\
\leq & \left(-m+e_{1} \lambda_{1} l_{1}+u e_{2} \lambda_{2} l_{2}\right) z(t) \\
= & B z(t) \quad t \neq n T  \tag{37}\\
& z\left(t^{+}\right)=z(t)+p \quad t=n T,
\end{align*}
$$

where $B=-m+e_{1} \lambda_{1} l_{1}+u e_{2} \lambda_{2} l_{2}$. Therefore, $z(t) \leq u(t)$ and $u(t) \rightarrow u^{*}(t)$ as $t \rightarrow \infty$, where $u(t)$ is the solution of the following equation:

$$
\begin{align*}
\dot{u}(t) & =B u(t) \quad t \neq n T \\
u\left(t^{+}\right) & =u(t)+p \quad t=n T  \tag{38}\\
u\left(0^{+}\right) & =z\left(0^{+}\right) \geq 0
\end{align*}
$$

and $u^{*}(t)=p \exp (B(t-n T)) /(1-\exp (B T))$. Then there exists a $T_{2}>0$, when $t \geq T_{2}$, such that

$$
\begin{equation*}
z(t) \leq u(t)<u^{*}(t)+\eta_{2} \tag{39}
\end{equation*}
$$

$$
\begin{gather*}
\dot{x}(t) \geq\left(r_{1}-b_{1} l_{1}-\lambda_{1}\left(u^{*}(t)+\eta_{2}\right)\right) x(t) \quad t \neq n T  \tag{40}\\
x\left(t^{+}\right)=x(t) \quad t=n T .
\end{gather*}
$$

Let $N_{2} \in N$ and $N_{2} T \geq T_{2}$. Integrating (40) on ( $n T$, $(n+$ 1)T], $n \geq N_{2}$, the following result can be obtained:

$$
\begin{align*}
& x((n+1) T) \\
& \quad \geq x(n T) \exp \left(\int_{n T}^{(n+1) T}\left(r_{1}-b_{1} l_{1}-\lambda_{1}\left(u^{*}+\eta_{2}\right)\right) d t\right) \\
& \quad=x(n T) \exp \left(\omega_{2}\right) . \tag{41}
\end{align*}
$$

Then $x\left(\left(N_{2}+k\right) T\right) \geq x\left(N_{2} T\right) \exp \left(k \omega_{2}\right) \rightarrow \infty$ as $k \rightarrow \infty$, which is a contradiction.

In conclusion, there exist $t_{1}>0$ and $t_{2}>0$ such that $x(t) \geq l_{1}, y(t) \geq l_{2}$.

Second, if $x\left(t_{1}\right) \geq l_{1}$, for all $t>t_{1}$, then the objective has been attained. Otherwise, there exists $t$ such that $x(t)<l_{1}$, for $t>t_{1}$. Let $t^{*}=\inf _{t<t^{*}}\left\{x(t)<l_{1}\right\}$. Then $x\left(t_{1}\right) \geq l_{1}$, for $t \in\left[t_{1}, t^{*}\right)$ and $t^{*} \in\left(n_{1} T,\left(n_{1}+1\right) T\right], n_{1} \in N$, and $x\left(t^{*}\right)=l_{1}$, because $x(t)$ is continuous. Choose $n_{2}, n_{3} \in N$ such that

$$
\begin{gather*}
n_{2} T>T_{2}=\frac{\ln \left(n_{1} /(M+p)\right)}{A}  \tag{42}\\
\exp \left(\delta\left(n_{2}+1\right) T\right) \exp \left(n_{3} \omega_{1}\right)>1
\end{gather*}
$$

where $\delta=r_{1}-b_{1} l_{1}-\lambda_{1} M<0$. Let $T^{\prime}=n_{2} T+n_{3} T$; then there must be a $t^{\prime} \in\left(\left(n_{1}+1\right) T,\left(n_{1}+1\right) T+T^{\prime}\right]$ such that $x\left(t^{\prime}\right) \geq l_{1}$; otherwise $x(t)<l_{1}, t \in\left(\left(n_{1}+1\right) T,\left(n_{1}+1\right) T+T^{\prime}\right]$. Considering (32) with $v\left(\left(n_{1}+1\right) T^{+}\right)=z\left(\left(n_{1}+1\right) T^{+}\right)$,

$$
\begin{align*}
v(t)= & \left(v\left(\left(n_{1}+1\right) T^{+}\right)-\frac{p}{1-\exp (A T)}\right)  \tag{43}\\
& \times \exp \left(A\left(t-\left(n_{1}+1\right) T\right)\right)+v^{*}(t)
\end{align*}
$$

for $t \in(n T,(n+1) T]$ and $n_{1}+1<n<n_{1}+1+n_{2}+n_{3}$. Then, for $\left(n_{1}+1+n_{2}\right) T \leq t \leq\left(n_{1}+1\right) T+T^{\prime}$,

$$
\begin{align*}
\left|v(t)-v^{*}(t)\right| & <(M+p) \exp \left(A\left(t-\left(n_{1}+1\right) T\right)\right)<\eta_{1} \\
& z(t) \leq v(t)<v^{*}(t)+\eta_{1} \tag{44}
\end{align*}
$$

It can be concluded that (34) holds for $\left(n_{1}+1+n_{2}\right) T \leq t \leq$ $\left(n_{1}+1\right) T+T^{\prime}$. As in the first step above, it is possible to obtain $x\left(\left(n_{1}+1+n_{2}+n_{3}\right) T\right) \geq x\left(\left(n_{1}+1+n_{2}\right) T\right) \exp \left(n_{3} \omega_{1}\right)$. There are two possible cases for $t \in\left(t^{*},\left(n_{1}+1\right) T\right]$.

Case (a). If $x(t)<l_{1}$ for $t \in\left(t^{*},\left(n_{1}+1\right) T\right]$, then $x(t)<l_{1}$ for $t \in\left(t^{*},\left(n_{1}+1+n_{2}\right) T\right]$. From System (2),

$$
\begin{align*}
\dot{x}(t) & \geq\left(r_{1}-b_{1} x(t)-\lambda_{1} z(t)\right) x(t)  \tag{45}\\
& \geq\left(r_{1}-b_{1} l_{1}-\lambda_{1} M\right) x(t)=\delta x(t) .
\end{align*}
$$

Integrating (45) on $\left(t^{*},\left(n_{1}+1+n_{2}\right) T\right], x\left(\left(n_{1}+1+n_{2}\right) T\right) \geq$ $l_{1} \exp \left(\delta\left(1+n_{2}\right) T\right)$.

Thus

$$
\begin{align*}
& x\left(\left(n_{1}+1+n_{2}+n_{3}\right) T\right) \\
& \quad \geq l_{1} \exp \left(\delta\left(1+n_{2}\right) T\right) \exp \left(n_{3} \omega_{1}\right)>l_{1} \tag{46}
\end{align*}
$$

which is a contradiction.
Let $\bar{t}=\inf _{t>t^{*}}\left\{x(t) \geq l_{1}\right\}$, so that $x(\bar{t})=l_{1}$ and (45) holds on $\left[t^{*}, \bar{t}\right)$. Integrating (45) on $\left[t^{*}, \bar{t}\right)$,

$$
\begin{align*}
x(t) & \geq x\left(t^{*}\right) \exp \left(\delta\left(t-t^{*}\right)\right) \\
& \geq l_{1} \exp \left(\delta\left(1+n_{2}+n_{3}\right) T\right) \xrightarrow{\Delta} \bar{l}_{1} . \tag{47}
\end{align*}
$$

For $t>\bar{t}$, the same arguments can be used because $x(\bar{t}) \geq l_{1}$.
Case (b). There exists a $t^{\prime \prime} \in\left(t^{*},\left(n_{1}+1\right) T\right]$ such that $x\left(t^{\prime \prime}\right) \geq$ $l_{1}$; let $\bar{t}=\inf _{t>t^{*}}\left\{x(t) \geq l_{1}\right\}$; then $x(t)<l_{1}$ for $\left[t^{*}, \bar{t}\right)$ and $x(\bar{t})=l_{1}$. For $t \in\left[t^{*}, \bar{t}\right)$, (45) holds true. Integrating (45) on $\left[t^{*}, \bar{t}\right)$,

$$
\begin{equation*}
x(t) \geq x\left(t^{*}\right) \exp \left(\delta\left(t-t^{*}\right)\right) \geq l_{1} \exp (\delta T)>\bar{l}_{1} \tag{48}
\end{equation*}
$$

For $t>\bar{t}$, the same arguments can be used because $x(\bar{t}) \geq l_{1}$.
In summary, $x(\bar{t}) \geq \bar{l}_{1}$ can be obtained for all $t>t_{1}$. In the same way, it can be proved that $y(\bar{t}) \geq \bar{l}_{2}$ for all $t>t_{2}$. This completes the proof.

## 3. Numerical Analysis

3.1. The Impulsive Effect and Optimal Foraging. To study the population dynamics of a three-species ecological model with impulsive effect, the solution of System (2) with initial conditions in the first quadrant is obtained numerically for a biologically feasible range of parametric values, and the bifurcation diagram provides a summary of the basic population dynamic behavior of the system.


Figure 1: Time series of System (2) when $u=0.8$ and $p=4.2$.

Now two different control parameters will be discussed, the number of predators released, $p$, and the probability $u$. Other parameters are set to

$$
\begin{array}{cccc}
a_{1}=0.35, & a_{2}=0.4, & r_{1}=0.9, & r_{2}=0.8, \\
k_{0}=15, & k_{1}=20, & k_{2}=12, & k_{3}=15, \\
b_{1}=0.045, & b_{2}=0.2, & h_{1}=0.8, & h_{2}=0.45, \\
e_{1}=0.5, & e_{2}=0.6, & m=0.1, & T=20 . \tag{49}
\end{array}
$$

From Theorem 7, it is known that the prey-eradication periodic solution $\left(0,0, z^{*}(t)\right)$ is locally asymptotically stable provided that $p>p_{\max } \approx 4.098648$. Figure 1 shows a typical prey-eradication periodic solution of System (2), in which it can be observed that the variable $z$ oscillates in a stable cycle. At the same time, the prey types $x$ and $y$ rapidly diminish and go to zero beyond $p_{\max } \approx 4.098648$. If the number of predators released, $p$, is less than $p_{\max }$, the prey-eradication solution becomes unstable. It is, however, possible that the two prey types and the predator can coexist in a stable positive periodic solution. In other words, the system can be permanent when the number of predators released, $p$, is less than $p_{\max }$.


Figure 2: Bifurcation diagram of System (2) with initial conditions $x(0)=0.3, y(0)=0.5, z(0)=0.3, u=0.8$, and $0.009 \leq p \leq 4.5$.

Next, the bifurcation diagrams for the control parameter $p$ will be examined. Figure 2 is plotted as a function of the bifurcation parameter $p$ and shows that the system has rich population dynamic behavior consistent with the theoretical analysis, such as period-halving bifurcation (see Figure 3), a chaotic band, a periodic window, and chaotic crises. Furthermore, Theorem 8 indicates that the system is permanent when the value of $p$ is less than some critical value. When the value of $p$ is in the interval [ $0.009,3.257895$ ], the two prey types and one predator can coexist. When the value of $p$ is in the interval [3.257895, 4.098648], the prey $x$ will become extinct rapidly, but the prey $y$ and the predator $z$ can coexist. These results may show that prey $x$ is inferior to prey $y$ in its ability to reproduce or prey $x$ is a favorite food of predator $z$. When the number of predators released is greater than some critical value, all species in the system will become extinct. All these results demonstrate the effectiveness of mathematical analysis for understanding such systems.

The next question is how $u$ impacts the complex population dynamics. In Figure 4, when prey and predator
populations are plotted as a function of the probability $u$, the value of $p$ is 1.45 . In the former case, it is assumed that the foraging behavior of predator $z$ follows optimal foraging theory [16-18] and prey $x$ is more beneficial for predator $z$ than prey $y$. In other words, the more beneficial prey $x$ is always included in the predator's diet, but if the density of prey $x$ falls below a critical threshold or goes to zero, prey $y$ is included with probability one. From Figure 4, it can be clearly observed that the two prey types and the predator can coexist in the intervals $[0,0.1136]$ and $[0.7215,0.8792]$, where the system dynamics can be chaotic, periodic, or nonperiodic. In the interval ( $0.1136,0.7215$ ), prey $x$ goes extinct, while prey $y$ and predator $z$ can coexist stably. This means that if the more beneficial prey $x$ disappears, prey $y$ alone can support the population of predator $z$. As $u$ increases, prey $y$ goes extinct, but prey $x$ and predator $z$ can coexist stably.
3.2. The Largest Lyapunov Exponent. Deterministic chaos is an important problem that is solved by measuring the largest


FIgure 3: Transition from chaos to period-halving in System (2): (a) chaos when $p=1.5$; (b) 8T-periodic solution when $p=1.73$; (c) 4T-periodic solution when $p=1.85$; (d) 2T-periodic solution when $p=2.5$.

Lyapunov exponent [31-36]. Based on research by various investigators, these results have confirmed the importance of detecting and exploring chaos. In this paper, the largest Lyapunov exponents for chaotic system (2) are examined. The largest Lyapunov exponents take into account the average exponential rates of divergence or convergence of nearby orbits in phase space [31,32]. If the attractor is chaotic, the largest Lyapunov exponent must be positive, which implies a stable or a periodic state. In Figure 2, the corresponding largest Lyapunov exponent ( $[0 \leq p \leq 3]$ ) can be calculated for System (2) (see Figure 5).
3.3. The Strange Attractor and Power Spectra. To study the properties of strange attractors, commonly used methods such as power spectra can be used $[35,36]$. A power spectrum was calculated using 4096 points corresponding to the time series of the variable $x$ with time increment $\Delta t=0.5[35,36]$. For strange attractors (a) and (b), it is known that the value of the largest Lyapunov exponent for the strange attractor (a) is 0.25603 , while for (b) the computed largest Lyapunov
exponent is 0.30567 . Therefore, strange attractors (a) and (b) are chaotic attractors. Moreover, the strange attractor (b) displays more chaotic dynamics than (a) because its positive exponent is larger than that of (a). In addition, the spectra of strange attractors (a) and (b) consist of strong broadband components and sharp peaks (Figures 6(c) and 6(d)) These results conform to the observation that strange attractors (a) and (b) arise from some weak limit cycles which can lose stability due to noise.

## 4. Conclusions and Remarks

Complex population dynamics of a three-species ecological model with optimal foraging and impulsive control strategy have been investigated both numerically and analytically. The periodic solution has been shown to be globally asymptotically stable by use of the Floquet theorem and small-amplitude perturbations, if $p>\max \left(r_{1} k_{0} T m / \lambda_{1} k_{1}, r_{2} k_{2} T m / u \lambda_{2} k_{3}\right)$ and $p \exp (-m T) /$ $(1-\exp (-m T))>\max \left(r_{1}\left(1+h_{1} \lambda_{1} M+u h_{2} \lambda_{2} M\right) / \lambda_{1}, r_{2}(1+\right.$


Figure 4: Bifurcation diagram of System (2) with initial conditions $p=1.45, x(0)=0.3, y(0)=0.5, z(0)=0.3$, and $0 \leq u \leq 1$.


Figure 5: The largest Lyapunov exponents for System (2) with $p$ varying from 0 to 3.
$\left.\left.h_{1} \lambda_{1} M+u h_{2} \lambda_{2} M\right) / u \lambda_{2}\right)$. At the same time, using the method of comparison involving multiple Lyapunov functions, the permanence of the system can be proved. Bifurcation diagrams of the impulsive perturbation $p$ and the probability parameter $u$ have also been obtained. The bifurcation diagrams of $p$ have shown that dynamic complexity exists in System (2), including chaotic behavior, periodic
windows, chaotic bands, chaotic crises, and period-halving bifurcations. The bifurcation diagrams of $u$ indicate that optimal foraging promotes species coexistence and that if the more beneficial prey goes extinct, the less beneficial prey can support the predator so that it will not die out. In addition, the presence of chaotic dynamics was confirmed, and the qualitative nature of strange attractors was investigated using


Figure 6: Strange attractors and power spectra: (a)strange attractor when $p=1.2$, (b) strange attractor when $p=1.51$, (c) power spectrum of attractor (a), and (d) power spectrum of attractor (b).
computer simulations of the largest Lyapunov exponents and Fourier spectra. All these results may be useful in the study of the dynamic complexity of ecosystems.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant nos. 31370381 and 31170338), by the Key Program of Zhejiang Provincial Natural Science Foundation of China (Grant no. LZ12C03001), and by the

National Key Basic Research Program of China (973 Program, Grant no. 2012CB426510).

## References

[1] R. M. May, "Will a large complex system be stable?" Nature, vol. 238, no. 5364, pp. 413-414, 1972.
[2] R. M. May, "Simple mathematical models with very complicated dynamics," Nature, vol. 261, no. 5560, pp. 459-467, 1976.
[3] R. M. May, "Biological populations with nonoverlapping generations: stable points, stable cycles, and chaos," Science, vol. 186, no. 4164, pp. 645-647, 1974.
[4] H. Zhu, S. A. Campbell, and G. S. K. Wolkowicz, "Bifurcation analysis of a predator-prey system with nonmonotonic functional response," SIAM Journal on Applied Mathematics, vol. 63, no. 2, pp. 636-682, 2002.
[5] S.-B. Hsu, T.-W. Hwang, and Y. Kuang, "Global analysis of the Michaelis-Menten-type ratio-dependent predator-prey system," Journal of Mathematical Biology, vol. 42, no. 6, pp. 489506, 2001.
[6] Y. Do, H. Baek, Y. Lim, and D. Lim, "A three-species food chain system with two types of functional responses," Abstract and Applied Analysis, vol. 2011, Article ID 934569, 16 pages, 2011.
[7] S. Lv and M. Zhao, "The dynamic complexity of a three species food chain model," Chaos, Solitons and Fractals, vol. 37, no. 5, pp. 1469-1480, 2008.
[8] W. Wang, J. Shen, and J. J. Nieto, "Permanence and periodic solution of predator-prey system with Holling type functional response and impulses," Discrete Dynamics in Nature and Society, vol. 2007, Article ID 81756, 15 pages, 2007.
[9] L. Zhang and M. Zhao, "Dynamic complexities in a hyperparasitic system with prolonged diapause for host," Chaos, Solitons and Fractals, vol. 42, no. 2, pp. 1136-1142, 2009.
[10] R. K. Upadhyay and R. K. Naji, "Dynamics of a three species food chain model with Crowley-Martin type functional response," Chaos, Solitons and Fractals, vol. 42, no. 3, pp. 13371346, 2009.
[11] M. Zhao, L. Zhang, and J. Zhu, "Dynamics of a host-parasitoid model with prolonged diapause for parasitoid," Commипications in Nonlinear Science and Numerical Simulation, vol. 16, no. 1, pp. 455-462, 2011.
[12] C. Wei and L. Chen, "A delayed epidemic model with pulse vaccination," Discrete Dynamics in Nature and Society, vol. 2008, Article ID 746951, 12 pages, 2008.
[13] J. Jiao, S. Cai, and L. Chen, "Analysis of a stage-structured predatory-prey system with birth pulse and impulsive harvesting at different moments," Nonlinear Analysis: Real World Applications, vol. 12, no. 4, pp. 2232-2244, 2011.
[14] R. Shi and L. Chen, "Stage-structured impulsive SI model for pest management," Discrete Dynamics in Nature and Society, vol. 2007, Article ID 97608, 11 pages, 2007.
[15] H.-F. Huo, Z.-P. Ma, and C.-Y. Liu, "Persistence and stability for a generalized Leslie-Gower model with stage structure and dispersal," Abstract and Applied Analysis, vol. 2009, Article ID 135843, 17 pages, 2009.
[16] H. Yu, S. Zhong, and R. P. Agarwal, "Mathematics analysis and chaos in an ecological model with an impulsive control strategy," Communications in Nonlinear Science and Numerical Simulation, vol. 16, no. 2, pp. 776-786, 2011.
[17] L. Wang, L. Chen, and J. J. Nieto, "The dynamics of an epidemic model for pest control with impulsive effect," Nonlinear Analysis: Real World Applications, vol. 11, no. 3, pp. 1374-1386, 2010.
[18] H. Yu, S. Zhong, R. P. Agarwal, and S. K. Sen, "Effect of seasonality on the dynamical behavior of an ecological system with impulsive control strategy," Journal of the Franklin Institute: Engineering and Applied Mathematics, vol. 348, no. 4, pp. 652670, 2011.
[19] J. Luo, "Permanence and extinction of a generalized Gause-type predator-prey system with periodic coefficients," Abstract and Applied Analysis, vol. 2010, Article ID 845606, 24 pages, 2010.
[20] X. Wang, W. Wang, and X. Lin, "Dynamics of a two-prey one-predator system with Watt-type functional response and impulsive control strategy," Chaos, Solitons and Fractals, vol. 40, no. 5, pp. 2392-2404, 2009.
[21] R. D. Holt, "Predation, apparent competition and the structure of prey communities," Theoretical Population Biology, vol. 12, no. 2, pp. 197-229, 1977.
[22] H. Werner, "Optimal foraging and the size selection of prey by the bluegill sunfish (Lepomis macrochirus)," Ecology, vol. 55, pp. 1042-1052, 1974.
[23] E. L. Charnov, "Optimal foraging: attack strategy of a mantid," The American Naturalist, vol. 110, no. 971, pp. 141-151, 1976.
[24] D. W. Stephens and J. R. Krebs, Foraging Theory, Princeton University Press, Princeton, NJ, USA, 1986.
[25] V. Křivan and J. Eisner, "Optimal foraging and predator-prey dynamics III," Theoretical Population Biology, vol. 63, no. 4, pp. 269-279, 2003.
[26] V. Křivan, "Optimal foraging and predator-prey dynamics," Theoretical Population Biology, vol. 49, no. 3, pp. 265-290, 1996.
[27] V. Křivan and A. Sikder, "Optimal foraging and predator-prey dynamics, II," Theoretical Population Biology, vol. 55, no. 2, pp. 111-126, 1999.
[28] V. Lakshmikantham, D. D. Bă̆nov, and P. S. Simeonov, Theory of Impulsive Differential Equations, vol. 6, World Scientific, Singapore, 1989.
[29] D. D. Baĭnov and P. S. Simeonov, Impulsive Differential Equations: Asymptotic Properties of the Solutions, World Scientific, Singapore, 1993.
[30] D. D. Bainov and V. C. Covachev, Impulsive Differential Equations with a Small Parameter, vol. 24, World Scientific, Singapore, 1994.
[31] S. Lv and M. Zhao, "The dynamic complexity of a hostparasitoid model with a lower bound for the host," Chaos, Solitons and Fractals, vol. 36, no. 4, pp. 911-919, 2008.
[32] M. Zhao and S. Lv, "Chaos in a three-species food chain model with a Beddington-DeAngelis functional response," Chaos, Solitons and Fractals, vol. 40, no. 5, pp. 2305-2316, 2009.
[33] M. Zhao and L. Zhang, "Permanence and chaos in a hostparasitoid model with prolonged diapause for the host," Communications in Nonlinear Science and Numerical Simulation, vol. 14, no. 12, pp. 4197-4203, 2009.
[34] L. Zhu and M. Zhao, "Dynamic complexity of a host-parasitoid ecological model with the Hassell growth function for the host," Chaos, Solitons and Fractals, vol. 39, no. 3, pp. 1259-1269, 2009.
[35] H. Yu, S. Zhong, R. P. Agarwal, and S. K. Sen, "Threespecies food web model with impulsive control strategy and chaos," Communications in Nonlinear Science and Numerical Simulation, vol. 16, no. 2, pp. 1002-1013, 2011.
[36] H. Yu, S. Zhong, and M. Ye, "Dynamic analysis of an ecological model with impulsive control strategy and distributed time delay," Mathematics and Computers in Simulation, vol. 80, no. 3, pp. 619-632, 2009.

## Research Article

# Successive Vaccination and Difference in Immunity of a Delay SIR Model with a General Incidence Rate 

Yongzhen Pei, ${ }^{1}$ Li Changguo, ${ }^{2}$ Qianyong $\mathrm{Wu},{ }^{1}$ and Yunfei $\mathrm{Lv}^{3}$<br>${ }^{1}$ School of Computer Science and Software Engineering, Tianjin Polytechnic University, Tianjin 300387, China<br>${ }^{2}$ Department of Basic Science, Military Transportation University, Tianjin 300161, China<br>${ }^{3}$ School of Science, Tianjin Polytechnic University, Tianjin 300387, China

Correspondence should be addressed to Yongzhen Pei; yongzhenpei@163.com
Received 28 December 2013; Revised 7 May 2014; Accepted 8 May 2014; Published 16 June 2014
Academic Editor: Kaifa Wang
Copyright © 2014 Yongzhen Pei et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

A delay SIR epidemic model with difference in immunity and successive vaccination is proposed to understand their effects on the disease spread. From theorems, it is obtained that the basic reproduction number governs the dynamic behavior of the system. The existence and stability of the possible equilibria are examined in terms of a certain threshold condition about the basic reproduction number. By use of new computational techniques for delay differential equations, we prove that the system is permanent. Our results indicate that the recovery rate and the vaccination rate are two factors for the dynamic behavior of the system. Numerical simulations are carried out to investigate the influence of the key parameters on the spread of the disease, to support the analytical conclusion, and to illustrate possible behavioral scenarios of the model.


## 1. Introduction

The current threat of some new type diseases has raised our awareness that curbing the spread of some emerging and reemerging human diseases is of public health importance such as H1N1. This emerging disease, which was first reported in Mexico, spread very quickly, due to the travel of infected persons by airplanes, trains, and buses to some other regions. It continued to spread around the world and caused about 5000 deaths. In recent years, many mathematical models have been developed for the transmission dynamics of infectious diseases such as SARS, HIV/AIDS, measles, and smallpox ([1-6], to name a few, and the references therein). These models have provided understanding of the underlying mechanisms which influence the spread of diseases and suggested some control strategies. Moreover, to our knowledge, the first effective control strategy for the elimination of infectious diseases is obtaining immunity. It has been reported that "People's immunity to A/H1N1 flu virus is greater than previously thought after access vaccines. The WHO is working to give more nations access vaccines to fight the H1N1 flu pandemic."

Besides these above studies, many authors formulated and analyzed SIR epidemic models for the control of diseases [7-20]. In particular, some authors have studied the effects of vaccination on the spread of diseases [7-11]; others have studied the effects of treatment on the spread of diseases [12-15]. Gao et al. have proposed an epidemic model with density-dependent birth pulses and seasonal prevention [16]. Recently, some works have investigated permanent and temporary immunity [17-20]. However, in these SIR models, an unrealistic assumption is that all the rest of infected individuals acquire immunity besides death. Measles encephalitis in adults in [21,22] shows that there is difference in immunity of infected individuals. That is, some infected individuals can acquire immunity after recovery, but some do not acquire immunity and can be infected once more. At the same time, vaccination is an important strategy for the elimination of infectious diseases [7-11].

Vaccinations have many types; impulsive vaccination and successive vaccination are two main policies. Successive vaccination is that people have been vaccinated at birth to protect themselves from disease; the studies can be found in
[23, 24]. Makinde in [23] studied a SIR model for the transmission dynamics of a childhood disease in the presence of a preventive vaccine and analyzed the vaccination reproductive number for disease control and eradication qualitatively. Impulsive vaccination (only at fixed time sequence we execute effectively the vaccination for the disease) is an important and effective strategy for the elimination of infectious diseases and has been studied in the literature. For example, see [10, 17, 25-27]. In above-mentioned papers, authors almost considered the vaccination of susceptible population. But, in fact, under a certain situation, the vaccine treatment also should be considered for the newborns of the susceptible, the exposed, and the removed. We find that there are few studies on the aspect of the vaccination of newborns. In this case, successive vaccination seems more reasonable than impulsive vaccination. Therefore, in this paper, a SIR model with difference in immunity and successive vaccination is considered.

As far as disease transmission is concerned, the incidence rate, defined as the rate of new infection, plays a very important role in modelling infectious diseases. Bilinear incidence rate $\beta S I$ in $[28,29]$ and standard incidence rates $\lambda S I / N$ in $[30,31]$ have often been used in epidemic models. However, it is unreasonable to consider the bilinear incidence rate (based on the law of mass action) as the number of susceptibles is large, owing to the number of susceptibles with which every infective contact within a certain time is limited. Standard incidence rate may be a good approximation if the number of available partners is large enough but it is not possible to make more contacts when the population $N$ is small. Combine the two previous approaches by assuming that if the number of available partners $N$ is low, the number of actual per capita partners is proportional to $N$, whereas if the number of available partners is large, there is a saturation effect which makes the number of actual partners constant. Considering this case, a saturation incidence rate of type $f(I) S$ with $f(I)=k I /(1+\alpha I)$ is being proposed in [32]. More general incidence rate used in the literature is the one for which $f(I)=k I^{l} /\left(1+\alpha I^{h}\right)[33,34]$, where $I^{l}$ measures the infection force of the disease and $f(I)=1 /\left(1+\alpha I^{h}\right)$ measures the inhibitory effect caused by behavioral changes. Note that if $f(I)$ is decreasing when $I$ is large, this may be interpreted as the fact that susceptibles tend to reduce their social contacts if the perceived number of infectives increases over a psychologically significant value. The above saturation incidence rate depends also on the size of the infectives $I$ termed as infectives-dependent. Particular examples of susceptibles-dependent incidence rate are $f(S)=k S /(1+\alpha S)$ [35]. Very general incidence rates which are not linear in $S$ are also used in Derrick and van den Driessche [36] $(f(S, I, N)=$ $I \Phi(S, I, N)$, where $N=S+I)$, Korobeinikov and Maini [37] $\left(f(S, I)=h_{1}(I) h_{2}(S)\right)$, and Moghadas and Alexander [38] $(f(S, I)=\beta(1+g(I, v)) I S)$.

Nie et al. [19] and Ji et al. [39] respectively considered a delay SIR epidemic model with nonlinear incidence rate and density-dependent birth and death rates. Motivated by the main idea described in $[6,39]$, in this paper, we consider a delay SIR model with difference in immunity and
successive vaccination and an abstract incidence rate. The main difference between our study and those described in [6, 39 ] is the difference in immunity and successive vaccination and an abstract incidence rate. An abstract incidence rate of type $f(I) S$ is employed to model the spread of the disease which is propagated through the infective individuals, under a few biologically feasible assumptions upon $f(I)$.

In view of above facts, we will formulate a mathematical model in Section 2. We provide the region of biologically feasible solutions in Section 3. Then, we study the existence and stability of the steady states in the next section, analyze the permanence result in Section 5, and give some numerical simulations in Section 6. Lastly, we end the paper with a brief discussion of our results in Section 6.

## 2. Model Formulation and Invariant Region

In this section, we will present a delay SIR epidemic model with a general nonlinear incidence rate. The total population $N(t)$ is divided into three subclasses, namely, the susceptibles $S(t)$, the infectives $I(t)$, and the recovered individuals $R(t)$. Based on the SIR model in [12, 39], we considered following system:

$$
\begin{align*}
\frac{d S}{d t}= & {\left[b-\frac{a r N}{K}\right] N-\beta e^{-d_{1} \tau} S f(I(t-\tau)) } \\
& -\left[d+\frac{(1-a) r N}{K}\right] S-\theta S+\mu_{1} I \\
\frac{d I}{d t}= & \beta e^{-d_{1} \tau} f(I(t-\tau)) S \\
& -\left[d+\frac{(1-a) r N}{K}\right] I-\mu_{1} I-e^{-d_{1} \omega} \delta I(t-\omega)  \tag{1}\\
\frac{d R}{d t}= & e^{-d_{1} \omega} \delta I(t-\omega)-\left[d+\frac{(1-a) r N}{K}\right] R+\theta S \\
& N(0)=S_{0}>0, \quad I(0)=\varphi(\theta) \geq 0 \\
& \forall \theta \in[-\bar{\tau}, 0], R(0)=R_{0} \geq 0
\end{align*}
$$

where $\bar{\tau}=\max \{\tau, \omega\}$ and $\varphi \in C([-\bar{\tau}, 0], R)$. We give the following useful assumptions.
(1) There are no disease induced deaths, and all the newborns are susceptible.
(2) $f(I)$ is the nonlinear incidence rate satisfying the following assumptions:

$$
\begin{gather*}
f(0)=0, \quad f^{\prime}(I)>0, \quad f^{\prime \prime}(I)<0, \\
\lim _{t \rightarrow \infty} f(I)=c<+\infty . \tag{2}
\end{gather*}
$$

(3) The force of infection at any time $t$ is dominated by $\beta e^{-d_{1} \tau} S(t) f(I(t-\tau))$, where $\tau$ is incubation period and $0<e^{-d_{1} \tau} \leq 1$ represents the survival probability of individuals in the population after time $\tau$ [20]. It is also assumed that $d_{1} \leq d$ in $[-\tau, 0]$, where $d$ is the death rate and $d_{1}$ is the death rate in the time interval $[-\tau, 0]$.
(4) The parameters $a$ are a convex combination constant, $r=b-d>0$ is the intrinsic growth rate ( $b$ is the birth rate), and $K>0$ is the carrying capacity of the population. The term $(b-(\operatorname{ar} N(t) / K))$ has a densitydependent per capita birth rate and the term $(d+((1-$ a) $r N(t) / K)$ ) has a density-dependent per capita death rate [39].
(5) For $0<a<1$, the birth and death rates are consistent with the limited resources associated with density dependence. The birth rate is density independent when $a=0$ and the death rate is density independent when $a=1$. Thus, the spread of the disease (animals such as rodents, etc.) is assumed to be governed by the following system of logistic equations with time delay.
(6) The total population is assumed to be large enough to be adequately described by a deterministic model and is divided into compartments based on the disease status [40].
(7) The successive vaccination rate $\theta$ is positive. The positive constant $\mu_{1}$ is the recovery rate of the infectious individuals from compartment $I$ to $S$. The parameters $\beta$ are the effective per capita contact rate constant of infected individuals. The parameters $\delta$ are the recovery rate of infected individuals.
(8) Models are formulated as functional differential and/or integral equations when time delay is included [40]. Ours follows the former with the assumption that the $I$-equation satisfies a certain integral condition [41].

Models with multiple delays are not common, but few authors have in the past considered these-Beretta et al. [42]to name but a few. Since $N(t)=S(t)+I(t)+R(t)$, thus the governing equation (1) can be rewritten as

$$
\begin{gathered}
\frac{d N}{d t}=r\left[1-\frac{N}{K}\right] N \\
\frac{d I}{d t}=\beta e^{-d_{1} \tau}(N-I-R) f(I(t-\tau)) \\
-\left[d+\frac{(1-a) r N}{K}\right] I-\mu_{1} I-e^{-d_{1} \omega} \delta I(t-\omega), \\
\frac{d R}{d t}=e^{-d_{1} \omega} \delta I(t-\omega) \\
-\left[d+\frac{(1-a) r N}{K}\right] R+\theta(N-I-R) .
\end{gathered}
$$

Let $\bar{\tau}=\max (\omega, \tau)$. Then (3) satisfies the following initial conditions

$$
\begin{array}{r}
N(0)=S_{0}>0, \quad I(0)=\varphi(\theta) \geq 0 \\
\forall \theta \in[-\bar{\tau}, 0], R(0)=R_{0} \geq 0 \tag{4}
\end{array}
$$

In this paper, we will consider two different delays $\tau, \omega$ which are important parameters on the dynamic behavior. So, the present study is continuation of the previous work $\tau=\omega$ by Naresh et al. [43].
Lemma 1. All solutions of the model system (3) starting in $R_{+}^{3}$ are bounded and eventually enter the compact attracting set

$$
\begin{equation*}
\Phi=\left\{(S, I, R) \in R_{+}^{3}: S(t)+I(t)+R(t)=N(t) \leq K\right\} . \tag{5}
\end{equation*}
$$

Lemma 2. Let the initial data be $N(0)=S_{0}>0, I(0)=$ $I_{0}(u) \geq 0$, for all $u \in[-\bar{\tau}, 0]$, with $I_{0}(0)>0, R(0)=R_{0} \geq$ 0 . Then, the solution $(S(t), I(t), R(t))$ of the model remains positive for all time $t>0$.

Lemma 3 (see [44]). For the characteristic equation in the form $p(\lambda)+q(\lambda) e^{-r \lambda}=0$, where $p$ and $q$ are polynomials with real coefficients and $r>0$ is the delay, suppose
(a) $p(\lambda) \neq 0, R(\lambda)>0$;
(b) $|q(i y)|<|p(i y)| ; 0 \leq y<\infty$;
(c) $\lim _{|\lambda| \rightarrow \infty, R(\lambda) \geq 0}|q(\lambda) / p(\lambda)|=0$.

Then $R(\lambda)<0$ for every root $\lambda$ and all $r>0$.

## 3. Equilibrium and Stability Analysis

In this section, we focus on the existence and local stability of equilibria. Let the right-hand side of equalities in model (3) be zero. Then, there are two equilibria; namely,
(i) $\left.E_{0}=(K, 0, p), p=K \theta /[d+(1-a) r+\theta]\right)$, disease-free equilibrium;
(ii) $E^{*}=\left(N^{*}, I^{*}, R^{*}\right)$, endemic equilibrium,
where the values of $N^{*}, I^{*}$, and $R^{*}$ are given in Section 3.2.
3.1. Community Matrix. Firstly, after computing the Jacobian or community matrix of model (3) at point ( $N, I, R$ ), the characteristic equation is given by

$$
\left|\begin{array}{ccc}
r-\frac{2 r}{K} N-\lambda & 0 & 0  \tag{6}\\
\beta e^{-d_{1} \tau} f(I)-\frac{(1-a) r}{K} I & m e^{-\left(d_{1}+\lambda\right) \tau}-\delta e^{-\left(d_{1}+\lambda\right) \omega}-n-\lambda & \beta e^{-d_{1} \tau} f(I) \\
\theta-\frac{(1-a) r}{K} R & \delta e^{-\left(d_{1}+\lambda\right) \omega}-\theta & -\left[d+\frac{(1-a) r}{K} N+\theta\right]-\lambda
\end{array}\right|=0
$$

where $m=\beta(N-I-R) f^{\prime}(I), n=\beta e^{-d_{1} \tau} f(I)+d+((1-$ a) $r / K) N+\mu_{1}$.

Now, we analyze the equilibria stability of system (3). Computing the Jacobian of system (3) evaluated at $E_{0}$, one gets the following matrix

$$
J\left(E_{0}\right)=\left(\begin{array}{ccc}
-r-\lambda & 0 & 0  \tag{7}\\
0 & a_{22}-\lambda & 0 \\
\frac{(d+\theta) \theta}{d+(1-a) r+\theta} & \delta e^{-\left(d_{1}+\lambda\right) \tau}-\theta & -[d+(1-a) r+\theta]-\lambda
\end{array}\right)
$$

where

$$
\begin{align*}
a_{22}= & \beta(K-p) f^{\prime}(0) e^{-\left(d_{1}+\lambda\right) \tau} \\
& -\delta e^{-\left(d_{1}+\lambda\right) \omega}-\left[d+(1-a) r+\mu_{1}\right] \tag{8}
\end{align*}
$$

Denote

$$
\begin{equation*}
A=\beta(K-p) f^{\prime}(0), \quad C=d+(1-a) r+\mu_{1} \tag{9}
\end{equation*}
$$

then

$$
\begin{equation*}
a_{22}=A e^{-\left(d_{1}+\lambda\right) \tau}-\delta e^{-\left(d_{1}+\lambda\right) \omega}-C \tag{10}
\end{equation*}
$$

Denote

$$
\begin{equation*}
h(\lambda)=A e^{-\left(d_{1}+\lambda\right) \tau}-\delta e^{-\left(d_{1}+\lambda\right) \omega}-C-\lambda . \tag{11}
\end{equation*}
$$

The eigenvalues of the system (3) about the steady state $E_{0}$ are $\lambda_{1}=-r, h(\lambda)=0$ and $\lambda_{3}=-[d+(1-a) r+\theta]$. All the parameters of the model are assumed to be nonnegative. Therefore, $\lambda_{1}$ and $\lambda_{3}$ are negative. Next, we discuss the roots of $h(\lambda)=0$ in five cases.

Case 1. For $\tau=\omega \neq 0$, from the second equation of the system (3), we can get the following.

Proposition 4. For $\tau=\omega>0, R(\lambda)<0$ for every root $\lambda$ of $h(\lambda)=0$ when

$$
\begin{equation*}
(A-\delta) e^{-d_{1} \tau}<C \tag{12}
\end{equation*}
$$

Proof. From the above analysis, $\lambda_{2}$ satisfies the following characteristic equation:

$$
\begin{equation*}
g(\lambda)=(A-\delta) e^{-\left(d_{1}+\lambda\right) \tau}-C-\lambda=0 \tag{13}
\end{equation*}
$$

(1) Clearly, $\lambda=0$ is not a root of $g(\lambda)=0$.
(2) From the fact that $g(0)<0, g^{\prime}(\lambda)<0$ for $\lambda>0$, it is obtained that $g(\lambda)=0$ has no positive real root.
(3) It is sufficient to show that $g(\lambda)=0$ does not admit a purely imaginary root. In fact, if $\lambda=i v(v>0)$ is a root of $(g(\lambda)=0)$, then by separating the real part, one gets

$$
\begin{equation*}
(A-\delta) e^{-d_{1} \tau} \cos (v \tau)=C \tag{14}
\end{equation*}
$$

Together with the condition of Proposition 4, we have

$$
\begin{equation*}
\cos (\nu \tau)>1 \tag{15}
\end{equation*}
$$

This is impossible.
(4) It is easy to show that $g(\lambda)=0$ has no imaginary root whose real part is positive. Otherwise, there is an imaginary root $\lambda=u+i v$ with $u>0$. Without any loss of generality, we consider $v>0$. Then, we take the real and imaginary parts of $g(\lambda)=0$; namely,

$$
\begin{equation*}
(A-\delta) e^{-\left(d_{1}+u\right) \tau} \cos (v \tau)=C+u \tag{16}
\end{equation*}
$$

Combined with (9), we have

$$
\begin{equation*}
C>(A-\delta) e^{-d_{1} \tau}>(A-\delta) e^{-\left(d_{1}+u\right) \tau} \cos (v \tau)=C+u \tag{17}
\end{equation*}
$$

This is a contradiction which implies that all eigenvalues roots of $g(\lambda)$ have negative real parts. Therefore, the disease-free equilibrium of the system (3) is locally asymptotically stable when (9) holds. The proof is completed.

Case 2. For $\tau \neq 0, \omega=0$, by the same way as in Case 1, one gets the following.

Proposition 5. For all $\tau \neq 0, \omega=0, R(\lambda)<0$ for every root $\lambda$ of $h(\lambda)=0$ when

$$
\begin{equation*}
A e^{-d_{1} \tau}-\delta<C \tag{18}
\end{equation*}
$$

Case 3. For $\omega \neq 0, \tau=0$, one gets the following.
Proposition 6. For $\omega \neq 0, \tau=0, R(\lambda)<0$ for every root $\lambda$ of $h(\lambda)=0$ when

$$
\begin{equation*}
\delta e^{-d_{1} \omega}<|A-C| \tag{19}
\end{equation*}
$$

Proof. By the fact that $h(\lambda)=0$ is equivalent to $p(\lambda)+$ $q(\lambda) e^{-\lambda \omega}=0$ with $p(\lambda)=A-C-\lambda, q(\lambda)=-\delta e^{-d_{1} \omega}$.
(i) Suppose $\lambda=u+i v(u>0)$. Then, $p(\lambda)=A-C-u-$ $i v \neq 0$.
(ii) By $|q(i v)|=\delta e^{-d_{1} \omega},|p(i v)|=\mid A-C-$ $i v \mid=\sqrt{(A-c)^{2}+v^{2}}$, together with the condition of Proposition 6, we know that $|q(i v)|<|p(i v)|$.
(iii) Suppose $\lambda=u+i v,(u>0)$. Then,

$$
\begin{gather*}
\lim _{u^{2}+\nu^{2} \rightarrow+\infty}\left|\frac{q(\lambda)}{p(\lambda)}\right|=\delta e^{-d_{1} \omega} \\
\lim _{u^{2}+v^{2} \rightarrow+\infty} \frac{1}{\sqrt{(A-C-u)^{2}+v^{2}}}=0 . \tag{20}
\end{gather*}
$$

Then, using Lemma 3, we have $R(\lambda)<0$ for all $\omega$.
Case 4. For $\omega>\tau>0$, let $\varepsilon=\omega-\tau$. Then, $\omega=\tau+\varepsilon$. For fixed $\tau$,

$$
\begin{equation*}
h(\lambda)=A e^{-\left(d_{1}+\lambda\right) \tau}-\delta e^{-\left(d_{1}+\lambda\right)(\tau+\varepsilon)}-C-\lambda . \tag{21}
\end{equation*}
$$

Let $\lambda=u+i v(u>0)$. Then, we take the real and imaginary parts of $h(\lambda)=0$; namely,

$$
\begin{gather*}
A e^{-\left(d_{1}+u\right) \tau} \cos (v \tau)-\delta e^{-\left(d_{1}+u\right)(\tau+\varepsilon)} \cos (v(\tau+\varepsilon)) \\
=C+u  \tag{22}\\
-A e^{-\left(d_{1}+u\right) \tau} \sin (v \tau)-\delta e^{-\left(d_{1}+u\right)(\tau+\varepsilon)} \sin (v(\tau+\varepsilon))=v .
\end{gather*}
$$

Sum of squares of the above equalities is

$$
\begin{align*}
& A^{2} e^{-2\left(d_{1}+u\right) \tau}+\delta^{2} e^{-2\left(d_{1}+u\right)(\tau+\varepsilon)}  \tag{23}\\
& \quad-2 A \delta e^{-\left(d_{1}+u\right)(2 \tau+\varepsilon)} \cos (v \varepsilon)-(C+u)^{2}-v^{2}=0
\end{align*}
$$

Then, we have

$$
\begin{equation*}
\left.\frac{\partial u}{\partial \varepsilon}\right|_{\varepsilon=0}=\frac{\delta\left(u+d_{1}\right)(A-\delta)}{(A-\delta)^{2} \tau+(C+u) e^{2 \tau(u+\delta)}} \tag{24}
\end{equation*}
$$

Obviously, $\partial u /\left.\partial \varepsilon\right|_{\varepsilon=0}<0$ when $A-\delta<0$. Combined with $(A-\delta) e^{-d_{1} \tau}<A e^{-d_{1} \tau}-\delta e^{-d_{1} \omega}$, we have the following.

Proposition 7. For $\omega>\tau>0, R(\lambda)<0$ for every root $\lambda$ of $h(\lambda)=0$ when

$$
\begin{equation*}
A e^{-d_{1} \tau}-\delta e^{-d_{1} \omega}<C, \quad A-\delta<0, \quad 0<\omega-\tau \ll 1 \tag{25}
\end{equation*}
$$

Case 5. For $0<\tau<\omega$, in the same way as in Case 4, we have the following.

Proposition 8. For $0<\tau<\omega, R(\lambda)<0$ for every root $\lambda$ of $h(\lambda)=0$ when

$$
\begin{equation*}
A e^{-d_{1} \tau}-\delta e^{-d_{1} \omega}<C, \quad A-\delta>0, \quad 0<\tau-\omega \ll 1 \tag{26}
\end{equation*}
$$

From what has been discussed above, we get the following.

Theorem 9. The disease-free equilibrium of the system (3) is locally asymptotically stable if one of the following conditions holds.
(a) $\tau=\omega \neq 0,(A-\delta) e^{-d_{1} \tau}<C$.
(b) $\tau \neq 0, \omega=0, A e^{-d_{1} \tau}-\delta<C$.
(c) $\tau=0, \omega \neq 0, \delta e^{-d_{1} \omega}<|A-C|$.
(d) $A e^{-d_{1} \tau}-\delta e^{-d_{1} \omega}<C, A-\delta<0,0<\omega-\tau \ll 1$.
(e) $A e^{-d_{1} \tau}-\delta e^{-d_{1} \omega}<C, A-\delta>0,0<\tau-\omega \ll 1$.
3.2. Existence of Endemic Equilibrium. Thus, by Theorem 9, we may define the basic reproduction number as

$$
\begin{equation*}
R_{0}=\frac{A e^{-d_{1} \tau}-\delta e^{-d_{1} \omega}}{C} \tag{27}
\end{equation*}
$$

This threshold $R_{0}$ defines the average number of secondary infections generated by a typical infectious individual in a completely susceptible population in a steady demographic state.

In Theorem 9, we have already shown that the system (3) has an infection-free steady state which is locally asymptotically stable under condition $R_{0}<1$. The disease-free equilibrium is unstable when $R_{0}>1$, and the system (3) has a nontrivial endemic equilibrium $E^{*}=\left(N^{*}, I^{*}, R^{*}\right)$ when $R_{0}>1$. From (3),

$$
\begin{gather*}
N^{*}=K>0 \\
R^{*}=\frac{\delta e^{-d_{1} \omega}-\theta}{d+(1-a) r+\theta} I^{*}  \tag{28}\\
+\frac{K \theta}{d+(1-a) r+\theta} \doteq q I^{*}+p
\end{gather*}
$$

where $q=\left(e^{-d_{1} \omega} \delta-\theta\right) /(d+(1-a) r+\theta)$. Substituting these values of $N^{*}$ and $R^{*}$ in the second equation of (3), we get the following equation for $I$ :

$$
\begin{equation*}
G(I)=\beta e^{-d_{1} \tau}(K-(1+q) I-p) f(I)-\left[C+\delta e^{-d_{1} \omega}\right] I \tag{29}
\end{equation*}
$$

Obviously, $I=0$ is one of the roots of $(29)$ as $f(0)=0$. Therefore, to exclude that root, choose

$$
\begin{equation*}
H(I)=\beta e^{-d_{1} \tau}(K-(1+q) I-p) \frac{f(I)}{I}-\left[C+\delta e^{-d_{1} \omega}\right] \tag{30}
\end{equation*}
$$

It can easily be seen that the function $H(I)$ is negative for large positive $I$; that is,

$$
\begin{equation*}
H(K)=-\beta e^{-d_{1} \tau}(K q+p) \frac{f(k)}{K}-\left[C+\delta e^{-d_{1} \omega}\right]<0 \tag{31}
\end{equation*}
$$

Next, we determine the sign of its derivative

$$
\begin{align*}
H^{\prime}(I)= & \beta e^{-d_{1} \tau}(K-p) \frac{f^{\prime}(I) I-f(I)}{I^{2}}  \tag{32}\\
& -\beta e^{-d_{1} \tau}(1+q) f^{\prime}(I)
\end{align*}
$$

It can easily be seen that $K>p$. In addition, from the properties of the function $f(I)$, in particular from $f(0)=0$ and $f^{\prime \prime}(0)<0$, it follows that $f(I)-f^{\prime}(I) I>0$, and consequently $H^{\prime}(I)<0$ for all $I>0$. Therefore, for a positive root of $H(I)=0$ to exist, $H(I)$ has to satisfy $H(0)>0$; that is,

$$
\begin{align*}
H(0) & =A e^{-d_{1} \tau}-\delta e^{-d_{1} \omega}-C \\
& =\left(\frac{A e^{-d_{1} \tau}-e^{-d_{1} \omega} \delta}{C}-1\right) C  \tag{33}\\
& =\left(R_{0}-1\right) C
\end{align*}
$$

Hence, one needs the requirement that $R_{0}>1$ to ensure the existence of the endemic equilibrium. From the above analysis, we have the following theorem.

Theorem 10. The system (3) has a nontrivial endemic equilib$\operatorname{rium} E^{*}=\left(N^{*}, I^{*}, R^{*}\right)$ when $R_{0}>1$.
3.3. Local Stability of the Endemic Equilibrium. In this section, we analyze the local stability of the endemic equilibrium $E^{*}$ for $\tau=\omega$. Its characteristic equation is given by

$$
\left|\begin{array}{ccc}
-r-\lambda & 0 & 0  \tag{34}\\
\beta e^{-d_{1} \tau} f\left(I^{*}\right)-\frac{(1-a) r I^{*}}{K} & m e^{-\left(d_{1}+\lambda\right) \tau}-n-\lambda & \beta e^{-d_{1} \tau} f\left(I^{*}\right) \\
\theta-\frac{(1-a) r R^{*}}{K} & \delta e^{-\left(d_{1}+\lambda\right) \tau}-\theta & -[d+(1-a) r+\theta]-\lambda
\end{array}\right|=0,
$$

where $m=\beta\left(K-I^{*}-R^{*}\right) f^{\prime}\left(I^{*}\right)-\delta, n=\beta e^{-d_{1} \tau} f\left(I^{*}\right)+d+$ $(1-a) r+\mu_{1}$.

The Jacobin matrix leads to the characteristic equation

$$
\begin{equation*}
(\lambda+r)\left[\lambda^{2}+m_{1} \lambda+m_{0}+\left(n_{1} \lambda+n_{0}\right) e^{-\lambda \tau}\right]=0 \tag{35}
\end{equation*}
$$

where

$$
\begin{align*}
m_{1}= & \beta e^{-d_{1} \tau} f\left(I^{*}\right)+2 d+2(1-a) r+\theta+\mu_{1}>0, \\
m_{0}= & {\left[\beta e^{-d_{1} \tau} f\left(I^{*}\right)+d+(1-a) r+\mu_{1}\right] } \\
& \times(d+(1-a) r+\theta)+\left(d+(1-a) r+\mu_{1}\right) \theta>0, \\
& n_{1}=-\left[\beta f^{\prime}\left(I^{*}\right)\left(K-I^{*}-R^{*}\right)-\delta\right] e^{-d_{1} \tau}, \\
n_{0}= & {\left[\beta f^{\prime}\left(I^{*}\right)\left(K-I^{*}-R^{*}\right)-\delta\right][d+(1-a) r+\theta] } \\
& \times e^{-d_{1} \tau}+\delta f\left(I^{*}\right) e^{-2 d_{1} \tau} . \tag{36}
\end{align*}
$$

Since all the model parameters are assumed to be nonnegative, it follows that one eigenvalue is negative; that is, $\lambda_{1}=-r$. Thus, the stability of $E^{*}$ depends on the roots of the quasipolynomial

$$
\begin{equation*}
\lambda^{2}+m_{1} \lambda+m_{0}+\left(n_{1} \lambda+n_{0}\right) e^{-\lambda \tau}=0 \tag{37}
\end{equation*}
$$

We note that $m_{1}>0$ and $m_{0}>0$, whereas $n_{1}$ and $n_{0}$ may be positive or negative. For $\tau=0$, we state the following results that follow directly from (39). The endemic steady state is locally asymptotically stable if the following conditions hold:

$$
\begin{align*}
& \beta f\left(I^{*}\right)+2 d+2(1-a) r+\theta+\mu_{1}+\delta \beta f^{\prime}\left(I^{*}\right) \\
& \quad>\beta\left(K-I^{*}-R^{*}\right) f^{\prime}\left(I^{*}\right),  \tag{1}\\
& {\left[\beta f\left(I^{*}\right)+d+(1-a) r+\mu_{1}+\delta-\beta\left(K-I^{*}-R^{*}\right) f^{\prime}\left(I^{*}\right)\right]} \\
& \quad \times[d+(1-a) r+\theta]+f\left(I^{*}\right) \delta>0 . \tag{2}
\end{align*}
$$

The main purpose of this paper is to study the stability behavior of $E^{*}$ in the case $\tau \neq 0$. Obviously, i $\eta(\eta>0)$ is the root of (29) if and only if $\eta$ satisfies

$$
\begin{equation*}
-\eta^{2}+m_{1} i \eta+m_{0}=-\left(n_{1} i \eta+n_{0}\right)(\cos \eta \tau-i \sin \eta \tau) . \tag{38}
\end{equation*}
$$

Separating the real and imaginary parts, we have

$$
\begin{gather*}
-\eta^{2}+m_{0}=-n_{0} \cos \eta \tau-n_{1} \eta \sin \eta \tau,  \tag{39}\\
m_{1} \eta=-n_{1} \eta \cos \eta \tau+n_{0} \sin \eta \tau . \tag{40}
\end{gather*}
$$

Eliminating $\tau$ by squaring and adding (39) and (40), we obtain a polynomial in $\eta$ as

$$
\begin{equation*}
\eta^{4}+\left(m_{1}^{2}-n_{1}^{2}-2 m_{0}\right) \eta^{2}+m_{0}^{2}-n_{0}^{2}=0 \tag{41}
\end{equation*}
$$

Suppose that the conditions

$$
\begin{equation*}
m_{1}^{2}>n_{1}^{2}+2 m_{0}, \quad m_{0}^{2}>n_{0}^{2} \tag{3}
\end{equation*}
$$

hold for all $\tau \geq 0$. Then, the infected steady state of the system (3) is locally asymptotically stable.

Theorem 11. For $\tau=\omega$, if $R_{0}>1$, then the endemic equilibrium of the system (3) is locally asymptotically stable, when conditions $\left(H_{1}\right)-\left(H_{3}\right)$ hold.

Corollary 12. For $\tau=\omega$, if $\mu_{1}<\mu_{1}^{*}$ or $\theta<\theta^{*}$, then the endemic equilibrium of the system (3) is locally asymptotically stable, when conditions $\left(H_{1}\right)-\left(H_{3}\right)$ hold.

## 4. Permanence

In this section, we investigate a permanence result [5]. The following is our main result of this paper. We will give the following result by using some techniques given in [8, 11]. The proof of the permanence with nonlinear incidence is a daunting task. Consequently, for simplicity and mathematical convenience, let us choose a linear incidence rate $f(I)=$ $I$. The result holds with the nonlinear incidence, as shown numerically, but the algebraic proof is long and tedious, and the conditions to impose on some of the parameters may be very restrictive. Now, let us firstly give the following theorem.

Theorem 13. If $R_{0}>1$ holds, then the system (1) with $\tau=\omega$ is permanent; that is, there are positive constants $c_{i}(i=1,2,3)$ such that

$$
\begin{align*}
& c_{1}<\lim _{t \rightarrow \infty} \inf S(t) \leq \lim _{t \rightarrow \infty} \sup S(t) \leq K, \\
& c_{2}<\lim _{t \rightarrow \infty} \inf I(t) \leq \lim _{t \rightarrow \infty} \sup I(t) \leq K,  \tag{42}\\
& c_{3}<\lim _{t \rightarrow \infty} \inf R(t) \leq \lim _{t \rightarrow \infty} \sup R(t) \leq K
\end{align*}
$$

hold for any solution of (1) with $\left(\phi_{1}(\theta), \phi_{2}(\theta), \phi_{3}(\theta)\right)$ in the interior of $\Phi$ for all $\theta \in[-\tau, 0]$. In fact, $c_{i}(i=1,2,3)$ can be chosen explicitly as

$$
\begin{gather*}
c_{1}=\frac{(b-a r) K}{\beta K e^{-d_{1} \tau}+d+(1-a) r+\theta}, \\
c_{2}=I^{*} e^{-\left(d+(1-a) r+\mu_{1}+e^{-d_{1} \tau} \delta\right)},  \tag{43}\\
c_{3}=\frac{\delta e^{-d_{1} \tau} c_{2}+\theta c_{1}}{d+(1-a) r}
\end{gather*}
$$

Proof. Note that $0<N(t)<K$ for all $t \geq 0$ and that $\lim _{t \rightarrow \infty} N(t)=K$. It is easy to see that $\lim _{t \rightarrow \infty} \inf S(t) \geq c_{1}$. In fact, let $\epsilon<K$ be arbitrary. Choose $T_{1}>\tau$ so large that $N(t)>K-\epsilon$ for $t>T_{1}$. We have the following inequality:

$$
\begin{align*}
\dot{S}(t)> & -\left[\beta K e^{-d_{1} \tau}+d+(1-a) r+\theta\right] S(t)  \tag{44}\\
& +(b-a r)(K-\epsilon)
\end{align*}
$$

for all $t \geq T_{1}$, which implies that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} S(t) \geq \frac{(b-a r)(K-\epsilon)}{\beta K e^{-d_{1} \tau}+d+(1-a) r+\theta} \tag{45}
\end{equation*}
$$

Note that $\epsilon$ may be arbitrarily small so that $\lim _{t \rightarrow \infty} \inf S(t) \geq$ $c_{1}$.

Next, we will show $\lim _{t \rightarrow \infty} \inf I(t) \geq c_{2}$. For any $\xi$ : $0<\xi<1$, we see the inequality $S^{*}<[(b-a r) K+$ $\left.\mu_{1} I^{*}\right] /\left(\beta e^{-d_{1} \tau} \xi I^{*}+d+(1-a) r+\theta\right)$. There exist sufficiently large $\rho \geq 1$ and sufficiently small $\epsilon$ such that $S^{*}<\{[(b-$ $\left.\left.\operatorname{ar})(K-\epsilon)+\mu_{1} I^{*}\right] /\left(\beta e^{-d_{1} \tau} \xi I^{*}+d+(1-a) r+\theta\right)\right\}(1-$ $\left.e^{-\left(\beta e^{-d_{1} \tau} \xi I^{*}+d+(1-a) r+\theta\right) \rho \tau}\right) \equiv S^{\Delta}$. We show that $I\left(t_{0}\right)>q I^{*}$ for some $t_{0} \geq \rho \tau$. In fact, if not, it follows from the first equation of (1) that, for all $t \geq \rho \tau+\tau \geq T_{1}+\tau$,

$$
\begin{align*}
\dot{S}(t) \geq & (b-a r)(K-\epsilon)+\mu_{1} I^{*} \\
& -\left[\beta e^{-d_{1} \tau} \xi I^{*}+d+(1-a) r+\theta\right] S(t) . \tag{46}
\end{align*}
$$

Hence, for $t \geq \rho \tau+\tau$,

$$
\begin{align*}
S(t) \geq & e^{-\left(\beta e^{-d_{1} \tau} \xi I^{*}+d+(1-a) r+\theta\right)(t-\rho \tau-\tau)} \\
& \times\left[S(\rho \tau+\tau)+(b-a r)(K-\epsilon)+\mu_{1} I^{*}\right] \\
& \times \int_{\rho \tau+\tau}^{t} e^{-\left(\beta e^{-d_{1} \tau} \xi I^{*}+d+(1-a) r+\theta\right)(t-\rho \tau-\tau)} d \theta  \tag{47}\\
> & \frac{\left[(b-a r)(K-\epsilon)+\mu_{1} I^{*}\right]}{\beta e^{-d_{1} \tau} \xi I^{*}+d+(1-a) r+\theta} \\
& \times\left(1-e^{-\left(\beta e^{-d_{1} \tau} \xi I^{*}+d+(1-a) r+\theta\right)(t-\rho \tau-\tau)}\right),
\end{align*}
$$

which gives us, for $t \geq 2 \rho \tau+\tau$,

$$
\begin{equation*}
S(t)>S^{\Delta}>S^{*} \tag{48}
\end{equation*}
$$

For $t \geq 0$, we define a positive differentiable function $V(t)$ as follows:

$$
\begin{equation*}
V(t)=I(t)+\frac{[\beta(K-p)-\delta] e^{-d_{1} \tau}}{R_{0}} \int_{t-\tau}^{t} I(s) d s \tag{49}
\end{equation*}
$$

We obtain the inequality, for $t \geq 2 \rho \tau+\tau$,

$$
\begin{align*}
\dot{V}(t)= & {\left[\beta e^{-d_{1} \tau}\left(S(t)-S^{*}\right) I(t-\tau)+(1-a) r\right] } \\
& \times\left(1-\frac{N(t)}{K}\right) I(t)  \tag{50}\\
> & \beta e^{-d_{1} \tau}\left(S(t)-S^{*}\right) I(t-\tau) \\
> & \beta e^{-d_{1} \tau}\left(S^{\Delta}-S^{*}\right) I(t-\tau)
\end{align*}
$$

Let $\underline{i}=\min _{\theta \in[-\tau, 0]} I(2 \rho \tau+2 \tau+\theta)$. Now, let us show that $I(t) \geq \underline{i}$ for all $t \geq 2 \rho \tau+\tau$. In fact, if there exists $T_{2} \geq 0$ such that $I(\bar{t}) \geq \underline{i}$ for $2 \rho \tau+\tau \leq t \leq 2 \rho \tau+2 \tau+T_{2}, I\left(2 \rho \tau+2 \tau+T_{2}\right)=\underline{i}$ and $\dot{I}\left(2 \rho \tau+2 \tau+T_{2}\right) \leq 0$. Direct calculation using the second equation of (1) and (48) gives

$$
\begin{align*}
\dot{I}(2 \rho \tau+ & \left.2 \tau+T_{2}\right) \\
> & {\left[\beta e ^ { - d _ { 1 } \tau } \left(S\left(2 \rho \tau+2 \tau+T_{2}\right)\right.\right.} \\
& \left.\quad-\left(d+(1-a) r+\mu_{1}+e^{-d_{1} \tau} \delta\right)\right] \underline{i}  \tag{51}\\
> & \left(d+(1-a) r+\mu_{1}+e^{-d_{1} \tau} \delta\right)\left[\frac{S^{\Delta}}{S^{*}}-1\right] \underline{i}>0 .
\end{align*}
$$

This contradicts the definition of $T_{2}$. Thus, we have shown that $I(t) \geq \underline{i}$ for all $t \geq 2 \rho \tau+\tau$. Hence, for all $t \geq 2 \rho \tau+2 \tau$,

$$
\begin{equation*}
\dot{V}(t)>\beta e^{-d_{1} \tau}\left(S^{\Delta}-S^{*}\right) \underline{i} \tag{52}
\end{equation*}
$$

which implies that $V(t) \rightarrow+\infty$ as $t \rightarrow+\infty$. This contradicts the boundedness of $V(t)$. Consequently, $I\left(t_{0}\right)>$ $\xi I^{*}$ for some $t_{0} \geq \rho \tau$.

In the rest, we now need to consider two cases:
(i) $I(t) \geq \xi I^{*}$ for all large $t$;
(ii) $I(t)$ oscillates about $\xi I^{*}$ for all large $t$.

We now need to show that $I(t) \geq \xi_{\mathcal{C}_{2}}$ for large $t$. Obviously, it suffices to show that it holds only for case (ii). We suppose that for any large $T$ there exists $t_{1}, t_{2}>T$ such that $I\left(t_{1}\right)=$ $I\left(t_{2}\right)=\xi I^{*}$ and $I(t)<\xi I^{*}$ for $t_{1}<t<t_{2}$. If $t_{2}-t_{1} \leq \tau$, the second equation (1) gives us $\dot{I}(t)>-\left(d+(1-a) r+\mu_{1}\right) I(t)$, which implies that $I(t)>I\left(t_{1}\right) e^{-\left(d+(1-a) r+\mu_{1}\right)\left(t-t_{1}\right)}$ on $\left(t_{1}, t_{2}\right)$. Thus, $I(t)>\xi c_{2}$. On the other hand, if $t_{2}-t_{1}>\tau$, applying the same manner gives $I(t) \geq \xi_{\mathcal{C}_{2}}$ on $\left[t_{1}, t_{1}+\tau\right]$, and hence the remaining work is to show $I(t) \geq \xi_{\mathcal{C}_{2}}$ on $\left[t_{1}+\tau, t_{2}\right]$. In fact, assuming that there exists $T_{3}>0$ such that $I(t) \geq \xi_{C_{2}}$ on $\left[t_{1}, t_{1}+\tau+T_{3}\right], I\left(t_{1}+\tau+T_{3}\right)=\xi \mathcal{C}_{2}$, and $\dot{I}\left(t_{1}+\tau+T_{3}\right) \leq 0$, it follows from (1) that

$$
\begin{align*}
& \dot{I}\left(t_{1}+\tau+T_{3}\right) \\
& \geq\left[\beta e^{-d_{1} \tau} S\left(t_{1}+\tau+T_{3}\right)\right. \\
& \left.\quad-\left(d+(1-a) r+\mu_{1}+e^{-d_{1} \tau} \delta\right)\right] \xi_{\mathcal{C}_{2}}  \tag{53}\\
& > \\
& \quad\left(d+(1-a) r+\mu_{1}+e^{-d_{1} \tau} \delta\right)\left[\frac{S^{\Delta}}{S^{*}}-1\right] \xi_{\mathcal{C}_{2}}>0 .
\end{align*}
$$

This contradicts the definition of $T_{3}$. Hence, $I(t) \geq \xi_{\mathcal{C}_{2}}$ on [ $t_{1}, t_{2}$ ]. Consequently, $I(t) \geq \xi_{\mathcal{C}_{2}}$ for large $t$ in the case (ii). Therefore, $\lim _{t \rightarrow \infty} \inf I(t) \geq \xi_{c_{2}}$. Note that $q$ may be so close to 1 that $\lim _{t \rightarrow \infty} \inf I(t) \geq c_{2}$.

Finally, let us show that $\lim _{t \rightarrow \infty} \inf R(t) \geq\left(\delta e^{-d_{1} \tau} c_{2}+\right.$ $\left.\theta c_{1}\right) /(d+(1-a) r)$. The third equation gives us

$$
\begin{align*}
\dot{R}(t) & \geq\left[\delta e^{-d_{1} \tau} I+\theta S-[d+(1-a) r] R\right. \\
& \geq\left[\delta e^{-d_{1} \tau} \xi_{\mathcal{C}_{2}}+\theta \xi_{c_{1}}-[d+(1-a) r] R\right. \tag{54}
\end{align*}
$$

for large $t$. Hence, $\lim _{t \rightarrow \infty} \inf R(t) \geq\left(\delta e^{-d_{1} \tau} \xi_{\mathcal{C}_{2}}+\right.$ $\left.\theta \xi c_{1}\right) /(d+(1-a) r)$. In a similar manner, we could show $\lim _{t \rightarrow \infty} \inf R(t) \geq c_{3}$. This proves the theorem.

Corollary 14. If $\mu_{1}<\mu_{1}^{*}$ and $\theta<\theta^{*}$, then the system (1) with $\tau=\omega$ is permanent.

## 5. Numerical Analysis

Since it is important to visualize the dynamical behavior of the model, the model system (3) is integrated numerically with the help of MATLAB 7.0 using the following set of parameters.
(1) Let $r=0.5, k=8, d=0.04, d_{1}=0.04, \beta=1$, $a=0.3, \delta=0.2, \mu_{1}=0.8, \theta=0.01$, and $\tau=5$. It is easy to compute that $E_{0}=(8,0,0.2)$ and $R_{0}=0.94<1$. In Figure 1, the infective population and recovered population, respectively, are plotted against the total population. We see from the figure that for any initial start the solution curves tend to the equilibrium $E_{0}$. Hence, we infer that the system (3) may be stable about the disease-free equilibrium point $E_{0}$, which satisfies Theorem 9.
(2) Let $r=0.5, k=8, d=0.04, d_{1}=0.04, \beta=1$, $a=0.3, \delta=0.2, \mu_{1}=0.2, \theta=0.02$, and $\tau=5$. We get $E^{*}=(8,2.23,1.11)$ and this set of parameter values satisfies the local asymptotic stability conditions of $E^{*}$. It is


Figure 1: The disease-free equilibrium $E_{0}$ is locally asymptotically stable. Variation of infective population $I(t)$ and recovered population $R(t)$ with total population $N(t)$.


Figure 2: The endemic equilibrium $E^{*}$ is locally asymptotically stable. Variation of infective population $I(t)$ and recovered population $R(t)$ with total population $N(t)$.


Figure 3: Variation of infective population $I(t)$ and recovered population $R(t)$ with time for different values of $\mu_{1}$.


Figure 4: Variation of infective population $I(t)$ and recovered population $R(t)$ with time for different values of $\theta$.
easy to verify that $R_{0}=1.83>1$ and all other conditions of Theorem 11 are satisfied. So, we can obtain from Figure 2 that the system (3) is stable at the endemic equilibrium point $E^{*}$.
(3) The results of numerical simulation are displayed graphically in Figures 3 and 4. In Figure 3, the variation of the infective population and recovered population is shown with time for different values of the removal rate constant from groups $I$ to $S, \mu_{1}$. It is found that both the infective population and the recovered population decrease as $\mu_{1}$ increases. Figure 4 depicts the variation of infective population and recovered population, respectively, with time for the different successive vaccination rate, $\theta$. As $\theta$ increases, the infective population decreases whereas the recovered population increases.

## 6. Discussion

In this paper, we will consider two different delays which are important parameters on the dynamic behavior. So, the present study is continuation of the previous work by [43]. Furthermore, from biological epidemic point of view, we investigate successive vaccination and difference in immunity in our system. From mathematical point of view, we study the stability of disease-free equilibrium and the existence of endemic equilibrium for different delay and consider the permanence of the system in the new paper.

In Theorems $9,10,11$, and 13 corresponding to their corollaries, when the effect of the successive vaccination rate and the transfer rate from the infectious group to the susceptible group after treatment is strong, that is, $\theta>\theta^{*}$ and $\mu_{1}>\mu_{1}^{*}$, the basic reproduction number $R_{0}$ being unity is a strict threshold for the control of the disease; the disease will be extinct or otherwise will tend to break out and persist. The other results are displayed graphically from our numerical simulation. We show the variation of
the infective population and recovered population with time for different values of $\mu_{1}$. It is found that both the infective population and the recovered population decrease as $\mu_{1}$ increases. The infective population decreases whereas the recovered population increases as the successive vaccination rate increases, $\theta$, respectively.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

This work is supported by National Natural Science Foundation of China (11101305).

## References

[1] S. Ruan, W. Wang, and S. A. Levin, "The effect of global travel on the spread of SARS," Mathematical Biosciences and Engineering, vol. 3, no. 1, pp. 205-218, 2006.
[2] W. Wang and S. Ruan, "Simulating the SARS outbreak in Beijing with limited data," Journal of Theoretical Biology, vol. 227, no. 3, pp. 369-379, 2004.
[3] S. M. Moghadas and A. B. Gumel, "Global stability of a twostage epidemic model with generalized non-linear incidence," Mathematics and Computers in Simulation, vol. 60, no. 1-2, pp. 107-118, 2002.
[4] X. Zhou, X. Song, and X. Shi, "Analysis of stability and Hopf bifurcation for an HIV infection model with time delay," Applied Mathematics and Computation, vol. 199, no. 1, pp. 23-38, 2008.
[5] A. Tripathi, R. Naresh, and D. Sharma, "Modelling the effect of screening of unaware infectives on the spread of HIV infection," Applied Mathematics and Computation, vol. 184, no. 2, pp. 10531068, 2007.
[6] N. Yoshida and T. Hara, "Global stability of a delayed SIR epidemic model with density dependent birth and death rates," Journal of Computational and Applied Mathematics, vol. 201, no. 2, pp. 339-347, 2007.
[7] X. Meng and L. Chen, "The dynamics of a new SIR epidemic model concerning pulse vaccination strategy," Applied Mathematics and Computation, vol. 197, no. 2, pp. 582-597, 2008.
[8] A. d'Onofrio, "On pulse vaccination strategy in the SIR epidemic model with vertical transmission," Applied Mathematics Letters, vol. 18, no. 7, pp. 729-732, 2005.
[9] F. H. Chen, "A susceptible-infected epidemic model with voluntary vaccinations," Journal of Mathematical Biology, vol. 53, no. 2, pp. 253-272, 2006.
[10] I. A. Moneim and D. Greenhalgh, "Threshold and stability results for an SIRS epidemic model with a general periodic vaccination strategy," Journal of Biological Systems, vol. 13, no. 2, pp. 131-150, 2005.
[11] E. Shim, Z. Feng, M. Martcheva, and C. Castillo-Chavez, "An age-structured epidemic model of rotavirus with vaccination," Journal of Mathematical Biology, vol. 53, no. 4, pp. 719-746, 2006.
[12] G. Zaman, Y. H. Kang, and I. H. Jung, "Optimal treatment of an SIR epidemic model with time delay," BioSystems, vol. 98, no. 1, pp. 43-50, 2009.
[13] Y. Pei, S. Liu, C. Li, and L. Chen, "The dynamics of an impulsive delay SI model with variable coefficients," Applied Mathematical Modelling, vol. 33, no. 6, pp. 2766-2776, 2009.
[14] S. Liu, Y. Pei, C. Li, and L. Chen, "Three kinds of TVS in a SIR epidemic model with saturated infectious force and vertical transmission," Applied Mathematical Modelling, vol. 33, no. 4, pp. 1923-1932, 2009.
[15] F. Brauer, "Epidemic models with heterogeneous mixing and treatment," Bulletin of Mathematical Biology, vol. 70, no. 7, pp. 1869-1885, 2008.
[16] S. Gao, L. Chen, and L. Sun, "Dynamic complexities in a seasonal prevention epidemic model with birth pulses," Chaos, Solitons \& Fractals, vol. 26, no. 4, pp. 1171-1181, 2005.
[17] Y. Pei and S. Liu, "Pulse vaccination strategy in a delayed SIRS epidemic model," in Proceedings of the 6th Conference of Biomathematics, Advanced in Biomathematics, vol. 2, pp. 775778, 2008.
[18] T. Zhang and Z. Teng, "Permanence and extinction for a nonautonomous SIRS epidemic model with time delay," Applied Mathematical Modelling, vol. 33, no. 2, pp. 1058-1071, 2009.
[19] L. Nie, Z. Teng, L. Hu, and J. Peng, "Permanence and stability in non-autonomous predator-prey Lotka-Volterra systems with feedback controls," Computers \& Mathematics with Applications, vol. 58, no. 3, pp. 436-448, 2009.
[20] Y. N. Kyrychko and K. B. Blyuss, "Global properties of a delayed SIR model with temporary immunity and nonlinear incidence rate," Nonlinear Analysis: Real World Applications, vol. 6, no. 3, pp. 495-507, 2005.
[21] Y. Baba, Y. Tsuboi, H. Inoue, T. Yamada, Z. K. Wszolek, and D. F. Broderick, "Acute measles encephalitis in adults," Journal of Neurology, vol. 253, no. 1, pp. 121-124, 2006.
[22] A. Schreurs, E. V. Stålberg, and A. R. Punga, "Indication of peripheral nerve hyperexcitability in adult-onset subacute sclerosing panencephalitis (SSPE)," Neurological Sciences, vol. 29, no. 2, pp. 121-124, 2008.
[23] O. D. Makinde, "Adomian decomposition approach to a SIR epidemic model with constant vaccination strategy," Applied Mathematics and Computation, vol. 184, no. 2, pp. 842-848, 2007.
[24] D. Greenhalgh, Q. J. A. Khan, and F. I. Lewis, "Hopf bifurcation in two SIRS density dependent epidemic models," Mathematical and Computer Modelling, vol. 39, no. 11-12, pp. 1261-1283, 2004.
[25] S. Gao, L. Chen, and Z. Teng, "Impulsive vaccination of an SEIRS model with time delay and varying total population size," Bulletin of Mathematical Biology, vol. 69, no. 2, pp. 731-745, 2007.
[26] Z. Lu, X. Chi, and L. Chen, "The effect of constant and pulse vaccination on SIR epidemic model with horizontal and vertical transmission," Mathematical and Computer Modelling, vol. 36, no. 9-10, pp. 1039-1057, 2002.
[27] A. d'Onofrio, "On pulse vaccination strategy in the SIR epidemic model with vertical transmission," Applied Mathematics Letters, vol. 18, no. 7, pp. 729-732, 2005.
[28] G. Röst and J. Wu, "SEIR epidemiological model with varying infectivity and infinite delay," Mathematical Biosciences and Engineering, vol. 5, no. 2, pp. 389-402, 2008.
[29] M. Y. Li, H. L. Smith, and L. Wang, "Global dynamics an SEIR epidemic model with vertical transmission," SIAM Journal on Applied Mathematics, vol. 62, no. 1, pp. 58-69, 2001.
[30] S. A. Gourley, Y. Kuang, and J. D. Nagy, "Dynamics of a delay differential equation model of hepatitis $B$ virus infection," Journal of Biological Dynamics, vol. 2, no. 2, pp. 140-153, 2008.
[31] J. Arino, J. R. Davis, D. Hartley, R. Jordan, J. M. Miller, and P. van den Driessche, "A multi-species epidemic model with spatial dynamics," Mathematical Medicine and Biology, vol. 22, no. 2, pp. 129-142, 2005.
[32] V. Capasso and G. Serio, "A generalization of the KermackMcKendrick deterministic epidemic model," Mathematical Biosciences, vol. 42, no. 1-2, pp. 43-61, 1978.
[33] S. Ruan and W. Wang, "Dynamical behavior of an epidemic model with a nonlinear incidence rate," Journal of Differential Equations, vol. 188, no. 1, pp. 135-163, 2003.
[34] W. M. Liu, S. A. Levin, and Y. Iwasa, "Influence of nonlinear incidence rates upon the behavior of SIRS epidemiological models," Journal of Mathematical Biology, vol. 23, no. 2, pp. 187204, 1986.
[35] S. Liu, Y. Pei, C. Li, and L. Chen, "Three kinds of TVS in a SIR epidemic model with saturated infectious force and vertical transmission," Applied Mathematical Modelling, vol. 33, no. 4, pp. 1923-1932, 2009.
[36] W. R. Derrick and P. van den Driessche, "A disease transmission model in a nonconstant population," Journal of Mathematical Biology, vol. 31, no. 5, pp. 495-512, 1993.
[37] A. Korobeinikov and P. K. Maini, "Non-linear incidence and stability of infectious disease models," Mathematical Medicine and Biology, vol. 22, no. 2, pp. 113-128, 2005.
[38] S. M. Moghadas and M. E. Alexander, "Bifurcations of an epidemic model with non-linear incidence and infectiondependent removal rate," Mathematical Medicine and Biology, vol. 23, no. 3, pp. 231-254, 2006.
[39] X. Ji, Y. Pei, and C. Li, "Two patterns of recruitment in an epidemic model with difference in immunity of individuals," Nonlinear Analysis: Real World Applications, vol. 11, no. 3, pp. 2078-2090, 2010.
[40] P. van den Driessche, "Time delay in epidemic models," in Mathematical Approaches for Emerging and Reemerging Infectious Diseases: An Introduction, C. Castillo-Chavez, S. Blower, P. van den Driessche, D. Kirschner, and A. Yakubu, Eds., vol. 126 of The IMA Volumes in Mathematics and its Applications, pp. 119-128, Springer, New York, NY, USA, 2002.
[41] H. W. Hethcote and P. van den Driessche, "Two SIS epidemiologic models with delays," Journal of Mathematical Biology, vol. 40, no. 1, pp. 3-26, 2000.
[42] W. Ma, Y. Takeuchi, T. Hara, and E. Beretta, "Permanence of an SIR epidemic model with distributed time delays," The Tohoku Mathematical Journal, vol. 54, no. 4, pp. 581-591, 2002.
[43] R. Naresh, A. Tripathi, J. M. Tchuenche, and D. Sharma, "Stability analysis of a time delayed SIR epidemic model with nonlinear incidence rate," Computers \& Mathematics with Applications, vol. 58, no. 2, pp. 348-359, 2009.
[44] H. Smith, An Introduction to Delay Differential Equations with Applications to the Life Sciences, vol. 57, Springer, New York, NY, USA, 2011.

## Research Article

# Spatial Complexity of a Predator-Prey Model with Holling-Type Response 

Lei Zhang ${ }^{1,2}$ and Zhibin Li ${ }^{1}$<br>${ }^{1}$ Computer Science and Technology Department, East China Normal University, Shanghai 200241, China<br>${ }^{2}$ Department of Mathematics, North University of China, Taiyuan, Shanxi 030051, China

Correspondence should be addressed to Lei Zhang; foxpujin@163.com
Received 21 January 2014; Accepted 1 February 2014; Published 1 June 2014
Academic Editor: Weiming Wang
Copyright © 2014 L. Zhang and Z. Li. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We focus on a spatially extended Holling-type IV predator-prey model that contains some important factors, such as noise (random fluctuations), external periodic forcing, and diffusion processes. By a brief stability and bifurcation analysis, we arrive at the Hopf and Turing bifurcation surface and derive the symbolic conditions for Hopf and Turing bifurcation on the spatial domain. Based on the stability and bifurcation analysis, we obtain spiral pattern formation via numerical simulation. Additionally, we study the model with a color noise and external periodic forcing. From the numerical results, we know that noise or external periodic forcing can induce instability and enhance the oscillation of the species density, and the cooperation between noise and external periodic forces inherent to the deterministic dynamics of periodically driven models gives rise to the appearance of a rich transport phenomenology. Our results show that modeling by reaction-diffusion equations is an appropriate tool for investigating fundamental mechanisms of complex spatiotemporal dynamics.


## 1. Introduction

Predation, a complex natural phenomenon, exists widely in the world, for example, the sea, the plain, the forest, the desert, and so on [1]. To model this phenomenon, the predator-prey model has been suggested for a long time since the pioneering works of Kendall [2]. Predator-prey model is a kind of "pursuit and evasion" system in which the prey trie to evade the predator and the predator tries to catch the prey if they interact [3]. Pursuit means the predator tries to shorten the spatial distance between the predator and the prey; evasion means the prey tries to widen this spatial distance. In fact, predatorprey model is a mathematical method to approximate some part of our real world. And the dynamic behavior of predatorprey model has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance $[4,5]$.

In general, a classical predator-prey model can be written as the form $[6,7]$

$$
\frac{d N}{d t}=N f(N)-m P g(N, P)
$$

$$
\begin{equation*}
\frac{d P}{d t}=P[c m g(N, P)-d] \tag{1}
\end{equation*}
$$

where $N$ and $P$ stand for prey and predator quantity, respectively, $f(N)$ is the per capita rate of increase of the prey in absence of predation, $d$ is the food-independent death rate of predator, $g(N, P)$ is the functional response, the prey consumption rate by an average single predator, which obviously increases with the prey consumption rate and can be influenced by the predator density, which refers to the change in the density of prey attached per unit time per predator as the prey density changes, $m g(N, P)$ is the amount of prey consumed per predator per unit time, and $\operatorname{cmg}(N, P)$ is the predator production per capita with predation.

In population dynamics, a functional response $g(N, P)$ describes the relationship between the predator and their prey, and the predator-prey model is always named after the corresponding functional response for its key position [6-9]. In the history of population ecology, both ecologists and mathematicians have a great interest in the Holling-type predator-prey models [ $3,8,10-21$ ], including Holling-types

I-III, originally due to Holling [22, 23], and Holling-type IV, suggested by Andrews [24]. The Holling-type functional responses are the so-called "prey-dependent" type [8], for $g(N, P)$ in (1) is a function only related to prey $N$. The classical expression of Holling-type II functional response is $g(N, P)=m N /(1+b N)$, and $g(N, P)=m N^{2} /\left(1+a N^{2}\right)$ is called Holling-type III. The Holling-type IV functional response is written as follows:

$$
\begin{equation*}
g(N, P)=\frac{m N}{1+b N+a N^{2}} \tag{2}
\end{equation*}
$$

Function (2) is called Monod-Haldane-type functional response too [25]. In addition, when $b=0$, a simplified form $g(N, P)=m N /\left(1+a N^{2}\right)$ is proposed by Sokol [26], and some scholars also called it as Holling-type IV [9, 25]. In this paper, we focus on the Holling-IV functional response taking the form (2), and the corresponding predator-prey model takes the form

$$
\begin{align*}
& \frac{d N}{d t}=r N\left(1-\frac{N}{K}\right)-\frac{m N P}{1+b N+a N^{2}} \\
& \frac{d P}{d t}=P\left(-q+\frac{c m N}{1+b N+a N^{2}}\right) \tag{3}
\end{align*}
$$

where $r>0$ stands for maximum per capita growth rate of the prey, $m>0$ is the capture rate, $c>0$ is the conversion rate of prey captured by predator, $q>0$ is the food-independent death rate of predator, $K>0$ is the carrying capacity, and $a>0$ is the so-called half-saturation constant; $b>-2 \sqrt{a}$ such that the denominator of above system does not vanish for nonnegative $N$.

On the other hand, the real world we live in is a spatial world, and spatial patterns are ubiquitous in nature, which modify the temporal dynamics and stability properties of population density at a range of spatial scales, whose effects must be incorporated in temporal ecological models that do not represent space explicitly [27]. And the spatial component of ecological interactions has been identified as an important factor in how ecological communities are shaped. Empirical evidence suggests that the spatial scale and structure of the environment can influence population interactions and the composition of communities [1].

The reaction-diffusion model is a typical spatially extended model. It considers not only time but also space and consists of several species which react with each other and diffuse within the spatial domain. It involves a pair of partial differential equations and represents the time course of reacting and diffusing process. In spatially extended predator-prey model, the interaction between the predator and the prey is the reaction item, and the diffusion item comes to being for the predator's "pursuit" and the prey's "evasion." Diffusion is a spatial process, and the whole model describes the evolution of the predator and the prey going with time.

Decades after Turing [28] demonstrated that spatial patterns could arise from the interaction of reactions or growth processes and diffusion; reaction-diffusion models have been studied in ecology to describe the population dynamics of predator-prey model for a long time since Segel
and Jackson applied Turing's idea [29]. Since then, a new field of ecology, pattern formation, came into being. The problem of pattern and scale is the central problem in ecology, unifying population biology and ecosystems science and marrying basic and applied ecology [30]. The study of spatial patterns in the distribution of organisms is a central issue in ecology, geology, chemistry, physics, and so on [1, 3, 11, $15,16,25,31-56]$. Theoretical work has shown that spatial and temporal pattern formation can play a very important role in ecological and evolutionary systems. Patterns can affect, for example, stability of ecosystems, the coexistence of species, invasion of mutants, and chaos. Moreover, the patterns themselves may interact, leading to selection on the level of patterns, interlocking eco-evolutionary time scales, evolutionary stagnation, and diversity.

Based on the above discussions, the spatially extended Holling-type IV predator-prey model with reaction diffusion takes as the form

$$
\begin{align*}
& \frac{\partial N}{\partial t}=r N\left(1-\frac{N}{K}\right)-\frac{m N P}{1+b N+a N^{2}}+d_{1} \nabla^{2} N \\
& \frac{\partial P}{\partial t}=P\left(-q+\frac{c m N}{1+b N+a N^{2}}\right)+d_{2} \nabla^{2} P \tag{4}
\end{align*}
$$

where $d_{1}$ and $d_{2}$ are the diffusion coefficients, respectively, and $\nabla^{2}=\partial / \partial x^{2}+\partial / \partial y^{2}$ is the usual Laplacian operator in twodimensional space; other parameters are the same definition as those in model (3).

It is easy to know that when a spatially extended predatorprey model is considered, the evolution of the model is decided by two sorts of sources (internal source and external source) which act together. The internal source is the dynamics of the individuals of the model, and the external source is the variability of environment. Some of the variability is periodic, such as temperature, water, food supply of the prey, and mating habits. It is necessary and important to consider models with periodic ecological parameters or perturbations which might be quite naturally exposed [57]. These periodic factors are regarded as external periodic forcing in the predator-prey systems. The external forcing can affect the population of predator and prey, respectively, which would go extinct in a deterministic environment. And some of the variability is irregular, such as the seasonal changes of the weather, food supply of the prey, and mating habits, and the effects of this variability are the so-called "noise." Ecological population dynamics are inevitably "noisy" [2]. In the predator-prey systems, the random fluctuations also are undeniably arising from either environmental variability or internal species. To quantify the relationship between fluctuations and species' concentration with spatial degrees of freedom, the consideration of these fluctuations supposes to deal with noisy quantities whose variance might at times be a sizable fraction of their mean levels. For example, the birth and death processes of individuals are intrinsically stochastic fluctuations which become especially pronounced when the number of individuals is small [16]. Moreover, there are many other stochastically factors causing predator-prey populations to change, such as effects of spatial structure of the habitat on the predator-prey ecosystem. The interactions
between the predator and prey, which are far from being uniformly distributed, also introduce randomness. And these processes can be regarded as a parameter that fluctuates irregularly in space and time.

External forcing and noise induce effects in population dynamics, such as pattern formation, stochastic resonance, delayed extinction, enhanced stability, and quasiperiodic oscillations, which have been investigated with increasing interest in the past decades [16, 34, 48, 56, 58-63]. And noise cannot systematically be neglected in models of population dynamics [63]. Zhou and Kurths [56] concluded these periodic variabilities as external forcing and investigated the interplay among noise, excitability, and mixing and external forcing in excitable media advected by a chaotic flow, in a twodimensional FitzHugh-Nagumo model described by a set of reaction-advection-diffusion equations. And Si et al. [61] studied the propagation of traveling waves in subexcitable systems driven; Liu et al. [59] considered a spatially extended phytoplankton-zooplankton system with additive noise and periodic forcing. Following these models they considered, the Holling-type IV predator-prey model with external periodic forcing and colored noise is as follows:

$$
\begin{align*}
& \frac{\partial N}{\partial t}=r N\left(1-\frac{N}{K}\right)-\frac{m N P}{1+b N+a N^{2}}+A \sin (\omega t)+d_{1} \nabla^{2} N \\
& \frac{\partial P}{\partial t}=P\left(-q+\frac{c m N}{1+b N+a N^{2}}\right)+\eta(\mathbf{r}, t)+d_{2} \nabla^{2} P \tag{5}
\end{align*}
$$

where $A \sin (\omega t)$ denotes the periodic forcing with amplitude $A$ and angular frequency $\omega$. The colored noise term $\eta(\mathbf{r}, t)(\mathbf{r}=(x, y))$ is introduced additively in space and time, referring to the fluctuations in the predator death rate, which partially results from the environmental factors such as epidemics, weather, and nature disasters and it is the Ornstein-Uhlenbeck process that obeys the following linear stochastic partial differential equation:

$$
\begin{equation*}
\frac{\partial \eta(\mathbf{r}, t)}{\partial t}=-\frac{1}{\tau} \eta(\mathbf{r}, t)+\frac{1}{\tau} \xi(\mathbf{r}, t), \tag{6}
\end{equation*}
$$

where $\xi(\mathbf{r}, t)$ is a Gaussian white noise or the so-called Markovian random telegraph process in both space and time with zero mean and correlation:

$$
\begin{equation*}
\langle\xi(\mathbf{r}, t)\rangle=0, \quad\left\langle\xi(\mathbf{r}, t) \xi\left(\mathbf{r}^{\prime}, t^{\prime}\right)\right\rangle=2 \varepsilon \delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \delta\left(t-t^{\prime}\right), \tag{7}
\end{equation*}
$$

where $\langle\cdot\rangle$ denotes averaging with respect to the noise $\xi(\mathbf{r}, t)$ and $\delta$ the Dirac delta-function and $\delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right)$ is the spatial correlation function of the Gaussian white noise $\xi(\mathbf{r}, t)$.

Integrating (6) with respect to time $t$, we get

$$
\begin{equation*}
\eta(\mathbf{r}, t)=\eta(\mathbf{r}, 0) e^{-t / \tau}+\frac{1}{\tau} e^{-t / \tau} \int_{0}^{t} e^{s / \tau} \xi(\mathbf{r}, s) d s \tag{8}
\end{equation*}
$$

The mean value of the colored noise is

$$
\begin{align*}
\langle\eta(\mathbf{r}, t)\rangle= & \langle\eta(\mathbf{r}, 0)\rangle e^{-t / \tau}+\frac{1}{\tau} e^{-t / \tau} \\
& \times \int_{0}^{t} e^{s / \tau}\langle\xi(\mathbf{r}, s)\rangle d s=\langle\eta(\mathbf{r}, 0)\rangle e^{-t / \tau} \tag{9}
\end{align*}
$$

and the correlation function of the colored noise is given by

$$
\begin{align*}
&\left\langle\eta(\mathbf{r}, t) \eta\left(\mathbf{r}^{\prime}, t^{\prime}\right)\right\rangle \\
&=\langle\eta(\mathbf{r}, 0)\rangle\left\langle\eta\left(\mathbf{r}^{\prime}, 0\right)\right\rangle e^{-\left(t+t^{\prime}\right) / \tau}+\frac{1}{\tau^{2}} e^{-\left(t+t^{\prime}\right) / \tau} \\
& \times \int_{0}^{t} \int_{0}^{t^{\prime}} e^{\left(s+s^{\prime}\right) / \tau}\left\langle\xi(\mathbf{r}, s) \xi\left(\mathbf{r}^{\prime}, s^{\prime}\right)\right\rangle d s d s^{\prime} \\
&=\langle\eta(\mathbf{r}, 0)\rangle\left\langle\eta\left(\mathbf{r}^{\prime}, 0\right)\right\rangle e^{-\left(t+t^{\prime}\right) / \tau}+\frac{\varepsilon}{\tau^{2}} e^{-\left(t+t^{\prime}\right) / \tau} \delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \\
& \times \int_{0}^{t} \int_{0}^{t^{\prime}} e^{\left(s+s^{\prime}\right) / \tau} \delta\left(t-t^{\prime}\right) d s d s^{\prime} \\
&=\langle\eta(\mathbf{r}, 0)\rangle\left\langle\eta\left(\mathbf{r}^{\prime}, 0\right)\right\rangle e^{-\left(t+t^{\prime}\right) / \tau}+\frac{\varepsilon}{\tau} \delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \\
& \times\left(e^{-\left(t+t^{\prime}\right) / \tau}-2 e^{-t / \tau}+e^{-\left(t-t^{\prime}\right) / \tau}\right) \tag{10}
\end{align*}
$$

Let $t \rightarrow+\infty$; then

$$
\begin{equation*}
\left\langle\eta(\mathbf{r}, t) \eta\left(\mathbf{r}^{\prime}, t^{\prime}\right)\right\rangle \longrightarrow \frac{\varepsilon}{\tau} e^{-\left(t-t^{\prime}\right) / \tau} \delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right) . \tag{11}
\end{equation*}
$$

The colored noise $\eta(\mathbf{r}, t)$ generated in this way represents a simple spatiotemporal structured noise that can be used to real mimic situations, which is temporally correlated and white in space, satisfying

$$
\begin{equation*}
\left\langle\eta(\mathbf{r}, t) \eta\left(\mathbf{r}^{\prime}, t^{\prime}\right)\right\rangle=\frac{\varepsilon}{\tau} e^{-\left|t-t^{\prime}\right| / \tau} \delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \tag{12}
\end{equation*}
$$

where the temporal memory of the stochastic process is controlled by $\tau$ and $\varepsilon$ is the intensity of noise. In this paper, we set $\tau=1$.

Based on these discussions above, in this paper, we mainly focus on the spatiotemporal dynamics of models (4) and (5). And the organization is as follows. In Section 2, we employ the method of stability analysis to derive the symbolic conditions for Hopf and Turing bifurcation in the spatial domain. In Section 3, we give the complex dynamics of models (4) and (5), involving pattern formation, phase portraits, time-series plots and resonant response, and so on, via numerical simulation. Then, in the last section, we give some discussions and remarks.

## 2. Hopf and Turing Bifurcation

The nonspatial model (3) has at least two equilibria (steady states) which correspond to spatially homogeneous equilibria of models (4) and (5), in the positive quadrant: $(0,0)$ (total extinct) is a saddle; $(K, 0)$ (extinct of the predator or preyonly) is a attracting node if $q>c m K /\left(1+K b+a K^{2}\right)$, a saddle
if $q<c m K /\left(1+K b+a K^{2}\right)$, or a saddle-node if $q=c m K /(1+$ $\left.K b+a K^{2}\right)$. When

$$
\begin{align*}
& (a, b, c, m, q, r, K) \in E_{1} \\
& \text { here, } E_{1}=\{(a, b, c, m, q, r, K) \mid m c>q b \\
& q^{2} a<(m c-q b)^{2} \\
& \sqrt{(m c-q b)^{2}-4 q^{2} a}  \tag{13}\\
& \quad>\left(-m^{2} c^{2}+2 q b m c+q a K m c\right. \\
& \left.\quad-q^{2} a K b+2 q^{2} a-q^{2} b^{2}\right) \\
& \\
& \times(-m c+q b+q a K)^{-1}>0 \\
& \left.\frac{b}{a}-\frac{m c}{q a}+K<0\right\}
\end{align*}
$$

there exists unique stationary coexistence state $\left(N_{1}^{*}, P_{1}^{*}\right)$, where

$$
\begin{align*}
& N_{1}^{*}=\frac{1}{2} \frac{-q b+m c-A}{q a},  \tag{14}\\
& P_{1}^{*}=\frac{c r\left((-m c+b q+q a K) N_{1}^{*}+q\right)}{a q^{2} K} .
\end{align*}
$$

On the other hand, when

$$
\begin{align*}
& (a, b, c, m, q, r, K) \in E_{2}, \\
& \text { here, } E_{2}=\left\{\begin{array}{l}
(a, b, c, m, q, r, K) \mid m c>q b, \\
\\
q^{2} a<(m c-q b)^{2}, \\
\sqrt{(m c-q b)^{2}-4 q^{2} a} \\
>-\left(-m^{2} c^{2}+2 q b m c+q a K m c\right. \\
\left.-q^{2} a K b+2 q^{2} a-q^{2} b^{2}\right) \\
\\
\times(-m c+q b+q a K)^{-1}>0 \\
\left.\frac{b}{a}-\frac{m c}{q a}+K>0\right\}
\end{array}\right.
\end{align*}
$$

there exists another unique stationary coexistence state ( $N_{2}^{*}, P_{2}^{*}$ ) implying

$$
\begin{align*}
& N_{2}^{*}=\frac{1}{2} \frac{-q b+m c+A}{q a}, \\
& P_{2}^{*}=\frac{c r\left((-m c+b q+q a K) N_{2}^{*}+q\right)}{a q^{2} K} . \tag{16}
\end{align*}
$$

It is worth mentioning that equilibria $\left(N_{1}^{*}, P_{1}^{*}\right)$ and $\left(N_{2}^{*}, P_{2}^{*}\right)$ cannot coexist. In this paper, we mainly focus on the dynamics of $\left(N_{1}^{*}, P_{1}^{*}\right)$ and rewrite it as $\left(N^{*}, P^{*}\right)$. The dynamics behavior of $\left(N_{2}^{*}, P_{2}^{*}\right)$ is similar to that of $\left(N_{1}^{*}, P_{1}^{*}\right)$.

To perform a linear stability analysis, we linearize model (3) around the stationary state $\left(N^{*}, P^{*}\right)$ for small space- and time-dependent fluctuations and expand them in Fourier space:

$$
\begin{array}{r}
N(\mathbf{r}, t) \sim N^{*} e^{\lambda t} e^{i \vec{k} \cdot \mathbf{r}}, \quad P(\mathbf{r}, t) \sim P^{*} e^{\lambda t} e^{i \vec{k} \cdot \mathbf{r}},  \tag{17}\\
\mathbf{r}=(x, y), \quad \vec{k}=\left(k_{x}, k_{y}\right),
\end{array}
$$

where $\lambda$ is the eigenvalue of the Jacobian matrix of model (3).
Hopf bifurcation is an instability induced by the transformation of the stability of a focus. Mathematically speaking, Hopf bifurcation occurs when $\operatorname{Im}(\lambda) \neq 0$ and $\operatorname{Re}(\lambda)=0$, at $k=0 ; \operatorname{Im}(\lambda)$ is the imaginary part, $\operatorname{Re}(\lambda)$ is the real part, and $k$ is the wave number. So we get the Hopf bifurcation surface:

$$
\begin{align*}
H=\{ & (a, b, c, m, q, r, K) \mid \operatorname{det}\left(J_{0}\right)>0, \\
& \left.\operatorname{trace}\left(J_{0}\right)=0\right\}, \tag{18}
\end{align*}
$$

where

$$
\begin{align*}
\operatorname{det}\left(J_{0}\right)= & -\left(r-2 \frac{r N^{*}}{K}\right) q \\
& +\frac{m q P^{*}+c m\left(r-2\left(r N^{*} / K\right)\right) N^{*}}{\left(1+b N^{*}+a N^{* 2}\right)} \\
& -\frac{m q N^{*} P^{*}\left(b+2 a N^{*}\right)}{\left(1+b N^{*}+a N^{* 2}\right)^{2}}, \\
\operatorname{trace}\left(J_{0}\right)= & r-2 \frac{r N^{*}}{K}-q \\
& +\frac{m\left(-P^{*}+c N^{*}+a N^{* 2} P^{*}+b c N^{* 2}+c N^{* 3} a\right)}{\left(1+b N^{*}+a N^{* 2}\right)^{2}}, \tag{19}
\end{align*}
$$

the frequency of periodic oscillations in time $\omega_{H}$ satisfies $\omega_{H}=\operatorname{Im}(\lambda)=\sqrt{\operatorname{det}\left(J_{0}\right)}$, and the corresponding wavelength $\lambda_{H}$ satisfies $\lambda_{H}=2 \pi / \omega_{H}=2 \pi / \sqrt{\operatorname{det}\left(J_{0}\right)}$. In particular, we take $K$ as the bifurcation parameter and can get the critical value of Hopf bifurcation from (18):

$$
\begin{gather*}
K_{H}=\left(-\left(a q^{2}(5 m c-3 q b)-(3 m c-q b)(m c-q b)^{2}\right)\right. \\
\times \sqrt{(m c-q b)^{2}-4 q^{2} a-4 q^{4} a^{2}+q^{2}(m c-q b)} \\
\left.\times(11 m c-5 q b) a-(3 m c-q b)(m c-q b)^{3}\right) \\
\times\left(\left(-a q\left(\left((2 m c-q b)(m c-q b)-2 q^{2} a\right)\right.\right.\right.  \tag{20}\\
\times \sqrt{(m c-q b)^{2}-4 q^{2} a}-2 a q^{2} \\
\times(3 m c-2 q b)+(2 m c-q b) \\
\left.\left.\left.\times(m c-q b)^{2}\right)\right)\right)^{-1} .
\end{gather*}
$$

Turing instability is induced only by "pursuit and evasion" if the predator can catch the prey by pursuit. We call the critical state of Turing instability as Turing bifurcation. Turing bifurcation occurs when $" \operatorname{Im}(\lambda)=0$ and $\operatorname{Re}(\lambda)=0$, at $k=$ $k_{T} \neq 0$," and the wavenumber $k_{T}$ satisfies $k_{T}^{2}=\sqrt{\operatorname{det}\left(J_{0}\right) / d_{1} d_{2}}$. In addition, at the Turing threshold, the spatial symmetry of the system is broken and the patterns are stationary in time and oscillatory in space with the wavelength $\lambda_{T}=2 \pi / k_{T}$. And the Turing bifurcation surface is given by

$$
\begin{align*}
& T=\left\{\left(a, b, c, m, q, r, d_{1}, d_{2}, K\right) \mid \operatorname{det}\left(J_{k}\right)=0\right.  \tag{21}\\
&\left.\operatorname{trace}\left(J_{k}\right)=0\right\}
\end{align*}
$$

where

$$
\begin{align*}
& \operatorname{det}\left(J_{k}\right) \\
& =-\left(r-2 \frac{r N^{*}}{K}-d_{1} k^{2}\right)\left(q+d_{2} k^{2}\right) \\
& \\
& \times\left(\left(q+d_{2} k^{2}\right) m P^{*}\right.  \tag{22}\\
& \\
& \left.\quad+\left(r-2\left(\frac{r N^{*}}{K}\right)-d_{1} k^{2}\right) c m N^{*}\right) \\
& \times \\
& \times\left(1+b N^{*}+a N^{* 2}\right)^{-1} \\
& \\
& -\frac{m\left(b+2 a N^{*}\right)\left(q+d_{2} k^{2}\right) N^{*} P^{*}}{\left(1+b N^{*}+a N^{* 2}\right)^{2}},
\end{align*}
$$

$$
\begin{align*}
\operatorname{trace}\left(J_{k}\right)= & r-2 \frac{r N^{*}}{K}-q-\left(d_{1}+d_{2}\right) k^{2} \\
& +\frac{m\left(-P^{*}+c N^{*}+a N^{* 2} P^{*}+b c N^{* 2}+c N^{* 3} a\right)}{\left(1+b N^{*}+a N^{* 2}\right)^{2}} \tag{23}
\end{align*}
$$

and the critical value of Turing bifurcation can be obtained from (21) as follows:

$$
\begin{equation*}
K_{T}=\frac{F_{1}}{F_{2}} \tag{24}
\end{equation*}
$$

where

$$
\begin{aligned}
F_{1}=r( & \left(4 q^{2} a(2 m c-q b)\right. \\
& \left.-(3 m c-q b)(m c-q b)^{2}\right) \\
& \times \sqrt{(m c-q b)^{2}-4 q^{2} a} \\
& +\left((m c-q b)^{2}-4 q^{2} a\right) \\
& \left.\cdot\left((3 m c-q b)(m c-q b)-2 q^{2} a\right)\right) \\
\times & \left((3 m c-q b)(m c-q b)-4 q^{2} a-(3 m c+q b)\right. \\
& \left.\times \sqrt{(m c-q b)^{2}-4 q^{2} a}\right) d_{2}
\end{aligned}
$$

$$
\begin{align*}
& F_{2}=q a\left(2 m c \left((m c-q b) \sqrt{(m c-q b)^{2}-4 q^{2} a}\right.\right. \\
& \left.+4 q^{2} a-(m c-q b)^{2}\right) B \\
& +\left((3 m c-q b) d_{2}\right. \\
& \cdot\left(2 m^{2} c^{2}-3 q b m c+q^{2} b^{2}-4 q^{2} a\right) r \\
& \left.+2 q d_{1} m c(m c-q b)^{2}\right) \\
& \times\left((m c-q b)^{2}-4 q^{2} a\right) \\
& +\left(-2 d_{2}\left((3 m c-q b)(2 m c-q b)(m c-q b)^{2}\right.\right. \\
& -2 q^{2} a\left(-4 q^{2} a+3 q^{2} b^{2}\right. \\
& \left.\left.-12 q b m c+11 m^{2} c^{2}\right)\right) r \\
& \left.-4 q c m(m c-q b) d_{1}\left((m c-q b)^{2}-4 q^{2} a\right)\right) \\
& \times \sqrt{(m c-q b)^{2}-4 q^{2} a} \\
& +\left(q^{2} b^{2}-2 q b m c+m^{2} c^{2}-4 q^{2} a\right) \\
& \times\left(( 2 m c - q b ) d _ { 2 } \left(3 m^{2} c^{2}-4 q b m c\right.\right. \\
& \left.+q^{2} b^{2}-4 q^{2} a\right) r \\
& \left.\left.+2 m c q d_{1}\left((m c-q b)^{2}-4 q^{2} a\right)\right)\right), \\
& B=\left(-2 d_{1} q\left(\left(\left(d_{1} b q^{2}-q d_{1} m c-q r b d_{2}+2 r m c d_{2}\right)\right.\right.\right. \\
& \times\left(4 q^{2} a-(m c-q b)^{2}\right) \\
& \left.-\operatorname{rmcd}_{2}(m c-q b)^{2}\right) \\
& \text { - } \sqrt{(m c-q b)^{2}-4 q^{2} a} \\
& +\left(8 q^{4} d_{2} r-8 q^{5} d_{1}\right) a^{2} \\
& +4(m c-q b) a q^{2} r m c d_{2} \\
& +\left(3 r m c d_{2}-q d_{1} m c+d_{1} b q^{2}-q r b d_{2}\right) \\
& \left.\left.\times\left((m c-q b)^{3}-6(m c-q b) a q^{2}\right)\right)\right)^{1 / 2} . \tag{25}
\end{align*}
$$

Linear stability analysis yields the bifurcation diagram with $r=1, a=0.125, b=1, c=0.7, m=0.625, q=0.18$, and $d_{2}=0.2$ as shown in Figure 1(a). In this case, parameters $(a, b, c, m, q, r, K) \in E_{1}$, and $\left(N^{*}, P^{*}\right)$ is the unique stationary coexistence state. From Figure 1(a), one can see that the Hopf bifurcation line and the Turing bifurcation curve separate
the parametric space into three distinct domains. In domain I, all two bifurcation lines are located below; the uniform steady state is the only stable solution of the model. Domain II is the region of pure Hopf instability. When the parameters correspond to domain III, which is located above all two bifurcation lines, both Hopf and Turing instability occur. Figure 1(b) illustrates the relation between the real and the imaginary parts of the eigenvalue $\lambda$ with $K=2.8>K_{H}=$ 2.279, which is located in domain II; one can see that when $k=0, \operatorname{Re}(\lambda(k))>0$ and $\operatorname{Im}(\lambda(k)) \neq 0$. Figure 1(c) displays the case of the critical value of Turing bifurcation $K=K_{T}=$ 3.499; in this case, $\operatorname{Re}(\lambda(k))=0$ and $\operatorname{Im}(\lambda(k))=0$ at $k=$ $k_{T}=2.080$. When $K=4.0$, parameters are located in domain III; Figure $1(\mathrm{~d})$ indicates that, at $k=0, \operatorname{Re}(\lambda(k))>0$ and $\operatorname{Im}(\lambda(k)) \neq 0$.

## 3. Spatiotemporal Dynamics of the Models

In this section, we perform extensive numerical simulations of the spatially extended models (4) and (5) in twodimensional space, and the qualitative results are shown here. All our numerical simulations employ the zero-flux Neumann boundary conditions with a system size of $200 \times 200$ space units. The parameters are $r=1, a=0.125, b=1, c=$ $0.7, m=0.625, q=0.18, d_{1}=0.02, d_{2}=0.2$, and $K=2.8$ or $K=4.0$, which satisfy $(a, b, c, m, q, r, K) \in E_{1}$. Models (4) and (5) are integrated initially in two-dimensional space from the homogeneous steady state; that is, we start with the unstable uniform solution $\left(N^{*}, P^{*}\right)$ with small random perturbation superimposed; in each, the initial condition is always a small amplitude random perturbation $\left( \pm 5 \times 10^{-4}\right)$, using a finite difference approximation for model (4) or Fourier transform method for model (5) for the spatial derivatives and an explicit Euler method for the time integration with a time stepsize of $\Delta t=1 / 24$ and space stepsize (lattice constant) of $\Delta x=\Delta y=1$. When the system reached a periodic oscillatory state, we took a snapshot with white corresponding to the high value of prey $N$ while black corresponding to the low one.

In the numerical simulations, different types of dynamics are observed and we have found that the distributions of predator and prey are always of the same type. Consequently, we can restrict our analysis of pattern formation to one distribution. In this section, we show, for instance, the distribution of prey $N$.
3.1. Pattern Formation of Model (4). Figure 2 shows the evolution of the spatial patterns of prey $N$ at $t=0,100$, $300,500,1000$, and 2000, with random small perturbation of the equilibrium $\left(N^{*}, P^{*}\right)=(0.748,2.132)$ of model (4) with $K=2.8$, located in domain II, more than the Hopf bifurcation threshold $K_{H}=2.279$ and less than the Turing bifurcation threshold $K_{T}=3.499$. In this case, pure Hopf instability occurs. One can see that, for model (4), the random initial distribution (cf. Figure 2(a)) leads to the formation of macroscopic spiral patterns (cf. Figures 2(d) to 2(f)). In other words, in this situation, spatially uniform steady-state predator-prey coexistence no longer exists. Small random
fluctuations will be strongly amplified by diffusion, leading to nonuniform population distributions. From the analysis in Section 2, we find that, with these parameters in domain II, the spiral pattern arises from the Hopf instability. The lower panel in Figure 2 shows the corresponding (g) time series and (h) phase portraits. Figure 2(g) illustrates the evolution process of prey $N$ and periodic oscillating in time finally; (h) exhibits the fact that a limit cycle arises, which is caused by the Hopf bifurcation.

When $K=4.0>K_{T}>K_{H}$, in this case, parameters in domain III (Figure 1(a)) and both Hopf and Turing instabilities occur. The nontrivial stationary state is $\left(N^{*}, P^{*}\right)=(0.748,2.365)$. As an example, the formation of a regular macroscopic two-dimensional spatial pattern is shown in Figure 3. The lower panel in Figure 3 shows the corresponding (g) time-series plots and (h) phase portraits.

Comparing this situation (Figure 3) with the one above (Figure 2), it is easy to see that the pattern formations are all spiral wave. From the analysis in Section 2, we know that when $K=2.8$, the wavelength $\lambda=3.100$ while, at $K=4.0$, $\lambda=3.021$. And the frequency of periodic oscillations in time is as inverse proportion with wavelength, so we can know that Turing instability has positive effect on the frequency while it has negative effect on wavelength. This is the reason why the spiral curves are more dense in Figure 3(f) than in Figure 2(f). On the other hand, one can see that when $K=4.0$, the time-series plots (cf. Figure 3(g)) indicate that when Turing instability occurs, the solution of model (4) is strongly oscillatory in time while with $K=2.8$ (pure Hopf bifurcation emerges) it is periodic (cf. Figure 2(g)). In addition, comparing Figure 2(g) with Figure 3(g), one can see that Turing instability has positive effects on the amplitude of prey $N$. And from Figure 3(h), one can see that a quasilimit cycle emerges while, in Figure 2(h), it is a cycle. Although there is some difference points between Figures 2 and 3, we can know that Turing instability cannot give birth to different type pattern. In our previous work [51], we find that Turing instability can change pattern type. This may be an important difference between the Holling-type IV and the ratio-dependent functional response of predator-prey model.

On the other hand, the basic idea of diffusion-driven instability in a reaction-diffusion system can be understood in terms of an activator-inhibitor system or predator-prey model (4). The functioning of this mechanism is based on three points [6]. First, a random increase of activator species (prey $N$ ) should have a positive effect on the creation rate of both activator (prey $N$ ) and inhibitor (prey $P$ ) species. Second, an increment in inhibitor species should have a negative effect on formation rate of both species. Finally, inhibitor species $P$ must diffuse faster than activator species $N$. Certainly, the reaction-diffusion predator-prey model (4), with Holling-type IV functional response and predators diffusing faster than prey (i.e., $d_{2}>d_{1}$ ), provides this mechanism. And spirals and curves are the most fascinating clusters to emerge from the predator-prey model. A spiral will form from a wave front when the prey line (which is leading the front) overlaps the pursuing line of predator [38]. The prey on the extreme end of the line stops moving as there is no predator in their immediate vicinity. However the prey $N$


Figure 1: (a) $K-d_{1}$ Bifurcation diagram for model (4) with $r=1, a=0.125, b=1, c=0.7, m=0.625, q=0.18$, and $d_{2}=0.2$. Hopf and Turing bifurcation lines separate the parameter space into three domains. The other parameters in (b)-(d) are $d_{1}=0.02$; the bifurcation parameter $K$ equals (b) $2.8>K_{H}=2.279$; (c) $3.499=K_{T}$; (d) $4.0>K_{T}>K_{H}$. The real parts $\operatorname{Re}(\lambda)$ and the imaginary parts $\operatorname{Im}(\lambda)$ are shown by solid curves and dashed curves, respectively.
and the predator $P$ in the center of the line continue moving forward. This forms a small trail of prey at one (or both) end of the front. This prey starts breeding and the trailing line of prey thickens and attracts the attention of predator at the end of the fox line that turns towards this new source of prey. Thus a spiral forms with predator $P$ on the inside and prey $N$ on the outside. If the original overlap of prey occurs at both ends of the line a double spiral will form. Spirals can also form as prey blob collapses after predator eats into it. This is the reason why the pattern formation of model (4) is spiral wave.
3.2. The Effect of Noise Only of Model (5). Now, we turn our focus on the effect of noise on the predator $P$ of stochastic model (5). In this case, $A=0$; that is, the periodic forcing is not present.

Figure 4 shows the dynamics of model (5) with noise on the predator. The first row of Figure 4, that is, (a), $\varepsilon=$ 0.0001 ; the second row, (b), $\varepsilon=0.01$; the third row, (c), $\varepsilon=0.05$; and the last row of Figure 4, (d), $\varepsilon=0.1$. And the first column of Figure 4, marked as (i), shows the snapshots of spatiotemporal pattern of model (5) at $t=$ 2000 with different intensity of noise, respectively. In this


Figure 2: Grey-scaled snapshots of spatiotemporal pattern of the prey $N$ of model (4) with $K=2.8$. (a) $t=0$, (b) $t=100$, (c) $t=300$, (d) $t=500$, (e) $t=1000$, and (f) $t=2000$. The lower panels show the corresponding (g) time-series plots and (h) phase portraits.
case, one can see that the pattern formation turns into spatial chaotic from spiral wave with the increase of noise intensity $\varepsilon$. And the second column of Figure 4, marked as (ii), displays the phase portraits of model (5) with different intensity of noise, respectively. We can see that, as noise intensity $\varepsilon$ increasing, the symmetry of the limit cycle is broken and gives rise to chaos. The last column of Figure 4, (iii), illustrates the time-series plots of prey $N$ with different intensity of noise, respectively. One can see that noise breaks the periodic oscillations in time and gives rise to drastically ruleless oscillations in time.
3.3. The Effect of Periodic Forcing of Model (5). In the previous subsection, we have shown the effect of noise on the predator $P$ of model (5). An interesting question is whether such noisesustained oscillations can be entrained by a weak external forcing, in this case, $\varepsilon=0$. This is investigated here.

When model (5) is noise free, there is a phenomenon of frequency locking or resonant response [56, 58-61]. That
is, without noise, the spatially homogeneous oscillation does not respond to the external periodic forcing when the amplitude $A$ is below a threshold whose value depends on the external period $T_{\text {in }}=2 \pi / \omega$. Above the threshold, model (5) may produce oscillations about period $T_{\text {out }}$ with respect to external period $T_{\text {in }}$, which is called frequency locking or resonant response. That is, the model produces one spike within each of the $M=T_{\text {out }} / T_{\text {in }}$ periods of the external force, called $M: 1$ resonant response $[56,61]$. The phenomenon of coherence resonance is of great importance [60]. Following Si et al. [61], in the present paper, the output period $T_{\text {out }}$ is defined as follows: $T_{i}$ is the time interval between the $i$ th spike and $(i+1)$ th spike. $m$ spikes are taken into account and the average value of them is $T_{\text {out }}=\sum_{i=1}^{m-1} T_{i} /(m-1)$.

As an example, with the amplitude $A=0.001$, Figure 5 shows $5: 1$ resonant response with $\omega=0.2 \pi$ (a) and $\omega=0.02 \pi$ (c), respectively. And Figures 5(b) and 5(d) are the phase portraits corresponding to (a) and (c). We can see that when $\omega=0.2 \pi$, there exists a periodic orbit, while, $\omega=0.02 \pi$,


FIgURE 3: Grey-scaled snapshots of spatiotemporal pattern of the prey $N$ of model (4) with $K=4.0$. (a) $t=0$, (b) $t=100$, (c) $t=300$, (d) $t=500$, (e) $t=1000$, and (f) $t=2000$. The lower panels show the corresponding (g) time-series plots and (h) phase portraits.
a periodic-2 orbit of model (5) emerges. Obviously, different $\omega$ can emerge from the same resonant response, and different phase orbits, that is, different numerical solution of model (5), may correspond to the same resonant response.
3.4. The Effect of Noise and Periodic Forcing of Model (5). Now, we consider the dynamics about resonant response of model (5) with both noise and periodic forcing. As depicted in Figure 6, the prey can generate $5: 1$ (a) and $4: 1$ (c)
locked oscillations, depending on the amplitude $A$ and angular frequency $\omega$. Figures 6(b) and 6(d) illustrate the spiral pattern at $t=2000$ corresponding to (a) and (c), respectively. In contrast, we change one of the parameters of Figure 6(c) $A=0.001$ to $A=0.01$ (e); one can see that the resonant response vanishes and the corresponding spiral pattern (f) is similar to (b). It indicates that the amplitude $A$ is a control factor for pattern formation. In addition, comparing Figure 6(b) with 6(d), one can see that the pattern formations are determined by noise intensity $\varepsilon$, too.


FIgure 4: Dynamics of model (5), for the following noise intensity. (a) $\varepsilon=0.0001$; (b) $\varepsilon=0.01$; (c) $\varepsilon=0.05$; (d) $\varepsilon=0.1$. (i) Snapshots of pattern formation at time 2000; (ii) phase portraits; (iii) time-series plots. $A=0$ and the other parameters are the same as those in Figure 2.

In Figure 7, we have shown a typical pattern formation process in the 5:1 frequency locking regime with $A=0.001$ and $\omega=0.2 \pi$. From $t=1870$ (a) to $t=1920$ (f), the pattern formation of prey $N$ is spiral wave and some small excitations already develop. One can see that, during the second period of the forcing, the prey is almost fully synchronized and relaxes slowly back to the state at moment (f). Obviously, the external periodic forcing at moment (e) repeats that at moment (a). However, the prey $N$ does not exactly repeat that due to a small fluctuation of the phase difference.

## 4. Conclusions and Remarks

In this paper, we present a spatial Holling-type IV predatorprey model containing some important factors, such as noise (random fluctuations), the external periodic forcing, and diffusion processes. And the numerical simulations were consistent with the predictions drawn from the bifurcation analysis, that is, Hopf bifurcation and Turing bifurcation.

If the parameter $K$, the carrying capacity, is located in domain II of Figure 1(a), the Hopf instability occurs and


FIGURE 5: External periodic forcing induced frequency locking of model (4). The solid curve is time series of prey $U$; the dash curve is the corresponding external periodic forcing. Other parameters are the same as those in Figure 3.
the destruction of the pattern begins from the prey $N$, while it begins from the predator $P$ if $K$ is located in domain III and both Hopf and Turing instabilities occur. From an ecological viewpoint, it shows that the initial and relatively rapid invasion of prey by predators can be followed by two subsequent invasions.

Furthermore, we demonstrate that noise and the external periodic forces play a key role in the predator-prey model (5) with the numerical simulations. We provoke qualitative transformations of the response of the model by changing noise intensity; noise can enhance the oscillation of the species density and format large clusters in the space. Periodic oscillations appear when the spatial noise and external periodic forcing are turned on; it also has been realized that model (5) is very sensitive to external periodic forcing through the natural annual variation of prey growth. In conclusion, we have shown that the cooperation between noise and external
periodic forces inherent to the deterministic dynamics of periodically driven models gives rise to the appearance of a rich transport phenomenology.

Significantly, model (5) exhibits oscillations when both noise and external forces are present. This means that the dynamics of the predator population may be partly determined not only by the deterministic factors but also by the external forcing and the stochastic factors. Therefore, the model for spatially extended systems composed of two species could be useful to explain spatiotemporal behaviors of populations whose dynamics are strongly affected by noise and the environmental physical variables, and the results of this paper are an important step toward providing the theoretical biology community with simple practical numerical methods, for investigating the key dynamics of realistic predator-prey models.


Figure 6: Dynamics of model (5) with both noise and periodic forcing. (b, d, and f) are snapshots at $t=2000$ corresponding to the left hand side resonant response. The other parameters are the same as those in Figure 3.


Figure 7: Typical pattern formation of the forced noisy prey in the $5: 1$ locking region at $A=0.001$ and $\varepsilon=0.0001$ corresponding to Figure 6(a). The lower panel shows the time series of the prey $N$ (the solid curve) and the corresponding external periodic forcing (the dash curve) corresponding to the snapshots of the patterns. The grey scale, from black to white, is in $[0,3.5]$ for all the snapshots.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

This research was supported by NSFC no. 11071273.

## References

[1] R. S. Cantrell and C. Cosner, Spatial Ecology via ReactionDiffusion Equations, Wiley Series in Mathematical and Computational Biology, John Wiley \& Sons, Chichester, UK, 2003.
[2] B. E. Kendall, "Cycles, chaos, and noise in predator-prey dynamics," Chaos, Solitons \& Fractals, vol. 12, no. 2, pp. 321-332, 2001.
[3] J. D. Murray, Mathematical Biology. II. Spatial Models and Biomedical Applications, vol. 18 of Interdisciplinary Applied Mathematics, Springer, New York, NY, USA, 3rd edition, 2003.
[4] A. A. Berryman, "The origins and evolution of predator-prey theory," Ecology, vol. 73, no. 5, pp. 1530-1535, 1992.
[5] Y. Kuang and E. Beretta, "Global qualitative analysis of a ratiodependent predator-prey system," Journal of Mathematical Biology, vol. 36, no. 4, pp. 389-406, 1998.
[6] D. Alonso, F. Bartumeus, and J. Catalan, "Mutual interference between predators can give rise to turing spatial patterns," Ecology, vol. 83, no. 1, pp. 28-34, 2002.
[7] R. Arditi and L. R. Ginzburg, "Coupling in predator-prey dynamics: ratio-dependence," Journal of Theoretical Biology, vol. 139, no. 3, pp. 311-326, 1989.
[8] P. A. Abrams and L. R. Ginzburg, "The nature of predation: prey dependent, ratio dependent or neither?" Trends in Ecology and Evolution, vol. 15, no. 8, pp. 337-341, 2000.
[9] S. Ruan and D. Xiao, "Global analysis in a predator-prey system with nonmonotonic functional response," SIAM Journal on Applied Mathematics, vol. 61, no. 4, pp. 1445-1472, 2000/01.
[10] Y. Chen, "Multiple periodic solutions of delayed predator-prey systems with type IV functional responses," Nonlinear Analysis. Real World Applications, vol. 5, no. 1, pp. 45-53, 2004.
[11] D. Estep and D. Neckels, "Fast methods for determining the evolution of uncertain parameters in reaction-diffusion equations," Computer Methods in Applied Mechanics and Engineering, vol. 196, no. 37-40, pp. 3967-3979, 2007.
[12] S. Gakkhar and B. Singh, "The dynamics of a food web consisting of two preys and a harvesting predator," Chaos, Solitons \& Fractals, vol. 34, no. 4, pp. 1346-1356, 2007.
[13] J.-C. Huang and D.-M. Xiao, "Analyses of bifurcations and stability in a predator-prey system with Holling type-IV functional response," Acta Mathematicae Applicatae Sinica, vol. 20, no. 1, pp. 167-178, 2004.
[14] W. Ko and K. Ryu, "Coexistence states of a predator-prey system with non-monotonic functional response," Nonlinear Analysis: Real World Applications, vol. 8, no. 3, pp. 769-786, 2007.
[15] T. Leppänen, Coputational studies of pattern formation in Turing systems [Ph.D. thesis], Helsinki University of Technology, 2004.
[16] H. Malchow, F. M. Hilker, and S. V. Petrovskii, "Noise and productivity dependence of spatiotemporal pattern formation in a prey-predator system," Discrete and Continuous Dynamical Systems. Series B, vol. 4, no. 3, pp. 705-711, 2004.
[17] K. Page, P. K. Maini, and N. A. M. Monk, "Pattern formation in spatially heterogeneous Turing reaction-diffusion models," Physica D, vol. 181, no. 1-2, pp. 80-101, 2003.
[18] G. T. Skalski and J. F. Gilliam, "Functional responses with predator interference: viable alternatives to the Holling type II model," Ecology, vol. 82, no. 11, pp. 3083-3092, 2001.
[19] S. Zhang, D. Tan, and L. Chen, "Chaos in periodically forced Holling type IV predator-prey system with impulsive perturbations," Chaos, Solitons \& Fractals, vol. 27, no. 4, pp. 980-990, 2006.
[20] W. Zhang, D. Zhu, and P. Bi, "Multiple positive periodic solutions of a delayed discrete predator-prey system with type IV functional responses," Applied Mathematics Letters, vol. 20, no. 10, pp. 1031-1038, 2007.
[21] H. Zhu, S. A. Campbell, and G. S. K. Wolkowicz, "Bifurcation analysis of a predator-prey system with nonmonotonic functional response," SIAM Journal on Applied Mathematics, vol. 63, no. 2, pp. 636-682, 2002.
[22] C. S. Holling, "The components of predation as revealed by a study of small mammal predation of the european pine sawfly," The Canadian Entomologist, vol. 91, pp. 293-320, 1959.
[23] C. S. Holling, "Some characteristics of simple types of predation and parasitism," The Canadian Entomologist, vol. 91, pp. 385395, 1959.
[24] J. F. Andrews, "A mathematical model for the continuous culture of microorganisms utilizing inhibitory substrates," Biotechnology and Bioengineering, vol. 10, pp. 707-723, 1968.
[25] P. Y. H. Pang and M. Wang, "Non-constant positive steady states of a predator-prey system with non-monotonic functional response and diffusion," Proceedings of the London Mathematical Society, vol. 88, no. 1, pp. 135-157, 2004.
[26] W. Sokol, "Oxidation of an inhibitory substrate by washed cells," Biotechnology and Bioengineering, vol. 30, no. 8, pp. 921-927, 1987.
[27] C. Neuhauser, "Mathematical challenges in spatial ecology," Notices of the American Mathematical Society, vol. 48, no. 11, pp. 1304-1314, 2001.
[28] A. M. Turing, "The chemical basis of morphogenisis", Philosophical Transactions of the Royal Society B, vol. 237, pp. 7-72, 1952.
[29] L. A. Segel and J. L. Jackson, "Dissipative structure: an explanation and an ecological example," Journal of Theoretical Biology, vol. 37, no. 3, pp. 545-559, 1972.
[30] S. A. Levin, "The problem of pattern and scale in ecology," Ecology, vol. 73, no. 6, pp. 1943-1967, 1992.
[31] A. Aotani, M. Mimura, and T. Mollee, "A model aided understanding of spot pattern formation in chemotactic E. Coli colonies," Japan Journal of Industrial and Applied Mathematics, vol. 27, no. 1, pp. 5-22, 2010.
[32] M. Baurmann, T. Gross, and U. Feudel, "Instabilities in spatially extended predator-prey systems: spatio-temporal patterns in the neighborhood of Turing-Hopf bifurcations," Journal of Theoretical Biology, vol. 245, no. 2, pp. 220-229, 2007.
[33] M. C. Cross and P. C. Hohenberg, "Pattern formation outside of equilibrium," Reviews of Modern Physics, vol. 65, no. 3, pp. 851-1112, 1993.
[34] J. García-Ojalvo and L. Schimansky-Geier, "Noise-induced spiral dynamics in excitable media," Europhysics Letters, vol. 47, no. 3, pp. 298-303, 1999.
[35] M. R. Garvie, "Finite-difference schemes for reaction-diffusion equations modeling predator-prey interactions in MATLAB," Bulletin of Mathematical Biology, vol. 69, no. 3, pp. 931-956, 2007.
[36] A. Gierer and H. Meinhardt, "A theory of biological pattern formation," Kybernetik, vol. 12, no. 1, pp. 30-39, 1972.
[37] D. A. Griffith and P. R. Peres-Neto, "Spatial modeling in ecology: the flexibility of eigenfunction spatial analyses," Ecology, vol. 87, no. 10, pp. 2603-2613, 2006.
[38] K. A. Hawick, H. A. James, and C. J. Scogings, "A zoology of emergent patterns in a predator-prey simulation model," in Proceedings of the 6th IASTED International Conference on Modelling, Simulation, and Optimizatiom (MSO '06), pp. 84-89, Gabarone, Botswana, September 2006.
[39] H. Katsuragi, "Diffusion-induced spontaneous pattern formation on gelation surfaces," Europhysics Letters, vol. 73, no. 5, pp. 793-799, 2006.
[40] C. A. Klausmeier, "Regular and irregular patterns in semiarid vegetation," Science, vol. 284, no. 5421, pp. 1826-1828, 1999.
[41] M. Li, B. Han, L. Xu, and G. Zhang, "Spiral patterns near Turing instability in a discrete reaction diffusion system," Chaos, Solitons \& Fractals, vol. 49, pp. 1-6, 2013.
[42] Z.-Z. Li, M. Gao, C. Hui, X.-Z. Han, and H. Shi, "Impact of predator pursuit and prey evasion on synchrony and spatial patterns in metapopulation," Ecological Modelling, vol. 185, no. 2-4, pp. 245-254, 2005.
[43] A. Luiz, C. Diomar, and P. Sergei, "Pattern formation in a spaceand time-discrete predator-prey system with a strong Allee effect," Theoretical Ecology, vol. 5, no. 3, pp. 341-362, 2012.
[44] A. Madzvamuse, R. D. K. Thomas, P. K. Maini, and A. J. Wathen, "A numerical approach to the study of spatial pattern formation in the ligaments of arcoid bivalves," Bulletin of Mathematical Biology, vol. 64, no. 3, pp. 501-530, 2002.
[45] P. K. Maini, R. E. Baker, and C.-M. Chuong, "The turing model comes of molecular age," Science, vol. 314, no. 5804, pp. 13971398, 2006.
[46] D. O. Maionchi, S. F. Dos Reis, and M. A. M. De Aguiar, "Chaos and pattern formation in a spatial tritrophic food chain," Ecological Modelling, vol. 191, no. 2, pp. 291-303, 2006.
[47] A. B. Medvinsky, S. V. Petrovskii, I. A. Tikhonova, H. Malchow, and B.-L. Li, "Spatiotemporal complexity of plankton and fish dynamics," SIAM Review, vol. 44, no. 3, pp. 311-370, 2002.
[48] S. S. Riaz, S. Dutta, S. Kar, and D. S. Ray, "Pattern formation induced by additive noise: a moment-based analysis", European Physical Journal B, vol. 47, no. 2, pp. 255-263, 2005.
[49] S. S. Riaz, S. Banarjee, S. Kar, and D. S. Ray, "Pattern formation in reaction-diffusion system in crossed electric and magnetic fields," European Physical Journal B, vol. 53, no. 4, pp. 509-515, 2006.
[50] K. Uriu and Y. Iwasa, "Turing pattern formation with two kinds of cells and a diffusive chemical," Bulletin of Mathematical Biology, vol. 69, no. 8, pp. 2515-2536, 2007.
[51] W. Wang, Q.-X. Liu, and Z. Jin, "Spatiotemporal complexity of a ratio-dependent predator-prey system," Physical Review E, vol. 75, no. 5, Article ID 051913, 9 pages, 2007.
[52] W. Wang, Y. Lin, F. Yang, L. Zhang, and Y. Tan, "Numerical study of pattern formation in an extended Gray-Scott model," Communications in Nonlinear Science and Numerical Simulation, vol. 16, no. 4, pp. 2016-2026, 2011.
[53] W. Wang, Y. Cai, Y. Zhu, and Z. Guo, "Allee-effect-induced instability in a reaction-diffusion predator-prey model," Abstract and Applied Analysis, vol. 2013, Article ID 487810, 10 pages, 2013.
[54] L. Yang, M. Dolnik, A. M. Zhabotinsky, and I. R. Epstein, "Pattern formation arising from interactions between turing and wave instabilities," Journal of Chemical Physics, vol. 117, no. 15, pp. 7259-7265, 2002.
[55] L. Yang, M. Dolnik, A. M. Zhabotinsky, and I. R. Epstein, "Spatial resonances and superposition patterns in a reactiondiffusion model with interacting turing modes," Physical Review Letters, vol. 88, no. 20, pp. 2083031-2083034, 2002.
[56] C. Zhou and J. Kurths, "Noise-sustained and controlled synchronization of stirred excitable media by external forcing," New Journal of Physics, vol. 7, article 18, 2005.
[57] J. M. Cushing, "Periodic time-dependent predator-prey systems," SIAM Journal on Applied Mathematics, vol. 32, no. 1, pp. 82-95, 1977.
[58] F. Rao and W. Wang, "Dynamics of a Michaelis-Menten-type predation model incorporating a prey refuge with noise and external forces," Journal of Statistical Mechanics: Theory and Experiment, vol. 2012, no. 3, Article ID P03014, 2012.
[59] Q. Liu, Z. Jin, and B. Li, "Resonance and frequency-locking phenomena in spatially extended phytoplankton-zooplankton system with additive noise and periodic forces," Journal of Statistical Mechanics: Theory and Experiment, vol. 2008, no. 5, Article ID P05011, 2008.
[60] R. Mankin, T. Laas, A. Sauga, and A. Ainsaar, "Colored-noiseinduced Hopf bifurcations in predator-prey communities," Physical Review E, vol. 74, no. 2, Article ID 021101, 10 pages, 2006.
[61] F. N. Si, Q. X. Liu, J. Z. Zhang, and L. Q. Zhou, "Propagation of travelling waves in sub-excitable systems driven by noise and periodic forcing," European Physical Journal B, vol. 60, no. 4, pp. 507-513, 2007.
[62] B. Spagnolo, D. Valenti, and A. Fiasconaro, "Noise in ecosystems: a short review," Mathematical Biosciences and Engineering, vol. 1, no. 1, pp. 185-211, 2004.
[63] J. M. G. Vilar and R. V. Solé, "Effects of noise in symmetric twospecies competition," Physical Review Letters, vol. 80, no. 18, pp. 4099-4102, 1998.

## Research Article

# Dynamical Behavior and Stability Analysis in a Hybrid Epidemiological-Economic Model with Incubation 

Chao Liu, ${ }^{1,2}$ Wenquan Yue, ${ }^{3}$ and Peiyong Liu ${ }^{2,4}$<br>${ }^{1}$ Institute of Systems Science, Northeastern University, Shenyang 110004, China<br>${ }^{2}$ State Key Laboratory of Integrated Automation of Process Industry, Northeastern University, Shenyang 110004, China<br>${ }^{3}$ Changli Institute of Fruit Forestry, Hebei Academy of Agricultural and Forestry Sciences, Changli 066600, China<br>${ }^{4}$ Institute of Biotechnology, College of Life and Health Sciences, Northeastern University, Shenyang 110004, China<br>Correspondence should be addressed to Chao Liu; singularsystem@163.com

Received 12 January 2014; Accepted 15 April 2014; Published 12 May 2014
Academic Editor: Weiming Wang
Copyright © 2014 Chao Liu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

A hybrid SIR vector disease model with incubation is established, where susceptible host population satisfies the logistic equation and the recovered host individuals are commercially harvested. It is utilized to discuss the transmission mechanism of infectious disease and dynamical effect of commercial harvest on population dynamics. Positivity and permanence of solutions are analytically investigated. By choosing economic interest of commercial harvesting as a parameter, dynamical behavior and local stability of model system without time delay are studied. It reveals that there is a phenomenon of singularity induced bifurcation as well as local stability switch around interior equilibrium when economic interest increases through zero. State feedback controllers are designed to stabilize model system around the desired interior equilibria in the case of zero economic interest and positive economic interest, respectively. By analyzing corresponding characteristic equation of model system with time delay, local stability analysis around interior equilibrium is discussed due to variation of time delay. Hopf bifurcation occurs at the critical value of time delay and corresponding limit cycle is also observed. Furthermore, directions of Hopf bifurcation and stability of the bifurcating periodic solutions are studied. Numerical simulations are carried out to show consistency with theoretical analysis.


## 1. Introduction

In recent decades, plenty of mathematical models describing the population dynamics of infectious disease have been extensively utilized to understand the transmission mechanism of infectious disease within population ecosystem (see [1-4] and references therein). Much research efforts have been paid to susceptible-infective-recovered (SIR) vector disease model and corresponding model dynamics (see [5-12] and references therein). Generally, in modelling of communicable disease, the incidence rate (the rate of new infections) is considered to play a vital role in ensuring that the model can provide a reasonable qualitative description of the infectious disease dynamics $[3,4]$.

In order to discuss the spread of an infectious disease transmitted by a vector (e.g., mosquitoes and rats), Takeuchi et al. [7] formulated a delayed SIR epidemic model with
a bilinear incidence rate. Beretta et al. [8] considered the global stability of disease free equilibrium and endemic equilibrium of model system; it was shown that the disease free equilibrium is globally stable for any time delay while the endemic equilibrium is not feasible. By constructing a suitable Lyapunov functional, sufficient conditions were derived to guarantee that if the endemic equilibrium is feasible, it is also globally stable for the delay being sufficiently small. Ruan and Wang [13] studied the global dynamics of an SIR model with vital dynamics and nonlinear incidence rate of saturated mass action and global qualitative and bifurcation analyses are carried out. Ma et al. [14] derived an explicit expression of lower bound of the infective individual of solution of model system, which was proposed as an open problem. They therefore gave an estimation of the length of time delay ensuring global asymptotic stability of the endemic equilibrium. Xu and Ma [15] proposed an SIR epidemic
model with nonlinear incidence rate and time delay. By analyzing the corresponding characteristic equations, local stability of an endemic equilibrium and a disease free equilibrium are discussed. An SIR model with distributed delay and a general incidence function is studied in McCluskey [9], and the global dynamics for the SIR epidemiological system is analyzed in Zhou and Cui [10]. Wang et al. [11] considered the asymptotic behavior of the following SIR vector model:

$$
\begin{align*}
& \dot{S}(t)=r\left(1-\frac{S(t)}{k}\right)-\beta S(t) I(t-\tau) \\
& \dot{I}(t)=\beta S(t) I(t-\tau)-\mu_{1} I(t)-m I(t)  \tag{1}\\
& \dot{R}(t)=m I(t)-\mu_{2} R(t)
\end{align*}
$$

where $S(t), I(t)$, and $R(t)$ represent the population density of susceptible, infective, and recovered host individuals at time $t$, respectively. It is assumed that the population growth of susceptible host individuals is governed by the logistic growth with a carrying capacity $k>0$ as well as intrinsic birth rate constant $r>0 . \beta>0$ is the average number of constants per infective per unit time and $\tau \geq 0$ denotes the incubation time, and $\mu_{1}>0$ and $\mu_{2}>0$ stand for the death rate of infective and recovered host individuals, respectively. $m>0$ represents the recovery rate of infective host individuals. The local stability of endemic equilibrium is investigated, and conditions for Hopf bifurcation to occur are derived in [11]. Along with the line of this research, Enatsu et al. [12] analyze stability of equilibria for a delayed SIR epidemic model, in which population growth is subject to logistic growth in absence of disease and the proposed model with a nonlinear incidence rate satisfying suitable monotonicity conditions.

Nowadays, biological resource within ecosystem is commercially harvested and sold with aim of achieving economic interest [16, 17]. It is well known that harvesting has a strong impact on the dynamic evolution of population and several mathematical models have been established to discuss dynamic effects of harvest effort on population in ecological-epidemiological system, which can be found in [18-21] and the references therein. The role of harvesting in a predator-prey-parasite system is discussed in [18]; theoretical results show that, using impulsive harvesting effort as control parameter, it is not only possible to control the cyclic behavior of the system populations leading to the persistence of all species but other desired stable equilibrium including disease free can be obtained. A ratio-dependent eco-epidemiological system is proposed in [19] where prey population is subject to harvesting. Positive invariance, boundedness, stability of equilibria, and permanence of system have been established. In [20], an eco-epidemiological model is studied where prey disease is modeled by a susceptible-infective scheme, and the role of harvesting and switching on the dynamics of disease propagation and/or eradication is discussed. An eco-epidemiological model with distributed time delay and impulsive control strategy is investigated in [21]; local stability and complex dynamical behavior are discussed. Under the system of market economy, harvest effort is usually influenced by variation of economic interest of commercial harvesting [16, 22]. It should be noted that the above mentioned
related work [18-21] only concentrate on the role of harvest effort on population dynamics, while the dynamic effect of economic interest on commercial harvesting and indirect dynamic effect on ecosystem are not considered. The work done in [12] is an extension of [11] with nonlinear incidence rate, while dynamic effect of harvest effort on population dynamics is not considered.

Recently, some hybrid dynamical models are proposed in [23-28], which are utilized to discuss the interaction mechanism of harvested ecosystem from an economic perspective. Compared with the traditional mathematical models (differential equations or difference equations) discussing the population dynamics in ecosystem, the hybrid mathematical models proposed in [23-28] are made up of differential equations and algebraic equations, where differential equations concentrate on coexistence and interaction mechanism of population and algebraic equations offer a simpler way to study the effect of harvest effort on ecosystem from an economic perspective. Complex dynamical behavior and stability analysis in prey-predator ecosystems with stagestructured population and gestation delay are considered in [23-28]. In general, differential-algebraic models exhibit more complicated dynamics than ordinary differential models. The differential-algebraic models have been applied widely in power systems, aerospace engineering, chemical processes, social management systems, biological systems, network analysis and oil catalysis, and cracking process (see [29-31] and references therein). With the help of differentialalgebraic model for the power systems and bifurcation theory, complex dynamical behaviors of the power systems, especially the bifurcation phenomena that reveal the instability mechanism of power systems have been extensively studied, which can be found in [32-34] and the references therein. Furthermore, some applications of differentialalgebraic models in the field of economy, which can be found in $[35,36]$.

It is well known that the recovered host individuals are naturally immune to vector disease [1], and its potential economic interest can be commercially exploited. Furthermore, harvest effort is usually influenced by variation of economic interest of commercial harvesting $[16,22]$ under the system of market economy. Consequently, it is necessary to discuss the coexistence and interaction mechanism of population within harvested epidemiological ecosystem as well as dynamical effect of harvest effort due to variation of economic interest. However, as far as knowledge goes, nobody has explicitly proposed a mathematical model to discuss the dynamic effect of commercial harvest on epidemiological system under the system of market economy. The main objective of this paper is to investigate the transmission mechanism of infectious disease and dynamical effect of commercial harvest on population dynamics, especially the complex dynamical behavior and stability switch due to variation of incubation and commercial harvest economic interest. The organisation of the rest section of this paper is as follows. By introducing commercial harvest effort into model system (1), a hybrid epidemiological-economic model is established in Section 2. Positivity and permanence of solutions of model system are discussed in Section 3. In Section 4, qualitative analyses of
model system are performed. Conditions for existence of interior equilibrium of model system are studied. Dynamical behavior of model system without incubation around the interior equilibrium is investigated due to variation of economic interest, and state feedback controllers are designed to stabilize model system around the desired interior equilibria. Furthermore, local stability analysis of model system with incubation is analyzed due to variation of time delay; directions of Hopf bifurcation and stability of the bifurcating periodic solutions are also studied. Numerical simulations are made in Section 5, which are utilized to support the theoretical findings obtained in this paper. Finally, this paper ends with a conclusion.

## 2. Model Formulation

In 1954, Gordon [22] proposed the economic theory of a common-property resource, which studies the effect of harvest effort on ecosystem from an economic perspective. In [22], an algebraic equation is proposed to investigate the economic interest of yield of the harvest effort, which takes form as follows:

$$
\begin{align*}
& \text { Net Economic Revenue (NER) }  \tag{2}\\
& \qquad=\text { Total Revenue (TR) - Total Cost (TC) }
\end{align*}
$$

Associated with model (1), an algebraic equation, which considers the economic interest $v$ of the harvest effort on recovered host individuals in epidemiological system, that is, $R(t)$, is established as follows:

$$
\begin{equation*}
E(t)(w R(t)-c)=v \tag{3}
\end{equation*}
$$

where $E(t)$ represents the harvest effort on recovered host individuals at time $t . v$ represents the economic interest of harvest effort on the recovered host individuals. $w$ and $c$ represent unit price of harvested population and cost of harvest effort, respectively.

Based on (1) and (3), a delayed hybrid model which consists of three differential equations and an algebraic equation can be established as follows:

$$
\begin{align*}
\dot{S}(t) & =r\left(1-\frac{S(t)}{k}\right)-\beta S(t) I(t-\tau) \\
\dot{I}(t) & =\beta S(t) I(t-\tau)-\mu_{1} I(t)-m I(t)  \tag{4}\\
\dot{R}(t) & =m I(t)-\mu_{2} R(t)-E(t) R(t) \\
0 & =E(t)(w R(t)-c)-v
\end{align*}
$$

where $S(t), I(t), R(t), E(t)$, and other parameters share the same interpretations mentioned in (1) and (3), and initial conditions $\psi=\left(\psi_{1}, \psi_{2}, \psi_{3}, \psi_{4}\right)$ for model system (4) are defined in the Banach space:

$$
\begin{align*}
\left\{\psi \in C^{+}\left([-\tau, 0], \mathbb{R}_{+}^{4}\right) \mid \psi_{1}(\theta)\right. & =S(\theta), \psi_{2}(\theta)=I(\theta) \\
\psi_{3}(\theta) & \left.=R(\theta), \psi_{4}(\theta)=E(\theta)\right\} \tag{5}
\end{align*}
$$

where $\mathbb{R}_{+}^{4}=\left\{(S, I, R, E) \in \mathbb{R}^{4}: S \geq 0, I \geq 0, R \geq 0, E \geq 0\right\}$. It is also assumed that $\psi_{i}(0)>0(i=1,2,3,4)$ for a biological reason.

Model system (4) can be expressed in the following form:

$$
\begin{equation*}
\Xi(t) \dot{X}(t)=F(X(t)) \tag{6}
\end{equation*}
$$

where

$$
\begin{align*}
X(t) & =(S(t), I(t), R(t), E(t))^{T}, \\
\Xi(t) & =\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{array}\right], \\
F(X(t)) & =\left[\begin{array}{l}
F_{1}(X(t)) \\
F_{2}(X(t)) \\
F_{3}(X(t)) \\
F_{4}(X(t))
\end{array}\right]  \tag{7}\\
& =\left[\begin{array}{c}
r\left(1-\frac{S(t)}{k}\right)-\beta S(t) I(t-\tau) \\
\beta S(t) I(t-\tau)-\mu_{1} I(t)-m I(t) \\
m I(t)-\mu_{2} R(t)-E(t) R(t) \\
E(t)(w R(t)-c)-v
\end{array}\right] .
\end{align*}
$$

Remark 1. The algebraic equation in model system (6) contains no differentiated variables; hence, the leading matrix $\Xi(t)$ in model system (6) has a corresponding zero row.

Remark 2. The model proposed in [11], which composed of differential equations, only discusses the interaction and coexistence mechanism of susceptible, infective, and recovered host individuals. Compared with the model proposed in [11], algebraic equations are incorporated into the model system (4), which focus on the economic interest of harvesting on recovered host individuals. Hence, the established model not only investigates interaction and coexistence mechanism of population in harvested ecosystem but also studies the dynamical behavior due to the variation of economic interest of commercial harvesting and incubation.

## 3. Positivity and Permanence

Theorem 3. Any solutions of model system (4) with initial conditions are positive.

Proof. For any solutions of model system (4), it is easy to show that $F_{i}: \mathbb{R}_{+}^{4+1} \rightarrow \mathbb{R}^{4}$ is locally Lipschitz and satisfies the condition, $\left.F_{i}(X(t))\right|_{X \in \mathbb{R}^{4}}>0$, where $F_{i}(X(t))(i=1,2,3,4)$ have been defined in model system (4).

Due to the lemma in [37] and Theorem A. 4 in [38], any solution of the model system (4) with positive initial conditions exists uniquely and each component of the solution remains within the interval $\left[0, A_{0}\right)$ for some $A_{0}>0$. Standard and simple arguments show that solutions of model system (4) always exist and stay positive. Hence, this completes the positivity of the solutions of model system (4).

From a viewpoint of biological and economic interest perspective, persistence of solutions of model system (4) in
the case of economic interest $v \geq 0$ will be investigated in this section. Some preliminaries are introduced as follows.

Definition 4 (see [39]). Model system (4) is said to be permanent if there exists a compact region $\Omega_{0} \in \operatorname{int} \Omega$ such that every solution of model system (4) with initial conditions will eventually enter and remain in region $\Omega_{0}$.

Definition 5 (see [39]). Consider a metric space $Q$ with metric $d$. The distance $d(x, y)$ of a point $x \in Q$ from a subset $Y$ of $Q$ is defined by

$$
\begin{equation*}
d(x, y)=\inf _{y \in Y} d(x, y) \tag{8}
\end{equation*}
$$

It is further assumed that $Q$ is the closure of an open set $Q^{0}$, and $Q^{0}=\partial Q^{0}$ is nonempty and is the boundary of $Q^{0}$. Consequently, $Q^{0} \cup Q_{0}=Q, Q^{0} \cap Q_{0}=\varnothing$. We will also suppose that $T(t)$ is a $Q^{0}$ semigroup on $Q$ satisfying

$$
\begin{equation*}
T(t): Q^{0} \longrightarrow Q^{0}, \quad T(t): Q_{0} \longrightarrow Q_{0} \tag{9}
\end{equation*}
$$

Let $T_{\partial}(t)=\left.T(t)\right|_{Q_{0}}$ and $A_{\partial}$ be the global attractor for $T_{\partial}(t)$.

Lemma 6 (see [39]). Suppose that $T(t)$ satisfies (9) and the following conditions hold.
(i) There is a $t_{0} \geq 0$ such that $T(t)$ is compact for $t>t_{0}$;
(ii) $T(t)$ is point dissipative in $Q$;
(iii) $\tilde{A}_{\partial}=\bigcup_{x \in A_{\partial}} \omega(x)$ is isolated and has an acyclic covering $Z$.

Then $T(t)$ is uniformly persistent if and only if for each $Z_{i} \in Z$, $W^{s}\left(Z_{i}\right) \cap Q^{0}=\emptyset$ for $i=1,2, \ldots, n$.

Lemma 7 (see [40]). Consider the following equation:

$$
\begin{equation*}
\dot{u}(t)=a u(t-\tau)-b u(t) \tag{10}
\end{equation*}
$$

where $a, b, \tau>0$ and $u(t)>0$ for all $-\tau \leq t \leq 0$; it derives the following:
(i) If $a<b$, then $\lim _{t \rightarrow+\infty} u(t)=0$,
(ii) If $a>b$, then $\lim _{t \rightarrow+\infty} u(t)=+\infty$.

Lemma 8. For any solutions of model system (4), we have

$$
\begin{gather*}
\limsup _{t \rightarrow+\infty} S(t) \leq k \\
\limsup _{t \rightarrow+\infty}(S(t)+I(t)) \leq \frac{k\left(r+\mu_{1}+m\right)^{2}}{4 r\left(\mu_{1}+m\right)} . \tag{11}
\end{gather*}
$$

Proof. By using Theorem 3, it follows from the first equation of model system (4) that

$$
\begin{equation*}
\dot{S}(t) \leq r\left(1-\frac{S(t)}{k}\right) S(t) \tag{12}
\end{equation*}
$$

which derives that $\lim \sup _{t \rightarrow+\infty} S(t) \leq k$.

According to Theorem 3 and the first and second equation of model system (4), it gives that

$$
\begin{equation*}
\dot{S}(t)+\dot{I}(t) \leq r\left(1-\frac{S(t)}{k}\right) S(t)-\left(\mu_{1}+m\right) I(t) \tag{13}
\end{equation*}
$$

which derives that $\lim \sup _{t \rightarrow+\infty}(S(t)+I(t)) \leq k\left(r+\mu_{1}+\right.$ $m)^{2} / 4 r\left(\mu_{1}+m\right)$.

Lemma 9. If $\mu_{1}+m<1$, then $(S(t), I(t))$ of solution of model system (4) with initial conditions satisfies

$$
\begin{equation*}
\liminf _{t \rightarrow+\infty} S(t) \geq S_{\eta}, \quad \liminf _{t \rightarrow+\infty} I(t) \geq I_{\eta}, \tag{14}
\end{equation*}
$$

where $S_{\eta}>0$ and $I_{\eta}>0$ are independent of corresponding initial values of model system (4).

Proof. Firstly, let $C^{+}\left([-\tau, 0], \mathbb{R}_{+}^{2}\right)$ denote space of continuous functions mapping $[-\tau, 0]$ into $\mathbb{R}_{+}^{2}$, where $\mathbb{R}_{+}^{2}=\{(x, y) \mid x \geq$ $0, y \geq 0\}$ :

$$
\begin{align*}
Q_{1}= & \left\{\left(\phi_{1}, \phi_{2}\right) \in C^{+}\left([-\tau, 0], \mathbb{R}_{+}^{2}\right) \mid \phi_{1}(\theta)=0, \theta \in[-\tau, 0]\right\} \\
Q_{2}= & \left\{\left(\phi_{1}, \phi_{2}\right) \in C^{+}\left([-\tau, 0], \mathbb{R}_{+}^{2}\right) \mid \phi_{1}(\theta)>0\right. \\
& \left.\phi_{2}(\theta)=0, \theta \in[-\tau, 0]\right\} . \tag{15}
\end{align*}
$$

Denote $Q_{0}=Q_{1} \cup Q_{2}, Q=C^{+}\left([-\tau, 0], \mathbb{R}_{+}^{2}\right)$ and $Q^{0}=$ $\operatorname{int} C^{+}\left([-\tau, 0], \mathbb{R}_{+}^{2}\right)$.

Next, all conditions in Lemma 6 will be checked. In order to facilitate the proof, we consider the following subsystem of model system (4):

$$
\begin{align*}
& \dot{S}(t)=r\left(1-\frac{S(t)}{k}\right)-\beta S(t) I(t-\tau)  \tag{16}\\
& \dot{I}(t)=\beta S(t) I(t-\tau)-\mu_{1} I(t)-m I(t)
\end{align*}
$$

where $S(\theta) \geq 0, I(\theta) \geq 0$ are continuous on $\theta \in[-\tau, 0]$ and $S(0)>0, I(0)>0$.

By Definition 5 and model system (16), it is easy to see that $Q^{0}$ and $Q_{0}$ are positively invariant, and conditions (i) and (ii) of Lemma 6 clearly hold.

Since model system (16) possesses two constant solutions in $Q_{0}: \widetilde{P}_{0} \in Q_{1}, \widetilde{P}_{1} \in Q_{2}$ with the following form:

$$
\begin{gather*}
\widetilde{P}_{0}=\left\{\left(\phi_{1}, \phi_{2}\right) \in C^{+}\left([-\tau, 0], \mathbb{R}_{+}^{2}\right) \mid \phi_{1}(\theta)=\phi_{2}(\theta)=0\right. \\
\\
\theta \in[-\tau, 0]\} \\
\widetilde{P}_{1}=\left\{\left(\phi_{1}, \phi_{2}\right) \in C^{+}\left([-\tau, 0], \mathbb{R}_{+}^{2}\right) \mid \phi_{1}(\theta)=1, \phi_{2}(\theta)=0\right.  \tag{17}\\
\theta \in[-\tau, 0]\}
\end{gather*}
$$

It follows from simple computation that

$$
\begin{equation*}
\left.\dot{S}(t)\right|_{\left(\phi_{1}, \phi_{2}\right) \in Q_{1}}=0,\left.\quad S(t)\right|_{\left(\phi_{1}, \phi_{2}\right) \in Q_{1}}=0 \quad \text { for } t \geq 0 \tag{18}
\end{equation*}
$$

Moreover, it follows from the second equation of model system (16) that

$$
\begin{equation*}
\left.\dot{I}(t)\right|_{\left(\phi_{1}, \phi_{2}\right) \in \mathrm{Q}_{1}}=-\left(\mu_{1}+m\right) I(t) \leq 0 \tag{19}
\end{equation*}
$$

which reveals that all points in $Q_{1}$ approach to $\widetilde{P}_{0}$; that is, $Q_{1}=W^{s}\left(\widetilde{P}_{0}\right)$. By using the similar analysis mentioned above, it can be also concluded that all points in $Q_{2}$ approach to $\widetilde{P}_{1}$; that is, $Q_{2}=W^{s}\left(\widetilde{P}_{1}\right)$. Based on the above analysis, it shows that invariant sets $\widetilde{P}_{0}$ and $\widetilde{P}_{1}$ are isolated invariant, and $\left\{\widetilde{P}_{0}, \widetilde{P}_{1}\right\}$ is isolated and an acyclic covering. It can be concluded that condition (iii) of Lemma 6 holds.

Finally, we will show that $W^{s}\left(\widetilde{P}_{i}\right) \cap Q^{0}=\emptyset$ for $i=0,1$. Based on the definition of $\widetilde{P}_{0}$, it is easy to show that $W^{s}\left(\widetilde{P}_{0}\right) \cap$ $Q^{0}=\emptyset$. We will show $W^{s}\left(\widetilde{P}_{1}\right) \cap Q^{0}=\varnothing$ in the following part.

If $W^{s}\left(\widetilde{P}_{1}\right) \cap Q^{0} \neq \emptyset$, then there exists a positive solution $(S(t), I(t))$ to model system (16) with $\lim _{t \rightarrow+\infty}(S(t), I(t))=$ $(1,0)$. If $\mu_{1}+m<1$, then $\mu_{1}+m<1-\epsilon$ holds for sufficiently small $\epsilon>0$ and there exists a positive constant $T=T(\epsilon)$ such that $S(t)>1-\epsilon>0$, and $0<I(t)<\epsilon$ for all $t \geq T$.

By the second equation of model system (16), it derives that

$$
\begin{equation*}
\dot{I}(t) \geq(1-\epsilon) I(t-\tau)-\left(\mu_{1}+m\right) I(t) \tag{20}
\end{equation*}
$$

holds for all $t \geq T+\tau$.
Consider the following equation:

$$
\begin{align*}
& \dot{x}(t)=(1-\epsilon) x(t-\tau)-\left(\mu_{1}+m\right) x(t), \quad t \geq T+\tau \\
& x(t)=I(t), \quad T \leq t \leq T+\tau . \tag{21}
\end{align*}
$$

Based on (21) and the comparison principle, it derives that $I(t) \geq x(t)$ for all $t>T$.

On the other hand, if $\mu_{1}+m<1$, then it follows from Lemma 7 that $\lim _{t \rightarrow+\infty} x(t)=+\infty$ for all solutions of (21). It can be concluded that $\lim _{t \rightarrow+\infty} I(t)=\infty$, which is a contradiction to $I(t)<\epsilon$. Consequently, it can be derived that $W^{s}\left(\widetilde{P}_{1}\right) \cap Q^{0}=\emptyset$.

According to the above analysis, all conditions of Lemma 6 hold. By using Lemma 6, it can be obtained that

$$
\begin{equation*}
\liminf _{t \rightarrow+\infty} S(t) \geq S_{\eta}, \quad \liminf _{t \rightarrow+\infty} I(t) \geq I_{\eta}, \tag{22}
\end{equation*}
$$

where $S_{\eta}>0$ and $I_{\eta}>0$ are independent of the corresponding initial values of model system (4).

Theorem 10. If $\mu_{1}+m<1, c \mu_{2}<m w I_{\eta}$, and $0 \leq v<$ $c \mu_{2}+m w I_{\eta}$, then all solutions of model system (4) with initial conditions are persistent.

Proof. According to Lemmas 8 and 9, it can be obtained that

$$
\begin{equation*}
S_{\eta} \leq S(t) \leq k, \quad S(t)+I(t) \leq \frac{k\left(r+\mu_{1}+m\right)^{2}}{4 r\left(\mu_{1}+m\right)} \tag{23}
\end{equation*}
$$

hold for all $t>0$, which derive that

$$
\begin{equation*}
I_{\eta} \leq I(t) \leq \frac{k\left(r+\mu_{1}+m\right)^{2}}{4 r\left(\mu_{1}+m\right)}-S_{\eta} \tag{24}
\end{equation*}
$$

When the economic interest $v=0$, it follows from Theorem 3 and the fourth equation of model system (4) that

$$
\begin{equation*}
R(t)=\frac{c}{w} \tag{25}
\end{equation*}
$$

and $\dot{R}(t)=0$. Based on the third equation of model system (4), it can be computed that $E(t)=(m w / c) I(t)-\mu_{2}$. Accordng to (24), it derives that

$$
\begin{align*}
0 & <\frac{m w I_{\eta}-c \mu_{2}}{c} \leq E(t) \\
& \leq \frac{m w\left[k\left(r+\mu_{1}+m\right)^{2}-4 S_{\eta} r\left(\mu_{1}+m\right)\right]}{4 c r\left(\mu_{1}+m\right)}-\mu_{2} \tag{26}
\end{align*}
$$

provided that $c \mu_{2}<m w I_{\eta}$.
In the case of $v>0$, it derives that $E(t)=v /(w R(t)-c)$ based on implicit function theory [41]. According to the third equation of model system (4), it can be obtained that

$$
\begin{equation*}
\dot{R}(t) \geq m I_{\eta}+\frac{c \mu_{2}-v}{w}-\mu_{2} R(t), \tag{27}
\end{equation*}
$$

which derives that

$$
\begin{equation*}
\liminf _{t \rightarrow+\infty} R(t) \geq \frac{m w I_{\eta}+c \mu_{2}-v}{w \mu_{2}}:=\underline{R}>0 \tag{28}
\end{equation*}
$$

provided that $0<v<c \mu_{2}+m w I_{\eta}$.
It follows from Theorem 3 and the third equation of model system (4) that

$$
\begin{equation*}
\dot{R}(t) \leq m I(t)-\mu_{2} R(t), \tag{29}
\end{equation*}
$$

which derives that

$$
\begin{equation*}
\limsup _{t \rightarrow+\infty} R(t) \leq \frac{m\left[k\left(r+\mu_{1}+m\right)^{2}-4 S_{\eta} r\left(\mu_{1}+m\right)\right]}{4 r \mu_{2}\left(\mu_{1}+m\right)}:=\bar{R} . \tag{30}
\end{equation*}
$$

Hence, it gives that $v /(w \bar{R}-c) \leq E(t) \leq v /(w \underline{R}-c)$, and it can be rewritten as follows:

$$
\begin{gather*}
\frac{4 r \mu_{2} v\left(\mu_{1}+m\right)}{w m\left[k\left(r+\mu_{1}+m\right)^{2}-4 S_{\eta} r\left(\mu_{1}+m\right)\right]-4 r c \mu_{2}\left(\mu_{1}+m\right)} \\
\leq E(t) \leq \frac{\mu_{2} v}{m w I_{\eta}-v} \tag{31}
\end{gather*}
$$

Based on (23), (24), (25), and (26), it can be concluded that all solutions of model system (4) with initial conditions are persistent in the case of $v=0$, and it follows from (23), (24), (28), (30) and (31) that all solutions of model system (4) with initial conditions are persistent in the case of $0<v<$ $c \mu_{2}+m w I_{\eta}$.

## 4. Qualitative Analysis of Model System

Dynamical effects of harvest effort and time delay on population dynamics are discussed in this section. It should be noted that the interior equilibrium biologically interprets that susceptible, infective, and recovered host individuals survive as well as harvest on recovered host individuals exists. Bifurcation phenomenon around the interior equilibria can reveal instability mechanism of model system, which are theoretically relevant to infectious disease control and sustainable yield on recovered host individuals in the real world. Consequently, we will mainly concentrate on dynamical behavior and local stability analysis around interior equilibrium of model system (4) in this paper.
4.1. Model System without Time Delay. In this section, dynamical behavior of model system (4) without time delay is investigated, and local stability analysis around the interior equilibrium is discussed due to variation of economic interest of commercial harvesting. Furthermore, state feedback controllers are designed to stabilize model system around the desired interior equilibria in the case of zero economic interest and positive economic interest, respectively.

### 4.1.1. Singularity Induced Bifurcation

Theorem 11. Model system (4) without time delay has a singularity induced bifurcation around the interior equilibrium, and $v=0$ is a bifurcation value. Furthermore, local stability switch occurs as $v$ increases through 0 .

Proof. Based on the economic theory of a common-property resource [22], there is a phenomenon of bioeconomic equilibrium in the case of zero harvest economic interest; that is, $v=0$. An interior equilibrium can be obtained as follows: $P^{*}\left(S^{*}, I^{*}, R^{*}, E^{*}\right)$, where $S^{*}=\left(\mu_{1}+m\right) / \beta, I^{*}=r(k \beta-$ $\left.\mu_{1}-m\right) / k \beta^{2}, R^{*}=c / w$, and $E^{*}=\left(w \operatorname{mr}\left(k \beta-\mu_{1}-m\right)-\right.$ $\left.c k \mu_{2} \beta^{2}\right) / c k \beta^{2}$.

According to biological interpretation of the interior equilibrium, it follows that $S^{*}>0, I^{*}>0, R^{*}>0$ and $E^{*}>$ 0 . In order to guarantee the existence of interior equilibrium, some inequalities are satisfied:

$$
\begin{gather*}
k \beta-\mu_{1}-m>0 \\
w m r\left(k \beta-\mu_{1}-m\right)-c k \mu_{2} \beta^{2}>0 \tag{32}
\end{gather*}
$$

Let $v$ be a bifurcation parameter, $H(t)=(S(t), I(t), R(t))^{T}$,

$$
\begin{align*}
& h_{1}(H(t), E(t), v)=\left[\begin{array}{c}
r\left(1-\frac{S(t)}{k}\right)-\beta S(t) I(t-\tau), \\
\beta S(t) I(t-\tau)-\mu_{1} I(t)-m I(t), \\
m I(t)-\mu_{2} R(t)-E(t) R(t)
\end{array}\right], \\
& h_{2}(H(t), E(t), v)=E(t)(w R(t)-c)-v . \tag{33}
\end{align*}
$$

It can be calculated that

$$
\begin{align*}
\operatorname{trace} & \left.\left(D_{E} h_{1} \operatorname{adj}\left(D_{E} h_{2}\right)\left(D_{H} h_{2}, D_{E} h_{2}\right)\right)\right|_{P^{*}} \\
= & -\frac{w m r\left(k \beta-\mu_{1}-m\right)-c k \mu_{2} \beta^{2}}{k \beta^{2}} \tag{34}
\end{align*}
$$

By virtue of (32), it can be obtained that

$$
\begin{equation*}
\left.\operatorname{trace}\left(D_{E} h_{1} \operatorname{adj}\left(D_{E} h_{2}\right)\left(D_{H} h_{2}, D_{E} h_{2}\right)\right)\right|_{P^{*}} \neq 0 \tag{35}
\end{equation*}
$$

Furthermore, it can be also calculated that
$\left|\begin{array}{ll}D_{H} h_{1} & D_{E} h_{1} \\ D_{H} h_{2} & D_{E} h_{2}\end{array}\right|_{P^{*}}$
$=\frac{r\left(\mu_{1}+m\right)\left(k \beta-\mu_{1}-m\right)\left[\omega m r\left(k \beta-\mu_{1}-m\right)-c k \mu_{2} \beta^{2}\right]}{k^{2} \beta^{3}}$.

It follows from (32) that

$$
\left|\begin{array}{cc}
D_{H} h_{1} & D_{E} h_{1}  \tag{37}\\
D_{H} h_{2} & D_{E} h_{2}
\end{array}\right|_{P^{*}} \neq 0
$$

Based on Section IV(A) in [42], $h_{3}(H(t), E(t), v)$ can be defined as follows:

$$
\begin{equation*}
h_{3}(H(t), E(t), v)=\operatorname{det}\left(D_{E} g\right)=w R(t)-c \tag{38}
\end{equation*}
$$

By simple computing,

$$
\left|\begin{array}{lll}
D_{H} h_{1} & D_{E} h_{1} & D_{v} h_{1}  \tag{39}\\
D_{H} h_{2} & D_{E} h_{2} & D_{v} h_{2} \\
D_{H} h_{3} & D_{E} h_{3} & D_{v} h_{3}
\end{array}\right|_{P^{*}}=\frac{c r\left(\mu_{1}+m\right)\left(k \beta-\mu_{1}+m\right)}{k \beta}
$$

According to (32), it derives that

$$
\left|\begin{array}{ccc}
D_{H} h_{1} & D_{E} h_{1} & D_{v} h_{1}  \tag{40}\\
D_{H} h_{2} & D_{E} h_{2} & D_{v} h_{2} \\
D_{H} h_{3} & D_{E} h_{3} & D_{v} h_{3}
\end{array}\right|_{P^{*}} \neq 0
$$

Based on the above analysis, four items (i-iv) can be obtained as follows.
(i) It is easy to show that $D_{E} h_{2}$ has a simple zero eigenvalue:

$$
\begin{equation*}
\left.h_{1}(H(t), E(t), v)\right|_{P^{*}}=0,\left.\quad h_{2}(H(t), E(t), v)\right|_{P^{*}}=0 \tag{41}
\end{equation*}
$$

and $\left.\operatorname{trace}\left(D_{E} h_{1} \operatorname{adj}\left(D_{E} h_{2}\right)\left(D_{H} h_{2}, D_{E} h_{2}\right)\right)\right|_{P^{*}} \neq 0$ based on (35).
(ii) It follows from (37) that $\left[\begin{array}{cc}D_{H} h_{1} & D_{\mathrm{E}} h_{1} \\ D_{H} h_{2} & D_{E} h_{2}\end{array}\right]$ is nonsingular around $P^{*}$.
(iii) By virtue of (40), it can be shown that $\left[\begin{array}{lll}D_{H} h_{1} & D_{E} h_{1} & D_{v} h_{1} \\ D_{H} h_{2} & D_{\mathrm{E}} h_{2} & D_{v} h_{2} \\ D_{H} h_{3} & D_{\mathrm{E}} h_{3} & D_{v} h_{3}\end{array}\right]$ is nonsingular around $P^{*}$; hence $\operatorname{rank}\left[\begin{array}{lll}D_{H} h_{3} & D_{E} h_{3} & D_{v} h_{3} \\ D_{H} h_{1} & D_{E} h_{1} & D_{v} h_{1} \\ D_{H} h_{2} & D_{E} h_{1} & D_{v} h_{2} \\ D_{H} h_{3} & D_{E} h_{3} & D_{v} h_{3}\end{array}\right]=5$.
(iv) It is easy to show rank $\left(h_{1}(H(t), E(t), v)\right)=3$ and $\operatorname{rank}\left(h_{2}(H(t), E(t), v)\right)=1$, which follows

$$
\begin{align*}
& \operatorname{rank} {\left[\begin{array}{lll}
D_{H} h_{1} & D_{E} h_{1} & D_{v} h_{1} \\
D_{H} h_{2} & D_{E} h_{2} & D_{v} h_{2} \\
D_{H} h_{3} & D_{E} h_{3} & D_{v} h_{3}
\end{array}\right] }  \tag{42}\\
&=\operatorname{rank}\left(h_{1}(H(t), E(t), v)\right) \\
& \quad+\operatorname{rank}\left(h_{2}(H(t), E(t), v)\right)+1 .
\end{align*}
$$

It should be noted that the conditions for singularity induced bifurcation, which is introduced in Section III (A) in [42], consist of three conditions, that is, SI1, SI2, and SI3. According to the above items (i)-(iv), SI1, SI2, and SI3 are all satisfied; hence model (4) without time delay has a singularity induced bifurcation around the interior equilibrium $P^{*}$ and the bifurcation value is $v=0$.

Along with the line of the above proof, for model (4) without time delay, it follows from simple computing that

$$
\begin{align*}
M & =-\left.\operatorname{trace}\left(D_{E} h_{1} \operatorname{adj}\left(D_{E} h_{2}\right)\left(D_{H} h_{2}, D_{E} h_{2}\right)\right)\right|_{P^{*}} \\
& =\frac{w m r\left(k \beta-\mu_{1}-m\right)-c k \mu_{2} \beta^{2}}{k \beta^{2}}, \\
N & =\left.\left[D_{v} h_{3}-\left[D_{H} h_{3}, D_{E} h_{3}\right]\left[\begin{array}{ll}
D_{H} h_{1} & D_{E} h_{1} \\
D_{H} h_{2} & D_{E} h_{2}
\end{array}\right]^{-1}\left[\begin{array}{l}
D_{v} h_{1} \\
D_{v} h_{2}
\end{array}\right]\right]\right|_{P^{*}} \\
& =\frac{c r\left(\mu_{1}+m\right)\left(k \beta-\mu_{1}-m\right)}{k \beta} . \tag{43}
\end{align*}
$$

It follows from (32) that

$$
\begin{equation*}
\frac{M}{N}=\frac{m r w\left(k \beta-\mu_{1}-m\right)-c k \mu_{2} \beta^{2}}{\operatorname{cr} \beta\left(\mu_{1}+m\right)\left(k \beta-\mu_{1}-m\right)}>0 \tag{44}
\end{equation*}
$$

Inequality (44) satisfies Theorem 3 of [42]. According to Theorem 3 of [42], when $v$ increases through 0 , one eigenvalue (denoted by $\lambda_{1}$ ) of model system (4) without time delay moves from $\mathbb{C}^{-}$to $\mathbb{C}^{+}$along the real axis by diverging through infinity; the movement behavior of this eigenvalue influences the stability of model system (4) without time delay.

Since the Jacobian of model system (4) without time delay evaluated around $P^{*}$ takes the following form:

$$
J_{P^{*}}=\left[\begin{array}{cccc}
-\frac{r S^{*}}{k} & -\beta S^{*} & 0 & 0  \tag{45}\\
\beta I^{*} & 0 & 0 & 0 \\
0 & m & -\left(\mu_{2}+E^{*}\right) & -R^{*} \\
0 & 0 & w E^{*} & 0
\end{array}\right]
$$

according to the leading matrix $\Xi(t)$ in model system (4) and $J_{P^{*}}$, the characteristic equation of the model system (4) without time delay around $P^{*}$ is

$$
\begin{equation*}
\operatorname{det}\left(\lambda \Xi-J_{P^{*}}\right)=0 \tag{46}
\end{equation*}
$$

Table 1: Signs of real parts of eigen values of model (4) without time delay around interior equilibrium $P^{*}$.

|  | $\operatorname{Re} \lambda_{1}$ | $\operatorname{Re} \lambda_{2}$ | $\operatorname{Re} \lambda_{3}$ |
| :---: | :---: | :---: | :---: |
| $v<0$ | - | - | - |
| $v>0$ | + | - | - |

By virtue of simple computation, the characteristic equation is as follows:

$$
\begin{equation*}
\lambda^{2}+\frac{r S^{*}}{k} \lambda+\beta^{2} S^{*} I^{*}=0 \tag{47}
\end{equation*}
$$

It can be concluded that the rest eigenvalues of model system (4) without time delay (denoted by $\lambda_{2}$ and $\lambda_{3}$ ) have negative real parts by using the Routh-Hurwitz criteria [43]. It follows from Theorem 3 in [42] that there is only one eigenvalue diverging to infinity as $v$ increases through 0 , and the rest eigenvalues are continuous and nonzero and cannot jump from one half open complex plane to another one as $v$ increases through 0 . It has been shown that $\lambda_{1}$ moves from $\mathbb{C}^{-}$to $\mathbb{C}^{+}$along the real axis by diverging through infinity. Therefore, $\lambda_{2}$ and $\lambda_{3}$ are continuous and bounded in the $\mathbb{C}^{-}$half plane as $v$ increases through 0 and their movement behaviors have no influence on the stability of model system (4) without time delay around the interior equilibrium $P^{*}$.

According to Table 1 and the stability theory, it can be concluded that model system (4) without time delay is stable around $P^{*}$ as $v<0$ and model system (4) without time delay is unstable around $P^{*}$ as $v>0$. Consequently, a stability switch occurs as $v$ increases through 0 .

Remark 12. Some preliminaries of singularity induced bifurcation are introduced below. Parameter dependent differen-tial-algebraic hybrid system of the form

$$
\begin{array}{rlrl}
\dot{x}(t) & =h(x(t), y(t), \lambda), & & h: R^{n} \times R^{m} \times R^{p} \longrightarrow R^{n}, \\
0 & =g(x(t), y(t), \lambda), & g: R^{n} \times R^{m} \times R^{p} \longrightarrow R^{m}, \tag{48}
\end{array}
$$

where $x(t), y(t)$, and $\lambda$ have appropriate dimensions. It has been shown recently that there are generically three types of codimension one local bifurcation associated with the differential-algebraic model (48), namely, saddle-node bifurcation, Hopf bifurcation, and singularity induced bifurcation (see [42]).

The singularity induced bifurcation is firstly introduced and analyzed in [42, 44]. It is a new type of bifurcation and does not occur in usual ordinary differential equation system, which has been characterized for differential-algebraic system, and later improved in [45, 46]. Roughly speaking, the singularity induced bifurcation refers to a stability change of the differential-algebraic hybrid model (48) owing to some eigenvalues of related linearization $h_{x}-h_{y} g_{y}^{-1} g_{x}$ diverging to infinity when Jacobian $g_{y}$ is singular.

One of the important consequences of the singularity induced bifurcation is that it leads to an impulse phenomenon, which may result in the collapse of the differentialalgebraic system (see [45]). More detailed introductions of the singularity induced bifurcation can be found in [42, 4446].

Remark 13. It follows from Theorem 11 that there is a phenomenon of singularity induced bifurcation around the interior equilibrium when economic interest increases through zero, which can cause local stability switch of model system (4). As stated in Remark 12, the singularity induced bifurcation can result in impulse phenomenon, which may lead to the collapse of the proposed model. In the harvested epidemiological-economic system, the impulse phenomenon is vividly reflected with the outbreak of infectious disease during a short period in the real world. Under this climate, the infected population will be beyond the carrying capacity of environment, which is disastrous for sustainable development of the harvested ecosystem as well as prosperous yield on recovered host individuals.
4.1.2. State Feedback Controller. In order to maintain the sustainable yield on recovered host individuals biological resource as well as economic interest of commercial harvesting at an ideal level, some corresponding control strategies should be taken to eliminate the impulse phenomenon caused by singularity induced bifurcation and stabilize model (4) without time delay. In this subsection, state feedback controllers are designed to stabilize model system (4) without time delay around corresponding interior equilibria in the case of $v=0$ and $v>0$, respectively.

According to the leading matrix $\Xi(t)$ in model system (4) and $J_{P^{*}}$ in (45) (the Jacobian of model system (4) without time delay around the interior equilibrium $P^{*}$ ), it can be calculated that $\operatorname{rank}\left(J_{P^{*}}, \Xi J_{P^{*}}, \Xi^{2} J_{P^{*}}, \Xi^{3} J_{P^{*}}\right)=4$. By using Theorem 2-2.1 in [47], it is easy to show that the model system (4) without time delay is locally controllable around the interior equilibrium $P^{*}$ in the case of $v=0$. Consequently, a state feedback controller can be applied to stabilize the model system (4) without time delay around $P^{*}$. By using Theorem 3-1.2 in [47], a state feedback controller $u(t)=l\left(E(t)-E^{*}\right)$ ( $l$ is a feedback gain and $E^{*}$ is the component of the interior equilibrium $P^{*}$ ) can be applied to stabilize model system (4) without time delay around $P^{*}$.

Furthermore, the controlled model system (4) without time delay takes the following form:

$$
\begin{align*}
\dot{S}(t) & =r\left(1-\frac{S(t)}{k}\right)-\beta S(t) I(t), \\
\dot{I}(t) & =\beta S(t) I(t)-\mu_{1} I(t)-m I(t),  \tag{49}\\
\dot{R}(t) & =m I(t)-\mu_{2} R(t)-E(t) R(t), \\
0 & =E(t)(w R(t)-c)-v+u(t) .
\end{align*}
$$

Theorem 14. When economic interest of harvesting is zero, $v=$ 0 , if the feedback gain $l$ satisfies the following inequality:

$$
\begin{gather*}
l>\max \left\{\frac{k w E^{*} R^{*}}{r S^{*}+k\left(\mu_{2}+E^{*}\right)}, \frac{w E^{*} R^{*}}{\mu_{2}+E^{*}}, \frac{r w E^{*} R^{*}}{k \beta^{2} I^{*}+r\left(\mu_{2}+E^{*}\right)},\right. \\
\left(w E^{*} R^{*}\left[2 k r\left(\mu_{2}+E^{*}\right)+2 k^{2} \beta^{2} I^{*}+r^{2} S^{*}\right]\right. \\
\left.\quad+\sqrt{4 k^{4} \beta^{4} I^{* 2}+8\left(\mu_{2}+E^{*}\right) r k^{3} \beta^{2} I^{*}+r^{4} S^{* 2}}\right) \\
\quad \times\left(2 r \left[r\left(\mu_{2}+E^{*}\right) S^{*}+k \beta^{2} I^{*} S^{*}\right.\right. \\
\left.\left.\left.+k\left(\mu_{2}+E^{*}\right)^{2}\right]\right)^{-1}\right\} \tag{50}
\end{gather*}
$$

then singularity induced bifurcation is eliminated and model system (49) is stable around $P^{*}$.

Proof. The Jacobian of the model system (49) evaluated at the interior equilibrium $P^{*}$ takes the form

$$
\tilde{J}_{P^{*}}=\left[\begin{array}{cccc}
-\frac{r S^{*}}{k} & -\beta S^{*} & 0 & 0  \tag{51}\\
\beta I^{*} & 0 & 0 & 0 \\
0 & m & -\left(\mu_{2}+E^{*}\right) & -R^{*} \\
0 & 0 & w E^{*} & l
\end{array}\right]
$$

According to the leading matrix $\Xi(t)$ in the model system (4) and $\tilde{J}_{P^{*}}$, the characteristic equation of model system (49) around $P^{*}$ is $\operatorname{det}\left(\lambda \Xi-\widetilde{J}_{P^{*}}\right)=0$, which can be expressed as follows:

$$
\begin{equation*}
\lambda^{3}+B_{1} \lambda^{2}+B_{2} \lambda+B_{3}=0 \tag{52}
\end{equation*}
$$

where $B_{1}=\mu_{2}+E^{*}+r S^{*} / k-w E^{*} R^{*} / l, B_{2}=\beta^{2} I^{*} S^{*}+$ $\left(r S^{*} / k\right)\left(\mu_{2}+E^{*}-w E^{*} R^{*} / l\right)$, and $B_{3}=\beta^{2} I^{*} S^{*}\left(\mu_{2}+E^{*}-\right.$ $\left.w E^{*} R^{*} / l\right)$.

By using the Routh-Hurwitz criteria [43], model (49) is locally stable around $P^{*}$ if and only if $l$ satisfies

$$
\begin{gather*}
l>\max \left\{\frac{k w E^{*} R^{*}}{r S^{*}+k\left(\mu_{2}+E^{*}\right)}, \frac{w E^{*} R^{*}}{\mu_{2}+E^{*}}, \frac{r w E^{*} R^{*}}{k \beta^{2} I^{*}+r\left(\mu_{2}+E^{*}\right)},\right. \\
\left(w E^{*} R^{*}\left[2 k r\left(\mu_{2}+E^{*}\right)+2 k^{2} \beta^{2} I^{*}+r^{2} S^{*}\right]\right. \\
\left.\quad+\sqrt{4 k^{4} \beta^{4} I^{* 2}+8\left(\mu_{2}+E^{*}\right) r k^{3} \beta^{2} I^{*}+r^{4} S^{* 2}}\right) \\
\times\left(2 r \left[r\left(\mu_{2}+E^{*}\right) S^{*}+k \beta^{2} I^{*} S^{*}\right.\right. \\
\left.\left.\left.+k\left(\mu_{2}+E^{*}\right)^{2}\right]\right)^{-1}\right\} \tag{53}
\end{gather*}
$$

Consequently, if the feedback gain satisfies the above inequality, then model system (49) is stable around $P^{*}$ in the case of zero interest of harvesting.

Let $\widetilde{P}^{*}\left(\widetilde{S}^{*}, \widetilde{I}^{*}, \widetilde{R}^{*}, \widetilde{E}^{*}\right)$ denote interior equilibrium of model (4) in the case of positive economic interest of harvesting $(v>0)$, where $\widetilde{S}^{*}=\left(\mu_{1}+m\right) / \beta, \widetilde{I}^{*}=r(k \beta-$ $\left.\mu_{1}-m\right) / k \beta^{2}, \widetilde{E}^{*}=v /\left(w R^{*}-c\right)$, and $\widetilde{R}^{*}$ satisfies the following equation:

$$
\begin{equation*}
\widetilde{R}^{* 2}+\widetilde{B}_{2} \widetilde{R}^{*}+\widetilde{B}_{3}=0 \tag{54}
\end{equation*}
$$

where $\widetilde{B}_{2}=\left(k \beta^{2}\left(v-c \mu_{2}\right)+w m r\left(m+\mu_{1}-k \beta\right)\right) / k w \mu_{2} \beta^{2}$, $\widetilde{B}_{3}=c m r\left(k \beta-m u_{1}-m\right) / k w \mu_{2} \beta^{2}$.

Based on Routh-Hurwitz criteria [43], (54) has two positive roots if economic interest $v$ satisfies the following inequalities:

$$
\begin{align*}
& 0<v<\min \left\{c \mu_{2}+\frac{w m r\left(k \beta-m-\mu_{1}\right)}{k \beta^{2}},\right. \\
&\left.(1-c) c \mu_{2}+\frac{w m r(c+1)\left(k \beta-\mu_{1}-m\right)}{k \beta^{2}}\right\} \\
&:=\widetilde{v} . \tag{55}
\end{align*}
$$

As analyzed above, there are two interior equilibria (denoted by $\widetilde{P}_{1}^{*}$ and $\widetilde{P}_{2}^{*}$ ) when $0<v^{*}<\widetilde{v}$. In this subsection, we only design the controller for the model (4) around the interior equilibrium $\widetilde{P}_{1}^{*}$. Some symmetric results about $\widetilde{P}_{2}^{*}$ can be also obtained, and $\widetilde{P}_{1}^{*}$ is denoted as $\widetilde{P}^{*}$ for simplicity in the following part.

Theorem 15. When economic interest of harvesting is positive, $0<v^{*}<\widetilde{\nu}$, if feedback gain $\widetilde{l}$ of controller $u(t)=\widetilde{l}\left(E(t)-\widetilde{E}^{*}\right)$ satisfies following inequality:

$$
\begin{gather*}
\tilde{l}>\max \left\{\frac{k w \widetilde{E}^{*} \widetilde{R}^{*}}{r \widetilde{S}^{*}+k\left(\mu_{2}+\widetilde{E}^{*}\right)}, \frac{w \widetilde{E}^{*} \widetilde{R}^{*}}{\mu_{2}+\widetilde{E}^{*}}, \frac{r w \widetilde{E}^{*} \widetilde{R}^{*}}{k \beta^{2} \widetilde{I}^{*}+r\left(\mu_{2}+\widetilde{E}^{*}\right)}\right. \\
\left(w \widetilde{E}^{*} \widetilde{R}^{*}\left[2 k r\left(\mu_{2}+\widetilde{E}^{*}\right)+2 k^{2} \beta^{2} \widetilde{I}^{*}+r^{2} \widetilde{S}^{*}\right]\right. \\
\left.\quad+\sqrt{4 k^{4} \beta^{4} \widetilde{I}^{* 2}+8\left(\mu_{2}+\widetilde{E}^{*}\right) r k^{3} \beta^{2} \widetilde{I}^{*}+r^{4} \widetilde{S}^{* 2}}\right) \\
\quad \times\left(2 r \left[r\left(\mu_{2}+\widetilde{E}^{*}\right) \widetilde{S}^{*}+k \beta^{2} \widetilde{I}^{*} \widetilde{S}^{*}\right.\right. \\
\left.\left.\left.\quad+k\left(\mu_{2}+\widetilde{E}^{*}\right)^{2}\right]\right)\right\} \tag{56}
\end{gather*}
$$

then model system (49) is stable around the interior equilibrium $\widetilde{P}^{*}\left(\widetilde{S}^{*}, \widetilde{I}^{*}, \widetilde{R}^{*}, \widetilde{E}^{*}\right)$.

Proof. The proof is similar to the proof of Theorem 14 of this paper.

Remark 16. It follows from (55) and Theorem 15 that economic interest of commercial harvesting should be regulated within certain interval $v \in(0, \widetilde{v})$, which guarantees the existence of interior equilibrium in the case of positive
economic interest. After applying the state feedback controller into model system (4) without time delay, model system can be stabilized around the corresponding interior equilibrium, respectively. The elimination of the singularity induced bifurcation means the harvested epidemiologicaleconomic system restores to ecological balance and avoidance of infectious disease outbreak.
4.2. Model System with Time Delay. By analyzing corresponding characteristic equation of model system with time delay, local stability analysis around the interior equilibrium due to variation of time delay is discussed. Conditions for existence of Hopf bifurcation are studied. Furthermore, directions of Hopf bifurcation and stability of periodic solutions are investigated.
4.2.1. Local Stability and Hopf Bifurcation. As analyzed in the above subsection, in the case of time delay and positive economic interest of harvesting $0<v^{*}<\widetilde{v}$ where $\widetilde{v}$ is defined in (55), there are two interior equilibria $\widetilde{P}_{1}^{*}$ and $\widetilde{P}_{2}^{*}$ for model system (4) with respect to the positive economic interest $v^{*}$.

In this subsection, we only investigate dynamical behavior of model system (4) around the interior equilibrium $\widetilde{P}_{1}^{*}$. Some symmetric results about the interior equilibrium $\widetilde{P}_{2}^{*}$ can be also obtained, and $\widetilde{P}_{1}^{*}$ is denoted as $\widetilde{P}^{*}$ for simplicity. According to Jacobian evaluated at the interior equilibrium $\widetilde{P}^{*}$ and the leading matrix $\Xi(t)$ in model system (4), we can obtain the characteristic equation of model system (4) around $\widetilde{P}^{*}$, which can be expressed as follows:

$$
\begin{align*}
& \left|\begin{array}{cccc}
\lambda+\frac{r \widetilde{S}^{*}}{k} & \beta \widetilde{S}^{*} e^{-\lambda \tau} & 0 & 0 \\
-\beta \widetilde{I}^{*} & \lambda-\beta \widetilde{S}^{*} e^{-\lambda \tau}+\mu_{1}+m & 0 & 0 \\
0 & -m & \lambda+\mu_{1}+\widetilde{E}^{*} & \widetilde{R}^{*} \\
0 & 0 & -w \widetilde{E}^{*} & -\frac{v^{*}}{\widetilde{E}^{*}}
\end{array}\right|=0  \tag{57}\\
& \Longrightarrow \\
&  \tag{58}\\
&
\end{align*}
$$

where

$$
\begin{aligned}
& M(\lambda)=\lambda^{3}+m_{1} \lambda^{2}+m_{2} \lambda+m_{3} \\
& N(\lambda)=n_{1} \lambda^{2}+n_{2} \lambda+n_{3} \\
& m_{1}=\mu_{2}+\left(\mu_{1}+m\right)\left(1+\frac{r}{k \beta}\right)-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}} \\
& m_{2}=\frac{\mu_{1}+m}{k \beta}\left[r\left(\mu_{1}+m+\mu_{2}-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}\right)\right. \\
& \\
& \left.\quad+k \beta\left(\mu_{2}-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}\right)\right]
\end{aligned}
$$

$$
\begin{align*}
m_{3}= & \frac{r\left(\mu_{1}+m\right)^{2}}{k \beta}\left[\mu_{2}-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}\right], \\
n_{1}= & -\mu_{1}-m, \\
n_{2}= & -\left(\mu_{1}+m\right) \\
& \times\left[\mu_{2}+\frac{2 r\left(\mu_{1}+m\right)-k r \beta}{k \beta}-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}\right], \\
n_{3}= & -r\left(\mu_{1}+m\right)\left[\frac{2\left(\mu_{1}+m\right)}{k \beta}-1\right] \\
& \times\left[\mu_{2}-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}\right] . \tag{59}
\end{align*}
$$

Now substituting $\lambda=i \sigma$ ( $\sigma$ is a positive real number) into (58) and separating the real and imaginary parts, two transcendental equations can be obtained as follows:

$$
\begin{align*}
\sigma^{3}-m_{2} \sigma & =n_{2} \sigma \cos (\sigma \tau)-\left(n_{3}-n_{1} \sigma^{2}\right) \sin (\sigma \tau)  \tag{60}\\
m_{1} \sigma^{2}-m_{3} & =\left(n_{3}-n_{1} \sigma^{2}\right) \cos (\sigma \tau)+n_{2} \sigma \sin (\sigma \tau) \tag{61}
\end{align*}
$$

By squaring and adding (60) and (61), it can be calculated that

$$
\begin{equation*}
\left(n_{3}-n_{1} \sigma^{2}\right)^{2}+n_{2}^{2} \sigma^{2}=\left(m_{1} \sigma^{2}-m_{3}\right)^{2}+\left(\sigma^{3}-m_{2} \sigma\right)^{2} \tag{62}
\end{equation*}
$$

$\Longrightarrow$

$$
\begin{equation*}
\sigma^{6}+C_{1} \sigma^{4}+C_{2} \sigma^{2}+C_{3}=0 \tag{63}
\end{equation*}
$$

where $C_{1}=m_{1}^{2}-2 m_{2}-n_{1}^{2}, C_{2}=m_{2}^{2}-2 m_{1} m_{3}+2 n_{1} n_{3}-n_{2}^{2}, C_{3}=$ $m_{3}^{2}-n_{3}^{2}$.

According to the values of $C_{j},(j=1,2,3)$ and the RouthHurwitz criteria [43], a simple assumption that (58) has at least one positive real root $\sigma_{0}$ is $C_{3}<0$, which derives that $k \beta>3\left(\mu_{1}+m\right)$. Hence, under this assumption, (58) will have a pair of purely imaginary roots of the form $\pm i \sigma_{0}$.

By eliminating $\sin (\sigma \tau)$ from (60) and (61), it can be calculated that the $\tau_{j}^{*}$ corresponding to $\sigma_{0}$ is as follows:

$$
\begin{align*}
\tau_{j}^{*}= & \frac{1}{\sigma_{0}} \\
& \times \arccos \left[\frac{n_{2} \sigma_{0}^{2}\left(\sigma_{0}^{2}-m_{2}\right)+\left(n_{3}-n_{1} \sigma_{0}^{2}\right)\left(m_{1} \sigma_{0}^{2}-m_{3}\right)}{\left(n_{3}-n_{1} \sigma_{0}^{2}\right)^{2}+\left(n_{2} \sigma_{0}\right)^{2}}\right] \\
& +\frac{2 j \pi}{\sigma_{0}} \tag{64}
\end{align*}
$$

where $j=0,1,2, \ldots$
By using Butler's lemma [48], model system (4) is locally stable around $\widetilde{P}^{*}$ for $\tau<\tau_{0}^{*}$. Subsequently, conditions for existence of Hopf bifurcation in [39] are utilized to investigate whether Hopf bifurcation occurs as $\tau$ increases through $\tau_{j}^{*}$.

Theorem 17. If $k \beta>3\left(\mu_{1}+m\right)$, then model system (4) undergoes Hopf bifurcation around the interior equilibrium $\widetilde{P}^{*}$ when $\tau=\tau_{j}^{*}, j=0,1,2, \ldots$. Furthermore, an attracting invariant closed curve bifurcates from interior equilibrium $\widetilde{P}^{*}$ when $\tau>\tau_{0}^{*}$ and $\left\|\tau-\tau_{0}^{*}\right\| \ll 1$.

Proof. As mentioned above, let $\lambda=i \sigma_{0}$ represent the purely imaginary root of (58). It follows from (58) that $\left|M\left(i \sigma_{0}\right)\right|=$ $\left|N\left(i \sigma_{0}\right)\right|$, which determines a set of possible values of $\sigma_{0}$.

In the following part, we determine the direction of motion of $\lambda=i \sigma_{0}$ as $\tau$ is varied; namely, we determine

$$
\begin{equation*}
\Theta=\operatorname{sign}\left[\frac{\mathrm{d}(\operatorname{Re} \lambda)}{\mathrm{d} \tau}\right]_{\lambda=i \sigma_{0}}=\operatorname{sign}\left[\operatorname{Re}\left(\frac{\mathrm{d} \lambda}{\mathrm{~d} \tau}\right)^{-1}\right]_{\lambda=i \sigma_{0}} \tag{65}
\end{equation*}
$$

By differentiating (58) with respect to $\tau$, it can be obtained that

$$
\begin{align*}
\left(\frac{\mathrm{d} \lambda}{\mathrm{~d} \tau}\right)^{-1}= & \frac{3 \lambda^{2}+2 m_{1} \lambda+m_{2}}{\lambda e^{-\lambda \tau}\left(n_{1} \lambda^{2}+n_{2} \lambda+n_{3}\right)}+\frac{2 n_{1} \lambda+n_{2}}{\lambda\left(n_{1} \lambda^{2}+n_{2} \lambda+n_{3}\right)} \\
& -\frac{\tau}{\lambda} \\
= & \frac{3 \lambda^{2}+2 m_{1} \lambda+m_{2}}{-\lambda\left(\lambda^{3}+m_{1} \lambda^{2}+m_{2} \lambda+m_{3}\right)} \\
& +\frac{2 n_{1} \lambda+n_{2}}{\lambda\left(n_{1} \lambda^{2}+n_{2} \lambda+n_{3}\right)}-\frac{\tau}{\lambda} \\
= & \frac{2 \lambda^{3}+m_{1} \lambda^{2}-m_{3}}{-\lambda^{2}\left(\lambda^{3}+m_{1} \lambda^{2}+m_{2} \lambda+m_{3}\right)} \\
& +\frac{n_{1} \lambda^{2}-n_{3}}{\lambda^{2}\left(n_{1} \lambda^{2}+n_{2} \lambda+n_{3}\right)}-\frac{\tau}{\lambda} . \tag{66}
\end{align*}
$$

From (62) and the above equation, it can be obtained that

$$
\begin{aligned}
& \Theta=\operatorname{sign}\left[\operatorname{Re}\left(\frac{\mathrm{d} \lambda}{\mathrm{~d} \tau}\right)^{-1}\right]_{\lambda=i \sigma_{0}} \\
& =\frac{1}{\sigma_{0}^{2}} \operatorname{sign}\left[\frac{\left(m_{3}+m_{1} \sigma_{0}^{2}\right)\left(m_{1} \sigma_{0}^{2}-m_{3}\right)+2 \sigma_{0}^{4}\left(\sigma_{0}^{2}-m_{2}\right)}{\left(m_{1} \sigma_{0}^{2}-m_{3}\right)^{2}+\left(\sigma_{0}^{3}-m_{2} \sigma_{0}\right)^{2}}\right. \\
& \left.+\frac{\left(n_{1} \sigma_{0}^{2}+n_{3}\right)\left(n_{3}-n_{1} \sigma_{0}^{2}\right)}{\left(n_{3}-n_{1} \sigma_{0}^{2}\right)^{2}+\left(n_{2} \sigma_{0}\right)^{2}}\right] \\
& =\operatorname{sign}\left[\left(\left(m_{3}+m_{1} \sigma_{0}^{2}\right)\left(m_{1} \sigma_{0}^{2}-m_{3}\right)+2 \sigma_{0}^{4}\left(\sigma_{0}^{2}-m_{2}\right)\right.\right. \\
& \left.+\left(n_{1} \sigma_{0}^{2}+n_{3}\right)\left(n_{3}-n_{1} \sigma_{0}^{2}\right)\right)
\end{aligned}
$$

$$
\begin{align*}
&\left.\times\left(\left(m_{1} \sigma_{0}^{2}-m_{3}\right)^{2}+\left(\sigma_{0}^{3}-m_{2} \sigma_{0}\right)^{2}\right)^{-1}\right] \\
&= \operatorname{sign}\left[\frac{2 \sigma_{0}^{6}+\left(m_{1}^{2}-2 m_{2}-n_{1}^{2}\right) \sigma_{0}^{4}+n_{3}^{2}-m_{3}^{2}}{\left(m_{1} \sigma_{0}^{2}-m_{3}\right)^{2}+\left(\sigma_{0}^{3}-m_{2} \sigma_{0}\right)^{2}}\right], \\
&=\operatorname{sign}\left[\frac{2 \sigma_{0}^{6}+C_{1} \sigma_{0}^{4}-C_{3}}{\left(m_{1} \sigma_{0}^{2}-m_{3}\right)^{2}+\left(\sigma_{0}^{3}-m_{2} \sigma_{0}\right)^{2}}\right] . \tag{67}
\end{align*}
$$

According to the values of $C_{j},(j=1,2,3)$ given in (58) of this paper, it is easy to show that $C_{1}=m_{1}^{2}-2 m_{2}-n_{1}^{2}=$ $r^{2}\left(\mu_{1}+m\right)^{2} / k^{2} \beta^{2}+\left[\mu_{2}-c v^{*} /(w \widetilde{R}-c)^{2}\right]^{2}>0$.

Furthermore, if $k \beta>3\left(\mu_{1}+m\right)$, then it can be shown that $C_{3}>0$. Hence, it can be concluded that $2 \sigma_{0}^{6}+C_{1} \sigma_{0}^{4}-$ $C_{3}>0$, which derives sign $[\mathrm{d}(\operatorname{Re} \lambda) / \mathrm{d} \tau]_{\tau=\tau_{j}^{*}, \sigma=\sigma_{0}}>0$. Consequently, the transversality condition holds and Hopf bifurcation occurs at $\sigma=\sigma_{0}, \tau=\tau_{j}^{*}$. Furthermore, an attracting invariant closed curve bifurcates from interior equilibrium $\widetilde{P}^{*}$ when $\tau>\tau_{0}^{*}$ and $\left\|\tau-\tau_{0}^{*}\right\| \ll 1$.

Remark 18. sign $[\mathrm{d}(\operatorname{Re} \lambda) / \mathrm{d} \tau]_{\tau=\tau_{j}^{*}}>0$ signifies that there exists at least one eigenvalue with positive real part for $\tau=$ $\tau_{j}^{*}$, and the conditions for Hopf bifurcation in [39] are also satisfied yielding the required periodic solution.
4.2.2. Properties of Hopf Bifurcation. By using normal theory and center manifold theorem [49], directions of Hopf bifurcation and stability of the bifurcating periodic solutions are discussed in this section. As analyzed in Section 4.1.2, when economic interest of harvesting $0<v^{*}<\widetilde{v}(\bar{v}$ is defined in (55)), it follows from implicit function theorem [41] and the fourth equation of model system (4) that $E(t)=v^{*} /(w R(t)-$ c). Furthermore, model system (4) can be transformed into the following form:

$$
\begin{align*}
& \dot{S}(t)=r\left(1-\frac{S(t)}{k}\right)-\beta S(t) I(t-\tau), \\
& \dot{I}(t)=\beta S(t) I(t-\tau)-\mu_{1} I(t)-m I(t),  \tag{68}\\
& \dot{R}(t)=m I(t)-\mu_{2} R(t)-\frac{v^{*} R(t)}{w R(t)-c} .
\end{align*}
$$

Firstly, some transformations associated with component $\left(\widetilde{S}^{*}, \widetilde{I}^{*}, \widetilde{R}^{*}\right)$ of interior equilibrium $\widetilde{P}^{*}$ are given as follows:

$$
\begin{array}{cc}
y_{1}=S-\widetilde{S}^{*}, & y_{2}=I-\widetilde{I}^{*}, \quad y_{3}=R-\widetilde{R}^{*}  \tag{69}\\
\bar{y}_{i}(t)=y_{i}(\tau t), & \tau=\rho+\tau_{j}, \rho \in \mathbb{R}=(-\infty,+\infty) .
\end{array}
$$

Then $\rho=0$ is the Hopf bifurcation value of model system (4). Bars of variables are dropped for simplicity of notations; model system (4) is transformed to a functional differential equation in $C=C\left([-1,0], \mathbb{R}^{3}\right)$ as

$$
\begin{equation*}
\dot{y}(t)=L_{\rho}\left(y_{t}\right)+f\left(\rho, y_{t}\right) \tag{70}
\end{equation*}
$$

where $C=C\left([-1,0], \mathbb{R}^{3}\right)$ is the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^{3}, y(t)=$ $\left(y_{1}(t), y_{2}(t), y_{3}(t)\right)^{T} \in \mathbb{R}^{3}, y_{t}(\theta)=y(t+\theta)$ for $\theta \in[-\tau, 0]$ and $L_{\rho}: C \rightarrow \mathbb{R}^{3}, f: \mathbb{R} \times C \rightarrow \mathbb{R}^{3}$ are defined as follows, respectively:

$$
\begin{align*}
L_{\rho}(\phi)= & \left(\tau_{j}+\rho\right) \\
& \times\left(\begin{array}{ccc}
-\frac{r \widetilde{S}^{*}}{k} & 0 & 0 \\
\beta \widetilde{I}^{*} & -\left(\mu_{1}+m\right) & 0 \\
0 & m & -\mu_{2}+\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}
\end{array}\right) \\
& \times\left(\begin{array}{l}
\phi_{1}(0) \\
\phi_{2}(0) \\
\phi_{3}(0)
\end{array}\right)+\left(\tau_{j}+\rho\right)\left(\begin{array}{ccc}
0 & -\beta \widetilde{S}^{*} & 0 \\
0 & \beta \widetilde{S}^{*} & 0 \\
0 & 0 & 0
\end{array}\right) \\
& \times\left(\begin{array}{l}
\phi_{1}(-1) \\
\phi_{2}(-1) \\
\phi_{3}(-1)
\end{array}\right) \tag{71}
\end{align*}
$$

$$
f(\rho, \phi)=\left(\tau_{j}+\rho\right)\left(\begin{array}{c}
-\frac{r}{k} \phi_{1}^{2}(0)-\beta \phi_{1}(0) \phi_{2}(-1)  \tag{72}\\
\beta \phi_{1}(0) \phi_{2}(-1) \\
\sum_{j=1}^{\infty} f_{j}^{(3)} \phi_{3}^{j}(0)
\end{array}\right)
$$

where $f_{j}^{(3)}=\left.\left(\mathrm{d}^{i}\left(-v^{*} R /(w R-c)\right) / \mathrm{d} R^{i}\right)\right|_{\left(\widetilde{S}^{*}, \tilde{I}^{*}, \widetilde{R}^{*}\right)}, j=1,2,3$.
It is easy to show that $L_{\rho}$ is a continuous linear function mapping $C$ into $\mathbb{R}^{3}$. According to Riesz representation theorem [40], there exists a $3 \times 3$ matrix function $\eta(\theta, \rho)$ of bounded variation for $\theta \in[-1,0]$ such that

$$
\begin{equation*}
L_{\rho}(\phi)=\int_{-1}^{0} \mathrm{~d} \eta(\theta, \rho) \phi(\theta) \tag{73}
\end{equation*}
$$

where $\phi \in C\left([-1,0], \mathbb{R}^{3}\right)$.
In fact, we can choose

$$
\begin{align*}
\eta(\theta, \rho)= & \left(\tau_{j}+\rho\right) \\
& \times\left(\begin{array}{ccc}
-\frac{r \widetilde{S}^{*}}{k} & 0 & 0 \\
\beta \widetilde{I}^{*} & -\left(\mu_{1}+m\right) & 0 \\
0 & m & -\mu_{2}+\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}
\end{array}\right) \\
& \times \delta(\theta) \\
& -\left(\tau_{j}+\rho\right)\left(\begin{array}{ccc}
0 & -\beta \widetilde{S}^{*} & 0 \\
0 & \beta \widetilde{S}^{*} & 0 \\
0 & 0 & 0
\end{array}\right) \delta(\theta+1) \tag{74}
\end{align*}
$$

where $\delta$ denotes the Dirac delta function.

If $\phi$ is any given function in $C\left([-1,0], \mathbb{R}^{3}\right)$ and $y(\phi)$ is the unique solution of the linearized equation $\dot{y}(t)=L_{\rho}\left(y_{t}\right)$ of (70) with initial function $\phi$ at zero, then the solution operator $\widetilde{T}(t): C \rightarrow C$ is defined by

$$
\begin{equation*}
\widetilde{T}(t) \phi=y_{t}(\phi), \quad t \geq 0 \tag{75}
\end{equation*}
$$

It follows from Lemma 7.1.1 in [39] that $\widetilde{T}(t), t \geq 0$ is a strongly continuous semigroup of linear transformation on $[0,+\infty)$ and the infinitesimal generator $A_{\rho}$ of $\widetilde{T}(t), t \geq 0$ is as follows:

$$
A_{\rho}(\phi)= \begin{cases}\frac{\mathrm{d} \phi(\theta)}{\mathrm{d} \theta}, & \theta \in[-1,0)  \tag{76}\\ \int_{-1}^{0} \mathrm{~d} \eta(\rho, s) \phi(s), & \theta=0\end{cases}
$$

for $\phi \in C^{1}\left([-1,0], \mathbb{R}^{3}\right)$, the space of functions mapping the interval $[-1,0]$ into $\mathbb{R}^{3}$ which have continuous first derivative and also define

$$
R(\rho)(\phi)= \begin{cases}0, & \theta \in[-1,0)  \tag{77}\\ f(\rho, \phi), & \theta=0\end{cases}
$$

then model system (70) is equivalent to

$$
\begin{equation*}
\dot{y}_{t}=A(\rho) y_{t}+R(\rho) y_{t} \tag{78}
\end{equation*}
$$

For $\psi \in C^{1}\left([0,1],\left(\mathbb{R}^{3}\right)^{*}\right)$, the space of functions mapping interval $[0,1]$ into the three-dimensional row vectors which have continuous first derivative, define

$$
A^{*} \phi(s)= \begin{cases}-\frac{\mathrm{d} \psi(s)}{\mathrm{d} s}, & s \in(0,1]  \tag{79}\\ \int_{-1}^{0} \mathrm{~d} \eta^{T}(t, 0) \psi(-t), & s=0\end{cases}
$$

and a bilinear inner product

$$
\begin{align*}
\langle\psi(s), \phi(\theta)\rangle= & \bar{\psi}(0) \phi(0) \\
& -\int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta) \mathrm{d} \eta(\theta) \phi(\xi) \mathrm{d} \xi \tag{80}
\end{align*}
$$

where $\eta(\theta)=\eta(\theta, 0)$. It follows from the above analysis $A(0)$ and $A^{*}$ are adjoint operators.

By virtue of discussion in Section 4.2.1, $\pm i \omega_{0} \tau_{j}$ are eigenvalues of $A(0)$. Hence, they are also eigenvalues of $A^{*}$. In the following, eigenvectors of $A(0)$ and $A^{*}$ are corresponding to $i \omega_{0} \tau_{j}$ and $-i \omega_{0} \tau_{j}$, respectively.

Suppose $q(\theta)=(1, a, b)^{T} e^{i \omega_{0} \tau_{j} \theta}$ is the eigenvectors of $A(0)$ corresponding to $i \omega_{0} \tau_{j}$, which derives that
$A(0) q(\theta)=i \omega_{0} \tau_{j} q(\theta)$. By using the definition of $A(0)$, (71) and (72), it gives that

$$
\begin{gather*}
\left(\begin{array}{ccc}
-\frac{r \widetilde{S}^{*}}{k} & 0 & 0 \\
\beta \widetilde{I}^{*} & -\left(\mu_{1}+m\right) & 0 \\
0 & m & -\mu_{2}+\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}
\end{array}\right) q(0)  \tag{81}\\
+\left(\begin{array}{ccc}
0 & -\beta \widetilde{S}^{*} & 0 \\
0 & \beta \widetilde{S}^{*} & 0 \\
0 & 0 & 0
\end{array}\right) q(-1)=i \omega_{0} q(0)
\end{gather*}
$$

For $q(-1)=q(0) e^{-i \omega_{0} \tau_{j}}$, then it can be obtained that

$$
\begin{align*}
& a=-\frac{r\left(\mu_{1}+m\right)+i k \beta \omega_{0}}{k \beta\left(\mu_{1}+m\right) e^{-i \omega_{0} \tau_{j}}} \\
& b=\frac{m\left(w \widetilde{R}^{*}-c\right)^{2}\left[r\left(m+\mu_{1}\right)+i k \beta \omega_{0}\right]}{k \beta\left(\mu_{1}+m\right)\left[c v^{*}-\left(\mu_{2}+i \omega_{0}\right)\left(w \widetilde{R}^{*}-c\right)^{2}\right] e^{-i \omega_{0} \tau_{j}}} . \tag{82}
\end{align*}
$$

Similarly, it follows from simple computation that eigenvector $q^{*}(s)=J\left(1, a^{*}, b^{*}\right) e^{i \omega_{0} \tau_{j} s}$ of $A^{*}$ is corresponding to $-i \omega_{0} \tau_{j}$, where

$$
\begin{align*}
a^{*} & =-\frac{r\left(\mu_{1}+m\right)-i k \beta \omega_{0}}{k \beta\left(\mu_{1}+m\right) e^{i \omega_{0} \tau_{j}}} \\
b^{*} & =\frac{m\left(w \widetilde{R}^{*}-c\right)^{2}\left[r\left(\mu_{1}+m\right)-i k \beta \omega_{0}\right]}{k \beta\left(\mu_{1}+m\right)\left[c v^{*}-\left(\mu_{2}-i \omega_{0}\right)\left(w \widetilde{R}^{*}-c\right)^{2}\right] e^{i \omega_{0} \tau_{j}}} \tag{83}
\end{align*}
$$

In order to assume $\left\langle q^{*}(s), q(\theta)\right\rangle=1$, we need to determine the value of $J$ in the following part.

By virtue of (80), it derives that

$$
\begin{align*}
& \left\langle q^{*}(s), q(\theta)\right\rangle \\
& \begin{aligned}
&= \bar{J}\left(1, \bar{a}^{*}, \bar{b}^{*}\right)(1, a, b)^{T} \\
&- \int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{J}\left(1, a^{*}, b^{*}\right) e^{-i \omega_{0} \tau_{j}(\xi-\theta)} \mathrm{d} \eta(\theta) \\
& \times(1, a, b)^{T} e^{i \omega_{0} \tau_{j} \xi} \mathrm{~d} \xi \\
&= \bar{J}\left[1+a \bar{a}^{*}+b \bar{b}^{*}\right. \\
&\left.\quad-\int_{-1}^{0}\left(1, \bar{a}^{*}, \bar{b}^{*}\right) \theta e^{i \omega_{0} \tau_{j} \theta} \mathrm{~d} \eta(\theta)(1, a, b)^{T}\right] \\
&= \bar{J}\left[1+a \bar{a}^{*}+b \bar{b}^{*}+k \bar{b}^{*} \tau_{j}\left(\widetilde{R}^{*}+b \widetilde{S}^{*}\right) e^{i \omega_{0} \tau_{j}}\right] .
\end{aligned}
\end{align*}
$$

Hence, we can choose $J$ as follows:

$$
\begin{equation*}
J=\frac{1}{1+\bar{a} a^{*}+\bar{b} b^{*}+k b^{*} \tau_{j}\left(\widetilde{R}^{*}+b \widetilde{S}^{*}\right) e^{i \omega_{0} \tau_{j}}} \tag{85}
\end{equation*}
$$

Next, we will compute the coordinate to describe the centre manifold $C_{0}$ at $\rho=0$. Let $y_{t}$ be the solution of (78) when $\rho=0$.

Define

$$
\begin{equation*}
z(t)=\left\langle q^{*}, y_{t}\right\rangle, \quad W(t, \theta)=y_{t}(\theta)-2 \operatorname{Re}\{z(t) q(\theta)\} \tag{86}
\end{equation*}
$$

On the center manifold $C_{0}$, it derives that

$$
\begin{equation*}
W(t, \theta)=W(z(t), \bar{z}(t), \theta) \tag{87}
\end{equation*}
$$

where
$W(z(t), \bar{z}(t), \theta)=W_{20}(\theta) \frac{z^{2}}{2}+W_{11}(\theta) z \bar{z}+W_{02} \frac{\bar{z}^{2}}{2}+\cdots$,
$z$ and $\bar{z}$ are local coordinates for center manifold $C_{0}$ in the direction of $q^{*}$ and $\bar{q}^{*}$.

It is noted that $W$ is real if $y_{t}$ is real, and we only consider real solutions. For solution $y_{t} \in C_{0}$ of (78), since $\rho=0$, it derives that

$$
\begin{align*}
\dot{z}(t) & =i \omega_{0} \tau_{j} z+\bar{q}^{*}(0) f(0, W(z, \bar{z}, 0)+2 \operatorname{Re}\{z q(\theta)\}) \\
& \triangleq i \omega_{0} \tau_{j} z+\bar{q}^{*}(0) f_{0}(z, \bar{z}) \tag{89}
\end{align*}
$$

The above equation can be rewritten as follows:

$$
\begin{equation*}
\dot{z}(t)=i \omega_{0} \tau_{j} z(t)+g(z, \bar{z}) \tag{90}
\end{equation*}
$$

where

$$
\begin{align*}
g(z, \bar{z}) & =\bar{q}^{*}(0) f_{0}(z, \bar{z}) \\
& =g_{20} \frac{z^{2}}{2}+g_{11} z \bar{z}+g_{02} \frac{\bar{z}^{2}}{2}+g_{21} \frac{z^{2} \bar{z}}{2}+\cdots \tag{91}
\end{align*}
$$

It follows from (86) and (88) that

$$
\begin{align*}
y_{t}(\theta)= & W(t, \theta)+2 \operatorname{Re}\{z(t) q(\theta)\} \\
= & W_{20}(\theta) \frac{z^{2}}{2}+W_{11}(\theta) z \bar{z}+W_{02}(\theta) \frac{\bar{z}^{2}}{2}  \tag{92}\\
& +(1, a, b)^{T} e^{i \omega_{0} \tau_{j} \theta} z+(1, \bar{a}, \bar{b})^{T} e^{-i \omega_{0} \tau_{j} \theta} \bar{z}+\cdots
\end{align*}
$$

By virtue of (72), (91), and (92), it derives that

$$
\left.+W_{02}^{(1)}(0) \frac{\bar{z}^{2}}{2}+o\left(|z, \bar{z}|^{3}\right)\right]
$$

$$
\times\left[e^{-i \omega_{0} \tau_{j}} z+e^{i \omega_{0} \tau_{j}} \bar{z}+W_{20}^{(2)}(-1) \frac{z^{2}}{2}\right.
$$

$$
\left.+W_{11}^{(2)}(-1) z \bar{z}+W_{02}^{(2)}(-1) \frac{\bar{z}^{2}}{2}+o\left(|z, \bar{z}|^{3}\right)\right]
$$

$$
+\tau_{j} \bar{J}^{*}\left[f _ { 1 } ^ { ( 3 ) } \left(z+\bar{z}+W_{20}^{(3)}(0) \frac{z^{2}}{2}+W_{11}^{(3)}(0) z \bar{z}\right.\right.
$$

$$
\left.+W_{02}^{(1)}(0) \frac{\bar{z}^{2}}{2}+o\left(|z, \bar{z}|^{3}\right)\right)
$$

$$
+f_{2}^{(3)}\left(z+\bar{z}+W_{20}^{(3)}(0) \frac{z^{2}}{2}\right.
$$

$$
+W_{11}^{(3)}(0) z \bar{z}+W_{02}^{(1)}(0) \frac{\bar{z}^{2}}{2}
$$

$$
\begin{equation*}
\left.\left.+o\left(|z, \bar{z}|^{3}\right)\right)+\cdots\right] \tag{93}
\end{equation*}
$$

$$
\begin{aligned}
& g(z, \bar{z})=\bar{q}^{*}(0) f_{0}(z, \bar{z}) \\
& =\bar{q}^{*}(0) f\left(0, y_{t}\right) \\
& =\tau_{j} \bar{J}\left(\begin{array}{c}
-\frac{r}{k} y_{1 t}^{2}(0)-\beta y_{1 t}(0) y_{2 t}(-1) \\
\beta y_{1 t}(0) y_{2 t}(-1) \\
\sum_{j=1}^{\infty} f_{j}^{(3)} y_{3 t}^{j}(0)
\end{array}\right) \\
& =\tau_{j} \bar{J}\left[-\frac{r}{k} y_{1 t}^{2}(0)-\beta y_{1 t}(0) y_{2 t}(-1)\right. \\
& \left.+\bar{a}^{*} \beta y_{1 t}(0) y_{2 t}(-1)+\sum_{j=1}^{\infty} \bar{b}^{*} f_{j}^{(3)} y_{3 t}^{j}(0)\right] \\
& =\tau_{j} \bar{J}\left[-\frac{r}{k} y_{1 t}^{2}(0)+\left(\bar{a}^{*}-1\right) \beta y_{1 t}(0) y_{2 t}(-1)\right. \\
& \left.+\sum_{j=1}^{\infty} \bar{b}^{*} f_{j}^{(3)} y_{3 t}^{j}(0)\right] \\
& =-\frac{r}{k} \tau_{j} \bar{J}\left[z+\bar{z}+W_{20}^{(1)}(0) \frac{z^{2}}{2}+W_{11}^{(1)}(0) z \bar{z}\right. \\
& \left.+W_{02}^{(1)}(0) \frac{\bar{z}^{2}}{2}+o\left(|(z, \bar{z})|^{3}\right)\right]^{2} \\
& +\left(\bar{a}^{*}-1\right) \beta \tau_{j} \bar{J} \\
& \times\left[z+\bar{z}+W_{20}^{(1)}(0) \frac{z^{2}}{2}+W_{11}^{(1)}(0) z \bar{z}\right.
\end{aligned}
$$

By comparing the coefficients with (91), it gives that

$$
\begin{align*}
& g_{20}=2 \tau_{j} \bar{J}[ -\frac{r}{k}+\left(\bar{a}^{*}-1\right) \beta e^{-i \omega_{0} \tau_{j}} \\
&\left.+\bar{b}^{*}\left(\frac{f_{1}^{(3)} W_{20}^{(3)}(0)}{2}+f_{2}^{(3)}\right)\right], \\
& g_{11}=\tau_{j} \bar{J}[ -\frac{2 r}{k}+2\left(\bar{a}^{*}-1\right) \beta \cos \omega_{0} \tau_{j} \\
&\left.+f_{1}^{(3)} W_{11}^{(3)}(0)+2 f_{2}^{(3)}\right], \\
& g_{02}=2 \tau_{j} \bar{J}[ -\frac{r}{k}+\left(\bar{a}^{*}-1\right) \beta e^{-i \omega_{0} \tau_{j}} \\
&\left.+\bar{b}^{*}\left(\frac{f_{1}^{(3)} W_{02}^{(3)}(0)}{2}+f_{2}^{(3)}\right)\right], \\
& g_{21}=2 \tau_{j} \bar{J} {\left[\begin{array}{rl}
\beta & \left(\bar{a}^{*}-1\right) \\
& \times\left(W_{11}^{(2)}(-1)+\frac{W_{20}^{(3)}(-1)+W_{20}^{(1)}(0) e^{i \omega_{0} \tau_{j}}}{2}\right. \\
& +\frac{r}{k}\left(W_{20}^{(1)}(0)+2 W_{11}^{(1)}(0)\right) \\
& \left.+f_{2}^{(3)}\left(W_{20}^{(3)}(0)+2 W_{11}^{(3)}(0)\right)\right]
\end{array}\right.} \\
&\left.\quad+W_{11}^{(1)}(0) e^{-i \omega_{0} \tau_{j}}\right) \\
&
\end{align*}
$$

Since $g_{21}$ is associated with $W_{20}(\theta)$ and $W_{11}(\theta)$, further attempts should be carried out to compute $W_{20}(\theta)$ and $W_{11}(\theta)$.

By virtue of (78) and (86), we have

$$
\begin{align*}
\dot{W} & =\dot{y}_{t}-\dot{z} q-\dot{\bar{z}} \bar{q} \\
& = \begin{cases}A W-2 \operatorname{Re}\left\{\bar{q}^{*}(0) f_{0} q(\theta)\right\}, & \theta \in[-1,0) \\
A W-2 \operatorname{Re}\left\{\bar{q}^{*}(0) f_{0} q(0)+f_{0}\right\}, & \theta=0 .\end{cases}  \tag{95}\\
& \triangleq A W+H(z, \bar{z}, \theta),
\end{align*}
$$

where

$$
\begin{equation*}
H(z, \bar{z}, \theta)=H_{20}(\theta) \frac{z^{2}}{2}+H_{11}(\theta) z \bar{z}+H_{02}(\theta) \frac{\bar{z}^{2}}{2}+\cdots \tag{96}
\end{equation*}
$$

By substituting the corresponding series into (95) and comparing the coefficients, we have

$$
\begin{gather*}
\left(A-2 i \omega_{0} \tau_{j}\right) W_{20}(\theta)=-H_{20}(\theta),  \tag{97}\\
A W_{11}(\theta)=-H_{11}(\theta), \ldots
\end{gather*}
$$

It follows from (95) that for $\theta \in[-1,0)$

$$
\begin{align*}
H(z, \bar{z}, \theta) & =-\bar{q}^{*}(0) f_{0} q(\theta)-q^{*}(0) \bar{f}_{0} \bar{q}(\theta)  \tag{98}\\
& =-g(z, \bar{z}) q(\theta)-\bar{g}(z, \bar{z}) \bar{q}(\theta)
\end{align*}
$$

By comparing coefficients in (96) with those in (94), it derives that

$$
\begin{align*}
& H_{20}(\theta)=-g_{20} q(\theta)-\bar{g}_{02} \bar{q}(\theta),  \tag{99}\\
& H_{11}(\theta)=-g_{11} q(\theta)-\bar{g}_{11} \bar{q}(\theta) . \tag{100}
\end{align*}
$$

Based on the definition of $A$ and (97) and (99), it can be obtained that

$$
\begin{align*}
& \dot{W}_{20}(\theta)=2 i \omega_{0} \tau_{j} W_{20}(\theta)+g_{20} q(\theta)+\bar{g}_{02} \bar{q}(\theta),  \tag{101}\\
& \text { For } q(\theta)=(1, a, b)^{T} e^{i \omega_{0} \tau_{j} \theta}, \\
& W_{20}(\theta)=\frac{i g_{20}}{\omega_{0} \tau_{j}} q(0) e^{i \omega_{0} \tau_{j} \theta}+\frac{i \bar{g}_{02}}{3 \omega_{0} \tau_{j}} \bar{q}(0) e^{-i \omega_{0} \tau_{j} \theta}+G_{1} e^{2 i \omega_{0} \tau_{j} \theta}, \tag{102}
\end{align*}
$$

where $G_{1}=\left(G_{1}^{(1)}, G_{1}^{(2)}, G_{1}^{(3)}\right)$ is a constant vector.
Similarly, it follows from (97) and (100) that

$$
\begin{equation*}
W_{11}(\theta)=-\frac{i g_{11}}{\omega_{0} \tau_{j}} q(0) e^{i \omega_{0} \tau_{j} \theta}+\frac{i \bar{g}_{11}}{\omega_{0} \tau_{j}} \bar{q}(0) e^{-i \omega_{0} \tau_{j} \theta}+G_{2} \tag{103}
\end{equation*}
$$

where $G_{2}=\left(G_{2}^{(1)}, G_{2}^{(2)}, G_{2}^{(3)}\right)$ is a constant vector.
Subsequently, values of $G_{1}$ and $G_{2}$ should be computed. By using the definition of $A$ and (95), we have

$$
\begin{gather*}
\int_{-1}^{0} \mathrm{~d} \eta w_{20}(\theta)=2 i \omega_{0} \tau_{j} W_{20}(0)-H_{20}(0),  \tag{104}\\
\int_{-1}^{0} \mathrm{~d} \eta(\theta) W_{11}(\theta)=-H_{11}(0), \tag{105}
\end{gather*}
$$

where $\eta(\theta)=\eta(0, \theta)$. Based on (95), it derives that in the case of $\theta=0$,

$$
\begin{align*}
H(z, \bar{z}, 0) & =-2 \operatorname{Re}\left\{\bar{q}^{*}(0) f_{0} q(0)\right\}+f(0) \\
& =-\bar{q}^{*}(0) f_{0} q(0)-q^{*}(0) \bar{f}_{0} \bar{q}(0)+f_{0}  \tag{106}\\
& =-g(z, \bar{z}) q(0)-\bar{g}(z, \bar{z}) \bar{q}(0)+f_{0}
\end{align*}
$$

which follows that

$$
\begin{aligned}
H_{20} & (\theta) \frac{z^{2}}{2}+H_{11}(\theta) z \bar{z}+H_{02}(\theta) \frac{\bar{z}^{2}}{2}+\cdots \\
\quad= & -q(0)\left(g_{20} \frac{z^{2}}{2}+g_{11} z \bar{z}+g_{02} \frac{\bar{z}^{2}}{2}+\cdots\right) \\
& -\bar{q}(0)\left(\bar{g}_{20} \frac{\bar{z}^{2}}{2}+\bar{g}_{11} z \bar{z}+\bar{g}_{02} \frac{z^{2}}{2}+\cdots\right)+f_{0} .
\end{aligned}
$$

By virtue of (72), it gives that

$$
f_{0}=\tau_{k}\left(\begin{array}{c}
-\frac{r}{k} y_{1 t}^{2}(0)-\beta y_{1 t}(0) y_{2 t}(-1)  \tag{108}\\
\beta y_{1 t}(0) y_{2 t}(-1) \\
\sum_{j=1}^{\infty} f_{j}^{(3)} y_{3 t}^{j}(0)
\end{array}\right)
$$

By virtue of (86), it can be obtained that

$$
\begin{align*}
y_{t}(\theta)= & W(t, \theta)+2 \operatorname{Re}\{z(t) q(\theta)\} \\
= & W(t, \theta)+z(t) q(\theta)+\bar{z}(t) \bar{q}(\theta) \\
= & W_{20}(\theta) \frac{z^{2}}{2}+W_{21}(\theta) z \bar{z}+z(t) q(\theta)  \tag{109}\\
& +\bar{z}(t) \bar{q}(\theta)+\cdots .
\end{align*}
$$

Then we have

$$
\begin{align*}
f_{0}= & \tau_{j}\left(\begin{array}{c}
-\frac{r}{k}-\beta e^{-i \omega_{0} \tau_{j}} \\
\beta e^{-i \omega_{0} \tau_{j}} \\
f_{1}^{(3)} \frac{W_{20}^{(3)}(0)}{2}+f_{2}^{(3)}
\end{array}\right) z^{2}  \tag{110}\\
& +\tau_{k}\left(\begin{array}{c}
-\frac{2 r}{k}-2 \beta \cos \omega_{0} \tau_{j} \\
2 \beta \cos \omega_{0} \tau_{j} \\
2 f_{2}^{(3)}+f_{1}^{(3)} W_{11}^{(3)}(0)
\end{array}\right) z \bar{z}+\cdots \tag{114}
\end{align*}
$$

where $I$ is identity matrix.
By substituting (102) and (111) into (104), it can be obtained that

$$
\begin{aligned}
& \left(2 i \omega_{0} \tau_{j} I-\int_{-1}^{0} e^{2 i \omega_{0} \tau_{j} \theta} \mathrm{~d} \eta(\theta)\right) G_{1} \\
& =2 \tau_{j}\left(\begin{array}{c}
-\frac{r}{k}-\beta e^{-i \omega_{0} \tau_{j}} \\
\beta e^{-i \omega_{0} \tau_{j}} \\
f_{1}^{(3)} \frac{W_{20}^{(3)}(0)}{2}+f_{2}^{(3)}
\end{array}\right)
\end{aligned}
$$

which can be rewritten as follows:

$$
\left(\begin{array}{ccc}
2 i \omega_{0}+\frac{r \widetilde{S}^{*}}{k} & \beta \widetilde{S}^{*} e^{-2 i \omega_{0} \tau_{j}} & 0  \tag{115}\\
-\beta \widetilde{I}^{*} & 2 i \omega_{0}+\left(\mu_{1}+m\right)-\beta \widetilde{S}^{*} e^{-2 i \omega_{0} \tau_{j}} & 0 \\
0 & -m & 2 i \omega_{0}+\mu_{2}-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}
\end{array}\right) G_{1}=2\left(\begin{array}{c}
-\frac{r}{k}-\beta e^{-i \omega_{0} \tau_{j}} \\
\beta e^{-i \omega_{0} \tau_{j}} \\
f_{1}^{(3)} \frac{W_{20}^{(3)}(0)}{2}+f_{2}^{(3)}
\end{array}\right)
$$

Based on Grammar Law [43], $G_{1}^{(1)}, G_{1}^{(2)}$, and $G_{1}^{(3)}$ can be obtained as follows:

$$
G_{1}^{(1)}=\frac{2}{U_{1}}\left|\begin{array}{ccc}
-\frac{r}{k}-\beta e^{-i \omega_{0} \tau_{j}} & \beta \widetilde{S}^{*} e^{-2 i \omega_{0} \tau_{j}} & 0 \\
\beta e^{-i \omega_{0} \tau_{j}} & 2 i \omega_{0}+\left(\mu_{1}+m\right)-\beta \widetilde{S}^{*} e^{-2 i \omega_{0} \tau_{j}} & 0 \\
f_{1}^{(3)} \frac{W_{20}^{(3)}(0)}{2}+f_{2}^{(3)} & -m & 2 i \omega_{0}+\mu_{2}-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}
\end{array}\right|,
$$

$$
\begin{align*}
& G_{1}^{(2)}=\frac{2}{U_{1}}\left|\begin{array}{ccc}
2 i \omega_{0}+\frac{r \widetilde{S}^{*}}{k} & -\frac{r}{k}-\beta e^{-i \omega_{0} \tau_{j}} & 0 \\
-\beta \widetilde{I}^{*} & \begin{array}{c}
\beta e^{-i \omega_{0} \tau_{j}} \\
0
\end{array} f_{1}^{(3)} \frac{W_{20}^{(3)}(0)}{2}+f_{2}^{(3)} & 2 i \omega_{0}+\mu_{2}-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}
\end{array}\right|, \\
& G_{1}^{(3)}=\frac{2}{U_{1}}\left|\begin{array}{ccc}
2 i \omega_{0}+\frac{r \widetilde{S}^{*}}{k} & \beta \widetilde{S}^{*} e^{-2 i \omega_{0} \tau_{j}} & -\frac{r}{k}-\beta e^{-i \omega_{0} \tau_{j}} \\
-\beta \widetilde{I}^{*} & 2 i \omega_{0}+\left(\mu_{1}+m\right)-\beta \widetilde{S}^{*} e^{-2 i \omega_{0} \tau_{j}} & \beta e^{-i \omega_{0} \tau_{j}} \\
0 & -m & f_{1}^{(3)} \frac{W_{20}^{(3)}(0)}{2}+f_{2}^{(3)}
\end{array}\right|, \tag{116}
\end{align*}
$$

where

$$
U_{1}=\left|\begin{array}{ccc}
2 i \omega_{0}+\frac{r \widetilde{S}^{*}}{k} & \beta \widetilde{S}^{*} e^{-2 i \omega_{0} \tau_{j}} & 0  \tag{117}\\
-\beta \widetilde{I}^{*} & 2 i \omega_{0}+\left(\mu_{1}+m\right)-\beta \widetilde{S}^{*} e^{-2 i \omega_{0} \tau_{j}} & 0 \\
0 & -m & 2 i \omega_{0}+\mu_{2}-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}
\end{array}\right|
$$

Similarly, substituting (103) and (112) into (105), it can be obtained that

$$
\left(\begin{array}{ccc}
\frac{r \widetilde{S}^{*}}{k} & \beta \widetilde{S}^{*} & 0  \tag{118}\\
-\beta \widetilde{I}^{*} & \mu_{1}+m-\beta \widetilde{S}^{*} & 0 \\
0 & -m & \mu_{2}-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}
\end{array}\right) G_{2}
$$

$$
=2\left(\begin{array}{c}
-\frac{2 r}{k}-2 \beta \cos \omega_{0} \tau_{j} \\
2 \beta \cos \omega_{0} \tau_{j} \\
2 f_{2}^{(3)}+f_{1}^{(3)} W_{11}^{(3)}(0)
\end{array}\right)
$$

Based on Grammar Law [43], $G_{2}^{(1)}, G_{2}^{(2)}$, and $G_{2}^{(3)}$ can be obtained as follows:

$$
\begin{aligned}
& G_{2}^{(1)} \\
& =\frac{1}{U_{2}}\left|\begin{array}{ccc}
-\frac{2 r}{k}-2 \beta \cos \omega_{0} \tau_{j} & \beta \widetilde{S}^{*} & 0 \\
2 \beta \cos \omega_{0} \tau_{j} & \mu_{1}+m-\beta \widetilde{S}^{*} & 0 \\
2 f_{2}^{(3)}+f_{1}^{(3)} W_{11}^{(3)}(0) & -m & \mu_{2}-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}
\end{array}\right|, \\
& G_{2}^{(2)} \\
& =\frac{1}{U_{2}}\left|\begin{array}{ccc}
\frac{r \widetilde{S}^{*}}{k} & -\frac{2 r}{k}-2 \beta \cos \omega_{0} \tau_{j} & 0 \\
-\beta \widetilde{I}^{*} & 2 \beta \cos \omega_{0} \tau_{j} & 0 \\
0 & 2 f_{2}^{(3)}+f_{1}^{(3)} W_{11}^{(3)}(0) & \mu_{2}-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}
\end{array}\right|
\end{aligned}
$$

$$
\begin{align*}
& G_{2}^{(3)} \\
& =\frac{1}{U_{2}}\left|\begin{array}{ccc}
\frac{r \widetilde{S}^{*}}{k} & \beta \widetilde{S}^{*} & -\frac{2 r}{k}-2 \beta \cos \omega_{0} \tau_{j} \\
-\beta \widetilde{I}^{*} & \mu_{1}+m-\beta \widetilde{S}^{*} & 2 \beta \cos \omega_{0} \tau_{j} \\
0 & -m & 2 f_{2}^{(3)}+f_{1}^{(3)} W_{11}^{(3)}(0)
\end{array}\right| \tag{119}
\end{align*}
$$

where

$$
U_{2}=\left|\begin{array}{ccc}
\frac{r \widetilde{S}^{*}}{k} & \beta \widetilde{S}^{*} & 0  \tag{120}\\
-\beta \widetilde{I}^{*} & \mu_{1}+m-\beta \widetilde{S}^{*} & 0 \\
0 & -m & \mu_{2}-\frac{c v^{*}}{\left(w \widetilde{R}^{*}-c\right)^{2}}
\end{array}\right|
$$

It follows from the above computation and analysis that $W_{20}(\theta)$ and $W_{11}(\theta)$ can be determined based on (102) and (103).

Furthermore, we can compute $g_{21}$ based on (94). Hence, the following values can be computed as follows:

$$
\begin{align*}
d_{1}(0) & =\frac{i}{2 \omega_{0} \tau_{j}}\left(g_{20} g_{11}-2\left|g_{11}\right|^{2}-\frac{\left|g_{02}\right|^{2}}{3}\right)+\frac{g_{21}}{2}, \\
\delta_{2} & =-\frac{\operatorname{Re}\left\{d_{1}(0)\right\}}{\operatorname{Re}\left\{\lambda^{\prime}\left(\tau_{j}\right)\right\}}  \tag{121}\\
\gamma_{2} & =2 \operatorname{Re}\left(d_{1}(0)\right) \\
T_{2} & =\frac{\operatorname{Im}\left\{d_{1}(0)\right\}+\delta_{2} \operatorname{Im}\left\{\lambda^{\prime}\left(\tau_{j}\right)\right\}}{\omega_{0} \tau_{j}} .
\end{align*}
$$

Table 2: Values of parameters for numerical simulation.

| Parameter | Value |
| :--- | :---: |
| $r$ | 0.1 |
| $k$ | 1 |
| $\beta$ | 1 |
| $m$ | 0.1 |
| $\mu_{1}$ | 0.1 |
| $\mu_{2}$ | 0.1 |
| $w$ | 20 |
| $c$ | 1 |

By using similar arguments in [49], some properties of bifurcating periodic solutions of model (4) in the center manifold at the critical values are discussed in this paper. Based on the analysis in Section 4.2.2 of this paper, the following theorem can be concluded.

Theorem 19. The properties of Hopf bifurcation are determined by values in (121).
(i) $\delta_{2}$ determines directions of Hopf bifurcation: if $\delta_{2}>$ $0\left(\delta_{2}<0\right)$, then Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau>\tau_{j}\left(\tau<\tau_{j}\right)$.
(ii) $\gamma_{2}$ determines the stability of bifurcating periodic solutions: bifurcating periodic solutions are stable (unstable) if $\gamma_{2}<0\left(\gamma_{2}>0\right)$.
(iii) $T_{2}$ determines the period of bifurcating periodic solutions: period increases (decreases) if $T_{2}>0\left(T_{2}<0\right)$.

## 5. Numerical Simulation

In this section, some numerical simulations are provided to substantiate the theoretical results obtained in Section 4 of this paper.
5.1. Numerical Simulation of State Feedback Controller for Singularity Induced Bifurcation and Local Stability Switch. In this subsection, values of parameters are partially taken from Section 5 of [11] and set in appropriate units, which can be found in Table 2. Numerical simulations are provided to illustrate the effectiveness of the state feedback controllers designed in Section 4.1 in the case of zero economic interest and positive economic interest, respectively.

In the case of zero economic interest, model (4) without time delay takes the following form:

$$
\begin{aligned}
\dot{S}(t) & =0.1(1-S(t))-S(t) I(t), \\
\dot{I}(t) & =S(t) I(t)-0.1 I(t)-0.1 I(t), \\
\dot{R}(t) & =0.1 I(t)-0.1 R(t)-E(t) R(t), \\
0 & =E(t)(20 R(t)-1) .
\end{aligned}
$$



Figure 1: Dynamical responses of model system (123) with state feedback controller, which shows that model system (123) is stable around ( $0.2,0.08,0.05,0.06$ ).

By using Theorem 11 of this paper, it can be shown that the model (4) without time delay has a singularity induced bifurcation around the interior equilibrium $P^{*}(0.2,0.08,0.05,0.06)$, and local stability switch occurs as $v$ increases through 0 .

Based on the analysis in Section 4.1.2 and Theorem 14 of this paper, a state feedback controller $u(t)=l(E(t)-0.06)$ can be applied to stabilize model system (122) around $P^{*}$, and then the model system (122) with the state feedback controller takes the following form:

$$
\begin{align*}
\dot{S}(t) & =0.1(1-S(t))-S(t) I(t) \\
\dot{I}(t) & =S(t) I(t)-0.1 I(t)-0.1 I(t)  \tag{123}\\
\dot{R}(t) & =0.1 I(t)-0.1 R(t)-E(t) R(t) \\
0 & =E(t)(20 R(t)-1)+l(E(t)-0.06) .
\end{align*}
$$

By using Theorem 14 of this paper, if the feedback gain $l$ satisfies $l>2.2429$, then model system (122) is stable around $P^{*}$ and singularity induced bifurcation of the model system (122) is also eliminated. The dynamical responses of model system (122) can be shown in Figure 1.

Furthermore, based on the analysis and inequality (55) in Section 4.1.2, there are two interior equilibria (denoted by $\widetilde{P}_{1}^{*}$ and $\widetilde{P}_{2}^{*}$ ) when $0<v<0.00668$. In the following part, we focus on the case of $0<v<0.00668$, and the economic interest is set as $v^{*}=0.005 \in(0,0.00668)$ in appropriate unit, which is arbitrarily selected within the interval $(0,0.00668)$ and is enough to merit the theoretical analysis obtained in Section 4.1.2. By virtue of the given values of parameters in Table 2 and (54), two interior equilibria can be obtained as follows: $\widetilde{P}_{1}^{*}(0.2,0.08,0.0557,0.0439)$ and $\widetilde{P}_{2}^{*}(0.2,0.08,0.07175,0.0115)$. By using Theorem 11 of this
paper, it can be shown that model system (124) is unstable around $\widetilde{P}_{1}^{*}$ and $\widetilde{P}_{2}^{*}$ :

$$
\begin{align*}
\dot{S}(t) & =0.1(1-S(t))-S(t) I(t), \\
\dot{I}(t) & =S(t) I(t)-0.1 I(t)-0.1 I(t),  \tag{124}\\
\dot{R}(t) & =0.1 I(t)-0.1 R(t)-E(t) R(t), \\
0 & =E(t)(20 R(t)-1)-0.005 .
\end{align*}
$$

Based on the analysis in Section 4.1.2 of this paper, state feedback controllers $u(t)=\widetilde{l}(E(t)-0.0439)$ and $u(t)=$ $\widetilde{l}(E(t)-0.0115)$ can be applied to stabilize the model system (124) around $\widetilde{P}_{1}^{*}$ and $\widetilde{P}_{2}^{*}$, respectively. The model system (124) with respective state feedback controller takes the following form:

$$
\begin{align*}
\dot{S}(t) & =0.1(1-S(t))-S(t) I(t) \\
\dot{I}(t) & =S(t) I(t)-0.1 I(t)-0.1 I(t) \\
\dot{R}(t) & =0.1 I(t)-0.1 R(t)-E(t) R(t) \\
0 & =E(t)(20 R(t)-1)-0.005+\widetilde{l}(E(t)-0.0439), \tag{125}
\end{align*}
$$

$$
\begin{align*}
\dot{S}(t) & =0.1(1-S(t))-S(t) I(t) \\
\dot{I}(t) & =S(t) I(t)-0.1 I(t)-0.1 I(t) \\
\dot{R}(t) & =0.1 I(t)-0.1 R(t)-E(t) R(t) \\
0 & =E(t)(20 R(t)-1)-0.005+\widetilde{l}(E(t)-0.0115) \tag{126}
\end{align*}
$$

By using Theorem 15 of this paper, if the feedback gain $\tilde{l}$ satisfies $\tilde{l}>3.753$, then model system (125) is stable around $\widetilde{P}_{1}^{*}$ and model system (126) is stable around $\widetilde{P}_{2}^{*}$. The dynamical responses of model system (125) and (126) can be shown in Figures 2 and 3, respectively.
5.2. Numerical Simulation for Hopf Bifurcation and Local Stability Switch. In this subsection,values of parameters are partially taken from Section 5 of [11] and set in appropriate units, which can be found in Table 3. Numerical simulations are provided to support the theoretical findings obtained in Section 4.2 of this paper.

Based on the analysis and inequality (55) in Section 4.1.2, there are two interior equilibria (denoted by $\widetilde{P}_{1}^{*}$ and $\widetilde{P}_{2}^{*}$ ) when $0<v<0.01831$. In the following part, we focus on the case of $0<v<0.01831$, and the economic interest is set as $v^{*}=0.012 \in(0,0.01831)$ in appropriate unit, which is arbitrarily selected within the interval $(0,0.01831)$ and is enough to merit the theoretical analysis obtained in Section 4.2. By virtue of the given values of parameters in Table 3 and (54), two interior equilibria can be obtained as follows: $\widetilde{P}_{1}^{*}(0.2,0.08,0.0907,0.0147)$ and $\widetilde{P}_{2}^{*}(0.2,0.08,0.0573,0.0821)$. Furthermore, it can be computed that $k \beta>3\left(\mu_{1}+m\right)$. Based on the analysis in


Figure 2: Dynamical responses of model system (125) with state feedback controller, which shows that model system (125) is stable around ( $0.2,0.08,0.0557,0.0439$ ).


Figure 3: Dynamical responses of model system (126) with state feedback controller, which shows that model system (126) is stable around ( $0.2,0.08,0.07175,0.0115$ ).

Table 3: Values of parameters for numerical simulation.

| Parameter | Value |
| :--- | :---: |
| $r$ | 0.1 |
| $k$ | 1 |
| $\beta$ | 1 |
| $m$ | 0.13 |
| $\mu_{1}$ | 0.1 |
| $\mu_{2}$ | 0.1 |
| $w$ | 20 |
| $c$ | 1 |

Section 4.2.1, it satisfies the assumption that (58) has a positive root, and then the corresponding $\tau_{0}^{*}=2.7814$ can be calculated by solving (64). It follows from (121) that $\delta_{2}=1.2963>0, \gamma_{2}=-0.3012<0$, and $T_{2}=$ $1.2467>0$. Consequently, the interior equilibrium $\widetilde{P}_{1}^{*}$


Figure 4: Dynamical responses of model system (4) with time delay $\tau=1.54$, which shows that model system (4) is stable around (0.2, 0.08, 0.0907, 0.0147).





Figure 5: Dynamical responses of model system (4) with time delay $\tau=3$, which shows that model system (4) is unstable around ( $0.2,0.08,0.0907,0.0147$ ).
remains stable for $\tau<\tau_{0}^{*}$, and dynamical responses of model system (4) with $\tau=1.54$ are plotted in Figure 4. It should be noted that $\tau=1.54$ in Figure 4 is ran-domly selected in the interval $(0,2.7814)$, which is enough to merit the above mathematical study. Only the dynamical responses and corresponding phase portrait of model (4) around $\widetilde{P}_{1}^{*}$


Figure 6: A limit cycle corresponding to the periodic solution in Figure 5 in the $S-I-R$ space.


Figure 7: Limit cycle corresponding to the periodic solution in Figure 5 in the $S-I-E$ space.
are plotted; some symmetric results about $\widetilde{P}_{2}^{*}$ can be also obtained. As $\tau$ increases through $\tau_{0}^{*}$, a periodic solution caused by the phenomenon of Hopf bifurcation occurs; that is, a family of periodic solutions bifurcate from the interior equilibrium $\widetilde{P}_{1}^{*}$. Since $\delta_{2}>0$ and $\gamma_{2}<0$, the Hopf bifurcation is supercritical, the directions of the Hopf bifurcation is $\tau>$ $\tau_{0}^{*}$, and these bifurcating periodic solutions from the interior equilibrium $\widetilde{P}_{1}^{*}$ at $\tau_{0}^{*}$ are stable. Dynamical responses of model (4) with $\tau=3>\tau_{0}^{*}$ are plotted in Figure 5. Figures 6 and 7 show a limit cycle corresponding to the periodic solution in Figure 5 in the $S-I-R$ and $S-I-E$ space, respectively.

## 6. Conclusion

It is well known that the recovered host individuals are naturally immune to vector disease [1], and its potential economic interest can be commercially exploited. Furthermore, harvest effort is usually influenced by variation of economic interest under market economy. Consequently, it is necessary to discuss the coexistence and interaction mechanism of population within harvested epidemiological ecosystem as well as dynamical effect of harvest effort due to variation of economic interest.

By introducing commercial harvest effort into model proposed in [11], a delayed hybrid mathematical model is established to investigate the dynamical effect of commercial harvesting and incubation time delay on epidemiological economic system, which extends the work done in [11] from a bioeconomic perspective. Positivity and persistence of solutions of model system are discussed in Theorems 3 and 10 , respectively. The economic interest of commercial harvesting should be restricted within certain interval that guarantees the existence of interior equilibrium, which can be found in Remark 16. Since the interior equilibrium biologically interprets that susceptible, infective, and recovered host individuals survive as well as harvest on recovered host individuals exist, the bifurcation phenomena around the interior equilibria can reveal the instability mechanism of model system, which are theoretically relevant to infectious disease control and sustainable yield on recovered host individuals. Consequently, we will mainly concentrate on dynamical behavior and local stability switch around interior equilibrium of model system (4) in this paper. As analyzed in Theorem 11 of this paper, a singularity induced bifurcation occurs which leads to local stability switch in the case of positive economic interest of harvesting. In the perspective of practical viewpoint, a direct damage done by the singularity induced bifurcation to the proposed model is impulse phenomenon, which may lead to outbreak of infectious disease and hamper prosperous harvesting on recovered host individual population resource in the harvested ecosystem, which can be found in Remark 13. With the purpose of maintaining the economic interest at an ideal level, state feedback controllers are designed to stabilize model system around the desirable interior equilibria in the case of zero economic interest and positive economic interest, respectively. The design of the state feedback controller can be found in Theorems 14 and 15 of this paper. The theoretical results and numerical simulations obtained in this paper suggest that incorporating harvest effort on recovered host individuals can not only prevent the stability switch of model system, but also drive model system to stable equilibrium, which will contribute to the persistence and sustainable yield of the harvested ecosystem.

Further attempts are made to understand the dynamical effect of incubation time delay and economic interest on local stability of model system around interior equilibrium. Local stability analysis reveals that incubation delay is responsible for local stability switch of the proposed model, and a family of periodic solutions bifurcate from the interior equilibrium which occurs as incubation delay increases through a critical threshold, which can be found in Theorem 17. The direction and stability of Hopf bifurcation are also discussed in Theorem 19 of this paper, which reveals that Hopf bifurcation is supercritical, the directions of Hopf bifurcation is $\tau>\tau_{0}^{*}$, and these bifurcating periodic solutions from the interior equilibrium are stable. The model proposed in [11] does not discuss the harvest effort on economic population. For the model proposed in [11], the threshold value of incubation delay where Hopf bifurcation occurs in [11] is $\widehat{\tau}_{0}=2.0842$. However, the harvest effort on recovered host individuals is considered in this paper. As calculated in Section 5.2 of this
paper, the Hopf bifurcation occurs at $\tau_{0}^{*}=2.7814$ in the case of positive economic interest. It is obvious that $\tau_{0}^{*}>\widehat{\tau}_{0}$, which implies that the harvesting has a stabilizing impact on the dynamical behavior of population dynamics; cyclic behavior caused by incubation delay can be deferred by introduction of commercial harvesting effort.

It should be noted that some hybrid dynamical models are proposed in [23-28], which are utilized to discuss the interaction mechanism of harvested ecosystem from an economic perspective in recent years. Complex dynamical behavior and stability analysis in prey-predator ecosystems with stagestructured population and gestation delay are considered. However, as far as knowledge goes, nobody has explicitly proposed a mathematical model to discuss the dynamic effect of commercial harvest on epidemiological system under the market economy environment. The main objective of this paper is to investigate the transmission mechanism of infectious disease and dynamical effect of commercial harvest on population dynamics, especially the complex dynamical behavior and stability switch due to variation of incubation and commercial harvest economic interest, which makes the work studied in this paper has some new and positive features.

## Conflict of Interests

All authors of this paper declare that there is no conflict of interests regarding the publication of this paper. They have no proprietary, financial, professional, or other personal interest of any nature or kind in any product, service, and/or company that could be construed as influencing the position presented in, or the review of, this paper.

## Acknowledgments

This work is supported by National Natural Science Foundation of China, Grant no. 61104003, Grant no. 61273008, and Grant no. 61104093; Research Foundation for Doctoral Program of Higher Education of Education Ministry, Grant no. 20110042120016; Hebei Province Natural Science Foundation, Grant no. F2011501023; Fundamental Research Funds for the Central Universities, Grant no. N120423009; and Research Foundation for Science and Technology Pillar Program of Northeastern University at Qinhuangdao, Grant no. XNK201301. This work is supported by State Key Laboratory of Integrated Automation of Process Industry, Northeastern University and supported by Hong Kong Admission Scheme for Mainland Talents and Professionals, Hong Kong Special Administrative Region.

## References

[1] R. M. Anderson and R. M. May, "Population biology of infectious diseases: part I," Nature, vol. 280, no. 5721, pp. 361367, 1979.
[2] S. A. Levin, T. G. Hallam, and J. J. Gross, Applied Mathematical Ecology, Springer, New York, NY, USA, 1990.
[3] H. W. Hethcote, "The mathematics of infectious diseases," SIAM Review, vol. 42, no. 4, pp. 599-653, 2000.
[4] Z. E. Ma, Y. Zhou, W. Wang, and Z. Jin, Mathematical Models and Dynamics OfInfectious Disease, Science Press, Beijing, China, 2004.
[5] K. L. Cooke, "Stability analysis for a vector disease model," The Rocky Mountain Journal of Mathematics, vol. 9, no. 1, pp. 3142, 1979, Conference on Deterministic Differential Equations and Stochastic Processes Models for Biological Systems (San Cristobal, N.M., 1977).
[6] W. M. Liu, H. W. Hethcote, and S. A. Levin, "Dynamical behavior of epidemiological models with nonlinear incidence rates," Journal of Mathematical Biology, vol. 25, no. 4, pp. 359380, 1987.
[7] Y. Takeuchi, W. Ma, and E. Beretta, "Global asymptotic properties of a delay SIR epidemic model with finite incubation times," Nonlinear Analysis: Theory, Methods \& Applications, vol. 42, no. 6, pp. 931-947, 2000.
[8] E. Beretta, T. Hara, W. Ma, and Y. Takeuchi, "Global asymptotic stability of an SIR epidemic model with distributed time delay," Nonlinear Analysis, vol. 47, pp. 4107-4115, 2001.
[9] C. C. McCluskey, "Global stability of an SIR epidemic model with delay and general nonlinear incidence," Mathematical Biosciences and Engineering, vol. 7, no. 4, pp. 837-850, 2010.
[10] X. Zhou and J. Cui, "Stability and Hopf bifurcation of a delay eco-epidemiological model with nonlinear incidence rate," Mathematical Modelling and Analysis, vol. 15, no. 4, pp. 547-569, 2010.
[11] J.-J. Wang, J.-Z. Zhang, and Z. Jin, "Analysis of an SIR model with bilinear incidence rate," Nonlinear Analysis: Real World Applications, vol. 11, no. 4, pp. 2390-2402, 2010.
[12] Y. Enatsu, E. Messina, Y. Muroya, Y. Nakata, E. Russo, and A. Vecchio, "Stability analysis of delayed SIR epidemic models with a class of nonlinear incidence rates," Applied Mathematics and Computation, vol. 218, no. 9, pp. 5327-5336, 2012.
[13] S. Ruan and W. Wang, "Dynamical behavior of an epidemic model with a nonlinear incidence rate," Journal of Differential Equations, vol. 188, no. 1, pp. 135-163, 2003.
[14] W. Ma, M. Song, and Y. Takeuchi, "Global stability of an SIR epidemic model with time delay," Applied Mathematics Letters, vol. 17, no. 10, pp. 1141-1145, 2004.
[15] R. Xu and Z. Ma, "Global stability of a SIR epidemic model with nonlinear incidence rate and time delay", Nonlinear Analysis: Real World Applications, vol. 10, no. 5, pp. 3175-3189, 2009.
[16] C. W. Clark, Mathematical Bioeconomics: The Optimal Management of Renewable Resource, John Wiley \& Sons, New York, NY, USA, 2nd edition, 1990.
[17] P. W. Dong, S. Y. Zhuang, X. H. Lin, and X. Z. Zhang, "Economic evaluation of forestay industry based on ecosystem coupling," Mathematical and Computer Modelling, vol. 58, no. 6, pp. 10101017, 2013.
[18] N. Bairagi, S. Chaudhuri, and J. Chattopadhyay, "Harvesting as a disease control measure in an eco-epidemiological system-a theoretical study," Mathematical Biosciences, vol. 217, no. 2, pp. 134-144, 2009.
[19] S. Chakraborty, S. Pal, and N. Bairagi, "Dynamics of a ratiodependent eco-epidemiological system with prey harvesting," Nonlinear Analysis: Real World Applications, vol. 11, no. 3, pp. 1862-1877, 2010.
[20] R. Bhattacharyya and B. Mukhopadhyay, "On an eco-epidemiological model with prey harvesting and predator switching: local and global perspectives," Nonlinear Analysis: Real World Applications, vol. 11, no. 5, pp. 3824-3833, 2010.
[21] L. Zou, Z. Xiong, and Z. Shu, "The dynamics of an eco-epidemic model with distributed time delay and impulsive control strategy," Journal of the Franklin Institute, vol. 348, no. 9, pp. 2332-2349, 2011.
[22] H. S. Gordon, "The economic theory of a common property resource: the fishery," Journal of Political Economy, vol. 62, no. 2, pp. 124-142, 1954.
[23] C. Liu, Q. Zhang, Y. Zhang, and X. Duan, "Bifurcation and control in a differential-algebraic harvested prey-predator model with stage structure for predator," International Journal of Bifurcation and Chaos in Applied Sciences and Engineering, vol. 18, no. 10, pp. 3159-3168, 2008.
[24] C. Liu, Q. Zhang, X. Zhang, and X. Duan, "Dynamical behavior in a stage-structured differential-algebraic prey-predator model with discrete time delay and harvesting," Journal of Computational and Applied Mathematics, vol. 231, no. 2, pp. 612-625, 2009.
[25] X. Zhang, Q.-L. Zhang, C. Liu, and Z.-Y. Xiang, "Bifurcations of a singular prey-predator economic model with time delay and stage structure," Chaos, Solitons \& Fractals, vol. 42, no. 3, pp. 1485-1494, 2009.
[26] G. Zhang, B. Chen, L. Zhu, and Y. Shen, "Hopf bifurcation for a differential-algebraic biological economic system with time delay," Applied Mathematics and Computation, vol. 218, no. 15, pp. 7717-7726, 2012.
[27] K. Chakraborty, M. Chakraborty, and T. K. Kar, "Bifurcation and control of a bioeconomic model of a prey-predator system with a time delay," Nonlinear Analysis: Hybrid Systems, vol. 5, no. 4, pp. 613-625, 2011.
[28] C. Liu, Q. Zhang, J. Huang, and W. Tang, "Dynamical analysis and control in a delayed differential-algebraic bio-economic model with stage structure and diffusion," International Journal of Biomathematics, vol. 5, no. 2, pp. 1-30, 2012.
[29] S. Campbell, Singular Systems of Differential Equations, Priman, London, UK, 1980.
[30] P. Muller, "Linear mechanical descriptor systems: identification, analysis and design," in Proceedings of the IFAC International Conference on Control of Industrial Systems, pp. 501-506, Belfort, France, May 1997.
[31] M. S. Silva and T. P. de Lima, "Looking for nonnegative solutions of a Leontief dynamic model," Linear Algebra and its Applications, vol. 364, no. 1, pp. 281-316, 2003.
[32] S. Ayasun, C. O. Nwankpa, and H. G. Kwatny, "Computation of singular and singularity induced bifurcation points of differential-algebraic power system model," IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 51, no. 8, pp. 15251538, 2004.
[33] M. Yue and R. Schlueter, "Bifurcation subsystem and its application in power system analysis," IEEE Transactions on Power Systems, vol. 19, no. 4, pp. 1885-1893, 2004.
[34] W. Marszalek and Z. W. Trzaska, "Singularity-induced bifurcations in electrical power systems," IEEE Transactions on Power Systems, vol. 20, no. 1, pp. 312-320, 2005.
[35] D. G. Luenberger, "Nonlinear descriptor systems," Journal of Economic Dynamics \& Control, vol. 1, no. 3, pp. 219-242, 1979.
[36] D. G. Luenberger and A. Arbel, "Singular dynamic Leontief systems," Econometrics, vol. 45, no. 32, pp. 991-995, 1997.
[37] X. Yang, L. Chen, and J. Chen, "Permanence and positive periodic solution for the single-species nonautonomous delay diffusive models," Computers \& Mathematics with Applications, vol. 32, no. 4, pp. 109-116, 1996.
[38] H. R. Thieme, Mathematics in Population Biology, Princeton University Press, Princeton, NJ, USA, 2003.
[39] J. K. Hale and S. M. Verduyn Lunel, Introduction to FunctionalDifferential Equations, vol. 99, Springer, New York, NY, USA, 1993.
[40] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, vol. 191, Academic Press, New York, NY, USA, 1993.
[41] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, vol. 42, Springer, New York, NY, USA, 1983.
[42] V. Venkatasubramanian, H. Schättler, and J. Zaborszky, "Local bifurcations and feasibility regions in differential-algebraic systems," IEEE Transactions on Automatic Control, vol. 40, no. 12, pp. 1992-2013, 1995.
[43] M. Kot, Elements of Mathematical Ecology, Cambridge University Press, Cambridge, UK, 2001.
[44] V. Venkatasubramanian, "Singularity induced bifurcation and the van der Pol oscillator," IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, vol. 41, no. 11, pp. 765-769, 1994.
[45] R. E. Beardmore, "The singularity-induced bifurcation and its Kronecker normal form," SIAM Journal on Matrix Analysis and Applications, vol. 23, no. 1, pp. 126-137, 2001.
[46] L. Yang and Y. Tang, "An improved version of the singularityinduced bifurcation theorem," IEEE Transactions on Automatic Control, vol. 46, no. 9, pp. 1483-1486, 2001.
[47] L. Dai, Singular Control Systems, vol. 118, Springer, New York, NY, USA, 1989.
[48] H. I. Freedman and V. Sree Hari Rao, "The trade-off between mutual interference and time lags in predator-prey systems," Bulletin of Mathematical Biology, vol. 45, no. 6, pp. 991-1004, 1983.
[49] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, Theory and Applications of Hopf Bifurcation, vol. 41, Cambridge University Press, Cambridge, UK, 1981.

## Research Article

# A Stochastic Predator-Prey System with Stage Structure for Predator 

Shufen Zhao ${ }^{1,2}$ and Minghui Song ${ }^{1}$<br>${ }^{1}$ Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China<br>${ }^{2}$ Department of Mathematics, Zhaotong University, Zhaotong 657000, China

Correspondence should be addressed to Shufen Zhao; zhaoshufen@gmail.com
Received 20 February 2014; Accepted 8 April 2014; Published 30 April 2014
Academic Editor: Weiming Wang
Copyright © 2014 S. Zhao and M. Song. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The authors introduce stochasticity into a predator-prey system with Beddington-DeAngelis functional response and stage structure for predator. We present the global existence and positivity of the solution and give sufficient conditions for the global stability in probability of the system. Numerical simulations are introduced to support the main theoretical results.

## 1. Introduction

The classical predator-prey model with BeddingtonDeAngelis type functional response can be denoted as

$$
\begin{gather*}
\dot{x}(t)=x\left[b_{1}-a_{11} x-\frac{a_{12} y}{1+m x+n y}\right]  \tag{1}\\
\dot{y}=y\left[-b_{2}+\frac{a_{21} x}{1+m x+n y}-a_{22} y\right]
\end{gather*}
$$

where $x(t)$ and $y(t)$ represent predator and prey densities at time $t$, respectively. $b_{i}, a_{i j}, m$, and $n$ are positive constants, $i$, $j=1,2$. For biological representation of each coefficient in (1) we refer the reader to [1, 2]. In model (1), it is assumed that all individuals of a single species have largely similar capabilities to hunt or to reproduce. But the life cycle of most animals consists of at least two stages, immature and mature, and the individuals in the first stage often can neither hunt or reproduce, being raised by their mature parents and there are recognizable morphological and behavioral differences that may exist between these stages. In [3], the authors studied the global properties of a predator-prey model with nonlinear functional response and stage structure for the predator, and the condition of the existence and the global stability of the positive steady states were established. However, May [4] pointed out that due to environment noise, the birth
rate, carrying capacity, competition coefficients, and other parameters involved in the system exhibit random fluctuation to a greater or lesser extent. So, a lot of authors introduced stochastic noise into deterministic models to reveal the effect of environmental variability on the population dynamics in mathematical ecology [5-7]. In Liu and Wang [5], the authors investigated the global stability of stage-structured predatorprey models with Beddington-DeAngelis type functional response and with stage structure for the prey. The authors [5] also pointed out that there are some technical obstacles that cannot be overcome at present to investigate the stage structure on predator model. So, in this paper we are going to do some work on this problem. The following model,

$$
\begin{gather*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=x\left(\gamma-\frac{q y_{2}}{1+m x+n y_{2}}-\beta x\right) \\
\frac{\mathrm{d} y_{1}}{\mathrm{~d} t}=\alpha y_{2}-d_{1} y_{1}-a_{1} y_{1}^{2}-b y_{1}  \tag{2}\\
\frac{\mathrm{~d} y_{2}}{\mathrm{~d} t}=b y_{1}-d_{2} y_{2}-a_{2} y_{2}^{2}+\frac{p y_{2} x}{1+m x+n y_{2}}
\end{gather*}
$$

is derived under the following assumptions.
(H1) The immature predator population $y_{1}$ : the birth rate into the immature population is proportional to
the existing mature predator population with probability $\alpha>0$; the death rate is proportional to the existing immature predator population with proportionality $d_{1}>0$; overcrowding rate of the immature predator population is $a_{1}>0$; the transformation rate from the immature predator to mature predator is proportional to the existing immature predator population with proportionality $b>0$.
(H2) The mature predator population $y_{2}: d_{2}>0$ and $a_{2}>0$ are the death rate and the overcrowding rate of the mature predator population, respectively, and only the mature predator population feeds the prey. It seems reasonable that a number of mammals, who are immature predators, are raised by their parents. $p / q$ is the rate of conversion of nutrients into the reproduction of the predator.
(H3) The prey population $x$ : the growth of the species is of the Lotka-Volterra nature and $\gamma>0$ is the birth rate; $\beta>0$ is the overcrowding rate. $q>0$ is the capturing rate of the predator.
System (2) is greatly different from the model investigated in [3] for we comprehend that the effect of the response function will diminish the death rate of the predator and the predator does not only feed on the prey.

Suppose that $\left(x^{*}, y_{1}^{*}, y_{2}^{*}\right)$ is a positive equilibrium of (2). If we take the environmental noise into account, we can replace the birth rate of prey population and death rate of predator population by an average value plus a random fluctuation, respectively,

$$
\begin{gather*}
\gamma+\sigma\left(x-x^{*}\right) \dot{B}(t) \\
d_{1}+\sigma_{1}\left(y_{1}-y_{1}^{*}\right) \dot{B}_{1}(t)  \tag{3}\\
d_{2}+\sigma_{2}\left(y_{2}-y_{2}^{*}\right) \dot{B}_{2}(t)
\end{gather*}
$$

where $\sigma^{2}, \sigma_{1}^{2}, \sigma_{2}^{2}$ represent the intensities of the noise and $\dot{B}(t)$, $\dot{B}_{1}(t), \dot{B}_{2}(t)$ are standard white noise; namely, $B(t), B_{1}(t), B_{2}(t)$ are standard Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, P)$ with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \in \mathbf{R}_{+}}$satisfying the usual condition (i.e., it is right continuous and increasing while $\mathscr{F}_{0}$ contains all $P$-null sets). So the corresponding stochastic system of (2) is

$$
\begin{align*}
\mathrm{d} x= & x\left(\gamma-\frac{q y_{2}}{1+m x+n y_{2}}-\beta x\right) \mathrm{d} t \\
& +\sigma x\left(x-x^{*}\right) \mathrm{d} B(t) \\
\mathrm{d} y_{1}= & \left(\alpha y_{2}-d_{1} y_{1}-a_{1} y_{1}^{2}-b y_{1}\right) \mathrm{d} t  \tag{4}\\
& +\sigma_{1} y_{1}\left(y_{1}-y_{1}^{*}\right) \mathrm{d} B_{1}(t) \\
\mathrm{d} y_{2}= & \left(b y_{1}-d_{2} y_{2}-a_{2} y_{2}^{2}+\frac{p x y_{2}}{1+m x+n y_{2}}\right) \mathrm{d} t \\
& -\sigma_{2} y_{2}\left(y_{2}-y_{2}^{*}\right) \mathrm{d} B_{2}(t)
\end{align*}
$$

with the initial condition $\left(x(0), y_{1}(0), y_{2}(0)\right) \in \mathbf{R}_{+}^{3}$ where $\mathbf{R}_{+}^{3}=\left\{x \in \mathbf{R}^{3} \mid x, y_{1}, y_{2}>0, i=1,2,3\right\}$.

The paper is organized as follows. In Section 2, we prove the existence, uniqueness, and the positivity of the solution to (4). In Section 3, we established the condition for the global stability of the positive equilibrium. We work out two simulation figures to illustrate our main results in Section 4. Section 5 gives the conclusions and future directions.

## 2. Existence of the Global Positive Solution

Theorem 1. For any initial value $\left(x(0), y_{1}(0), y_{2}(0)\right) \in \mathbf{R}_{+}^{3}$, system (4) has a unique global positive solution $\left(x(t), y_{1}(t), y_{2}(t)\right)$ on $t>0$ with probability one.

Proof. We see that the coefficients of the system are locally Lipschitz continuous, so, for any given initial values $x(0)>0$, $y_{1}(0)>0, y_{2}(0)>0$, there is a unique maximal local solution $\left(x(t), y_{1}(t), y_{2}(t)\right)$ on $t \in\left[0, \tau_{e}\right)$, where $\tau_{e}$ is the explosion time [8]. To show this solution is global, we need to show that $\tau_{e}=$ $\infty$. Define the stopping time by

$$
\begin{gather*}
\tau_{k}=\inf \left\{t \in(0, \infty), y_{i}(t)=\frac{1}{k}\right. \\
\text { or } y_{i}(t)=k, \quad i=1,2,  \tag{5}\\
\\
\text { or } \left.x(t)=\frac{1}{k} \text { or } x(t)=k\right\},
\end{gather*}
$$

where, throughout this paper, we set $\inf \emptyset=\infty$. Clearly, $\tau_{k}$ is increasing as $k \rightarrow \infty$. Set $\tau_{\infty}=\lim _{k \rightarrow \infty} \tau_{k}$, whence $\tau_{\infty} \leq \tau_{e}$ a.s. If we can show that $\tau_{\infty}=\infty$ a.s., then $\tau_{e}=\infty$ a.s. Namely, to complete the proof, it is sufficient to show that $\tau_{\infty}=\infty$ a.s. If this statement is false, then there is a pair of constants $T>0$ and $\epsilon \in(0,1)$, such that $P\left\{\tau_{\infty} \leq T\right\}>\epsilon$. Hence, there is an integer $k_{1} \geq k_{0}$ such that

$$
\begin{equation*}
P\left\{\tau_{k} \leq T\right\} \geq \epsilon \quad \forall k \geq k_{1} \tag{6}
\end{equation*}
$$

where $k_{0}$ is satisfying $1 / k_{0}<y_{i}(0)<k_{0}, i=1,2$, and $1 / k_{0}<$ $x(0)<k_{0}$.

Define a $C^{2}$-function $V: \mathbf{R}_{+}^{3} \rightarrow \mathbf{R}_{+}$by

$$
\begin{align*}
V\left(x, y_{1}, y_{2}\right)= & (\sqrt{x}-1-0.5 \ln x)+\left(\sqrt{y_{1}}-1-0.5 \ln y_{1}\right) \\
& +\left(\sqrt{y_{2}}-1-0.5 \ln y_{2}\right) . \tag{7}
\end{align*}
$$

Using Itô's formula, we get

$$
\begin{align*}
\mathrm{d} V= & \frac{1}{2}\left(x^{-1 / 2}-x^{-1}\right) \cdot \mathrm{d} x+\frac{1}{2}\left(y_{1}^{-1 / 2}-y_{1}^{-1}\right) \cdot \mathrm{d} y_{1} \\
& +\frac{1}{2}\left(y_{2}^{-1 / 2}-y_{2}^{-1}\right) \cdot \mathrm{d} y_{2} \\
& +\frac{1}{2}\left(-\frac{1}{4} x^{-3 / 2}+\frac{1}{2} x^{-2}\right) \cdot(\mathrm{d} x)^{2}  \tag{8}\\
& +\frac{1}{2}\left(-\frac{1}{4} y_{1}^{-3 / 2}+\frac{1}{2} y_{1}^{-2}\right) \cdot\left(\mathrm{d} y_{1}\right)^{2} \\
& +\frac{1}{2}\left(-\frac{1}{4} y_{2}^{-3 / 2}+\frac{1}{2} y_{2}^{-2}\right) \cdot\left(\mathrm{d} y_{2}\right)^{2}
\end{align*}
$$

$$
\begin{aligned}
L V= & \frac{1}{2}\left(x^{1 / 2}-1\right)\left(\gamma-\frac{q y_{2}}{1+m x+n y_{2}}-\beta x\right) \\
& +\frac{1}{2}\left(-\frac{1}{4} x^{1 / 2}+\frac{1}{2}\right) \sigma^{2}\left(x-x^{*}\right)^{2} \\
& +\frac{1}{2}\left(y_{1}^{-1 / 2}-y_{1}^{-1}\right)\left(\alpha y_{2}-d_{1} y_{1}-a_{1} y_{1}^{2}-b y_{1}\right) \\
& +\frac{1}{2}\left(-\frac{1}{4} y_{1}^{1 / 2}+\frac{1}{2}\right) \sigma_{1}^{2}\left(y_{1}-y_{1}^{*}\right)^{2} \\
& +\frac{1}{2}\left(y_{2}^{-1 / 2}-y_{2}^{-1}\right) \\
& \times\left(b y_{1}-d_{2} y_{2}-a_{2} y_{2}^{2}+\frac{p x y_{2}}{1+m x+n y_{2}}\right) \\
& +\frac{1}{2}\left(-\frac{1}{4} y_{2}^{1 / 2}+\frac{1}{2}\right) \sigma_{2}^{2}\left(y_{2}-y_{2}^{*}\right)^{2} \\
= & \frac{1}{2}\left(x^{1 / 2}-1\right)\left(\gamma-\frac{q y_{2}}{1+m x+n y_{2}}-\beta x\right) \\
& +\frac{1}{2}\left(y_{2}^{-1 / 2}-y_{2}^{-1}\right) b y_{1}+\frac{1}{2}\left(y_{1}^{-1 / 2}-y_{1}^{-1}\right) \alpha y_{2} \\
& +\frac{1}{2}\left(y_{1}^{1 / 2}-1\right)\left(-d_{1}-a_{1} y_{1}-b\right) \\
& +\frac{1}{2}\left(y_{2}^{1 / 2}-1\right)\left(-d_{2}-a_{2} y_{2}+\frac{p y_{2}}{1+m x+n y_{2}}\right) \\
& +\frac{1}{2}\left(-\frac{1}{4} x^{1 / 2}+\frac{1}{2}\right) \sigma^{2}\left(x-x^{*}\right)^{2} \\
& +\frac{1}{2}\left(-\frac{1}{4} y_{1}^{1 / 2}+\frac{1}{2}\right) \sigma_{1}^{2}\left(y_{1}-y_{1}^{*}\right)^{2} \\
& +\frac{1}{2}\left(-\frac{1}{4} y_{2}^{1 / 2}+\frac{1}{2}\right) \sigma_{2}^{2}\left(y_{2}-y_{2}^{*}\right)^{2} .
\end{aligned}
$$

Now, we pay attention to the term $(1 / 2)\left(y_{1}^{-1 / 2}-y_{1}^{-1}\right) \alpha y_{2}$. If $y_{1}<1$, then $(1 / 2)\left(y_{1}^{-1 / 2}-y_{1}^{-1}\right) \alpha y_{2}<0$, so, this term can be omitted from the right side of the inequality. If $y_{1}>1$, then $(1 / 2)\left(y_{1}^{-1 / 2}-y_{1}^{-1}\right) \alpha y_{2}<(1 / 2)\left(y_{1}^{1 / 2}-1\right) \alpha y_{2}$. The similar argument can be taken on $(1 / 2)\left(y_{2}^{-1 / 2}-y_{2}^{-1}\right) b y_{1}$. So we get the following inequality:

$$
\begin{aligned}
L V \leq & \frac{1}{2}\left[-\beta x^{3 / 2}+\beta x+\gamma x^{1 / 2}+\frac{q}{n}-\gamma\right] \\
& +\frac{b}{2} y_{1}\left(\sqrt{y_{2}}-1\right)+\frac{\alpha}{2} y_{2}\left(\sqrt{y_{1}}-1\right) \\
& +\frac{1}{2}\left[-a_{1} y_{1}^{3 / 2}-\left(d_{1}+b\right) y_{1}^{1 / 2}+a_{1} y_{1}+d_{1}+b\right] \\
& +\frac{1}{2}\left[-a_{2} y_{2}^{3 / 2}+\left(\frac{p}{n}-d\right) y_{2}^{1 / 2}+a_{2} y_{2}+d_{2}\right]
\end{aligned}
$$

$$
\begin{gathered}
+\frac{1}{2} \sigma^{2}\left(-\frac{1}{4} x^{5 / 2}+\frac{1}{2} x^{2}+\frac{1}{2} x^{*} x^{3 / 2}\right. \\
\left.-x^{*} x-\frac{1}{4} x_{2}^{* 2} x^{1 / 2}+\frac{1}{2} x_{2}^{* 2}\right) \\
+\frac{1}{2} \sigma_{1}^{2}\left(-\frac{1}{4} y_{1}^{5 / 2}+\frac{1}{2} y_{1}^{2}+\frac{1}{2} y_{1}^{*} y_{1}^{3 / 2}\right. \\
\left.-y_{1}^{*} y_{1}-\frac{1}{4} y_{1}^{* 2} y_{1}^{1 / 2}+\frac{1}{2} y_{1}^{* 2}\right) \\
+\frac{1}{2} \sigma_{2}^{2}\left(-\frac{1}{4} y_{2}^{5 / 2}+\frac{1}{2} y_{2}^{2}+\frac{1}{2} y_{2}^{*} y_{2}^{3 / 2}\right. \\
\left.-y_{2}^{*} y_{2}-\frac{1}{4} y_{2}^{* 2} y_{2}^{1 / 2}+\frac{1}{2} y_{2}^{* 2}\right) \\
\leq \frac{1}{2}\left[-\beta x^{3 / 2}+\beta x+\gamma x^{1 / 2}+\frac{q}{n}\right] \\
+\frac{1}{2}\left[\frac{b}{2} y_{1}^{2}-a_{1} y_{1}^{3 / 2}-\left(d_{1}+b\right) y_{1}^{1 / 2}\right. \\
\left.+\left(a_{1}-b+\frac{\alpha}{2}\right) y_{1}+d_{1}+b\right] \\
+\frac{1}{2}\left[\frac{\alpha}{2} y_{2}^{2}-a_{2} y_{2}^{3 / 2}+\left(\frac{p}{n}-d\right) y_{2}^{1 / 2}\right. \\
\left.+\left(a_{2}-\alpha+\frac{b}{2}\right) y_{2}+d_{2}\right] \\
+\frac{1}{2} \sigma^{2}\left(-\frac{1}{4} x^{5 / 2}+\frac{1}{2} x^{2}+\frac{1}{2} x^{*} x^{3 / 2}\right. \\
\left.\quad-x^{*} x-\frac{1}{4} x_{2}^{* 2} x^{1 / 2}+\frac{1}{2} x_{2}^{* 2}\right) \\
+\frac{1}{2} \sigma_{1}^{2}\left(-\frac{1}{4} y_{1}^{5 / 2}+\frac{1}{2} y_{1}^{2}+\frac{1}{2} y_{1}^{2} y_{1}^{3 / 2}\right. \\
\left.\quad-y_{1}^{*} y_{1}-\frac{1}{4} y_{1}^{* 2} y_{1}^{1 / 2}+\frac{1}{4} y_{2}^{5 / 2}+\frac{1}{2} y_{2}^{2}+\frac{1}{2} y_{2}^{* 2} y_{2}^{3 / 2}\right) \\
\left.\quad-y_{2}^{*} y_{2}-\frac{1}{4} y_{2}^{* 2} y_{2}^{1 / 2}+\frac{1}{2} y_{2}^{* 2}\right)
\end{gathered}
$$

$$
\begin{equation*}
\leq K \tag{10}
\end{equation*}
$$

where $K$ is positive constant. Integrating both sides of (8) from 0 to $\tau_{k} \wedge T$ and then taking the expectations lead to

$$
\begin{align*}
& V\left(x\left(\tau_{k} \wedge T\right), y_{1}\left(\tau_{k} \wedge T\right), y_{2}\left(\tau_{k} \wedge T\right)\right) \\
& \quad \leq V\left(x(0), x_{1}(0), y_{2}(0)\right)+K T . \tag{11}
\end{align*}
$$

Setting $\Omega_{k}=P\left\{\tau_{k} \leq T\right\}$ for $k \geq k_{1}$, then by the inequality (6) we have $P\left(\Omega_{k}\right) \geq \epsilon$; note that for every $\omega \in \Omega_{k}$, $V\left(x\left(\tau_{k}, \omega\right), y_{1}\left(\tau_{k}, \omega\right), y_{2}\left(\tau_{k}, \omega\right)\right)$ is no less than $\min \{\sqrt{k}-1-$ $0.5 \ln k, \sqrt{1 / k}-1-0.5 \ln (1 / k)\}$.

It then follows from (11) that

$$
\begin{align*}
& E\left[I_{\Omega_{k}} V\left(x\left(\tau_{k}\right), x_{1}\left(\tau_{k}\right), y_{2}\left(\tau_{k}\right)\right)\right] \\
& \quad \leq V\left(x(0), y_{1}(0), y_{2}(0)\right)+K T \tag{12}
\end{align*}
$$

where $I_{\Omega_{k}}$ is the indicator function of $\Omega_{k}$ and $E\left[I_{\Omega_{k}} V\left(x\left(\tau_{k}\right), y_{1}\left(\tau_{k}\right), y_{2}\left(\tau_{k}\right)\right)\right] \geq \epsilon \min \{\sqrt{k}-1-0.5 \ln k$, $\sqrt{1 / k}-1-0.5 \ln (1 / k)\}$. So,

$$
\begin{gather*}
\epsilon \min \left\{\sqrt{k}-1-0.5 \ln k, \sqrt{\frac{1}{k}}-1-0.5 \ln \frac{1}{k}\right\}  \tag{13}\\
\leq V\left(x, y_{1}(0)(0), y_{2}(0)\right)+K T .
\end{gather*}
$$

Letting $k \rightarrow \infty$ leads to the contradiction

$$
\begin{equation*}
\infty>V\left(x(0), y_{1}(0), y_{2}(0)\right)+K T \geq \infty . \tag{14}
\end{equation*}
$$

This contradiction shows that $\tau_{\infty}=\infty$, which completes the proof.

## 3. Global Behavior

Suppose $z=z(t)$ is the solution of the following $n$ dimensional stochastic differential equation:

$$
\begin{equation*}
\mathrm{d} z(t)=f(z(t), t) \mathrm{d} t+g(z(t), t) \mathrm{d} B(t) \tag{15}
\end{equation*}
$$

and $z^{*}$ is the equilibrium position of (15).
From the stability theory of stochastic differential equations, we only need to find a Lyapunov function $V(z)$ satisfying $L V(z) \leq 0$ and the identity holds if and only if $z=z^{*}$ [9], where $z=z(t)$ is the solution of the $n$-dimensional stochastic differential equation (15) and $d V(x(t), t)=L V d t+$ $V_{x}(x(t), t) g(t) d B(t)$.

Theorem 2. If $a_{1}-\left(\sigma_{1}^{2} / 2\right)>0, a_{2}-\left(\sigma_{2}^{2} / 2\right)>0$, and $\beta-\left(\sigma^{2} / 2\right)-$ $\left(q m y_{2}^{*} /\left(1+m x^{*}+n y_{2}^{*}\right)\right)>0$, then the positive equilibrium $\left(x^{*}, y_{1}^{*}, y_{2}^{*}\right)$ of model (4) is globally asymptotically stable with probability one.

Proof. System (4) can be rewritten as

$$
\begin{aligned}
\mathrm{d} x= & \frac{q x\left[m y_{2}^{*}\left(x-x^{*}\right)-\left(1+m x^{*}\right)\left(y_{2}-y_{2}^{*}\right)\right]}{\left(1+m x+n y_{2}\right)\left(1+m x^{*}+n y_{2}^{*}\right)} \mathrm{d} t \\
& -\beta x\left(x-x^{*}\right) \mathrm{d} t+\sigma x\left(x-x^{*}\right) \mathrm{d} B(t), \\
\mathrm{d} y_{1}= & \frac{\alpha}{y_{1}^{*}}\left[y_{1}\left(y_{2}-y_{2}^{*}\right)-y_{2}\left(y_{1}-y_{1}^{*}\right)\right] \mathrm{d} t \\
& -a_{1} y_{1}\left(y_{1}-y_{1}^{*}\right) \mathrm{d} t+\sigma_{1} y_{1}\left(y_{1}-y_{1}^{*}\right) \mathrm{d} B_{1}(t), \\
\mathrm{d} y_{2}= & \frac{b}{y_{2}^{*}}\left[y_{2}\left(y_{1}-y_{1}^{*}\right)-y_{1}\left(y_{2}-y_{2}^{*}\right)\right] \mathrm{d} t \\
& +\frac{p y_{2}\left[\left(1+n y_{2}^{*}\right)\left(x-x^{*}\right)-n x^{*}\left(y_{2}-y_{2}^{*}\right)\right]}{\left(1+m x+n y_{2}\right)\left(1+m x^{*}+n y_{2}^{*}\right)} \mathrm{d} t \\
& -a_{2} y_{2}\left(y_{2}-y_{2}^{*}\right) \mathrm{d} t+\sigma_{2} y_{2}\left(y_{2}-y_{2}^{*}\right) \mathrm{d} B_{2}(t) .
\end{aligned}
$$

Define

$$
\begin{align*}
V\left(x_{1}, x_{2}, y\right)= & c_{1}\left(x-x^{*}-x^{*} \ln \left(\frac{x}{x^{*}}\right)\right) \\
& +c_{2}\left(y_{1}-y_{1}^{*}-y_{1}^{*} \ln \left(\frac{y_{1}}{y_{1}^{*}}\right)\right)  \tag{17}\\
& +c_{3}\left(y_{2}-1-y_{2}^{*} \ln \frac{y_{2}}{y_{2}^{*}}\right)
\end{align*}
$$

where $c_{i}(i=1,2,3)$ are positive numbers to be determined. Applying Itô's formula to system (16) gives

$$
\begin{align*}
L V= & c_{2}\left\{\left(y_{1}-y_{1}^{*}\right) \frac{\alpha}{y_{1}^{*}}\left[\left(y_{2}-y_{2}^{*}\right)-\frac{y_{2}}{y_{1}}\left(y_{1}-y_{1}^{*}\right)\right]\right. \\
& \left.-a_{1}\left(y_{1}-y_{1}^{*}\right)^{2}+\frac{\sigma_{1}^{2}}{2}\left(y_{1}-y_{1}^{*}\right)^{2}\right\} \\
+ & c_{3}\left\{\left(y_{2}-y_{2}^{*}\right) \frac{b}{y_{2}^{*}}\left[\left(y_{1}-y_{1}^{*}\right)-\frac{y_{1}}{y_{2}}\left(y_{2}-y_{2}^{*}\right)\right]\right. \\
& \left.\quad-a_{2}\left(y_{2}-y_{2}^{*}\right)^{2}+\frac{\sigma_{2}^{2}}{2}\left(y_{2}-y_{2}^{*}\right)^{2}\right\} \\
+ & c_{3} p\left(y_{2}-y_{2}^{*}\right) \frac{\left(1+n x^{*}\right)\left(x-x^{*}\right)-n x^{*}\left(y_{2}-y_{2}^{*}\right)}{\left(1+m x+n y_{2}\right)\left(1+m x^{*}+n y_{2}^{*}\right)} \\
+ & c_{1} q\left(x-x^{*}\right) \frac{m y_{2}^{*}\left(x-x^{*}\right)-\left(1+m x^{*}\right)\left(y_{2}-y_{2}^{*}\right)}{\left(1+m x+n y_{2}\right)\left(1+m x^{*}+n y_{2}^{*}\right)} \\
& -c_{1} \beta\left(x-x^{*}\right)^{2}+c_{1} \frac{\sigma^{2}}{2}\left(x-x^{*}\right)^{2} . \tag{18}
\end{align*}
$$

Set $c_{2}=y_{1}^{*} / \alpha, c_{3}=y_{2}^{*} / b, c_{1}=c_{3} p\left(1+n x_{2}^{*}\right) / q\left(1+m y^{*}\right)=$ $\left(y_{2}^{*} / b\right)(p / q)\left(\left(1+m x^{*}\right) /\left(1+n y^{*}\right)\right)$. Then we have

$$
\begin{aligned}
L V= & \left\{-\frac{y_{2}}{y_{1}}\left(y_{1}-y_{1}^{*}\right)^{2}+2\left(y_{1}-y_{1}^{*}\right)\left(y_{2}-y_{2}^{*}\right)\right. \\
& \left.-\frac{y_{1}}{y_{2}}\left(y_{2}-y_{2}^{*}\right)^{2}\right\}-c_{2}\left(a_{1}-\frac{\sigma_{1}^{2}}{2}\right)\left(y_{1}-y_{1}^{*}\right)^{2} \\
& -c_{3}\left(a_{2}-\frac{\sigma_{2}^{2}}{2}\right)\left(y_{2}-y_{2}^{*}\right)^{2} \\
& -\frac{c_{3} p n x^{*}\left(y_{2}-y_{2}\right)^{2}}{\left(1+m x+n y_{2}\right)\left(1+m x^{*}+n y_{2}^{*}\right)} \\
& -c_{1}\left(\beta-\frac{\sigma^{2}}{2}\right)\left(x-x^{*}\right)^{2} \\
& +\frac{c_{1} q y_{2}^{*} m\left(x-x^{*}\right)^{2}}{\left(1+m x+n y_{2}\right)\left(1+m x^{*}+n y_{2}^{*}\right)}
\end{aligned}
$$

$$
\begin{align*}
\leq & -\left[\sqrt{\frac{y_{2}}{y_{1}}}\left(y_{1}-y_{1}^{*}\right)-\sqrt{\frac{y_{1}}{y_{2}}}\left(y_{2}-y_{2}^{*}\right)\right]^{2} \\
& -\frac{y_{1}^{*}}{\alpha}\left(a_{1}-\frac{\sigma_{1}^{2}}{2}\right)\left(y_{1}-y_{1}^{*}\right)^{2} \\
& -\left(\frac{y_{2}^{*}}{b} a a_{2}-\frac{y_{2}^{*}}{b} \frac{\sigma_{2}^{2}}{2}\right)\left(y_{2}-y_{2}^{*}\right)^{2}-\frac{y_{2}^{*}}{b} \frac{p}{q} \frac{1+n x^{*}}{1+m y^{*}} \\
& \times\left(\beta-\frac{\sigma^{2}}{2}-\frac{q m y_{2}^{*}}{1+m x^{*}+n y_{2}^{*}}\right)\left(x-x^{*}\right)^{2} \\
\leq & -\frac{y_{1}^{*}}{\alpha}\left(a_{1}-\frac{\sigma_{1}^{2}}{2}\right)\left(y_{1}-y_{1}^{*}\right)^{2}-\frac{y_{2}^{*}}{b}\left(a_{2}-\frac{\sigma_{2}^{2}}{2}\right) \\
& \times\left(y_{2}-y_{2}^{*}\right)^{2}-\frac{y_{2}^{*}}{b} \frac{p}{q} \frac{1+m x^{*}}{1+n y^{*}} \\
& \times\left(\beta-\frac{\sigma^{2}}{2}-\frac{q m y_{2}^{*}}{1+m x^{*}+n y_{2}^{*}}\right)\left(x-x^{*}\right)^{2} . \tag{19}
\end{align*}
$$

The condition in Theorem 2 implies $L V \leq 0$, and the identity holds if and only if $\left(x, y_{1}, y_{2}\right)=\left(x^{*}, y_{1}^{*}, y_{2}^{*}\right)$. By Theorem 2.1 in [9] and the description of that theorem, we get the conclusion.

## 4. Numerical Simulations

In this section, we will use the Euler method and the Milstein method mentioned in [10] to substantiate the analytical findings. For system (4), consider the discretization equations

$$
\begin{aligned}
x^{(k+1)}-x^{(k)}= & x^{(k)}\left(\gamma-\frac{q y_{2}^{(k)}}{1+m x^{(k)}+n y_{2}^{(k)}}-\beta x^{(k)}\right) \\
& +\sigma x^{(k)}\left(x^{(k)}-x^{*}\right) \sqrt{\Delta t} \xi^{(k)} \\
& +\frac{\sigma^{2}}{2}\left(x^{(k)}-x^{*}\right)\left[\left(\xi^{(k)}\right)^{2}-1\right] \\
y_{1}^{(k+1)}-y_{1}^{(k)}= & {\left[\alpha y_{2}^{(k)}-d_{1} y_{1}^{(k)}-a_{1}\left(y_{1}^{(k)}\right)^{2}-b y_{1}^{(k)}\right] \Delta t } \\
& -\sigma_{1} y_{1}^{(k)}\left(y_{1}^{(k)}-y_{1}^{*}\right) \sqrt{\Delta t} \zeta^{(k)} \\
& +\frac{\sigma_{1}^{2}}{2}\left(y_{1}^{(k)}-y_{1}^{*}\right)\left[\left(\zeta^{(k)}\right)^{2}-1\right] \\
y_{2}^{(k+1)}-y_{2}^{(k)}= & {\left[b y_{1}^{(k)}-d_{2} y_{2}^{(k)}-a_{2}\left(y_{2}^{(k)}\right)^{2}\right.} \\
& \left.+\frac{p y_{2}^{(k)} x^{(k)}}{1+m x^{(k)}+n y_{2}^{(k)}}\right] \Delta t
\end{aligned}
$$



Figure 1: The solution of (2) with initial values $x(0)=10, y_{1}(0)=$ $10, y_{2}(0)=12$.

$$
\begin{align*}
& -\sigma_{2} y_{2}^{(k)}\left(y_{2}^{(k)}-y_{2}^{*}\right) \sqrt{\Delta t} \eta^{(k)} \\
& +\frac{\sigma_{2}^{2}}{2}\left(y_{2}^{(k)}-y_{2}^{*}\right)\left[\left(\eta^{(k)}\right)^{2}-1\right] \tag{20}
\end{align*}
$$

where $\xi^{(k)}, \zeta^{(k)}, \eta^{(k)}, k=1,2, \ldots, n$ are the Gaussian random variables which follow $N(0,1)$.

In Figure 1, we show the dynamics of the deterministic model with parameters $\alpha=1.2, d_{1}=0.26, b=0.58, a_{1}=$ $0.01, q=1.5, p=1, \beta=0.375, d_{2}=0.32, a_{2}=0.13, m=1$, $n=0.081, \gamma=5.9, \sigma=\sigma_{1}=\sigma_{2}=0$; then $x^{*}=12.37$, $y_{1}^{*}=13.1, y_{2}^{*}=10.6$.

In Figure 2, we choose the same parameter values as Figure 1 except that $\sigma=0.4, \sigma_{1}=0.45, \sigma_{2}=0.375$, which satisfy the condition in Theorem 2, so Figure 2 clearly supports the conclusion of Theorem 2.

In Figure 3, we choose $\alpha=0.9, d_{1}=0.34, b=0.4, q=0.9$, $p=0.4, \beta=0.15, d_{2}=0.12, a_{1}=0.1, a_{2}=0.6, m=0.8, n=$ $0.5, \gamma=0.75, \sigma=\sigma_{1}=\sigma_{2}=0$; then $x^{*}=3.066, y_{1}=1.157$, $y_{2}=1.1$.; in Figure 4, we choose the same parameter values as Figure 1 except that $\sigma_{1}=0.2 ; \sigma_{2}=0.42 ; \sigma_{3}=0.25$. So the conditions of our theoretical results hold. Obviously, the numerical simulations are indeed confirming our analytical results.

## 5. Discussion

In this paper, a stochastic predator-prey model with stage structure for the predator has been proposed and investigated. We discuss the biological significance of the model and establish sufficient conditions for global asymptotic stability of the model. These results are important because from the biological point of view, a global stable positive equilibrium


Figure 2: The solution of the stochastic model (4) with the same parameters as in Figure 1.


Figure 3: The solution of (2) with initial values $x(0)=3, y_{1}(0)=$ 3.6, $y_{2}(0)=3.2$.
means that the community consisting of two species is a stable biotic community in which all species will coexist. To the best of our knowledge, the present paper is the first attempt to study system (4).

Although we only consider the global stability of the positive equilibrium, some interesting questions deserved investigation, like the stage structure effect on the long term behavior of the system. In fact, in (4) we have supposed that the predator is not only feeding on prey; we can also discuss the case in which the predator feeds on prey only. We want to


Figure 4: The solution of the stochastic model (4) with the same parameters as in Figure 3.
mention that we are unable to give the sufficient conditions under which system (4) or (2) has a positive equilibrium, for there are some technical obstacles that cannot be overcome at present stage. However, the values in Figure 1 show that the system (2) has the positive equilibrium position in some case, and we leave this for future work.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## References

[1] J. R. Beddington, "Mutual interference between parasities and its effect on searching efficiency," Journal of Animal Ecology, vol. 44, pp. 331-341, 1975.
[2] D. L. DeAngelis, R. A. Goldsten, and R. Neil, "A model for trophic interaction," Ecology, vol. 56, pp. 88-92, 1975.
[3] P. Georgescu and Y.-H. Hsieh, "Global dynamics of a predatorprey model with stage structure for the predator," SIAM Journal on Applied Mathematics, vol. 67, no. 5, pp. 1379-1395, 2007.
[4] R. M. May, Stability and Complexity in Model Ecosystems, Princeton University Press, Princeton, NJ, USA, 2001.
[5] M. Liu and K. Wang, "Global stability of stage-structured predator-prey models with Beddington-DeAngelis functional response," Communications in Nonlinear Science and Numerical Simulation, vol. 16, no. 9, pp. 3792-3797, 2011.
[6] M. Liu and K. Wang, "Analysis of a stochastic autonomous mutualism model," Journal of Mathematical Analysis and Applications, vol. 402, pp. 392-403, 2013.
[7] M. Liu and K. Wang, "Stochastic Lotka-Volterra systems with Lévy noise," Journal of Mathematical Analysis and Applications, vol. 410, no. 2, pp. 750-763, 2014.
[8] X. Mao, Stochastic Differential Equations and Their Applications, Horwood, Chichester, UK, 1997.
[9] X. Mao, "Stochastic versions of the LaSalle theorem," Journal of Differential Equations, vol. 153, no. 1, pp. 175-195, 1999.
[10] D. J. Higham, "An algorithmic introduction to numerical simulation of stochastic differential equations," SIAM Review, vol. 43, no. 3, pp. 525-546, 2001.

# Research Article <br> Traveling Waves in a Diffusive Predator-Prey Model Incorporating a Prey Refuge 

Xiujuan Wu, Yong Luo, and Yizheng Hu<br>College of Mathematics and Information Science, Wenzhou University, Zhejiang 325035, China<br>Correspondence should be addressed to Yong Luo; luoyong_china@yahoo.com

Received 24 January 2014; Accepted 9 March 2014; Published 24 April 2014
Academic Editor: Weiming Wang
Copyright © 2014 Xiujuan Wu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We establish the existence of traveling wave solutions and small amplitude traveling wave train solutions for a reaction-diffusion system based on a predator-prey model incorporating a prey refuge. By using the shooting argument, invariant manifold theory, and the Hopf bifurcation theorem, we analyze the dynamic behavior of this model in the three-dimensional phase space. Numerical results are also presented to illustrate the theoretical results.


## 1. Introduction

In mathematical biology, one interesting and dominant theme is the dynamic relationship between predators and their prey [1-3]. Predator-prey models have been studied mathematically since the pioneering work of Lotka and Volterra. In recent years, Leslie-Gower model $[4,5]$, an important predator-prey model, has been extensively modified and studied by many authors [6-11]. A modified LeslieGower predator-prey model is read as

$$
\begin{gather*}
\frac{\mathrm{d} H}{\mathrm{~d} t}=H(r-a H)-\frac{\beta_{1} H P}{b+H}  \tag{1}\\
\frac{\mathrm{~d} P}{\mathrm{~d} t}=P\left(d-\frac{\beta_{2} P}{b+H}\right),
\end{gather*}
$$

where function values $H$ and $P$ represent prey and predator population densities, respectively, at any time $t$. The model parameters $r, a, b, \beta_{1}, \beta_{2}$, and $d$ are positive constants. $r$ describes the growth rate of prey H. a measures the strength of competition among individuals of species $H . b$ measures the extent to which environment provides protection to prey $H . d$ is the growth rate of predators $P . \beta_{1}$ is the maximum value of per capita reduction of $H$ due to $P . \beta_{2}$ has a similar meaning to $\beta_{1}$.

As the authors of [6] said, we live in a spatial world, and spatial component of ecological interaction has been identified as an important factor in how ecological communities are shaped. Mite predator-prey interactions often exhibit spatial refugia, which means the prey received some degree of protection from predation and reduces the chance of extinction due to predation [6, 9-15]. A great deal of researches on the effects of prey refuges on the population dynamic has been studied. Kar [12] indicated that the increasing refuge can increase prey densities and lead to population outbreaks. Chen et al. [9] showed that the prey refuge could greatly influence the densities of both prey and predator species, while it has no influence on the species' persistence property. In [13-15] it was obtained that the refuges protecting a constant number of prey have a stronger stabilizing effect on population dynamic than the refuges protecting a constant proportion of prey.

On the other hand, the existence of traveling solutions has been wildly studied by many researchers [16-24]. A traveling wave solution is a spatial translation invariant solution of differential equations with spatial-diffusion. Dunbar [16] proved the existence of traveling wave solutions of diffusive Lotka-Volterra and used the methods of a shooting argument and a Lyapunov function. Zhang [19] showed the existence of traveling wave solutions in a modified vector-disease model by using the geometric singular perturbation theory. Hou
and Leung [20] used the method of upper-lower solutions to prove the existence of traveling solutions of a competitive reaction-diffusive system. Ahmad et al. [21, 22] used only functional analysis, without constructing a Lyapunov function, to prove the existence of such solutions for a class of reaction-diffusion equations. Huang et al. [23] and Li and Wu [24] used Dunbar' method to study the existence of traveling solutions of diffusive predator-prey models with Holling type-II and Holling type-III, respectively.

In this paper, based on the above discussion, we are interested in the existence of traveling wave solutions of a reaction-diffusion Leslie-Gower-type model incorporating a prey refuge, which is modified from model (1). Taking $P^{\prime}=$ $\beta_{1} P, \beta=\beta_{2} / \beta_{1}$ and dropping the stars on $P$, we will extend model (1) by incorporating a prey refuge into the following system:

$$
\begin{gather*}
\frac{\partial H}{\partial t}=D_{1} \Delta H+H(r-a H)-\frac{(1-m) H P}{b+(1-m) H} \\
\frac{\partial P}{\partial t}=D_{2} \Delta P+P\left(d-\frac{\beta P}{b+(1-m) H}\right) \tag{2}
\end{gather*}
$$

where $\Delta=\nabla^{2}=\left(\partial^{2} / \partial x^{2}+\partial^{2} / \partial y^{2}\right)$ is the usual Laplacian operator in two-dimensional space. $D_{1}$ and $D_{2}$ are the diffusion coefficients of prey and predator, respectively. $m \in$ $[0,1)$ is constant. $m H$ is a refuge protecting of the prey, which means $(1-m) H$ of prey available to the predator. To ensure system (2) has a positive equilibrium point, we require that $r>d(1-m)$. Obviously, system (2) has four equilibrium points:

$$
\begin{equation*}
E_{0}(0,0), \quad E_{1}\left(\frac{r}{a}, 0\right), \quad E_{2}\left(0, \frac{a b}{\beta}\right), \quad E\left(H^{*}, P^{*}\right), \tag{3}
\end{equation*}
$$

where

$$
\begin{align*}
H^{*} & =\frac{d m-d+\beta r}{a \beta}, \\
P^{*} & =\frac{a b d \beta+(d m-d+\beta r)(1-m) d}{a \beta^{2}}  \tag{4}\\
& =\frac{d\left(b+(1-m) H^{*}\right)}{\beta} .
\end{align*}
$$

The equilibrium point $E_{0}$ corresponds to absence of both species, $E_{1}$ corresponds to the prey at the environment carrying capacity in the absence of the predator, $E_{2}$ means the extinct of prey, and $E$ corresponds to coexistence of the two species. From [6], we know $E_{0}$ and $E_{1}$ are two saddle points and $E$ is globally asymptotical stable when $d H^{*}(1-m)^{3}<$ $a d \beta\left(b+(1-m) H^{*}\right)$, which indicates that system (2) may have traveling waves.

For mathematical simplicity, we assume that $D_{1}=0$ (considered as the $D_{1}$ is sufficient small which indicates the prey disperse very slowly relative to the mobile herbivore
predator [16]). Then system (2) can be converted to the system:

$$
\begin{gather*}
\frac{\partial H}{\partial t}=H(r-a H)-\frac{(1-m) H P}{b+(1-m) H},  \tag{5}\\
\frac{\partial P}{\partial t}=D \Delta P+P\left(d-\frac{\beta P}{b+(1-m) H}\right) .
\end{gather*}
$$

We will establish the existence of traveling wave solutions and small amplitude traveling wave train solutions of this system. The method used here is a shooting argument in $\mathbb{R}^{3}$ together with a Lyapunov function, LaSalle's invariant principle, and Hopf bifurcation theorem.

Remark that although the methods we use to prove the existence are similar to these in [16, 23, 24], there are several differences. For one thing, it is a different model, a modified Leslie-Gower model incorporating a prey refuge. For the other thing, we construct a different Wazewski set $W$ and a new Lyapunov function [25-27].

The rest of the paper is organized as follows. In Section 2, main results on the existence of traveling wave solutions and small amplitude wave train solutions are stated. In Section 3, we give the proofs of the main results. In Section 4, some numerical results are presented.

## 2. Main Results

A traveling wave solution is a spatial translation invariant solution. In order to establish the existence of traveling wave solutions of system (5), we assume the system has a solution of the special form $H(x, t)=H(x+c t), P(x, t)=P(x+c t)$, where parameter $c(>0)$ is the wave speed. Substituting $H(x, t)=$ $H(s), P(x, t)=P(s), s=x+c t$ into (5), the corresponding wave equations become

$$
c H^{\prime}=H(r-a H)-\frac{(1-m) H P}{b+(1-m) H},
$$

$$
\begin{equation*}
c P^{\prime}=D P^{\prime \prime}+P\left(d-\frac{\beta P}{b+(1-m) H}\right) \tag{6}
\end{equation*}
$$

Here $\left({ }^{\prime}\right)$ denotes the differentiation with respect to the traveling wave variable $s$. Due to ecological motivation, we require that the traveling wave solutions $H$ and $P$ are nonnegative and satisfying the following boundary conditions:

$$
\begin{array}{cl}
H(-\infty)=\frac{r}{a}, & H(+\infty)=H^{*}  \tag{7}\\
P(-\infty)=0, & P(+\infty)=P^{*}
\end{array}
$$

Rewrite the system (6) as a system of first order equation in $\mathbb{R}^{3}$ :

$$
\begin{array}{r}
H^{\prime}=\frac{1}{c} H(r-a H)-\frac{1}{c} \frac{(1-m) H P}{b+(1-m) H}, \\
P^{\prime}=U,  \tag{8}\\
U^{\prime}=\frac{c}{D} U-\frac{1}{D} P\left(d-\frac{\beta P}{b+(1-m) H}\right) .
\end{array}
$$

Lemma 1. Let $f(H)=(r-a H)(b+(1-m) H)-(1-m) P^{*}$, and then $f\left(H^{*}\right)=0$ and $f(H)=0$ has two real roots when $r>((d(1-m)) / \beta)$ (i.e., $\left.b r-(1-m) P^{*}>0\right)$. Furthermore, the following results hold:
(a) if $0<H<H^{*}$, then $f(H)>0$;
(b) if $H>H^{*}$, then $f(H)<0$.

Now we state the main results as follows.
Theorem 2. (i) If $0<c<\sqrt{4 D \bar{d}}$, then there are no nonnegative solutions of system (8) satisfying the boundary conditions (7).
(ii) If $c>\sqrt{4 D d}, r>((d(1-m)) / \beta)$, then there are nonnegative solutions of system (8) satisfying the boundary conditions (7), which correspond to traveling wave solutions of system (5).

Theorem 3. Let $P(\lambda)=\lambda^{3}-(M / c+c / D) \lambda^{2}+((M-d) / D) \lambda-$ $\left(a d H^{*} / c D\right)=0$, where $M=-a H^{*}+\left((1-m)^{2} H^{*} d^{2} / \beta^{2} P^{*}\right)$.
(a) If $P(\lambda)_{\text {maximum }}<0$, then $H, P$ spreads to the positive equilibrium point $\left(H^{*}, P^{*}\right)$ nonmonotonously for traveling wave variable s.
(b) If $P(\lambda)_{\text {maximum }} \geq 0$, then $H$, $P$ spreads to the positive equilibrium point $\left(H^{*}, P^{*}\right)$ monotonously for traveling wave variable s.

Theorem 4. Let $p=M-d$ and $q=a d H^{*}$. If

$$
\begin{equation*}
\max \left\{(1-m), \frac{d(1-m)}{r}\right\}<\beta<\frac{a b d(1-m)^{2}}{(r(1-m)+(3 / 2) a b)^{2}} \tag{9}
\end{equation*}
$$

then, as the parameter $\beta$ crosses the bifurcation curve $c^{2}=$ $D[q / p-p-d]$ at $\beta_{0}$ in the $(\beta, c)$-parameter plane, system (8) undergoes a Hopf bifurcation to a small amplitude periodic solution at the equilibrium point $\left(H^{*}, P^{*}, 0\right)$, which corresponds to a small amplitude traveling wave train solution of system (5).

## 3. Proofs of the Main Results

3.1. Proof of Theorem 2. In this section, we subdivide the proof into several Sections 3.1.1-3.1.4 for convenience. In Section 3.1.1, we recall some notations used throughout this section and state the well-known Wazewski Theorem. Section 3.1.2 contains a Wazewski set $W$ and the exit set $W^{-}$. In Section 3.1.3, the behavior of trajectories on the strongly unstable manifold at $((r / a), 0,0)$ is presented by some technical lemmas. In Section 3.1.4, we finish the proof of existence of traveling wave solutions by constructing a Lyapunov function.
3.1.1. Recall the Wazewski Theorem [16, 17]. Consider the differential equation:

$$
\begin{equation*}
\frac{\mathrm{d} y}{\mathrm{~d} s}=f(y), \quad y \in \mathbb{R}^{\mathbb{N}} \tag{10}
\end{equation*}
$$

where $f: \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}$ is a continuous function and satisfying a Lipschitz condition. Let $y\left(0, y_{0}\right)$ be the unique solution of (10) satisfying $y\left(0, y_{0}\right)=y_{0}$. For convenience, set $y\left(s, y_{0}\right)=$ $y_{0} \cdot s$. Let $U \cdot S$ be the set of points $y_{0} \cdot s$, where $y_{0} \in U$ and $s \in S$.

Given $W \subseteq \mathbb{R}^{\mathbb{N}}$, define

$$
\begin{equation*}
W^{-}=\left\{y_{0} \in W: \forall s>0, y_{0} \cdot[0, s) \nsubseteq W\right\} \tag{11}
\end{equation*}
$$

$W^{-}$is called the immediate exit set of $W$. Given $\Sigma \subseteq W$, let

$$
\begin{equation*}
\Sigma^{0}=\left\{y_{0} \in \Sigma: \exists s_{0}=s_{0}\left(y_{0}\right) \text { such that } y_{0} \cdot s_{0} \notin W\right\} . \tag{12}
\end{equation*}
$$

For $y_{0} \in \Sigma^{0}$, define

$$
\begin{equation*}
T\left(y_{0}\right)=\sup \left\{s: y_{0} \cdot[0, s) \subseteq W\right\} \tag{13}
\end{equation*}
$$

$T\left(y_{0}\right)$ is called an exit time. Note that $y_{0} \cdot T\left(y_{0}\right) \in W^{-}$and $T\left(y_{0}\right)=0$ if and only if $y_{0} \in W^{-}$. The notation $\mathrm{cl}(W)$ denotes the closure of $W$.

## Lemma 5. Suppose that

(i) if $y_{0} \in \Sigma$ and $y_{0} \cdot[0, s] \subseteq \operatorname{cl}(W)$, then $y_{0} \cdot[0, s] \subseteq W$;
(ii) if $y_{0} \in \Sigma, y_{0} \cdot s \in W$ and $y_{0} \cdot s \notin W^{-}$, then there is an open set $V_{s}$ about $y_{0} \cdot s$ disjoint from $W^{-}$;
(iii) $\Sigma=\Sigma^{0}, \Sigma$ is a compact set and intersects a trajectory of (10) only once. Then the mapping $F\left(y_{0}\right)=y_{0} \cdot T\left(y_{0}\right)$ is a homeomorphism from $\Sigma$ to its image on $W^{-}$.

A set $W \subseteq \mathbb{R}^{\mathbb{N}}$ satisfying the conditions (i) and (ii) is called a Wazewski set.
3.1.2. Construct $W$ and $W^{-}$. Evaluating the Jacobin of system (8) at the equilibrium $E_{1}((r / a), 0,0)$ gives

$$
J\left(E_{1}\right)=\left(\begin{array}{ccc}
-\frac{r}{c} & -\frac{(1-m) r}{c(a b+(1-m) r)} & 0  \tag{14}\\
0 & 0 & 1 \\
0 & -\frac{d}{D} & \frac{c}{D}
\end{array}\right)
$$

The corresponding eigenvalues of (14) are

$$
\begin{align*}
& \lambda_{1}=-\frac{r}{c} \\
& \lambda_{2}=\frac{c / D-\sqrt{c^{2} / D^{2}-4 d / D}}{2}  \tag{15}\\
& \lambda_{3}=\frac{c / D+\sqrt{c^{2} / D^{2}-4 d / D}}{2} .
\end{align*}
$$

If $0<c<\sqrt{4 D d}$, then $\lambda_{2}$ and $\lambda_{3}$ are a pair of complex conjugate eigenvalues with positive real part. By Theorems 6.1 and 6.2 in [25], there exists a 2 -dimensional unstable manifold based at $((r / a), 0,0)$, the point is a spiral point on this unstable manifold, and the trajectory approaching $((r / a), 0,0)$ as $s \rightarrow-\infty$ must have $P(s)<0$ for some $s$. This violates the requirement that the traveling wave solution must be nonnegative. So the first part of Theorem 2 is proved.

We only need to account for the case $c>\sqrt{4 D d}$ in the following. It is obvious that $\lambda_{1}<0<\lambda_{2}<\lambda_{3}$, the eigenvectors $\mathbf{x}_{1}, \mathbf{x}_{2}, \mathbf{x}_{3}$ associated with $\lambda_{1}, \lambda_{2}, \lambda_{3}$, respectively, are

$$
\begin{equation*}
\mathbf{x}_{i}=\left(-1, p\left(\lambda_{i}\right), \lambda_{i} p\left(\lambda_{i}\right)\right), \quad i=1,2,3 \tag{16}
\end{equation*}
$$

where $p\left(\lambda_{i}\right)=((c(a b+(1-m) r)) /((1-m) r)) \cdot\left(\lambda_{i}+\right.$ $r / c)$. Applying Theorems 6.1 and 6.2 in [25], we get a onedimension strongest unstable manifold $\mathbf{u}_{1}$ tangent to $\mathbf{x}_{3}$ at ( $(r / a), 0,0)$ and a two-dimension strongly unstable manifold $\mathbf{u}_{2}$ tangent to the span of $\mathbf{x}_{2}, \mathbf{x}_{3}$ at point $((r / a), 0,0)$. In a small neighborhood of $((r / a), 0,0)$, points on $\mathbf{u}_{1}$ are parametrically represented by a function $f_{1}(m)\left(\mathbb{R}^{1} \rightarrow \mathbb{R}^{2}\right)$ :

$$
\begin{equation*}
f_{1}(m)=\left(\frac{r}{a}, 0,0\right)^{T}+m \mathbf{x}_{3}+o(|m|) \tag{17}
\end{equation*}
$$

and points on $\mathbf{u}_{2}$ also could be represented by a function $f_{2}(m)\left(\mathbb{R}^{2} \rightarrow \mathbb{R}^{3}\right):$

$$
\begin{equation*}
f_{2}(m)=\left(\frac{r}{a}, 0,0\right)^{T}+m \mathbf{x}_{3}+n \mathbf{x}_{2}+o(|m|+|n|) . \tag{18}
\end{equation*}
$$

Obviously, $\mathbf{u}_{1} \subseteq \mathbf{u}_{2}$.
The motivation and method of constructing the Wazewski set $W$ are similar to that in Dunbar [17]: it will be the complement of two blocks of $\mathbb{R}^{3}$ and the two blocks are chosen so that $U^{\prime}$ has the same sign as $U$. Thus, solutions would not have $U \rightarrow 0$ as $s \rightarrow \infty$ when entering these blocks. In this paper, the Wazewski set $W$ is defined as follows:

$$
\begin{equation*}
W=\mathbb{R}^{3} \backslash(T \cup Q) \tag{19}
\end{equation*}
$$

where

$$
\begin{align*}
& T=\left\{(H, P, U): U>0, H<H^{*}, P>P^{*}\right\}, \\
& Q=\left\{(H, P, U): U<0, H>H^{*}, P<P^{*}\right\} . \tag{20}
\end{align*}
$$

Note that $T \cap Q=\varnothing$ and $W$ is a closed set. We obtain

$$
\begin{gather*}
\partial W=\partial T \cup \partial Q \\
W^{-}=\partial W \backslash\left(J \cup\left\{\left(H^{*}, P^{*}, U^{*}\right)\right\}\right), \tag{21}
\end{gather*}
$$

where

$$
\begin{equation*}
J=\left\{(H, P, U): H \geq H^{*}, P \leq 0, U=0\right\} \tag{22}
\end{equation*}
$$

Obviously, $W^{-}$is not a connected set. Actually, one component of $W^{-}$is $\partial P \backslash\left\{\left(H^{*}, P^{*}, U^{*}\right)\right\}$ and the other is $\partial Q \backslash(J \cup$ $\left.\left\{\left(H^{*}, P^{*}, U^{*}\right)\right\}\right)$.

As the details of proving that $W^{-}$is the set described above are tedious, we just prove the portion $\partial Q$ of $\partial W$ to show why the set $J$ must be excluded from $\partial W$ to $W^{-}$.
(1) $H=H^{*}, P<P^{*}, U<0$. Then we have

$$
\begin{align*}
H^{\prime} & =\frac{H}{c}\left(r-a H-\frac{(1-m) P}{b+(1-m) H}\right)_{H=H^{*}, P<P^{*}}  \tag{23}\\
& >\frac{H^{*}}{c}\left(r-a H^{*}-\frac{(1-m) P^{*}}{b+(1-m) H^{*}}\right)=0 .
\end{align*}
$$

Then the trajectory enters $Q$.
(2) $H>H^{*}, P=P^{*}, U<0$. Then

$$
\begin{equation*}
P^{\prime}=U<0 \tag{24}
\end{equation*}
$$

Thus, $P$ is decreasingly entering $Q$.
(3) $H>H^{*}, P<P^{*}, U=0$. Then

$$
\begin{equation*}
U^{\prime}=\frac{P}{D}\left(\frac{\beta P}{b+(1-m) H}-d\right) \tag{25}
\end{equation*}
$$

(i) $0<P<P^{*}$, and thus $\beta P /(b+(1-m) H)-d<$ $\beta P^{*} /\left(b+(1-m) H^{*}\right)-d=0$ and the trajectory enters the $Q$.
(ii) $P<0$, then $U^{\prime}>0$. This implies $H>H^{*}, P<$ $P^{*}, U>0$. The trajectory does not enter $T$ and Q.
(iii) $P=0$, and then $U^{\prime}=P^{\prime}=0, U^{\prime \prime}=P^{\prime \prime}=0$; furthermore, $U^{(n)}=P^{(n)}=0$. This implies the trajectory does not enter the inner of $Q$.
(4) $H=H^{*}, P=P^{*}, U=0$. This is a singular point not in the immediate exit set.
(5) $H=H^{*}, P=P^{*}, U<0$. Then $P^{\prime}=U<0, H^{\prime}=0$ and

$$
\begin{equation*}
H^{\prime \prime}=\frac{H^{*}}{c}\left(-\frac{(1-m) U}{b+(1-m) H^{*}}\right)<0, \tag{26}
\end{equation*}
$$

which implies $P$ and $H$ both decrease. The trajectory enters $Q$.
(6) $H>H^{*}, P=P^{*}, U=0$. Then

$$
\begin{gather*}
U^{\prime}=\frac{1}{D}\left(c U-P\left(d-\frac{\beta P}{b+(1-m) H}\right)\right)_{H>H^{*}, P=P^{*}, U=0}<0, \\
P^{\prime}=U=0, \quad P^{\prime \prime}=U^{\prime}<0 . \tag{27}
\end{gather*}
$$

Hence, the trajectory enters $Q$.
(7) $H=H^{*}, P<P^{*}, U=0$. Then

$$
\begin{align*}
H^{\prime} & =\frac{H}{c}\left(r-a H-\frac{(1-m) P}{b+(1-m) H}\right)_{H=H^{*}, P<P^{*}} \\
& =\frac{H^{*}}{c}\left(r-a H^{*}-\frac{(1-m) P}{b+(1-m) H^{*}}\right)  \tag{28}\\
& >\frac{H^{*}}{c}\left(r-a H^{*}-\frac{(1-m) P^{*}}{b+(1-m) H^{*}}\right)=0 .
\end{align*}
$$

(i) $P<0$, and then $U^{\prime}>0$. This implies $H>H^{*}$, $P<P^{*}, U>0$. The trajectory does not enter $P$ and $Q$.
(ii) $P=0$, and then $U^{(n)}=P^{(n)}=0,(n=1,2, \ldots)$. This implies the trajectory does not enter the inner of $Q$.
(iii) $0<P<P^{*}$, and then

$$
\begin{align*}
U^{\prime} & =\frac{1}{D}\left(c U-P\left(d-\frac{\beta P}{b+(1-m) H}\right)\right)_{H=H^{*}, U=0} \\
& =\frac{1}{D}\left(-P\left(d-\frac{\beta P}{b+(1-m) H^{*}}\right)\right)  \tag{29}\\
& <\frac{1}{D}\left(-P^{*}\left(d-\frac{\beta P^{*}}{b+(1-m) H^{*}}\right)\right)=0 .
\end{align*}
$$

Hence, it implies $H>H^{*}, P<P^{*}, U<0$, which ensures the trajectory enters the $Q$.

Based on the above analysis, $J=\{(H, P, U): H \geq$ $\left.H^{*}, P \leq 0, U=0\right\}$ and $\left(H^{*}, P^{*}, 0\right)$ must be excluded from $\partial W$ to $W^{-}$.
3.1.3. Construct the Set $\Sigma$. We need to construct the set $\Sigma$ before using Lemma 5 . By a series of lemmas (Lemmas 5-9), we obtain set $\Sigma$ will be an arc of a sufficient small circle surrounding $((r / a), 0,0)$ on the unstable manifold $\mathbf{u}_{2}$. Furthermore, one endpoint of the arc is the intersection of the circle with the strongly unstable manifold $\mathbf{u}_{1}$, and the other endpoint is the intersection of the circle with the plane defined by $U=0$. Lemmas also show that the first endpoint is carried by the strongly unstable manifold into $T$ while the other is carried into $P$.

We take a notation $\Omega_{1}=\{(H, P, U): H \leq(r / a), P \geq$ $0, U \geq 0\}$.

Lemma 6. Let $c>\sqrt{4 D d}$. Any solutions of (8) having a point $s_{0}$ such that $0<H\left(s_{0}\right), P\left(s_{0}\right)>0$, and $U\left(s_{0}\right)>(c / 2 D) P\left(s_{0}\right)$ will have $P(s)>0$ and $U(s)>(c / 2 D) P(s)$ for all $s>s_{0}$. This is particularly true for trajectories on the branch of strongly unstable manifold $\mathbf{u}_{1}$ in the octant $\Omega_{1}$.

Proof. Take $s_{0}=0$ without loss of generality. Suppose, on the contrary, that there exists an $s>0$ such that $U(s)<$ $(c / 2 D) P(s)$. Let

$$
\begin{equation*}
s_{1}=\inf \left\{s>0: U(s) \leq \frac{c}{2 D} P(s)\right\} . \tag{30}
\end{equation*}
$$

For $0 \leq s \leq s_{1}, P^{\prime}(s)=U(s) \geq(c / 2 D) P(s)$ and $P(0)>0$, so $P\left(s_{1}\right)>0$. Also $U\left(s_{1}\right)=(c / 2 D) P\left(s_{1}\right)$ and $U(s)>(c / 2 D) P(s)$ for $0 \leq s<s_{1}$. Thus $(c / 2 D) P^{\prime}\left(s_{1}\right) \geq U^{\prime}\left(s_{1}\right)$ (i.e., $U^{\prime}\left(s_{1}\right)-$ $\left.(c / 2 D) P^{\prime}\left(s_{1}\right) \leq 0\right)$. Then, from (8), we have

$$
\begin{equation*}
\left(\frac{c}{D} U-\frac{P}{D}\left(d-\frac{\beta P}{b+(1-m) H}\right)-\frac{c}{2 D} U\right)_{s_{1}} \leq 0 \tag{31}
\end{equation*}
$$

Then

$$
\begin{align*}
& \frac{c}{2 D} U\left(s_{1}\right)-\frac{d}{D} P\left(s_{1}\right) \\
& \quad \leq \frac{c}{2 D} U\left(s_{1}\right)-\frac{1}{D} P\left(s_{1}\right)\left(d-\frac{\beta P\left(s_{1}\right)}{b+(1-m) H\left(s_{1}\right)}\right) \leq 0 . \tag{32}
\end{align*}
$$

Since $U\left(s_{1}\right)=(c / 2 D) P\left(s_{1}\right)$, we have $c^{2} \leq 4 D d$.

It must be the case that $0<H\left(s_{1}\right)<(r / a)$. The plane defined by $U=0$ is an invariant manifold, so $H\left(s_{1}\right)>0$ is obvious. We just verify that $H\left(s_{1}\right)<(r / a)$. If this is not true, then there exists $0<s_{2} \leq s_{1}$ such that $H\left(s_{2}\right)=(r / a)$ and $H^{\prime}\left(s_{2}\right) \geq 0$. But then

$$
\begin{align*}
0 \leq H^{\prime}\left(s_{2}\right) & =\left(\frac{1}{c} H(r-a H)-\frac{1}{c} \frac{(1-m) H P}{b+(1-m) H}\right)_{s=s_{2}}  \tag{33}\\
& =-\left.\frac{1}{c} \frac{(1-m) H P}{b+(1-m) H}\right|_{s=s_{2}}<0,
\end{align*}
$$

so $0<H\left(s_{1}\right)<(r / a)$ for $0 \leq s \leq s_{1}$. So $c^{2} \leq 4 D d$, which is a contradiction. Thus $U(s)>(c / 2 D) P(s)$ for all $s>0$. Then also $P(s)>0$ for all $s>0$.

A trajectory on the branch of the strongly unstable manifold $\mathbf{u}_{1}$ in the octant $\Omega_{1}$ approaches ( $\left.(r / a), 0,0\right)$ tangent to $\mathbf{x}_{3}$. From subset $B$, the second and third components of this tangent vector satisfy $U=\lambda_{3} P$. Thus there exists a point $s_{0}$ on the trajectory whose components satisfy $0<H\left(s_{0}\right)<$ $(r / a), P\left(s_{0}\right)>0$, and $U\left(s_{0}\right)=\lambda_{3} P\left(s_{0}\right)>(c / 2 D) P\left(s_{0}\right)$. This completes the proof.

Lemma 7. Assume that $c>\sqrt{4 D d}$; then a trajectory on the portion of the strongly unstable manifold $\mathbf{u}_{1}$ in the octant $\Omega_{1}$ must satisfy

$$
\begin{equation*}
P(s) \geq-\left(H(s)-\frac{r}{a}\right)\left(\frac{c(a b+(1-m) r)}{(1-m) r} \frac{c^{2}+2 D r}{2 D c}\right) \tag{34}
\end{equation*}
$$

for alls.
Proof. A trajectory on the portion of the strongly unstable manifold $\mathbf{u}_{1}$ in the octant $\Omega_{1}$ could be written as $P(s)=$ $-p\left(\lambda_{3}\right)(H(s)-r / a)$, where

$$
\begin{align*}
p\left(\lambda_{3}\right)= & \frac{c(a b+(1-m) r)}{(1-m) r}\left(\lambda_{3}+\frac{r}{c}\right) \\
= & \frac{c(a b+(1-m) r)}{(1-m) r} \\
& \times\left(\frac{c / D+\sqrt{c^{2} / D^{2}-4 d / D}}{2}+\frac{r}{c}\right)  \tag{35}\\
> & \frac{c(a b+(1-m) r)}{(1-m) r} \frac{c^{2}+2 D r}{2 D c} .
\end{align*}
$$

Lemma 8. Let $l>(c / D)$ be a fixed number. A solutions of (8) having a point $s_{0}$ such that $0<H\left(s_{0}\right)<(r / a), P\left(s_{0}\right)>0$, and $U\left(s_{0}\right)<l P\left(s_{0}\right)$ will have $U(s)<l P(s)$ for all $s>s_{0}$ such that $P(s)>0$. In particular, this is true for trajectories on branch of the strongly unstable manifold $\mathbf{u}_{1}$ in the octant $\Omega_{1}$.

The proof is similar to that of Lemma 6, so it is omitted.
Lemma 9. If a solution of (8) has a point, taking to $s=0$ without loss of generality, such that $H(0)<(r / a), 0<P(0)<$ $-\left((c(a b+(1-m) r)) /\left(a H^{*}(1-m)\right)\right)(l+(r / c))(H(s)-(r / a))$,
and $U(0)<l P(0)$, then for all $s>0$, as long as $H(s)>H^{*}$, $P(s)>0$ the trajectory must have that $P(s)<-(c(a b+(1-$ $\left.m) r) / a H^{*}(1-m)\right)(l+(r / c))(H(s)-(r / a))$. In particular, this is true for a trajectory on the branch of the strongly unstable manifold $\mathbf{u}_{1}$ in the octant $\Omega_{1}$.

Proof. We first prove that $H(s)<(r / a)$ for all $s>0$ such that $P(s)>0$. If this is not true, then there exists a first time $s_{1}>0$ such that $H(s)=(r / a), H^{\prime}\left(s_{1}\right) \geq 0$ and $P\left(s_{1}\right)>0$. But then,

$$
\begin{equation*}
0 \leq H^{\prime}\left(s_{1}\right)=\left(\frac{1}{c} H(r-a H)-\frac{1}{c} \frac{(1-m) H P}{b+(1-m) H}\right)_{s=s_{1}}<0 \tag{36}
\end{equation*}
$$

This is a contradiction. Thus $H(s)<(r / a)$ for all $s>0$ such that $P(s)>0$.

Now we show that $P(s)<-A_{0}(H(s)-(r / a))$ for all $s>0$ as long as $H(s)>H^{*}$ and $P(s)>0$. Let $A_{0}=(c(a b+(1-$ $\left.m) r) / a H^{*}(1-m)\right)(l+(r / c))$. Suppose on the contrary that there exists a first time $s_{2}$ such that $H\left(s_{2}\right)>H^{*}, P\left(s_{2}\right)>0$, but $P\left(s_{2}\right)=-A_{0}\left(H\left(s_{2}\right)-(r / a)\right)$. Then $P^{\prime}\left(s_{2}\right) \geq-A_{0}\left(H^{\prime}\left(s_{2}\right)\right)$. By Lemma $8, U(s)<l P(s)$ for all $s>s_{0}$ such that $P(s)>0$. Then

$$
\begin{align*}
l P\left(s_{2}\right) \geq U\left(s_{2}\right) & =P^{\prime}\left(s_{2}\right) \geq-A_{0}\left(H^{\prime}\left(s_{2}\right)\right) \\
& =-A_{0} \frac{H}{c}\left(r-a H-\frac{(1-m) P}{b+(1-m) H}\right)_{s=s_{2}} . \tag{37}
\end{align*}
$$

For $P\left(s_{2}\right)=-A_{0}\left(H\left(s_{2}\right)-(r / a)\right)$ and $H^{*}<H\left(s_{2}\right)<(r / a)$, we have

$$
\begin{align*}
l & \geq \frac{-A_{0}(H / c)(r-a H-((1-m) P) /(b+(1-m) H))_{s=s_{2}}}{-A_{0}\left(H\left(s_{2}\right)-(r / a)\right)} \\
& =-\frac{1}{c}\left(\frac{A_{0} H(r-a H)}{-A_{0}(H-(r / a))}-\frac{(1-m) P}{b+(1-m) H} \cdot \frac{A_{0} H}{P}\right)_{s=s_{2}} \\
& =-\frac{1}{c}\left(a H-\frac{A_{0}(1-m) H}{b+(1-m) H}\right)_{s=s_{2}} \\
& >\frac{1}{c}\left(\frac{A_{0}(1-m) H^{*}}{b+(1-m)(r / a)}-r\right) \\
& =\frac{a A_{0}(1-m) H^{*}}{c(a b+(1-m) r)}-\frac{r}{c} \\
& =l \tag{38}
\end{align*}
$$

which is a contradiction. This completes the proof.

Now combine all the results of Lemmas 6-9 to follow the trajectory of a solution of (8) on the strongly unstable manifold $\mathbf{u}_{1}$. Let

$$
\begin{align*}
\mathscr{R}=\{ & (H, P, U): H^{*}<H<\frac{r}{a}, \\
& -\frac{c(a b+(1-m) r)}{(1-m) r} \cdot \frac{c^{2}+2 d r}{2 d c}\left(H-\frac{r}{a}\right)<P  \tag{39}\\
& <-\frac{c(a b+(1-m) r)}{a H^{*}(1-m)}\left(l+\frac{r}{c}\right)\left(H-\frac{r}{a}\right), \\
& \left.\frac{c}{2 D} P<U<l P\right\} .
\end{align*}
$$

Then the trajectory of a solution of (8) on the strongly unstable manifold $\mathbf{u}_{1}$ is contained in $\mathscr{R}$. Since $0<m<1$, we obtain

$$
\begin{align*}
P & \geq-\frac{c(a b+(1-m) r)}{(1-m) r} \cdot \frac{c^{2}+2 D r}{2 D c}\left(H-\frac{r}{a}\right) \\
& =\left(\frac{r}{a}-H\right) \cdot \frac{c(a b+(1-m) r)}{(1-m) r} \cdot\left(\frac{c}{2 D}+\frac{r}{c}\right) \\
& \geq\left(\frac{r}{a}-H\right) \cdot \frac{c(a b+(1-m) r)}{(1-m) r} \cdot \frac{r}{c}  \tag{40}\\
& =(r-a H) \frac{b+(1-m)(r / a)}{(1-m)} \\
& >(r-a H) \frac{b+(1-m) H}{(1-m)} .
\end{align*}
$$

This shows the region $\mathscr{R}$ lies in the region defined by $H>0$ and $P>(r-a H)(b+(1-m) H /(1-m))$. Then, on the strongly unstable manifold $\mathbf{u}_{1}, H^{\prime}=H((r-a H)-((1-m) P / b+(1-$ $m) H)$ ) 0 . So, for a solution of (8) on $\mathbf{u}_{1}, H(s)$ decreases until $H\left(s_{0}\right)=H^{*}$ for some finite $s_{0}$. And at the time $s_{0}$, we have

$$
\begin{equation*}
P>\left(r-a H^{*}\right) \frac{b+(1-m) H^{*}}{(1-m)}=\frac{d\left(b+(1-m) H^{*}\right)}{\beta}=P^{*} \tag{41}
\end{equation*}
$$

Thus the trajectory of this solution hits $\partial W$ on the face $H=$ $H^{*}, P>P^{*}$, and $U>0$. Therefore, the vector field shows that the solution of (8) on $\mathbf{u}_{1}$ enters the region $T$ at some finite time.

Lemma 10. In a sufficient small neighborhood of $((r / a), 0,0)$ the two-dimensional unstable manifold $\mathbf{u}_{2}$ intersects the plane defined by $U=0$ in a smooth $\mathscr{C}^{1}$ curve $\Gamma$, given by $P=\mathscr{M}(H)$, $U=0$, where

$$
\begin{align*}
P & =\mathscr{M}(H) \\
& =-\frac{\lambda_{3} p\left(\lambda_{3}\right)}{\lambda_{2} p\left(\lambda_{2}\right)}\left(H-\frac{r}{a}\right)  \tag{42}\\
& =-\frac{\lambda_{3}\left(r+c \lambda_{3}\right)}{\lambda_{2}\left(r+c \lambda_{2}\right)}\left(H-\frac{r}{a}\right) .
\end{align*}
$$

Proof. The proof is similar to Lemma 5 in [16] and is omitted.

Remark 11. The portion of the curve $\Gamma$ is in the region $H<$ $(r / a)$. Obviously, the $P$ component of points along the curve $\Gamma$ satisfies $P>0$ from Lemma 10 . From the direction of the vector filed on the quarter plane, $H>H^{*}, P>0$, and $U=0$, any trajectory passing through a point of $\Gamma$ near $((r / a), 0,0)$ will immediately enter the region $Q$.

Now, we place a sufficiently small circle about ( $(r / a), 0,0)$ on the two-dimensional unstable manifold $\mathbf{u}_{2}$. The circle is contained in the neighborhood of $((r / a), 0,0)$ given in Lemma 10 and satisfies the conditions of Lemmas 6-9. Then the circle intersects the curve $\Gamma$. Define $\Sigma$ to be arc of this circle contained in the octant $\Omega_{1}$ whose endpoints are the intersections of the circle with $\mathbf{u}_{1}$ and the curve $\Gamma$.
3.1.4. Proof of (ii) of Theorem 2. In this section, we firstly use Lemma 5 to produce a trajectory which remains in the region W. Second, we construct a Lyapunov function to demonstrate that the trajectory approaches $\left(H^{*}, P^{*}, 0\right)$. For simplicity, we denote $N=\{(H, P, U): P=U=0\}, L=\{(H, P, U): H=0\}$.

Lemma 12. There exists a point $y^{*} \in \Sigma$ such that the solution $y\left(s, y^{*}\right)=\left(H_{1}(s), P_{1}(s), U_{1}(s)\right)$ of (8) remains in the region $W$ for alls.

Proof. It is obvious that the set $W$ is closed satisfying the (i) of Lemma 5. Before using Lemma 5 to prove this conclusion, we also need to check the conditions (ii) and (iii) of it. Suppose $y_{0} \in \Sigma, s<T\left(y_{0}\right), y\left(s, y_{0}\right) \in W \backslash W^{-}$. Then $y\left(s, y_{0}\right) \in \operatorname{int} W \cup$ $J$. As $s<T\left(y_{0}\right)$, we easily verify that

$$
\begin{equation*}
y\left(s, y_{0}\right) \notin\left\{(H, P, U): H \geq H^{*}, P<0, U=0\right\} \tag{43}
\end{equation*}
$$

Moreover, as $N$ is an invariant manifold,

$$
\begin{equation*}
y\left(s, y_{0}\right) \notin\left\{(H, P, U): H \geq H^{*}, P=0, U=0\right\} \tag{44}
\end{equation*}
$$

Thus $y\left(s, y_{0}\right) \in$ int $W$ and there exists an open set $V$ around $y\left(s, y_{0}\right)$ disjoint from $\partial W$. So (ii) of Lemma 5 is satisfied.

From the previous 5 lemmas, we know that the image of one endpoint of $\Sigma$ lies in the portion $\partial T \backslash\left\{\left(H^{*}, P^{*}, 0\right)\right\}$ of $W^{-}$; and the image of the other endpoint is in the component $\partial Q \backslash$ $\left(J \cup\left\{\left(H^{*}, P^{*}, 0\right)\right\}\right)$ of $W^{-}$. Thus $\Sigma$ is compact, intersects any trajectory of (8) only once, and is simple connected. If $\Sigma=\Sigma^{0}$, then $F$ would be a homeomorphism of the connected set $\Sigma$ to its image in the disconnected set $W^{-}$. This is impossible. So $\Sigma \neq \Sigma^{0}$. Thus there exists some point $y^{*}$ such that $y\left(s, y^{*}\right) \in$ $W$ for all $s$.

Lemma 13. The solution $y\left(s, y^{*}\right)$ remains in the region

$$
\begin{gather*}
\Omega=\left\{(H, P, U): 0<H<\frac{r}{a}, 0<P<k(H),\right.  \tag{45}\\
\left.-\frac{\beta P^{2}}{c b}<U<l P\right\}
\end{gather*}
$$

for all $s$, where

$$
k(H)=\left\{\begin{array}{r}
-\frac{c(a b+(1-m) r)}{a H^{*}(1-m)}\left(l+\frac{r}{c}\right)\left(H-\frac{r}{a}\right)  \tag{46}\\
H^{*}<H<\frac{r}{a} \\
-\frac{c(a b+(1-m) r)}{a H^{*}(1-m)}\left(l+\frac{r}{c}\right)\left(H^{*}-\frac{r}{a}\right) \\
0<H^{*} \leq H^{*}
\end{array}\right.
$$

Proof. Firstly, $y\left(s, y^{*}\right)$ must have $H_{1}(s)>0$ for all $s$, as $L$ is an invariant manifold.

Secondly, we prove $P_{1}(s)>0$. If it is not true, then $y\left(s, y^{*}\right)$ enters region $N_{1}=\{(H, P, U): P<0\}$. Let $s_{1}=\inf \{s:$ $\left.y\left(s, y^{*}\right) \in N_{1}\right\}$. Then $P_{1}\left(s_{1}\right)=0$ and $P_{1}^{\prime}\left(s_{1}\right) \leq 0$, so $U_{1}\left(s_{1}\right) \leq 0$. As $N$ is an invariant manifold, $U_{1}\left(s_{1}\right)<0$. And $H_{1}\left(s_{1}\right)<H^{*}$ for $y\left(s, y^{*}\right) \notin Q$. From (8), $H_{1}^{\prime}\left(s_{1}\right)>0$, which means $H_{1}(s)$ is increasing for $s>s_{1}$. Then the solution enters

$$
\begin{equation*}
N_{2}=\left\{(H, P, U): H_{1}\left(s_{1}\right)<H<H^{*}, P<0, U<0\right\} . \tag{47}
\end{equation*}
$$

Obviously, in $N_{2}, P^{\prime}=U<0$, so $P_{1}(s)$ is decreasing. Thus, we have

$$
\begin{equation*}
H_{1}^{\prime}(s) \geq \frac{1}{c} \min \left\{H_{1}\left(s_{1}\right)\left(r-a H_{1}\left(s_{1}\right)\right), H^{*}\left(r-a H^{*}\right)\right\} \tag{48}
\end{equation*}
$$

So $H_{1}(s)$ increases to $H^{*}$ in the finite time $s_{2}$; that is, $H_{2}(s)=$ $H^{*}$. Then also $P_{1}\left(s_{2}\right)<0, U_{1}\left(s_{2}\right)<0$. So $y\left(s, y^{*}\right)$ enter $Q$. This is a contradiction. Therefore, $P_{1}(s)>0$ for all time.

By Lemma 9, we know

$$
\begin{align*}
& P_{1}<-\frac{c(a b+(1-m) r)}{a H^{*}(1-m)}\left(l+\frac{r}{c}\right)\left(H_{1}-\frac{r}{a}\right),  \tag{49}\\
& \text { for } H^{*}<H_{1} \leq \frac{r}{a}
\end{align*}
$$

As $P_{1}(s)>0$, so $H_{1}(s)<(r / a)$ for all $s$.
Suppose, on the contrary, there exists $s$ such that $P_{1}(s) \geq$ $-A_{0}\left(H^{*}-(r / a)\right)$ for $0<H_{1} \leq H^{*}$, where $A_{0}=(c(a b+(1-$ $\left.m) r) / a H^{*}(1-m)\right)(l+r / c)$. Take

$$
\begin{equation*}
s_{2}=\inf \left\{s: P_{1}(s) \geq-A_{0}\left(H^{*}-\frac{r}{a}\right)\right\} \tag{50}
\end{equation*}
$$

Then $H_{1}\left(s_{2}\right) \leq H^{*}, P_{1}\left(s_{2}\right)>P^{*}$, and $U_{1}\left(s_{2}\right)=P_{1}^{\prime}\left(s_{2}\right) \geq 0$. Then either $y\left(s, y^{*}\right) \in T$ or $y\left(s, y^{*}\right)$ immediately enter $T$, which is impossible. So $P_{1}(s) \leq-A_{0}\left(H^{*}-(r / a)\right)$ for $0<$ $H_{1} \leq H^{*}$.

At last, we prove $-\left(\beta P_{1}^{2} / c b\right)<U_{1}<l P_{1} . U_{1}<l P_{1}$ is obvious. Because a trajectory starting on $\Sigma$ approaches $((r / a), 0,0)$ tangent to $\mathbf{x}_{2}$ or $\mathbf{x}_{3}$ has $U=\lambda_{2} P$ or $U=\lambda_{3} P$. Since $\lambda_{2}, \lambda_{3}<l$, from Lemma 8, we know $U_{1}(s)<l P_{1}(s)$ for all s. We only need to prove $-\left(\beta P_{1}^{2} / c b\right)<U_{1}$. Suppose, on the contrary,
that there exists a $s_{3}$ such that $U_{1}\left(s_{3}\right)<-\left(\beta P_{1}^{2}\left(s_{3}\right) / c b\right)<0$; then $U_{1}\left(s_{3}\right)<-\left(\beta P_{1}^{2}\left(s_{3}\right) / c b\right)$ for all $s>s_{3}$. If this is not true, there exists a $s_{4}>s_{3}$ such that $U_{1}\left(s_{4}\right)=-\left(\beta P_{1}^{2}\left(s_{4}\right) / c b\right)$, and thus $U_{1}^{\prime}\left(s_{4}\right)+\left(\beta P_{1}^{2}\left(s_{4}\right) / c b\right) \geq 0$. Then from (8) we have

$$
\begin{equation*}
\left(\frac{c}{D} U-\frac{d}{D} P+\frac{\beta P^{2}}{D(b+(1-m) H)}+\frac{2 \beta P U}{c b}\right)_{s=s_{4}} \geq 0 \tag{51}
\end{equation*}
$$

Then after some calculation, we obtain

$$
\begin{equation*}
\left(-\frac{\beta^{2} P}{D}\left(\frac{1}{b}-\frac{1}{b+(1-m) H}\right)-\frac{d}{D} P-\frac{2 \beta^{2} P^{3}}{c^{2} b}\right)_{s=s_{4}} \geq 0 \tag{52}
\end{equation*}
$$

this is a contradiction. So if $U_{1}\left(s_{3}\right)<-\left(\beta P_{1}^{2}\left(s_{3}\right) / c b\right)$, then $U_{1}\left(s_{3}\right)<-\left(\beta P_{1}^{2}\left(s_{3}\right) / c b\right)$ for all $s>s_{3}$. Thus,

$$
\begin{align*}
U_{1}^{\prime} & =\frac{c}{D} U_{1}-\frac{d}{D} P_{1}+\frac{\beta P_{1}^{2}}{D(b+(1-m) H)}  \tag{53}\\
& <-\frac{\beta P_{1}^{2}}{D}\left(\frac{1}{b}-\frac{1}{b+(1-m) H_{1}}\right)-\frac{d}{D} P_{1}<0
\end{align*}
$$

for all $s>s_{3}$. So $U_{1}(s)<U_{1}\left(s_{3}\right)$ for all $s>s_{3}$. Thus $P_{1}^{\prime}(s)=U_{1}(s)<0$ and bounded away from zero by $U_{1}\left(s_{3}\right)$. Therefore $P_{1}(s)<0$ for some finite $s$, which is a contradiction. So $-\left(\beta P_{1}^{2} / c b\right)<U_{1}$.

This completes the proof.
Lemma 14. The trajectory $y\left(s, y^{*}\right) \rightarrow\left(H^{*}, P^{*}, 0\right)$ as $s \rightarrow$ $-\infty$.

Proof. Define following Lyapunov function:

$$
\begin{align*}
V(H, P, U)= & \frac{d c}{D}\left[H-H^{*} \ln H\right]+\left[c\left(\frac{P}{D}-P^{*}\right)-U\right] \\
& +P^{*}\left[\frac{U}{P}-\frac{c}{D} \ln \frac{P}{P^{*}}\right] \tag{54}
\end{align*}
$$

Then $V(H, P, U)$ is continuous and bounded below on $\Omega$, and

$$
\begin{aligned}
\frac{d V}{d s}= & \frac{\partial V}{\partial H} \cdot H_{t}+\frac{\partial V}{\partial P} \cdot P_{t}+\frac{\partial V}{\partial U} \cdot U_{t} \\
= & \frac{d c\left(H-H^{*}\right)}{D H} \cdot \frac{H}{c}\left[r-a H-\frac{(1-m) P}{b+(1-m) H}\right] \\
& +\left[\frac{c}{D}\left(1-\frac{P^{*}}{P}\right)-\frac{P^{*} U}{P^{2}}\right] \cdot U
\end{aligned}
$$

$$
\begin{align*}
& +\left(\frac{P^{*}}{P}-1\right) \cdot \frac{1}{D}\left[c U+\frac{\beta P^{2}}{b+(1-m) H}-P d\right] \\
& =\frac{d\left(H-H^{*}\right)}{D}\left[r-a H-\frac{(1-m) P}{b+(1-m) H}\right] \\
& +\frac{P^{*}-P}{D}\left[\frac{\beta P}{b+(1-m) H}-d\right]-\frac{P^{*} U^{2}}{P^{2}} \\
& =\frac{d\left(H-H^{*}\right)}{D}\left[r-a H-\frac{(1-m) P}{b+(1-m) H}\right]-\frac{P^{*} U^{2}}{P^{2}} \\
& +\frac{P^{*}-P}{D}\left[\frac{\beta P}{b+(1-m) H}-\frac{\beta P^{*}}{b+(1-m) H}\right. \\
& \left.+\frac{\beta P^{*}}{b+(1-m) H}-d\right] \\
& =\frac{d\left(H-H^{*}\right)}{D}\left[r-a H-\frac{(1-m) P}{b+(1-m) H}\right] \\
& +\frac{P^{*}-P}{D}\left[\frac{(1-m) P^{*}}{b+(1-m) H}-d\right] \\
& -\frac{\beta\left(P^{*}-P\right)^{2}}{D(b+(1-m) H)}-\frac{P^{*} U^{2}}{P^{2}} \\
& =\frac{d\left(H-H^{*}\right)}{D}\left[r-a H-\frac{(1-m) P}{b+(1-m) H}\right] \\
& +\frac{P^{*}-P}{D} \cdot \frac{d(1-m)\left(H-H^{*}\right)}{b+(1-m) H} \\
& -\frac{\beta\left(P^{*}-P\right)^{2}}{D(b+(1-m) H)}-\frac{P^{*} U^{2}}{P^{2}} \\
& =\frac{d\left(H-H^{*}\right)}{D}\left[r-a H-\frac{(1-m) P^{*}}{b+(1-m) H}\right] \\
& -\frac{\beta\left(P^{*}-P\right)^{2}}{D(b+(1-m) H)}-\frac{P^{*} U^{2}}{P^{2}}, \tag{55}
\end{align*}
$$

where $g(H)=f(H) /(b+(1-m) H)$ is defined. Obviously, $b+(1-m) H>0$ in $\Omega$ and $g\left(H^{*}\right)=0$. According to Lemma 1, when $r>(d(1-m) / \beta)$, the following result always holds:

$$
\begin{equation*}
\frac{d\left(H-H^{*}\right)}{D}\left[r-a H-\frac{(1-m) P^{*}}{b+(1-m) H}\right] \leq 0 . \tag{56}
\end{equation*}
$$

Therefore, the $d V / d s$ is always nonpositive in $\Omega$. Moreover, $d V / d s=0$ if and only if $H=H^{*}, P=P^{*}$, and $U=0$; the largest invariant subset of this segment is the single point $\left(H^{*}, P^{*}, 0\right)$. By LaSalle's Invariance Principle, $y\left(s, y^{*}\right) \rightarrow$ $\left(H^{*}, P^{*}, 0\right)$ as $s \rightarrow-\infty$. This completes the proof.
3.2. Proof of Theorem 3. The Jacobin of system (8) at the equilibrium $E\left(H^{*}, P^{*}, 0\right)$ is

$$
\begin{align*}
& J(E) \\
& =\left(\begin{array}{ccc}
-\frac{1}{c}\left(-a H^{*}+\frac{(1-m)^{2} H^{*} d^{2}}{\beta^{2} P^{*}}\right) & -\frac{d(1-m) H^{*}}{c \beta P^{*}} & 0 \\
0 & 0 & 1 \\
-\frac{d^{2}(1-m)}{\beta D} & -\frac{d}{D} & \frac{c}{D}
\end{array}\right) . \tag{57}
\end{align*}
$$

Let $M=-a H^{*}+\left((1-m)^{2} H^{*} d^{2} / \beta^{2} P^{*}\right)$; then the corresponding characteristic equation of (57) is given by

$$
\begin{equation*}
P(\lambda)=\lambda^{3}-\left(\frac{M}{c}+\frac{c}{D}\right) \lambda^{2}+\frac{M-d}{D} \lambda-\frac{a d H^{*}}{c D}=0 . \tag{58}
\end{equation*}
$$

In order to get the sign of the roots of characteristic equation (58), we will use Routh-Hurwitz analysis [25]. The RouthHurwitz range of (58) is

$$
\begin{array}{c|cc}
\lambda^{3} & a_{3}=1 & a_{1}=\frac{(M-d)}{D}  \tag{59}\\
\lambda^{2} & a_{2}=-\left(\frac{M}{c}+\frac{c}{D}\right) & a_{0}=-\frac{a d H^{*}}{c D} \\
\lambda^{1} & b_{1} & b_{2} \\
\lambda^{0} & c_{1} & c_{2}
\end{array}
$$

where

$$
\begin{align*}
b_{1} & =-\frac{1}{a_{2}}\left|\begin{array}{ll}
a_{3} & a_{1} \\
a_{2} & a_{0}
\end{array}\right| \\
& =-\frac{1}{M / c+c / D}\left|\begin{array}{cc}
1 & \frac{M-d}{D} \\
-\left(\frac{M}{c}+\frac{c}{D}\right) & -\frac{a d H^{*}}{c D}
\end{array}\right| \\
& =-\frac{a d H^{*}}{M D+c^{2}}+\frac{M-d}{D},  \tag{60}\\
b_{2} & =-\frac{1}{a_{2}}\left|\begin{array}{ll}
a_{3} & 0 \\
a_{2} & 0
\end{array}\right|=0, \\
c_{1} & =-\frac{1}{b_{1}}\left|\begin{array}{ll}
a_{2} & a_{0} \\
b_{1} & b_{2}
\end{array}\right|=a_{0}=-\frac{a d H^{*}}{c D}, \\
c_{2} & =-\frac{1}{b_{1}}\left|\begin{array}{ll}
a_{2} & 0 \\
b_{1} & 0
\end{array}\right|=0 .
\end{align*}
$$

In the above range, we easily know that $a_{3}>0, c_{1}<0$. When $\beta>1-m$, (i) if $M / c+c / D<0\left(a_{2}>0\right)$, then no matter the sigh of $b_{1}$, the sigh of the first arrange of (59) will change once, and the no row of (59) is full zero. So character equation (59) always has a real root and two complex roots with negative real part; (ii) if $M / c+c / D>0\left(a_{2}<0\right)$, we obtain ( $(M-$ d) $/ D)<0$ with $\beta>1-m$, and then $b_{1}<0$. Thus, the sigh of the first arrange of (59) will change once and the no row of (59) is full zero. So character equation (58) has a real root and two complex roots with negative real part.

Therefore, there is a 2-dimensional stable manifold and 1-dimensional unstable manifold based at $\left(H^{*}, P^{*}, 0\right)$ when $\beta>1-m$.

The differentiation of (58) is

$$
\begin{equation*}
P^{\prime}(\lambda)=3 \lambda^{2}-2\left(\frac{M}{c}+\frac{c}{D}\right) \lambda+\frac{M-d}{D} . \tag{61}
\end{equation*}
$$

Let $P^{\prime}(\lambda)=0$; then we obtain

$$
\begin{equation*}
\lambda_{ \pm}=\frac{2(M / c+c / D) \pm \sqrt{4(M / c+c / D)-12((M-d) / D)}}{6} \tag{62}
\end{equation*}
$$

Thus, $P(\lambda)$ get the maximum at $\lambda=\lambda_{-}, P(\lambda)$ get the minimum at $\lambda=\lambda_{+}$, and $P(\lambda)_{\text {minimum }}<0$. So we just consider

$$
\begin{equation*}
P(\lambda)_{\text {maximum }}=\lambda_{-}^{3}-\left(\frac{M}{c}+\frac{c}{D}\right) \lambda_{-}^{2}+\frac{M-d}{D} \lambda_{-}-\frac{a d H^{*}}{c D} \tag{63}
\end{equation*}
$$

If $P(\lambda)_{\text {maximum }}>0,(58)$ has two negative roots and a positive root. If $P(\lambda)_{\text {maximum }}=0$, (58) has a negative root and a positive root. If $P(\lambda)_{\text {maximum }}<0$, (58) has a positive root and two complex roots with negative real part. So the solution of (8) satisfying (7) spreads to the positive equilibrium $\left(H^{*}, P^{*}, 0\right)$ monotonously when $P(\lambda)_{\text {maximum }} \geq$ 0 , and it spreads to the positive equilibrium $\left(H^{*}, P^{*}, 0\right)$ nonmonotonously when $P(\lambda)_{\text {maximum }}<0$.
3.3. Proof of Theorem 4. In order to prove Theorem 4, we take $D, r, a, m$, and $d$ as fixed and $\beta$ and $c$ as parameters. It means we only allow the predator effectiveness to vary. We search for purely imaginary roots of the characteristic equation

$$
\begin{equation*}
\lambda^{3}-\left(\frac{p+d}{c}+\frac{c}{D}\right) \lambda^{2}+\frac{p}{D} \lambda-\frac{q}{c D}=0 \tag{64}
\end{equation*}
$$

where $p=M-d, q=a d H^{*}, M=-a H^{*}+((1-$ $\left.m)^{2} H^{*} d^{2} / \beta^{2} P^{*}\right)$, and $H^{*}=((d m-d+\beta r) / a \beta)$.

It is easy to see that $p<0, q>0$ and $0<H^{*}<r / a$. Substituting $\lambda=k i$ into (64) and simplifying it, we have

$$
\begin{gather*}
k^{2}=\frac{p}{D} \\
k^{2}=\frac{q}{D(p+d)+c^{2}} \tag{65}
\end{gather*}
$$

Thus, a pair of imaginary eigenvalues exists if the parameters $\beta$ and $c$ satisfy the condition

$$
\begin{equation*}
c^{2}=D\left(\frac{q}{p}-p-d\right) \tag{66}
\end{equation*}
$$



Figure 1: The traveling wave solution of system (8) from $E_{1}(1,0)$ tends to $E\left(H^{*}, P^{*}\right)$ monotonously with the parameters $D=0.8, r=1$, $a=1, m=0.3, b=6.65, d=0.5$, and $\beta=0.5$.

Regarding $\lambda$ as a function of $\beta$ and differentiating the characteristic equation (64) with respect to $\beta$, we obtain

$$
\begin{equation*}
\frac{\mathrm{d} \lambda(\beta)}{\mathrm{d} \beta}=\frac{\left(p^{\prime} / c\right) \lambda^{2}-\left(p^{\prime} / D\right) \lambda+\left(q^{\prime} / c D\right)}{3 \lambda^{2}-2(((p+d) / c)+(c / D)) \lambda+(p / D)} . \tag{67}
\end{equation*}
$$

Here $\left({ }^{\prime}\right)$ denotes the differentiation with respect to $\beta$. Substituting $\lambda=k i$ into (66), we have

$$
\begin{equation*}
\frac{\mathrm{d} \lambda(\beta)}{\mathrm{d} \beta}=\frac{\left(-\left(p^{\prime} / c\right) k^{2}+\left(q^{\prime} / c D\right)\right)-\left(p^{\prime} / D\right) k i}{\left(3 k^{2}+(p / D)\right)-2(((p+d) / c)+c / D) k i} \tag{68}
\end{equation*}
$$

After some calculation, we have that the sign of the real part of $d \lambda(\beta) / d \beta$ is determined by the sign of

$$
\begin{equation*}
\frac{-2}{c D}\left(p q^{\prime}-p^{2} p^{\prime}-p^{\prime} q\right) \tag{69}
\end{equation*}
$$

From (64), we know $\left(\mathrm{d} c^{2} / \mathrm{d} \beta\right)=\left(D / p^{2}\right)\left(p q^{\prime}-p^{2} p^{\prime}-p^{\prime} q\right)$. Thus, it is obvious that

$$
\begin{equation*}
-\frac{2 p^{2}}{c D^{2}} \frac{\mathrm{~d} c^{2}}{\mathrm{~d} \beta}=\frac{-2}{c D}\left(p q^{\prime}-p^{2} p^{\prime}-p^{\prime} q\right) \tag{70}
\end{equation*}
$$


(a) The traveling wave of prey


- $t=400$

$t=420$$\quad$ - | $t=460$ |
| ---: |
| $t=500$ |

(c) The traveling wave of predator at different time

(b) The traveling wave of predator

(d) The traveling wave of predator with different time

Figure 2: The traveling wave solution of system (8) from $E_{1}(10,0)$ tends to $E\left(H^{*}, P^{*}\right)$ nonmonotonously with the parameters $D=0.8, r=1.5$, $a=0.15, m=0.35, b=0.15, d=1$, and $\beta=0.75$.

So the sign of $\operatorname{Re}(\mathrm{d} \lambda(\beta) / \mathrm{d} \beta)$ is opposite to that of $d c^{2} / d \beta$. In fact, $q^{\prime}=\operatorname{ad}\left(H^{*}\right)_{\beta}^{\prime}=d(1-m) / a \beta^{2}>0$, while

$$
\begin{aligned}
p^{\prime}= & \{M-d\}_{\beta}^{\prime} \\
= & \left\{-a H^{*}+\frac{d(1-m)^{2} H^{*}}{\beta\left(b+(1-m) H^{*}\right)}-d\right\}_{\beta}^{\prime} \\
= & -\frac{d(1-m)}{\beta^{2}} \\
& +\left(\frac{d^{2}(1-m)^{3}}{a \beta}\left[b+(1-m) H^{*}\right]\right.
\end{aligned}
$$

$$
\left.-d(1-m)^{2} H^{*}\left[b+(1-m) H^{*}+\frac{d(1-m)^{2}}{a \beta}\right]\right)
$$

$$
\begin{align*}
& \times\left(\beta^{2}\left[b+(1-m) H^{*}\right]^{2}\right)^{-1} \\
=d(1-m) & \left\{\frac{b d(1-m)^{2}}{a \beta}\right. \\
& \left.-\left[(1-m)^{2}\left(H^{*}\right)^{2}+3 b(1-m) H^{*}+b^{2}\right]\right\} . \tag{71}
\end{align*}
$$

Define function $h\left(H^{*}\right)=(1-m)^{2}\left(H^{*}\right)^{2}+3 b(1-m) H^{*}+b^{2}$, and then $h^{\prime}\left(H^{*}\right)=2(1-m) H^{*}+3 b(1-m)>0$ if $0<H^{*}<$ $r / a$, where $\left(^{\prime}\right)$ denotes the differentiation with respect to $H^{*}$. So $h\left(H^{*}\right)$ is increasing with respect to $H^{*}$. Thus,

$$
\begin{align*}
h\left(H^{*}\right) & <h\left(\frac{r}{a}\right)=\frac{(1-m)^{2} r^{2}+3 a b r(1-m)+a^{2} b^{2}}{a^{2}}  \tag{72}\\
& <\frac{((1-m) r+(3 / 2) a b)^{2}}{a^{2}} .
\end{align*}
$$

So if

$$
\begin{equation*}
\beta<\frac{a b d(1-m)^{2}}{((1-m) r+(3 / 2) a b)^{2}}, \tag{73}
\end{equation*}
$$

we have

$$
\begin{equation*}
\frac{b d(1-m)^{2}}{a \beta}-h\left(H^{*}\right)>0, \quad \text { that is } p^{\prime}>0 \tag{74}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
\frac{\mathrm{d} c^{2}}{\mathrm{~d} \beta}=\frac{D}{p^{2}}\left(p q^{\prime}-p^{\prime}\left(p^{2}+q\right)\right)<0 \tag{75}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\operatorname{Re}\left(\frac{\mathrm{d} \lambda(\beta)}{\mathrm{d} \beta}\right)>0 \tag{76}
\end{equation*}
$$

By the Hopf bifurcation Theorem, we obtain that when the parameter $\beta$ crosses the bifurcation curve $c^{2}=D((q / p)-p-$ d) at $\beta_{0}$ in the $\beta-c$ parameter plane, system (8) undergoes a Hopf bifurcation to a small amplitude periodic solution at the equilibrium point $\left(H^{*}, P^{*}, 0\right)$. It corresponds to a small amplitude traveling wave train solution of system (5). This completes the proof.

## 4. Numerical Simulations

In this section, we will give numerical examples to illustrate the results of Theorems 2 and 3. All the numerical simulations are under the Neumann boundary conditions.

Figure 1 shows that there exists traveling wave solution and it from $E_{1}((r / a), 0)$ tends to $E\left(H^{*}, P^{*}\right)$ monotonously. In Figure 1, we consider the following parameters $D=0.8, r=1$, $a=1, m=0.3, b=6.65, d=0.5$, and $\beta=0.5$. Figure 2 shows that there exists traveling wave solution and it from $E_{1}((r / a), 0)$ tends to $E\left(H^{*}, P^{*}\right)$ nonmonotonously with the parameters $D=0.8, r=1.5, a=0.15, m=0.35, b=0.15$, $d=1$, and $\beta=0.75$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The authors would like to thank the reviewers for their helpful suggestions and comments. This research is supported by the National Science Foundation of China (11001204).

## References

[1] Y. Kuang and E. Beretta, "Global qualitative analysis of a ratiodependent predator-prey system," Journal of Mathematical Biology, vol. 36, no. 4, pp. 389-406, 1998.
[2] D. Xiao and S. Ruan, "Global dynamics of a ratio-dependent predator-prey system," Journal of Mathematical Biology, vol. 43, no. 3, pp. 268-290, 2001.
[3] Y.-H. Fan and W.-T. Li, "Global asymptotic stability of a ratiodependent predator-prey system with diffusion," Journal of Computational and Applied Mathematics, vol. 188, no. 2, pp. 205-227, 2006.
[4] P. H. Leslie, "Some further notes on the use of matrices in population mathematics," Biometrika, vol. 35, pp. 213-245, 1948.
[5] P. H. Leslie, "A stochastic model for studying the properties of certain biological systems by numerical methods," Biometrika, vol. 45, pp. 16-31, 1958.
[6] X. Guan, W. Wang, and Y. Cai, "Spatiotemporal dynamics of a Leslie-Gower predator-prey model incorporating a prey refuge," Nonlinear Analysis: Real World Applications, vol. 12, no. 4, pp. 2385-2395, 2011.
[7] M. A. Aziz-Alaoui, "Study of a Leslie-Gower-type tritrophic population model," Chaos, Solitons \& Fractals, vol. 14, no. 8, pp. 1275-1293, 2002.
[8] L. Chen and F. Chen, "Global stability of a Leslie-Gower predator-prey model with feedback controls," Applied Mathematics Letters, vol. 22, no. 9, pp. 1330-1334, 2009.
[9] F. Chen, L. Chen, and X. Xie, "On a Leslie-Gower predator-prey model incorporating a prey refuge," Nonlinear Analysis: Real World Applications, vol. 10, no. 5, pp. 2905-2908, 2009.
[10] S. Yuan and Y. Song, "Stability and Hopf bifurcations in a delayed Leslie-Gower predator-prey system," Journal of Mathematical Analysis and Applications, vol. 355, no. 1, pp. 82-100, 2009.
[11] P. Aguirre, E. González-Olivares, and E. Sáez, "Three limit cycles in a Leslie-Gower predator-prey model with additive Allee effect," SIAM Journal on Applied Mathematics, vol. 69, no. 5, pp. 1244-1262, 2009.
[12] T. K. Kar, "Stability analysis of a prey-predator model incorporating a prey refuge," Communications in Nonlinear Science and Numerical Simulation, vol. 10, no. 6, pp. 681-691, 2005.
[13] V. Křivan, "Effects of optimal antipredator behavior of prey on predator-prey dynamics: the role of refuges," Theoretical Population Biology, vol. 53, no. 2, pp. 131-142, 1998.
[14] E. González-Olivares and R. Ramos-Jiliberto, "Dynamic consequences of prey refuges in a simple model system: more prey, fewer predators and enhanced stability," Ecological Modelling, vol. 166, no. 1-2, pp. 135-146, 2003.
[15] Y. Huang, F. Chen, and L. Zhong, "Stability analysis of a preypredator model with Holling type III response function incorporating a prey refuge," Applied Mathematics and Computation, vol. 182, no. 1, pp. 672-683, 2006.
[16] S. R. Dunbar, "Travelling wave solutions of diffusive LotkaVolterra equations," Journal of Mathematical Biology, vol. 17, no. 1, pp. 11-32, 1983.
[17] S. R. Dunbar, "Traveling wave solutions of diffusive LotkaVolterra equations: a heteroclinic connection in $\mathbf{R}^{4}$," Transactions of the American Mathematical Society, vol. 286, no. 2, pp. 557-594, 1984.
[18] S. R. Dunbar, "Traveling waves in diffusive predator-prey equations: periodic orbits and point-to-periodic heteroclinic orbits," SIAM Journal on Applied Mathematics, vol. 46, no. 6, pp. 1057-1078, 1986.
[19] J. Zhang, "Existence of travelling waves in a modified vectordisease model," Applied Mathematical Modelling, vol. 33, no. 2, pp. 626-632, 2009.
[20] X. Hou and A. W. Leung, "Traveling wave solutions for a competitive reaction-diffusion system and their asymptotics," Nonlinear Analysis: Real World Applications, vol. 9, no. 5, pp. 2196-2213, 2008.
[21] S. Ahmad, A. C. Lazer, and A. Tineo, "Traveling waves for a system of equations," Nonlinear Analysis: Theory, Methods \& Applications, vol. 68, no. 12, pp. 3909-3912, 2008.
[22] B. I. Camara, "Waves analysis and spatiotemporal pattern formation of an ecosystem model," Nonlinear Analysis: Real World Applications, vol. 12, no. 5, pp. 2511-2528, 2011.
[23] J. Huang, G. Lu, and S. Ruan, "Existence of traveling wave solutions in a diffusive predator-prey model," Journal of Mathematical Biology, vol. 46, no. 2, pp. 132-152, 2003.
[24] W.-T. Li and S.-L. Wu, "Traveling waves in a diffusive predatorprey model with Holling type-III functional response," Chaos, Solitons \& Fractals, vol. 37, no. 2, pp. 476-486, 2008.
[25] P. Hartman, Ordinary Differential Equations, John Wiley \& Sons, New York, NY, USA, 1964.
[26] J. P. LaSalle, "Stability theory for ordinary differential equations," Journal of Differential Equations, vol. 4, pp. 57-65, 1968.
[27] R. Seydel, Practical Bifurcation and Stability Analysis: From Equilibrium to Chaos, vol. 5 of Interdisciplinary Applied Mathematics, Springer, New York, NY, USA, 2nd edition, 1994.

## Research Article

# Modeling Peer-to-Peer Botnet on Scale-Free Network 

Liping Feng, ${ }^{1}$ Hongbin Wang, ${ }^{1}$ Qi Han, ${ }^{2}$ Qingshan Zhao, ${ }^{1}$ and Lipeng Song ${ }^{3}$<br>${ }^{1}$ Department of Computer Science and Technology, Xinzhou Teachers University, Xinzhou 034000, China<br>${ }^{2}$ School of Electronic and Information Engineering, Chongqing University of Science and Technology, Chongqing 401331, China<br>${ }^{3}$ Department of Computer Science and Technology, North University of China, Taiyuan 030051, China<br>Correspondence should be addressed to Lipeng Song; slp880@gmail.com

Received 23 January 2014; Accepted 27 March 2014; Published 23 April 2014
Academic Editor: Yun Kang
Copyright © 2014 Liping Feng et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

Peer-to-peer (P2P) botnets have emerged as one of the serious threats to Internet security. To prevent effectively P2P botnet, in this paper, a mathematical model which combines the scale-free trait of Internet with the formation of P2P botnet is presented. Explicit mathematical analysis demonstrates that the model has a globally stable endemic equilibrium when infection rate is greater than a critical value. Meanwhile, we find that, in scale-free network, the critical value is very little. Hence, it is unrealistic to completely dispel the P2P botnet. Numerical simulations show that one can take effective countermeasures to reduce the scale of P2P botnet or delay its outbreak. Our findings can provide meaningful instruction to network security management.


## 1. Introduction

A botnet is a network of thousands of compromised computers (bots) under the control of botmaster, which usually recruits new vulnerable computers by running all kinds of malicious software, such as Trojan horses, worms, and computer viruses [1]. For nefarious profits, the botnetmaster which operates a botnet manipulates remotely zombie computers to work on various malicious activities, such as distributed denial-of-service attacks (DDoS), email spam, and password cracking. Nowadays, botnets have become one of the most serious threats to Internet.

According to operating mechanism of botnets, there are two kinds of botnets. One is the traditional botnet using Internet relay chat (IRC) as a form of communication for centralized command and control (C\&C) structure (see Figure 1 [2]). The other is peer-to-peer botnet utilizing a distributed command-and-control structure (see Figure 2 [2]). Traditional botnets are easily checked and cracked by defenders, and the threats of botnets can be mitigated and eliminated if the central of $\mathrm{C} \& \mathrm{C}$ is unavailable [3]. By contrast, P2P botnets employing a decentralized command-and-control structure are more robust and are much harder for security community to dismantle [4]. Therefore, P2P botnets, such as Trojan.Peacomm and Storm botnet [5], have
emerged and gradually escalated in recent years. Moreover, P2P botnets are increasingly sophisticated and thus their potential damage is much greater than traditional botnets. Further, the potential for more damage exits in the future.

Therefore, threats of P2P botnets to Internet security have drawn widespread attention [6-12]. Yan et al. [6] mathematically analyzed the performance of Antbot-a new type of P2P botnets-from the perfectives of resilience, reachability, and scalability, and the authors developed a distributed P2P botnet simulator to evaluate the effectiveness of Antbot against pollution-based mitigation in practice. Kolesnichenko et al. [7] developed the mean-field model to analyze behaviors of P2P botnet and compared it with simulations obtained from the Mobius tool (a software tool for modeling the behavior of complex systems). Results show that the mean-field method is much faster than simulation for predicting the behavior of P2P botnet. van Ruitenbeek and Sanders [8] presented a stochastic model of Storm Worm P2P botnet to examine how different factors, such as the removal rate and the initial infection rate, impact the total propagation bots. To be well prepared for future botnet attacks, Wang et al. [9] studied advanced botnet attack techniques that could be developed by botmasters in the future and proposed the design of an advanced hybrid P2P botnet. Results show that a honeypot, in computer terminology, is a trap set to detect, deflect, or,


Figure 1: Centralized botnet.


Figure 2: P2P botnet.
in some manner, counteract attempts at unauthorized use of information systems. Generally, a honeypot consists of a computer, data, or a network site that appears to be part of a network, but is actually isolated and monitored, and which seems to contain information or a resource of value to attackers-play an important role to defend against an advanced botnet.

Nevertheless, few people studied the dynamical behaviors of P2P botnets. In [7], the authors proposed a mean-field model of P2P botnet, but the model has not been analyzed mathematically. In fact, explicit mathematical analysis contributes to understand deeply the prevalent characteristics of P2P botnet. Aiming at describing the dynamics of P2P botnets in a more effective way, in this paper, we employ the dynamical model of computer worms, which has been widely used by many researchers to study Internet malware propagation [13-22]. As many botnets are created by computer worms [23], it is reasonable to describe the prevalence of P2P botnets with the model of worm propagation. In addition, by analyzing data from real computer virus epidemics, the authors [24] pointed out the importance of incorporating the peculiar topology of scale-free network in the theoretical description of computer worm propagation. In biological epidemic areas, there is much valuable research which considers the effect
of complex network on pathophoresis [25, 26]. However, we have not seen the report which considers the effect of complex network on prevalence of P2P botnet. Hence, it is necessary to examine the effect of the topology of the network on the propagation of P 2 P botnet.

In this paper, the dynamics of leaching P2P botnets are investigated. In a leaching P2P botnet, botmasters recruit new zombies on the Internet. For constructing this kind of P2P botnet, there are two steps: the first step is trying to infect new vulnerable hosts throughout the Internet, and the second step is newly compromised hosts joining the botnet and connecting with other bots [2]. In SF network, taking into account the heterogeneity induced by the hosts with different degree $k$, we divide the hosts into different states where the hosts in each state have the same degree $k$.

## 2. The Model

To model the propagation of the P2P botnet on the Internet, we assume that the total number of nodes on Internet is a constant $N$. Each node changes over time among four states: susceptible $(S)$, exposed $(E)$, infected $(I)$, and recovered $(R)$ due to the spread of computer worm. We describe these four states in detail as follows.
(1) Susceptible ( $S$ ): a node has the software vulnerability that the bot program can exploit.
(2) Exposed (E): a node has been infected by the bot program, but it has not become a member of P2P botnet.
(3) Infected $(I)$ : a node is a formal member of P2P botnet, which means the node can infect its neighbors with the bot program.
(4) Removed ( $R$ ): a node has installed a detection tool that can identify and remove the bot program, or a node has installed a software patch to eliminate the node vulnerability exploited by the bot program.

There are five state transitions among these four states.
(1) Propagating the bot program: nodes in the "susceptible" state will change to the "exposed" state with the infection rate $\beta$.
(2) Joining the P2P botnet from exposed state: nodes in the "exposed" state will join the P2P botnet under the control of the botmaster and change to "infected" state at the proportion $\delta$.
(3) Immunizing nodes from susceptible state: nodes in the "susceptible" state will change to the "recovered" state at the proportion $r_{s}$ if corresponding nodes take countermeasures, for example, antivirus software, patching, firewall, and intrusion detection system (IDS). The immune rate is affected by many factors, for example, user vigilance.
(4) Immunizing nodes from exposed state: nodes in the "exposed" state will change to the "recovered" state at the proportion $r_{1}$ if corresponding nodes take antivirus countermeasures.
(5) Immunizing nodes from infected state: nodes in the "infected" state will change to the "recovered" state at the proportion $r_{2}$ if corresponding nodes take antivirus countermeasures.

Let $S_{k}(t), E_{k}(t), I_{k}(t)$, and $R_{k}(t)$ be the number of degree $k$ in states $S, E, I$, and $R$ at time $t$, respectively. Then one has

$$
\begin{equation*}
S_{k}(t)+E_{k}(t)+I_{k}(t)+R_{k}(t)=N . \tag{1}
\end{equation*}
$$

The dynamic equations can be written as

$$
\begin{gather*}
\frac{d S_{k}(t)}{d t}=\mu-\alpha k \theta(I(t)) S_{k}(t)-\left(\mu+r_{s}\right) S_{k}(t) \\
\frac{d E_{k}(t)}{d t}=\alpha k \theta(I(t)) S_{k}(t)-\left(\mu+r_{1}+\delta\right) E_{k}(t) \\
\frac{d I_{k}(t)}{d t}=\delta E_{k}(t)-\left(\mu+r_{2}\right) I_{k}(t)  \tag{2}\\
\frac{d R_{k}(t)}{d t}= \\
r_{s} S_{k}(t)+r_{1} E_{k}(t)+r_{2} I_{k}(t)-\mu R_{k}(t)
\end{gather*}
$$

where the probability $0 \leq \theta(I(t)) \leq 1$ describes a link pointing to an infected host, which satisfies the relation

$$
\begin{equation*}
\theta(I(t))=\frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k}(t), \tag{3}
\end{equation*}
$$

and $I(t)=\sum_{k} P(k) I_{k}$ is the density of infected hosts in the whole network at time $t ; P(k)$ is a degree distribution. Other parameters can be explained as follows. $\mu$ is the replacement rate of the hosts per hour; $\alpha$ is infection rate per hour; $r_{s}$ is the state transition rate from $S_{k}$ to $R_{k}$ due to immune measures; $r_{i}(i=1,2)$ is the recovery rate from exposed state $E_{k}$ and infected state $I_{k}$, respectively; and $\delta$ is transition rate from $E_{k}$ to $I_{k}$.

## 3. Model Analysis

In this subsection, we solve the equilibria of system (2) and investigate their stability.

The first three equations in system (2) do not depend on the fourth equation, and, therefore, this equation may be omitted without loss of generality. Hence, system (2) can be rewritten as

$$
\begin{gather*}
\frac{d S_{k}(t)}{d t}=\mu-\alpha k \theta(I(t)) S_{k}(t)-\left(\mu+r_{s}\right) S_{k}(t) \\
\frac{d E_{k}(t)}{d t}=\alpha k \theta(I(t)) S_{k}(t)-\left(\mu+r_{1}+\delta\right) E_{k}(t)  \tag{4}\\
\frac{d I_{k}(t)}{d t}=\delta E_{k}(t)-\left(\mu+r_{2}\right) I_{k}(t)
\end{gather*}
$$

The equilibria of system (7) are determined by setting

$$
\begin{gathered}
\mu-\alpha k \theta(I(t)) S_{k}(t)-\left(\mu+r_{s}\right) S_{k}(t)=0, \\
\alpha k \theta(I(t)) S_{k}(t)-\left(\mu+r_{1}+\delta\right) E_{k}(t)=0, \\
\delta E_{k}(t)-\left(\mu+r_{2}\right) I_{k}(t)=0 .
\end{gathered}
$$

There is always a disease-free equilibrium (DFE) $Q_{0}=$ $\left(\mu /\left(\mu+r_{s}\right), 0,0\right)$. Furthermore, solving the endemic equilibrium of (5), one can obtain $Q_{1}=\left(S_{k}^{*}, E_{k}^{*}, I_{k}^{*}\right)$, where

$$
\begin{gather*}
S_{k}^{*}=\frac{\mu}{\alpha k \theta+\mu+r_{s}}, \\
E_{k}^{*}=\frac{\mu \alpha k \theta}{\left(\alpha k \theta+\mu+r_{s}\right)\left(\mu+r_{1}+\delta\right)},  \tag{6}\\
I_{k}^{*}=\frac{\delta \mu \alpha k \theta}{\left(\alpha k \theta+\mu+r_{s}\right)\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)} .
\end{gather*}
$$

Substituting $I_{k}^{*}$ into (3), we have

$$
\begin{align*}
\theta & =\frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k}(t)  \tag{7}\\
& =\frac{1}{\langle k\rangle} \sum_{k} k P(k) \frac{\delta \mu \alpha k \theta}{\left(\alpha k \theta+\mu+r_{s}\right)\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)} .
\end{align*}
$$

Obviously, if the endemic equilibrium exists, there must be $0<\theta \leq 1$. That is, it must satisfy

$$
\begin{align*}
& \left.\frac{d}{d \theta}\left[\frac{1}{\langle k\rangle} \sum_{k} k P(k) \frac{\delta \mu \alpha k \theta}{\left(\alpha k \theta+\mu+r_{s}\right)\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)}\right]\right|_{\theta=0} \\
& \quad \geq 1 \tag{8}
\end{align*}
$$

and it equals

$$
\begin{aligned}
\frac{1}{\langle k\rangle} \sum_{k} k P(k)\{ & \left(\delta \mu \alpha k\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)\left(\alpha k \theta+\mu+r_{s}\right)\right. \\
& \left.-\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right) \delta \mu \alpha^{2} k^{2} \theta\right) \\
& \left.\times\left(\left(\alpha k \theta+\mu+r_{s}\right)\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)\right)^{-1}\right\}\left.\right|_{\theta=0}
\end{aligned}
$$

$\geq 1$.

Let $\alpha_{c}$ be the minimum value of $\alpha$ satisfying the above inequality. Then,

$$
\begin{equation*}
\frac{\delta \mu \alpha_{c}}{\langle k\rangle\left(\mu+r_{s}\right)\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)} \sum_{k} k^{2} P(k)=1 \tag{10}
\end{equation*}
$$

that is

$$
\begin{equation*}
\frac{\left\langle k^{2}\right\rangle \delta \mu \alpha_{c}}{\langle k\rangle\left(\mu+r_{s}\right)\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)}=1, \tag{11}
\end{equation*}
$$

where $\left\langle k^{2}\right\rangle=\sum_{k} k^{2} P(k)$.
Hence,

$$
\begin{equation*}
\alpha_{c}=\frac{\langle k\rangle\left(\mu+r_{s}\right)\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)}{\left\langle k^{2}\right\rangle \delta \mu} . \tag{12}
\end{equation*}
$$

Summarizing the above analysis, one can get the following theorem.

Theorem 1. If $\alpha<\alpha_{c}$, then system (4) has only one freeequilibrium $Q_{0}$; if $\alpha>\alpha_{c}$, then system (4) has endemicequilibrium $Q^{*}$ except $Q_{0}$.

In what follows, the endemic-equilibrium point $Q^{*}$ will be analyzed.

The Jacobian matrix of system (4) at $Q^{*}$ is

$$
J=\left(\begin{array}{ccc}
-\alpha k \frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k}^{*} & 0 & \alpha k S_{k}^{*} \frac{1}{\langle k\rangle} \sum_{k} k P(k)  \tag{13}\\
\alpha k \frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k}^{*} & -\left(\mu+\gamma_{1}+\delta\right) & \alpha k S_{k}^{*} \frac{1}{\langle k\rangle} \sum_{k} k P(k) \\
0 & \delta & -\left(\mu+\gamma_{2}\right)
\end{array}\right),
$$

and the associated characteristic equation is

$$
\begin{equation*}
\lambda^{3}+a \lambda^{2}+b \lambda+c=0 \tag{14}
\end{equation*}
$$

where

$$
\begin{align*}
& a=\mu+r_{1}+\delta+\mu+r_{2}+\alpha k \frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k}^{*}, \\
& b=\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)-\delta \alpha k S_{k}^{*} \frac{1}{\langle k\rangle} \sum_{k} k P(k) \\
&+\left(\alpha k \frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k}^{*}+\mu+r_{s}\right)\left(r_{1}+\delta+2 \mu+r_{2}\right), \\
& c=\left(\alpha k \frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k}^{*}+\mu+r_{s}\right)\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right) \\
&-\left(\delta \alpha k S_{k}^{*} \frac{1}{\langle k\rangle} \sum_{k} k P(k)\right) \\
& \times\left(\alpha k \frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k}^{*}+\mu+r_{s}+\alpha k \frac{1}{\langle k\rangle} \sum_{k} k P(k)\right) . \tag{15}
\end{align*}
$$

According to Hurwitz criteria [27],

$$
\begin{gather*}
H_{1}=\mu+r_{1}+\delta+\mu+r_{2}+\alpha k \frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k}^{*}>0  \tag{16}\\
H_{2}=H_{1} b-c, \quad H_{3}=H_{2} c
\end{gather*}
$$

Hence, one can obtain the following lemmas.
Lemma 2. For system (4), if $\mathrm{H}_{2}>0$ and $\mathrm{H}_{3}>0$ hold, then the endemic-equilibrium $Q^{*}$ is locally asymptotically stable.

For depicting the globally asymptotical stability of $Q^{*}$, firstly, one can introduce three preliminary results.

Lemma 3 (see [28, 29]). Suppose that the initial relative infected density $0<I_{k}(0)<1$ satisfies $\sum_{k} k P(k) I_{k}(0)>0$. Then, for all $t>0$, the solution of system (4) satisfies $0<$ $\theta(I(t))<1$ and $0<I_{k}(t)<1$.

Proposition 4 (see $[28,29]$ ). Suppose that the solution $I_{k}(t)$ of system (4) satisfies $\lim \sup _{t \rightarrow \infty} I_{k} \leq U_{k}$ and $\liminf _{t \rightarrow \infty} I_{k} \geq$ $\ell_{k}$, where $U_{k} \geq 0$ and $\ell_{k} \geq 0$. Then,

$$
\begin{align*}
\lim _{t \rightarrow \infty} \sup I_{k} \leq & \left(\alpha \delta \mu k \frac{1}{\langle k\rangle} \sum_{k} k P(k) U_{k}\right) \\
& \times\left(\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)\right. \\
& \left.\times\left(\mu+r_{s}+\alpha k \frac{1}{\langle k\rangle} \sum_{k} k P(k) U_{k}\right)\right)^{-1}, \\
\lim _{t \rightarrow \infty} \inf I_{k} \geq & \left(\alpha \delta \mu k \frac{1}{\langle k\rangle} \sum_{k} k P(k) \ell_{k}\right) \\
& \times\left(\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)\right. \\
& \left.\times\left(\mu+r_{s}+\alpha k \frac{1}{\langle k\rangle} \sum_{k} k P(k) U_{k}\right)\right)^{-1} . \tag{17}
\end{align*}
$$

Proposition 5 (see [28, 29]). Suppose that the initial relative infected densities $0<I_{k}(0)<1$ satisfy $\alpha>\alpha_{c}$ and $\sum_{k} k P(k) I_{k}(0)>0$. Then, the solution of system (4) satisfies $\lim _{t \rightarrow \infty} \inf \theta(I(t))>0$ and $\lim _{t \rightarrow \infty} \inf I_{k}(t)>0$.

The proofs of the above conclusions are similar to those presented in [28, 29]. Here, we will omit them.

Next, main results will be presented.
Lemma 6. Suppose that the initial relative infected densities $0<I_{k}(0)<1$ satisfy $\alpha>\alpha_{c}$ and $\sum_{k} k P(k) I_{k}(0)>$ 0 . Then, the solution of system (4) satisfies $\lim _{t \rightarrow \infty} I_{k}(t)=$ $I_{k}, \lim _{t \rightarrow \infty} E_{k}(t)=E_{k}$, and $\lim _{t \rightarrow \infty} S_{k}(t)=S_{k}$, where $I_{1}, I_{2}, I_{3}, \ldots, I_{n}\left(E_{1}, E_{2}, E_{3}, \ldots, E_{n} ; S_{1}, S_{2}, S_{3}, \ldots, S_{n}\right)$ are the unique nonzero stationary points of system (4).

The proof is completed in the appendix
Combining Lemma 2 with Lemma 6, one can conclude the following conclusion.

Theorem 7. If the endemic-equilibrium $Q^{*}$ exists, then it is globally asymptotically stable.

## 4. Numerical Analysis and Control Strategies

4.1. Numerical Examples. In this subsection we present the results of numerical experiments investigating the effectiveness of theoretic analysis. In order to observe the effects of parameters on transmission process, we use system (4) to simulate the evolution behavior of P2P botnet for given parameters on SF network with $\langle k\rangle=8$ and $N=100000$. Here, we set the parameter values of system (4) which are, respectively, $\mu=0.01, r_{s}=0.01, r_{1}=0.06, r_{2}=0.06$, and $\delta=0.6$. By calculation, one can obtain $\alpha_{c}=1.49 \times 10^{-5}$.


Figure 3: The density of infected nodes with parameters $\mu=0.01$, $r_{s}=0.01, r_{1}=0.06, r_{2}=0.06, \alpha=0.005>\alpha_{c},\langle k\rangle=8$, and $N=100000$.


Figure 4: The density of infected nodes with parameters $\mu=0.01$, $r_{s}=0.01, r_{1}=0.06, r_{2}=0.06, \alpha=1.5 \times 10^{-8}\left\langle\alpha_{c},\langle k\rangle=8\right.$, and $N=100000$.

Figures 3 and 4 show the simulation results with $\alpha=0.005>$ $\alpha_{c}$ and $\alpha=1.5 \times 10^{-6}<\alpha_{c}$, respectively, which are consistent with theoretical analysis.

From the conclusion of Theorem 7, we learn that it is necessary for eliminating P2P botnet on the Internet to let $\alpha<\alpha_{c}$ by corresponding countermeasures. Meanwhile, the simulation results show that the critical value of infection $\alpha_{c}$ is very little, and this means that it is difficult to destroy completely the P2P botnet in reality.
4.2. Control Strategies. In what follows, we consider mainly the effect of the real-time immune measurement and antivirus software on the scale of the P2P botnet.
(i) For fixed model parameters, $\mu=0.01, r_{1}=0.06, r_{2}=$ $0.06, \delta=0.6$, and $\alpha=0.005$, we investigate the effect of different real-time immunity $\left(r_{s}\right)$ on the scale of P2P botnet. Simulation result is depicted in Figure 5. From Figure 5, it can be observed that enhancing


Figure 5: An illustration of the impact of real-time immune measure ( $r_{s}$ ) on the density of infected nodes.


Figure 6: An illustration of the impact of antivirus software ( $\delta$ ) on the density of infected nodes.
real-time immune measures contributes to reduce the scale of P2P botnet and delay its outbreak. Hence, it is strongly advised that network users should install patches for bugs in time and update antivirus software to the latest version.
(ii) For fixed model parameters, $\mu=0.01, r_{1}=0.06$, $r_{2}=0.06, r_{s}=0.01$, and $\alpha=0.005$, we investigate the effect of antivirus software ( $\delta$ ) on the scale of P2P botnet. Simulation results are depicted in Figure 6. The profile of Figure 6 demonstrates that the larger percent conversion from $E$ to $I$ there is, the bigger scale a P2P botnet has. Thus, it is proposed that malware is killed when the node is infected by the bot program but does not join botnet.


Figure 7: An illustration of the impact of average degree $(\langle k\rangle)$ on the density of infected nodes.

Additionally, the effect of average degree $\langle k\rangle$ on prevalent behavior of P2P botnet is depicted in Figure 7. From Figure 7, we find that the scale of P2P botnet will increase when $\langle k\rangle$ becomes larger. So decreasing the average degree of network can also control the massive outbreak of P2P botnet.

## 5. Conclusions

As a new kind of attack platform to network security, P2P botnets have attracted considerable attention. Research is necessary to fully understand the threat and prepare to defend against it. To better exploit the spreading behavior of P2P botnet, in this paper, we present a mathematical model of creation of P2P botnet, which combines the scale-free character of Internet with the formation trait of P2P botnet. Hence, the model can portrait more accurately the dynamical features of P2P botnet propagation. Theoretical analysis shows that the model has a globally stable endemic equilibrium. The influence of some parameters to the scale of P2P botnet has been investigated. Simulation results demonstrate that it is difficult to destroy completely the P2P botnet in reality. This is the reason that many malwares saturate to a very low level of persistence [30]. However, Figures 6 and 7 show that we can reduce the scale of P2P botnet and delay its outbreak by efficient countermeasures, such as real-time immunity or autorunning of antivirus software.

The dynamical model we present could be extended to study the growth possibilities of P2P botnets in future work. The model is also possible to predict how botnetmasters could create more potent and aggressive botnets. Such predictions could ultimately be useful to antimalware developers as well.

## Appendix

Proof of Lemma 6. Substituting (3)into $I_{k}^{*}$, we can obtain

$$
\begin{align*}
I_{k}^{*}= & \left(\alpha \delta \mu k \frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k}\right) \\
& \times\left(\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)\right.  \tag{A.1}\\
& \left.\times\left(\mu+r_{s}+\alpha k \frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k}\right)\right)^{-1}
\end{align*}
$$

Let $U_{k}^{(1)}=1$, and define the following sequence:

$$
\begin{align*}
U_{k}^{(m+1)}= & \left(\alpha \delta \mu k \frac{1}{\langle k\rangle} \sum_{k} k P(k) U_{k}^{(m)}\right) \\
& \times\left(\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)\right. \\
& \left.\times\left(\mu+r_{s}+\alpha k \frac{1}{\langle k\rangle} \sum_{k} k P(k) U_{k}^{(m)}\right)\right)^{-1} \tag{A.2}
\end{align*}
$$

Then, according to Lemma 3, for $1 \leq k \leq n$, $\lim _{t \rightarrow \infty} \sup I_{k}(t) \leq 1=U_{k}^{(1)}$. By applying Proposition 4, we obtain

$$
\begin{equation*}
\lim \sup _{t \rightarrow \infty} I_{k}(t) \leq U_{k}^{(m)}, \quad 0 \leq k \leq n, m=1,2, \ldots \tag{A.3}
\end{equation*}
$$

In what follows, consider the convergence of the sequence defined in (A.2). By (A.2), for all $k, U_{k}^{(2)} \leq 1=U_{k}^{(1)}$. If for all $k, U_{k}^{(m+1)} \leq U_{k}^{(m)}$, then it is easy to obtain $U_{k}^{(m+2)} \leq U_{k}^{(m+1)}$.

By induction, for all $k$, the sequence $U_{k}^{(m)}$ is decreasing, so its limit exists, denoted by $U_{k}=\lim _{m \rightarrow \infty} U_{k}^{(m)}$. Then it is easy to show that $U_{k}=\lim _{t \rightarrow \infty} \sup I_{k}(t) \leq U_{k}$.

On the other hand, substituting (A.1) into (3), we can get the following equation:

$$
\begin{align*}
\theta(t)= & \frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k} \\
= & \frac{1}{\langle k\rangle} \sum_{k} k P(k)\left(\alpha \delta \mu k \frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k}\right) \\
& \times\left(\left(\mu+r_{1}+\delta\right)\left(\mu+r_{2}\right)\right.  \tag{A.4}\\
& \left.\times\left(\mu+r_{s}+\alpha k \frac{1}{\langle k\rangle} \sum_{k} k P(k) I_{k}\right)\right)^{-1} .
\end{align*}
$$

From (7), $\theta=F(\theta)$, so by letting $\hbar(x)=F(x)-x$, one can obtain that $\hbar(0)=0$ and $\hbar^{\prime}(0)>0$. By the definition of derivative, if $x>0$ is sufficiently small, then $\hbar(x)>\hbar(0)=0$.

According to Proposition 5, we can take $\ell_{k}^{(1)}$ such that, for all $k, 0<\ell_{k}^{(1)}<\lim _{t \rightarrow \infty} \inf I_{k}(t)$.

Let

$$
\begin{equation*}
x=\frac{1}{\langle k\rangle} \sum_{k} k P(k) \ell_{k}^{(1)}, \quad \hbar\left(\frac{1}{\langle k\rangle} \sum_{k} k P(k) \ell_{k}^{(1)}\right)>0 \tag{A.5}
\end{equation*}
$$

we have

$$
\begin{equation*}
\frac{1}{\langle k\rangle} \sum_{k} k P(k) \ell_{k}^{(2)}>\frac{1}{\langle k\rangle} \sum_{k} k P(k) \ell_{k}^{(1)} \tag{A.6}
\end{equation*}
$$

If for all $k, \ell_{k}^{(m+1)}>\ell_{k}^{(m)}$, it is easy to obtain $\ell_{k}^{(m+2)}>$ $\ell_{k}^{(m+1)}$.

Thus, by induction, for each $k$, the sequence $\ell_{k}^{(m)}$ is increasing, so its limit exists, denoted by $\ell_{k}=\lim _{m \rightarrow \infty} \ell_{k}^{(m)}$. Thus, it is easy to verify that $\ell_{k}<\lim _{t \rightarrow \infty}$ inf $I_{k}(t)$.

Both $U_{k}$ and $\ell_{k}$ are positive stationary points of system (4). Therefore, by the uniqueness of the positive stationary point of the differential equation, we have $U_{k}=\ell_{k}=I_{k}$ and $I_{k} \leq \lim _{t \rightarrow \infty} \inf I_{k}(t) \leq \lim _{t \rightarrow \infty} \sup I_{k}(t) \leq I_{k}, 1 \leq k \leq n ;$ that is, $\lim _{t \rightarrow \infty} I_{k}(t)=I_{k}$.

Substituting $I_{k}$ into (5), we will obtain $\lim _{t \rightarrow \infty} E_{k}(t)=E_{k}$ and $\lim _{t \rightarrow \infty} S_{k}(t)=S_{k}$.

Lemma 6 is proven.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work is supported by the National Natural Science Foundation of China (61379125), Program for Basic Research of Shan'xi Province (2012011015-3), Higher School of Science and Technology Innovation Project of Shan'xi Province (2013148), Key Construction Disciplines of Xinzhou Teachers University (ZDXK201204, XK201307), Research Project of Chongqing University of Science and Technology (CK2013B15), and Research Program of Chongqing Municipal Education Commission (KJ131401).

## References

[1] L.-P. Song, Z. Jin, and G.-Q. Sun, "Modeling and analyzing of botnet interactions," Physica A, vol. 390, no. 2, pp. 347-358, 2011.
[2] P. Wang, B. Aslam, and C. C. Zou, Peer-To-Peer Botnets: The Next Generation of Botnet Attacks, School of Electrical Engineering and Computer Science, University of Central Florida, Orlando, Fla, USA, 2010.
[3] J. B. Grizzard, V. Sharma, C. Nunnery, and B. B. H. Kang, "Peer-to-peer botnet: overview and case study," in Proceedings of the 1st Workshop on Hot Topics in Understanding Botnets, pp. 1-8, 2007.
[4] Q. T. Han, W. Q. Yu, Y. Y. Zhang, and Z. W. Zhao, "Modeling and evaluating of typical advanced peer-to-peer botnet," Performance Evaluation, vol. 72, pp. 1-15, 2014.
[5] T. Holz, M. Steiner, F. Dahl, E. Biersack, and F. Freiling, "Measurements and mitigation of peer-to- peer-based botnets: a case study on storm worm," in Proceedings of the 1st Usenix Workshop on Large-Scale Exploits and Emergent Threats, 2008.
[6] G. Yan, D. T. Ha, and S. Eidenbenz, "AntBot: anti-pollution peer-to-peer botnets," Computer Networks, vol. 55, no. 8, pp. 19411956, 2011.
[7] A. Kolesnichenko, A. Remke, P. T. Boer, and B. R. Haverkort, "Comparison of the mean-field approach and simulation in a peer-to-peer botnet case study," in Computer Performance Engineering, vol. 6977, pp. 133-147, 2011.
[8] E. van Ruitenbeek and W. H. Sanders, "Modeling peer-to-peer botnets," in Proceedings of the 5th International Conference on the Quantitative Evaluation of Systems (QEST '08), pp. 307-316, September 2008.
[9] P. Wang, S. Sparks, and C. C. Zou, "An advanced hybrid peer-to-peer botnet," IEEE Transactions on Dependable and Secure Computing, vol. 7, no. 2, pp. 113-127, 2010.
[10] G. P. Schaffer, "Worms and viruses and botnets, Oh My! Rational responses to emerging Internet threats," IEEE Security and Privacy, vol. 4, no. 3, pp. 52-58, 2006.
[11] H. L. Jiang and X. X. . L. Shao, "Detecting P2P botnets by discovering flow dependency in C\&C traffic," in Peer-To-Peer Networking and Applications, pp. 1-12, 2012.
[12] M. Khosroshahy, M. K. Ali, and D. Y. Qiu, "The SIC botnet lifecycle model: a step beyond traditional epidemiological models," Computer Networks, vol. 57, pp. 404-421, 2013.
[13] X. Han, Y.-H. Li, L.-P. Feng, and L.-P. Song, "Influence of removable devices' heterouse on the propagation of malware," Abstract and Applied Analysis, vol. 2013, Article ID 296940, 6 pages, 2013.
[14] Y. Li, J. Pan, L. Song, and Z. Jin, "The influence of user protection behaviors on the control of internet worm propagation," Abstract and Applied Analysis, vol. 2013, Article ID 531781, 13 pages, 2013.
[15] L.-P. Song, X. Han, D.-M. Liu, and Z. Jin, "Adaptive human behavior in a two-worm interaction model," Discrete Dynamics in Nature and Society, vol. 2012, Article ID 828246, 13 pages, 2012.
[16] L.-P. Song, Z. Jin, G.-Q. Sun, J. Zhang, and X. Han, "Influence of removable devices on computer worms: dynamic analysis and control strategies," Computers \& Mathematics with Applications, vol. 61, no. 7, pp. 1823-1829, 2011.
[17] L.-X. Yang and X. Yang, "Propagation behavior of virus codes in the situation that infected computers are connected to the internet with positive probability," Discrete Dynamics in Nature and Society, vol. 2012, Article ID 693695, 13 pages, 2012.
[18] Q. Zhu, X. Yang, L.-X. Yang, and X. Zhang, "A mixing propagation model of computer viruses and countermeasures," Nonlinear Dynamics, vol. 73, no. 3, pp. 1433-1441, 2013.
[19] Q. Zhu, X. Yang, and J. Ren, "Modeling and analysis of the spread of computer virus," Communications in Nonlinear Science and Numerical Simulation, vol. 17, no. 12, pp. 5117-5124, 2012.
[20] Q. Zhu, X. Yang, L.-X. Yang, and C. Zhang, "Optimal control of computer virus under a delayed model," Applied Mathematics and Computation, vol. 218, no. 23, pp. 11613-11619, 2012.
[21] L.-X. Yang and X. Yang, "The effect of infected external computers on the spread of viruses: a compartment modeling study," Physica A, vol. 392, no. 24, pp. 6523-6535, 2013.
[22] L.-X. Yang and X. Yang, "The spread of computer viruses over a reduced scale-free network," Physica A, vol. 396, pp. 173-184, 2014.
[23] D. Dagon, C. C. Zou, and W. K. Lee, "Modeling botnet propagation using time and zones," in Proceedings of the 13th Annual Network and Distributed System Security Symposium (NDSS '06), 2006.
[24] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, "Complex networks: structure and dynamics," Physics Reports, vol. 424, no. 4-5, pp. 175-308, 2006.
[25] Y. Moreno, R. Pastor-Satorras, and A. Vespignani, "Epidemic outbreaks in complex heterogeneous networks," The European Physical Journal B: Condensed Matter and Complex Systems, vol. 26, no. 4, pp. 521-529, 2002.
[26] R. Yang, B.-H. Wang, J. Ren et al., "Epidemic spreading on heterogeneous networks with identical infectivity," Physics Letters A, vol. 364, no. 3-4, pp. 189-193, 2007.
[27] R. C. Robinson, An introduction to Dynamical Systems: Continuous and Discrete, Pearson Prentice Hall, Upper Saddle River, NJ, USA, 2004.
[28] L. Wang and G.-Z. Dai, "Global stability of virus spreading in complex heterogeneous networks," SIAM Journal on Applied Mathematics, vol. 68, no. 5, pp. 1495-1502, 2008.
[29] M. Yang, X. C. Fu, and Q. C. Wu, "Global stability of SIS epidemic model with infective medium on complex networks," Journal of Systems Engineering, vol. 25, pp. 767-772, 2011 (Chinese).
[30] J. O. Kephart, G. B. Sorkin, D. M. Chess, and S. R. White, "Fightingcomputer viruses," Scientific American, vol. 277, no. 5, pp. 88-93, 1997.

## Research Article

# Stability Analysis of a Multigroup SEIR Epidemic Model with General Latency Distributions 

Nan Wang, Jingmei Pang, and Jinliang Wang<br>School of Mathematical Science, Heilongjiang University, Harbin 150080, China<br>Correspondence should be addressed to Jinliang Wang; jinliangwang@aliyun.com

Received 11 February 2014; Accepted 27 March 2014; Published 17 April 2014
Academic Editor: Kaifa Wang
Copyright © 2014 Nan Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

The global stability of a multigroup SEIR epidemic model with general latency distribution and general incidence rate is investigated. Under the given assumptions, the basic reproduction number $\Re_{0}$ is defined and proved as the role of a threshold; that is, the diseasefree equilibrium $P_{0}$ is globally asymptotically stable if $\Re_{0} \leq 1$, while an endemic equilibrium $P^{*}$ exists uniquely and is globally asymptotically stable if $\Re_{0}>1$. For the proofs, we apply the classical method of Lyapunov functionals and a recently developed graph-theoretic approach.


## 1. Introduction

Mathematical models have become important tools in analyzing the spread and control of infectious diseases. The SIR model is one of the most popular ones in this field, for which the total population is subdivided into three compartments: susceptible, infectious, and removed. For some diseases, it is reasonable to include a latent (or exposed) class for those susceptible individuals who are infected with the disease but are not yet infectious, which leads to SEIR model [1-6]. Let $S(t), E(t), I(t)$, and $R(t)$ be the numbers of individuals in the susceptible, exposed, infectious, and removed compartments, respectively, with the total population $N(t)=S(t)+E(t)+$ $I(t)+R(t)$. Suppose that $d>0$ represents the constant recruitment rate and the natural mortality rate. Assuming mass action for the disease transmission and letting $\beta>0$ denote the effective contact rate, the rate of change of $S(t)$ is

$$
\begin{equation*}
S^{\prime}(t)=d-\beta S(t) I(t)-d S(t) \tag{1}
\end{equation*}
$$

Taking into consideration a general exposed distribution, van den Driessche et al. [5] formulated and studied the following model:

$$
\begin{aligned}
& S^{\prime}(t)=d-\beta S(t) I(t)-d S(t) \\
& E(t)=\int_{0}^{t} \beta S(u) I(u) e^{-d(t-u)} P(t-u) \mathrm{d} u
\end{aligned}
$$

$$
\begin{align*}
R^{\prime}(t) & =r I(t)-d R(t) \\
I(t) & =N-S(t)-E(t)-R(t) \tag{2}
\end{align*}
$$

where $r \geq 0$ is the rate at which infective individuals recover. $N$ is constant total populations. It is assumed in [5] that individuals rarely die of the disease and the disease-induced death is negligible, which ensures a constant population; that is, $N(t)=N \cdot P(t)$ denotes the probability (without taking death into account) that an exposed individual still remains in the exposed class $t$ time units after entering the exposed class and it satisfies the following.
$\left(\mathbf{A}_{1}\right) P:[0, \infty) \rightarrow[0,1]$ is nonincreasing, piecewise continuous with possibly finitely many jumps and satisfies $P\left(0_{+}\right)=1, \lim _{t \rightarrow \infty} P(t)=0$ with $\int_{0}^{\infty} P(u) d u$ being positive and finite.

In fact, the integral term in model (2) is in the sense of Riemann-Stieltjes integrals; the second equation of (2) takes the following form:

$$
\begin{align*}
E^{\prime}(t)= & \beta S(t) I(t)-d E(t) \\
& +\int_{0}^{t} \beta S(u) I(u) e^{-d(t-u)} d_{t} P(t-u) \mathrm{d} u \tag{3}
\end{align*}
$$

where $d_{t} P(t-u)=d P(t-u) / d t$. It follows from total population size $N$ which is constant that the rate of change of $I$ is governed by

$$
\begin{equation*}
I^{\prime}(t)=-\int_{0}^{t} \beta S(u) I(u) e^{-d(t-u)} d_{t} P(t-u) \mathrm{d} u-(d+r) I(t) \tag{4}
\end{equation*}
$$

Thus, model (2) can be written as the system

$$
\begin{align*}
S^{\prime}(t)= & d-\beta S(t) I(t)-d S(t) \\
E^{\prime}(t)= & \beta S(t) I(t)-d E(t) \\
& +\int_{0}^{t} \beta S(u) I(u) e^{-d(t-u)} d_{t} P(t-u) \mathrm{d} u  \tag{5}\\
I^{\prime}(t)= & -\int_{0}^{t} \beta S(u) I(u) e^{-d(t-u)} d_{t} P(t-u) \mathrm{d} u \\
& -(d+r) I(t) \\
R^{\prime}(t)= & r I(t)-d R(t)
\end{align*}
$$

Recently, a model of this type including the possibility of disease relapse has been proposed in $[5,6]$ to study the transmission and spread of some infectious diseases such as herpes, and its global dynamics have been completely investigated in [5, 7].

Heterogeneity in the host population can result from different contact modes such as those among children and adults for childhood diseases (e.g., measles and mumps) or different behaviors such as the numbers of sexual partners for some sexually transmitted infections (e.g., herpes and condyloma acuminatum). Taking into consideration different contact patterns, distinct number of sexual partners, or different geography and so forth, it is more proper to divide individual hosts into groups. Therefore, lots of multigroup models have been proposed in the literature to describe the transmission of infectious disease in heterogeneity environment (see [8-17] and references cited therein).

In multigroup epidemic models, a heterogeneous host population is divided into several homogeneous groups according to modes of transmission, contact patterns, or geographic distributions, so that within-group and intergroup interactions can be modeled separately. In this paper, we formulate a multigroup SEIR epidemic model with general exposed distribution and general incidence rates. The population is divided into $n$ distinct groups $(n \geq 2)$. For $1 \leq$ $k \leq n$, the $k$ th group is further partitioned into four compartments: susceptible, exposed, infectious, and recovered, whose numbers of individuals at time $t$ are denoted by $S_{k}(t), E_{k}(t)$, $I_{k}(t)$, and $R_{k}(t)$, respectively. Within the $k$ th group, $\varphi_{k}\left(S_{k}\right)$ represents the growth rate of $S_{k}$, which includes both the production and the natural death of susceptible individuals.

In [18], Zhang et al. studied a multigroup SEIR epidemic model with general exposed distribution and general incidence rates. By using the well-known "linear chain trick," the authors reformulate the model into an equivalent ordinary differential equations system. The global stability
results of equilibria are obtained by constructing suitable Lyapunov functionals for general incidence rate function $f_{k j}\left(S_{k}(t), I_{j}(t)\right)$. In [19], Hattaf et al. introduced a general incidence rate $f(S, I) I$ in a delayed SIR epidemic model.

Motivated by these facts, in this paper, we incorporate the general incidence rate presented in [19] to the following system of differential and integral equations:

$$
\begin{gather*}
S_{k}^{\prime}(t)=\varphi_{k}\left(S_{k}(t)\right)-\sum_{j=1}^{n} f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t), \\
E_{k}^{\prime}(t) \\
=\sum_{j=1}^{n} f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t) \\
\quad-\sum_{j=1}^{n} \int_{0}^{t} f_{k j}\left(S_{k}(u), I_{j}(u)\right) I_{j}(u) e^{-\delta_{k}(t-u)} g_{k}(t-u) \mathrm{d} u \\
-\delta_{k} E_{k}(t), \\
I_{k}^{\prime}(t)=\sum_{j=1}^{n} \int_{0}^{t} f_{k j}\left(S_{k}(u), I_{j}(u)\right) I_{j}(u) e^{-\delta_{k}(t-u)} g_{k}(t-u) \mathrm{d} u \\
\quad-\left(\delta_{k}+\gamma_{k}\right) I_{k}(t), \\
R_{k}^{\prime}(t)=\gamma_{k} I_{k}(t)-\delta_{k} R_{k}(t), \tag{6}
\end{gather*}
$$

where $g_{j}(t)=-P_{j}^{\prime}(t)$, the nonlinear term $f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t)$ represents the cross-infection from group $j$ to group $k, \delta_{k}$ denotes the natural death rates of exposed and infectious classes in the $k$ th group, and $\gamma_{k}$ denotes the production of the recovered individuals from infectious ones in the $k$ th group. All constants $\delta_{k}, \gamma_{k}, k=1,2, \ldots, n$, are assumed to be positive.

The organization of this paper is as follows; in the next section, we give some preliminaries of our main model. In Section 3, we prove the global asymptotic stability of the disease-free equilibrium $P_{0}$ for $\Re_{0} \leq 1$ using the classical method of Lyapunov. The existence of endemic equilibrium is also proved. In Section 4, we prove global asymptotic stability of an endemic equilibrium $P^{*}$ for $\Re_{0}>1$ using the graphtheoretic approach.

## 2. Preliminaries

Since the variables $E_{k}$ and $R_{k}$ do not appear in the first and third equations of (6), we can only consider the reduced system as follows:

$$
\begin{align*}
& S_{k}^{\prime}(t)=\varphi_{k}\left(S_{k}(t)\right)-\sum_{j=1}^{n} f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t) \\
& I_{k}^{\prime}(t)=\sum_{j=1}^{n} \int_{0}^{t} f_{k j}\left(S_{k}(u), I_{j}(u)\right) I_{j}(u) e^{-\delta_{k}(t-u)} g_{k}(t-u) \mathrm{d} u \\
& \quad-\left(\delta_{k}+\gamma_{k}\right) I_{k}(t) \tag{7}
\end{align*}
$$

The incidence function $f_{k j}\left(S_{k}, I_{j}\right)$ in (7) is assumed to be continuously differentiable in the interior of $\mathbb{R}_{+}^{2}$ and to satisfy the following hypotheses:
$\left(\mathrm{S}_{1}\right) f_{k j}\left(0, I_{j}\right)=0$, for all $I_{j} \geq 0$;
$\left(S_{2}\right) \partial f_{k j}\left(S_{k}, I_{j}\right) / \partial S_{k}>0$, for all $S_{k}>0$ and $I_{j} \geq 0$;
$\left(S_{3}\right) \partial f_{k j}\left(S_{k}, I_{j}\right) / \partial I_{j} \leq 0$, for all $S_{k} \geq 0$ and $I_{j} \geq 0$;
assume that the functions $\varphi_{k}$ satisfy the following conditions:
$\left(\mathrm{S}_{4}\right) \varphi_{k}$ are local Lipschitz on $[0, \infty)$ with $\varphi_{k}(0)>0$, and there is a unique positive solution $\xi=S_{k}^{0}$ for the equation $\varphi_{k}(\xi)=0 ; \varphi_{k}\left(S_{k}\right)>0$ for $0 \leq S_{k}<S_{k}^{0}$, and $\varphi_{k}\left(S_{k}\right)<0$ for $S_{k}>S_{k}^{0}$.

Typical examples of $f_{k j}\left(S_{k}, I_{j}\right)$ satisfying $\left(\mathrm{S}_{1}\right)-\left(\mathrm{S}_{3}\right)$ include common incidence functions such as

$$
\begin{gather*}
f_{k j}\left(S_{k}, I_{j}\right)=S_{k} I_{j}[20,2,3], \quad f_{k j}\left(S_{k}, I_{j}\right)=S_{k}^{q} I_{j} \\
f_{k j}\left(S_{k}, I_{j}\right)=\frac{\eta S_{k} I_{j}}{1+\theta S_{k}}[1] . \tag{8}
\end{gather*}
$$

The class of $\varphi_{k}\left(S_{k}\right)$ that satisfies $\left(S_{4}\right)$ includes both $\lambda_{k}-d_{k}^{S} S_{k}$ and $\lambda_{k}-d_{k}^{S} S_{k}+r_{k} S_{k}\left(1-S_{k} / N_{k}\right)$, which have been widely used in the literature of population dynamics [1, 8].

For model (7), the existence, uniqueness, and continuity of solutions follow from the theory for integrodifferential equations in [22]. It can be easily verified that every solution of (7) with nonnegative initial conditions remains nonnegative. It follows from $\left(S_{4}\right)$ and the first equation in (7) that $S_{k}^{\prime}(t) \leq \varphi_{k}\left(S_{k}(t)\right)$, and thus

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} S_{k}(t) \leq S_{k}^{0}, \quad \text { for } 1 \leq k \leq n \tag{9}
\end{equation*}
$$

From the biological significance, we only need to consider (7) in the following region:

$$
\begin{align*}
\Gamma:=\{ & \left(S_{1}, I_{1}, S_{2}, I_{2}, \ldots, S_{n}, I_{n}\right)  \tag{10}\\
& \left.\in \mathbb{R}_{+}^{2 n}: S_{k}, I_{k} \geq 0, S_{k}+I_{k} \leq S_{k}^{0}, 1 \leq k \leq n\right\} .
\end{align*}
$$

Indeed, one can easily verify that the set $\Gamma$ is positively invariant with respect to (7).

It is clear that system (7) has a disease-free equilibrium $P_{0}=\left(S_{1}^{0}, 0, S_{1}^{0}, 0, \ldots, S_{n}^{0}, 0\right)$ in $\Gamma$. Next, we will give some notations which will be useful for our main results.

Let

$$
\begin{align*}
J(\xi) & =\int_{\xi}^{\infty} g_{k}(u) e^{-\delta_{k} u} \mathrm{~d} u  \tag{11}\\
Q_{k} & =J(0)=\int_{0}^{\infty} g_{k}(u) e^{-\delta_{k} u} \mathrm{~d} u
\end{align*}
$$

It can be verified that $Q_{k} \in(0,1)$.

For finite time $t$, system (7) may not have an endemic equilibrium. If system (7) has an endemic equilibrium, the endemic equilibrium must satisfy the limiting system

$$
\begin{gather*}
S_{k}^{\prime}(t)=\varphi_{k}\left(S_{k}(t)\right)-\sum_{j=1}^{n} f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t), \\
I_{k}^{\prime}(t)=\sum_{j=1}^{n} \int_{0}^{\infty} f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right)  \tag{12}\\
\times I_{j}(t-u) e^{-\delta_{k} u} g_{k}(u) \mathrm{d} u \\
-\left(\delta_{k}+\gamma_{k}\right) I_{k}(t) .
\end{gather*}
$$

Since the limiting system (12) contains an infinite delay, its associated initial condition needs to be restricted in an appropriate fading memory space. For any $\sigma_{k} \in\left(0, \delta_{k}\right)$, define the following Banach space of fading memory type (see $[23,24]$ and references therein):

$$
C_{k}=\left\{\phi_{k} \in C((-\infty, 0], \mathbb{R}): \phi_{k}(s) e^{\sigma_{k} s}\right.
$$

is uniformly continuous on $(-\infty, 0]$,

$$
\begin{align*}
& \left.\sup _{s \leq 0}\left|\phi_{k}(s)\right| e^{\sigma_{k} s}<\infty\right\},  \tag{13}\\
& Y_{\Delta}=\left\{\phi_{k} \in C_{k}: \phi_{k}(s) \geq 0 \quad \forall s \leq 0\right\}
\end{align*}
$$

with norm $\|\phi\|_{k}=\sup _{s \leq 0}|\phi(s)| e^{\sigma_{k} s}$. Let $\psi_{t} \in C_{i}$ and $t>0$ be such that $\psi_{t}(s)=\psi(t+s), s \in(-\infty, 0]$.

Let $\phi_{k}, \psi_{k} \in C_{k}$ such that $\phi_{k}(s), \psi_{k}(s) \geq 0$ for all $s \in(-\infty, 0]$. We consider solutions of system (12), $\left(S_{1 t}, I_{1 t}, \ldots, S_{n t}, I_{n t}\right)$, with initial conditions

$$
\begin{equation*}
\left(\phi_{1}, \psi_{1}, \phi_{2}, \psi_{2}, \ldots, \phi_{n}, \psi_{n}\right) \tag{14}
\end{equation*}
$$

The standard theory of functional differential equations [24] implies $\left(S_{1 t}, I_{1 t}, \ldots, S_{n t}, I_{n t}\right) \in C_{k}$ for all $t>0$. We study system (12) in the following phase space:

$$
\begin{equation*}
\mathbb{X}_{\mathfrak{g}}=\prod_{k=1}^{n}\left(\mathbb{R} \times C_{k}\right) \tag{15}
\end{equation*}
$$

It can be verified that solutions of (12) in $\mathbb{X}_{\mathrm{g}}$ with initial conditions (14) remain nonnegative.

An equilibrium $P^{*}=\left(S_{1}^{*}, I_{1}^{*}, S_{2}^{*}, I_{2}^{*}, \ldots, S_{n}^{*}, I_{n}^{*}\right)$ in the interior of $\Gamma$ is called an endemic equilibrium of system (12), where $S_{k}^{*}, I_{k}^{*}>0$ satisfy the equilibrium equations

$$
\begin{gather*}
\varphi_{k}\left(S_{k}^{*}\right)=\sum_{j=1}^{n} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*},  \tag{16}\\
\sum_{j=1}^{n} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} Q_{k}=\left(\delta_{k}+\gamma_{k}\right) I_{k}^{*} .
\end{gather*}
$$

Set $R_{0}=\rho\left(M^{0}\right)$ to denote the special radius of the matrix $M^{0}$, where

$$
\begin{equation*}
M^{0}=\left(\frac{f_{k j}\left(S_{k}^{0}, 0\right) Q_{k}}{\delta_{k}+\gamma_{k}}\right)_{n \times n} \tag{17}
\end{equation*}
$$

The parameter $R_{0}$ is defined as the basic reproduction number [25, 26]. Since it can be verified that system (7) satisfies conditions $\left(A_{1}\right)-\left(A_{5}\right)$ of Theorem 2 of [26], we have the following lemma.

Lemma 1. For system (7), the disease-free equilibrium $P_{0}$ is locally asymptotically stable if $\mathfrak{R}_{0}<1$ while it is unstable if $\boldsymbol{R}_{0}>1$.

## 3. Global Stability of the Disease-Free Equilibrium

Theorem 2. Assume that the functions $\varphi_{k}$ and $f_{k j}$ satisfy $\left(S_{1}\right)-\left(S_{4}\right)$, and $M^{0}$ is irreducible.
(i) If $\Re_{0} \leq 1$, then $P_{0}$ is the unique equilibrium of system (7), and $P_{0}$ is globally asymptotically stable in $\Gamma$.
(ii) If $\Re_{0}>1$, then $P_{0}$ is unstable and system (7) is uniformly persistent.

Proof. It follows from the Perron-Frobenius theorem (see Theorem 2.1.4 in [27]) that the nonnegative irreducible matrix $M^{0}$ has a positive eigenvector $\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)$ such that

$$
\begin{equation*}
\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right) \rho\left(M^{0}\right)=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right) M^{0} \tag{18}
\end{equation*}
$$

Now, we construct a Lyapunov functional

$$
\begin{equation*}
V_{P_{0}}=\sum_{k=1}^{n} \frac{\omega_{k}}{\delta_{k}+\gamma_{k}} I_{k} . \tag{19}
\end{equation*}
$$

Differentiating $V_{P_{0}}$ along the solution of system (7) and under $\left(\mathrm{S}_{2}\right)$ and $\left(\mathrm{S}_{3}\right)$, we obtain

$$
\left.\begin{array}{rl}
V_{P_{0}}^{\prime}= & \sum_{k=1}^{n} \omega_{k}\left[\frac{1}{\delta_{k}+\gamma_{k}}\right. \\
& \times \sum_{j=1}^{n} \int_{0}^{t} f_{k j}\left(S_{k}(u), I_{j}(u)\right) I_{j}(u) \\
\times e^{-\delta_{k}(t-u)} g_{k}(t-u) \mathrm{d} u
\end{array}\right] \begin{aligned}
&\left.-I_{k}(t)\right] \\
& \leq \sum_{k=1}^{n} \omega_{k}\left[\frac{1}{\delta_{k}+\gamma_{k}} \omega_{j=1}^{n} \omega_{k j}\left(\frac{1}{\delta_{k}+\gamma_{k}} \sum_{j=1}^{n} f_{k j}\left(S_{k}^{0}, 0\right) I_{j}(t) Q_{k}-I_{k}(t)\right] Q_{k}-I_{k}(t)\right] \\
&=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)\left[M^{0} I-I\right] \\
&= {\left[\rho\left(M^{0}\right)-1\right]\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right) I }
\end{aligned}
$$

where $I=\left(I_{1}, I_{2}, \ldots, I_{n}\right)^{T}$. Suppose that $\rho\left(M^{0}\right)<1$. Then, $V_{P_{0}}^{\prime}=0$ if and only if $I=0$. Suppose that $\rho\left(M^{0}\right)=1$. Then, it follows from (20) that $V_{P_{0}}^{\prime}=0$ implies

$$
\begin{equation*}
\sum_{k=1}^{n} \omega_{k}\left[\frac{1}{\delta_{k}+\gamma_{k}} \sum_{j=1}^{n} f_{k j}\left(S_{k}, 0\right) I_{j}(t) Q_{k}\right]=\sum_{k=1}^{n} \omega_{k} I_{k}(t) \tag{21}
\end{equation*}
$$

If $S_{k} \neq S_{k}^{0}$, then

$$
\begin{align*}
& \sum_{k=1}^{n} \omega_{k}\left[\frac{1}{\delta_{k}+\gamma_{k}} \sum_{j=1}^{n} f_{k j}\left(S_{k}, 0\right) I_{j}(t) Q_{k}\right] \\
& \quad \leq \sum_{k=1}^{n} \omega_{k}\left[\frac{1}{\delta_{k}+\gamma_{k}} \sum_{j=1}^{n} f_{k j}\left(S_{k}^{0}, 0\right) I_{j}(t) Q_{k}\right]  \tag{22}\\
& \\
& \quad \leq\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right) M^{0} I \\
& \quad=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right) \rho\left(M^{0}\right) I \\
& \quad=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right) I
\end{align*}
$$

which implies that (21) has only the trivial solution $I=0$. Therefore, $V_{P_{0}}^{\prime}=0$ if and only if $I_{k}=0$ or $S_{k}=S_{k}^{0}$ provided $\rho\left(M^{0}\right)=1$. It can be verified that the only compact invariant subset of the set where $V_{P_{0}}^{\prime}=0$ is the singleton $\left\{P_{0}\right\}$. By LaSalle's Invariance Principle, $P_{0}$ is globally asymptotically stable in $\Gamma$ if $\rho\left(M^{0}\right) \leq 1$.

If $\Re_{0}>1$ and $I \neq 0$, it is easy to see that

$$
\begin{equation*}
\left[\rho\left(M^{0}\right)-1\right]\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right) I>0 \tag{23}
\end{equation*}
$$

It follows from the continuity that $V_{P_{0}}^{\prime}>0$ holds in a small neighborhood of $P_{0}$. This implies that $P_{0}$ is unstable. Using a uniform persistence result from [28] and similar arguments as in $[4,10,13,16,17]$, we know that, if $\Re_{0}>1$, the instability of $P_{0}$ implies the uniform persistence of (7) in $\Gamma$; that is, there exists a positive constant $\epsilon>0$ such that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} S_{k}(t) \geq \epsilon, \quad \liminf _{t \rightarrow \infty} I_{k}(t) \geq \epsilon, \quad k=1,2, \ldots, n . \tag{24}
\end{equation*}
$$

The uniform persistence of system (7) together with the uniform boundedness of solutions in $\Gamma$, which follows from the positive invariance of $\Gamma$, implies the existence of an endemic equilibrium $P^{*}$ in $\Gamma$ (see Theorem 2.8.6 of [29] or Theorem D. 3 of [30]). Summarizing the statements above, if $\mathfrak{R}_{0}>1$, system (7) is uniformly persistent and there exists at least one endemic equilibrium $P^{*}$ in $\Gamma$. This completes the proof.

## 4. Global Stability of an Endemic Equilibrium

Denote

$$
\begin{equation*}
H(u)=u-1-\ln u, \quad \forall u>0 . \tag{25}
\end{equation*}
$$

Obviously, $H: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$attains its strict global minimum at $u=1$ and $H(1)=0$.

To get the global stability of $P^{*}$, we make the following assumptions:

$$
\begin{aligned}
& \left(\mathrm{S}_{5}\right)\left(\varphi_{k}\left(S_{k}\right)-\varphi_{k}\left(S_{k}^{*}\right)\right)\left(S_{k}-S_{k}^{*}\right) \leq 0 \text { for } S_{k} \geq 0 ; \\
& \left(\mathrm{S}_{6}\right)\left(\varphi_{k}\left(S_{k}\right)-\varphi_{k}\left(S_{k}^{*}\right)\right)\left[f_{k k}\left(S_{k}, I_{k}^{*}\right)-f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right)\right]<0 \text { for } \\
& \quad S_{k} \neq S_{k}^{*} ; \\
& \left(S_{7}\right)\left(\left(\left(f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right) f_{k j}\left(S_{k}, I_{j}\right) I_{j}\right) /\left(f_{k k}\left(S_{k}, I_{k}^{*}\right) f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}\right)\right)\right. \\
& \quad-1)\left(1-\left(\left(f_{k k}\left(S_{k}, I_{k}^{*}\right) f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right)\right) /\left(f _ { k k } ( S _ { k } ^ { * } , I _ { k } ^ { * } ) f _ { k j } \left(S_{k},\right.\right.\right.\right. \\
& \left.\left.\left.\left.\quad I_{j}\right)\right)\right)\right) \leq 0 \text { for } S_{k}, I_{j}>0 .
\end{aligned}
$$

Theorem 3. Assume that the functions $\varphi_{k}$ and $f_{k j}$ satisfy $\left(S_{1}\right)-\left(S_{7}\right)$, and the matrix $M^{0}$ is irreducible. If $\mathfrak{R}_{0}>1$, then there is a unique endemic equilibrium $P^{*}$ for system (12), and $P^{*}$ is globally asymptotically stable in the interior of $\Gamma$.

Proof. Define a Lyapunov functional as

$$
\begin{align*}
V_{P^{*}}= & Q_{k} \int_{S_{k}^{*}}^{S_{k}(t)} \frac{f_{k k}\left(\eta, I_{k}^{*}\right)-f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right)}{f_{k k}\left(\eta, I_{k}^{*}\right)} \mathrm{d} \eta  \tag{26}\\
& +I_{k}^{*} H\left(\frac{I_{k}(t)}{I_{k}^{*}}\right)+V_{+}
\end{align*}
$$

where

$$
\begin{align*}
& V_{+}=\sum_{j=1}^{n} \int_{0}^{\infty} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} J(u) \\
& \times H\left(\frac{f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right) I_{j}(t-u)}{f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}}\right) \mathrm{d} u \tag{27}
\end{align*}
$$

First, we calculate the derivative of $V_{+}$; then, we have

$$
\begin{aligned}
& V_{+}^{\prime} \\
& \begin{aligned}
&=\sum_{j=1}^{n} \int_{0}^{\infty} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} J(u) \frac{\mathrm{d}}{\mathrm{~d} t} \\
& \times H\left(\frac{f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right) I_{j}(t-u)}{f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}}\right) \mathrm{d} u \\
&=-\sum_{j=1}^{n} \int_{0}^{\infty} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} J(u) \frac{\mathrm{d}}{\mathrm{~d} u} \\
& \quad \times H\left(\frac{f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right) I_{j}(t-u)}{f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}}\right) \mathrm{d} u \\
&=-\sum_{j=1}^{n} f_{k j}\left(\mathrm{~S}_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} J(u) \\
& \quad \times\left. H\left(\frac{f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right) I_{j}(t-u)}{f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}}\right)\right|_{u=0} ^{\infty}
\end{aligned}
\end{aligned}
$$

$$
\begin{align*}
& +\sum_{j=1}^{n} \int_{0}^{\infty} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} \\
& \times H\left(\frac{f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right) I_{j}(t-u)}{f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}}\right) \mathrm{d} J(u) \\
& =\sum_{j=1}^{n} Q_{k} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} H\left(\frac{f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t)}{f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}}\right) \\
& -\sum_{j=1}^{n} \int_{0}^{\infty} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} g_{k}(u) e^{-\delta_{k} u} \\
& \times H\left(\frac{f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right) I_{j}(t-u)}{f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}}\right) \mathrm{d} u \\
& =\sum_{j=1}^{n} Q_{k}\left(f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t)-f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}\right. \\
& \left.\times \ln \frac{f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t)}{f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}}\right) \\
& -\sum_{j=1}^{n} \int_{0}^{\infty} g_{k}(u) e^{-\delta_{k} u} \\
& \times\left[f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right) I_{j}(t-u)\right. \\
& -f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} . \\
& \left.\times \ln \frac{f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right) I_{j}(t-u)}{f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}}\right] \mathrm{d} u . \tag{28}
\end{align*}
$$

Calculating the time derivative of $V_{P^{*}}$ along the solution of system (12), we have

$$
\begin{align*}
V_{P^{*}}^{\prime}= & Q_{k}\left(1-\frac{f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right)}{f_{k k}\left(S_{k}(t), I_{k}^{*}\right)}\right) \\
& \times\left[\varphi_{k}\left(S_{k}(t)\right)-\sum_{j=1}^{n} f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t)\right] \\
& +\left(1-\frac{I_{k}^{*}}{I_{k}(t)}\right) \\
\times & \times\left[\sum _ { j = 1 } ^ { n } \int _ { 0 } ^ { \infty } f _ { k j } \left(S_{k}(u),\right.\right. \\
& \left.\quad I_{j}(u)\right) I_{j}(u) e^{-\delta_{k}(t-u)} g_{k}(t-u) \mathrm{d} u \\
& \left.-\left(\delta_{k}+\gamma_{k}\right) I_{k}(t)\right]+V_{+}^{\prime} . \tag{29}
\end{align*}
$$

Using equilibrium equations (16), we have

$$
\begin{align*}
V_{P^{*}}^{\prime}= & Q_{k}\left(\varphi_{k}\left(S_{k}(t)\right)-\varphi_{k}\left(S_{k}^{*}\right)\right)\left(1-\frac{f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right)}{f_{k k}\left(S_{k}(t), I_{k}^{*}\right)}\right) \\
& +\sum_{j=1}^{n} Q_{k} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}-\sum_{j=1}^{n} Q_{k} f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t) \\
& -\sum_{j=1}^{n} Q_{k} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} \frac{f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right)}{f_{k k}\left(S_{k}(t), I_{k}^{*}\right)} \\
& +\sum_{j=1}^{n} Q_{k} f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t) \frac{f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right)}{f_{k k}\left(S_{k}(t), I_{k}^{*}\right)} \\
& +\sum_{j=1}^{n} \int_{0}^{\infty} f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right) \\
& -\frac{I_{k}(t)}{I_{k}^{*}} \sum_{j=1}^{n} \int_{0}^{\infty} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} e^{-\delta_{k} u} g_{k}(u) \mathrm{d} u \\
& -\frac{I_{k}^{*}}{I_{k}(t)} \sum_{j=1}^{n} \int_{0}^{\infty} f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right) \\
& +\sum_{j=1}^{n} \int_{0}^{\infty} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} e^{-\delta_{k} u} g_{k}(u) \mathrm{d} u+V_{+}^{\prime}
\end{align*}
$$

Using $V_{+}^{\prime}$, we rewrite (30) as

$$
\begin{gathered}
V_{P^{*}}^{\prime}=Q_{k}\left(\varphi_{k}\left(S_{k}(t)\right)-\varphi_{k}\left(S_{k}^{*}\right)\right)\left(1-\frac{f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right)}{f_{k k}\left(S_{k}(t), I_{k}^{*}\right)}\right) \\
+\sum_{j=1}^{n} Q_{k} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} \\
\times\left[2-\frac{f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right)}{f_{k k}\left(S_{k}(t), I_{k}^{*}\right)}\right. \\
\quad+\frac{f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right) f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t)}{f_{k k}\left(S_{k}(t), I_{k}^{*}\right) f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}} \\
\left.\quad-\frac{I_{k}(t)}{I_{k}^{*}}\right] \\
-\sum_{j=1}^{n} \int_{0}^{\infty} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} g_{k}(u) e^{-\delta_{k} u}
\end{gathered}
$$

$$
\begin{align*}
& {\left[\frac{I_{k}^{*} f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right) I_{j}(t-u)}{I_{k}(t) f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}}\right.} \\
& \left.\quad-\ln \frac{f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right) I_{j}(t-u)}{f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t)}\right] \mathrm{d} u . \tag{31}
\end{align*}
$$

Therefore,
$V_{P^{*}}^{\prime}$
$=Q_{k}\left(\varphi_{k}\left(S_{k}(t)\right)-\varphi_{k}\left(S_{k}^{*}\right)\right)\left(1-\frac{f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right)}{f_{k k}\left(S_{k}(t), I_{k}^{*}\right)}\right)$
$-\sum_{j=1}^{n} Q_{k} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}$
$\times\left[H\left(\frac{f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right)}{f_{k k}\left(S_{k}(t), I_{k}^{*}\right)}\right)\right.$
$\left.+H\left(\frac{f_{k k}\left(S_{k}(t), I_{k}^{*}\right) f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right)}{f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right) f_{k j}\left(S_{k}(t), I_{j}(t)\right)}\right)\right]$
$+\sum_{j=1}^{n} Q_{k} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right)$
$\times\left(\frac{f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right) f_{k j}\left(S_{k}(t), I_{j}(t)\right) I_{j}(t)}{f_{k k}\left(S_{k}(t), I_{k}^{*}\right) f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}}-1\right)$
$\times\left(1-\frac{f_{k k}\left(S_{k}(t), I_{k}^{*}\right) f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right)}{f_{k k}\left(S_{k}^{*}, I_{k}^{*}\right) f_{k j}\left(S_{k}(t), I_{j}(t)\right)}\right)$
$-\sum_{j=1}^{n} \int_{0}^{\infty} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*} g_{k}(u) e^{-\delta_{k} u}$
$\times H\left(\frac{I_{k}^{*} f_{k j}\left(S_{k}(t-u), I_{j}(t-u)\right) I_{j}(t-u)}{I_{k}(t) f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) I_{j}^{*}}\right) \mathrm{d} u$
$+\sum_{j=1}^{n} Q_{k} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right)$
$\times I_{j}^{*}\left[\frac{I_{j}(t)}{I_{j}^{*}}-\frac{I_{k}(t)}{I_{k}^{*}}-\ln \frac{I_{j}(t)}{I_{j}^{*}}+\ln \frac{I_{k}(t)}{I_{k}^{*}}\right]$.

Furthermore, under $\left(S_{5}\right)-\left(S_{7}\right)$, we have

$$
\begin{align*}
V_{P^{*}}^{\prime} \leq \sum_{j=1}^{n} & Q_{k} f_{k j}\left(S_{k}^{*}, I_{j}^{*}\right) \\
& \times I_{j}^{*}\left[\frac{I_{j}(t)}{I_{j}^{*}}-\frac{I_{k}(t)}{I_{k}^{*}}-\ln \frac{I_{j}(t)}{I_{j}^{*}}+\ln \frac{I_{k}(t)}{I_{k}^{*}}\right] . \tag{33}
\end{align*}
$$

Obviously, the equalities in (33) hold if and only if $S_{k}=S_{k}^{*}$ and $I_{k}=I_{k}^{*}, k=1,2, \ldots, n$. Therefore, the functional $V=$ $\sum_{k=1}^{n} v_{k} V_{P^{*}}$ as defined in Theorem 3.1 of [12] is a Lyapunov function for system (12). Using similar arguments as in [4, $8-13,16,17]$, one can show that the largest invariant subset where $V_{p^{*}}^{\prime}=0$ is the singleton $\left\{P^{*}\right\}$. By LaSalle's Invariance Principle, $P^{*}$ is globally asymptotically stable in the interior of $\Gamma$. This completes the proof of Theorem 3.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The authors would like to thank the anonymous referees and the editor for very helpful suggestions and comments which led to improvements of our original paper. J. Wang is supported by National Natural Science Foundation of China (no. 11201128), the Science and Technology Research Project of the Department of Education of Heilongjiang Province (no. 12531495), the Natural Science Foundation of Heilongjiang Province (no. A201211), and the Science and Technology Innovation Team in Higher Education Institutions of Heilongjiang Province.

## References

[1] R. M. Anderson and R. M. May, "Population biology of infectious diseases I," Nature, vol. 280, pp. 361-367, 1979.
[2] S. J. Gao, Z. D. Teng, and D. H. Xie, "The effects of pulse vaccination on SEIR model with two time delays," Applied Mathematics and Computation, vol. 201, no. 1-2, pp. 282-292, 2008.
[3] A. Korobeinikov, "Global properties of SIR and SEIR epidemic models with multiple parallel infectious stages," Bulletin of Mathematical Biology, vol. 71, no. 1, pp. 75-83, 2009.
[4] M. Y. Li, J. R. Graef, L. Wang, and J. Karsai, "Global dynamics of a SEIR model with varying total population size," Mathematical Biosciences, vol. 160, no. 2, pp. 191-213, 1999.
[5] P. van den Driessche, L. Wang, and X. Zou, "Modeling diseases with latency and relapse," Mathematical Biosciences and Engineering, vol. 4, no. 2, pp. 205-219, 2007.
[6] P. van den Driessche and X. Zou, "Modeling relapse in infectious diseases," Mathematical Biosciences, vol. 207, no. 1, pp. 89103, 2007.
[7] S. Liu, S. Wang, and L. Wang, "Global dynamics of delay epidemic models with nonlinear incidence rate and relapse," Nonlinear Analysis. Real World Applications, vol. 12, no. 1, pp. 119-127, 2011.
[8] H. Guo, M. Y. Li, and Z. Shuai, "Global stability of the endemic equilibrium of multigroup SIR epidemic models," Canadian Applied Mathematics Quarterly, vol. 14, no. 3, pp. 259-284, 2006.
[9] H. Guo, M. Y. Li, and Z. Shuai, "A graph-theoretic approach to the method of global Lyapunov functions," Proceedings of the American Mathematical Society, vol. 136, no. 8, pp. 2793-2802, 2008.
[10] T. Kuniya, "Global stability of a multi-group SVIR epidemic model," Nonlinear Analysis. Real World Applications, vol. 14, no. 2, pp. 1135-1143, 2013.
[11] M. Y. Li, Z. S. Shuai, and C. C. Wang, "Global stability of multi-group epidemic models with distributed delays," Journal of Mathematical Analysis and Applications, vol. 361, no. 1, pp. 38-47, 2010.
[12] M. Y. Li and Z. S. Shuai, "Global-stability problem for coupled systems of differential equations on networks," Journal of Differential Equations, vol. 248, no. 1, pp. 1-20, 2010.
[13] R. Sun, "Global stability of the endemic equilibrium of multigroup SIR models with nonlinear incidence," Computers \& Mathematics with Applications, vol. 60, no. 8, pp. 2286-2291, 2010.
[14] R. Sun and J. Shi, "Global stability of multigroup epidemic model with group mixing and nonlinear incidence rates," Applied Mathematics and Computation, vol. 218, no. 2, pp. 280286, 2011.
[15] H. Shu, D. Fan, and J. Wei, "Global stability of multi-group SEIR epidemic models with distributed delays and nonlinear transmission," Nonlinear Analysis. Real World Applications, vol. 13, no. 4, pp. 1581-1592, 2012.
[16] J. Wang, J. Zu, X. Liu, G. Huang, and J. Zhang, "Global dynamics of a multi-group epidemic model with general relapse distribution and nonlinear incidence rate," Journal of Biological Systems, vol. 20, no. 3, pp. 235-258, 2012.
[17] J. Wang, Y. Takeuchi, and S. Liu, "A multi-group SVEIR epidemic model with distributed delay and vaccination," International Journal of Biomathematics, vol. 5, no. 3, Article ID 1260001, 2012.
[18] L. Zhang, J. Pang, and J. Wang, "Stability analysis of a multigroup epidemic model with general exposed distribution and nonlinear incidence rates," Abstract and Applied Analysis, vol. 2013, Article ID 354287, 11 pages, 2013.
[19] K. Hattaf, A. A. Lashari, Y. Louartassi, and N. Yousfi, "A delayed SIR epidemic model with general incidence rate," Electronic Journal of Qualitative Theory of Differential Equations, vol. 3, pp. 1-9, 2013.
[20] P. Georgescu, Y.-H. Hsieh, and H. Zhang, "A Lyapunov functional for a stage-structured predator-prey model with nonlinear predation rate," Nonlinear Analysis. Real World Applications, vol. 11, no. 5, pp. 3653-3665, 2010.
[21] X. Wang, Y. D. Tao, and X. Y. Song, "Pulse vaccination on SEIR epidemic model with nonlinear incidence rate," Applied Mathematics and Computation, vol. 210, no. 2, pp. 398-404, 2009.
[22] R. K. Miller, Nonlinear Volterra Integral Equations, W. A. Benjamin, New York, NY, USA, 1971.
[23] J. K. Hale and S. M. Verduyn Lunel, Introduction to FunctionalDifferential Equations, vol. 99 of Applied Mathematical Sciences, Springer, New York, NY, USA, 1993.
[24] Y. Hino, S. Murakami, and T. Naito, Functional-Differential Equations with Infinite Delay, vol. 1473 of Lecture Notes in Mathematics, Springer, 1991.
[25] O. Diekmann, J. A. P. Heesterbeek, and J. A. J. Metz, "On the definition and the computation of the basic reproduction ratio $R_{0}$ in models for infectious diseases in heterogeneous populations," Journal of Mathematical Biology, vol. 28, no. 4, pp. 365-382, 1990.
[26] P. van den Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," Mathematical Biosciences, vol. 180, pp. 29-48, 2002.
[27] A. Berman and R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, NY, USA, 1979.
[28] H. I. Freedman, S. G. Ruan, and M. X. Tang, "Uniform persistence and flows near a closed positively invariant set," Journal of Dynamics and Differential Equations, vol. 6, no. 4, pp. 583-600, 1994.
[29] N. P. Bhatia and G. P. Szegő, Dynamical Systems: Stability Theory and Applications, Springer, Berlin, Germany, 1967.
[30] H. L. Smith and P. Waltman, The Theory of the Chemostat, Cambridge University Press, 1995.

## Research Article

# Spatiotemporal Patterns in a Ratio-Dependent Food Chain Model with Reaction-Diffusion 

Lei Zhang<br>Computer Science and Technology Department, East China Normal University, Shanghai 200241, China<br>Correspondence should be addressed to Lei Zhang; foxpujin@163.com

Received 30 January 2014; Accepted 8 March 2014; Published 17 April 2014
Academic Editor: Weiming Wang
Copyright © 2014 Lei Zhang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

Predator-prey models describe biological phenomena of pursuit-evasion interaction. And this interaction exists widely in the world for the necessary energy supplement of species. In this paper, we have investigated a ratio-dependent spatially extended food chain model. Based on the bifurcation analysis (Hopf and Turing), we give the spatial pattern formation via numerical simulation, that is, the evolution process of the system near the coexistence equilibrium point $\left(u_{2}^{*}, v_{2}^{*}, w_{2}^{*}\right)$, and find that the model dynamics exhibits complex pattern replication. For fixed parameters, on increasing the control parameter $c_{1}$, the sequence "holes $\rightarrow$ holes-stripe mixtures $\rightarrow$ stripes $\rightarrow$ spots-stripe mixtures $\rightarrow$ spots" pattern is observed. And in the case of pure Hopf instability, the model exhibits chaotic wave pattern replication. Furthermore, we consider the pattern formation in the case of which the top predator is extinct, that is, the evolution process of the system near the equilibrium point $\left(u_{1}^{*}, v_{1}^{*}, 0\right)$, and find that the model dynamics exhibits stripes-spots pattern replication. Our results show that reaction-diffusion model is an appropriate tool for investigating fundamental mechanism of complex spatiotemporal dynamics. It will be useful for studying the dynamic complexity of ecosystems.


## 1. Introduction

Predator-prey models are studied in detail in the focus on equilibria, stability, asymptotic behavior, persistence, bifurcation, chaos, and so on [1-8]. In the past 40 years, with the idea of Turing [9], spatial extended models, in which not only the species evolve through time but also distribute in space, and pattern formation are one of the hot spots $[6,8,10-20]$.

Food web models describe the same phenomena as predator-prey models, but the former description is more actual than the latter since our real world is so complex. Until recently, food webs models are widely studied as predatorprey models [12-14, 17, 21-29]. But as far as we know, spatially extended models seem rare and not regarded. In fact, we live in a spatial world, and the spatial component of ecological interactions has been identified as an important factor in how ecological communities are shaped. Understanding the role of space is challenging both theoretically and empirically [30]. And the issue of spatial and spatiotemporal pattern formation in biological communities is probably one of the most exciting problems in modern biology and ecology [31, 32].

And the food web models with spatial distribution will do better job than the classical models.

In general, a classical food chain model with the nondimensional form can be written as follows:

$$
\begin{align*}
& \frac{d u}{d t}=u g(u)-c_{1} f_{1}(u, v) v \\
& \frac{d v}{d t}=\left(m_{1} f_{1}(u, v)-q_{1}\right) v-c_{2} p_{2}(v, w) w,  \tag{1}\\
& \frac{d w}{d t}=\left(m_{2} f_{2}(v, w)-q_{2}\right) w,
\end{align*}
$$

where $u$ stands for prey density, $w$ is the top predator density, and $v$-the density of the intermediate predator-describes the predator of $u$ and the prey of $w ; g(u)$ is the per capita rate of increase of the prey in the absence of predation. And all coefficients are positive constants, $c_{1}$ and $c_{2}$ are the maximum ingestion rates of intermediate predator and top predator, $m_{1}$ is the conversion factor of prey to intermediate predator, $m_{2}$ is the conversion factor of intermediate predator
to top predator, $q_{1}$ is the food-independent death rate of the intermediate predator, and $q_{2}$ is the food-independent death rate of the top predator. $f_{i}$ is the functional response. The functional response is the prey consumption rate by an average single predator. It can be influenced by the prey consumption rate and the predator density. $c_{i} f_{i}$ is the amount of prey consumed per predator per unit time, $m_{i} f_{i}$ is the predator production per capita with predation.

In this paper, we focus on the following ratio-dependent food chain model [28]:

$$
\begin{align*}
\frac{d u}{d t} & =u(1-u)-c_{1} \frac{u}{u+v} v \\
\frac{d v}{d t} & =\left(m_{1} \frac{u}{u+v}-q_{1}\right) v-c_{2} \frac{v}{v+w} w  \tag{2}\\
\frac{d w}{d t} & =\left(m_{2} \frac{v}{v+w}-q_{2}\right) w
\end{align*}
$$

The necessary condition of the persistence of $v$ and $w$ is $m_{1}>$ $q_{1}$ and $m_{2}>q_{2}$, respectively.

When all the species distribute randomly in the space, model (2) can be rewritten with a supplement:

$$
\begin{align*}
& \frac{d u}{d t}=u(1-u)-c_{1} \frac{u}{u+v} v+d_{1} \nabla^{2} u \\
& \frac{d v}{d t}=\left(m_{1} \frac{u}{u+v}-q_{1}\right) v-c_{2} \frac{v}{v+w} w+d_{2} \nabla^{2} v  \tag{3}\\
& \frac{d w}{d t}=\left(m_{2} \frac{v}{v+w}-q_{2}\right) w+d_{3} \nabla^{2} w
\end{align*}
$$

where $d_{1}, d_{2}$, and $d_{3}$ are the diffusion coefficients of the three species, respectively, $\nabla^{2}=\partial / \partial x^{2}+\partial / \partial y^{2}$ is the usual Laplacian operator in two-dimensional space, and other parameters have the same definitions as those above.

Model (3) is to be analyzed under the nonzero initial condition and Neumann, or zero flux, boundary conditions:

$$
\begin{gather*}
u(x, y, 0) \geq 0, \quad v(x, y, 0) \geq 0, \quad w(x, y, 0) \geq 0, \\
(x, y) \in \Omega \subset \mathbb{R}^{2},  \tag{4}\\
\frac{\partial u}{\partial \mathbf{n}}=\frac{\partial v}{\partial \mathbf{n}}=\frac{\partial v}{\partial \mathbf{n}}=0, \quad(x, y) \in \partial \Omega .
\end{gather*}
$$

In the above, $\mathbf{n}$ is the outward unit normal vector of the boundary $\partial \Omega$ which we will assume is smooth. The main reason for choosing such boundary conditions is that we are interested in the self-organization of pattern; zero-flux conditions imply no external input [17].

This paper is organized as follows. In the next section, we give a local stability analysis of model (3). Then, we present the pattern formation of model (3) via numerical simulations, which is followed by Section 2. Finally, we give some discussions in Section 4.


Figure 1: $c_{1}-q_{1}$ bifurcation diagram for model 3 with $m_{1}=1.5, m_{2}=$ $2, q_{2}=1, c_{2}=0.5, d_{1}=0.01, d_{2}=0.1, d_{3}=1$, and $q_{1}$ a variational parameter. Hopf and Turing bifurcation curves separate the coexistence parameter space into four domains. . . .: the dividing line of coexistence and noncoexistence of prey and their predators.

## 2. Linear Stability Analysis

There are two equilibria (steady states) in model (2), which correspond to spatially homogeneous equilibria of model (3):

$$
\begin{align*}
E_{1}^{*}= & \left(u_{1}^{*}, v_{1}^{*}, 0\right) \\
= & \left(-\frac{-m_{1}+c_{1} m_{1}-c_{1} q_{1}}{m_{1}},\right.  \tag{5}\\
& \left.-\frac{\left(-m_{1}+c_{1} m_{1}-c_{1} q_{1}\right)\left(m_{1}-q_{1}\right)}{m_{1} q_{1}}, 0\right),
\end{align*}
$$

corresponding to top-predator extinction when $m_{1}\left(c_{1}-\right.$ $1) / c_{1}<q_{1}$. Consider

$$
\begin{equation*}
E_{2}^{*}=\left(u_{2}^{*}, v_{2}^{*}, w_{2}^{*}\right), \tag{6}
\end{equation*}
$$

corresponding to coexistence of prey and predators when

$$
\begin{equation*}
c_{1}<1, \quad \frac{m_{2} c_{2}-q_{2} c_{2}}{m_{2}}<q_{1}<\frac{m_{2} c_{2}-q_{2} c_{2}}{m_{2}}-m_{1}+\frac{m_{1}}{c_{1}} \tag{7}
\end{equation*}
$$

or

$$
\begin{equation*}
c_{1}=1, \quad q_{1}<\frac{m_{2} c_{2}-q_{2} c_{2}}{m_{2}} \tag{8}
\end{equation*}
$$

or

$$
\begin{equation*}
c_{1}>1, \quad \frac{m_{2} c_{2}-q_{2} \mathcal{c}_{2}}{m_{2}}-m_{1}+\frac{m_{1}}{c_{1}}<q_{1}<\frac{m_{2} c_{2}-q_{2} c_{2}}{m_{2}}, \tag{9}
\end{equation*}
$$

where

$$
u_{2}^{*}=\frac{m_{1} m_{2}-c_{1} m_{1} m_{2}+c_{1} m_{2} q_{1}+c_{1} m_{2} c_{2}-c_{1} q_{2} c_{2}}{m_{1} m_{2}}
$$



Figure 2: Holes pattern of the prey $u$ obtained with model (3) with $c_{1}=1.84$ and $q_{1}=0.6$. Iterations: (a) 0 , (b) 20000 , (c) 60000 , and (d) 200000 .


FIgURe 3: Pattern formation of the prey $u$ of model (3) with $q_{1}=0.6, c_{1}=1.87$ (a) and $c_{1}=1.95$ (b). Iterations: (a) 100000 and (b) 20000.

$$
\begin{align*}
v_{2}^{*}= & -\left(m_{1} m_{2}-c_{1} m_{1} m_{2}+c_{1} m_{2} q_{1}+c_{1} m_{2} c_{2}-c_{1} q_{2} c_{2}\right) \\
& \times\left(-m_{1} m_{2}+m_{2} q_{1}+m_{2} c_{2}-q_{2} c_{2}\right) \\
& \times\left(m_{1} m_{2}\left(m_{2} q_{1}+m_{2} c_{2}-q_{2} c_{2}\right)\right)^{-1}, \tag{10}
\end{align*}
$$

$$
\begin{aligned}
w_{2}^{*}= & -\left(m_{1} m_{2}-c_{1} m_{1} m_{2}+c_{1} m_{2} q_{1}+c_{1} m_{2} c_{2}-c_{1} q_{2} c_{2}\right) \\
& \times\left(-m_{1} m_{2}+m_{2} q_{1}+m_{2} c_{2}-q_{2} c_{2}\right)\left(m_{2}-q_{2}\right) \\
& \times\left(m_{1} m_{2}\left(m_{2} q_{1}+m_{2} c_{2}-q_{2} c_{2}\right) q_{2}\right)^{-1} .
\end{aligned}
$$



Figure 4: Pattern formation of the prey $u$ of model (3) with $q_{1}=0.6, c_{1}=2.1$ (a) and $c_{1}=2.21$ (b). Iterations: (a) 100000 and (b) 20000.

And in the presence of diffusion, set $u=u^{*}+\tilde{u}, v=$ $v^{*}+\widetilde{v}, w=w^{*}+\widetilde{w}$, and the standard linear analysis predicts exponentially growing solutions of model (3) in the form

$$
\begin{array}{r}
\widetilde{u}(\mathbf{r}, t) \sim e^{\lambda t} t e^{i \vec{k} \cdot \mathbf{r}}, \quad \widetilde{v}(\mathbf{r}, t) \sim e^{\lambda t} t e^{i \vec{k} \cdot \mathbf{r}}, \quad \widetilde{w}(\mathbf{r}, t) \sim e^{\lambda t} t e^{i \vec{k} \cdot \mathbf{r}}, \\
\mathbf{r}=(x, y), \tag{11}
\end{array}
$$

where $\vec{k} \cdot \vec{k}=k^{2}, k$, and $\lambda$ are the wave-number and frequency, respectively.

And the eigenvalue equation then reads

$$
\begin{equation*}
\left|\lambda I+k^{2} D-J\right|=0 \tag{12}
\end{equation*}
$$

where the diffusion matrix $D=\operatorname{diag}\left(d_{1}, d_{2}, d_{3}\right)$, and $J$ the Jacobian matrix

$$
\begin{align*}
J & =\left[\begin{array}{ccc}
1-2 u-\frac{c_{1} v}{u+v}+\frac{c_{1} u v}{(u+v)^{2}} & -\frac{c_{1} u}{u+v}+\frac{c_{1} u v}{(u+v)^{2}} & 0 \\
\frac{m_{1} v}{u+v}-\frac{m_{1} u v}{(u+v)^{2}} & \frac{m_{1} u}{u+v}-\frac{m_{1} u v}{(u+v)^{2}}-q_{1}-\frac{c_{2} w}{v+w}+\frac{c_{2} v w}{(v+w)^{2}} & -\frac{c_{2} v}{v+w}+\frac{c_{2} v w}{(v+w)^{2}} \\
0 & \frac{m_{2} w}{v+w}-\frac{m_{2} v w}{(v+w)^{2}} & \frac{m_{2} v}{v+w}-\frac{m_{2} v w}{(v+w)^{2}}-q_{2}
\end{array}\right]_{\left(u_{2}^{*}, v_{2}^{*}, w_{2}^{*}\right)}  \tag{13}\\
& =\left[\begin{array}{lll}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{array}\right]
\end{align*}
$$

Then we can obtain the eigenvalues $\lambda(k)$ as functions of the wave number $k$ as the roots of

$$
\begin{equation*}
\lambda^{3}+p\left(k^{2}\right) \lambda^{2}+q\left(k^{2}\right) \lambda+r\left(k^{2}\right)=0 \tag{14}
\end{equation*}
$$

where

$$
\begin{gather*}
p\left(k^{2}\right)=\left(d_{1}+d_{2}+d_{3}\right) k^{2}-\left(J_{11}+J_{22}+J_{33}\right), \\
q\left(k^{2}\right)=\left(d_{1} d_{2}+d_{1} d_{3}+d_{2} d_{3}\right) k^{4} \tag{15}
\end{gather*}
$$

$$
\begin{aligned}
& -\left(d_{1} J_{22}+d_{1} J_{33}+d_{2} J_{11}\right. \\
& \left.\quad+d_{2} J_{33}+d_{3} J_{11}+d_{3} J_{22}\right) k^{2} \\
& +\left(J_{11} J_{22}+J_{11} J_{33}-J_{12} J_{21}+J_{22} J_{33}-J_{23} J_{32}\right) \\
& r\left(k^{2}\right)= \\
& d_{1} d_{2} d_{3} k^{6}-\left(d_{1} d_{2} J_{33}+d_{1} d_{3} J_{22}+d_{2} d_{3} J_{11}\right) k^{4} \\
& \quad+\left(d_{1} J_{22} J_{33}-d_{1} J_{23} J_{32}+d_{2} J_{11} J_{33}\right. \\
& \left.\quad+d_{3} J_{11} J_{22}-d_{3} J_{12} J_{21}\right) k^{2} \\
& \quad+\left(-J_{11} J_{22} J_{33}+J_{11} J_{23} J_{32}+J_{12} J_{21} J_{33}\right)
\end{aligned}
$$



Figure 5: Chaotic wave pattern of the prey $u$ obtained with model (3) with $c_{1}=5.15$ and $q_{1}=1.07$. Iterations: (a) 0 , (b) 50000 , (c) 100000 , and (d) 200000.

And one type of bifurcation will break one type of symmetry of a system; that is, in the bifurcation point, two equilibrium states intersect and exchange their stability. Biologically speaking, this bifurcation corresponds to a smooth transition between equilibrium states [33]. The reactiondiffusion systems have led to the characterization of two basic types of symmetry-breaking bifurcations-Hopf and Turing bifurcation, which are responsible for the emergence of spatiotemporal patterns.

The onset of Hopf instability corresponds to the case when a pair of imaginary eigenvalues cross the real axis from the negative to the positive side. And this situation occurs only when the diffusion vanishes. Mathematically speaking, the Hopf bifurcation occurs when $\mathfrak{R e}\left(\lambda\left(k^{2}\right)\right)=0$, $\mathfrak{J} \mathfrak{m}\left(\lambda\left(k^{2}\right)\right) \neq 0$ at the wavenumber $k=0$. For unstable steady states to heterogeneous perturbations leading to Turing patterns, the real part of the eigenvalue, $\mathfrak{R e}\left(\lambda\left(k^{2}\right)\right)$, has to be greater than zero. Mathematically speaking, the Turing bifurcation occurs when $\mathfrak{J}(\lambda(k))=0, \mathfrak{R}(\lambda(k))=0$ at the wavenumber $k \neq 0$.

Here, we take $c_{1}$ as the bifurcation parameter; linear stability analysis yields the bifurcation diagram with $m_{1}=$ $1.5, m_{2}=2, q_{2}=1, c_{2}=0.5, d_{1}=0.01, d_{2}=0.1$,
$d_{3}=1$, and $q_{1}$ is a variational parameter (c.f., Figure 1). In Figure 1, the spotted curve is critical state in which above the spotted curve, the three species cannot both be positive; under the spotted curve, they are both positive. The $c_{1}-q_{1}$ bifurcation diagram shows the two bifurcation curves separate the coexistence space into four domains. In domain I, located below all two bifurcation lines, the uniform steady state is the only stable solution of the model. Domain II is the region of pure Turing instability. Domain III is the region of pure Hopf instability. When the parameters correspond to domain IV, which is located above all two bifurcation lines, both Hopf instability and Turing instability occur.

In Figure 1, the stationary state in the parameter domains II and IV (sometimes called the "Turing space") is unstable only to a nonuniform perturbation. As expected, this domain exists only when the inhibitor species (for predator-prey system, predator $u$ ) diffuses faster than the activator species (for predator-prey system, prey $v, w$ ) and the area of this Turing space increases with $d_{3}>d_{2}>d_{1}$.

## 3. Pattern Formation

In this section, we perform extensive numerical simulations of the spatially extended model (3) in two-dimensional


Figure 6: Dynamical behaviors of model (3). (a) Time-series plot of $u$, (b) time-series plot of $v$, (c) time-series plot of $w$; (d) phase portrait. The parameters are the same as those in Figure 5.
spaces, and the qualitative results are shown here. The parameters are $m_{1}=1.5, m_{2}=2, q_{2}=1, c_{2}=0.5, d_{1}=$ $0.01, d_{2}=0.1$, and $d_{3}=1$. Model (3) is integrated initially in two-dimensional space from the homogeneous steady state; that is, we start with the unstable uniform solution $E_{2}^{*}=$ $\left(u_{2}^{*}, v_{2}^{*}, w_{2}^{*}\right)$ with small random perturbation superimposed; in each, the initial condition is always a small amplitude random perturbation ( $\pm 5 \times 10^{-4}$ ), using an explicit Euler method for the time integration with a time stepsize of $\Delta t=0.01$. We use the standard five-point approximation for the Laplacian operator with the Zero-flux boundary conditions and the system size is $50 \times 50$ space units with space stepsize (lattice constant) $h=\Delta x=\Delta y=0.25$, discretized through $x \rightarrow\left(x_{0}, x_{1}, \ldots, x_{i}, \ldots, x_{200}\right)$ and $y \rightarrow$ $\left(y_{0}, y_{1}, \ldots, y_{j}, \ldots, y_{200}\right)$. The form of the Laplacian operator is taken as follows:

$$
\begin{equation*}
\Delta_{h} u_{i, j}^{n}=\frac{u_{i+1, j}^{n}+u_{i-1, j}^{n}+u_{i, j+1}^{n}+u_{i, j-1}^{n}-4 u_{i, j}^{n}}{h^{2}} . \tag{16}
\end{equation*}
$$

The concentrations $\left(u_{i, j}^{n+1}, v_{i, j}^{n+1}, w_{i, j}^{n+1}\right)$ at the moment ( $n+$ 1) $\tau$ at the mesh position $(i, j)$ are given by

$$
\begin{gather*}
u_{i, j}^{n+1}=u_{i, j}^{n}+\tau d_{1} \Delta_{h} u_{i, j}^{n}+\tau f_{1}\left(u_{i, j}^{n}, v_{i, j}^{n}, w_{i, j}^{n}\right) \\
v_{i, j}^{n+1}=v_{i, j}^{n}+\tau d_{2} \Delta_{h} v_{i, j}^{n}+\tau f_{2}\left(u_{i, j}^{n}, v_{i, j}^{n}, w_{i, j}^{n}\right)  \tag{17}\\
w_{i, j}^{n+1}=w_{i, j}^{n}+\tau d_{3} \Delta_{h} w_{i, j}^{n}+\tau f_{3}\left(u_{i, j}^{n}, v_{i, j}^{n}, w_{i, j}^{n}\right) .
\end{gather*}
$$

When the evolution processes reached steady state, we took a snapshot with white corresponding to the high value of prey $u$ while black corresponding to the low one.

In the numerical simulations, different types of dynamics are observed and it is found that the distributions of predator and prey are always of the same type. Consequently, we can restrict our analysis of pattern formation to one distribution. In this section, we show the distribution of prey $u$, for instance.

From the bifurcation diagram in the above section (cf., Figure 1), the results of numerical simulations show that the type of the system dynamics is determined by the values of
$c_{1}$ and $q_{1}$. And for different sets of parameters, the features of the spatial patterns become essentially different if $c_{1}$ exceeds the bifurcation curves which depend on $q_{1}$.

First, we consider the pattern formation for the parameters $\left(q_{1}, c_{1}\right)$ located in domain II (c.f., Figure 1); the region of pure Turing instability occurs while Hopf stability occurs. As an example, we show the time evolution of three typical patterns when $q_{1}=0.6$. With the parameters set, one can conclude that the critical value of Hopf bifurcation is $c_{1}=$ 1.81365 and Turing bifurcation value is $c_{1}=2.12057$. So, the values of $c_{1}$ that we adopt are between 1.81365 and 2.12057.

As an example, in Figure 2, we show the time evolution of holes pattern of prey $u$ at $0,20000,60000$, and 200000 iterations for $\left(q_{1}, c_{1}\right)=(0.6,1.84)$. In this case, one can see that for model (1), the pattern takes a long time to settle down, starting with a homogeneous state $\left(u^{*}, v^{*}, w^{*}\right)=(0.20267,0.15498,0.15498)($ c.f., Figure 5(a)), and the random initial distribution leads to the formation of regular holes (c.f., Figure 2(d)). This pattern (c.f., Figure 2(d)) consists of black (minimum density of $u$ ) hexagons on a white (maximum density of $u$ ) background, that is, isolated zones with low population densities. Baurmann et al. [33] called this type pattern "cold spots" and von Hardenberg et al. [34] called it "holes." In this paper, we adopt the name "holes."

When increasing $c_{1}$ to 1.87 , a few of stripes emerge, and the remainder of the holes pattern remains time independent (Figure 3(a)). And while increasing $c_{1}$ to 1.95 , model dynamics exhibits a transition from stripe-hole growth to stripes replication; that is, holes decay and the stripes pattern emerges (Figure 3(b)).

Next, we consider the pattern formation in domain IV in Figure 1; both Hopf and Turing instability occur in this domain. We adopt $q_{1}=0.6$ and $2.12057<c_{1}<2.30769-$ the maximized value of the coexistence of prey and their predators. The model dynamics exhibits two typical pattern formations.

In Figure 4, with the increasing of $c_{1}$ to 2.1, a few of white hexagons (i.e., spots, associate with high population densities) fill in the stripes; that is, the stripes-spots pattern emerges (c.f., Figure 4(a)). And while increasing $c_{1}$ to 2.21, model dynamics exhibits a transition from stripe-spots growth to spots replication; that is, stripes decay and the spots pattern emerges(c.f., Figure 4(b)).

From Figures 2-4, one can see that, with fixed parameters, on increasing the control parameter $c_{1}$, the sequence "holes $\rightarrow$ holes-stripes mixtures $\rightarrow$ stripes $\rightarrow$ spots-stripes mixtures $\rightarrow$ spots" pattern is observed.

In addition, we consider the pattern formation when $\left(c_{1}, q_{1}\right)$ locates in domain III in Figure 1, pure Hopf instability occurs. Figure 5 shows the evolution of the chaotic wave pattern of prey $u$ at $0,50000,100000$, and 200000 iterations with $\left(q_{1}, c_{1}\right)=(1.07,5.15)$. With these fixed parameters, the critical value of Hopf bifurcation is $c_{1}=5.13108$ and the Turing bifurcation values equal $c_{1}=5.24545$. In order to make it clearer, in Figure 6, we show oscillate time series plots of $u, v, w$ (c.f., Figures 6(a), 6(b), and 6(c)), respectively. And phase portrait (c.f., Figure 6(d)) shows that there exhibits the "local" phase plane of the system obtained in a fixed point


Figure 7: $c_{1}-q_{1}$ bifurcation diagram for model (3) with $m_{1}=$ $1.5, m_{2}=2, q_{2}=1, c_{2}=0.5, d_{1}=0.01, d_{2}=0.1, d_{3}=1$ and $q_{1}$; the corresponding steady state is $\left(u_{1}^{*}, v_{1}^{*}, 0\right)=(0.04667,0.30333,0)$. .... the dividing line of coexistence and noncoexistence of prey and their predators.
$E_{2}^{*}=(0.38200,0.05209,0.05209)$ inside the region invaded by the irregular spatiotemporal oscillations.

Furthermore, we restrict our attention to the case when the top predator vanishes. Extinction of the top predator is studied by Chiu and coworkers; they gave a criterion for the extinction of top predator [35]. Here, we will illustrate the pattern formation about this case.

According to food chain model, $E_{1}^{*}=\left(u_{1}^{*}, v_{1}^{*}, 0\right)$ describes extinction of the top predator. With the same method and the same parameters in Section 2, the bifurcation diagram is shown in Figure 7. In Figure 7, the spotted curve is critical state in which the domain above the spotted curve is noncoexistence space; the domain under the spotted curve is coexistence space. Only Turing curve intersects with the spotted curve, and it separates the coexistence space into two domains. When $\left(c_{1}, q_{1}\right)$ locates in domain I , under the Turing curve, the steady state is only stable solution of model (3); when $\left(c_{1}, q_{1}\right)$ locates in domain II in Figure 7, pure Turing instability occurs. That is to say, domain II is the "Turing space" only.

Figure 8 shows the evolution of the spatial pattern of prey at $0,10000,100000$, and 300000 iterations with $c_{1}=1.1$ and $q_{1}=0.2$; that is, $\left(c_{1}, q_{1}\right)$ point locates in domain II in Figure 7. The random initial distribution around the steady state $E_{1}^{*}=$ $(0.04667,0.30333,0)$ leads to the formation of stripes-spots pattern (c.f., Figure 8(d)).

## 4. Conclusions and Remarks

In summary, we have investigated a ratio-dependent spatially extended food chain model. Based on the bifurcation analysis (Hopf and Turing), we give the spatial pattern formation via numerical simulation. For the coexistence equilibrium


FIGURE 8: Spatiotemporal pattern of the prey $u$ of model (3) with $c_{1}=1.07$ and $q_{1}=0.2$; the initial condition is the random initial distribution around the steady state $\left(u_{1}^{*}, v_{1}^{*}, 0\right)=(0.04667,0.30333,0)$. Iterations: (a) 0 , (b) 10000, (c) 100000, and (d) 300000 .
point $E_{2}^{*}=\left(u_{2}^{*}, v_{2}^{*}, w_{2}^{*}\right)$, we find that the model dynamics exhibits complex pattern replication, such as holes, holesstripes, stripes, spots-stripes, spots, and chaotic wave pattern. And for the extinction of the top predator equilibrium point $E_{1}^{*}=\left(u_{1}^{*}, v_{1}^{*}, 0\right)$, we find that the model dynamics exhibits stripes-spots pattern replication.

In fact, in our world, every day, hundreds of species are extinct, and the extinction of a species is a fearful thing. And the top predator is extinct because there is a balance between the prey $u$ and the intermediate predator $v$. In the case we considered, the density of the intermediate predator $v$ is not small, but very big. The intermediate predator $v$ is strong enough to fight back the top predator $w$.

On the other hand, in the analysis of bifurcations (i.e., Hopf and Turing), we find that huge-sized computations are required, so we have to obtain more help via computers. In fact, computer-aided analysis is useful for nonlinear analysis. And computers have played an important role throughout the history of ecology. Today, numerical simulations also play an important role in spatial ecology. There are some international mathematical softwares, such as Matlab, Maple, and Mathematica, all of which have powerful function library
and can provide scientific calculation and programming with friendly platform. We have finished all our symbolic computations in Maple and obtained our pattern snapshots (i.e., numerical simulations) in Matlab as Maple is more superior in symbolic computations while Matlab is more superior in numerical computations.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

This research was supported by NSFC no. 11071273.

## References

[1] P. A. Abrams and L. R. Ginzburg, "The nature of predation: prey dependent, ratio dependent or neither?" Trends in Ecology and Evolution, vol. 15, no. 8, pp. 337-341, 2000.
[2] R. Arditi and L. R. Ginzburg, "Coupling in predator-prey dynamics: ratio-dependence," Journal of Theoretical Biology, vol. 139, no. 3, pp. 311-326, 1989.
[3] C. Jost, Comparing predator-prey models qualitatively and quantitatively with ecological timeseries data [Ph.D. thesis], Institute National Agronomique, Paris, France, 1998.
[4] C. Jost, O. Arino, and R. Arditi, "About deterministic extinction in ratio-dependent predator-prey models," Bulletin of Mathematical Biology, vol. 61, no. 1, pp. 19-32, 1999.
[5] Y. Kuang and E. Beretta, "Global qualitative analysis of a ratiodependent predator-prey system," Journal of Mathematical Biology, vol. 36, no. 4, pp. 389-406, 1998.
[6] F. Rao and W. Wang, "Dynamics of a Michaelis-Menten-type predation model incorporating a prey refuge with noise and external forces," Journal of Statistical Mechanics: Theory and Experiment, vol. 2012, no. 3, Article ID P03014, 2012.
[7] S. Ruan and D. Xiao, "Global analysis in a predator-prey system with nonmonotonic functional response," SIAM Journal on Applied Mathematics, vol. 61, no. 4, pp. 1445-1472, 2001.
[8] W. Wang, Y. Cai, Y. Zhu, and Z. Guo, "Allee-effect-induced instability in a reaction-diffusion predator-prey model," Abstract and Applied Analysis, vol. 2013, Article ID 487810, 10 pages, 2013.
[9] A. M. Turing, "The chemical basis of morphogenisis," Philosophical Transactions of the Royal Society B, vol. 237, pp. 7-72, 1952.
[10] A. Aotani, M. Mimura, and T. Mollee, "A model aided understanding of spot pattern formation in chemotactic E. Coli colonies," Japan Journal of Industrial and Applied Mathematics, vol. 27, no. 1, pp. 5-22, 2010.
[11] D. Alonso, F. Bartumeus, and J. Catalan, "Mutual interference between predators can give rise to turing spatial patterns," Ecology, vol. 83, no. 1, pp. 28-34, 2002.
[12] S. A. Levin, "The problem of pattern and scale in ecology", Ecology, vol. 73, no. 6, pp. 1943-1967, 1992.
[13] M. Li, B. Han, L. Xu, and G. Zhang, "Spiral patterns near Turing instability in a discrete reaction diffusion system," Chaos, Solitons \& Fractals, vol. 49, pp. 1-6, 2013.
[14] L. A. Díaz Rodrigues, D. C. Mistro, and S. Petrovskii, "Pattern formation in a space- and time-discrete predator-prey system with a strong Allee effect," Theoretical Ecology, vol. 5, no. 3, pp. 341-362, 2012.
[15] P. K. Maini, "Using mathematical models to help understand biological pattern formation," Comptes Rendus-Biologies, vol. 327, no. 3, pp. 225-234, 2004.
[16] Z. Mei, Numerical Bifurcation Analysis for Reaction-Diffusion Equations, Springer, Berlin, Germany, 2000.
[17] J. D. Murray, Mathematical Biology II, Spatial Models and Biomedical Applications, vol. 18 of Interdisciplinary Applied Mathematics, Springer, New York, NY, USA, 3rd edition, 2003.
[18] N. Sapoukhina, Y. Tyutyunov, and R. Arditi, "The role of prey taxis in biological control: a spatial theoretical model," American Naturalist, vol. 162, no. 1, pp. 61-76, 2003.
[19] W. Wang, Q.-X. Liu, and Z. Jin, "Spatiotemporal complexity of a ratio-dependent predator-prey system," Physical Review E: Statistical, Nonlinear, and Soft Matter Physics, vol. 75, no. 5, Article ID 051913, 2007.
[20] W. Wang, L. Zhang, H. Wang, and Z. Li, "Pattern formation of a predator-prey system with Ivlev-type functional response," Ecological Modelling, vol. 221, no. 2, pp. 131-140, 2010.
[21] A. Klebanoff and A. Hastings, "Chaos in three-species food chains," Journal of Mathematical Biology, vol. 32, no. 5, pp. 427451, 1994.
[22] M. P. Boer, B. W. Kooi, and S. A. L. M. Kooijman, "Homoclinic and heteroclinic orbits to a cycle in a tri-trophic food chain," Journal of Mathematical Biology, vol. 39, no. 1, pp. 19-38, 1999.
[23] D. O. Maionchi, S. F. dos Reis, and M. A. M. de Aguiar, "Chaos and pattern formation in a spatial tritrophic food chain," Ecological Modelling, vol. 191, no. 2, pp. 291-303, 2006.
[24] S. Gakkhar and B. Singh, "The dynamics of a food web consisting of two preys and a harvesting predator," Chaos, Solitons and Fractals, vol. 34, no. 4, pp. 1346-1356, 2007.
[25] J. P. Keener, "Oscillatory coexistence in a food chain model with competing predators," Journal of Mathematical Biology, vol. 22, no. 2, pp. 123-135, 1985.
[26] B. W. Kooi, M. P. Boer, and S. A. L. M. Kooijman, "Complex dynamic behaviour of autonomous microbial food chains," Journal of Mathematical Biology, vol. 36, no. 1, pp. 24-40, 1997.
[27] J. López-Gómez and R. Pardo San Gil, "Coexistence in a simple food chain with diffusion," Journal of Mathematical Biology, vol. 30, no. 7, pp. 655-668, 1992.
[28] S.-B. Hsu, T.-W. Hwang, and Y. Kuang, "A ratio-dependent food chain model and its applications to biological control," Mathematical Biosciences, vol. 181, no. 1, pp. 55-83, 2003.
[29] T. Lindström, "On the dynamics of discrete food chains: lowand high-frequency behavior and optimality of chaos," Journal of Mathematical Biology, vol. 45, no. 5, pp. 396-418, 2002.
[30] C. Neuhauser, "Mathematical challenges in spatial ecology," Notices of the American Mathematical Society, vol. 48, no. 11, pp. 1304-1314, 2001.
[31] R. S. Cantrell and C. Cosner, Spatial Ecology via ReactionDiffusion Equations, John Wiley \& Sons, New York, NY, USA, 2003.
[32] S. V. Petrovskii and H. Malchow, "Wave of chaos: new mechanism of pattern formation in spatio-temporal population dynamics," Theoretical Population Biology, vol. 59, no. 2, pp. 157174, 2001.
[33] M. Baurmann, T. Gross, and U. Feudel, "Instabilities in spatially extended predator-prey systems: spatio-temporal patterns in the neighborhood of Turing-Hopf bifurcations," Journal of Theoretical Biology, vol. 245, no. 2, pp. 220-229, 2007.
[34] J. von Hardenberg, E. Meron, M. Shachak, and Y. Zarmi, "Diversity of vegetation patterns and desertification," Physical Review Letters, vol. 87, no. 19, Article ID 198101, 2001.
[35] C.-H. Chiu and S.-B. Hsu, "Extinction of top-predator in a three-level food-chain model," Journal of Mathematical Biology, vol. 37, no. 4, pp. 372-380, 1998.

## Research Article

# Bifurcations of Tumor-Immune Competition Systems with Delay 

Ping Bi and Heying Xiao<br>Department of Mathematics, Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200241, China<br>Correspondence should be addressed to Ping Bi; pbi@math.ecnu.edu.cn

Received 5 November 2013; Accepted 6 January 2014; Published 16 April 2014
Academic Editor: Kaifa Wang
Copyright © 2014 P. Bi and H. Xiao. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

A tumor-immune competition model with delay is considered, which consists of two-dimensional nonlinear differential equation. The conditions for the linear stability of the equilibria are obtained by analyzing the distribution of eigenvalues. General formulas for the direction, period, and stability of the bifurcated periodic solutions are given for codimension one and codimension two bifurcations, including Hopf bifurcation, steady-state bifurcation, and B-T bifurcation. Numerical examples and simulations are given to illustrate the bifurcations analysis and obtained results.


## 1. Introduction

In this century, cancer remains one of the most dangerous killers of humankind; every year millions of people suffer from cancer and die from this disease throughout the world; see Boyle et al. [1]. Recently, there has been much interest in mathematical modeling of immune response with the intruder (see, e.g., Liu et al. [2, 3], Yafia [4], d'Onofrio et al. [ 5,6$]$, and the references cited therein). In fact, mathematical models are feasible to propose simple models which are capable of displaying some of the essential immunological phenomena. The delayed models of tumor and immune response interactions have been studied extensively; we refer to Bi and Ruan [7], Yafia [8], Mayer et al. [9], Yafia [10], and the references cited therein, which have shown that various bifurcations can occur in such models. It is interesting to consider the nonlinear dynamics of the delayed tumorimmune model.

In 1994, Kuznetsov et al. [11] took into account the penetration of tumor cells (TCs) by effector cells (ECs) and proposed a model describing the response of ECs to the growth of TCs. They assumed that interactions between ECs and TCs in vitro can be described by the kinetic scheme shown in Figure 1, where $E, T, C, T^{*}$, and $E^{*}$ are the local concentrations of ECs, TCs, EC-TC complexes, inactivated

ECs, and lethally hit TCs, respectively. Then the Kuznetsov and Taylor model is as follows:

$$
\begin{gather*}
\frac{\mathrm{d} E}{\mathrm{~d} t}=c+F(C, T)-d_{1} E-k_{1} E T+\left(k_{-1}+k_{3}\right) C \\
\frac{\mathrm{~d} T}{\mathrm{~d} t}=a T\left(1-b T_{\mathrm{tot}}\right)-k_{1} E T+\left(k_{-1}+k_{2}\right) C \\
\frac{\mathrm{~d} C}{\mathrm{~d} t}=k_{1} E T-\left(k_{-1}+k_{2}+k_{3}\right) C  \tag{1}\\
\frac{\mathrm{~d} T^{*}}{\mathrm{~d} t}=k_{3} C-d_{2} T^{*} \\
\frac{\mathrm{~d} E^{*}}{\mathrm{~d} t}=k_{2} C-d_{3} E^{*}
\end{gather*}
$$

where $c$ is the normal rate of the flow of adult ECs into the tumor site, $F(C, T)$ describes the accumulation of effector cells in the tumor cells localization due to the presence of the tumor, $d_{1}, d_{2}$, and $d_{3}$ are the coefficients of the processes of destruction and migration for $\mathrm{E}, \mathrm{EC}$, and TC, respectively, $a$ is the coefficient of the maximal growth of tumor, and $b$ is the environment capacity. Kuznetsov et al. [11] claimed that experimental observations motivate the approximation $d C / d t \approx 0$; therefore, it is reasonable to assume that $C \approx K E T$

$$
E+T \underset{k_{-1}}{\stackrel{k_{1}}{\rightleftarrows}} C{\underset{k}{3}>T^{*}+E}_{\stackrel{k_{2}}{\leftrightarrows}}^{E^{*}+T}
$$

Figure 1: Kinetic scheme describing interactions between ECs and TCs.
with $K=k_{1} /\left(k_{1}+k_{2}+k_{3}\right)$. Kuznetsov et al. [11] also suggested that the function $F$ is in the Michaelis-Menten form $F(C, T)=$ $F(E, T)=(f C /(g+T))(f, g>0)$. In 2003, Gałach [12] suggested that the function $F$ should be in a Lotka-Volterra form $F(C, T)=F(E, T)=n_{1} E T$; then the model (1) can be reduced to

$$
\begin{gather*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=c+n_{1} x y-m_{1} x y-d_{1} x \\
\frac{\mathrm{~d} y}{\mathrm{~d} t}=a y(1-b y)-m_{2} x y \tag{2}
\end{gather*}
$$

where $x$ denotes the dimensionless density of ECs, $y$ stands for the dimensionless density of the population of TCs, $m_{1}=$ $k k_{2}, m_{2}=k k_{3}$, and all coefficients are positive. Set $x=x_{0} x^{\prime}$, $y=y_{0} y^{\prime}, t=\left(1 / m_{2} x_{0}\right) t^{\prime}, x_{0}>0, y_{0}>0$. Replace $x$ with $x^{\prime}$ and $y$ with $y^{\prime}$. Then (2) can be written as

$$
\begin{gather*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=\sigma+n x y-m x y-\delta x \\
\frac{\mathrm{~d} y}{\mathrm{~d} t}=\alpha y(1-\beta y)-x y \tag{3}
\end{gather*}
$$

where $\sigma=s / m_{2} x_{0}^{2}, \quad n=n_{1} y_{0} / m_{2} x_{0}, \quad m=m_{1} y_{0} / m_{2} x_{0}, \delta=$ $d_{1} / m_{2} x_{0}, \alpha=a / m_{2} x_{0}$, and $\beta=b y_{0}$.

Mayer et al. [9] and Asachenkov et al. [13] pointed out that the delays should be taken into account to describe the times necessary for molecule production, proliferation, differentiation of cells, transport, and so forth. In fact, the immune system needs time to develop a suitable response after the invasion of tumor cells; the binding of EC and TC also needs time. Therefore, we introduce time delays into the model of immune response. Integrating models [9-11], we will consider the model as follows:

$$
\begin{gather*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=\sigma+\zeta x\left(t-\tau_{1}\right) y\left(t-\tau_{1}\right)-\delta x \\
\frac{\mathrm{~d} y}{\mathrm{~d} t}=\alpha y\left(1-\beta_{2} y\right)-x\left(t-\tau_{2}\right) y\left(t-\tau_{2}\right) \tag{4}
\end{gather*}
$$

where $\zeta=n-m$; if the stimulation coefficient of the immune system exceeds the neutralization coefficient of ECs in the process of the formation of EC-TC complexes, then $\zeta>0$. Yafia [4] studied the linear stability of the equilibria and the existence of Hopf bifurcation for model (4) with $\tau_{1}=\tau_{2}=0$. Yafia [10] and Gałach [12] obtained similar results as those of Yafia [4] for (4) with $\tau_{2}=0$. Recently, Bi and Xiao [14] give conditions for the properties of Hopf bifurcated periodic solution and existence of the global Hopf bifurcation for (4) with $\tau_{2}=0$.

In this paper, we will consider the dynamical behaviors of model (4) with $\tau_{1}=\tau_{2}=\tau$. The rest of this paper
is organized as follows. In Section 2, the linear analysis of the model is carried out and local stability of the equilibria and the conditions of Hopf bifurcation are given. Section 3 is devoted to the analysis of Hopf, steady-state bifurcations, and B-T bifurcation. Numerical results and simulations are carried out to illustrate the main results. A brief discussion and more numerical simulations are given in Section 4.

## 2. Local Analysis

In this section, we will study the local stability of the equilibria and the Hopf bifurcations of system

$$
\begin{gather*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=\sigma+\zeta x(t-\tau) y(t-\tau)-\delta x  \tag{5}\\
\frac{\mathrm{~d} y}{\mathrm{~d} t}=\alpha y\left(1-\beta_{2} y\right)-x(t-\tau) y(t-\tau)
\end{gather*}
$$

It is easy to obtain that system (5) have three equilibria $P_{0}(\sigma / \delta, 0), P_{1}\left(x_{1}, y_{1}\right)$, and $P_{2}\left(x_{2}, y_{2}\right)$, where

$$
\begin{align*}
& x_{1}=\frac{-\alpha(\beta \delta-\zeta)-\sqrt{\Delta}}{2 \zeta} \\
& y_{1}=\frac{\alpha(\beta \delta+\zeta)+\sqrt{\Delta}}{2 \alpha \beta \zeta}  \tag{6}\\
& x_{2}=\frac{-\alpha(\beta \delta-\zeta)+\sqrt{\Delta}}{2 \zeta}, \\
& y_{2}=\frac{\alpha(\beta \delta+\zeta)-\sqrt{\Delta}}{2 \alpha \beta \zeta}
\end{align*}
$$

$\Delta=\alpha^{2}(\beta \delta-\zeta)^{2}+4 \alpha \beta \zeta \sigma>0$. It is easy to see that $x_{1}<0$. Because the number of tumor cells or effect cells is positive, we only consider the dynamical behaviors of the equilibria $P_{0}$ (tumor-free point) and $P_{2}$ in the rest of the paper.

Let $z_{1}(t)=x(t)-x^{*}, z_{2}(t)=y(t)-y^{*}$. System (5) can be written as

$$
\begin{align*}
z_{1}^{\prime}(t)= & \alpha_{1} z_{1}(t)+\alpha_{2} z_{1}(t-\tau) \\
& +\alpha_{3} z_{2}(t-\tau)+\zeta z_{1}(t-\tau) z_{2}(t-\tau) \\
z_{2}^{\prime}(t)= & \beta_{1} z_{1}(t-\tau)+\beta_{2} z_{2}(t)+\beta_{3} z_{2}(t-\tau)  \tag{7}\\
& -\alpha \beta z_{2}^{2}(t)-z_{1}(t-\tau) z_{2}(t-\tau),
\end{align*}
$$

where $\alpha_{1}=-\delta<0, \alpha_{2}=\zeta y^{*} \geq 0, \alpha_{3}=\zeta x^{*}>0, \beta_{1}=$ $-y^{*} \leq 0, \beta_{2}=\alpha-2 \alpha \beta y^{*}, \beta_{3}=-x^{*}<0$ and $\left(x^{*}, y^{*}\right)$ is the coordinate of the equilibrium.

It is easy to see that the linear system of system (7) is

$$
\begin{align*}
& z_{1}^{\prime}(t)=\alpha_{1} z_{1}(t)+\alpha_{2} z_{1}(t-\tau)+\alpha_{3} z_{2}(t-\tau) \\
& z_{2}^{\prime}(t)=\beta_{1} z_{1}(t-\tau)+\beta_{2} z_{2}(t)+\beta_{3} z_{2}(t-\tau) \tag{8}
\end{align*}
$$

where $\alpha_{1}, \alpha_{2}, \alpha_{3}, \beta_{1}, \beta_{2}$, and $\beta_{3}$ are the same as those in (7).
2.1. Tumor-Free Point. The characteristic equation of system (8) at the tumor-free equilibrium $P_{0}$ is

$$
\begin{equation*}
\Delta(\lambda)=(\lambda+\delta)\left(\lambda-\alpha+\frac{\sigma}{\delta} e^{-\lambda \tau}\right)=0 \tag{9}
\end{equation*}
$$

Then we have the following results.
Lemma 1. (I) If $\alpha=\sigma / \delta$, then
(1) Equation (9) has a simple zero root, and all other roots have negative real parts as $0 \leq \tau<\delta / \sigma$;
(2) Equation (9) has a double zero root, and all other roots have negative real parts as $\tau=\delta / \sigma$;
(3) Equation (9) has at least one root with positive real parts as $\tau>\delta / \sigma$.
(II) If $\alpha<\sigma / \delta$, then
(1) all roots of (9) have negative real parts as $0 \leq \tau<\tau_{0}$;
(2) Equation (9) has a pair of conjugate purely imaginary roots $\pm i \omega_{+}$, and all other roots have negative real parts as $\tau=\tau_{0}$;
(3) Equation (9) has at least one root with positive real parts as $\tau>\tau_{0}$.
(III) Equation (9) has a negative root $-\delta$, and all other roots have positive real parts as $\alpha>\sigma / \delta$.

Proof. $\lambda=0$ is a root of (9) if and only if $\alpha=\sigma / \delta$. If $\tau=0$, (9) has two roots $\lambda_{1}=-\delta$ and $\lambda_{2}=\alpha-\sigma / \delta$. Then there are three cases: (1) $\lambda_{2}=0$ as $\alpha=\sigma / \delta$; (2) $\lambda_{2}<0$ as $\alpha<\sigma / \delta$; (3) $\lambda_{2}>0$ as $\alpha>\sigma / \delta$.

We will consider the case $\tau>0$ as follows. If $\alpha=\sigma / \delta$, $\tau=\delta / \sigma$, then

$$
\begin{equation*}
\Delta^{\prime}(\lambda)=2 \lambda-\alpha+\frac{\sigma}{\delta} e^{-\lambda \tau}-\tau \lambda \frac{\sigma}{\delta} e^{-\lambda \tau}+\delta-\sigma \tau e^{-\lambda \tau} \tag{10}
\end{equation*}
$$

hence, $\left.\Delta^{\prime}(\lambda)\right|_{\lambda=0}=0$, and $\left.\Delta^{\prime \prime}(\lambda)\right|_{\lambda=0}=\delta^{2} / \sigma>0$. Thus $\lambda=0$ is the double zero root of (9).

If (9) has purely imaginary roots, then the roots must be the solution of

$$
\begin{equation*}
\Delta_{0}(\lambda)=\lambda-\alpha+\frac{\sigma}{\delta} e^{-\lambda \tau}=0 \tag{11}
\end{equation*}
$$

Assume that $\lambda=i \omega(\omega>0)$ is the root of (11); that is,

$$
\begin{align*}
& -\alpha+\frac{\sigma}{\delta} \cos \omega \tau=0 \\
& \omega-\frac{\sigma}{\delta} \sin \omega \tau=0 \tag{12}
\end{align*}
$$

that is $\omega^{2}=\sigma^{2} / \delta^{2}-\alpha^{2}$. Hence (11) has a positive root $\omega_{+}=$ $\sqrt{\sigma^{2} / \delta^{2}-\alpha^{2}}$ if and only if $\alpha<\sigma / \delta$, and the corresponding critical values are

$$
\begin{equation*}
\tau_{k}=\frac{1}{\omega_{+}}\left\{\operatorname{arc} \cos \frac{\alpha \delta}{\sigma}+2 k \pi\right\}, \quad k=0,1,2, \ldots . \tag{13}
\end{equation*}
$$

Using Rouché theorem, we know that conclusions (II)(1), (II)(2), and (III) hold.

If $0<\tau<\delta / \sigma, \alpha=\sigma / \delta$, we can obtain $\left.\Delta_{0}^{\prime}(\lambda)\right|_{\lambda=0}=$ $1-(\sigma / \delta) \tau>0, \Delta_{0}(0)=0$. Noting the continuous of the function $\Delta_{0}(\lambda)$, we know that there is at least a $\lambda<0$ such that $\Delta_{0}(\lambda)<0$. On the other hand, it is easy to see that $\lim _{\lambda \rightarrow-\infty} \Delta_{0}(\lambda)=+\infty$; then there exists $\lambda<0$ such that $\Delta_{0}(\lambda)=0$.

Differentiating both sides of (11) with respect to $\tau$, we have

$$
\begin{equation*}
\frac{\mathrm{d} \lambda}{\mathrm{~d} \tau}=\frac{\lambda(\sigma / \delta) e^{-\lambda \tau}}{1-(\sigma / \delta) \tau e^{-\lambda \tau}} \tag{14}
\end{equation*}
$$

If $\tau \neq \delta / \sigma$, then

$$
\begin{equation*}
\left.\frac{\mathrm{d} \lambda}{\mathrm{~d} \tau}\right|_{\lambda=0}=\left.\frac{\lambda(\sigma / \delta) e^{-\lambda \tau}}{1-(\sigma / \delta) \tau e^{-\lambda \tau}}\right|_{\lambda=0}=0 \tag{15}
\end{equation*}
$$

Using Rouché theorem, we know that the conclusions of (I)(1) and (I)(2) are true.

If $\tau=\delta / \sigma$, then

$$
\begin{equation*}
\lim _{\lambda \rightarrow 0} \frac{\mathrm{~d} \tau}{\mathrm{~d} \lambda}=\lim _{\lambda \rightarrow 0} \frac{-\tau(\sigma / \delta)(-\tau) e^{-\lambda \tau}}{(\sigma / \delta) e^{-\lambda \tau}-(\sigma / \delta) \tau \lambda e^{-\lambda \tau}}=\tau^{2} \tag{16}
\end{equation*}
$$

Thus
$\operatorname{sgn}\left\{\left.\operatorname{Re}\left(\frac{\mathrm{d} \lambda}{\mathrm{d} \tau}\right)\right|_{\lambda=0}\right\}=\operatorname{sgn}\left\{\left.\operatorname{Re}\left(\frac{\mathrm{d} \tau}{\mathrm{d} \lambda}\right)\right|_{\lambda=0}\right\}=1>0$.

Hence the conclusion (I)(3) is true.
Noting

$$
\begin{align*}
\operatorname{sgn}\left\{\left.\operatorname{Re}\left(\frac{\mathrm{d} \lambda}{\mathrm{~d} \tau}\right)\right|_{\lambda=i \omega_{+}}\right\} & =\operatorname{sgn}\left\{\left.\operatorname{Re}\left(\frac{\mathrm{d} \tau}{\mathrm{~d} \lambda}\right)\right|_{\lambda=i \omega_{+}}\right\} \\
& =\operatorname{sgn}\left\{\frac{\sin \omega_{+} \tau}{(\sigma / \delta) \omega_{+}}\right\}  \tag{18}\\
& =\operatorname{sgn}\left\{\left(\frac{\delta}{\sigma}\right)^{2}\right\}=1>0
\end{align*}
$$

then (II)(3) is proved. Then all the proof is finished.
Thus the following results can be obtained by Lemma 1.
Theorem 2. (I) If $\alpha=\sigma / \delta$, then
(1) system (5) undergoes a codimension one steady-state bifurcation at the tumor-free equilibrium $P_{0}$ as $0<\tau<$ $\delta / \sigma$;
(2) the tumor-free equilibrium $P_{0}$ is a B-T singular equilibrium as $\tau=\delta / \sigma$.
(II) If $\alpha<\sigma / \delta$, then
(1) the tumor-free equilibrium $P_{0}$ is asymptotically stable as $0 \leq \tau<\tau_{0}$;
(2) the tumor-free equilibrium $P_{0}$ is unstable as $\tau>\tau_{0}$;
(3) system (5) undergoes Hopf bifurcation at the tumor-free equilibrium $P_{0}$ as $\tau=\tau_{k}$.
(III) The tumor-free equilibrium $P_{0}$ is unstable as $\alpha>\sigma / \delta$.
2.2. Positive Equilibrium. If $\alpha \delta-\sigma>0$, then the positive equilibrium $P_{2}$ exists. The characteristic equation of system (8) at the point $P_{2}$ is

$$
\begin{equation*}
\lambda^{2}+p \lambda+q \lambda e^{-\lambda \tau}+r+l e^{-\lambda \tau}=0 \tag{19}
\end{equation*}
$$

where

$$
\begin{gather*}
p=-\left(\alpha-\delta-2 \alpha \beta y_{2}\right), \\
q=x_{2}-\zeta y_{2}  \tag{20}\\
r=-\delta\left(\alpha-2 \alpha \beta y_{2}\right) \\
l=\delta x_{2}+\zeta y_{2}\left(\alpha-2 \alpha \beta y_{2}\right)
\end{gather*}
$$

Lemma 3. If $\alpha \delta-\sigma>0, \beta<\alpha \delta / 2(\alpha \delta-\sigma)$, then
(1) all roots of (19) have negative real parts as $0 \leq \tau<\tau_{0}^{\prime}$;
(2) Equation (19) has a pair of conjugate purely imaginary roots $i \widetilde{\omega}_{+}$, and all other roots have negative real parts as $\tau=\tau_{0}^{\prime} ;$
(3) Equation (19) has at least one root with positive real parts as $\tau>\tau_{0}^{\prime}$.

Proof. Noting $\beta<\alpha \delta / 2(\alpha \delta-\sigma), 1-2 \beta y_{2}>0$, one has

$$
\begin{equation*}
r+l=\zeta \alpha y_{2}\left(1-2 \beta y_{2}\right)+\delta \alpha \beta y_{2}>0 \tag{21}
\end{equation*}
$$

thus (19) has no zero root.
If $\tau=0$, then (19) can be written as

$$
\begin{equation*}
\lambda^{2}+(p+q) \lambda+r+l=0 \tag{22}
\end{equation*}
$$

It is easy to see

$$
\begin{equation*}
p+q=\delta-\alpha\left(1-2 \beta y_{2}\right)+x_{2}-\zeta y_{2}=\delta+(\alpha \beta-\zeta) y_{2}>0 \tag{23}
\end{equation*}
$$

and then all roots of (22) have negative real parts.
If $\tau>0$, we assume that (19) has a pair of purely imaginary roots $\lambda=i \omega(\omega>0)$; thus

$$
\begin{gather*}
-\omega^{2}+l \cos \omega \tau+q \omega \sin \omega \tau+r=0  \tag{24}\\
p \omega+q \omega \cos \omega \tau-l \sin \omega \tau=0
\end{gather*}
$$

and hence

$$
\begin{equation*}
\omega^{4}+\left(p^{2}-2 r-q^{2}\right) \omega^{2}+r^{2}-l^{2}=0 \tag{25}
\end{equation*}
$$

Noting $r+l>0, r<0$, then we have $r-l<0$ and $r^{2}-l^{2}=$ $(r+l)(r-l)<0$. That is to say, (25) has only one positive root

$$
\begin{equation*}
\widetilde{\omega}_{+}=\sqrt{\frac{-\left(p^{2}-2 r-q^{2}\right)+\sqrt{\left(p^{2}-2 r-q^{2}\right)^{2}-4\left(r^{2}-l^{2}\right)}}{2}} \tag{26}
\end{equation*}
$$

and the corresponding critical value is

We can also give the following transversal condition:

$$
\begin{aligned}
& \operatorname{sgn}\left\{\left.\operatorname{Re}\left(\frac{\mathrm{d} \lambda}{\mathrm{~d} \tau}\right)\right|_{\lambda=i \widetilde{\omega}_{+}}\right\} \\
& =\operatorname{sgn}\left\{\left.\operatorname{Re}\left(\frac{\mathrm{d} \tau}{\mathrm{~d} \lambda}\right)\right|_{\lambda=i \widetilde{\omega}_{+}}\right\} \\
& =\operatorname{sgn}\left\{\left.\operatorname{Re}\left(\frac{2 \lambda+p}{\lambda(q \lambda+l) e^{-\lambda \tau}}+\frac{q}{\lambda(q \lambda+l)}\right)\right|_{\lambda=i \widetilde{\omega}_{+}}\right\} \\
& =\operatorname{sgn}\left\{2\left(\widetilde{\omega}_{+}^{2}-r\right)+p^{2}-q^{2}\right\} \\
& = \\
& \operatorname{sgn}\left\{-\left(p^{2}-2 r-q^{2}\right)+\left(p^{2}-2 r-q^{2}\right)\right. \\
& \left.\quad+\sqrt{\left(p^{2}-2 r-q^{2}\right)^{2}-4\left(r^{2}-l^{2}\right)}\right\}
\end{aligned}
$$

$$
\begin{equation*}
=1>0 . \tag{28}
\end{equation*}
$$

Then all results of this theorem have been proven.

From Lemma 3, the following theorem can be obtained directly.

Theorem 4. Suppose that $\alpha \delta-\sigma>0, \beta<\alpha \delta / 2(\alpha \delta-\sigma)$; then
(1) the positive equilibrium $P_{2}$ is stable as $0 \leq \tau<\tau_{0}^{\prime}$;
(2) the positive equilibrium $P_{2}$ is unstable as $\tau>\tau_{0}^{\prime}$;
(3) system (5) undergoes a Hopf bifurcation at the equilibrium $P_{2}$ as $\tau=\tau_{k}^{\prime}$.

## 3. Direction and Stability of the Bifurcations

3.1. Hopf Bifurcation. In the previous section, we know that system (5) undergoes Hopf bifurcation at the tumor-free equilibrium $P_{0}$ and positive equilibrium $P_{2}$ under certain conditions. In this section, we will study the stability and direction of the Hopf bifurcated periodic solution by using the center manifold reduction and normal form theory of retarded functional differential equations due to the ideals of Faria and Magalhães [15, 16]. Throughout this section, we always assume that system (5) undergoes Hopf bifurcations at the equilibrium $P\left(P_{0}\right.$ or $\left.P_{2}\right)$ as the critical parameter $\tau=\tau_{k}$ and the corresponding purely imaginary roots are $\pm i \omega_{k}$.

Normalizing the delay $\tau$ in system (7) by the time-scaling $t \rightarrow t / \tau$, then (7) is transformed into

$$
\begin{align*}
z_{1}^{\prime}(t)= & \tau\left[\alpha_{1} z_{1}(t)+\alpha_{2} z_{1}(t-1)\right. \\
& \left.\quad+\alpha_{3} z_{2}(t-1)+\zeta z_{1}(t-1) z_{2}(t-1)\right] \\
z_{2}^{\prime}(t)=\tau[ & \beta_{1} z_{1}(t-1)+\beta_{2} z_{2}(t)+\beta_{3} z_{2}(t-1)  \tag{29}\\
& \left.-\alpha \beta z_{2}^{2}(t)-z_{1}(t-1) z_{2}(t-1)\right] .
\end{align*}
$$

This scaling is irrelevant for the study of the stability of the equilibrium but will be crucial for the Hopf bifurcation analysis.

Let $z(t)=\left\{\begin{array}{l}z_{1}(t) \\ z_{2}(t)\end{array}\right.$. we transformed (29) into an FDE in $C\left([-1,0], \mathbb{R}^{2}\right)$ :

$$
\begin{equation*}
\dot{z}(t)=N(\tau)\left(z_{t}\right)+F\left(z_{t}, \tau\right) \tag{30}
\end{equation*}
$$

where $N(\varphi): C\left([-1,0], \mathbb{R}^{2}\right) \rightarrow \mathbb{R}^{2}, F(\varphi): C\left([-1,0] \mathbb{R}^{2}\right) \rightarrow$ $\mathbb{R}^{2}$, are given by

$$
\begin{gather*}
N(\tau)(\varphi)=\tau\binom{\alpha_{1} \varphi_{1}(0)+\alpha_{2} \varphi_{1}(-1)+\alpha_{3} \varphi_{2}(-1)}{\beta_{1} \varphi_{1}(-1)+\beta_{2} \varphi_{2}(0)+\beta_{3} \varphi_{2}(-1)},  \tag{31}\\
F(\varphi, \tau)=\tau\binom{\zeta \varphi_{1}(-1) \varphi_{2}(-1)}{-\alpha \beta \varphi_{2}^{2}(0)-\varphi_{1}(-1) \varphi_{2}(-1)},
\end{gather*}
$$

where $\varphi=\operatorname{col}\left(\varphi_{1}, \varphi_{2}\right) \in C\left([-1,0], \mathbb{R}^{2}\right)$. Let $\Lambda=\left\{i \omega_{k},-i \omega_{k}\right\}$. Setting the new parameter $\gamma=\tau-\tau_{k}$, then (30) can be written as

$$
\begin{equation*}
\dot{z}(t)=N\left(\tau_{k}\right)\left(z_{t}\right)+\widetilde{F}\left(z_{t}, \gamma\right) \tag{32}
\end{equation*}
$$

where $\widetilde{F}\left(z_{t}, \gamma\right)=N(\gamma)\left(z_{t}\right)+F\left(z_{t}, \tau_{k}+\gamma\right)$.
Assume that $A$ is the infinitesimal generator of $\dot{z}(t)=$ $N\left(\tau_{k}\right)\left(z_{t}\right)$ satisfying $A \Phi=\Phi B$ with

$$
B=\left(\begin{array}{cc}
i \omega_{k} & 0  \tag{33}\\
0 & -i \omega_{k}
\end{array}\right)
$$

and $A$ has a pair of conjugate purely imaginary roots $\pm i \omega_{k}$. Denote that $P$ is the invariant space of $A$ associated with $\Lambda$; then $\operatorname{dim} P=2$. We can decompose $C:=C\left([-1,0], \mathbb{R}^{2}\right)$ to $C=P \bigoplus Q$ by the formal adjoint theory for FDEs by Hale [17]. Considering complex coordinates, we still denote $C$ as $\left([-1,0], C^{2}\right)$. Let $\Phi=\left(\Phi_{1}, \Phi_{2}\right)$ be the bases of $P$, where

$$
\begin{equation*}
\Phi_{1}=e^{i \omega_{k} \theta} v, \quad \Phi_{2}=\bar{\Phi}_{1}, \quad \theta \in[-1,0], \tag{34}
\end{equation*}
$$

$v=\binom{v_{1}}{v_{2}}$ is a vector in $C^{2}$ and $N\left(\tau_{k}\right) \Phi_{1}=i \omega_{k} v$.
Choose a basis $\Psi$ for the adjoint space $P^{*}$, such that $(\Psi, \Phi)=I_{2}$, where $(\cdot, \cdot)$ is the bilinear form on $C^{*} \times C$ associated with the adjoint equation. Thus, $\Psi=\operatorname{col}\left(\Psi_{1}, \Psi_{2}\right)$ with

$$
\begin{align*}
\Psi_{1}=e^{-i \omega_{k} \tilde{\theta}} u^{T}, \quad \Psi_{2}=\bar{\Psi}_{1}, \quad u & =\binom{u_{1}}{u_{2}}  \tag{35}\\
\tilde{\theta} & \in[0,1]
\end{align*}
$$

such that $\left(\Psi_{1}, \Phi_{1}\right)=1,\left(\Psi_{1}, \Phi_{2}\right)=0$. Then

$$
\left.\begin{array}{c}
v=\left(\frac{1}{i \omega_{k}-\left(\alpha_{1}+\alpha_{2} e^{-i \omega_{k}}\right) \tau_{k}}\right. \\
\tau_{k} \alpha_{3} e^{-i \omega_{k}} \tag{36}
\end{array}\right), ~\left(\frac{1}{u=u_{1}\left(\frac{i \omega_{k}-\left(\alpha_{1}+\alpha_{2} e^{-i \omega_{k}}\right) \tau_{k}}{\tau_{k} \beta_{1} e^{-i \omega_{k}}}\right)} .\right.
$$

and $1 / u_{1}=1+\left(1+\beta_{3} e^{-i \omega_{k}}\right) v_{2}\left(\left(i \omega_{k}-\left(\alpha_{1}+\alpha_{2} e^{-i \omega_{k}}\right) \tau_{k}\right) /\right.$ $\left.\tau_{k} \beta_{1} e^{-i \omega_{k}}\right)+\left(\alpha_{2}+2 \alpha_{3} v_{2}\right) e^{-i \omega_{k}}$.

Take the enlarged phase space $B C$, defined as

$$
B C:=\left\{\varphi:[-1,0] \longrightarrow \mathbb{C}^{2} \mid \varphi \text { is continuous on }[-1,0)\right.
$$

$$
\left.\lim _{\theta \rightarrow 0^{-}} \varphi(\theta) \text { exists }\right\}
$$

The projection $\pi: B C \rightarrow P$ is defined as

$$
\begin{equation*}
\pi\left(\varphi+X_{0} b\right)=\Phi[(\Psi, \varphi)+\Psi(0) b], \quad \forall \varphi \in C, b \in \mathbb{R}^{2} \tag{38}
\end{equation*}
$$

thus we have the decomposition $B C=P \bigoplus \operatorname{Ker} \pi$. Let $z_{t}=$ $\Phi x+y, x \in \mathbb{C}^{2}, y \in \operatorname{ker}(\pi) \cap C^{1}:=Q^{1}$, we can decompose (32) to

$$
\begin{gather*}
\dot{x}=B x+\Psi(0) \widetilde{F}(\Phi x+y, \gamma) \\
\frac{\mathrm{d} y}{\mathrm{~d} x}=A_{\mathrm{Q}^{\mathrm{1}}} y+(I-\pi) X_{0} \widetilde{F}(\Phi x+y, \gamma) \tag{39}
\end{gather*}
$$

where

$$
X_{0}(\theta)= \begin{cases}I, & \theta=0  \tag{40}\\ 0, & -1 \leq \theta<0\end{cases}
$$

We write the Taylor expansion as follows:

$$
\begin{align*}
& \Psi(0) \widetilde{F}(\Phi x+y, \gamma)=\frac{1}{2} f_{2}^{1}(x, y, \gamma)+\frac{1}{3!} f_{3}^{1}(x, y, \gamma)+\text { h.o.t., } \\
& \begin{aligned}
(I-\pi) X_{0} \widetilde{F}(\Phi x+y, \gamma)= & \frac{1}{2} f_{2}^{2}(x, y, \gamma) \\
& +\frac{1}{3!} f_{3}^{2}(x, y, \gamma)+\text { h.o.t., }
\end{aligned}
\end{align*}
$$

where $f_{k}^{1}$ and $f_{k}^{2}$ are homogeneous polynomials in $x, y$, and $\gamma$ of degree $k, k=2,3$, with coefficients in $\mathbb{C}^{2}$ and $\operatorname{Ker} \pi$, h.o.t. stands for higher order terms. The normal form method implies a normal form on the center manifold of the origin for (32) which is

$$
\begin{equation*}
\dot{x}=B x+\frac{1}{2} g_{2}^{1}(x, 0, \gamma)+\frac{1}{3!} g_{3}^{1}(x, 0, \gamma)+\text { h.o.t. } \tag{42}
\end{equation*}
$$

where $g_{2}^{1}(x, 0, \gamma)$ and $g_{3}^{1}(x, 0, \gamma)$ are homogeneous polynomials in $x$ and $\gamma$, respectively.

From (39), it follows that

$$
\begin{equation*}
f_{2}^{1}(x, 0, \gamma)=2 \Psi(0)\left[N(\gamma)(\Phi x)+F\left(\Phi x, \tau_{k}\right)\right] \tag{43}
\end{equation*}
$$

that is,

$$
\begin{align*}
& f_{2}^{1}(x, 0, \gamma) \\
& =2\binom{A_{1} x_{1} \gamma+A_{2} x_{2} \gamma+a_{20} x_{1}^{2}+a_{11} x_{1} x_{2}+a_{02} x_{2}^{2}}{\bar{A}_{1} x_{2} \gamma+\bar{A}_{2} x_{1} \gamma+\bar{a}_{02} x_{1}^{2}+\bar{a}_{11} x_{1} x_{2}+\bar{a}_{20} x_{2}^{2}} \tag{44}
\end{align*}
$$

where

$$
\begin{gather*}
A_{1}=\frac{i \omega_{k}}{\tau_{k}} u^{T} v, \quad A_{2}=\frac{-i \omega_{k}}{\tau_{k}} u^{T} \bar{v}, \\
a_{20}=\tau_{k}\left[u_{1} e^{-2 i \omega_{k}} \zeta v_{1} v_{2}+u_{2}\left(-\alpha \beta v_{2}^{2}-e^{-2 i \omega_{k}} v_{1} v_{2}\right)\right], \\
a_{11}=\tau_{k}\left[\zeta u_{1}\left(v_{1} \bar{v}_{2}+\bar{v}_{1} v_{2}\right)\right.  \tag{45}\\
\left.-u_{2}\left(2 \alpha \beta v_{2} \bar{v}_{2}+v_{1} \bar{v}_{2}+\bar{v}_{1} v_{2}\right)\right], \\
a_{02}=\tau_{k}\left[u_{1} e^{2 i \omega_{k}} \zeta \bar{v}_{1} \bar{v}_{2}+u_{2}\left(-\alpha \beta \bar{v}_{2}^{2}-e^{2 i \omega_{k}} \bar{v}_{1} \bar{v}_{2}\right)\right]
\end{gather*}
$$

Thus

$$
\begin{equation*}
g_{2}^{1}(x, 0, \gamma)=\operatorname{Proj}_{\operatorname{Ker}\left(M_{2}^{1}\right)} f_{2}^{1}(x, 0, \gamma)=\binom{2 A_{1} x_{1} \gamma}{2 \bar{A}_{1} x_{2} \gamma} . \tag{46}
\end{equation*}
$$

We will compute the cubic terms $g_{3}^{1}(x, 0, \gamma)$ as follows.
Since $O\left(|x| \gamma^{2}\right)$ are irrelevant to determine the generic Hopf bifurcation, then

$$
\begin{equation*}
J=\operatorname{span}\left\{\binom{x_{1}^{2} x_{2}}{0},\binom{0}{x_{1} x_{2}^{2}}\right\} ; \tag{47}
\end{equation*}
$$

hence

$$
\begin{equation*}
g_{3}^{1}(x, 0, \gamma)=\operatorname{Proj}_{J} \bar{f}_{3}^{1}(x, 0,0)+o\left(|x| \gamma^{2}\right) \tag{48}
\end{equation*}
$$

where $\bar{f}_{3}^{1}(x, 0,0)=(3 / 2)\left[\left(D_{x} f_{2}^{1}\right) u_{2}^{1}-\left(D_{x} u_{2}^{1}\right) g_{2}^{1}\right]_{(x, 0,0)}+$ $(3 / 2)\left[\left(D_{y} f_{2}^{1}\right) u_{2}^{2}\right]_{(x, 0,0)}$. In order to obtain $g_{3}^{1}(x, 0, \gamma)$, we need to compute $f_{3}^{1}(x, 0,0)$; that is, $\operatorname{Proj}_{J}\left[\left(D_{x} f_{2}^{1}\right) u_{2}^{1}\right]_{(x, 0,0)}$, $\operatorname{Proj}_{J}\left[\left(D_{x} u_{2}^{1}\right) g_{2}^{1}\right]_{(x, 0,0)}$, and $\operatorname{Proj}_{J}\left[\left(D_{y} f_{2}^{1}\right) u_{2}^{2}\right]_{(x, 0,0)}$ should be given; we will compute them as follows.

Firstly, knowing that

$$
\begin{align*}
& f_{2}^{1}(x, 0,0)=2\binom{a_{20} x_{1}^{2}+a_{11} x_{1} x_{2}+a_{02} x_{2}^{2}}{\bar{a}_{02} x_{1}^{2}+\bar{a}_{11} x_{1} x_{2}+\bar{a}_{20} x_{2}^{2}}, \\
& u_{2}^{1}(x, 0)=\frac{2}{i \omega_{k}}\binom{a_{20} x_{1}^{2}-a_{11} x_{1} x_{2}-\frac{1}{3} a_{02} x_{2}^{2}}{\frac{1}{3} \bar{a}_{02} x_{1}^{2}+\bar{a}_{11} x_{1} x_{2}-\bar{a}_{20} x_{2}^{2}}, \tag{49}
\end{align*}
$$

then

$$
\begin{align*}
& \operatorname{Proj}_{J}\left[\left(D_{x} f_{2}^{1}\right) u_{2}^{1}\right]_{(x, 0,0)} \\
& \quad=\frac{4}{i \omega_{k}}\binom{\left(-a_{20} a_{11}+\frac{2}{3}\left|a_{02}\right|^{2}+\left|a_{11}\right|^{2}\right) x_{1}^{2} x_{2}}{\left(-\frac{2}{3}\left|a_{02}\right|^{2}-\left|a_{11}\right|^{2}+\overline{a_{20} a_{11}}\right) x_{1} x_{2}^{2}}  \tag{50}\\
& \quad=4\binom{A_{3} x_{1}^{2} x_{2}}{\bar{A}_{3} x_{1} x_{2}^{2}}
\end{align*}
$$

Secondly, noting (46), we know that $g_{2}^{1}(x, 0,0)=0$; then $\operatorname{Proj}_{J}\left[\left(D_{x} u_{2}^{1}\right) g_{2}^{1}\right]_{(x, 0,0)}=0$.

Lastly, we will compute $\operatorname{Proj}_{j}\left[\left(D_{y} f_{2}^{1}\right) u_{2}^{2}\right]_{(x, 0,0)}$ as follow.
Let $h=u_{2}^{2}=h_{200} x_{1}^{2}+h_{020} x_{2}^{2}+h_{002} \gamma^{2}+h_{110} x_{1} x_{2}+h_{101} x_{1} \gamma+$ $h_{011} x_{2} \gamma$. Noting $g_{2}^{2}=0$, one has

$$
\begin{align*}
M_{2}^{2} h(x, \gamma) & =f_{2}^{2}=2(I-\pi) X_{0} \widetilde{F}(\Phi x, \gamma) \\
& =2(I-\pi) X_{0}\left[N(\gamma)(\Phi x)+F\left(\Phi x, \tau_{k}\right)\right] \tag{51}
\end{align*}
$$

On the other hand, we know that

$$
\begin{array}{rl}
M_{2}^{2} & h(x, \gamma) \\
= & D_{x} h(x, \gamma) B x-A_{\mathrm{Q}^{1}} h(x, \gamma) \\
= & D_{x} h(x, \gamma) B x \\
& -\left[\dot{h}(x, \gamma)+X_{0}\left(L\left(\tau_{k}\right)(h(x, \gamma))-\dot{h}(x, \gamma)(0)\right)\right] . \tag{52}
\end{array}
$$

If $\gamma=0$, then

$$
\begin{gather*}
\dot{h}(x)-D_{x} h(x) B x=2 \Phi \Psi(0) F\left(\Phi x, \tau_{k}\right), \\
\dot{h}(x)(0)-L\left(\tau_{k}\right)(h(x))=2 F\left(\Phi x, \tau_{k}\right) . \tag{53}
\end{gather*}
$$

Let

$$
\begin{align*}
W(\theta) & =\Phi x+y=\Phi_{1} x_{1}+\Phi_{2} x_{2}+y(\theta) \\
& =e^{i \omega_{k} \theta} v x_{1}+e^{-i \omega_{k} \theta} \bar{v} x_{2}+y(\theta)  \tag{54}\\
\widetilde{W}(\theta)=\Phi x & =\Phi_{1} x_{1}+\Phi_{2} x_{2}=e^{i \omega_{k} \theta} v x_{1}+e^{-i \omega_{k} \theta} \bar{v} x_{2} .
\end{align*}
$$

From

$$
\begin{align*}
& f_{2}^{1}(x, y, 0) \\
& \quad=2 \tau_{k}\left(\begin{array}{c}
\zeta W_{1}(-1) W_{2}(-1) \\
u^{T}\left(\begin{array}{c}
\zeta \beta W_{2}^{2}(0)-W_{1}(-1) W_{2}(-1)
\end{array}\right) \\
\zeta W_{1}(-1) W_{2}(-1) \\
\bar{u}^{T}\left(\begin{array}{c} 
\\
-\alpha \beta W_{2}^{2}(0)-W_{1}(-1) W_{2}(-1)
\end{array}\right)
\end{array}\right), \tag{55}
\end{align*}
$$

we obtain

$$
\left[\left(D_{y} f_{2}^{1}\right) h\right]_{(x, 0,0)}=2\left(\begin{array}{c}
\tau_{k} u^{T}\binom{\zeta \widetilde{W}_{2}(-1) h^{1}(-1)+\zeta \widetilde{W}_{1}(-1) h^{2}(-1)}{-\widetilde{W}_{2}(-1) h^{1}(-1)-\widetilde{W}_{1}(-1) h^{2}(-1)-2 \alpha \beta \widetilde{W}_{2}(0) h^{2}(0)} \\
\zeta \widetilde{W}_{2}(-1) h^{1}(-1)+\zeta \widetilde{W}_{1}(-1) h^{2}(-1)  \tag{57}\\
\tau_{k} \bar{u}^{T}\left(\begin{array}{c}
\widetilde{W}_{1}(-1) h^{2}(-1)-2 \alpha \beta \widetilde{W}_{2}(0) h^{2}(0)
\end{array}\right) \\
-\widetilde{W}_{2}(-1) h^{1}(-1)-\widetilde{W}_{1}\left(-2 x^{2}\right.
\end{array}\right),
$$

where

$$
\begin{gather*}
A_{4}=\tau_{k}\left[u _ { 1 } \zeta \left(e^{-i \omega_{k}} v_{2} h_{110}^{1}(-1)+e^{i \omega_{k}} \bar{v}_{2} h_{200}^{1}(-1)\right.\right. \\
\left.\left.+e^{-i \omega_{k}} v_{1} h_{110}^{2}(-1)+e^{i \omega_{k}} \bar{v}_{1} h_{200}^{2}(-1)\right)\right] \\
+u_{2} \tau_{k}\left[-e^{-i \omega_{k}} v_{2} h_{110}^{1}(-1)-e^{i \omega_{k}} \bar{v}_{2} h_{200}^{1}(-1)\right. \\
\left.-e^{-i \omega_{k}} v_{1} h_{110}^{2}(-1)-e^{i \omega_{k}} \bar{v}_{1} h_{200}^{2}(-1)\right] \\
-u_{2} \tau_{k}\left[2 \alpha \beta\left(v_{2} h_{110}^{2}(0)+\bar{v}_{2} h_{200}^{2}(0)\right)\right] . \tag{58}
\end{gather*}
$$

In order to obtain $A_{4}$, we need to compute $h_{110}(\theta)$, $h_{200}(\theta)$. From (53), it follows that

$$
\begin{gather*}
\dot{h}_{110}=2\left(\Phi_{1}, \Phi_{2}\right)\binom{a_{11}}{a_{11}} \\
\dot{h}_{110}(0)-L\left(\tau_{k}\right)\left(h_{110}\right)=\tau_{k}\binom{a_{1}}{b_{1}}, \\
\dot{h}_{200}-2 i \omega_{k} h_{200}=2\left(\Phi_{1}, \Phi_{2}\right)\binom{a_{20}}{a_{02}},  \tag{59}\\
\dot{h}_{200}(0)-L\left(\tau_{k}\right)\left(h_{200}\right)=\tau_{k}\binom{a_{2}}{b_{2}},
\end{gather*}
$$

where $a_{1}=2\left[\zeta\left(v_{1} \bar{v}_{2}+\bar{v}_{1} v_{2}\right)\right], b_{1}=2\left[-2 \alpha \beta v_{2} \bar{v}_{2}-\left(v_{1} \bar{v}_{2}+\bar{v}_{1} v_{2}\right)\right]$, $a_{2}=2\left[\zeta v_{1} v_{2} e^{-2 i \omega_{k}}\right]$, and $b_{2}=2\left[-\alpha \beta v_{2}^{2}-e^{-2 i \omega_{k}} v_{1} v_{2}\right]$. Solving (59), we can obtain

$$
\begin{gather*}
h_{110}=2\left[\frac{a_{11}}{i \omega_{k}} \Phi_{1}-\frac{\bar{a}_{11}}{i \omega_{k}} \Phi_{2}\right]+C_{1}, \\
h_{200}=2\left[\frac{a_{20}}{-i \omega_{k}} \Phi_{1}+\frac{\bar{a}_{02}}{-3 i \omega_{k}} \Phi_{2}\right]+C_{2} e^{2 i \omega_{k} \theta}, \tag{60}
\end{gather*}
$$

where

$$
\begin{gathered}
C_{1}=\binom{C_{1}^{1}}{C_{1}^{2}} \\
C_{1}^{1}=\frac{\left|\begin{array}{cc}
a_{1} & -\alpha_{3} \\
b_{1} & -\left(\beta_{2}+\beta_{3}\right)
\end{array}\right|}{\left|\begin{array}{cc}
-\left(\alpha_{1}+\alpha_{2}\right) & -\alpha_{3} \\
-\beta_{1} & -\left(\beta_{2}+\beta_{3}\right)
\end{array}\right|},
\end{gathered}
$$

$$
\begin{gather*}
C_{1}^{2}=\frac{\left|\begin{array}{cc}
-\left(\alpha_{1}+\alpha_{2}\right) & a_{1} \\
-\beta_{1} & b_{1}
\end{array}\right|}{\left|\begin{array}{cc}
-\left(\alpha_{1}+\alpha_{2}\right) & -\alpha_{3} \\
-\beta_{1} & -\left(\beta_{2}+\beta_{3}\right)
\end{array}\right|}, \\
C_{2}=\binom{C_{2}^{1}}{C_{2}^{2}}, \\
C_{2}^{1}=\frac{\left|\begin{array}{cc}
\tau_{k} a_{2} & -\tau_{k} \alpha_{3} e^{-2 i \omega_{k}} \\
\tau_{k} b_{2} & 2 i \omega_{k}+\tau_{k} \beta_{2}+\tau_{k} \beta_{3} e^{-2 i \omega_{k}}
\end{array}\right|}{\left|\begin{array}{cc}
2 i \omega_{k}-\tau_{k} \alpha_{1}-\tau_{k} \alpha_{2} e^{-2 i \omega_{k}} & -\tau_{k} \alpha_{3} e^{-2 i \omega_{k}} \\
-\tau_{k} \beta_{1} e^{-2 i \omega_{k}} & 2 i \omega_{k}+\tau_{k} \beta_{2}+\tau_{k} \beta_{3} e^{-2 i \omega_{k}}
\end{array}\right|}, \\
C_{2}^{2}=\frac{\left|\begin{array}{cc}
2 i \omega_{k}-\tau_{k} \alpha_{1}-\tau_{k} \alpha_{2} e^{-2 i \omega_{k}} & \tau_{k} a_{2} \\
-\tau_{k} \beta_{1} e^{-2 i \omega_{k}} & \tau_{k} b_{2}
\end{array}\right|}{\left|\begin{array}{cc}
2 i \omega_{k}-\tau_{k} \alpha_{1}-\tau_{k} \alpha_{2} e^{-2 i \omega_{k}} & -\tau_{k} \alpha_{3} e^{-2 i \omega_{k}} \\
-\tau_{k} \beta_{1} e^{-2 i \omega_{k}} & 2 i \omega_{k}+\tau_{k} \beta_{2}+\tau_{k} \beta_{3} e^{-2 i \omega_{k}}
\end{array}\right|} \tag{61}
\end{gather*}
$$

Hence

$$
\begin{equation*}
g_{3}^{1}(x, 0,0)=\binom{\left(6 A_{3}+3 A_{4}\right) x_{1}^{2} x_{2}}{\left(6 \bar{A}_{3}+3 \bar{A}_{4}\right) x_{1} x_{2}^{2}} \tag{62}
\end{equation*}
$$

Thus, the normal form of system (42) has the form

$$
\begin{align*}
\dot{x}= & B x+\binom{A_{1} x_{1} \gamma}{\bar{A}_{1} x_{2} \gamma}+\frac{1}{3!}\binom{\left(6 A_{3}+3 A_{4}\right) x_{1}^{2} x_{2}}{\left(6 \bar{A}_{3}+3 \bar{A}_{4}\right) x_{1} x_{2}^{2}}  \tag{63}\\
& +o\left(|x|^{4}+|x| \gamma^{2}\right) .
\end{align*}
$$

Let $x_{1}=\xi_{1}-i \xi_{2}, x_{2}=\xi_{1}+i \xi_{2}, \xi_{1}=\rho \cos \omega$, and $\xi_{2}=$ $\rho \sin \omega$. Then the normal form becomes

$$
\begin{gather*}
\dot{\rho}=r_{1} \gamma \rho+r_{2} \rho^{3}+O\left(\gamma^{2} \rho+|(\rho, \gamma)|^{4}\right) \\
\dot{\omega}=-\omega_{k}-\operatorname{Im}\left(A_{1}\right) \gamma-\operatorname{Im}\left(A_{3}+\frac{1}{2} A_{4}\right) \rho^{2}+o\left(\left|\left(\rho^{2}, \gamma\right)\right|\right), \tag{64}
\end{gather*}
$$

where $r_{1}=\operatorname{Re} A_{1}, r_{2}=\operatorname{Re}\left(A_{3}+(1 / 2) A_{4}\right)$.
Summarizing all above, we have the following theorem.
Theorem 5. The flow on the center manifold of the equilibrium $P$ at $\gamma=0$ is given by (64). Also the following results hold:
(1) the Hopf bifurcation is supercritical if $r_{1} r_{2}<0$ and subcritical if $r_{1} r_{2}>0$;


Figure 2: (a) The equilibrium (1.3344, 92.1911) is stable when $\tau=0.2$. (b) The oscillation solutions $x(t)$ and $y(t)$ in terms of time $t$ when $\tau=0.2<\tau_{0}$. (c) The oscillation solution $x(t)$ in terms of $t$ when $\tau=0.372892$. (d) The oscillation solution $y(t)$ in terms of $t$.
(2) the bifurcated periodic solution is stable if $r_{2}<0$ and unstable if $r_{2}>0$;
(3) the period of the bifurcated periodic solution is

$$
\begin{align*}
p(\gamma)= & \frac{2 \pi}{\omega_{k}}-\frac{2 \pi \gamma\left(r_{2} \operatorname{Im}\left(A_{1}\right)-r_{1} \operatorname{Im}\left(A_{3}+(1 / 2) A_{4}\right)\right)}{r_{2} \omega_{k}^{2}} \\
& +O\left(\gamma^{2}\right) \tag{65}
\end{align*}
$$

In the following, we will give some simulations to illustrate the results of Theorems 4 and 5 for model (4). We cite the parameters in [11], that is, $\sigma=0.1181, \zeta=$ $0.0031, \delta=0.3743, \alpha=1.636$, and $\beta=0.002$. Then (4) has a tumor-free equilibrium $(0.3155,0)$, which is unstable, and a positive equilibrium $(1.33435,92.1911)$, which is locally asymptotically stable. We only simulate local properties of the
stable equilibrium ( $1.33435,92.1911$ ) here in Figures 2(a) and 2(b).

Remark 6. From Figures 2(c) and 2(d), we can see that the amplitude vibration for $x(t)$ is much bigger than that of $y(t)$; also both $x(t)$ and $y(t)$ with respect to $t$ are not so smooth. Then the Hopf bifurcated periodic solution on $(x(t), y(t))$ plan is not given here. At the same time, we can see that the dynamical behaviors of the system have been changed although $\tau$ is small.
3.2. Steady-State Bifurcation. From Section 2, we know that system (5) undergoes a steady-state bifurcation at the tumorfree equilibrium $P_{0}$ as $\alpha=\sigma / \delta, 0<\tau<\delta / \sigma$. In this section, we will discuss the properties of the steady-state bifurcation by using the center manifold reduction and normal form theories of retarded functional differential equations.

At the tumor-free equilibrium $P_{0}$, we write system (5) as an FDE:

$$
\begin{equation*}
\dot{z}(t)=N\left(z_{t}\right)+F\left(z_{t}\right), \tag{66}
\end{equation*}
$$

where

$$
\begin{gather*}
N(\varphi)=\tau\binom{-\delta \varphi_{1}(0)+\zeta \frac{\sigma}{\delta} \varphi_{2}(-1)}{\alpha \varphi_{2}(0)-\frac{\sigma}{\delta} \varphi_{2}(-1)},  \tag{67}\\
F(\varphi)=\tau\binom{\zeta \varphi_{1}(-1) \varphi_{2}(-1)}{-\alpha \beta \varphi_{2}^{2}(0)-\varphi_{1}(-1) \varphi_{2}(-1)} .
\end{gather*}
$$

Letting $\alpha=(\sigma / \delta)+\gamma$, then (66) can be written as

$$
\begin{equation*}
\dot{z}(t)=N\left(\frac{\sigma}{\delta}\right)\left(z_{t}\right)+\widetilde{F}\left(z_{t}, \gamma\right) \tag{68}
\end{equation*}
$$

where $\widetilde{F}\left(z_{t}, \gamma\right)=N(\gamma)\left(z_{t}\right)+F\left(z_{t}, \gamma\right)$.
Assuming that $A$ is the infinitesimal generator of $\dot{z}(t)=$ $N(\sigma / \delta)\left(z_{t}\right)$, then $A$ has a simple zero root. Set $\Lambda=\{0\}$ and we denote by $P$ the invariant space of $A$ associated with $\Lambda$; then $\operatorname{dim} P=1$. We can decompose $C:=C\left([-1,0], \mathbb{R}^{2}\right)$ to $C=P \bigoplus Q$ by the formal adjoint theory for FDEs by Hale [17]. Let $P=\operatorname{span}(\Phi)$ be the bases for $P$, where $\Phi=\binom{v_{1}}{v_{2}}$, which is a vector in $\mathbb{R}^{2}$ satisfying

$$
\begin{equation*}
N\left(\frac{\sigma}{\delta}\right) \Phi=\dot{\Phi}(0) \tag{69}
\end{equation*}
$$

Choose a basis $\Psi$ for the adjoint space $P^{*}$, where $\Psi=\left(u_{1}, u_{2}\right)$, which is a vector in $\mathbb{R}^{2 *}$ satisfying $\Psi N(\sigma / \delta)=-\dot{\Psi}(0)$. Thus we can obtain

$$
\begin{equation*}
\Phi=\binom{1}{\frac{\delta^{2}}{\zeta \sigma}}, \quad \Psi=\left(0, \frac{1}{\left(\delta^{2} / \zeta \sigma\right)-(\tau \delta / \zeta)}\right) \tag{70}
\end{equation*}
$$

According to the method of Faria and a similar computation in the last section, we can obtain

$$
\begin{align*}
f_{2}^{1}(x, 0, \gamma) & =2 \Psi(0)\left[N(\gamma)(\Phi x)+F\left(\Phi x, \tau_{k}\right)\right] \\
& =2 \frac{\tau}{\delta / \sigma-\tau}\left[\gamma \frac{\delta}{\sigma} x-\left(\left(\beta \delta^{2}\right) / \zeta \sigma+\delta / \sigma\right) x^{2}\right] \tag{71}
\end{align*}
$$

Noting

$$
\begin{equation*}
\operatorname{Ker}\left(M_{2}^{1}\right)=\operatorname{span}\left\{x^{2}, x \gamma, \gamma^{2}\right\} \tag{72}
\end{equation*}
$$

one has

$$
\begin{align*}
g_{2}^{1}(x, 0, \gamma) & =\operatorname{Proj}_{\operatorname{Ker}\left(M_{2}^{1}\right)} f_{2}^{1}(x, 0, \gamma) \\
& =2 \frac{\tau}{\delta / \sigma-\tau}\left[\gamma \frac{\delta}{\sigma} x-\left(\frac{\beta \delta^{2}}{\zeta \sigma}+\frac{\delta}{\sigma}\right) x^{2}\right] . \tag{73}
\end{align*}
$$

Thus, the normal form of system (5) is

$$
\begin{equation*}
\dot{x}=\frac{\tau}{\delta / \sigma-\tau}\left[\gamma \frac{\delta}{\sigma} x-\left(\frac{\beta \delta^{2}}{\zeta \sigma}+\frac{\delta}{\sigma}\right) x^{2}\right]+o\left(x^{2}\right) \tag{74}
\end{equation*}
$$

Then the following two results are obvious.
Theorem 7. If $\alpha=\sigma / \delta, 0<\tau<\delta / \sigma$, then the tumor-free equilibrium $P_{0}$ is stable.

Theorem 8. If $0<\tau<\delta / \sigma, \alpha=\sigma / \delta+\gamma$, and $\gamma$ is small enough, then
(1) the tumor-free equilibrium $P_{0}$ is stable as $\gamma>0$ and unstable as $\gamma<0$;
(2) system (5) undergoes transcritical bifurcation at the tumor-free equilibrium $P_{0}$.
3.3. Bogdanov-Takens Bifurcation. From Theorem 2 we know that the tumor-free equilibrium $P_{0}$ is a $\mathrm{B}-\mathrm{T}$ singular equilibrium of the system (5) as $\alpha=\sigma / \delta, \tau=\delta / \sigma$. In this section, we will discuss the bifurcations of the system (5) at $P_{0}$.

At tumor-free equilibrium $P_{0}$, we can write (5) as an FDE:

$$
\begin{equation*}
\dot{z}(t)=N\left(z_{t}\right)+F\left(z_{t}\right), \tag{75}
\end{equation*}
$$

where

$$
\begin{gather*}
N(\varphi)=\tau\binom{-\delta \varphi_{1}(0)+\zeta \frac{\sigma}{\delta} \varphi_{2}(-1)}{\alpha \varphi_{2}(0)-\frac{\sigma}{\delta} \varphi_{2}(-1)},  \tag{76}\\
F(\varphi)=\tau\binom{\zeta \varphi_{1}(-1) \varphi_{2}(-1)}{-\alpha \beta \varphi_{2}^{2}(0)-\varphi_{1}(-1) \varphi_{2}(-1)} .
\end{gather*}
$$

Let $\alpha=\sigma / \delta+\gamma_{1}, \tau=\delta / \sigma+\gamma_{2}$. Then system (75) can be written as

$$
\begin{equation*}
\dot{z}(t)=N\left(\frac{\sigma}{\delta}\right)\left(z_{t}\right)+\widetilde{F}\left(z_{t}, \gamma\right) \tag{77}
\end{equation*}
$$

where $\widetilde{F}\left(z_{t}, \gamma\right)=N(\gamma)\left(z_{t}\right)+F\left(z_{t}, \gamma\right), \gamma=\binom{\gamma_{1}}{\gamma_{2}}$.
Assuming that $A$ is the infinitesimal generator of $\dot{z}(t)=$ $N(\sigma / \delta)\left(z_{t}\right)$, then $A$ has double zero roots. Set $\Lambda=\{0\}$ and denote by $P$ the invariant space of $A$ associated with $\Lambda$; then $\operatorname{dim} P=2$. We can decompose $C:=C\left([-1,0], \mathbb{R}^{2}\right)$ as $C=$ $P \bigoplus Q$ by the formal adjoint theory for FDEs by Hale [17]. Assume that $P=\operatorname{span}(\Phi)$ and $P^{*}=\operatorname{span}(\Psi)$. On the other hand, we know that $A \Phi=\Phi B$, where

$$
B=\left(\begin{array}{ll}
0 & 1  \tag{78}\\
0 & 0
\end{array}\right)
$$

that is, $N \Phi=\dot{\Phi}(0)=\Phi B, \Psi N=-\dot{\Psi}(0)=-B \Psi$, and $(\Psi, \Phi)=$ $I_{2}$.

Let

$$
A^{\prime}=\left(\begin{array}{cc}
-\frac{\delta^{2}}{\sigma} & 0  \tag{79}\\
0 & 1
\end{array}\right), \quad B^{\prime}=\left(\begin{array}{cc}
0 & \zeta \\
0 & -1
\end{array}\right)
$$

From Lemma 3.1 by Xu and Huang [18], we can get

$$
\Phi(\theta)=\left(\begin{array}{cc}
\frac{\sigma}{\delta^{2}} & \frac{\sigma}{\delta^{2}} \theta-\frac{\sigma^{2}}{\delta^{4}}  \tag{80}\\
\frac{1}{\zeta} & \frac{1}{\zeta}(\theta+1)
\end{array}\right)
$$

$$
\Psi(\widetilde{\theta})=\left(\begin{array}{cc}
0 & -\frac{4}{3} \zeta-2 \zeta \widetilde{\theta}  \tag{81}\\
0 & 2 \zeta
\end{array}\right)
$$

Using the method of Faria and the last section, we can obtain

$$
\begin{align*}
f_{2}^{1}(x, 0, \gamma) & =2 \Psi(0)\left[N(\gamma)(\Phi x)+F\left(\Phi x, \tau_{k}\right)\right] \\
& =2\binom{-\frac{4}{3}\left[\frac{\delta}{\sigma} \gamma_{1} x_{1}+\left(\frac{\delta}{\sigma} \gamma_{1}+\frac{\sigma}{\delta} \gamma_{2}\right) x_{2}-\left(\frac{\beta}{\zeta}+\frac{1}{\delta}\right) x_{1}^{2}+\left(-\frac{2 \beta}{\zeta}+\frac{\sigma}{\delta^{3}}+\frac{1}{\delta}\right) x_{1} x_{2}-\frac{\beta}{\zeta} x_{2}^{2}\right]}{2\left[\frac{\delta}{\sigma} \gamma_{1} x_{1}+\left(\frac{\delta}{\sigma} \gamma_{1}+\frac{\sigma}{\delta} \gamma_{2}\right) x_{2}-\left(\frac{\beta}{\zeta}+\frac{1}{\delta}\right) x_{1}^{2}+\left(-\frac{2 \beta}{\zeta}+\frac{\sigma}{\delta^{3}}+\frac{1}{\delta}\right) x_{1} x_{2}-\frac{\beta}{\zeta} x_{2}^{2}\right]} \tag{82}
\end{align*}
$$

On the other hand, the basis of $\operatorname{Ker}\left(M_{2}^{1}\right)$ is

$$
\binom{0}{x_{1} \gamma_{2}}, \quad\binom{0}{x_{2} \gamma_{2}}
$$

$$
\begin{array}{ll}
\binom{0}{x_{1} \gamma_{1}}, & \binom{0}{x_{2} \gamma_{1}},  \tag{83}\\
\binom{0}{x_{1}^{2}}, & \binom{0}{x_{1} x_{2}}
\end{array}
$$

$$
\binom{0}{\gamma_{1}^{2}}, \quad\binom{0}{\gamma_{2}^{2}}, \quad\binom{0}{\gamma_{1} \gamma_{2}}
$$

and then we can obtain

$$
\begin{align*}
g_{2}^{1}(x, 0, \gamma) & =\operatorname{Proj}_{\operatorname{Ker}\left(M_{2}^{1}\right)} f_{2}^{1}(x, 0, \gamma) \\
& =2\left(2\left[\frac{\delta}{\sigma} \gamma_{1} x_{1}+\left(\frac{\delta}{\sigma} \gamma_{1}+\frac{\sigma}{\delta} \gamma_{2}\right) x_{2}-\left(\frac{\beta}{\zeta}+\frac{1}{\delta}\right) x_{1}^{2}+\left(-\frac{2 \beta}{\zeta}+\frac{\sigma}{\delta^{3}}+\frac{1}{\delta}\right) x_{1} x_{2}\right]\right) . \tag{84}
\end{align*}
$$

Thus, the normal form of the system (5) is

$$
\begin{gather*}
\dot{x}_{1}=x_{2} \\
\dot{x}_{2}=a_{1} x_{1}+a_{2} x_{2}+b_{1} x_{1}^{2}+b_{2} x_{1} x_{2}+\text { h.o.t } \tag{85}
\end{gather*}
$$

where

$$
\begin{gather*}
a_{1}=2 \frac{\delta}{\sigma} \gamma_{1}, \quad a_{2}=2\left(\frac{\delta}{\sigma} \gamma_{1}+\frac{\sigma}{\delta} \gamma_{2}\right), \\
b_{1}=-2\left(\frac{\beta}{\zeta}+\frac{1}{\delta}\right), \quad b_{2}=2\left(-\frac{2 \beta}{\zeta}+\frac{\sigma}{\delta^{3}}+\frac{1}{\delta}\right) . \tag{86}
\end{gather*}
$$

From above, we know that the following result can be obtained with the help of the theories of Xu and Huang [18] and Chow and Hale [19].

Theorem 9. Assume that $\alpha=\sigma / \delta+\gamma_{1}, \tau=\delta / \sigma+\gamma_{2}$, $b_{2}>0$, and $\gamma$ is small enough; then system (5) undergoes Bogdanov-Takens bifurcation at the tumor-free equilibrium $P_{0}$. Furthermore, on the $\left(a_{1}, a_{2}\right)$-parameter plane, both $H$ and $H L$
are located in the area $a_{1}>0, a_{2}<0$ and $H$ is on the left of HL, where H is Hopf bifurcation curve defined by

$$
\begin{align*}
H=\{ & \left(\gamma_{1}, \gamma_{2}\right) \\
& \left.: a_{2}\left(\gamma_{1}, \gamma_{2}\right)=\frac{b_{2}}{b_{1}} a_{1}\left(\gamma_{1}, \gamma_{2}\right)+\text { h.o.t., } a_{1}\left(\gamma_{1}, \gamma_{2}\right)>0\right\}, \tag{87}
\end{align*}
$$

HL is the homoclinic bifurcation curve defined by

$$
\begin{align*}
H L=\{ & 0\left(\gamma_{1}, \gamma_{2}\right) \\
& : a_{2}\left(\gamma_{1}, \gamma_{2}\right)=\mu\left(\sqrt{a_{1}\left(\gamma_{1}, \gamma_{2}\right)}\right) a_{1}\left(\gamma_{1}, \gamma_{2}\right)  \tag{88}\\
& \left.+ \text { h.o.t., } a_{1}\left(\gamma_{1}, \gamma_{2}\right)>0\right\}
\end{align*}
$$

and $\mu$ is a continuously differentiable function with $\mu(0)=$ $6 b_{2} / 7 b_{1}$.

Take the same parameter in last section, that is, $\sigma=$ $0.1181, \zeta=0.0031, \delta=0.3743$, and $\beta=0.002$; then


Figure 3: The bifurcation diagram of system (5) at the tumor-free equilibrium $P_{0}$.
the tumor-free equilibrium is $(0.3155,0)$. From Theorem 9, we can obtain that the Hopf bifurcation curve is $a_{2}=$ $-1.0955 a_{1}$ and the homoclinic bifurcation is $a_{2}=-0.939 a_{1}$. Then on ( $a_{1}, a_{2}$ )-parameter plane, the bifurcation diagram of system (5) at the equilibrium $P_{0}$ is in Figure 3.

## 4. Discussion

We have studied the nonlinear dynamics of Kuznetsov, Makalkin, and Taylor's model with delay, which is a twodimensional model of tumor cells and immune system. We first provided linear analysis of the model with delays at the possible equilibria, namely, the tumor-free and positive equilibria, and discussed the existence of Hopf bifurcation at the equilibria. We investigated the Hopf bifurcation, Bogdanov-Takens bifurcation, and steady-state bifurcation in the model. Numerical simulations were presented to illustrate the theoretical analysis and results.

Our analysis on the existence and stability of the tumorfree equilibrium corresponds to this elimination process and on the existence and stability of the positive equilibrium corresponds to coexistence of the immune system and the tumor system. Our results on the existence and stability of the Hopf bifurcated periodic solutions of $P_{2}$ describe the equilibrium process. When a stable periodic orbit exists, it can be understood that the tumor and the immune system can coexist although the cancer is not eliminated. The conditions for the parameters provide theories basis to control the development or progression of the tumors. The phenomena have been observed in some models as d'Onofrio [5], Kuznetsov et al. [11], and Bi and Xiao [14]. In particular, Bi and Ruan [7] have shown that various bifurcations, including Hopf bifurcation, Bautin bifurcation, and HopfHopf bifurcation, can occur in such models. Our results on the existence and stability of the bifurcated (Hopf, BogdanovTakens, and steady-state) periodic solutions describe rich dynamical behaviors of $P_{0}$, which show that the elimination process is so complex and difficult to control.

Finally, we should point out that we have studied the local dynamical behaviors of $P_{0}$ and $P_{2}$. As the example in our paper showed these two equilibria may coexist. Correspondingly, the system can exhibit more degenerate bifurcations including Hopf-Hopf and resonant higher codimension bifurcations. It would be interesting to consider these dynamics of the delayed model.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work is supported by the National Natural Science Foundation of China (no. 11171110), Shanghai Leading Academic Discipline Project (no. B407), and 211 Project of Key Academic Discipline of East China Normal University.

## References

[1] P. Boyle, A. d'Onofrio, P. Maisonneuve et al., "Measuring progress against cancer in Europe: has the 15\% decline targeted for 2000 come about?" Annals of Oncology, vol. 14, no. 8, pp. 1312-1325, 2003.
[2] D. Liu, S. Ruan, and D. Zhu, "Bifurcation analysis in models of tumor and immune system interactions," Discrete and Continuous Dynamical Systems B, vol. 12, no. 1, pp. 151-168, 2009.
[3] D. Liu, S. Ruan, and D. Zhu, "Stable periodic oscillations in a two-stage cancer model of tumor and immune system interactions," Mathematical Biosciences and Engineering, vol. 9, no. 2, pp. 347-368, 2012.
[4] R. Yafia, "Hopf bifurcation analysis and numerical simulations in an ODE model of the immune system with positive immune response," Nonlinear Analysis. Real World Applications, vol. 8, no. 5, pp. 1359-1369, 2007.
[5] A. d'Onofrio, "A general framework for modeling tumorimmune system competition and immunotherapy: mathematical analysis and biomedical inferences," Physica D, vol. 208, no. 3-4, pp. 220-235, 2005.
[6] A. d'Onofrio, "Tumor-immune system interaction: modeling the tumor-stimulated proliferation of effectors and immunotherapy," Mathematical Models \& Methods in Applied Sciences, vol. 16, no. 8, pp. 1375-1401, 2006.
[7] P. Bi and S. Ruan, "Bifurcations in delay differential equations and applications to tumor and immune system interaction models," SIAM Journal on Applied Dynamical Systems, vol. 12, no. 4, pp. 1847-1888, 2013.
[8] R. Yafia, "Hopf bifurcation in differential equations with delay for tumor-immune system competition model," SIAM Journal on Applied Mathematics, vol. 67, no. 6, pp. 1693-1703, 2007.
[9] H. Mayer, K. S. Zaenker, and U. An Der Heiden, "A basic mathematical model of the immune response," Chaos, vol. 5, no. 1, pp. 155-161, 1995.
[10] R. Yafia, "Hopf bifurcation in a delayed model for tumorimmune system competition with negative immune response," Discrete Dynamics in Nature and Society, vol. 2006, Article ID 95296, 9 pages, 2006.
[11] V. A. Kuznetsov, I. A. Makalkin, M. A. Taylor, and A. S. Perelson, "Nonlinear dynamics of immunogenic tumors: parameter estimation and global bifurcation analysis," Bulletin of Mathematical Biology, vol. 56, no. 2, pp. 295-321, 1994.
[12] M. Gałach, "Dynamics of the tumor-immune system competition-the effect of time delay," International Journal of Applied Mathematics and Computer Science, vol. 13, no. 3, pp. 395-406, 2003.
[13] A. L. Asachenkov, G. I. Marchuk, R. R. Mohler, and S. M. Zuev, "Immunology and disease control: a systems approach," IEEE Transactions on Biomedical Engineering, vol. 41, no. 10, pp. 943953, 1994.
[14] P. Bi and H. Xiao, "Hopf bifurcation for tumor-immune competition systems with delay," Electronic Journal of Differential Equations, vol. 2014, no. 27, pp. 1-13, 2014.
[15] T. Faria and L. T. Magalhães, "Normal forms for retarded functional-differential equations and applications to BogdanovTakens singularity," Journal of Differential Equations, vol. 122, no. 2, pp. 201-224, 1995.
[16] T. Faria and L. T. Magalhães, "Normal forms for retarded functional-differential equations with parameters and applications to Hopf bifurcation," Journal of Differential Equations, vol. 122, no. 2, pp. 181-200, 1995.
[17] J. Hale, Theory of Functional Differential Equations, Springer, New York, NY, USA, 1977.
[18] Y. Xu and M. Huang, "Homoclinic orbits and Hopf bifurcations in delay differential systems with T-B singularity," Journal of Differential Equations, vol. 244, no. 3, pp. 582-598, 2008.
[19] S. N. Chow and J. K. Hale, Methods of Bifurcation Theory, Springer, New York, NY, USA, 1982.

## Research Article

# Stability and Hopf Bifurcation Analysis of a Gene Expression Model with Diffusion and Time Delay 

Yahong Peng ${ }^{1}$ and Tonghua Zhang ${ }^{2}$<br>${ }^{1}$ Department of Mathematics, Donghua University, Shanghai 200051, China<br>${ }^{2}$ Department of Mathematics, Swinburne University of Technology, Melbourne, VIC 3122, Australia<br>Correspondence should be addressed to Yahong Peng; pengyh.mail@163.com

Received 24 February 2014; Accepted 13 March 2014; Published 14 April 2014
Academic Editor: Weiming Wang
Copyright © 2014 Y. Peng and T. Zhang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We consider a model for gene expression with one or two time delays and diffusion. The local stability and delay-induced Hopf bifurcation are investigated. We also derive the formulas determining the direction and the stability of Hopf bifurcations by calculating the normal form on the center manifold.


## 1. Introduction

The study on dynamics of a biological model is one of the dominant subjects in mathematical biology due to its universal existence and importance. In this paper, we consider a mathematical model of intracellular regulatory system which began with the work of Goodwin [1]. Since then, many researchers developed this work [2-5]. However, it is Monk who developed the first mathematical model for the Hesl system and validated it with biological data [6]. In order to describe the intracellular process more precisely, he introduced time delays to account for the processes of transcription and translation. If we denote by $M(t)$ and $P(t)$ the concentrations of Hes1 mRNA and Hesl protein, respectively, the basic reaction kinetics for this system can be expressed in the form of

$$
\begin{gather*}
\frac{d M(t)}{d t}=f\left(P\left(t-\tau_{m}\right)\right)-c M(t)  \tag{1}\\
\frac{d P(t)}{d t}=a M\left(t-\tau_{p}\right)-b P(t)
\end{gather*}
$$

where $f(P)=\alpha /\left(1+\left(P / P_{0}\right)^{n}\right)$. The parameter $a$ is the rate at which Hesl protein is produced from Hesl mRNA and $b$ and $c$ are the decay rates of Hesl protein and Hesl mRNA, respectively. $f(P)$ is the rate of production of new mRNA molecules, with $\alpha$ and $P_{0}$ as constants, to represent the rate of transcript initiation in the absence of Hesl protein and
the reference concentration of protein, respectively, and $n$ is the Hill coefficient. $\tau_{m}$ and $\tau_{p}$ represent the transcriptional and translational time delays. The units of the parameters are as follows. $a$ is measured in protein molecules per mRNA molecule per minute; $b$ and $c$ are measured in molecules per minute; $\alpha$ is measured in mRNA molecules per diploid cell per minute; $P_{0}$ is measured in molecules and time delays $\tau_{m}$ and $\tau_{p}$ are measured in minute.

In order to reduce the number of parameters, [7] introduced transformations

$$
\begin{equation*}
m=\frac{M}{\alpha}, \quad p=\frac{P}{\alpha a}, \quad p_{0}=\frac{P_{0}}{\alpha a} \tag{2}
\end{equation*}
$$

under which system (1) takes the following form:

$$
\begin{gather*}
\frac{d m(t)}{d t}=\frac{1}{1+\left(p\left(t-\tau_{m}\right) / p_{0}\right)^{n}}-c m(t)  \tag{3}\\
\frac{d p(t)}{d t}=m\left(t-\tau_{p}\right)-b p(t)
\end{gather*}
$$

Recently, Zhang et al. [8] investigated the stability and Hopf bifurcation of the equilibrium of system (3). By using the method of multiple time scales, they also obtained the normal form on the center manifold of delay differential equations (3).

The diffusion process comes naturally in biology as pointed out by Murray in [9]. Experimental data also suggests
that many pathways exhibit oscillation in concentrations of substance involved, both temporally and spatially. For example, a type diffusion process appears in a process of assemblage of particles, which is due to the particles spread out as a result of an irregular individual particle's motion. In these cases, a partial differential equation or system of partial differential equations will be employed to describe the processes. Nowadays, models involving delays and also spatial diffusion are increasingly applied to the study due to more appropriate biological justification. Reference [10] gave plausible biological explanations for the delays appearing in the model (3). In what follows, spatial diffusion will be applied to the model (3) to gain new information about the precise spatiotemporal dynamics of mRNA and proteins. In fact, Sturrock et al. [11] derived a system of partial differential equations to capture the evolution in space and time of the variables in the Hes1 and p53-Mdm2 systems. In this paper, we extend previous mathematical model (3) into its spatial version by considering spatial interactions within the cell explicitly (of course, the clear biological explanation will be given behind the model (4) of promotion); namely, we consider the delayed reaction-diffusion system with the following initial and boundary conditions:

$$
\begin{align*}
& \frac{\partial m(x, t)}{\partial t}=d_{1} \frac{\partial^{2} m(x, t)}{\partial x^{2}}+\frac{1}{1+\left(p\left(x, t-\tau_{m}\right) / p_{0}\right)^{n}} \\
& -c m(x, t), \quad t>0, \quad x \in(0, \pi), \\
& \frac{\partial p(x, t)}{\partial t}=d_{2} \frac{\partial^{2} p(x, t)}{\partial x^{2}}+m\left(x, t-\tau_{p}\right)-b p(x, t), \\
& t>0, x \in(0, \pi), \\
& \frac{\partial m(x, t)}{\partial x}=\frac{\partial p(x, t)}{\partial x}=0, \quad x=0, \pi, t \geq 0 \\
& m(x, t)=\phi(x, t) \geq 0, \quad p(x, t)=\psi(x, t) \geq 0 \\
& \quad(x, t) \in[0, \pi] \times[-\tau, 0] \tag{4}
\end{align*}
$$

where $d_{1}$ and $d_{2}$ are the diffusion coefficients for Hesl mRNA and Hesl protein, respectively, with unit such as $\mathrm{cm} / \mathrm{min}$. The initial function $\phi(t, x), \psi(t, x) \in \mathscr{C}:=C\left([-\tau, 0], L^{2}([0, \pi])\right)$. The imposed Neumann boundary condition here implies that mRNA and protein are not exported across the nuclear membrane or the cell membrane.

Obviously, (4) is a system of reaction-diffusion equations modeling the spatiotemporal evolution of the Hesl system. The same reaction kinetics from the ODE model (3) are retained but are now also coupled with diffusion to model explicitly protein and mRNA transport in a cell. That is to say, molecules move from the nucleus to the cytoplasm and from cytoplasm to nucleus across the nuclear membrane. Here, we use a system of PDEs (4) to reflect the reality that the mRNA is transcribed from DNA exclusively in the nucleus and that protein is translated from mRNA exclusively in the cytoplasm. The main advantage of using systems of PDEs (4) to model intracellular reactions is that the PDEs enable spatial
effects to be examined explicitly. The main object of this paper is to investigate the effect of the delay and diffusion on the dynamics of system (4). In addition, in order to determine the direction and the stability of Hopf bifurcations, we use the normal form procedure for functional differential equations (FDEs) with diffusion due to Faria [12, 13].

This paper is organized as follows. In Section 2, stability of positive equilibrium and existence of Hopf bifurcation are studied using $\tau$ as a parameter. In Section 3, the effect of diffusion on the Hopf bifurcation will be investigated. Using the normal form technique for partial functional differential equations, the formulas for determining the direction and stability of Hopf bifurcation are presented in Section 4. Finally, in Section 5, we will illustrate the theoretical results by numerical simulations along with some discussion.

## 2. Stability of Positive Equilibrium and Existence of Hopf Bifurcation

It is easy to verify that system (4) has a unique positive equilibrium $E^{*}\left(m^{*}, p^{*}\right)$ determined by

$$
\begin{gather*}
\frac{1}{1+\left(p^{*} / p_{0}\right)^{n}}-c m^{*}=0  \tag{5}\\
m^{*}-b p^{*}=0
\end{gather*}
$$

The object here is then to relate the dynamics of (3) and (4) in the neighborhood of $E^{*}$, at the first critical point of the parameter $\tau$. To this end, we let $\bar{m}=m-m^{*}, \bar{p}=p-p^{*}$. With the help of (5), after dropping the bars for simplicity of notation, (4) is transformed into the following system:

$$
\begin{align*}
\frac{\partial m(x, t)}{\partial t}= & d_{1} \frac{\partial^{2} m(x, t)}{\partial x^{2}}+\frac{1}{1+\left(\left(p\left(x, t-\tau_{m}\right)+p *\right) / p_{0}\right)^{n}} \\
& -c\left(m(x, t)+m^{*}\right), \\
\frac{\partial p(x, t)}{\partial t} & =d_{2} \frac{\partial^{2} p(x, t)}{\partial x^{2}}+m\left(x, t-\tau_{p}\right)-b p(x, t) \tag{6}
\end{align*}
$$

with the origin as its equilibrium.
Let

$$
\begin{align*}
f^{(1)}(m, p)= & \frac{1}{1+\left(\left(p\left(x, t-\tau_{m}\right)+p *\right) / p_{0}\right)^{n}} \\
& -c\left(m(x, t)+m^{*}\right)  \tag{7}\\
f^{(2)}(m, p) & =m\left(x, t-\tau_{p}\right)-b p(x, t) .
\end{align*}
$$

Define $f_{i j}^{(1)}(i+j \geq 1)$ and $f_{i j}^{(2)}(i+j \geq 1)$ as follows:

$$
\begin{equation*}
f_{i j}^{(1)}=\frac{\partial^{i+j} f^{(1)}(0,0)}{\partial m^{i} \partial p^{j}}, \quad f_{i j}^{(2)}=\frac{\partial^{i+j} f^{(2)}(0,0)}{\partial m^{i} \partial p^{j}} \tag{8}
\end{equation*}
$$

where $i$ and $j$ are the nonnegative integers. Then system (6) can be rewritten as

$$
\begin{align*}
\frac{\partial m(x, t)}{\partial t}= & d_{1} \frac{\partial^{2} m(x, t)}{\partial x^{2}}+a_{11} m(x, t)+a_{12} p\left(x, t-\tau_{m}\right) \\
& +\sum_{i+j \geq 2} \frac{1}{i!j!} f_{i j}^{(1)} m^{i}(x, t) p^{j}\left(x, t-\tau_{m}\right), \\
\frac{\partial p(x, t)}{\partial t}= & d_{2} \frac{\partial^{2} p(x, t)}{\partial x^{2}}+a_{21} m\left(x, t-\tau_{p}\right)+a_{22} p(x, t) \\
& +\sum_{i+j \geq 2} \frac{1}{i!j!} f_{i j}^{(2)} m^{i}\left(x, t-\tau_{p}\right) p^{j}(x, t), \tag{9}
\end{align*}
$$

where

$$
\begin{align*}
a_{11} & =f_{10}^{(1)}=\frac{\partial f^{(1)}}{\partial m}(0,0)=-c \\
a_{12}=f_{01}^{(1)} & =\frac{\partial f^{(1)}}{\partial p}(0,0)=-\frac{n c^{2} b^{2}\left(p^{*}\right)^{n+1}}{p_{0}^{n}}  \tag{10}\\
a_{21} & =f_{10}^{(2)}=\frac{\partial f^{(2)}}{\partial m}(0,0)=1 \\
a_{22} & =f_{01}^{(2)}=\frac{\partial f^{(2)}}{\partial p}(0,0)=-b
\end{align*}
$$

For simplification of notation, we use $m(t)$ for $m(\cdot, t)$ and $p(t)$ for $p(\cdot, t)$ and $(m(t), p(t))=(m(\cdot, t), p(\cdot, t))$ is in a suitable Hilbert space $X$

$$
\begin{equation*}
X=\left\{(m, p): m, p \in W^{2,2}(0, \pi), \frac{\partial m}{\partial x}=\frac{\partial p}{\partial x}=0 \text { at } x=0, \pi\right\} . \tag{11}
\end{equation*}
$$

By setting $U(t)=(m(t), p(t)) \in X$, we further write (9) as an abstract equation in $\mathscr{C}:=C([-\tau, 0], X)$ :

$$
\begin{equation*}
\frac{d U(t)}{d t}=d \Delta U(t)+L\left(U_{t}\right)+F\left(U_{t}\right) \tag{12}
\end{equation*}
$$

where $d \Delta=\left(d_{1} \Delta, d_{2} \Delta\right)$ and $L: \mathscr{C} \rightarrow R^{2}$, and $F: \mathscr{C} \times R \rightarrow$ $R^{2}$ are given by

$$
\begin{gather*}
L(\varphi)=\binom{a_{11} \varphi_{1}(0)+a_{12} \varphi_{2}\left(-\tau_{m}\right)}{a_{21} \varphi_{1}\left(-\tau_{p}\right)+a_{22} \varphi_{2}(0)}, \\
F(\varphi)=\binom{\sum_{i+j \geq 2} \frac{1}{i!j!} f_{i j}^{(1)} \varphi_{1}^{i}(0) \varphi_{2}^{j}\left(-\tau_{m}\right)}{\sum_{i+j \geq 2} \frac{1}{i!j!} f_{i j}^{(2)} \varphi_{1}^{i}\left(-\tau_{p}\right) \varphi_{2}^{j}(0)}, \tag{13}
\end{gather*}
$$

for $\varphi=\left(\varphi_{1}, \varphi_{2}\right) \in \mathscr{C}$. Obviously, $L$ is a linear operator. The linearization of (12) is

$$
\begin{equation*}
\frac{d U(t)}{d t}=d \Delta U(t)+L\left(U_{t}\right) \tag{14}
\end{equation*}
$$

It has a characteristic equation given by

$$
\begin{equation*}
\Delta_{k}(\lambda, \tau)=\lambda^{2}+p_{k} \lambda+r_{k}+K e^{-2 \lambda \tau}=0, \quad k=0,1,2, \ldots, \tag{15}
\end{equation*}
$$

where $p_{k}=\left(d_{1}+d_{2}\right) k^{2}+(b+c), r_{k}=\left(d_{1} k^{2}+c\right)\left(d_{2} k^{2}+b\right)$, $K=n c^{2} b^{2}\left(p^{*}\right)^{n+1} / p_{0}^{n}$, and $2 \tau=\tau_{m}+\tau_{p}$ is the total time delay. When there is no diffusion effect, namely, $d_{1}=d_{2}=0$, (15) can be written as

$$
\begin{equation*}
\lambda^{2}+(b+c) \lambda+b c+K e^{-2 \lambda \tau}=0 \tag{16}
\end{equation*}
$$

which is equivalent to (12) of Zhang et al.'s work in [8], and the related stability and Hopf bifurcation have been investigated.

In what follows, we will analyze the effect of diffusion terms by the distribution of the roots of (15) with $d_{1}>0$, $d_{2}>0$. We first consider the case when the delay is zero. For (15), if $\tau=0$, then we have

$$
\begin{equation*}
\lambda^{2}+p_{k} \lambda+r_{k}+K=0, \quad k=0,1,2 \tag{17}
\end{equation*}
$$

Since $p_{k}>0, r_{k}>0$ for any $k \in N$ and $K>0$, it is easy to verify that (17) has a pair of roots with negative real parts. And, for $\tau>0$, we have the following lemma.

Lemma 1. Assume that

$$
\begin{equation*}
\left(d_{1}+c\right)\left(d_{2}+b\right) \geq K \tag{H}
\end{equation*}
$$

holds. Then all the roots of the characteristic equation (15) have negative real part for $\tau>0$.

Proof. If the conclusion is not true, namely, (15) admits at least one root $\lambda=\mu+i \omega$ with $\mu \geq 0$, then we obtain

$$
\begin{equation*}
(\mu+i \omega)^{2}+p_{k}(\mu+i \omega)+r_{k}+K e^{-2 \tau(\mu+i \omega)}=0 \tag{18}
\end{equation*}
$$

Separating the real and imaginary parts yields

$$
\begin{gather*}
\mu^{2}-\omega^{2}+p_{k} \mu+r_{k}+K e^{-\mu \cdot 2 \tau} \cos (\omega \cdot 2 \tau)=0 \\
2 \mu \omega+p_{k} \omega-K e^{-\mu \cdot 2 \tau} \sin (\omega \cdot 2 \tau)=0 \tag{19}
\end{gather*}
$$

It implies that

$$
\begin{equation*}
\left(\mu^{2}-\omega^{2}+p_{k} \mu+r_{k}\right)^{2}+\left(2 \mu \omega+p_{k} \omega\right)^{2}=K^{2} e^{-2 \mu \cdot 2 \tau} \tag{20}
\end{equation*}
$$

namely,

$$
\begin{align*}
\mu^{4}+ & \omega^{4}+p_{k}^{2} \mu^{2}+r_{k}^{2}+2 \mu^{2} \omega^{2}+2 p_{k} \mu^{3} \\
& +2 \mu^{2} \omega^{2}+\left(p_{k}^{2}-2 r_{k}\right) \omega^{2}+2 p_{k} \mu r_{k}=K^{2} e^{-2 \mu \cdot 2 \tau} \tag{21}
\end{align*}
$$

Since $p_{k}^{2}-2 r_{k}=k^{4}\left(d_{1}^{2}+d_{2}^{2}\right)+2\left(b d_{2}+c d_{1}\right) k^{2}+b^{2}+c^{2}>0$, we can easily verify

$$
\begin{equation*}
r_{k}^{2}<K^{2} e^{-2 \mu \cdot 2 \tau}<K^{2} \tag{22}
\end{equation*}
$$

That is, $r_{k}<K$. Notice that $\left(d_{1}+c\right)\left(d_{2}+b\right) \leq r_{k}$, for $k \geq 1$, $k \in N$, gives

$$
\begin{equation*}
\left(d_{1}+c\right)\left(d_{2}+b\right)<K \tag{23}
\end{equation*}
$$

It is a contradiction. Thus the conclusion follows.

Notice that when $\tau=0$, all roots of (15) have negative real part and the roots of (15) continuously depend on the parameter $\tau$, and we can summarize our conclusion as follows.

Theorem 2. Assume that $(H)$ holds. Then the equilibrium point $E^{*}$ of system (4) is asymptotically stable for $\tau \geq 0$.

## 3. Effects of Diffusion on the Hopf Bifurcation

Assume that $i \omega(\omega>0)$ is a purely imaginary root of (15); then we have

$$
\begin{gather*}
-\omega^{2}+r_{k}+K \cos (\omega \cdot 2 \tau)=0,  \tag{24}\\
p_{k} \omega-K \sin (\omega \cdot 2 \tau)=0,
\end{gather*}
$$

which implies that

$$
\begin{equation*}
\omega^{4}+P_{k} \omega^{2}+R_{k}=0, \quad k=0,1,2, \ldots \tag{25}
\end{equation*}
$$

where

$$
\begin{align*}
P_{k}= & p_{k}^{2}-2 r_{k} \\
= & \left(d_{1}^{2}+d_{2}^{2}\right) k^{4}+2\left(b d_{2}+c d_{1}\right) k^{2}+b^{2}+c^{2}>0, \\
R_{k}=r_{k}^{2}-K^{2}= & \left(r_{k}+K\right)\left(r_{k}-K\right) \\
= & {\left[\left(d_{1} k^{2}+c\right)\left(d_{2} k^{2}+c\right)+K\right] } \\
& \times\left[\left(d_{1} k^{2}+c\right)\left(d_{2} k^{2}+c\right)-K\right] . \tag{26}
\end{align*}
$$

Equation (25) implies that $R_{k}$ should be negative for some $k \in$ N . It is equivalent to the fact that

$$
\begin{equation*}
\tilde{R}_{k}=\left(d_{1} k^{2}+c\right)\left(d_{2} k^{2}+c\right)-K \tag{27}
\end{equation*}
$$

should be less than 0 as it is easy to see that $R_{k}$ has the same sign as that of $\widetilde{R}_{k}$. Rewrite $\widetilde{R}_{k}$ into the following form:

$$
\begin{equation*}
\widetilde{R}_{k}=d_{1} d_{2} k^{4}+\left(d_{1} b+d_{2} c\right) k^{2}+b c-K \tag{28}
\end{equation*}
$$

It is obviously a quadratic polynomial in terms of $k^{2}$. Equation (28) implies that there exists a $k_{1} \in \mathbf{N}$ such that $\widetilde{R}_{k_{1}}<0$ if and only if $b c-K<0$. Furthermore, we have

$$
\begin{gather*}
\widetilde{R}_{k}<0 \quad \text { for } 0 \leq k \leq k_{1} \\
\widetilde{R}_{k}>0 \quad \text { for } k>k_{1}, k \in \mathbb{N} . \tag{29}
\end{gather*}
$$

From (29), we obtain that, for each $k \in\left\{0,1, \ldots, k_{1}\right\}$, (25) has only one positive real root $\omega_{k}$, which is given by

$$
\begin{equation*}
\omega_{k}=\frac{\sqrt{2}}{2} \sqrt{-P_{k}+\sqrt{P_{k}^{2}-4 R_{k}}} . \tag{30}
\end{equation*}
$$

We now can make the following conclusion.

Lemma 3. Assume that $(H)$ is not true. Then (15) has a pair of purely imaginary roots $\pm i \omega_{k}$ for each $k \in\left\{0,1, \ldots, k_{1}\right\}$ and has no purely imaginary roots for $k_{1}<k \in \mathbf{N}$, where $k_{1}$ and $\omega_{k}$ are defined as above.

In the rest of this section, we will discuss the case of $k<$ $k_{1}$.

From (24), we have

$$
\begin{equation*}
\sin (\omega \cdot 2 \tau)=\frac{p_{k} \omega}{K}, \quad \cos (\omega \cdot 2 \tau)=\frac{\omega^{2}-r_{k}}{K} \tag{31}
\end{equation*}
$$

Then, for $k \in\left\{0,1, \ldots, k_{1}\right\}$, define

$$
\begin{equation*}
\tau_{j}^{k}=\frac{1}{\omega_{k}}\left(\arccos \frac{\omega_{k}^{2}-r_{k}}{K}+2 j \pi\right), \quad j=0,1,2, \ldots \tag{32}
\end{equation*}
$$

In the following, we will order the sequence of $\tau_{j}^{k}$ depending on the diffusion coefficients $d_{1}$ and $d_{2}$ for $k \in\left\{0,1, \ldots, k_{1}\right\}$.

Lemma 4. If $d_{1}=d_{2}$ and $(\bar{H})$ holds, then

$$
\begin{equation*}
\tau_{0}^{0}=\min \left\{\tau_{j}^{k}\right\}_{k \in\left\{0,1, \ldots, k_{1}\right\}}, \quad j=0,1,2, \ldots \tag{33}
\end{equation*}
$$

Proof. If $d_{1}=d_{2}$, from (30), we have

$$
\begin{gather*}
\omega_{k}^{2}=\frac{1}{2}\left[-(b+c)^{2}-2(b+c) d k^{2}-2 d^{2} k^{4}+2 b c\right. \\
\\
\left.+\sqrt{(b-c)^{2}\left[2 d k^{2}+(b+c)\right]^{2}+4 K^{2}}\right],  \tag{34}\\
\omega_{k}^{2}-r_{k}=\frac{1}{2}\left[-(b+c)^{2}-4(b+c) d k^{2}-4 d^{2} k^{4}\right. \\
\\
\left.+\sqrt{(b-c)^{2}\left[2 d k^{2}+(b+c)\right]^{2}+4 K^{2}}\right] .
\end{gather*}
$$

Let $x=\sqrt{(b-c)^{2}\left[2 d k^{2}+(b+c)\right]^{2}+4 K^{2}}$; it is easy to verify

$$
\begin{gather*}
x>(b-c)^{2},  \tag{35}\\
\omega_{k}^{2}=\frac{1}{2}\left[x-\frac{x^{2}-4 K^{2}}{2(b-c)^{2}}+\frac{(b-c)^{2}}{2}\right], \\
\omega_{k}^{2}-r_{k}=\frac{1}{2}\left[x-\frac{x^{2}-4 K^{2}}{2(b-c)^{2}}\right] \tag{36}
\end{gather*}
$$

Thus, according to (32), we obtain

$$
\begin{align*}
\tau_{j}^{k} & =\tau(x) \\
& =\frac{\arccos \left[\left(x-\left(\left(x^{2}-4 K^{2}\right) /(b-c)^{2}\right) / 2 K\right)+2 j \pi\right]}{(\sqrt{2} / 2)\left[x-\left(\left(x^{2}-4 K^{2}\right) / 2(b-c)^{2}\right)+\left((b-c)^{2} / 2\right)\right]^{1 / 2}} . \tag{37}
\end{align*}
$$

In addition,

$$
\begin{equation*}
\frac{d\left(x-\left(\left(x^{2}-4 K^{2}\right) / 2(b-c)^{2}\right)\right)}{d x}=1-\frac{x}{(b-c)^{2}} \tag{38}
\end{equation*}
$$

From (35), we know that $x-\left(\left(x^{2}-4 K^{2}\right) / 2(b-c)^{2}\right)$ is decreasing with respect to $x$. Then we obtain $\tau_{j}^{k+1}>\tau_{j}^{k}$ if $d_{1}=d_{2}$. Clearly, $\tau_{j+1}^{k}>\tau_{j}^{k}$, so we have

$$
\begin{equation*}
\tau_{0}^{0}=\min \left\{\tau_{j}^{k}\right\}_{k \in\{0,1,2, \ldots\}}, \quad j=0,1,2, \ldots \tag{39}
\end{equation*}
$$

According to Lemma 4 and the continuous dependence of $\tau_{j}^{k}$ on $d_{1}$ and $d_{2}$, we summarize the following lemma.

Lemma 5. For any $d>0$, there exists an $\epsilon(d)>0$ such that, for any $d_{1}, d_{2} \in(d-\epsilon, d+\epsilon)$ satisfying $(\bar{H}), \tau_{0}^{0}=$ $\min \left\{\tau_{j}^{k}\right\}_{k \in\left\{0,1, \ldots ., k_{1}\right\}}, j=0,1,2, \ldots$.

Let $\lambda_{k}(\tau)=\mu_{k}(\tau)+i \omega_{k}(\tau)$ be the roots of (15) near $2 \tau=\tau_{j}^{k}$ satisfying $\mu_{k}\left(\tau_{j}^{k}\right)=0, \omega_{k}\left(\tau_{j}^{k}\right)=\omega_{k}$. By using the method in [14, 15], we can prove the following transversality condition.

Lemma 6. If $d_{1}$ and $d_{2}$ satisfy the condition in Lemma 5, then, for $k \in\left\{0,1, \ldots, k_{1}\right\}$ and $j \in \mathbb{N}_{0}, d \operatorname{Re}(\lambda) /\left.d \tau\right|_{2 \tau=\tau_{j}^{k}}>0$.

Proof. Differentiating equation (15) with respect to $\tau$, we obtain

$$
\begin{equation*}
\left(\frac{d \lambda}{d \tau}\right)^{-1}=\frac{\left(2 \lambda+p_{k}\right) e^{2 \lambda \tau}}{2 \lambda K}-\frac{\tau}{\lambda} \tag{40}
\end{equation*}
$$

From (24), we have

$$
\begin{align*}
\operatorname{Re}\left(\left.\frac{d \lambda}{d \tau}\right|_{2 \tau=\tau_{j}^{k}}\right)^{-1} & =\frac{2 \omega_{k} \cos \left(\omega_{k} \tau_{j}^{k}\right)+p_{k} \sin \left(\omega_{k} \tau_{j}^{k}\right)}{2 k \omega_{k}} \\
& =\frac{2 \omega_{k} \cdot\left(\left(\omega_{k}^{2}-r_{k}\right) / K\right)+p_{k} \cdot\left(p_{k} \omega_{k} / K\right)}{2 k \omega_{k}} \\
& =\frac{2 \omega_{k}^{2}+\left(p_{k}^{2}-2 r_{k}\right)}{2 K^{2}}>0 \tag{41}
\end{align*}
$$

since $p_{k}^{2}-2 r_{k}>0$.
Combining the above analysis and the qualitative theory of partial functional differential equations in [16], we have the following results on the stability of equilibrium $E^{*}$ of system (4) and existence of Hopf bifurcation near $E^{*}$.

Theorem 7. Assume that $d_{1}, d_{2}$ satisfy the condition in Lemma 5. Then, for system (4),
(i) the positive equilibrium $E^{*}$ is asymptotically stable for $\tau \in\left[0, \tau_{0}^{0}\right)$ and unstable for $\tau \in\left(\tau_{0}^{0},+\infty\right)$;
(ii) it undergoes Hopf bifurcations near the positive equilibrium $E^{*}$ at $2 \tau=\tau_{j}^{k}$ for $k \in\left\{0,1, \ldots, k_{1}\right\}$ and $j \in \mathbb{N}_{0}$.

Remark 8. If $d_{1}=d_{2}=0$, Theorem 2 and Theorem 7 are the conclusion of Theorem 1 in [8]. That is, assuming either condition $(H)$ or $(\bar{H})$, Theorem 2 and Theorem 7 show that
the local stability of $E^{*}$ for $0 \leq \tau<\tau_{0}^{0}$ is the same for system (3) and system (4). Here we can know the effect of the diffusion coefficients $d_{1}$ and $d_{2}$, or, in order to more clearly understand the effect of diffusion, we take $b=c=0.03$, $p_{0}=0.4, n=2$, and then we get $p^{*}=5.6134, K \approx 17.91 \times 10^{-4}$ by calculation. In the absence of diffusion, $b \times c=9 \times 10^{-4}<K$, it is known that the equilibrium $E^{*}$ of system (3) is stable when $\tau<\tau_{0}^{0}$ according to Theorem 1 in [8]. In the presence of diffusion, for example, taking $d_{1}=0.008, d_{2}=0.02$, which implies $\left(d_{1}+c\right)\left(d_{2}+b\right)=19 \times 10^{-4}>K$, we know that the equilibrium $E^{*}$ of system (4) is stable for $\tau \geq 0$ by Theorem 2 .

## 4. The Direction and Stability of Hopf Bifurcation

In this section, we assume the hypotheses of Theorem 7 hold and $\tau_{m}=\tau_{p}=\tau$. For the case of $\tau_{m} \neq \tau_{p}$, which is not our concern in this paper, the calculation of the normal form should follow the method developed in [17]. By using the normal form method in [12] for partial differential equations with time delay, we will investigate the stability of these Hopf bifurcations. For standard notations and classical results on partial functional differential equations, please refer to [12, $13,17]$. More details on techniques for computing the normal form can also be found in recent work [18].

Now, normalizing by the time-scaling $t \rightarrow t / \tau$, then (12)(13) can be rewritten as

$$
\begin{equation*}
\frac{d U(t)}{d t}=\tau d \Delta U(t)+L(\tau)\left(U_{t}\right)+f\left(U_{t}, \tau\right) \tag{42}
\end{equation*}
$$

where

$$
\begin{gather*}
L(\tau)(\varphi)=\tau\binom{a_{11} \varphi_{1}(0)+a_{12} \varphi_{2}(-1)}{a_{21} \varphi_{1}(-1)+a_{22} \varphi_{2}(0)}, \\
f(\varphi, \tau)=\tau\binom{\sum_{i+j \geq 2} \frac{1}{i!j!} f_{i j}^{(1)} \varphi_{1}^{i}(0) \varphi_{2}^{j}(-1)}{\sum_{i+j \geq 2} \frac{1}{i!j!} f_{i j}^{(2)} \varphi_{1}^{i}(-1) \varphi_{2}^{j}(0)} . \tag{43}
\end{gather*}
$$

In the following, we denote any one of these critical values by $\tau_{*}$ at which the characteristic equation (15) has a pair of simply purely imaginary roots $\pm i \omega_{*}$. Let $\tau=\tau_{*}+\alpha, \alpha \in$ $\mathbb{R}$, and consider only the case $\Lambda_{0}=\left\{-i \tau_{*} \omega_{*}, i \tau_{*} \omega_{*}\right\}$ is the set of eigenvalues on the imaginary axis of the infinitesimal generator associated with the flow of

$$
\begin{equation*}
\frac{d U(t)}{d t}=\tau^{*} d \Delta U(t)+L\left(\tau_{*}\right)\left(U_{t}\right) \tag{44}
\end{equation*}
$$

Equation (42) is now written as

$$
\begin{equation*}
\frac{d U(t)}{d t}=\tau d \Delta U(t)+L(\tau)\left(U_{t}\right)+F\left(U_{t}, \alpha\right) \tag{45}
\end{equation*}
$$

where $F(\varphi, \alpha)=\alpha d \Delta \varphi(0)+L(\alpha)(\varphi)+f\left(\varphi, \tau_{*}+\alpha\right)$, for $\varphi \in \mathscr{C}$. The eigenvalues of $\tau_{*} d \Delta$ on $X$ are $\mu_{k}^{i}=-d_{i} \tau_{*} k^{2}, i=1,2$,
$k \in \mathbb{N}_{0}$, with corresponding normalized eigenfunctions $\beta_{k}^{i}$, where

$$
\begin{gather*}
\beta_{k}^{1}(x)=\binom{\gamma_{k}(x)}{0}, \quad \beta_{k}^{2}(x)=\binom{0}{\gamma_{k}(x)} \\
\gamma_{k}(x)=\frac{\cos (k x)}{\|\cos (k x)\|_{2,2}}  \tag{46}\\
k \in \mathbb{N}_{0}
\end{gather*}
$$

Let $\mathscr{B}_{k}=\operatorname{span}\left\{\left\langle v(\cdot), \beta_{k}^{i}\right\rangle \beta_{k}^{i} \mid v \in \mathscr{C}, i=1,2\right\}$. Assume that $z_{t}(\theta) \in C=C\left([-1,0], \mathbb{R}^{2}\right)$ and

$$
\begin{equation*}
z_{t}^{T}(\theta)\binom{\beta_{k}^{1}}{\beta_{k}^{2}} \in \mathscr{B}_{k} . \tag{47}
\end{equation*}
$$

Then linear PFDE (44) restricted to $\mathscr{B}_{k}$ is equivalent to the FDE on $C\left([-1,0], \mathbb{R}^{2}\right)$

$$
\dot{z}(t)=\left(\begin{array}{cc}
\mu_{k}^{1} & 0  \tag{48}\\
0 & \mu_{k}^{2}
\end{array}\right) z(t)+L\left(\tau_{*}\right)\left(z_{t}\right)
$$

with the characteristic equation given by (15).
Suppose that there exists a $k \in \mathbb{N}_{0}$ such that when $\tau=\tau_{*}$, (15) for fixed $k$ has a pair of purely imaginary roots $\pm i \omega_{*}$ and all other roots of (15) have negative real parts. Define $\eta(\theta) \in$ $B V([-1,0] ; R)$ such that

$$
\begin{equation*}
\mu_{k} \psi(0)+L\left(\tau_{*}\right) \psi=\int_{-1}^{0} d \eta(\theta) \psi(\theta), \quad \psi \in C \tag{49}
\end{equation*}
$$

and the adjoint bilinear form on $C^{*} \times C, C^{*}=C\left([0,1], \mathbb{R}^{2 *}\right)$,

$$
(\psi(s), \phi(\theta))=\psi(0) \phi(0)-\int_{-1}^{0} \int_{0}^{\theta} \psi(\xi-\theta) d \eta(\theta) \phi(\xi) d \xi
$$

$$
\begin{equation*}
\text { for } \psi \in C^{*}, \phi \in C \tag{50}
\end{equation*}
$$

For (48) with fixed $k$, choose a basis $\Psi_{k}$ for the adjoint space $P^{*}$ and a basis $\Phi_{k}$ for its the eigenspace $P$ as follows:

$$
\begin{gather*}
\Phi_{k}=\left(p e^{i \omega_{*} \tau_{*} \theta}, \bar{p} e^{-i \omega_{*} \tau_{*} \theta}\right) \\
\Psi_{k}=\operatorname{col}\left(q^{T} e^{-i \omega_{*} \tau_{*} s}, \bar{q}^{T} e^{i \omega_{*} \tau_{*} s}\right) \tag{51}
\end{gather*}
$$

such that $\left(\Phi_{k}, \Phi_{k}\right)=I_{2}$, where $I_{2}$ is a $2 \times 2$ identity matrix. Then we can easily have

$$
\begin{gather*}
p=\binom{p_{1}}{p_{2}}=\left(\frac{i \omega_{*}+d_{1} k^{2}-a_{11}}{a_{12}} e^{i \omega_{*} \tau_{*}}\right)  \tag{52}\\
q=\binom{q_{1}}{q_{2}}=q_{1}\left(\frac{i \omega_{*}+d_{1} k^{2}-a_{11}}{a_{21}} e^{i \omega_{*} \tau_{*}}\right) \tag{53}
\end{gather*}
$$

with

$$
\begin{align*}
& q_{1}=\left(1+2 \tau_{*}\left(i \omega_{*}+d_{1} k^{2}-a_{11}\right)\right. \\
&\left.+\frac{\left(i \omega_{*}+d_{1} k^{2}-a_{11}\right)^{2} e^{i \omega_{*} \tau_{*}}}{a_{12} a_{21}}\right)^{-1} . \tag{54}
\end{align*}
$$

Following the standard procedure in [12], especially [18], using the decomposition $\varphi(t)=\left(\Phi_{k} z\right)^{T}\binom{\beta_{k}^{1}}{\beta_{k}^{2}}+y, z(t)=$ $\left(\Psi_{k},\binom{\left\langle\varphi(\cdot), \beta_{k}^{1}\right\rangle}{\left\langle\varphi(\cdot), \beta_{k}^{2}\right\rangle}\right) \in \mathbb{R}^{2}, y(t) \in \mathscr{C}_{0}^{1} \cap \operatorname{Ker} \pi=\mathscr{C}_{0}^{1} \cap \mathbb{Q}:=\mathbb{Q}^{1}$, we decompose (45) as

$$
\begin{gather*}
\dot{z}=B z+\Psi_{k}(0)\binom{\left\langle F\left(\left(\Phi_{k} z\right)^{T}\binom{\beta_{k}^{1}}{\beta_{k}^{2}}+y, \alpha\right), \beta_{k}^{1}\right\rangle}{\left\langle F\left(\left(\Phi_{k} z\right)^{T}\binom{\beta_{k}^{1}}{\beta_{k}^{2}}+y, \alpha\right), \beta_{k}^{2}\right\rangle}, \\
\frac{d}{d t} y=A_{Q^{1}} y+(I-\pi) X_{0} F\left(\left(\Phi_{k} z\right)^{T}\binom{\beta_{k}^{1}}{\beta_{k}^{2}}+y, \alpha\right) \tag{55}
\end{gather*}
$$

where here and throughout this section we refer to $[12,18]$ for results and explanations of several notations involved.

Consider the formal Taylor expansion

$$
\begin{equation*}
F(v, \alpha)=\sum_{j \geq 2} \frac{1}{j!} F_{j}(v, \alpha), \tag{56}
\end{equation*}
$$

where $F_{j}$ is the $j$ th Fréchet derivative of $F$. Then (55) can be written as

$$
\begin{gather*}
\dot{z}=B z+\sum_{j \geq 2} \frac{1}{j!} f_{j}^{1}(z, y, \alpha) \\
\frac{d}{d t} y=A_{Q^{1}} y+\sum_{j \geq 2} \frac{1}{j!} f_{j}^{2}(z, y, \alpha), \tag{57}
\end{gather*}
$$

where $f_{j}^{1}(z, y, \alpha)$ and $f_{j}^{2}(z, y, \alpha)$ are given by (4.8) in [18]. Then (45) has a normal form on the center manifold of the origin at $\alpha=0$, written as

$$
\begin{equation*}
\dot{z}=B z+\frac{1}{2} g_{2}^{1}(z, 0, \alpha)+\frac{1}{3!} g_{3}^{1}(z, 0,0)+O\left(\alpha^{2}|z|+\alpha|z|^{2}\right) \tag{58}
\end{equation*}
$$

where $B=\operatorname{diag}\left\{i \omega_{*} \tau_{*},-i \omega_{*} \tau_{*}\right\}$ and $g_{j}^{1}, j=2,3$, are given by (4.9) in [18]. The normal form procedure will show that these terms have the form

$$
\begin{equation*}
\frac{1}{2} g_{2}^{1}(z, 0, \alpha)=\binom{A_{k 1} z_{1} \alpha}{A_{k 1} z_{2} \alpha} \tag{59}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{k 1}=-k^{2}\left(d_{1} q_{1} p_{1}+d_{2} q_{2} p_{2}\right)+i \omega_{*} q^{T} p \tag{60}
\end{equation*}
$$

$\bar{A}_{k 1}$ is the conjugate of $A_{k 1}$. Consider the following:

$$
\begin{equation*}
\frac{1}{3!} g_{3}^{1}(z, 0,0)=\binom{A_{k 2} z_{1}^{2} z_{2}}{\bar{A}_{k 2} z_{1} z_{2}^{2}} \tag{61}
\end{equation*}
$$

where

$$
\begin{align*}
A_{k 2}= & \frac{i}{2 \omega_{*} \tau_{*}}\left(a_{k 20} a_{k 11}-2\left|a_{k 11}\right|^{2}-\frac{1}{3}\left|a_{k 02}\right|^{2}\right)  \tag{62}\\
& +\frac{1}{2}\left(a_{k 21}+\frac{1}{2} b_{k 21}\right)
\end{align*}
$$

with

$$
\begin{aligned}
& a_{k 20}=\tau_{*} \int_{0}^{\pi} \gamma_{k}^{3}(x) d x\left(b_{1} q_{1}\right) \\
& =\left\{\begin{array}{ll}
\frac{\tau_{*}}{\sqrt{\pi}}\left(b_{1} q_{1}\right), & k=0, \\
0, & k \neq 0,
\end{array} \quad b_{1}=f_{02}^{(1)} p_{2}^{2} e^{-2 i \omega_{*} \tau_{*}},\right. \\
& a_{k 11}=\tau_{*} \int_{0}^{\pi} \gamma_{k}^{3}(x) d x\left(b_{3} q_{1}\right) \\
& =\left\{\begin{array}{ll}
\frac{\tau_{*}}{\sqrt{\pi}}\left(b_{3} q_{1}\right), & k=0, \\
0, & k \neq 0,
\end{array} \quad b_{3}=f_{02}^{(1)}\left|p_{2}\right|^{2},\right. \\
& a_{k 02}=\tau_{*} \int_{0}^{\pi} \gamma_{k}^{3}(x) d x\left(\bar{b}_{1} q_{1}\right) \\
& = \begin{cases}\frac{\tau_{*}}{\sqrt{\pi}}\left(\bar{b}_{1} q_{1}\right), & k=0, \\
0, & k \neq 0,\end{cases} \\
& a_{k 21}=\tau_{*} \int_{0}^{\pi} \gamma_{k}^{4}(x) d x\left(q_{1} f_{03}^{(1)} p_{2}\left|p_{2}\right|^{2} e^{-i \omega_{*} \tau_{*}}\right) \\
& = \begin{cases}\frac{\tau_{*}}{\pi}\left(q_{1} f_{03}^{(1)} p_{2}\left|p_{2}\right|^{2} e^{-i \omega_{*} \tau_{*}}\right), & k=0, \\
\frac{3 \tau_{*}}{2 \pi}\left(q_{1} f_{03}^{(1)} p_{2}\left|p_{2}\right|^{2} e^{-i \omega_{*} \tau_{*}}\right), & k \neq 0,\end{cases} \\
& b_{k 21}= \begin{cases}M_{0}, & k=0, \\
M_{0}+\frac{\sqrt{2}}{2} M_{2 k}, & k \neq 0,\end{cases}
\end{aligned}
$$

where, for $j=0,2 k, M_{j}=\left(2 \tau_{*} / \sqrt{\pi}\right) q_{1}\left(f_{02}^{(1)} P_{2} e^{i \omega_{*} \tau_{*}} h_{j 11}^{(2)}(-1)+\right.$ $\left.f_{02}^{(1)} \bar{P}_{2} e^{-i \omega_{*} \tau_{*}} h_{j 11}^{(2)}(-1)\right)$, while $h_{k 20}(\theta)=\bar{h}_{k 02}(\theta)$ and $h_{k 20}(\theta)$, $h_{k 11}(\theta)$ are determined by the following equations:

$$
\begin{gathered}
\dot{h}_{k 20}(\theta)-2 i \tau_{*} \omega_{*} h_{k 20}(\theta)=\Phi_{k}\binom{a_{k 20}}{\bar{a}_{k 02}} \\
\dot{h}_{k 20}(0)-L\left(\tau_{*}\right)\left(h_{k 20}\right)=\tau_{*} c_{k j}\binom{b_{1}}{0} \\
\dot{h}_{k 11}(\theta)=\Phi_{k}\binom{2 a_{k 11}}{2 \bar{a}_{k 11}} \\
\dot{h}_{k 11}(0)-L\left(\tau_{*}\right)\left(h_{k 11}\right)=\tau_{*} c_{k j}\binom{b_{3}}{0}
\end{gathered}
$$

where

$$
c_{k j}=\int_{0}^{\pi} \gamma_{k}^{2}(x) \gamma_{j}(x) d x= \begin{cases}\frac{1}{\sqrt{\pi}}, & j=k=0  \tag{67}\\ \frac{1}{\sqrt{\pi}}, & j=0, k \neq 0 \\ \frac{1}{\sqrt{2 \pi}}, & j=2 k \neq 0 \\ 0 . & \text { otherwise }\end{cases}
$$

So the normal form (45) on the center manifold has the form

$$
\begin{equation*}
\dot{z}=B z+\binom{A_{k 1} z_{1} \mu}{\bar{A}_{k 1} z_{2} \mu}+\binom{A_{k 2} z_{1}^{2} z_{2}}{\bar{A}_{k 2} z_{1} z_{2}^{2}}+O\left(|z| \alpha^{2}+\left|z^{4}\right|\right) \tag{68}
\end{equation*}
$$

Next we will derive the normal form in the real coordinates. To this end, let $z_{1}=w_{1}-i w_{2}, z_{2}=w_{1}+i w_{2}$, and then the polar coordinates $w_{1}=\rho \cos \xi, w_{2}=\rho \sin \xi$. We finally reach

$$
\begin{gather*}
\dot{\rho}=\iota_{k 1} \alpha \rho+\iota_{k 2} \rho^{3}+O\left(\alpha^{2} \rho+|(\rho, \alpha)|^{4}\right),  \tag{69}\\
\dot{\xi}=-\omega_{*} \tau_{*}+O(|(\rho, \alpha)|)
\end{gather*}
$$

here $t_{k 1}=\operatorname{Re} A_{k 1}, \iota_{k 2}=\operatorname{Re} A_{k 2}$. Then, from [19], we know that the number $t_{k 2}$ tells the bifurcation direction and the stability of bifurcating periodic solution.
(i) When $t_{k 2}<0$, it is a supercritical bifurcation and the bifurcating periodic solution is stable.
(ii) When $t_{k 2}>0$, it is a subcritical bifurcation and the bifurcating periodic solution is unstable.

## 5. Numerical Simulation and Discussion

In this section, we present some numerical simulations to system (4). These simulations are used to support our theoretical results. Similar to $[6,7,20]$, the values of parameters are taken from the published experimental and theoretical results in our simulations where, $n \in[2,10], P_{0} \in[40,100]$, and $\mu \in[0.01,1] . \alpha=33$ and $a=4.5$ are taken as in [10] for original model (1).


FIGURE 1: Numerical simulations of system (4) with $d_{1}=0.002, d_{2}=0.02, b=c=0.03, n=2, p_{0}=0.4$, and $\tau=25<\tau_{0}^{0}$. The initial values are $m_{0}(x)=0.2+0.1 \cos x ; p_{0}(x)=5.6-0.01 \cos x$. The positive equilibrium $E^{*}(0.1684,5.6134)$ of system (4) is asymptotically stable for $\in\left[0, \tau_{0}^{0}\right)$.


Figure 2: Numerical simulations system (4) with $d_{1}=0.002, d_{2}=0.02, b=c=0.03, n=2, p_{0}=0.4$, and $\tau=30>\tau_{0}^{0}$. The initial values are $m_{0}(x)=0.2+0.1 \cos x ; p_{0}(x)=5.6-0.01 \cos x$. The positive equilibrium $E^{*}(0.1684,5.6134)$ of system (4) becomes unstable and there exist stable spatially homogeneous periodic solutions.

Taking $d_{1}=0.002, d_{2}=0.02, n=2, b=c=\mu=0.03$, and $p_{0}=0.4\left(P_{0}=59.4\right)$, then the positive equilibrium $E^{*}\left(m^{*}, p^{*}\right)=(0.1684,5.6134)$. From (30) and (32), we obtain the critical value for time delay, $\tau_{0}^{0} \doteq 26.3983$. In this case, the parameters satisfy $(\bar{H})$. By Theorem 7 , the positive equilibrium $E^{*}(0.1684,5.6134)$ is asymptotically stable for $\tau=25<\tau_{0}^{0}$. Figure 1 is the numerical simulation of system (4) for $\tau=25$.

When the delay increasingly crosses through the critical value $\tau_{0}^{0} \doteq 26.3983$, the positive equilibrium $E^{*}$ loses its stability and the Hopf bifurcation occurs. Taking $\tau=30>\tau_{0}^{0}$,

Figure 2 is the numerical simulation results of system (4). It is consistent with the theoretical results.

In Figure 2, we fix $x=1.5708$ and the other parameter values are the same as Figure 2. Then we get Figure 3 which shows that the oscillation will sustain when the time delay $\tau=30$ is much greater than the critical value $\tau_{0}^{0} \doteq$ 26.3983.

Remark 9. Comparing Figure 3 with Figure 3 in [8], we know that the oscillation is still sustained when $\tau$ is much greater than its critical value $\tau_{0}^{0}$ in the presence of diffusion.


Figure 3: Sustained oscillation when $\tau=30$ is much larger than its critical value $\tau_{0}^{0} \doteq 26.3983$ with $d_{1}=0.002, d_{2}=0.02, b=c=0.03$, $n=2, p_{0}=0.4$ and the initial values are $m_{0}(x)=0.2+0.1 \cos x$; and $p_{0}(x)=5.6-0.01 \cos x$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The research is supported by the National Natural Science Foundation of China (nos. 11101076 and 11371087), the Shanghai Committee of Science and Technology (nos. 11ZR1400200 and 12ZR1400100), the Fundamental Research Funds for the Central Universities (2232011D-35), and China Scholarship Council.

## References

[1] B. C. Goodwin, "Oscillatory behavior in enzymatic control processes," Advances in Enzyme Regulation, vol. 3, pp. 425-428, 1965.
[2] S. Busenberg and J. Mahaffy, "Interaction of spatial diffusion and delays in models of genetic control by repression," Journal of Mathematical Biology, vol. 22, no. 3, pp. 313-333, 1985.
[3] J. M. Mahaffy, "Genetic control models with diffusion and delays," Mathematical Biosciences, vol. 90, no. 1-2, pp. 519-533, 1988.
[4] J. M. Mahaffy and C. V. Pao, "Models of genetic control by repression with time delays and spatial effects," Journal of Mathematical Biology, vol. 20, no. 1, pp. 39-57, 1984.
[5] M. Sturrock, A. J. Terry, D. P. Xirodimas, A. M. Thompson, and M. A. J. Chaplain, "Spatio-temporal modelling of the Hes1 and NF- $\kappa$ B driven by transcriptional time delays," Current Biology, vol. 13, pp. 1409-1413, 2003.
[6] N. A. M. Monk, "Oscillatory expression of Hes1, p53, and NF$\kappa \mathrm{B}$ driven by transcriptional time delays," Current Biology, vol. 13, no. 16, pp. 1409-1413, 2003.
[7] A. Verdugo and R. Rand, "Hopf bifurcation in a DDE model of gene expression," Communications in Nonlinear Science and Numerical Simulation, vol. 13, no. 2, pp. 235-242, 2008.
[8] T. H. Zhang, Y. Song, and H. Zang, "The stability and Hopf bifurcation analysis of a gene expression model," Journal of Mathematical Analysis and Applications, vol. 395, no. 1, pp. 103113, 2012.
[9] J. D. Murray, Mathematical Biology-I. An Introduction, Springer, Berlin, Germany, 3rd edition, 2002.
[10] J. Lewis, "Autoinhibition with transcriptional delay: a simple mechanism for the zebrafish somitogenesis oscillator," Current Biology, vol. 13, no. 16, pp. 1398-1408, 2003.
[11] M. Sturrock, A. J. Terry, D. P. Xirodimas, A. M. Thompson, and M. A. J. Chaplain, "Spatio-temporal modelling of the Hesl and p53-Mdm2 intracellular signalling pathways," Journal of Theoretical Biology, vol. 273, pp. 15-31, 2011.
[12] T. Faria, "Normal forms and Hopf bifurcation for partial differential equations with delays," Transactions of the American Mathematical Society, vol. 352, no. 5, pp. 2217-2238, 2000.
[13] T. Faria, "Bifurcation aspects for some delayed population models with diffusion," in Differential Equations with Applications to Biology, vol. 21 of Fields Institute Communications, pp. 143-158, 1999.
[14] E. Beretta and Y. Kuang, "Geometric stability switch criteria in delay differential systems with delay dependent parameters," SIAM Journal on Mathematical Analysis, vol. 33, no. 5, pp. 11441165, 2002.
[15] Y. L. Song and J. J. Wei, "Local Hopf bifurcation and global periodic solutions in a delayed predator-prey system," Journal of Mathematical Analysis and Applications, vol. 301, no. 1, pp. 121, 2005.
[16] J. Wu, Theory and Applications of Partial Functional-Differential Equations, Springer, New York, NY, USA, 1996.
[17] T. Faria, "Stability and bifurcation for a delayed predator-prey model and the effect of diffusion," Journal of Mathematical Analysis and Applications, vol. 254, no. 2, pp. 433-463, 2001.
[18] Y. L. Song, Y. H. Peng, and X. F. Zou, "Persistence, stability and Hopf bifurcation in a diffusive ratio-dependent predatorprey model with delay," International Journal of Bifurcation and Chaos. In press.
[19] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer, New York, NY, USA, 2nd edition, 2003.
[20] M. H. Jensen, K. Sneppen, and G. Tiana, "Sustained oscillations and time delays in gene expression of protein Hesl", FEBS Letters, vol. 541, no. 1-3, pp. 176-177, 2003.

## Research Article

# Traveling Wave Solutions for a Delayed SIRS Infectious Disease Model with Nonlocal Diffusion and Nonlinear Incidence 

Xiaohong Tian and Rui Xu<br>Institute of Applied Mathematics, Shijiazhuang Mechanical Engineering College, No. 97 Heping West Road, Shijiazhuang, Hebei 050003, China<br>Correspondence should be addressed to Xiaohong Tian; tianxh-2008@163.com

Received 17 February 2014; Accepted 8 March 2014; Published 10 April 2014
Academic Editor: Weiming Wang
Copyright © 2014 X. Tian and R. Xu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A delayed SIRS infectious disease model with nonlocal diffusion and nonlinear incidence is investigated. By constructing a pair of upper-lower solutions and using Schauder's fixed point theorem, we derive the existence of a traveling wave solution connecting the disease-free steady state and the endemic steady state.

## 1. Introduction

Mathematical modeling has been proven to be valuable in studying the transmission dynamics of infectious diseases in a host population. We note that in disease progression, the spatial content of the environment plays a crucial role; the spread of germs, bacteria, and pathogen in the area is the main reason which leads to the spread of infectious disease. Thus, due to the large mobility of people within a country or even worldwide, spatially uniform models are not sufficient to give a realistic picture of a disease's diffusion. Considering the spatial effects, Gan et al. [1] considered the following SIRS epidemic model with spatial diffusion and time delay:

$$
\begin{gather*}
\frac{\partial S}{\partial t}= \\
D_{S} \frac{\partial^{2} S}{\partial x^{2}}+A-d S(x, t) \\
\\
-\beta S(x, t) I(x, t-\tau)+\delta R(x, t)  \tag{1}\\
\frac{\partial I}{\partial t}=D_{I} \frac{\partial^{2} I}{\partial x^{2}}+\beta S(x, t) I(x, t-\tau)-(d+\gamma+a) I(x, t), \\
\frac{\partial R}{\partial t}= \\
D_{R} \frac{\partial^{2} R}{\partial x^{2}}+\gamma I(x, t)-(d+\delta) R(x, t)
\end{gather*}
$$

where $S(t)$ represents the number of individuals who are susceptible to the disease, $I(t)$ represents the number of infected
individuals who are infectious and are able to spread the disease by contact with susceptible individuals, and $R(t)$ represents the number of individuals who have been removed from the possibility of infection through full immunity. The parameters $A, a, d, \beta, \gamma, \delta$ are positive constants in which $A$ is the recruitment rate of the population, $a$ is the death rate due to disease, $d$ is the natural death rate of the population, $\beta$ is the transmission rate, $\gamma$ is the recovery rate of the infective individuals, and $\delta$ is the rate at which recovered individuals lose immunity and return to the susceptible class. $\tau>0$ is a fixed time during which the infectious agents develop in the vector and it is only after that time that the infected vector can infect a susceptible human. $D_{S}, D_{I}$, and $D_{R}$ denote the corresponding diffusion rates for the susceptible, infected, and removed populations, respectively. In [1], by constructing a pair of upper-lower solutions, the existence of a traveling wave solution connecting the disease-free steady state and the endemic steady state was given. In recent years, there has been a fair amount of work on epidemiological models with spatial diffusion (see, e.g., [2-6]).

In system (1), the Laplacian operator $\partial^{2} / \partial x^{2}$ has been used to model the diffusion of the species, which suggests that the population at the location $x$ can only be influenced by the variation of the population near the location $x$. However, in dynamics of infectious diseases, dispersal is better described as a long range process rather than as a local one. At the same time, studies of disease infections have also shown that
reaction-diffusion equation does not accurately describe the spatial and temporal behavior of some diseases, for example, in the incubation period of SARS patients, who can move freely and the movement may transmit the disease to other people. Since the long range effect is taken into account, nonlocal diffusion equations have received great interest and have been recently intensively studied to analyze the long range effects of the dispersal (see, e.g., [7-12]). A basic nonlocal diffusion equation is of the form [13]

$$
\begin{equation*}
\frac{\partial}{\partial t} u(x, t)=J * u-u(x, t)+f(u) \tag{2}
\end{equation*}
$$

where the kernel $J$ of the convolution $(J * u)(x, t)=\int_{\mathbb{R}} J(x-$ $y) u(y, t) d y$ is a nonnegative function of mass one and $f$ a given nonlinearity. As stated in [9], if $u(x, t)$ represents the density of a species at the point $x$ and time $t$ and $J(x-y)$ is regarded as the probability distribution of jumping from location $y$ to location $x$, then $\int_{\mathbb{R}} J(x-y) u(y, t) d y$ is the rate at which individuals arrive at position $x$ from all other places and $-u(x, t)=-\int_{\mathbb{R}} J(x-y) u(y, t) d y$ is the rate at which they leave location $x$ to travel to all other sites. The diffusion is modeled by a convolution operator which looks to be biologically reasonable.

We note that in system (1), Gan et al. used a bilinear incidence rate $\beta S I$ based on the law of mass action. If the number of susceptible individuals is very large, it is unreasonable to consider the bilinear incidence within a certain limited time, because the number of effective contacts between infective individuals and susceptible individuals may saturate at high infective levels due to crowding of infective individuals or due to the protection measures by the susceptible individuals. After a study of the cholera epidemic spread in Bari in 1973, Capasso and Serio [14] introduced a saturated incidence rate $g(I) S$ into epidemic models, where $g(I)$ tends to a saturation level when $I$ gets large; that is, $g(I)=\beta I /(1+\alpha I)$; here $\beta I$ measures the force of infection of the disease, and $1 /(1+\alpha I)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the susceptible individuals. This incidence rate seems more reasonable than the bilinear incidence rate, because it includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters [15].

Motivated by the works of Capasso and Serio [14], Gan et al. [1], and Li et al. [13], in this paper, we study the following delayed SIRS infectious disease model with nonlocal diffusion:

$$
\begin{aligned}
\frac{\partial S}{\partial t}= & D[(J * S)(x, t)-S(x, t)]+A-d S(x, t) \\
& -\frac{\beta S(x, t) I(x, t-\tau)}{1+\alpha I(x, t-\tau)}+\delta R(x, t),
\end{aligned}
$$

$$
\begin{align*}
\frac{\partial I}{\partial t}= & D[(J * I)(x, t)-I(x, t)]+\frac{\beta S(x, t) I(x, t-\tau)}{1+\alpha I(x, t-\tau)} \\
& -(d+\gamma+a) I(x, t), \\
\frac{\partial R}{\partial t}= & D[(J * R)(x, t)-R(x, t)]+\gamma I(x, t) \\
& -(d+\delta) R(x, t), \tag{3}
\end{align*}
$$

where the parameter $D$ denotes the corresponding diffusion rates for the three populations, respectively. Here, for simplicity, we assume $D_{S}=D_{I}=D_{R}=D . J(z)$ is a kernel function which is continuous satisfying
(A1) $\int_{\mathbb{R}} J(x) d x=1, J(x) \geq 0$ and $J(x)=J(-x)$, for $x \in \mathbb{R}$. For any fixed $\mu>0, J_{\mu}:=\int_{-\infty}^{+\infty} J(x) e^{\mu|x|} d x<\infty$ and

$$
\begin{equation*}
\lim _{\mu \rightarrow \infty} \int_{-\infty}^{+\infty} J(x) e^{\mu|x|} d x=\infty \tag{4}
\end{equation*}
$$

The initial conditions for system (3) take the form

$$
\begin{array}{r}
S(x, t)=\rho_{1}(x, t), \quad I(x, t)=\rho_{2}(x, t), \quad R(x, t)=\rho_{3}(x, t), \\
t \in[-\tau, 0] . \tag{5}
\end{array}
$$

In the biological context, it is important to analyse the epidemic waves which are described by traveling wave solutions propagating with a certain speed. In this paper, our focus is on the existence of traveling wave solutions to the SIRS infectious disease model (3).

The rest of this paper is organized as follows. In Section 2, by constructing a pair of upper-lower solutions and using Schauder's fixed point theorem, the existence of traveling wave solutions connecting the disease-free steady state and the endemic steady state of system (3) is established. In Section 3, a brief concluding remark is given to end this work.

## 2. Existence of Traveling Waves

In this section, we apply Schauder's fixed point theorem, the method of cross-iteration scheme associated with upperlower solutions, to study the existence of traveling wave solutions of system (3) connecting the disease-free steady state and the endemic steady state.

Denote

$$
\begin{equation*}
\mathscr{R}_{0}=\frac{A \beta}{d(d+\gamma+a)} \tag{6}
\end{equation*}
$$

$\mathscr{R}_{0}$ is called the basic reproduction ratio of system (3), which describes the average number of newly infected cells generated from one infected cell at the beginning of the infectious process. This quantity determines the thresholds for disease transmissions. It is easy to show that system (3) always has a disease-free steady state $E^{0}(A / d, 0,0)$. Further,
if $\mathscr{R}_{0}>1$, system (3) has a unique endemic steady state $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$, where

$$
\begin{align*}
S^{*} & =\frac{(d+\gamma+a)\left(1+\alpha I^{*}\right)}{\beta}, \\
I^{*} & =\frac{d(d+\delta)(d+\gamma+a)\left(\mathscr{R}_{0}-1\right)}{(\alpha d+\beta)(d+\delta)(d+\gamma+a)-\beta \delta \gamma}  \tag{7}\\
R^{*} & =\frac{\gamma}{d+\delta} I^{*}
\end{align*}
$$

Denoting $N=S+I+R$, then system (3) is equivalent to the following system:

$$
\begin{align*}
\frac{\partial N}{\partial t}= & D[(J * N)(x, t)-N(x, t)]+A-d N(x, t) \\
& -a I(x, t), \\
\frac{\partial I}{\partial t}= & D[(J * I)(x, t)-I(x, t)] \\
& +\frac{\beta[N(x, t)-I(x, t)-R(x, t)] I(x, t-\tau)}{1+\alpha I(x, t-\tau)}  \tag{8}\\
& -(d+\gamma+a) I(x, t), \\
\frac{\partial R}{\partial t}= & D[(J * R)(x, t)-R(x, t)]+\gamma I(x, t) \\
& -(d+\delta) R(x, t) .
\end{align*}
$$

By making a change of variables $\widetilde{N}=A / d-N, \widetilde{I}=I, \widetilde{R}=$ $R$ and dropping the tildes, system (8) becomes

$$
\begin{align*}
\frac{\partial N}{\partial t}= & D[(J * N)(x, t)-N(x, t)]-d N(x, t)+a I(x, t) \\
\frac{\partial I}{\partial t}= & D[(J * I)(x, t)-I(x, t)] \\
& +\frac{\beta[A / d-N(x, t)-I(x, t)-R(x, t)] I(x, t-\tau)}{1+\alpha I(x, t-\tau)} \\
& -(d+\gamma+a) I(x, t), \\
& \frac{\partial R}{\partial t}= \\
& D[(J * R)(x, t)-R(x, t)]+\gamma I(x, t)  \tag{9}\\
& -(d+\delta) R(x, t) .
\end{align*}
$$

It is easy to show that if $\mathscr{R}_{0}>1$, system (9) has two steady states $(0,0,0)$ and $\left(k_{1}, k_{2}, k_{3}\right)$, where $k_{1}=A / d-S^{*}-I^{*}-R^{*}$ and $k_{2}=I^{*}, k_{3}=R^{*}$.

A traveling wave solution of (9) is a special translation invariant solution of the form $(N(x, t), I(x, t), R(x, t))=$ $(\phi(x+c t), \varphi(x+c t), \psi(x+c t))$, where $(\phi, \varphi, \psi) \in C\left(\mathbb{R}, \mathbb{R}^{3}\right)$ is the profile of the wave that propagates through onedimensional spatial domain at a constant speed $c>0$. On substituting $N(x, t)=\phi(x+c t), I(x, t)=\varphi(x+c t)$,
$R(x, t)=\psi(x+c t)$ into (9) and denoting the traveling wave coordinate $x+c t$ still by $t$, we derive from (9) that

$$
\begin{align*}
& c \phi^{\prime}(t)=D \int_{R} J(t-y) \phi(y) d y-D \phi(t)+f_{c 1}\left(\phi_{t}, \varphi_{t}, \psi_{t}\right), \\
& c \varphi^{\prime}(t)=D \int_{R} J(t-y) \varphi(y) d y-D \phi(t)+f_{c 2}\left(\phi_{t}, \varphi_{t}, \psi_{t}\right), \\
& c \psi^{\prime}(t)=D \int_{R} J(t-y) \psi(y) d y-D \psi(t)+f_{c 3}\left(\phi_{t}, \varphi_{t}, \psi_{t}\right), \tag{10}
\end{align*}
$$

where

$$
\begin{gather*}
f_{c 1}\left(\phi_{t}, \varphi_{t}, \psi_{t}\right)=-d \phi(t)+a \varphi(t), \\
f_{c 2}\left(\phi_{t}, \varphi_{t}, \psi_{t}\right) \\
=\frac{\beta[A / d-\phi(t)-\varphi(t)-\psi(t)] \varphi(t-c \tau)}{1+\alpha \varphi(t-c \tau)}  \tag{11}\\
-(d+\gamma+a) \varphi(t), \\
f_{c 3}\left(\phi_{t}, \varphi_{t}, \psi_{t}\right)=\gamma \varphi(t)-(d+\delta) \psi(t) .
\end{gather*}
$$

Equation (10) will be solved subject to the following boundary value conditions:

$$
\begin{align*}
& \lim _{t \rightarrow-\infty}(\phi(t), \varphi(t), \psi(t))=(0,0,0), \\
& \lim _{t \rightarrow+\infty}(\phi(t), \varphi(t), \psi(t))=\left(k_{1}, k_{2}, k_{3}\right) . \tag{12}
\end{align*}
$$

Now, we give the definition of upper and lower solutions of system (10) as follows.

Definition 1. A pair of continuous functions $\bar{\Phi}=(\bar{\phi}, \bar{\varphi}, \bar{\psi})$ and $\Phi=(\phi, \varphi, \psi)$ are called a pair of upper-lower solutions of system (10), if there exist constants $T_{i}(i=1,2, \ldots, m)$ such that $\bar{\Phi}$ and $\underline{\Phi}$ are twice differential on $\mathbb{R} \backslash\left\{T_{i}: i=1,2, \ldots, m\right\}$ and satisfy

$$
\begin{aligned}
& D \int_{R} J(t-y) \bar{\phi}(y) d y-D \bar{\phi}(t)-c \bar{\phi}^{\prime}(t) \\
& \quad+f_{c 1}\left(\bar{\phi}_{t}, \bar{\varphi}_{t}, \bar{\psi}_{t}\right) \leq 0, \\
& D \int_{R} J(t-y) \bar{\varphi}(y) d y-D \bar{\varphi}(t)-c \bar{\varphi}^{\prime}(t) \\
& \quad+f_{c 2}\left(\underline{\phi}_{t}, \bar{\varphi}_{t}, \underline{\psi}_{t}\right) \leq 0, \\
& D \int_{R} J(t-y) \bar{\psi}(y) d y-D \bar{\psi}(t)-c \bar{\psi}^{\prime}(t) \\
& \quad+f_{c 3}\left(\bar{\phi}_{t}, \bar{\varphi}_{t}, \bar{\psi}_{t}\right) \leq 0,
\end{aligned}
$$

$$
\begin{align*}
& D \int_{R} J(t-y) \underline{\phi}(y) d y-D \underline{\phi}(t)-c \underline{\phi}^{\prime}(t) \\
& \quad+f_{c 1}\left(\underline{\phi}_{t}, \underline{\varphi}_{t}, \underline{\psi}_{t}\right) \geq 0, \\
& D \int_{R} J(t-y) \underline{\varphi}(y) d y-D \underline{\varphi}(t)-c \underline{\varphi}^{\prime}(t) \\
& \quad+f_{c 2}\left(\bar{\phi}_{t}, \underline{\varphi}_{t}, \bar{\psi}_{t}\right) \geq 0, \\
& D \int_{R} J(t-y) \underline{\psi}(y) d y-D \underline{\psi}(t)-c \underline{\psi}^{\prime}(t) \\
& \quad+f_{c 3}\left(\underline{\phi}_{t}, \underline{\varphi}_{t}, \underline{\psi}_{t}\right) \geq 0, \tag{13}
\end{align*}
$$

for $t \in \mathbb{R} \backslash\left\{T_{i}: i=1,2, \ldots, m\right\}$.
In what follows, we assume that there exist an upper solution $\bar{\Phi}(t)=(\bar{\phi}, \bar{\varphi}, \bar{\psi})(t)$ and a lower solution $\underline{\Phi}(t)=$ $(\phi, \underline{\varphi}, \underline{\psi})(t)$ of system (10) satisfying (P1)-(P2):
(P1) $\mathbf{0} \leq \underline{\Phi} \leq \bar{\Phi} \leq M=\left(M_{1}, M_{2}, M_{3}\right)$;
(P2) $\lim _{t \rightarrow-\infty} \bar{\Phi}(t)=\mathbf{0}, \lim _{t \rightarrow+\infty} \underline{\Phi}(t)=\lim _{t \rightarrow+\infty} \bar{\Phi}(t)=$ $\mathbf{K}=\left(k_{1}, k_{2}, k_{3}\right)$.
Let

$$
\begin{align*}
C_{[0, M]}\left(\mathbb{R}, \mathbb{R}^{3}\right)=\{ & \left\{\Phi(t)=(\phi, \varphi, \psi)(t) \in C\left(\mathbb{R}, \mathbb{R}^{3}\right)\right. \\
& : \mathbf{0} \leq \Phi(t) \leq M, t \in \mathbb{R}\} \tag{14}
\end{align*}
$$

where $M_{i}>k_{i}(i=1,2,3)$ satisfy

$$
\begin{gather*}
\frac{A \beta}{d}-(\gamma+a)>a \frac{M_{2}}{M_{1}}>d \\
\frac{A \beta}{d}-(\gamma+a)>\gamma \frac{M_{2}}{M_{3}}-\delta>d \\
\frac{A}{d} \geq M_{1}+M_{2}+M_{3} \tag{15}
\end{gather*}
$$

We look for traveling wave solutions to system (10) in the following profile set:

$$
\begin{align*}
\Gamma=\{ & (\phi, \varphi, \psi)(t) \in C_{[0, M]}\left(\mathbb{R}, \mathbb{R}^{3}\right) \\
& :(\underline{\phi}, \underline{\varphi}, \underline{\psi})(t) \leq(\phi, \varphi, \psi)(t) \leq(\bar{\phi}, \bar{\varphi}, \bar{\psi})(t)\} . \tag{16}
\end{align*}
$$

Obviously, $\Gamma$ is nonempty, convex, closed, and bounded.
Furthermore, corresponding to (10), we make the following hypotheses.
(A2) There exist three positive constants $L_{i}>0(i=1,2,3)$ such that

$$
\begin{equation*}
\left|f_{c i}\left(\phi_{1}, \varphi_{1}, \psi_{1}\right)-f_{c i}\left(\phi_{2}, \varphi_{2}, \psi_{2}\right)\right| \leq L_{i}\|\Phi(t)-\Psi(t)\|, \tag{17}
\end{equation*}
$$

for $\Phi(t)=\left(\phi_{1}, \varphi_{1}, \psi_{1}\right)(t)$ and $\Psi(t)=\left(\phi_{2}, \varphi_{2}, \psi_{2}\right)(t) \in$ $C\left([-\tau, 0], \mathbb{R}^{3}\right)$ with $(0,0,0) \leq\left(\phi_{j}(t), \varphi_{j}(t), \psi_{j}(t)\right) \leq$ $\left(M_{1}, M_{2}, M_{3}\right), j=1,2, t \in[-\tau, 0], M_{i} \geq k_{i}$, are positive constants.

For $\Phi=(\phi, \varphi, \psi) \in C_{[0, M]}\left(\mathbb{R}, \mathbb{R}^{3}\right)$, we define two operators $H=\left(H_{1}, H_{2}, H_{3}\right)$ and $F=\left(F_{1}, F_{2}, F_{3}\right)$ from $C_{[0, M]}\left(\mathbb{R}, \mathbb{R}^{3}\right)$ to $C\left(\mathbb{R}, \mathbb{R}^{3}\right)$ by

$$
\begin{align*}
& H_{1}(\phi, \varphi, \psi)(t)=D \int_{R} J(t-y) \phi(y) d y+a \varphi(t), \\
& H_{2}(\phi, \varphi, \psi)(t) \\
& =D \int_{R} J(t-y) \varphi(y) d y+\beta M_{2} \varphi(t) \\
& \quad+\frac{\beta[A / d-\phi(t)-\varphi(t)-\psi(t)] \varphi(t-c \tau)}{1+\alpha \varphi(t-c \tau)} \\
& H_{3}(\phi, \varphi, \psi)(t)=D \int_{R} J(t-y) \psi(y) d y+\gamma \varphi(t) \\
& F_{i}(\phi, \varphi, \psi)(t)=\frac{1}{c} e^{-\left(\beta_{i} / c\right) t} \int_{-\infty}^{t} e^{\left(\beta_{i} / c\right) s} H_{i}(\phi, \varphi, \psi)(s) d s \\
& \quad(i=1,2,3) . \tag{18}
\end{align*}
$$

Letting $\beta_{1}=D+d, \beta_{2}=D+d+\gamma+a$, and $\beta_{3}=D+d+\delta$, then system (10) can be rewritten as

$$
\begin{align*}
& c \phi^{\prime}(t)=-\beta_{1} \phi(t)+H_{1}(\phi, \varphi, \psi)(t), \\
& c \varphi^{\prime}(t)=-\beta_{2} \varphi(t)+H_{2}(\phi, \varphi, \psi)(t),  \tag{19}\\
& c \psi^{\prime}(t)=-\beta_{3} \psi(t)+H_{3}(\phi, \varphi, \psi)(t),
\end{align*}
$$

and then $F$ is well defined such that

$$
\begin{array}{r}
c F_{i}^{\prime}(\phi, \varphi, \psi)(t)=-\beta_{i} F_{i}(\phi, \varphi, \psi)(t)+H_{i}(\phi, \varphi, \psi)(t) \\
(i=1,2,3) . \tag{20}
\end{array}
$$

Hence, a fixed point of $F$ is a solution of (10), which is a traveling wave solution of (9) connecting $\mathbf{0}=(0,0,0)$ with $\mathbf{K}=\left(k_{1}, k_{2}, k_{3}\right)$ if it satisfies (P2).

In the following, we introduce some lemmas to support our main results.

For $\mu>0$, define

$$
\begin{align*}
& B_{\mu}\left(\mathbb{R}, \mathbb{R}^{3}\right)=\{ (\phi, \varphi, \psi) \in C\left(\mathbb{R}, \mathbb{R}^{3}\right) \\
&\left.: \sup _{t \in \mathbb{R}}|(\phi, \varphi, \psi)(t)| e^{-\mu|t|}<\infty\right\}, \\
&|(\phi, \varphi, \psi)(t)|_{\mu}=\sup _{t \in \mathbb{R}}|(\phi, \varphi, \psi)(t)| e^{-\mu|t|} \tag{21}
\end{align*}
$$

Then it is easy to check that $\left(B_{\mu}\left(\mathbb{R}, \mathbb{R}^{3}\right),|\cdot|\right)$ is a Banach space.
In view of the definition of $H$ and $F$, we can easily see that they admit the following properties.

Lemma 2. Let $\left(A_{1}\right)$ hold. One has
(i)

$$
\begin{gather*}
H_{1}(\phi, \varphi, \psi)(t) \geq 0, \quad H_{3}(\phi, \varphi, \psi)(t) \geq 0, \\
H_{1}\left(\phi_{2}, \varphi_{2}, \psi_{2}\right)(t) \leq H_{1}\left(\phi_{1}, \varphi_{1}, \psi_{1}\right)(t), \\
H_{2}\left(\phi_{1}, \varphi_{1}, \psi_{1}\right)(t) \leq H_{2}\left(\phi_{2}, \varphi_{1}, \psi_{1}\right)(t),  \tag{22}\\
H_{2}\left(\phi_{1}, \varphi_{2}, \psi_{1}\right)(t) \leq H_{2}\left(\phi_{1}, \varphi_{1}, \psi_{1}\right)(t), \\
H_{2}\left(\phi_{1}, \varphi_{1}, \psi_{1}\right)(t) \leq H_{2}\left(\phi_{1}, \varphi_{1}, \psi_{2}\right)(t), \\
H_{3}\left(\phi_{2}, \varphi_{2}, \psi_{2}\right)(t) \leq H_{3}\left(\phi_{1}, \varphi_{1}, \psi_{1}\right)(t),
\end{gather*}
$$

(ii)

$$
\begin{align*}
& F_{1}\left(\phi_{2}, \varphi_{2}, \psi_{2}\right)(t) \leq F_{1}\left(\phi_{1}, \varphi_{1}, \psi_{1}\right)(t), \\
& F_{2}\left(\phi_{1}, \varphi_{1}, \psi_{1}\right)(t) \leq F_{2}\left(\phi_{2}, \varphi_{1}, \psi_{1}\right)(t), \\
& F_{2}\left(\phi_{1}, \varphi_{2}, \psi_{1}\right)(t) \leq F_{2}\left(\phi_{1}, \varphi_{1}, \psi_{1}\right)(t),  \tag{23}\\
& F_{2}\left(\phi_{1}, \varphi_{1}, \psi_{1}\right)(t) \leq F_{2}\left(\phi_{1}, \varphi_{1}, \psi_{2}\right)(t), \\
& F_{3}\left(\phi_{2}, \varphi_{2}, \psi_{2}\right)(t) \leq F_{3}\left(\phi_{1}, \varphi_{1}, \psi_{1}\right)(t),
\end{align*}
$$

for $t \in \mathbb{R}$ with $0 \leq \phi_{2}(t) \leq \phi_{1}(t) \leq M_{1}, 0 \leq \varphi_{2}(t) \leq \varphi_{1}(t) \leq$ $M_{2}, 0 \leq \psi_{2}(t) \leq \psi_{1}(t) \leq M_{3}$.

By using a similar argument as in the proof of Lemmas 3.3-3.6 in [16], one can show the following lemmas.

Lemma 3. Assume that (A2) holds. $F=\left(F_{1}, F_{2}, F_{3}\right)$ is continuous with respect to the norm $|\cdot|_{\mu}$ in $B_{\mu}\left(\mathbb{R}, \mathbb{R}^{3}\right)$.

Lemma 4. $F(\Gamma((\underline{\phi}, \underline{\varphi}, \underline{\psi}),(\bar{\phi}, \bar{\varphi}, \bar{\psi}))) \subset \Gamma((\underline{\phi}, \underline{\varphi}, \underline{\psi}),(\bar{\phi}, \bar{\varphi}, \bar{\psi}))$, where $F=\left(F_{1}, F_{2}, \overline{F_{3}}\right)$.

Lemma 5. $F: \Gamma((\underline{\phi}, \underline{\varphi}, \underline{\psi}),(\bar{\phi}, \bar{\varphi}, \bar{\psi})) \rightarrow \Gamma((\underline{\phi}, \underline{\varphi}, \underline{\psi}),(\bar{\phi}, \bar{\varphi}, \bar{\psi}))$ is compact.

We now consider the following equations:

$$
\begin{align*}
& \Delta_{1}(\lambda, c):=D \int_{\mathbb{R}} J(y) e^{-\lambda y} d y-D-c \lambda-d+a \frac{M_{2}}{M_{1}}, \\
& \Delta_{2}(\lambda, c):=D \int_{\mathbb{R}} J(y) e^{-\lambda y} d y-D-c \lambda+\frac{A \beta}{d}-(d+\gamma+a), \\
& \Delta_{3}(\lambda, c):=D \int_{\mathbb{R}} J(y) e^{-\lambda y} d y-D-c \lambda-(d+\delta)+\gamma \frac{M_{2}}{M_{3}} . \tag{24}
\end{align*}
$$

Since (A1) and (15) hold, direct calculations show that

$$
\begin{gather*}
\Delta_{1}(0, c)=-d+a \frac{M_{2}}{M_{1}}>0, \quad \Delta_{1}(\lambda,+\infty)=-\infty, \\
\forall \lambda>0 ; \\
\frac{\partial \Delta_{1}(\lambda, c)}{\partial c}=-\lambda<0, \quad \forall \lambda>0 ;  \tag{25}\\
\frac{\partial^{2} \Delta_{1}(\lambda, c)}{\partial \lambda^{2}}=D \int_{\mathbb{R}} y^{2} J(y) e^{-\lambda y} d y>0
\end{gather*}
$$

Therefore, we obtain that there exist $c_{1}^{*}>0$ and $\lambda_{1 *}>0$ such that $\partial \Delta_{1}(\lambda, c) /\left.\partial \lambda\right|_{\left(\lambda_{1 *}, c_{1}^{*}\right)}=0$ and $\Delta_{1}\left(\lambda_{1 *}, c_{1}^{*}\right)=0$. Further, if $c>c_{1}^{*}$, there exist $\lambda_{1}(c)>0$ and $\lambda_{2}(c)>0$ satisfying

$$
\begin{equation*}
0<\lambda_{1}(c)<\lambda_{1 *}<\lambda_{2}(c) \tag{26}
\end{equation*}
$$

Similarly, we can show that there exist $c_{i}^{*}, \lambda_{i *}>0$ such that $\Delta_{i}\left(\lambda_{i *}, c_{i}^{*}\right)=0$. If $c>c_{i}^{*}$, there exist $\lambda_{j}(c)>0$ satisfying $0<\lambda_{3}(c)<\lambda_{2 *}<\lambda_{4}(c)$ and $0<\lambda_{5}(c)<\lambda_{3 *}<\lambda_{6}(c)(i=$ $2,3 ; j=3,4,5,6)$.

Lemma 6. Let $c^{*}=\max \left\{c_{1}^{*}, c_{2}^{*}, c_{3}^{*}\right\}$. Assume that $\mathscr{R}_{0}>1$; then one has $\lambda_{1}(c)<\lambda_{3}(c)$ and $\lambda_{5}(c)<\lambda_{3}(c)$.

Proof. Define

$$
\begin{align*}
& h(\lambda)=D \int_{\mathbb{R}} J(y) e^{-\lambda y} d y, \\
& g_{1}(\lambda)=c \lambda+D+d-a \frac{M_{2}}{M_{1}},  \tag{27}\\
& g_{2}(\lambda)=c \lambda+D+d+\gamma+a-\frac{A \beta}{d}, \\
& g_{3}(\lambda)=c \lambda+D+d+\delta-\gamma \frac{M_{2}}{M_{3}} .
\end{align*}
$$

It is easy to show that

$$
\begin{gather*}
h\left(\lambda_{1}\right)=g_{1}\left(\lambda_{1}\right), \quad h\left(\lambda_{3}\right)=g_{2}\left(\lambda_{3}\right) \\
h\left(\lambda_{5}\right)=g_{3}\left(\lambda_{5}\right) \tag{28}
\end{gather*}
$$

If $\mathscr{R}_{0}>1$, then by (9), we see that $g_{2}(0)<g_{1}(0)$ and $g_{2}(0)<$ $g_{3}(0)$. Note that $h^{\prime \prime}(\lambda)>0$ for all $\lambda$; hence, we have $\lambda_{1}(c)<$ $\lambda_{3}(c)$ and $\lambda_{5}(c)<\lambda_{3}(c)$.

Suppose that $\mathscr{R}_{0}>1$ and $a / d+\gamma /(d+\delta)<1$; we can choose $\varepsilon_{i}>0(i=1,2, \ldots, 6)$, and $\varepsilon_{1}, \varepsilon_{2} \in\left(0, k_{1}\right), \varepsilon_{3}, \varepsilon_{4} \in$ $\left(0, k_{2}\right), \varepsilon_{5}, \varepsilon_{6} \in\left(0, k_{3}\right)$ satisfying

$$
\begin{gather*}
a\left(k_{2}+\varepsilon_{3}\right)-d\left(k_{1}+\varepsilon_{1}\right)<0, \\
\beta\left(\frac{A}{d}-k_{1}+\varepsilon_{2}-k_{2}-\varepsilon_{3}-k_{3}+\varepsilon_{6}\right) \\
<(d+\gamma+a)\left(1+\alpha\left(k_{2}+\varepsilon_{3}\right)\right), \\
\gamma\left(k_{2}+\varepsilon_{3}\right)-(d+\delta)\left(k_{3}+\varepsilon_{5}\right)<0,  \tag{29}\\
-d\left(k_{1}-\varepsilon_{2}\right)+a\left(k_{2}-\varepsilon_{4}\right)>0, \\
\beta\left(\frac{A}{d}-k_{1}-\varepsilon_{1}-k_{2}+\varepsilon_{4}-k_{3}-\varepsilon_{5}\right) \\
>(d+\gamma+a)\left(1+\alpha\left(k_{2}-\varepsilon_{4}\right)\right), \\
\gamma\left(k_{2}-\varepsilon_{4}\right)-(d+\delta)\left(k_{3}-\varepsilon_{6}\right)>0 .
\end{gather*}
$$

In fact, noting that $d k_{1}=a k_{2}$, for $\varepsilon_{3}, \varepsilon_{4} \in\left(0, k_{2}\right)$, there exist $\varepsilon_{1}, \varepsilon_{2} \in\left(0, k_{1}\right)$ and $\varepsilon_{5}, \varepsilon_{6} \in\left(0, k_{3}\right)$ such that

$$
\begin{align*}
& k_{1}>\varepsilon_{1}>\frac{a}{d} \varepsilon_{3}=\frac{a}{d} \varepsilon_{3}+\frac{a}{d} k_{2}-k_{1}  \tag{30}\\
& k_{1}>\varepsilon_{2}>\frac{a}{d} \varepsilon_{4}=\frac{a}{d} \varepsilon_{4}+k_{1}-\frac{a}{d} k_{2}
\end{align*}
$$

which yield

$$
\begin{align*}
& a\left(k_{2}+\varepsilon_{3}\right)-d\left(k_{1}+\varepsilon_{1}\right)<0 \\
& -d\left(k_{1}-\varepsilon_{2}\right)+a\left(k_{2}-\varepsilon_{4}\right)>0 \tag{31}
\end{align*}
$$

Since $\gamma k_{2}=(d+\delta) k_{3}$, for $\varepsilon_{3}, \varepsilon_{4} \in\left(0, k_{2}\right)$, we can find $\varepsilon_{1}, \varepsilon_{2} \in\left(0, k_{1}\right)$ and $\varepsilon_{5}, \varepsilon_{6} \in\left(0, k_{3}\right)$ such that

$$
\begin{align*}
k_{3} & >\varepsilon_{5}>\frac{\gamma}{d+\delta} \varepsilon_{3}=\frac{\gamma}{d+\delta} \varepsilon_{3}+\frac{\gamma}{d+\delta} k_{2}-k_{3} \\
& \Longrightarrow \gamma\left(k_{2}+\varepsilon_{3}\right)-(d+\delta)\left(k_{3}+\varepsilon_{5}\right)<0,  \tag{32}\\
k_{3} & >\varepsilon_{6}>\frac{\gamma}{d+\delta} \varepsilon_{4}=\frac{\gamma}{d+\delta} \varepsilon_{4}+k_{3}-\frac{\gamma}{d+\delta} k_{2} \\
& \Longrightarrow \gamma\left(k_{2}-\varepsilon_{4}\right)-(d+\delta)\left(k_{3}-\varepsilon_{6}\right)>0 .
\end{align*}
$$

If $a / d+\gamma /(d+\delta)<1$, then we can choose suitable values of $\varepsilon_{3}, \varepsilon_{4} \in\left(0, k_{2}\right)$ such that

$$
\begin{equation*}
\left(\frac{a}{d}+\frac{\gamma}{d+\delta}\right) \varepsilon_{3}<\varepsilon_{4}, \quad\left(\frac{a}{d}+\frac{\gamma}{d+\delta}\right) \varepsilon_{4}<\varepsilon_{3} . \tag{33}
\end{equation*}
$$

Furthermore, we can choose $\varepsilon_{1}, \varepsilon_{2} \in\left(0, k_{1}\right), \varepsilon_{3}, \varepsilon_{4} \in$ $\left(0, k_{2}\right), \varepsilon_{5}, \varepsilon_{6} \in\left(0, k_{3}\right)$ satisfying

$$
\begin{array}{lll}
\varepsilon_{2}>\frac{a}{d} \varepsilon_{4}, & \varepsilon_{6}>\frac{\gamma}{d+\delta} \varepsilon_{4}, & \frac{a}{d} \varepsilon_{4}+\frac{\gamma}{d+\delta} \varepsilon_{4}<\varepsilon_{3}, \\
\varepsilon_{1}>\frac{a}{d} \varepsilon_{3}, & \varepsilon_{5}>\frac{\gamma}{d+\delta} \varepsilon_{3}, & \frac{a}{d} \varepsilon_{3}+\frac{\gamma}{d+\delta} \varepsilon_{3}<\varepsilon_{4} . \tag{34}
\end{array}
$$

Accordingly, there exist suitable constants $\varepsilon_{i}>0(i=$ $1, \ldots, 6$ ) such that

$$
\begin{equation*}
\varepsilon_{2}+\varepsilon_{6}-\varepsilon_{3}<0, \quad \varepsilon_{1}+\varepsilon_{5}-\varepsilon_{4}<0 \tag{35}
\end{equation*}
$$

By the second equation of system (10), we have $\beta\left(A / d-k_{1}-\right.$ $\left.k_{2}-k_{3}\right)-(d+\gamma+a)\left(1+\alpha k_{2}\right)=0$. It then follows from (35) that

$$
\begin{align*}
& \beta\left(\frac{A}{d}-k_{1}+\varepsilon_{2}-k_{2}-\varepsilon_{3}-k_{3}+\varepsilon_{6}\right) \\
& <(d+\gamma+a)\left(1+\alpha\left(k_{2}+\varepsilon_{3}\right)\right), \\
& \beta\left(\frac{A}{d}-k_{1}-\varepsilon_{1}-k_{2}+\varepsilon_{4}-k_{3}-\varepsilon_{5}\right)  \tag{36}\\
& \quad>(d+\gamma+a)\left(1+\alpha\left(k_{2}-\varepsilon_{4}\right)\right) .
\end{align*}
$$

Now, we define the continuous functions $\bar{\Phi}(t)=$ $(\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t))$ and $\underline{\Phi}(t)=(\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t))$ as follows:

$$
\begin{align*}
& \bar{\phi}(t)= \begin{cases}k_{1} e^{\lambda_{1} t}, & t \leq t_{1}, \\
k_{1}+\varepsilon_{1} e^{-\lambda t}, & t>t_{1},\end{cases} \\
& \underline{\phi}(t)= \begin{cases}0, & t \leq t_{2}, \\
k_{1}-\varepsilon_{2} e^{-\lambda t}, & t>t_{2},\end{cases} \\
& \bar{\varphi}(t)= \begin{cases}l k_{2} e^{\lambda_{3} t}, & t \leq t_{3}, \\
k_{2}+\varepsilon_{3} e^{-\lambda t}, & t>t_{3},\end{cases} \\
& \underline{\varphi}(t)= \begin{cases}0, & t \leq t_{4}, \\
k_{2}-\varepsilon_{4} e^{-\lambda t}, & t>t_{4},\end{cases}  \tag{37}\\
& \bar{\psi}(t)= \begin{cases}k_{3} e^{\lambda_{5} t}, & t \leq t_{5}, \\
k_{3}+\varepsilon_{5} e^{-\lambda t}, & t>t_{5},\end{cases} \\
& \underline{\psi}(t)= \begin{cases}0, & t \leq t_{6}, \\
k_{3}-\varepsilon_{6} e^{-\lambda t}, & t>t_{6},\end{cases}
\end{align*}
$$

where $t_{1}, t_{3}, t_{5}>0, t_{2}, t_{4}, t_{6}<0$ and $\lambda>0$ is a constant sufficiently small to be chosen later. Then we can choose $\lambda>$ 0 to be sufficiently small such that $t_{1}>0, t_{3}>0, t_{5}>0$ satisfying

$$
\begin{align*}
& k_{1}+\varepsilon_{1}>M_{1}=\sup _{t \in \mathbb{R}} \bar{\phi}(t)=k_{1} e^{\lambda_{1} t_{1}}>k_{1}, \\
& k_{2}+\varepsilon_{3}>M_{2}=\sup _{t \in \mathbb{R}} \bar{\varphi}(t)=k_{2} e^{\lambda_{3} t_{3}}>k_{2},  \tag{38}\\
& k_{3}+\varepsilon_{5}>M_{3}=\sup _{t \in \mathbb{R}} \bar{\psi}(t)=k_{3} e^{\lambda_{5} t_{5}}>k_{3},
\end{align*}
$$

where $M_{1}, M_{2}, M_{3}$ are defined in (15). Furthermore, we can choose $l \in(0,1)$ such that $t_{3} \geq \max \left\{t_{1}, t_{5}\right\}$. If $a / d+\gamma /(\delta+d)<$ 1 , it is easy to show that $t_{4} \leq \min \left\{t_{2}, t_{6}\right\}$. Clearly, $\bar{\Phi}(t)$ and $\Phi(t)$ satisfy (P1) and (P2).

Lemma 7. $\bar{\Phi}(t)=(\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t))$ is an upper solution of system (10).

## Proof. Denote

$$
\begin{align*}
P_{1}(t)= & D \int_{\mathbb{R}} J(t-y) \bar{\phi}(y) d y-D \bar{\phi}(t)-c \bar{\phi}^{\prime}(t) \\
& -d \bar{\phi}(t)+a \bar{\varphi}(t), \\
P_{2}(t)= & D \int_{\mathbb{R}} J(t-y) \bar{\varphi}(y) d y-D \bar{\varphi}(t)-c \bar{\varphi}^{\prime}(t) \\
& +\frac{\beta[A / d-\phi(t)-\bar{\varphi}(t)-\psi(t)] \bar{\varphi}(t-c \tau)}{1+\alpha \bar{\varphi}(t-c \tau)}  \tag{39}\\
& -(d+\gamma+a) \bar{\varphi}(t), \\
P_{3}(t)= & D \int_{\mathbb{R}} J(t-y) \bar{\psi}(y) d y-D \bar{\psi}(t)-c \bar{\psi}^{\prime}(t) \\
& +\gamma \bar{\varphi}(t)-(d+\delta) \bar{\psi}(t) .
\end{align*}
$$

If $t \leq t_{1}, \bar{\phi}(t)=k_{1} e^{\lambda_{1} t}$ and $\bar{\varphi}(t)=l k_{2} e^{\lambda_{3} t}$. By Lemma 6, it follows that

$$
\left.\left.\begin{array}{rl}
P_{1}(t)= & k_{1} e^{\lambda_{1} t}[D
\end{array} \int_{\mathbb{R}} J(y) e^{-\lambda_{1} y} d y-D-c \lambda_{1}-d\right] \text { al } \frac{k_{2}}{k_{1}} e^{\left(\lambda_{3}-\lambda_{1}\right) t}\right] .
$$

If $t_{1}<t \leq t_{3}, \bar{\phi}(t)=k_{1}+\varepsilon_{1} e^{-\lambda t}$ and $\bar{\varphi}(t)=l k_{2} e^{\lambda_{3} t}$. Then, we have

$$
\begin{align*}
P_{1}(t)= & {\left[D \varepsilon_{1} \int_{\mathbb{R}} J(y) e^{\lambda y} d y-D \varepsilon_{1}+c \lambda \varepsilon_{1}\right] e^{-\lambda t} }  \tag{41}\\
& -d\left(k_{1}+\varepsilon_{1} e^{-\lambda t}\right)+a l k_{2} e^{\lambda_{3} t}
\end{align*}
$$

Note that $l \in(0,1)$ and $a k_{2} / d=k_{1}$. Hence, for $\lambda$ sufficiently small, there exists $\lambda_{1}^{*}>0$ such that $P_{1}(t)<0$ for all $\lambda \in$ ( $0, \lambda_{1}^{*}$ ).

If $t>t_{3}, \bar{\phi}(t)=k_{1}+\varepsilon_{1} e^{-\lambda t}$ and $\bar{\varphi}(t)=k_{2}+\varepsilon_{3} e^{-\lambda t}$. We obtain that

$$
\begin{align*}
P_{1}(t) \leq & {\left[D \varepsilon_{1} \int_{\mathbb{R}} J(y) e^{\lambda y} d y-D \varepsilon_{1}+c \lambda \varepsilon_{1}\right] e^{-\lambda t} }  \tag{42}\\
& -d\left(k_{1}+\varepsilon_{1} e^{-\lambda t}\right)+a\left(k_{2}+\varepsilon_{3} e^{-\lambda t}\right):=I_{1}(\lambda) .
\end{align*}
$$

For $\lambda$ sufficiently small, $a\left(k_{2}+\varepsilon_{3}\right)<d\left(k_{1}+\varepsilon_{1}\right)$ implies that $I_{1}(0)<0$, and there exists $\lambda_{2}^{*}>0$ such that $P_{1}(t) \leq I_{1}(\lambda)<0$ for all $\lambda \in\left(0, \lambda_{2}^{*}\right)$.

If $t \leq t_{3}, \bar{\varphi}(t)=l k_{2} e^{\lambda_{3} t}$ and $\bar{\varphi}(t-c \tau)=l k_{2} e^{\lambda_{3}(t-c \tau)}$. It follows that

$$
\begin{align*}
& P_{2}(t) \leq l k_{2} e^{\lambda_{3} t}[D \int_{\mathbb{R}} J(y) e^{-\lambda_{3} y} d y-D-c \lambda_{3}+\frac{A \beta}{d} \\
&-(d+\gamma+a)]  \tag{43}\\
&=l k_{2} e^{\lambda_{3} t} \Delta_{2}\left(\lambda_{3}, c\right)=0
\end{align*}
$$

If $t_{3}<t \leq t_{3}+c \tau, \bar{\varphi}(t)=k_{2}+\varepsilon_{3} e^{-\lambda t}, \bar{\varphi}(t-c \tau)=l k_{2} e^{\lambda_{3}(t-c \tau)}$, $\underline{\phi}(t)=k_{1}-\varepsilon_{2} e^{-\lambda t}$, and $\underline{\psi}(t)=k_{3}-\varepsilon_{6} e^{-\lambda t}$. We obtain that

$$
\begin{equation*}
P_{2}(t)=\varepsilon_{3} e^{-\lambda t}\left[D \int_{\mathbb{R}} J(y) e^{\lambda y} d y-D+c \lambda\right]+I_{2}(\lambda) \tag{44}
\end{equation*}
$$

where

$$
\begin{align*}
I_{2}(\lambda)= & \left(\beta \left[\frac{A}{d}-\left(k_{1}-\varepsilon_{2} e^{-\lambda t}\right)-\left(k_{2}+\varepsilon_{3} e^{-\lambda t}\right)\right.\right. \\
& \left.\left.-\left(k_{3}-\varepsilon_{6} e^{-\lambda t}\right)\right] l k_{2} e^{\lambda_{3}(t-c \tau)}\right)  \tag{45}\\
& \times\left(1+\alpha l k_{2} e^{\lambda_{3}(t-c \tau)}\right)^{-1} \\
& -(d+\gamma+a)\left(k_{2}+\varepsilon_{3} e^{-\lambda \tau}\right)
\end{align*}
$$

Then by (29), we have

$$
\begin{align*}
I_{2}(0)= & \frac{\beta\left(A / d-k_{1}+\varepsilon_{2}-k_{2}-\varepsilon_{3}-k_{3}+\varepsilon_{6}\right) l k_{2}}{1+\alpha l k_{2}} \\
& -(d+\gamma+a)\left(k_{2}+\varepsilon_{3}\right)  \tag{46}\\
\leq & \frac{d+\gamma+a}{1+\alpha l k_{2}}\left((l-1) k_{2}-\varepsilon_{3}\right) .
\end{align*}
$$

Since $l \in(0,1)$, for $\lambda$ sufficiently small, it is easy to show that $I_{2}(0)<0$ and there exists $\lambda_{3}^{*}>0$ such that $P_{2}(t)<0$ for all $\lambda \in\left(0, \lambda_{3}^{*}\right)$.

If $t>t_{3}+c \tau, \bar{\varphi}(t)=k_{2}+\varepsilon_{3} e^{-\lambda t}, \bar{\varphi}(t-c \tau)=k_{2}+\varepsilon_{3} e^{-\lambda(t-c \tau)}$, $\underline{\phi}(t)=k_{1}-\varepsilon_{2} e^{-\lambda t}$, and $\underline{\psi}(t)=k_{3}-\varepsilon_{6} e^{-\lambda t}$. It follows that

$$
\begin{equation*}
P_{2}(t)=\varepsilon_{3} e^{-\lambda t}\left[D \int_{\mathbb{R}} J(y) e^{\lambda y} d y-D+c \lambda\right]+I_{3}(\lambda) \tag{47}
\end{equation*}
$$

where

$$
\begin{align*}
I_{3}(\lambda)= & \left(\beta \left[\frac{A}{d}-\left(k_{1}-\varepsilon_{2} e^{-\lambda t}\right)-\left(k_{2}+\varepsilon_{3} e^{-\lambda t}\right)\right.\right. \\
& \left.\left.-\left(k_{3}-\varepsilon_{6} e^{-\lambda t}\right)\right]\left(k_{2}+\varepsilon_{3} e^{-\lambda(t-c \tau)}\right)\right)  \tag{48}\\
& \times\left(1+\alpha\left(k_{2}+\varepsilon_{3} e^{-\lambda(t-c \tau)}\right)\right)^{-1} \\
& -(d+\gamma+a)\left(k_{2}+\varepsilon_{3} e^{-\lambda t}\right)
\end{align*}
$$

For $\lambda$ sufficiently small, by (29), we see that $I_{3}(0)<0$ and there exists $\lambda_{4}^{*}>0$ such that $P_{2}(t) \leq I_{3}(\lambda)<0$ for all $\lambda \in$ $\left(0, \lambda_{4}^{*}\right)$.

If $t \leq t_{5}, \bar{\psi}(t)=k_{3} e^{\lambda_{5} t}$ and $\bar{\varphi}(t)=l k_{2} e^{\lambda_{3}(t)}$. Then, by Lemma 6, we have

$$
\left.\begin{array}{rl}
P_{3}(t)= & k_{3} e^{\lambda_{5} t}[D
\end{array} \int_{\mathbb{R}} J(y) e^{-\lambda_{5} t} d y-D-c \lambda_{5}, ~(d+\delta)+\frac{\gamma l k_{2} e^{\lambda_{3} t}}{k_{3} e^{\lambda_{5} t}}\right] .
$$

If $t_{5}<t \leq t_{3}, \bar{\psi}(t)=k_{3}+\varepsilon_{5} e^{-\lambda t}$ and $\bar{\varphi}(t)=l k_{2} e^{\lambda_{3} t}$. We derive that

$$
\begin{align*}
P_{3}(t)= & \varepsilon_{3} e^{-\lambda t}\left[D \int_{\mathbb{R}} J(y) e^{\lambda y} d y-D+c \lambda\right]+\gamma l k_{2} e^{\lambda_{3} t}  \tag{50}\\
& -(d+\delta)\left(k_{3}+\varepsilon_{5} e^{-\lambda t}\right):=I_{4}(\lambda) .
\end{align*}
$$

Note that $l \in(0,1)$; then we have $l k_{2} e^{\lambda_{3} t} \leq k_{2} e^{\lambda_{3} t_{3}}<k_{2}+\varepsilon_{3}$. By (29), for $\lambda$ sufficiently small, it is easy to show that $I_{4}(0)<0$ and there exists $\lambda_{5}^{*}>0$ such that $P_{3}(t)<0$ for all $\lambda \in\left(0, \lambda_{5}^{*}\right)$.

If $t>t_{3}, \bar{\psi}(t)=k_{3}+\varepsilon_{5} e^{-\lambda t}$ and $\bar{\varphi}(t)=k_{2}+\varepsilon_{3} e^{-\lambda t}$. We obtain that

$$
\begin{align*}
P_{3}(t) \leq & \varepsilon_{5} e^{-\lambda t}\left[D \int_{\mathbb{R}} J(y) e^{\lambda y} d y-D+c \lambda\right] \\
& +\gamma\left(k_{2}+\varepsilon_{3} e^{-\lambda t}\right)-(d+\delta)\left(k_{3}+\varepsilon_{5} e^{-\lambda t}\right)  \tag{51}\\
: & =I_{5}(\lambda)
\end{align*}
$$

For $\lambda$ sufficiently small, by (29), we see that $\gamma\left(k_{2}+\varepsilon_{3}\right)-(d+$ $\delta)\left(k_{3}+\varepsilon_{5}\right)<0$ implies that $I_{5}(0)<0$ and there exists $\lambda_{6}^{*}>0$ such that $P_{3}(t) \leq I_{5}(\lambda)<0$ for all $\lambda \in\left(0, \lambda_{6}^{*}\right)$.

Clearly, for all $\lambda \in\left(0, \min \left\{\lambda_{i}^{*}, i=1, \ldots, 6\right\}\right), P_{i}(t) \leq 0(i=$ $1,2,3)$. This completes the proof.

Lemma 8. $\underline{\Phi}(t)=(\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t))$ is a lower solution of system (10).

## Proof. Denote

$$
\begin{aligned}
Q_{1}(t)= & D \int_{\mathbb{R}} J(t-y) \underline{\phi}(y) d y-D \underline{\phi}(t)-c \underline{\phi}^{\prime}(t)-d \underline{\phi}(t) \\
& +a \underline{\varphi}(t), \\
Q_{2}(t)= & D \int_{\mathbb{R}} J(t-y) \underline{\varphi}(y) d y-D \underline{\varphi}(t)-c \underline{\varphi}^{\prime}(t)
\end{aligned}
$$

$$
\begin{align*}
& +\frac{\beta[A / d-\bar{\phi}(t)-\underline{\varphi}(t)-\bar{\psi}(t)] \underline{\varphi}(t-c \tau)}{1+\alpha \underline{\varphi}(t-c \tau)} \\
& -(d+\gamma+a) \underline{\varphi}(t), \\
Q_{3}(t)= & D \int_{\mathbb{R}} J(t-y) \underline{\psi}(y) d y-D \underline{\psi}(t)-c \underline{\psi}^{\prime}(t) \\
& +\gamma \underline{\varphi}(t)-(d+\delta) \underline{\psi}(t) . \tag{52}
\end{align*}
$$

If $t \leq t_{2}, \underline{\phi}(t)=0$. It is easy to see that $Q_{1}(t)=a \varphi(t) \geq 0$.
If $t>t_{2}, \bar{\phi}(t)=k_{1}-\varepsilon_{2} e^{-\lambda t}$ and $\underline{\varphi}(t)=k_{2}-\varepsilon_{4} e^{-\bar{\lambda} t}$. Then, we have

$$
\begin{align*}
Q_{1}(t)= & \varepsilon_{2} e^{-\lambda t}\left[-D \int_{\mathbb{R}} J(y) e^{\lambda y} d y+D-c \lambda\right]  \tag{53}\\
& -d\left(k_{1}-\varepsilon_{2} e^{-\lambda t}\right)+a\left(k_{2}-\varepsilon_{4} e^{-\lambda t}\right)
\end{align*}
$$

For $\lambda$ sufficiently small, $-d\left(k_{1}-\varepsilon_{2}\right)+a\left(k_{2}-\varepsilon_{4}\right)>0$ implies that $Q_{1}(0)>0$ and there exists $\lambda_{7}^{*}>0$ such that $Q_{1}(t)>0$ for all $\lambda \in\left(0, \lambda_{7}^{*}\right)$.

If $t \leq t_{4}, \underline{\varphi}(t)=0$ and $\underline{\varphi}(t-c \tau)=0$. Hence, $Q_{2}(t)=0$.
If $t>t_{4}, \underline{\varphi}(t)=k_{2}-\varepsilon_{4} e^{-\lambda \tau}$. Noting that $\underline{\varphi}(t-c \tau) \geq$ $k_{2}-\varepsilon_{4} e^{-\lambda t}, \bar{\phi}(\bar{t}) \leq k_{1}+\varepsilon_{1} e^{-\lambda \tau}$ and $\bar{\psi}(t) \leq k_{3}+\varepsilon_{5} e^{-\lambda t}$. We obtain that

$$
\begin{equation*}
Q_{2}(t) \geq \varepsilon_{4} e^{-\lambda t}\left[-D \int_{\mathbb{R}} J(y) e^{\lambda y} d y+D-c \lambda\right]+I_{6}(\lambda), \tag{54}
\end{equation*}
$$

where

$$
\begin{align*}
I_{6}(\lambda)= & \left(\beta \left[\frac{A}{d}-k_{1}-\varepsilon_{1} e^{-\lambda t}-k_{2}+\varepsilon_{4} e^{-\lambda t}-k_{3}\right.\right. \\
& \left.\left.-\varepsilon_{5} e^{-\lambda t}\right]\left(k_{2}-\varepsilon_{4} e^{-\lambda t}\right)\right)  \tag{55}\\
& \times\left(1+\alpha\left(k_{2}-\varepsilon_{4} e^{-\lambda t}\right)\right)^{-1} \\
& -(d+\gamma+a)\left(k_{2}-\varepsilon_{4} e^{-\lambda t}\right) .
\end{align*}
$$

By (29), we have $I_{6}(0)>0$. Accordingly, for $\lambda$ sufficiently small, there exists $\lambda_{8}^{*}>0$ such that $Q_{2}(t) \geq I_{6}(\lambda)>0$ for all $\lambda \in\left(0, \lambda_{8}^{*}\right)$.

If $t \leq t_{6}, \psi(t)=0$. Then, we have $Q_{3}(t)=\gamma \varphi(t) \geq 0$.
If $t>t_{6}, \underline{\psi}(t)=k_{3}-\varepsilon_{6} e^{-\lambda t}$ and $\underline{\varphi}(t)=k_{2}^{-}-\varepsilon_{4} e^{-\lambda t}$. We obtain that

$$
\begin{align*}
Q_{3}(t)= & \varepsilon_{6} e^{-\lambda t}\left[-D \int_{\mathbb{R}} J(y) e^{\lambda y} d y+D-c \lambda\right]  \tag{56}\\
& +\gamma\left(k_{2}-\varepsilon_{4} e^{-\lambda t}\right)-(d+\delta)\left(k_{3}-\varepsilon e^{-\lambda t}\right) .
\end{align*}
$$

For $\lambda$ sufficiently small, then, by (29), it is readily seen that $Q_{3}(0)>0$ and there exists $\lambda_{9}^{*}>0$ such that $Q_{3}(t)>0$ for all $\lambda \in\left(0, \lambda_{9}^{*}\right)$.

Obviously, for all $\lambda \in\left(0, \min \left\{\lambda_{7}^{*}, \lambda_{8}^{*}, \lambda_{9}^{*}\right\}\right), Q_{i}(t) \geq 0(i=$ $1,2,3)$. This completes the proof.

Applying Lemmas 2-8 and Schauder's fixed point theorem, we know that if $\mathscr{R}_{0}>1$ and $a / d+\gamma /(d+\delta)<1$, system (9) has a traveling wave solution with speed $c>c^{*}$ connecting the steady states $(0,0,0)$ and $\left(k_{1}, k_{2}, k_{3}\right)$. Accordingly, we have the following conclusion.

Theorem 9. Let $\mathscr{R}_{0}>1$. Assume that (A1) and $a / d+\gamma /(d+$ $\delta)<1$ hold. For every $c>c^{*}$, system (3) always has a traveling wave solution with speed $c$ connecting the disease-free steady state $E^{0}(A / d, 0,0)$ and the endemic steady state $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$.

## 3. Concluding Remark

In this paper, we have discussed a delayed SIRS infectious disease model with nonlocal diffusion and nonlinear incidence. By constructing a pair of upper-lower solutions and using Schauder's fixed point theorem, we investigated the existence of a traveling wave solution connecting the diseasefree steady state $E^{0}$ and the endemic steady state $E^{*}$. We now study the influence of the nonlocal diffusion terms and time delay describing the incubation period on the spreading speed $c^{*}$. From the second equation of system (3), we have a linearized equation at $E^{0}$ that takes the form

$$
\begin{align*}
c I^{\prime}(\xi)= & D[J * I(\xi)-I(\xi)]+\frac{A \beta}{d} I(\xi-c \tau)  \tag{57}\\
& -(d+\gamma+a) I(\xi)
\end{align*}
$$

Letting $I(\xi)=e^{\lambda \xi}$ yields the following characteristic equation:

$$
\begin{align*}
\Delta(\lambda, c)= & D \int_{\mathbb{R}} J(y)\left[e^{-\lambda y}-1\right] d y-c \lambda+\frac{A \beta}{d} e^{-\lambda c \tau}  \tag{58}\\
& -(d+\gamma+a)
\end{align*}
$$

By direct calculations we have

$$
\begin{align*}
\frac{\mathrm{d} c^{*}}{\mathrm{~d} D} & =\frac{\int_{\mathbb{R}} J(y)\left[e^{-\lambda_{*} y}-1\right] d y}{\lambda_{*}\left(1+(A \beta \tau / d) e^{-\lambda_{*} c^{*} \tau}\right)}>0  \tag{59}\\
\frac{\mathrm{~d} c^{*}}{\mathrm{~d} \tau} & =-\frac{A \beta c^{*}}{d e^{-c^{*} \lambda_{*} \tau}+A \beta \tau}<0
\end{align*}
$$

It is easy to show that the spreading speed $c^{*}$ is monotonically increasing for the nonlocal diffusion rate $D$ and is monotonically decreasing for the time delay $\tau$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work was supported by the National Natural Science Foundation of China (nos. 11371368, 11071254), the Natural Science Foundation of Hebei Province of China under Grant (no. A2014506015), the Natural Science Foundation for

Young Scientists of Hebei Province (no. A2013506012), and the Science Research Foundation of Mechanical Engineering College (nos. YJJXM12010, YJJXM13008).

## References

[1] Q. Gan, R. Xu, and P. Yang, "Travelling waves of a delayed SIRS epidemic model with spatial diffusion," Nonlinear Analysis, vol. 12, no. 1, pp. 52-68, 2011.
[2] K. B. Blyuss, "On a model of spatial spread of epidemics with long-distance travel," Physics Letters A: General, Atomic and Solid State Physics, vol. 345, no. 1-3, pp. 129-136, 2005.
[3] M. Cui, T. Ma, and X. Li, "Spatial behavior of an epidemic model with migration," Nonlinear Dynamics, vol. 64, no. 4, pp. 331-338, 2011.
[4] Y. Lou and X. Zhao, "A reaction-diffusion malaria model with incubation period in the vector population," Journal of Mathematical Biology, vol. 62, no. 4, pp. 543-568, 2011.
[5] Z. Wang, W. Li, and S. Ruan, "Travelling wave fronts in reactiondiffusion systems with spatio-temporal delays," Journal of Differential Equations, vol. 222, no. 1, pp. 185-232, 2006.
[6] P. Weng and X. Zhao, "Spreading speed and traveling waves for a multi-type SIS epidemic model," Journal of Differential Equations, vol. 229, no. 1, pp. 270-296, 2006.
[7] X. Chen, "Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations," Advances in Differential Equations, vol. 2, no. 1, pp. 125-160, 1997.
[8] J. Coville, J. Dávila, and S. Martínez, "Nonlocal anisotropic dispersal with monostable nonlinearity," Journal of Differential Equations, vol. 244, no. 12, pp. 3080-3118, 2008.
[9] P. Fife, "Some nonclassical trends in parabolic and paraboliclike evolutions," in Trends in Nonlinear Analysis, pp. 153-191, Springer, Berlin, Germany, 2003.
[10] Y. Sun, W. Li, and Z. Wang, "Entire solutions in nonlocal dispersal equations with bistable nonlinearity," Journal of Differential Equations, vol. 251, no. 3, pp. 551-581, 2011.
[11] Y. Sun, W. Li, and Z. Wang, "Traveling waves for a nonlocal anisotropic dispersal equation with monostable nonlinearity," Nonlinear Analysis A: Theory and Methods, vol. 74, no. 3, pp. 814-826, 2011.
[12] G. Zhang and Y. Wang, "Critical exponent for nonlocal diffusion equations with Dirichlet boundary condition," Mathematical and Computer Modelling, vol. 54, no. 1-2, pp. 203-209, 2011.
[13] W. T. Li, Y. J. Sun, and Z. C. Wang, "Entire solutions in the Fisher-KPP equation with nonlocal dispersal," Nonlinear Analysis, vol. 11, no. 4, pp. 2302-2313, 2010.
[14] V. Capasso and G. Serio, "A generalization of the KermackMcKendrick deterministic epidemic model," Mathematical Biosciences, vol. 42, no. 1-2, pp. 43-61, 1978.
[15] R. Xu and Z. Ma, "Stability of a delayed SIRS epidemic model with a nonlinear incidence rate," Chaos, Solitons \& Fractals, vol. 41, no. 5, pp. 2319-2325, 2009.
[16] X. Yu, C. Wu, and P. Weng, "Traveling waves for a SIRS model with nonlocal diffusion," International Journal of Biomathematics, vol. 5, no. 5, 26 pages, 2012.

## Research Article

# Hopf Bifurcation of a Delayed Epidemic Model with Information Variable and Limited Medical Resources 

Caijuan Yan and Jianwen Jia<br>School of Mathematics and Computer Science, Shanxi Normal University, Linfen, Shanxi 041004, China<br>Correspondence should be addressed to Jianwen Jia; jiajw.2008@163.com

Received 7 January 2014; Revised 4 March 2014; Accepted 10 March 2014; Published 9 April 2014
Academic Editor: Kaifa Wang
Copyright © 2014 C. Yan and J. Jia. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We consider SIR epidemic model in which population growth is subject to logistic growth in absence of disease. We get the condition for Hopf bifurcation of a delayed epidemic model with information variable and limited medical resources. By analyzing the corresponding characteristic equations, the local stability of an endemic equilibrium and a disease-free equilibrium is discussed. If the basic reproduction ratio $\mathscr{R}_{0}<1$, we discuss the global asymptotical stability of the disease-free equilibrium by constructing a Lyapunov functional. If $\mathscr{R}_{0}>1$, we obtain sufficient conditions under which the endemic equilibrium $E^{*}$ of system is locally asymptotically stable. And we also have discussed the stability and direction of Hopf bifurcations. Numerical simulations are carried out to explain the mathematical conclusions.


## 1. Introduction

From an epidemiological viewpoint, it is important to investigate the global dynamics of the disease transmission. In the literature, many authors have researched various epidemic models $[1,2]$, in which the stability analyses have been carried out extensively. In the recent years, based on SIR epidemic model, in order to investigate the spread of an infectious disease transmitted by a vector, Wang et al. [3] have considered the asymptotic behavior of the following delayed SIR epidemic model:

$$
\begin{align*}
& \frac{d S(t)}{d t}=r S\left(1-\frac{S}{k}\right)-\beta S I(t-\tau) \\
& \frac{d I(t)}{d t}=\beta S I(t-\tau)-\left(\mu_{1}+\gamma\right) I  \tag{1}\\
& \frac{d R(t)}{d t}=\gamma I-\mu_{2} R
\end{align*}
$$

Since nonlinearity in the incidence rates has been observed in disease transmission dynamics, it has been suggested that the standard bilinear incidence rate will be modified into a nonlinear incidence rate by many authors [46]. In [7], incidence rate $\beta S I(t-\tau)$ in (1) was replaced by
a nonlinear incidence rate of the form $\beta S G(I(t-\tau))$ with the following system:

$$
\begin{align*}
& \frac{d S(t)}{d t}=r S\left(1-\frac{S}{k}\right)-\beta S G(I(t-\tau)) \\
& \frac{d I(t)}{d t}=\beta S G(I(t-\tau))-\left(\mu_{1}+\gamma\right) I  \tag{2}\\
& \frac{d R(t)}{d t}=\gamma I-\mu_{2} R
\end{align*}
$$

In order to control the spread of epidemic, we consider the new variable $Z$ :

$$
\begin{equation*}
Z(t)=\int_{-\infty}^{t} S \frac{1}{T} \exp \left(-\frac{1}{T}(t-\tau)\right) d \tau \tag{3}
\end{equation*}
$$

called information variable which summarizes information about the current state of the disease, that is, depending on current values of state variables, and also summarizes information about past values of state variables. Many authors have used this variable in their models (see, e.g., [8-10]).

In this paper, we consider the information variable $Z(t)$, nonlinear incidence rate of the form $\beta S G(I(t-\tau))$, and limited
medical resources $h(I)=b I /(\omega+I)$. The model can be described by the following system of equations:

$$
\begin{align*}
\frac{d S(t)}{d t} & =r S\left(1-\frac{S}{k}\right)-\beta S G(I(t-\tau)) \\
\frac{d I(t)}{d t} & =\beta Z G(I(t-\tau))-\left(\mu_{1}+\gamma+\varepsilon\right) I-\frac{b I}{\omega+I} \\
Z(t) & =\int_{-\infty}^{t} S \frac{1}{T} \exp \left(-\frac{1}{T}(t-\tau)\right) d \tau  \tag{4}\\
\frac{d R(t)}{d t} & =\gamma I-\mu_{2} R+\frac{b I}{\omega+I}
\end{align*}
$$

where $S(t), I(t), R(t)>0$ and $S(t), I(t), R(t)$ denote the numbers of susceptible, infective, and recovered individuals at time $t$, respectively. $r$ is the intrinsic growth rate of susceptibles, $k$ is the carrying capacity of susceptibles, $\alpha$ is the saturation factor that measures the inhibitory effect, $\beta$ is the transmission or contact rate, $\mu_{1}, \mu_{2}$ are the natural death rate of the infective and recovered individuals, $\gamma$ is the natural recovery rate, $\varepsilon$ is the disease-related mortality, $b \geq 0$ is the maximal medical resources supplied per unit time, and $\omega>0$ is half-saturation constant. $r, b, \mu_{1}, \mu_{2}, \gamma, \alpha, \beta, k, \omega$ are all positive.

We further assume that the function $G$ is continuous on $[0,+\infty)$ and continuously differentiable on $(0,+\infty)$ satisfying the following hypotheses:
(1) $G(I)$ is strictly monotone increasing on $[0,+\infty)$ with $G(0)=0$;
(2) $I / G(I)$ is monotone increasing on $(0,+\infty)$ with $\lim _{I \rightarrow 0+}(I / G(I))=1$.

The organization of this paper is as follows. In Section 2, we explore the existence of disease-free equilibria point and the unique existence of the endemic equilibrium point. In Section 3, we analyze the stability of the disease-free equilibria. In Section 4, we obtain sufficient conditions under which the endemic equilibrium $E^{*}$ of system is locally asymptotically stable. In Section 5, we also have discussed the stability and direction of Hopf bifurcations. A numerical analysis and a simple discussion are given to conclude this paper in Section 6.

## 2. The Existence of Equilibria

The nonlinear integrodifferential system (4) can be transformed into the following set of nonlinear ordinary differential questions:

$$
\begin{aligned}
\frac{d S(t)}{d t} & =r S\left(1-\frac{S}{k}\right)-\beta S G(I(t-\tau)) \\
\frac{d I(t)}{d t} & =\beta Z G(I(t-\tau))-\left(\mu_{1}+\gamma+\varepsilon\right) I-\frac{b I}{\omega+I} \\
\frac{d Z(t)}{d t} & =\frac{1}{T}(S-Z) \\
\frac{d R(t)}{d t} & =\gamma I-\mu_{2} R+\frac{b I}{\omega+I}
\end{aligned}
$$

Since the dynamical behavior of the last equation of the system (5), that is, the dynamics of $R$, depends only the dynamics of $I$, we do not consider that equation in our discussion. Here we will study the following nonlinear ordination differential equations:

$$
\begin{align*}
\frac{d S(t)}{d t} & =r S\left(1-\frac{S}{k}\right)-\beta S G(I(t-\tau)) \\
\frac{d I(t)}{d t} & =\beta Z G(I(t-\tau))-\left(\mu_{1}+\gamma+\varepsilon\right) I-\frac{b I}{\omega+I}  \tag{6}\\
\frac{d Z(t)}{d t} & =\frac{1}{T}(S-Z)
\end{align*}
$$

For simplicity, we nondimensionalize system (6) by defining

$$
\begin{align*}
& \widetilde{S}(\widetilde{t})=\frac{S(t)}{k}, \quad \widetilde{I}(\widetilde{t})=\frac{I(t)}{k}, \quad \widetilde{Z}(\widetilde{t})=\frac{Z(t)}{k}, \\
& \widetilde{t}=\beta k t, \quad \widetilde{T}=\beta k T, \quad \widetilde{\omega}=\frac{\omega}{k}, \quad \widetilde{r}=\frac{r}{\beta k}, \\
& \widetilde{G}(\widetilde{I}(\widetilde{t}))=\frac{G(I(t))}{k}, \quad \widetilde{\mu}_{1}=\frac{\mu_{1}}{\beta k}, \quad \widetilde{\mu}_{2}=\frac{\mu_{2}}{\beta k}, \\
& \tilde{\gamma}=\frac{\gamma}{\beta k}, \quad \widetilde{\varepsilon}=\frac{\varepsilon}{\beta k}, \quad \widetilde{b}=\frac{b}{\beta k^{2}} . \tag{7}
\end{align*}
$$

We note that $\widetilde{G}$ also satisfies the hypotheses (1) and (2). Dropping the ${ }^{\sim}$ for convenience of readers, system (6) can be written in the following form:

$$
\begin{align*}
\frac{d S(t)}{d t} & =r S(1-S)-S G(I(t-\tau)) \\
\frac{d I(t)}{d t} & =Z G(I(t-\tau))-\left(\mu_{1}+\gamma+\varepsilon\right) I-\frac{b I}{\omega+I}  \tag{8}\\
\frac{d Z(t)}{d t} & =\frac{1}{T}(S-Z)
\end{align*}
$$

The basic reproduction is $\mathscr{R}_{0}=1 /\left(\mu_{1}+\gamma+\varepsilon+b / \omega\right)$.
Theorem 1. (1) The system (8) has a trivial equilibrium $E_{0}=$ $(0,0,0)$ and the disease-free equilibrium $E_{1}=(1,0,1)$.
(2) If $\mathscr{R}_{0}>1$, the system (8) has one endemic equilibrium $E^{*}=\left(S^{*}, I^{*}, Z^{*}\right)$ except the disease-free equilibria $E_{0}$ and $E_{1}$.

Proof. (1) Let $I=0$; we have $S=Z=0$, or $S=Z=1$; it is not easy to find that the system has a trivial equilibrium and the disease-free equilibria $E_{0}=(0,0,0)$ and $E_{1}=(1,0,1)$.
(2) If $\mathscr{R}_{0}>1$, from the third question of (8), we have $Z^{*}=S^{*}$; from the second question of (8), we have

$$
\begin{equation*}
S^{*}=Z^{*}=\frac{\left(\mu_{1}+\gamma+\varepsilon+b /\left(\omega+I^{*}\right)\right) I^{*}}{G\left(I^{*}\right)} . \tag{9}
\end{equation*}
$$

Then substituting them into the first question of (8) yields

$$
\begin{equation*}
r\left[1-\frac{\left(\mu_{1}+\gamma+\varepsilon+b /\left(\omega+I^{*}\right)\right) I^{*}}{G\left(I^{*}\right)}\right]-G\left(I^{*}\right)=0 . \tag{10}
\end{equation*}
$$

Let $F(I)=r\left[1-\left(\mu_{1}+\gamma+\varepsilon+b /\left(\omega+I^{*}\right)\right) I / G(I)\right]-G(I)$. By hypothesis (2), we obtain

$$
\begin{align*}
\lim _{I \rightarrow+0} F(I) & =r\left[1-\left(\mu_{1}+\gamma+\varepsilon+\frac{b}{\omega+I^{*}}\right)\right] \\
& >r\left[1-\left(\mu_{1}+\gamma+\varepsilon+\frac{b}{\omega}\right)\right]=r\left(1-\frac{1}{R_{0}}\right)>0 . \tag{11}
\end{align*}
$$

Since $F(I)$ is strictly monotone decreasing function on $(0,+\infty)$, it suffices to show that $F(I)<0$ holds for $I$ sufficiently large. From (1), $G(I)$ is either unbounded above or bounded above on $[0,+\infty)$.

First, we suppose that $G(I)$ is unbounded above. Then there exists an $I_{1}>0$ such that $G\left(I_{1}\right)=r$, from which we have $F(I)<0$ for all $I \geq I_{1}$. Second we suppose that
$G(I)$ is bounded above. Then, from (2), $I / G(I)$ is unbounded above on $[0,+\infty)$; that is, there exists an $I_{2}>0$ such that $\mu_{1}+\gamma+\varepsilon+b /\left(\omega+I_{2}\right)=G\left(I_{2}\right) / I_{2}$. This yields $F(I)<0$ for all $I>I_{2}$. Therefore, for the both cases, there exists a unique endemic $I^{*}>0$ such that $F\left(I^{*}\right)=0$. By the second and third equations of (8), there exists a unique endemic equilibrium $E^{*}$ of system (8) if $\mathscr{R}_{0}>1$.

Second, we assume $\mathscr{R}_{0} \leq 1$; then it is obvious that system (8) has no equilibria. Hence the proof is complete.

## 3. The Stability Analysis of Disease-Free Equilibrium Point

In this section, we will examine the local stability of the equilibria by analyzing the eigenvalues of the Jacobian matrices of (8) at the equilibria and using Routh-Hurwitz criterion.

Let $\bar{E}=(\bar{S}, \bar{I}, \bar{R})$ be the arbitrarily equilibrium point of system (8); then the Jacobian matrix of (8) at $\bar{E}$ is

$$
\left.J(\bar{S}, \bar{I}, \bar{Z})=\left(\begin{array}{ccc}
r(1-2 \bar{S})-G(\bar{I}) & -\bar{S} G^{\prime}(\bar{I}) e^{-\lambda \tau} & 0  \tag{12}\\
0 & \bar{Z} G^{\prime}(\bar{I}) e^{-\lambda \tau}-\left(\mu_{1}+\gamma+\varepsilon+\frac{b \omega}{(\omega+\bar{I})^{2}}\right.
\end{array}\right) \underset{(\bar{I})}{ } \begin{array}{ccc}
\frac{1}{T} & 0 & -\frac{1}{T}
\end{array}\right)
$$

Then the characteristic equation of the system (8) at equilibrium $\bar{E}$ is

$$
\begin{align*}
& |\lambda E-J(\bar{E})| \\
& = \\
& \quad\left(\lambda+\frac{1}{T}\right)[\lambda+G(\bar{I})-r(1-2 \bar{S})]  \tag{13}\\
& \\
& \times\left[\lambda+\mu_{1}+\gamma+\varepsilon+\frac{b \omega}{(\omega+\bar{I})^{2}}\right] \\
& \\
& \quad-\left[\left(\lambda+\frac{1}{T}\right)(\lambda+G(\bar{I})-r(1-2 \bar{S}))-\frac{1}{T} G(\bar{I})\right] \\
& \\
& \quad \times \bar{Z} G^{\prime}(\bar{I}) e^{-\lambda \tau}=0 .
\end{align*}
$$

Theorem 2. The trivial equilibrium $E_{0}$ of system (8) is always unstable.

Proof. The characteristic equation (13) at $E_{0}=(0,0,0)$ becomes as follows:

$$
\begin{equation*}
\left(\lambda+\frac{1}{T}\right)(\lambda-r)\left(\lambda+\mu_{1}+\gamma+\varepsilon+\frac{b}{\omega}\right)=0 \tag{14}
\end{equation*}
$$

Since (14) has a positive root $\lambda=r>0, E_{0}$ is unstable.
Theorem 3. The disease-free equilibrium $E_{1}$ of system (8) is locally asymptotically stable if $\mathscr{R}_{0}<1$ and it is unstable if $\mathscr{R}_{0}>1$.

Proof. For $E_{1}=(1,0,1)$, the characteristic equation (13) at $E_{1}$ becomes as follows:

$$
\begin{equation*}
\left(\lambda+\frac{1}{T}\right)(\lambda+r)\left[\lambda+\left(\mu_{1}+\gamma+\varepsilon+\frac{b}{\omega}\right)-e^{-\lambda \tau}\right]=0 . \tag{15}
\end{equation*}
$$

It is clear that both $\lambda=-1 / T$ and $\lambda=-r$ are all the negative root of (15). Then the other root of (15) is determined as the following equation:

$$
\begin{equation*}
\lambda+\left(\mu_{1}+\gamma+\varepsilon+\frac{b}{\omega}\right)-e^{-\lambda \tau}=0 \tag{16}
\end{equation*}
$$

For the case $\mathscr{R}_{0}<1$, we suppose on the contrary that $E_{1}$ is not locally asymptotically stable; that is, $\operatorname{Re} \tilde{\lambda}>0$. Then, there exists a root $\lambda=\widetilde{\lambda}$, such that $\operatorname{Re} \widetilde{\lambda} \geq 0$. However, from (16), we obtain

$$
\begin{align*}
\operatorname{Re} \tilde{\lambda} & =\left(\mu_{1}+\gamma+\varepsilon+\frac{b}{\omega}\right)\left(\mathscr{R}_{0} e^{-\operatorname{Re} \tilde{\lambda} \tau} \cos (\operatorname{Im} \tilde{\lambda} \tau)-1\right) \\
& \leq\left(\mu_{1}+\gamma+\varepsilon+\frac{b}{\omega}\right)\left(\mathscr{R}_{0}-1\right)<0 \tag{17}
\end{align*}
$$

which is a contradiction. Hence, if $\mathscr{R}_{0}<1$, the disease-free equilibrium $E_{1}$ of system (8) is locally asymptotically stable.

Now, we put

$$
\begin{equation*}
P(\lambda)=\lambda+\left(\mu_{1}+\gamma+\varepsilon+\frac{b}{\omega}\right)-e^{-\lambda \tau}=0 \tag{18}
\end{equation*}
$$

For the case $\mathscr{R}_{0}>1$, we have $P(0)=\mu_{1}+\gamma+\varepsilon+b / \omega-1<0$ and $\lim _{\lambda \rightarrow+\infty} P(\lambda)=+\infty$; then $P(\lambda)=0$ has at least one positive root. Hence, $E_{1}$ is unstable if and only if $\mathscr{R}_{0}>1$. The proof is complete.

## 4. The Stability Analysis of the Endemic Equilibrium Point

Theorem 4. If $\tau=0, \mathscr{R}_{0}>1$, and $r-G^{\prime}\left(I^{*}\right)>$ $r T G\left(I^{*}\right)$, then the positive equilibrium $E^{*}$ of system (8) is locally asymptotically stable.

Proof. The characteristic equation of (13) at $E^{*}$ becomes as follows:

$$
\begin{align*}
(\lambda+ & \left.\frac{1}{T}\right)\left(\lambda+r S^{*}\right)\left[\lambda+\mu_{1}+\gamma+\varepsilon+\frac{b \omega}{\left(\omega+I^{*}\right)^{2}}\right] \\
& -\left[\left(\lambda+\frac{1}{T}\right)\left(\lambda+r S^{*}\right)-\frac{1}{T} G\left(I^{*}\right)\right] Z^{*} G^{\prime}\left(I^{*}\right) e^{-\lambda \tau}=0 . \tag{19}
\end{align*}
$$

The above equation can be rewritten as

$$
\begin{equation*}
P(\lambda)+Q(\lambda) e^{-\lambda \tau}=0 \tag{20}
\end{equation*}
$$

where $P(\lambda)=\lambda^{3}+b_{1} \lambda^{2}+b_{2} \lambda+b_{3}, Q(\lambda)=b_{4} \lambda^{2}+b_{5} \lambda+b_{6}$, and

$$
\begin{align*}
& b_{1}= \frac{1}{T}+r S^{*}+\mu_{1}+\gamma+\varepsilon+\frac{b \omega}{\left(\omega+I^{*}\right)^{2}}, \\
& b_{2}= \frac{1}{T} r S^{*}+\frac{1}{T}\left(\mu_{1}+\gamma+\varepsilon+\frac{b \omega}{\left(\omega+I^{*}\right)^{2}}\right) \\
&+r S^{*}\left(\mu_{1}+\gamma+\varepsilon+\frac{b \omega}{\left(\omega+I^{*}\right)^{2}}\right), \\
& b_{3}= \frac{1}{T} r S^{*}\left(\mu_{1}+\gamma+\varepsilon+\frac{b \omega}{\left(\omega+I^{*}\right)^{2}}\right),  \tag{21}\\
& b_{4}=-S^{*} G^{\prime}\left(I^{*}\right), \\
& b_{5}=-S^{*} G^{\prime}\left(I^{*}\right)\left[\frac{1}{T}+r S^{*}\right], \\
& b_{6}=-\frac{1}{T} S^{*} G^{\prime}\left(I^{*}\right)\left(r S^{*}-G\left(I^{*}\right)\right) .
\end{align*}
$$

Let $C=\mu_{1}+\gamma+\varepsilon+b \omega /\left(\omega+I^{*}\right)^{2}-S^{*} G^{\prime}\left(I^{*}\right)$.
Then if $\tau=0,(20)$ becomes $P(\lambda)+Q(\lambda)=0$; that is,

$$
\begin{equation*}
\lambda^{3}+a_{1} \lambda^{2}+a_{2} \lambda+a_{3}=0 \tag{22}
\end{equation*}
$$

where

$$
\begin{align*}
& a_{1}=b_{1}+b_{4}=\frac{1}{T}+r S^{*}+C \\
& a_{2}=b_{2}+b_{5}=\frac{1}{T}\left(r S^{*}+C\right)+r S^{*} C  \tag{23}\\
& a_{3}=b_{3}+b_{6}=\frac{1}{T}\left[r S^{*} C+S^{*} G\left(I^{*}\right) G^{\prime}\left(I^{*}\right)\right] .
\end{align*}
$$

Let $r-G^{\prime}\left(I^{*}\right)>r T G\left(I^{*}\right)$; we have $a_{1}>0, a_{3}>0$, and

$$
\begin{aligned}
a_{1} a_{2}-a_{3}= & \left(\frac{1}{T}+r S^{*}+C\right)\left[\frac{1}{T}\left(r S^{*}+C\right)+r S^{*} C\right] \\
& -\frac{1}{T}\left[r S^{*} C+S^{*} G\left(I^{*}\right) G^{\prime}\left(I^{*}\right)\right]
\end{aligned}
$$

$$
\begin{align*}
= & \frac{1}{T}\left(r S^{*}+C\right)\left(\frac{1}{T}+r S^{*}+C\right) \\
& +\left(r S^{*}+C\right) r S^{*} C-\frac{1}{T} S^{*} G\left(I^{*}\right) G^{\prime}\left(I^{*}\right) \\
> & \frac{1}{T}\left(r S^{*}+C\right)\left(\frac{1}{T}+r S^{*}+C\right) \\
& +\left(r S^{*}+C\right) r S^{*} C-\frac{1}{T} r S^{*} G\left(I^{*}\right) \\
= & \frac{1}{T^{2}}\left(r S^{*}+C\right) \\
& +\frac{1}{T}\left(r S^{*}+C\right)\left(r S^{*}+C+r T S^{*} C\right) \\
& -\frac{1}{T} r S^{*} G\left(I^{*}\right) \\
= & \frac{1}{T^{2}}\left(\mu_{1}+\gamma+\varepsilon+\frac{b \omega}{\left(\omega+I^{*}\right)^{2}}\right) \\
& +\frac{1}{T}\left(r S^{*}+C\right)\left(r S^{*}+C+r T S^{*} C\right) \\
& +\frac{1}{T^{2}} S^{*}\left(r-G^{\prime}\left(I^{*}\right)\right)-\frac{1}{T} r S^{*} G\left(I^{*}\right)>0 . \tag{24}
\end{align*}
$$

By using the Routh-Hurwitz theorem, $\lambda$ has negative real part for $\tau=0$. So the positive equilibrium $E^{*}$ is locally asymptotically stable.

In the following, we investigate the existence of purely imaginary roots $\lambda=i \omega(\omega>0)$ to (19). Equation (19) takes the form of a third-degree exponential polynomial in $\lambda$, with all the coefficients of $P$ and $Q$ depending on $\tau$. Beretta and Kuang [11] established a geometrical criterion which gives the existence of purely imaginary root of a characteristic equation with delay dependent coefficients.

Now we let $\lambda=i \omega(\omega>0)$ be a root of (20) from which we have that

$$
\begin{align*}
P(\lambda)+ & Q(\lambda) e^{-\lambda \tau} \\
= & -i \omega^{3}-b_{1} \omega^{2}+b_{2} \omega i+b_{3} \\
& +\left(-b_{4} \omega^{2}+b_{5} \omega i+b_{6}\right)(\cos (\omega \tau)-i \sin (\omega \tau)) \\
= & -b_{1} \omega^{2}+b_{3}  \tag{25}\\
& -\left[b_{4} \omega^{2} \cos (\omega \tau)-b_{5} \omega \sin (\omega \tau)-b_{6} \cos (\omega \tau)\right] \\
& +i\left[-\omega^{3}+b_{2} \omega+b_{4} \omega^{2} \sin (\omega \tau)\right. \\
& \left.\quad+b_{5} \omega \cos (\omega \tau)-b_{6} \sin (\omega \tau)\right]=0 .
\end{align*}
$$

Hence, we have that

$$
\begin{aligned}
-b_{1} \omega^{2}+b_{3} & =b_{4} \omega^{2} \cos (\omega \tau)-b_{5} \omega \sin (\omega \tau)-b_{6} \cos (\omega \tau) \\
& =\left(b_{4} \omega^{2}-b_{6}\right) \cos (\omega \tau)-b_{5} \omega \sin (\omega \tau),
\end{aligned}
$$

$$
\begin{align*}
\omega^{3}-b_{2} \omega & =b_{4} \omega^{2} \sin (\omega \tau)+b_{5} \omega \cos (\omega \tau)-b_{6} \sin (\omega \tau) \\
& =\left(b_{4} \omega^{2}-b_{6}\right) \sin \omega \tau+b_{5} \omega \cos (\omega \tau) \tag{26}
\end{align*}
$$

From (26), it follows that

$$
\begin{align*}
& \cos (\omega \tau)=\frac{b_{5} \omega\left(\omega^{3}-b_{2} \omega\right)+\left(b_{3}-b_{1} \omega^{2}\right)\left(b_{4} \omega^{2}-b_{6}\right)}{\left(b_{4} \omega^{2}-b_{6}\right)^{2}+\left(b_{5} \omega\right)^{2}},  \tag{27a}\\
& \sin (\omega \tau)=\frac{\left(\omega^{3}-b_{2} \omega\right)\left(b_{4} \omega^{2}-b_{6}\right)-b_{5} \omega\left(b_{3}-b_{1} \omega^{2}\right)}{\left(b_{4} \omega^{2}-b_{6}\right)^{2}+\left(b_{5} \omega\right)^{2}} \tag{27b}
\end{align*}
$$

By the definitions of $P(\lambda), Q(\lambda)$ is as in (20), and applying the property (1), (27a) and (27b) can be written as

$$
\begin{equation*}
\sin (\omega \tau)=\operatorname{Im} \frac{P(i \omega)}{Q(i \omega)}, \quad \cos (\omega \tau)=-\operatorname{Re} \frac{P(i \omega)}{Q(i \omega)}, \tag{28}
\end{equation*}
$$

which yields $|P(i \omega)|^{2}=|Q(i \omega)|^{2}$.
Assume that $D \in R_{0^{+}}$is the set where $\omega \tau$ is a positive root of

$$
\begin{equation*}
F(\omega)=|P(i \omega)|^{2}-|Q(i \omega)|^{2} \tag{29}
\end{equation*}
$$

From
$|P(i \omega)|^{2}$

$$
\begin{align*}
& =\left|-i \omega^{3}-b_{1} \omega^{2}+b_{2} \omega i+b_{3}\right|^{2} \\
& =\left|\left(b_{3}-b_{1} \omega^{2}\right)+\left(b_{2} \omega-\omega^{3}\right)\right|^{2} \\
& =\omega^{6}+\left(b_{1}^{2}-2 b_{2}\right) \omega^{4}+\left(b_{2}^{2}-2 b_{1} b_{3}\right) \omega^{2}+b_{3}^{2} \tag{30}
\end{align*}
$$

$|Q(i \omega)|^{2}$

$$
\begin{aligned}
& =\left|\left(-b_{4} \omega^{2}+b_{5} \omega i+b_{6}\right)(\cos (\omega \tau)-\sin (\omega \tau))\right|^{2} \\
& =\left(b_{6}-b_{4} \omega^{2}\right)^{2}+\left(b_{5} \omega\right)^{2} \\
& =b_{4}^{2} \omega^{4}+\left(b_{5}^{2}-2 b_{4} b_{6}\right) \omega^{2}+b_{6}^{2},
\end{aligned}
$$

we have $F(\omega)=\omega^{6}+a_{1} \omega^{4}+a_{2} \omega^{2}+a_{3}$, where $a_{1}=b_{1}^{2}-2 b_{2}-b_{4}^{2}$, $a_{2}=b_{2}^{2}-2 b_{1} b_{3}-b_{5}^{2}+2 b_{4} b_{6}, a_{3}=b_{3}^{2}-b_{6}^{2}$, and, for $\tau \in D, \omega \tau$ is not defined. Then, for all $\tau$ in $D, \omega \tau$ satisfied $F(\omega)=0$.

Let $\omega^{2}=h$; then we have that

$$
\begin{equation*}
F(h)=h^{3}+a_{1} h^{2}+a_{2} h+a_{3}=0 . \tag{31}
\end{equation*}
$$

Assume that $F(\omega)$ has only one positive real root; we denote by $h^{+}$this positive real root. Thus, (29) has only one positive real root $\omega=\sqrt{h^{+}}$. And the critical values of $\tau$ and $\omega(\tau)$ are impossible to solve explicitly, so we will use the procedure described in Beretta and Kuang [11] and Song et al. [12]. According to this procedure, we define $\theta(\tau) \in$ $[0,2 \pi)$ such that $\sin \theta(\tau)$ and $\cos \theta(\tau)$ are given by the righthand sides of (27a) and (27b), respectively, with $\theta(\tau)$ given by (19).

And the relation between the argument $\theta$ and $\omega(\tau)$ in (28) for $\tau>0$ must be

$$
\begin{equation*}
\omega(\tau)=\theta+2 n \pi, \quad n=0,1,2, \ldots \tag{32}
\end{equation*}
$$

Hence we can define the maps $\tau_{n}: D \rightarrow R_{+0}$ given by

$$
\begin{equation*}
\tau_{n}=\frac{\theta(\tau)+2 n \pi}{\omega(\tau)}, \quad \tau_{n}>0, n=1,2, \ldots \tag{33}
\end{equation*}
$$

where a positive root $\omega(\tau)$ of (31) exists in $D$. Let us introduce the functions

$$
\begin{array}{r}
S_{n}(\tau): D \longrightarrow R, \quad S_{n}(\tau)=\tau-\frac{\theta(\tau)+2 n \pi}{\omega(\tau)},  \tag{34}\\
n=0,1,2, \ldots
\end{array}
$$

which are continuous and differentiable in $\tau$. Thus, we give the following theorem which is due to Beretta and Kuang [11].

Theorem 5. Assume that $\omega(\tau)$ is a positive root of (19) defined for $\tau \in D, D \subseteq R_{+0}$, and, at some $\tau^{*} \in D, S_{n}\left(\tau^{*}\right)=0$ for some $n \in N_{0}$. Then a pair of simple conjugate pure imaginary roots $\lambda= \pm i \omega$ exists at $\tau=\tau^{*}$ which crosses the imaginary axis from left to right if

$$
\begin{equation*}
\delta\left(\tau^{*}\right)=\operatorname{sign}\left\{F_{\omega}^{\prime}\left(\omega \tau^{*}, \tau^{*}\right)\right\} \operatorname{sign}\left\{\left.\frac{d S_{n}(\tau)}{d \tau}\right|_{\tau=\tau^{*}}\right\} . \tag{35}
\end{equation*}
$$

Applying Theorems 2 and 3 and the Hopf bifurcation theorem for functional differential equation [13], we can conjecture the existence of a Hopf bifurcation as stated in Theorem 6.

Theorem 6 (a conjecture). For system (2), there exists $\tau^{*} \in D$, such that the equilibrium $E^{*}$ is asymptotically stable for $0 \leq$ $\tau<\tau^{*}$, and it becomes unstable for $\tau$ staying in some tight neighborhood of $\tau^{*}$, with a Hopf bifurcation occurring when $\tau=\tau^{*}$.

## 5. Stability and Direction of Hopf Bifurcations

In this section, we will study the direction of the Hopf bifurcation and stability of bifurcating periodic solutions by using the normal theory and center manifold theorem due to Hassard et al. [14]. Letting $u_{1}=S-S^{*}, u_{2}=I-I^{*}, u_{3}=Z-Z^{*}$, $\tilde{u}_{i}(t)=u_{i}(\tau t)(i=1,2,3), \tau=v+\tau^{*}$, and dropping the bars for simplification of notations, system (8) becomes functional differential equations in $C=C\left([-1,0], \mathbb{R}^{3}\right)$ as

$$
\begin{aligned}
& u_{1}^{\prime}(t)=\left(\tau^{*}+\nu\right) \\
& \times {\left[r\left(1-2 S^{*}\right) u_{1}(t)-G\left(I^{*}\right) u_{1}(t)\right.} \\
&\left.-S^{*} G^{\prime}\left(I^{*}\right) u_{2}(t-1)-G^{\prime}\left(I^{*}\right) u_{1}(t) u_{2}(t-1)\right] \\
& u_{2}^{\prime}(t)=\left(\tau^{*}+v\right) \\
& \times {\left[G^{\prime}\left(I^{*}\right) u_{3}(t) u_{2}(t-1)+G\left(I^{*}\right) u_{3}(t)\right.}
\end{aligned}
$$



FIgure 1: (a)-(d) showed that the equilibrium $E_{1}$ of system (8) with initial condition $S(0)=3 ; I(0)=1 ; Z(0)=1 ; R_{0}=0.3745<1 ;$ and $T=4$ is locally asymptotically stable.

$$
\begin{align*}
& +Z^{*} G^{\prime}\left(I^{*}\right) u_{2}(t-1)-\left(\mu_{1}+\gamma+\varepsilon\right) u_{2}(t) \\
& \left.-\frac{b \omega}{\left(\omega+I^{*}\right)^{2}} u_{2}(t)+\frac{b \omega}{\left(\omega+I^{*}\right)^{3}} u_{2}^{2}(t)-\cdots\right], \\
& u_{3}^{\prime}(t)=\frac{1}{T}\left(\tau^{*}+v\right)\left(u_{1}(t)-u_{3}(t)\right) . \tag{36}
\end{align*}
$$

Then system (36) is equivalent to

$$
\begin{equation*}
u^{\prime}(t)=L_{v} u(t)+f(v, u(t)), \tag{37}
\end{equation*}
$$

where $u(t)=\left(u_{1}(t), u_{2}(t), u_{3}(t)\right)^{T} \in \mathbb{R}^{3}$ and $L_{v}: C \rightarrow \mathbb{R}^{3}$, $f: \mathbb{R} \times C \rightarrow \mathbb{R}^{3}$ are given, respectively, by

$$
\begin{aligned}
& L_{\nu}(\phi) \\
& =\left(\tau^{*}+\nu\right)
\end{aligned}
$$

$$
\times\left(\begin{array}{ccc}
r\left(1-2 S^{*}\right)-G\left(I^{*}\right) & 0 & 0 \\
0 & -\left(\mu_{1}+\gamma+\varepsilon\right)-\frac{b \omega}{\left(\omega+I^{*}\right)^{2}} & G\left(I^{*}\right) \\
\frac{1}{T} & 0 & -\frac{1}{T}
\end{array}\right)
$$



Figure 2: (a)-(d) showed that equilibrium $E^{*}$ of system (8) with initial condition $S(0)=2 ; I(0)=2 ; Z(0)=4.5 ; R_{0}=3.0303>1$; and $\tau=1.56<\tau^{*}$ is locally asymptotically stable; that is, the trajectory converges to the positive equilibrium at $\tau=1.56$.

$$
\times\left(\begin{array}{l}
\phi_{1}(0)  \tag{38}\\
\phi_{2}(0) \\
\phi_{3}(0)
\end{array}\right)+\left(\tau^{*}+\nu\right)\left(\begin{array}{ccc}
0 & -S^{*} G^{\prime}\left(I^{*}\right) & 0 \\
0 & S^{*} G^{\prime}\left(I^{*}\right) & 0 \\
0 & 0 & 0
\end{array}\right)\left(\begin{array}{c}
\phi_{1}(-1) \\
\phi_{2}(-1) \\
\phi_{3}(-1)
\end{array}\right),
$$

$$
\begin{align*}
& f(v, \phi) \\
&=\left(\tau^{*}+\nu\right) \\
& \times\left\{\left(\begin{array}{c}
-G^{\prime}\left(I^{*}\right) \phi_{1}(0) \phi_{2}(-1) \\
G^{\prime}\left(I^{*}\right) \phi_{3}(0) \phi_{2}(-1) \\
0
\end{array}\right)+\binom{\frac{b \omega}{\left(\omega+I^{*}\right)^{3}} \phi_{2}^{2}(0)}{0}+\cdots\right\} . \tag{39}
\end{align*}
$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \nu)$ of bounded variation for $\theta \in[-1,0]$, such that

$$
\begin{equation*}
L_{\nu}(\phi)=\int_{-1}^{0} d \eta(\theta, \nu) \phi(\theta), \quad \text { for } \theta \in C \tag{40}
\end{equation*}
$$

In fact, we can choose

$$
\begin{align*}
& \eta(\theta, \nu) \\
& =\left(\tau^{*}+\nu\right) \\
& \\
& \times\left(\begin{array}{ccc}
r\left(1-2 S^{*}\right)-G\left(I^{*}\right) & 0 & 0 \\
0 & -\left(\mu_{1}+\gamma+\varepsilon\right)-\frac{b \omega}{\left(\omega+I^{*}\right)^{2}} & G\left(I^{*}\right) \\
\frac{1}{T} & 0 & -\frac{1}{T}
\end{array}\right)  \tag{41}\\
& \quad \times \delta(\theta)-\left(\tau^{*}+\nu\right)\left(\begin{array}{ccc}
0 & -S^{*} G^{\prime}\left(I^{*}\right) & 0 \\
0 & S^{*} G^{\prime}\left(I^{*}\right) & 0 \\
0 & 0 & 0
\end{array}\right) \delta(\theta+1),
\end{align*}
$$

where $\delta$ denote the Dirac delta function. For $\phi \in C([-1,0]$, $\mathbb{R}^{3}$ ), define

$$
\begin{aligned}
A(v) \phi & = \begin{cases}\frac{d \phi(\theta)}{d \theta}, & \theta \in[-1,0) \\
\int_{-1}^{0} d \eta(\theta, \nu) \phi(\theta), & \theta=0\end{cases} \\
R(\nu)(\phi) & = \begin{cases}0, & \theta \in[-1,0) \\
f(\nu, \phi), & \theta=0\end{cases}
\end{aligned}
$$

Then system (37) is equivalent to

$$
\begin{equation*}
\dot{u}(t)=A(v) u_{t}+R(v) u_{t} \tag{43}
\end{equation*}
$$

where $u_{t}=u(t+\theta)$ for $\theta \in[-1,0]$.
For $\psi \in C\left([0,1],\left(\mathbb{R}^{3}\right)^{*}\right)$, define

$$
A^{*} \psi(s)= \begin{cases}-\frac{d \psi(s)}{d s}, & s \in(0,1]  \tag{44}\\ \int_{-1}^{0} \psi(-t) d \eta(t, 0), & s=0\end{cases}
$$

and a bilinear inner product

$$
\begin{align*}
& \langle\psi(s), \phi(\theta)\rangle \\
& =\bar{\psi}(0) \phi(0)-\int_{-1}^{0} \int_{\sigma=0}^{\theta} \bar{\psi}(\sigma-\theta) d \eta(\theta) \phi(\sigma) d \sigma \tag{45}
\end{align*}
$$

where $\eta(\theta)=\eta(\theta, 0)$. Then $A(0)$ and $A^{*}$ are adjoint operators. By the discussion in (20), we know that $\pm i \omega^{*} \tau^{*}$ are eigenvalues of $A(0)$. Hence, they are also eigenvalues of $A^{*}$. We first need to compute the eigenvectors of $A(0)$ and $A^{*}$ corresponding to $i \omega^{*} \tau^{*}$ and $-i \omega^{*} \tau^{*}$, respectively.

Suppose that $q(\theta)=\left(1, q_{1}, q_{2}\right)^{T} e^{i \omega^{*} \tau^{*} \theta}$ is the eigenvectors of $A(0)$ corresponding to $i \omega^{*} \tau^{*}$; then $A(0) q(\theta)=i \omega^{*} \tau^{*} q(\theta)$. Then, from the definition of $A(0)$ and (38), (40), (41), and $q(-1)=q(0) e^{-i \omega^{*} \tau^{*}}$, we have

$$
\begin{align*}
& \left(\begin{array}{ccc}
r\left(1-2 S^{*}\right)-G\left(I^{*}\right) & -S^{*} G^{\prime}\left(I^{*}\right) & 0 \\
0 & S^{*} G^{\prime}\left(I^{*}\right)-\left(\mu_{1}+\gamma+\varepsilon\right)-\frac{b \omega}{\left(\omega+I^{*}\right)^{2}} & G\left(I^{*}\right) \\
\frac{1}{T} & 0 & -\frac{1}{T}
\end{array}\right) \\
& \times\left(\begin{array}{c}
1 \\
q_{1}(0) \\
q_{2}(0)
\end{array}\right)=i \omega^{*}\left(\begin{array}{c}
1 \\
q_{1}(0) \\
q_{2}(0)
\end{array}\right) . \tag{46}
\end{align*}
$$

We obtain

$$
\begin{equation*}
q_{1}=\frac{r-2 r S^{*}-G\left(I^{*}\right)-i \omega^{*}}{S^{*} G^{\prime}\left(I^{*}\right)}, \quad q_{2}=\frac{1}{1+T i \omega^{*}} \tag{47}
\end{equation*}
$$

Similarly, we can obtain the eigenvector $q^{*}(s)=$ $D\left(1, q_{1}^{*}, q_{2}^{*}\right)^{T} e^{i \omega^{*} \tau^{*}}$ of $A^{*}$ corresponding to $-i \omega^{*} \tau^{*}$, where

$$
\begin{gather*}
q_{1}^{*}=\frac{S^{*} G^{\prime}\left(I^{*}\right)}{S^{*} G^{\prime}\left(I^{*}\right)-\left(\mu_{1}+\gamma+\varepsilon\right)-b \omega /\left(\omega+I^{*}\right)^{2}+i \omega^{*}}  \tag{48}\\
q_{2}^{*}=T\left[-i \omega-r\left(1-2 S^{*}\right)+G\left(I^{*}\right)\right]
\end{gather*}
$$

In order to assure that $\left\langle q^{*}(s), q(\theta)\right\rangle=1,\left\langle q^{*}(s), \bar{q}(\theta)\right\rangle=0$, we need to determine the value of $D$. By (45), we have

$$
\begin{align*}
& \left\langle q^{*}(s), q(\theta)\right\rangle \\
& =\bar{D}\left(1, \bar{q}_{1}^{*}, \bar{q}_{2}^{*}\right)\left(1, q_{1}, q_{2}\right)^{T} \\
& -\int_{-1}^{0} \int_{\sigma=0}^{\theta} \bar{D}\left(1, \bar{q}_{1}^{*}, \bar{q}_{2}^{*}\right) e^{-i \omega^{*} \tau^{*}(\sigma-\theta)} d \eta \\
& \quad \times(\theta)\left(1, q_{1}, q_{2}\right)^{T} e^{i \omega^{*} \tau^{*} \sigma} d \sigma \\
& =\bar{D}\left\{1+q_{1} \bar{q}_{1}^{*}+q_{2} \bar{q}_{2}^{*}\right. \\
& \left.\quad-\int_{-1}^{0}\left(1, \bar{q}_{1}^{*}, \bar{q}_{2}^{*}\right) \theta e^{i \omega^{*} \tau^{*} \theta} d \eta(\theta)\left(1, q_{1}, q_{2}\right)^{T}\right\} \\
& =\bar{D}\left\{1+q_{1} \bar{q}_{1}^{*}+q_{2} \bar{q}_{2}^{*}+\tau^{*} q_{1} S^{*} G^{\prime}\left(I^{*}\right)\left(-1+\bar{q}_{1}^{*}\right) e^{-i \omega^{*} \tau^{*}}\right\} \tag{49}
\end{align*}
$$

Therefore, we can choose $D$ as

$$
\begin{equation*}
D=\frac{1}{1+\bar{q}_{1} q_{1}^{*}+\bar{q}_{2} q_{2}^{*}+\tau^{*} \bar{q}_{1} S^{*} G^{\prime}\left(I^{*}\right)\left(-1+q_{1}^{*}\right) e^{-i \omega^{*} \tau^{*}}} \tag{50}
\end{equation*}
$$

We use the way of [14] and similarly way of [3]; we obtain that the coefficients are

$$
\begin{align*}
g_{20}= & 2 \tau^{*} \bar{D}\left(\bar{q}_{1}^{*}-1\right) G^{\prime}\left(I^{*}\right) q_{1}+2 \tau^{*} \bar{D} \bar{q}_{1}^{*} \frac{b \omega}{\left(\omega+I^{*}\right)^{3}} q_{1}^{2}, \\
g_{11}= & 2 \tau^{*} \bar{D}\left(\bar{q}_{1}^{*}-1\right) G^{\prime}\left(I^{*}\right) \operatorname{Re}\left\{q_{1}\right\} \\
& +2 \tau^{*} \bar{D} \bar{q}_{1}^{*} \frac{b \omega}{\left(\omega+I^{*}\right)^{3}}\left|q_{1}\right|^{2} ; \\
g_{02}= & 2 \tau^{*} \bar{D}\left(\bar{q}_{1}^{*}-1\right) G^{\prime}\left(I^{*}\right) \bar{q}_{1}+2 \tau^{*} \bar{D} \bar{q}_{1}^{*} \frac{b \omega}{\left(\omega+I^{*}\right)^{3}} \bar{q}_{1}^{2} ; \\
g_{21}= & \tau^{*} \bar{D}\left(\bar{q}_{1}^{*}-1\right) G^{\prime}\left(I^{*}\right) \\
+ & {\left[W_{20}^{(1)}(0) \bar{q}_{1}+2 q_{1} W_{11}^{(1)}(0)+W_{20}^{(2)}(0)+2 W_{11}^{(2)}(0)\right] } \\
+ & 2 \tau^{*} \bar{D} \bar{q}_{1}^{*} \frac{b \omega}{\left(\omega+I^{*}\right)^{3}}\left[\bar{q}_{1} W_{20}^{(2)}(0)+2 q_{1} W_{11}^{(2)}(0)\right] \tag{51}
\end{align*}
$$

where

$$
\begin{aligned}
W_{20}(\theta)= & \frac{i g_{20}}{\omega^{*} \tau^{*}} q(0) e^{i \omega^{*} \tau^{*} \theta} \\
& +\frac{i \bar{g}_{02}}{3 \omega^{*} \tau^{*}} \bar{q}(0) e^{-i \omega^{*} \tau^{*} \theta} \\
& +\left[W_{20}(0)+\frac{g_{20}}{i \omega^{*} \tau^{*}} q(0)+\frac{\bar{g}_{02}}{3 i \omega^{*} \tau^{*}} \bar{q}(0)\right] e^{2 i \omega^{*} \tau^{*} \theta}
\end{aligned}
$$



FIgure 3: (a)-(d) showed that equilibrium $E^{*}$ of system (8) with initial condition $S(0)=2 ; I(0)=2 ; Z(0)=4.5 ; R_{0}=3.0303>1$; and $\tau=2.56>\tau^{*}$ is unstable, that is, a periodic behavior at $\tau=2.56$.

$$
\begin{align*}
\triangleq & \frac{i g_{20}}{\omega^{*} \tau^{*}} q(0) e^{i \omega^{*} \tau^{*} \theta}+\frac{i \bar{g}_{02}}{3 \omega^{*} \tau^{*}} \bar{q}(0) e^{-i \omega^{*} \tau^{*} \theta} \\
& +E_{1} e^{2 i \omega^{*} \tau^{*} \theta} \tag{52}
\end{align*}
$$

$$
W_{11}(\theta)=\frac{i g_{11}}{\omega^{*} \tau^{*}} \bar{q}(0) e^{-i \omega^{*} \tau^{*} \theta}+E_{2}
$$

Besides, $E_{1}, E_{2}$ are satisfied with the following equation:

$$
\left(\begin{array}{ccc}
2 i \omega^{*}-r\left(1-2 S^{*}\right)+G\left(I^{*}\right) & S^{*} G^{\prime}\left(I^{*}\right) & 0 \\
0 & 2 i \omega^{*}-S^{*} G^{\prime}\left(I^{*}\right)+\left(\mu_{1}+\gamma+\varepsilon\right)+\frac{b \omega}{\left(\omega+I^{*}\right)^{2}} & -G\left(I^{*}\right) \\
-\frac{1}{T} & 0 & 2 i \omega^{*}+\frac{1}{T}
\end{array}\right) E_{1}
$$

$$
\begin{gather*}
\left(\begin{array}{ccc}
-r\left(1-2 S^{*}\right)+G\left(I^{*}\right) & S^{*} G^{\prime}\left(I^{*}\right) & 0 \\
0 & -S^{*} G^{\prime}\left(I^{*}\right)+\left(\mu_{1}+\gamma+\varepsilon\right)+\frac{b \omega}{\left(\omega+I^{*}\right)^{2}} & -G\left(I^{*}\right) \\
-\frac{1}{T} & 0 & \frac{1}{T}
\end{array}\right) E_{2} \\
=2 \tau^{*} G^{\prime}\left(I^{*}\right) q_{1}(-1,1,0)^{T}+2 \tau^{*} \frac{b \omega}{\left(\omega+I^{*}\right)^{3}} q_{1}^{2}(0,1,0)^{T} \tag{53}
\end{gather*}
$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (52). Furthermore, we can compute $g_{21}$ by (51). Thus we can compute the following values:

$$
\begin{gather*}
c_{1}(0)=\frac{i}{2 \omega^{*} \tau^{*}}\left(g_{20} g_{11}-2\left|g_{11}\right|^{2}-\frac{\left|g_{02}\right|}{3}\right)+\frac{g_{21}}{2} \\
v_{2}=-\frac{\operatorname{Re}\left\{c_{1}(0)\right\}}{\operatorname{Re}\left\{d \lambda\left(\tau^{*}\right) / d \tau\right\}}  \tag{54}\\
\beta_{2}=2 \operatorname{Re}\left\{c_{1}(0)\right\} \\
T_{2}=-\frac{\operatorname{Re}\left\{c_{1}(0)\right\}+v_{2} \operatorname{Re}\left\{d \lambda\left(\tau^{*}\right) / d \tau\right\}}{\omega^{*} \tau^{*}}
\end{gather*}
$$

By the result of Hassard et al. [14], we have the following.
Theorem 7. In (54), the following results hold:
(i) the sign of $\nu_{2}$ determines the directions of the Hopf bifurcation: if $\nu_{2}>0\left(v_{2}<0\right)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau>\tau^{*}\left(\tau<\tau^{*}\right)$;
(ii) the sign of $\beta_{2}$ determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_{2}<0\left(\beta_{2}>0\right)$;
(iii) the sign of $T_{2}$ determines the period of the bifurcating periodic solutions: the period is increasing (decreasing) if $\beta_{2}>0\left(\beta_{2}<0\right)$.

## 6. Numerical Simulations

To demonstrate the theoretical results obtained from this paper, letting $G(I(t-\tau))=I(t-\tau) /(1+\alpha I(t-\tau))$, we will give some numerical simulations. We consider the hypothetical set of parameter values as follows.
(1) Consider $\mu_{1}=0.1 ; r=3 ; b=1 ; \gamma=0.05 ; T=4$; $\varepsilon=0.02 ; \alpha=0.2 ; \omega=0.4$. By directly computing, we obtain $R_{0}=0.1786<1$. According to Theorem 4, we know that the disease-free equilibrium of system (8) is locally asymptotically stable for this case (see Figures $1(\mathrm{a})-1(\mathrm{~d})$ ).
(2) Consider $\mu_{1}=0.01 ; r=3 ; b=0.1 ; \gamma=0.05 ; T=10$; $\varepsilon=0.02 ; \alpha=0.2 ; \omega=0.4 ; \tau=1.56$. By directly computing, we obtain $R_{0}=3.0303>1$. According to Theorem 4, we know that the disease-free equilibrium
of system (8) is locally asymptotically stable for this case (see Figures 2(a)-2(d)).
(3) Consider $\mu_{1}=0.01 ; r=3 ; b=0.1 ; \gamma=0.05$; $T=10 ; \varepsilon=0.02 ; \alpha=0.2 ; \omega=0.4, \tau=2.56$. By directly computing, we obtain $R_{0}=3.0303>1$. According to Theorem 6, we know that the diseasefree equilibrium of system (8) is unstable for this case (see Figures 3(a)-3(d)).

## 7. Conclusion

In this paper, we formulate and analyze a new delayed epidemic model with information variable and limited medical resources, the conditions for Hopf bifurcation to occur are derived. By analyzing the model, we have found the existence of disease-free equilibria $E_{0}$ and $E_{1}$ and have a unique positive equilibrium $E^{*}$. The basic reproduction number $\mathscr{R}_{0}$ changes the stability of the disease-free equilibrium. When $\mathscr{R}_{0}<$ 1, we discuss the stability of the disease-free equilibrium by analyzing the corresponding characteristic equations and constructing a Lyapunov functional, respectively. The conclusion reveals that the disease dies out and when $\mathscr{R}_{0}>1$, we also get the sufficient criteria of stability switch at the positive equilibrium. Using the time delay (i.e., incubation time) as a bifurcation parameter, the local stability of the endemic equilibrium is investigated, and the conditions for Hopf bifurcation to occur are derived. Using the normal form theory and the center manifold theorem introduced by Hassard et al., we have studied the direction and stability of the bifurcating periodic solutions. Our theoretical results show that the time delay $\tau$ must be responsible for the observed regular cycles of disease incidence.

Lastly, a numerical simulation provided that when $\mathscr{R}_{0}$ is less than 1 , the disease-free equilibrium is stable and while $\mathscr{R}_{0}$ is more than 1 , the disease-free equilibrium is unstable; that is, the endemic equilibrium exists (see Figure 1). Further, for $\tau>$ 0 , there will exist $\tau^{*} \in I$, such that the endemic equilibrium is asymptotically stable for $0<\tau<\tau^{*}$ (see Figure 2) and becomes unstable for $\tau$ staying in some right neighborhood of $\tau^{*}$, with a Hopf bifurcation occurring when $\tau=\tau^{*}$. If $\tau>\tau^{*}$, the endemic equilibrium is unstable (see Figure 3).

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The authors would like to thank the anonymous referees for their careful reading of the original paper and their many valuable comments and suggestions that greatly improved the presentation of this work. This work is supported by the Natural Science Foundation of Shanxi province (20130110022).

## References

[1] J. W. Jia and Q. Y. Li, "Qualitative analysis of an SIR epidemic model with stage structure," Applied Mathematics and Computation, vol. 193, no. 1, pp. 106-115, 2007.
[2] J.-Q. Li, Z.-E. Ma, and J. Zhang, "Global analysis of some epidemic models with general contact rate and constant immigration," Applied Mathematics and Mechanics, vol. 25, no. 4, pp. 396-404, 2004.
[3] J. J. Wang, J. Z. Zhang, and Z. Jin, "Analysis of an SIR model with bilinear incidence rate," Nonlinear Analysis: Real World Applications, vol. 11, no. 4, pp. 2390-2402, 2010.
[4] G. Huang and Y. Takeuchi, "Global analysis on delay epidemiological dynamic models with nonlinear incidence," Journal of Mathematical Biology, vol. 63, no. 1, pp. 125-139, 2011.
[5] A. Korobeinikov, "Global properties of infectious disease models with nonlinear incidence," Bulletin of Mathematical Biology, vol. 69, no. 6, pp. 1871-1886, 2007.
[6] R. Xu and Z. E. Ma, "Global stability of a SIR epidemic model with nonlinear incidence rate and time delay," Nonlinear Analysis: Real World Applications, vol. 10, no. 5, pp. 3175-3189, 2009.
[7] Y. Enatsu, E. Messina, Y. Muroya, Y. Nakata, E. Russo, and A. Vecchio, "Stability analysis of delayed SIR epidemic models with a class of nonlinear incidence rates," Applied Mathematics and Computation, vol. 218, no. 9, pp. 5327-5336, 2012.
[8] A. d'Onofrio, P. Manfredi, and P. Manfredi, "Bifurcation thresholds in an SIR model with information-dependent vaccination," Mathematical Modelling of Natural Phenomena, vol. 2, no. 1, pp. 26-43, 2007.
[9] A. d'Onofrio, P. Manfredi, and E. Salinelli, "Vaccinating behaviour, information, and the dynamics of SIR vaccine preventable diseases," Theoretical Population Biology, vol. 71, no. 3, pp. 301-317, 2007.
[10] B. Buonomo, A. d'Onofrio, and D. Lacitignola, "Global stability of an SIR epidemic model with information dependent vaccination," Mathematical Biosciences, vol. 216, no. 1, pp. 9-16, 2008.
[11] E. Beretta and Y. Kuang, "Geometric stability switch criteria in delay differential systems with delay dependent parameters," SIAM Journal on Mathematical Analysis, vol. 33, no. 5, pp. 11441165, 2002.
[12] X. Y. Song, S. L. Wang, and J. Dong, "Stability properties and Hopf bifurcation of a delayed viral infection model with lytic immune response," Journal of Mathematical Analysis and Applications, vol. 373, no. 2, pp. 345-355, 2011.
[13] J. Hale and S. M. V. Lunel, Introduction to the Theory of Functional Differential Equations Methods and Applications, Spring, 1993.
[14] B. Hassard, N. Kazarinoff, and Y. Wan, Theory and Applications of Hopf Bifurcation, Cambridge University Press, Cambridge, UK, 1981.

## Research Article

# The Space-Jump Model of the Movement of Tumor Cells and Healthy Cells 

Meng-Rong Li, ${ }^{1}$ Yu-Ju Lin, ${ }^{1}$ and Tzong-Hann Shieh ${ }^{2}$<br>${ }^{1}$ Department of Mathematical Sciences, National Chengchi University, No. 64, Section 2, ZhipNan Road, Wenshan District, Taipei 11605, Taiwan<br>${ }^{2}$ Department of Aerospace and Systems Engineering, Feng Chia University, No. 100 Wenhwa Road, Seatwen, Taichung 40724, Taiwan<br>Correspondence should be addressed to Tzong-Hann Shieh; thshieh@fcu.edu.tw

Received 24 January 2014; Accepted 23 March 2014; Published 9 April 2014
Academic Editor: Malay Banerjee
Copyright © 2014 Meng-Rong Li et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We establish the interaction model of two cell populations following the concept of the random-walk, and assume the cell movement is constrained by space limitation primarily. Furthermore, we analyze the model to obtain the behavior of two cell populations as time is closed to initial state and far into the future.


## 1. Introduction

In the 1980 s , the movement of isolated single cells was researched and was modelled by a range of authors (Oster [1]; Oster and Perelson [2]; Bottino and Fauci [3]; and Bottino, et al. [4]). In mathematics and biomedicine, not only of one-cell population but of multiple cell populations, there are many researches on the movement.

A consequential early paper written by Keller and Segel [5] modelled a partial differential equation to study the biochemical regulation of bacteria movement; their research has been the basis for models of the movement of diversified cell populations, such as slime mould aggregation (Höfer et al. [6]), tumor angiogenesis (Chaplain and Stuart [7]), primitive streak formation (Painter et al. [8]), and wound repair (Pettet et al. [9]).

In the recent years, most of the researches on cell movement focused on the interaction of multiple cell populations, precise cell behavior, and the development of the mathematics modelling. In this study we follow the contour of two-cell interaction developed by Painter and Sherratt [10]. The modelling of interaction of tumor- and healthycell populations was developed with the concept of randomwalk (space-jump). Assuming the movement is according to space limitation and the diffusion coefficients of two cell populations are the same, we develop a system of partial
differential equations (PDEs). Through some calculations, the system of PDEs is simplified to a system of ordinary differential equations (o.d.es.). Analyzing the system of o.d.es., it is obtained that the number of two cell populations per unit area in a unit amount of time is finite no matter when; namely, the density of each cell population does not blow up.

To model the motion of biological organisms, there are three major concepts which would be used:
(a) the space-jump process in which the individual jumps between sites on a lattice,
(b) the velocity-jump process in which discontinuous changes in the speed or direction of an individual are generated by a Poisson process,
(c) the flux motion in which the movement of cells are treated as the flux motion.

In this work we adopt space-jump concept to establish our model and from it we show how a PDE of cell movement could be deduced. Then we use the same concept and expand the PDE which has been deduced to reason a system of PDEs describing the interaction of two cell population.

## 2. Movement of One-Cell Population

We will deduce an equation of cell movement on a lattice from the space-jump concept; moreover, we translate that equation
into a PDE of cell movement through changing variables. First, we list the functions and variables that will be used in this content and call the considering cell population by $u$-cell as follows:
$u\left(x_{i}, t\right) \equiv u_{i}$ number of $u$-cell at site $x_{i}$ at time $t$ per unit area in a unit amount of time (the density of $u$ cell at site $x_{i}$ at time $t$ ),
$E\left(x_{i}, t\right) \equiv E_{i}$ the information of $u$-cell at site $x_{i}$ at time $t$,
$g\left(E_{i+1}\right)$ the probability of $u$-cell moving from $x_{i}$ to $x_{i+1}$ (to right),
$g\left(E_{i-1}\right)$ the probability of $u$-cell moving from $x_{i}$ to $x_{i-1}$ (to left).

Moreover, the meaning of $g\left(E_{i+1}\right)$ is that the probability of the cell moving to the target would be influenced by the information of the cell's jumping target.

For example, we choose that the cell density on position $x_{i+1}$ at time $t$ is the information of cells on $x_{i+1}$ at $t$; then the probability of cells moving from $x_{i}$ to $x_{i+1}$ would be influenced by $E_{i+1}$, which is the density of cell population on position $x_{i+1}$ at time $t$. Reasonably, a decreasing function $g\left(E_{i+1}\right)$ with respect to $E_{i+1}$ implies that a lower probability results from the more crowded target.

Supposing that cells move continuously in time on a lattice (discrete space), a PDE of $u$-cell movement would be modelled.

In the lattice space, the $u$-cells' movement at time $t$ can be modelled as

$$
\begin{align*}
\frac{\partial u_{i}}{\partial t}= & g\left(E_{i}\right)\left(D_{u}\left(x_{i-1}, t\right) u\left(x_{i-1}, t\right)+D_{u}\left(x_{i+1}, t\right) u\left(x_{i+1}, t\right)\right) \\
& -D_{u}\left(x_{i}, t\right) u\left(x_{i}, t\right)\left(g\left(E_{i-1}\right)+g\left(E_{i+1}\right)\right) \tag{1}
\end{align*}
$$

We explain our idea as shown in Figure 1.
Figure 1 shows the movement of cells; the function on the figure is the moving probability. The changing of the $u$ cell density at site $x_{i}$ at time $t$ is equal to that of the $u$-cell number jumping from site $x_{i-1}$ and site $x_{i+1}$ minus the $u$ cell number jumping to site $x_{i-1}$ and site $x_{i+1} . \partial u_{i} / \partial t$ means the changing of $u$-cell density at site $x_{i}$ and time $t$. The function $g\left(E_{i}\right) D_{u}\left(x_{i-1}, t\right) u\left(x_{i-1}, t\right)+g\left(E_{i}\right) D_{u}\left(x_{i+1}, t\right) u\left(x_{i+1}, t\right)$ is the increase of $u$-cell density at site $x_{i}$ at time $t$ with cells moving from site $x_{i-1}$ and site $x_{i+1}$ to site $x_{i}$, where $D_{u}\left(x_{i}, t\right)$ is the jumping (diffusion) coefficient of $u$-cell at site $x_{i}$ at time $t$. And $-D_{u}\left(x_{i}, t\right) u\left(x_{i}, t\right)\left(g\left(E_{i-1}\right)+g\left(E_{i+1}\right)\right)$ is the decrease of $u$-cell density at site $x_{i}$ at time $t$ with cells moving to site $x_{i-1}$ and site $x_{i+1}$ from site $x_{i}$. Thus, (1) is obtained.

The model of $u$-cell movement in continuous space can be deduce from (1) in a lattice space through changing variables. Let $x_{i+k}=x+k h, k \in \mathbb{Z} . x_{i}=x, x_{i+1}=x+h, x_{i-1}=x-h$, and $E_{i}=E(x, t)$; hence, (1) becomes

$$
\begin{aligned}
\frac{\partial u(x, t)}{\partial t}=g(E(x, t)) & \left(D_{u}(x-h, t) u(x-h, t)\right. \\
& \left.+D_{u}(x+h, t) u(x+h, t)\right)
\end{aligned}
$$



Figure 1: The movement of cells.

$$
\begin{align*}
& -(g(E(x-h, t))+g(E(x+h, t)) \\
& \left.\quad \times D_{u}(x, t) u(x, t)\right) . \tag{2}
\end{align*}
$$

For a continuum flow we consider that the jumping coefficient $D_{u}(x, t)=D_{u}$ is a constant. Denote $u(x-h, t)$ and $u(x+h, t)$ by Taylor's series

$$
\begin{align*}
u(x-h, t)= & u(x, t)+\frac{\partial u}{\partial x}(x-h-x) \\
& +\frac{1}{2!} \frac{\partial^{2} u}{\partial x^{2}}(x-h-x)^{2}+\cdots  \tag{3}\\
u(x+h, t)= & u(x, t)+\frac{\partial u}{\partial x}(x+h-x) \\
& +\frac{1}{2!} \frac{\partial^{2} u}{\partial x^{2}}(x+h-x)^{2}+\cdots
\end{align*}
$$

In consequence, $u(x-h, t)+u(x+h, t)=2 u(x, t)+$ $\left(\partial^{2} u / \partial x^{2}\right) h^{2}+O\left(h^{4}\right)$; similarly,

$$
\begin{align*}
g(E(x-h, t))+g(E(x+h, t))= & 2 g(E(x, t)) \\
& +\frac{\partial^{2} g}{\partial x^{2}} h^{2}+O\left(h^{4}\right) . \tag{4}
\end{align*}
$$

Consequently,

$$
\begin{align*}
\frac{\partial u}{\partial t}(x, t)= & g(E) D_{u} \frac{\partial^{2} u}{\partial x^{2}} h^{2}+O\left(h^{4}\right) g(E) \\
& -D_{u} u \frac{\partial^{2} g}{\partial x^{2}} h^{2}-D_{u} u O\left(h^{4}\right) \tag{5}
\end{align*}
$$

and then we get

$$
\begin{equation*}
\frac{\partial u}{\partial t}=D_{u} \frac{\partial}{\partial x}\left(g(E) \frac{\partial u}{\partial x}-u \frac{\partial g(E)}{\partial x}\right) h^{2}+O\left(h^{4}\right) . \tag{6}
\end{equation*}
$$

Therefore, we consider (1) as the following.
The $u$-cell movement can be modelled as

$$
\begin{equation*}
\frac{\partial u(x, t)}{\partial t}=D_{u} \frac{\partial}{\partial x}\left(g(E) \frac{\partial u}{\partial x}-u \frac{\partial g(E)}{\partial x}\right), \tag{7}
\end{equation*}
$$

where $D_{u}$ is a diffusion coefficient and $E(x, t) \equiv E$ is the information of $u$-cell on position $x$ at time $t$.

## 3. Interaction of Two Cell Populations

Now we show how to deduce a system of PDEs which describes the interaction of two cell populations. Here the two considered cell populations are called by $u$-cell and $v$-cell. What the variables and functions $(E(x, t)$ and $g(E))$ mean is as above; moreover, denote the density of $u$-cell and $v$ cell populations on position $x$ at time $t$ by $u(x, t)$ and $v(x, t)$, respectively. On the other hand, we write $w(x, t):=u(x, t)+$ $v(x, t)$ to describe the total cell density. There is also another vague function, $g(E)$, which needs to be defined clearly.

Given that space limitation influences the movement of cells, the probability of cells moving to position $x$ decreases with how the position is crowded with cells. We choose $w(x, t)$, the total cell density, to express the information of cells on position $x$, namely, $E(x, t)=w(x, t)$. Hence $g(E)=$ $g(w)=1-(w / T)$ shows that the probability of cells moving to position $x$ decreases with the total cell density on position $x$, where $T \gg w$ initially and $T$ is a constant. Here the assumption on $g(E)$ follows the paper written by Painter and Sherratt (2003) [10].

After defining those variables, the model of interaction of two cell populations ( $u$-cell and $v$-cell) can be deduced. According to (7), replacing $g(E)$ by $1-(w(x, t) / T) \equiv 1-$ $(w / T)$, then

$$
\begin{align*}
\frac{\partial u}{\partial t} & =D_{u} \frac{\partial}{\partial x}\left(\left(1-\frac{w}{T}\right) \frac{\partial u}{\partial x}+u \frac{\partial}{\partial x}\left(1-\frac{w}{T}\right)\right) \\
& =D_{u} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}-\frac{w}{T} \frac{\partial u}{\partial x}+\frac{u}{T} \frac{\partial w}{\partial x}\right) \\
& =D_{u} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}-\frac{v}{T} \frac{\partial u}{\partial x}+\frac{u}{T} \frac{\partial v}{\partial x}\right)  \tag{8}\\
& =D_{u}\left(\frac{\partial^{2} u}{\partial x^{2}}-\frac{v}{T} \frac{\partial^{2} u}{\partial x^{2}}+\frac{u}{T} \frac{\partial^{2} v}{\partial x^{2}}\right),
\end{align*}
$$

where $D_{u}$ is a constant. Similarly, the same processes are applied to $v$. We obtain the following equation:

$$
\begin{equation*}
\frac{\partial v}{\partial t}=D_{v}\left(\frac{\partial^{2} v}{\partial x^{2}}-\frac{u}{T} \frac{\partial^{2} v}{\partial x^{2}}+\frac{v}{T} \frac{\partial^{2} u}{\partial x^{2}}\right) \tag{9}
\end{equation*}
$$

Consequently, we get the interaction of two cell populations.
Following space limitation, the interaction of two cell populations can be modelled as

$$
\begin{align*}
& \frac{\partial u}{\partial t}=D_{u}\left(\left(1-\frac{v}{T}\right) \frac{\partial^{2} u}{\partial x^{2}}+\frac{u}{T} \frac{\partial^{2} v}{\partial x^{2}}\right) \\
& \frac{\partial v}{\partial t}=D_{v}\left(\left(1-\frac{u}{T}\right) \frac{\partial^{2} v}{\partial x^{2}}+\frac{v}{T} \frac{\partial^{2} u}{\partial x^{2}}\right) \tag{10}
\end{align*}
$$

where $D_{u}$ and $D_{v}$ are diffusion coefficients with respect to $u$ cell and $v$-cell ( $D_{u}$ and $D_{v}$ are constants), respectively.

Furthermore, through changing variables,

$$
\begin{equation*}
\mu \equiv \mu(x, t)=\frac{u(x, t)}{T}, \quad \nu \equiv \nu(x, t)=\frac{v(x, t)}{T} \tag{11}
\end{equation*}
$$

with the consequence that

$$
\begin{equation*}
\frac{\partial u}{\partial t}=T \frac{\partial \mu}{\partial t}, \quad \frac{\partial v}{\partial t}=T \frac{\partial v}{\partial t}, \quad \frac{\partial^{2} u}{\partial x^{2}}=T \frac{\partial^{2} \mu}{\partial x^{2}}, \quad \frac{\partial^{2} v}{\partial x^{2}}=T \frac{\partial^{2} v}{\partial x^{2}} \tag{12}
\end{equation*}
$$

Rewriting system (10) as

$$
\begin{align*}
& T \frac{\partial \mu}{\partial t}=D_{\mu}\left((1-v) \frac{\partial^{2} \mu}{\partial x^{2}}+\mu \frac{\partial^{2} v}{\partial x^{2}}\right) T  \tag{13}\\
& T \frac{\partial v}{\partial t}=D_{v}\left((1-\mu) \frac{\partial^{2} v}{\partial x^{2}}+v \frac{\partial^{2} \mu}{\partial x^{2}}\right) T
\end{align*}
$$

the system of P.D.Es (10) can be simplified as

$$
\begin{align*}
& \frac{\partial \mu}{\partial t}=D_{\mu}\left((1-v) \frac{\partial^{2} \mu}{\partial x^{2}}+\mu \frac{\partial^{2} v}{\partial x^{2}}\right),  \tag{14}\\
& \frac{\partial v}{\partial t}=D_{v}\left((1-\mu) \frac{\partial^{2} v}{\partial x^{2}}+\nu \frac{\partial^{2} \mu}{\partial x^{2}}\right),
\end{align*}
$$

where $D_{\mu}$ and $D_{\nu}$ are diffusion coefficients.
Now, the interaction of $u$-cell and $v$-cell has been modelled. Model (14) will be used frequently in the following context, and some properties of two cell populations can be deduced from analyzing model (14). We show the analyzing procedures and some results in the next section.

## 4. The Behavior and the Meaning of $\nu(x, t)=\nu(z)$ as $z \rightarrow 0$

We have got the system of PDEs (14) which shows the interaction of two cell populations. In this section, model (14) will be transformed to a system of o.d.es. and then analyzed to obtain some properties of $v(x, t)=\nu(z)$ as $z$ approaches to zero and infinite; furthermore, the properties of $\mu(x, t)=$ $\mu(z)$ will be deduced from the properties of $\nu(z)$ and $\omega(z)$, where $\omega(z)$ is $\mu(z)+\nu(z)$.

Our purpose is to obtain a simpler form of (14) in order to analyze the model conveniently. Supposing that $u$ cell and $v$-cell have the same diffusion coefficient $\left(D_{\mu}\right.$ is equal to $\left.D_{\nu}\right), k$ denotes the diffusion coefficients $D_{\mu}$ and $D_{\nu}$. Through changing variables, the system of PDEs (14) could be transformed to a system of o.d.es.

Lemma 1. Given two cell populations with the same diffusion coefficient, the system of PDEs (14) can be shown as a system of o.d.es. as follows:

$$
\begin{align*}
& -\frac{1}{2} z \mu^{\prime}(z)=k\left((1-\nu) \mu^{\prime \prime}(z)+\mu \nu^{\prime \prime}(z)\right)  \tag{15}\\
& -\frac{1}{2} z \nu^{\prime}(z)=k\left((1-\mu) \nu^{\prime \prime}(z)+\nu \mu^{\prime \prime}(z)\right)
\end{align*}
$$

where $z=x / \sqrt{t}, k \equiv D_{\mu}=D_{\gamma}$.

Proof. According to the system of PDEs (14), we could obtain

$$
\begin{align*}
& \frac{\partial \mu}{\partial t}=k\left((1-v) \frac{\partial^{2} \mu}{\partial x^{2}}+\mu \frac{\partial^{2} v}{\partial x^{2}}\right),  \tag{16}\\
& \frac{\partial v}{\partial t}=k\left((1-\mu) \frac{\partial^{2} v}{\partial x^{2}}+\nu \frac{\partial^{2} \mu}{\partial x^{2}}\right) .
\end{align*}
$$

Let $\mu(z)=\mu(x / \sqrt{t}) \equiv \mu(x, t)$ and $\nu(z)=\nu(x / \sqrt{t}) \equiv$ $\nu(x, t)$, with the consequence that

$$
\begin{align*}
\frac{\partial \mu(x, t)}{\partial t} & \equiv-\frac{1}{2} x t^{-3 / 2} \mu^{\prime}\left(\frac{x}{\sqrt{t}}\right), \\
\frac{\partial \nu(x, t)}{\partial t} & \equiv-\frac{1}{2} x t^{-3 / 2} \nu^{\prime}\left(\frac{x}{\sqrt{t}}\right), \\
\frac{\partial^{2} \mu(x, t)}{\partial x^{2}} & \equiv t^{-1} \mu^{\prime \prime}\left(\frac{x}{\sqrt{t}}\right),  \tag{17}\\
\frac{\partial^{2} \nu(x, t)}{\partial x^{2}} & \equiv t^{-1} v^{\prime \prime}\left(\frac{x}{\sqrt{t}}\right) .
\end{align*}
$$

The system of PDEs (16) can be written as model (15).
In that case, the simpler form (model (15)) will be analyzed in the following subsections in order to obtain some properties of $\nu(z)$.

Before deducing that $v(x, t)=\nu(z)$ is bounded for $z$ in $[0, \delta]$ ( $\delta$ is very small), we must know the behavior of total cells.

Lemma 2. The movement of total cells (u-cell and v-cell) can be modelled as a classical diffusion equation $\omega^{\prime \prime}(z)+$ $(z / 2 k) \omega^{\prime}(z)=0$.

Proof. Adding the two equations in the system (15), we obtain

$$
\begin{equation*}
\mu^{\prime \prime}(z)+\frac{z}{2 k} \mu^{\prime}(z)+\nu^{\prime \prime}(z)+\frac{z}{2 k} \nu^{\prime}(z)=0 \tag{18}
\end{equation*}
$$

Imposing $\omega(z)$ upon (18), equation (18) could be rewritten as follows:

$$
\begin{equation*}
\omega^{\prime \prime}(z)+\frac{z}{2 k} \omega^{\prime}(z)=0 \tag{19}
\end{equation*}
$$

In consequence,

$$
\begin{equation*}
\omega(z)=\omega\left(z_{0}\right)+\omega^{\prime}\left(z_{0}\right) \int_{z_{0}}^{z} e^{-r^{2} / 4 k} d r \tag{20}
\end{equation*}
$$

where $z_{0}=x_{0} / \sqrt{t_{0}}$, for some site $x_{0}$ at initial time $t_{0}$.
According to above assumptions, $\omega(x, t) \equiv \omega(z)=\mu(z)+$ $\nu(z)$ and $\mu(z)=u(z) / T$ and $\nu(z)=v(z) / T, \omega(z)$ can be restored to $(u(z) / T)+(v(z) / T)$, where $T$ is a constant. In that case, equation (20) can be transformed into the form

$$
\begin{equation*}
\left(\frac{u+v}{T}\right)(z)=\left(\frac{u+v}{T}\right)\left(z_{0}\right)+\left(\frac{u+v}{T}\right)^{\prime}(z) \int_{z_{0}}^{z} e^{-r^{2} / 4 k} d r \tag{21}
\end{equation*}
$$

and then written as

$$
\begin{equation*}
(u+v)(z)=(u+v)\left(z_{0}\right)+(u+v)^{\prime}\left(z_{0}\right) \int_{z_{0}}^{z} e^{-r^{2} / 4 k} d r \tag{22}
\end{equation*}
$$

where $z$ is $x / \sqrt{t}$ and $k$ is a constant. The last equation shows the behavior of total cells; moreover, that is the classical representation of the solution of the fundamental diffusion equation.

After describing the behavior of total cells, following (15), we replace $\mu$ by $\omega-\nu$ in the equation

$$
\begin{equation*}
-\frac{1}{2} z \mu^{\prime}(z)=k\left((1-\nu) \mu^{\prime \prime}(z)+\mu \nu^{\prime \prime}(z)\right) . \tag{23}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
-\frac{1}{2} z(\omega-\nu)^{\prime}(z)=k\left((1-v)(\omega-\nu)^{\prime \prime}(z)+(\omega-v) \nu^{\prime \prime}(z)\right) \tag{24}
\end{equation*}
$$

Given that $-(z / 2 k) \omega^{\prime}=\omega^{\prime \prime}$, the equation (24) is simplified as

$$
\begin{equation*}
(\omega(z)-1) \nu^{\prime \prime}(z)-\frac{z}{2 k} v^{\prime}(z)-\omega^{\prime \prime}(z) v(z)=0 \tag{25}
\end{equation*}
$$

where $\omega(z)$ is as (20), with the consequence that

$$
\begin{equation*}
v^{\prime \prime}(z)+\frac{(-z)}{2 k(\omega(z)-1)} v^{\prime}(z)+\frac{\left(-\omega^{\prime \prime}(z)\right)}{\omega(z)-1} \nu(z)=0 \tag{26}
\end{equation*}
$$

Lemma 3. Equation (26) can be transformed to

$$
\begin{equation*}
\bar{v}^{\prime \prime}(z)+a(z) \bar{\nu}(z)=0, \tag{27}
\end{equation*}
$$

where

$$
\begin{equation*}
a(z)=\frac{1+2 \omega^{\prime}\left(z_{0}\right) z e^{-z^{2} / 4 k}}{4 k(\omega(z)-1)}-\frac{4 k \omega^{\prime}\left(z_{0}\right) z e^{-z^{2} / 4 k}+z^{2}}{16 k^{2}(\omega(z)-1)^{2}} . \tag{28}
\end{equation*}
$$

Proof. Assuming that $\nu(z)=\bar{\nu}(z) \exp \left((1 / 2) \int^{z} p(r) d r\right)$, equation (26) is transformed as follows:

$$
\begin{equation*}
\bar{\nu}^{\prime \prime}(z)+\left(q(z)-\frac{1}{2} p^{\prime}(z)-\frac{1}{4} p^{2}(z)\right) \bar{\nu}(z)=0 \tag{29}
\end{equation*}
$$

where $p(z)=-z / 2 k(\omega(z)-1)$ and $q(z)=-\omega^{\prime \prime}(z) /(\omega(z)-1)$. Hence we denote $a(z)$ as $q(z)-(1 / 2) p^{\prime}(z)-(1 / 4) p^{2}(z)$.

Therefore,

$$
\begin{align*}
a(z)= & \frac{-\omega^{\prime \prime}(z)}{\omega(z)-1}-\frac{1}{2} \frac{-2 k(\omega(z)-1)+z 2 k \omega^{\prime}(z)}{4 k^{2}(\omega(z)-1)^{2}} \\
& -\frac{1}{4} \frac{(-z)^{2}}{4 k^{2}(\omega(z)-1)^{2}} \\
= & \frac{\omega^{\prime}\left(z_{0}\right) z e^{-z^{2} / 4 k}}{2 k(\omega(z)-1)}  \tag{30}\\
& +\frac{4 k(\omega(z)-1)-4 k \omega^{\prime}\left(z_{0}\right) z e^{-z^{2} / 4 k}-z^{2}}{16 k^{2}(\omega(z)-1)^{2}} .
\end{align*}
$$

Hence, $\bar{\nu}^{\prime \prime}(z)+a(z) \bar{\nu}(z)=0$, where $v(z)=$ $\bar{\nu}(z) e^{(1 / 2) \int^{z} p(r) d r}$.

In order to simplify the representation of the following equations, we let

$$
\begin{align*}
& a_{1}(z)=\frac{\omega^{\prime}\left(z_{0}\right) z e^{-z^{2} / 4 k}}{2 k(\omega(z)-1)} \\
& a_{2}(z)=\frac{1}{4 k(\omega(z)-1)}  \tag{31}\\
& a_{3}(z)=-\frac{\omega^{\prime}\left(z_{0}\right) z e^{-z^{2} / 4 k}}{4 k(\omega(z)-1)^{2}} \\
& a_{4}(z)=-\frac{z^{2}}{16 k^{2}(\omega(z)-1)^{2}} \tag{32}
\end{align*}
$$

The following theorem would show that $\bar{\nu}(z)$ and $\nu(z)$ are bounded on $[0, \delta]$ for some small $\delta$.

Before we make the following theorem complete, the substantiation of the next lemma must be finished.

Theorem 4. The solution of $\bar{\nu}^{\prime \prime}(z)+\left(-M_{0}^{2}+b(z)\right) \bar{\nu}(z)=0$ is bounded where $M_{0}$ is a constant and $b(z)$ is closed to zero as $z \ll 1$ if the solution of $\bar{\nu}^{\prime \prime}(z)+\left(-M_{0}^{2}\right) \bar{\nu}(z)=0$ is bounded as $z \ll 1$.

Proof. Assume $z \ll 1$; the solution of $\bar{\nu}^{\prime \prime}(z)+\left(-M_{0}^{2}\right) \bar{\nu}(z)=0$ is given by

$$
\begin{equation*}
\bar{\nu}(z)=c_{1} e^{M_{0} z}+c_{2} e^{-M_{0} z} \tag{33}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are constants.
We say that $\bar{\nu}_{1}(z)$ is the solution of $\bar{\nu}^{\prime \prime}(z)+\left(-M_{0}^{2}\right) \bar{\nu}(z)=0$ and $\bar{\nu}_{2}(z)$ is the solution of $\bar{\nu}^{\prime \prime}(z)+\left(-M_{0}^{2}+b(z)\right) \bar{\nu}(z)=0$. Then we have

$$
\begin{align*}
\left|\bar{v}_{1}\right| & =\left|c_{1} e^{M_{0} z}+c_{2} e^{-M_{0} z}\right| \\
& \leq\left|c_{1}\right| e^{M_{0} z}+\left|c_{2}\right| e^{-M_{0} z}  \tag{34}\\
& \leq\left|c_{1}\right| e^{M_{0} \delta}+\left|c_{2}\right|, \quad \forall z \in[0, \delta], \delta<1
\end{align*}
$$

Let $\bar{\nu}_{21}(z)=\bar{\nu}_{2}(z), \bar{\nu}_{22}(z)=\bar{\nu}_{2}^{\prime}(z)$, and

$$
\begin{array}{ll}
\bar{V}(z)=\left[\begin{array}{l}
\bar{\nu}_{21}(z) \\
\bar{\nu}_{22}(z)
\end{array}\right], \quad A=\left[\begin{array}{cc}
0 & 1 \\
M_{0}^{2} & 0
\end{array}\right], \\
B(z)=\left[\begin{array}{cc}
0 & 0 \\
-b(z) & 0
\end{array}\right] . \tag{35}
\end{array}
$$

The equation $\bar{\nu}^{\prime \prime}(z)+\left(-M_{0}^{2}+b(z)\right) \bar{\nu}(z)=0$ can be written as

$$
\begin{equation*}
\frac{d}{d z} \bar{V}(z)=A \bar{V}(z)+B(z) \bar{V} \tag{36}
\end{equation*}
$$

Let $\Phi(z)$ be a fundamental solution matrix of $\Phi^{\prime}(z)=$ $A \Phi(z)$. Then

$$
\begin{aligned}
\bar{V}= & \Phi(z) \Phi^{-1}\left(z_{0}\right) \bar{V}\left(z_{0}\right) \\
& +\Phi(z) \int_{z_{0}}^{z} \Phi^{-1}(r) B(r) \bar{V}(r) d r
\end{aligned}
$$

$$
\begin{align*}
\|\bar{V}\| \leq & \left\|\Phi(z) \Phi^{-1}\left(z_{0}\right) \bar{V}\left(z_{0}\right)\right\| \\
& +\int_{z_{0}}^{z}\left\|\Phi\left(z-r+z_{0}\right) \Phi^{-1}\left(z_{0}\right) B(r) \bar{V}(r)\right\| d r \\
\leq & M_{1} M_{2}+\int_{z_{0}}^{z} M_{1}\|B(r)\|\|\bar{V}\| d r, \tag{37}
\end{align*}
$$

where $\|\cdot\|$ is the super norm and $M_{1}=\left\|\Phi(z) \Phi^{-1}\left(z_{0}\right)\right\|, M_{2}=$ $\left\|\bar{V}\left(z_{0}\right)\right\|$.

By Granwall's inequality and $\int_{z_{0}}^{z} M_{1}\|B(r)\| d r \leq$ $M_{1}\|B(z)\| \delta$ for all $z$ in $[0, \delta]$, then

$$
\begin{align*}
\|\bar{V}\| & \leq M_{1} M_{2} \exp \left(\int_{z_{0}}^{z} M_{1}\|B(r)\| d r\right)  \tag{38}\\
& \leq M_{1} M_{2} \exp \left(M_{1}\|B(z)\| \delta\right)<\infty
\end{align*}
$$

for all $z$ in $[0, \delta]$.
Hence, the solution of $\bar{\nu}^{\prime \prime}(z)+\left(-M_{0}^{2}+b(z)\right) \bar{\nu}(z)=0$ is bounded as $z \ll 1$.

Theorem 5. $\bar{\nu}(z)$ is bounded on $[0, \delta]$ for some small $\delta$; moreover, $\nu(z)$ is bounded on $[0, \delta]$.

Proof. Supposing that $\omega(z)=\omega\left(z_{0}\right)+\omega^{\prime}\left(z_{0}\right) \int_{z_{0}}^{z} e^{-r^{2} / 4 k} d r$ is closed to $\omega\left(z_{0}\right)$ as $z \rightarrow 0^{+}$and $\omega\left(z_{0}\right)<1$, then $\omega(z)-1<0$ when $z \rightarrow 0^{+}$.

According to the above assumptions, we have

$$
\begin{align*}
& a_{1}(z)=\frac{\omega^{\prime}(z) z e^{-z^{2} / 4 k}}{2 k(\omega(z))-1} \sim 0 \quad \text { as } z \sim 0, \\
& a_{2}(z)=\frac{1}{4 k(\omega(z)-1)} \sim-M_{0}^{2} \quad \text { as } z \sim 0, \\
& a_{3}(z)=\frac{-\omega^{\prime}\left(z_{0}\right) z e^{-z^{2} / 4 k}}{4 k(\omega(z)-1)^{2}} \sim 0 \quad \text { as } z \sim 0,  \tag{39}\\
& a_{4}(z)=\frac{-z^{2}}{16 k^{2}(\omega(z)-1)^{2}} \sim 0 \quad \text { as } z \sim 0
\end{align*}
$$

For $z \ll 1, a_{1}(z)+a_{3}(z)+a_{4}(z)=b(z), a(z)=-M_{0}^{2}+b(z)$ can be estimated immediately.

Thus the equation $\bar{\nu}^{\prime \prime}(z)+a(z) \bar{v}(z)=0$ can be written as

$$
\begin{equation*}
\bar{\nu}^{\prime \prime}(z)+\left(-M_{0}^{2}+b(z)\right) \bar{\nu}(z)=0 \tag{40}
\end{equation*}
$$

for all $z \ll 1$.
Because the solution of $\bar{\nu}^{\prime \prime}(z)+\left(-M_{0}^{2}\right) \bar{\nu}(z)=0$ is bounded as $z \rightarrow 0$, the solution of $\bar{\nu}^{\prime \prime}(z)+\left(-M_{0}^{2}+b(z)\right) \bar{\nu}(z)=0$ is also bounded as $z \rightarrow 0$. Consequently, $\bar{\nu}(z)$, the solution of $\bar{\nu}^{\prime \prime}(z)+\left(-M_{0}^{2}+b(z)\right) \bar{\nu}(z)=0$ for all $z \ll 1$, is bounded on $[0, \delta]$ for some small $\delta$, saying that $|\bar{\nu}(z)| \leq M$ and $M$ is a constant.

Hence,

$$
\begin{align*}
\nu(z) & =\bar{\nu}(z) \exp \left(\frac{-1}{2} \int^{z} p(r) d r\right) \\
& \leq M \exp \left(\frac{-1}{2} \int^{z} p(r) d r\right) \tag{41}
\end{align*}
$$

where $p(z)=(-z) /(2 k(\omega(z)-1))>0$ for some $k>0$; moreover, since $p(z)>0, e^{((-1) / 2) \int^{z} p(r) d r} \leq 1$ for all $z$ in $[0, \delta]$ and for some $k>0$. In consequence, $v(z)$ is bounded by $M \exp \left((-1 / 2) \int^{z} p(r) d r\right)$ where $M$ is a constant and $p(z)=$ $-z / 2 k(\omega(z)-1)$ on $[0, \delta]$ for some $k>0$.

It is verified that $\nu(z)$ is bounded by $M \exp \left((-1 / 2) \int^{z} p(r) d r\right)$ where $M$ is a constant and $p(z)=-z / 2 k(\omega(z)-1)$ on $[0, \delta]$, where $z$ is $x / \sqrt{t}$ and $\delta$ is very small. Furthermore, we restore $\nu(z)$ to $v(x / \sqrt{t}) / T$, where $T$ is a positive constant. $z \rightarrow 0$ expresses that time $t$ approximates infinite. Therefore, Theorem 5 indicates that the density of $v$-cell population approximates finite number as time approaches infinite. Through writing $u(x / \sqrt{t})$ as $w(x / \sqrt{t})-v(x / \sqrt{t})$, it could be deduced immediately that the density of $u$-cell population is finite no matter how long time passes.

## 5. The Behavior and the Meaning of <br> $\nu(x, t)=\nu(z)$ as $z \rightarrow \infty$

Near $z=0$ (namely, $x / \sqrt{t}$ approaches zero), the boundedness of $v(z)$ has been shown. Hence, we obtain that the density of $u$-cell and $v$-cell populations would not blow up when time approached infinity. In this section, through justifying that $\bar{\nu}(z)$ is bounded by $e^{z^{2} / 8 k \delta}$ first, we will show that $\nu(z)$ is also bounded when $z$ approaches $\infty$.

Theorem 6. The solution of $\bar{\nu}^{\prime \prime}(z)+a(z) \bar{\nu}(z)=0$, got by Lemma 3, is bounded by $e^{z^{2} / 8 k \delta}$ as $z$ approaches $\infty$, where $\delta>0$.

Proof. Supposing $\omega(z)=\omega\left(z_{0}\right)+\omega^{\prime}\left(z_{0}\right) \int^{z} \exp \left(-r^{2} / 4 k\right) d r$ approaches $1^{-}$, there is a $\delta>0$ such that $\omega-1$ approaches $-\delta$ as $z \rightarrow \infty$. As $z$ tends to infinity, $a(z)$ could be rewritten as the following asymptotic form:

$$
\begin{align*}
a(z) & =\frac{2 \omega^{\prime}\left(z_{0}\right) z e^{-z^{2} / 4 k}+1}{4 k(\omega(z)-1)}-\frac{4 k \omega^{\prime}\left(z_{0}\right) z e^{-z^{2} / 4 k}+z^{2}}{16 k^{2}(\omega(z)-1)^{2}}  \tag{42}\\
& \sim \frac{-1}{4 k \delta}-\left(\frac{z}{4 k \delta}\right)^{2}, \quad \text { as } z \longrightarrow \infty
\end{align*}
$$

Consider

$$
\begin{equation*}
\bar{\nu}^{\prime \prime}(z)+\left(\frac{-1}{4 k \delta}-\left(\frac{z}{4 k \delta}\right)^{2}\right) \bar{\nu}(z)=0 \tag{43}
\end{equation*}
$$

and let $\bar{\nu}_{1}(z)=e^{f(z)}$ be a solution of (43). Immediately,

$$
\begin{equation*}
f^{\prime \prime}(z)+\left(f^{\prime}(z)\right)^{2}=\frac{1}{4 k \delta}+\left(\frac{z}{4 k \delta}\right)^{2} \tag{44}
\end{equation*}
$$

is obtained. Assume $f(z)=b_{0} z^{2}+b_{1} z+b_{2}$, where $b_{0}, b_{1}$, and $b_{2}$ are constants; then

$$
\begin{equation*}
4 b_{0}^{2} z^{2}+4 b_{0} b_{1} z+b_{1}^{2}+2 b_{0}=\frac{1}{4 k \delta}+\left(\frac{1}{4 k \delta}\right)^{2} z^{2} \tag{45}
\end{equation*}
$$

Consequently, $b_{0}=1 / 8 k \delta$ and $b_{1}=0$; then $f(z)=$ $\left(z^{2} / 8 k \delta\right)+b_{2}$. Hence, we get $\bar{\nu}_{1}(z)=b e^{z^{2} / 8 k \delta}$, where $b \in \mathbb{R}$.

Now let $\bar{v}_{2}$ be another solution of (43). Assume that $\bar{v}_{2}=$ $g(z) e^{z^{2} / 8 k \delta}, g^{\prime \prime}(z)+(z / 2 k \delta) g^{\prime}(z)=0$, with the consequence that $g(z)=g\left(z_{0}\right)+g^{\prime}\left(z_{0}\right) \int_{z_{0}}^{z} e^{\left(-r^{2}\right) / 4 k \delta} d r$. We get

$$
\begin{equation*}
\bar{\nu}_{2}(z)=g\left(z_{0}\right) e^{z^{2} / 8 k \delta}+g^{\prime}\left(z_{0}\right) \int_{z_{0}}^{z} e^{\left(\left(z^{2} / 8 k \delta\right)+\left(-r^{2} / 4 k \delta\right)\right)} d r \tag{46}
\end{equation*}
$$

Moreover, $\quad \int^{z} e^{\left(\left(z^{2} / 8 k \delta\right)+\left(-r^{2} / 4 k \delta\right)\right)} d r$ is convergent since $\left(z^{2} / 8 k \delta\right)+\left(-r^{2} / 4 k \delta\right)=\left(z^{2}-2 r^{2}\right) / 8 k \delta<0$, as $r>z / \sqrt{2}$. Therefore, the solution of $\bar{\nu}^{\prime \prime}(z)+a(z) \bar{\nu}(z)=0$ is

$$
\begin{align*}
b e^{z^{2} / 8 k \delta}+(g & \left(z_{0}\right) \exp \left(\frac{z^{2}}{8 k \delta}\right)+g^{\prime}\left(z_{0}\right) \\
& \left.\times \int_{z_{0}}^{z} \exp \left(\frac{z^{2}}{8 k \delta}+\frac{-r^{2}}{4 k \delta}\right) d r\right) \tag{47}
\end{align*}
$$

and then

$$
\begin{equation*}
\bar{\nu}(z) \leq\left(b+g\left(z_{0}\right)\right) e^{z^{2} / 8 k \delta}+M \tag{48}
\end{equation*}
$$

where $b$ is a constant and $M$ is defined as $g^{\prime}\left(z_{0}\right)\left(\int_{z_{0}}^{z} \exp \left(\left(z^{2} / 8 k \delta\right)+\left(-r^{2} / 4 k \delta\right)\right) d r\right)$.

After substantiating that $\bar{\nu}(z)$ is bounded by $e^{z^{2} / 8 k \delta}$ as $z$ approaches $\infty$, where $\delta>0$, it is not difficult to verify that $\nu(z)$ is also bounded as $z$ approaches $\infty$.

Theorem 7. $v(z)$ is bounded when $z$ approaches $\infty$.
Proof. Given $z \gg 1$, in above Theorem 6, we have transformed

$$
\begin{equation*}
v^{\prime \prime}(z)+\frac{(-z)}{2 k(\omega(z)-1)} v^{\prime}(z)+\frac{\left(-\omega^{\prime \prime}(z)\right)}{\omega(z)-1} v(z)=0 \tag{49}
\end{equation*}
$$

to $\bar{\nu}^{\prime \prime}(z)+a(z) \bar{\nu}(z)=0$ through changing $\nu(z)$ to $\bar{\nu}(z) e^{(-1 / 2) \int^{z}(-z / 2 k(\omega(z)-1)) d r}$, and

$$
\begin{align*}
& \bar{\nu}(z) \exp \left(\frac{-1}{2} \int^{z} \frac{-z}{2 k(\omega(z)-1)} d r\right) \\
& \quad \leq\left(\left(b+g\left(z_{0}\right)\right) e^{z^{2} / 8 k \delta}+M\right) \exp \left(\frac{-1}{2} \int^{z} \frac{r}{2 k \delta} d r\right) . \tag{50}
\end{align*}
$$

In consequence,

$$
\begin{align*}
v(z) & \leq\left(\left(b+g\left(z_{0}\right)\right) e^{z^{2} / 8 k \delta}+M\right) e^{-z^{2} / 8 k \delta}  \tag{51}\\
& =\left(b+g\left(z_{0}\right)\right)+M e^{-z^{2} / 8 k \delta},
\end{align*}
$$

where $b \in \mathbb{R}$ and $M \equiv g^{\prime}\left(z_{0}\right)\left(\int^{z} e^{\left(\left(z^{2} / 8 k \delta\right)+\left(-r^{2} / 4 k \delta\right)\right)} d r\right)$. Hence, $\nu(z)$ is bounded by

$$
\begin{equation*}
\left(b+g\left(z_{0}\right)\right)+M e^{-z^{2} / 8 k \delta} \tag{52}
\end{equation*}
$$

as $z \rightarrow \infty$.
Restoring $z$ to $x / \sqrt{t}$, according to Theorem 7, we know that $v(x / \sqrt{t})$ is bounded by $\left(b+g\left(z_{0}\right)\right)+M \exp \left(-x^{2} /(8 k \delta t)\right)$ as $x / \sqrt{t}$ approaches $\infty$; namely, $t$ approaches initial time. In consequence, it is obtained immediately that the density of $v$-cell population which is denoted by $\nu(x / \sqrt{t}) / T$ tends to a finite number as $v$-cell population has begun moving for a fleeting time. Furthermore, the density of $u$-cell population would also approximate a finite number for the same time.

If it is possible, we hope the solutions of (10) could be obtained by using our methods that were analytically used in [11-18] or numerically used in $[19,20]$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

Thanks are due to Professor Long-Yi Tsai, Professor Tai-Ping Liu, Professor Ton Yang, and Professor Shih-Shien Yu for their continuous encouragement and discussions over this work, to Metta Education, Grand Hall, and Auria Solarfor for their financial assistance, and to the referee for his interest and helpful comments on this paper.

## References

[1] G. F. Oster, "On the crawling of cells," Journal of Embryology and Experimental Morphology, vol. 83, pp. 329-364, 1984.
[2] G. F. Oster and A. S. Perelson, "Cell spreading and motility: a model lamellipod," Journal of Mathematical Biology, vol. 21, no. 3, pp. 383-388, 1985.
[3] D. C. Bottino and L. J. Fauci, "A computational model of ameboid deformation and locomotion," European Biophysics Journal, vol. 27, no. 5, pp. 532-539, 1998.
[4] D. Bottino, A. Mogilner, T. Roberts, M. Stewart, and G. Oster, "How nematode sperm crawl," Journal of Cell Science, vol. 115, no. 2, pp. 367-384, 2002.
[5] E. F. Keller and L. A. Segel, "Initiation of slime mold aggregation viewed as an instability," Journal of Theoretical Biology, vol. 26, no. 3, pp. 399-415, 1970.
[6] T. Höfer, J. A. Sherratt, and P. K. Maini, "Dyctyostelium discoideum: cellular self-organisation in an excitable biological medium," Proceedings of the Royal Society of London B, vol. 259, no. 1356, pp. 249-257, 1995.
[7] M. A. J. Chaplain and A. M. Stuart, "A model mechanism for the chemotactic response of endothelial cells to tumour angiogenesis factor," IMA Journal of Mathematics Applied in Medicine and Biology, vol. 10, no. 3, pp. 149-168, 1993.
[8] K. J. Painter, P. K. Maini, and H. G. Othmer, "A chemotactic model for the advance and retreat of the primitive streak in
avian development," Bulletin of Mathematical Biology, vol. 62, no. 3, pp. 501-525, 2000.
[9] G. J. Pettet, H. M. Byrne, D. L. S. Mcelwain, and J. Norbury, "A model of wound-healing angiogenesis in soft tissue," Mathematical Biosciences, vol. 136, no. 1, pp. 35-63, 1996.
[10] K. J. Painter and J. A. Sherratt, "Modelling the movement of interacting cell populations," Journal of Theoretical Biology, vol. 225, no. 3, pp. 327-339, 2003.
[11] R. Duan, M.-R. Li, and T. Yang, "Propagation of singularities in the solutions to the Boltzmann equation near equilibrium," Mathematical Models and Methods in Applied Sciences, vol. 18, no. 7, pp. 1093-1114, 2008.
[12] M.-R. Li and Y.-L. Chang, "On a particular Emden-Fowler equation with non-positive energy $u^{\prime \prime}-u^{3}=0$ : mathematical model of enterprise competitiveness and performance," Applied Mathematics Letters, vol. 20, no. 9, pp. 1011-1015, 2007.
[13] M.-R. Li, "Estimates for the life-span of the solutions for some semilinear wave equations," Communications on Pure and Applied Analysis, vol. 7, no. 2, pp. 417-432, 2008.
[14] M.-R. Li, "Blow-up solutions to the nonlinear second order differential equation $u^{\prime \prime}=u^{p}\left(c_{1}+c_{2} u^{\prime}(t)^{q}\right)$," Taiwanese Journal of Mathematics, vol. 12, no. 3, pp. 599-622, 2008.
[15] M.-R. Li and J.-T. Pai, "Quenching problem in some semilinear wave equations," Acta Mathematica Scientia, vol. 28, no. 3, pp. 523-529, 2008.
[16] M.-R. Li, "On the Emden-Fowler equation $u(t) u^{\prime \prime}(t)=c_{1}+$ $c_{2} u^{\prime}(t)^{2}$," Acta Mathematica Scientia B, vol. 30, no. 4, pp. 12271234, 2010.
[17] M.-R. Li, Y.-J. Lin, and T.-H. Shieh, "The flux model of the movement of tumor cells and healthy cells using a system of nonlinear heat equations," Journal of Computational Biology, vol. 18, no. 12, pp. 1831-1839, 2011.
[18] T.-H. Shieh, T.-M. Liou, M.-R. Li, C.-H. Liu, and W.-J. Wu, "Analysis on numerical results for stage separation with different exhaust holes," International Communications in Heat and Mass Transfer, vol. 36, no. 4, pp. 342-345, 2009.
[19] T.-H. Shieh and M.-R. Li, "Numeric treatment of contact discontinuity with multi-gases," Journal of Computational and Applied Mathematics, vol. 230, no. 2, pp. 656-673, 2009.
[20] M.-R. Li, Y.-T. Li, T.-H. Shieh, C. J. Yue, and P. Lee, "Parabola method in ordinary differential equation," Taiwanese Journal of Mathematics, vol. 15, no. 4, pp. 1841-1857, 2011.

## Research Article

# Stability of Virus Infection Models with Antibodies and Chronically Infected Cells 

Mustafa A. Obaid and A. M. Elaiw<br>Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia<br>Correspondence should be addressed to Mustafa A. Obaid; drmobaid@gmail.com

Received 9 December 2013; Revised 18 February 2014; Accepted 6 March 2014; Published 3 April 2014
Academic Editor: Malay Banerjee
Copyright © 2014 M. A. Obaid and A. M. Elaiw. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

Two virus infection models with antibody immune response and chronically infected cells are proposed and analyzed. Bilinear incidence rate is considered in the first model, while the incidence rate is given by a saturated functional response in the second one. One main feature of these models is that it includes both short-lived infected cells and chronically infected cells. The chronically infected cells produce much smaller amounts of virus than the short-lived infected cells and die at a much slower rate. Our mathematical analysis establishes that the global dynamics of the two models are determined by two threshold parameters $R_{0}$ and $R_{1}$. By constructing Lyapunov functions and using LaSalle's invariance principle, we have established the global asymptotic stability of all steady states of the models. We have proven that, the uninfected steady state is globally asymptotically stable (GAS) if $R_{0}<1$, the infected steady state without antibody immune response exists and it is GAS if $R_{1}<1<R_{0}$, and the infected steady state with antibody immune response exists and it is GAS if $R_{1}>1$. We check our theorems with numerical simulation in the end.


## 1. Introduction

In recent years, many mathematical models have been proposed to study the dynamics of viral infections such as the human immunodeficiency virus (HIV), the hepatitis C virus (HCV), and the hepatitis B virus (HBV) (see, e.g., [1-17]). Such virus infection models can be very useful in the control of epidemic diseases and provide insights into the dynamics of viral load in vivo. Therefore, mathematical analysis of the virus infection models can play a significant role in the development of a better understanding of diseases and various drug therapy strategies. Most of the mathematical models of viral infection presented in the literature did not differentiate between the short-lived infected cells and chronically infected cells. The chronically infected cells produce much smaller amounts of virus than the short-lived infected cells and die at a much slower rate [18]. The virus dynamics model with chronically infected cells and under the effect of antiviral drug therapy was introduced in [18] as

$$
\begin{aligned}
\dot{T} & =\lambda-d T-(1-\varepsilon) k T V \\
\dot{T}^{*} & =(1-\alpha)(1-\varepsilon) k T V-\delta T^{*},
\end{aligned}
$$

$$
\begin{align*}
\dot{C}^{*} & =\alpha(1-\varepsilon) k T V-a C^{*} \\
\dot{V} & =N_{T} \delta T^{*}+N_{C} a C^{*}-c V \tag{1}
\end{align*}
$$

where $T, T^{*}, C^{*}$, and $V$ are the concentration of the uninfected cells, short-lived infected cells, chronically infected cells, and free virus particles, respectively. The constant $\lambda$ is the rate at which new uninfected cells are generated and $d$ is the natural death rate constant of uninfected cells. $k$ is the infection rate constant. The fractions $(1-\alpha)$ and $\alpha$ with $0<\alpha<1$ are the probabilities that, upon infection, an uninfected cell will become either short-lived infected or chronically infected. $\delta$ and $a$ are the death rate constants of the short-lived infected cells and chronically infected cells, respectively. $N_{T}$ and $N_{C}$ are the average number of virions produced in the lifetime of the short-lived infected and chronically infected cells, respectively. The chronically infected cells produce much smaller amounts of virus than the short-lived infected cells and die at a much slower rate (i.e., $N_{T}>N_{C}$ and $\delta>a$ ). The free viruses are cleared with rate constant $c$. The drug efficacy is denoted by $\varepsilon$ and $0 \leq \varepsilon \leq 1$.

It is observed that the basic and global properties of model (1) are not studied in the literature. Moreover, model (1) did not take into consideration the immune response. During viral infections, the host immune system reacts with antigenspecific immune response. The immune system is described as having two "arms": the cellular arm, which depends on T cells to mediate attacks on virally infected or cancerous cells, and the humoral arm, which depends on B cells. The B cell is a type of blood cell which belongs to a group of white blood cells (WBCs) called lymphocytes. WBCs protect the body from infection. The main job of B cells is to fight infection. $B$ cells get activated when an infection occurs and they produce molecules called antibodies that attach to the surface of the infectious agent. These antibodies either kill the infection causing organism or make it prone to attack by other WBCs. They play a major role in the immune system, which guards the body against infection. Virus infection models with antibody immune response have been analyzed by many researchers (see [19-28]). However, in all of these works, the chronically infected cells have been neglected.

In this paper, we propose two virus infection models with antibody immune response and chronically infected cells. In the first model, bilinear incidence rate which is based on the law of mass-action is considered. The second model generalizes the first one where the incidence rate is given by a saturation functional response. The global stability of all equilibria of the models is established using the method of Lyapunov function. We prove that the global dynamics of the models are determined by two threshold parameters $R_{0}$ and $R_{1}$. If $R_{0} \leq 1$, then the infection-free equilibrium is globally asymptotically stable (GAS), if $R_{1} \leq 1<R_{0}$, then the infected equilibrium without antibody immune response exists and it is GAS, and if $R_{1}>1$ then the infected equilibrium with antibody immune response exists and it is GAS.

## 2. Model with Bilinear Incidence Rate

In this section we propose a viral dynamics model with antibody immune response, taking into consideration the chronically infected cells. Based on the mass-action principle, we assume that the incidence rate of infection is bilinear; that is, the infection rate per virus and per uninfected cell is constant:

$$
\begin{align*}
\dot{T} & =\lambda-d T-(1-\varepsilon) k T V,  \tag{2}\\
\dot{T}^{*} & =(1-\alpha)(1-\varepsilon) k T V-\delta T^{*},  \tag{3}\\
\dot{C}^{*} & =\alpha(1-\varepsilon) k T V-a C^{*},  \tag{4}\\
\dot{V} & =N_{T} \delta T^{*}+N_{C} a C^{*}-c V-r V Z,  \tag{5}\\
\dot{Z} & =g V Z-\mu Z, \tag{6}
\end{align*}
$$

where $Z$ is the concentration of antibody immune cells. The viruses are attacked by the antibodies with rate $r V Z$. The antibody immune cells are proliferated at rate $g V Z$ and die at rate $\mu \mathrm{Z}$. All the other variables and parameters of the model have the same meanings as given in (1).
2.1. Positive Invariance. We note that model (2)-(6) are biologically acceptable in the sense that no population goes negative. It is straightforward to check the positive invariance of the nonnegative orthant $\mathbb{R}_{+}^{5}$ by model (2)-(6) (see, e.g., [6]). In the following, we show the boundedness of the solution of model (2)-(6).

Proposition 1. There exist positive numbers $L_{i}, i=1,2,3$, such that the compact set

$$
\begin{align*}
\Omega=\{ & \left(T, T^{*}, C^{*}, V, Z\right) \in \mathbb{R}_{+}^{4}: 0 \leq T, T^{*}, C^{*} \leq L_{1},  \tag{7}\\
& \left.0 \leq V \leq L_{2}, 0 \leq Z \leq L_{3}\right\}
\end{align*}
$$

is positively invariant.
Proof. To show the boundedness of the solutions we let $G_{1}(t)=T(t)+T^{*}(t)+C^{*}(t)$; then

$$
\begin{align*}
\dot{G}_{1}(t)= & \lambda-d T(t)-(1-\varepsilon) k T(t) V(t) \\
& +(1-\alpha)(1-\varepsilon) k T(t) V(t)-\delta T^{*}  \tag{8}\\
& +\alpha(1-\varepsilon) k T(t) V(t)-a C^{*}(t) \\
\leq & \lambda-s_{1} G_{1}(t),
\end{align*}
$$

where $s_{1}=\min \{d, a, \delta\}$. Hence $G_{1}(t) \leq L_{1}$, if $G_{1}(0) \leq L_{1}$ where $L_{1}=\lambda / s_{1}$. Since $T(t)>0, T^{*}(t) \geq 0$, and $C^{*}(t) \geq 0$, then $0 \leq T(t), T^{*}(t), C^{*}(t) \leq L_{1}$ if $0 \leq T(0)+T^{*}(0)+C^{*}(0) \leq$ $L_{1}$. Let $G_{2}(t)=V(t)+(r / g) Z(t)$; then

$$
\begin{align*}
\dot{G}_{2}(t) & =N_{T} \delta T^{*}(t)+N_{C} a C^{*}(t)-c V(t)-\frac{r \mu}{g} Z(t) \\
& \leq\left(N_{T} \delta+N_{C} a\right) L_{1}-s_{2}\left(V(t)+\frac{r}{g} Z(t)\right)  \tag{9}\\
& =\left(N_{T} \delta+N_{C} a\right) L_{1}-s_{2} G_{2}(t)
\end{align*}
$$

where $s_{2}=\min \{c, \mu\}$. Hence $G_{2}(t) \leq L_{2}$, if $G_{2}(0) \leq L_{2}$, where $L_{2}=\left(N_{T} \delta+N_{C} a\right) L_{1} / s_{2}$. Since $V(t) \geq 0$ and $Z(t) \geq 0$ then $0 \leq V(t) \leq L_{2}$ and $0 \leq Z(t) \leq L_{3}$ if $0 \leq V(0)+(r / g) Z(0) \leq$ $L_{2}$, where $L_{3}=g L_{2} / r$.
2.2. Equilibria. System (2)-(6) always admits an infectionfree equilibrium $E_{0}=\left(T_{0}, 0,0,0,0\right)$, where $T_{0}=\lambda / d$. In addition to $E_{0}$, the system can have an infected equilibrium without antibody immune response $E_{1}\left(T_{1}, T_{1}^{*}, C_{1}^{*}, V_{1}, 0\right)$ and an infected equilibrium with antibody immune response $E_{2}\left(T_{2}, T_{2}^{*}, C_{2}^{*}, V_{2}, Z_{2}\right)$ where

$$
\begin{aligned}
T_{1} & =\frac{c}{(1-\varepsilon) k\left[(1-\alpha) N_{T}+\alpha N_{C}\right]} \\
T_{1}^{*} & =\frac{(1-\alpha) \lambda\left\{(1-\varepsilon) k T_{0}\left[(1-\alpha) N_{T}+\alpha N_{C}\right]-c\right\}}{\delta(1-\varepsilon) k T_{0}\left[(1-\alpha) N_{T}+\alpha N_{C}\right]} \\
C_{1}^{*} & =\frac{\alpha \lambda\left\{(1-\varepsilon) k T_{0}\left[(1-\alpha) N_{T}+\alpha N_{C}\right]-c\right\}}{a(1-\varepsilon) k T_{0}\left[(1-\alpha) N_{T}+\alpha N_{C}\right]} \\
V_{1} & =\frac{d\left\{(1-\varepsilon) k T_{0}\left[(1-\alpha) N_{T}+\alpha N_{C}\right]-c\right\}}{(1-\varepsilon) k c}
\end{aligned}
$$

$$
\begin{align*}
& T_{2}=\frac{\lambda g}{g d+(1-\varepsilon) k \mu}, \quad T_{2}^{*}=\frac{(1-\alpha)(1-\varepsilon) k \lambda \mu}{\delta(d g+(1-\varepsilon) k \mu)}, \\
& C_{2}^{*}=\frac{\alpha(1-\varepsilon) k \lambda \mu}{a(d g+(1-\varepsilon) k \mu)}, \quad V_{2}=\frac{\mu}{g} \\
& Z_{2}=\frac{c}{r}\left(\frac{d g(1-\varepsilon) k T_{0}\left[(1-\alpha) N_{T}+\alpha N_{C}\right]}{c(d g+(1-\varepsilon) k \mu)}-1\right) . \tag{10}
\end{align*}
$$

We discuss the local stability of the infection-free equilibrium $E_{0}$. At the infection-free equilibrium $E_{0}\left(T_{0}, 0,0,0,0\right)$, the system has the Jacobian matrix given by

$$
J_{E_{0}}=\left[\begin{array}{ccccc}
-d & 0 & 0 & -(1-\varepsilon) k T_{0} & 0  \tag{11}\\
0 & -\delta & 0 & (1-\alpha)(1-\varepsilon) k T_{0} & 0 \\
0 & 0 & -a & \alpha(1-\varepsilon) k T_{0} & 0 \\
0 & \delta N_{T} & a N_{C} & -c & 0 \\
0 & 0 & 0 & 0 & -\mu
\end{array}\right]
$$

The characteristic equation of the Jacobian matrix evaluated at $E_{0}$ is

$$
\begin{equation*}
(s+d)(s+\mu)\left(s^{3}+a_{1} s^{2}+a_{2} s+a_{3}\right)=0 \tag{12}
\end{equation*}
$$

where

$$
\begin{gather*}
a_{1}=a+c+\delta, \\
a_{2}=a c+a \delta+c \delta-(1-\alpha)(1-\varepsilon) k T_{0} N_{T} \delta \\
-\alpha(1-\varepsilon) k T_{0} N_{C} a  \tag{13}\\
a_{3}=a c \delta\left(1-\frac{(1-\varepsilon) k T_{0}\left[(1-\alpha) N_{T}+\alpha N_{C}\right]}{c}\right) .
\end{gather*}
$$

We observe that (12) has two negative eigenvalues $s_{1}=-d$ and $s_{2}=-\mu$. By the Routh-Hurwitz criterion, the remaining three eigenvalues of (12) have negative real parts if $a_{1}>0, a_{3}>0$, and $a_{1} a_{2}-a_{3}>0$. We have $a_{1}>0$ and if $(1-\varepsilon) k T_{0}\left[(1-\alpha) N_{T}+\right.$ $\left.\alpha N_{C}\right] / c<1$, then $a_{3}>0$ and

$$
\begin{align*}
a_{1} a_{2}-a_{3}= & a \delta^{2}+a^{2} \delta+2 a c \delta \\
& +a(a+c)\left[c-\alpha(1-\varepsilon) k T_{0} N_{C}\right]  \tag{14}\\
& +\delta(\delta+c)\left[c-(1-\alpha)(1-\varepsilon) k T_{0} N_{T}\right] \\
> & 0
\end{align*}
$$

Now we define the basic reproduction number for system (2)-(6) as

$$
\begin{equation*}
R_{0}=\frac{(1-\varepsilon) k T_{0}\left[(1-\alpha) N_{T}+\alpha N_{C}\right]}{c} \tag{15}
\end{equation*}
$$

It follows that the equilibria $E_{1}$ and $E_{2}$ can be written as

$$
\begin{gathered}
T_{1}=\frac{T_{0}}{R_{0}}, \quad T_{1}^{*}=\frac{(1-\alpha) \lambda}{\delta} \frac{\left(R_{0}-1\right)}{\mathrm{R}_{0}}, \\
C_{1}^{*}=\frac{\alpha \lambda}{a} \frac{\left(R_{0}-1\right)}{R_{0}}, \quad V_{1}=\frac{d}{(1-\varepsilon) k}\left(R_{0}-1\right), \\
T_{2}=\frac{\lambda g}{g d+(1-\varepsilon) k \mu}, \quad T_{2}^{*}=\frac{(1-\alpha)(1-\varepsilon) k \lambda \mu}{\delta(d g+(1-\varepsilon) k \mu)},
\end{gathered}
$$

$$
\begin{gather*}
C_{2}^{*}=\frac{\alpha(1-\varepsilon) k \lambda \mu}{a(d g+(1-\varepsilon) k \mu)}, \quad V_{2}=\frac{\mu}{g} \\
Z_{2}=\frac{c}{r}\left(\frac{d g R_{0}}{d g+(1-\varepsilon) k \mu}-1\right) \tag{16}
\end{gather*}
$$

We note that $T_{1}, T_{1}^{*}, C_{1}^{*}$, and $V_{1}$ are positive when $R_{0}>1$ and that $Z_{2}>0$ when $d g R_{0} /(d g+(1-\varepsilon) k \mu)>1$. Now we define another threshold parameter $R_{1}$ as

$$
\begin{equation*}
R_{1}=\frac{R_{0}}{1+((1-\varepsilon) k \mu / d g)} \tag{17}
\end{equation*}
$$

Clearly $R_{1}<R_{0}$.
From (2.2) we have the following statements:
(i) if $R_{0} \leq 1$, then there exists only positive equilibrium $E_{0}$;
(ii) if $R_{1} \leq 1<R_{0}$, then there exist two positive equilibria $E_{0}$ and $E_{1}$;
(iii) if $R_{1}>1$, then there exist three positive equilibria $E_{0}$, $E_{1}$, and $E_{2}$.
2.3. Global Stability Analysis. In this section, we study the global stability of all the equilibria of system (2)-(6) employing the method of Lyapunov function.

Theorem 2. For system (2)-(6), if $R_{0} \leq 1$, then $E_{0}$ is GAS.
Proof. Define a Lyapunov function $U_{0}$ as follows:

$$
\begin{equation*}
U_{0}=T_{0}\left(\frac{T}{T_{0}}-1-\ln \left(\frac{T}{T_{0}}\right)\right)+\eta_{1} T^{*}+\eta_{2} C^{*}+\eta_{3} V+\eta_{4} Z \tag{18}
\end{equation*}
$$

where $\eta_{i}, i=1, \ldots, 4$, are positive constants to be determined below. Calculating the derivative of $U_{0}$ along the solutions of the system (2)-(6) and applying $\lambda=T_{0} d$, we obtain

$$
\begin{align*}
\frac{d U_{0}}{d t}= & \left(1-\frac{T_{0}}{T}\right)(\lambda-d T-(1-\varepsilon) k T V) \\
& +\eta_{1}\left((1-\alpha)(1-\varepsilon) k T V-\delta T^{*}\right) \\
& +\eta_{2}\left(\alpha(1-\varepsilon) k T V-a C^{*}\right)  \tag{19}\\
& +\eta_{3}\left(N_{T} \delta T^{*}+N_{C} a C^{*}-c V-r V Z\right) \\
& +\eta_{4}(g V Z-\mu Z)
\end{align*}
$$

Let $\eta_{i}, i=1, \ldots, 4$, be chosen such as

$$
\begin{gather*}
(1-\alpha) \eta_{1}+\alpha \eta_{2}=1, \quad \eta_{1}-N_{T} \eta_{3}=0  \tag{20}\\
\eta_{2}-N_{C} \eta_{3}=0, \quad g \eta_{4}-r \eta_{3}=0
\end{gather*}
$$

The solution of (20) is given by

$$
\begin{array}{ll}
\eta_{1}=\frac{N_{T}}{(1-\alpha) N_{T}+\alpha N_{C}}, & \eta_{2}=\frac{N_{C}}{(1-\alpha) N_{T}+\alpha N_{C}} \\
\eta_{3}=\frac{1}{(1-\alpha) N_{T}+\alpha N_{C}}, & \eta_{4}=\frac{r}{g\left[(1-\alpha) N_{T}+\alpha N_{C}\right]} \tag{21}
\end{array}
$$

The values of $\eta_{i}, i=1, \ldots, 4$, given by (21) will be used throughout the paper. Then

$$
\begin{align*}
\frac{d U_{0}}{d t} & =\left(1-\frac{T_{0}}{T}\right)(\lambda-d T)+(1-\varepsilon) k T_{0} V-\eta_{3} c V-\eta_{4} \mu Z \\
& =-d \frac{\left(T-T_{0}\right)^{2}}{T}+\eta_{3} c\left(R_{0}-1\right) V-\eta_{4} \mu Z . \tag{22}
\end{align*}
$$

If $R_{0} \leq 1$ then $d U_{0} / d t \leq 0$ for all $T, V, Z>0$. Thus the solutions of system (2)-(6) limit to $M$, the largest invariant subset of $\left\{d U_{0} / d t=0\right\}$. Clearly, it follows from (22) that $d U_{0} / d t=0$ if and only if $T=T_{0}, V=0$, and $Z=0$. Noting that $M$ is invariant, for each element of $M$ we have $V=0$ and $Z=0$, and then $\dot{V}=0$. From (5) we derive that

$$
\begin{equation*}
0=\dot{V}=N_{T} \delta T^{*}+N_{C} a C^{*} \tag{23}
\end{equation*}
$$

Since $T^{*}, C^{*} \geq 0$, then $T^{*}=C^{*}=0$. Hence $d U_{0} / d t=0$ if and only if $T=T_{0}, T^{*}=0, C^{*}=0, V=0$, and $Z=0$. It follows from LaSalle's invariance principle that the infectionfree equilibrium $E_{0}$ is GAS when $R_{0} \leq 1$.

Theorem 3. For system (2)-(6), if $R_{1} \leq 1<R_{0}$, then $E_{1}$ is GAS.

Proof. Define the following Lyapunov function:

$$
\begin{align*}
U_{1}= & T_{1}\left(\frac{T}{T_{1}}-1-\ln \left(\frac{T}{T_{1}}\right)\right)+\eta_{1} T_{1}^{*}\left(\frac{T^{*}}{T_{1}^{*}}-1-\ln \left(\frac{T^{*}}{T_{1}^{*}}\right)\right) \\
& +\eta_{2} C_{1}^{*}\left(\frac{C^{*}}{C_{1}^{*}}-1-\ln \left(\frac{C^{*}}{C_{1}^{*}}\right)\right) \\
& +\eta_{3} V_{1}\left(\frac{V}{V_{1}}-1-\ln \left(\frac{V}{V_{1}}\right)\right)+\eta_{4} Z \tag{24}
\end{align*}
$$

The time derivative of $U_{1}$ along the trajectories of (2)-(6) is given by

$$
\begin{align*}
\frac{d U_{1}}{d t}= & \left(1-\frac{T_{1}}{T}\right)(\lambda-d T-(1-\varepsilon) k T V) \\
& +\eta_{1}\left(1-\frac{T_{1}^{*}}{T^{*}}\right)\left((1-\alpha)(1-\varepsilon) k T V-\delta T^{*}\right) \\
& +\eta_{2}\left(1-\frac{C_{1}^{*}}{C^{*}}\right)\left(\alpha(1-\varepsilon) k T V-a C^{*}\right) \\
& +\eta_{3}\left(1-\frac{V_{1}}{V}\right)\left(N_{T} \delta T^{*}+N_{C} a C^{*}-c V-r V Z\right) \\
& +\eta_{4}(g V Z-\mu Z) \tag{25}
\end{align*}
$$

Applying $\lambda=d T_{1}+(1-\varepsilon) k T_{1} V_{1}$ we get

$$
\begin{aligned}
\frac{d U_{1}}{d t}= & \left(1-\frac{T_{1}}{T}\right)\left(d T_{1}-d T\right)+(1-\varepsilon) k T_{1} V_{1}\left(1-\frac{T_{1}}{T}\right) \\
& +(1-\varepsilon) k T_{1} V-\eta_{1}(1-\alpha)(1-\varepsilon) k T V \frac{T_{1}^{*}}{T^{*}}+\eta_{1} \delta T_{1}^{*}
\end{aligned}
$$

$$
\begin{align*}
& -\eta_{2} \alpha(1-\varepsilon) k T V \frac{C_{1}^{*}}{C^{*}}+\eta_{2} a C_{1}^{*}-\delta \eta_{1} \frac{V_{1} T^{*}}{V} \\
& -a \eta_{2} \frac{V_{1} C^{*}}{V}-c \eta_{3} V+c \eta_{3} V_{1}+r \eta_{3} V_{1} Z-\mu \eta_{4} Z \tag{26}
\end{align*}
$$

Using the following equilibrium conditions for $E_{1}$,

$$
\begin{gather*}
(1-\alpha)(1-\varepsilon) k T_{1} V_{1}=\delta T_{1}^{*} \\
\alpha(1-\varepsilon) k T_{1} V_{1}=a C_{1}^{*}  \tag{27}\\
c V_{1}=N_{T} \delta T_{1}^{*}+N_{C} a C_{1}^{*}
\end{gather*}
$$

then we have $(1-\varepsilon) k T_{1} V_{1}=\eta_{1} \delta T_{1}^{*}+\eta_{2} a C_{1}^{*}$ and

$$
\begin{align*}
\frac{d U_{1}}{d t}= & -d \frac{\left(T-T_{1}\right)^{2}}{T}+\eta_{1} \delta T_{1}^{*}\left(1-\frac{T_{1}}{T}\right)+\eta_{2} a C_{1}^{*}\left(1-\frac{T_{1}}{T}\right) \\
& -\eta_{1} \delta T_{1}^{*} \frac{T V T_{1}^{*}}{T_{1} V_{1} T^{*}}+\eta_{1} \delta T_{1}^{*}-\eta_{2} a C_{1}^{*} \frac{T V C_{1}^{*}}{T_{1} V_{1} C^{*}} \\
& +\eta_{2} a C_{1}^{*}-\eta_{1} \delta T_{1}^{*} \frac{V_{1} T^{*}}{V T_{1}^{*}}-\eta_{2} a C_{1}^{*} \frac{V_{1} C^{*}}{V C_{1}^{*}} \\
& +\eta_{1} \delta T_{1}^{*}+\eta_{2} a C_{1}^{*}+r \eta_{3}\left(V_{1}-\frac{\mu}{g}\right) Z \\
= & -d \frac{\left(T-T_{1}\right)^{2}}{T}+\eta_{1} \delta T_{1}^{*}\left[3-\frac{T_{1}}{T}-\frac{T_{1}^{*} T V}{T^{*} T_{1} V_{1}}-\frac{V_{1} T^{*}}{V T_{1}^{*}}\right] \\
& +\eta_{2} a C_{1}^{*}\left[3-\frac{T_{1}}{T}-\frac{C_{1}^{*} T V}{C^{*} T_{1} V_{1}}-\frac{C^{*} V_{1}}{C_{1}^{*} V}\right] \\
& +r \eta_{3}\left(\frac{d g+(1-\varepsilon) k \mu}{g(1-\varepsilon) k}\right)\left(R_{1}-1\right) Z . \tag{28}
\end{align*}
$$

We have that if $R_{0}>1$, then $T_{1}, T_{1}^{*}, C_{1}^{*}, V_{1}>0$. Since the arithmetical mean is greater than or equal to the geometrical mean, then if $R_{1} \leq 1$ then $d U_{1} / d t \leq 0$ for all $T, T^{*}, C^{*}, V, Z>$ 0 . It can be seen that $d U_{1} / d t=0$ if and only if $T=T_{1}$, $T^{*}=T_{1}^{*}, C^{*}=C_{1}^{*}, V=V_{1}$, and $Z=0$. LaSalle's invariance principle implies global stability of $E_{1}$.

Theorem 4. For system (2)-(6), if $R_{0} \leq 1$, then $E_{0}$ is GAS.
Proof. We consider a Lyapunov function

$$
\begin{align*}
U_{2}= & T_{2}\left(\frac{T}{T_{2}}-1-\ln \left(\frac{T}{T_{2}}\right)\right)+\eta_{1} T_{2}^{*}\left(\frac{T^{*}}{T_{2}^{*}}-1-\ln \left(\frac{T^{*}}{T_{2}^{*}}\right)\right) \\
& +\eta_{2} C_{2}^{*}\left(\frac{C^{*}}{C_{2}^{*}}-1-\ln \left(\frac{C^{*}}{C_{2}^{*}}\right)\right) \\
& +\eta_{3} V_{2}\left(\frac{V}{V_{2}}-1-\ln \left(\frac{V}{V_{2}}\right)\right) \\
& +\eta_{4} Z_{2}\left(\frac{Z}{Z_{2}}-1-\ln \left(\frac{Z}{Z_{2}}\right)\right) \tag{29}
\end{align*}
$$

Further, function $U_{2}$ along the trajectories of system (2)-(6) satisfies

$$
\begin{align*}
\frac{d U_{2}}{d t}= & \left(1-\frac{T_{2}}{T}\right)(\lambda-d T-(1-\varepsilon) k T V) \\
& +\eta_{1}\left(1-\frac{T_{2}^{*}}{T^{*}}\right)\left((1-\alpha)(1-\varepsilon) k T V-\delta T^{*}\right) \\
& +\eta_{2}\left(1-\frac{C_{2}^{*}}{C^{*}}\right)\left(\alpha(1-\varepsilon) k T V-a C^{*}\right) \\
& +\eta_{3}\left(1-\frac{V_{2}}{V}\right)\left(N_{T} \delta T^{*}+N_{C} a C^{*}-c V-r V Z\right) \\
& +\eta_{4}\left(1-\frac{Z_{2}}{Z}\right)(g V Z-\mu Z) \tag{30}
\end{align*}
$$

Using the following equilibrium conditions for $E_{2}$,

$$
\begin{gather*}
\lambda=d T_{2}+(1-\varepsilon) k T_{2} V_{2}, \quad(1-\alpha)(1-\varepsilon) k T_{2} V_{2}=\delta T_{2}^{*} \\
\alpha(1-\varepsilon) k T_{2} V_{2}=a C_{2}^{*} \\
c V_{2}+r V_{2} Z_{2}=N_{T} \delta T_{2}^{*}+N_{C} a C_{2}^{*} \tag{31}
\end{gather*}
$$

we get

$$
\begin{align*}
\frac{d U_{2}}{d t}= & -d \frac{\left(T-T_{2}\right)^{2}}{T}+(1-\varepsilon) k T_{2} V_{2}\left(1-\frac{T_{2}}{T}\right) \\
& +(1-\varepsilon) k T_{2} V-\eta_{1}(1-\alpha)(1-\varepsilon) k T V \frac{T_{2}^{*}}{T^{*}} \\
& +\delta \eta_{1} T_{2}^{*}-\eta_{2} \alpha(1-\varepsilon) k T V \frac{C_{2}^{*}}{C^{*}}+a \eta_{2} C_{2}^{*} \\
& -\delta \eta_{1} \frac{V_{2} T^{*}}{V}-a \eta_{2} \frac{V_{2} C^{*}}{V}-c \eta_{3} V+c \eta_{3} V_{2} \\
& +r \eta_{4} V_{2} Z-r \eta_{4} Z_{2} V+\mu \eta_{4} Z_{2}-\mu \eta_{4} Z \\
= & -d \frac{\left(T-T_{2}\right)^{2}}{T}+\eta_{1} \delta T_{2}^{*}\left(1-\frac{T_{2}}{T}\right) \\
& +\eta_{2} a C_{2}^{*}\left(1-\frac{T_{2}}{T}\right)-\eta_{1} \delta T_{2}^{*} \frac{T V T_{2}^{*}}{T_{2} V_{2} T^{*}}+\eta_{1} \delta T_{2}^{*} \\
& -\eta_{2} a C_{2}^{*} \frac{T V C_{2}^{*}}{T_{2} V_{2} C^{*}}+\eta_{2} a C_{2}^{*}-\eta_{1} \delta T_{2}^{*} \frac{V_{2} T^{*}}{V T_{2}^{*}} \\
& -\eta_{2} a C_{2}^{*} \frac{V_{2} C^{*}}{V C_{2}^{*}}+\eta_{1} \delta T_{2}^{*}+\eta_{2} a C_{2}^{*} \\
= & -d \frac{\left(T-T_{2}\right)^{2}}{T}+\eta_{1} \delta T_{2}^{*}\left[3-\frac{T_{2}}{T}-\frac{T_{2}^{*} T V}{T^{*} T_{2} V_{2}}-\frac{V_{2} T^{*}}{V T_{2}^{*}}\right] \\
& +\eta_{2} a C_{2}^{*}\left[3-\frac{T_{2}}{T}-\frac{C_{2}^{*} T V}{C^{*} T_{2} V_{2}}-\frac{C^{*} V_{2}}{C_{2}^{*} V}\right] \tag{32}
\end{align*}
$$

Thus, if $R_{1}>1$, then $T_{2}, T_{2}^{*}, C_{2}^{*}, V_{2}$ and $Z_{2}>0$. Since the arithmetical mean is greater than or equal to the geometrical mean, then $d U_{2} / d t \leq 0$. It can be seen that $d U_{2} / d t=0$ if and only if $T=T_{2}, T^{*}=T_{2}^{*}, C^{*}=C_{2}^{*}$, and $V=V_{2}$. From (5), if $V=V_{2}$, then $\dot{V}=0$ and $0=N_{T} \delta T_{2}^{*}+N_{C} a C_{2}^{*}-c V-r V_{2} Z=0$, so $Z=Z_{2}$ and hence $d U_{2} / d t$ is equal to zero at $E_{2}$. So, the global stability of the equilibrium $E_{2}$ follows from LaSalle's invariance principle.

## 3. Model with Saturation Incidence Rate

In model (2)-(6), the infection process is characterized by bilinear incidence rate $(1-\varepsilon) k x v$. However, there are a number of reasons why this bilinear incidence can be insufficient to describe infection process in detail (see, e.g., [29-31]). For example, a less than linear response in $v$ could occur when the concentration of viruses becomes higher, where the infectious fraction is high so that exposure is very likely [29]. Experiments reported in [32] strongly suggested that the infection rate of microparasitic infections is an increasing function of the parasite dose and is usually sigmoidal in shape (see, e.g., [33]). In [33], to place the model on more sound biological grounds, Regoes et al. replaced the mass-action infection rate with a dose-dependent infection rates. In this section, the incidence rate is given by a saturation functional response:

$$
\begin{align*}
\dot{T} & =\lambda-d T-\frac{(1-\varepsilon) k T V}{1+\beta V}  \tag{33}\\
\dot{T}^{*} & =\frac{(1-\alpha)(1-\varepsilon) k T V}{1+\beta V}-\delta T^{*}  \tag{34}\\
\dot{C}^{*} & =\frac{\alpha(1-\varepsilon) k T V}{1+\beta V}-a C^{*}  \tag{35}\\
\dot{V} & =N_{T} \delta T^{*}+N_{C} a C^{*}-c V-r V Z  \tag{36}\\
\dot{Z} & =g V Z-\mu Z \tag{37}
\end{align*}
$$

where $\beta>0$ is a constant, which represents the saturation infection rate constant.

All the variables and parameters have the same meanings as given in model (2)-(6).
3.1. Equilibria. Similar to the previous section, we can define two threshold parameters $R_{0}$ and $R_{1}$ for system (33)-(37) as

$$
\begin{align*}
& R_{0}=\frac{(1-\varepsilon) k T_{0}\left[(1-\alpha) N_{T}+\alpha N_{C}\right]}{c}, \\
& R_{1}=\frac{R_{0}}{1+(d \beta \mu+(1-\varepsilon) k \mu / d g)} . \tag{38}
\end{align*}
$$

Clearly $R_{1}<R_{0}$. It is clear that system (33)-(37) has an infection-free equilibrium $E_{0}=\left(T_{0}, 0,0,0,0\right)$, where $T_{0}=\lambda / d$. In addition to $E_{0}$, the system can have an
infected equilibrium without antibody immune response $E_{1}\left(T_{1}, T_{1}^{*}, C_{1}^{*}, V_{1}, 0\right)$, where

$$
\begin{align*}
T_{1} & =\frac{\beta \lambda\left[(1-\alpha) N_{T}+\alpha N_{C}\right]+c}{((1-\varepsilon) k+d \beta)\left[(1-\alpha) N_{T}+\alpha N_{C}\right]}, \\
T_{1}^{*} & =\frac{(1-\alpha) c d}{\delta((1-\varepsilon) k+d \beta)\left[(1-\alpha) N_{T}+\alpha N_{C}\right]}\left(R_{0}-1\right), \\
C_{1}^{*} & =\frac{\alpha c d}{a((1-\varepsilon) k+d \beta)\left[(1-\alpha) N_{T}+\alpha N_{C}\right]}\left(R_{0}-1\right), \\
V_{1} & =\frac{d}{(1-\varepsilon) k+d \beta}\left(R_{0}-1\right), \tag{39}
\end{align*}
$$

and infected equilibrium with antibody immune response $E_{2}\left(T_{2}, T_{2}^{*}, C_{2}^{*}, V_{2}, Z_{2}\right)$, where

$$
\begin{align*}
T_{2} & =\frac{\lambda(g+\beta \mu)}{g d+(1-\varepsilon) k \mu+d \beta \mu}, \\
T_{2}^{*} & =\frac{(1-\alpha)(1-\varepsilon) k \lambda \mu}{\delta(d g+(1-\varepsilon) k \mu+d \beta \mu)},  \tag{40}\\
C_{2}^{*} & =\frac{\alpha(1-\varepsilon) k \lambda \mu}{a(d g+(1-\varepsilon) k \mu+d \beta \mu)}, \quad V_{2}=\frac{\mu}{g}, \\
Z_{2} & =\frac{c}{r}\left(R_{1}-1\right) .
\end{align*}
$$

It is clear from (39) and (40) that
(i) if $R_{0} \leq 1$, then there exists only positive equilibrium $E_{0}$;
(ii) if $R_{1} \leq 1<R_{0}$, then there exist two positive equilibria $E_{0}$ and $E_{1}$;
(iii) if $R_{1}>1$, then there exist three positive equilibria $E_{0}$, $E_{1}$, and $E_{2}$.
3.2. Global Stability Analysis. In this section, we study the global stability of all the equilibria of system (33)-(37) employing the method of Lyapunov function and LaSalle's invariance principle.

Theorem 5. For system (33)-(37), if $R_{0} \leq 1$, then $E_{0}$ is GAS.
Proof. Define a Lyapunov function $U_{0}$ as follows:

$$
\begin{equation*}
U_{0}=T_{0}\left(\frac{T}{T_{0}}-1-\ln \left(\frac{T}{T_{0}}\right)\right)+\eta_{1} T^{*}+\eta_{2} C^{*}+\eta_{3} V+\eta_{4} Z \tag{41}
\end{equation*}
$$

Calculating the derivative of $U_{0}$ along the solutions of system (33)-(37) and applying $\lambda=T_{0} d$, we obtain

$$
\begin{aligned}
\frac{d U_{0}}{d t}= & \left(1-\frac{T_{0}}{T}\right)\left(\lambda-d T-\frac{(1-\varepsilon) k T V}{1+\beta V}\right) \\
& +\eta_{1}\left(\frac{(1-\alpha)(1-\varepsilon) k T V}{1+\beta V}-\delta T^{*}\right)
\end{aligned}
$$

$$
\begin{align*}
& +\eta_{2}\left(\frac{\alpha(1-\varepsilon) k T V}{1+\beta V}-a C^{*}\right) \\
& +\eta_{3}\left(N_{T} \delta T^{*}+N_{C} a C^{*}-c V-r V Z\right) \\
& +\eta_{4}(g V Z-\mu Z) \\
= & \left(1-\frac{T_{0}}{T}\right)(\lambda-d T)+\frac{(1-\varepsilon) k T_{0} V}{1+\beta V} \\
& -c \eta_{3} V-\mu \eta_{4} Z \\
= & -\left[d \frac{\left(T-T_{0}\right)^{2}}{T}+\eta_{3} \frac{c \beta R_{0} V^{2}}{(1+\beta V)}+\mu \eta_{4} Z\right] \\
& +c \eta_{3}\left(R_{0}-1\right) V . \tag{42}
\end{align*}
$$

Similar to the proof of Theorem 2, one can easily show that $E_{0}$ is GAS when $R_{0} \leq 1$.

Theorem 6. For system (33)-(37), if $R_{1} \leq 1<R_{0}$, then $E_{1}$ is GAS.

Proof. Construct a Lyapunov function as follows:

$$
\begin{align*}
U_{1}= & T_{1}\left(\frac{T}{T_{1}}-1-\ln \left(\frac{T}{T_{1}}\right)\right) \\
& +\eta_{1} T_{1}^{*}\left(\frac{T^{*}}{T_{1}^{*}}-1-\ln \left(\frac{T^{*}}{T_{1}^{*}}\right)\right) \\
& +\eta_{2} C_{1}^{*}\left(\frac{C^{*}}{C_{1}^{*}}-1-\ln \left(\frac{C^{*}}{C_{1}^{*}}\right)\right)  \tag{43}\\
& +\eta_{3} V_{1}\left(\frac{V}{V_{1}}-1-\ln \left(\frac{V}{V_{1}}\right)\right)+\eta_{4} Z
\end{align*}
$$

The derivative of $U_{1}$ along the trajectories of system (33)-(37) is given by

$$
\begin{align*}
\frac{d U_{1}}{d t}= & \left(1-\frac{T_{1}}{T}\right)\left(\lambda-d T-\frac{(1-\varepsilon) k T V}{1+\beta V}\right) \\
& +\eta_{1}\left(1-\frac{T_{1}^{*}}{T^{*}}\right)\left(\frac{(1-\alpha)(1-\varepsilon) k T V}{1+\beta V}-\delta T^{*}\right) \\
& +\eta_{2}\left(1-\frac{C_{1}^{*}}{C^{*}}\right)\left(\frac{\alpha(1-\varepsilon) k T V}{1+\beta V}-a C^{*}\right) \\
& +\eta_{3}\left(1-\frac{V_{1}}{V}\right)\left(N_{T} \delta T^{*}+N_{C} a C^{*}-c V-r V Z\right) \\
& +\eta_{4}(g V Z-\mu Z) \tag{44}
\end{align*}
$$

Applying $\lambda=d T_{1}+\left((1-\varepsilon) k T_{1} V_{1} /\left(1+\beta V_{1}\right)\right)$ we get

$$
\begin{aligned}
\frac{d U_{1}}{d t}= & \left(1-\frac{T_{1}}{T}\right)\left(d T_{1}-d T\right) \\
& +\frac{(1-\varepsilon) k T_{1} V_{1}}{1+\beta V_{1}}\left(1-\frac{T_{1}}{T}\right)+\frac{(1-\varepsilon) k T_{1} V}{1+\beta V}
\end{aligned}
$$

$$
\begin{align*}
& -\eta_{1}(1-\alpha) \frac{(1-\varepsilon) k T V}{1+\beta V} \frac{T_{1}^{*}}{T^{*}}+\eta_{1} \delta T_{1}^{*} \\
& -\eta_{2} \alpha \frac{(1-\varepsilon) k T V}{1+\beta V} \frac{C_{1}^{*}}{C^{*}}+\eta_{2} a C_{1}^{*} \\
& -\eta_{1} \delta \frac{V_{1} T^{*}}{V}-\eta_{2} a \frac{V_{1} C^{*}}{V}-c \eta_{3} V \\
& +c \eta_{3} V_{1}+r \eta_{3} V_{1} Z-\mu \eta_{4} Z \tag{45}
\end{align*}
$$

Using the following equilibrium conditions for $E_{1}$,

$$
\begin{gather*}
\frac{(1-\alpha)(1-\varepsilon) k T_{1} V_{1}}{1+\beta V_{1}}=\delta T_{1}^{*} \\
\frac{\alpha(1-\varepsilon) k T_{1} V_{1}}{1+\beta V_{1}}=a C_{1}^{*}, \quad c V_{1}=N_{T} \delta T_{1}^{*}+N_{C} a C_{1}^{*} \tag{46}
\end{gather*}
$$

we get

$$
\begin{aligned}
& \frac{d U_{1}}{d t}=-d \frac{\left(T-T_{1}\right)^{2}}{T}+\eta_{1} \delta T_{1}^{*}\left(1-\frac{T_{1}}{T}\right)+\eta_{2} a C_{1}^{*}\left(1-\frac{T_{1}}{T}\right) \\
& +\frac{(1-\varepsilon) k T_{1} V_{1}}{1+\beta V_{1}}\left[\frac{V\left(1+\beta V_{1}\right)}{V_{1}(1+\beta V)}-\frac{V}{V_{1}}\right] \\
& -\eta_{1} \delta T_{1}^{*} \frac{T V T_{1}^{*}\left(1+\beta V_{1}\right)}{T_{1} V_{1} T^{*}(1+\beta V)}+\eta_{1} \delta T_{1}^{*} \\
& -\eta_{2} a C_{1}^{*} \frac{T V C_{1}^{*}\left(1+\beta V_{1}\right)}{T_{1} V_{1} C^{*}(1+\beta V)}+\eta_{2} a C_{1}^{*} \\
& -\eta_{1} \delta T_{1}^{*} \frac{V_{1} T^{*}}{V T_{1}^{*}}-\eta_{2} a C_{1}^{*} \frac{V_{1} C^{*}}{V C_{1}^{*}} \\
& +\eta_{1} \delta T_{1}^{*}+\eta_{2} a C_{1}^{*}+r \eta_{3}\left(V_{1}-\frac{\mu}{g}\right) Z \\
& =-d \frac{\left(T-T_{1}\right)^{2}}{T} \\
& +\frac{(1-\varepsilon) k T_{1} V_{1}}{1+\beta V_{1}}\left[-1+\frac{V\left(1+\beta V_{1}\right)}{V_{1}(1+\beta V)}-\frac{V}{V_{1}}+\frac{1+\beta V}{1+\beta V_{1}}\right] \\
& +\eta_{1} \delta T_{1}^{*}\left[4-\frac{T_{1}}{T}-\frac{T V T_{1}^{*}\left(1+\beta V_{1}\right)}{T_{1} V_{1} T^{*}(1+\beta V)}-\frac{V_{1} T^{*}}{V T_{1}^{*}}\right. \\
& \left.-\frac{1+\beta V}{1+\beta V_{1}}\right] \\
& +\eta_{2} a C_{1}^{*}\left[4-\frac{T_{1}}{T}-\frac{T V C_{1}^{*}\left(1+\beta V_{1}\right)}{T_{1} V_{1} C^{*}(1+\beta V)}-\frac{C^{*} V_{1}}{C_{1}^{*} V}\right. \\
& \left.-\frac{1+\beta V}{1+\beta V_{1}}\right]+r \eta_{3}\left(V_{1}-\frac{\mu}{g}\right) Z
\end{aligned}
$$

$$
\begin{aligned}
& =-d \frac{\left(T-T_{1}\right)^{2}}{T} \\
& -\frac{(1-\varepsilon) k T_{1} V_{1}}{1+\beta V_{1}}\left[\frac{\beta\left(V-V_{1}\right)^{2}}{V_{1}(1+\beta V)\left(1+\beta V_{1}\right)}\right] \\
& +\eta_{1} \delta T_{1}^{*}\left[4-\frac{T_{1}}{T}-\frac{T V T_{1}^{*}\left(1+\beta V_{1}\right)}{T_{1} V_{1} T^{*}(1+\beta V)}-\frac{V_{1} T^{*}}{V T_{1}^{*}}\right. \\
& \left.\quad-\frac{1+\beta V}{1+\beta V_{1}}\right]
\end{aligned}
$$

$$
+\eta_{2} a C_{1}^{*}\left[4-\frac{T_{1}}{T}-\frac{T V C_{1}^{*}\left(1+\beta V_{1}\right)}{T_{1} V_{1} C^{*}(1+\beta V)}-\frac{C^{*} V_{1}}{C_{1}^{*} V}\right.
$$

$$
\left.-\frac{1+\beta V}{1+\beta V_{1}}\right]
$$

$$
\begin{equation*}
+r \eta_{3}\left(\frac{d g+(1-\varepsilon) k \mu+d \beta \mu}{g(1-\varepsilon) k+d g \beta}\right)\left(R_{1}-1\right) Z . \tag{47}
\end{equation*}
$$

We have that if $R_{1} \leq 1<R_{0}$, then $d U_{1} / d t \leq 0$ where equality occurs at $E_{1}$. LaSalle's invariance principle implies global stability of $E_{1}$.

Theorem 7. For system (33)-(37), if $R_{1}>1$, then $E_{2}$ is GAS.
Proof. We consider a Lyapunov function as follows:

$$
\begin{align*}
U_{2}= & T_{2}\left(\frac{T}{T_{2}}-1-\ln \left(\frac{T}{T_{2}}\right)\right)+\eta_{1} T_{2}^{*}\left(\frac{T^{*}}{T_{2}^{*}}-1-\ln \left(\frac{T^{*}}{T_{2}^{*}}\right)\right) \\
& +\eta_{2} C_{2}^{*}\left(\frac{C^{*}}{C_{2}^{*}}-1-\ln \left(\frac{C^{*}}{C_{2}^{*}}\right)\right) \\
& +\eta_{3} V_{2}\left(\frac{V}{V_{2}}-1-\ln \left(\frac{V}{V_{2}}\right)\right) \\
& +\eta_{4} Z_{2}\left(\frac{Z}{Z_{2}}-1-\ln \left(\frac{Z}{Z_{2}}\right)\right) \tag{48}
\end{align*}
$$

Further, function $U_{2}$ along the trajectories of system (33)-(37) satisfies

$$
\begin{align*}
\frac{d U_{2}}{d t}= & \left(1-\frac{T_{2}}{T}\right)\left(\lambda-d T-\frac{(1-\varepsilon) k T V}{1+\beta V}\right) \\
& +\eta_{1}\left(1-\frac{T_{2}^{*}}{T^{*}}\right)\left(\frac{(1-\alpha)(1-\varepsilon) k T V}{1+\beta V}-\delta T^{*}\right) \\
& +\eta_{2}\left(1-\frac{C_{2}^{*}}{C^{*}}\right)\left(\frac{\alpha(1-\varepsilon) k T V}{1+\beta V}-a C^{*}\right) \\
& +\eta_{3}\left(1-\frac{V_{2}}{V}\right)\left(N_{T} \delta T^{*}+N_{C} a C^{*}-c V-r V Z\right) \\
& +\eta_{4}\left(1-\frac{Z_{2}}{Z}\right)(g V Z-\mu Z) \tag{49}
\end{align*}
$$

Using the following equilibrium conditions for $E_{2}$,

$$
\begin{gather*}
\lambda=d T_{2}+\frac{(1-\varepsilon) k T_{2} V_{2}}{1+\beta V_{2}}, \\
\delta T_{2}^{*}=\frac{(1-\alpha)(1-\varepsilon) k T_{2} V_{2}}{1+\beta V_{2}},  \tag{50}\\
a C_{2}^{*}=\frac{\alpha(1-\varepsilon) k T_{2} V_{2}}{1+\beta V_{2}}, \\
c V_{2}+r V_{2} Z_{2}=N_{T} \delta T_{2}^{*}+N_{C} a C_{2}^{*},
\end{gather*}
$$

we get

$$
\begin{align*}
& \frac{d U_{2}}{d t}=-d \frac{\left(T-T_{2}\right)^{2}}{T}+\frac{(1-\varepsilon) k T_{2} V_{2}}{1+\beta V_{2}}\left(1-\frac{T_{2}}{T}\right) \\
& +\frac{(1-\varepsilon) k T_{2} V}{1+\beta V}-\eta_{1}(1-\alpha) \frac{(1-\varepsilon) k T V}{1+\beta V} \frac{T_{2}^{*}}{T^{*}} \\
& +\eta_{1} \delta T_{2}^{*}-\eta_{2} \alpha \frac{(1-\varepsilon) k T V}{1+\beta V} \frac{C_{2}^{*}}{C^{*}}+\eta_{2} a C_{2}^{*} \\
& -\eta_{1} \delta \frac{V_{2} T^{*}}{V}-\eta_{2} a \frac{V_{2} C^{*}}{V}-\eta_{3} c V+\eta_{3} c V_{2} \\
& +\eta_{3} r V_{2} Z-\eta_{4} g Z_{2} V+\mu \eta_{4} Z_{2}-\mu \eta_{4} Z \\
& =-d \frac{\left(T-T_{2}\right)^{2}}{T}+\eta_{1} \delta T_{2}^{*}\left(1-\frac{T_{2}}{T}\right)+\eta_{2} a C_{2}^{*}\left(1-\frac{T_{2}}{T}\right) \\
& +\frac{(1-\varepsilon) k T_{2} V_{2}}{1+\beta V_{2}}\left[\frac{V\left(1+\beta V_{2}\right)}{V_{2}(1+\beta V)}-\frac{V}{V_{2}}\right] \\
& -\eta_{1} \delta T_{2}^{*} \frac{T V T_{2}^{*}\left(1+\beta V_{2}\right)}{T_{2} V_{2} T^{*}(1+\beta V)}+\eta_{1} \delta T_{2}^{*} \\
& -\eta_{2} a C_{2}^{*} \frac{T V C_{2}^{*}\left(1+\beta V_{2}\right)}{T_{2} V_{2} C^{*}(1+\beta V)}+\eta_{2} a C_{2}^{*} \\
& -\eta_{1} \delta T_{2}^{*} \frac{V_{2} T^{*}}{V T_{2}^{*}}-\eta_{2} a C_{2}^{*} \frac{V_{2} C^{*}}{V C_{2}^{*}}+\eta_{1} \delta T_{2}^{*}+\eta_{2} a C_{2}^{*} \\
& =-d \frac{\left(T-T_{2}\right)^{2}}{T} \\
& -\frac{(1-\varepsilon) k T_{2} V_{2}}{1+\beta V_{2}}\left[\frac{\beta\left(V-V_{2}\right)^{2}}{V_{2}(1+\beta V)\left(1+\beta V_{2}\right)}\right] \\
& +\eta_{1} \delta T_{2}^{*}\left[4-\frac{T_{2}}{T}-\frac{T V T_{2}^{*}\left(1+\beta V_{2}\right)}{T_{2} V_{2} T^{*}(1+\beta V)}-\frac{V_{2} T^{*}}{V T_{2}^{*}}\right. \\
& \left.-\frac{1+\beta V}{1+\beta V_{2}}\right] \\
& +\eta_{2} a C_{2}^{*}\left[4-\frac{T_{2}}{T}-\frac{T V C_{2}^{*}\left(1+\beta V_{2}\right)}{T_{2} V_{2} C^{*}(1+\beta V)}-\frac{C^{*} V_{2}}{C_{2}^{*} V}\right. \\
& \left.-\frac{1+\beta V}{1+\beta V_{2}}\right] \text {. } \tag{51}
\end{align*}
$$

Similar to the proof of Theorem 4, one can show that $E_{2}$ is GAS.

## 4. Numerical Simulations

We now use simple numerical simulations to illustrate our theoretical results for the two models. In both models we will fix the following data: $\lambda=10 \mathrm{~mm}^{-3}$ day $^{-1}, d=0.01$ day $^{-1}$, $k=0.001 \mathrm{~mm}^{3}$ day $^{-1}, \delta=0.5$ day $^{-1}, \alpha=0.5, a=0.1$ day $^{-1}$, $c=3$ day $^{-1}, N_{T}=10, N_{C}=5, r=0.01 \mathrm{~mm}^{3}$ day $^{-1}$, and $\mu=0.1 \mathrm{day}^{-1}$. The other parameters will be chosen below. All computations were carried out by MATLAB.
4.1. Model with Bilinear Incidence Rate. In this section, we perform simulation results for model (2)-(6) to check our theoretical results given in Theorems 2-4. We have the following cases.
(i) $R_{0} \leq 1$. We choose $\varepsilon=0.63$ and $g=0.01 \mathrm{~mm}^{3}$ day $^{-1}$. Using these data we compute $R_{0}=0.92$ and $R_{1}=$ 0.672 . Figures $1,2,3,4$, and 5 show that the numerical results are consistent with Theorem 2 . We can see that, the concentration of uninfected cells is increased and converges to its normal value $\lambda / d=1000 \mathrm{~mm}^{-3}$, while the concentrations of short-lived infected cells, chronically infected cells, free viruses, and antibody immune cells are decaying and tend to zero.
(ii) $R_{1} \leq 1<R_{0}$. We take $\varepsilon=0$ and $g=0.005 \mathrm{~mm}^{3}$ day $^{-1}$. In this case, $R_{0}=2.5$ and $R_{1}=0.833$. Figures $1-5$ show that the numerical results are consistent with Theorem 3. We can see that the trajectory of the system will tend to the infected equilibrium without antibody immune response $E_{1}(400,6,27.77,15,0)$. In this case, the infection becomes chronic but with no persistent antibody immune response.
(iii) $R_{1}>1$. We choose $\varepsilon=0$ and $g=0.01 \mathrm{~mm}^{3}$ day $^{-1}$. Then we compute $R_{0}=2.5$ and $R_{1}=1.25$. From Figures 1-5 we can see that our simulation results are consistent with the theoretical results of Theorem 4. We observe that the trajectory of the system will tend to the infected equilibrium with antibody immune response $E_{2}(500.04,5,23.15,10,57.03)$. In this case, the infection becomes chronic but with persistent antibody immune response.

We note that the values of the parameters $g, r$, and $\mu$ have no impact on the value of $R_{0}$, since $R_{0}$ is independent of those parameters. This fact seems to suggest that antibodies do not play a role in eliminating the viruses. From the definition of $R_{1}$, we can see that $R_{1}$ can be increased by increasing $g$ or decreasing $\mu$.

Figures 1 and 4 show that the presence of antibody immune response (i.e., $R_{1}>1$ ) reduces the concentration of free viruses and increases the concentration of uninfected cells. This can be seen by comparing the virus and uninfected cell components in the equilibria $E_{1}$ and $E_{2}$ under the


Figure 1: The evolution of uninfected cells for model (2)-(6).


Figure 2: The evolution of short-lived infected cells for model (2)(6).
condition $R_{1}>1$. For model (2)-(6), simple calculation shows that

$$
\begin{equation*}
V_{1}-V_{2}=\left(\frac{d g+(1-\varepsilon) k \mu}{g(1-\varepsilon) k}\right)\left(R_{1}-1\right) \tag{52}
\end{equation*}
$$

It follows that if $R_{1}>1$, then $V_{2}<V_{1}$. From (2) and at any equilibrium point $\bar{E}\left(\bar{T}, \bar{T}^{*}, \bar{C}^{*}, \bar{V}, \bar{Z}\right)$ we have

$$
\begin{equation*}
\bar{T}=\frac{\lambda}{d+(1-\varepsilon) k \bar{V}} . \tag{53}
\end{equation*}
$$



Figure 3: The evolution of chronically infected cells for model (2)(6).


Figure 4: The evolution of free viruses for model (2)-(6).

Clearly, $\bar{T}$ is a decreasing function of $\bar{V}$. This yields that if $R_{1}>$ 1 , then $V_{2}<V_{1}$ and $T_{2}>T_{1}$.
4.2. Model with Saturation Functional Response. In this section, we perform simulation results to check Theorems 5-7. The parameter $\beta$ is chosen as $\alpha=0.2 \mathrm{~mm}^{3}$. We have the following cases.
(i) $R_{0} \leq 1$. We take $\varepsilon=0.63$ and $g=0.01 \mathrm{~mm}^{3}$ day $^{-1}$. Using these data, we compute $R_{0}=0.92$ and $R_{1}=$ 0.273 . The simulation results of this case are shown in Figures 6, 7, 8, 9, and 10. We can see that the numerical


FIGURE 5: The evolution of antibody immune cells for model (2)-(6).


Figure 6: The evolution of uninfected cells for model (33)-(37).
results are consistent with Theorem 5. It is observed that the viruses will be cleared and the uninfected cells will return to their normal value.
(ii) $R_{1} \leq 1<R_{0}$. To satisfy this condition, we take $\varepsilon=0$ and $g=0.005 \mathrm{~mm}^{3}$ day $^{-1}$. This will give $R_{0}=2.5$ and $R_{1}=0.833$. Figures $6-10$ show that the numerical results are consistent with Theorem 6 . We see that the infected equilibrium $E_{1}(800,2,9.25,5,0)$ is GAS, and the infection becomes chronic but with no persistent antibody immune response.


Figure 7: The evolution of short-lived infected cells for model (33)(37).


Figure 8: The evolution of chronically infected cells for model (33)(37).
(iii) $R_{1}>1$. This condition is satisfied by choosing $\varepsilon=$ 0 and $g=0.01 \mathrm{~mm}^{3}$ day $^{-1}$. This yields $R_{0}=2.5$ and $R_{1}=1.25$. Figures 6-10 demonstrate the global stability of $E_{2}(832.58,1.67,7.71,3.34,74.55)$. Then, the infection becomes chronic but with persistent antibody immune response.

From the definition of the parameter $R_{0}$, we can see that the value of the saturation infection rate constant $\beta$ has no impact on the value of $R_{0}$. This means that saturation does not play a role in eliminating the virus. From the definition


Figure 9: The evolution of free viruses for model (33)-(37).
of $R_{1}$, we can see that $R_{1}$ can be increased by increasing $g$ or decreasing $\mu$ and $\beta$.

Figures 6 and 9 show that if $R_{1}>1$ the antibody immune response reduces the concentration of free viruses and increases the concentration of uninfected cells. For model (33)-(37), simple calculation shows that

$$
\begin{equation*}
V_{1}-V_{2}=\left(\frac{d g+(1-\varepsilon) k \mu+d \beta \mu}{g(1-\varepsilon) k+d g \beta}\right)\left(R_{1}-1\right) . \tag{54}
\end{equation*}
$$

As a result, if $R_{1}>1$, then $V_{2}<V_{1}$. From (33) and at any equilibrium point $\bar{E}\left(\bar{T}, \bar{T}^{*}, \bar{C}^{*}, \bar{V}, \bar{Z}\right)$ we have

$$
\begin{align*}
\bar{T} & =\frac{(1+\beta \bar{V}) \lambda}{d+(1-\varepsilon) k \bar{V}+d \bar{V} \beta}, \\
\frac{d \bar{T}}{d \bar{V}} & =\frac{-(1-\varepsilon) k \lambda}{(d+(1-\varepsilon) k \bar{V}+d \bar{V} \beta)^{2}} . \tag{55}
\end{align*}
$$

Then, $\bar{T}$ is a decreasing function of $\bar{V}$. It follows that if $R_{1}>1$ then $V_{2}<V_{1}$ and $T_{2}>T_{1}$.

## 5. Conclusions

In this paper, we have proposed two virus infection models with antibody immune response taking into account the chronically infected cells. In the first model we have assumed that the incidence rate of infection is bilinear while in the second model the incidence rate is given by saturation functional response. We have shown that the dynamics of the models are fully determined by two threshold parameters $R_{0}$ and $R_{1}$. The parameter $R_{0}$ determines whether a chronic infection can be established while $R_{1}$ determines whether a persistent antibody response can be established. By constructing


Figure 10: The evolution of antibody immune cells for model (33)(37).

Lyapunov function and using LaSalle's invariance principle, we have investigated the global stability of all equilibria of the two models. We have proven that if $R_{0} \leq 1$ then the infection-free equilibrium $E_{0}$ is GAS, and the viruses are cleared. If $R_{1} \leq 1<R_{0}$, then the infected equilibrium without antibody immune response $E_{1}$ exists and it is GAS, and the infection becomes chronic but with no persistent antibody immune response. If $R_{1}>1$, then the infected equilibrium with antibody immune response $E_{2}$ exists and it is GAS, and the infection is chronic with persistent antibody immune response. Numerical simulations have been performed for the two models. Our simulation results confirm the analytic results given in Theorems 2-7.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under Grant no. 130-078-D1434. The authors, therefore, acknowledge with thanks DSR technical and financial support. The authors are also grateful to Professor Malay Banerjee and to the anonymous reviewers for constructive suggestions and valuable comments, which improve the quality of the paper.

## References

[1] M. A. Nowak and R. M. May, Virus Dynamics: Mathematical Principles of Immunology and Virology, University of Oxford, Oxford, UK, 2000.
[2] M. A. Nowak and C. R. M. Bangham, "Population dynamics of immune responses to persistent viruses," Science, vol. 272, no. 5258, pp. 74-79, 1996.
[3] A. S. Perelson and P. W. Nelson, "Mathematical analysis of HIV1 dynamics in vivo," SIAM Review, vol. 41, no. 1, pp. 3-44, 1999.
[4] A. S. Perelson, A. U. Neumann, M. Markowitz, J. M. Leonard, and D. D. Ho, "HIV-1 dynamics in vivo: virion clearance rate, infected cell life-span, and viral generation time," Science, vol. 271, no. 5255, pp. 1582-1586, 1996.
[5] A. U. Neumann, N. P. Lam, H. Dahari et al., "Hepatitis C viral dynamics in vivo and the antiviral efficacy of interferon$\alpha$ therapy," Science, vol. 282, no. 5386, pp. 103-107, 1998.
[6] A. S. Perelson, D. E. Kirschner, and R. De Boer, "Dynamics of HIV infection of CD4 ${ }^{+}$T cells," Mathematical Biosciences, vol. 114, no. 1, pp. 81-125, 1993.
[7] A. M. Elaiw, "Global properties of a class of HIV models," Nonlinear Analysis: Real World Applications, vol. 11, no. 4, pp. 2253-2263, 2010.
[8] A. M. Elaiw, "Global properties of a class of virus infection models with multitarget cells," Nonlinear Dynamics, pp. 1-13, 2011.
[9] A. M. Elaiw and S. A. Azoz, "Global properties of a class of HIV infection models with Beddington- DeAngelis functional response," Mathematical Methods in the Applied Sciences, vol. 36, no. 4, pp. 383-394, 2013.
[10] A. M. Elaiw and A. S. Alsheri, "Global dynamics of HIV infection of $\mathrm{CD} 4^{+} \mathrm{T}$ cells and macrophages," Discrete Dynamics in Nature and Society, vol. 2013, Article ID 264759, 8 pages, 2013.
[11] A. M. Elaiw, I. A. Hassanien, and S. A. Azoz, "Global stability of HIV infection models with intracellular delays," Journal of the Korean Mathematical Society, vol. 49, no. 4, pp. 779-794, 2012.
[12] A. M. Elaiw and M. A. Alghamdi, "Global properties of virus dynamics models with multitarget cells and discrete-time delays," Discrete Dynamics in Nature and Society, vol. 2011, Article ID 201274, 19 pages, 2011.
[13] A. M. Elaiw, "Global dynamics of an HIV infection model with two classes of target cells and distributed delays," Discrete Dynamics in Nature and Society, vol. 2012, Article ID 253703, 13 pages, 2012.
[14] M. A. Obaid, "Global analysis of a virus infection model with multitarget cells and distributed intracellular delays," Life Science Journal, vol. 9, pp. 1500-1508, 2012.
[15] K. Wang, A. Fan, and A. Torres, "Global properties of an improved hepatitis B virus model," Nonlinear Analysis: Real World Applications, vol. 11, no. 4, pp. 3131-3138, 2010.
[16] X. Wang, A. Elaiw, and X. Song, "Global properties of a delayed HIV infection model with CTL immune response," Applied Mathematics and Computation, vol. 218, no. 18, pp. 9405-9414, 2012.
[17] J. Li, K. Wang, and Y. Yang, "Dynamical behaviors of an HBV infection model with logistic hepatocyte growth," Mathematical and Computer Modelling, vol. 54, no. 1-2, pp. 704-711, 2011.
[18] D. S. Callaway and A. S. Perelson, "HIV-1 infection and low steady state viral loads," Bulletin of Mathematical Biology, vol. 64, no. 1, pp. 29-64, 2002.
[19] R. M. Anderson, R. M. May, and S. Gupta, "Non-linear phenomena in host-parasite interactions," Parasitology, vol. 99, pp. S59-S79, 1989.
[20] A. Murase, T. Sasaki, and T. Kajiwara, "Stability analysis of pathogen-immune interaction dynamics," Journal of Mathematical Biology, vol. 51, no. 3, pp. 247-267, 2005.
[21] D. Wodarz, R. M. May, and M. A. Nowak, "The role of antigenindependent persistence of memory cytotoxic T lymphocytes," International Immunology, vol. 12, no. 4, pp. 467-477, 2000.
[22] C. Chiyaka, W. Garira, and S. Dube, "Modelling immune response and drug therapy in human malaria infection," Computational and Mathematical Methods in Medicine, vol. 9, no. 2, pp. 143-163, 2008.
[23] A. S. Perelson, "Modelling viral and immune system dynamics," Nature Reviews Immunology, vol. 2, no. 1, pp. 28-36, 2002.
[24] S. Wang and D. Zou, "Global stability of in-host viral models with humoral immunity and intracellular delays," Applied Mathematical Modelling, vol. 36, no. 3, pp. 1313-1322, 2012.
[25] H. F. Huo, Y. L. Tang, and L. X. Feng, "A virus dynamics model with saturation infection and humoral immunity," Journal of Mathematical Analysis and Applications, vol. 6, no. 40, pp. 19771983, 2012.
[26] A. M. Elaiw, A. Alhejelan, and M. A. Alghamdi, "Global dynamics of virus infection model with antibody immune response and distributed delays," Discrete Dynamics in Nature and Society, vol. 2013, Article ID 781407, 9 pages, 2013.
[27] T. Wang, Z. Hu, and F. Liao, "Stability and Hopf bifurcation for a virus infection model with delayed humoral immunity response," Journal of Mathematical Analysis and Applications, vol. 411, no. 1, pp. 63-74, 2014.
[28] X. Wang and S. Liu, "A class of delayed viral models with saturation infection rate and immune response," Mathematical Methods in the Applied Sciences, vol. 36, no. 2, pp. 125-142, 2013.
[29] X. Song and A. U. Neumann, "Global stability and periodic solution of the viral dynamics," "Journal of Mathematical Analysis and Applications, vol. 329, no. 1, pp. 281-297, 2007.
[30] P. Georgescu and Y.-H. Hsieh, "Global stability for a virus dynamics model with nonlinear incidence of infection and removal," SIAM Journal on Applied Mathematics, vol. 67, no. 2, pp. 337-353, 2006.
[31] A. Korobeinikov, "Global asymptotic properties of virus dynamics models with dose-dependent parasite reproduction and virulence and non-linear incidence rate," Mathematical Medicine and Biology, vol. 26, no. 3, pp. 225-239, 2009.
[32] D. Ebert, C. D. Zschokke-Rohringer, and H. J. Carius, "Dose effects and density-dependent regulation of two microparasites of Daphnia magna," Oecologia, vol. 122, no. 2, pp. 200-209, 2000.
[33] R. R. Regoes, D. Ebert, and S. Bonhoeffer, "Dose-dependent infection rates of parasites produce the Allee effect in epidemiology," Proceedings of the Royal Society B: Biological Sciences, vol. 269, no. 1488, pp. 271-279, 2002.

## Research Article

# Dynamic Analysis of Nonlinear Impulsive Neutral Nonautonomous Differential Equations with Delays 

Jinxian Li<br>School of Mathematical Science, Shanxi University, Taiyuan 030006, China<br>Correspondence should be addressed to Jinxian Li; lijinxian@sxu.edu.cn

Received 6 January 2014; Accepted 26 February 2014; Published 1 April 2014
Academic Editor: Weiming Wang
Copyright © 2014 Jinxian Li. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

A class of neural networks described by nonlinear impulsive neutral nonautonomous differential equations with delays is considered. By means of Lyapunov functionals and differential inequality technique, criteria on global exponential stability of this model are derived. Many adjustable parameters are introduced in criteria to provide flexibility for the design and analysis of the system. The results of this paper are new and they supplement previously known results. An example is given to illustrate the results.


## 1. Introduction

Many evolution processes in nature exhibit abrupt changes of states at certain moments. That was the reason for the development of the theory of impulsive differential equations and impulsive delay differential equations; see the monographs [1, 2]. But the theory of impulsive neutral differential equations is not well developed due to some theoretical and technical difficulties. For impulsive neutral differential equations, some existence results and oscillation criteria are obtained in [3-5] and some stability conditions are derived in [6]; for neural networks described by impulsive neutral differential equations with delays, the exponential stability results are obtained in [7-11], but their work focuses on the autonomous system. So in this paper, the exponential stability for neural networks described by nonlinear impulsive neutral nonautonomous differential equations with delays is considered.

The purpose of this paper is to study the stability of the following impulsive neural networks with variable coefficients and several time-varying delays:

$$
\begin{aligned}
\dot{x}_{i}(t)= & -b_{i}(t) x_{i}(t)+\sum_{j=1}^{n} a_{i j}(t) f_{i j}\left(x_{j}(t)\right) \\
& +\sum_{j=1}^{n} c_{i j}(t) g_{i j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)
\end{aligned}
$$

$$
\begin{align*}
& \quad+\sum_{j=1}^{n} d_{i j}(t) h_{i j}\left(x_{j}^{\prime}\left(t-\widehat{\tau}_{i j}(t)\right)\right) \\
& +k_{i}(t), \quad \text { a.e. } t>0, t \neq t_{k},  \tag{1a}\\
& x_{i}\left(t^{+}\right)=I_{i k}\left(x_{i}(t)\right)+J_{i k}\left(x_{i}\left(t-\varsigma_{i}(t)\right)\right)+K_{i k}(t), \\
& \quad t=t_{k}, \quad i=1,2, \ldots, n ; \quad k=1,2, \ldots, \tag{lb}
\end{align*}
$$

where $n$ corresponds to the number of units in a neural network; for $i, j=1,2, \ldots, n, x_{i}(t)$ denotes the potential of cell $i$ at time $t ; 0 \leq \tau_{i j}(t), \widehat{\tau}_{i j}(t), \varsigma_{i}(t) \leq \tau$ correspond to the transmission delays. (la) (called continuous part) describes the continuous evolution processes of the neural networks. For $i, j=1,2, \ldots, n, a_{i j}(t), c_{i j}(t)$, and $d_{i j}(t)$ denote the strengths of connectivity between cells $i$ and $j$ at time $t$, respectively; $f_{i j}, g_{i j}, h_{i j}$ show how the $i$ th neuron reacts to the input; $k_{i}(t)$ is the external bias on the $i$ th at time $t$. (lb) (called discrete part) describes that the evolution processes experience abrupt change of states at the moments of $t_{k}$ (called impulsive moments); for $i=1,2, \ldots, n, k=1,2, \ldots$, the fixed moment $t_{k}$ satisfies $t_{1}<t_{2}<\cdots<t_{k}<\cdots$, and $\lim _{k \rightarrow \infty} t_{k}=\infty ; I_{i k}$ represents impulsive perturbations of $i$ th unit at time $t_{k} ; J_{i k}$ represents impulsive perturbations of $i$ th unit at time $t_{k}$, which is caused by the transmission delays; $K_{i k}\left(t_{k}\right)$ represents the external impulsive input at time $t_{k}$.

The theory on linear matrix inequality (LMI) or MMatrix provides effective methods for the analysis of exponential stability of autonomous neural networks. See [7, 9, 10] and the reference therein. But for nonautonomous neural networks, it is invalid. Differential inequalities are important tools for investigating the stability of impulsive differential equations. See $[7,8,12,13]$ and the reference therein. The method in this paper is partially motivated by the work in [7].

In this paper, we will investigate the global exponential stability of the nonautonomous neural networks and focus on the effect of impulse on the dynamic behavior of (1a) and (1b). The results do not require the boundedness of $\left\{t_{k}-t_{k-1}\right\}$ and the differentiability of $\tau_{i j}$. So they are new and complement previously known results.

For a continuous function $a(t)$, we denote

$$
\begin{array}{cc}
a^{+}(t)=\max \{0, a(t)\}, & a^{-}(t)=\min \{0, a(t)\}, \\
a\left(t^{+}\right)=\lim _{s \rightarrow t^{+}} a(s), & a\left(t^{-}\right)=\lim _{s \rightarrow t^{-}} a(s) . \tag{2}
\end{array}
$$

## Define

$$
\begin{gathered}
R^{+}=[0, \infty), \quad N=\{1,2, \ldots, n\}, \quad N^{*}=\{1,2, \ldots\}, \\
C(\Omega, R)=\{\psi: \Omega \longrightarrow R \mid \psi \text { is continous, } \Omega \subset R\}, \\
C B(\Omega, R)=\{\psi \in C(\Omega, R) \mid \psi \text { is bounded }\}, \\
P C([-\tau, 0], R) \\
=\{\psi:[-\tau, 0] \longrightarrow R \\
\mid \psi\left(t^{-}\right)=\psi(t), \text { for } t \in[-\tau, 0], \psi\left(t^{+}\right) \\
\text {exists on } R \text { and } \psi\left(t^{+}\right)=\psi(t)
\end{gathered}
$$

for all but at most a finite
number of points on $[-\tau, 0]$.$\} ,$

$$
\begin{aligned}
P C^{1} & ([-\tau, 0], R) \\
\quad= & \{\psi \in P C([-\tau, 0], R)
\end{aligned}
$$

$$
\psi^{\prime}\left(t^{+}\right) \text {and } \psi^{\prime}\left(t^{-}\right) \text {exist, } \psi^{\prime}(t)=\psi^{\prime}\left(t^{-}\right)
$$

$$
\text { for } t \in[-\tau, 0] \text { and } \psi^{\prime}\left(t^{+}\right)=\psi^{\prime}(t)
$$

for all but at most a finite
number of points on $[-\tau, 0]$.$\} ,$

$$
\begin{aligned}
P C & \left([-\tau, 0], R^{n}\right) \\
& =\left\{\widehat{\psi}=\left(\psi_{1}, \psi_{2}, \ldots, \psi_{n}\right)^{T}\right.
\end{aligned}
$$

$$
\left.\mid \psi_{i} \in P C([-\tau, 0], R), i \in N .\right\},
$$

$$
\begin{align*}
& P C^{1}\left([-\tau, 0], R^{n}\right) \\
& \quad=\left\{\widehat{\psi}=\left(\psi_{1}, \psi_{2}, \ldots, \psi_{n}\right)^{T}\right. \\
& \left.\quad \mid \psi_{i} \in P C^{1}([-\tau, 0], R), i \in N .\right\} . \tag{3}
\end{align*}
$$

For any $\phi \in P C([-\tau, 0], R), \widehat{\phi} \in P C^{1}([-\tau, 0], R), \psi=\left(\psi_{1}\right.$, $\left.\psi_{2}, \ldots, \psi_{n}\right)^{T} \in P C\left([-\tau, 0], R^{n}\right)$, and $\widehat{\psi}=\left(\widehat{\psi}_{1}, \widehat{\psi}_{2}, \ldots, \widehat{\psi}_{n}\right)^{T} \in$ $P C^{1}\left([-\tau, 0], R^{n}\right)$, define $\|\cdot\|_{\tau},\|\cdot\|_{1 \tau},\|\cdot\|_{\tau}^{n}$, and $\|\cdot\|_{1 \tau}^{n}$ as

$$
\begin{gather*}
\|\phi\|_{\tau}=\sup _{-\tau \leq s \leq 0}|\phi(s)|, \quad\|\hat{\phi}\|_{1 \tau}=\max \left\{\|\hat{\phi}\|_{\tau^{\prime}}\left\|\hat{\phi}^{\prime}\right\|_{\tau}\right\},  \tag{4}\\
\|\psi\|_{\tau}^{n}=\max _{1 \leq i \leq n}\left\|\psi_{i}\right\|_{\tau}, \quad\|\widehat{\psi}\|_{1 \tau}^{n}=\max _{1 \leq i \leq n}\left\|\widehat{\psi}_{i}\right\|_{1 \tau},
\end{gather*}
$$

respectively.
For convenience, the following conditions are listed.
$\left(\mathrm{H}_{1}\right)$ For $i, j \in N, b_{i} \in C\left(R^{+}, R^{+}\right), a_{i j} \in C\left(R^{+}, R\right)$, and $c_{i j}, d_{i j} \in C B\left(R^{+}, R\right), f_{i j}, g_{i j}, h_{i j} \in C(R, R)$.
$\left(\mathrm{H}_{2}\right)$ There are positive constants $F_{i j}, G_{i j}, H_{i j}, i, j \in N$, such that

$$
\begin{align*}
& \left|f_{i j}(u)-f_{i j}(v)\right| \leq F_{i j}|u-v|, \\
& \left|g_{i j}(u)-g_{i j}(v)\right| \leq G_{i j}|u-v|,  \tag{5}\\
& \left|h_{i j}(u)-h_{i j}(v)\right| \leq H_{i j}|u-v|,
\end{align*}
$$

for all $u, v \in R$.
$\left(\mathrm{H}_{3}\right)$ There exist positive constants $I_{i k}^{*}$ and $J_{i k}^{*}, i \in N, k \in$ $N^{*}$, such that

$$
\begin{align*}
& \left|I_{i k}(u)-I_{i k}(v)\right| \leq I_{i k}^{*}|u-v|, \\
& \left|J_{i k}(u)-J_{i k}(v)\right| \leq J_{i k}^{*}|u-v|,  \tag{6}\\
& \max _{i \in N, k \in N^{*}} I_{i k}^{*}+\max _{i \in N, k \in N^{*}} J_{i k}^{*}<1,
\end{align*}
$$

for all $u, v \in R$.
$\left(\mathrm{H}_{4}\right)$ There exist positive constants $p_{i}, q_{i}, i \in N$ and $\sigma$ such that

$$
\begin{align*}
& p_{i} b_{i}(t)-\sum_{j=1}^{n} p_{j} F_{i j} a_{i j}^{+}(t) \\
& \quad-\sum_{j=1}^{n}\left(p_{j} G_{i j} c_{i j}^{+}(t)+q_{j} H_{i j} d_{i j}^{+}(t)\right) \geq \sigma>0, \\
& q_{i}-p_{i} b_{i}(t)-\sum_{j=1}^{n} p_{j} F_{i j} a_{i j}^{+}(t)  \tag{7}\\
& \quad-\sum_{j=1}^{n}\left(p_{j} G_{i j} c_{i j}^{+}(t)+q_{j} H_{i j} d_{i j}^{+}(t)\right) \geq \sigma>0
\end{align*}
$$

for $t \in[0, \infty), i \in N$.

We assume that (1a) and (lb) are with the following initial conditions:

$$
\begin{equation*}
x(s)=\phi(s), \quad s \in[-\tau, 0] \tag{8}
\end{equation*}
$$

where $\phi \in P C\left([-\tau, 0], R^{n}\right)$. According to [13], the initial value problems (1a), (1b), and (8) have the unique solution $x(t, \phi)$ under assumptions $\left(\mathrm{H}_{2}\right)$ and $\left(\mathrm{H}_{3}\right)$.

Definition 1. A function $x(t)=\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T}$ is said to be a solution of (1a) and (lb) on $[-\tau, \infty)$ if for $i \in N$,
(i) $x_{i}(t)$ is absolutely continuous on each interval ( $0, t_{1}$ ) and $\left(t_{k}, t_{k+1}\right), k \in N^{*}$;
(ii) for any $t_{k}, k \in N^{*}, x_{i}\left(t_{k}^{+}\right)$and $x_{i}\left(t_{k}^{-}\right)$exist and $x_{i}\left(t_{k}^{-}\right)=$ $x_{i}\left(t_{k}\right)$;
(iii) $x(t)$ satisfies (la) for almost everywhere in $[0, \infty)$ and satisfies (1b) for every $t=t_{k}, k \in N^{*}$.

Obviously, a solution $X(t)=\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T}$ of (1a) and (lb) is continuous at $t \neq t_{k}$ and discontinuous at $t=t_{k}$. Furthermore, $X^{\prime}(t)=\left(x_{1}^{\prime}(t), x_{2}^{\prime}(t), \ldots, x_{n}^{\prime}(t)\right)^{T}$ has discontinuities of the first kind at the fixed impulsive moments $t_{k}$ and some moments $\bar{t} \in\left(t_{k}, t_{k+1}\right), k \in N^{*}$. Denote $X^{\prime}\left(t_{k}\right)=X^{\prime}\left(t_{k}^{-}\right), X^{\prime}(\bar{t})=X^{\prime}(\bar{t})$.

Definition 2. Let $X(t)=\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T}$ and $Y(t)=$ $\left(y_{1}(t), y_{2}(t), \ldots, y_{n}(t)\right)^{T}$ be two solutions of (1a), (lb), and (8) with $\phi=\varphi$ and $\phi=\psi$, respectively, where $\varphi$ and $\psi \in$ $P C\left([-\tau, 0], R^{n}\right)$. If there exist $\alpha>0$ and $M>1$ such that

$$
\begin{equation*}
\left|x_{i}(t)-y_{i}(t)\right| \leq M\|\varphi-\psi\|_{1 \tau}^{n} e^{-\alpha t}, \quad \forall t>0, i \in N \tag{9}
\end{equation*}
$$

then (la) and (lb) are said to be globally exponentially stable.

## 2. The Main Result

To study the exponential stability of (1a) and (1b), we need the following lemma.

Lemma 3. Assume that $\left(H_{1}\right)$ and $\left(H_{4}\right)$ hold and there exist nonnegative vector functions $\left(V_{1}(t), V_{2}(t), \ldots, V_{n}(t)\right)^{T}$ and
$\left(W_{1}(t), W_{2}(t), \ldots, W_{n}(t)\right)^{T} \in P C\left([-\tau, 0], R^{n}\right)$, where $V_{i}(t)$ is continuous at $t \neq t_{k}\left(k \in N^{*}\right)$, such that

$$
\begin{align*}
D^{-} V_{i}\left(t^{-}\right) \leq & -b_{i}(t) V_{i}\left(t^{-}\right)+\sum_{j=1}^{n} a_{i j}^{+}(t) F_{i j} V_{j}\left(t^{-}\right) \\
& +\sum_{j=1}^{n} c_{i j}^{+}(t) G_{i j}\left\|V_{j t^{-}}\right\|_{\tau}+\sum_{j=1}^{n} d_{i j}^{+}(t) H_{i j}\left\|W_{j t^{-}}\right\|_{\tau^{\prime}} \tag{10a}
\end{align*}
$$

$$
\begin{align*}
W_{i}\left(t^{+}\right) \leq & b_{i}(t) V_{i}\left(t^{+}\right)+\sum_{j=1}^{n} a_{i j}^{+}(t) F_{i j} V_{j}\left(t^{+}\right) \\
& +\sum_{j=1}^{n} c_{i j}^{+}(t) G_{i j}\left\|V_{j t^{+}}\right\|_{\tau}+\sum_{j=1}^{n} d_{i j}^{+}(t) H_{i j}\left\|W_{j t^{+}}\right\|_{\tau^{\prime}} \tag{10b}
\end{align*}
$$

$$
\begin{equation*}
V_{i}\left(t_{k}^{+}\right) \leq I_{i k}^{*} V_{i}\left(t_{k}\right)+J_{i k}^{*} V_{i}\left(t_{k}-\varsigma_{i}\left(t_{k}\right)\right), \tag{10c}
\end{equation*}
$$

for $t>0, i \in N, k \in N^{*}$. Then for all $t \geq 0$ and $i \in N$, there exists a positive constant $L$ such that

$$
\begin{equation*}
V_{i}(t) \leq L \sum_{l=1}^{n} \max \left\{\left\|V_{l 0}\right\|_{\tau},\left\|W_{l 0}\right\|_{\tau}\right\} e^{-\left(\lambda^{*}-\mu\right) t} \tag{11}
\end{equation*}
$$

where $\lambda^{*}$ and $\mu$ are defined, respectively, as

$$
\begin{gather*}
\lambda^{*}=\min \left\{\lambda_{i}^{*}, \hat{\lambda}_{i}^{*} \mid i \in N\right\}, \\
\lambda^{*}+\frac{1}{\tau} \ln \frac{\max _{i \in N, k \in N^{*} J_{i k}^{*}}^{1-\max _{i \in N, k \in N^{*}} I_{i k}^{*}} \leq \mu \leq \lambda^{*},}{\lambda_{i}^{*}=\inf _{t \geq 0}\{\lambda(t)>0, \lambda(t)} \begin{array}{r}
-\left[b_{i}(t)-\frac{1}{p_{i}} \sum_{j=1}^{n} p_{j} F_{i j} a_{i j}^{+}(t)\right] \\
+\frac{1}{p_{i}} \sum_{j=1}^{n}\left(p_{j} G_{i j} c_{i j}^{+}(t)+q_{j} H_{i j} d_{i j}^{+}(t)\right) \\
\hat{\lambda}_{i}^{*}=\inf _{t \geq 0}\{\lambda(t)>0, \\
-\left[1-\frac{p_{i}}{q_{i}} b_{i}(t)-\frac{1}{q_{i}} \sum_{j=1}^{n} p_{j} F_{i j} a_{i j}^{+}(t)\right] \\
\\
+\frac{1}{q_{i}} \sum_{j=1}^{n}\left(p_{j} G_{i j} c_{i j}^{+}(t)+q_{j} H_{i j} d_{i j}^{+}(t)\right)>0, \\
\left.\times e^{\lambda(t) \tau}=0\right\}>0 .
\end{array} \tag{12}
\end{gather*}
$$

Proof. By the similar analysis in [14, Lemma 4.1], we can deduce that $\lambda_{i}^{*}$ and $\hat{\lambda}_{i}^{*}$ exist uniquely and $\lambda_{i}^{*}>0, \hat{\lambda}_{i}^{*}>0$ under the assumption of $\left(\mathrm{H}_{1}\right)$ and $\left(\mathrm{H}_{4}\right)$. Consequently, $\lambda^{*}>$

0 . Choose a positive constant $\theta$ such that $\min \left\{p_{i}, q_{i} \mid i \in\right.$ $N\} \theta>1$. Let

$$
\begin{gather*}
\Phi_{i}(t)=\max \left\{\frac{1}{p_{i}} V_{i}(t), \frac{1}{q_{i}} W_{i}(t)\right\}, \\
\Psi(t)=\theta \sum_{l=1}^{n} \max \left\{\left\|V_{l 0}\right\|_{\tau},\left\|W_{l 0}\right\|_{\tau}\right\} e^{-\left(\lambda^{*}-\mu\right) t}, \tag{14}
\end{gather*}
$$

$$
i \in N
$$

Then for all $t \in[-\tau, 0]$ and $\gamma>1$, we have

$$
\begin{equation*}
\gamma \Psi(t)=\gamma \theta \sum_{l=1}^{n} \max \left\{\left\|V_{l 0}\right\|_{\tau},\left\|W_{l 0}\right\|_{\tau}\right\} e^{-\left(\lambda^{*}-\mu\right) t}>\Phi_{i}(t) \tag{15}
\end{equation*}
$$

Then

$$
\begin{equation*}
\Phi_{i}(t)<\gamma \Psi(t), \quad \forall t \in[0, \infty), \quad i \in N \tag{16}
\end{equation*}
$$

For the sake of contradiction, assume that there exist $i \in N$ and $\bar{t}>0$ such that

$$
\begin{align*}
\Phi_{i}\left(\bar{t}^{+}\right) \geq \gamma \Psi(\bar{t}), \quad & \Phi_{j}(t)<\gamma \Psi(t)  \tag{17}\\
& \text { for } t \in[0, \bar{t}), \quad j \in N .
\end{align*}
$$

From (17), we have

$$
\begin{align*}
\left\|V_{j \bar{t}}\right\|_{\tau} & =p_{j} \sup _{-\tau \leq \theta \leq 0} \frac{1}{p_{j}} V_{j}(\bar{t}+\theta)  \tag{18}\\
& \leq p_{j} \sup _{-\tau \leq \theta \leq 0} \gamma \Psi(\bar{t}+\theta) \leq \gamma p_{j} \Psi(\bar{t}-\tau)
\end{align*}
$$

similarly,

$$
\begin{equation*}
\left\|W_{j \hbar}\right\|_{\tau} \leq \gamma q_{j} \Psi(\bar{t}-\tau) . \tag{19}
\end{equation*}
$$

Then we have the following cases.
(I) $\left(1 / p_{i}\right) V_{i}\left(\bar{t}^{+}\right) \geq \gamma \Psi(\bar{t})$; then we have the following subcases.
(i) $\bar{t} \neq t_{k}, k \in N^{*}$. So $V_{i}(t)$ is continuous at $\bar{t}$. By (17), we have

$$
\begin{equation*}
\frac{1}{p_{i}} V_{i}(\bar{t})=\gamma \Psi(\bar{t}), \quad \frac{1}{p_{i}} D^{-} V_{i}(\bar{t})>\gamma \Psi^{\prime}(\bar{t}) . \tag{20}
\end{equation*}
$$

From $\left(\mathrm{H}_{4}\right),(17)-(19)$, and the definition of $\lambda^{*}$, we have

$$
\begin{align*}
& \frac{1}{p_{i}} D^{-} V_{i}(\bar{t})-\gamma \Psi^{\prime}(\bar{t}) \\
& \leq-\gamma b_{i}(\bar{t}) \Psi(\bar{t})+\sum_{j=1}^{n} \frac{p_{j}}{p_{i}} \gamma a_{i j}^{+}(\bar{t}) F_{i j} \Psi(\bar{t})  \tag{21}\\
& \quad+\sum_{j=1}^{n} \gamma\left(\frac{p_{j}}{p_{i}} c_{i j}^{+}(\bar{t}) G_{i j}+\frac{q_{j}}{p_{i}} d_{i j}^{+}(\bar{t}) H_{i j}\right) \\
& \quad \times \Psi(\bar{t}-\tau)+\gamma \lambda^{*} \Psi(\bar{t})<0,
\end{align*}
$$

which is a contradiction with (20).
(ii) There exists a $k_{0} \in N^{*}$ such that $\bar{t}=t_{k_{0}}$. By (17), we have

$$
\begin{equation*}
\frac{1}{p_{i}} V_{i}(\bar{t}) \leq \gamma \Psi(\bar{t}) \leq \frac{1}{p_{i}} V_{i}\left(\bar{t}^{+}\right) \tag{22}
\end{equation*}
$$

Noting $\left(1 / p_{i}\right) V_{i}\left(\bar{t}^{+}\right) \neq\left(1 / p_{i}\right) V_{i}\left(\bar{t}^{-}\right)$, we have $\left(1 / p_{i}\right) V_{i}(\bar{t})<$ $\gamma \Psi(\bar{t})$ or $\gamma \Psi(\bar{t})<\left(1 / p_{i}\right) V_{i}\left(\bar{t}^{+}\right)$. Without loss of generality, we assume that $\gamma \Psi(\bar{t})<\left(1 / p_{i}\right) V_{i}\left(\bar{t}^{+}\right)$. From (10c) and (22), we get that

$$
\begin{equation*}
\gamma \Psi(\bar{t})<\frac{1}{p_{i}} V_{i}\left(\bar{t}^{+}\right) \leq \gamma\left(I_{i k_{0}}^{*}+J_{i k_{0}}^{*} e^{\left(\lambda^{*}-\mu\right) \tau}\right) \Psi(\bar{t}) . \tag{23}
\end{equation*}
$$

Simplifying (23), we obtain $\mu<\lambda^{*}+(1 / \tau) \ln \left(J_{i k_{0}}^{*} /\left(1-I_{i k_{0}}^{*}\right)\right)$, which contradict (12).

If (I) does not hold, then
(II)

$$
\begin{align*}
\frac{1}{q_{i}} W_{i}\left(\bar{t}^{+}\right) & \geq \gamma \Psi(\bar{t}), \quad \frac{1}{q_{j}} W_{j}(t)<\gamma \Psi(t) \\
\frac{1}{p_{j}} V_{j}(t) & <\gamma \Psi(t) \tag{24}
\end{align*}
$$

$$
\text { for } t \in[0, \bar{t}), \quad j \in N
$$

Then from (10b) and (17)-(19), we have

$$
\begin{align*}
0 \leq & -W_{i}\left(\bar{t}^{+}\right)+b_{i}(\bar{t}) V_{i}\left(\bar{t}^{+}\right)+\sum_{j=1}^{n} a_{i j}^{+}(\bar{t}) F_{i j} V_{j}\left(\bar{t}^{+}\right) \\
& +\sum_{j=1}^{n} c_{i j}^{+}(\bar{t}) G_{i j}\left\|V_{j t^{+}}\right\|_{\tau}+\sum_{j=1}^{n} d_{i j}^{+}(\bar{t}) H_{i j}\left\|W_{j t^{+}}\right\|_{\tau^{\prime}} \\
\leq & \gamma \Psi(\bar{t})\left[-q_{i}+p_{i} b_{i}(\bar{t})+\sum_{j=1}^{n} p_{j} a_{i j}^{+}(\bar{t}) F_{i j}\right. \\
& \left.+\sum_{j=1}^{n}\left(p_{j} c_{i j}^{+}(\bar{t}) G_{i j}+q_{j} d_{i j}^{+}(\bar{t}) H_{i j}\right) e^{\lambda^{*} \tau}\right]<0 \tag{25}
\end{align*}
$$

which is a contradiction.
From (I) and (II), (16) holds. Letting $\gamma \rightarrow 1^{+}$in (16), we have

$$
\begin{equation*}
\Phi_{i}(t) \leq \Psi(t), \quad \forall t \in[0, \infty), \quad i \in N \tag{26}
\end{equation*}
$$

So $\left(1 / p_{i}\right) V_{i}(t) \leq \Psi(t)$ for all $t \in[0, \infty), i \in N$. Let $L=$ $\max _{i \in N}\left\{\theta p_{i}\right\}$; then for $t \geq 0$ and $i \in N$, we have

$$
\begin{equation*}
V_{i}(t) \leq L \sum_{l=1}^{n} \max \left\{\left\|V_{l 0}\right\|_{\tau},\left\|W_{l 0}\right\|_{\tau}\right\} e^{-\left(\lambda^{*}-\mu\right) t} \tag{27}
\end{equation*}
$$

The proof of Lemma 3 is complete.
Theorem 4. Assume that $\left(H_{1}\right)-\left(H_{4}\right)$ hold. Then systems (1a) and (1b) are globally exponentially stable.

Proof. Let $X(t)=\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T}$ and $Y(t)=$ $\left(y_{1}(t), y_{2}(t), \ldots, y_{n}(t)\right)^{T}$ be solutions of (la), (1b), and (8) with $\phi=\varphi$ and $\phi=\psi$, respectively. Let

$$
\begin{align*}
& V_{i}(t)=\left|x_{i}(t)-y_{i}(t)\right|, \quad W_{i}(t)=\left|x_{i}^{\prime}(t)-y_{i}^{\prime}(t)\right|,  \tag{28}\\
& t \in R^{+}, \quad i \in N .
\end{align*}
$$

By (la) and (lb), for $i \in N$, we have

$$
\begin{align*}
D^{-} V_{i}\left(t^{-}\right) \leq & -b_{i}(t) V_{i}\left(t^{-}\right)+\sum_{j=1}^{n} a_{i j}^{+}(t) F_{i j} V_{j}\left(t^{-}\right) \\
& +\sum_{j=1}^{n} c_{i j}^{+}(t) G_{i j}\left\|V_{j t^{-}}\right\|_{\tau}  \tag{29}\\
& +\sum_{j=1}^{n} d_{i j}^{+}(t) H_{i j}\left\|W_{j t^{-}}\right\|_{\tau^{\prime}} \quad t>0, \\
W_{i}\left(t^{+}\right) \leq & b_{i}(t) V_{i}\left(t^{+}\right) \\
& +\sum_{j=1}^{n} a_{i j}^{+}(t) F_{i j} V_{j}\left(t^{+}\right)+\sum_{j=1}^{n} c_{i j}^{+}(t) G_{i j}\left\|V_{j t^{+}}\right\|_{\tau}  \tag{30}\\
& +\sum_{j=1}^{n} d_{i j}^{+}(t) H_{i j}\left\|W_{j t^{+}}\right\|_{\tau^{\prime}} \quad t>0 .
\end{align*}
$$

By (1b) and $\left(\mathrm{H}_{3}\right)$, we have

$$
\begin{equation*}
V_{i}\left(t_{k}^{+}\right)=\left|x_{i}\left(t_{k}^{+}\right)-y_{i}\left(t_{k}^{+}\right)\right| \leq I_{i k}^{*} V_{i}\left(t_{k}\right)+J_{i k}^{*} V_{i}\left(t_{k}-\varsigma_{i}\left(t_{k}\right)\right) . \tag{31}
\end{equation*}
$$

By (29)-(31) and Lemma 3, there exists a positive constant $M$ such that

$$
\begin{align*}
V_{i}(t) & \leq M \sum_{l=1}^{n} \max \left\{\left\|V_{l 0}\right\|_{\tau},\left\|W_{l 0}\right\|_{\tau}\right\} e^{-\left(\lambda^{*}-\mu\right) t}  \tag{32}\\
& \leq M n\|\phi-\psi\|_{1 \tau}^{n} e^{-\left(\lambda^{*}-\mu\right) t}
\end{align*}
$$

where $\lambda^{*}$ and $\mu$ are defined in (12).
Remark 5. For autonomous system, the exponential stability of the zero solution of (la) with $x_{i}\left(t_{k}^{+}\right)=I_{i k}\left(x_{1}\left(t_{k}\right)\right.$, $\left.\ldots, x_{n}\left(t_{k}\right)\right), k \in N^{*}$, is considered in [7]. But the results require that $\left\{t_{k}-t_{k-1}\right\}$ is bounded.

When there is no impulse in systems (1a) and (1b), (1a) and (1b) reduce to the following model which has been studied in $[9,10]$ :

$$
\begin{align*}
\dot{x}_{i}(t)= & -b_{i}(t) x_{\mathrm{i}}(t)+\sum_{j=1}^{n} a_{i j}(t) f_{i j}\left(x_{j}(t)\right) \\
& +\sum_{j=1}^{n} c_{i j}(t) g_{i j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)  \tag{33}\\
& +\sum_{j=1}^{n} d_{i j}(t) h_{i j}\left(x_{j}^{\prime}\left(t-\widehat{\tau}_{i j}(t)\right)\right) \\
& +k_{i}(t), \quad t>0, \quad i \in N
\end{align*}
$$

Corollary 6. Assume that $\left(H_{1}\right),\left(H_{2}\right)$, and $\left(H_{4}\right)$ hold. (33) is globally exponentially stable.

Remark 7. For autonomous system, the stability of (33) with $h_{i j}(x)=x, f_{i j}=g_{i j}$, is considered in [10]. However, the authors assume that $f_{i j}, i, j=1,2, \ldots, n$, are monotonic, bounded and $\tau_{i j}, i, j=1,2, \ldots, n$, are constants.

Remark 8. The stability results about the zero solution of $x^{\prime}(t)=-b(t) x(t)+c(t) x(t-\tau(t))+d(t) x^{\prime}(t-\tau(t))$ are obtained by the fixed-point theory in [15]. But the differentiability of $\tau$ is needed.

## 3. An Illustrative Example

To show the effectiveness of Theorem 4, consider the following nonautonomous neural networks with impulse:

$$
\begin{align*}
& \dot{x}_{i}(t)=-b_{i}(t) x_{i}(t)+\sum_{j=1}^{2} a_{i j}(t) f_{i j}\left(x_{j}(t)\right) \\
&+\sum_{j=1}^{2} c_{i j}(t) g_{i j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)  \tag{34a}\\
&+\sum_{j=1}^{2} d_{i j}(t) h_{i j}\left(x_{j}^{\prime}\left(t-\widehat{\tau}_{i j}(t)\right)\right) \\
&+k_{i}(t), \quad \text { a.e. } t>0, \\
& x_{i}\left(t_{k}^{+}\right)=g_{i} x_{i}\left(t_{k}\right)+I_{i}, \\
& t_{k}=5 k, \quad i=1,2 ; \quad k=1,2, \ldots, \tag{34b}
\end{align*}
$$

where

$$
\begin{gathered}
\binom{b_{1}(t)}{b_{2}(t)}=\binom{7+\sin t}{5-\cos t}, \quad\binom{k_{1}(t)}{k_{2}(t)}=\binom{e^{-t}}{e^{-2 t}}, \\
\binom{g_{1}}{g_{2}}=\binom{0.6}{0.3}, \quad\binom{I_{1}}{I_{2}}=\binom{0.3}{-0.1}, \\
\left(a_{i j}(t)\right)_{2 \times 2}=\left(\begin{array}{cc}
0, & \frac{1}{3} \cos 3 t \\
\frac{\cos 2 t}{2}, & 0
\end{array}\right), \\
\left(c_{i j}(t)\right)_{2 \times 2}=\left(\begin{array}{cc}
\sin 2 t, & 0 \\
0, & \frac{\cos t}{2}
\end{array}\right) \\
\left(d_{i j}(t)\right)_{2 \times 2}=\left(\begin{array}{cc}
\frac{1}{6} \sin 3 t, & \frac{1}{8} \sin t \\
\frac{1}{9} \cos t, & \frac{1}{10} \cos 2 t
\end{array}\right),
\end{gathered}
$$


(b)

Figure 1: (a) Time response of state variables $x_{1}$, $u_{1}$ without impulsive effects. (b) Time response of state variables $x_{1}, u_{1}$ with impulsive effects.


Figure 2: (a) Time response of state variables $x_{2}$, $u_{2}$ without impulsive effects. (b) Time response of state variables $x_{2}$, $u_{2}$ with impulsive effects.

$$
\begin{gather*}
\left(f_{i j}(x)\right)_{2 \times 2}=\left(\begin{array}{cc}
0, & \frac{|x+1|-|x-1|}{2} \\
\frac{|x+1|+|x-1|}{2}, & 0
\end{array}\right), \\
\left(g_{i j}(x)\right)_{2 \times 2}=\left(\begin{array}{cc}
\frac{|x+1|+|x-1|}{3}, & 0 \\
0, & \frac{|x+1|-|x-1|}{3}
\end{array}\right),  \tag{35}\\
\left(h_{i j}(x)\right)_{2 \times 2}=\left(\begin{array}{cc}
\sin x, & \cos x \\
\cos x, & \sin x
\end{array}\right) \\
\left(\tau_{i j}(t)\right)_{2 \times 2}=\left(\begin{array}{cc}
2 \sin ^{2} t, & 0 \\
0, & 2|\cos t|
\end{array}\right)
\end{gather*}
$$

$$
\left(\widehat{\tau}_{i j}(t)\right)_{2 \times 2}=\left(\begin{array}{cc}
0, & \frac{1-\sin t}{2} \\
\frac{1+\cos t}{2}, & 0
\end{array}\right)
$$

Obviously, $\left(F_{i j}\right)_{2 \times 2}=\left(\begin{array}{ll}0, & 1 \\ 1, & 0\end{array}\right),\left(G_{i j}\right)_{2 \times 2}=\left(\begin{array}{cc}2 / 3, & 0 \\ 0, & 2 / 3\end{array}\right)$, and $\left(H_{i j}\right)_{2 \times 2}=\binom{1,1}{1,1}$.

Let $p_{1}=p_{2}=1$ and $q_{1}=18, q_{2}=10$. From the above assumption, the conditions of Theorem 4 are satisfied. Therefore, (34a) and (34b) are globally exponentially stable. $\left(x_{1}(t), x_{2}(t)\right)^{T}$ and $\left(u_{1}(t), u_{2}(t)\right)^{T}$ are the solutions of (34a) and $(34 \mathrm{~b})$ with $x_{1}(0)=0.5, x_{2}(0)=-0.8$ and $u_{1}(0)=$ $-0.5, u_{2}(0)=0.8$, respectively. Figures $1(a)$ and $1(b)$ depict


Figure 3: (a) Phase plot in space $\left(t, x_{1}, x_{2}\right),\left(t, u_{1}, u_{2}\right)$ without impulsive effects. (b) Phase plot in space $\left(t, x_{1}, x_{2}\right),\left(t, u_{1}, u_{2}\right)$ with impulsive effects.
time response of state variables $x_{1}, u_{1}$ without and with impulse effects; Figures 2(a) and 2(b) depict time response of state variables $x_{2}, u_{2}$ without and with impulse effects; Figures 3(a) and 3(b) depict the phase plot in the space $\left(t, x_{1}, x_{2}\right),\left(t, u_{1}, u_{2}\right)$ without and with impulse effects.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work was supported by the Science Foundation of Shanxi Province (no. 2010021001-1) and the National Natural Science Foundation of China (nos. 11101251 and 11001157).

## References

[1] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, Theory of Impulsive Differential Equations, vol. 6 of Series in Modern Applied Mathematics, World Scientific, Singapore, 1989.
[2] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, vol. 74 of Mathematics and its Applications, Kluwer Academic, Dordrecht, The Netherlands, 1992.
[3] H.-F. Huo, "Existence of positive periodic solutions of a neutral delay Lotka-Volterra system with impulses," Computers \& Mathematics with Applications, vol. 48, no. 12, pp. 1833-1846, 2004.
[4] Z. Luo and J. Shen, "Oscillation for solutions of nonlinear neutral differential equations with impulses," Computers \& Mathematics with Applications, vol. 42, no. 10-11, pp. 1285-1292, 2001.
[5] J. R. Graef, J. H. Shen, and I. P. Stavroulakis, "Oscillation of impulsive neutral delay differential equations," Journal of Mathematical Analysis and Applications, vol. 268, no. 1, pp. 310333, 2002.
[6] X. Liu and J. Shen, "Asymptotic behavior of solutions of impulsive neutral differential equations," Applied Mathematics Letters, vol. 12, no. 7, pp. 51-58, 1999.
[7] D. Xu, Z. Yang, and Z. Yang, "Exponential stability of nonlinear impulsive neutral differential equations with delays," Nonlinear Analysis: Theory, Methods \& Applications, vol. 67, no. 5, pp. 1426-1439, 2007.
[8] H. Chen, C. Zhu, and Y. Zhang, "A note on exponential stability for impulsive neutral stochastic partial functional differential equations," Applied Mathematics and Computation, vol. 227, no. 15, pp. 139-147, 2014.
[9] Y. Chen, A. Xue, R. Lu, and S. Zhou, "On robustly exponential stability of uncertain neutral systems with time-varying delays and nonlinear perturbations," Nonlinear Analysis: Theory, Methods \& Applications, vol. 68, no. 8, pp. 2464-2470, 2008.
[10] S. Xu, J. Lam, D. W. C. Ho, and Y. Zou, "Delay-dependent exponential stability for a class of neural networks with time delays," Journal of Computational and Applied Mathematics, vol. 183, no. 1, pp. 16-28, 2005.
[11] A. Bellen and N. Guglielmi, "Solving neutral delay differential equations with state-dependent delays," Journal of Computational and Applied Mathematics, vol. 229, no. 2, pp. 350-362, 2009.
[12] F. Jiang and J. Sun, "Asymptotic behavior of neutral delay differential equation of Euler form with constant impulsive jumps," Applied Mathematics and Computation, vol. 219, no. 19, pp. 9906-9913, 2013.
[13] G. Ballinger and X. Liu, "Existence and uniqueness results for impulsive delay differential equations," Dynamics of Continuous, Discrete and Impulsive Systems, vol. 5, no. 1-4, pp. 579-591, 1999.
[14] W. Zhao, "Dynamics of Cohen-Grossberg neural network with variable coefficients and time-varying delays," Nonlinear Analysis: Real World Applications, vol. 9, no. 3, pp. 1024-1037, 2008.
[15] Y. N. Raffoul, "Stability in neutral nonlinear differential equations with functional delays using fixed-point theory," Mathematical and Computer Modelling, vol. 40, no. 7-8, pp. 691-700, 2004.

## Research Article

# Minimal Wave Speed of Bacterial Colony Model with Saturated Functional Response 

Tianran Zhang ${ }^{1}$ and Qingming Gou ${ }^{2}$<br>${ }^{1}$ School of Mathematics and Statistics, Southwest University, Chongqing 400715, China<br>${ }^{2}$ College of Mathematics \& Computer Science, Yangtze Normal University, Chongqing 408100, China

Correspondence should be addressed to Tianran Zhang; zhtr0123@126.com
Received 12 January 2014; Accepted 24 February 2014; Published 31 March 2014
Academic Editor: Weiming Wang
Copyright © 2014 T. Zhang and Q. Gou. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

By considering bacterium death and general functional response we develop previous model of bacterial colony which focused on the traveling speed of bacteria. The minimal wave speed for our model is expressed by parameters and the necessary and sufficient conditions for traveling wave solutions (TWSs) are given. To prove the existence of TWSs, an auxiliary system is introduced and the existence of TWSs for this auxiliary system is proved by Schauder's fixed point theorem. The limit arguments show the existence of TWSs for original system. By introducing negative one-sided Laplace transform, we prove the nonexistence of TWSs.


## 1. Introduction

Experiments show that bacterial colonies on agar plates with nutrients exhibit a variety of sizes and shapes [1-7]. According to the substrate softness and nutrient concentration, the colony patterns are divided into five types $[6,8]$. Why were so many rich diffusive patterns observed in bacterial experiments? To answer this question, lots of diffusive mathematical models have been proposed and studied [4, 7, 9-16]. In these mathematical models, the colony patterns are proved or simulated on bounded domains. For bacterial colony, the colony speed is one of the most important focuses and traveling wave solution (TWS) can foresee such speed. Thus many researches studied the bacterial colony speeds through TWSs [17-24].

To more exactly anticipate the traveling speed of bacterial colony, we develop above TWS models to a more accurate bacterial colony model with bacterium death and general functional response, which is more complex compared with above TWS models. Let $N(t, x)$ and $B(t, x)$ denote the concentrations of nutrients and bacteria at time $t$ and position $x$, respectively. Then our model is as follows:

$$
\begin{align*}
N_{t} & =d_{N} N_{x x}-f(N) B,  \tag{1}\\
B_{t} & =d_{B} B_{x x}+\kappa f(N) B-d B,
\end{align*}
$$

where parameters $d_{N}$ and $d_{B}$ denote the motility of the nutrients and bacteria. $\kappa$ is the conversion rate of nutrients to bacteria and $d$ is the death rate of bacteria. Function $f(N)$ is the functional response to nutrients. For simplicity, we assume $f(N)=k_{1} N /\left(1+k_{2} N\right)$ with $k_{1}>0$ and $k_{2}>0$. Actually, in the following proof we only use the monotonicity and boundedness of $f(N)$.

In this paper, the minimal wave speed $c^{*}$ is given and the necessary and sufficient conditions for the existence of TWSs are obtained. To arrive at such aim, the existence of TWSs is proved by Schauder's fixed point theorem and the nonexistence is finished by negative one-sided Laplace transform proposed firstly by us. To apply Schauder's fixed point theorem, a bounded invariant cone is needed. Such cone is constructed generally by a pair of upper and lower solutions. However, it is difficult for us to construct such solutions for model (1). Consequently, an auxiliary system is introduced, for which the upper and lower solutions can be easily constructed and are very simple. Such type of upper and lower solutions is motivated by Diekmann [25]. Then limit arguments give the existence of TWSs of model (1). Twosided Laplace transform was firstly introduced by Carr and Chmaj [26] to prove nonexistence of TWSs and was further applied by [27-29]. However, the introduction of negative one-sided Laplace transform simplifies the proof.

This paper is organized as follows. In the next section, an auxiliary system is firstly introduced and the existence of TWSs is proved by Schauder's fixed point theorem. Then limit arguments give the existence of TWSs for original system. In Section 3, the negative one-sided Laplace transform is defined and then the nonexistence of TWSs is obtained.

## 2. Existence of Traveling Wave Solution

A traveling wave solution of system (1) is a nonnegative nontrivial solution of the form

$$
\begin{equation*}
N(t, x)=U(\xi), \quad B(t, x)=V(\xi), \quad \xi=x+c t \tag{2}
\end{equation*}
$$

satisfying boundary condition

$$
\begin{align*}
& (U(-\infty), V(-\infty))=\left(N^{0}, 0\right) \\
& (U(+\infty), V(+\infty))=\left(N^{1}, 0\right) \tag{3}
\end{align*}
$$

where $N^{0}>0$ is initial density of nutrients. It is obvious that $N^{0}>N^{1} \geq 0$.

Define $c^{*}=2 \sqrt{d_{B}\left[\kappa f\left(N^{0}\right)-d\right]}$. The existence of traveling wave solutions is given as follows.

Theorem 1. Suppose $f\left(N^{0}\right)>d / \kappa$. For any $c \geq c^{*}$ system (1) has a traveling wave solution $(U(x+c t), V(x+c t))$ satisfying boundary conditions (3) such that $U(\xi)$ is nonincreasing in $\mathbb{R}$ and $f\left(N^{1}\right)<d / \kappa$. Furthermore, one has that

$$
\begin{equation*}
\int_{-\infty}^{+\infty} V(\eta) d \eta=\frac{\kappa c}{d}\left(N^{0}-N^{1}\right), \quad 0 \leq V(\xi) \leq \kappa\left(N^{0}-N^{1}\right), \tag{4}
\end{equation*}
$$

for any $\xi \in \mathbb{R}$.
Substituting wave profile $N(t, x)=U(\xi), B(t, x)=$ $V(\xi), \quad \xi=x+c t$ into system (1) yields the following equations:

$$
\begin{align*}
& c U^{\prime}=d_{N} U^{\prime \prime}-f(U) V,  \tag{5}\\
& c V^{\prime}=d_{B} V^{\prime \prime}+\kappa f(U) V-d V
\end{align*}
$$

where ' denotes the derivative with respect to $\xi$.
To prove the existence of solutions of (5) satisfying (3), we construct an auxiliary system:

$$
\begin{align*}
& c U^{\prime}=d_{N} U^{\prime \prime}-f(U) V \\
& c V^{\prime}=d_{B} V^{\prime \prime}+\kappa f(U) V-d V-\gamma V^{2} \tag{6}
\end{align*}
$$

where $\gamma$ is a positive constant and can be supposed to be small enough according to what we will need. Next, an invariant cone will be constructed and Schauder's fixed point theorem will be used to prove the existence of traveling wave solutions. We firstly linearize the second equation of (6) at $\left(N^{0}, 0\right)$ and obtain

$$
\begin{equation*}
c \phi^{\prime}=d_{B} \phi^{\prime \prime}+\kappa f\left(N^{0}\right) \phi-d \phi \tag{7}
\end{equation*}
$$

Obviously, the characteristic equation is

$$
\begin{equation*}
H(\lambda)=d_{B} \lambda^{2}-c \lambda+\kappa f\left(N^{0}\right)-d=0 . \tag{8}
\end{equation*}
$$

Denote $\lambda_{1}=\left(c-\sqrt{c^{2}-c^{* 2}}\right) /\left(2 d_{B}\right)$ and $\lambda_{2}=(c+$ $\left.\sqrt{c^{2}-c^{* 2}}\right) /\left(2 d_{B}\right)$. In the remainder of this section, we always suppose $\kappa f\left(N^{0}\right)>d$ and $c>c^{*}$ hold unless other conditions are specified. Define

$$
\begin{align*}
& \underline{U}(\xi)=\max \left\{N^{0}-\sigma e^{\alpha \xi}, 0\right\} \\
& \bar{V}(\xi)=\min \left\{e^{\lambda_{1} \xi}, V^{0}\right\}  \tag{9}\\
& \underline{V}(\xi)=\max \left\{e^{\lambda_{1} \xi}\left(1-M e^{\varepsilon \xi}\right), 0\right\}
\end{align*}
$$

where $V^{0}=\left(\kappa f\left(N^{0}\right)-d\right) / \gamma$ and $\gamma<\kappa f\left(N^{0}\right)-d$.
Lemma 2. The function $\bar{V}(\xi)$ satisfies inequality

$$
\begin{equation*}
c \bar{V}^{\prime} \geq d_{B} \bar{V}^{\prime \prime}+\kappa f\left(N^{0}\right) \bar{V}-d \bar{V}-\gamma \bar{V}^{2} \tag{10}
\end{equation*}
$$

for any $\xi \neq \ln V^{0} / \lambda_{1}$.
Proof. Firstly, assume $\xi<\ln V^{0} / \lambda_{1}$ and, therefore, $\bar{V}(\xi)=$ $e^{\lambda_{1} \xi}$. Since $\bar{V}(\xi)$ satisfies (7), we have

$$
\begin{equation*}
c \bar{V}^{\prime}-d_{B} \bar{V}^{\prime \prime}-\kappa f\left(N^{0}\right) \bar{V}+d \bar{V}+\gamma \bar{V}^{2}=\gamma \bar{V}^{2} \geq 0 . \tag{11}
\end{equation*}
$$

Secondly, let $\xi>\ln V^{0} / \lambda_{1}$, which implies $\bar{V}(\xi)=V^{0}$. We have that

$$
\begin{align*}
c \bar{V}^{\prime}- & d_{B} \bar{V}^{\prime \prime}-\kappa f\left(N^{0}\right) \bar{V}+d \bar{V}+\gamma \bar{V}^{2} \\
& =-\kappa f\left(N^{0}\right) V^{0}+d V^{0}+\gamma V^{0^{2}}=0 . \tag{12}
\end{align*}
$$

The proof is completed.
Lemma 3. For $0<\alpha<\min \left\{c / d_{N}, \lambda_{1}\right\}$ and $\sigma>$ $\max \left\{N^{0}, f\left(N^{0}\right) /\left(c-d_{N} \alpha\right)\right\}$, the function $\underline{U}(\xi)$ satisfies

$$
\begin{equation*}
c \underline{U}^{\prime} \leq d_{N} \underline{U}^{\prime \prime}-f(\underline{U}(\xi)) \bar{V}(\xi) \tag{13}
\end{equation*}
$$

for any $\xi \neq 1 / \alpha \ln \left(N^{0} / \sigma\right)$.
Proof. It is easy to show that $1 / \alpha \ln \left(N^{0} / \sigma\right)<0 \leq$ $\min \left\{0, \ln V^{0} / \lambda_{1}\right\}$. When $\xi>1 / \alpha \ln \left(N^{0} / \sigma\right)$, then $\underline{U}(\xi)=0$ and the lemma is obviously true. Now, suppose $\xi<1 / \alpha \ln \left(N^{0} / \sigma\right)$. Then $\underline{U}(\xi)=N^{0}-\sigma e^{\alpha \xi}$ and

$$
\begin{align*}
-c \underline{U}^{\prime} & +d_{N} \underline{U}^{\prime \prime}-f(\underline{U}(\xi)) \bar{V}(\xi) \\
& =c \sigma \alpha e^{\alpha \xi}-d_{N} \sigma \alpha^{2} e^{\alpha \xi}-f\left(N^{0}-\sigma e^{\alpha \xi}\right) e^{\lambda_{1} \xi} \\
& =\left[c \sigma \alpha-d_{N} \sigma \alpha^{2}-f\left(N^{0}-\sigma e^{\alpha \xi}\right) e^{\left(\lambda_{1}-\alpha\right) \xi}\right] e^{\alpha \xi}  \tag{14}\\
& \geq\left[\left(c-d_{N} \alpha\right) \alpha \sigma-f\left(N^{0}\right)\right] e^{\alpha \xi} \\
& \geq 0 .
\end{align*}
$$

Thus the proof is completed.

Lemma 4. Let $\varepsilon<\alpha<\min \left\{\lambda_{1}, \lambda_{2}-\lambda_{1}\right\} / 2$. Then for $M>0$ large enough, the function $\underline{V}(\xi)$ satisfies

$$
\begin{equation*}
c \underline{V^{\prime}} \leq d_{B} \underline{V}^{\prime \prime}+\kappa f(\underline{U}) \underline{V}-d \underline{V}-\gamma \underline{V^{2}} \tag{15}
\end{equation*}
$$

for any $\xi \neq 1 / \varepsilon \ln (1 / M)$.
Proof. It is clear that $\underline{U}(\xi)=0$ if and only if $\xi=1 / \alpha \ln \left(N^{0} / \sigma\right)$, that $\underline{V}(\xi)=0$ if and only if $\xi=1 / \varepsilon \ln (1 / M)$, and that $1 / \varepsilon \ln (1 / M)<1 / \alpha \ln \left(N^{0} / \sigma\right)$ if and only if $M>\left(\sigma / N^{0}\right)^{(\varepsilon / \alpha)}$. Assume $M>\left(\sigma / N^{0}\right)^{(\varepsilon / \alpha)}$. When $\xi>1 / \varepsilon \ln (1 / M)$, then $e^{\lambda_{1} \xi}\left(1-M e^{\varepsilon \xi}\right)<0, \underline{V}(\xi)=0$, and Lemma 4 holds.

In this paragraph, assume $\xi<1 / \varepsilon \ln (1 / M)$. Then $\xi<$ $1 / \alpha \ln \left(N^{0} / \sigma\right), \underline{U}(\xi)=N^{0}-\sigma e^{\alpha \xi}>0$, and $\underline{V}(\xi)=e^{\lambda_{1} \xi}(1-$ $\left.M e^{\varepsilon \xi}\right)>0$. To prove this lemma, it is enough to show

$$
\begin{align*}
0 \leq & e^{-\lambda_{1} \xi}\left[d_{B} \underline{V}^{\prime \prime}-c \underline{V}^{\prime}+\kappa f(\underline{U}) \underline{V}-d \underline{V}-\gamma \underline{V}^{2}\right] \\
= & d_{B} \lambda_{1}^{2}-d_{B} M\left(\lambda_{1}+\varepsilon\right)^{2} e^{\varepsilon \xi}-c \lambda_{1}+c M\left(\lambda_{1}+\varepsilon\right) e^{\varepsilon \xi} \\
& -d+d M e^{\varepsilon \xi} \\
& +\kappa\left[f\left(N^{0}\right)-f^{\prime}\left(\underline{U}^{0}\right) \sigma e^{\alpha \xi}\right]\left(1-M e^{\varepsilon \xi}\right) \\
& -\gamma e^{\lambda_{1} \xi}\left(1-M e^{\varepsilon \xi}\right)^{2} \\
= & d_{B} \lambda_{1}^{2}-c \lambda_{1}+\kappa f\left(N^{0}\right)-d \\
& +M\left[-d_{B}\left(\lambda_{1}+\varepsilon\right)^{2}+c\left(\lambda_{1}+\varepsilon\right)-\kappa f\left(N^{0}\right)+d\right] e^{\varepsilon \xi} \\
& -\kappa f^{\prime}\left(\underline{U^{0}}\right) \sigma e^{\alpha \xi}-\gamma e^{\lambda_{1} \xi}\left(1-M e^{\varepsilon \xi}\right)^{2}+M \kappa f^{\prime}\left(\underline{U^{0}}\right) \sigma e^{\alpha \xi} e^{\varepsilon \xi} \\
= & {\left[-M H\left(\lambda_{1}+\varepsilon\right)-\kappa f^{\prime}\left(\underline{U}^{0}\right) \sigma e^{(\alpha-\varepsilon) \xi}\right.} \\
& \left.-\gamma\left(1-M e^{\varepsilon \xi}\right)^{2} e^{\left(\lambda_{1}-\varepsilon\right) \xi}\right] e^{\varepsilon \xi} \\
& +M \kappa f^{\prime}\left(\underline{U}^{0}\right) \sigma e^{\alpha \xi} e^{\varepsilon \xi}, \tag{16}
\end{align*}
$$

where $\underline{U}(\xi)<\underline{U}^{0}<N^{0}$. Since $f^{\prime}\left(\underline{U}^{0}\right)>0$, we only need to show

$$
\begin{equation*}
-M H\left(\lambda_{1}+\varepsilon\right) \geq \kappa f^{\prime}\left(\underline{U}^{0}\right) \sigma e^{(\alpha-\varepsilon) \xi}+\gamma\left(1-M e^{\varepsilon \xi}\right)^{2} e^{\left(\lambda_{1}-\varepsilon\right) \xi} \tag{17}
\end{equation*}
$$

Since $\xi<1 / \alpha \ln \left(N^{0} / \sigma\right)<0$ by $\sigma>N^{0}$ and $0<f^{\prime}(N)<k_{1}$ for any $N \geq 0$, we have

$$
\begin{align*}
& \kappa k_{1} \sigma>\kappa f^{\prime}\left(\underline{U}^{0}\right) \sigma e^{(\alpha-\varepsilon) \xi}  \tag{18}\\
& \gamma \geq \gamma\left(1-M e^{\varepsilon \xi}\right)^{2} e^{\left(\lambda_{1}-\varepsilon\right) \xi}
\end{align*}
$$

Since $H\left(\lambda_{1}+\varepsilon\right)<0$, inequality (17) is satisfied if

$$
\begin{equation*}
M>-\frac{\kappa k_{1} \sigma+\gamma}{H\left(\lambda_{1}+\varepsilon\right)} \tag{19}
\end{equation*}
$$

The proof is completed.

To apply Schauder's fixed point theorem, we will introduce a topology in $C\left(\mathbb{R}, \mathbb{R}^{2}\right)$. Let $\Lambda_{11}<0<\Lambda_{12}$ be the roots of

$$
\begin{equation*}
d_{N} \Lambda^{2}-c \Lambda-\beta_{1}=0 \tag{20}
\end{equation*}
$$

and $\Lambda_{21}<0<\Lambda_{22}$ the roots of

$$
\begin{equation*}
d_{B} \Lambda^{2}-c \Lambda-\beta_{2}=0 \tag{21}
\end{equation*}
$$

where $\beta_{1}$ and $\beta_{2}$ are positive constants that will be determined later. Let $\mu$ be a positive constant which can be small enough. For $\Phi(\xi)=\left(\phi_{1}(\xi), \phi_{2}(\xi)\right)$, define

$$
\begin{gather*}
|\Phi(\cdot)|_{\mu}=\max \left\{\sup _{\xi \in \mathbb{R}}\left|\phi_{1}(\xi)\right| e^{-\mu|\xi|}, \sup _{\xi \in \mathbb{R}}\left|\phi_{2}(\xi)\right| e^{-\mu|\xi|}\right\}  \tag{22}\\
B_{\mu}\left(\mathbb{R}, \mathbb{R}^{2}\right)=\left\{\Phi(\cdot) \in C\left(\mathbb{R}, \mathbb{R}^{2}\right):|\Phi(\cdot)|_{\mu}<+\infty\right\}
\end{gather*}
$$

We will find the traveling wave solution in the following profile set:

$$
\begin{align*}
& \Gamma=\{ (U(\cdot), V(\cdot)) \in C\left(\mathbb{R}, \mathbb{R}^{2}\right): \underline{U}(\xi) \leq U(\xi) \leq N^{0}, \\
&\underline{V}(\xi) \leq V(\xi) \leq \bar{V}(\xi) \text { for any } \xi \in \mathbb{R}\} . \tag{23}
\end{align*}
$$

Obviously, $\Gamma$ is closed and convex in $C\left(\mathbb{R}, \mathbb{R}^{2}\right)$. Firstly, we change system (6) into the following form:

$$
\begin{align*}
& -d_{N} U^{\prime \prime}+c U^{\prime}+\beta_{1} U=H_{1}(U, V)(\xi) \\
& -d_{B} V^{\prime \prime}+c V^{\prime}+\beta_{2} V=H_{2}(U, V)(\xi) \tag{24}
\end{align*}
$$

where $\beta_{1} \geq V^{0}, \beta_{2} \geq 2 \gamma V^{0}+d=2\left[\kappa f\left(N^{0}\right)-d\right]+d$, and

$$
\begin{align*}
& H_{1}(U, V)(\xi)=\beta_{1} U(\xi)-f(U(\xi)) V(\xi) \\
& H_{2}(U, V)(\xi)=\left[\beta_{2}-d+\kappa f(U(\xi))-\gamma V(\xi)\right] V(\xi) . \tag{25}
\end{align*}
$$

Furthermore, define $F=\left(F_{1}, F_{2}\right): \Gamma \rightarrow C\left(\mathbb{R}, \mathbb{R}^{2}\right)$ by

$$
\begin{aligned}
& F_{1}(U(\cdot), V(\cdot))(\xi) \\
&=\frac{1}{d_{N} \Lambda_{1}} {\left[\int_{-\infty}^{\xi} e^{\Lambda_{11}(\xi-t)} H_{1}(U, V)(t) d t\right.} \\
&\left.+\int_{\xi}^{+\infty} e^{\Lambda_{12}(\xi-t)} H_{1}(U, V)(t) d t\right]
\end{aligned}
$$

$$
F_{2}(U(\cdot), V(\cdot))(\xi)
$$

$$
\begin{align*}
=\frac{1}{d_{B} \Lambda_{2}} & {\left[\int_{-\infty}^{\xi} e^{\Lambda_{21}(\xi-t)} H_{2}(U, V)(t) d t\right.} \\
& \left.+\int_{\xi}^{+\infty} e^{\Lambda_{22}(\xi-t)} H_{2}(U, V)(t) d t\right] \tag{26}
\end{align*}
$$

where $\Lambda_{1}=\Lambda_{12}-\Lambda_{11}, \Lambda_{2}=\Lambda_{22}-\Lambda_{21}$.
Lemma 5. Consider $F(\Gamma) \subset \Gamma$.

Proof. Suppose $(U(\cdot), V(\cdot)) \in \Gamma$; that is, $\underline{U}(\xi) \leq U(\xi) \leq$ $N^{0}, \underline{V}(\xi) \leq V(\xi) \leq \bar{V}(\xi)$ for any $\xi \in \mathbb{R}$. Then we will prove that

$$
\begin{gather*}
\underline{U}(\xi) \leq F_{1}(U(\cdot), V(\cdot))(\xi) \leq N^{0}  \tag{27}\\
\underline{V}(\xi) \leq F_{2}(U(\cdot), V(\cdot))(\xi) \leq \bar{V}(\xi),
\end{gather*}
$$

for any $\xi \in \mathbb{R}$.
If $\xi \geq \xi_{0} \triangleq 1 / \varepsilon \ln (1 / M)$, then $\underline{V}(\xi)=0$, which implies that $F_{2}(U(\cdot), V(\cdot))(\xi) \geq \underline{V}(\xi)$ since $U(\xi) \geq \underline{U}(\xi) \geq 0, V(\xi) \geq$ $\underline{V}(\xi) \geq 0$. Assume $\xi<\xi_{0}$. From Lemma 4 and $\beta_{2} \geq 2 \gamma V^{0}+d$, $\overline{\text { it }}$ is clear that

$$
\begin{align*}
& -d_{B} \underline{V}^{\prime \prime}+c \underline{V^{\prime}}+\beta_{2} \underline{V}(\xi) \\
& \quad \leq\left[\beta_{2}-d+\kappa f(\underline{U}(\xi))-\gamma \underline{V}(\xi)\right] \underline{V}(\xi)  \tag{28}\\
& \quad \leq\left[\beta_{2}-d+\kappa f(U(\xi))-\gamma V(\xi)\right] V(\xi) \\
& \quad=H_{2}(U, V)(\xi)
\end{align*}
$$

which implies that

$$
\begin{align*}
& F_{2}(U(\cdot), V(\cdot))(\xi) \\
&= \frac{1}{d_{B} \Lambda_{2}}\left[\int_{-\infty}^{\xi} e^{\Lambda_{21}(\xi-t)} H_{2}(U, V)(t) d t\right. \\
&\left.+\int_{\xi}^{+\infty} e^{\Lambda_{22}(\xi-t)} H_{2}(U, V)(t) d t\right] \\
& \geq \frac{1}{d_{B} \Lambda_{2}} \int_{-\infty}^{\xi} e^{\Lambda_{21}(\xi-t)}\left[-d_{B} \underline{V}^{\prime \prime}(t)+c \underline{V}^{\prime}(t)+\beta_{2} \underline{V}(t)\right] d t \\
&+\frac{1}{d_{B} \Lambda_{2}} \int_{\xi}^{\xi_{0}} e^{\Lambda_{22}(\xi-t)}\left[-d_{B} \underline{V}^{\prime \prime}(t)+c \underline{V^{\prime}}(t)+\beta_{2} \underline{V}(t)\right] d t \\
&+\frac{1}{d_{B} \Lambda_{2}} \int_{\xi_{0}}^{+\infty} e^{\Lambda_{22}(\xi-t)}\left[-d_{B} \underline{V}^{\prime \prime}(t)+c \underline{V}^{\prime}(t)+\beta_{2} \underline{V}(t)\right] d t \\
&= \underline{V}(\xi)+\frac{1}{\Lambda_{2}} e^{\Lambda_{22}\left(\xi-\xi_{0}\right)}\left[\underline{V^{\prime}}\left(\xi_{0}+0\right)-\underline{V^{\prime}}\left(\xi_{0}-0\right)\right] \\
& \geq \underline{V}(\xi), \tag{29}
\end{align*}
$$

where the final inequality is due to $\underline{V}^{\prime}\left(\xi_{0}+0\right)=0$ and $\underline{V}^{\prime}\left(\xi_{0}-\right.$ $0)<0$. In conclusion, $F_{2}(U(\cdot), V(\cdot))(\xi) \geq \underline{V}(\xi)$ for any $\bar{\xi} \in \mathbb{R}$. Similarly, it can be proved that

$$
\begin{gather*}
\underline{U}(\xi) \leq F_{1}(U(\cdot), V(\cdot))(\xi) \leq N^{0} \\
\quad F_{2}(U(\cdot), V(\cdot))(\xi) \leq \bar{V}(\xi) \tag{30}
\end{gather*}
$$

for any $\xi \in \mathbb{R}$. The proof is completed.
Lemma 6. For $\mu$ small enough, map $F=\left(F_{1}, F_{2}\right): \Gamma \rightarrow$ $C\left(\mathbb{R}, \mathbb{R}^{2}\right)$ is continuous with respect to the norm $|\cdot|_{\mu}$ in $B_{\mu}\left(\mathbb{R}, \mathbb{R}^{2}\right)$.

Proof. Suppose $\Phi_{i}(\cdot)=\left(U_{i}(\cdot), V_{i}(\cdot)\right) \in \Gamma$, which implies

$$
\begin{equation*}
0 \leq U_{i}(\xi) \leq N^{0}, \quad 0 \leq V_{i}(\xi) \leq V^{0} \tag{31}
\end{equation*}
$$

for any $\xi \in \mathbb{R}$, where $i=1,2$. Then we have

$$
\begin{align*}
& \left|H_{2}\left(\Phi_{1}\right)(\xi)-H_{2}\left(\Phi_{2}\right)(\xi)\right| e^{-\mu|\xi|} \\
& =\mid\left(\beta_{2}-d\right)\left[V_{1}(\xi)-V_{2}(\xi)\right]-\gamma\left[V_{1}(\xi)+V_{2}(\xi)\right] \\
& \quad \times\left[V_{1}(\xi)-V_{2}(\xi)\right]+\kappa f\left(U_{1}(\xi)\right)\left[V_{1}(\xi)-V_{2}(\xi)\right] \\
& \quad+\kappa V_{2}(\xi)\left[f\left(U_{1}(\xi)\right)-f\left(U_{2}(\xi)\right)\right] \mid e^{-\mu|\xi|} \\
& \leq\left[\beta_{2}-d+2 \gamma V^{0}+\kappa f\left(N^{0}\right)\right]\left|\Phi_{1}(\cdot)-\Phi_{2}(\cdot)\right|_{\mu} \\
& \quad+\kappa V_{2}(\xi) f^{\prime}\left(U^{*}\right)\left|U_{1}(\xi)-U_{2}(\xi)\right| e^{-\mu|\xi|} \\
& \leq\left[\beta_{2}-d+2 \gamma V^{0}+\kappa f\left(N^{0}\right)+\kappa V^{0} f^{\prime}(0)\right]\left|\Phi_{1}(\cdot)-\Phi_{2}(\cdot)\right|_{\mu} \\
& =M_{1}\left|\Phi_{1}(\cdot)-\Phi_{2}(\cdot)\right|_{\mu} \tag{32}
\end{align*}
$$

where $U^{*}$ is between $U_{1}(\xi)$ and $U_{2}(\xi)$ and

$$
\begin{equation*}
M_{1}=\beta_{2}-d+2 \gamma V^{0}+\kappa f\left(N^{0}\right)+\kappa V^{0} f^{\prime}(0)>0 \tag{33}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
& \left|F_{2}\left(\Phi_{1}(\cdot)\right)(\xi)-F_{2}\left(\Phi_{2}(\cdot)\right)(\xi)\right| e^{-\mu|\xi|} \\
& \begin{aligned}
\leq \frac{e^{-\mu|\xi|}}{d_{B} \Lambda_{2}}[ & \int_{-\infty}^{\xi} e^{\Lambda_{21}(\xi-t)}\left|H_{2}\left(\Phi_{1}\right)(t)-H_{2}\left(\Phi_{2}\right)(t)\right| d t \\
& \left.+\int_{\xi}^{+\infty} e^{\Lambda_{22}(\xi-t)}\left|H_{2}\left(\Phi_{1}\right)(t)-H_{2}\left(\Phi_{2}\right)(t)\right| d t\right] \\
\leq \frac{M_{1} e^{-\mu|\xi|}}{d_{B} \Lambda_{2}} & {\left[\int_{-\infty}^{\xi} e^{\Lambda_{21}(\xi-t)+\mu|t|} d t\right.} \\
& \left.\quad+\int_{\xi}^{+\infty} e^{\Lambda_{22}(\xi-t)+\mu|t|} d t\right]\left|\Phi_{1}(\cdot)-\Phi_{2}(\cdot)\right|_{\mu} .
\end{aligned}
\end{align*}
$$

Set $\mu<\min \left\{-\Lambda_{21}, \Lambda_{22}\right\}$. If $\xi<0$, it holds that

$$
\begin{align*}
& \left|F_{2}\left(\Phi_{1}(\cdot)\right)(\xi)-F_{2}\left(\Phi_{2}(\cdot)\right)(\xi)\right| e^{-\mu|\xi|} \\
& \leq \frac{M_{1} e^{\mu \xi}}{d_{B} \Lambda_{2}}\left[e^{\Lambda_{21} \xi} \int_{-\infty}^{\xi} e^{-\left(\Lambda_{21}+\mu\right) t} d t+e^{\Lambda_{22} \xi} \int_{\xi}^{0} e^{-\left(\Lambda_{22}+\mu\right) t} d t\right. \\
& \\
& \left.\quad+e^{\Lambda_{22} \xi} \int_{0}^{+\infty} e^{\left(\mu-\Lambda_{22}\right) t} d t\right]\left|\Phi_{1}(\cdot)-\Phi_{2}(\cdot)\right|_{\mu} \\
& = \\
& \quad \frac{M_{1}}{d_{B} \Lambda_{2}}\left[\frac{1}{-\Lambda_{21}-\mu}+\frac{1-e^{\left(\Lambda_{22}+\mu\right) \xi}}{\Lambda_{22}+\mu}+\frac{e^{\left(\Lambda_{22}+\mu\right) \xi}}{\Lambda_{22}-\mu}\right] \\
& \quad \times\left|\Phi_{1}(\cdot)-\Phi_{2}(\cdot)\right|_{\mu}  \tag{35}\\
& \leq \\
& \quad \frac{M_{1}}{d_{B} \Lambda_{2}}\left(\frac{1}{-\Lambda_{21}-\mu}+\frac{1}{\Lambda_{22}+\mu}+\frac{1}{\Lambda_{22}-\mu}\right) \\
& \quad \times\left|\Phi_{1}(\cdot)-\Phi_{2}(\cdot)\right|_{\mu} \cdot
\end{align*}
$$

If $\xi \geq 0$, we have

$$
\begin{align*}
& \left|F_{2}\left(\Phi_{1}(\cdot)\right)(\xi)-F_{2}\left(\Phi_{2}(\cdot)\right)(\xi)\right| e^{-\mu|\xi|} \\
& \begin{aligned}
\leq & \frac{M_{1} e^{-\mu \xi}}{d_{B} \Lambda_{2}}\left[e^{\Lambda_{21} \xi} \int_{-\infty}^{0} e^{-\left(\Lambda_{21}+\mu\right) t} d t+e^{\Lambda_{21} \xi} \int_{0}^{\xi} e^{\left(\mu-\Lambda_{21}\right) t} d t\right. \\
& \left.\quad+e^{\Lambda_{22} \xi} \int_{\xi}^{+\infty} e^{\left(\mu-\Lambda_{22}\right) t} d t\right]\left|\Phi_{1}(\cdot)-\Phi_{2}(\cdot)\right|_{\mu} \\
= & \frac{M_{1}}{d_{B} \Lambda_{2}}\left[\frac{e^{\left(\Lambda_{21}-\mu\right) \xi}}{-\Lambda_{21}-\mu}+\frac{1-e^{\left(\Lambda_{21}-\mu\right) \xi}}{\mu-\Lambda_{21}}+\frac{1}{\Lambda_{22}-\mu}\right] \\
& \times\left|\Phi_{1}(\cdot)-\Phi_{2}(\cdot)\right|_{\mu} \\
\leq & \frac{M_{1}}{d_{B} \Lambda_{2}}\left(\frac{1}{-\Lambda_{21}-\mu}+\frac{1}{\mu-\Lambda_{21}}+\frac{1}{\Lambda_{22}-\mu}\right) \\
& \times\left|\Phi_{1}(\cdot)-\Phi_{2}(\cdot)\right|_{\mu} \cdot
\end{aligned}
\end{align*}
$$

Consequently, we conclude that

$$
\begin{equation*}
\left|F_{2}\left(\Phi_{1}(\cdot)\right)(\cdot)-F_{2}\left(\Phi_{2}(\cdot)\right)(\cdot)\right|_{\mu} \leq M_{2}\left|\Phi_{1}(\cdot)-\Phi_{2}(\cdot)\right|_{\mu}, \tag{37}
\end{equation*}
$$

where

$$
\begin{align*}
M_{2}=\frac{M_{1}}{d_{B} \Lambda_{2}} \max \{ & \frac{1}{-\Lambda_{21}-\mu}+\frac{1}{\Lambda_{22}+\mu}+\frac{1}{\Lambda_{22}-\mu} \\
& \left.\frac{1}{-\Lambda_{21}-\mu}+\frac{1}{\mu-\Lambda_{21}}+\frac{1}{\Lambda_{22}-\mu}\right\} . \tag{38}
\end{align*}
$$

Thus $F_{2}: \Gamma \rightarrow C(\mathbb{R}, \mathbb{R})$ is continuous with respect to the norm $|\cdot|_{\mu}$ in $B_{\mu}(\mathbb{R}, \mathbb{R})$. Similarly, it can be proved that $F_{1}$ : $\Gamma \rightarrow C(\mathbb{R}, \mathbb{R})$ is also continuous with respect to the norm $|\cdot|_{\mu}$ in $B_{\mu}(\mathbb{R}, \mathbb{R})$. The proof is completed.

Lemma 7. Map $F=\left(F_{1}, F_{2}\right): \Gamma \rightarrow \Gamma$ is compact with respect to the norm $|\cdot|_{\mu}$ in $B_{\mu}\left(\mathbb{R}, \mathbb{R}^{2}\right)$.

Proof. Assume $\Phi(\cdot)=(U(\cdot), V(\cdot)) \in \Gamma$. Then we have

$$
\begin{equation*}
\left|H_{2}(\Phi)(\xi)\right|=\left|\left[\beta_{2}-d+\kappa f(U(\xi))-\gamma V(\xi)\right] V(\xi)\right| \leq M_{3}, \tag{39}
\end{equation*}
$$

where

$$
\begin{equation*}
M_{3}=\left(\beta_{2}+d+\frac{\kappa k_{1}}{k_{2}}+\gamma V^{0}\right) V^{0} \tag{40}
\end{equation*}
$$

Then

$$
\begin{align*}
& \left\lvert\, \begin{array}{l}
\left|\frac{d}{d \xi} F_{2}(\Phi(\cdot))(\xi)\right| \\
= \\
\left.\quad \frac{1}{d_{B} \Lambda_{2}} \right\rvert\, \Lambda_{21} \int_{-\infty}^{\xi} e^{\Lambda_{21}(\xi-t)} H_{2}(\Phi)(t) d t \\
\\
\quad+\Lambda_{22} \int_{\xi}^{+\infty} e^{\Lambda_{22}(\xi-t)} H_{2}(\Phi)(t) d t \mid \\
\leq \frac{M_{3}}{d_{B} \Lambda_{2}}\left[\left|\Lambda_{21}\right| \int_{-\infty}^{\xi} e^{\Lambda_{21}(\xi-t)} d t+\Lambda_{22} \int_{\xi}^{+\infty} e^{\Lambda_{22}(\xi-t)} d t\right] \\
=\frac{2 M_{3}}{d_{B} \Lambda_{2}},
\end{array}\right.
\end{align*}
$$

which implies

$$
\begin{equation*}
\left|\frac{d}{d \xi} F_{2}(\Phi(\cdot))(\cdot)\right|_{\mu}<\frac{2 M_{3}}{d_{B} \Lambda_{2}} . \tag{42}
\end{equation*}
$$

Consequently, $\left|(d / d \xi) F_{2}(\Phi(\cdot))(\cdot)\right|_{\mu}$ is bounded. Similarly, $\left|(d / d \xi) F_{1}(\Phi(\cdot))(\cdot)\right|_{\mu}$ is also bounded, which shows that $F(\Gamma)$ is uniformly bounded and equicontinuous with respect to the norm $|\cdot|_{\mu}$.

Furthermore, for any positive integer $n$, we define

$$
F^{n}(\Phi(\cdot))(\xi)= \begin{cases}F(\Phi(\cdot))(\xi), & \xi \in[-n, n]  \tag{43}\\ F(\Phi(\cdot))(-n), & \xi \in(-\infty,-n] \\ F(\Phi(\cdot))(n), & \xi \in[n,+\infty)\end{cases}
$$

Obviously, for fixed $n, F^{n}(\Gamma)$ is uniformly bounded and equicontinuous with respect to the norm $|\cdot|_{\mu}$ in $B_{\mu}\left(\mathbb{R}, \mathbb{R}^{2}\right)$, implying that $F^{n}: \Gamma \rightarrow \Gamma$ is a compact operator. Since

$$
\begin{align*}
\mid F_{2} & (\Phi(\cdot))(\xi) \mid \\
& \leq \frac{M_{3}}{d_{B} \Lambda_{2}}\left[\int_{-\infty}^{\xi} e^{\Lambda_{21}(\xi-t)} d t+\int_{\xi}^{+\infty} e^{\Lambda_{22}(\xi-t)} d t\right] \\
& =\frac{M_{3}}{d_{B}\left|\Lambda_{21}\right| \Lambda_{22}} \tag{44}
\end{align*}
$$

we have

$$
\begin{align*}
& \left|F_{2}^{n}(\Phi(\cdot))(\cdot)-F_{2}(\Phi(\cdot))(\cdot)\right|_{\mu} \\
& =\sup _{\xi \in \mathbb{R}}\left|F_{2}^{n}(\Phi(\cdot))(\xi)-F_{2}(\Phi(\cdot))(\xi)\right| e^{-\mu|\xi|} \\
& =\sup _{\xi \in(-\infty,-n] \cup[n,+\infty)}\left|F_{2}^{n}(\Phi(\cdot))(\xi)-F_{2}(\Phi(\cdot))(\xi)\right| e^{-\mu|\xi|} \\
& \leq \frac{2 M_{3}}{d_{B}\left|\Lambda_{21}\right| \Lambda_{22}} e^{-\mu n} \longrightarrow 0, \quad \text { as } n \longrightarrow+\infty . \tag{45}
\end{align*}
$$

Similarly, we can prove that

$$
\begin{equation*}
\left|F_{1}^{n}(\Phi(\cdot))(\cdot)-F_{1}(\Phi(\cdot))(\cdot)\right|_{\mu} \longrightarrow 0 \tag{46}
\end{equation*}
$$

when $n \rightarrow+\infty$. Thus, $\left|F^{n}(\Phi(\cdot))(\cdot)-F(\Phi(\cdot))(\cdot)\right|_{\mu} \quad \rightarrow \quad 0$ when $n \rightarrow+\infty$. By Proposition 2.1 in Zeidler [30] we see that $F^{n}$ converges to $F$ in $\Gamma$ with respect to the norm $|\cdot|_{\mu}$. Consequently, $F: \Gamma \rightarrow \Gamma$ is compact with respect to the norm $|\cdot|_{\mu}$. The proof is completed.

Lemma 8. Let $c>c^{*}$; then (6) has a solution $(U(\xi), V(\xi))$ satisfying (3):

$$
\begin{array}{r}
\int_{-\infty}^{+\infty}\left[d V(\eta)+\gamma V^{2}(\eta)\right] d \eta=\kappa c\left(N^{0}-N^{1}\right)  \tag{47}\\
0 \leq V(\xi) \leq \kappa\left(N^{0}-N^{1}\right)
\end{array}
$$

for any $\xi \in \mathbb{R}$.
Proof. Combination of Schauder's fixed point theorem, Lemmas 5, 6 , and 7 shows that there exists a nonnegative traveling wave solution $\left(U_{c}(\cdot), V_{c}(\cdot)\right) \in \Gamma$ such that $\left(U_{c}(\xi), V_{c}(\xi)\right) \rightarrow$ $\left(N^{0}, 0\right)$ when $\xi \rightarrow-\infty$. Since $\left(U_{c}(\cdot), V_{c}(\cdot)\right)$ is the fixed point of $F$, L'Hospital principal shows that $U^{\prime}(-\infty)=0, V^{\prime}(-\infty)=$ 0 . Then from (6) we have that $U^{\prime \prime}(-\infty)=0, V^{\prime \prime}(-\infty)=0$. Since $\left(U_{c}(\xi), V_{c}(\xi)\right)$ is the solution of (6), thus

$$
\begin{align*}
& c U_{c}^{\prime}=d_{N} U_{c}^{\prime \prime}-f\left(U_{c}\right) V_{c}  \tag{48}\\
& c V_{c}^{\prime}=d_{B} V_{c}^{\prime \prime}+\kappa f\left(U_{c}\right) V_{c}-d V_{c}-\gamma V_{c}^{2}
\end{align*}
$$

The first equation of (48) can be changed into

$$
\begin{equation*}
\frac{c}{d_{N}} U_{c}^{\prime}-U_{c}^{\prime \prime}=-\frac{1}{d_{N}} f\left(U_{c}\right) V_{c} \tag{49}
\end{equation*}
$$

Multiplying this equation by $e^{-c / d_{N} \xi}$ yields

$$
\begin{equation*}
-\left[e^{-c / d_{N} \xi} U_{c}^{\prime}(\xi)\right]^{\prime}=-\frac{1}{d_{N}} f\left(U_{c}\right) V_{c} e^{-c / d_{N} \xi} \tag{50}
\end{equation*}
$$

From the proof of Lemma 7, we have $U_{c}^{\prime}(\xi)=$ $F_{1}^{\prime}\left(U_{c}(\cdot), V_{c}(\cdot)\right)(\xi)$ is bounded in $\mathbb{R}$. Then integrating above equality from $\xi$ to $+\infty$, we have

$$
\begin{equation*}
U_{c}^{\prime}(\xi)=-\frac{1}{d_{N}} e^{c / d_{N} \xi} \int_{\xi}^{+\infty} f\left(U_{c}(\eta)\right) V_{c}(\eta) e^{-c / d_{N} \eta} d \eta \leq 0 \tag{51}
\end{equation*}
$$

which implies that $U_{c}(\xi)$ is nonincreasing in $\mathbb{R}$ and has limit $N^{1}$ as $\xi \rightarrow+\infty$. By the definition of $\underline{U}(\xi)$ and $\underline{V}(\xi)$ there is a $\xi_{0}<0$ such that $\underline{U}(\xi)>0$ and $\underline{V}(\xi)>0$ when $\xi<\xi_{0}$. Therefore, if $\xi<\xi_{0}$, we have that $U_{c}^{\prime}(\xi)<0$ which implies that $N^{0}>N^{1} \geq 0$.

Integrating the first equation of (48) from $-\infty$ to $\xi$ gives

$$
\begin{equation*}
\int_{-\infty}^{\xi} f\left(U_{c}(\eta)\right) V_{c}(\eta) d \eta=d_{N} U_{c}^{\prime}(\xi)-c\left[U_{c}(\xi)-N^{0}\right] \tag{52}
\end{equation*}
$$

which implies that $\int_{-\infty}^{+\infty} f\left(U_{c}(\eta)\right) V_{c}(\eta) d \eta<+\infty$. Integrating the second equation of (48) from $-\infty$ to $\xi$ gives

$$
\begin{align*}
c V_{c}(\xi)= & d_{B} V_{c}^{\prime}(\xi)+\int_{-\infty}^{\xi} \kappa f\left(U_{c}(\eta)\right) V_{c}(\eta) d \eta \\
& -d \int_{-\infty}^{\xi} V_{c}(\eta) d \eta-\gamma \int_{-\infty}^{\xi} V_{c}^{2}(\eta) d \eta \tag{53}
\end{align*}
$$

Thus $\int_{-\infty}^{+\infty} V_{c}(\eta) d \eta<+\infty$ and $\lim _{\xi \rightarrow+\infty} V_{c}(\xi)=0$ since $V_{c}^{\prime}(\xi)$ is bounded in $\mathbb{R}$. By (51) and L'Hospital principal, it follows $U_{c}^{\prime}(+\infty)=0$. Then using (52) and (53) shows that

$$
\begin{equation*}
\int_{-\infty}^{+\infty}\left[d V_{c}(\eta)+\gamma V_{c}^{2}(\eta)\right] d \eta=\kappa c\left(N^{0}-N^{1}\right) \tag{54}
\end{equation*}
$$

Next, we prove that $0 \leq V_{c}(\xi) \leq d\left(N^{0}-N^{1}\right) /\left(d-\alpha_{2}\right)$. Let

$$
\begin{align*}
R(\xi)= & \frac{1}{c} \int_{-\infty}^{\xi}\left[d V_{c}(\eta)+\gamma V_{c}^{2}(\eta)\right] d \eta \\
& +\frac{1}{c} \int_{\xi}^{+\infty} e^{c(\xi-\eta) / d_{B}}\left[d V_{c}(\eta)+\gamma V_{c}^{2}(\eta)\right] d \eta \tag{55}
\end{align*}
$$

It is clear that $R(-\infty)=0$ and $R(+\infty)=\kappa\left(N^{0}-N^{1}\right)$. Define $S(\xi)=V_{c}(\xi)+R(\xi)$. Calculations show that

$$
\begin{equation*}
c S^{\prime}(\xi)-d_{B} S^{\prime \prime}(\xi)=\kappa f\left(U_{c}(\xi)\right) V_{c}(\xi) \tag{56}
\end{equation*}
$$

Multiplying this equality by $e^{-c \xi / d_{B}}$ and then integrating from $\xi$ to $+\infty$ show that

$$
\begin{equation*}
S^{\prime}(\xi)=\frac{\kappa}{d_{B}} \int_{\xi}^{+\infty} e^{c(\xi-\zeta) / d_{B}}\left[f\left(U_{c}(\zeta)\right) V_{c}(\zeta) d \eta\right] d \zeta \geq 0 \tag{57}
\end{equation*}
$$

for any $\xi \in \mathbb{R}$. Consequently, $S(\xi)$ is nondecreasing in $\mathbb{R}$. Since

$$
\begin{equation*}
S(+\infty)=R(+\infty)=\kappa\left(N^{0}-N^{1}\right) \tag{58}
\end{equation*}
$$

we have that $0 \leq V_{c}(\xi) \leq \kappa\left(N^{0}-N^{1}\right)$ for any $\xi \in \mathbb{R}$. The proof is completed.

Proof of Theorem 1. Firstly, we consider the case $c>c^{*}$. Let $\left\{\varepsilon_{n}\right\}$ be a sequence such that $0<\varepsilon_{i+1}<\varepsilon_{i}<1$ and $\varepsilon_{n} \rightarrow 0$. By Lemma 8, there exists a traveling wave solution
$\Phi_{n}(\xi)=\left(U_{n}(\xi), V_{n}(\xi)\right)$ of system (6) for $\gamma=\varepsilon_{n}$ satisfying the conclusion of Theorem 1. From (51), we have

$$
\begin{align*}
\left|U_{n}^{\prime}(\xi)\right| & =\frac{1}{d_{N}} e^{c / d_{N} \xi} \int_{\xi}^{+\infty} f\left(U_{n}(\eta)\right) V_{n}(\eta) e^{-c / d_{N} \eta} d \eta \leq 0 \\
& \leq \frac{f\left(N^{0}\right) \kappa\left(N^{0}-N^{1}\right)}{d_{N}} e^{c / d_{N} \xi} \int_{\xi}^{+\infty} e^{-c / d_{N} \eta} d \eta \\
& =\frac{f\left(N^{0}\right) \kappa\left(N^{0}-N^{1}\right)}{c} \tag{59}
\end{align*}
$$

Similarly, it can be shown that $\left|V_{n}^{\prime}(\xi)\right| \leq M_{0}$, where $M_{0}$ is independent of $\varepsilon_{n}$ due to $\varepsilon_{n}<1$. By (6), there is a positive constant $M_{4}$ independent of $\varepsilon_{n}$ such that $\left|U_{n}^{\prime \prime}(\xi)\right|,\left|V_{n}^{\prime \prime}(\xi)\right|$, $\left|U_{n}^{\prime \prime \prime}(\xi)\right|$, and $\left|V_{n}^{\prime \prime \prime}(\xi)\right|$ are bounded in $\xi \in \mathbb{R}$ by $M_{4}$.

Therefore, $\left\{\Phi_{n}(\xi)\right\},\left\{\Phi_{n}^{\prime}(\xi)\right\},\left\{\Phi_{n}^{\prime \prime}(\xi)\right\}$ are equicontinuous and uniformly bounded in $\mathbb{R}$. Then Arzela-Ascoli's theorem implies that there exists a subsequence $\left\{\varepsilon_{n_{k}}\right\}$ such that

$$
\begin{gather*}
\Phi_{n_{k}}(\xi) \longrightarrow \Psi(\xi), \quad \Phi_{n_{k}}^{\prime}(\xi) \longrightarrow \Psi^{\prime}(\xi), \\
\Phi_{n_{k}}^{\prime \prime}(\xi) \longrightarrow \Psi^{\prime \prime}(\xi) \tag{60}
\end{gather*}
$$

uniformly in any bounded closed interval when $k \rightarrow \infty$ and pointwise on $\mathbb{R}$, where $\Psi(\xi)=\left(\psi_{1}(\xi), \psi_{2}(\xi)\right)$. Since $\Phi_{n_{k}}(\xi)$ is the solution of (6) and $\varepsilon_{n} \rightarrow 0$, we get

$$
\begin{align*}
& c \psi_{1}^{\prime}(\xi)=d_{N} \psi_{1}^{\prime \prime}(\xi)-f\left(\psi_{1}(\xi)\right) \psi_{2}(\xi) \\
& c \psi_{2}^{\prime}(\xi)=d_{B} \psi_{2}^{\prime \prime}(\xi)+\kappa f\left(\psi_{1}(\xi)\right) \psi_{2}(\xi)-d \psi_{2}(\xi) \tag{61}
\end{align*}
$$

That is, $\Psi(\xi)$ is a solution of (5) satisfying (3):

$$
\begin{array}{r}
\int_{-\infty}^{+\infty} \psi_{2}(\eta) d \eta=\frac{\kappa c}{d}\left(N^{0}-N^{1}\right)  \tag{62}\\
0 \leq \psi_{2}(\xi) \leq \kappa\left(N^{0}-N^{1}\right)
\end{array}
$$

To complete the proof of case $c>c^{*}$, we need to prove $f\left(N^{1}\right)<d / \kappa$. Integrating the second equation of system (5) from $-\infty$ to $+\infty$ and noting that $U(\xi)$ is decreasing from $N^{0}$ to $N^{1}$, we have

$$
\begin{align*}
& d \int_{-\infty}^{+\infty} V(\xi) d \xi \\
& \quad=\kappa \int_{-\infty}^{+\infty} f(U(\xi)) V(\xi) d \xi>\kappa f\left(N^{1}\right) \int_{-\infty}^{+\infty} V(\xi) d \xi \tag{63}
\end{align*}
$$

which implies $f\left(N^{1}\right)<d / \kappa$.
To prove case $c=c^{*}$, let the parameter $c=c_{n}$ in system (5), $c^{*}<c_{n}<c^{*}+1$, and $c_{n} \rightarrow c^{*}$. Similar to above proof about case $c>c^{*}$, we can finish the proof.

## 3. Nonexistence of Traveling Wave Solution

In this section, we give the conditions on which system (1) has no traveling wave solutions.

Theorem 9. (I) Assume $f\left(N^{0}\right)>d / \kappa$. Then for any $0<$ $c<c^{*}$, system (1) has no nonnegative traveling wave solutions ( $U(x+c t), V(x+c t)$ ) satisfying boundary condition (3).
(II) Suppose $f\left(N^{0}\right) \leq d / \kappa$. Then for any $c>0$, system (1) has no traveling wave solutions $(U(x+c t), V(x+c t))$ satisfying boundary condition (3).

Proof of Theorem 9(I). Suppose (I) fails. That is, system (5) has a nonnegative nontrivial traveling wave solution $(U(\xi), V(\xi))$ satisfying boundary condition (3). Since $U(-\infty)=N^{0}$ and $f\left(N^{0}\right)>d / \kappa$, there exists a $\xi_{0}<0$ such that $f(U(\xi)) \geq\left[f\left(N^{0}\right)+d / \kappa\right] / 2$ for any $\xi<\xi_{0}$. Thus, we get

$$
\begin{align*}
c V^{\prime}(\xi) & =d_{B} V^{\prime \prime}(\xi)+\kappa f(U(\xi)) V(\xi)-d V(\xi) \\
& \geq d_{B} V^{\prime \prime}(\xi)+\frac{\kappa f\left(N^{0}\right)+d}{2} V(\xi)-d V(\xi)  \tag{64}\\
& =d_{B} V^{\prime \prime}(\xi)+\frac{\kappa f\left(N^{0}\right)-d}{2} V(\xi),
\end{align*}
$$

for any $\xi \leq \xi_{0}$. That is,

$$
\begin{equation*}
\frac{\kappa f\left(N^{0}\right)-d}{2} V(\xi) \leq c V^{\prime}(\xi)-d_{B} V^{\prime \prime}(\xi) \tag{65}
\end{equation*}
$$

for any $\xi<\xi_{0}$. Now we show $V^{\prime}(-\infty)=0$. Denote $W(\xi) \triangleq$ $V^{\prime}(\xi)$. From the second equation of (5), we have

$$
\begin{equation*}
d_{B} W^{\prime}(\xi)=c W(\xi)+G(\xi) \tag{66}
\end{equation*}
$$

where $G(\xi)=d V(\xi)-\kappa f(U(\xi)) V(\xi)$. Since $(U(\xi), V(\xi))$ satisfies boundary condition (3), it follows $G(-\infty)=0$. If $W(-\infty) \neq 0$, then $W(-\infty)=+\infty$ or $W(-\infty)=-\infty$, which imply $V(-\infty)=-\infty$ or $V(-\infty)=+\infty$ contradicting $V(-\infty)=0$.

Defining $J(\xi)=\int_{-\infty}^{\xi} V(\eta) d \eta$ and integrating (65) from $-\infty$ to $\xi$, we have that

$$
\begin{equation*}
\frac{\kappa f\left(N^{0}\right)-d}{2} J(\xi) \leq c V(\xi)-d_{B} V^{\prime}(\xi) \tag{67}
\end{equation*}
$$

Integrating (67) from $-\infty$ to $\xi$ with $\xi \leq \xi_{0}$ yields

$$
\begin{equation*}
\frac{\kappa f\left(N^{0}\right)-d}{2} \int_{-\infty}^{\xi} J(\eta) d \eta+d_{B} V(\xi) \leq c J(\xi) \tag{68}
\end{equation*}
$$

Therefore, we get

$$
\begin{equation*}
\frac{\kappa f\left(N^{0}\right)-d}{2} \int_{-\infty}^{0} J(\xi+\eta) d \eta \leq c J(\xi) \tag{69}
\end{equation*}
$$

for any $\xi \leq \xi_{0}$. Since $J(\eta)$ is increasing in $\mathbb{R}$, it is clear that

$$
\begin{equation*}
\frac{\kappa f\left(N^{0}\right)-d}{2} \eta J(\xi-\eta) \leq c J(\xi) \tag{70}
\end{equation*}
$$

for any $\xi \leq \xi_{0}$ and $\eta>0$. Therefore, there is $\eta_{0}>0$ large enough such that

$$
\begin{equation*}
J\left(\xi-\eta_{0}\right) \leq \frac{1}{2} J(\xi) \tag{71}
\end{equation*}
$$

for any $\xi \leq \xi_{0}$. Let $p(\xi)=J(\xi) e^{-\mu_{0} \xi}$ and $\mu_{0}=\left(1 / \eta_{0}\right) \ln 2$. We get that

$$
\begin{equation*}
p\left(\xi-\eta_{0}\right)=J\left(\xi-\eta_{0}\right) e^{-\mu_{0}\left(\xi-\eta_{0}\right)} \leq \frac{1}{2} J(\xi) e^{-\mu_{0}\left(\xi-\eta_{0}\right)}=p(\xi) \tag{72}
\end{equation*}
$$

for any $\xi \leq \xi_{0}$. Since $J(\xi)$ is bounded in $\mathbb{R}$, thus $p(\xi) \rightarrow 0$ as $\xi \rightarrow+\infty$, which implies that there exists $p_{0}>0$ such that $p(\xi) \leq p_{0}$ for any $\xi \in \mathbb{R}$. Hence, we have that

$$
\begin{equation*}
J(\xi) \leq p_{0} e^{\mu_{0} \xi}, \tag{73}
\end{equation*}
$$

for $\xi \in \mathbb{R}$ and that there exists $q_{0}>0$ such that $\int_{-\infty}^{\xi} J(\eta) d \eta \leq$ $q_{0} e^{\mu_{0} \xi}$. In addition, inequalities (65)-(68) imply that

$$
\begin{align*}
& \sup _{\xi \in \mathbb{R}}\left\{V(\xi) e^{-\mu_{0} \xi}\right\}<+\infty, \\
& \sup _{\xi \in \mathbb{R}}\left\{\left|V^{\prime}(\xi)\right| e^{-\mu_{0} \xi}\right\}<+\infty,  \tag{74}\\
& \sup _{\xi \in \mathbb{R}}\left\{\left|V^{\prime \prime}(\xi)\right| e^{-\mu_{0} \xi}\right\}<+\infty .
\end{align*}
$$

To complete the proof, we define negative one-sided Laplace transform as follows:

$$
\begin{equation*}
\mathscr{V}(\lambda)=\mathscr{N}[V(\cdot)](\lambda):=\int_{-\infty}^{0} e^{-\lambda \xi} V(\xi) d \xi \tag{75}
\end{equation*}
$$

for $\lambda \geq 0$. Obviously $\mathscr{V}(\lambda)$ is increasing in $\left[0, \lambda^{*}\right)$ such that $\lambda^{*}<+\infty$ satisfying $\lim _{\lambda \rightarrow \lambda^{*-}} \mathscr{V}(\lambda)=+\infty$ or $\lambda^{*}=+\infty$. Since $\sup _{\xi \in \mathbb{R}}\left\{V(\xi) e^{-\mu_{0} \xi}\right\}<+\infty$, we have $\lambda^{*} \geq \mu_{0}$. Trivial calculations show that $\mathcal{N}[\cdot]$ satisfies

$$
\begin{align*}
\mathscr{N}\left[V^{\prime}(\cdot)\right](\lambda) & =\lambda \mathscr{V}(\lambda)+V(0), \\
\mathscr{N}\left[V^{\prime \prime}(\cdot)\right](\lambda) & =\lambda^{2} \mathscr{V}(\lambda)+\lambda V(0)+V^{\prime}(0) \tag{76}
\end{align*}
$$

The second equation of (5) can be rewritten as

$$
\begin{equation*}
L[V(\cdot)](\xi)=\kappa\left[f\left(N^{0}\right)-f(U(\xi))\right] V(\xi) \tag{77}
\end{equation*}
$$

where

$$
\begin{equation*}
L[V(\cdot)](\xi)=d_{B} V^{\prime \prime}(\xi)-c V^{\prime}(\xi)+\left[\kappa f\left(N^{0}\right)-d\right] V(\xi) \tag{78}
\end{equation*}
$$

Define $\rho=\min \{H(\lambda): \lambda \geq 0\}$. Noticing $0<c<c^{*}$ yields $\rho>0$. Since (5) is autonomous, then for any $a \in$ $\mathbb{R},(U(\xi-a), V(\xi-a))$ is also a solution of (5) satisfying boundary condition (3) and $U(\xi-a) \rightarrow N^{0}$ as $a \rightarrow+\infty$. Hence, without losing generality we can assume

$$
\begin{equation*}
\kappa\left[f\left(N^{0}\right)-f(U(\xi))\right]<\frac{\rho}{2}, \tag{79}
\end{equation*}
$$

for all $\xi \leq 0$. That is,

$$
\begin{equation*}
L[V(\cdot)](\xi) \leq \frac{\rho}{2} V(\xi) \tag{80}
\end{equation*}
$$

Applying the operator $\mathcal{N}[\cdot]$ to this inequality and using the properties of $\mathscr{N}[\cdot]$ concluded above yield that

$$
\begin{equation*}
\frac{\rho}{2} \mathscr{V}(\lambda) \geq \mathscr{N}[L[V(\cdot)](\cdot)](\lambda) \geq H(\lambda) \mathscr{V}(\lambda)+q(\lambda) \tag{81}
\end{equation*}
$$

where $H(\lambda)$ is the characteristic function of (7) and

$$
\begin{equation*}
q(\lambda)=d_{B} V^{\prime}(0)+\left(d_{B} \lambda-c\right) V(0) . \tag{82}
\end{equation*}
$$

Consequently, we have

$$
\begin{equation*}
\mathscr{H}(\lambda):=\left[H(\lambda)-\frac{\rho}{2}\right] \mathscr{V}(\lambda)+q(\lambda) \leq 0 . \tag{83}
\end{equation*}
$$

If $\lambda^{*}<+\infty$, then $\lim _{\lambda \rightarrow \lambda^{*-}} \mathscr{V}(\lambda)=+\infty$ and, therefore, $\lim _{\lambda \rightarrow \lambda^{*-}} \mathscr{H}(\lambda)=+\infty$, which is a contradiction. If $\lambda^{*}=+\infty$, we have that $\lim _{\lambda \rightarrow+\infty} \mathscr{H}(\lambda)=+\infty$ by the monotonicity of $\mathscr{V}(\lambda)$ and the definitions of $H(\lambda)$ and $q(\lambda)$, which is still a contradiction. The proof of Theorem 9(I) is completed.

Proof of Theorem 9(II). Suppose $(U(\xi), V(\xi))$ is a nontrivial solution of system (5) satisfying boundary condition (3). Similar to the arguments about (66), it is easy to show that $V^{\prime}( \pm \infty)=0$. Then integrating the second equation of (5) from $-\infty$ to $+\infty$ yields

$$
\begin{align*}
& d \int_{-\infty}^{+\infty} V(\xi) d \xi \\
& \quad=\kappa \int_{-\infty}^{+\infty} f(U(\xi)) V(\xi) d \xi<\kappa f\left(N^{0}\right) \int_{-\infty}^{+\infty} V(\xi) d \xi \\
& \quad \leq d \int_{-\infty}^{+\infty} V(\xi) d \xi \tag{84}
\end{align*}
$$

which is a contradiction. The proof is completed.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

The authors were supported by the Fundamental Research Funds for the Central Universities (XDJK2012C042 and SWU113048).

## References

[1] A. Aotani, M. Mimura, and T. Mollee, "A model aided understanding of spot pattern formation in chemotactic E. coli colonies," Japan Journal of Industrial and Applied Mathematics, vol. 27, no. 1, pp. 5-22, 2010.
[2] T. Banitz, K. Johst, L. Y. Wick, I. Fetzer, H. Harms, and K. Frank, "The relevance of conditional dispersal for bacterial colony growth and biodegradation," Microbial Ecology, vol. 63, no. 2, pp. 339-347, 2012.
[3] J. A. Bonachela, C. D. Nadell, J. B. Xavier, and S. A. Levin, "Universality in bacterial colonies," Journal of Statistical Physics, vol. 144, no. 2, pp. 303-315, 2011.
[4] J. Lega and T. Passot, "Hydrodynamics of bacterial colonies," Nonlinearity, vol. 20, no. 1, pp. C1-C16, 2007.
[5] A. Marrocco, H. Henry, I. B. Holland, M. Plapp, S. J. Séror, and B. Perthame, "Models of self-organizing bacterial communities and comparisons with experimental observations," Mathematical Modelling of Natural Phenomena, vol. 5, no. 1, pp. 148-162, 2010.
[6] M. Mimura, H. Sakaguchi, and M. Matsushita, "Reactiondiffusion modelling of bacterial colony patterns," Physica A, vol. 282, no. 1, pp. 283-303, 2000.
[7] M. K. Roy, P. Banerjee, T. K. Sengupta, and S. Dattagupta, "Glucose induced fractal colony pattern of Bacillus thuringiensis," Journal of Theoretical Biology, vol. 265, no. 3, pp. 389-395, 2010.
[8] Y. Yamazaki, T. Ikeda, H. Shimada et al., "Periodic growth of bacterial colonies," Physica D, vol. 205, no. 1-4, pp. 136-153, 2005.
[9] O. A. Croze, G. P. Ferguson, M. E. Cates, and W. C. K. Poon, "Migration of chemotactic bacteria in soft agar: role of gel concentration," Biophysical Journal, vol. 101, no. 3, pp. 525-534, 2011.
[10] B. Grammaticos, M. Badoual, and M. Aubert, "An (almost) solvable model for bacterial pattern formation," Physica D, vol. 234, no. 2, pp. 90-97, 2007.
[11] J. F. Leyva, C. Málaga, and R. G. Plaza, "The effects of nutrient chemotaxis on bacterial aggregation patterns with non-linear degenerate cross diffusion," Physica A, vol. 392, no. 22, pp. 56445662, 2013.
[12] A. F. Miguel, "Constructal pattern formation in stony corals, bacterial colonies and plant roots under different hydrodynamics conditions," Journal of Theoretical Biology, vol. 242, no. 4, pp. 954-961, 2006.
[13] A. Nishiyama, T. Tokihiro, M. Badoual, and B. Grammaticos, "Modelling the morphology of migrating bacterial colonies," Physica D, vol. 239, no. 16, pp. 1573-1580, 2010.
[14] A. M. A. El-Sayed, S. Z. Rida, and A. A. M. Arafa, "On the solutions of the generalized reaction-diffusion model for bacterial colony," Acta Applicandae Mathematicae, vol. 110, no. 3, pp. 1501-1511, 2010.
[15] M. Torrisi and R. Tracinà, "Exact solutions of a reactiondiffusion systems for Proteus mirabilis bacterial colonies," Nonlinear Analysis: Real World Applications, vol. 12, no. 3, pp. 1865-1874, 2011.
[16] L. Zhang, "Positive steady states of an elliptic system arising from biomathematics," Nonlinear Analysis: Real World Applications, vol. 6, no. 1, pp. 83-110, 2005.
[17] P. Feng and Z. Zhou, "Finite traveling wave solutions in a degenerate cross-diffusion model for bacterial colony," Communications on Pure and Applied Analysis, vol. 6, no. 4, pp. 11451165, 2007.
[18] M. B. A. Mansour, "Traveling wave solutions of a reactiondiffusion model for bacterial growth," Physica A, vol. 383, no. 2, pp. 466-472, 2007.
[19] M. B. A. Mansour, "Traveling wave solutions of a nonlinear reaction-diffusion-chemotaxis model for bacterial pattern formation," Applied Mathematical Modelling, vol. 32, no. 2, pp. 240247, 2008.
[20] M. B. A. Mansour, "Analysis of propagating fronts in a nonlinear diffusion model with chemotaxis," Wave Motion, vol. 50, no. 1, pp. 11-17, 2013.
[21] J. Müller and W. van Saarloos, "Morphological instability and dynamics of fronts in bacterial growth models with nonlinear
diffusion," Physical Review E, vol. 65, no. 6, Article ID 061111, 2002.
[22] R. A. Satnoianu, P. K. Maini, F. S. Garduno, and J. P. Armitage, "Travelling waves in a nonlinear degenerate diffusion model for bacterial pattern formation," Discrete and Continuous Dynamical Systems B, vol. 1, no. 3, pp. 339-362, 2001.
[23] S. Thanarajah and H. Wang, "Competition of motile and immotile bacterial strains in a petri dish," Mathematical Biosciences and Engineering, vol. 10, no. 2, pp. 399-424, 2013.
[24] J. Y. Wakano, A. Komoto, and Y. Yamaguchi, "Phase transition of traveling waves in bacterial colony pattern," Physical Review $E$, vol. 69, no. 5, Article ID 051904, 9 pages, 2004.
[25] O. Diekmann, "Thresholds and travelling waves for the geographical spread of infection," Journal of Mathematical Biology, vol. 6, no. 2, pp. 109-130, 1978.
[26] J. Carr and A. Chmaj, "Uniqueness of travelling waves for nonlocal monostable equations," Proceedings of the American Mathematical Society, vol. 132, no. 8, pp. 2433-2439, 2004.
[27] Z.-C. Wang, W.-T. Li, and S. Ruan, "Traveling fronts in monostable equations with nonlocal delayed effects," Journal of Dynamics and Differential Equations, vol. 20, no. 3, pp. 573-607, 2008.
[28] Z.-C. Wang, W.-T. Li, and S. Ruan, "Entire solutions in bistable reaction-diffusion equations with nonlocal delayed nonlinearity," Transactions of the American Mathematical Society, vol. 361, no. 4, pp. 2047-2084, 2009.
[29] Z.-C. Wang and J. Wu, "Travelling waves of a diffusive KermackMcKendrick epidemic model with non-local delayed transmission," Proceedings of The Royal Society of London A, vol. 466, no. 2113, pp. 237-261, 2010.
[30] E. Zeidler, Nonlinear Functional Analysis and Its applications I, Springer, New York, NY, USA, 1986.

# Modeling Saturated Diagnosis and Vaccination in Reducing HIV/AIDS Infection 

Can Chen and Yanni Xiao<br>Department of Applied Mathematics, Xian Jiaotong University, Xian 710049, China<br>Correspondence should be addressed to Yanni Xiao; yxiao@mail.xjtu.edu.cn

Received 18 January 2014; Accepted 26 February 2014; Published 30 March 2014
Academic Editor: Kaifa Wang
Copyright © 2014 C. Chen and Y. Xiao. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

A mathematical model is proposed to consider the effects of saturated diagnosis and vaccination on HIV/AIDS infection. By employing center manifold theory, we prove that there exists a backward bifurcation which suggests that the disease cannot be eradicated even if the basic reproduction number is less than unity. Global stability of the disease-free equilibrium is investigated for appropriate conditions. When the basic reproduction number is greater than unity, the system is uniformly persistent. The proposed model is applied to describe HIV infection among injecting drug users (IDUs) in Yunnan province, China. Numerical studies indicate that new cases and prevalence are sensitive to transmission rate, vaccination rate, and vaccine efficacy. The findings suggest that increasing vaccination rate and vaccine efficacy and enhancing interventions like reducing share injectors can greatly reduce the transmission of HIV among IDUs in Yunnan province, China.


## 1. Introduction

Acquired immunodeficiency syndrome (AIDS) is spreading rapidly in the world ever since it was firstly detected in 1981 and continues to threaten the health of human seriously, especially among sex workers and injecting drug users. Furthermore, AIDS also influences the economy of many countries which has attracted great attention of governments. For such a severe scenario, the governments have taken intervention measures to reduce HIV transmission.

Mathematical models play a vital role in gaining a quantitative insight into HIV transmission dynamics and suggesting the effective control strategies. In order to study the effect of various intervention strategies on HIV transmission, extensive mathematical models have been formulated. Traditionally, models of HIV/AIDS dynamics often incorporate staged progression (see, e.g., [1-3]), but these did not include any control measures. Hyman et al. [4] extended these models to consider screening and contact tracing and discussed which strategy would slow infectiousness. Compartmental models with staged progression that incorporate the imperfect vaccine were constructed in [5] to predict HIV epidemic, but they did not consider diagnosis. Elbasha and Gumel [6] considered that a proportion of new recruits are vaccinated
and upon becoming infected with HIV, susceptible and vaccinated individuals enter the classes of infected and vaccine infected people, separately. They showed the existence of backward bifurcation via numerical simulations. Sharomi et al. [7] explored the role of the choice of incidence function in HIV models formulated in [6] and obtained that the phenomenon of backward bifurcation can be removed by substituting the standard incidence function with a mass action incidence. In South Africa, testing and screening campaign was launched for HIV; Nyabadza and Mukandavire [8] analyzed their effects by developing HIV models. More recently, a model of HIV/AIDS with diagnosis was presented in [9]. The authors estimated parameter values and predicted its transmission in China in the next few years.

The majority of mathematical models consider only one control strategy, vaccination or diagnosis, for instance [5, 9]; however, curbing HIV/AIDS infection needs comprehensive strategies, since, under the serious threat of HIV, it may be more rational to adopt various measures for different high risk groups. These motivate us to consider two combined intervention measures, vaccination and diagnosis. Furthermore, due to the limited resources, we then choose a nonlinear function which can be used to describe saturation effect. We use a parameter $h$, representing the half saturation
constant, in the diagnosis function to measure the effect of HIV individuals being late for diagnosis [10]. When the number of infected individuals $I$ is low, the number of actual per capita diagnosed individuals is proportional to $I$, whereas when the number of infected individuals $I$ is sufficiently large, there is a saturation effect which makes the number of diagnosed individuals approach to be constant due to the limitation of human and economic power. The number of new diagnosed cases per unit time is saturated with the total infected population.

The paper is organized as follows. The model is formulated in Section 2. The existence of backward bifurcation and the stability of the disease-free equilibrium are discussed in Section 3. In Section 4, uniform persistence of the model is investigated. Numerical simulation results are concluded in Section 5. In Section 6, we give a brief summary.

## 2. The Model

The model describes the spread of HIV/AIDS in a high risk population. The total high risk population size at time $t$, denoted by $(N(t))$, is subdivided into susceptible individuals $(S(t))$, vaccinated susceptible individuals $(V(t))$, HIV infected but not yet diagnosed individuals $(I(t))$, diagnosed HIVpositive individuals $(D(t))$, and those who have developed AIDS $(A(t))$, so that $N(t)=S(t)+V(t)+I(\mathrm{t})+D(t)+A(t)$.

The equations of the model are

$$
\begin{align*}
S^{\prime} & =\Pi+\omega V-\lambda(t) S-\xi S-\mu S, \\
V^{\prime} & =\xi S-(1-\epsilon) \lambda(t) V-\omega V-\mu V, \\
I^{\prime} & =\lambda(t)(S+(1-\epsilon) V)-\sigma_{1} I-\frac{q I}{1+h I}-\mu I,  \tag{1}\\
D^{\prime} & =\frac{q I}{1+h I}-\sigma_{2} D-\mu D, \\
A^{\prime} & =\sigma_{1} I+\sigma_{2} D-(\mu+\psi) A,
\end{align*}
$$

where the incidence rate $\lambda(t)=\beta\left(\left(I(t)+\eta_{1} D(t)+\right.\right.$ $\left.\left.\eta_{2} A(t)\right) / N(t)\right), \beta$ denotes the transmission rate, and $\eta_{1}$ and $\eta_{2}$ illustrate the modification factors in the transmission coefficient of diagnosed HIV-positive individuals and AIDS patients, respectively. People enter into the susceptible class at a rate $\Pi$, become infected at a rate $\lambda(t) S$, and become vaccinated at a rate $\xi$. Also $\mu$ is the natural death rate; $\omega$ denotes the waning rate of vaccine; $\epsilon$ represents the vaccine efficacy; $q$ is the diagnosis rate; $\sigma_{1}$ and $\sigma_{2}$ are the progression rate to diagnose HIV-positive individuals and AIDS patients, respectively; $\psi$ is the disease-induced death rate.

Since the model monitors change in the human population, the variables and parameters are assumed to be nonnegative for all $t \geq 0$. The system will be analyzed in a suitable feasible region $\Omega \subseteq R_{+}^{5}$, where $\Omega=\{(S, V, I, D, A) \in$ $\left.R_{+}^{5} \mid S+V+I+D+A \leq \Pi / \mu\right\}$. We can easily prove that the solutions of system (1) with nonnegative initial conditions remain nonnegative, and the feasible region $\Omega$ is positively invariant and attracting with respect to system (1) for all $t>0$.

## 3. Model Analysis

3.1. Disease-Free Equilibrium and the Basic Reproduction Number. Model (1) has a disease-free equilibrium (DFE), obtained by setting the right-hand sides of system (1) to zero, represented as

$$
\begin{align*}
E_{0}:(\bar{S}, \bar{V}, \bar{I}, \bar{D}, \bar{A}) & =\left(\frac{\Pi(\mu+\omega)}{\mu(\mu+\xi+\omega)}, \frac{\Pi \xi}{\mu(\mu+\xi+\omega)}, 0,0,0\right) \\
\bar{N} & =\bar{S}+\bar{V}+\bar{I}+\bar{D}+\bar{A}=\frac{\Pi}{\mu} \tag{2}
\end{align*}
$$

Following [11], the reproduction number can be established by using the next generation operator approach. The matrices for new infection and transition terms, respectively, given by $F$ and $V$, are

$$
\begin{align*}
F & =\left(\begin{array}{ccc}
\beta m & \eta_{1} \beta m & \eta_{2} \beta m \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \\
V & =\left(\begin{array}{ccc}
\sigma_{1}+q+\mu & 0 & 0 \\
-q & \sigma_{2}+\mu & 0 \\
-\sigma_{1} & -\sigma_{2} & \mu+\psi
\end{array}\right), \tag{3}
\end{align*}
$$

where $m=(\bar{S}+(1-\epsilon) \bar{V}) / \bar{N}=(\mu+\omega+(1-\epsilon) \xi) /(\mu+\omega+\xi)$.
Denote by $R_{0}$ the basic reproduction number as

$$
\begin{align*}
R_{0} & =\rho\left(F V^{-1}\right) \\
& =\frac{\beta m}{\sigma_{1}+q+\mu}\left(1+\frac{\eta_{1} q}{\sigma_{2}+\mu}+\frac{\eta_{2}}{\mu+\psi}\left(\frac{\sigma_{2} q}{\sigma_{2}+\mu}+\sigma_{1}\right)\right), \tag{4}
\end{align*}
$$

that is, the spectral radius of the next generation matrix $F V^{-1}$. Biologically speaking, $R_{0}$ is the average number of new secondary infections generated by a single HIV infected individual, introduced into a susceptible population in which some individuals have been vaccinated.
3.2. Existence of Backward Bifurcation. Employing the center manifold theory as described in [12], we investigate the existence of backward bifurcation. In order to apply the center manifold theory, we make the following changes of variables:

$$
\begin{equation*}
S=x_{1}, \quad V=x_{2}, \quad I=x_{3}, \quad D=x_{4}, \quad A=x_{5} \tag{5}
\end{equation*}
$$

so that $N=\sum_{n=1}^{5} x_{n}$. We now use the vector notation $X=$ $\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right)^{T}$, where $(\cdot)^{T}$ denotes a matrix transpose. System (1) can then be written as $\dot{X}=F=\left(f_{1}, f_{2}, f_{3}, f_{4}, f_{5}\right)^{T}$, so that

$$
\begin{aligned}
& \dot{x}_{1}(t)=f_{1}=\Pi+\omega x_{2}-\lambda x_{1}-\xi x_{1}-\mu x_{1} \\
& \dot{x}_{2}(t)=f_{2}=\xi x_{1}-(1-\epsilon) \lambda x_{2}-\omega x_{2}-\mu x_{2}
\end{aligned}
$$

$$
\begin{align*}
& \dot{x}_{3}(t)=f_{3}=\lambda\left(x_{1}+(1-\epsilon) x_{2}\right)-\left(\sigma_{1}+\frac{q}{1+h x_{3}}+\mu\right) x_{3}, \\
& \dot{x}_{4}(t)=f_{4}=\frac{q x_{3}}{1+h x_{3}}-\left(\sigma_{2}+\mu\right) x_{4}, \\
& \dot{x}_{5}(t)=f_{5}=\sigma_{1} x_{3}+\sigma_{2} x_{4}-(\psi+\mu) x_{5}, \tag{6}
\end{align*}
$$

where $\lambda=\beta\left(\left(x_{3}+\eta_{1} x_{4}+\eta_{2} x_{5}\right) / N\right)$.

If $\beta$ is taken as the bifurcation parameter and we consider the case $R_{0}=1$, solving for $\beta$ gives $\beta=\beta^{*}$; that is,

$$
\begin{equation*}
\beta^{*} m\left(1+\frac{\eta_{1} q}{\sigma_{2}+\mu}+\frac{\eta_{2}}{\mu+\psi}\left(\frac{\sigma_{2} q}{\sigma_{2}+\mu}+\sigma_{1}\right)\right)=\sigma_{1}+q+\mu \tag{7}
\end{equation*}
$$

First of all, observe that the eigenvalues of the Jacobian matrix $J\left(E_{0}\right)$ at $\beta=\beta^{*}[13]$, that is, $\left.J\left(E_{0}\right)\right|_{\beta=\beta^{*}}$,

$$
\left.J\left(E_{0}\right)\right|_{\beta=\beta^{*}}=\left(\begin{array}{ccccc}
-\mu-\xi & \omega & -\beta^{*} \frac{\bar{S}}{\bar{N}} & -\eta_{1} \beta^{*} \frac{\bar{S}}{\bar{N}} & -\eta_{2} \beta^{*} \frac{\bar{S}}{\bar{N}}  \tag{8}\\
\xi & -\mu-\omega & -\beta^{*}(1-\epsilon) \frac{\bar{V}}{\bar{N}} & -\eta_{1} \beta^{*}(1-\epsilon) \frac{\bar{V}}{\bar{N}} & -\eta_{2} \beta^{*}(1-\epsilon) \frac{\bar{V}}{\bar{N}} \\
0 & 0 & \beta^{*} m-\sigma_{1}-q-\mu & \eta_{1} \beta^{*} m & \eta_{2} \beta^{*} m \\
0 & 0 & q & -\sigma_{2}-\mu & 0 \\
0 & 0 & \sigma_{1} & \sigma_{2} & -\mu-\psi
\end{array}\right)
$$

are given by

$$
\begin{equation*}
\lambda_{1}=-\mu, \quad \lambda_{2}=-(\mu+\omega+\xi), \quad \lambda_{3}=0 \tag{9}
\end{equation*}
$$

The other two eigenvalues satisfy the following equation:

$$
\begin{aligned}
= & \left(\sigma_{2}+\mu\right)(\mu+\psi) \\
& \times\left\{1+\beta^{*} m\left[\frac{\eta_{2}}{(\mu+\psi)^{2}}\left(\frac{\sigma_{2} q}{\sigma_{2}+\mu}+\sigma_{1}\right)\right.\right.
\end{aligned}
$$

$$
\begin{equation*}
\left.\left.+\frac{\eta_{1} q}{\left(\sigma_{2}+\mu\right)^{2}}+\frac{\eta_{2} \sigma_{2} q}{(\mu+\psi)\left(\sigma_{2}+\mu\right)^{2}}\right]\right\} \tag{12}
\end{equation*}
$$

$$
\begin{equation*}
\lambda^{2}+b \lambda+c=0 \tag{10}
\end{equation*}
$$

Clearly, $b$ and $c$ are positive. Equation (10) has two roots with negative real parts. Hence, $\lambda_{3}=0$ is a simple zero eigenvalue and all other eigenvalues have negative real parts. The assumptions in [12] are satisfied. Therefore, the center manifold theory can be used to analyze the dynamics of system (1) near $\beta=\beta^{*}$ (or, equivalently, $R_{0}=1$ ). The Jacobian matrix of system (1) at $\beta=\beta^{*}$ has a right eigenvector $w$, given by $w=\left(w_{1}, w_{2}, w_{3}, w_{4}, w_{5}\right)^{T}$. And it can be computed from the system $\left(\left.J\left(E_{0}\right)\right|_{\beta=\beta^{*}}\right) \cdot w=0$; that is,

$$
\begin{gather*}
0=-(\mu+\xi) w_{1}+\omega w_{2}-\beta^{*} \frac{\overline{x_{1}}}{\bar{N}} w_{3}-\eta_{1} \beta^{*} \frac{\overline{x_{1}}}{\bar{N}} w_{4} \\
-\eta_{2} \beta^{*} \frac{\overline{x_{1}}}{\bar{N}} w_{5}, \\
0=  \tag{13}\\
\xi w_{1}-(\mu+\omega) w_{2}-(1-\epsilon) \beta^{*} \frac{\overline{x_{2}}}{\bar{N}} w_{3} \\
\quad-(1-\epsilon) \eta_{1} \beta^{*} \frac{\overline{x_{2}}}{\bar{N}} w_{4}-(1-\epsilon) \eta_{2} \beta^{*} \frac{\overline{x_{2}}}{\bar{N}} w_{5}, \\
0=\left(\beta^{*} m-\sigma_{1}-q-\mu\right) w_{3}+\eta_{1} \beta^{*} m w_{4}+\eta_{2} \beta^{*} m w_{5}, \\
0=q w_{3}-\left(\sigma_{2}+\mu\right) w_{4}, \\
0=\sigma_{1} w_{3}+\sigma_{2} w_{4}-(\mu+\psi) w_{5} ;
\end{gather*}
$$

from (13), we derive the following solutions:

$$
\begin{gather*}
w_{1}=-\frac{(\mu+w)^{2}+(1-\epsilon) w \xi}{\mu(\mu+w+\xi)^{2}} \beta^{*}\left(w_{3}+\eta_{1} w_{4}+\eta_{2} w_{5}\right)<0 \\
w_{2}=-\frac{\xi(\mu+w+(1-\epsilon)(\mu+\xi))}{\mu(\mu+w+\xi)^{2}} \beta^{*}\left(w_{3}+\eta_{1} w_{4}+\eta_{2} w_{5}\right) \\
<0, \\
w_{3}=w_{3}>0, \quad w_{4}=\frac{q}{\sigma_{2}+\mu} w_{3}>0 \\
w_{5}=\frac{\sigma_{1} w_{3}+\sigma_{2} w_{4}}{\mu+\psi}>0 \tag{14}
\end{gather*}
$$

The left eigenvector of $\left.J\left(E_{0}\right)\right|_{\beta=\beta^{*}}$ is $v$, denoted by $v=$ $\left(v_{1}, v_{2}, v_{3}, v_{4}, v_{5}\right)$. And it can be computed from the system $\left(\left.J\left(E_{0}\right)\right|_{\beta=\beta^{*}}\right)^{T} \cdot v=0$; that is,

$$
\begin{gather*}
0=-(\mu+\xi) v_{1}+\xi v_{2}, \\
0=\omega v_{1}-(\mu+\omega) v_{2}, \\
0=-\beta^{*} \frac{\overline{x_{1}}}{\bar{N}} v_{1}-(1-\epsilon) \beta^{*} \frac{\overline{x_{2}}}{\bar{N}} v_{2}+\left(\beta^{*} m-\sigma_{1}-q-\mu\right) v_{3} \\
+q v_{4}+\sigma_{1} v_{5}, \\
0=-\eta_{1} \beta^{*} \frac{\overline{x_{1}}}{\bar{N}} v_{1}-(1-\epsilon) \eta_{1} \beta^{*} \frac{\overline{x_{2}}}{\bar{N}} v_{2}+\eta_{1} \beta^{*} m v_{3} \\
-\left(\sigma_{2}+\mu\right) v_{4}+\sigma_{2} v_{5}, \\
0= \\
-\eta_{2} \beta^{*} \frac{\overline{x_{1}}}{\bar{N}} v_{1}-(1-\epsilon) \eta_{2} \beta^{*} \frac{\overline{x_{2}}}{\bar{N}} v_{2}+\eta_{2} \beta^{*} m v_{3}  \tag{15}\\
\\
-(\mu+\psi) v_{5},
\end{gather*}
$$

with the following solutions:

$$
\begin{gather*}
v_{1}=0, \quad v_{2}=0, \quad v_{3}=v_{3}>0 \\
v_{4}=\frac{\beta^{*} m}{\sigma_{2}+\mu}\left(\eta_{1}+\frac{\sigma_{2} \eta_{2}}{\mu+\psi}\right) v_{3}>0, \quad v_{5}=\frac{\eta_{2} \beta^{*} m}{\mu+\psi} v_{3}>0 \tag{16}
\end{gather*}
$$

The local bifurcation analysis near $\beta=\beta^{*}\left(R_{0}=1\right)$ is then determined by the signs of two associated constants, denoted by $a$ and $b$, respectively, as

$$
\begin{gather*}
a=\sum_{k, i, j=1}^{5} v_{k} w_{i} w_{j} \frac{\partial^{2} f_{k}}{\partial x_{i} \partial x_{j}}(0,0), \\
b=\sum_{k, i=1}^{5} v_{k} w_{i} \frac{\partial^{2} f_{k}}{\partial x_{i} \partial \beta^{*}}(0,0) . \tag{17}
\end{gather*}
$$

The computations of $a$ and $b$ are done as follows: for system (6) the associated nonzero partial derivatives of $F$ at the disease-free equilibrium are

$$
\begin{gather*}
\frac{\partial^{2} f_{3}}{\partial x_{1} \partial x_{3}}=\frac{\beta^{*}}{\bar{N}}(1-m), \quad \frac{\partial^{2} f_{3}}{\partial x_{1} \partial x_{4}}=\frac{\eta_{1} \beta^{*}}{\bar{N}}(1-m), \\
\frac{\partial^{2} f_{3}}{\partial x_{1} \partial x_{5}}=\frac{\eta_{2} \beta^{*}}{\bar{N}}(1-m), \quad \frac{\partial^{2} f_{3}}{\partial x_{2} \partial x_{3}}=\frac{\beta^{*}}{\bar{N}}(1-\epsilon-m), \\
\frac{\partial^{2} f_{3}}{\partial x_{2} \partial x_{4}}=\frac{\eta_{1} \beta^{*}}{\bar{N}}(1-\epsilon-m), \\
\frac{\partial^{2} f_{3}}{\partial x_{2} \partial x_{5}}=\frac{\eta_{2} \beta^{*}}{\bar{N}}(1-\epsilon-m), \\
\frac{\partial^{2} f_{3}}{\partial x_{3} \partial x_{3}}=-\frac{2 \beta^{*}}{\bar{N}} m+2 q h, \quad \frac{\partial^{2} f_{3}}{\partial x_{3} \partial x_{4}}=-\frac{\beta^{*}\left(1+\eta_{1}\right)}{\bar{N}} m, \\
\frac{\partial^{2} f_{3}}{\partial x_{3} \partial x_{5}}=-\frac{\beta^{*}\left(1+\eta_{2}\right)}{\bar{N}} m, \quad \frac{\partial^{2} f_{3}}{\partial x_{4} \partial x_{4}}=-\frac{2 \eta_{1} \beta^{*}}{\bar{N}} m, \\
\frac{\partial^{2} f_{3}}{\partial x_{4} \partial x_{5}}=-\frac{\left(\eta_{1}+\eta_{2}\right) \beta^{*}}{\bar{N}} m, \quad \frac{\partial^{2} f_{3}}{\partial x_{5} \partial x_{5}}=-\frac{2 \eta_{2} \beta^{*}}{\bar{N}} m, \\
\frac{\partial^{2} f_{3}}{\partial x_{3} \partial \beta^{*}}=m, \quad \frac{\partial^{2} f_{3}}{\partial x_{4} \partial \beta^{*}}=\eta_{1} m, \quad \frac{\partial^{2} f_{3}}{\partial x_{5} \partial \beta^{*}}=\eta_{2} m .
\end{gather*}
$$

Substituting (18) into (17), we get

$$
\begin{gather*}
a=\frac{2 \beta^{*}}{\bar{N}} v_{3}\left\{\left(w_{3}+\eta_{1} w_{4}+\eta_{2} w_{5}\right)\right. \\
\times\left(\beta^{*}\left(w_{3}+\eta_{1} w_{4}+\eta_{2} w_{5}\right) \frac{(1-\epsilon) \epsilon \xi}{(\mu+w+\xi)^{2}}\right. \\
 \tag{19}\\
\left.\left.\quad-m\left(w_{3}+w_{4}+w_{5}\right)\right)\right\} \\
+2 q h w_{3}^{2} v_{3}\left(1-\frac{\beta^{*} m}{\sigma_{2}+\mu}\left(\eta_{1}+\frac{\sigma_{2} \eta_{2}}{\mu+\psi}\right)\right) \\
b= \\
\quad v_{3}\left(w_{3}+\eta_{1} w_{4}+\eta_{2} w_{5}\right) m
\end{gather*}
$$

From [14, 15], we know that if $a>0, b>0$, there exists a backward bifurcation. Since the bifurcation coefficient, $b$, is always positive, then we establish the following result.

Theorem 1. If $a>0$, system (1) exhibits a backward bifurcation when $R_{0}=1$.

Due to existence of backward bifurcation we know that, for positive $a$, there exists another critical value $R_{c}$, which is less than unity, for model (1). Moreover, there is no endemic equilibrium for $R_{0}<R_{c}$; there are two distinct endemic


FIGURE 1: (a) Plot of the function $F(I)$ with different values of the transmission coefficient $\beta$. (b) Backward bifurcation when the transmission coefficient $\beta$ varies. Other parameters: $\Pi=100, \mu=1 / 45, w=1 / 20, \epsilon=0.4, \xi=0.6, \sigma_{1}=0.7, \sigma_{2}=0.4, \psi=0.5, q=1, h=1, \eta_{1}=0.4$, and $\eta_{2}=0.7$.
equilibria for $R_{c}<R_{0}<1$, and a unique endemic equilibrium exists for $R_{0}=R_{c}<1$ or $R_{0}>1$. Numerical studies will confirm this in the end of this subsection.

We now analyze the endemic equilibrium of model (1). The equilibrium of model (1) can be obtained as follows:

$$
\begin{align*}
& S^{*}=\frac{\Pi\left((1-\epsilon) \lambda^{*}+\mu+\omega\right)}{\left((1-\epsilon) \lambda^{*}+\mu+\omega\right)\left(\lambda^{*}+\mu+\xi\right)-\omega \xi}, \\
& V^{*}=\frac{\Pi \xi}{\left((1-\epsilon) \lambda^{*}+\mu+\omega\right)\left(\lambda^{*}+\mu+\xi\right)-\omega \xi}, \\
& D^{*}=\frac{1}{\sigma_{2}+\mu} \frac{q I^{*}}{1+h I^{*}},  \tag{20}\\
& A^{*}=\frac{\sigma_{1}}{\mu+\psi} I^{*}+\frac{\sigma_{2}}{\mu+\psi} \frac{1}{\sigma_{2}+\mu} \frac{q I^{*}}{1+h I^{*}}, \\
& N^{*}=\frac{\Pi-\psi A^{*}}{\mu}
\end{align*}
$$

where

$$
\begin{align*}
\lambda^{*}= & \lambda\left(I^{*}\right)=\beta \frac{I^{*}+\eta_{1} D^{*}+\eta_{2} A^{*}}{N^{*}} \\
= & \beta \frac{I^{*}}{N^{*}}\left\{1+\frac{\eta_{1}}{\sigma_{2}+\mu} \frac{q}{1+h I^{*}}\right.  \tag{21}\\
& \left.+\eta_{2}\left(\frac{\sigma_{1}}{\mu+\psi}+\frac{\sigma_{2}}{\mu+\psi} \frac{1}{\sigma_{2}+\mu} \frac{q}{1+h I^{*}}\right)\right\} .
\end{align*}
$$

Substituting (20) into the third equation of system (1), it is easy to derive the following equation:

$$
\begin{equation*}
f(I)=g(I), \tag{22}
\end{equation*}
$$

where

$$
\begin{align*}
f(I)= & (S+(1-\epsilon) V) \frac{\beta}{N} \\
& \times\left\{1+\frac{\eta_{1}}{\sigma_{2}+\mu} \frac{q}{1+h I}\right.  \tag{23}\\
+ & \left.\eta_{2}\left(\frac{\sigma_{1}}{\mu+\psi}+\frac{\sigma_{2}}{\mu+\psi} \frac{1}{\sigma_{2}+\mu} \frac{q}{1+h I}\right)\right\} \\
& g(I)=\sigma_{1}+\mu+\frac{q}{1+h I} \tag{24}
\end{align*}
$$

Clearly, $I^{*}=0$ is a fixed point, which corresponds to the disease-free equilibrium $E_{0}$. For $I=0$, we can obtain

$$
\begin{align*}
g(0) & =\sigma_{1}+q+\mu \\
f(0) & =\beta m\left\{1+\frac{\eta_{1} q}{\sigma_{2}+\mu}+\frac{\sigma_{2}}{\mu+\psi}\left(\sigma_{1}+\frac{\sigma_{2} q}{\sigma_{2}+\mu}\right)\right\}  \tag{25}\\
R_{0} & =\frac{f(0)}{g(0)}
\end{align*}
$$

Define

$$
\begin{equation*}
F(I)=\frac{f(I)}{g(I)} . \tag{26}
\end{equation*}
$$

From model (1), it can be shown that if $I^{*}$ is a positive solution of $F(I)=1$, then $S^{*}, V^{*}, D^{*}$, and $A^{*}$ are positive. Thus, the equilibrium is biologically relevant. Unfortunately, it is hard to solve the equation $F(I)=1$ analytically; in the following we numerically show that this equation can have two positive roots, which confirms the existence of backward bifurcation. In Figure 1(a), $F(I)$ is plotted versus $I$ for different values of $\beta$ and all other parameters are fixed. Figure 1(a) shows that an increase in $\beta$ would lead to curve $F(I)$ becoming tangent
to line 1 and defining a critical value $\left.F\left(I^{*}\right)\right|_{\beta=\beta_{c}}=1$ and $\left.F^{\prime}\left(I^{*}\right)\right|_{\beta=\beta_{c}}=0$ hold true. Figure $1(\mathrm{~b})$ shows the occurrence of the backward bifurcation as parameter $\beta$ varies. We write $R_{0}(\beta)$ as the threshold value to indicate $\beta$ as the bifurcation parameter while all other parameters are fixed. Define [16]

$$
\begin{equation*}
R_{c}=R_{0}\left(\beta_{c}\right), \tag{27}
\end{equation*}
$$

below which the disease-free equilibrium is unique equilibrium.
3.3. Stability Analysis of Equilibria. First, we have the following result on the local stability of $E_{0}$.

Theorem 2. The disease-free equilibrium $E_{0}$ of system (1) is locally asymptotically stable if $R_{0}<1$ and unstable otherwise.

Proof. By checking the Jacobian matrix of system (1) evaluated at $E_{0}$, we know that the characteristic equation for $J\left(E_{0}\right)$ has two eigenvalues as

$$
\begin{equation*}
\lambda_{1}=-\mu, \quad \lambda_{2}=-(\mu+\omega+\xi) \tag{28}
\end{equation*}
$$

and the others satisfy the following equation:

$$
\begin{align*}
h(\lambda)= & \frac{\beta m}{\lambda+\sigma_{1}+q+\mu}+\frac{\eta_{1} \beta m q}{\left(\lambda+\sigma_{1}+q+\mu\right)\left(\lambda+\sigma_{2}+\mu\right)} \\
& +\frac{\eta_{2} \beta m q \sigma_{2}}{\left(\lambda+\sigma_{1}+q+\mu\right)\left(\lambda+\sigma_{2}+\mu\right)(\lambda+\mu+\psi)} \\
& +\frac{\eta_{2} \beta m \sigma_{1}}{\left(\lambda+\sigma_{1}+q+\mu\right)(\lambda+\mu+\psi)}=1 . \tag{29}
\end{align*}
$$

If the real parts of the roots of the equation $h(\lambda)=1$ are nonnegative, that is, $\Re(\lambda) \geq 0$, then [17]

$$
\begin{equation*}
|h(\lambda)| \leq h(0)=R_{0} . \tag{30}
\end{equation*}
$$

Hence, if $R_{0}<1, \forall \lambda$ such that $\Re(\lambda) \geq 0$, then $|h(\lambda)| \leq R_{0}<1$, showing that there are no solutions to $h(\lambda)=1$ with positive real part. Hence, $E_{0}$ is locally asymptotically stable if $R_{0}<1$. This proof is completed.

Then, using Lyapunov function we can get global stability of $E_{0}$.

Theorem 3. The disease-free equilibrium $E_{0}$ of system (1) is globally asymptotically stable if $R_{0}<\min \left\{R_{c},\left(\sigma_{1}+\mu\right) m /\left(\sigma_{1}+\right.\right.$ $q+\mu)\}$.

Proof. We note that no endemic equilibrium exists for $R_{0}<$ $R_{c}$. Then, $E_{0}$ is a unique equilibrium of system (1). We now consider a Lyapunov function:

$$
\begin{equation*}
V=I+\frac{\beta}{\sigma_{2}+\mu}\left(\eta_{1}+\frac{\sigma_{2} \eta_{2}}{\mu+\psi}\right) D+\frac{\eta_{2} \beta}{\mu+\psi} A \tag{31}
\end{equation*}
$$

The time derivative of $V$ is given by

$$
\begin{align*}
\dot{V}= & \dot{I}+\frac{\beta}{\sigma_{2}+\mu}\left(\eta_{1}+\frac{\sigma_{2} \eta_{2}}{\mu+\psi}\right) \dot{D}+\frac{\eta_{2} \beta}{\mu+\psi} \dot{A} \\
= & \left\{\beta\left(I+\eta_{1} D+\eta_{2} A\right) \frac{S+(1-\epsilon) V}{N}-\sigma_{1} I-\frac{q I}{1+h I}-\mu I\right\} \\
& +\frac{\beta}{\sigma_{2}+\mu}\left(\eta_{1}+\frac{\sigma_{2} \eta_{2}}{\mu+\psi}\right)\left(\frac{q I}{1+h I}-\sigma_{2} D-\mu D\right) \\
& +\frac{\eta_{2} \beta}{\mu+\psi}\left(\sigma_{1} I+\sigma_{2} D-(\mu+\psi) A\right) \\
\leq & \left\{\beta\left(I+\eta_{1} D+\eta_{2} A\right)-\sigma_{1} I-\frac{q I}{1+h I}-\mu I\right\} \\
& +\frac{\beta}{\sigma_{2}+\mu}\left(\eta_{1}+\frac{\sigma_{2} \eta_{2}}{\mu+\psi}\right)\left(q I-\sigma_{2} D-\mu D\right) \\
= & I\left\{\beta-\sigma_{1}-\frac{q}{1+h I}-\mu+\frac{\beta}{\sigma_{2}+\mu}\left(\sigma_{1} I+\frac{\left.\sigma_{2} D-(\mu+\psi) A\right)}{\mu+\psi}\right) q\right. \\
\leq & I\left\{R_{0} \frac{\sigma_{1} \eta_{2}+q+\mu}{m}-\left(\sigma_{1}+\mu\right)\right\} \\
= & \frac{\left(\sigma_{1}+q+\mu\right) I}{m}\left\{R_{0}-\frac{\left(\sigma_{1}+\mu\right) m}{\sigma_{1}+q+\mu}\right\} .
\end{align*}
$$

Note that $\dot{V} \leq 0$ if $R_{0}<\left(\sigma_{1}+\mu\right) m /\left(\sigma_{1}+q+\mu\right)$. Furthermore, $\dot{V}=0$ if and only if $I=0$. Therefore, the largest compact invariant set in $\Omega$ : $\dot{V}=0$, when $R_{0}<\min \left\{R_{c},\left(\sigma_{1}+\right.\right.$ $\left.\mu) m /\left(\sigma_{1}+q+\mu\right)\right\}$, is the singleton $E_{0}$. Thus, $E_{0}$ is globally asymptotically stable if $R_{0}<\min \left\{R_{c},\left(\sigma_{1}+\mu\right) m /\left(\sigma_{1}+q+\mu\right)\right\}$. This completes the proof.

## 4. Persistence of the Model

In this section, we will prove that system (1) is uniformly persistent. First, we present the following definition that is similar to that in $[18,19]$.

Definition 4. Model (1) is said to be uniformly persistent if there exists a positive constant $\varepsilon>0$ (independent of initial data) such that every solution with positive initial conditions satisfies $\liminf _{t \rightarrow \infty} S(t) \geq \varepsilon, \liminf _{t \rightarrow \infty} V(t) \geq$ $\varepsilon, \liminf _{t \rightarrow \infty} I(t) \geq \varepsilon, \liminf _{t \rightarrow \infty} D(t) \geq \varepsilon$, and $\liminf _{t \rightarrow \infty} A(t) \geq \varepsilon$.

Theorem 5. If $R_{0}>1$, system (1) is uniformly persistent; that is, there exists a positive constant $\varepsilon$, such that, for all initial values

$$
\begin{equation*}
(S(0), V(0), I(0), D(0), A(0)) \in R_{+}^{2} \times \operatorname{Int}\left(R_{+}^{3}\right), \tag{33}
\end{equation*}
$$

the solutions of system (1) satisfy. $\liminf _{t \rightarrow \infty} S(t) \geq \varepsilon$, $\liminf _{t \rightarrow \infty} V(t) \geq \varepsilon, \liminf _{t \rightarrow \infty} I(t) \geq \varepsilon, \liminf _{t \rightarrow \infty} D(t) \geq$ $\varepsilon$, and $\lim \inf _{t \rightarrow \infty} A(t) \geq \varepsilon$.

## Proof. Define

$$
\begin{align*}
X & =\{(S, V, I, D, A) \mid S \geq 0, V \geq 0, I \geq 0, D \geq 0, A \geq 0\}, \\
X_{0} & =\{(S, V, I, D, A) \mid S \geq 0, V \geq 0, I>0, D>0, A>0\}, \\
\partial X_{0} & =X \backslash X_{0} . \tag{34}
\end{align*}
$$

It then suffices to show that system (1) is uniformly persistent with respect to $\left(X_{0}, \partial X_{0}\right)$. First, by the form of (1), it is easy to see that both $X$ and $X_{0}$ are positively invariant. Clearly, $\partial X_{0}$ is relatively closed in $X$. Furthermore, system (1) is point dissipative. Set

$$
\begin{align*}
M_{\partial}=\{ & (S(0), V(0), I(0), D(0), A(0)): \\
& (S(t), V(t), I(t), D(t), A(t)) \text { satisfies (1) } \\
& \left.(S(t), V(t), I(t), D(t), A(t)) \in \partial X_{0}, \forall t \geq 0\right\} \tag{35}
\end{align*}
$$

We now prove that

$$
\begin{equation*}
M_{\partial}=\{(S, V, 0,0,0): S \geq 0, V \geq 0\} . \tag{36}
\end{equation*}
$$

Assume that $(S(0), V(0), I(0), D(0), A(0)) \in M_{\partial}$. It suffices to show that

$$
\begin{equation*}
I(t)=0, \quad D(t)=0, \quad A(t)=0, \quad \forall t \geq 0 \tag{37}
\end{equation*}
$$

If this is not true, then there exists a $t_{0} \geq 0$ such that

$$
\begin{equation*}
I\left(t_{0}\right)>0, \quad D\left(t_{0}\right)=0, \quad A\left(t_{0}\right)=0 \tag{38}
\end{equation*}
$$

or

$$
\begin{equation*}
I\left(t_{0}\right)>0, \quad D\left(t_{0}\right)>0, \quad A\left(t_{0}\right)=0 \tag{39}
\end{equation*}
$$

or

$$
\begin{equation*}
I\left(t_{0}\right)>0, \quad D\left(t_{0}\right)=0, \quad A\left(t_{0}\right)>0 \tag{40}
\end{equation*}
$$

or

$$
\begin{equation*}
I\left(t_{0}\right)=0, \quad D\left(t_{0}\right)>0, \quad A\left(t_{0}\right)=0 \tag{41}
\end{equation*}
$$

or

$$
\begin{equation*}
I\left(t_{0}\right)=0, \quad D\left(t_{0}\right)=0, \quad A\left(t_{0}\right)>0 \tag{42}
\end{equation*}
$$

or

$$
\begin{equation*}
I\left(t_{0}\right)=0, \quad D\left(t_{0}\right)>0, \quad A\left(t_{0}\right)>0 \tag{43}
\end{equation*}
$$

For $I\left(t_{0}\right)>0, D\left(t_{0}\right)=0$, and $A\left(t_{0}\right)=0$, we get

$$
\begin{equation*}
D^{\prime}\left(t_{0}\right)=\frac{q I\left(t_{0}\right)}{1+h I\left(t_{0}\right)}>0, \quad A^{\prime}\left(t_{0}\right)=\sigma_{1} I\left(t_{0}\right)>0 . \tag{44}
\end{equation*}
$$

It follows that there is an $\varepsilon_{0}>0$ such that $D(t)>0, A(t)>0$, for $t_{0}<t<t_{0}+\varepsilon_{0}$. This proves that

$$
\begin{equation*}
(S(t), V(t), I(t), D(t), A(t)) \notin \partial X_{0} \quad \text { for } t_{0}<t<t_{0}+\varepsilon_{0} \tag{45}
\end{equation*}
$$

which contradicts the assumption that $(S(0), V(0), I(0)$, $D(0), A(0)) \in M_{\partial}$. Similarly, we can obtain contradictions for other cases. This proves that

$$
\begin{equation*}
M_{\partial}=\{(S, V, 0,0,0): S \geq 0, V \geq 0\} \tag{46}
\end{equation*}
$$

Note that $E_{0}$ is globally asymptotically stable in Int $M_{\partial}$, and $E_{0}$ is an isolated invariant set in $X$. That is to say, $W^{s}\left(E_{0}\right) \cap$ $X_{0}=\emptyset$. Every orbit in $M_{\partial}$ converges to $E_{0}$, and $E_{0}$ is acyclic in $M_{\partial}$. We claim that $W^{s}\left(E_{0}\right) \cap X_{0}=\emptyset$ for $R_{0}>1$. If this is false, then we have $W^{s}\left(E_{0}\right) \cap X_{0} \neq \emptyset$. The system has a positive solution

$$
\begin{equation*}
(S(t), V(t), I(t), D(t), A(t)) \tag{47}
\end{equation*}
$$

where $(S(0), V(0), I(0), D(0), A(0)) \in X_{0}$. Then

$$
\begin{array}{r}
(S(t), V(t), I(t), D(t), A(t)) \longrightarrow E_{0}  \tag{48}\\
\text { as } t \longrightarrow \infty \text { for } R_{0}>1 .
\end{array}
$$

For $R_{0}>1$, we can choose an $\eta>0$ small enough such that $R_{0}(1-\eta)>1$. Then, when $t$ is sufficiently large, we have

$$
\begin{gather*}
m-\eta m \leq \frac{S(t)+(1-\epsilon) V(t)}{N(t)} \leq m+\eta m,  \tag{49}\\
I^{\prime} \geq \beta m(1-\eta)\left(I+\eta_{1} D+\eta_{2} A\right)-\sigma_{1} I-\frac{q I}{1+h I}-\mu I, \\
D^{\prime} \geq \frac{q I}{1+h I}-\sigma_{2} D-\mu D,  \tag{50}\\
A^{\prime} \geq \sigma_{1} I+\sigma_{2} D-(\mu+\psi) A .
\end{gather*}
$$

Define
M

$$
=\left(\begin{array}{ccc}
\beta m(1-\eta)-\sigma_{1}-q-\mu & \eta_{1} \beta m(1-\eta) & \eta_{2} \beta m(1-\eta)  \tag{51}\\
q & -\sigma_{2}-\mu & 0 \\
\sigma_{1} & \sigma_{2} & -\mu-\psi
\end{array}\right) .
$$

Recall that the stability modulus of an $n \times n$ matrix $M$, denoted by $s(M)$, is defined as

$$
\begin{equation*}
s(M)=\max \{\operatorname{Re} \lambda: \lambda \text { is an eigenvalue of } M\} . \tag{52}
\end{equation*}
$$

Note that $M$ is irreducible and has nonnegative offdiagonal elements. It then follows that $s(M)$ is a simple eigenvalue of $M$ with a (componentwise) positive eigenvector. Thus,

$$
\begin{equation*}
|\lambda I-M|=\lambda^{3}+a_{1} \lambda^{2}+a_{2} \lambda+a_{3} \tag{53}
\end{equation*}
$$

Table 1: Parameter description and values.

| Parameters | Description | Estimated values | Source |
| :--- | :--- | :---: | :---: |
| $\Pi$ | Recruitment rate | 4348 | $[21]$ |
| $\beta$ | Transmission coefficient | 0.304 | $[26]$ |
| $w$ | Per capita waning rate of vaccine | $1 / 20$ | $[5]$ |
| $\xi$ | Per capita vaccination rate | 0.4 | Variable |
| $\varepsilon$ | Vaccine efficacy | 0.4 | Variable |
| $\mu$ | Natural death rate | 0.0246 | $[27]$ |
| $\psi$ | Disease-induced death rate | 0.7114 | $[21]$ |
| $\sigma_{1}$ | Progression rate to AIDS stage for the infection stage | 0.0413 | $[9]$ |
| $\sigma_{2}$ | Progression rate to AIDS stage for the diagnosed stage | 0.116 | $[9]$ |
| $q$ | Diagnosis rate | 0.304 | $[9]$ |
| $\eta_{1}$ | Modification factor in transmission coefficient of diagnosed HIV-positive individuals | 0.491 | $[9]$ |
| $\eta_{2}$ | Modification factor in transmission coefficient of AIDS patients | 0.1 | Variable |

where

$$
\begin{align*}
& a_{1}=\left(\sigma_{1}+q+\mu\right)+\left(\sigma_{2}+\mu\right)+(\mu+\psi)-\beta m, \\
& a_{2}=\left(\sigma_{1}+q+\mu-\beta m\right)\left(\sigma_{2}+\mu\right)+\left(\sigma_{2}+\mu\right)(\mu+\psi) \\
&+\left(\sigma_{1}+q+\mu-\beta m\right)(\mu+\psi)-\eta_{1} \beta m q-\eta_{2} \beta m \sigma_{1}, \\
& a_{3}=\left(\sigma_{1}+q+\mu\right)\left(\sigma_{2}+\mu\right)(\mu+\psi)\left(1-R_{0}(1-\eta)\right) . \tag{54}
\end{align*}
$$

For $R_{0}(1-\eta)>1$, we obtain $a_{3}<0$. Thus, $s(M)$ is a simple positive eigenvalue of $M$ with a (componentwise) positive eigenvector. By comparison theorem, we get

$$
\begin{array}{r}
I(t) \longrightarrow \infty, \quad D(t) \longrightarrow \infty, \quad A(t) \longrightarrow \infty  \tag{55}\\
\quad \text { as } t \longrightarrow \infty
\end{array}
$$

which contradicts the assumption that

$$
\begin{equation*}
(S(t), V(t), I(t), D(t), A(t)) \longrightarrow E_{0} \quad \text { as } t \longrightarrow \infty \tag{56}
\end{equation*}
$$

This proves that $W^{s}\left(E_{0}\right) \cap X_{0}=\emptyset$ for $R_{0}>1$. By [20], system (1) is uniformly persistent. Thus, the proof of the theorem is completed.

## 5. Numerical Simulations

5.1. Numerical Results. We initially investigate variation in $R_{0}$ with different vaccine efficacy, vaccination rate, and diagnosis rate to compare the impact of these intervention measures on HIV transmission. The parameter values in Table 1 are chosen based on HIV/AIDS transmission among IDUs in Yunnan province, China. For simplicity, we choose
$h=1$. Differentiating partially $R_{0}$ with respect to $\xi$ and $\epsilon$, respectively, we obtain

$$
\begin{align*}
\frac{\partial R_{0}}{\partial \xi}= & -\frac{\beta}{\sigma_{1}+q+\mu} \frac{\epsilon(\mu+\omega)}{(\mu+\omega+\xi)^{2}} \\
& \times\left(1+\frac{\eta_{1} q}{\sigma_{2}+\mu}+\frac{\eta_{2}}{\mu+\psi}\left(\frac{\sigma_{2} q}{\sigma_{2}+\mu}+\sigma_{1}\right)\right)<0, \\
\frac{\partial R_{0}}{\partial \epsilon}= & -\frac{\beta}{\sigma_{1}+q+\mu} \frac{\xi}{\mu+\omega+\xi} \\
& \times\left(1+\frac{\eta_{1} q}{\sigma_{2}+\mu}+\frac{\eta_{2}}{\mu+\psi}\left(\frac{\sigma_{2} q}{\sigma_{2}+\mu}+\sigma_{1}\right)\right)<0, \tag{57}
\end{align*}
$$

which implies that an increase of vaccination rate and vaccine efficacy leads to the basic reproduction number decline, as shown in Figure 2(a), in which the contour plots of $R_{0}$ versus vaccine efficacy $\epsilon$ and vaccination rate $\xi$ were plotted. It also shows that the basic reproduction number is more sensitive to vaccine efficacy than vaccination rate. Figure 2(b) shows the contour plot of $R_{0}$ with diagnosis rate and vaccination rate, which implies a decrease in $R_{0}$ with increasing diagnosis rate $q$ and vaccination rate $\xi$. Furthermore, when $50 \%$ of HIV individuals are diagnosed, vaccination level of at least $60 \%$ would be needed to achieve $R_{0}<1$. This suggests that the strategies of diagnosis and vaccination should be stringent enough to reduce $R_{0}$.

Next, we consider the effect of different transmission rate, vaccination rate, vaccine efficacy, and recruitment rate on transmission of HIV/AIDS. We take the year 2004 as starting time; since then the policy of diagnosis is consistent. In [21], we get that the number of diagnosed HIV-positive individuals and AIDS patients in Yunnan province was 27168 and 1223 in year 2004, respectively. Besides, $22.6 \%$ of these HIV individuals were transmitted by share injectors [22]. Hence, $D(0)=27168 \times 22.6 \%=6140, A(0)=1223 \times 22.6 \%=$ 276. Note that the diagnosis rate is estimated to be 0.304 [9]; then we have $I(0)=D(0) / 0.304=20197$. We have no reliable data on the number of susceptible individuals, that is, number of IDUs in Yunnan province. However, we know


FIgure 2: Contour plots of $R_{0}$ versus (a) vaccine efficacy $\epsilon$ and vaccination rate $\xi$ and (b) diagnosis rate $q$ and vaccination rate $\xi$.
that 3.2 million blood samples were tested in Yunnan in [23]. We then assume that in these blood samples the fraction of share injectors is the same as fraction of transmission via share injectors (i.e., $22.6 \%$ ). Then the number of susceptible individuals who share injectors is $S(0)+V(0)=3.2 \times 10^{6} \times$ $22.6 \%=723200$. If the vaccination rate is assumed to be 0.4 , then $V(0)=S(0) \times 0.4$. We obtain the initial values

$$
\begin{gather*}
S(0)=516571, \quad V(0)=206628, \quad I(0)=20197 \\
D(0)=6140, \quad A(0)=276 . \tag{58}
\end{gather*}
$$

Figure 3 shows the variation in the number of HIV infected individuals with different transmission rates, vaccination rates, vaccine efficacy, and recruitment rates. It follows from Figure 3(a) that decreasing transmission rate could lead to the number of HIV-positive individuals decline. The effect of increasing vaccination rate on HIV transmission is shown in Figure 3(b) and it is seen that the number of HIVpositive individuals becomes much smaller if vaccination rate increases more. Figure 3(c) illustrates that, with increasing vaccine efficacy, the number of HIV-positive individuals decreases. Figure 3(d) shows that if the inflow of susceptible individuals into the community is restricted due to education, the disease spread will slow down.
5.2. Sensitivity Analysis. In this section, we use sensitivity analysis method [24] to investigate the impact of various intervention measures on HIV transmission in Yunnan province, China. We hope that these results obtained here could improve the knowledge of the effects of different interventions.

Figures 4(a) and 4(b) show the comparison of sensitivity coefficients of new cases and prevalence against parameters $\beta$, $\epsilon, \xi, q$, and $\Pi$, separately. Note that the sensitivity coefficient of new cases and prevalence can be interpreted as the percentage change in the number of new cases and prevalence for $1 \%$ decline in the parameters $\beta$ and $\Pi$ or $1 \%$ increase in $\epsilon, \xi$, and $q$,
respectively [25]. In particular, let function $f$ be new cases or prevalence; the sensitivity coefficients (SC) of new cases and prevalence are given by
$\mathrm{SC}=\frac{f(\text { perturbed variables })-f(\text { original variable })}{f(\text { original variable })} \times 100 \%$.

It follows from Figure 4 that a decrease in transmission coefficient $\beta$ causes new cases and prevalence decline substantially. Besides, an increase in vaccine efficacy $\epsilon$ and vaccination rate $\xi$ can lead to a decrease in new cases and prevalence, whereas the change of both diagnosis rate $q$ and recruitment rate $\Pi$ slightly affects the new cases or prevalence. Thus, new cases and prevalence are sensitive to transmission coefficient, vaccine efficacy, and vaccination rate. Then, reducing the transmission coefficient and increasing the vaccine efficacy and vaccination rate can greatly reduce new cases and prevalence.

## 6. Conclusion

In this paper, we established an epidemic model to investigate effects of saturated diagnosis and vaccination on HIV transmission. It proved that backward bifurcation occurs by employing center manifold theory, which causes the diseasefree equilibrium to be locally asymptotically stable instead of globally asymptotically stable for $R_{0}<1$. Thus, making the basic reproduction number less than unity is not enough to eliminate the HIV infection. We note that $R_{0}<1$ is equivalent to

$$
\begin{aligned}
\xi>((\mu+\omega)[ & \beta\left(1+\frac{\eta_{1} q}{\sigma_{2}+\mu}+\frac{\eta_{2}}{\mu+\psi}\left(\frac{\sigma_{2} q}{\sigma_{2}+\mu}+\sigma_{1}\right)\right) \\
& \left.\left.-\left(\sigma_{1}+q+\mu\right)\right]\right)
\end{aligned}
$$



FIgure 3: Variation in the number of HIV-positive individuals with different (a) transmission coefficient, (b) vaccination rate, (c) vaccine efficacy, and (d) recruitment rate.

$$
\begin{align*}
& \times\left(\left(\sigma_{1}+q+\mu\right)\right. \\
& \left.\quad-\beta(1-\epsilon)\left(1+\frac{\eta_{1} q}{\sigma_{2}+\mu}+\frac{\eta_{2}}{\mu+\psi}\left(\frac{\sigma_{2} q}{\sigma_{2}+\mu}+\sigma_{1}\right)\right)\right)^{-1} \\
:= & \xi_{c}, \tag{60}
\end{align*}
$$

which means that only the vaccination rate is greater than $\xi_{c}$; HIV infection might be eliminated, depending on initial data. There exists the critical threshold $R_{c}$, which cannot
be explicitly expressed due to nonlinearity, such that when $R_{0}<\min \left\{R_{c},\left(\sigma_{1}+\mu\right) m /\left(\sigma_{1}+q+\mu\right)\right\}<1$, the diseasefree equilibrium is globally asymptotically stable. However, if $R_{0}>1$, the disease uniformly persists.

It is interesting to note that if the diagnosis is described linearly backward bifurcation does not happen. This implies that nonlinear diagnosis due to limited medical resources leads to backward bifurcation, and consequently complete elimination of HIV infection becomes difficult. That is, HIV infection might be extinct only by improving integrated interventions, which ensures that $R_{0}$ is less than $R_{c}$ and ( $\sigma_{1}+$ $\mu) m /\left(\sigma_{1}+q+\mu\right)$.


Figure 4: Sensitivity coefficients of new cases (a) and prevalence (b) on $\beta, \epsilon, \xi, q$, and $\Pi$ over time $t$. All other parameters are shown in Table 1.

Since several candidate HIV vaccines are in development, it is useful to study the effectiveness. Moreover, the detection of HIV-positive individuals is limited due to medical resources. We then applied the proposed model with nonlinear diagnosis and vaccination to examine HIV infection among IDUs in Yunnan province, China. Sensitivity analysis shows that new cases and prevalence are sensitive to transmission rate, vaccine efficacy, and vaccination rate, whereas diagnosis rate and recruitment rate slightly affect both of them. Therefore, enlarging vaccination rate, improving vaccine efficacy, and lowering transmission rate by reducing sterile injecting equipment are beneficial to reduce transmission of HIV infection. In order to efficiently reduce HIV transmission, combined intervention strategies are suggested to be implemented simultaneously.

Effective antiretroviral therapy (ART) is an important strategy to slow down the progression to AIDS due to great reduction in viral loads and is not included in our model. Note that when HIV infected individuals are diagnosed and CD4 T cell counts decrease to 350 copies $/ \mu \mathrm{L}$, they will accept treatment. We will include treatment strategy to construct HIV/AIDS models to investigate the transmission of HIV/AIDS in the future work and provide policy makers with effective suggestions.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The authors are supported by the National Megaproject of Science Research no. 2012ZX10001-001, the National Natural Science Foundation of China (NSFC, 11171268 (YX)), the Fundamental Research Funds for the Central Universities (GK 08143042 (YX)), and the International Development Research Center, Ottawa, Canada (104519-010).

## References

[1] J. M. Hyman, J. Li, and E. A. Stanley, "The differential infectivity and staged progression models for the transmission of HIV," Mathematical Biosciences, vol. 155, no. 2, pp. 77-109, 1999.
[2] R. Naresh, A. Tripathi, and D. Sharma, "Modelling and analysis of the spread of AIDS epidemic with immigration of HIV infectives," Mathematical and Computer Modelling, vol. 49, no. 5-6, pp. 880-892, 2009.
[3] F. Nyabadza, Z. Mukandavire, and S. D. Hove-Musekwa, "Modelling the HIV/AIDS epidemic trends in South Africa: insights from a simple mathematical model," Nonlinear Analysis: Real World Applications, vol. 12, no. 4, pp. 2091-2104, 2011.
[4] J. M. Hyman, J. Li, and E. A. Stanley, "Modeling the impact of random screening and contact tracing in reducing the spread of HIV", Mathematical Biosciences, vol. 181, no. 1, pp. 17-54, 2003.
[5] A. B. Gumel, C. C. McCluskey, and P. van den Driessche, "Mathematical study of a staged-progression HIV model with imperfect vaccine," Bulletin of Mathematical Biology, vol. 68, no. 8, pp. 2105-2128, 2006.
[6] E. H. Elbasha and A. B. Gumel, "Theoretical assessment of public health impact of imperfect prophylactic HIV-1 vaccines
with therapeutic benefits," Bulletin of Mathematical Biology, vol. 68, no. 3, pp. 577-614, 2006.
[7] O. Sharomi, C. N. Podder, A. B. Gumel, E. H. Elbasha, and J. Watmough, "Role of incidence function in vaccine-induced backward bifurcation in some HIV models," Mathematical Biosciences, vol. 210, no. 2, pp. 436-463, 2007.
[8] F. Nyabadza and Z. Mukandavire, "Modelling HIV/AIDS in the presence of an HIV testing and screening campaign," Journal of Theoretical Biology, vol. 280, no. 1, pp. 167-179, 2011.
[9] Y. Xiao, S. Tang, Y. Zhou, R. J. Smith, J. Wu, and N. Wang, "Predicting the HIV/AIDS epidemic and measuring the effect of mobility in mainland China," Journal of Theoretical Biology, vol. 317, pp. 271-285, 2013.
[10] X. Zhang and X. Liu, "Backward bifurcation of an epidemic model with saturated treatment function," Journal of Mathematical Analysis and Applications, vol. 348, no. 1, pp. 433-443, 2008.
[11] P. van den Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," Mathematical Biosciences, vol. 180, no. 1-2, pp. 29-48, 2002.
[12] C. Castillo-Chavez and B. Song, "Dynamical models of tuberculosis and their applications," Mathematical Biosciences and Engineering, vol. 1, no. 2, pp. 361-404, 2004.
[13] B. Buonomo and D. Lacitignola, "On the backward bifurcation of a vaccination model with nonlinear incidence," Nonlinear Analysis: Modelling and Control, vol. 16, no. 1, pp. 30-46, 2011.
[14] S. D. Hove-Musekwa, F. Nyabadza, and H. MambiliMamboundou, "Modelling hospitalization, home-based care, and individual withdrawal for people living with HIV/AIDS in high prevalence settings," Bulletin of Mathematical Biology, vol. 73, no. 12, pp. 2888-2915, 2011.
[15] L. Zhou and M. Fan, "Dynamics of an SIR epidemic model with limited medical resources revisited," Nonlinear Analysis: Real World Applications, vol. 13, no. 1, pp. 312-324, 2012.
[16] J. Arino, C. C. McCluskey, and P. van den Driessche, "Global results for an epidemic model with vaccination that exhibits backward bifurcation," SIAM Journal on Applied Mathematics, vol. 64, no. 1, pp. 260-276, 2003.
[17] Z. Mukandavire and W. Garira, "Effects of public health educational campaigns and the role of sex workers on the spread of HIV/AIDS among heterosexuals," Theoretical Population Biology, vol. 72, no. 3, pp. 346-365, 2007.
[18] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, vol. 191 of Mathematics in Science and Engineering, Academic Press, Boston, Mass, USA, 1993.
[19] Y. Xiao and L. Chen, "Modeling and analysis of a predator-prey model with disease in the prey," Mathematical Biosciences, vol. 171, no. 1, pp. 59-82, 2001.
[20] X.-Q. Zhao, Dynamical Systems in Population Biology, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 16, Springer, New York, NY, USA, 2003.
[21] T. Zhang, M. Jia, H. Luo, Y. Zhou, and N. Wang, "Study on a HIV/AIDS model with application to Yunnan province, China," Applied Mathematical Modelling, vol. 35, no. 9, pp. 4379-4392, 2011.
[22] L. Lu, M. Jia, J. Lu, H. Luo, and X. Zhang, "Analysis of HIV/AIDS prevalence in Yunnan province," Journal for China AIDS/STD, vol. 11, no. 3, pp. 172-175, 2005.
[23] L. Lu, M. Jia, Y. Ma et al., "The changing face of HIV in China," Nature, vol. 455, no. 7213, pp. 609-611, 2008.
[24] X. Xu, Y. Xiao, and N. Wang, "Modeling sexual transmission of HIV/AIDS in Jiangsu province, China," Mathematical Methods in the Applied Sciences, vol. 36, no. 2, pp. 234-248, 2013.
[25] L. Liu, X.-Q. Zhao, and Y. Zhou, "A tuberculosis model with seasonality," Bulletin of Mathematical Biology, vol. 72, no. 4, pp. 931-952, 2010.
[26] L. Han, J. Lou, Y. Ruan, and Y. Shao, "The analysis of the HIV/AIDS mathematical model for the injection drug use population," Journal of Biomathematics, vol. 23, no. 3, pp. 429434, 2008.
[27] N. Bacaër, X. Abdurahman, and J. Ye, "Modeling the HIV/AIDS epidemic among injecting drug users and sex workers in Kunming, China," Bulletin of Mathematical Biology, vol. 68, no. 3, pp. 525-550, 2006.

# Comparison of Three Measures to Promote National Fitness in China by Mathematical Modeling 

Pan Tang, ${ }^{1}$ Daqing Xu, ${ }^{1}$ Qing Dai, ${ }^{1}$ and Tingting Huang ${ }^{2}$<br>${ }^{1}$ Department of Public Physical Education, Hefei University, Hefei 230601, China<br>${ }^{2}$ School of Mathematical Sciences, Anhui University, Hefei 230601, China

Correspondence should be addressed to Pan Tang; pan@hfuu.edu.cn
Received 22 January 2014; Accepted 27 January 2014; Published 26 March 2014
Academic Editor: Weiming Wang
Copyright © 2014 Pan Tang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

In this paper we established a mathematical model for national fitness in China. Based on a questionnaire and data of the General Administration of Sport of China and the National Bureau of Statistics of China, the dynamics for three classes of people are expressed by a system of three-dimensional ordinary equations. Model parameters are estimated from the data. This study indicated that national fitness put out by the Chinese government is reasonable. By finding the key parameter, the best measure to promote national fitness is put forward. In order to increase the number of people who frequently participate in sport exercise in a short period of time, if only one measure can be chosen, guiding people who never take part in physical exercise will be the best measure.


## 1. Introduction

To promote the development of mass sports in the new era of reform and opening-up in China and to improve national physical quality, the State Council launched a national fitness program on June 20, 1995 [1]. Now national fitness has got remarkable achievement. The survey shows that the number of people who often participate in sports activities in 2004 is more than $5.7 \%$ the number in 1997 and the country has $28.2 \%$ of population who often take part in physical exercise in 2007 [1].

To better implement the national fitness program, the government takes some measures mainly on three aspects. The first one is conducting propaganda work, to make people aware of the importance of sports fitness and the implementation of the national fitness program and to arouse enthusiasm of the masses to participate in fitness [1]. The second one is guidance, that is, generally establishing institutions responsible for mass sports management work, such as social sports guidance center, national fitness guidance center, and social sports instructors [1]. The purpose is to help guide people to exercise. The last one is improvement of sports facilities. China has built and renovated various sports
stadiums and actively promoted open stadiums. These have greatly improved material conditions of national fitness [1].

In [1] General Administration of Sport of China (GASC) also pointed out that, although we have got many achievements and experience, the mass sports in our country's overall development level is not high and the proportion of the population participating in regular physical exercise is not high. We still face many challenges [1].

According to the report of GASC [2], population can be separated into three classes. The first one is people who never take part in physical exercise. We call it as never exercise for short. The second one is people who occasionally take part in physical exercise. We call it as occasionally exercise for short. This class of people does weekly exercise activities of 1 to 2 times. The third one is people who often take part in physical exercise. We call it as frequently exercise for short. This class of people does weekly exercise activities of not less than 3 times, each time not less than 30 minutes and the exercise intensity above the average of person. Of course, the social sports instructors are included in the third-class of people.

So far, China has carried out three investigations of the current situation of mass sports in 1997, 2001, and 2008. According to the three surveys and the data of National

Bureau of Statistics of China [3], the proportion of the three classes of people is listed in Table 1. Based on these data, we can see that the proportion of the third-class of people is not high. In the national fitness program (2011-2015), Chinese government put forward the target that the proportion of the third-class of people should be more than $32 \%$ [2]. It is a natural question whether the goal can be attained. This is the first problem that we want to prove.

In addition, all of these programs cost an enormous sum of money every year, for example, over one year a total investment of sports funds $191,450,000,000$ RMB on construction of site [1]. It is natural to ask how we can economically and quickly increase the number of frequently exercise people. Based on the national fitness program [1], there are some measures including propagation, guidance, and sports facilities. There is no doubt that the three measures together can promote national fitness. However, in order to save manpower and wealth of the country, we want to know which measure should be the best one if only one measure can be chosen. This is the second problem that we want to prove.

Currently, although some literatures reported the importance of national fitness, they only state or compare the achievement after the implementation of the national fitness program [4, 5, 8, 9]. There are few literatures discussing the best strategy to improve the national fitness program. There are not any mathematical models to study how to improve the national fitness program. In this paper, based on a questionnaire and the data of GASC and the National Bureau of Statistics of China, we established a mathematical model and incorporated three measures and three classes of parameters. By numerical simulation and analysis, the key parameters are found and used to develop the best measure. Hopefully, it could provide the theory reference for the mass sports policy basis.

The paper is organized as follows. In Section 2 we establish a mathematical model based on a questionnaire and the data of the General Administration of Sport of China and the National Bureau of Statistics of China. Section 3 is devoted to parameter estimation, numerical simulations, and analysis of the system. The goal is to find the key parameter. We end the paper with a brief discussion.

## 2. Modeling

Based on the surveys, the population can be separated into three classes $x, y$, and $z[4-6] . x, y$, and $z$ present the number of individuals who never, occasionally, and frequently participate in physical exercise at time $t$, respectively. Each class can be transformed to other classes. In order to better understand how to do transformation between the three-class groups and in order to get real data, we conducted a questionnaire survey. According to the questionnaire, by contacting the people who occasionally participate in exercise, that is, $y$, there is $10.57 \%$ of the first-class of people $x$ who become the second-class of people $y$. By contacting the people who frequently participate


Figure 1: The diagram of class transformation. The red line represents transformation by contact, the blue line represents automatically transformation, and the green line represents reduction of the times of exercise.


Figure 2: The fitting between the real number of three classes of people and the solution of the system. The red is real number and the blue line is the solution.
in exercise, that is, $z$, there are $8.89 \%$ and $11.2 \%$ of the firstclass of people $x$ who become the second-class of people $y$ and the third-class of people $z$, respectively; and there is also $17.6 \%$ of the second-class of people $y$ who become the thirdclass of people $z$. Because of propaganda, there are 26.03\% and $21.2 \%$ of the first-class of people $x$ who automatically become the second-class of people $y$ and the third-class of people $z$, respectively. At the same time, some people reduced the times of exercise because of the lack of sport facilities. For the third-class of people $z$, there are $24.52 \%$ and $15.93 \%$ of the class of people who become the second-class of people $y$ and the first-class of people $x$, respectively. And there is $15.04 \%$ of the second-class of people $y$ who become the first-class of

Table 1: Data of survey for the proportion of the three classes of people.

| Year | First-class (\%) | Second-class (\%) | Third-class (\%) | The total number of <br> populations (million) | References |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 1996 | 65.65 | 18.85 | 15.5 | 1223.89 | $[3-5]$ |
| 2000 | 65 | 16.7 | 18.3 | 1267.43 | $[3-5]$ |
| 2007 | 61.5 | 10.3 | 28.2 | 1321.29 | $[3,6]$ |

Table 2: Parameters.

| Parameters | Description |
| :--- | :--- |
| $r$ | Intrinsic rate of increase of human |
| $K$ | Carrying capacity of population |
| $\alpha_{1}$ | Automatic transformation rate of individual from the class $x$ to the class $y$ because of propaganda |
| $\alpha_{2}$ | Automatic transformation rate of individual from the class $x$ to the class $z$ because of propaganda |
| $\beta_{1}$ | Transformation rate of individual from the class $x$ to the class $y$ by contacting individuals at the class $y$ |
| $\beta_{2}$ | Transformation rate of individual from the class $x$ to the class $y$ by contacting individuals at the class $z$ |
| $\beta_{3}$ | Transformation rate of individual from the class $x$ to the class $z$ by contacting individuals at the class $z$ |
| $\beta_{4}$ | Transformation rate of individual from the class $y$ to the class $z$ by contacting individuals at the class $z$ |
| $\gamma_{1}$ | Automatic transformation rate of individual from the class $z$ to the class $y$ because of the lack of sports facilities |
| $\gamma_{2}$ | Automatic transformation rate of individual from the class $z$ to the class $x$ because of the lack of sports facilities |
| $\gamma_{3}$ | Automatic transformation rate of individual from the class $y$ to the class $x$ because of the lack of sports facilities |
| $\mu$ | Death rate of human |

Table 3: Birth rate and death rate.

| Year | Birth rate (per year) | Death rate (per year) |
| :--- | :---: | :---: |
| 1996 | 0.01698 | 0.00656 |
| 2000 | 0.01403 | 0.00645 |
| 2007 | 0.0121 | 0.00693 |
| Mean | 0.01437 | 0.00665 |

Table 4: Value of parameters.

| Parameters | Values (per year) | Reference |
| :--- | :---: | :---: |
| $r$ | 0.00772 | $[3]$ |
| $K$ | $1.6 * 10^{9}$ | $[7]$ |
| $\alpha$ | 0.02614 | Estimated |
| $\beta$ | 0.0089 | Estimated |
| $\gamma$ | 0.01512 | Estimated |
| $\mu$ | 0.00665 | $[3]$ |

people $x$. The diagram of class transformation is in Figure 1. Then we obtain the following system:

$$
\begin{align*}
\frac{d x}{d t}= & r x\left(1-\frac{x}{K}\right)-\beta_{1} x y-\beta_{2} x z-\beta_{3} x z-\alpha_{1} x \\
& +\gamma_{2} z+\gamma_{3} y-\alpha_{2} x \\
\frac{d y}{d t}= & \beta_{1} x y+\beta_{2} x z+\alpha_{1} x+\gamma_{1} z-\gamma_{3} y-\beta_{4} y z-\mu y  \tag{1}\\
\frac{d z}{d t}= & -\gamma_{1} z-\gamma_{2} z+\beta_{3} x z+\beta_{4} y z+\alpha_{2} x-\mu z
\end{align*}
$$

## 3. Simulation and Analysis

Since the system is very complex, theoretical analysis is very difficult and we use numerical simulation to analyze


Figure 4: Increasing the value of $\alpha$ and $\beta$ and reducing the value of $\gamma$ can all increase the number of $z$. The impact of $\beta$ on the system is the biggest. The impact of $\gamma$ on the system is little bigger than the impact of $\alpha$.
the dynamical behaviors based on the real data. According to the data of the National Bureau of Statistics of China [3], the birth rate and the death rate of population in China are listed in the Table 3. In 1991, China Academy of Sciences published a report about the productivity and population carrying capacity of the land resource in China. The carrying capacity is 1600000000 [7]. To obtain the value of other parameters, we performed some estimations based on the data of GASC and the National Bureau of Statistics of China. For convenience, we assume that $\alpha_{1}=\alpha_{2}=\alpha, \beta_{1}=$ $\beta_{2}=\beta_{3}=\beta_{4}=\beta$, and $\gamma_{1}=\gamma_{2}=\gamma_{3}=\gamma$. Then, we estimate the three parameters based on the minimal error method and obtain the fitting figure (Figure 2). Finally,
the value of all parameters is listed in Table 4. Based on these values of parameters, the system approaches a stable positive equilibrium state (Figure 3). From Figure 3, we can see that the number of people who frequently participate in exercise will increase. Hence, the goal of Chinese government in the national fitness program (2011-2015) can be attained.

Now we want to solve the second question. By changing the value of three parameters $\alpha, \beta$, and $\gamma$, we investigate the impact of these three measures on the number of three classes of people $x, y$, and $z$ (Figure 4). From Figure 4(a), we can see that $\alpha$ cannot infect the dynamics of the system. This means that intensifying propaganda is not obvious to increase the number of people who frequently participate in sport


Figure 5: The impact of four different $\beta$ on the system.
exercise. However, Figure 4(b) displays the fact that $\beta$ can strongly infect the dynamics of the system. In other words, by providing guidance for the first- and second-class of people, more and more people will frequently participate in sport exercises. At last, improving sport facility can increase the number of the people who frequently participate in sport exercise, which can be seen from Figure 4(c).

Now we want to know which class of people provided guidance that can lead to the best effect. By changing the four different parameters $\beta_{1}, \beta_{2}, \beta_{3}$, and $\beta_{4}$ in the original system (1), we find that $\beta_{3}$ and $\beta_{4}$ have stronger effect in the system (Figure 5). This means that only the third- class of people $z$
guiding the other classes of people $x$ and $y$ can lead to better effect.

Next, by changing the values of $\beta_{3}$ and $\beta_{4}$, we compare the effect of guiding between the first- and the second-class of people (Figure 6) and we can obtain the following results. First, from the black line we can see that guiding both the first- and second-class of people will increase the number of the third-class of people quickly, which means the effect by guiding the first two classes of people is the best one. Second, if we want to increase the number of people who frequently participate in sport exercise in a short period of time, guiding the first-class of people will be the best measure. Lastly, if we


Figure 6: Comparing the effect of guiding for different classes of people.
want to increase the number of the third-class of people over a long period of time, guiding the second-class of people will be the best measure.

## 4. Discussion

As we know, national fitness can improve national physical quality. National fitness is promoted all over the world. Furthermore, Chinese government put forward the target that the proportion of the third-class of people should be more than $32 \%$ [2]. It is needed to consider whether the goal can be attained and how to better and faster achieve the anticipated goal. In previous literatures and reports, all conclusions are from reports or experiences. However, there are not any results based on mathematical theory. In this paper, based on a questionnaire and the data of the General Administration of Sport of China and the National Bureau of Statistics of China, we established a mathematical model for three classes of people and represented three measures by three classes of parameters. By numerical simulation and analysis, the key parameter $\beta$ is found. Furthermore, the two previous problems can be answered.

By simulation and analysis based on our mathematical model, the following conclusions can be obtained. First, the number of people who frequently participate in sport exercises will increase definitely and then the goal can be attained. Second, we found that conducting propaganda cannot increase obviously the number of people who frequently participate in sport exercises. In reality, people can be impressed by much propaganda. However, it is always difficult to act due to various reasons. Third, sports facilities are not so important as people imagine. From the figures we can see that the effect of reducing the value of $\gamma$ is not stronger.

In fact, there are many ways to participate in physical exercise without sports facilities.

Lastly, it is very important to provide guidance for all people.From the simulation of the model, we can see that the effect of parameter $\beta$ is the biggest. Furthermore, increasing the value of $\beta$ by five times can obtain better effect than increasing the value by ten times. Hence, proper guidance will enable more people to participate in physical exercise frequently. Currently, there are many ways to provide guidance, for example, establishing national fitness guidance center and providing social sports instructor. The report of the General Administration of Sport of China in 2009 [10] also fully affirmed that social sports instructors play a very important role in national fitness. In addition, by comparing the effect between $\beta_{3}$ and $\beta_{4}$, we found some different results. If we want to increase the number of people who frequently participate in sport exercise in a short period of time, guiding the first-class of people will be the best measure. If we want to increase the number of the third-class of people in a long period of time, guiding the second- class of people will be the best measure. Hence, we can provide more and more social sports instructors only for the first- and the second-class of people in the future.

In summary, these conclusions are also consistent with the actual situation. Furthermore, based on the mathematical modeling and the real data, we think that guidance is the best measure for national fitness. Hopefully, these conclusions could provide the theory reference for the mass sports policy basis.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This research is supported by Sports and Social Science Foundation of Sports Bureau of Anhui Province (ASS2012238). The authors would like to specially thank Professor Huaiping Zhu, York University of Canada, for his guidance and help. Also, they would like to thank anonymous reviewers for the very helpful suggestions which improved greatly this paper.

## References

[1] General Administration of Sport of China, "Outline of the national fitness program (The implementation of fifteen years)," pp. 1-66, 2011.
[2] General Administration of Sport of China, "The national fitness program (2011-2015)," 2011, http://www.chinasfa. net/ArtDetail.aspx?aid=6537.
[3] National Bureau of Statistics of China, http://www.stats.gov.cn/.
[4] Z. J. Zhang and C. Liu, "Commentary on the development of Chinese sports population since reform and opening," Journal of Physical Education Institute of Shanxi Normal University, vol. 22, no. 1, pp. 19-22, 2007.
[5] X. R. Zhou and M. Y. Tan, "Comparative research on our mass sports condition during two times investigation," Sport Science, vol. 24, no. 7, pp. 12-15, 2004.
[6] General Administration of Sport of China, "In 2007 China urban and rural residents to participate in physical exercise situation survey bulletin," 2012, http://www.gov.cn/ test/2012-04/19/content2117453.htm.
[7] China Academy of Sciences, "Population carrying capacity of China," 1991, http://zh.wikipedia.org.
[8] X. P. Lin, "The research on China mass sports developing tendency toward 2010," China Sport Science and Technology, vol. 37, no. 11, pp. 10-13, 2001.
[9] P. Tang, "Analysis of the characteristics of the national sports meeting," Sports Culture Guide, vol. 12, pp. 9-11, 2010.
[10] General Administration of Sport of China, "Active in the mass sports, social sports instructor was generally recognized," 2009, http://www.chinasfa.net/ArtDetail.aspx?aid=5306.

## Research Article

# Multiple Positive Periodic Solutions for Functional Differential Equations with Impulses and a Parameter 

Zhenguo Luo ${ }^{1,2}$<br>${ }^{1}$ Department of Mathematics, Hengyang Normal University, Hengyang, Hunan 421008, China<br>${ }^{2}$ China Department of Mathematics, National University of Defense Technology, Changsha, Hunan 410073, China<br>Correspondence should be addressed to Zhenguo Luo; robert186@163.com<br>Received 19 November 2013; Revised 24 December 2013; Accepted 7 January 2014; Published 26 March 2014<br>Academic Editor: Yun Kang<br>Copyright © 2014 Zhenguo Luo. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We apply the Krasnoselskii fixed-point theorem to investigate the existence of multiple positive periodic solutions for a class of impulsive functional differential equations with a parameter; some verifiable sufficient results are established easily. In particular, our results extend and improve some previous results.


## 1. Introduction

It is well known that impulsive differential equations arise naturally from a wide variety of applications such as aircraft control, the inspection processes in operations research, drug administration, and threshold theory in biology. Therefore, the impulsive differential equations represent a more natural framework for the mathematical model of many real world phenomena than differential equations (see [1-7]). In recent years, many researchers have obtained some properties of impulsive differential equations, such as oscillation, asymptotic behavior, stability and existence of solutions (see [8-16]). However, there are a little work discussing the existence of multiple positive periodic solutions for the high-dimensional functional differential equations with impulse and parameters. Motivated by this, in this paper, we mainly consider the following impulsive functional differential equations with a parameter:

$$
\begin{gather*}
x^{\prime}(t)=A(t, x(t)) x(t)+\lambda B(t, x(t)) f\left(t, x_{t}\right), \\
t \in R, \quad t \neq t_{k},  \tag{1}\\
\Delta x\left(t_{k}\right)=I_{k}\left(t_{k}, x\left(t_{k}\right)\right), \quad k \in Z_{+},
\end{gather*}
$$

where $\lambda>0$ is a parameter, $A(t, x(t))=\operatorname{diag}\left[a_{1}(t, x(t))\right.$, $\left.a_{2}(t, x(t)), \ldots, a_{n}(t, x(t))\right], \quad B(t, x(t))=\operatorname{diag}\left[b_{1}(t, x(t))\right.$, $\left.b_{2}(t, x(t)), \ldots, b_{n}(t, x(t))\right], a_{j}, b_{j} \in C\left(R \times R^{+}, R^{+}\right)(j=1, \ldots, n)$ are $\omega$-periodic. $f=\left(f_{1}, \ldots, f_{n}\right)^{T}, f\left(t, x_{t}\right)$ is an operator on
$R \times B C\left(R, R^{n}\right)$ (here $B C\left(R, R^{n}\right)$ denoting the Banach space of bounded continuous operator $\varphi: R \rightarrow R^{n}$ with the norm $\|\varphi\|=\sum_{i+1}^{n} \sup _{\theta \in R}|\varphi(\theta)|$, where $\left.\varphi=\left(\varphi_{1}, \ldots, \varphi_{n}\right)^{T}\right)$ ); $f_{i}\left(t+\omega, x_{t}\right)=f_{i}\left(t, x_{t}\right)$. If $x \in B C\left(R, R^{n}\right)$, then $x_{t} \in B C\left(R, R^{n}\right)$ for any $t \in R$, where $x_{t}$ is defined by $x_{t}(\theta)=x(t+\theta)$ for $\theta \in R$ and $\Delta x\left(t_{k}\right)=x\left(t_{k}^{+}\right)-x\left(t_{k}\right)$ (here $x\left(t_{k}^{+}\right)$representing the right limit of $x(t)$ at the point $\left.t_{k}\right)$. Consider that $I_{k}=$ $\left(I_{k}^{1}, I_{k}^{2}, \ldots, I_{k}^{n}\right) \in C\left(R_{+}^{n}, R_{-}^{n}\right)$; that is, $x$ changes decreasingly suddenly at times $t_{k} . \omega>0$ is a constant, $Z_{+}=\{1,2,3, \ldots\}$, $R=(-\infty,+\infty), R_{+}=[0,+\infty)$, and $R_{-}=(-\infty, 0]$. We assume that there exists an integer $q>0$ such that $t_{k+q}=t_{k}+\omega$, $I_{k+q}=I_{k}$, where $0<t_{1}<t_{2}<\cdots<t_{q}<\omega$.

It is well known that the functional differential system (1) includes many mathematical ecological models for example:
in the case (a), $B(t, x(t)) \equiv 1, I_{k}\left(t_{k}, x\left(t_{k}\right)\right) \equiv 0$ in [17], Zeng et al. studied the existence of multiple positive periodic solutions of (1) by applying the Krasnoselskii fixed-point theorem.
in the case (b), $A(t, x(t))=A(t), \lambda B(t, x(t)) \equiv$ 1 ; in [18], Zhang et al. established the existence of positive periodic solutions of (1) by using the fixedpoint theorem in cones.
in the case (c), $A(t, x(t))=A(t), B(t, x(t)) \equiv 1$, and $I_{k}\left(t_{k}, x\left(t_{k}\right)\right) \equiv 0$; in [19], Jiang et al. investigated the existence, multiplicity, and nonexistence of positive periodic solutions of (1).

In this paper, we will study the existence of positive periodic solutions in more cases than the previously mentioned papers and obtain some easily verifiable sufficient criteria.

Throughout the paper, we make the following assumptions:
$\left(H_{1}\right) a_{i}, b_{i}: R \times R_{+} \rightarrow \quad R_{+}$satisfy Caratheodory conditions; that is, $a_{i}(t, x), b_{i}(t, x)$ are locally Lebesgue measurable in $t$ for each fixed $x$, are continuous in $x$ for each fixed $t$, and are $\omega$-periodic functions in $t$. Moreover, there exist $\omega$-periodic functions $a_{1 i}, a_{2 i}, b_{1 i}$, $b_{2 i}: R \rightarrow R_{+}$which are locally bounded Lebesgue measurable such that $a_{1 i}(t) \leq a_{i}(t, x(t)) \leq a_{2 i}(t)$, $b_{1 i}(t) \leq b_{1 i}(t, x(t)) \leq b_{2 i}(t)$ and $\int_{0}^{\omega} a_{1 i}(t) d t>0$, $\int_{0}^{\omega} b_{1 i}(t) d t>0 ;$
$\left(H_{2}\right) f\left(t, \varphi_{t}\right) \leq 0$ for all $(t, \varphi) \in R \times B C\left(R, R_{+}^{n}\right)$, and $f_{i}\left(t, \varphi_{t}\right)$ is a continuous function of $t$ for each $\varphi \in B C\left(R, R_{+}^{n}\right)$, $i=1,2, \ldots, n$;
$\left(H_{3}\right)$ for any $L>0$ and $\epsilon>0$, there exists $\delta>0$ such that for $\phi, \psi \in B C\left(R, R_{+}^{n}\right),|\phi| \leq L,|\psi| \leq L$, and $|\phi-\psi|<\delta$ imply that $\left|f_{i}\left(s, \phi_{s}\right)-f_{i}\left(s, \psi_{s}\right)\right|<\epsilon, s \in[0, \omega](i=$ $1,2, \ldots, n$ );
$\left(H_{4}\right)\left\{t_{k}\right\}, k \in Z_{+}$satisfies $0<t_{1}<t_{2}<\cdots<t_{k}<$ $\cdots$ and $\lim _{k \rightarrow+\infty} t_{k}=+\infty ; I_{k}: R \times R_{+} \rightarrow R$, $k \in Z_{+}$, satisfy Caratheodory conditions and are $\omega$ periodic functions in $t$ and, moreover, $I_{k}(t, 0)=0$ for all $k \in Z^{+}$. There exists a positive constant $q$ such that $t_{k+q}=t_{k}+\omega, I_{k+q}\left(t_{k+q}, x\left(t_{k+q}\right)\right)=I_{k}\left(t_{k}, x\left(t_{k}\right)\right), k \in Z_{+}$. Without loss of generality, we can assume that $t_{k} \neq 0$ and $[0, \omega] \cap\left\{t_{k}, k \in Z^{+}\right\}=\left\{t_{1}, t_{2}, \ldots, t_{q}\right\}$.

In addition, the parameters in this paper are assumed to be not identically equal to zero.

Furthermore, we will use the following notation. Let $J \subset$ $R$ denote by $P C\left(J, R^{n}\right)$ the set of operators $\varphi: J \rightarrow R^{n}$ which are continuous for $t \in J, t \neq t_{k}$ and have discontinuities of the first kind at the points $t_{k} \in J\left(k \in Z_{+}\right)$but are continuous from the left at these points. For each $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)^{T} \in R^{n}$, the norm of $x$ is defined as $|x|=\sum_{i=1}^{n}\left|x_{i}\right|$. The matrix $A>$ $B(A \leq B)$ means that each pair of corresponding elements of $A$ and $B$ satisfies the inequality " $>$ " (" $\leq$ "). In particular, $A$ is called a positive matrix if $A>0$.

The paper is organized as follows. In Section 2, we give some definitions and lemmas to prove the main results of this paper. In Section 3, existence theorems for one or two positive periodic solutions of (1) are established by using the Krasnoselskii fixed-point theorem under some conditions.

## 2. Preliminaries

In this section, we make some preparations for the following sections. For $(t, s) \in R^{2}, 1 \leq i \leq n$, we define

$$
\begin{gather*}
G_{i}(t, s)=\frac{e^{-\int_{t}^{s} a_{i}(\xi, x(\xi))} d \xi}{e^{-\int_{0}^{\omega} a_{i}(\xi, x(\xi))} d \xi-1}  \tag{2}\\
G(t, s)=\operatorname{diag}\left[G_{1}(t, s), G_{2}(t, s), \ldots, G_{n}(t, s)\right]
\end{gather*}
$$

It is clear that $G_{i}(t+\omega, s+\omega)=G_{i}(t, s), \partial G_{i}(t, s) / \partial t=$ $a_{i}(t, x(t)) G_{i}(t, s), G_{i}(t, t+\omega)-G_{i}(t, t)=1$. For all $(t, s) \in R^{2}$ and by $\left(H_{2}\right)$, we have

$$
\begin{array}{r}
G_{i}(t, s) f_{i}\left(s, \varphi_{s}\right) \geq 0, \quad \text { for any }(t, s) \in R^{2}  \tag{3}\\
\left(s, \varphi_{s}\right) \in R \times B C\left(R, R_{+}^{n}\right)
\end{array}
$$

In view of $\left(H_{1}\right)$, we also define for $1 \leq i \leq n$ the following:

$$
\begin{gather*}
\alpha_{i}:=\min _{0 \leq t \leq s \leq \omega}\left|G_{i}(t, s)\right|=\frac{e^{-\int_{0}^{\omega} a_{2 i}(\xi, x(\xi)) d \xi}}{1-e^{-\int_{0}^{\omega} a_{2 i}(\xi, x(\xi)) d \xi}}, \\
\beta_{i}:=\max _{0 \leq t \leq s \leq \omega}\left|G_{i}(t, s)\right|=\frac{e^{-\int_{0}^{\omega} a_{1 i}(\xi, x(\xi)) d \xi}}{1-e^{-\int_{0}^{\omega} a_{1 i}(\xi, x(\xi)) d \xi}}, \\
\alpha=\min _{1 \leq i \leq n} \alpha_{i}, \quad \beta=\max _{1 \leq i \leq n} \beta_{i}, \quad \sigma=\frac{\alpha}{\beta} \in(0,1),  \tag{4}\\
B_{i}(t)=\max \left\{\left|b_{1 i}(t)\right|,\left|b_{2 i}(t)\right|\right\}, \\
B_{i}^{\prime}(t)=\min \left\{\left|b_{1 i}(t)\right|,\left|b_{2 i}(t)\right|\right\}, \\
B(t)=\max _{1 \leq i \leq n}\left\{B_{i}(t)\right\}, \quad B^{\prime}(t)=\min _{1 \leq i \leq n}\left\{B_{i}^{\prime}(t)\right\}
\end{gather*}
$$

Let $X=\left\{x=\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T} \in P C\left(R, R^{n}\right) \mid\right.$ $x(t+\omega)=x(t)\}$ with the norm $\|x\|=\sum_{i=1}^{n}\left|x_{i}\right|_{0},\left|x_{i}\right|_{0}=$ $\sup _{t \in[0, \omega]}\left|x_{i}(t)\right|$. It is easy to verify that $(X,\|\cdot\|)$ is a Banach space. Define $E$ as a cone in $X$ by

$$
\begin{align*}
& E=\{ x  \tag{5}\\
&=\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T} \in X: x_{i}(t) \\
&\left.\geq \sigma\left\|x_{i}\right\|_{0}, \quad t \in[0, \omega]\right\} .
\end{align*}
$$

We easily verify that $E$ is a cone in $X$. We define an operator $T: X \rightarrow X$ as follows:

$$
\begin{equation*}
(T x)(t)=\left(\left(T_{1} x\right)(t),\left(T_{2} x\right)(t), \ldots,\left(T_{n} x\right)(t)\right)^{T} \tag{6}
\end{equation*}
$$

where

$$
\begin{align*}
\left(T_{i} x\right)(t)= & \lambda \int_{t}^{t+\omega} G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right) d s  \tag{7}\\
& +\sum_{t \leq t_{k}<t+\omega} G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)
\end{align*}
$$

The proofs of the main results in this paper are based on an application of the Krasnoselskii fixed-point theorem in cones. To make use of the fixed-point theorem in cones, firstly, we need to introduce some definitions and lemmas.

Definition 1 (see [20]). A function $x: R \rightarrow(0,+\infty)$ is said to be a positive solution of (1), if the following conditions are satisfied:
(a) $x(t)$ is absolutely continuous on each $\left(t_{k}, t_{k+1}\right)$;
(b) for each $k \in Z_{+}, x\left(t_{k}^{+}\right)$and $x\left(t_{k}^{-}\right)$exist, and $x\left(t_{k}^{-}\right)=$ $x\left(t_{k}\right)$;
(c) $x(t)$ satisfies the first equation of (1) for almost everywhere in $R$ and $x\left(t_{k}\right)$ satisfies the second equation of (1) at impulsive point $t_{k}, k \in Z_{+}$.

Definition 2 (see [21]). Let $X$ be a real Banach space; $E$ is a cone of $X$. The semiorder induced by the cone $E$ is denoted by " $\leq$ "; that is, $x \leq y$ if and only if $y-x \in P$ for any $x, y \in E$.

Secondly, let us introduce the Krasnoselskii point theorem in cones which will be used in this paper.

Lemma 3 (for the Krasnoselskii fixed-point theorem; see [2224]). Let $E$ be a cone in a real Banach space $X$. Assume that $\Omega_{1}$ and $\Omega_{2}$ are open subsets of $X$ with $0 \in \Omega_{1} \subset \overline{\Omega_{1}} \subset \Omega_{2}$, where $\Omega_{i}=\left\{x \in X:\|x\|<r_{i}\right\}(i=1,2)$. Let $T: E \cap\left(\Omega_{2} \backslash \overline{\Omega_{1}}\right) \rightarrow E$ be a completely continuous operator and satisfy either
(1) $\|T x\| \geq\|x\|$, for any $x \in E \cap \partial \Omega_{1}$ and $\|T x\| \leq\|x\|$, for any $x \in E \cap \partial \Omega_{2}$, or
(2) $\|T x\| \leq\|x\|$, for any $x \in E \cap \partial \Omega_{1}$ and $\|T x\| \geq\|x\|$, for any $x \in E \cap \partial \Omega_{2}$.

Then $T$ has a fixed point in $E \cap\left(\Omega_{2} \backslash \overline{\Omega_{1}}\right)$.
Lemma 4 (see [25]). Assume that $f(t)$ and $g(t)$ are continuous nonnegative functions defined on the interval $[\alpha, \beta]$; then there exists $\xi \in[\alpha, \beta]$ such that

$$
\begin{equation*}
\int_{\alpha}^{\beta} f(t) g(t) d t=f(\xi) \int_{\alpha}^{\beta} g(t) d t \tag{8}
\end{equation*}
$$

Lemma 5. Assume that $\left(H_{1}\right)-\left(H_{4}\right)$ hold. The existence of positive $\omega$-periodic solution of (1) is equivalent to that of nonzero fixed point of $T$ in $E$.

Proof. Assume that $x=\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T} \in X$ is a periodic solution of (1). Then, we have

$$
\begin{align*}
& {\left[x_{i}(t) e^{-\int_{0}^{t} a_{i}(s, x(s)) d s}\right]^{\prime}} \\
& \quad=\lambda e^{-\int_{0}^{t} a_{i}(s, x(s)) d s} b_{i}(t, x(t)) f_{i}\left(t, x_{t}\right),  \tag{9}\\
& \quad t \neq t_{k}, \quad i=1,2, \ldots, n .
\end{align*}
$$

Integrating the above equation over $[t, t+\omega]$, we can have

$$
\begin{align*}
& \left.x_{i}(u) e^{-\int_{0}^{u} a_{i}(s, x(s)) d s}\right|_{t} ^{t_{m_{1}}+n \omega} \\
& \quad+\left.x_{i}(u) e^{-\int_{0}^{u} a_{i}(s, x(s)) d s}\right|_{t_{m_{1}}+n \omega} ^{t_{m_{2}}+n \omega}+\cdots \\
& \quad+\left.x_{i}(u) e^{-\int_{0}^{u} a_{i}(s, x(s)) d s}\right|_{t_{m_{q}}+n \omega} ^{t+\omega}  \tag{10}\\
& = \\
& \\
& \quad \lambda \int_{t}^{t+\omega} e^{-\int_{0}^{u} a_{i}(s, x(s)) d s} b_{i}(u, x(u)) f_{i}\left(u, x_{u}\right) d u
\end{align*}
$$

where $t_{m_{k}}+n \omega \in(t, t+\omega), m_{k} \in\{1,2, \ldots, q\}, k=1,2, \ldots, q$, and $n \in Z_{+}$. Therefore,

$$
\begin{align*}
& x_{i}(t) e^{-\int_{0}^{t} a_{i}(s, x(s)) d s}\left[e^{-\int_{t}^{t+\omega} a_{i}(s, x(s)) d s}-1\right] \\
& \quad-\sum_{t \leq t_{k}<t+\omega} \Delta x_{i}\left(t_{m_{k}}\right) e^{-\int_{0}^{t_{m_{k}}+n \omega} a_{i}(s, x(s)) d s}  \tag{11}\\
& = \\
& =\lambda \int_{t}^{t+\omega} e^{-\int_{0}^{u} a_{i}(s, x(s)) d s} b_{i}(u, x(u)) f_{i}\left(u, x_{u}\right) d u,
\end{align*}
$$

which can be transformed into

$$
\begin{align*}
x_{i}(t)= & \lambda \int_{t}^{t+\omega} G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right) d s  \tag{12}\\
& +\sum_{t \leq t_{k}<t+\omega} G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)=\left(T_{i} x\right)(t) .
\end{align*}
$$

Thus, $x_{i}$ is a periodic solution for (7).
If $x=\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T} \in E$ and $T x=$ $\left(T_{1} x, T_{2} x, \ldots, T_{n} x\right)^{T}=x$ with $x \neq 0$, then, for any $t=t_{k}$ we can get the derivation of (7) about $t$,

$$
\begin{align*}
\left(T_{i} x\right)^{\prime}(t)= & \frac{d}{d t}\left[\lambda \int_{t}^{t+\omega} G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right) d s\right] \\
= & \lambda\left[G_{i}(t, t+\omega) b_{i}(t+\omega, x(t+\omega))\right. \\
& \times f_{i}\left(t+\omega, x_{t+\omega}\right)-G_{i}(t, t) \\
& \left.\times b_{i}(t, x(t)) f_{i}\left(t, x_{t}\right)\right]+a_{i}(t, x(t)) x_{i}(t) \\
= & a_{i}(t, x(t)) x_{i}(t)+\lambda b_{i}(t, x(t)) f_{i}\left(t, x_{t}\right)=x_{i}^{\prime}(t) . \tag{13}
\end{align*}
$$

For any $t=t_{j}, j \in Z_{+}$, we have from (7) that

$$
\begin{align*}
x_{i}\left(t_{j}^{+}\right)- & x_{i}\left(t_{j}\right) \\
= & \lambda \int_{t_{j}}^{t_{j}+\omega}\left[G_{i}\left(t_{j}^{+}, s\right)-G_{i}\left(t_{j}, s\right)\right] \\
& \times b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right) d s \\
& +\sum_{t_{j}^{+} \leq t_{k}<t_{j}+\omega} G_{i}\left(t_{j}^{+}, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)  \tag{14}\\
& -\sum_{t_{j} \leq t_{k}<t_{j}+\omega} G_{i}\left(t_{j}, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right) \\
= & I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right) .
\end{align*}
$$

Hence $x(t)=\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)^{T}$ is a positive $\omega$-periodic solution of (1). Thus we complete the proof of Lemma 5.

Lemma 6. Assume that $\left(H_{1}\right)-\left(H_{4}\right)$ hold. Then $T: E \rightarrow E$ is well defined.

Proof. From (7), it is easy to verify that $(T x)(t)$ is continuous in $\left(t_{k}, t_{k+1}\right),(T x)\left(t_{k}^{+}\right)$and $(T x)\left(t_{k}^{-}\right)$exist, and $(T x)\left(t_{k}^{-}\right)=$ (Tx) $\left(t_{k}\right)$ for each $k \in Z_{+}$. Moreover, for any $x \in E$,

$$
\begin{align*}
(T x)(t+\omega)= & \lambda \int_{t+\omega}^{t+2 \omega} G(t+\omega, s) b(s, x(s)) f\left(s, x_{s}\right) d s \\
& +\sum_{t+\omega \leq t_{k}<t+2 \omega} G\left(t+\omega, t_{k}\right) I_{k}\left(t_{k}, x\left(t_{k}\right)\right) \\
= & \lambda \int_{t}^{t+\omega} G(t+\omega, u+\omega) \\
& \times b(u+\omega, x(u+\omega)) \\
& \times \sum_{t \leq t_{k}<t+\omega} G\left(u+\omega, x_{u+\omega}\right) d u \\
= & \lambda \int_{t}^{t+\omega} G(t, s) b(s, x(s)) f\left(s, x_{s}\right) d s \\
& +\sum_{t \leq t_{k}<t+\omega} G\left(t, t_{k}\right) I_{k}\left(t_{k}, x\left(t_{k}\right)\right) \\
& \left.\left.\quad t_{k}\right)\right)=(T x)(t) . \tag{15}
\end{align*}
$$

Therefore, $(T x) \in X$. From (7), we have

$$
\begin{gather*}
\left|T_{i} x\right|_{0} \leq \beta_{i}\left[\lambda \int_{0}^{\omega}\left|b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right)\right| d s\right. \\
\left.+\sum_{t \leq t_{k}<t+\omega} I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right] \tag{16}
\end{gather*}
$$

Noticing that $G_{i}(t, s) f_{i}\left(s, x_{s}\right) \geq 0$, we obtain

$$
\begin{align*}
\left(T_{i} x\right)(t) \geq & \geq \alpha_{i}\left[\lambda \int_{0}^{\omega}\left|b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right)\right| d s\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega} I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right]  \tag{17}\\
\geq & \frac{\alpha_{i}}{\beta_{i}}\left|T_{i} x\right|_{0} \geq \sigma\left|T^{i} x\right|_{0}
\end{align*}
$$

Therefore, $T x \in E$. This completes the proof of Lemma 6.
Lemma 7. Assume that $\left(H_{1}\right)-\left(H_{4}\right)$ hold. Then $T: E \rightarrow E$ is completely continuous.

Proof. We first show that $T$ is continuous. By $\left(H_{3}\right)-\left(H_{4}\right), f$ and $I_{k}$ are continuous in $x$; it follows that, for any $\epsilon>0$, let $\delta>0$ be small enough to satisfy that, if $x, y \in E$, with $|x-y|<$ $\delta$,

$$
\begin{gather*}
\left|f\left(s, x_{s}\right)-f\left(s, y_{s}\right)\right|<\frac{\epsilon}{2 \bar{B} \lambda \beta \omega}, \quad s \in R ;  \tag{18}\\
\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)-I_{k}\left(t_{k}, y\left(t_{k}\right)\right)\right|<\frac{\epsilon}{2 \beta q}, \quad k \in Z_{+} .
\end{gather*}
$$

Therefore,

$$
\begin{align*}
& \|(T x)(t)-(T y)(t)\| \\
& \left.\begin{array}{l}
=\sum_{i=1}^{n}\left|T_{i} x-T_{i} y\right|_{0} \\
\leq
\end{array}\right] \sum_{i=1}^{n} \lambda \int_{t}^{t+\omega} \mid b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right) \\
& \quad-b_{i}(s, y(s)) f_{i}\left(s, y_{s}\right) \mid d s \\
& +\beta \sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega} \mid I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)  \tag{19}\\
& \quad-I_{k}^{i}\left(t_{k}, y\left(t_{k}\right)\right) \mid \\
& <\beta \lambda \bar{B} \omega \frac{\epsilon}{2 \bar{B} \lambda \beta \omega}+\beta q \frac{\epsilon}{2 \beta q}=\epsilon,
\end{align*}
$$

which implies that $T$ is continuous on $E$.
Next we show that $T$ maps a bounded set into a bounded set. Indeed, let $C \subset E$ be a bounded set. For any $t \in R$ and $x \in C$, by (7), we have

$$
\begin{align*}
\|(T x)(t)\|= & \sum_{i=1}^{n}\left|T_{i} x\right|_{0} \\
\leq & \beta\left[\lambda \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right)\right| d s\right. \\
& \left.+\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right]  \tag{20}\\
= & \beta\left[\int_{0}^{\omega} b_{2 i}(s) f\left(s, x_{s}\right) d s\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right]
\end{align*}
$$

Since $C$ is bounded, in view of the continuity of $T$, it follows from (19) that $T x$ is bounded and $\{T x: x \in C\}$ is uniformly bounded. Finally, we show that the family of functions $\{T x$ : $x \in C\}$ is equicontinuous on $[0, \omega]$. Let $\theta_{1}, \theta_{2} \in[0, \omega]$ with $\theta_{1}<\theta_{2}$. From (7), for any $x \in C$, we have

$$
\begin{aligned}
& \left\|(T x)\left(\theta_{2}\right)-(T x)\left(\theta_{1}\right)\right\| \\
& \qquad \begin{aligned}
\leq \lambda \sum_{i=1}^{n}[ & \int_{\theta_{1}}^{\theta_{2}}\left(G_{i}\left(\theta_{2}, s\right)-G_{i}\left(\theta_{1}, s\right)\right) \\
& \quad \times b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right) d s \\
& +\int_{\theta_{2}}^{\theta_{1}+\omega}\left(G_{i}\left(\theta_{2}, s\right)-G_{i}\left(\theta_{1}, s\right)\right)
\end{aligned}
\end{aligned}
$$

$$
\begin{align*}
& \quad \times b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right) d s \\
& +\sum_{\theta_{1} \leq t_{k}<\theta_{2}}\left(G_{i}\left(\theta_{2}, t_{k}\right)-G_{i}\left(\theta_{1}, t_{k}\right)\right) \\
& \times\left|I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
& +\sum_{\theta_{2} \leq t_{k}<\theta_{1}+\omega}\left(G_{i}\left(\theta_{2}, t_{k}\right)-G_{i}\left(\theta_{1}, t_{k}\right)\right) \\
& \left.\times\left|I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] . \tag{21}
\end{align*}
$$

Since for $x \in C, t \in[0, \omega], 0 \leq k \leq q, b_{i}(t, x(t))$, $f_{i}\left(t, x\left(t-\tau_{1}(t, x(t))\right), \ldots, x\left(t-\tau_{m}(t, x(t))\right)\right)$, and $I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)$ are uniformly bounded in $X$; in view of (21), it is easy to see that when $\theta_{2}-\theta_{1}$ tends to zero, $\left|(T x)\left(\theta_{2}\right)-(T x)\left(\theta_{1}\right)\right|$ tends uniformly to zero in $X$. Hence, $\{T x: x \in C\}$ is a family of uniformly bounded and equicontinuous functions on $[0, \omega]$. By Ascoli-Arzelà theorem, the operator $T$ is completely continuous. The proof of Lemma 7 is complete.

For convenience in the following discussion, we introduce the following notations:

$$
\begin{align*}
& f^{a}=\lim _{x \in P\|,\| x \| \rightarrow a} \operatorname{mup}_{\max [0, \omega]} \frac{\int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t}{\|x\|}, \\
& f_{a}=\lim _{x \in P,\|x\| \rightarrow a t \in[0, \omega]} \min _{0} \frac{\int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t}{\|x\|}, \\
& \bar{f}_{r}=\max _{0<x \leq r \in[\in[0, \omega]} \frac{\int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t}{\|x\|}, \\
& \underline{f}_{r}=\min _{0<x \leq r} \min _{t \in[0, \omega]} \frac{\int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t}{\|x\|},  \tag{22}\\
& I^{a}=\lim _{x \in P,\|x\| \rightarrow a} \max _{a \in[0, \omega]} \frac{\sum_{t \leq t_{k}<t+\omega}\left|I_{k}(t, x)\right|}{\|x\|}, \\
& I_{a}=\lim _{x \in P,\|x\| \rightarrow a} \inf _{a t \in[0, \omega]} \frac{\sum_{t \leq t_{k}<t+\omega}\left|I_{k}(t, x)\right|}{\|x\|}, \\
& \bar{I}_{r}=\max _{0<x \leq r t \in[0, \omega], k \in[1, q]} \frac{\sum_{t \leq t_{k}<t+\omega}\left|I_{k}(t, x)\right|}{\|x\|}, \\
& \underline{I}_{r}=\min _{0<x \leq r \in[0[0,0], k \in[1, q]} \min \frac{\sum_{t \leq t_{k}<t+\omega}\left|I_{k}(t, x)\right|}{\|x\|},
\end{align*}
$$

where $a$ denotes either 0 or $\infty, r$ denotes a positive number, and $\|x\|=\max \left\{\left|x_{1}\right|,\left|x_{2}\right|, \ldots,\left|x_{m}\right|\right\}$.

## 3. Main Results

Our main results of this paper are as follows.

Theorem 8. Assume that $\left(H_{1}\right)-\left(H_{4}\right)$ and the following conditions:

$$
\begin{aligned}
& \left(H_{5}\right) \alpha \sigma\left(\lambda B^{\prime}(\xi) \underline{f}_{r}+\underline{I}_{r}\right)>1, \xi \in[0, \omega] ; \\
& \left(H_{6}\right) f^{0}=I^{0}=f^{\infty}=I^{\infty}=0
\end{aligned}
$$

hold. Then (1) has two positive $\omega$-periodic solutions.
Proof. First, we define $\Omega_{r}=\{x \in X:\|x\|<r\}$; then $\Omega_{r}$ is an open subset of $X$. From (7), ( $H_{5}$ ), and Lemma 4, for any $x \in E \cap \partial \Omega_{r}$, we have

$$
\begin{align*}
&\|(T x)(t)\|= \sum_{i=1}^{n}\left|T_{i} x\right|_{0} \\
&= \sum_{i=1}^{n}\left[\lambda \int_{t}^{t+\omega}\left|G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right)\right| d s\right. \\
&\left.+\sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
& \geq \lambda B^{\prime}(\xi) \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|G_{i}(t, s) f_{i}\left(s, x_{s}\right)\right| d s \\
&+\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
&=\alpha {\left[\lambda B^{\prime}(\xi) \int_{0}^{\omega}\left|f\left(s, x_{s}\right)\right| d s\right.} \\
&\left.\quad+\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
& \geq \alpha\left(\lambda B^{\prime}(\xi) \underline{f}_{r}+\underline{I}_{r}\right)\|x\|>\|x\| . \tag{23}
\end{align*}
$$

This yields

$$
\begin{equation*}
\|(T x)(t)\|>\|x\|, \quad \text { for any } x \in E \cap \partial \Omega_{r} \tag{24}
\end{equation*}
$$

On the other hand, if $f^{0}=I^{0}=0$ holds, then we can choose $0<r_{1}<r$, such that $\int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t \leq \epsilon\|x\|$ and $\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right| \leq \epsilon\|x\|$ for $x \in\left[0, r_{1}\right], t \in[0, \omega]$, and $1 \leq k<q$, where constant $\epsilon>0$ satisfies $\epsilon \beta(\lambda B(\xi)+1) \leq 1$. By (7) and Lemma 4, we can obtain

$$
\begin{aligned}
(T x)(t) & =\sum_{i=1}^{n}\left(T_{i} x\right) \\
& =\sum_{i=1}^{n}\left[\lambda \int_{t}^{t+\omega} G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right) d s\right.
\end{aligned}
$$

$$
\left.+\sum_{t \leq t_{k}<t+\omega} G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right]
$$

$$
\begin{align*}
& \leq \lambda B(\xi) \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|G_{i}(t, s) f_{i}\left(s, x_{s}\right)\right| d s \\
&+\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
&= \beta\left[\lambda B(\xi) \int_{0}^{\omega}\left|f\left(s, x_{s}\right)\right| d s\right. \\
&\left.\quad+\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
& \leq \epsilon \beta(\lambda B(\xi)+1)\|x\| \leq\|x\| . \tag{25}
\end{align*}
$$

This yields

$$
\begin{equation*}
\|(T x)(t)\| \leq\|x\|, \quad \text { for any } x \in E \cap \partial \Omega_{r_{1}} \tag{26}
\end{equation*}
$$

In view of (24) and (26), by Lemma 3, it follows that $T$ has a fixed point $x_{1} \in E \cap\left(\Omega_{r} \backslash \overline{\Omega_{r 1}}\right)$ with $r_{1}<\left\|x_{1}\right\|<r$, which is a positive $\omega$-periodic solution of (1).

Likewise, if $f^{\infty}=I^{\infty}=0$ holds, then there is $N>0$ such that $\int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t \leq \epsilon\|x\|$ and $\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right| \leq \epsilon\|x\|$ for $x \geq N, t \in[0, \omega]$, and $1 \leq k<q$, where constant $\epsilon>0$ satisfies $\epsilon \beta(\lambda B(\xi)+1) \leq 1$. Let $r_{2}=\max \{2 r, N / \sigma\}$ and it follows that $x(t) \geq \sigma\|x\|>N$ for $x \in \Omega_{r_{2}}, t \in[0, \omega]$, and $0<k<q$. Thus

$$
\begin{align*}
& \int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t \leq \epsilon\|x\| \\
& \sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right| \leq \epsilon\|x\|, \tag{27}
\end{align*}
$$

$$
\text { for } x \in \Omega_{r_{2}}, t \in[0, \omega], 1 \leq k<q .
$$

By (7) and Lemma 4, we have

$$
\begin{equation*}
(T x)(t) \leq \epsilon \beta(\lambda B(\xi)+1)\|x\| \leq\|x\| ; \tag{28}
\end{equation*}
$$

this yields

$$
\begin{equation*}
\|(T x)(t)\| \leq\|x\|, \quad \text { for any }, x \in E \cap \partial \Omega_{r_{2}} . \tag{29}
\end{equation*}
$$

In view of (24) and (29), by Lemma 3, it follows that $T$ has a fixed point $x_{2} \in E \cap\left(\Omega_{r_{2}} \backslash \overline{\Omega_{r}}\right)$ with $r<\left\|x_{2}\right\|<r_{2}$, which is a positive $\omega$-periodic solution of (1). Therefore (1) has at least two positive periodic solutions; that is, $r_{1}<\left\|x_{1}\right\|<r<$ $\left\|x_{2}\right\|<r_{2}$. This proves Theorem 8.

Remark 9. Assume that $\left(H_{1}\right)-\left(H_{4}\right)$ and the following conditions:

$$
\begin{aligned}
& \left(H_{5}\right) \alpha \sigma\left(\lambda B^{\prime}(\xi) \underline{f}_{r}+\underline{I}_{r}\right)>1 \\
& \left(H_{7}\right) f^{0}=I^{0}=0, \text { or } f^{\infty}=I^{\infty}=0
\end{aligned}
$$

hold. Then (1) has a positive $\omega$-periodic solution.

Corollary 10. Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ and $\lambda>1 / \alpha \sigma B^{\prime}(\xi) f_{r}$ hold.
$\left(H_{6}\right)$ is satisfied; then (1) has two positive $\omega$-periodic solutions;
$\left(H_{7}\right)$ is satisfied; then (1) has a positive $\omega$-periodic solution.
Theorem 11. Assume that $\left(H_{1}\right)-\left(H_{4}\right)$ and the following conditions:

$$
\begin{aligned}
& \left(H_{8}\right) \beta\left(\lambda B(\xi) \bar{f}_{r}+\bar{I}_{r}\right)<1 ; \\
& \left(H_{9}\right) f_{0}=I_{0}=f_{\infty}=I_{\infty}=\infty
\end{aligned}
$$

hold. Then (1) has two positive $\omega$-periodic solutions.
Proof. We define $\Omega_{r}=\{x \in X:\|x\|<r\}$, for a positive number $r$. Then $\Omega_{r}$ is an open subset of $X$ and $0 \in \Omega_{r}$. By (7), $\left(H_{8}\right)$, and Lemma 4, for any $x \in E \cap \partial \Omega_{r}$, we have

$$
\begin{align*}
\|(T x)(t)\|= & \sum_{i=1}^{n}\left|T_{i} x\right|_{0} \\
= & \sum_{i=1}^{n}\left[\lambda \int_{t}^{t+\omega}\left|G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right)\right| d s\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
\leq & \lambda B(\xi) \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|G_{i}(t, s) f_{i}\left(s, x_{s}\right)\right| d s \\
& +\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
= & \beta\left[\lambda B(\xi) \int_{0}^{\omega}\left|f\left(s, x_{s}\right)\right| d s\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
\leq & \beta\left(\lambda B(\xi) \bar{f}_{r}+\bar{I}_{r}\right)\|x\|<\|x\| . \tag{30}
\end{align*}
$$

This implies that for any $x \in E \cap \partial \Omega_{r}$

$$
\begin{equation*}
\|(T x)(t)\|<\|x\| . \tag{31}
\end{equation*}
$$

On the one hand, since $f_{0}=I_{0}=\infty$, there exists $0<r_{1}<r$ and small enough $0<\epsilon$ satisfies $\alpha \delta\left[\lambda B^{\prime}(\xi)\left(f_{0}-\epsilon\right)+\left(I_{0}-\epsilon\right)\right]>$ 1 such that, for any $x$ with $\|x\| \in\left[0, r_{1}\right]$,

$$
\begin{gather*}
\int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t \geq\left(f_{0}-\varepsilon\right)\|x\| ; \\
\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right| \geq\left(I_{0}-\varepsilon\right)\|x\| . \tag{32}
\end{gather*}
$$

Define $\Omega_{r_{2}}=\left\{x \in X:\|x\|<r_{2}\right\}$; then $\Omega_{r_{1}}$ is an open subset of $X$. For any $x \in E \cap \partial \Omega_{r_{1}}$, by (7) and Lemma 4, we have

$$
\begin{align*}
&\|(T x)(t)\|= \sum_{i=1}^{n}\left\|\left(T_{i} x\right)\right\|_{0} \\
&= \sum_{i=1}^{n}\left[\lambda \int_{t}^{t+\omega}\left|G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right)\right| d s\right. \\
&\left.+\sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
& \geq \lambda B^{\prime}(\xi) \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|G_{i}(t, s) f_{i}\left(s, x_{s}\right)\right| d s \\
&+\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
&= {\left[\lambda B^{\prime}(\xi) \int_{0}^{\omega}\left|f\left(s, x_{s}\right)\right| d s\right.} \\
&\left.+\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
& \geq \alpha \delta\left[\lambda B^{\prime}(\xi)\left(f_{0}-\epsilon\right)+\left(I_{0}-\epsilon\right)\right]\|x\| \geq\|x\| . \tag{33}
\end{align*}
$$

This yields

$$
\begin{equation*}
\|(T x)(t)\| \geq\|x\|, \quad \text { for any } x \in E \cap \partial \Omega_{r_{1}} \tag{34}
\end{equation*}
$$

In view of (31) and (34), by Lemma 3, it follows that $T$ has a fixed point $x_{1} \in E \cap\left(\Omega_{r} \backslash \overline{\Omega_{r 1}}\right)$ with $r_{1}<\left\|x_{1}\right\|<r$, which is a positive $\omega$-periodic solution of (1). On the other hand, if $f_{\infty}=I_{\infty}=\infty$, we can find small enough $0<\epsilon$ that satisfies $\alpha \delta\left[\lambda B^{\prime}(\xi)\left(f_{\infty}-\epsilon\right)+\left(I_{\infty}-\epsilon\right)\right]>1$ and large enough $\eta>r>0$, such that $\|x\| \geq \eta$,

$$
\begin{gather*}
\int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t \geq\left(f_{\infty}-\epsilon\right)\|x\|  \tag{35}\\
\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right| \geq\left(I_{\infty}-\epsilon\right)\|x\|
\end{gather*}
$$

Define $r_{2}=\eta / \sigma>r$ and $\Omega_{r_{2}}=\left\{x \in X:\|x\|<r_{2}\right\}$; then $\Omega_{r_{2}}$ is an open subset of $X$. For any $x \in E \cap \partial \Omega_{r_{2}}$, from (7) and Lemma 3, we have

$$
\begin{aligned}
\|(T x)(t)\|= & \sum_{i=1}^{n}\left\|\left(T_{i} x\right)\right\|_{0} \\
= & \sum_{i=1}^{n}\left[\lambda \int_{t}^{t+\omega}\left|G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right)\right| d s\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right]
\end{aligned}
$$

$$
\begin{align*}
& \geq \lambda B^{\prime}(\xi) \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|G_{i}(t, s) f_{i}\left(s, x_{s}\right)\right| d s \\
&+\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
&= \alpha\left[\lambda B^{\prime}(\xi) \int_{0}^{\omega}\left|f\left(s, x_{s}\right)\right| d s\right. \\
& \quad\left.\quad \sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
& \geq \alpha \delta\left[\lambda B^{\prime}(\xi)\left(f_{\infty}-\epsilon\right)+\left(I_{\infty}-\epsilon\right)\right]\|x\| \geq\|x\| \tag{36}
\end{align*}
$$

This yields

$$
\begin{equation*}
\|(T x)(t)\| \geq\|x\|, \quad \text { for any } x \in E \cap \partial \Omega_{r_{2}} \tag{37}
\end{equation*}
$$

In view of (31) and (37), by Lemma 3, it follows that $T$ has a fixed point $x_{2} \in E \cap\left(\Omega_{r_{2}} \backslash \overline{\Omega_{r}}\right)$ with $r<\left\|x_{2}\right\|<r_{2}$, which is a positive $\omega$-periodic solution of (1). Therefore, (1) has at least two positive periodic solutions; that is, $r_{1}<\left\|x_{1}\right\|<r<$ $\left\|x_{2}\right\|<r_{2}$. This proves Theorem 11.

Remark 12. Assume that $\left(H_{1}\right)-\left(H_{4}\right)$ and the following conditions:

$$
\begin{aligned}
& \left(H_{8}\right) \beta\left(\lambda B(\xi) \bar{f}_{r}+\bar{I}_{r}\right)<1 \\
& \left(H_{10}\right) f_{0}=I_{0}=\infty, \text { or } f_{\infty}=I_{\infty}=\infty
\end{aligned}
$$

hold. Then (1) has a positive $\omega$-periodic solution.
Corollary 13. Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ and $\lambda<1 / \beta B(\xi) \bar{f}_{r}$ hold.
$\left(H_{9}\right)$ is satisfied; then (1) has two positive $\omega$-periodic solu-
tions;
$\left(H_{10}\right)$ is satisfied; then (1) has a positive $\omega$-periodic solution.
Theorem 14. Assume that $\left(H_{1}\right)-\left(H_{4}\right)$ and

$$
\begin{aligned}
& \left(H_{11}\right) \beta\left(\lambda B(\xi) f^{0}+I^{0}\right)<1 \\
& \left(H_{12}\right) \alpha \sigma\left(\lambda B^{\prime}(\xi) f_{\infty}+I_{\infty}\right)>1
\end{aligned}
$$

hold. Then (1) has a positive $\omega$-periodic solution, where $f^{0}, f_{\infty}$, $I^{0}$, and $I_{\infty}$ are positive constants.

Proof. From $\left(H_{11}\right)$, we can choose $\epsilon>0$ such that $\beta\left(\lambda B(x i)\left(f^{0}+\epsilon\right)+\left(I^{0}+\epsilon\right)\right)<1$. Thus there exists $r>0$ such that, for $x \in[0, r], t \in[0, \omega]$ and $1 \leq k<q$,

$$
\begin{gather*}
\int_{0}^{\omega} f\left(t, x_{t}\right) d t \leq\left(f^{0}+\epsilon\right)\|x\| \\
\sum_{t \leq t_{k}<t+\omega} I_{k}\left(t_{k}, x\left(t_{k}\right)\right) \leq\left(I^{0}+\epsilon\right)\|x\| \tag{38}
\end{gather*}
$$

by (7), $\left(H_{11}\right)$, and Lemma 4 , we have

$$
\begin{align*}
\|(T x)(t)\|= & \sum_{i=1}^{n}\left|T_{i} x\right|_{0} \\
= & \sum_{i=1}^{n}\left[\lambda \int_{t}^{t+\omega}\left|G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right)\right|\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
\leq & \lambda B(\xi) \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|G_{i}(t, s) f_{i}\left(s, x_{s}\right)\right| d s  \tag{39}\\
& +\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
= & \beta\left[\lambda B(\xi) \int_{0}^{\omega}\left|f\left(s, x_{s}\right)\right| d s\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
\leq & \beta\left(\lambda B(\xi)\left(f^{0}+\epsilon\right)+\left(I^{0}+\epsilon\right)\right)\|x\|<\|x\| .
\end{align*}
$$

This implies that for any $x \in E \cap \partial \Omega_{r}$

$$
\begin{equation*}
\|(T x)(t)\|<\|x\| . \tag{40}
\end{equation*}
$$

On the other hand, choose $\varepsilon>0$ such that $f_{\infty}-\epsilon>0$ and $I_{\infty}-\epsilon>0$, and from $\left(H_{12}\right)$, we can obtain

$$
\begin{equation*}
\alpha \sigma\left[\lambda B^{\prime}(\xi)\left(f_{\infty}-\epsilon\right)+\left(I_{\infty}-\epsilon\right)\right]>1 \tag{41}
\end{equation*}
$$

It is easy to see that there exists large enough $\eta>r>0$, such that $\|x\| \geq \eta$,

$$
\begin{gather*}
\int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t \geq\left(f_{\infty}-\epsilon\right)\|x\| \\
\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right| \geq\left(I_{\infty}-\epsilon\right)\|x\| . \tag{42}
\end{gather*}
$$

Define $R=\eta / \sigma>r$ and $\Omega_{R}=\{x \in X:\|x\|<R\}$; then $\Omega_{R}$ is an open subset of $X$. From (7), ( $H_{12}$ ), and Lemma 4, for any $x \in E \cap \partial \Omega_{R}$, we have

$$
\begin{aligned}
\|(T x)(t)\|= & \sum_{i=1}^{n}\left|T_{i} x\right|_{0} \\
= & \sum_{i=1}^{n}\left[\lambda \int_{t}^{t+\omega}\left|G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right)\right| d s\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right]
\end{aligned}
$$

$$
\begin{align*}
& \geq \lambda B^{\prime}(\xi) \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|G_{i}(t, s) f_{i}\left(s, x_{s}\right)\right| d s \\
&+\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
&= \alpha\left[\lambda B^{\prime}(\xi) \int_{0}^{\omega}\left|f\left(s, x_{s}\right)\right| d s\right. \\
&\left.\quad+\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
& \geq \alpha \sigma\left[\lambda B^{\prime}(\xi)\left(f_{\infty}-\epsilon\right)+\left(I_{\infty}-\epsilon\right)\right]\|x\|>\|x\| \tag{43}
\end{align*}
$$

This yields

$$
\begin{equation*}
\|T x\|>\|x\|, \quad \text { for any } x \in E \cap \partial \Omega_{R} \tag{44}
\end{equation*}
$$

In view of (40) and (44), by Lemma 3, it follows that $T$ has a fixed point $x^{*} \in E \cap\left(\Omega_{R} \backslash \overline{\Omega_{r}}\right)$ with $r<\left\|x^{*}\right\|<R$, which is a positive $\omega$-periodic solution of (1). This proves Theorem 14.

Corollary 15. Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ and the following condition:

$$
\left(H_{13}\right) 1 / \alpha \sigma B^{\prime}(\xi) f_{\infty}<\lambda<1 / \beta B(\xi) f^{0}
$$

hold. Then (1) has a positive $\omega$-periodic solution.
Similarly, we can prove the following theorem and corollary.

Theorem 16. Assume that $\left(H_{1}\right)-\left(H_{4}\right)$ and the following conditions:

$$
\begin{aligned}
& \left(H_{14}\right) \beta\left(\lambda B(\xi) f^{\infty}+I^{\infty}\right)<1 \\
& \left(H_{15}\right) \alpha \sigma\left(\lambda B^{\prime}(\xi) f_{0}+I_{0}\right)>1
\end{aligned}
$$

hold. Then (1) has a positive $\omega$-periodic solution, where $f_{0}, f^{\infty}$, $I_{0}$, and $I^{\infty}$ are positive constants.

Corollary 17. Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ and the following condition:

$$
\left(H_{16}\right) 1 / \alpha \sigma B^{\prime}(\xi) f_{0}<\lambda<1 / \beta B(\xi) f^{\infty}
$$

hold. Then (1) has a positive $\omega$-periodic solution.
Theorem 18. Assume that $\left(H_{1}\right)-\left(H_{4}\right),\left(H_{5}\right)$, and the following condition:

$$
\begin{aligned}
& \left(H_{17}\right) 0<f^{0}<1 / 2 \lambda \beta B(\xi), 0<I^{0}<1 / 2 \beta \text { and } 0<f^{\infty}< \\
& 1 / 2 \lambda \beta B(\xi), 0<I^{\infty}<1 / 2 \beta
\end{aligned}
$$

hold. Then (1) has two positive $\omega$-periodic solutions.

Proof. First, we define $\Omega_{r}=\{x \in X:\|x\|<r\}$; then $\Omega_{r}$ is an open subset of $X$. From (7), ( $H_{5}$ ), and Lemma 4, for any $x \in E \cap \partial \Omega_{r}$, we have

$$
\begin{align*}
&\|(T x)(t)\|= \sum_{i=1}^{n}\left|T_{i} x\right|_{0} \\
&= \sum_{i=1}^{n}\left[\lambda \int_{t}^{t+\omega}\left|G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right)\right| d s\right. \\
&\left.+\sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
& \geq \lambda B^{\prime}(\xi) \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|G_{i}(t, s) f_{i}\left(s, x_{s}\right)\right| d s \\
&+\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
&=\alpha {\left[\lambda B^{\prime}(\xi) \int_{0}^{\omega}\left|f\left(s, x_{s}\right)\right| d s\right.} \\
&\left.+\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
& \geq \alpha\left(\lambda B^{\prime}(\xi) \underline{f}_{r}+\underline{I}_{r}\right)\|x\|>\|x\| \tag{45}
\end{align*}
$$

This yields

$$
\begin{equation*}
\|(T x)(t)\|>\|x\|, \quad \text { for any } x \in E \cap \partial \Omega_{r} \tag{46}
\end{equation*}
$$

On the one hand, since $0<f^{0}<1 / 2 \lambda \beta B(\xi)$ and $0<I^{0}<$ $1 / 2 \beta$, there exists $0<r_{1}<r$ such that for $0<\|x\|<r_{1}$

$$
\begin{align*}
& \int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t \leq \frac{r_{1}}{2 \lambda \beta B(\xi)} \\
& \sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right| \leq \frac{r_{1}}{2 \beta} \tag{47}
\end{align*}
$$

Set $\Omega_{r_{1}}=\left\{x \in X:\|x\|<r_{1}\right\}$; then $\Omega_{r_{1}}$ is an open subset of $X$. From (7), $\left(H_{17}\right)$, and Lemma 4, for any $x \in E \cap \partial \Omega_{r_{1}}$, $t \in[0, \omega]$, and $1 \leq k<q$, we have

$$
\begin{aligned}
(T x)(t)= & \sum_{i=1}^{n}\left(T_{i} x\right) \\
= & \sum_{i=1}^{n}\left[\lambda \int_{t}^{t+\omega} G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right) d s\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega} G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right]
\end{aligned}
$$

$$
\begin{align*}
\leq & \lambda B(\xi) \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|G_{i}(t, s) f_{i}\left(s, x_{s}\right)\right| d s \\
& +\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
= & \beta\left[\lambda B(\xi) \int_{0}^{\omega}\left|f\left(s, x_{s}\right)\right| d s\right. \\
& \left.\quad+\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
\leq & \beta\left[\lambda B(\xi) \frac{1}{2 \lambda \beta B(\xi)}+\frac{r_{1}}{2 \beta}\right]\|x\|=\|x\| \tag{48}
\end{align*}
$$

This yields

$$
\begin{equation*}
\|(T x)(t)\| \leq\|x\|, \quad \text { for any } x \in E \cap \partial \Omega_{r_{1}} \tag{49}
\end{equation*}
$$

In view of (46) and (49), by Lemma 3, it follows that $T$ has a fixed point $x_{1} \in E \cap\left(\Omega_{r} \backslash \overline{\Omega_{r 1}}\right)$ with $r_{1}<\left\|x_{1}\right\|<r$, which is a positive $\omega$-periodic solution of (1).

On the other hand, if $0<f^{\infty}<1 / 2 \lambda \beta B(\xi)$, and $0<I^{\infty}<$ $1 / 2 \beta$ hold, then there is $N>0$ such that

$$
\begin{align*}
& \int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t \leq \frac{N}{2 \lambda \beta B(\xi)} \\
& \sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right| \leq \frac{N}{2 \beta}, \tag{50}
\end{align*}
$$

for $x \geq N, t \in[0, \omega], 1 \leq k<q$. Let $r_{2}=\max \{2 r, N / \sigma\}$ and it follows that $x(t) \geq \sigma\|x\|>N$ for $x \in \Omega_{r_{2}}, t \in[0, \omega]$, and $0<k<q$. Thus

$$
\begin{gather*}
\int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t \leq \frac{1}{2 \lambda \beta B(\xi)}\|x\| \\
\quad \sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right| \leq \frac{1}{2 \beta}\|x\| \tag{51}
\end{gather*}
$$

for $x \in \Omega_{r_{2}}, t \in[0, \omega], 1 \leq k<q$.
By (7), $\left(H_{17}\right)$, and Lemma 4, we have

$$
\begin{aligned}
(T x)(t)= & \sum_{i=1}^{n}\left(T_{i} x\right) \\
= & \sum_{i=1}^{n}\left[\lambda \int_{t}^{t+\omega} G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right) d s\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega} G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right]
\end{aligned}
$$

$$
\begin{align*}
\leq & \lambda B(\xi) \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|G_{i}(t, s) f_{i}\left(s, x_{s}\right)\right| d s \\
& +\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
= & \beta\left[\lambda B(\xi) \int_{0}^{\omega}\left|f\left(s, x_{s}\right)\right| d s\right. \\
& \left.\quad+\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
\leq & \beta\left[\lambda B(\xi) \frac{1}{2 \lambda \beta B(\xi)}+\frac{r_{1}}{2 \beta}\right]\|x\|=\|x\| . \tag{52}
\end{align*}
$$

This yields

$$
\begin{equation*}
\|(T x)(t)\| \leq\|x\|, \quad \text { for any } x \in E \cap \partial \Omega_{r_{2}} . \tag{53}
\end{equation*}
$$

In view of (46) and (53), by Lemma 3, it follows that $T$ has a fixed point $x_{2} \in E \cap\left(\Omega_{r_{2}} \backslash \overline{\Omega_{r}}\right)$ with $r<\left\|x_{2}\right\|<r_{2}$, which is a positive $\omega$-periodic solution of (1). Therefore (1) has at least two positive periodic solutions; that is, $r_{1}<\left\|x_{1}\right\|<r<$ $\left\|x_{2}\right\|<r_{2}$. This proves Theorem 18.

Remark 19. Assume that $\left(H_{1}\right)-\left(H_{4}\right),\left(H_{5}\right)$, and the following condition:

$$
\begin{aligned}
& \left(H_{18}\right) 0<f^{0}<1 / 2 \lambda \beta B(\xi), 0<I^{0}<1 / 2 \beta \text { or } 0<f^{\infty}< \\
& 1 / 2 \lambda \beta B(\xi), 0<I^{\infty}<1 / 2 \beta
\end{aligned}
$$

hold. Then (1) has a positive $\omega$-periodic solution.
Corollary 20. Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ and $\lambda>1 / \alpha \sigma B^{\prime}(\xi) \underline{f}_{r}$ hold.
$\left(H_{17}\right)$ is satisfied; then (1) has two positive $\omega$-periodic solutions.
$\left(H_{18}\right)$ is satisfied; then (1) has a positive $\omega$-periodic solution.
From the arguments in the previous proof, we have the following consequences immediately.

Theorem 21. Assume that $\left(H_{1}\right)-\left(H_{4}\right),\left(H_{8}\right)$, and the following condition:

$$
\begin{aligned}
&\left(H_{19}\right) \infty>f_{0}>1 / 2 \lambda \alpha \sigma B^{\prime}(\xi), \infty>I_{0}>1 / 2 \alpha \sigma \text { and } \infty> \\
& f_{\infty}>1 / 2 \lambda \alpha \sigma B^{\prime}(\xi), \infty>I_{\infty}>1 / 2 \alpha \sigma
\end{aligned}
$$

hold. Then (1) has two positive $\omega$-periodic solutions.
Remark 22. Assume that $\left(H_{1}\right)-\left(H_{4}\right),\left(H_{8}\right)$, and the following condition:

$$
\begin{aligned}
&\left(H_{20}\right) \infty>f_{0}>1 / 2 \lambda \alpha \sigma B^{\prime}(\xi), \infty>I_{0}>1 / 2 \alpha \sigma \text { or } \infty> \\
& f_{\infty}>1 / 2 \lambda \alpha \sigma B^{\prime}(\xi), \infty>I_{\infty}>1 / 2 \alpha \sigma
\end{aligned}
$$

hold. Then (1) has a positive $\omega$-periodic solution.

Corollary 23. Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ and $\lambda<1 / \beta B(\xi) \bar{f}_{r}$ hold.
$\left(H_{19}\right)$ is satisfied; then (1) has two positive $\omega$-periodic solutions.
$\left(H_{20}\right)$ is satisfied; then (1) has a positive $\omega$-periodic solution.
Theorem 24. Assume that $\left(H_{1}\right)-\left(H_{4}\right),\left(H_{5}\right)$, and one of the following conditions

$$
\begin{aligned}
& \left(H_{21}\right) f^{0}=I^{0}=0 \text { and } 0<f^{\infty}<1 / 2 \lambda \beta B(\xi), 0<I^{\infty}< \\
& 1 / 2 \beta, \\
& \left(H_{22}\right) 0<f^{0}<1 / 2 \lambda \beta B(\xi), 0<I^{0}<1 / 2 \beta \text { and } f^{\infty}=I^{\infty}= \\
& 0
\end{aligned}
$$

hold. Then (1) has two positive $\omega$-periodic solutions.
Proof. We only consider the case $\left(H_{21}\right)$. When the case $\left(H_{22}\right)$ holds, the conclusion remains true by a similar proof and we will omit it. We define $\Omega_{r}=\{x \in X:\|x\|<r\}$; then $\Omega_{r}$ is an open subset of $X$. From (7), $\left(H_{5}\right)$, and Lemma 4, for any $x \in E \cap \partial \Omega_{r}$, we have

$$
\begin{align*}
&\|(T x)(t)\|= \sum_{i=1}^{n}\left|T_{i} x\right|_{0} \\
&= \sum_{i=1}^{n}\left[\lambda \int_{t}^{t+\omega}\left|G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right)\right| d s\right. \\
&\left.+\sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
& \geq \lambda B^{\prime}(\xi) \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|G_{i}(t, s) f_{i}\left(s, x_{s}\right)\right| d s \\
&+\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
&= {\left[\lambda B^{\prime}(\xi) \int_{0}^{\omega}\left|f\left(s, x_{s}\right)\right| d s\right.} \\
&\left.+\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
& \geq \alpha\left(\lambda B^{\prime}(\xi) \underline{f}_{r}+\underline{I}_{r}\right)\|x\|>\|x\| . \tag{54}
\end{align*}
$$

This yields

$$
\begin{equation*}
\|(T x)(t)\|>\|x\|, \quad \text { for any } x \in E \cap \partial \Omega_{r} \tag{55}
\end{equation*}
$$

On the one hand, if $f^{0}=I^{0}=0$ holds, then we can choose $0<r_{1}<r$, such that $\int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t \leq \epsilon\|x\|$ and $\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right| \leq \epsilon\|x\|$ for $x \in\left[0, r_{1}\right], t \in[0, \omega]$, and
$1 \leq k<q$, where constant $\epsilon>0$ satisfies $\epsilon \beta(\lambda B(\xi)+1) \leq 1$.
By (7), ( $\mathrm{H}_{21}$ ), and Lemma 4, we can obtain

$$
\begin{aligned}
(T x)(t)= & \sum_{i=1}^{n}\left(T_{i} x\right) \\
= & \sum_{i=1}^{n}\left[\lambda \int_{t}^{t+\omega} G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right) d s\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega} G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right] \\
\leq & \lambda B(\xi) \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|G_{i}(t, s) f_{i}\left(s, x_{s}\right)\right| d s \\
& +\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
= & \beta\left[\lambda B(\xi) \int_{0}^{\omega}\left|f\left(s, x_{s}\right)\right| d s\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
\leq & \epsilon \beta(\lambda B(\xi)+1)\|x\| \leq\|x\|
\end{aligned}
$$

This yields

$$
\begin{equation*}
\|(T x)(t)\| \leq\|x\|, \quad \text { for any } x \in E \cap \partial \Omega_{r_{1}} . \tag{57}
\end{equation*}
$$

In view of (55) and (57), by Lemma 3, it follows that $T$ has a fixed point $x_{1} \in E \cap\left(\Omega_{r} \backslash \overline{\Omega_{r 1}}\right)$ with $r_{1}<\left\|x_{1}\right\|<r$, which is a positive $\omega$-periodic solution of (1).

On the other hand, if $0<f^{\infty}<1 / 2 \lambda \beta B(\xi)$ and $0<I^{\infty}<$ $1 / 2 \beta$ hold, then there is $N>0$ such that

$$
\begin{align*}
& \int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t \leq \frac{N}{2 \lambda \beta B(\xi)} \\
& \sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right| \leq \frac{N}{2 \beta} \tag{58}
\end{align*}
$$

for $x \geq N, t \in[0, \omega]$, and $1 \leq k<q$; let $r_{2}=\max \{2 r, N / \sigma\}$ and it follows that $x(t) \geq \sigma\|x\|>N$ for $x \in \Omega_{r_{2}}, t \in[0, \omega]$, and $0<k<q$. Thus

$$
\begin{align*}
& \int_{0}^{\omega}\left|f\left(t, x_{t}\right)\right| d t \leq \frac{1}{2 \lambda \beta B(\xi)}\|x\| \\
& \sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right| \leq \frac{1}{2 \beta}\|x\| \tag{59}
\end{align*}
$$

By (7), $\left(H_{21}\right)$, and Lemma 4, we have

$$
\begin{align*}
(T x)(t)= & \sum_{i=1}^{n}\left(T_{i} x\right) \\
= & \sum_{i=1}^{n}\left[\lambda \int_{t}^{t+\omega} G_{i}(t, s) b_{i}(s, x(s)) f_{i}\left(s, x_{s}\right) d s\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega} G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right] \\
\leq & \lambda B(\xi) \sum_{i=1}^{n} \int_{t}^{t+\omega}\left|G_{i}(t, s) f_{i}\left(s, x_{s}\right)\right| d s  \tag{60}\\
& +\sum_{i=1}^{n} \sum_{t \leq t_{k}<t+\omega}\left|G_{i}\left(t, t_{k}\right) I_{k}^{i}\left(t_{k}, x\left(t_{k}\right)\right)\right| \\
= & \beta\left[\lambda B(\xi) \int_{0}^{\omega}\left|f\left(s, x_{s}\right)\right| d s\right. \\
& \left.+\sum_{t \leq t_{k}<t+\omega}\left|I_{k}\left(t_{k}, x\left(t_{k}\right)\right)\right|\right] \\
\leq & \beta\left[\lambda B(\xi) \frac{1}{2 \lambda \beta B(\xi)}+\frac{r_{1}}{2 \beta}\right]\|x\|=\|x\|
\end{align*}
$$

This yields

$$
\begin{equation*}
\|(T x)(t)\| \leq\|x\|, \quad \text { for any } x \in E \cap \partial \Omega_{r_{2}} \tag{61}
\end{equation*}
$$

In view of (55) and (61), by Lemma 3, it follows that $T$ has a fixed point $x_{2} \in E \cap\left(\Omega_{r_{2}} \backslash \overline{\Omega_{r}}\right)$ with $r<\left\|x_{2}\right\|<r_{2}$, which is a positive $\omega$-periodic solution of (1). Therefore (1) has at least two positive periodic solutions; that is, $r_{1}<\left\|x_{1}\right\|<r<$ $\left\|x_{2}\right\|<r_{2}$. This proves Theorem 24.

Remark 25. Assume that $\left(H_{1}\right)-\left(H_{4}\right),\left(H_{5}\right)$, and one of the following conditions:

$$
\begin{aligned}
& \left(H_{23}\right) f^{0}=I^{0}=0 \text { or } 0<f^{\infty}<1 / 2 \lambda \beta B(\xi), 0<I^{\infty}<1 / 2 \beta \\
& \left(H_{24}\right) 0<f^{0}<1 / 2 \lambda \beta B(\xi), 0<I^{0}<1 / 2 \beta \text { or } f^{\infty}=I^{\infty}=0
\end{aligned}
$$

hold. Then (1) has a positive $\omega$-periodic solution.
Corollary 26. Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ and $\lambda>1 / \alpha \sigma B^{\prime}(\xi) \underline{f}_{r}$ hold.

Either $\left(H_{21}\right)$ or $\left(H_{22}\right)$ is satisfied; then (1) has two positive $\omega$-periodic solutions.
Either $\left(H_{23}\right)$ or $\left(H_{24}\right)$ is satisfied; then (1) has a positive $\omega$-periodic solution.

From the arguments in the previous proof, we have the following consequences immediately.

Theorem 27. Assume that $\left(H_{1}\right)-\left(H_{4}\right),\left(H_{8}\right)$, and one of the following conditions:

$$
\begin{aligned}
& \left(H_{25}\right) \begin{array}{l}
f_{0}=I_{0}=0 \text { and } \infty>f_{\infty}>1 / 2 \lambda \alpha \sigma B^{\prime}(\xi), \infty>I_{\infty}> \\
1 / 2 \alpha \sigma, \\
\left(H_{26}\right) \infty>f_{0}>1 / 2 \lambda \alpha \sigma B^{\prime}(\xi), \infty>I_{0}>1 / 2 \alpha \sigma \text { and } f_{\infty}= \\
I_{\infty}=0
\end{array}
\end{aligned}
$$

hold. Then (1) has two positive $\omega$-periodic solutions.
Remark 28. Assume that $\left(H_{1}\right)-\left(H_{4}\right),\left(H_{8}\right)$, and one of the following conditions

$$
\begin{aligned}
& \left(H_{27}\right) \begin{array}{l}
f_{0}=I_{0}=0 \text { or } \infty>f_{\infty}>1 / 2 \lambda \alpha \sigma B^{\prime}(\xi), \infty>I_{\infty}> \\
1 / 2 \alpha \sigma, \\
\left.\left(H_{28}\right)\right)^{\infty>f_{0}>1 / 2 \lambda \alpha \sigma B^{\prime}(\xi), \infty>I_{0}>1 / 2 \alpha \sigma \text { or } f_{\infty}=} \\
I_{\infty}=0
\end{array}
\end{aligned}
$$

hold. Then (1) has a positive $\omega$-periodic solution.
Corollary 29. Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ and $\lambda<1 / \beta B(\xi) \bar{f}_{r}$ hold.

Either $\left(H_{25}\right)$ or $\left(H_{26}\right)$ is satisfied; then (1) has two positive $\omega$-periodic solutions.
Either $\left(H_{27}\right)$ or $\left(H_{28}\right)$ is satisfied; then (1) has a positive $\omega$-periodic solution.

Remark 30. Suppose that $B(t, x(t))=1$ and $I_{k}\left(t_{k}, x\left(t_{k}\right)\right)=$ 0 , under some conditions; we can obtain the corresponding results of [17]. Hence, our results generalize and improve the corresponding results of [17].

Remark 31. Assume that $A(t, x(t))=A(t), B(t, x(t))=1, \lambda=$ 1 under some conditions; we can obtain the corresponding results of [18]. Hence, our results generalize and improve the corresponding results of [18].

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This research is supported by NSF of China (nos. 11161015, 11371367, and 11361012), PSF of China (nos. 2012M512162 and 2013T60934), NSF of Hunan Province (nos. 11JJ900, 12JJ9001, and 13JJ4098), the Science Foundation of Hengyang Normal University (no. 11B36), and the Construct Program of the Key Discipline in Hunan Province.

## References

[1] D. Y. Xu and Z. C. Yang, "Impulsive delay differential inequality and stability of neural networks," Journal of Mathematical Analysis and Applications, vol. 305, no. 1, pp. 107-120, 2005.
[2] A. M. Zhao and J. R. Yan, "Asymptotic behavior of solutions of impulsive delay differential equations," Journal of Mathematical Analysis and Applications, vol. 201, no. 3, pp. 943-954, 1996.
[3] Y. P. Xing and M. A. Han, "A new approach to stability of impulsive functional differential equations," Applied Mathematics and Computation, vol. 151, no. 3, pp. 835-847, 2004.
[4] Z. G. Luo and L. P. Luo, "Global positive periodic solutions of generalized $n$-species competition systems with multiple delays and impulses," Abstract and Applied Analysis, vol. 2013, Article ID 980974, 12 pages, 2013.
[5] W. T. Li and H. F. Huo, "Existence and global attractivity of positive periodic solutions of functional differential equations with impulses," Nonlinear Analysis: Theory, Methods \& Applications, vol. 59, no. 6, pp. 857-877, 2004.
[6] J. S. Yu, "Stability for nonlinear delay differential equations of unstable type under impulsive perturbations," Applied Mathematics Letters, vol. 14, no. 7, pp. 849-857, 2001.
[7] X. Y. Li, X. N. Lin, D. Q. Jiang, and X. Y. Zhang, "Existence and multiplicity of positive periodic solutions to functional differential equations with impulse effects," Nonlinear Analysis: Theory, Methods \& Applications, vol. 62, no. 4, pp. 683-701, 2005.
[8] G. Ballinger and X. Z. Liu, "Existence, uniqueness and boundedness results for impulsive delay differential equations," Applicable Analysis, vol. 74, no. 1-2, pp. 71-93, 2000.
[9] J. R. Yan, "Global attractivity for impulsive population dynamics with delay arguments," Nonlinear Analysis: Theory, Methods \& Applications, vol. 71, no. 11, pp. 5417-5426, 2009.
[10] M. Liu and K. Wang, "Asymptotic behavior of a stochastic nonautonomous Lotka-Volterra competitive system with impulsive perturbations," Mathematical and Computer Modelling, vol. 57, no. 3-4, pp. 909-925, 2013.
[11] A. M. Samoikleno and N. A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.
[12] Z. G. Luo, B. X. Dai, and Q. H. Zhang, "Existence of positive periodic solutions for an impulsive semi-ratio-dependent predator-prey model with dispersion and time delays," Applied Mathematics and Computation, vol. 215, no. 9, pp. 3390-3398, 2010.
[13] X. Q. Ding, C. W. Wang, and P. Chen, "Permanence for a twospecies Gause-type ratio-dependent predator-prey system with time delay in a two-patch environment," Applied Mathematics and Computation, vol. 219, no. 17, pp. 9099-9105, 2013.
[14] Y. F. Shao and B. X. Dai, "The dynamics of an impulsive delay predator-prey model with stage structure and Beddington-type functional response," Nonlinear Analysis: Real World Applications, vol. 11, no. 5, pp. 3567-3576, 2010.
[15] X. D. Li and X. L. Fu, "On the global exponential stability of impulsive functional differential equations with infinite delays or finite delays," Communications in Nonlinear Science and Numerical Simulation, vol. 19, no. 3, pp. 442-447, 2014.
[16] Z. G. Luo, L. P. Luo, and Y. H. Zeng, "Positive periodic solutions for impulsive functional differential equations with infinite delay and two parameters," Journal of Applied Mathematics, vol. 2014, Article ID 751612, 17 pages, 2014.
[17] Z. J. Zeng, B. Li, and M. Fan, "Existence of multiple positive periodic solutions for functional differential equations," Journal of Mathematical Analysis and Applications, vol. 325, no. 2, pp. 1378-1389, 2007.
[18] N. Zhang, B. X. Dai, and X. Z. Qian, "Periodic solutions for a class of higher-dimension functional differential equations with impulses," Nonlinear Analysis: Theory, Methods \& Applications, vol. 68, no. 3, pp. 629-638, 2008.
[19] D. Q. Jiang, D. O'Regan, R. P. Agarwal, and X. J. Xu, "On the number of positive periodic solutions of functional differential equations and population models," Mathematical Models \& Methods in Applied Sciences, vol. 15, no. 4, pp. 555-573, 2005.
[20] D. J. Guo, Nonlinear Functional Analysis, Shandong Science and Technology Press, Shandong, China, 2001, (in Chinese).
[21] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[22] M. A. Krasnoselskii, Positive Solution of Operator Equation, Noordhoff, Gröningen, The Netherlands, 1964.
[23] K. Deimling, Nonlinear Functional Analysis, Springer, Berlin, Germany, 1985.
[24] D. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones, vol. 5, Academic Press, Orlando, Fla, USA, 1988.
[25] Y. H. Fan, W. T. Li, and L. L. Wang, "Periodic solutions of delayed ratio-dependent predator-prey models with monotonic or nonmonotonic functional responses," Nonlinear Analysis: Real World Applications, vol. 5, no. 2, pp. 247-263, 2004.

## Research Article

# Existence of Traveling Wave Solutions for Cholera Model 

Tianran Zhang, ${ }^{1}$ Qingming Gou, ${ }^{2}$ and Xiaoli Wang ${ }^{1}$<br>${ }^{1}$ School of Mathematics and Statistics, Southwest University, Chongqing 400715, China<br>${ }^{2}$ College of Mathematics \& Computer Science, Yangtze Normal University, Chongqing 408100, China

Correspondence should be addressed to Tianran Zhang; zhtr0123@126.com
Received 28 December 2013; Accepted 25 January 2014; Published 26 March 2014
Academic Editor: Kaifa Wang
Copyright © 2014 Tianran Zhang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

To investigate the spreading speed of cholera, Codeço's cholera model (2001) is developed by a reaction-diffusion model that incorporates both indirect environment-to-human and direct human-to-human transmissions and the pathogen diffusion. The two transmission incidences are supposed to be saturated with infective density and pathogen density. The basic reproduction number $R_{0}$ is defined and the formula for minimal wave speed $c^{*}$ is given. It is proved by shooting method that there exists a traveling wave solution with speed $c$ for cholera model if and only if $c \geq c^{*}$.


## 1. Introduction

Cholera has been a serious threat to human health in the past and at present, which is an acute, diarrheal illness caused by infection of the intestine with the bacterium Vibrio cholera. An estimated 3-5 million cases and over 100,000 deaths occur each year around the world [1]. The cholera bacterium is usually found in water or food sources that have been contaminated by feces from a person infected with cholera. Cholera is most likely to be found and to spread in places with inadequate water treatment, poor sanitation, and inadequate hygiene. Therefore, cholera outbreaks have occurred in developing countries, for example, Iraq (20072008), Guinea Bissau (2008), Zimbabwe (2008-2009), Haiti (2010), Democratic Republic of Congo (2011-2012), and Sierra Leone (2012) [2].

To understand the propagation mechanism of cholera, many mathematical models were proposed, whose earlier one was established by Capasso and Paveri-Fontana [3] to study the 1973 cholera epidemic in the Mediterranean region as follows:

$$
\begin{equation*}
\frac{d I}{d t}=g(B)-a_{22} I, \quad \frac{d B}{d t}=-a_{11} B+a_{12} I \tag{1}
\end{equation*}
$$

where $B(t)$ and $I(t)$ denote the concentrations of the pathogen and the infective populations, respectively. In addition, Codeço [4] investigated the role of the aquatic pathogen
in dynamics of cholera through the following susceptible-infective-pathogen model:

$$
\begin{gather*}
\frac{d S}{d t}=n(H-S)-a \frac{S B}{K+B}, \quad \frac{d I}{d t}=a \frac{S B}{K+B}-r I, \\
\frac{d B}{d t}=e I-(m b-n b) B \tag{2}
\end{gather*}
$$

where $S(t)$ is the susceptible individuals. In this model, human is divided into two groups: the susceptible group and the infective group. As pointed out in [4-8], bacterium Vibrio cholera can spread by direct human-to-human and indirect environment-to-human modes. To understand the complex dynamics of cholera, model (2) is extended by [8-15] and so forth.

In all previous models the influences of space distribution of human on the transmission of cholera are omitted. Cholera usually spreads in spatial wave [16]. Cholera bacteria live in rivers and interact with the plankton on the surface of the water [17]. When individuals drink contaminated water and are infected, they will release cholera bacteria through excretion [18]. Capasso et al. [19-23] developed model (1) by incorporating the bacterium diffusion in a bounded area and studied the existence and stability of solutions. To deeply investigate the interaction of transmission modes and bacterium diffusion, Bertuzzo et al. [24, 25] incorporated patchy structure into model (2) and supposed that pathogen
in water could diffuse among these patches. Furthermore, Mari et al. [26] studied the influence of diffusion of both human and pathogen on cholera dynamics through a patchy model.

Infectious case is usually found firstly at some location and then spreads to other areas. Consequently, the most important question for cholera is what the spreading speed of cholera is. However, the above spatial models mainly focus on the stability of solutions not the spreading speed. Traveling wave solution is an important tool used to study the spreading speed of infectious diseases [27-29]. Based on Capasso's model (1), Zhao and Wang [30], Xu and Zhao [31], Jin and Zhao [32], and Hsu and Yang [33] studied the influences of pathogen diffusion on the spread speed of cholera.

The studies of traveling wave solutions of Capasso's model (1) incorporating pathogen diffusion provide insight into the spreading speed of cholera. However, some pieces of information are omitted, such as the interaction of direct human-tohuman and indirect environment-to-human transmissions. In this paper, a reaction-diffusion model with pathogen diffusion and both transmission paths is proposed by developing Codeço's model (2). Based on model (2) and ignoring the disease-related death, a general diffusive cholera model can be formulated as the following reaction-diffusion system:

$$
\begin{gather*}
\frac{\partial S}{\partial t}=b(N-S)-f(I) S-g(B) S \\
\frac{\partial I}{\partial t}=f(I) S+g(B) S-b I  \tag{3}\\
\frac{\partial B}{\partial t}=d \frac{\partial^{2} B}{\partial x^{2}}+e I-m B
\end{gather*}
$$

where $S=S(x, t)$ and $I=I(x, t)$ denote the concentrations of susceptible and infected individuals, respectively, and $B=$ $B(x, t)$ is the concentration of the infectious agents. $N$ is the total human population, $b$ stands for the natural birth and death rate, $e$ denotes the contribution of each infected person to the concentration of cholera, and $m$ is the net death rate of vibrio cholera. $f(I)$ and $g(B)$ are the human-tohuman and environment-to-human transmission incidences, respectively. Similar to [10], we assume that $f(I)$ and $g(B)$ satisfy
(A1) $f(0)=0, f^{\prime}(I) \geq 0, f^{\prime \prime}(I) \leq 0$;
(A2) $g(0)=0, g^{\prime}(0)>0, g^{\prime}(B) \geq 0, g^{\prime \prime}(B) \leq 0$, and $g(B)$ is strictly monotonously increasing in $[0,+\infty)$.
It is easy to conclude that $f(I) \leq f^{\prime}(0) I, g(B) \leq g^{\prime}(0) B$, and $f(I) / I$ and $g(B) / B$ are nonincreasing. Obviously, hypotheses (A1) and (A2) imply that the two transmission paths are saturated. In Tian and Wang [10], $f(I)$ and $g(B)$ have the following expressions:

$$
\begin{equation*}
f(I)=\beta_{1} I, \quad g(B)=\frac{\beta_{2} B}{K+B} . \tag{4}
\end{equation*}
$$

Obviously, as a special case, such selections satisfy (A1) and (A2).

Shooting method is very important in proving the existence of traveling wave solutions, which was proposed by

Dunbar [34, 35] and was applied to many models (e.g., [3640]). In this paper, the existence of traveling wave solutions of system (3) will be proved by shooting method and the formula for minimal wave speed will be given.

This paper is organized as follows. In next section, the main theorem and the formula for minimal wave speed will be given. In Section 3, the nonexistence of the traveling wave solutions for $c<c^{*}$ is proved by geometric method. Section 4 is devoted to shooting arguments and the construction of Wazewski set. In Section 5, we prove the existence of traveling wave solutions for $c>c^{*}$ and then give the existence of traveling wave solution for $c=c^{*}$ by limit arguments. The final section is devoted to the simulations.

## 2. Main Results

For convenience, we introduce dimensionless variables and parameters. By setting

$$
\begin{equation*}
u_{1}=\frac{S}{b N}, \quad u_{2}=\frac{I}{b N}, \quad u_{3}=\frac{m}{e b N} B, \quad y=\frac{x}{\sqrt{d}}, \tag{5}
\end{equation*}
$$

model (3) has the form

$$
\begin{gather*}
u_{1, t}=1-b u_{1}-f_{1}\left(u_{2}\right) u_{1}-g_{1}\left(u_{3}\right) u_{1}, \\
u_{2, t}=f_{1}\left(u_{2}\right) u_{1}+g_{1}\left(u_{3}\right) u_{1}-b u_{2},  \tag{6}\\
u_{3, t}=u_{3, y y}+m\left(u_{2}-u_{3}\right),
\end{gather*}
$$

where $f_{1}\left(u_{2}\right)=f\left(b N u_{2}\right)$ and $g_{1}\left(u_{3}\right)=g\left(e b N u_{3} / m\right)$.
Denote $R_{0}=\left[f_{1}^{\prime}(0)+g_{1}^{\prime}(0)\right] / b^{2}$, which is the basic reproduction number of (6). Then hypotheses (A1) and (A2) imply that system (6) has two nonnegative constant solutions $P_{1}(1 / b, 0,0)$ and $P_{2}\left(1 / b-u^{*}, u^{*}, u^{*}\right)$ if and only if $R_{0}>1$, where $u^{*}$ is the only one positive root of equation

$$
\begin{equation*}
\left[f_{1}\left(u^{*}\right)+g_{1}\left(u^{*}\right)\right]\left(\frac{1}{b}-u^{*}\right)=b u^{*} \tag{7}
\end{equation*}
$$

and $0<u^{*}<1 / b$. Biologically, $P_{1}$ corresponds to disease-free equilibrium and $P_{2}$ corresponds to endemic equilibrium. To study the spreading wave of cholera, it is assumed that $R_{0}>1$ holds in this paper; that is

$$
\begin{equation*}
f_{1}^{\prime}(0)+g_{1}^{\prime}(0)>b^{2} \tag{8}
\end{equation*}
$$

A traveling wave solution of system (6) with speed $c$ is a nonnegative solution of the form

$$
\begin{gather*}
u_{1}(y, t)=u_{1}(s), \quad u_{2}(y, t)=u_{2}(s),  \tag{9}\\
u_{3}(y, t)=u_{3}(s), \quad s=y+c t .
\end{gather*}
$$

Substituting traveling profile ( $u_{1}(s), u_{2}(s), u_{3}(s)$ ) into system (6) yields the following equations:

$$
\begin{gather*}
c u_{1}^{\prime}=1-b u_{1}-f_{1}\left(u_{2}\right) u_{1}-g_{1}\left(u_{3}\right) u_{1}, \\
c u_{2}^{\prime}=f_{1}\left(u_{2}\right) u_{1}+g_{1}\left(u_{3}\right) u_{1}-b u_{2},  \tag{10}\\
c u_{3}^{\prime}=u_{3}^{\prime \prime}+m\left(u_{2}-u_{3}\right),
\end{gather*}
$$

where I denotes $d / d s$. To investigate invasion question by cholera, we will study the positive solutions of (10) such that

$$
\begin{gather*}
\left(u_{1}(+\infty), u_{2}(+\infty), u_{3}(+\infty)\right)=\left(\frac{1}{b}-u^{*}, u^{*}, u^{*}\right)  \tag{11}\\
\left(u_{1}(-\infty), u_{2}(-\infty), u_{3}(-\infty)\right)=\left(\frac{1}{b}, 0,0\right)
\end{gather*}
$$

Before giving the main theorem, we introduce the equation for minimal wave speed

$$
\begin{equation*}
\Delta(c):=b_{3} c^{6}+b_{2} c^{4}+b_{1} c^{2}+b_{0}=0 \tag{12}
\end{equation*}
$$

where

$$
\begin{gather*}
\epsilon=f_{1}^{\prime}(0)-b^{2}, \\
b_{3}=b^{2} \epsilon^{2}+2 b^{3} m\left(f_{1}^{\prime}(0)+g_{1}^{\prime}(0)-b^{2}\right) \\
+2 b^{3} m g_{1}^{\prime}+b^{4} m^{2}, \\
b_{2}=-2 b \epsilon^{3}+2 b^{2} m \epsilon^{2}+\left(8 b^{3} m^{2}-6 b^{2} m g_{1}^{\prime}\right) \epsilon \\
+4 m^{3} b^{4}+18 b^{3} m^{2} g_{1}^{\prime}(0),  \tag{13}\\
b_{1}=\epsilon^{4}-8 m b \epsilon^{3}-\left(8 b^{2} m^{2}+6 b m g_{1}^{\prime}\right) \epsilon^{2} \\
-36 b^{2} m^{2} g_{1}^{\prime}(0) \epsilon-27 m^{2} b^{2} g_{1}^{\prime}(0)^{2}, \\
b_{0}=4 m\left(b^{2}-f_{1}^{\prime}(0)\right)^{3}\left(b^{2}-f_{1}^{\prime}(0)-g_{1}^{\prime}(0)\right) .
\end{gather*}
$$

Theorem 1. There exists a constant $c^{*}>0$ which is the greatest positive root of (12). When $c \geq c^{*}$, system (6) has a traveling wave solution satisfying boundary condition (11). When $0<$ $c<c^{*}$, system (6) has no traveling wave solutions satisfying boundary condition (11).

## 3. Nonexistence of Traveling Wave Solutions for $c<c^{*}$

From (10), we have

$$
\begin{equation*}
\left[u_{1}(s)+u_{2}(s)\right]^{\prime}=\frac{\left[1-b\left(u_{1}(s)+u_{2}(s)\right)\right]}{c} \tag{14}
\end{equation*}
$$

Consequently, if $u_{1}(0)+u_{2}(0) \neq 1 / b$, then

$$
\begin{equation*}
\left|u_{1}(s)+u_{2}(s)\right| \longrightarrow \infty \quad \text { when } s \longrightarrow-\infty . \tag{15}
\end{equation*}
$$

Hence, the traveling profile $\left(u_{1}(s), u_{2}(s), u_{3}(s)\right)$ with boundary condition (11) must satisfy

$$
\begin{equation*}
u_{1}(s)+u_{2}(s)=\frac{1}{b} \quad \text { for any } s \in R \tag{16}
\end{equation*}
$$

Therefore, to study traveling wave solutions we assume (16) satisfies. Setting $u_{3}^{\prime}=z$ in system (10) and noticing (16), it follows

$$
\begin{gather*}
u_{2}^{\prime}=\frac{\left[\left(f_{1}\left(u_{2}\right)+g_{1}\left(u_{3}\right)\right)\left((1 / b)-u_{2}\right)-b u_{2}\right]}{c} \\
u_{3}^{\prime}=z  \tag{17}\\
z^{\prime}=c z+m\left(u_{3}-u_{2}\right)
\end{gather*}
$$

If $u_{1}(s)=0$, then $u_{1}^{\prime}(s)=1 / c>0$ by system (10). Therefore, we suppose $u_{1}(s)=1 / b-u_{2}(s)>0$ for any $s$; that is, $u_{2}(s)<1 / b$.

Obviously, system (17) has two equilibria $E_{1}(0,0,0)$ and $E_{2}\left(u^{*}, u^{*}, 0\right)$. A profile solution of (10) which satisfies boundary condition (11) corresponds to the positive solution ( $\left.u_{2}(s), u_{3}(s), z(s)\right)$ of system (17) which satisfies

$$
\begin{equation*}
u(+\infty)=\left(u_{2}^{*}, u_{2}^{*}, 0\right), \quad u(-\infty)=(0,0,0) \tag{18}
\end{equation*}
$$

where $u(s)=\left(u_{2}(s), u_{3}(s), z(s)\right)$. Therefore, to study the solutions of (10), it is sufficient to study those of system (17) satisfying boundary condition (18).

Firstly, we investigate the dynamics near $E_{1}$. Simple calculations show that the characteristic equation of the linearization of system (17) at $E_{1}$ is

$$
\begin{equation*}
H(\lambda)=\lambda^{3}+a_{2} \lambda^{2}+a_{1} \lambda+a_{0}=0 \tag{19}
\end{equation*}
$$

where

$$
\begin{gather*}
a_{0}=\frac{m\left(f_{1}^{\prime}(0)+g_{1}^{\prime}(0)-b^{2}\right)}{b c}, \quad a_{1}=\frac{f_{1}^{\prime}(0)-b^{2}-m b}{b}, \\
a_{2}=\frac{b^{2}-f_{1}^{\prime}(0)-b c^{2}}{b c} . \tag{20}
\end{gather*}
$$

Because $a_{0}>0$ (19) has a negative real root, which is denoted by $\lambda_{3}$. Let $\lambda_{1}$ and $\lambda_{2}$ be the other two eigenvalues of (19) and suppose that $\operatorname{Re} \lambda_{1} \geq \operatorname{Re} \lambda_{2}$. To investigate the distribution of roots of (19), denote

$$
\begin{equation*}
p=a_{1}-\frac{a_{2}^{2}}{3}, \quad q=\frac{2 a_{2}^{3}}{27}-\frac{a_{1} a_{2}}{3}+a_{0}, \quad \Delta_{0}=\frac{q^{2}}{4}+\frac{p^{3}}{27} \tag{21}
\end{equation*}
$$

and introduce the following lemma [41].
Lemma 2. (a) If $\Delta_{0}>0$, (19) has one real root and two nonreal complex conjugate roots.
(b) If $\Delta_{0}=0$, (19) has a multiple root and all its roots are real.
(c) If $\Delta_{0}<0$, (19) has three distinct real roots.

Direct calculations show that $\Delta_{0}=-\Delta /\left(108 b^{4} c^{4}\right)$, where $\Delta$ is defined by (12).

Lemma 3. (a) The real parts of $\lambda_{1}$ and $\lambda_{2}$ are positive.
(b) Assume $f_{1}^{\prime}(0) \leq b^{2}$. Then, there exists $c^{*}>0$ which is the only positive root of $\Delta(c)=0$. When $c \geq c^{*}, \lambda_{1}$, and $\lambda_{2}$ are real. When $0<c<c^{*}, \lambda_{1}$, and $\lambda_{2}$ are complex and nonreal.
(c) Assume that $f_{1}^{\prime}(0)>b^{2}$. Then, there exist two positive constants $c_{1}^{*}<c^{*}$ which are all positive roots of $\Delta(c)=0 . \lambda_{1}$ and $\lambda_{2}$ are complex and nonreal if and only if $c_{1}^{*}<c<c^{*}$. If $c>c^{*}$, then $\lambda^{*}<\lambda_{2}<\lambda_{1}$; if $0<c \leq c_{1}^{*}$, then $\lambda_{2} \leq \lambda_{1}<\lambda^{*}$, where $\lambda^{*}=\left(f_{1}^{\prime}(0)-b^{2}\right) /(b c)$.
(d) $\lambda_{1}=\lambda_{2}$ if and only if $c=c^{*}$ or $c_{1}^{*}$.

Proof. Suppose $\lambda=\beta i \neq 0$ is the root of (19). Substituting $\lambda=\beta i$ into (19) and comparing real and imaginary parts
show that $a_{1}=\beta^{2}>0$ and $a_{0}=a_{1} a_{2}$. Since $a_{0}>0$, then $a_{2}>0$. However, it is impossible that $a_{1}>0$ and $a_{2}>0$ by the expressions of $a_{1}$ and $a_{2}$. Therefore, the real parts of $\lambda_{1}$ and $\lambda_{2}$ are not zero. Furthermore, since it is impossible that $a_{1}>0$ and $a_{2}>0$, Routh-Hurwitz theorem implies that it is impossible that the real parts of both $\lambda_{1}$ and $\lambda_{2}$ are negative. Consequently, there are two cases: (i) $\lambda_{1}$ and $\lambda_{2}$ are complex conjugate roots with positive real parts; (ii) $\lambda_{1}$ and $\lambda_{2}$ are real and at least one is positive. However, Descartes' rule of signs shows that the number of positive roots of (19) is zero or two. Thus, if case (ii) is true, both of $\lambda_{1}$ and $\lambda_{2}$ are real and positive. Therefore, (a) is proved.

In this paragraph, we consider the case $f_{1}^{\prime}(0) \leq b^{2}$. Firstly, suppose that $f_{1}^{\prime}(0)<b^{2}$. Obviously, $b_{0}<0$ and $b_{3}>0$. By the expression of $b_{2}$, we have

$$
\begin{align*}
b_{2}= & -2 b \epsilon^{3}+2 b^{2} m \epsilon^{2}-6 b^{2} m g_{1}^{\prime} \epsilon \\
& +8 b^{3} m^{2}\left(f_{1}^{\prime}(0)+g_{1}^{\prime}(0)-b^{2}\right)+4 m^{3} b^{4}+10 b^{3} m^{2} g_{1}^{\prime}(0) \\
> & 0 \tag{22}
\end{align*}
$$

since $\epsilon=f_{1}^{\prime}(0)-b^{2}<0$. Now, assume $f_{1}^{\prime}(0)=b^{2}$; that is, $\epsilon=0$. Then. $b_{3}>0, b_{2}>0, b_{1}<0$, and $b_{0}=0$. Then, if $f_{1}^{\prime}(0) \leq b^{2}$, Descartes' rule of signs shows that there exists $c^{*}>0$ which is the only positive root of $\Delta(c)=0$, where $\Delta(c)<0$ for $0<c<c^{*}$ and $\Delta(c)>0$ for $c>c^{*}$. Using Lemma 2 completes the proof of (b).

Suppose that $f_{1}^{\prime}(0)>b^{2}$ in this paragraph and, thus, $\epsilon>0$. Calculations show that

$$
\begin{gather*}
H\left(\lambda^{*}\right)=\frac{m g_{1}^{\prime}}{b c}>0, \\
H^{\prime}\left(\lambda^{*}\right)=\frac{\epsilon^{2}-b c^{2} \epsilon-m b^{2} c^{2}}{b^{2} c^{2}} \tag{23}
\end{gather*}
$$

and that $H^{\prime}(\lambda)=0$ has two roots $\lambda_{1}^{*}$ and $\lambda_{2}^{*}$, where

$$
\begin{align*}
& \lambda_{1}^{*}=\frac{b c^{2}+\epsilon+\sqrt{b^{2} c^{4}+\left(3 m b^{2}-b \epsilon\right) c^{2}+\epsilon^{2}}}{3 b c}  \tag{24}\\
& \lambda_{2}^{*}=\frac{b c^{2}+\epsilon-\sqrt{b^{2} c^{4}+\left(3 m b^{2}-b \epsilon\right) c^{2}+\varepsilon^{2}}}{3 b c}
\end{align*}
$$

and $\lambda_{1}^{*}>\lambda_{2}^{*}$. By letting $c_{0} \triangleq \epsilon / \sqrt{b \epsilon+m b^{2}}$ and using trivial calculations, we get (see Figure 1)

$$
\begin{align*}
& \lambda^{*}=\lambda_{1}^{*} \Longleftrightarrow c=c_{0} \Longleftrightarrow H^{\prime}\left(\lambda^{*}\right)=0, \\
& \lambda^{*}>\lambda_{1}^{*} \Longleftrightarrow c<c_{0} \Longleftrightarrow H^{\prime}\left(\lambda^{*}\right)>0,  \tag{25}\\
& \lambda^{*}<\lambda_{1}^{*} \Longleftrightarrow c>c_{0} \Longleftrightarrow H^{\prime}\left(\lambda^{*}\right)<0 .
\end{align*}
$$

Therefore, if $c=c_{0}$, then $H\left(\lambda_{1}^{*}\right)=H\left(\lambda^{*}\right)>0$. Since $\lambda_{1}^{*}$ is the only minimum-value point of $H(\lambda)$, and then $H(\lambda)>0$ for any $\lambda>0$ and both of $\lambda_{1}$ and $\lambda_{2}$ are not real. Lemma 2 shows that $\Delta\left(c_{0}\right)<0$. Thus, since $b_{0}>0$ and $b_{3}>0$, there
exist two positive roots $c_{1}^{*}<c^{*}$ for equation $\Delta(c)=0$ such that $c_{1}^{*}<c_{0}<c^{*}$. Then, using (25) and Lemma 2 completes the proof of (c) and (d).

Direct calculations show that corresponding eigenvectors of eigenvalue $\lambda_{i}$ are

$$
\begin{equation*}
e_{i}=\left(1-\frac{\lambda_{i}\left(\lambda_{i}-c\right)}{m}, 1, \lambda_{i}\right) \tag{26}
\end{equation*}
$$

where $i=1,2,3$. Since

$$
\begin{align*}
H\left(\lambda_{i}\right) & =-m\left[1-\frac{\lambda_{i}\left(\lambda_{i}-c\right)}{m}\right]\left[\lambda_{i}-\frac{f_{1}^{\prime}(0)-b^{2}}{b c}\right]+\frac{m g_{1}^{\prime}}{b c} \\
& =0 \tag{27}
\end{align*}
$$

and thus

$$
\begin{equation*}
1-\frac{\lambda_{i}\left(\lambda_{i}-c\right)}{m}=\frac{g_{1}^{\prime}(0)}{b c \lambda_{i}+b^{2}-f_{1}^{\prime}(0)}=\frac{g_{1}^{\prime}(0)}{b c\left(\lambda_{i}-\lambda^{*}\right)} \tag{28}
\end{equation*}
$$

Then, we have the following lemma.
Lemma 4. If $0<c<c^{*}$, there exist no traveling wave solutions which satisfy boundary condition (11).

Proof. Assume that $f_{1}^{\prime}(0) \leq b^{2}$ and $0<c<c^{*}$. Then, (b) of Lemma 3 implies that $\lambda_{1}$ and $\lambda_{2}$ are complex conjugate eigenvalues and there exits locally unstable manifold $\mathscr{W}^{u}$ and locally stable manifold $\mathscr{W}^{s}$. If a solution of (17) tends to $E_{1}$ when $s \rightarrow-\infty$, then it will be spiral on $\mathscr{W}^{u}$. By the structures of $e_{1}$ and $e_{2}, u_{2}(s)<0$ at some time $s<0$, which shows that there exist no traveling wave solutions departing from $E_{1}$.

Suppose that $f_{1}^{\prime}(0)>b^{2}$. If $c_{1}^{*}<c<c^{*}$, (c) of Lemma 3 shows that $\lambda_{1}$ and $\lambda_{2}$ are complex conjugate eigenvalues and similar arguments to that of previous paragraph finish the proof. If $0<c \leq c_{1}^{*}$, (c) of Lemma 3 shows that $\lambda_{1}$ and $\lambda_{2}$ are real; however, $\lambda_{2} \leq \lambda_{1}<\lambda^{*}$. If a solution of (17) tends to $E_{1}$ when $s \rightarrow-\infty$, structures of $e_{1}$ and $e_{2}$ indicate that there is an $s<0$ such that $u_{2}(s)<0$. The proof is completed.

From Section 4 to Section 5.2, we suppose that $c>c^{*}$, which implies $\lambda^{*}<\lambda_{2}<\lambda_{1}$.

## 4. Shooting Method and Wazewski Set

To prove the existence of traveling wave, shooting method developed by Dunbar [34] is used. Firstly, we give the shooting arguments.

Consider the differential equation

$$
\begin{equation*}
\frac{d y}{d s}=f(y) \tag{29}
\end{equation*}
$$

where $f(y)$ from $R^{n}$ to $R^{n}$ satisfies Lipschitz condition about $y$. Let $y\left(s ; y_{0}\right)$ denote the unique solution of (29) with initial


Figure 1: Distribution of eigenvalues of (19) when $f_{1}^{\prime}(0)>b^{2}$, (a) for $c>c^{*}$ and (b) for $c<c_{1}^{*}$.
value $y(0)=y_{0}$. It is convenient to give the notations $y_{0} \cdot s \triangleq$ $y\left(s ; y_{0}\right)$ and $y_{0} \cdot S \triangleq\left\{y_{0} \cdot s \mid s \in S \subset R\right\}$. To describe the shooting method (or Wazewski theorem), some definitions are necessary.

Definition 5. (a) For $W \subseteq R^{n}$, define immediate exit set $W^{-}$ of $W$ as

$$
\begin{equation*}
W^{-} \triangleq\left\{y_{0} \in W \mid \forall s>0, y_{0} \cdot[0, s) \nsubseteq W\right\} \tag{30}
\end{equation*}
$$

(b) For $\Sigma \subseteq W$, let $\Sigma^{0} \triangleq\left\{y_{0} \in \Sigma \mid \exists s_{0}>0\right.$ such that $y_{0}$. $\left.s_{0} \notin W\right\}$.
(c) Given $y_{0} \in \Sigma^{0}$, define exit time $T\left(y_{0}\right)$ of $y_{0}$ by

$$
\begin{equation*}
T\left(y_{0}\right) \triangleq \sup \left\{s \mid y_{0} \cdot[0, s) \subseteq W\right\} \tag{31}
\end{equation*}
$$

Then, Wazewski theorem is formulated as follows.
Lemma 6 (see [34]). Suppose that
(1) if $y_{0} \in \Sigma$ and $y_{0} \cdot[0, s] \subseteq \operatorname{cl}(W)$, then $y_{0} \cdot[0, s] \subseteq W$.
(2) If $y_{0} \in \Sigma, y_{0} \cdot s \in W$ and $y_{0} \cdot s \notin W^{-}$, then there exists an open set $V_{s}$ about $y_{0} \cdot s$ disjoint from $W^{-}$.
(3) If $\Sigma=\Sigma^{0}, \Sigma$ is compact and $\Sigma$ intersects a trajectory of (29) only once.

Then, the mapping $H\left(y_{0}\right)=y_{0} \cdot T\left(y_{0}\right)$ is a homeomorphism from $\Sigma$ to its image on $W^{-}$.

A set $W \subseteq R^{n}$ satisfying conditions (1) and (2) of Lemma 6 is called a Wazewski set. In the following, we first construct the Wazewski set $W$. Fundamental idea to construct a Wazewski set is that the characteristic vectors corresponding eigenvalues with positive real parts should be removed from $W$ and that those characteristic vectors corresponding eigenvalues with negative real parts should be included. Therefore, we set

$$
\begin{equation*}
W=\mathbb{R}^{3} \backslash(P \cup Q) \tag{32}
\end{equation*}
$$

where

$$
\begin{gather*}
P=\left\{\left(u_{1}, u_{2}, u_{3}, z\right): u_{3}>u_{2}>u^{*}, z>0\right\} \\
Q=\left\{\left(u_{1}, u_{2}, u_{3}, z\right): 0<u_{3}<u_{2}<u^{*}, z<0\right\} . \tag{33}
\end{gather*}
$$



Figure 2: The construction of $W$ and $W^{-}$.

It is obvious that $\partial W=\partial P \cup \partial Q$. Firstly, we give the construction of $W^{-}$, which is described in Figure 2.

Lemma 7. The construction of $W^{-}$is as follows:

$$
\begin{equation*}
W^{-}=\partial W \backslash\left(J \cup E_{2}\right) \tag{34}
\end{equation*}
$$

where $J=\left\{\left(u_{2}, u_{3}, z\right): 0 \leq u_{2} \leq u^{*}, u_{3}=0, z \leq 0\right\}$.
Proof. It is enough to analyze the behavior of solution on $\partial P \cup \partial Q$. We only study $\partial Q$ and omit the proof of $\partial P$ since the analysis of $\partial P$ is similar to that of $\partial Q$ and is simpler. In the process of this proof, we use some notations to simplify the proof. Set

$$
\begin{gather*}
u_{i}^{\prime}=\left.\frac{d u_{i}}{d s}\right|_{\left(u_{2}, u_{3}, z\right) \in \partial Q}, \quad z^{\prime}=\left.\frac{d z}{d s}\right|_{\left(u_{2}, u_{3}, z\right) \in \partial Q}, \quad i=2,3, \\
h\left(u_{2}\right)=\left[\frac{f_{1}\left(u_{2}\right)}{u_{2}}+\frac{g_{1}\left(u_{2}\right)}{u_{2}}\right]\left(\frac{1}{b}-u_{2}\right)-b . \tag{35}
\end{gather*}
$$

From hypotheses (A1) and (A2), we find that $f_{1}\left(u_{2}\right) / u_{2}$ and $g_{1}\left(u_{2}\right) / u_{2}$ are monotonously decreasing, $h\left(u_{2}\right)$ is strictly monotonously decreasing for $u_{2} \in(0,1 / b)$, and $u^{*}$ is the only positive root of $h\left(u_{2}\right)=0$. The set $\partial Q$ is classified into two cases according to variable $z$.
(a) Case $z<0$. This case is classified as follows.
(1) Case $0=u_{3}<u_{2}<u^{*}$. Then $u_{3}^{\prime}=z<0$ and the solution of (17) will enter $\operatorname{int}(W)$.
(2) Case $0<u_{3}=u_{2}<u^{*}$. Then

$$
\begin{equation*}
\left(u_{3}-u_{2}\right)^{\prime}=\frac{z-h\left(u_{2}\right) u_{2}}{c}<0 \tag{36}
\end{equation*}
$$

The solution of (17) will enter $Q$.
(3) Case $0<u_{3}<u_{2}=u^{*}$. Then

$$
\begin{align*}
u_{2}^{\prime} & =\frac{\left[\left(f_{1}\left(u^{*}\right) / u^{*}+g_{1}\left(u_{3}\right) / u^{*}\right)\left(1 / b-u^{*}\right)-b\right] u^{*}}{c} \\
& <\frac{h\left(u^{*}\right) u^{*}}{c}=0 . \tag{37}
\end{align*}
$$

The solution of (17) will enter $Q$.
(4) Case $0=u_{3}=u_{2}<u^{*}$. Then $u_{3}^{\prime}=z<0$ and the solution of (17) will enter $\operatorname{int}(W)$.
(5) Case $0=u_{3}$ and $u_{2}=u^{*}$. The solution of (17) will enter int $(W)$.
(6) Case $u_{3}=u_{2}=u^{*}$. Then $u_{2}^{\prime}=0$,

$$
\begin{gather*}
u_{2}^{\prime \prime}=\left[\left(f_{1}^{\prime}(0)\left(u^{*}\right) u_{2}^{\prime}+g_{1}^{\prime}(0)\left(u^{*}\right) z\right)\left(\frac{1}{b}-u^{*}\right)\right. \\
\left.\quad-\left(f_{1}\left(u^{*}\right)+g_{1}\left(u^{*}\right)\right) u_{2}^{\prime}-b\right] \times(c)^{-1} \\
=\frac{\left[g_{1}^{\prime}(0)\left(u^{*}\right) z\left((1 / b)-u^{*}\right)-b\right]}{c}<0, \tag{38}
\end{gather*}
$$

and $\left(u_{3}-u_{2}\right)^{\prime}=z<0$. Therefore, the solution of (17) will enter $Q$.
(b) Case $z=0$. This case is classified as follows.
(1) Case $0<u_{3}<u_{2}<u^{*}$. Then $z^{\prime}=m\left(u_{3}-u_{2}\right)<0$ and the solution of (17) will enter $Q$.
(2) Case $0=u_{3}<u_{2}<u^{*}$. Then $u_{3}^{\prime}=z=0$, $u_{3}^{\prime \prime}=z^{\prime}=-m u_{2}<0$. The solution of (17) will enter $\operatorname{int}(W)$.
(3) Case $0<u_{3}=u_{2}<u^{*}$. Then $\left(u_{3}-u_{2}\right)^{\prime}=$ $-h\left(u_{2}\right) u_{2} / c<0, z^{\prime}=0$, and $z^{\prime \prime}=c z^{\prime}+$ $m\left(u_{3}-u_{2}\right)^{\prime}<0$. The solution of (17) will enter Q.
(4) Case $0<u_{3}<u_{2}=u^{*}$. Then $u_{2}^{\prime}<0$ and $z^{\prime}=$ $m\left(u_{3}-u_{2}\right)<0$. The solution of (17) will enter Q.
(5) Case $0=u_{3}=u_{2}<u^{*}$. In this case, $(0,0,0)$ is equilibrium and is constant.
(6) Case $0=u_{3}$ and $u_{2}=u^{*}$. Then $u_{3}^{\prime}=z=0$ and $u_{3}^{\prime \prime}=z^{\prime}=-m u_{2}<0$. The solution of (17) will enter $\operatorname{int}(W)$.
(7) Case $u_{3}=u_{2}=u^{*}$. Then $\left(u^{*}, u^{*}, 0\right)$ is equilibrium and is constant.

## 5. Existence of Traveling Wave <br> Solution for $c \geq c^{*}$

In this section, we prove the existence of traveling wave solution for $c \geq c^{*}$. Firstly, we study the behaviors of solutions near $E_{1}$.

### 5.1. Behaviors of Solutions Near $E_{1}$

Lemma 8. Suppose $\left(u_{2}(s), u_{3}(s), z(s)\right)$ is a solution of (17) satisfying initial conditions

$$
\begin{equation*}
z(0)>k u_{3}(0), \quad u_{3}(0)>\frac{b c k+b^{2}-f_{1}^{\prime}(0)}{g_{1}^{\prime}(0)} u_{2}(0)>0, \tag{39}
\end{equation*}
$$

where $k=\left(\lambda_{1}+\lambda_{2}\right) / 2$. Then, for everys $>0$, we have

$$
\begin{equation*}
z(s)>k u_{3}(s), \quad u_{3}(s)>\frac{b c k+b^{2}-f_{1}^{\prime}(0)}{g_{1}^{\prime}(0)} u_{2}(s)>0 . \tag{40}
\end{equation*}
$$

Proof. From Lemma 3, we have $\left(b c k+b^{2}-f_{1}^{\prime}(0)\right) / g_{1}^{\prime}(0)>0$. To finish the proof, it is sufficient to prove that the set

$$
\begin{equation*}
\Psi=\left\{\left(u_{2}, u_{3}, z\right): z>k u_{3}, u_{3}>\frac{b c k+b^{2}-f_{1}^{\prime}(0)}{g_{1}^{\prime}(0)} u_{2}>0\right\} \tag{41}
\end{equation*}
$$

is positively invariant. It is obvious that

$$
\begin{equation*}
\partial \Psi=\partial \Psi_{1} \cup \partial \Psi_{2} \cup \partial \Psi_{3} \cup E_{1} \tag{42}
\end{equation*}
$$

where

$$
\begin{gather*}
\partial \Psi_{1}=\left\{\left(u_{2}, u_{3}, z\right): z=k u_{3}, u_{3} \geq \frac{b c k+b^{2}-f_{1}^{\prime}(0)}{g_{1}^{\prime}(0)} u_{2}>0\right\}, \\
\partial \Psi_{2}=\left\{\left(u_{2}, u_{3}, z\right): z>k u_{3}, u_{3}=\frac{b c k+b^{2}-f_{1}^{\prime}(0)}{g_{1}^{\prime}(0)} u_{2} \geq 0\right\}, \\
\partial \Psi_{3}=\left\{\left(u_{2}, u_{3}, z\right): z \geq k u_{3}, u_{3}>u_{2}=0\right\} . \tag{43}
\end{gather*}
$$

Suppose that $\left(u_{2}\left(s_{0}\right), u_{3}\left(s_{0}\right), z\left(s_{0}\right)\right) \in \partial \Psi_{1}$. Then, $z\left(s_{0}\right)=$ $k u_{3}\left(s_{0}\right)$ and

$$
\begin{align*}
\frac{d}{d s} & {\left[z(s)-k u_{3}(s)\right]_{s=s_{0}} } \\
& =c z\left(s_{0}\right)+m\left[u_{3}\left(s_{0}\right)-u_{2}\left(s_{0}\right)\right]-k z\left(s_{0}\right) \\
& =(c-k) z\left(s_{0}\right)+m\left[u_{3}\left(s_{0}\right)-u_{2}\left(s_{0}\right)\right] \\
& =(c-k) k u_{3}\left(s_{0}\right)+m\left[u_{3}\left(s_{0}\right)-u_{2}\left(s_{0}\right)\right] \\
& =[(c-k) k+m] u_{3}\left(s_{0}\right)-m u_{2}\left(s_{0}\right) \\
& \geq\left\{[(c-k) k+m] \frac{b c k+b^{2}-f_{1}^{\prime}(0)}{g_{1}^{\prime}(0)}-m\right\} u_{2}\left(s_{0}\right) \\
& =-\frac{b c}{g_{1}^{\prime}(0)} H(k) u_{2}\left(s_{0}\right)>0 . \tag{44}
\end{align*}
$$

The last inequality is given since $\lambda_{2}<k<\lambda_{1}$. Suppose that $\left(u_{2}\left(s_{0}\right), u_{3}\left(s_{0}\right), z\left(s_{0}\right)\right) \in \partial \Psi_{2}$. If $u_{2}\left(s_{0}\right)>0$, then

$$
\begin{align*}
\frac{d}{d s} & {\left[u_{3}(s)-\frac{b c k+b^{2}-f_{1}^{\prime}(0)}{g_{1}^{\prime}(0)} u_{2}(s)\right]_{s=s_{0}} } \\
= & \left\{z-\frac{b c k+b^{2}-f_{1}^{\prime}(0)}{g_{1}^{\prime}(0)} \cdot \frac{1}{c}\right. \\
& \left.\times\left[\left(f_{1}\left(u_{2}\right)+g_{1}\left(u_{3}\right)\right)\left(\frac{1}{b}-u_{2}\right)-b u_{2}\right]\right\}_{s=s_{0}} \\
> & \left\{k u_{3}-\frac{b c k+b^{2}-f_{1}^{\prime}(0)}{c g_{1}^{\prime}}\right. \\
= & \left.\times\left[\frac{f_{1}^{\prime}(0) u_{2}+g_{1}^{\prime}(0) u_{3}}{b}-b u_{2}\right]\right\}_{s=s_{0}} \\
= & 0 .
\end{align*}
$$

If $u_{2}\left(s_{0}\right)=0$, we have

$$
\begin{align*}
& \frac{d}{d s}\left[u_{3}(s)-\frac{b c k+b^{2}-f_{1}^{\prime}(0)}{g_{1}^{\prime}(0)} u_{2}(s)\right]_{s=s_{0}}  \tag{46}\\
& \quad=z\left(s_{0}\right)>0
\end{align*}
$$

Consequently, the solution of system (17) departing from $\Psi$ cannot intersect $\partial \Psi_{1} \cup \partial \Psi_{2}$. If $\left(u_{2}\left(s_{0}\right), u_{3}\left(s_{0}\right), z\left(s_{0}\right)\right) \in \partial \Psi_{3}$, then $u_{2}^{\prime}\left(s_{0}\right)=g_{1}\left(u_{3}\left(s_{0}\right)\right) /(b c)>0$. Since $E_{1}$ is equilibrium, in summary, $\Psi$ is positive invariant.

Since $\lambda_{1}>\lambda_{2}>0$, stable manifold theorem implies that there exists a one-dimensional strong unstable manifold $\mathscr{W}_{1}$ tangent to $e_{1}$ at $E_{1}$ such that the point on $\mathscr{W}_{1}$ near $E_{1}$ can be expressed by

$$
\begin{equation*}
G_{1}(\varepsilon)=\varepsilon e_{1}+o(\varepsilon) \tag{47}
\end{equation*}
$$

Furthermore, there is a two-dimensional unstable manifold $\mathscr{W}_{2}$ tangent to span $\left\{e_{1}, e_{2}\right\}$ at $E_{1}$ such that $\mathscr{W}_{2}$ near $E_{1}$ can be expressed by

$$
\begin{equation*}
G_{2}\left(\varepsilon_{1}, \varepsilon_{2}\right)=\varepsilon_{1} e_{1}+\varepsilon_{2} e_{2}+o\left(\sqrt{\varepsilon_{1}^{2}+\varepsilon_{2}^{2}}\right) \tag{48}
\end{equation*}
$$

Lemma 9. Suppose that $u(s) \triangleq\left(u_{2}(s), u_{3}(s), z(s)\right)$ is a solution of (17) such that $u(0) \in \mathscr{W}_{1}$ for small $\varepsilon>0$. Then, $u(s)$ will leave $W$ and enter $P$.

Proof. Obviously, $u(s)$ satisfies initial condition (39) by the structure of $e_{1}$, and Lemma 8 implies $u(s)>0(u(s)>0$ means that $u_{i}(s)>0$ and $\left.z(s)>0, i=2,3\right)$ for every $s>0$.

Furthermore, Lemma 8 shows that $u_{3}^{\prime}(s)=z(s)>k u_{3}(s)$, implying $\lim _{s \rightarrow+\infty} u_{3}(s)=+\infty$. Since $u_{2}(s)<1 / b$, it follows
$\lim _{s \rightarrow+\infty} z(s)=+\infty$. Suppose that $u_{2}(s)<u^{*}$ for every $s>0$. Then

$$
\begin{align*}
u_{2}^{\prime} & >\frac{\left[\left(\left(f_{1}\left(u^{*}\right) / u^{*}\right)+\left(g_{1}\left(u_{3}\right) / u^{*}\right)\right)\left((1 / b)-u^{*}\right)-b\right] u_{2}}{c} \\
& >\frac{\left[\left(\left(f_{1}\left(u^{*}\right) / u^{*}\right)+\left(g_{1}\left(2 u^{*}\right) / u^{*}\right)\right)\left((1 / b)-u^{*}\right)-b\right] u_{2}}{c} \\
& =\frac{M u_{2}}{c}>0 \tag{49}
\end{align*}
$$

for large $s$ since $u_{3}(s)$ and $g_{1}\left(u_{3}\right)$ are strictly monotonous increasing with respect to $s$ and $u_{3}$, respectively. Thus, we have that $\lim _{s \rightarrow+\infty} u_{2}(s)=+\infty$, contradicting $u_{2}(s)<1 / b$ for any $s \in R$. Therefore, there exists $s_{1}>0$ such that $u_{2}\left(s_{1}\right)=u^{*}$. Without losing generality, let $s_{1}=\inf \left\{s>0: u_{2}(s)=u^{*}\right\}$. Obviously, we have $u_{2}^{\prime}\left(s_{1}\right) \geq 0$. If $u_{3}\left(s_{1}\right)<u^{*}$, then

$$
\begin{align*}
& u_{2}^{\prime}\left(s_{1}\right) \\
& =\frac{\left[\left(\left(f_{1}\left(u^{*}\right) / u^{*}\right)+\left(g_{1}\left(u_{3}\left(s_{1}\right)\right) / u^{*}\right)\right)\left((1 / b)-u^{*}\right)-b\right] u^{*}}{c} \\
& <0, \tag{50}
\end{align*}
$$

which is a contradiction. Therefore, $u_{3}\left(s_{1}\right) \geq u^{*}$ and $u\left(s_{1}\right) \in$ $\partial P$. Then, the construction of $W^{-}$shows that $u(s)$ will leave $W$ and enter $P$.

Let $C$ be a small circle on $\mathscr{W}_{2}$ centered at $E_{1}$. Then, points on $C$ can be expressed in terms of local coordinate by

$$
\begin{equation*}
F(\theta) \triangleq G_{2}(\varepsilon \cos \theta, \varepsilon \sin \theta)=\varepsilon\left[e_{1} \cos \theta+e_{2} \sin \theta+O(\varepsilon)\right] \tag{51}
\end{equation*}
$$

where $\theta \in\left[\theta_{1}, 2 \pi+\theta_{1}\right), \varepsilon>0$, and $\theta_{1}$ is chosen such that $F\left(\theta_{1}\right)$ lies on $\mathscr{W}_{1}$ with $z>0$. Then, stable manifold theorem shows that $\theta_{1} \rightarrow 0$ when $\varepsilon \rightarrow 0$. Denote $F(\theta) \triangleq$ $\left(\bar{u}_{2}(\theta), \bar{u}_{3}(\theta), \bar{z}(\theta)\right)$.

Lemma 10. There exists a $\theta_{2} \in(\pi / 2,3 \pi / 4)$ such that

$$
\begin{equation*}
\bar{z}\left(\theta_{2}\right)=0, \quad 0<\bar{u}_{3}\left(\theta_{2}\right)<\bar{u}_{2}\left(\theta_{2}\right)<u^{*} \tag{52}
\end{equation*}
$$

and that

$$
\begin{equation*}
\bar{z}(\theta)>0, \quad 0<\bar{u}_{2}\left(\theta_{2}\right)<u^{*}, \quad 0<\bar{u}_{3}(\theta)<u^{*} \tag{53}
\end{equation*}
$$

for $\theta \in\left[\theta_{1}, \theta_{2}\right)$.
Proof. From (51), we have

$$
\begin{align*}
\bar{z}(\theta)= & \varepsilon\left[\lambda_{1} \cos \theta+\lambda_{2} \sin \theta+O(\varepsilon)\right] \\
= & \varepsilon \sqrt{\lambda_{1}^{2}+\lambda_{2}^{2}} \\
& \times\left[\frac{\lambda_{1}}{\sqrt{\lambda_{1}^{2}+\lambda_{2}^{2}}} \cos \theta+\frac{\lambda_{2}}{\sqrt{\lambda_{1}^{2}+\lambda_{2}^{2}}} \sin \theta+O(\varepsilon)\right] \\
= & \varepsilon \sqrt{\lambda_{1}^{2}+\lambda_{2}^{2}}\left[\sin \left(\varphi_{0}+\theta\right)+O(\varepsilon)\right] \tag{54}
\end{align*}
$$

where $\sin \left(\varphi_{0}\right)=\lambda_{1} / \sqrt{\lambda_{1}^{2}+\lambda_{2}^{2}}, \cos \left(\varphi_{0}\right)=\lambda_{2} / \sqrt{\lambda_{1}^{2}+\lambda_{2}^{2}}$, and $\varphi_{0} \in(\pi / 4, \pi / 2)$ since $\lambda_{1}>\lambda_{2}$. Therefore, $\bar{z}\left(\theta_{2}\right)=0$ and $\theta_{2} \in$ $[0, \pi]$ imply that $\theta_{2}=\pi-\varphi_{0}+O(\varepsilon) \in(\pi / 2,3 \pi / 4)$. Obviously, $\bar{z}(\theta)>0$ for any $\theta \in\left[\theta_{1}, \theta_{2}\right)$. However,

$$
\begin{gather*}
\bar{u}_{2}(\theta)=\frac{\varepsilon g_{1}^{\prime}(0)}{b c}\left[\frac{\cos \theta}{\lambda_{1}-\lambda^{*}}+\frac{\sin \theta}{\lambda_{2}-\lambda^{*}}+O(\varepsilon)\right] \\
=\frac{\varepsilon}{m}\left[\left(-\lambda_{1}^{2}+\lambda_{1} c+m\right) \cos \theta\right. \\
\left.\quad+\left(-\lambda_{2}^{2}+\lambda_{2} c+m\right) \sin \theta+O(\varepsilon)\right], \\
\bar{u}_{3}(\theta)=\varepsilon[\cos \theta+\sin \theta+O(\varepsilon)], \\
\bar{u}_{2}(\theta)-\bar{u}_{3}(\theta)=\frac{\varepsilon}{m}\left[\lambda_{1}\left(c-\lambda_{1}\right) \cos \theta+\lambda_{2}\left(c-\lambda_{2}\right) \sin \theta+O(\varepsilon)\right] . \tag{55}
\end{gather*}
$$

Then, equality $\bar{z}\left(\theta_{2}\right)=\varepsilon\left[\lambda_{1} \cos \theta_{2}+\lambda_{2} \sin \theta_{2}+O(\varepsilon)\right]=0$, together with the last of (55), reveals $\bar{u}_{2}\left(\theta_{2}\right)-\bar{u}_{3}\left(\theta_{2}\right)=\varepsilon\left[\left(\lambda_{1}-\right.\right.$ $\left.\left.\lambda_{2}\right) \lambda_{2} \sin \theta_{2}+O(\varepsilon)\right] / m>0$; that is, $\bar{u}_{2}\left(\theta_{2}\right)>\bar{u}_{3}\left(\theta_{2}\right)$. For $\theta \in$ $\left[\theta_{1}, \theta_{2}\right]$, the first and second equalities of (55) imply that $0<$ $\bar{u}_{i}(\theta)<u^{*}$ where $i=2,3$ since $\lambda_{1}>\lambda_{2}$ and $0<\varepsilon \ll 1$.

Let

$$
\begin{equation*}
\Sigma=\left\{F(\theta) \mid \theta \in\left[\theta_{1}, \theta_{2}\right], \varepsilon \text { is small enough }\right\} . \tag{56}
\end{equation*}
$$

By Lemma 10, $\Sigma$ is an arc of circle, $\Sigma \subseteq W$, and the solution of (17) with initial value being the endpoint $F\left(\theta_{2}\right)$ will enter $Q$ since $F\left(\theta_{2}\right) \in W^{-} \cap \partial Q$. From Lemma 9, the solution of (17) with initial value being the endpoint $F\left(\theta_{1}\right)$ will enter $P$.

### 5.2. Traveling Wave Solution for $c>c^{*}$

Lemma 11. Let $u(s)=\left(u_{2}(s), u_{3}(s), z(s)\right)$ be a solution of (17) such that $u(0) \in \Lambda$. If $u(s) \in W$ for any $s \geq 0$, then $u(s) \in \Lambda$ for anys $>0$, where

$$
\begin{equation*}
\Lambda=\left\{\left(u_{2}, u_{3}, z\right): 0<u_{2}<u^{*}, u_{3}>0,0<z<k u_{3}\right\} \tag{57}
\end{equation*}
$$

and $k=c+\sqrt{c^{2}+4 m}$.
Proof. Set $s_{0}=\inf \{s: u(s) \notin \Lambda, s \geq 0\}$. Suppose the conclusion is false; that is, $s_{0}<+\infty$. Obviously, $s_{0}>0$ and $u\left(s_{0}\right) \in$ $\partial \Lambda$ where

$$
\begin{align*}
\partial \Lambda & =\left(\cup_{i=1}^{7} \partial \Lambda_{i}\right) \cup E_{2}, \\
\partial \Lambda_{1} & =\left\{\left(u_{2}, u_{3}, z\right): u_{2}=u^{*}, u_{3} \geq u^{*}, 0 \leq z \leq k u_{3}\right\} \backslash E_{2}, \\
\partial \Lambda_{2} & =\left\{\left(u_{2}, u_{3}, z\right): 0<u_{3} \leq u_{2} \leq u^{*}, z=0\right\} \backslash E_{2}, \\
\partial \Lambda_{3} & =\left\{\left(u_{2}, u_{3}, z\right): u_{2}=0, u_{3}>0,0 \leq z \leq k u_{3}\right\}, \\
\partial \Lambda_{4} & =\left\{\left(u_{2}, u_{3}, z\right): u_{2}=u^{*}, 0<u_{3}<u^{*}, 0<z \leq k u_{3}\right\}, \\
\partial \Lambda_{5} & =\left\{\left(u_{2}, u_{3}, z\right): 0 \leq u_{2} \leq u^{*}, u_{3}>0, z=k u_{3}\right\}, \\
\partial \Lambda_{6} & =\left\{\left(u_{2}, u_{3}, z\right): u_{3}>u_{2}, 0 \leq u_{2}<u^{*}, u_{3}>0, z=0\right\}, \\
\partial \Lambda_{7} & =\left\{\left(u_{2}, u_{3}, z\right): 0 \leq u_{2} \leq u^{*}, u_{3}=z=0\right\} . \tag{58}
\end{align*}
$$



Figure 3: The construction of $\partial \Lambda$.

In Figure 3, we find $\partial \Lambda_{1}=\left\{\right.$ unbounded area $\left.B E_{2} C D\right\}$, $\partial \Lambda_{2}=$ \{triangle $\left.O A E_{2} O\right\}, \partial \Lambda_{3}=$ \{unbounded cone $\left.u_{3} O E\right\}, \partial \Lambda_{4}=\left\{\right.$ triangle $\left.A E_{2} C A\right\}, \partial \Lambda_{5}=\{$ unbounded area $D C A O E\}, \partial \Lambda_{6}=\left\{\right.$ unbounded area $\left.B E_{2} O u_{3}\right\}$, and $\partial \Lambda_{7}=$ \{segment $O A\}$.

Since $\partial \Lambda_{1} \cup \partial \Lambda_{2} \subset W^{-}$, thus $u\left(s_{0}\right) \notin \partial \Lambda_{1} \cup \partial \Lambda_{2}$. If $u\left(s_{0}\right) \in$ $\partial \Lambda_{3}$, we have $u_{2}^{\prime}\left(s_{0}\right) \leq 0$ because $u_{2}(s)>0$ for $0<s<s_{0}$ and $u_{2}\left(s_{0}\right)=0$. However, $u_{2}^{\prime}\left(s_{0}\right)=g_{1}\left(u_{3}\left(s_{0}\right)\right) /(b c)>0$ which is a contradiction. Therefore, $u\left(s_{0}\right) \notin \partial \Lambda_{3}$. If $u\left(s_{0}\right) \in \partial \Lambda_{4}$, then

$$
\begin{align*}
& u_{2}^{\prime}\left(s_{0}\right) \\
& \quad=\frac{\left[\left(\left(f_{1}\left(u^{*}\right) / u^{*}\right)+\left(g_{1}\left(u_{3}\left(s_{0}\right)\right) / u^{*}\right)\right)\left((1 / b)-u^{*}\right)-b\right] u^{*}}{c} \\
& \quad<0, \tag{59}
\end{align*}
$$

contradicting $u_{2}^{\prime}\left(s_{0}\right) \geq 0$. If $u\left(s_{0}\right) \in \partial \Lambda_{5}$, then

$$
\begin{align*}
& {\left[z(s)-k u_{3}(s)\right]_{s=s_{0}}^{\prime}} \\
& \quad=(c-k) z\left(s_{0}\right)+m\left[u_{3}\left(s_{0}\right)-u_{2}\left(s_{0}\right)\right]  \tag{60}\\
& \quad=[(c-k) k+m] u_{3}\left(s_{0}\right)-m u_{2}\left(s_{0}\right)<0
\end{align*}
$$

since $(c-k) k+m<0$, contradicting $\left[z(s)-k u_{3}(s)\right]_{s=s_{0}}^{\prime} \geq 0$. If $u\left(s_{0}\right) \in \partial \Lambda_{6}$, then $z^{\prime}\left(s_{0}\right)=m\left[u_{3}\left(s_{0}\right)-u_{2}\left(s_{0}\right)\right]>0$ which is a contradiction. In conclusion, $u\left(s_{0}\right) \notin \partial \Lambda_{4} \cup \partial \Lambda_{5} \cup \partial \Lambda_{6}$. If $u\left(s_{0}\right) \in \partial \Lambda_{7}$, then $u_{3}(s)>0$ and $z(s)>0$ for any $0<$ $s<s_{0}$. Hence, $u_{3}^{\prime}(s)=z(s)>0$ for any $0<s<s_{0}$, which implies that $u_{3}\left(s_{0}\right)>u_{3}(0)>0$. From this contradiction we find $u\left(s_{0}\right) \notin \partial \Lambda_{7}$. Because $E_{2}$ is a constant solution, we get $u\left(s_{0}\right) \neq E_{2}$. In summary, $u\left(s_{0}\right) \notin \partial \Lambda$ and $s_{0}=+\infty$. The proof is completed.

Lemma 12. There exists a point $u_{0}=\left(u_{20}, u_{30}, z_{0}\right) \in \Sigma$ such that the solution $u\left(s ; u_{0}\right)=\left(u_{2}(s), u_{3}(s), z(s)\right)$ of (17) with initial value being $u_{0}$ will stay in $W$ for any $s>0$.

Proof. It is sufficient to prove $\Sigma \neq \Sigma_{0}$. Suppose that $\Sigma=$ $\Sigma_{0}$. Firstly, we verify Conditions (1) and (2) of Lemma 6. Condition (1) of Lemma 6 is valid since $W$ is closed.

Suppose $u_{0}=\left(u_{20}, u_{30}, z_{0}\right) \in \Sigma, s<T\left(u_{0}\right)$ and $u\left(s ; u_{0}\right) \in$ $W \backslash W^{-}$. Then, $u\left(s ; u_{0}\right) \in \operatorname{int} W \cup J$ and $u_{0} \neq F\left(\theta_{2}\right)$ since $F\left(\theta_{2}\right) \in W^{-}$. The structure of $\Sigma$ implies that $u_{20}>0, u_{30}>0$, and $z_{0}>0$. By the proof of Lemma 11, we have that $u\left(s ; u_{0}\right)>$ 0 for $s<T\left(u_{0}\right)$. Therefore, $u\left(s ; u_{0}\right) \notin J$ and $u\left(s ; u_{0}\right) \in \operatorname{int} W$. Condition (2) of Lemma 6 holds.

Lemma 6 shows that $\Sigma$ is homeomorphic to $H(\Sigma)$. Since

$$
\begin{equation*}
H\left(F\left(\theta_{1}\right)\right) \in \partial P \cap W^{-}, \quad H\left(F\left(\theta_{2}\right)\right) \in \partial Q \cap W^{-} \tag{61}
\end{equation*}
$$

and $W^{-}$is disconnected, we have that $H(\Sigma)$ is disconnected, contradicting the connection of $\Sigma$. Thus, $\Sigma \neq \Sigma_{0}$ and the proof is completed.

Lemma 13. Let $c>c^{*}$. Then, there exists a positive solution $u(s)=\left(u_{2}(s), u_{3}(s), z(s)\right)$ of (17) such that

$$
\begin{equation*}
u(+\infty)=E_{2}, \quad u(-\infty)=E_{1} . \tag{62}
\end{equation*}
$$

Proof. By Lemma 12 there exists a point $u_{0}=\left(u_{20}, u_{30}, z_{0}\right) \in$ $\Sigma$ such that the solution $u\left(s ; u_{0}\right)=\left(u_{2}(s), u_{3}(s), z(s)\right)$ of (17) with initial value being $u_{0}$ will stay in $W$ for any $s>0$. Furthermore, Lemma 11 shows $u\left(s ; u_{0}\right)>0$ for any $s \geq 0$. Stable manifold theorem implies that $u\left(s ; u_{0}\right)>0$ for any $s \leq 0$ and $\lim _{s \rightarrow-\infty} u\left(s ; u_{0}\right)=E_{1}$. Therefore, $u\left(s ; u_{0}\right)$ is a positive solution.

To complete the proof, it is sufficient to show that $\lim _{s \rightarrow+\infty} u\left(s ; u_{0}\right)=E_{2}$. By Lemma 11, we know that $u_{2}(s)<$ $u^{*}$ for any $s>0$ since $u\left(s ; u_{0}\right)$ remains in $W$ for all $s$. Because $u_{3}^{\prime}(s)=z(s)>0$, then the limit of $u_{3}(s)$ exists; that is, $\lim _{s \rightarrow+\infty} u_{3}(s)=u_{3}^{*}$ and $0<u_{3}^{*} \leq+\infty$. Suppose that $u^{*}<u_{3}^{*} \leq+\infty$. The first equation of (17) shows that

$$
\begin{align*}
u_{2}^{\prime} & >\frac{\left[\left(\left(f_{1}\left(u^{*}\right) / u^{*}\right)+\left(g_{1}\left(u_{3}\right) / u^{*}\right)\right)\left((1 / b)-u^{*}\right)-b\right] u_{2}}{c} \\
& >\left(\left[\left(\frac{f_{1}\left(u^{*}\right)}{u^{*}}+\frac{g_{1}\left(\left(u_{3}^{*}+u^{*}\right) / 2\right)}{u^{*}}\right)\left(\frac{1}{b}-u^{*}\right)-b\right] u_{2}\right) \times(c)^{-1} \\
& =\frac{M u_{2}}{c}>0 \tag{63}
\end{align*}
$$

for large $s$, which implies that there is an $s^{*}>0$ such that $u_{2}\left(s^{*}\right)>u^{*}$. This is a contradiction, and thus $0<u_{3}^{*} \leq u^{*}$. From the first equation of (17), we have $\lim _{s \rightarrow+\infty} u_{2}(s)=u_{2}^{*}$ where $u_{2}^{*}$ is the only positive root of algebra equation

$$
\begin{equation*}
\left[f_{1}\left(u_{2}\right)+g_{1}\left(u_{3}^{*}\right)\right]\left(\frac{1}{b}-u_{2}\right)-b u_{2}=0 \tag{64}
\end{equation*}
$$

At the same time, the third equation of (17) implies $\lim _{s \rightarrow+\infty} z(s)=z^{*}$ and $z^{*}=m\left(u_{2}^{*}-u_{3}^{*}\right) / c$ or $\pm \infty$. It is impossible that $z^{*}= \pm \infty$ due to the boundedness of $u_{3}(s)$. In conclusion, the limit $\lim _{s \rightarrow+\infty} u(s)=\left(u_{2}^{*}, u_{3}^{*}, z^{*}\right)$ exists and is finite. By [42], $\left(u_{2}^{*}, u_{3}^{*}, z^{*}\right)$ must be equilibrium. Since $u_{3}^{*}>0$, then $\left(u_{2}^{*}, u_{3}^{*}, z^{*}\right)=E_{2}$.

Noticing the relation of systems (17) and (10) completes the proof of Theorem 1 for case $c>c^{*}$.
5.3. Traveling Wave Solution for $c=c^{*}$. Firstly, suppose $c>$ $c^{*}$ and let $u\left(s ; u_{0}\right)=\left(u_{2}(s), u_{3}(s), z(s)\right)$ be the traveling wave solution of (17). Then, Lemma 11 implies that $u\left(s ; u_{0}\right) \in \Lambda$ for all $s$. From the proof of Lemma 13, we find $u_{3}(s) \leq u^{*}$. Therefore, for all $s$, we have $u\left(s ; u_{0}\right) \in \Pi$ where

$$
\begin{equation*}
\Pi=\left\{\left(u_{2}, u_{3}, z\right): 0<u_{2}<u^{*}, 0<u_{3} \leq u^{*}, 0<z<k u_{3}\right\} . \tag{65}
\end{equation*}
$$

Let $\left\{c_{n}\right\}$ be a sequence such that $c^{*}<c_{n}<c_{n+1}$ for any $n$ and $\lim _{n \rightarrow \infty} c_{n}=c^{*}$. Set $k_{n}=c_{n}+\sqrt{c_{n}^{2}+4 m}$ and

$$
\begin{equation*}
\Pi_{n}=\left\{\left(u_{2}, u_{3}, z\right): 0<u_{2} \leq u^{*}, 0<u_{3} \leq u^{*}, 0<z \leq k_{n} u_{3}\right\} \tag{66}
\end{equation*}
$$

Then, $\Pi_{n} \subseteq \Pi_{1}$ for any $n$.
Lemma 13 shows that there is a positive solution $w_{n}(s)=$ ( $\left.u_{2, n}(s), u_{3, n}(s), z_{n}(s)\right)$ for system

$$
\begin{gather*}
u_{2}^{\prime}=\frac{\left[\left(f_{1}\left(u_{2}\right)+g_{1}\left(u_{3}\right)\right)\left((1 / b)-u_{2}\right)-b u_{2}\right]}{c_{n}} \\
u_{3}^{\prime}=z  \tag{67}\\
z^{\prime}=c_{n} z+m\left(u_{3}-u_{2}\right)
\end{gather*}
$$

satisfying boundary condition (62) such that $w_{n}(s) \in \Pi_{n} \subseteq$ $\Pi_{1}$ for any $s$.

Lemma 14. Let $c=c^{*}$. Then, there exists a traveling wave solution for system (6) satisfying boundary condition (11).

Proof. It is sufficient to prove that there exists a positive solution $u(s)=\left(u_{2}(s), u_{3}(s), z(s)\right)$ of (17) satisfying boundary condition (62).

Firstly, we show that sequences $\left\{u_{2, n}\right\},\left\{u_{3, n}\right\},\left\{z_{n}\right\},\left\{u_{2, n}^{\prime}\right\}$, $\left\{u_{3, n}^{\prime}\right\}$, and $\left\{z_{n}^{\prime}\right\}$ are uniformly bounded and equicontinuous. The idea of Lemma 11 in [34] is used. Obviously, $\left\{u_{2, n}\right\},\left\{u_{3, n}\right\}$, and $\left\{z_{n}\right\}$ are uniformly bounded since $w_{n}(s) \subseteq \Pi_{1}$ for any $s$. Because $w_{n}(s)=\left(u_{2, n}(s), u_{3, n}(s), z_{n}(s)\right)$ is the solution of (67), $\left\{u_{2, n}^{\prime}\right\},\left\{u_{3, n}^{\prime}\right\}$, and $\left\{z_{n}^{\prime}\right\}$ are also uniformly bounded. Since $\left|z_{n}\left(s_{1}\right)-z_{n}\left(s_{2}\right)\right|=z_{n}^{\prime}\left(s_{3}\right)\left|s_{1}-s_{2}\right|$ where $s_{1}<s_{3}<s_{2}$, then $\left\{z_{n}\right\}$ is equicontinuous. Similarly, $\left\{u_{2, n}\right\}$ and $\left\{u_{3, n}\right\}$ are also equicontinuous. By differentiating the equations of (67) and using the previous bounds, we can get that $\left\{u_{2, n}^{\prime \prime}\right\},\left\{u_{3, n}^{\prime \prime}\right\}$, and $\left\{z_{n}^{\prime \prime}\right\}$ are uniformly bounded, and hence $\left\{u_{2, n}^{\prime}\right\},\left\{u_{3, n}^{\prime}\right\}$, and $\left\{z_{n}^{\prime}\right\}$ are equicontinuous.

The previous paragraph and Arzelà-Ascoli theorem imply that there exist subsequences, again denoted by $\left\{u_{2, n}\right\},\left\{u_{3, n}\right\}$, and $\left\{z_{n}\right\}$ and functions $u_{2}, u_{3}$, and $z$ such that

$$
\begin{equation*}
u_{2, n} \longrightarrow u_{2}, \quad u_{3, n} \longrightarrow u_{3}, \quad z_{n} \longrightarrow z \tag{68}
\end{equation*}
$$

uniformly on compact subsets of $\mathbb{R}$, thus pointwise on $\mathbb{R}$. Same arguments imply that $\left\{u_{2, n}^{\prime}\right\},\left\{u_{3, n}^{\prime}\right\}$, and $\left\{z_{n}^{\prime}\right\}$ are also uniformly convergent on compact subsets of $\mathbb{R}$ and pointwise convergent on $\mathbb{R}$. Consequently, we get

$$
\begin{equation*}
u_{2, n}^{\prime} \longrightarrow u_{2}^{\prime}, \quad u_{3, n}^{\prime} \longrightarrow u_{3}^{\prime}, \quad z_{n}^{\prime} \longrightarrow z^{\prime} \tag{69}
\end{equation*}
$$



Figure 4: The wave profiles for $S$ and $I$ and their movements.

Since $\left(u_{2, n}, u_{3, n}, z_{n}\right)$ is the solution of (67), then $u(s)=$ $\left(u_{2}(s), u_{3}(s), z(s)\right)$ is the solution of (17) for $c=c^{*}$ and $u(s) \in$ $\mathrm{cl}\left(\Pi_{1}\right)$, where $\mathrm{cl}\left(\Pi_{1}\right)$ is the closer of $\Pi_{1}$. Because system (67) is autonomous and ( $u_{2, n}, u_{3, n}, z_{n}$ ) satisfies boundary condition (62), we can assume that $u_{3, n}(0)=u^{*} / 2$ for all $n$; thus, $u_{3}(0)>$ 0 . Then, similar to the proof of Lemma 13, we have that the solution $u(s)$ satisfies boundary condition (62).

## 6. Simulations

In this section, we present some simulations to confirm the theoretical results. Set

$$
\begin{equation*}
f(I)=\frac{\beta_{1} I}{K_{h}+I}, \quad g(B)=\frac{\beta_{2} B}{K_{e}+B} \tag{70}
\end{equation*}
$$

and assign numerical values to parameters as follows:

$$
\begin{gather*}
b=0.01, \quad e=1, \quad m=0.5, \quad K_{e}=6, \quad K_{h}=2, \\
\beta_{1}=0.62, \quad \beta_{2}=0.001, \quad N=200, \quad d=2 . \tag{71}
\end{gather*}
$$

Obviously, such selection for $f(I)$ and $g(B)$ satisfies (A1) and (A2). Then, the traveling wave solution is described in Figure 4.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The authors are supported by the Fundamental Research Funds for the Central Universities (Grants nos. XDJK2012C042 and SWU113048) and NSFC (Grant no. 11201380).

## References

[1] Centers for Disease Control and Prevention, USA, http://www .cdc.gov/cholera/general/.
[2] World Health Organization, http://www.who.int/csr/don/ archive/disease/cholera/en/.
[3] V. Capasso and S. L. Paveri-Fontana, "Mathematical model for the 1973 cholera epidemic in the European Mediterranean region," Revue d'Epidemiologie et de Sante Publique, vol. 27, no. 2, pp. 121-132, 1979.
[4] C. T. Codeço, "Endemic and epidemic dynamics of cholera: the role of the aquatic reservoir," BMC Infectious Diseases, vol. 1, no. 1, article 1, 2001.
[5] J. R. Andrews and S. Basu, "Transmission dynamics and control of cholera in Haiti: an epidemic model," The Lancet, vol. 377, no. 9773, pp. 1248-1255, 2011.
[6] K. T. Goh, S. H. Teo, S. Lam, and M. K. Ling, "Person-to-person transmission of cholera in a psychiatric hospital," Journal of Infection, vol. 20, no. 3, pp. 193-200, 1990.
[7] A. A. Weil, A. I. Khan, F. Chowdhury et al., "Clinical outcomes in household contacts of patients with cholera in Bangladesh," Clinical Infectious Diseases, vol. 49, no. 10, pp. 1473-1479, 2009.
[8] J. H. Tien and D. J. D. Earn, "Multiple transmission pathways and disease dynamics in a waterborne pathogen model," Bulletin of Mathematical Biology, vol. 72, no. 6, pp. 1506-1533, 2010.
[9] Z. Mukandavire, S. Liao, J. Wang, H. Gaff, D. L. Smith, and J. G. Morris Jr., "Estimating the reproductive numbers for the 2008-2009 cholera outbreaks in Zimbabwe," Proceedings of the National Academy of Sciences of the United States of America, vol. 108, no. 21, pp. 8767-8772, 2011.
[10] J. P. Tian and J. Wang, "Global stability for cholera epidemic models," Mathematical Biosciences, vol. 232, no. 1, pp. 31-41, 2011.
[11] D. M. Hartley, J. G. Morris Jr., and D. L. Smith, "Hyperinfectivity: a critical element in the ability of V. cholerae to cause epidemics?" PLoS Medicine, vol. 3, no. 1, pp. 63-69, 2006.
[12] Z. Shuai and P. van den Driessche, "Global dynamics of cholera models with differential infectivity," Mathematical Biosciences, vol. 234, no. 2, pp. 118-126, 2011.
[13] Z. Shuai, J. H. Tien, and P. van den Driessche, "Cholera models with hyperinfectivity and temporary immunity", Bulletin of Mathematical Biology, vol. 74, no. 10, pp. 2423-2445, 2012.
[14] M. A. Jensen, S. M. Faruque, J. J. Mekalanos, and B. R. Levin, "Modeling the role of bacteriophage in the control of cholera outbreaks," Proceedings of the National Academy of Sciences of the United States of America, vol. 103, no. 12, pp. 4652-4657, 2006.
[15] R. P. Sanches, C. P. Ferreira, and R. A. Kraenkel, "The role of immunity and seasonality in cholera epidemics," Bulletin of Mathematical Biology, vol. 73, no. 12, pp. 2916-2931, 2011.
[16] A. Mutreja, D. W. Kim, N. R. Thomson et al., "Evidence for several waves of global transmission in the seventh cholera pandemic," Nature, vol. 477, no. 7365, pp. 462-465, 2011.
[17] R. R. Colwell, "Global climate and infectious disease: the cholera paradigm," Science, vol. 274, no. 5295, pp. 2025-2031, 1996.
[18] S. M. Faruque, M. J. Islam, Q. S. Ahmad et al., "Self-limiting nature of seasonal cholera epidemics: role of host-mediated amplification of phage," Proceedings of the National Academy of Sciences of the United States of America, vol. 102, no. 17, pp. 61196124, 2005.
[19] S. Anița and V. Capasso, "Stabilization of a reaction-diffusion system modelling a class of spatially structured epidemic systems via feedback control," Nonlinear Analysis. Real World Applications, vol. 13, no. 2, pp. 725-735, 2012.
[20] V. Capasso, "Asymptotic stability for an integro-differential reaction-diffusion system," Journal of Mathematical Analysis and Applications, vol. 103, no. 2, pp. 575-588, 1984.
[21] V. Capasso and K. Kunisch, "A reaction-diffusion system arising in modelling man-environment diseases," Quarterly of Applied Mathematics, vol. 46, no. 3, pp. 431-450, 1988.
[22] V. Capasso and L. Maddalena, "Convergence to equilibrium states for a reaction-diffusion system modelling the spatial spread of a class of bacterial and viral diseases," Journal of Mathematical Biology, vol. 13, no. 2, pp. 173-184, 1981-1982.
[23] V. Capasso and R. E. Wilson, "Analysis of a reaction-diffusion system modeling man-environment-man epidemics," SIAM Journal on Applied Mathematics, vol. 57, no. 2, pp. 327-346, 1997.
[24] E. Bertuzzo, S. Azaele, A. Maritan, M. Gatto, I. RodriguezIturbe, and A. Rinaldo, "On the space-time evolution of a cholera epidemic," Water Resources Research, vol. 44, no. 1, Article ID W01424, 2008.
[25] E. Bertuzzo, R. Casagrandi, M. Gatto, I. Rodriguez-Iturbe, and A. Rinaldo, "On spatially explicit models of cholera epidemics," Journal of the Royal Society Interface, vol. 7, no. 43, pp. 321-333, 2010.
[26] L. Mari, E. Bertuzzo, L. Righetto et al., "Modelling cholera epidemics: the role of waterways, human mobility and sanitation," Journal of the Royal Society Interface, vol. 9, no. 67, pp. 376-388, 2012.
[27] O. Diekmann, "Thresholds and travelling waves for the geographical spread of infection," Journal of Mathematical Biology, vol. 6, no. 2, pp. 109-130, 1978.
[28] M. Lewis, J. Rencławowicz, and P. van den Driessche, "Traveling waves and spread rates for a West Nile virus model," Bulletin of Mathematical Biology, vol. 68, no. 1, pp. 3-23, 2006.
[29] J. Radcliffe and L. Rass, "The asymptotic speed of propagation of the deterministic nonreducible $n$-type epidemic," Journal of Mathematical Biology, vol. 23, no. 3, pp. 341-359, 1986.
[30] X.-Q. Zhao and W. Wang, "Fisher waves in an epidemic model," Discrete and Continuous Dynamical Systems B, vol. 4, no. 4, pp. 1117-1128, 2004.
[31] D. Xu and X.-Q. Zhao, "Bistable waves in an epidemic model," Journal of Dynamics and Differential Equations, vol. 16, no. 3, pp. 679-707, 2004.
[32] Y. Jin and X.-Q. Zhao, "Bistable waves for a class of cooperative reaction-diffusion systems," Journal of Biological Dynamics, vol. 2, no. 2, pp. 196-207, 2008.
[33] C.-H. Hsu and T.-S. Yang, "Existence, uniqueness, monotonicity and asymptotic behaviour of travelling waves for epidemic models," Nonlinearity, vol. 26, no. 1, pp. 121-139, 2013.
[34] S. R. Dunbar, "Travelling wave solutions of diffusive LotkaVolterra equations," Journal of Mathematical Biology, vol. 17, no. 1, pp. 11-32, 1983.
[35] S. R. Dunbar, "Traveling wave solutions of diffusive LotkaVolterra equations: a heteroclinic connection in $R^{4}$," Transactions of the American Mathematical Society, vol. 286, no. 2, pp. 557-594, 1984.
[36] W.-T. Li and S.-L. Wu, "Traveling waves in a diffusive predatorprey model with Holling type-III functional response," Chaos, Solitons \& Fractals, vol. 37, no. 2, pp. 476-486, 2008.
[37] C.-H. Hsu, C.-R. Yang, T.-H. Yang, and T.-S. Yang, "Existence of traveling wave solutions for diffusive predator-prey type systems," Journal of Differential Equations, vol. 252, no. 4, pp. 3040-3075, 2012.
[38] W. Huang, "Traveling wave solutions for a class of predator-prey systems," Journal of Dynamics and Differential Equations, vol. 24, no. 3, pp. 633-644, 2012.
[39] J. Huang, G. Lu, and S. Ruan, "Existence of traveling wave solutions in a diffusive predator-prey model," Journal of Mathematical Biology, vol. 46, no. 2, pp. 132-152, 2003.
[40] X. Lin, P. Weng, and C. Wu, "Traveling wave solutions for a predator-prey system with sigmoidal response function," Journal of Dynamics and Differential Equations, vol. 23, no. 4, pp. 903-921, 2011.
[41] R. S. Irving, Integers, Polynomials, and Rings, Springer, New York, NY, USA, 2004.
[42] Z. Ma and Y. Zhou, Qualitative and Stable Method of Ordinary Differential Equations, Science Press, Beijing, China, 2005.

## Research Article

# Dynamics of a Stochastic Functional System for Wastewater Treatment 

Xuehui Ji and Sanling Yuan<br>College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China<br>Correspondence should be addressed to Sanling Yuan; sanling@usst.edu.cn

Received 20 January 2014; Accepted 2 February 2014; Published 24 March 2014
Academic Editor: Weiming Wang
Copyright © 2014 X. Ji and S. Yuan. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

The dynamics of a delayed stochastic model simulating wastewater treatment process are studied. We assume that there are stochastic fluctuations in the concentrations of the nutrient and microbes around a steady state, and introduce two distributed delays to the model describing, respectively, the times involved in nutrient recycling and the bacterial reproduction response to nutrient uptake. By constructing Lyapunov functionals, sufficient conditions for the stochastic stability of its positive equilibrium are obtained. The combined effects of the stochastic fluctuations and delays are displayed.


## 1. Introduction

In the last few years, the use of mathematical models describing wastewater treatment is gaining attention as a promising method [1-6]. A basic chemostat model describing substrate-microbe interaction in an activated sludge process is as follows:

$$
\begin{gather*}
\frac{d S}{d t}=\frac{\mathrm{Q}\left(S^{0}-S\right)}{V}-\frac{k x S}{K_{S}+S} \frac{D_{\mathrm{O}}}{K_{\mathrm{O}}+D_{\mathrm{O}}},  \tag{1}\\
\frac{d x}{d t}=x\left(\frac{k Y S}{K_{S}+S}-K_{d}\right) \frac{D_{\mathrm{O}}}{K_{\mathrm{O}}+D_{\mathrm{O}}}-\frac{\mathrm{Q}_{w} x}{V},
\end{gather*}
$$

where $S(t)$ and $x(t)$ represent the concentrations of the substrate (biochemical oxygen demand) and microbes in an aeration tank at time $t$, respectively. $Q$ is the washout rate, $S^{0}$ is the input concentration of the substrate, and $V$ is the effective volume of the aeration tank; $k$ is the maximum uptake rate of the substrate; $K_{S}$ and $K_{\mathrm{O}}$ are the half-saturation constants of the substrate and oxygen; respectively, $K_{d}$ is the decay rate of microbes and $Q_{w}$ is the emission rate of the sludge; $D_{\mathrm{O}}$ is the concentration of the dissolved oxygen and $D_{\mathrm{O}} /\left(K_{\mathrm{O}}+D_{\mathrm{O}}\right)$ is a switching function describing the effect of $D_{\mathrm{O}}$ on the uptake rate $k$ and the decay rate $K_{d} ; Y \in(0,1)$ is the ratio of the concentration of mixed liquor suspended solids to the
substrate. Some extensions and generalizations of the model have been proposed by many researchers (see [7-27], etc.).

Even though deterministic model (1) has a stable positive equilibrium ( $S^{*}, x^{*}$ ) under certain conditions, oscillations have been observed frequently in the growth of microbes during the experiments [28, 29], which have also been confirmed by many mathematical works for some extended chemostat models incorporating factors such as time delay [15-18, 3032], periodic nutrient input [19-21, 33-35], feedback control [22-24], and stochastic environmental perturbations [2527]. For a better understanding of microbial population dynamics in the activated sludge process, we take two steps towards developing model (1).

On the one hand, we take into account time delays that may exist in the process of wastewater treatment. By the death regeneration theory of Dold and Marais [36], the active biomass dies at a certain rate; of the biomass lost, the biodegradable portion adds to the slowly biodegradable organic matter which passes through the various stages to be utilised for active biomass synthesis, which requires some time for the completion of the regeneration. Also there is a time delay that accounts for the time lapse between the uptakes of substrates and the incorporation of these substrates, which has ever been observed from chemostat experiments with microalgae Chlamidomonas Reinhardii even when the limiting nutrient is at undetectable small
concentration (see [37, 38], etc.). In the recent years, chemostat models with such time delays have been given much attention (see, e.g., $[9,14,16-18,39]$, etc.). In this paper, we will use distributed delays to describe the nutrient recycling and the time lapse between the uptakes of nutrient and the incorporation of this nutrient with delay kernels $f(s)$ and $g(s)$, respectively.

On the other hand, in a real process of wastewater treatment there will be fluctuations in concentration of the substrate and microbe population due to stochastic perturbations from external sources such as temperature, light, and the like, or inherent sources in the chemical-physical and biological processes [40]. So we assume that model (1) is exposed by stochastic perturbations which are of white noise type and are proportional to the distances $S(t), x(t)$ from values of the positive equilibrium $S^{*}, x^{*}$, influence on the $\dot{S}(t)$ and $\dot{x}(t)$, respectively. By this way, model (1) becomes in the following form:

$$
\begin{align*}
d \mathrm{~S}= & {\left[\frac{Q\left(S^{0}-S\right)}{V}-k U(S) \frac{D_{\mathrm{O}}}{K_{\mathrm{O}}+D_{\mathrm{O}}}\right.} \\
& \left.+\mu K_{d} \frac{D_{\mathrm{O}}}{K_{\mathrm{O}}+D_{\mathrm{O}}} \int_{0}^{\infty} f(s) x(t-s) d s\right] d t \\
& +\sigma_{1}\left(S-S^{*}\right) d B_{1}(t)  \tag{2}\\
d x= & {\left[x\left(Y k \int_{0}^{\infty} g(s) U(S(t-s)) d s-K_{d}\right)\right.} \\
& \left.\times \frac{D_{\mathrm{O}}}{K_{\mathrm{O}}+D_{\mathrm{O}}}-\frac{Q_{w} x}{V}\right] d t+\sigma_{2}\left(x-x^{*}\right) d B_{2}(t),
\end{align*}
$$

where $B_{i}(t)(i=1,2)$ are standard independent Wiener processes and $\sigma_{i} \geq 0(i=1,2)$ represent the intensities of the noises. $\mu \in(0,1)$ is the fraction of the substrate regenerated from the dead biomass; $U(S)$ is a general specific growth function.

Recently, stochastic biological systems and stochastic epidemic models have been studied by many authors; see, for example, Mao et al. [41, 42], Jiang et al. [43, 44], Liu and Wang [45, 46], and the references cited therein. But, as far as we know, there are few works on model (2). In this paper, our main purpose is to study the combined effect of the noises and delays on the dynamics of model (2), that is, whether and how the noises and delays affect the stability of $E^{*}$. By the construction of appropriate Lyapunov functionals, we will show that the positive equilibrium keeps stochastically stable if the noises and delays are small. Furthermore, the sensitivities of the stability of $E^{*}$ with respect to the delays and noises are also discussed.

The paper is organized as follows. We first establish some preliminary results in Section 2. By constructing Lyapunov function(al)s, sufficient conditions for the stochastic stability of the positive equilibrium of the model without and with delays are obtained, respectively, in Sections 3 and 4. Numerical simulations and discussions are finally presented in Section 5.

## 2. Some Preliminaries

Define $\mathrm{Q} / V=D, \mathrm{Q}_{w} / V=D_{w}, k\left(D_{\mathrm{O}} /\left(K_{\mathrm{O}}+D_{\mathrm{O}}\right)\right)=m$, $K_{d}\left(D_{\mathrm{O}} /\left(K_{\mathrm{O}}+D_{\mathrm{O}}\right)\right)=D_{1}$, and $Y=\gamma$. Then model (2) can be simplified as follows:

$$
\begin{align*}
d S= & {\left[D\left(S^{0}-S\right)-m U(S) x\right.} \\
& \left.+\mu D_{1} \int_{0}^{\infty} f(s) x(t-s) d s\right] d t \\
& +\sigma_{1}\left(S-S^{*}\right) d B_{1}, \\
d x=x & {\left[-\left(D_{w}+D_{1}\right)+\gamma m \int_{0}^{\infty} g(s) U(S(t-s)) d s\right] d t } \\
& +\sigma_{2}\left(x-x^{*}\right) d B_{2} \tag{3}
\end{align*}
$$

with initial value conditions

$$
\begin{array}{r}
S(\theta, \omega)=\varphi_{1}(\theta) \geq 0, \quad x(\theta, \omega)=\varphi_{2}(\theta) \geq 0 \\
\theta \in(-\infty, 0] \tag{4}
\end{array}
$$

where $\varphi_{1}(\theta), \varphi_{2}(\theta) \in \mathscr{B} \mathscr{C}\left((-\infty, 0], \mathbb{R}_{+}\right)$, the families of bounded continuous functions from $(-\infty, 0]$ to $\mathbb{R}_{+}$.

The corresponding deterministic model of (3) is

$$
\begin{gather*}
\dot{S}=D\left(S^{0}-S\right)-m U(S) x+\mu D_{1} \int_{0}^{\infty} f(s) x(t-s) d s  \tag{5}\\
\dot{x}=-\left(D_{w}+D_{1}\right) x+\gamma m x \int_{0}^{\infty} g(s) U(S(t-s)) d s
\end{gather*}
$$

the special case of which when $D=D_{w}$ has ever been investigated by He et al. [18]. It is easy to see that model (5) has a positive equilibrium $E^{*}\left(S^{*}, x^{*}\right)$ provided that

$$
\begin{equation*}
D_{w}+D_{1}<\gamma m, \quad S^{0}>S^{*} \tag{6}
\end{equation*}
$$

where

$$
\begin{equation*}
S^{*}=U^{-1}\left(\frac{D_{w}+D_{1}}{\gamma m}\right), \quad x^{*}=\frac{D\left(S^{0}-S^{*}\right)}{m U\left(S^{*}\right)-\mu D_{1}} . \tag{7}
\end{equation*}
$$

$E^{*}\left(S^{*}, x^{*}\right)$ is globally asymptotically stable provided that the average delays are sufficiently small. Obviously, $E^{*}$ is still an equilibrium of stochastic model (3) if condition (6) holds.

We assume that function $U(S)$ is nonnegative satisfying

$$
\begin{gather*}
U(0)=0, \quad U^{\prime}(S)>0 \\
U^{\prime \prime}(S)<0 \quad \text { for } S>0, \quad \lim _{S \rightarrow \infty} U(S)=1 \tag{8}
\end{gather*}
$$

And we extend the function $U(S)$ by defining

$$
\begin{equation*}
U(S)=U^{\prime}(0) S+\frac{1}{2} U^{\prime \prime}(0) S^{2} \quad \text { for } S \leq 0 \tag{9}
\end{equation*}
$$

so that $U$ is well defined in $\mathbb{R}$ and is still of class $\mathscr{C}^{2}$ in $\mathbb{R}$. Thus one can write

$$
\begin{equation*}
U(S)=a+b\left(S-S^{*}\right)+F\left(S-S^{*}\right), \tag{10}
\end{equation*}
$$

where $F$ represents terms of order $\geq 2$ in $S-S^{*}$. Noting also that $a=U\left(S^{*}\right)$ and $b=U^{\prime}\left(S^{*}\right)$, by condition (6), it follows that $m a>\mu D_{1}$.

Introduce new variables $u_{1}=S-S^{*}, u_{2}=x-x^{*}$; then model (3) can be rewritten as follows:

$$
\begin{align*}
d u_{1}= & {\left[-\left(D+m b x^{*}\right) u_{1}+\mu D_{1} \int_{0}^{\infty} f(s) u_{2}(t-s) d s\right.} \\
& \left.-m a u_{2}+F_{1}\left(u_{1}, u_{2}\right)\right] d t+\sigma_{1} u_{1} d B_{1} \\
d u_{2}= & {\left[\gamma m b x^{*} \int_{0}^{\infty} g(s) u_{1}(t-s) d s\right.}  \tag{11}\\
& \left.+\widetilde{F}_{2}\left(u_{1}, u_{2}\right)\right] d t+\sigma_{2} u_{2} d B_{2}
\end{align*}
$$

where

$$
\begin{align*}
& F_{1}=-m b u_{1} u_{2}-m F\left(u_{1}\right) u_{2}-m x^{*} F\left(u_{1}\right) \\
& \widetilde{F}_{2}=\gamma m b u_{2} \int_{0}^{\infty} g(s) u_{1}(t-s) d s  \tag{12}\\
& \quad+\left(u_{2}+x^{*}\right) \gamma m \int_{0}^{\infty} g(s) F\left(u_{1}(t-s)\right) d s
\end{align*}
$$

Note that if $f(s)=g(s)=\delta(0)$, then model (11) has the form

$$
\begin{gather*}
d u_{1}=\left[-\left(D+m b x^{*}\right) u_{1}+\left(\mu D_{1}-m a\right) u_{2}\right. \\
\left.+F_{1}\left(u_{1}, u_{2}\right)\right] d t+\sigma_{1} u_{1} d B_{1}  \tag{13}\\
d u_{2}=\left[\gamma m b x^{*} u_{1}+F_{2}\left(u_{1}, u_{2}\right)\right] d t+\sigma_{2} u_{2} d B_{2}
\end{gather*}
$$

where

$$
\begin{equation*}
F_{2}=\gamma m b u_{1} u_{2}+\left(u_{2}+x^{*}\right) \gamma m F\left(u_{1}\right) . \tag{14}
\end{equation*}
$$

Obviously, model (13) has the same equilibrium $(0,0)$ as model (11), and the stochastic stability of the positive equilibrium $E^{*}$ of model (3) is equivalent to the zero solution of model (11). We wonder how the stochastic perturbations and delays affect the dynamics of model (3) or (11).

Before starting our analysis, we first give some basic theories in stochastic differential equations and stochastic functional differential equations [47-49]. Let $\left(\Omega, \mathscr{F},\left\{\mathscr{F}_{t}\right\}_{t \geq 0}, P\right)$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_{0}$ contains all $P$-null sets). Let $B_{i}(i=1,2, \ldots, n)$ be the Brownian motions defined on this probability space. Consider the following $n$-dimensional stochastic differential equation:

$$
\begin{equation*}
d x(t)=f(x(t), t) d t+g(x(t), t) d B(t), \quad t \geq t_{0} \tag{15}
\end{equation*}
$$

Definition 1. The trivial solution of system (15) is said to be as follows:
(i) stochastically stable or stable in probability if for every pair of $\varepsilon \in(0,1)$ and $r>1$, there exists a $\delta=$ $\delta\left(\varepsilon, r, t_{0}\right)>0$ such that

$$
\begin{equation*}
P\left\{\left|x\left(t ; t_{0}, x_{0}\right)\right|<r \forall t \geq t_{0}\right\} \geq 1-\varepsilon \tag{16}
\end{equation*}
$$

whenever $\left|x_{0}\right|<\delta$. Otherwise, it is said to be stochastically unstable,
(ii) stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\varepsilon \in(0,1)$, there exists a $\delta_{0}=\delta_{0}\left(\varepsilon, t_{0}\right)>0$ such that

$$
\begin{equation*}
P\left\{\lim _{t \rightarrow \infty} x\left(t ; t_{0}, x_{0}\right)=0\right\} \geq 1-\varepsilon \tag{17}
\end{equation*}
$$

whenever $\left|x_{0}\right|<\delta_{0}$,
(iii) globally asymptotically stable in probability if it is stochastically asymptotically stable and, moreover, for all $x_{0} \in \mathbb{R}^{n}$

$$
\begin{equation*}
P\left\{\lim _{t \rightarrow \infty} x\left(t ; t_{0}, x_{0}\right)=0\right\}=1 \tag{18}
\end{equation*}
$$

Lemma 2. If there exists a nonnegative function $V(x, t) \in$ $C^{2,1}\left(\mathbb{R}^{n} \times\left[t_{0}, \infty\right] ; \mathbb{R}_{+}\right)$, two continuous functions $\psi_{1}, \psi_{2}$ : $\mathbb{R}_{+}^{0} \rightarrow \mathbb{R}_{+}^{0}$, and a positive constant $K$ such that, for $|x|<K$,

$$
\begin{equation*}
\psi_{1}(|x|) \leq V(x, t) \leq \psi_{2}(|x|) \tag{19}
\end{equation*}
$$

hold.
(i) If

$$
\begin{equation*}
L V \leq 0, \quad \text { for }|x| \in[0, K] \tag{20}
\end{equation*}
$$

then the trivial solution of system (A.1) is stochastically stable.
(ii) If there exists a continuous function $\psi_{3}: \mathbb{R}_{+}^{0} \rightarrow \mathbb{R}_{+}^{0}$ such that

$$
\begin{equation*}
L V \leq-\psi_{3}(|x|) \tag{21}
\end{equation*}
$$

holds, then the trivial solution of system (15) is stochastically asymptotically stable.
(iii) If (ii) holds and moreover

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \psi_{1}(r)=+\infty \tag{22}
\end{equation*}
$$

then the trivial solution of system (15) is globally asymptotically stable in probability.

For the stability of the equilibrium of a nonlinear stochastic system, it can be reduced to problems concerning stability of solutions of the linear associated system. The linear form of (15) is defined as follows:

$$
\begin{equation*}
d x(t)=F(t) \cdot x(t) d t+G(t) \cdot x(t) d B(t), \quad t \geq t_{0} \tag{23}
\end{equation*}
$$

Lemma 3. If the trivial solution is stochastically stable for the linear system (23) with constant coefficients $(F(t)=F$, $G(t)=G)$ and the coefficients of systems (15) and (23) satisfy the following inequality:

$$
\begin{equation*}
|f(x, t)-F \cdot x|+|g(x, t)-G \cdot x|<\rho|x| \tag{24}
\end{equation*}
$$

in a sufficiently small neighborhood of $x=0$, with a sufficiently small constant $\rho$, then the trivial solution of system (15) is asymptotically stable in probability.

Consider the following $n$-dimensional stochastic functional differential equation

$$
\begin{equation*}
d x=f\left(t, x_{t}\right) d t+g\left(t, x_{t}\right) d B(t) \tag{25}
\end{equation*}
$$

with initial condition $x_{0}=\varphi \in \mathscr{H}$, where $\mathscr{H}$ is the space of $\mathscr{F}_{0}$-adapted random variables $\varphi$, with $\varphi(s) \in \mathbb{R}^{n}$ for $s \leq 0$, and

$$
\begin{equation*}
\|\varphi\|=\sup _{s \leq 0}|\varphi(s)|, \quad\|\varphi\|_{1}^{2}=\sup E\left(|\varphi(s)|^{2}\right) \tag{26}
\end{equation*}
$$

Definition 4. The trivial solution of system (25) is said to be
(i) mean square stable if, for each $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that for any initial process $\varphi(\theta)$,

$$
\begin{equation*}
E\left(|x(t, \varphi(\theta))|^{2}\right)<\varepsilon \tag{27}
\end{equation*}
$$

for any $t \geq 0$ provided that $\sup _{\theta \leq 0} E\left(|\varphi(\theta)|^{2}\right)<\delta(\varepsilon)$,
(ii) asymptotically mean square stable if it is mean square stable and

$$
\begin{equation*}
\lim _{t \rightarrow \infty} E\left(|x(t, \varphi)|^{2}\right)=0 \tag{28}
\end{equation*}
$$

(iii) stochastically stable if for any $\varepsilon_{1}>0$ and $\varepsilon_{2}>0$, there exists a $\delta>0$ such that

$$
\begin{equation*}
P\left\{\sup _{t \geq 0}|x(t, \varphi)| \leq \varepsilon_{1}\right\} \geq 1-\varepsilon_{2} \tag{29}
\end{equation*}
$$

provided that $P\{\|\varphi\| \leq \delta\}=1$.

## 3. Dynamical Behavior of the System without Delays

We first study the stochastic stability of the equilibria $(0,0)$ of model (13). Throughout the paper, we assume that the basic hypotheses given in the Section 2 are satisfied. The linearized system of model (13) is

$$
\begin{align*}
& d u_{1}= {\left[-\left(D+m b x^{*}\right) u_{1}+\left(\mu D_{1}-m a\right) u_{2}\right] d t } \\
&+\sigma_{1} u_{1} d B_{1},  \tag{30}\\
& d u_{2}=\gamma m b x^{*} u_{1} d t+\sigma_{2} u_{2} d B_{2} .
\end{align*}
$$

For convenience, let

$$
\begin{equation*}
p=\frac{\gamma m b x^{*}}{2\left(m a-\mu D_{1}\right)}, \quad q=\frac{\gamma m b x^{*}-p\left(m a-\mu D_{1}\right)}{\gamma^{2}\left(m a-\mu D_{1}\right)+\gamma D} . \tag{31}
\end{equation*}
$$

For linearized system (30), we have the following theorem.
Theorem 5. Let condition (6) hold. If

$$
\begin{equation*}
\sigma_{1}^{2}<2 D+2 m b x^{*}, \quad \sigma_{2}^{2}<\frac{2 q}{1+q} \gamma\left(m a-\mu D_{1}\right) \tag{32}
\end{equation*}
$$

then the trivial solution of system (30) is globally asymptotically stable in probability.

Proof. Define a smooth function $V: \mathbb{R}^{2} \rightarrow \mathbb{R}_{+}$by

$$
\begin{equation*}
V\left(u_{1}, u_{2}\right)=p u_{1}^{2}+u_{2}^{2}+q\left(\gamma u_{1}+u_{2}\right)^{2} \tag{33}
\end{equation*}
$$

Then using Itô's formula, for all $\left(u_{1}, u_{2}\right) \neq(0,0)$, we have

$$
\begin{align*}
d V\left(u_{1}, u_{2}\right)= & 2 p u_{1} d u_{1}+p\left(d u_{1}\right)^{2}+2 u_{2} d u_{2}+\left(d u_{2}\right)^{2} \\
& +2 q\left(\gamma u_{1}+u_{2}\right) d\left(\gamma u_{1}+u_{2}\right) \\
& +q\left(d\left(\gamma u_{1}+u_{2}\right)\right)^{2} \\
= & L V\left(u_{1}, u_{2}\right) d t+2 p \sigma_{1} u_{1}^{2} d B_{1}+2 \sigma_{2} u_{2}^{2} d B_{2} \\
& +2 q\left(\gamma u_{1}+u_{2}\right)\left(\gamma \sigma_{1} u_{1} d B_{1}+\sigma_{2} u_{2} d B_{2}\right) \tag{34}
\end{align*}
$$

where

$$
\begin{align*}
L V\left(u_{1}, u_{2}\right)= & 2 p u_{1}\left[-\left(D+m b x^{*}\right) u_{1}+\left(\mu D_{1}-m a\right) u_{2}\right] \\
& +p \sigma_{1}^{2} u_{1}^{2}+2 \gamma m b x^{*} u_{1} u_{2}+\sigma_{2}^{2} u_{2}^{2} \\
& +2 q\left(\gamma u_{1}+u_{2}\right) \\
\times & {\left[-\gamma D u_{1}+\gamma\left(\mu D_{1}-m a\right) u_{2}\right] } \\
& +q\left(\gamma^{2} \sigma_{1}^{2} u_{1}^{2}+\sigma_{2}^{2} u_{2}^{2}\right) \\
= & -\left[2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2}\right. \\
& \left.+2 q \gamma^{2} D-q \gamma^{2} \sigma_{1}^{2}\right] u_{1}^{2} \\
& -\left[2 q \gamma\left(m a-\mu D_{1}\right)-(1+q) \sigma_{2}^{2}\right] u_{2}^{2} \\
& -2\left[p\left(m a-\mu D_{1}\right)-\gamma m b x^{*}\right. \\
& \left.+q \gamma^{2}\left(m a-\mu D_{1}\right)+q \gamma D\right] u_{1} u_{2} . \tag{35}
\end{align*}
$$

By (31), we obtain

$$
\begin{align*}
L V & \left(u_{1}, u_{2}\right) \\
= & -\left[2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2}+2 q \gamma^{2} D-q \gamma^{2} \sigma_{1}^{2}\right] u_{1}^{2}  \tag{36}\\
& -\left[2 q \gamma\left(m a-\mu D_{1}\right)-(1+q) \sigma_{2}^{2}\right] u_{2}^{2} .
\end{align*}
$$

We take $\psi_{i}: R_{+}^{0} \rightarrow R_{+}^{0}(i=1,2,3)$ by

$$
\begin{aligned}
& \psi_{1}(|u|)=\min \{p, 1, q\}|u|^{2}, \\
& \psi_{2}(|u|)=\max \{p, 1, q\}|u|^{2},
\end{aligned}
$$

$$
\begin{align*}
\psi_{3}(|u|)=\min \{ & 2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2}+2 q \gamma^{2} D \\
& \left.-q \gamma^{2} \sigma_{1}^{2}, 2 q \gamma\left(m a-\mu D_{1}\right)-(1+q) \sigma_{2}^{2}\right\}|u|^{2} \tag{37}
\end{align*}
$$

thus the thesis follows by Lemma 2. This completes the proof of Theorem 5 .

Now, we are in a position to prove the stability of the trivial solution $(0,0)$ of model (13).

Theorem 6. Let condition (6) hold. If the conditions in (32) are satisfied, then the trivial solution of model (13) is stochastically asymptotically stable.

Proof. For a sufficiently small constant $\epsilon>0,\left(u_{1}, u_{2}\right) \epsilon$ $(-\epsilon, \epsilon) \times(-\epsilon, \epsilon)$, we have

$$
\begin{gather*}
|f(t, X)-F \cdot X|+|g(t, X)-G \cdot X| \\
\quad=\sqrt{F_{1}^{2}\left(u_{1}, u_{2}\right)+F_{2}^{2}\left(u_{1}, u_{2}\right)} . \tag{38}
\end{gather*}
$$

Note that $F_{1}, F_{2}$ are the terms of order $\geq 2$ in $u_{1}$ and $u_{2}$; then we have

$$
\begin{equation*}
\lim _{u_{1}^{2}+u_{2}^{2} \rightarrow 0} \frac{F_{1}^{2}\left(u_{1}, u_{2}\right)+F_{2}^{2}\left(u_{1}, u_{2}\right)}{u_{1}^{2}+u_{2}^{2}}=0 \tag{39}
\end{equation*}
$$

Thus for a sufficiently small constant $\rho>0$, we have

$$
\begin{equation*}
F_{1}^{2}\left(u_{1}, u_{2}\right)+F_{2}^{2}\left(u_{1}, u_{2}\right)<\rho^{2}\left(u_{1}^{2}+u_{2}^{2}\right) \tag{40}
\end{equation*}
$$

provided $u_{1}^{2}+u_{2}^{2}<\epsilon^{2}$. Therefore,

$$
\begin{equation*}
|f(t, X)-F \cdot X|+|g(t, X)-G \cdot X|<\rho|u| . \tag{41}
\end{equation*}
$$

Applying Lemma 3 and Theorem 5, we obtain the conclusion.

## 4. Dynamical Behavior of the System with Delays

We now study the stability in probability of the equilibria $(0,0)$ of system (11). Its corresponding linearized system is

$$
\begin{align*}
d u_{1}= & {\left[-\left(D+m b x^{*}\right) u_{1}\right.} \\
& \left.+\mu D_{1} \int_{0}^{\infty} f(s) u_{2}(t-s) d s-m a u_{2}\right] d t  \tag{42}\\
+ & \sigma_{1} u_{1} d B_{1} \\
d u_{2}= & \gamma m b x^{*} \int_{0}^{\infty} g(s) u_{1}(t-s) d s d t+\sigma_{2} u_{2} d B_{2} .
\end{align*}
$$

Define the average time lags as

$$
\begin{equation*}
T_{f}=\int_{0}^{\infty} s f(s) d s, \quad T_{g}=\int_{0}^{\infty} s g(s) d s \tag{43}
\end{equation*}
$$

and let $q, p$ be defined in (31). For linearized system (42) we have the following theorem.

Theorem 7. Let condition (6) hold. If

$$
\begin{align*}
\sigma_{1}^{2}+ & 2 \mu D_{1} \gamma m b x^{*} T_{f}+\frac{1+q}{p+q \gamma^{2}}\left(D+m b x^{*}\right) \gamma m b x^{*} T_{g} \\
& <2 D+\frac{2 p m b x^{*}}{p+q \gamma^{2}}  \tag{44}\\
\sigma_{2}^{2}+ & \left(D+m b x^{*}+2 m a+2 \mu D_{1}\right) \gamma m b x^{*} T_{g} \\
& <\frac{2 q}{1+q} \gamma\left(m a-\mu D_{1}\right)
\end{align*}
$$

then the trivial solution of system (42) is asymptotically mean square stable.

Proof. Consider the function $V_{1}\left(u_{1}, u_{2}\right)$ defined in (33). It follows from (42) and Itô's formula that

$$
\begin{align*}
& d V_{1}\left(u_{1}, u_{2}\right)= 2 p u_{1} d u_{1}+p\left(d u_{1}\right)^{2}+2 u_{2} d u_{2} \\
&+\left(d u_{2}\right)^{2}+2 q\left(\gamma u_{1}+u_{2}\right) d\left(\gamma u_{1}+u_{2}\right) \\
&+q\left(d\left(\gamma u_{1}+u_{2}\right)\right)^{2} \\
&=\left\{2 p u _ { 1 } \left[-\left(D+m b x^{*}\right) u_{1}\right.\right. \\
&+\mu D_{1} \int_{0}^{\infty} f(s) u_{2}(t-s) d s \\
&+p \sigma_{1}^{2} u_{1}^{2}+2 \gamma m b x^{*} u_{2} \int_{0}^{\infty} g(s) u_{1}(t-s) d s \\
&+\sigma_{2}^{2} u_{2}^{2}+2 q\left(\gamma u_{1}+u_{2}\right) \\
& \times\left[-\gamma\left(D+m b x^{*}\right) u_{1}\right. \\
& \quad+\gamma \mu D_{1} \int_{0}^{\infty} f(s) u_{2}(t-s) d s-\gamma m a u_{2} \\
&\left.+\gamma m b x^{*} \int_{0}^{\infty} g(s) u_{1}(t-s) d s\right] \\
&\left.+q\left(\gamma^{2} \sigma_{1}^{2} u_{1}^{2}+\sigma_{2}^{2} u_{2}^{2}\right)\right\} d t \\
&+2 \sigma_{2} u_{2}^{2} d B_{2}+2 q\left(\gamma u_{1}+u_{2}\right) \\
& \times\left(\gamma \sigma_{1} u_{1} d B_{1}+\sigma_{2} u_{2} d B_{2}\right)+2 p \sigma_{1} u_{1}^{2} d B_{1} .
\end{align*}
$$

Straightforward computations lead to

$$
\begin{align*}
L V_{1}\left(u_{1}, u_{2}\right)= & -\left[2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2}\right. \\
& \left.+2 q \gamma^{2}\left(D+m b x^{*}\right)-q \gamma^{2} \sigma_{1}^{2}\right] u_{1}^{2} \\
- & {\left[2 q \gamma m a-(1+q) \sigma_{2}^{2}\right] u_{2}^{2} } \\
- & 2\left[p m a+q \gamma^{2} m a+q \gamma\left(D+m b x^{*}\right)\right] u_{1} u_{2} \\
+ & 2(1+q) \gamma m b x^{*} u_{2} \int_{0}^{\infty} g(s) u_{1}(t-s) d s \\
+ & 2 q \gamma^{2} m b x^{*} u_{1} \int_{0}^{\infty} g(s) u_{1}(t-s) d s \\
& +2\left(p+q \gamma^{2}\right) \mu D_{1} u_{1} \int_{0}^{\infty} f(s) u_{2}(t-s) d s \\
& +2 q \gamma \mu D_{1} u_{2} \int_{0}^{\infty} f(s) u_{2}(t-s) d s . \tag{46}
\end{align*}
$$

From the terms of the right-hand side of (46), we have

$$
\begin{align*}
& u_{1} \int_{0}^{\infty} g(s) u_{1}(t-s) d s \leq \frac{1}{2}\left(u_{1}^{2}+\int_{0}^{\infty} g(s) u_{1}^{2}(t-s) d s\right) \\
& u_{2} \int_{0}^{\infty} f(s) u_{2}(t-s) d s \leq \frac{1}{2}\left(u_{2}^{2}+\int_{0}^{\infty} f(s) u_{2}^{2}(t-s) d s\right) \tag{47}
\end{align*}
$$

For the term $u_{1} \int_{0}^{\infty} f(s) u_{2}(t-s) d s$, it is clear that

$$
\begin{align*}
u_{1} \int_{0}^{\infty} & f(s) u_{2}(t-s) d s \\
= & u_{1} u_{2}-u_{1} \int_{0}^{t} f(s) \int_{t-s}^{t} d u_{2}(\tau) d s+h_{1}(t)  \tag{48}\\
= & u_{1} u_{2}-\gamma m b x^{*} H_{1}\left(u_{1}, u_{2}\right)+h_{1}(t) \\
& -u_{1} \int_{0}^{t} f(s) \int_{t-s}^{t} \sigma_{2} u_{2}(\tau) d B_{2}(\tau) d s
\end{align*}
$$

where

$$
\begin{aligned}
h_{1}(t)= & -u_{1} \int_{t}^{\infty} f(s)\left(u_{2}(t)-u_{2}(t-s)\right) d s, \\
H_{1}\left(u_{1}, u_{2}\right)= & u_{1} \int_{0}^{t} f(s) \int_{t-s}^{t} \int_{0}^{\infty} g(v) u_{1}(\tau-v) d v d \tau d s \\
\leq & \frac{1}{2} \int_{0}^{\infty} f(s) \\
& \times \int_{t-s}^{t} \int_{0}^{\infty} g(v) \\
& \times\left(u_{1}^{2}(t)+u_{1}^{2}(\tau-v)\right) d v d \tau d s
\end{aligned}
$$

$$
\begin{align*}
=\frac{1}{2} T_{f} u_{1}^{2}+\frac{1}{2} \int_{0}^{\infty} & f(s) \\
& \times \int_{t-s}^{t} \int_{0}^{\infty} g(v) \\
& \times u_{1}^{2}(\tau-v) d v d \tau d s \tag{50}
\end{align*}
$$

For the term $u_{2} \int_{0}^{\infty} g(s) u_{1}(t-s) d s$, we have that

$$
\begin{align*}
& u_{2} \int_{0}^{\infty} g(s) u_{1}(t-s) d s \\
&= u_{1} u_{2}-u_{2} \int_{0}^{t} g(s) \int_{t-s}^{t} d u_{1}(\tau) d s+h_{2}(t) \\
&= u_{1} u_{2}+\left(D+m b x^{*}\right) H_{2}\left(u_{1}, u_{2}\right)  \tag{51}\\
& \quad+m a H_{3}\left(u_{1}, u_{2}\right)-\mu D_{1} H_{4}\left(u_{1}, u_{2}\right) \\
& \quad+u_{2} \int_{0}^{\infty} g(s) \int_{t-s}^{t} \sigma_{2} u_{2}(\tau) d B_{2}(\tau) d s+h_{2}(t)
\end{align*}
$$

where

$$
\begin{gather*}
h_{2}(t)=-u_{2} \int_{t}^{\infty} g(s)\left(u_{1}(t)-u_{1}(t-s)\right) d s,  \tag{52}\\
H_{2}\left(u_{1}, u_{2}\right)= \\
u_{2} \int_{0}^{t} g(s) \int_{t-s}^{t} u_{1}(\tau) d \tau d s \\
\leq \frac{1}{2} T_{g} u_{2}^{2}+\frac{1}{2} \int_{0}^{\infty} g(s) \int_{t-s}^{t} u_{1}^{2}(\tau) d \tau d s, \\
H_{3}\left(u_{1}, u_{2}\right)= \\
u_{2} \int_{0}^{t} g(s) \int_{t-s}^{t} u_{2}(\tau) d \tau d s \\
H_{4}\left(u_{1}, u_{2}\right)= \\
\leq \frac{1}{2} T_{g} \int_{0}^{2}+\frac{1}{2} \int_{0}^{\infty} g(s) \int_{t-s}^{t} \int_{0}^{\infty} f(s) \int_{t-s}^{t} u_{2}^{2}(\tau) d \tau d s, u_{2}(\tau-v) d v d \tau d s \\
u_{2}^{2}+\frac{1}{2} \int_{0}^{\infty} g(s)  \tag{53}\\
\times \int_{t-s}^{t} \int_{0}^{\infty} f(v) \\
\times u_{2}^{2}(\tau-v) d v d \tau d s .
\end{gather*}
$$

Substituting (47)-(48) together with (51) into (46), we obtain

$$
\begin{aligned}
L V_{1}\left(u_{1}, u_{2}\right) \leq- & {\left[2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2}\right.} \\
& +2 q \gamma^{2}\left(D+m b x^{*}\right)-q \gamma^{2} \sigma_{1}^{2}-q \gamma^{2} m b x^{*} \\
& \left.-\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f}\right] u_{1}^{2} \\
- & {\left[2 q \gamma m a-(1+q) \sigma_{2}^{2}-(1+q)\right.}
\end{aligned}
$$

$$
\begin{align*}
& \quad \times\left(D+m b x^{*}+m a+\mu D_{1}\right) \gamma m b x^{*} T_{g} \\
& \left.\quad-q \gamma \mu D_{1}\right] u_{2}^{2} \\
& -2\left[p m a+q \gamma^{2} m a+q \gamma\left(D+m b x^{*}\right)\right. \\
& \left.\quad-(1+q) \gamma m b x^{*}-\left(p+q \gamma^{2}\right) \mu D_{1}\right] u_{1} u_{2} \\
& +(1+q) \gamma m b x^{*} \\
& \times\left[\mu D_{1} \int_{0}^{\infty} g(s)\right. \\
& \quad \times \int_{t-s}^{t} \int_{0}^{\infty} f(v) u_{2}^{2}(\tau-v) d v d \tau d s \\
& \quad+\left(D+m b x^{*}\right) \int_{0}^{\infty} g(s) \int_{t-s}^{t} u_{1}^{2}(\tau) d \tau d s \\
& +q \gamma^{2} m b x^{*} \int_{0}^{\infty} g(s) u_{1}^{2}(t-s) d s \\
& \\
& \left.+q(s) \int_{t-s}^{t} u_{2}^{2}(\tau) d \tau d s\right] \\
& +q \gamma \mu D_{1} \int_{0}^{\infty} f(s) u_{2}^{2}(t-s) d s \\
& +\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} \\
& \times \int_{0}^{\infty} f(s) \int_{t-s}^{t} \int_{0}^{\infty} g(v) u_{1}^{2}(\tau-v) d v d \tau d s \\
& +2(1+q) \gamma m b x^{*} h_{2}(t)  \tag{54}\\
& +2\left(p+q \gamma^{2}\right) \mu D_{1} h_{1}(t)
\end{align*}
$$

For technical reasons, we assume that $\int_{0}^{\infty} s^{2} f(s) d s<\infty$ and $\int_{0}^{\infty} s^{2} g(s) d s<\infty$. Then the function

$$
\begin{aligned}
& V_{2}\left(u_{1}, u_{2}\right) \\
& \qquad \begin{array}{l}
=(1+q) \gamma m b x^{*} \\
\quad \times\left[\mu D_{1} \int_{0}^{\infty} g(s)\right. \\
\quad \times \int_{t-s}^{t} \int_{r}^{t} \int_{0}^{\infty} f(v) u_{2}^{2}(\tau-v) d v d \tau d r d s \\
\quad+\left(D+m b x^{*}\right) \\
\quad \times \int_{0}^{\infty} g(s) \int_{t-s}^{t} \int_{r}^{t} u_{1}^{2}(\tau) d \tau d r d s \\
\left.\quad+m a \int_{0}^{\infty} g(s) \int_{t-s}^{t} \int_{r}^{t} u_{2}^{2}(\tau) d \tau d r d s\right]
\end{array}
\end{aligned}
$$

$$
\begin{align*}
& +q \gamma^{2} m b x^{*} \int_{0}^{\infty} g(s) \int_{t-s}^{t} u_{1}^{2}(\tau) d \tau d s \\
& +q \gamma \mu D_{1} \int_{0}^{\infty} f(s) \int_{t-s}^{t} u_{2}^{2}(\tau) d \tau d s \\
& +\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} \\
& \times \int_{0}^{\infty} f(s) \\
& \quad \times \int_{t-s}^{t} \int_{r}^{t} \int_{0}^{\infty} g(v) u_{1}^{2}(\tau-v) d v d \tau d r d s \tag{55}
\end{align*}
$$

is well defined. Using Itô's formula, we have

$$
\begin{align*}
L\left(V_{1}+V_{2}\right) \leq- & {\left[2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2}\right.} \\
& +2 q \gamma^{2}\left(D+m b x^{*}\right)-q \gamma^{2} \sigma_{1}^{2}-2 q \gamma^{2} m b x^{*} \\
& -\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f} \\
& \left.-(1+q)\left(D+m b x^{*}\right) \gamma m b x^{*} T_{g}\right] u_{1}^{2} \\
- & {\left[2 q \gamma m a-(1+q) \sigma_{2}^{2}-(1+q)\right.} \\
& \times\left(D+m b x^{*}+2 m a+\mu D_{1}\right) \gamma m b x^{*} T_{g} \\
& \left.-2 q \gamma \mu D_{1}\right] u_{2}^{2} \\
- & 2\left[p m a+q \gamma^{2} m a+q \gamma\left(D+m b x^{*}\right)\right. \\
& \left.-(1+q) \gamma m b x^{*}-\left(p+q \gamma^{2}\right) \mu D_{1}\right] u_{1} u_{2} \\
+ & (1+q) \gamma m b x^{*} \mu D_{1} T_{g} \\
\times & \int_{0}^{\infty} f(s) u_{2}^{2}(t-s) d s \\
+ & \left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f} \\
\times & \int_{0}^{\infty} g(s) u_{1}^{2}(t-s) d s \\
+ & 2(1+q) \gamma m b x^{*} h_{2}(t) \\
+ & 2\left(p+q \gamma^{2}\right) \mu D_{1} h_{1}(t) . \tag{56}
\end{align*}
$$

We now consider the function

$$
\begin{align*}
V_{3}\left(u_{1}, u_{2}\right)= & (1+q) \gamma m b x^{*} \mu D_{1} T_{g} \\
& \times \int_{0}^{\infty} f(s) \int_{t-s}^{t} u_{2}^{2}(\tau) d \tau d s  \tag{57}\\
& +\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f} \\
& \times \int_{0}^{\infty} g(s) \int_{t-s}^{t} u_{1}^{2}(\tau) d \tau d s
\end{align*}
$$

It follows from (56) and (57) that

$$
\begin{aligned}
L\left(V_{1}+\right. & \left.V_{2}+V_{3}\right) \\
\leq-[ & 2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2} \\
& +2 q \gamma^{2}\left(D+m b x^{*}\right)-q \gamma^{2} \sigma_{1}^{2}-2 q \gamma^{2} m b x^{*} \\
& -2\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f} \\
& -(1+q)\left(D+m b x^{*}\right) \gamma m b x^{*} T_{g} \\
& -2\left(p+q \gamma^{2}\right) \mu D_{1} \int_{t}^{\infty} f(s) d s \\
& \left.-(1+q) \gamma m b x^{*} \int_{t}^{\infty} g(s) d s\right] u_{1}^{2} \\
- & 2 q \gamma m a-(1+q) \sigma_{2}^{2}-(1+q) \\
& \times\left(D+m b x^{*}+2 m a+2 \mu D_{1}\right) \gamma m b x^{*} T_{g} \\
& -2 q \gamma \mu D_{1}-\left(p+q \gamma^{2}\right) \mu D_{1} \int_{t}^{\infty} f(s) d s \\
& \left.-2(1+q) \gamma m b x^{*} \int_{t}^{\infty} g(s) d s\right] u_{2}^{2} \\
- & {\left[p m a+q \gamma^{2} m a+q \gamma\left(D+m b x^{*}\right)\right.} \\
& \left.\quad-(1+q) \gamma m b x^{*}-\left(p+q \gamma^{2}\right) \mu D_{1}\right] u_{1} u_{2} \\
+ & \left.p+q \gamma^{2}\right) \mu D_{1} \int_{t}^{\infty} f(s) u_{2}^{2}(t-s) d s \\
+ & (1+q) \gamma m b x^{*} \int_{t}^{\infty} g(s) u_{1}^{2}(t-s) d s .
\end{aligned}
$$

Therefore, for the function

$$
\begin{equation*}
V\left(u_{1}, u_{2}\right)=V_{1}\left(u_{1}, u_{2}\right)+V_{2}\left(u_{1}, u_{2}\right)+V_{3}\left(u_{1}, u_{2}\right) \tag{59}
\end{equation*}
$$

we have

$$
\begin{aligned}
L V \leq- & {\left[2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2}\right.} \\
& +2 q \gamma^{2}\left(D+m b x^{*}\right)-q \gamma^{2} \sigma_{1}^{2}-2 q \gamma^{2} m b x^{*} \\
& -2\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f} \\
& -(1+q)\left(D+m b x^{*}\right) \gamma m b x^{*} T_{g} \\
& -2\left(p+q \gamma^{2}\right) \mu D_{1} \int_{t}^{\infty} f(s) d s \\
& \left.-(1+q) \gamma m b x^{*} \int_{t}^{\infty} g(s) d s\right] u_{1}^{2}
\end{aligned}
$$

$$
\begin{align*}
& -\left[2 q \gamma m a-(1+q) \sigma_{2}^{2}-(1+q)\right. \\
& \quad \times\left(D+m b x^{*}+2 m a+2 \mu D_{1}\right) \gamma m b x^{*} T_{g} \\
& -2 q \gamma \mu D_{1}-\left(p+q \gamma^{2}\right) \mu D_{1} \int_{t}^{\infty} f(s) d s \\
& \left.\quad-2(1+q) \gamma m b x^{*} \int_{t}^{\infty} g(s) d s\right] u_{2}^{2} \\
& + \\
& +\left(p+q \gamma^{2}\right) \mu D_{1} \int_{t}^{\infty} f(s) u_{2}^{2}(t-s) d s  \tag{60}\\
& + \\
& \hline(1+q) \gamma m b x^{*} \int_{t}^{\infty} g(s) u_{1}^{2}(t-s) d s .
\end{align*}
$$

By (44), we choose $\varepsilon>0$ such that

$$
\begin{align*}
& 2 p\left(D+m b x^{*}\right)+2 q \gamma^{2} D \\
& \quad>\left(p+q \gamma^{2}\right) \sigma_{1}^{2}+2\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f} \\
& \quad+(1+q)\left(D+m b x^{*}\right) \gamma m b x^{*} T_{g} \\
& \quad+2\left(p+q \gamma^{2}\right) \mu D_{1} \varepsilon+(1+q) \gamma m b x^{*} \varepsilon,  \tag{61}\\
& \frac{2 q}{1+q} \gamma\left(m a-\mu D_{1}\right) \\
& \quad>\sigma_{2}^{2}+\left(D+m b x^{*}+2 m a+2 \mu D_{1}\right) \gamma m b x^{*} T_{g} \\
& \quad+\left(p+q \gamma^{2}\right) \mu D_{1} \varepsilon+2(1+q) \gamma m b x^{*} \epsilon .
\end{align*}
$$

Let $T=T(\varepsilon)$ such that $\int_{t}^{\infty} f(s) d s<\varepsilon$ and $\int_{t}^{\infty} g(s) d s<\varepsilon$ for all $t \geq T$. Then for all $t \geq T$, one has

$$
L V \leq-\left[2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2}+2 q \gamma^{2}\left(D+m b x^{*}\right)\right.
$$

$$
-q \gamma^{2} \sigma_{1}^{2}-2 q \gamma^{2} m b x^{*}
$$

$$
-2\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f}-(1+q)
$$

$$
\times\left(D+m b x^{*}\right) \gamma m b x^{*} T_{g}
$$

$$
\left.-2\left(p+q \gamma^{2}\right) \mu D_{1} \varepsilon-(1+q) \gamma m b x^{*} \varepsilon\right] u_{1}^{2}
$$

$$
-\left[2 q \gamma m a-(1+q) \sigma_{2}^{2}-(1+q)\right.
$$

$$
\times\left(D+m b x^{*}+2 m a+2 \mu D_{1}\right) \gamma m b x^{*} T_{g}
$$

$$
-2 q \gamma \mu D_{1}-\left(p+q \gamma^{2}\right) \mu D_{1} \varepsilon
$$

$$
\left.-2(1+q) \gamma m b x^{*} \varepsilon\right] u_{2}^{2}
$$

$$
+\left(p+q \gamma^{2}\right) \mu D_{1}\left\|\varphi_{2}\right\|^{2} \int_{t}^{\infty} f(s) d s
$$

$$
\begin{equation*}
+(1+q) \gamma m b x^{*}\left\|\varphi_{1}\right\|^{2} \int_{t}^{\infty} g(s) d s \tag{62}
\end{equation*}
$$

## For convenience, let

$$
\begin{aligned}
Q=\min \{ & 2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2} \\
& +2 q \gamma^{2}\left(D+m b x^{*}\right)-q \gamma^{2} \sigma_{1}^{2}-2 q \gamma^{2} m b x^{*} \\
& -2\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f}-(1+q) \\
& \times\left(D+m b x^{*}\right) \gamma m b x^{*} T_{g} \\
& -2\left(p+q \gamma^{2}\right) \mu D_{1} \varepsilon-(1+q) \gamma m b x^{*} \varepsilon \\
& 2 q \gamma m a-(1+q) \sigma_{2}^{2}-(1+q) \\
& \times\left(D+m b x^{*}+2 m a+2 \mu D_{1}\right) \gamma m b x^{*} T_{g} \\
& -2 q \gamma \mu D_{1}-\left(p+q \gamma^{2}\right) \mu D_{1} \varepsilon \\
& \left.-2(1+q) \gamma m b x^{*} \varepsilon\right\} .
\end{aligned}
$$

Integrating both sides of (62) from $T$ to $t \geq T$, we have

$$
\begin{align*}
& E(V(t))+Q \int_{T}^{t} E\left(u_{1}^{2}(s)+u_{2}^{2}(s)\right) d s \\
& \leq V(T)+\left(p+q \gamma^{2}\right) \mu D_{1}\left\|\varphi_{2}\right\|^{2} \int_{T}^{t} \int_{s}^{\infty} f(u) d u d s \\
&+(1+q) \gamma m b x^{*}\left\|\varphi_{1}\right\|^{2} \int_{T}^{t} \int_{t}^{\infty} g(u) d u d s \\
& \leq V(T)+\left(p+q \gamma^{2}\right) \mu D_{1}\left\|\varphi_{2}\right\|^{2} \int_{0}^{\infty} s f(s) d s  \tag{64}\\
&+(1+q) \gamma m b x^{*}\left\|\varphi_{1}\right\|^{2} \int_{0}^{\infty} s g(s) d s \\
&= V(T)+\left(p+q \gamma^{2}\right) \mu D_{1}\left\|\varphi_{2}\right\|^{2} T_{f} \\
&+(1+q) \gamma m b x^{*}\left\|\varphi_{1}\right\|^{2} T_{g}<\infty
\end{align*}
$$

Discussing as that in He et al. [18], by the Barbălat lemma, we conclude $E\left(u_{1}^{2}(t)+u_{2}^{2}(t)\right) \rightarrow 0$ as $t \rightarrow \infty$. Applying Definition 4, we obtain the conclusion.

Now, we are in a position to prove the stability of the trivial solution $(0,0)$ of nonlinear system (11) using the Lyapunov functionals constructed above.

Theorem 8. Let condition (6) hold. If conditions (44) are satisfied, then the trivial solution $(0,0)$ of the system (11) or the equilibrium $\left(S^{*}, x^{*}\right)$ of system (6) is stochastically stable.

Proof. Consider the Lyapunov function $V_{1}\left(u_{1}, u_{2}\right)$ defined in (33). It follows from (11) and Itô's formula that

$$
\begin{aligned}
d V_{1}\left(u_{1}, u_{2}\right)= & 2 p u_{1} d u_{1}+p\left(d u_{1}\right)^{2}+2 u_{2} d u_{2} \\
& +\left(d u_{2}\right)^{2}+2 q\left(\gamma u_{1}+u_{2}\right) d\left(\gamma u_{1}+u_{2}\right)
\end{aligned}
$$

$$
\begin{align*}
& +q\left(d\left(\gamma u_{1}+u_{2}\right)\right)^{2} \\
& =\left\{2 p u _ { 1 } \left[-\left(D+m b x^{*}\right) u_{1}\right.\right. \\
& +\mu D_{1} \int_{0}^{\infty} f(s) u_{2}(t-s) d s \\
& \left.-m a u_{2}+F_{1}\right]+p \sigma_{1}^{2} u_{1}^{2} \\
& +2 \gamma m b x^{*} u_{2} \int_{0}^{\infty} g(s) u_{1}(t-s) d s \\
& +2 u_{2} \widetilde{F}_{2}+\sigma_{2}^{2} u_{2}^{2}+2 q\left(\gamma u_{1}+u_{2}\right) \\
& \times\left[-\gamma\left(D+m b x^{*}\right) u_{1}\right. \\
& +\gamma \mu D_{1} \int_{0}^{\infty} f(s) u_{2}(t-s) d s \\
& -\gamma m a u_{2}+\gamma F_{1} \\
& \left.+\gamma m b x^{*} \int_{0}^{\infty} g(s) u_{1}(t-s) d s+\widetilde{F}_{2}\right] \\
& \left.+q\left(\gamma^{2} \sigma_{1}^{2} u_{1}^{2}+\sigma_{2}^{2} u_{2}^{2}\right)\right\} d t \\
& +2 \sigma_{2} u_{2}^{2} d B_{2} \\
& +2 q\left(\gamma u_{1}+u_{2}\right)\left(\gamma \sigma_{1} u_{1} d B_{1}+\sigma_{2} u_{2} d B_{2}\right) \\
& +2 p \sigma_{1} u_{1}^{2} d B_{1}=L V_{1}\left(u_{1}, u_{2}\right) d t+2 \sigma_{2} u_{2}^{2} d B_{2} \\
& +2 q\left(\gamma u_{1}+u_{2}\right)\left(\gamma \sigma_{1} u_{1} d B_{1}+\sigma_{2} u_{2} d B_{2}\right) \\
& +2 p \sigma_{1} u_{1}^{2} d B_{1} \text {, } \tag{65}
\end{align*}
$$

where

$$
\begin{aligned}
L V_{1}\left(u_{1}, u_{2}\right)= & -\left[2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2}\right. \\
& \left.+2 q \gamma^{2}\left(D+m b x^{*}\right)-q \gamma^{2} \sigma_{1}^{2}\right] u_{1}^{2} \\
- & {\left[2 q \gamma m a-(1+q) \sigma_{2}^{2}\right] u_{2}^{2} } \\
- & 2\left[p m a+q \gamma^{2} m a+q \gamma\left(D+m b x^{*}\right)\right] u_{1} u_{2} \\
+ & 2(1+q) \gamma m b x^{*} u_{2} \int_{0}^{\infty} g(s) u_{1}(t-s) d s \\
+ & 2 q \gamma^{2} m b x^{*} u_{1} \int_{0}^{\infty} g(s) u_{1}(t-s) d s \\
+ & 2\left(p+q \gamma^{2}\right) \mu D_{1} u_{1} \int_{0}^{\infty} f(s) u_{2}(t-s) d s \\
& +2 q \gamma \mu D_{1} u_{2} \int_{0}^{\infty} f(s) u_{2}(t-s) d s
\end{aligned}
$$

$$
\begin{align*}
& +2 p u_{1} F_{1}+2 u_{2} \widetilde{F}_{2} \\
& +2 q\left(\gamma u_{1}+u_{2}\right)\left(\gamma F_{1}+\widetilde{F}_{2}\right) . \tag{66}
\end{align*}
$$

From the terms of the right-hand side of (66), we observe that

$$
\begin{align*}
& u_{1} \int_{0}^{\infty} f(s) u_{2}(t-s) d s \\
&= u_{1} u_{2}-u_{1} \int_{0}^{t} f(s) \int_{t-s}^{t} d u_{2}(\tau) d s+h_{1}(t) \\
&= u_{1} u_{2}-\gamma m b x^{*} H_{1}\left(u_{1}, u_{2}\right)-u_{1} \int_{0}^{t} f(s) \int_{t-s}^{t} \widetilde{F}_{2} d \tau d s \\
&-u_{1} \int_{0}^{t} f(s) \int_{t-s}^{t} \sigma_{2} u_{2}(\tau) d B_{2}(\tau) d s+h_{1}(t) \tag{67}
\end{align*}
$$

where $h_{1}(t)$ is defined in (49), and

$$
\begin{align*}
& u_{2} \int_{0}^{\infty} g(s) u_{1}(t-s) d s \\
&= u_{1} u_{2}-u_{2} \int_{0}^{t} g(s) \int_{t-s}^{t} d u_{1}(\tau) d s+h_{2}(t) \\
&= u_{1} u_{2}+\left(D+m b x^{*}\right) H_{2}\left(u_{1}, u_{2}\right) \\
&+m a H_{3}\left(u_{1}, u_{2}\right)-\mu D_{1} H_{4}\left(u_{1}, u_{2}\right)  \tag{68}\\
& \quad-u_{2} \int_{0}^{t} g(s) \int_{t-s}^{t} F_{1} d \tau d s \\
&+u_{2} \int_{0}^{t} g(s) \int_{t-s}^{t} \sigma_{2} u_{2}(\tau) d B_{2}(\tau) d s+h_{2}(t)
\end{align*}
$$

where $h_{2}(t)$ is defined in (52). Substituting (67) and (68) into (46), we get

$$
\begin{aligned}
& L V_{1}\left(u_{1}, u_{2}\right) \\
& \leq-\left[2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2}\right. \\
& +2 q \gamma^{2}\left(D+m b x^{*}\right)-q \gamma^{2} \sigma_{1}^{2}-q \gamma^{2} m b x^{*} \\
& \left.-\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f}\right] u_{1}^{2} \\
& -\left[2 q \gamma m a-(1+q) \sigma_{2}^{2}-(1+q)\right. \\
& \times\left(D+m b x^{*}+m a+\mu D_{1}\right) \gamma m b x^{*} T_{g} \\
& \left.-q \gamma \mu D_{1}\right] u_{2}^{2} \\
& -2\left[p m a+q \gamma^{2} m a\right. \\
& +q \gamma\left(D+m b x^{*}\right)-(1+q) \gamma m b x^{*} \\
& \left.-\left(p+q \gamma^{2}\right) \mu D_{1}\right] u_{1} u_{2}+(1+q) \gamma m b x^{*}
\end{aligned}
$$

$$
\begin{align*}
& \times\left[\mu D_{1} \int_{0}^{\infty} g(s) \int_{t-s}^{t} \int_{0}^{\infty} f(v) u_{2}^{2}(\tau-v) d v d \tau d s\right. \\
& \quad+\left(D+m b x^{*}\right) \int_{0}^{\infty} g(s) \int_{t-s}^{t} u_{1}^{2}(\tau) d \tau d s \\
& \left.\quad+m a \int_{0}^{\infty} g(s) \int_{t-s}^{t} u_{2}^{2}(\tau) d \tau d s\right] \\
& +q \gamma^{2} m b x^{*} \int_{0}^{\infty} g(s) u_{1}^{2}(t-s) d s \\
& +q \gamma \mu D_{1} \int_{0}^{\infty} f(s) u_{2}^{2}(t-s) d s \\
& +\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} \\
& \times \int_{0}^{\infty} f(s) \int_{t-s}^{t} \int_{0}^{\infty} g(v) u_{1}^{2}(\tau-v) d v d \tau d s \\
& +2 p u_{1} F_{1}+2 u_{2} \widetilde{F}_{2}+2 q\left(\gamma u_{1}+u_{2}\right)\left(\gamma F_{1}+\widetilde{F}_{2}\right) \\
& -2\left(p+q \gamma^{2}\right) \mu D_{1} u_{1} \int_{0}^{\infty} f(s) \int_{t-s}^{t} \widetilde{F}_{2} d \tau d s \\
& +2\left(p+q \gamma^{2}\right) \mu D_{1} h_{1}(t) \\
& +2(1+q) \gamma m b x^{*} u_{2} \int_{0}^{\infty} g(s) \int_{t-s}^{t} F_{1} d \tau d s \\
& +2(1+q) \gamma m b x^{*} h_{2}(t) \tag{69}
\end{align*}
$$

For the functions $V_{2}\left(u_{1}, u_{2}\right)$ and $V_{3}\left(u_{1}, u_{2}\right)$ defined in (55) and (57), one has

$$
\begin{aligned}
L\left(V_{1}+\right. & \left.V_{2}+V_{3}\right) \\
\leq- & {\left[2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2}+2 q \gamma^{2}\left(D+m b x^{*}\right)\right.} \\
& -q \gamma^{2} \sigma_{1}^{2}-2 q \gamma^{2} m b x^{*} \\
& -2\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f}-(1+q) \\
& \left.\times\left(D+m b x^{*}\right) \gamma m b x^{*} T_{g}\right] u_{1}^{2} \\
- & {\left[2 q \gamma m a-(1+q) \sigma_{2}^{2}-(1+q)\right.} \\
& \times\left(D+m b x^{*}+2 m a+2 \mu D_{1}\right) \gamma m b x^{*} T_{g} \\
& \left.-2 q \gamma \mu D_{1}\right] u_{2}^{2} \\
- & 2\left[p m a+q \gamma^{2} m a+q \gamma\left(D+m b x^{*}\right)\right. \\
& \left.\quad(1+q) \gamma m b x^{*}-\left(p+q \gamma^{2}\right) \mu D_{1}\right] u_{1} u_{2} \\
+ & 2 p u_{1} F_{1}+2 u_{2} \widetilde{F}_{2}+2 q\left(\gamma u_{1}+u_{2}\right)\left(\gamma F_{1}+\widetilde{F}_{2}\right) \\
- & 2\left(p+q \gamma^{2}\right) \mu D_{1} u_{1} \int_{0}^{\infty} f(s) \int_{t-s}^{t} \widetilde{F}_{2} d \tau d s \\
+ & 2\left(p+q \gamma^{2}\right) \mu D_{1} h_{1}(t)
\end{aligned}
$$

$$
\begin{align*}
& -2(1+q) \gamma m b x^{*} u_{2} \int_{0}^{\infty} g(s) \int_{t-s}^{t} F_{1} d \tau d s \\
& +2(1+q) \gamma m b x^{*} h_{2}(t) \tag{70}
\end{align*}
$$

It follows from the expression of $h_{1}(t)$ and $h_{2}(t)$ that

$$
\begin{align*}
& h_{1}(t) \leq 2 u_{1}^{2} \int_{t}^{\infty} f(s) d s+\left(u_{2}^{2}+\left\|\varphi_{2}\right\|^{2}\right) \int_{t}^{\infty} f(s) d s \\
& h_{2}(t) \leq 2 u_{2}^{2} \int_{t}^{\infty} g(s) d s+\left(u_{1}^{2}+\left\|\varphi_{1}\right\|^{2}\right) \int_{t}^{\infty} g(s) d s \tag{71}
\end{align*}
$$

For $V\left(u_{1}, u_{2}\right)=V_{1}\left(u_{1}, u_{2}\right)+V_{2}\left(u_{1}, u_{2}\right)+V_{3}\left(u_{1}, u_{2}\right)$, one has

$$
\begin{aligned}
& L V\left(u_{1}, u_{2}\right) \\
& \leq-\left[2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2}\right. \\
& +2 q \gamma^{2}\left(D+m b x^{*}\right)-q \gamma^{2} \sigma_{1}^{2}-2 q \gamma^{2} m b x^{*} \\
& -2\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f}-(1+q) \\
& \times\left(D+m b x^{*}\right) \gamma m b x^{*} T_{g} \\
& -2\left(p+q \gamma^{2}\right) \mu D_{1} \int_{t}^{\infty} f(s) d s \\
& \left.-(1+q) \gamma m b x^{*} \int_{t}^{\infty} g(s) d s\right] u_{1}^{2} \\
& -\left[2 q \gamma m a-(1+q) \sigma_{2}^{2}-(1+q)\right. \\
& \times\left(D+m b x^{*}+2 m a+2 \mu D_{1}\right) \gamma m b x^{*} T_{g} \\
& -2 q \gamma \mu D_{1}-\left(p+q \gamma^{2}\right) \mu D_{1} \int_{t}^{\infty} f(s) d s \\
& \left.-2(1+q) \gamma m b x^{*} \int_{t}^{\infty} g(s) d s\right] u_{2}^{2} \\
& +\left(p+q \gamma^{2}\right) \mu D_{1}\left\|\varphi_{2}\right\|^{2} \int_{t}^{\infty} f(s) d s \\
& +(1+q) \gamma m b x^{*}\left\|\varphi_{1}\right\|^{2} \int_{t}^{\infty} g(s) d s \\
& +2 p u_{1} F_{1}+2 u_{2} \widetilde{F}_{2}+2 q\left(\gamma u_{1}+u_{2}\right)\left(\gamma F_{1}+\widetilde{F}_{2}\right) \\
& -2(1+q) \gamma m b x^{*} u_{2} \int_{0}^{\infty} g(s) \int_{t-s}^{t} F_{1} d \tau d s \\
& -2\left(p+q \gamma^{2}\right) \mu D_{1} u_{1} \int_{0}^{\infty} f(s) \int_{t-s}^{t} \widetilde{F}_{2} d \tau d s .
\end{aligned}
$$

Since $F_{1}$ and $\widetilde{F}_{2}$ are terms of order $\geq 2$ in $u_{1}, u_{2}$, then we have

$$
\begin{equation*}
\lim _{u_{1}, u_{2} \rightarrow 0} \frac{F_{1}\left(u_{1}, u_{2}\right)}{\sqrt{u_{1}^{2}+u_{2}^{2}}}=\lim _{u_{1}, u_{2} \rightarrow 0} \frac{\widetilde{F}_{2}\left(u_{1}, u_{2}\right)}{\sqrt{u_{1}^{2}+u_{2}^{2}}}=0 \tag{73}
\end{equation*}
$$

For $\varepsilon>0$, we can find a constant $\zeta \in(0,1)$ such that

$$
\begin{equation*}
F_{1}\left(u_{1}, u_{2}\right) \leq \frac{\varepsilon}{\sqrt{2}} \sqrt{u_{1}^{2}+u_{2}^{2}}, \quad \widetilde{F}_{2}\left(u_{1}, u_{2}\right) \leq \frac{\varepsilon}{\sqrt{2}} \sqrt{u_{1}^{2}+u_{2}^{2}} \tag{74}
\end{equation*}
$$

provided that $u_{1}^{2}+u_{2}^{2} \leq 2 \zeta^{2}$. Now consider the class of processes

$$
\begin{equation*}
\Psi=\left\{\varphi \in \mathscr{H} \mid P\left\{\sup _{-\infty \leq s \leq 0}|\varphi(s)|<\zeta\right\}=1\right\} \tag{75}
\end{equation*}
$$

Notice that for $u_{t} \in \Psi$,

$$
\begin{align*}
& \left|\int_{0}^{\infty} g(s) \int_{t-s}^{t} F_{1}(\tau) d \tau d s\right| \leq \varepsilon T_{g} \zeta  \tag{76}\\
& \left|\int_{0}^{\infty} f(s) \int_{t-s}^{t} \widetilde{F}_{2}(\tau) d \tau d s\right| \leq \varepsilon T_{f} \zeta
\end{align*}
$$

are valid. Substituting (74)-(76) into (72), we obtain

$$
\begin{aligned}
& L V\left(u_{1}, u_{2}\right) \\
& \leq-\left[2 p\left(D+m b x^{*}\right)-p \sigma_{1}^{2}\right. \\
& +2 q \gamma^{2}\left(D+m b x^{*}\right)-q \gamma^{2} \sigma_{1}^{2}-2 q \gamma^{2} m b x^{*} \\
& -2\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f} \\
& -(1+q)\left(D+m b x^{*}\right) \gamma m b x^{*} T_{g} \\
& -2\left(p+q \gamma^{2}\right) \mu D_{1} \int_{t}^{\infty} f(s) d s \\
& \left.-(1+q) \gamma m b x^{*} \int_{t}^{\infty} g(s) d s\right] u_{1}^{2} \\
& -\left[2 q \gamma m a-(1+q) \sigma_{2}^{2}-(1+q)\right. \\
& \times\left(D+m b x^{*}+2 m a+2 \mu D_{1}\right) \gamma m b x^{*} T_{g} \\
& -2 q \gamma \mu D_{1}-\left(p+q \gamma^{2}\right) \mu D_{1} \int_{t}^{\infty} f(s) d s \\
& \left.-2(1+q) \gamma m b x^{*} \int_{t}^{\infty} g(s) d s\right] u_{2}^{2} \\
& +\left(p+q \gamma^{2}\right) \mu D_{1}\left\|\varphi_{2}\right\|^{2} \int_{t}^{\infty} f(s) d s \\
& +(1+q) \gamma m b x^{*}\left\|\varphi_{1}\right\|^{2} \int_{t}^{\infty} g(s) d s \\
& +2 \varepsilon\left[p+1+q(\gamma+1)^{2}+(1+q) \gamma m b x^{*} T_{g}\right. \\
& \left.+\left(p+q \gamma^{2}\right) \mu D_{1} T_{f}\right] \zeta^{2} .
\end{aligned}
$$

Integrating both sides of the above formula from $T$ to $t \wedge T_{\varepsilon_{1}}$ yields

$$
\begin{align*}
E\left(V\left(t \wedge T_{\varepsilon_{1}}\right)\right) \leq & V(T)+\left(p+q \gamma^{2}\right) \mu D_{1}\left\|\varphi_{2}\right\|^{2} \\
& \times \int_{0}^{t \wedge T_{\varepsilon_{1}}} \int_{s}^{\infty} f(\tau) d \tau d s \\
& +(1+q) \gamma m b x^{*}\left\|\varphi_{1}\right\|^{2} \\
& \times \int_{0}^{t \wedge T_{\varepsilon_{1}}} \int_{s}^{\infty} g(\tau) d \tau d s+2 \varepsilon k_{1} \zeta^{2} \\
\leq & V(T)+\left(p+q \gamma^{2}\right) \mu D_{1}\left\|\varphi_{2}\right\|^{2} \int_{0}^{\infty} s f(s) d s \\
& +(1+q) \gamma m b x^{*}\left\|\varphi_{1}\right\|^{2} \\
& \times \int_{0}^{\infty} s g(s) d s+2 \varepsilon k_{1} \zeta^{2} \tag{78}
\end{align*}
$$

where

$$
\begin{align*}
k_{1}= & p+1+q(\gamma+1)^{2}+(1+q) \gamma m b x^{*} T_{g}  \tag{79}\\
& +\left(p+q \gamma^{2}\right) \mu D_{1} T_{f} .
\end{align*}
$$

By the definition of function $V\left(u_{1}, u_{2}\right)$, we can find a constant $k_{2}>0$ such that

$$
\begin{equation*}
V(T) \leq k_{2}\left(\left\|\varphi_{1}\right\|^{2}+\left\|\varphi_{2}\right\|^{2}\right) \tag{80}
\end{equation*}
$$

Obviously,

$$
\begin{equation*}
E\left(V\left(t \wedge T_{\varepsilon_{1}}\right)\right) \leq k_{3}\left(\left\|\varphi_{1}\right\|^{2}+\left\|\varphi_{2}\right\|^{2}\right)+2 \varepsilon k_{1} \zeta^{2} \tag{81}
\end{equation*}
$$

where $k_{3}=\max \left\{k_{2}+(1+q) \gamma m b x^{*}, k_{2}+\left(p+q \gamma^{2}\right) \mu D_{1}\right\}$. Now for $\varepsilon_{1}, \varepsilon_{2} \in(0,1)$, let

$$
\begin{equation*}
\delta=\min \left\{\left(\frac{1 \wedge p}{2 \varepsilon k_{1}+k_{3}} \varepsilon_{2}\right)^{1 / 2} \varepsilon_{1}, \frac{\varepsilon_{1}}{2}, \frac{\zeta}{2}\right\} \tag{82}
\end{equation*}
$$

and $\left\|\varphi_{1}\right\|^{2}+\left\|\varphi_{2}\right\|^{2}<\delta^{2}$. Then it follows that

$$
\begin{equation*}
E\left(V\left(t \wedge T_{\varepsilon_{1}}\right)\right) \leq\left(2 \varepsilon k_{1}+k_{3}\right) \delta^{2} \leq(1 \wedge p) \varepsilon_{1}^{2} \varepsilon_{2} \tag{83}
\end{equation*}
$$

On the other hand, we have

$$
\begin{align*}
E\left(V\left(t \wedge T_{\varepsilon_{1}}\right)\right) & \geq E\left[1_{\left\{T_{\varepsilon_{1}} \leq t\right\}} V\left(t \wedge T_{\varepsilon_{1}}\right)\right] \\
& =E\left[1_{\left\{T_{\varepsilon_{1}} \leq t\right\}} V\left(T_{\varepsilon_{1}}\right)\right]  \tag{84}\\
& =P\left\{T_{\varepsilon_{1}} \leq t\right\} V\left(T_{\varepsilon_{1}}\right) \\
& \geq(1 \wedge p) \varepsilon_{1}^{2} P\left\{T_{\varepsilon_{1}} \leq t\right\}
\end{align*}
$$

Hence, we have $P\left\{T_{\varepsilon_{1}} \leq t\right\} \leq \varepsilon_{2}$. Let $t \rightarrow \infty$; then

$$
\begin{equation*}
P\left\{T_{\varepsilon_{1}}<\infty\right\} \leq \varepsilon_{2} \tag{85}
\end{equation*}
$$

Equivalently,

$$
\begin{equation*}
P\left\{u_{1}^{2}+u_{2}^{2}<\varepsilon_{1}^{2}\right\} \geq 1-\varepsilon_{2} . \tag{86}
\end{equation*}
$$

Applying Definition 4, we obtain the conclusion.

## 5. Simulations and Discussions

In this paper, we have considered a stochastic chemostat model simulating the process of wastewater treatment. The model incorporates a general nutrient uptake function and two distributed delays. The first delay models the fact that nutrient is partially recycled after the death of the biomass by bacterial decomposition and the second indicates that the growth of the species depends on the past concentration of the nutrient. Furthermore, we consider the stochastic perturbations which are of white noise type and are proportional to the distances of $S(t), x(t)$ from the values of the positive equilibrium $S^{*}, x^{*}$. By constructing appropriate Liapunovlike functionals, some sufficient conditions for the stochastic stability of the positive equilibrium have been obtained.

For model (3), we have first analyzed the stochastic stability of the positive equilibrium $E^{*}$ in the case when the delays are ignored, that is, the average delays $T_{f}=T_{g}=0$. Our findings in Theorem 6 reveal that $E^{*}$ is stochastically stable provided that the intensities of noises are small. When at least one of the average delays $T_{f}$ and $T_{g}$ is not equal to zero, our results in Theorem 8 reveal that $E^{*}$ is stochastically stable provided that the average delays $T_{f}$ and $T_{g}$ are both small. Obviously, Theorem 8 reduces to Theorem 6 when $T_{f}=T_{g}=0$, which indicates that if the average delays are sufficiently small, $E^{*}$ is still stochastically stable; and in the case of $\sigma_{i}=0(i=1,2)$, Theorem 8 reduces to He et al. [18, Theorem 3.1]; that is to say, the equilibrium $E^{*}$ of model (3) is still stable if $\sigma_{1}$ and $\sigma_{2}$ are sufficient small, which preserves the dynamics of its corresponding deterministic counterpart (5).

To illustrate the results obtained above, some numerical simulations are carried out by using Milstein scheme [50]. Here we assume that the specific growth function $U(S)$ is of Michaelis-Menten type

$$
\begin{equation*}
U(S)=\frac{S}{a_{1}+S}, \tag{87}
\end{equation*}
$$

where $a_{1}$ is the half-saturation constant. For the kernel functions $f(s)$ and $g(s)$, we consider two special cases: (1) $f(s)=g(s)=\delta(0)$; (2) $f(s)=\alpha e^{-\alpha s}$ and $g(s)=$ $\beta e^{-\beta s}$. For case (1), the discretization of model (3) for $t=$ $0, \Delta t, 2 \Delta t, \ldots, n \Delta t$ takes the form

$$
\begin{align*}
S_{i+1}= & S_{i}+\left[D\left(S^{0}-S_{i}\right)-m U\left(S_{i}\right) x_{i}+\mu D_{1} x_{i}\right] \Delta t \\
& +\sigma_{1}\left(S_{i}-S^{*}\right) \sqrt{\Delta t} \xi_{i},  \tag{88}\\
x_{i+1}= & x_{i}+x_{i}\left[-\left(D_{w}+D_{1}\right)+\gamma m U\left(S_{i}\right)\right] \Delta t \\
& +\sigma_{2}\left(x_{i}-x^{*}\right) \sqrt{\Delta t} \xi_{i},
\end{align*}
$$

where time increment $\Delta t>0$ and $\xi_{i}$ is $N(0,1)$-distributed independent random variables which can be generated


FIGURE 1: The dynamics of stochastic model compared with deterministic model with $\sigma_{1}=0.1$ and $\sigma_{2}=0.08$. Here $S(0)=0.3, x(0)=0.5$.


Figure 2: The dynamics of stochastic model with different values of $\sigma_{1}$ and $\sigma_{2}$. Here $S(0)=0.3, x(0)=0.5$.
numerically by pseudorandom number generators. For case (2), define

$$
\begin{gather*}
y(t)=\int_{0}^{\infty} \alpha e^{-\alpha s} x(t-s) d s  \tag{89}\\
z(t)=\int_{0}^{\infty} \beta e^{-\beta s} U(S(t-s)) d s
\end{gather*}
$$

then the discretization of model (3) for $t=0, \Delta t, 2 \Delta t, \ldots, n \Delta t$ takes the form

$$
\begin{aligned}
S_{i+1}= & S_{i}+\left[D\left(S^{0}-S_{i}\right)-m U\left(S_{i}\right) x_{i}+\mu D_{1} y_{i}\right] \Delta t \\
& +\sigma_{1}\left(S_{i}-S^{*}\right) \sqrt{\Delta t} \xi_{i}
\end{aligned}
$$

$$
\begin{gather*}
x_{i+1}=x_{i}+x_{i}\left[-\left(D_{w}+D_{1}\right)+\gamma m z_{i}\right] \Delta t \\
+\sigma_{2}\left(x_{i}-x^{*}\right) \sqrt{\Delta t} \xi_{i}, \\
y_{i+1}=y_{i}+\left(-\alpha y_{i}+\alpha x_{i}\right) \Delta t, \\
z_{i+1}=z_{i}+\left(-\beta z_{i}+\beta U\left(S_{i}\right)\right) \Delta t . \tag{90}
\end{gather*}
$$

Let in model (3) $D=D_{w}=0.3, D_{1}=0.1, S^{0}=5, m=$ $0.7, a_{1}=0.4, \mu=0.3, \gamma=0.8$. It is easy to compute that $a \doteq 0.7143, b \doteq 0.2041, p \doteq 0.3100, q \doteq 0.2694$, and $E^{*}=$ $(1,2.55)$.

The first two examples given below concern case (1) when the delays are ignored; that is to say, it is assumed that the


Figure 3: The dynamics of stochastic functional model with different $\alpha, \beta$. Here $S(0)=0.3, x(0)=0.5, y(0)=0.3, z(0)=0.5$.



$$
\sigma_{1}=0.1, \sigma_{2}=0.08
$$

$$
\sigma_{1}=0.1, \sigma_{2}=0.08
$$

$$
\alpha=1, \beta=5
$$

$$
\sigma_{1}=1, \sigma_{2}=0.8
$$

$$
\alpha=0.1, \beta=0.1
$$

Figure 4: The dynamics of stochastic functional model with different $\sigma_{1}, \sigma_{2}$ and $\alpha, \beta$. Here $S(0)=0.3, x(0)=0.5, y(0)=0.3, z(0)=0.5$.
process of nutrient recycling and the growth response of the species are immediate and, therefore, $T_{f}=T_{g}=0$. Example 1 verifies the results obtained in Theorem 6.
Example 1. Let $\sigma_{1}=0.1$ and $\sigma_{2}=0.08$, then by straightforward computations, we have that $0.01=\sigma_{1}^{2}<$ $2 D+2 m b x^{*} \doteq 1.3285,0.0064=\sigma_{2}^{2}<(2 q /(1+q)) \gamma(m a-$ $\left.\mu D_{1}\right) \doteq 0.1596$. In view of Theorem 6 , the equilibrium $E^{*}$ of (3) is stochastically asymptotically stable, which is consistent with the simulation results as shown in Figure 1.

To further study the combined effects of $\sigma_{i}, i=1,2$ when $T_{f}=T_{g}=0$, we need to consider four situations: (a) $\sigma_{1}$ increases, $\sigma_{2}$ increases; (b) $\sigma_{1}$ increases, $\sigma_{2}$ decreases; (c) $\sigma_{1}$
decreases, $\sigma_{2}$ increases; (d) $\sigma_{1}$ decreases, $\sigma_{2}$ decreases. Here we only give one example about situation (a); other situations can be considered similarly.

Example 2. Let the intensities $\sigma_{i}, i=1,2$ increase from $\sigma_{1}=0.1, \sigma_{2}=0.08$ to $\sigma_{1}=1, \sigma_{2}=0.12$, respectively. Simulations show that the trajectories of model (3) still approach ultimately to the positive equilibrium $E^{*}$, but they need to go through more oscillations and more time to return to $E^{*}$ (see Figure 2).

The next two examples concern case (2) when $f(s)$ and $g(s)$ take weak kernels; that is, $f(s)=\alpha e^{-\alpha s}$ and $g(s)=\beta e^{-\beta s}$,


Figure 5: The positive equilibrium $E^{*}$ is stochastically stable provided that $\left(\sigma_{1}, T_{f}\right) \in \Omega_{\sigma_{1}, T_{f}}$. Here $\sigma_{2}=0.08$ and $T_{g}=0.2$.


Figure 6: The positive equilibrium $E^{*}$ is stochastically stable provided that $\left(\sigma_{2}, T_{f}\right) \in \Omega_{\sigma_{2}, T_{f}}$. Here $\sigma_{1}=0.1$ and $T_{g}=0.2$.


Figure 7: The positive equilibrium $E^{*}$ is stochastically stable provided that $\left(T_{f}, T_{g}\right) \in \Omega_{T_{f}, T_{g}}$. Here $\sigma_{1}=0.1$ and $\sigma_{2}=0.08$.
which means that $T_{f}=1 / \alpha$ and $T_{g}=1 / \beta$. Example 3 verifies the results obtained in Theorem 8.

Example 3. Let $\sigma_{1}=0.1, \sigma_{2}=0.08, \alpha=1$ and $\beta=5$. It is easy to compute that $\left(p+q \gamma^{2}\right) \sigma_{1}^{2}+2\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f}+(1+$ $q)\left(D+m b x^{*}\right) \gamma m b x^{*} T_{g} \doteq 0.0624,2 p\left(D+m b x^{*}\right)+2 q \gamma^{2} D \doteq$ 0.5154 and $\sigma_{2}^{2}+\left(D+m b x^{*}+2 m a+2 \mu D_{1}\right) \gamma m b x^{*} T_{g} \doteq 0.1069$, $(2 q /(1+q)) \gamma\left(m a-\mu D_{1}\right) \doteq 0.1596$; thus conditions (44) are satisfied. By Theorem 8 , the equilibrium $E^{*}$ of model (3) is stochastically stable. Our simulation supports this result as shown in Figure 3.

To examine the combined effects of the noise intensities and the delays on the dynamics of model (3), we first consider the case when the values of $\sigma_{i}, i=1,2$ in Example 3 are fixed and the values of $\alpha$ and $\beta$ are reduced from 1 and 5 to 0.1 and 0.1 , respectively. That is to say, the average delays $T_{f}$ and $T_{g}$ increase from 1 and 0.2 to 10 and 10 , respectively. Simulation results show that the solution of (3) will suffer more oscillations and more time to approach the equilibrium $E^{*}$ when delays increase (see Figure 3). When both the values of the noise intensities and the delays vary, the dynamics of model (3) may become more complicated. Here we only consider the case when $\sigma_{i}(i=1,2), T_{f}$ and $T_{g}$ (i.e., $1 / \alpha$ and $1 / \beta$ ) all increase. See the following Example.

Example 4. Let $\sigma_{i}(i=1,2), T_{f}$ and $T_{g}$ (i.e., $1 / \alpha$ and $1 / \beta$ ) increase from $0.1,0.08,1$, and 0.2 (i.e., $\alpha=1$ and $\beta=5$ ) to $1,0.8,10$, and 10 (i.e., $\alpha=0.1$ and $\beta=0.1$ ), respectively. It is found that the trajectories of model (3) fluctuate wildly and suffer more oscillations and need more time to approach the equilibrium $E^{*}$; please see Figure 4.

Notice also that conditions (44) in Theorem 8 are only sufficient conditions to insure the stochastic stability of $E^{*}$, which are dependent on parameters $\sigma_{1}, \sigma_{2}, T_{f}$, and $T_{g}$. Define

$$
\begin{align*}
M_{0}= & \left(\left(p+q \gamma^{2}\right) \sigma_{1}^{2}+2\left(p+q \gamma^{2}\right) \mu D_{1} \gamma m b x^{*} T_{f}\right. \\
& \left.+(1+q)\left(D+m b x^{*}\right) \gamma m b x^{*} T_{g}\right) \\
& \times\left(2 p\left(D+m b x^{*}\right)+2 q \gamma^{2} D\right)^{-1}  \tag{91}\\
M_{1}= & \frac{\sigma_{2}^{2}+\left(D+m b x^{*}+2 m a+2 \mu D_{1}\right) \gamma m b x^{*} T_{g}}{(2 q /(1+q)) \gamma\left(m a-\mu D_{1}\right)}
\end{align*}
$$

Thus, conditions (44) are equivalent to those when parameters $\sigma_{1}, \sigma_{2}, T_{f}$, and $T_{g}$ are seated in the following parameter set:

$$
\begin{align*}
\Omega=\{ & \left(\sigma_{1}, \sigma_{2}, T_{f}, T_{g}\right) \mid \max \left\{M_{0}, M_{1}\right\}<1, \\
& \left.\sigma_{i} \geq 0, T_{f} \geq 0, T_{g} \geq 0\right\} \tag{92}
\end{align*}
$$

from which we can further perform some approximate sensitivity analysis of the stochastic stability of $E^{*}$ with respect to these parameters. To do this, we can let two of the parameters (e.g., $\sigma_{1}$ and $T_{f}$ ) vary and the other two ( $\sigma_{2}$ and $T_{g}$ ) be fixed, which have six cases in all.

Let us first consider the case when $\sigma_{2}=0.08$ and $T_{g}=0.2$; then $M_{0}$ and $M_{1}$ are both functions of $\sigma_{1}$ and $T_{f}$. Then $\Omega$ defined in (92) is equivalent to

$$
\begin{equation*}
\Omega_{\sigma_{1}, T_{f}}=\left\{\left(\sigma_{1}, T_{f}\right) \mid\left(\sigma_{1}, 0.08, T_{f}, 0.2\right) \in \Omega\right\} \tag{93}
\end{equation*}
$$

which is the projection of surfaces $M_{0}=M_{0}\left(\sigma_{1}, T_{f}\right)$ and $M_{1}=M_{1}\left(\sigma_{1}, T_{f}\right)$ in the first octant such that $\max \left\{M_{0}, M_{1}\right\}<$ 1 (see Figure 5). The positive equilibrium $E^{*}$ is stochastically stable provided that $\left(\sigma_{1}, T_{f}\right) \in \Omega_{\sigma_{1}, T_{f}}$.

To better observe the dependence of the stochastic stability of $E^{*}$ on all parameters, we further consider another two cases when $\sigma_{1}=0.1$ and $T_{g}=0.2$ are fixed and $\sigma_{1}=0.1$ and $\sigma_{2}=0.08$ are fixed. Accordingly, $\Omega$ defined in (92) is equivalent, respectively, to

$$
\begin{gather*}
\Omega_{\sigma_{2}, T_{f}}=\left\{\left(\sigma_{2}, T_{f}\right) \mid\left(0.1, \sigma_{2}, T_{f}, 0.2\right) \in \Omega\right\} \\
\Omega_{T_{f}, T_{g}}=\left\{\left(T_{f}, T_{g}\right) \mid\left(0.1,0.08, T_{f}, T_{g}\right) \in \Omega\right\} \tag{94}
\end{gather*}
$$

which are plotted, respectively, in Figures 6 and 7 (other three cases can be considered similarly). From Figures 5-7, we find that the stochastic stability of $E^{*}$ is greatly affected by $\sigma_{1}, \sigma_{2}$, and $T_{g}$ and less affected by $T_{f}$ (which is consistent with the results observed in [13, 17]). We would like to point out here that $E^{*}$ may also be stable when the parameters are seated outside of the set $\Omega$, since (44) are only sufficient conditions ensuring the stochastic stability of $E^{*}$.

In conclusion, this paper presents an investigation on the combined effect of the noises and delays on a bottom-microbe model. Our findings are useful for better understanding of the dynamics of microbial population in the activated sludge process. We should point out that there are still some other interesting topics about the wastewater treatment deserving further investigation, for example, membrane reactor, and so forth. We leave these for future considerations.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work is supported by the National Natural Science Foundation of China (no. 11271260), Shanghai Leading Academic Discipline Project (no. XTKX2012), and the Innovation Program of Shanghai Municipal Education Commission (no. 13ZZ116).

## References

[1] M. Henze, C. P. L. Grady Jr., and W. Gujer, "A general model for single-sludge wastewater treatment systems," Water Research, vol. 21, no. 5, pp. 505-515, 1987.
[2] J. C. Kabouris and A. P. Georgakakos, "Parameter and state estimation of the activated sludge process-I. Model development," Water Research, vol. 30, no. 12, pp. 2853-2865, 1996.
[3] M. Zaiat, F. H. Passig, and E. Foresti, "A mathematical model and criteria for designing horizontal-flow anaerobic immobilized biomass reactors for wastewater treatment," Bioresource Technology, vol. 71, no. 3, pp. 235-243, 2000.
[4] G. Gehlert and J. Hapke, "Mathematical modeling of a continuous aerobic membrane bioreactor for the treatment of different kinds of wastewater," Desalination, vol. 146, no. 1-3, pp. 405-412, 2002.
[5] B. Beran and F. Kargi, "A dynamic mathematical model for wastewater stabilization ponds," Ecological Modelling, vol. 181, no. 1, pp. 39-57, 2005.
[6] Y. Peng, B. Wang, and S. Wang, "Multivariable optimal control of activated sludge process: I. Basic theory and effect of DO on operation cost," Acta Scientiae Circumstantiae, vol. 18, no. 1, pp. 11-19, 1998.
[7] Z. Chen and T. Zhang, "Long time behaviour of a stochastic model for continuous flow bioreactor," Journal of Mathematical Chemistry, vol. 51, no. 2, pp. 451-464, 2013.
[8] Z. Chen and T. Zhang, "Dynamics of a stochastic model for continuous flow bioreactor with Contois growth rate," Journal of Mathematical Chemistry, vol. 51, no. 3, pp. 1076-1091, 2013.
[9] S. Yuan, D. Xiao, and M. Han, "Competition between plasmidbearing and plasmid-free organisms in a chemostat with nutrient recycling and an inhibitor," Mathematical Biosciences, vol. 202, no. 1, pp. 1-28, 2006.
[10] J. Luo, S. Yuan, and W. Zhang, "Competition between two microorganisms in the chemostat with general variable yields and general growth rates," International Journal of Biomathematics, vol. 1, no. 4, pp. 463-474, 2008.
[11] H. L. Smith and P. Waltman, The Theory of the Chemostat: Dynamics of Microbial Competition, Cambridge University Press, Cambridge, UK, 1995.
[12] H. I. Freedman and Y. T. Xu, "Models of competition in the chemostat with instantaneous and delayed nutrient recycling," Journal of Mathematical Biology, vol. 31, no. 5, pp. 513-527, 1993.
[13] S. G. Ruan, "The effect of delays on stability and persistence in plankton models," Nonlinear Analysis. Theory, Methods \& Applications A, vol. 24, no. 4, pp. 575-585, 1995.
[14] S. Yuan, W. Zhang, and M. Han, "Global asymptotic behavior in chemostat-type competition models with delay," Nonlinear Analysis. Real World Applications, vol. 10, no. 3, pp. 1305-1320, 2009.
[15] S. Ruan and G. S. K. Wolkowicz, "Bifurcation analysis of a chemostat model with a distributed delay," Journal of Mathematical Analysis and Applications, vol. 204, no. 3, pp. 786-812, 1996.
[16] B. Li, G. S. K. Wolkowicz, and Y. Kuang, "Global asymptotic behavior of a chemostat model with two perfectly complementary resources and distributed delay," SIAM Journal on Applied Mathematics, vol. 60, no. 6, pp. 2058-2086, 2000.
[17] E. Beretta, G. I. Bischi, and F. Solimano, "Stability in chemostat equations with delayed nutrient recycling," Journal of Mathematical Biology, vol. 28, no. 1, pp. 99-111, 1990.
[18] X.-Z. He, S. Ruan, and H. Xia, "Global stability in chemostattype equations with distributed delays," SIAM Journal on Mathematical Analysis, vol. 29, no. 3, pp. 681-696, 1998.
[19] J. Jiao, X. Yang, L. Chen, and S. Cai, "Effect of delayed response in growth on the dynamics of a chemostat model with impulsive input," Chaos, Solitons and Fractals, vol. 42, no. 4, pp. 2280-2287, 2009.
[20] X. Meng, Q. Gao, and Z. Li, "The effects of delayed growth response on the dynamic behaviors of the Monod type chemostat model with impulsive input nutrient concentration," Nonlinear Analysis. Real World Applications, vol. 11, no. 5, pp. 44764486, 2010.
[21] Z. Zhao, L. Chen, and X. Song, "Extinction and permanence of chemostat model with pulsed input in a polluted environment," Communications in Nonlinear Science and Numerical Simulation, vol. 14, no. 4, pp. 1737-1745, 2009.
[22] Z. Li, L. Chen, and Z. Liu, "Periodic solution of a chemostat model with variable yield and impulsive state feedback control," Applied Mathematical Modelling, vol. 36, no. 3, pp. 1255-1266, 2012.
[23] Y. Tian, K. Sun, L. Chen, and A. Kasperski, "Studies on the dynamics of a continuous bioprocess with impulsive state feedback control," Chemical Engineering Journal, vol. 157, no. 23, pp. 558-567, 2010.
[24] O. Tagashira and T. Hara, "Delayed feedback control for a chemostat model," Mathematical Biosciences, vol. 201, no. 1-2, pp. 101-112, 2006.
[25] L. Imhof and S. Walcher, "Exclusion and persistence in deterministic and stochastic chemostat models," Journal of Differential Equations, vol. 217, no. 1, pp. 26-53, 2005.
[26] F. Campillo, M. Joannides, and I. Larramendy-Valverde, "Stochastic modeling of the chemostat," Ecological Modelling, vol. 222, no. 15, pp. 2676-2689, 2011.
[27] B. S. Khatri, A. Free, and R. J. Allen, "Oscillating microbial dynamics driven by small populations, limited nutrient supply and high death rates," Journal of Theoretical Biology, vol. 314, pp. 120-129, 2012.
[28] S. R. Hansen and S. P. Hubbell, "Single-nutrient microbial competition: qualitative agreement between experimental and theoretically forecast outcomes," Science, vol. 207, no. 4438, pp. 1491-1493, 1980.
[29] D. Porro, E. Martegani, B. M. Ranzi, and L. Alberghina, "Oscillation in continuous cultures of budding yeast: a segregated parameter analysis," Biotechnology and Bioengineering, vol. 32, no. 4, pp. 411-417, 1988.
[30] S. Yuan and M. Han, "Bifurcation analysis of a chemostat model with two distributed delays," Chaos, Solitons \& Fractals, vol. 20, no. 5, pp. 995-1004, 2004.
[31] S. Yuan, Y. Song, and M. Han, "Direction and stability of bifurcating periodic solutions of a chemostat model with two distributed delays," Chaos, Solitons \& Fractals, vol. 21, no. 5, pp. 1109-1123, 2004.
[32] S. Yuan, P. Li, and Y. Song, "Delay induced oscillations in a turbidostat with feedback control," Journal of Mathematical Chemistry, vol. 49, no. 8, pp. 1646-1666, 2011.
[33] S. Yuan and T. Zhang, "Dynamics of a plasmid chemostat model with periodic nutrient input and delayed nutrient recycling," Nonlinear Analysis. Real World Applications, vol. 13, no. 5, pp. 2104-2119, 2012.
[34] S. Yuan, W. Zhang, and Y. Zhao, "Bifurcation analysis of a model of plasmid-bearing, plasmid-free competition in a pulsed chemostat with an internal inhibitor," IMA Journal of Applied Mathematics, vol. 76, no. 2, pp. 277-297, 2011.
[35] S. Yuan, Y. Zhao, A. Xiao, and T. Zhang, "Bifurcation and chaos in a pulsed plankton model with instantaneous nutrient recycling," The Rocky Mountain Journal of Mathematics, vol. 42, no. 4, pp. 1387-1409, 2012.
[36] P. L. Dold and G. v. R. Marais, "Evaluation of the general activated sludge model proposed by the IAWPRC task group," Water Science and Technology, vol. 18, no. 6, pp. 63-89, 1986.
[37] J. Caperon, "Time lag in population growth response of isochrysis galbana to a variable nitrate environment," Ecology, vol. 50, no. 2, pp. 188-192, 1969.
[38] A. Cunningham and R. M. Nisbet, "Time lag and co-operativity in the transient growth dynamics of microalgae," Journal of Theoretical Biology, vol. 84, no. 2, pp. 189-203, 1980.
[39] Y. Kuang, Delay Differential Equations with Applications in Populatin Dynamics, Academic Press, New York, NY, USA, 1993.
[40] L. Yu, X. Hu, R. Lin, H. Zhang, Z. Nan, and F. Li, "The effects of environmental conditions on the growth of petroleum microbes by microcalorimetry," Thermochimica Acta, vol. 359, no. 2, pp. 95-101, 2000.
[41] X. Mao, Y. Shen, and C. Yuan, "Almost surely asymptotic stability of neutral stochastic differential delay equations with Markovian switching," Stochastic Processes and Their Applications, vol. 118, no. 8, pp. 1385-1406, 2008.
[42] X. Mao, "Stationary distribution of stochastic population systems," Systems \& Control Letters, vol. 60, no. 6, pp. 398-405, 2011.
[43] D. Jiang, N. Shi, and X. Li, "Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation," Journal of Mathematical Analysis and Applications, vol. 340, no. 1, pp. 588-597, 2008.
[44] D. Jiang, C. Ji, N. Shi, and J. Yu, "The long time behavior of DI SIR epidemic model with stochastic perturbation," Journal of Mathematical Analysis and Applications, vol. 372, no. 1, pp. 162-180, 2010.
[45] M. Liu and K. Wang, "Asymptotic behavior of a stochastic nonautonomous Lotka-Volterra competitive system with impulsive perturbations," Mathematical and Computer Modelling, vol. 57, no. 3-4, pp. 909-925, 2013.
[46] M. Liu and K. Wang, "Global asymptotic stability of a stochastic Lotka-Volterra model with infinite delays," Communications in Nonlinear Science and Numerical Simulation, vol. 17, no. 8, pp. 3115-3123, 2012.
[47] R. Z. KHasminskii, Stochastic Stability of Differential Equations, Sijthoff \& Noordhoff, Alphen aan den Rijn, The Netherlands, 1980.
[48] X. Mao, Stochastic Differential Equations and Their Applications, Horwood Publishing, Chichester, UK, 1997.
[49] V. B. Kolmanovskiĭ and V. R. Nosov, Stability of Functional Differential Equations, Academic Press, New York, NY, USA, 1986.
[50] D. J. Higham, "An algorithmic introduction to numerical simulation of stochastic differential equations," SIAM Review, vol. 43, no. 3, pp. 525-546, 2001.

## Research Article

# Pattern Formation in a Bacterial Colony Model 

Xinze Lian, ${ }^{1,2}$ Guichen Lu, ${ }^{3}$ and Hailing Wang ${ }^{4}$<br>${ }^{1}$ School of Mathematics, Wenzhou University, Wenzhou 325000, China<br>${ }^{2}$ Chengdu Institute of Computer Application, Chinese Academy of Sciences, Chengdu 610041, China<br>${ }^{3}$ School of Mathematics and Statistics, Chongqing University of Technology, Chongqing 400054, China<br>${ }^{4}$ Department of Mathematics, Hubei Minzu University, Enshi, Hubei 445000, China

Correspondence should be addressed to Xinze Lian; xinzelian@163.com and Guichen Lu; bromn006@gmail.com
Received 2 January 2014; Accepted 8 February 2014; Published 23 March 2014
Academic Editor: Kaifa Wang
Copyright © 2014 Xinze Lian et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We investigate the spatiotemporal dynamics of a bacterial colony model. Based on the stability analysis, we derive the conditions for Hopf and Turing bifurcations. Furthermore, we present novel numerical evidence of time evolution of patterns controlled by parameters in the model and find that the model dynamics exhibit a diffusion controlled formation growth to spots, holes and stripes pattern replication, which show that the bacterial colony model is useful in revealing the spatial predation dynamics in the real world.


## 1. Introduction

Spatial patterns which are formed by some kinds of bacterial colonies present an interesting structure during their growth conditions. In particular, colonies of bacterium bacillus subtilis can present a rich variety of structures [1-13]. The nature of the pattern exhibited depends on the particular bacterial species used and the environmental conditions imposed. Ohgiwari et al. [11] have shown that for a nutrient-poor solid agar, the bacterium colonies exhibit fractal morphogenesis similar to diffusion-limited aggregation (DLA). For softer agar medium, the colonies tend to show a dense-branching morphology (DBM) [7]. If both the nutrient concentration and the agar's softness further increase, simple circular colonies grow almost homogeneously in space [14].

There are many mathematical models for explaining each characteristic colony pattern. Kawasaki et al. [7] have developed a reaction-diffusion model and have shown the patterns by using the computer simulations. Since in Kawasaki et al.s model, all the nutrients must be consumed; L. Braverman and E. Braverman [4] have introduced a model of prey-predator type with Holling-II functional response under the situation of a renewable nutrient. In the present paper, motivated by the work of L. Braverman and E. Braverman, we consider
the model with the consumption term of nutrient in a Holling III functional response.

Let us denote by $u(t, x, y)$ and $v(t, x, y)$ the nutrient concentration and the density of the bacterial cells at point $(x, y)$, respectively. We consider the following system:

$$
\begin{align*}
& \frac{\partial u}{\partial t}=D_{u} \nabla^{2} u-\frac{\kappa u^{2} v}{v^{2}+\gamma_{0}^{2} u^{2}}+r u\left(1-\frac{u}{M}\right)  \tag{1}\\
& \frac{\partial v}{\partial t}=\nabla \cdot\left(D_{v} \nabla v\right)+\theta \frac{\kappa u^{2} v}{v^{2}+\gamma_{0}^{2} u^{2}}-\gamma v
\end{align*}
$$

where $r$ is the intrinsic nutrient growth rate, $M$ is the carrying capacity of the environment for the nutrient (prey), $\gamma$ is the bacteria (predator) mortality rate, $\kappa, \theta$, and $\gamma_{0}$ are parameters of the Holling Type III functional response, and $D_{v}$ is the nutrient diffusion coefficient. Following [4, 7], we assumed that the diffusion coefficient is proportional to both nutrient and bacteria densities

$$
\begin{equation*}
D_{v}=\sigma u v . \tag{2}
\end{equation*}
$$

Here we try to model the situation of a renewable nutrient. Then the system involves two reaction-diffusion equations of a predator-prey type with a Holling Type III
functional response. Diffusive predator-prey systems were extensively studied; we mention here the recent papers [1519], the monograph [20], and the references therein. In the present paper, it is to investigate the spatial pattern formation of system (1) which means the convergence of solutions to some stable spatially-in-homogeneous pattern as time tends to infinity. And in natural science, the pattern formation can reveal the evolution process of the species; it is, perhaps, the most challenging in modern ecology, biology, chemistry, and many other fields of science [21-38]. Thus, our basic concern is to find, if any, a spatially inhomogeneous equilibrium and periodic solutions that are stable in a certain sense. From the pioneer work by Turing [12], it is widely known that a reaction-diffusion system exhibits Turing instability if the homogenous steady state is stable to small perturbations in the absence of diffusion but unstable to small spatial perturbations when diffusion is present which implies the existence of spatially in-homogenous solutions. From the Hopf bifurcation analysis and the phrase transition theory developed by Ma and Wang [39-42], it is shown that the periodic solutions exist [43].

The paper is organized as follows. In Section 2, we give the analysis of the model and mathematical setup. In Section 3, we analyze the spatial model, we derive the conditions of the Turing bifurcation and Hopf bifurcation, and we give the existence of periodic solution. We give some computer simulations to illustrate the emergence of pattern formation in Section 4. Finally, some conclusions are given.

## 2. Modeling Analysis and Mathematical Setup

To obtain the dimensionless form of the system (1), we introduce the following:

$$
\begin{gather*}
u=M u^{\prime}, \quad v=M \gamma_{0} v^{\prime}, \quad t=\frac{1}{r} t^{\prime}, \quad x=\left(\frac{1}{r}\right)^{1 / 2} x^{\prime} \\
y=\left(\frac{1}{r}\right)^{1 / 2} y^{\prime}, \quad \gamma=\frac{1}{r} \gamma^{\prime}, \quad \sigma=\frac{1}{\gamma_{0} M^{2}} \sigma^{\prime} \tag{3}
\end{gather*}
$$

Omitting the primes, we obtain the following nondimensional form of (1):

$$
\begin{align*}
& \frac{\partial u}{\partial t}=D_{u} \nabla^{2} u-\alpha \frac{u^{2} v}{v^{2}+u^{2}}+u(1-u)  \tag{4}\\
& \frac{\partial v}{\partial t}=\sigma \nabla \cdot(u v \nabla v)+\beta \frac{u^{2} v}{v^{2}+u^{2}}-\gamma v
\end{align*}
$$

with $\alpha=\kappa / r \gamma_{0}, \beta=\theta \kappa / r \gamma_{0}$.
Model (4) is to be analyzed under the following nonzero initial conditions:

$$
\begin{align*}
& u(t, x, y)>0, \quad v(t, x, y)>0 \\
& (x, y) \in \Omega=\left(0, L_{x}\right) \times\left(0, L_{y}\right) \tag{5}
\end{align*}
$$

and Neumann boundary conditions:

$$
\begin{equation*}
\left.\frac{\partial u}{\partial \nu}\right|_{\partial \Omega}=\left.\frac{\partial v}{\partial v}\right|_{\partial \Omega}=0 \tag{6}
\end{equation*}
$$

In the above, $L_{x}$ and $L_{y}$ denote the size of the system in square domain and $\nu$ is the outward unit normal vector of the boundary $\partial \Omega$. The main reason for choosing such boundary conditions is that we are interested in the self-organization of the pattern and the Neumann conditions imply no external input [22].

It is known that only nonnegative solutions of (4) have biological significance. System (4) has two spatially homogeneous stationary solutions:
(1) the bacteria-free equilibrium $U_{0}=(1,0)$ which implies that the nutrient is at the carrying capacity level;
(2) coexistence equilibrium $U^{*}=\left(u^{*}, v^{*}\right)$ which represents a uniform distribution of bacteria, where

$$
\begin{equation*}
u^{*}=\frac{\beta-S \alpha}{\beta}, \quad v^{*}=\frac{S(\beta-S \alpha)}{\beta \gamma}, \tag{7}
\end{equation*}
$$

and $S=\sqrt{\gamma(\beta-\gamma)}$ with $\beta>\gamma$ and $\beta^{2}-\alpha^{2} \gamma \beta+\alpha^{2} \gamma^{2}>0$.
To consider the pattern formation of (4) from $\left(u^{*}, v^{*}\right)$ we make the translation

$$
\begin{equation*}
u \longrightarrow u_{1}+u^{*}, \quad v \longrightarrow u_{2}+v^{*} \tag{8}
\end{equation*}
$$

Then, (4) are rewritten as

$$
\begin{gather*}
\frac{\partial u_{1}}{\partial t}=D_{u} \nabla^{2} u_{1}+a_{11} u_{1}+a_{12} u_{2}+G_{1}\left(u_{1}, u_{2}\right) \\
\frac{\partial u_{2}}{\partial t}=\mu \nabla^{2} u_{2}+a_{21} u_{1}+a_{22} u_{2}+g\left(u_{1}, u_{2}\right)+G_{2}\left(u_{1}, u_{2}\right) \tag{9}
\end{gather*}
$$

where

$$
\begin{align*}
& a_{11}=\frac{-\beta^{2}+2 S \alpha \gamma}{\beta^{2}}, \quad a_{12}=-\frac{(2 \gamma-\beta) \gamma \alpha}{\beta^{2}}  \tag{10}\\
& a_{21}=-2 \frac{(\gamma-\beta) S \alpha}{\alpha \beta},
\end{align*} \quad a_{22}=2 \frac{\gamma(\gamma-\beta)}{\beta},
$$

and $\mu=u^{*} v^{*} \sigma, g\left(u_{1}, u_{2}\right)=\sigma\left[\nabla \cdot\left(u_{1} u_{2} \nabla u_{2}\right)+v^{*} \nabla \cdot\left(u_{1} \nabla u_{2}\right)+\right.$ $\left.u^{*} \nabla \cdot\left(u_{2} \nabla u_{2}\right)\right], G_{1}\left(u_{1}, u_{2}\right)$, and $G_{2}\left(u_{1}, u_{2}\right)$ are terms of high order.

Define two Hilbert spaces

$$
\begin{gather*}
X=H^{2}(\Omega) \\
X_{1}=\left\{u \in H^{2}(\Omega, R) \left\lvert\, \frac{\partial u}{\partial v}\right. \text { on } \partial \Omega\right\} . \tag{11}
\end{gather*}
$$

Then $X_{1} \rightarrow X$ is dense and compact inclusion.

$$
\begin{equation*}
L_{\lambda}:=-B_{\lambda}+A, \tag{12}
\end{equation*}
$$

where

$$
\begin{align*}
-B_{\lambda} u & =\left(D_{u} \Delta u_{1}, \mu \Delta u_{2}\right)^{T}, \\
A u & =\left(\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right)\binom{u_{1}}{u_{2}} \tag{13}
\end{align*}
$$

for $u=\left(u_{1}, u_{2}\right)^{T} \in X_{1}$.

## Furthermore, denote that

$$
\begin{align*}
& G(u, \lambda) \\
& \qquad=\left(G_{1}^{2}(u, \lambda)+G_{1}^{3}(u, \lambda)+g_{1}\left(u_{1}, u_{2}\right),\right. \\
& \left.\quad G_{2}^{2}(u, \lambda)+G_{2}^{3}(u, \lambda)+g\left(u_{1}, u_{2}\right)+g_{2}\left(u_{1}, u_{2}\right)\right)^{T} \tag{14}
\end{align*}
$$

with

$$
\begin{array}{r}
\binom{G_{1}^{2}(u, \lambda)}{G_{2}^{2}(u, \lambda)}=\binom{a_{20} u_{1}^{2}+a_{11} u_{1} u_{2}+a_{02} u_{2}^{2}}{b_{20} u_{1}^{2}+b_{11} u_{1} u_{2}+b_{02} u_{2}^{2}}, \\
\binom{G_{1}^{3}(u, \lambda)}{G_{2}^{3}(u, \lambda)}=\binom{a_{30} u_{1}^{3}+a_{21} u_{1}^{2} u_{2}+a_{12} u_{1} u_{2}^{2}+a_{03} u_{2}^{3}}{b_{30} u_{1}^{3}+b_{21} u_{1}^{2} u_{2}+b_{12} u_{1} u_{2}^{2}+b_{03} u_{2}^{3}}, \tag{15}
\end{array}
$$

where

$$
\begin{aligned}
& a_{20}=-\frac{-\beta^{3}-4 S \alpha \gamma^{2}+5 S \alpha \beta \gamma}{\beta^{2}(-\beta+S \alpha)}, \\
& a_{11}=-2 \frac{\alpha \gamma(-\gamma+\beta)(\beta-4 \gamma)}{\beta^{2}(-\beta+S \alpha)}, \\
& a_{02}=\frac{(\beta-4 \gamma) S \alpha \gamma}{\beta^{2}(-\beta+S \alpha)}, \\
& a_{30}=4 \frac{(\beta-2 \gamma)(\beta-\gamma) S \alpha \gamma}{\beta^{2}(-\beta+S \alpha)^{2}}, \\
& a_{21}=\frac{\alpha \gamma(\beta-\gamma)\left(24 \gamma^{2}-16 \gamma \beta+\beta^{2}\right)}{\beta^{2}(-\beta+S \alpha)^{2}}, \\
& a_{12}=-2 \frac{\left(-10 \gamma \beta+\beta^{2}+12 \gamma^{2}\right) S \alpha \gamma}{\beta^{2}(-\beta+S \alpha)^{2}}, \\
& a_{03}=\frac{\alpha \gamma^{2}\left(-8 \gamma \beta+\beta^{2}+8 \gamma^{2}\right)}{\beta^{2}(-\beta+S \alpha)^{2}}, \\
& b_{20}=-\frac{S \alpha(\beta-\gamma)(\beta-4 \gamma)}{\alpha \beta(-\beta+S \alpha)}, \\
& b_{11}=\frac{2 \gamma(\beta-\gamma)(\beta-4 \gamma)}{\beta(-\beta+S \alpha)}, \\
& b_{02}=-\frac{(\beta-4 \gamma) S \gamma}{\beta(-\beta+S \alpha)}, \\
& b_{30}=-\frac{4(\beta-2 \gamma)(\beta-\gamma) S \gamma}{\beta(-\beta+S \alpha)^{2}}, \\
& \beta(\beta-\gamma)\left(24 \gamma^{2}-16 \gamma \beta+\beta^{2}\right) \\
& b_{21}(-\beta+S \alpha)^{2}
\end{aligned},
$$

$$
\begin{align*}
& b_{12}=\frac{2\left(-10 \gamma \beta+\beta^{2}+12 \gamma^{2}\right) S \gamma}{\beta(-\beta+S \alpha)^{2}} \\
& b_{03}=-\frac{\gamma^{2}\left(-8 \gamma \beta+\beta^{2}+8 \gamma^{2}\right)}{\beta(-\beta+S \alpha)^{2}} \tag{16}
\end{align*}
$$

Here $g_{1}\left(u_{1}, u_{2}\right)$ and $g_{2}\left(u_{1}, u_{2}\right)$ are terms of high order.
Then $G(\cdot, \lambda): X_{1} \rightarrow X$ are a family of parameterized $\mathbb{C}^{\infty}$ bounded operators continuously depending on the parameter $\lambda$ such that $G(u, \lambda)=o(\|u\|)$.

Then (9) can be written in the following operator form:

$$
\begin{equation*}
\frac{d u}{d t}=F(u)=L_{\lambda} u+G(u, \lambda) . \tag{17}
\end{equation*}
$$

## 3. Bifurcation Analysis

Unless otherwise specified, in this section, we require that $U^{*}=\left(u^{*}, v^{*}\right)$ always exist; that is, $\beta>\gamma$ and $\beta^{2}-\alpha^{2} \gamma \beta+$ $\alpha^{2} \gamma^{2}>0$.

Consider the following eigenvalue problem of system (9):

$$
\begin{equation*}
L_{\lambda} \varphi=\lambda \varphi, \quad \varphi \in H_{1} \tag{18}
\end{equation*}
$$

with the Neumann boundary condition (6).
Let $\rho_{k}$ and $e_{k}$ be the $k$ th eigenvalue and eigenvector of the Laplacian $\nabla^{2}$ with Neumann boundary condition and

$$
\begin{align*}
-\nabla^{2} e_{k} & =\rho_{k} e_{k} \\
\left.\frac{\partial e_{k}}{\partial \nu}\right|_{\partial \Omega} & =0 \tag{19}
\end{align*}
$$

with $\rho_{0}=0, e_{0}=(1,1)^{T}$.
Denote by $M_{k}$ the matrix given by

$$
M_{k}=\left(\begin{array}{cc}
a_{11}-D_{u} \rho_{k} & a_{12}  \tag{20}\\
a_{21} & a_{22}-\mu \rho_{k}
\end{array}\right), \quad k=0,1,2, \ldots
$$

Thus, all eigenvalues $\lambda=\beta_{k}^{ \pm}$of (18) satisfy

$$
\begin{equation*}
M_{k} \xi_{k}^{ \pm}=\beta_{k} \xi_{k}^{ \pm}, \quad k=0,1,2, \ldots \tag{21}
\end{equation*}
$$

where $\xi_{k}^{ \pm} \in \mathbb{R}^{2}$ is the eigenvector of $M_{k}$ corresponding to $\beta_{k}^{ \pm}$ and $\beta_{k}^{ \pm}$is expressed as

$$
\begin{equation*}
\beta_{k}^{ \pm}=\frac{1}{2}\left(\operatorname{tr}\left(M_{k}\right) \pm \sqrt{\operatorname{tr}\left(M_{k}\right)^{2}-4 \operatorname{det}\left(M_{k}\right)}\right) \tag{22}
\end{equation*}
$$

with

$$
\begin{align*}
\operatorname{tr}\left(M_{k}\right) & =\left(D_{u} \rho_{k}-a_{11}\right)+\left(\mu \rho_{k}-a_{22}\right) \\
\operatorname{det}\left(M_{k}\right) & =\left(D_{u} \rho_{k}-a_{11}\right)\left(\mu \rho_{k}-a_{22}\right)-a_{12} a_{21} \tag{23}
\end{align*}
$$

Hence, the eigenvector $\phi_{k}^{ \pm}$of (18) corresponding to $\beta_{k}^{ \pm}$is

$$
\begin{equation*}
\phi_{k}^{ \pm}=\xi_{k}^{ \pm} e_{k}, \tag{24}
\end{equation*}
$$

where $e_{k}$ is as in (19).
3.1. Hopf Bifurcation Analysis. It is clear that $\beta_{k}^{ \pm}(\alpha)= \pm i \sigma_{k}(\alpha)$ with $\sigma_{k} \neq 0$ if and only if

$$
\begin{gather*}
\operatorname{tr}\left(M_{k}\right)=\left(a_{11}-D_{u} \rho_{k}\right)+\left(a_{22}-\mu \rho_{k}\right)=0,  \tag{25}\\
\operatorname{det}\left(M_{k}\right)=\left(D_{u} \rho_{k}-a_{11}\right)\left(\mu \rho_{k}-a_{22}\right)-a_{12} a_{21}>0 .
\end{gather*}
$$

Thus, we introduce one critical number

$$
\begin{equation*}
\alpha_{0}=\frac{\beta\left(\beta-2 \gamma^{2}+2 \gamma \beta\right)}{2 S \gamma} \tag{26}
\end{equation*}
$$

where $\rho_{k}=\rho_{0}=0$ such that $\chi(\alpha)$ attains its minimum values. Consider

$$
\begin{align*}
\chi(\alpha) & =\min _{\rho_{k}}\left\{\left(D_{u} \rho_{k}+a_{11}\right)\left(\mu \rho_{k}+a_{22}\right)-a_{12} a_{21}\right\}  \tag{27}\\
& =a_{11} a_{22}-a_{12} a_{21} .
\end{align*}
$$

Theorem 1. Let $\alpha_{0}$ be the number given in (26) such that (27) is satisfied. Then $\beta_{0}^{+}(\lambda)$ and $\beta_{0}^{-}(\lambda)$ are a pair of first complex eigenvalues of (18) near $\lambda=\alpha_{0}$, and

$$
\begin{gathered}
\operatorname{Re} \beta_{0}^{+}(\lambda)=\operatorname{Re} \beta_{0}^{-}(\lambda) \begin{cases}<0, & \lambda<\alpha_{0} \\
=0, & \lambda=\alpha_{0} \\
>0, & \lambda>\alpha_{0}\end{cases} \\
\operatorname{Im} \beta_{0}^{ \pm}\left(\alpha_{0}\right) \neq 0,
\end{gathered}
$$

$$
\operatorname{Re} \beta_{k}^{ \pm}\left(\alpha_{0}\right)<0, \quad \forall k>0
$$

3.2. Periodic Solution from Hopf Bifurcation. By Theorem 1, problem (4) undergoes a dynamic transition to a periodic solution from $\alpha=\alpha_{0}$. To determine the types of transition we introduced a parameter as follows:

$$
\begin{equation*}
b=\frac{F_{1}}{F_{2}}, \tag{29}
\end{equation*}
$$

where

$$
\begin{aligned}
& F_{1}=\pi\left(\alpha^{2} \gamma S \beta-\alpha^{2} S \gamma^{2}-2 \beta \gamma^{2} \alpha+2 \alpha \gamma \beta^{2}+S \beta^{2}\right) \\
& \times( -8 \beta^{5} \gamma^{4} \alpha^{4}-20 \beta^{9} \gamma^{2}+512 \beta^{2} \gamma^{7} \alpha^{4} \\
&-416 \beta \gamma^{8} \alpha^{4}-66 \beta^{8} \gamma^{3} \omega^{2}+80 \beta^{7} \gamma^{4} \omega^{2} \\
&-32 \beta^{6} \omega^{2} \gamma^{5}+66 \beta^{8} \gamma^{3}+20 \beta^{9} \gamma^{2} \omega^{2} \\
&-2 \beta^{10} \gamma \omega^{2}+464 \beta^{5} \alpha^{2} \gamma^{5}-88 \beta^{6} \gamma^{5} \alpha^{2} \\
&+8 \beta^{7} \gamma^{4} \alpha^{2}-456 \beta^{4} \alpha^{2} \gamma^{6}+160 \beta^{3} \alpha^{2} \gamma^{7} \\
&+128 \alpha^{4} \gamma^{9}-296 \beta^{3} \gamma^{6} \alpha^{4}+80 \beta^{4} \gamma^{5} \alpha^{4} \\
&-202 \beta^{6} \alpha^{2} \gamma^{4}+2 \beta^{10} \gamma-16 \gamma^{5} \beta^{5} \omega \alpha
\end{aligned}
$$

$$
\begin{align*}
& -2 \gamma \beta^{9} \omega \alpha+16 \gamma^{2} \beta^{8} \omega \alpha-42 \gamma^{3} \beta^{7} \omega \alpha \\
& +44 \gamma^{4} \beta^{6} \omega \alpha-64 \alpha^{3} \beta^{2} \gamma^{6} \omega^{2} S+18 \alpha^{3} \beta^{5} \gamma^{3} \omega^{2} S \\
& +120 \alpha^{3} \beta^{3} \gamma^{5} \omega^{2} S-76 \alpha^{3} \beta^{4} \gamma^{4} \omega^{2} S \\
& -\alpha^{3} \beta^{6} \gamma^{2} \omega^{2} S+252 \alpha^{2} \beta^{5} \gamma^{4} \omega S-304 \alpha^{2} \beta^{4} \gamma^{5} \omega S \\
& +128 \alpha^{2} \beta^{3} \gamma^{6} \omega S-86 \alpha^{2} \beta^{6} \gamma^{3} \omega S+10 \alpha^{2} \beta^{7} \gamma^{2} \omega S \\
& -16 \alpha \beta^{7} \gamma^{2} \omega^{2} S+44 \alpha \beta^{6} \gamma^{3} \omega^{2} S-32 \alpha \beta^{5} \gamma^{4} \omega^{2} S \\
& +\alpha \beta^{8} \gamma \omega^{2} S-2 \beta^{8} \alpha^{2} \gamma^{2}+2 \beta^{10} \omega S \\
& -128 \alpha^{2} \beta^{2} \gamma^{9}-584 \alpha^{2} \beta^{4} \gamma^{7}+344 \alpha^{2} \beta^{5} \gamma^{6} \\
& +448 \alpha^{2} \beta^{3} \gamma^{8}+36 \beta^{7} \gamma^{3} \alpha^{2}-80 \beta^{7} \gamma^{4} \\
& +32 \beta^{6} \gamma^{5}-22 \beta^{9} \gamma \omega S+84 \beta^{8} \gamma^{2} \omega S \\
& -128 \beta^{7} \gamma^{3} \omega S+64 \beta^{6} \gamma^{4} \omega S+340 \alpha^{3} \beta^{4} \gamma^{5} S \\
& +4 \alpha^{3} \beta^{6} S \gamma^{3}-760 \alpha^{3} \beta^{3} \gamma^{6} S-72 \alpha^{3} \beta^{3} \gamma^{5} S \\
& +2 \alpha^{3} \beta^{6} S \gamma^{2}-256 \alpha^{3} \beta \gamma^{8} S+56 \alpha^{3} \beta^{4} S \gamma^{4} \\
& -18 \alpha^{3} \beta^{5} S \gamma^{3}-64 \alpha^{3} \beta^{5} S \gamma^{4}+32 \alpha^{3} \beta^{2} \gamma^{6} S \\
& +736 \alpha^{3} \beta^{2} \gamma^{7} S-64 \alpha^{2} \beta^{3} \omega^{2} \gamma^{7}-120 \alpha^{2} \beta^{5} \omega^{2} \gamma^{5} \\
& +152 \alpha^{2} \beta^{4} \omega^{2} \gamma^{6}+34 \alpha^{2} \beta^{6} \gamma^{4} \omega^{2} \\
& -2 \alpha^{2} \beta^{7} \gamma^{3} \omega^{2}-4 \alpha \beta^{9} \omega \gamma^{2}+48 \alpha \beta^{8} \gamma^{3} \omega \\
& -212 \alpha \beta^{7} \omega \gamma^{4}+424 \alpha \beta^{6} \gamma^{5} \omega \\
& -384 \alpha \gamma^{6} \omega \beta^{5}+128 \alpha \gamma^{7} \omega \beta^{4} \\
& -10 \alpha \beta^{8} \gamma^{2} \omega^{3}+46 \alpha \beta^{7} \gamma^{3} \omega^{3} \\
& -68 \alpha \beta^{6} \gamma^{4} \omega^{3}+32 \alpha \beta^{5} \gamma^{5} \omega^{3}-28 \alpha \beta^{6} S \gamma^{3} \\
& +14 \alpha \beta^{7} S \gamma^{2}-8 \alpha \beta^{8} \gamma^{2} S+80 \alpha \beta^{7} \gamma^{3} S \\
& -264 \alpha \beta^{6} \gamma^{4} S+320 \alpha \beta^{5} \gamma^{5} S-128 \alpha \beta^{4} \gamma^{6} S \\
& \left.+16 \alpha \beta^{5} S \gamma^{4}-2 \alpha \beta^{8} S \gamma\right), \\
& F_{2}=4 \omega^{2}(-2 \gamma+\beta)^{2}\left(-\alpha^{2} \gamma^{2}-\beta^{2}+\alpha^{2} \gamma \beta\right)^{2} \\
& \times \beta^{3} \alpha^{2} \gamma^{2}(-\gamma+\beta) . \tag{30}
\end{align*}
$$

Theorem 2. Let $b$ be the number given by (29), then the problem undergoes a transition to periodic solutions at $\lambda=\lambda_{0}$, and the following assertions hold true.
(1) When $b<0$, the transition is continuous and the system bifurcates to a periodic solution on $\alpha<\alpha_{0}$ which is an attractor.
(2) When $b>0$, the transition is jump and the system bifurcates to a periodic solution on $\alpha>\alpha_{0}$ which is a repeller.

Proof. We will verify this theorem by using Theorem A. 3 in [44]. The eigenvalues $\beta_{1}^{ \pm}$at $\lambda=\alpha_{0}$ in are given by $\beta_{1}^{+}=\bar{\beta}_{1}^{-}=$ $i \omega$. The eigenvectors $\xi$ and $\eta$ corresponding to $\beta_{1}^{ \pm}\left(\alpha_{0}\right)$ satisfy

$$
\begin{gather*}
A \xi=\omega \eta \\
A \eta=-\omega \xi \tag{31}
\end{gather*}
$$

It is easy to see that

$$
\begin{align*}
& \xi=\left(\xi_{1}, \xi_{2}\right)=\left(a_{11}, a_{12}\right)  \tag{32}\\
& \eta=\left(\eta_{1}, \eta_{2}\right)=(-\omega, 0)
\end{align*}
$$

The conjugate eigenvectors $\xi^{*}$ and $\eta^{*}$ satisfy

$$
\begin{align*}
A \xi^{*} & =\omega \eta^{*}  \tag{33}\\
A \eta^{*} & =-\omega \xi^{*}
\end{align*}
$$

It is easy to check that

$$
\begin{align*}
& \xi^{*}=\left(\xi_{1}^{*}, \xi_{2}\right)=\left(a_{11}, a_{21}\right), \\
& \eta^{*}=\left(\eta_{1}^{*}, \eta_{2}^{*}\right)=(-\omega, 0) . \tag{34}
\end{align*}
$$

It is known that functions $\psi_{1}^{*}$ and $\psi_{2}^{*}$ are given by

$$
\begin{align*}
\psi_{1}^{*}= & \frac{1}{\left(\xi, \xi^{*}\right)}\left[\left(\xi, \xi^{*}\right) \xi^{*}+\left(\xi, \eta^{*}\right) \eta^{*}\right]=\left(0, a_{21}\right) \\
\psi_{2}^{*} & =\frac{1}{\left(\eta, \eta^{*}\right)}\left[\left(\eta, \xi^{*}\right) \xi^{*}+\left(\eta, \eta^{*}\right) \eta^{*}\right]  \tag{35}\\
& =\left(\frac{a_{12} a_{21}}{\omega},-\frac{a_{11} a_{21}}{\omega}\right)
\end{align*}
$$

Because the first eigenvector space $E=\operatorname{span}\{\xi, \eta\}$ of (18) with (6) is invariant for the equations (4) with (6), the center manifold function $\Phi$ vanishes; that is,

$$
\begin{equation*}
\Phi(x, y) \equiv 0 \tag{36}
\end{equation*}
$$

Therefore, we derive from (32) to (35) that

$$
\begin{aligned}
& \frac{G(x \xi+y \eta+\Phi), \psi_{1}^{*}}{\left(\xi, \psi_{1}^{*}\right)} \\
& =\bar{a}_{20} x^{2}+\bar{a}_{11} x y+\bar{a}_{02} y^{2} \\
& \quad+\bar{a}_{30} x^{3}+\bar{a}_{21} x^{2} y+\bar{a}_{12} x y^{2}+\bar{a}_{03} y^{3} \\
& \frac{G(x \xi+y \eta+\Phi), \psi_{2}^{*}}{\left(\eta, \psi_{2}^{*}\right)} \\
& = \\
& \bar{b}_{20} x^{2}+\bar{b}_{11} x y+\bar{b}_{02} y^{2}+\bar{b}_{30} x^{3} \\
& \quad+\bar{b}_{21} x^{2} y+\bar{b}_{12} x y^{2}+\bar{b}_{03} y^{3}
\end{aligned}
$$

where

$$
\begin{align*}
& \bar{a}_{20}=\frac{1}{a_{12}}\left(b_{11} a_{11} a_{12}+b_{02} a_{12}^{2}+b_{20} a_{11}^{2}\right), \\
& \bar{a}_{11}=\frac{\omega}{a_{12}}\left(2 b_{20} a_{11}+b_{11} a_{12}\right) \bar{a}_{02}=\frac{b_{20} \omega^{2}}{a_{12}}, \\
& \bar{a}_{30}=\frac{1}{a_{12}}\left(b_{21} a_{11}^{2} a_{12}+b_{30} a_{11}^{3}\right. \\
& \left.+b_{12} a_{11} a_{12}^{2}+b_{03} a_{12}^{3}\right), \\
& \bar{a}_{12}=\frac{\omega^{2}}{a_{12}}\left(b_{21} a_{12}+3 b_{30} a_{11}\right), \\
& \bar{a}_{21}=-\frac{\omega}{a_{12}}\left(3 b_{30} a_{11}^{2}+b_{12} a_{12}^{2}+2 b_{21} a_{11} a_{12}\right), \\
& \bar{a}_{03}=-\frac{b_{30} \omega^{3}}{a_{12}}, \\
& \bar{b}_{11}=-\frac{a_{11}}{a_{12}}\left(-2 a_{12} a_{20}-a_{12}^{2}+2 b_{20} a_{11}+b_{11} a_{12}\right), \\
& \bar{b}_{02}=\frac{\omega}{a_{12}}\left(-a_{12} a_{20}+b_{20} a_{11}\right), \\
& \bar{b}_{20}=-\frac{1}{a_{12} \omega}\left(a_{11}^{2} a_{12}^{2}+a_{02} a_{12}^{3}+a_{12} a_{20} a_{11}^{2}\right. \\
& \left.-b_{11} a_{11}^{2} a_{12}-a_{11} b_{02} a_{12}^{2}-b_{20} a_{11}^{3}\right), \\
& \bar{b}_{30}=-\frac{1}{a_{12} \omega}\left(a_{11}^{2} a_{12}^{2} a_{21}+a_{12} a_{30} a_{11}^{3}+a_{12}^{4} a_{11}\right. \\
& +a_{03} a_{12}^{4}-b_{21} a_{11}^{3} a_{12}-b_{30} a_{11}^{4} \\
& \left.-b_{12} a_{11}^{2} a_{12}^{2}-a_{11} b_{03} a_{12}^{3}\right), \\
& \bar{b}_{21}=-\frac{1}{a_{12}}\left(-3 a_{12} a_{30} a_{11}^{2}-a_{12}^{4}\right. \\
& -2 a_{21} a_{11} a_{12}^{2}+3 b_{30} a_{11}^{3} \\
& \left.+b_{12} a_{11} a_{12}^{2}+2 b_{21} a_{11}^{2} a_{12}\right), \\
& \bar{b}_{12}=\frac{\omega}{a_{12}}\left(-a_{12}^{2} a_{21}-3 a_{12} a_{30} a_{11}\right. \\
& \left.+b_{21} a_{11} a_{12}+3 b_{30} a_{11}^{2}\right), \\
& \bar{b}_{03}=-\frac{\omega^{2}}{a_{12}}\left(-a_{12} a_{30}+b_{30} a_{11}\right) . \tag{38}
\end{align*}
$$

From the focus values in [39, 40, 43], we have that
$b=\frac{3 \pi}{4}\left(\bar{a}_{30}+\bar{b}_{03}\right)+\frac{\pi}{4}\left(\bar{a}_{12}+\bar{b}_{21}\right)$

$$
\begin{align*}
& +\frac{\pi}{2 \omega}\left(\bar{a}_{02} \bar{b}_{02}-\bar{a}_{20} \bar{b}_{20}\right) \\
& +\frac{\pi}{4 \omega}\left(\bar{a}_{11} \bar{a}_{20}+\bar{a}_{11} \bar{a}_{02}-\bar{b}_{11} \bar{b}_{20}-\bar{b}_{11} \bar{b}_{02}\right) \tag{39}
\end{align*}
$$

is the same as in (29). Hence, by Theorem A. 3 in [3] the system bifurcates from $(u, \alpha)=\left(0, \alpha_{0}\right)$ to a periodic solution; thus the proof is complete.

Remark 3. As an example, let $D_{u}=1, D_{v}=16, \gamma=1$, $\beta=5 / 4$, and $\alpha_{0}=35 / 16$, then from (29), we compute $b=$ $\pi((1141012 / 2205)-(235072 / 3675) \sqrt{5}) \doteq 1176.323160>0$. From Theorem 2, we can conclude that the transition is jump and the system bifurcates to a periodic solution on $\alpha>\alpha_{0}$ which is a repeller (see Figure 1).
3.3. Turing Bifurcation Analysis. In this subsection, we will state the Turing instability for the positive equilibrium $E^{*}$ of model (1). Mathematically speaking, the positive equilibrium $E^{*}$ is Turing instability, which was emphasized by Turing in his pioneering work in 1952 [12]. The Turing bifurcation occurs when

$$
\begin{equation*}
\operatorname{Im}\left(\beta_{k}^{ \pm}\right)=0, \quad \operatorname{Re}\left(\beta_{k}^{ \pm}\right)=0 \quad \text { at } \rho_{k}=\rho_{T} \neq 0 \tag{40}
\end{equation*}
$$

and the wave-number $\sqrt{\rho_{T}}$ satisfies

$$
\begin{equation*}
\rho_{T}=\sqrt{\frac{\operatorname{det}\left(M_{0}\right)}{\mu D_{u}}} . \tag{41}
\end{equation*}
$$

Hence, Turing instability occurs when the condition either $\operatorname{tr}\left(M_{k}\right)<0$ or $\operatorname{det}\left(M_{k}\right)>0$ is violated.

Since the positive equilibrium $E^{*}$ is stable without diffusion means that $\operatorname{tr}\left(M_{0}\right)<0$ and $\operatorname{det}\left(M_{0}\right)>0$ hold, then $\operatorname{tr}\left(M_{k}\right)<0$ is always true. Therefore, for the emergency of the diffusion-driven instability in model (1), it is needed $\operatorname{det}\left(M_{k}\right)<0$ for some $\rho_{k}>0$. A necessary condition is

$$
\begin{equation*}
a_{11} \mu+d_{22} D_{u}>0 \tag{42}
\end{equation*}
$$

otherwise $\operatorname{det}\left(M_{k}\right)>0$ for all $k>0$ since $\mu D_{u}>0$ and $a_{11} a_{22}-a_{12} a_{21}>0$. And we notice that $\operatorname{det}\left(M_{k}\right)$ achieves its minimum

$$
\begin{equation*}
\min _{k \in \mathbf{R}^{+}} \operatorname{det}\left(M_{k}\right)=a_{11} a_{22}-a_{12} a_{21}-\frac{\left(\mu a_{11}+D_{u} a_{22}\right)^{2}}{4 D_{u} \mu} \tag{43}
\end{equation*}
$$

at the critical value $\mu_{c}^{2}>0$ where

$$
\begin{equation*}
\mu_{c}^{2}=\frac{\mu a_{11}+D_{u} a_{22}}{2 D_{u} \mu} . \tag{44}
\end{equation*}
$$

Summarizing the above calculation, we conclude.
Theorem 4. If

$$
\begin{gather*}
a_{11}+a_{22}<0 \\
a_{11} a_{22}-a_{12} a_{21}>0 \\
\mu a_{11}+D_{u} a_{22}>0  \tag{45}\\
\left(\mu a_{11}+D_{u} a_{22}\right)^{2}>4 D_{u} \mu\left(a_{11} a_{22}-a_{12} a_{21}\right),
\end{gather*}
$$

then the positive equilibrium $E^{*}$ of model (1) is Turing unstable.


Figure 1: The phrase diagram with $D_{u}=1, D_{v}=16, \alpha=35 / 16, \beta=$ $5 / 4$, and $\gamma=1$ illustrating system (4) admits an unstable periodic solution.


Figure 2: The dispersal relation of $\gamma$ with $\beta$. Parameters: $\alpha=1.8$, $D_{u}=0.02, \sigma=18$. The blue and red curves represent Hopf and Turing bifurcation curves, respectively. They separate the parametric space into three domains, and domain(III) is called Turing space.

In Figure 2, based on the results of Theorem 4, we show the dispersal relation of $\gamma$ with $\alpha$. The blue and red curves represent Hopf and Turing bifurcation curves, respectively. They separate the parametric space into three domains. The outside domain of the Hopf bifurcation curve is stable and the inside domain of the Turing bifurcation curve is unstable. Hence, among these domains, only the domain(III) satisfies the conditions of Theorem 4 and we call domain(III) as Turing space, where the Turing instability occurs and the Turing patterns may be undergone.


Figure 3: The process of formation of spiral pattern for $u$ for $(\beta, \gamma)=(0.55,0.44)$; the other parameters are fixed as in (46). Times: (a) $t=0$, (b) $t=100$, (c) $t=500$, and (d) $t=2000$.

## 4. Pattern Formation

In this section, we perform extensive numerical simulations of the spatially extended model (4) in 2-dimensional spaces, and the qualitative results are shown here. Our numerical simulations employ the nonzero initial (5) and the zero-flux boundary conditions (6) with a system size of $L_{x} \times L_{y}$, with $L_{x}=L_{y}=25$ discretized through $x \rightarrow\left(x_{0}, x_{1}, x_{2}, \ldots, x_{n}\right)$ and $y \rightarrow\left(y_{0}, y_{1}, y_{2}, \ldots, y_{n}\right)$, with $n=100$. Other parameters are fixed as

$$
\begin{equation*}
\alpha=1.8, \quad D_{u}=0.02, \quad \sigma=18, \quad h=\frac{1}{4} . \tag{46}
\end{equation*}
$$

The numerical integration of (4) was performed by fourthorder Runge-Kutta scheme integration [45], with a time step of $\tau=0.01$, and by using the standard five-point approximation for the 2D Laplacian with the zero-flux boundary conditions [46, 47]. More precisely, the concentrations $\left(u_{i, j}^{n+1}\right)$
at the moment $(n+1) \tau$ at the mesh position $\left(x_{i}, y_{j}\right)$ are given by

$$
\begin{align*}
u_{i, j}^{(1)}= & u_{i, j}^{n}+\frac{1}{2} \tau F\left(u_{i, j}^{n}\right) \\
u_{i, j}^{(2)}= & u_{i, j}^{n}+\frac{1}{2} \tau F\left(u_{i, j}^{(1)}\right) \\
u_{i, j}^{(3)}= & u_{i, j}^{n}+\tau F\left(u_{i, j}^{(2)}\right)  \tag{47}\\
u^{(n+1)}= & \frac{1}{3}\left(-u_{i, j}^{n}+u_{i, j}^{(1)}+2 u_{i, j}^{(2)}+u_{i, j}^{(3)}\right) \\
& +\frac{1}{6} \tau F\left(u_{i, j}^{(3)}\right),
\end{align*}
$$

where $F(u)$ is defined in (17).
Initially, the entire system is placed in the steady state $\left(u^{*}, v^{*}\right)$, and the propagation velocity of the initial perturbation is thus on the order of $5 \times 10^{-4}$ space units per time unit. And the system is then integrated for 200000 time steps,


Figure 4: Spots-stripes and holes-stripes patterns obtained with model (4) for $(\mathrm{a})(\beta, \gamma)=(0.60,0.46)$ and $(\mathrm{b})(\beta, \gamma)=(0.60,0.55)$ at 200000 iterations. Other parameters are fixed as (46).


FIGURE 5: Spots and holes patterns obtained with model (4) for $(\mathrm{a})(\beta, \gamma)=(0.55,0.4)$ and $(\mathrm{b})(\beta, \gamma)=(0.79,0.67)$ at 200000 iterations. Other parameters are fixed as (46).
and the last images are saved. After the initial period during which the perturbation spreads, either the system goes into a time-dependent state or to an essentially steady state (time independent).

In the numerical simulations, different types of dynamics are observed and it is found that the distributions of predator and prey are always of the same type. Consequently, we can restrict our analysis of pattern formation to one distribution. In this section, we show the distribution of prey $u$, for instance. And the parameters are located in the Turing space (cf., Figure 2), the region where Turing instability occurs. We have taken some snapshots with red (blue) corresponding to the high (low) value of prey $u$.

Figure 3 shows the process of pattern formation for model (4) with $\beta=0.55$ and $\gamma=0.44$. In this case, the pattern takes a long time to settle down, starting with a homogeneous state $\left(u^{*}, v^{*}\right)=(0.2800,0.1400)$ (cf., Figure 3(a)), and the random perturbation leads to the formation of stripes and
spots (cf., Figures 3(b) and 3(c)) and ends with stripes only (cf., Figure 3(d)), which is time independent.

In Figure 4, we show two spots-stripes patterns obtained with model (4) at 100000 iterations; that is, $t=5000$. These two patterns are similar to each other. With $(\beta, \gamma)=$ $(0.60,0.46)$, in this case, the equilibrium is $\left(u^{*}, v^{*}\right)=$ $(0.2386,0.1316)$ and the spots-stripes pattern is relatively high (cf., Figure $4(\mathrm{a})$ ), while with $(\beta, \gamma)=(0.60,0.50)$, the equilibrium is $\left(u^{*}, v^{*}\right)=(0.3291,0.1472)$, at low prey densities (c.f., Figure 4(b)).

In Figure 5, we show the interesting and similartimeindependent patterns which obtained by model (4) at 200 000 iterations. They consist of blue/red spots on a red/blue background. We refer to them as spots (cf., Figure 5(a)) and holes (cf., Figure 5(b)), respectively. In Figure 5(a), with $(\beta, \gamma)=(0.55,0.40),\left(u^{*}, v^{*}\right)=(0.1983,0.1214)$, the hot spots are isolated zones with high prey densities. In this case, the predators are in low density obviously. While with
$(\beta, \gamma)=(0.79,0.67),\left(u^{*}, v^{*}\right)=(0.3539,0.1497)$, holes are isolated zones with low prey density (Figure 5(b)). In this case, the predators are in high density. From Figure 5(b), one can see that the predators apparently almost occupy the whole spatial domain.

## 5. Concluding and Remarks

In this paper, pattern formation of a spatial model for the growth of bacterial colonies with the two-dimensional space is investigated. Based on both mathematical analysis and numerical simulations, we have found that its spatial pattern includes periodic solutions from Hopf bifurcation and the spotted and striped patterns from Turing bifurcation.

It should be noticed that, if considered in a somewhat broader ecological perspective, our results have an intuitively clear meaning; there has been a growing understanding in the past regarding the dynamics of the system's parameter. From this standpoint, it seems interesting to know that the dynamics vary when the parameter moves across the diagram. From our analysis, the parameters $\gamma$ and $\beta$ play an important role in pattern formation. Our results show that the pattern formation formed by the bacterial colonies model represents rich spatial dynamics which will be useful for studying the dynamic complexity of bacterial ecosystems.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgements

This research was supported by the Natural Science Foundation of China ( 61373005 and 11371386), the Doctoral Program of Higher Education of China (20115134110001), the Scientific Research Foundation of CQUT(2012ZD37), and the Fund Project of Zhejiang Provincial Education Department (Y201223449).

## References

[1] E. O. Budrene and H. C. Berg, "Complex patterns formed by motile cells of Escherichia coli," Nature, vol. 349, no. 6310, pp. 630-633, 1991.
[2] E. O. Budrene and H. C. Berg, "Dynamics of formation of symmetrical patterns by chemotactic bacteria," Nature, vol. 376, no. 6535, pp. 49-53, 1995.
[3] M. Badoual, P. Derbez, M. Aubert, and B. Grammaticos, "Simulating the migration and growth patterns of Bacillus subtilis," Physica A, vol. 388, no. 4, pp. 549-559, 2009.
[4] L. Braverman and E. Braverman, "Stability analysis and bifurcations in a diffusive predator-prey system," Discrete and Continuous Dynamical Systems Supplements, pp. 92-100, 2009.
[5] H. Fujikawa and M. Matsushita, "Fractal growth of Bacillus subtilis on agar plates," Journal of the Physical Society of Japan, vol. 58, no. 11, pp. 3875-3878, 1989.
[6] D. Hartmann, "Pattern formation in cultures of Bacillus subtilis," Journal of Biological Systems, vol. 12, no. 2, pp. 179-199, 2004.
[7] K. Kawasaki, A. Mochizuki, M. Matsushita, T. Umeda, and N. Shigesada, "Modeling spatio-temporal patterns generated by Bacillus subtilis," Journal of Theoretical Biology, vol. 188, no. 2, pp. 177-185, 1997.
[8] E. F. Keller and L. A. Segel, "Initiation of slime mold aggregation viewed as an instability," Journal of Theoretical Biology, vol. 26, no. 3, pp. 399-415, 1970.
[9] A. M. Lacasta, I. R. Cantalapiedra, C. E. Auguet, A. Peñaranda, and L. Ramírez-Piscina, "Modeling of spatiotemporal patterns in bacterial colonies," Physical Review E, vol. 59, no. 6, pp. 70367041, 1999.
[10] M. Mimura, H. Sakaguchi, and M. Matsushita, "Reactiondiffusion modelling of bacterial colony patterns," Physica A, vol. 282, no. 1, pp. 283-303, 2000.
[11] M. Ohgiwari, M. Matsushita, and T. Matsuyama, "Morphological changes in growth phenomena of bacterial colony patterns," Journal of the Physical Society of Japan, vol. 61, no. 3, pp. 816-822, 1992.
[12] A. M. Turing, "The chemical basis of morphogenesis," Philosophical Transactions of the Royal Society of London B, vol. 237, no. 641, pp. 37-72, 1952.
[13] R. Tyson, S. R. Lubkin, and J. D. Murray, "A minimal mechanism for bacterial pattern formation," Proceedings of the Royal Society B, vol. 266, no. 1416, pp. 299-304, 1999.
[14] J. Wakita, K. Komatsu, A. Nakahara, T. Matsuyama, and M. Matsushita, "Experimental investigation on the validity of population dynamics approach to bacterial colony formation," Journal of the Physical Society of Japan, vol. 63, no. 3, pp. 12051211, 1994.
[15] B. Chen and M. X. Wang, "Qualitative analysis for a diffusive predator-prey model," Computers \& Mathematics with Applications, vol. 55, no. 3, pp. 339-355, 2008.
[16] E. N. Dancer and Y. H. Du, "Effects of certain degeneracies in the predator-prey model," SIAM Journal on Mathematical Analysis, vol. 34, no. 2, pp. 292-314, 2002.
[17] Y. H. Du and S. B. Hsu, "A diffusive predator-prey model in heterogeneous environment," Journal of Differential Equations, vol. 203, no. 2, pp. 331-364, 2004.
[18] Y. N. Zhu, Y. L. Cai, S. L. Yan, and W. M. Wang, "Dynamical analysis of a delayed reaction-diffusion predator-prey system," Abstract and Applied Analysis, vol. 2012, Article ID 323186, 23 pages, 2012.
[19] S. L. Yan, X. Z. Lian, W. M. Wang, and Y. B. Wang, "Bifurcation analysis in a delayed diffusive Leslie-Gower model," Discrete Dynamics in Nature and Society, vol. 2013, Article ID 170501, 11 pages, 2013.
[20] X. Q. Zhao, Dynamical Systems in Population Biology, Springer, New York, NY, USA, 2003.
[21] G. Nicolis and I. Prigogine, Self-Organization in Nonequibibrium System, John Wiley \& Sons, New York, NY, USA, 1997.
[22] J. D. Murray, "Discussion: turing's theory of morphogenesis-its influence on modelling biological pattern and form," Bulletin of Mathematical Biology, vol. 52, no. 1-2, pp. 119-152, 1990.
[23] H. Malchow and S. V. Petrovskii, "Dynamical stabilization of an unstable equilibrium in chemical and biological systems," Mathematical and Computer Modelling, vol. 36, no. 3, pp. 307319, 2002.
[24] S. L. Yan, X. Z. Lian, W. M. Wang, and R. K. Upadhyay, "Spatiotemporal dynamics in a delayed diffusive predator model," Applied Mathematics and Computation, vol. 224, pp. 524-534, 2013.
[25] X. N. Guan, W. M. Wang, and Y. L. Cai, "Spatiotemporal dynamics of a Leslie-Gower predator-prey model incorporating a prey refuge," Nonlinear Analysis: Real World Applications, vol. 12, no. 4, pp. 2385-2395, 2011.
[26] X. Z. Lian, H. L. Wang, and W. M. Wang, "Delay-driven pattern formation in a reaction-diffusion predatorprey model incorporating a prey refuge," Journal of Statistical Mechanics: Theory and Experiment, vol. 2013, no. 4, Article ID P04006, 2013.
[27] W. M. Wang, Y. Z. Lin, L. Zhang, F. Rao, and Y. J. Tan, "Complex patterns in a predator-prey model with self and cross-diffusion," Communications in Nonlinear Science and Numerical Simulation, vol. 16, no. 4, pp. 2006-2015, 2011.
[28] W. M. Wang, W. J. Wang, Y. Z. Lin, and Y. J. Tan, "Pattern selection in a predation model with self and cross diffusion," Chinese Physics B, vol. 20, no. 3, Article ID 034702, 2011.
[29] X. Z. Lian, Y. H. Yue, and H. L. Wang, "Pattern formation in a cross-diffusive ratio-dependent predator-prey model," Discrete Dynamics in Nature and Society, vol. 2012, Article ID 814069, 13 pages, 2012.
[30] R. K. Upadhyay, W. M. Wang, and N. K. Thakur, "Spatiotemporal dynamics in a spatial plankton system," Mathematical Modelling of Natural Phenomena, vol. 5, no. 5, pp. 102-122, 2010.
[31] W. M. Wang, Z. G. Guo, R. K. Upadhyay, and Y. Z. Lin, "Pattern formation in a cross-diffusive Holling-Tanner model," Discrete Dynamics in Nature and Society, vol. 2012, Article ID 828219, 12 pages, 2012.
[32] Y. L. Cai, W. B. Liu, Y. B. Wang, and W. M. Wang, "Complex dynamics of a diffusive epidemic model with strong Allee effect," Nonlinear Analysis: Real World Applications, vol. 14, no. 4, pp. 1907-1920, 2013.
[33] S. Wang, W. B. Liu, Z. G. Guo, and W. M. Wang, "Traveling wave solutions in a reaction-diffusion epidemic model," Abstract and Applied Analysis, vol. 2013, Article ID 216913, 13 pages, 2013.
[34] W. M. Wang, Y. L. Cai, Y. N. Zhu, and Z. G. Guo, "Allee-effect-induced instability in a reaction-diffusion predator-prey model," Abstract and Applied Analysis, vol. 2013, Article ID 487810, 10 pages, 2013.
[35] Y. Yuan, H. L. Wang, and W. M. Wang, "The existence of positive nonconstant steady states in a reaction: diffusion epidemic model," Abstract and Applied Analysis, vol. 2013, Article ID 921401, 7 pages, 2013.
[36] W. M. Wang, Y. N. Zhu, Y. L. Cai, and W. J. Wang, "Dynamical complexity induced by Allee effect in a predator-prey model," Nonlinear Analysis: Real World Applications, vol. 16, pp. 103-119, 2014.
[37] Y. L. Cai, C. D. Zhao, and W. M. Wang, "Spatiotemporal complexity of a leslie-gower predator-prey model with the weak allee effect," Journal of Applied Mathematics, vol. 2013, Article ID 535746, 16 pages, 2013.
[38] W. M. Wang, Y. L. Cai, M. J. Wu, K. F. Wang, and Z. Q. Li, "Complex dynamics of a reaction-diffusion epidemic model," Nonlinear Analysis: Real World Applications, vol. 13, no. 5, pp. 2240-2258, 2012.
[39] T. Ma and S. H. Wang, Bifurcation Theory and Applications, World Scientific, Singapore, 2005.
[40] T. Ma and S. H. Wang, Stability and Bifurcation of Nonlinear Evolution Equations, 2007.
[41] T. Ma and S. Wang, "Dynamic phase transition theory in PVT systems," Indiana University Mathematics Journal, vol. 57, no. 6, pp. 2861-2889, 2008.
[42] T. Ma and S. H. Wang, "Cahn-Hilliard equations and phase transition dynamics for binary systems," Discrete and Continuous Dynamical Systems B, vol. 11, no. 3, pp. 741-784, 2009.
[43] T. Ma and S. H. Wang, "Phase transitions for the Brusselator model," Journal of Mathematical Physics, vol. 52, no. 3, Article ID 033501, 23 pages, 2011.
[44] C. H. Hsia, T. Ma, and S. Wang, "Rotating Boussinesq equations: dynamic stability and transitions," Discrete and Continuous Dynamical Systems A, vol. 28, no. 1, pp. 99-130, 2010.
[45] G. S. Jiang and C. W. Shu, "Efficient implementation of weighted ENO schemes," Journal of Computational Physics, vol. 126, no. 1, pp. 202-228, 1996.
[46] M. R. Garvie, "Finite-difference schemes for reaction-diffusion equations modeling predator-prey interactions in MATLAB," Bulletin of Mathematical Biology, vol. 69, no. 3, pp. 931-956, 2007.
[47] A. Munteanu and R. V. Solé, "Pattern formation in noisy self-replicating spots," International Journal of Bifurcation and Chaos, vol. 16, no. 12, pp. 3679-3685, 2006.

# Research Article Threshold Dynamics of a Huanglongbing Model with Logistic Growth in Periodic Environments 

Jianping Wang, Shujing Gao, Yueli Luo, and Dehui Xie<br>Key Laboratory of Jiangxi Province for Numerical Simulation and Emulation Techniques, Gannan Normal University, Ganzhou 341000, China<br>Correspondence should be addressed to Shujing Gao; gaosjmath@tom.com

Received 10 January 2014; Accepted 10 February 2014; Published 20 March 2014
Academic Editor: Kaifa Wang
Copyright © 2014 Jianping Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We analyze the impact of seasonal activity of psyllid on the dynamics of Huanglongbing (HLB) infection. A new model about HLB transmission with Logistic growth in psyllid insect vectors and periodic coefficients has been investigated. It is shown that the global dynamics are determined by the basic reproduction number $R_{0}$ which is defined through the spectral radius of a linear integral operator. If $R_{0}<1$, then the disease-free periodic solution is globally asymptotically stable and if $R_{0}>1$, then the disease persists. Numerical values of parameters of the model are evaluated taken from the literatures. Furthermore, numerical simulations support our analytical conclusions and the sensitive analysis on the basic reproduction number to the changes of average and amplitude values of the recruitment function of citrus are shown. Finally, some useful comments on controlling the transmission of HLB are given.


## 1. Introduction

Plant disease is an important constraint to crop production. Due to plant diseases, more than $10 \%$ of global food production is lost and 800 million people do not have adequate food in the world [1-3]. Plant pathologists cannot ignore the juxtaposition of these figures for food shortage and the reduction of crops caused by plant disease.

Nowadays, Huanglongbing (HLB) which is a century old disease caused by the bacteria Candidatus Liberibacter spp is one of the most serious problems of citrus worldwide [4]. HLB has been responsible for the near destruction of citrus industries in Asia and Africa [4]. The main symptoms on HLB-infected citrus trees are yellow shoots, leaves with blotchy mottle, and small lopsided fruits [4,5]. The HLB is a phloem-restricted, noncultured, Gram-negative bacterium causing crippling diseases denoting "greening" in South Africa, "mottle leaf" in the Philippines, "dieback" in India, and "vein phloem degeneration" in Indonesia. The infected citrus orchards are usually destroyed or become unproductive in 5 to 8 years [4].

Most of the known plant viruses are transmitted by insect vectors and entirely dependent on the behaviour and
dispersal capacity of their vectors to spread from plant to plant. HLB, a destructive disease of citrus, can be transmitted by grafting from citrus to citrus and by dodder to periwinkle. The citrus psyllid (Diaphorina Citri Kuwayama) is natural and mainly vector [4]. In this paper, we mainly consider that HLB transmitted from tree to tree by Asian citrus psyllid insect vectors.

Mathematical models play an important role in understanding the epidemiology of vector-transmitted plant diseases. Since the introduction of HLB, a lot of researches have been conducted on the epidemiology of the disease and on the vector, but the result of these two lines of inquiry integrated is very few. Analytical models have also been developed for the spread of citrus canker [6], but models for vectortransmitted bacterial pathogens are still preliminary [7]. In [8], the authors proposed a deterministic compartmental mathematic model to analyze HLB spread between citrus plants. They assumed that all coefficients of the model are constant (autonomous systems). However, in the real world, actual data and evidence show that dynamics of disease transmission are not as simple as shown in the model. In [9], Hall and Hentz have studied seasonal activity of psyllid insect vectors which is correlated with humidity. Seasonal
fluctuations in the transmission of infectious diseases imply that the corresponding mathematical models may admit periodic solutions. It is interesting and important to study the globally dynamics which are determined by threshold parameter $R_{0}$ in periodic epidemiological models.

Based on above introduction, we propose a model with periodic transmission rates to investigate the seasonal HLB epidemics [10, 11]. In this model, we consider Logistic growth term for dynamics of susceptible psyllid vector. Furthermore, we assumed that the infective citrus population is generated through susceptible citrus which was bit by infective psyllid and the susceptible psyllid bit the infective citrus which will become infective psyllid. Then, the periodic system is as follows:

$$
\begin{align*}
\frac{d S_{h}(t)}{d t}= & \Lambda(t)-\beta_{1}(t) S_{h}(t) I_{v}(t)-\mu_{1}(t) S_{h}(t) \\
\frac{d I_{h}(t)}{d t}= & \beta_{1}(t) S_{h}(t) I_{v}(t)-\mu_{1}(t) I_{h}(t)-d(t) I_{h}(t) \\
\frac{d S_{v}(t)}{d t}= & b(t)\left(S_{v}(t)+I_{v}(t)\right)\left[1-\frac{S_{v}(t)+I_{v}(t)}{m\left(S_{h}(t)+I_{h}(t)\right)}\right] \\
& -\beta_{2}(t) S_{v}(t) I_{h}(t) \\
\frac{d I_{v}(t)}{d t}= & \beta_{2}(t) S_{v}(t) I_{h}(t)-\mu_{2}(t) I_{v}(t) \tag{1}
\end{align*}
$$

with initial condition

$$
\begin{equation*}
S_{h}(0)>0, \quad I_{h}(0)>0, \quad S_{v}(0)>0, \quad I_{v}(0)>0 . \tag{2}
\end{equation*}
$$

Here, $S_{h}(t), I_{h}(t), S_{v}(t)$, and $I_{v}(t)$ represent susceptible citrus host, infected citrus host, susceptible psyllid, and infected psyllid, respectively. We can easily see that $N_{h}(t)=S_{h}(t)+$ $I_{h}(t)$ and $N_{v}(t)=S_{v}(t)+I_{v}(t)$ are the number of citrus population and psyllid population, respectively. $\Lambda(t)$ is the recruitment rate of citrus at time $t, \beta_{1}(t)$ is the infected rate of citrus host at time $t, \mu_{1}(t)$ and $d(t)$ are the nature death and disease induced death rate of citrus host at time $t$, respectively, $b(t)$ is the intrinsic growth rate of psyllid at time $t, \beta_{2}(t)$ and $\mu_{2}(t)$ are the infected rate and the nature death rate of psyllid at time $t$, respectively, and $m(>0)$ is the maximum abundance of psyllid per citrus. $\Lambda(t), \beta_{1}(t), \mu_{1}(t), d(t), b(t), \beta_{2}(t)$, and $\mu_{2}(t)$ are continuous, positive $\omega$-periodic functions.

The paper is organized as follows. In the next section, we give the basic reproduction number of (1). In Sections 3 and 4 , the results show that the dynamical properties of the model are completely determined by $R_{0}$. That is, if $R_{0}<1$, the disease-free periodic solution is globally asymptotically stable, and if $R_{0}>1$, the model is permanence. In Section 5, we present numerical simulations which demonstrate the theoretical analysis and a brief discussion of our main results.

## 2. Basic Reproduction Number

In the following, we introduce some notations and lemmas which will be used for our further argument.

Let $\left(R^{k}, R_{+}^{k}\right)$ be the standard ordered $k$-dimensional Euclidean space with a norm $\|\cdot\|$. For $u, v \in R^{k}$; we denote $u \geq v$ if $u-v \in R_{+}^{k}, u>v$ if $u-v \in R_{+}^{k} \backslash\{0\}$, and $u \gg v$ if $u-v \in \operatorname{Int}\left(R_{+}^{k}\right)$, respectively.

Define $g^{L}=\max _{t \in[0, \omega)} g(t)$ and $g^{M}=\min _{t \in[0, \omega)} g(t)$, where $g(t)$ is a continuous, positive, $\omega$-periodic function.

Consider the following linear ordinary differential system:

$$
\begin{equation*}
\frac{d x(t)}{d t}=A(t) x(t) \tag{3}
\end{equation*}
$$

where $A(t)$ is a continuous, cooperative, irreducible, and $\omega$-periodic $k \times k$ matrix function. Denote $\Phi_{A}(t)$ be the fundamental solution matrix of (3) and $r\left(\Phi_{A}(\omega)\right)$ be the spectral radius of $\Phi_{A}(\omega)$. By the Perron-Frobenius Theorem, we know that $r\left(\Phi_{A}(\omega)\right)$ is the principle eigenvalue of $\Phi_{A}(\omega)$; that is, it is simple and admits an eigenvector $v^{*} \gg 0$.

Lemma 1 (see [12]). Let $p=(1 / \omega) \ln r\left(\Phi_{A(\cdot)}(\omega)\right)$. Then there exists a positive $\omega$-periodic function $v(t)$ such that $\exp (p t) v(t)$ is a solution of (3).

Consider the following nonautonomous linear equation:

$$
\begin{equation*}
\frac{d S_{h}(t)}{d t}=\Lambda(t)-\mu_{1}(t) S_{h}(t), \tag{4}
\end{equation*}
$$

where $\Lambda(t)$ and $\mu_{1}(t)$ are the same as in System (1). From Zhang and Teng ([13, Lemma 2.1]) and simple calculation, we have the following lemma.

Lemma 2. System (4) has a unique positive $\omega$-periodic solution $S_{h}^{*}(t)$ which is globally asymptotically stable.

Consider the following nonautonomous Logistic equation:

$$
\begin{equation*}
\frac{d S_{v}(t)}{d t}=b(t) S_{v}(t)\left(1-\frac{S_{v}(t)}{m S_{h}(t)}\right) \tag{5}
\end{equation*}
$$

where $b(t)$ and $m$ are the same as in system (1). From Teng and Li ([14, Lemma 2]) and simple calculation, we can obtain the following lemma.

Lemma 3. System (5) has a unique positive $\omega$-periodic solution $S_{v}^{*}(t)$ which is globally asymptotically stable, where $S_{v}^{*}(t)=$ $m S_{h}^{*}(t)$.

According to Lemmas 2 and 3, it is easy to see that (1) has a unique disease-free periodic solution $\left(S_{h}^{*}(t), 0, S_{v}^{*}(t), 0\right)$.

Now, we use the generation operator approach (see [15]) to derive the basic reproduction number. Applying the symbol of the theory in Wang and Zhao [15], for system (1), we have

$$
\mathscr{F}(t, x)=\left(\begin{array}{c}
\beta_{1}(t) S_{h}(t) I_{v}(t) \\
\beta_{2}(t) S_{v}(t) I_{h}(t) \\
0 \\
0
\end{array}\right)
$$

$$
\begin{gather*}
\mathscr{V}^{+}(t, x)=\left(\begin{array}{c}
0 \\
0 \\
\Lambda(t) \\
b(t)\left(S_{v}(t)+I_{v}(t)\right)
\end{array}\right), \\
\mathscr{V}^{-}(t, x)=\left(\begin{array}{c}
\left(\mu_{1}(t)+d(t)\right) I_{h}(t) \\
\mu_{2}(t) I_{v}(t) \\
\mu_{1}(t) S_{h}+\beta_{1}(t) S_{h}(t) I_{v}(t) \\
b(t) \frac{\left(S_{v}(t)+I_{v}(t)\right)^{2}}{m\left(S_{h}(t)+I_{h}(t)\right)}+\beta_{2}(t) S_{v}(t) I_{h}(t)
\end{array}\right) \tag{6}
\end{gather*}
$$

where $x=\left(I_{h}(t), I_{v}(t), S_{h}(t), S_{v}(t)\right)^{T}$. Then System (1) can be written as the following form:

$$
\begin{equation*}
\frac{d x(t)}{d t}=\mathscr{F}(t, x(t))-\mathscr{V}(t, x(t)) \tag{7}
\end{equation*}
$$

where $\mathscr{V}(t, x(t))=\mathscr{V}^{-}(t, x(t))-\mathscr{V}^{+}(t, x(t))$.
It is easy to obtain that the conditions (A1)-(A5) in [15] hold. In the following, we will check the conditions (A6) and (A7) in [15].

We know that $x^{*}(t)=\left(0,0, S_{h}^{*}(t), S_{v}^{*}(t)\right)$ is the diseasefree periodic solution of system (7). Denote

$$
\begin{align*}
f(t, x(t)) & =\mathscr{F}(t, x(t))-\mathscr{V}(t, x(t)), \\
M(t) & =\left(\frac{\partial f_{i}\left(t, x^{*}(t)\right)}{\partial x_{j}}\right)_{3 \leq i, j \leq 4}, \tag{8}
\end{align*}
$$

where $f_{i}(t, x(t))$ and $x_{i}$ are the $i$ th components of $f(t, x(t))$ and $x$, respectively. According to (6), we have

$$
M(t)=\left(\begin{array}{cc}
-\mu_{1}(t) & 0  \tag{9}\\
\frac{b(t) S_{v}^{* 2}(t)}{m S_{h}^{* 2}(t)} & -b(t)
\end{array}\right)
$$

It is easy to see that $r\left(\Phi_{M}(\omega)\right)<1$, where $r\left(\Phi_{M}(\omega)\right)$ is the spectral radius of $\Phi_{M}(\omega)$. This implies that $x^{*}(t)$ is linearly asymptotically stable in the disease-free subspace $X_{S}=\left\{\left(0,0, S_{h}, S_{v}\right) \in R_{+}^{4}\right\}$. Thus, condition (A6) in [15] holds.

We further define

$$
\begin{align*}
& F(t)=\left(\frac{\partial \mathscr{F}_{i}\left(t, x^{*}(t)\right)}{\partial x_{j}}\right)_{1 \leq i, j \leq 2},  \tag{10}\\
& V(t)=\left(\frac{\partial \mathscr{V}_{i}\left(t, x^{*}(t)\right)}{\partial x_{j}}\right)_{1 \leq i, j \leq 2}
\end{align*}
$$

where $\mathscr{F}_{i}(t, x)$ and $\mathscr{V}_{i}(t, x)$ are the $i$ th components of $\mathscr{F}(t, x)$ and $\mathscr{V}(t, x)$, respectively. Then, from (6), we obtain that

$$
\begin{aligned}
& F(t)=\left(\begin{array}{cc}
0 & \beta_{1}(t) S_{h}^{*}(t) \\
\beta_{2}(t) S_{v}^{*}(t) & 0
\end{array}\right) \\
& V(t)=\left(\begin{array}{cc}
\mu_{1}(t)+d(t) & 0 \\
0 & \mu_{2}(t)
\end{array}\right)
\end{aligned}
$$

Let $Y(t, s)$ be a $2 \times 2$ matrix solution of the system:

$$
\begin{gather*}
\frac{d Y(t, s)}{d t}=-V(t) Y(t, s), \quad \forall t \geq s  \tag{12}\\
Y(s, s)=I
\end{gather*}
$$

where $I$ is $2 \times 2$ identity matrix. From (11) and (12), we have $r\left(\Phi_{-V}(\omega)\right)<1$. Therefore, the condition (A7) in [15] also holds.

Let $C_{\omega}$ be the ordered Banach space of all $\omega$-periodic function from $R \rightarrow R^{2}$, which is equipped with maximum norm $\|\cdot\|_{\infty}$ and the positive cone $C_{\omega}^{+}=\left\{\phi \in C_{\omega}: \phi(t) \geq 0\right.$, for all $t \in R\}$. Define the following linear operator $L: C_{\omega} \rightarrow$ $C_{\omega}$ by

$$
\begin{array}{r}
(L \phi)(t)=\int_{0}^{+\infty} Y(t, t-a) F(t-a) \phi(t-a) d a  \tag{13}\\
\forall t \in R, \quad \phi \in C_{\omega}
\end{array}
$$

Based on the assumptions above and the results of Wang and Zhang [15], we can derive the basic reproduction number $R_{0}$ of system (1) as follows:

$$
\begin{equation*}
R_{0}=r(L) \tag{14}
\end{equation*}
$$

and obtain the following conclusion.
Theorem 4. For system (1), the following statements are valid:
(i) $R_{0}=1$ if and only if $r\left(\Phi_{F-V}(\omega)\right)=1$,
(ii) $R_{0}>1$ if and only if $r\left(\Phi_{F-V}(\omega)\right)>1$,
(iii) $R_{0}<1$ if and only if $r\left(\Phi_{F-V}(\omega)\right)<1$,
where $F(t)$ and $V(t)$ are defined in (11).
It follows from Theorem 4 that the disease-free periodic solution $\left(S_{h}^{*}(t), 0, S_{v}^{*}(t), 0\right)$ of system (1) is asymptotically stable if $R_{0}<1$, and it is unstable if $R_{0}>1$.

In order to calculate $R_{0}$, we consider the following linear $\omega$-periodic system:

$$
\begin{equation*}
\frac{d w}{d t}=\left(-V(t)+\frac{1}{\lambda} F(t)\right) w, \quad \lambda \in(0, \infty) \tag{15}
\end{equation*}
$$

Let $W(t, s, \lambda), t \geqslant s, s \in R$, be the evolution operator of the System (15) on $R^{2}$. Since $F(t)$ is nonnegative and $-V(t)$ is cooperative, then $r(W(\omega, 0, \lambda))$ is continuous and nonincreasing for $\lambda \in(0, \infty)$, and $\lim _{\lambda \rightarrow \infty} r(W(\omega, 0, \lambda))<1$. Thus, we have the following result, which will be used in our numerical calculation of the basic reproduction ratio $R_{0}$ in Section 5.

Lemma 5 (see [15]). The following statements are valid.
(i) If $r(W(\omega, 0, \lambda))=1$ has a positive solution, $\lambda_{0}$ is an eigenvalue of $L$, and hence $R_{0}>0$.
(ii) If $R_{0}>0$, then $\lambda=R_{0}$ is the unique solution of $r(W)$, $0, \lambda))=1$.
(iii) $R_{0}=0$ if and only if $r(W(\omega, 0, \lambda))<1$ for all $\lambda>0$.

## 3. Global Stability of Disease-Free Periodic Solution

In this section we will prove the global asymptotical stability of the disease-free periodic solution $\left(S_{h}^{*}(t), 0, S_{v}^{*}(t), 0\right)$.

$$
\text { Let } N_{h}(t)=S_{h}(t)+I_{h}(t), N_{v}(t)=S_{v}(t)+I_{v}(t) \text {. Denote }
$$

$$
\begin{gather*}
\Omega=\left\{\left(S_{h}, I_{h}, S_{v}, I_{v}\right) \in R_{+}^{4} \mid 0 \leq S_{h}+I_{h} \leq N_{1}<+\infty,\right. \\
\left.0 \leq S_{v}+I_{v} \leq N_{2}<+\infty\right\}, \tag{16}
\end{gather*}
$$

where $N_{1}=\Lambda^{L} / \mu_{1}^{M}$ and $N_{2}=m N_{1}$. Similar to [16, 17], we firstly prove the following lemmas.

Lemma 6. $\Omega$ is a positively invariant set for (1).
Proof. From the equations in (1), we have

$$
\begin{align*}
\frac{d N_{h}(t)}{d t} & =\Lambda(t)-\mu_{1}(t) N_{h}(t) \\
& \leq \Lambda^{L}-\mu_{1}^{M} N_{h}(t) \\
& \leq 0 \quad \text { if } N_{h}(t) \geq N_{1} \\
\frac{d N_{v}(t)}{d t} & =b(t) N_{v}\left(1-\frac{N_{v}}{m N_{h}}\right)-\mu_{2}(t) I_{v}(t)  \tag{17}\\
& \leq b(t) N_{v}(t)\left(1-\frac{N_{v}(t)}{m N_{h}}\right) \\
& \leq 0 \quad \text { if } N_{v}(t) \geq N_{2}
\end{align*}
$$

which implies that $\Omega$ is a positive invariant compact set for (1). The proof is completed.

Lemma 7. Let $\left(S_{h}(t), I_{h}(t), S_{v}(t), I_{v}(t)\right)$ be any solution of system (1). It holds that

$$
\begin{align*}
& \lim _{t \rightarrow+\infty}\left(N_{h}(t)-S_{h}^{*}(t)\right)=0,  \tag{18}\\
& \lim _{t \rightarrow+\infty}\left(N_{v}(t)-S_{v}^{*}(t)\right)=0,
\end{align*}
$$

where $S_{h}^{*}(t), S_{v}^{*}(t)$ are defined in Lemmas 2 and 3, respectively.
Proof. We denote that $y_{1}(t)=N_{h}(t)-S_{h}^{*}(t)$. It follows from the first equation of (17) that $d y_{1}(t) / d t \leq-\mu_{1}(t) y_{1}(t)$, which implies that $\lim _{t \rightarrow+\infty} y_{1}(t)=\lim _{t \rightarrow+\infty}\left(N_{h}(t)-S_{h}^{*}(t)\right)=0$. Further, from Lemma 6, we obtain that for any $\varepsilon>0$, there exists a $T>0$ such that

$$
\begin{equation*}
S_{h}^{*}(t)-\varepsilon \leq N_{h}(t) \leq S_{h}^{*}(t)+\varepsilon, \quad N_{v}(t)<N_{2}, \quad \forall t \geq T . \tag{19}
\end{equation*}
$$

Let $y_{2}(t)=N_{v}(t)-S_{v}^{*}(t)$. From the second equation of (17) and (19), we get

$$
\begin{aligned}
\frac{d y_{2}(t)}{d t}= & b(t) N_{v}(t)\left[1-\frac{N_{v}(t)}{m N_{h}(t)}\right]-\mu_{2}(t) I_{v}(t) \\
& -b(t) S_{v}^{*}(t)\left[1-\frac{S_{v}^{*}(t)}{m S_{h}^{*}(t)}\right]
\end{aligned}
$$

$$
\begin{align*}
\leq & b(t) N_{v}(t)\left[1-\frac{N_{v}(t)}{m\left(S_{h}^{*}(t)+\varepsilon\right)}\right] \\
& -b(t) S_{v}^{*}(t)\left[1-\frac{S_{v}^{*}(t)}{m S_{h}^{*}(t)}\right] \\
= & b(t)\left(N_{v}(t)-S_{v}^{*}(t)\right)\left[1-\frac{S_{h}^{*}(t)}{S_{h}^{*}(t)+\varepsilon}\right] \\
& -b(t)\left(N_{v}^{*}(t)-S_{v}^{*}(t)\right) \frac{N_{v}(t)}{m\left(S_{h}^{*}(t)+\varepsilon\right)} \\
& +b(t) \frac{S_{v}^{* 2}(t) \varepsilon}{m S_{h}^{*}(t)\left(S_{h}^{*}(t)+\varepsilon\right)} \\
= & -b(t) \frac{N_{v}(t)}{m\left(S_{h}^{*}(t)+\varepsilon\right)} y_{2}(t)+\Delta(\varepsilon) \tag{20}
\end{align*}
$$

for all $t>T$, where

$$
\begin{align*}
\Delta(\varepsilon)= & b(t)\left(N_{v}(t)-S_{v}^{*}(t)\right)\left[1-\frac{S_{h}^{*}(t)}{S_{h}^{*}+\varepsilon}\right]  \tag{21}\\
& +b(t) \frac{S_{v}^{* 2}(t) \varepsilon}{m S_{h}^{*}(t)\left(S_{h}^{*}+\varepsilon\right)}
\end{align*}
$$

Obviously, $\lim _{\varepsilon \rightarrow 0} \Delta(\varepsilon)=0$. Because $\varepsilon$ is arbitrarily small, then $\lim _{t \rightarrow+\infty} y_{2}(t)=\lim _{t \rightarrow+\infty}\left(N_{v}(t)-S_{v}^{*}(t)\right)=0$. Hence, the proof is completed.

Theorem 8. The disease-free periodic solution $\left(S_{h}^{*}(t), 0, S_{v}^{*}(t)\right.$, 0 ) is globally asymptotically stable if $R_{0}<1$, whereas it is unstable if $R_{0}>1$.

Proof. From Theorem 4, we have that $\left(S_{h}^{*}(t), 0, S_{v}^{*}(t), 0\right)$ is unstable if $R_{0}>1$, and $\left(S_{h}^{*}(t), 0, S_{v}^{*}(t), 0\right)$ is locally stable if $R_{0}<1$. Therefore, we only need to show the global attractivity of $\left(S_{h}^{*}(t), 0, S_{v}^{*}(t), 0\right)$ for $R_{0}<1$.

Since $R_{0}<1$, by Theorem 4, we can choose $\epsilon_{1}>0$ sufficiently small such that

$$
\begin{equation*}
r\left(\Phi_{F-V+M_{\varepsilon_{1}}}(\omega)\right)<1 \tag{22}
\end{equation*}
$$

where

$$
M_{\epsilon_{1}}(t)=\left(\begin{array}{cc}
0 & \epsilon_{1}  \tag{23}\\
\epsilon_{1} & 0
\end{array}\right)
$$

From Lemma 6 and (18), we have that, for above mentioned $\epsilon_{1}>0$, there exists a $T_{1}>0$ such that $S_{h}(t) \leq S_{h}^{*}(t)+\epsilon_{1}$, $S_{v}(t) \leq S_{v}^{*}(t)+\epsilon_{1}$ for $t>T_{1}$. It follows from the second and fourth equations that for $t>T_{1}$,

$$
\begin{align*}
& \frac{d I_{h}(t)}{d t} \leq \beta_{1}(t)\left(S_{h}^{*}(t)+\epsilon_{1}\right) I_{v}(t)-\left(\mu_{1}(t)+d(t)\right) I_{h}(t) \\
& \frac{d I_{v}(t)}{d t} \leq \beta_{2}(t)\left(S_{v}^{*}(t)+\epsilon_{1}\right) I_{h}(t)-\mu_{2}(t) I_{v}(t) \tag{24}
\end{align*}
$$

Consider the following comparison system:

$$
\begin{align*}
& \frac{d \widetilde{I}_{h}(t)}{d t}=\beta_{1}(t)\left(S_{h}^{*}(t)+\epsilon_{1}\right) \widetilde{I}_{v}(t)-\left(\mu_{1}(t)+d(t)\right) \widetilde{I}_{h}(t) \\
& \frac{d \widetilde{I}_{v}(t)}{d t}=\beta_{2}(t)\left(S_{v}^{*}(t)+\epsilon_{1}\right) \widetilde{I}_{h}(t)-\mu_{2}(t) \widetilde{I}_{v}(t) \tag{25}
\end{align*}
$$

In view of Lemma 1, we know that there exists a positive $\omega$ periodic function $v_{1}(t)$ such that $J(t) \leq v_{1}(t) \exp \left(p_{1} t\right)$, where $J(t)=\left(\widetilde{I}_{h}(t), \widetilde{I}_{v}(t)\right)^{T}$ and $p_{1}=(1 / \omega) \ln r\left(\Phi_{F-V+M_{e}}(\omega)\right)<$ 0 . It follows from (22) that $\lim _{t \rightarrow+\infty} \widetilde{I}_{h}(t)=0$ and $\lim _{t \rightarrow+\infty} \widetilde{I}_{v}(t)=0$. By the comparison of theorem [18], we have $\lim _{t \rightarrow+\infty} I_{h}(t)=0$ and $\lim _{t \rightarrow+\infty} I_{v}(t)=0$. From (18), we have

$$
\begin{align*}
& \lim _{t \rightarrow+\infty}\left(S_{h}(t)-S_{h}^{*}(t)\right)=0 \\
& \lim _{t \rightarrow+\infty}\left(S_{v}(t)-S_{v}^{*}(t)\right)=0 . \tag{26}
\end{align*}
$$

Hence, the disease free periodic solution $\left(S_{h}^{*}(t), 0, S_{v}^{*}(t), 0\right)$ is globally attractive. This completes the proof.

## 4. Permanence

In this section, we show that if $R_{0}>1$, then the disease persists.

Firstly, we define $X=\left\{\left(S_{h}, I_{h}, S_{v}, I_{v}\right) \in R_{+}^{4}\right\}, X_{0}=\left\{\left(S_{h}, I_{h}\right.\right.$, $\left.\left.S_{v}, I_{v}\right) \in X: S_{h} \geq 0, I_{h}>0, S_{v} \geq 0, I_{v}>0\right\}$, and $\partial X_{0}=$ $X \backslash X_{0}$, and we denote $u\left(t, x_{0}\right)$ as the unique solution of System (1) with the initial value $x_{0}=\left(S_{h}^{0}, I_{h}^{0}, S_{v}^{0}, I_{v}^{0}\right)$.

Define Poincaré map $P: X \rightarrow X$ associated with System (1) as follows:

$$
\begin{equation*}
P\left(x_{0}\right)=u\left(\omega, x_{0}\right), \quad \forall x_{0} \in X \tag{27}
\end{equation*}
$$

By Lemma 6, it is easy to see that both $X$ and $X_{0}$ are positively invariant and $P$ is point dissipative. Set

$$
\begin{align*}
& M_{\partial}=\left\{\left(S_{h}^{0}, I_{h}^{0}, S_{v}^{0} I_{v}^{0}\right) \in \partial X_{0} \mid P^{m}\left(S_{h}^{0}, I_{h}^{0}, S_{v}^{0}, I_{v}^{0}\right) \in \partial X_{0}\right. \\
& \left.\quad m \in Z^{+}\right\} \tag{28}
\end{align*}
$$

where $Z^{+}=\{0,1,2, \ldots\}$. We claim that

$$
\begin{equation*}
M_{\partial}=\left\{\left(S_{h}, 0, S_{v}, 0\right), S_{h} \geq 0, S_{v} \geq 0\right\} \tag{29}
\end{equation*}
$$

Obviously, $M_{\partial} \supseteq\left\{\left(S_{h}, 0, S_{v}, 0\right), S_{h} \geq 0, S_{v} \geq 0\right\}$. Next we want to show $M_{\partial} \backslash\left\{\left(S_{h}, 0, S_{v}, 0\right), S_{h} \geq 0, S_{v} \geq 0\right\}=\emptyset$. If it does not hold, then there exists a point $\left(S_{h}^{0}, I_{h}^{0}, S_{v}^{0}, I_{v}^{0}\right) \in M_{\partial} \backslash$ $\left\{\left(S_{h}, 0, S_{v}, 0\right), S_{h} \geq 0, S_{v} \geq 0\right\}$.

Case 1. $I_{h}^{0}=0$ and $I_{v}^{0}>0$. It is obvious that $I_{v}(t)>0$ and $S_{h}(t)>0$ for any $t>0$. Then, from the second equation of System (1), $d I_{h}(t) /\left.d t\right|_{t=0}=\beta_{1}(0) S_{h}(0) I_{v}(0)>0$ holds. It follows that $\left(S_{h}(t), I_{h}(t), S_{v}(t), I_{v}(t)\right) \notin \partial X_{0}$ for $0<t \ll 1$. This is a contradiction.

Case 2. $I_{h}^{0}>0$ and $I_{v}^{0}=0$. It is obvious that $I_{h}(t)>0$ and $S_{v}(t)>0$ for any $t>0$. Then, from the fourth equation of System (1), $d I_{v}(t) /\left.d t\right|_{t=0}=\beta_{2}(0) S_{v}(0) I_{h}(0)>0$ holds. It follows that $\left(S_{h}(t), I_{h}(t), S_{v}(t), I_{v}(t)\right) \notin \partial X_{0}$ for $0<t \ll 1$. This is a contradiction.

That is to say, for any $\left(S_{h}^{0}, I_{h}^{0}, S_{v}^{0}, I_{v}^{0}\right) \notin\left\{\left(S_{h}, 0, S_{v}, 0\right): S_{h} \geq\right.$ $\left.0, S_{v} \geq 0\right\}$, then $\left(S_{h}^{0}, I_{h}^{0}, S_{v}^{0}, I_{v}^{0}\right) \notin M_{\partial}$. Therefore we have $M_{\partial}=$ $\left\{\left(S_{h}, 0, S_{v}, 0\right): S_{h} \geq 0, S_{v} \geq 0\right\}$.

Next, we present the following result of the uniform persistence of the disease.

Theorem 9. Suppose $R_{0}>1$. Then there is a positive constant $\epsilon>0$ such that each positive solution $\left(S_{h}(t), I_{h}(t), S_{v}(t), I_{v}(t)\right)$ of System (1) satisfies

$$
\begin{equation*}
\liminf _{t \rightarrow+\infty} I_{h}(t) \geq \epsilon, \quad \liminf _{t \rightarrow+\infty} I_{v}(t) \geq \epsilon \tag{30}
\end{equation*}
$$

Proof. By Theorem 4, we obtain $r\left(\Phi_{F-V}(\omega)\right)>1$. So we can choose $\eta>0$ small enough such that $r\left(\Phi_{F-V-M_{\eta}}\right)>1$, where

$$
M_{\eta}=\left(\begin{array}{ll}
0 & \eta  \tag{31}\\
\eta & 0
\end{array}\right)
$$

Put $P_{0}=\left\{S_{h}^{*}(0), 0, S_{v}^{*}(0), 0\right\}$. Now we proceed by contradiction to prove that

$$
\begin{equation*}
\limsup _{m \rightarrow+\infty} d\left(P^{m}\left(S_{h}^{0}, I_{h}^{0}, S_{v}^{0}, I_{v}^{0}\right), P_{0}\right) \geq \delta \tag{32}
\end{equation*}
$$

If it does not hold, then

$$
\begin{equation*}
\limsup _{m \rightarrow+\infty} d\left(P^{m}\left(S_{h}^{0}, I_{h}^{0}, S_{v}^{0}, I_{v}^{0}\right), P_{0}\right)<\delta \tag{33}
\end{equation*}
$$

for some $\left(S_{h}^{0}, I_{h}^{0}, S_{v}^{0}, I_{v}^{0}\right) \in X_{0}$. Without loss of generality, suppose that

$$
\begin{equation*}
d\left(P^{m}\left(S_{h}^{0}, I_{h}^{0}, S_{v}^{0}, I_{v}^{0}\right), P_{0}\right)<\delta, \quad \forall m \in Z_{+} \tag{34}
\end{equation*}
$$

By the continuity of the solutions with respect to the initial values, we obtain

$$
\begin{array}{r}
\left\|u\left(t, P^{m}\left(S_{h}^{0}, I_{h}^{0}, S_{v}^{0}, I_{v}^{0}\right)\right)-u\left(t, P_{0}\right)\right\| \leq \eta  \tag{35}\\
\forall t \in[0, \omega], \quad \forall m \in Z_{+}
\end{array}
$$

For any $t \geq 0$, there exists a $m \in Z_{+}$such that $t=m \omega+t_{1}$, where $t_{1} \in[0, \omega]$. Then we have

$$
\begin{align*}
& \left\|u\left(t,\left(S_{h}^{0}, I_{h}^{0}, S_{v}^{0}, I_{v}^{0}\right)\right)-u\left(t, P_{0}\right)\right\|  \tag{36}\\
& \quad=\left\|u\left(t_{1}, P^{m}\left(S_{h}^{0}, I_{h}^{0}, S_{v}^{0}, I_{v}^{0}\right)\right)-u\left(t_{1}, P_{0}\right)\right\| \leq \eta
\end{align*}
$$

for all $t \geq 0$, which implies that $S_{h}^{*}(t)-\eta \leq S_{h}(t) \leq S_{h}^{*}(t)+\eta$, $S_{v}^{*}(t)-\eta \leq S_{v}(t) \leq S_{v}^{*}(t)+\eta$. Then from (1) we have

$$
\begin{align*}
\frac{d I_{h}(t)}{d t} & \geq \beta_{1}(t)\left(S_{h}^{*}(t)-\eta\right) I_{v}(t)-\left(\mu_{1}(t)+d(t)\right) I_{h}(t) \\
\frac{d I_{v}(t)}{d t} & \geq \beta_{2}(t)\left(S_{v}^{*}(t)-\eta\right) I_{h}(t)-\mu_{2}(t) I_{v}(t) \tag{37}
\end{align*}
$$

Table 1: Parameter definitions and values used for numerical simulations of the Huanglongbing model.

| Parameter | Definition | Average value | Unit | Reference |
| :--- | :---: | :---: | :---: | :---: |
| $\Lambda$ | The recruitment rate of citrus | - | month $^{-1}$ | Estimate |
| $\beta_{1}$ | Infected rate of citrus | - | month $^{-1}$ | Estimate |
| $\mu_{1}$ | Nature death rate of citrus | $0.00275-0.004167$ | month $^{-1}$ | $[20]$ |
| $d$ | Disease induced death rate of citrus | $0.016667-0.027775$ | month $^{-1}$ | $[21]$ |
| $D$ | Birth rate of psyllid | $3.78327-33.526137$ | month $^{-1}$ | month |
| $\beta_{2}$ | Infected rate of psyllid | - | month $^{-1}$ | Estimate |
| $\mu_{2}$ | Nature death rate of psyllid | $0.1169825-0.95052$ | $[23]$ |  |
| $m$ | Max abundance of psyllid per citrus | $120-1000$ | - | $[24]$ |

Table 2: Parameter functions for model (1) according to the values of Table 1.

| Parameter functions | Value | Reference |
| :--- | :---: | :---: |
| $\beta_{1}(t)$ | $0.0042925+0.003543 \cos (2 \pi t / 12)$ | Estimate |
| $\mu_{1}(t)$ | $0.0034585+0.0007085 \cos (2 \pi t / 12)$ | $[20]$ |
| $d(t)$ | $0.022221+0.005554 \cos (2 \pi t / 12)$ | $[21]$ |
| $D(t)$ | $18.6547035+14.8714335 \cos (2 \pi t / 12)$ | $[20,22]$ |
| $\beta_{2}(t)$ | $0.008779171+0.004838437 \cos (2 \pi t / 12)$ | Estimate |
| $\mu_{2}(t)$ | $0.53375125+0.41676875 \cos (2 \pi t / 12)$ | $[23]$ |
| $b(t)=D(t)-\mu_{2}(t)$ | $18.120952+14.45466475 \cos (2 \pi t / 12)$ | $[20,22,23]$ |
| $m$ | 560 | $[24]$ |

Consider the linear system

$$
\begin{align*}
& \frac{d \widehat{I}_{h}(t)}{d t}=\beta_{1}(t)\left(S_{h}^{*}(t)-\eta\right) \widehat{I}_{v}(t)-\left(\mu_{1}(t)+d(t)\right) \widehat{I}_{h}(t) \\
& \frac{d \widehat{I}_{v}(t)}{d t}=\beta_{2}(t)\left(S_{v}^{*}(t)-\eta\right) \widehat{I}_{h}(t)-\mu_{2}(t) \widehat{I}_{v}(t) \tag{38}
\end{align*}
$$

By Lemma 1 and the standard comparison principle, we have that there exists a positive $\omega$-periodic function $v_{2}(t)$ such that $J(t)=\exp \left(p_{2} t\right) v_{2}(t)$ is a solution of System (38), where $J(t)=$ $\left(\widehat{I}_{h}(t), \widehat{I}_{v}(t)\right)^{T}$ and

$$
\begin{equation*}
p_{2}=\frac{1}{\omega} \ln r\left(\Phi_{F-V-M_{\eta}}(\omega)\right) . \tag{39}
\end{equation*}
$$

It follows from $r\left(\Phi_{F-V-M_{\eta}}(\omega)\right)>1$ that $p_{2}>0$ and $J(t) \rightarrow$ $+\infty$ as $t \rightarrow+\infty$. Applying the comparison principle [18], we know that $I_{h}(t) \rightarrow+\infty$ and $I_{v}(t) \rightarrow+\infty$ as $t \rightarrow+\infty$. This is a contradiction. Thus, we have proved that (32) holds and $P$ is weakly uniformly persistent with respect to ( $X_{0}, \partial X_{0}$ ).

According to the results of Lemma 7, we can easily obtain that $P$ has a global attractor $P_{0}$. It is easy to obtain that $P_{0}$ is an isolated invariant set in $X$ and $W^{s}\left(P_{0}\right) \cap X_{0}=\emptyset$. We know that $P_{0}$ is acyclic in $M_{\partial}$ and every solution in $M_{\partial}$ converges to $P_{0}$. According to Zhao [19], we have that $P$ is uniformly persistent with respect to $\left(X_{0}, \partial X_{0}\right)$. This implies that the solution of (1) is uniformly persistent with respect to ( $X_{0}, \partial X_{0}$ ). Thus we have that there exists a $\epsilon>0$ such that $\liminf _{t \rightarrow+\infty} I_{h}(t) \geq \epsilon$, $\liminf _{t \rightarrow+\infty} I_{v}(t) \geq \epsilon$.

## 5. Numerical Simulations and Sensitivity Analysis

In this section, we will make numerical simulations by means of Matlab 7.1 to support our theoretical results, to predict the trend of the disease, and to explore some control and prevention measures. Numerical values of parameters of system (1) are given in Table 1 (most of the data are taken from the literatures ([20-24])).

According to the periodicity of System (1) and Table 1, we set $\mu_{1}(t)=\alpha_{1}^{0}+\alpha_{2}^{0} \cos (2 \pi t / 12)$, where $\alpha_{2}^{0}=(0.004167-$ $0.00275) / 2=0.0007085$ and $\alpha_{1}^{0}=0.00275+\alpha_{2}^{0}=0.0034585$. By the similar method, we can obtain the other parameter functions of model (1) (see Table 2). For the simulations that follows, we apply the parameters in Table 2 unless otherwise stated.

Choose $\Lambda(t)=0.00265+0.00235 \cos (2 \pi t / 12)$. Then from Lemma 5, we can compute $R_{0}=0.9844<1$ by means of Matlab 7.1. From Theorem 8 we obtain that the infected citrus population $I_{h}(t)$ and the infected psyllid population $I_{v}(t)$ of system (1) are extinct (see Figures 1 and 2).

Choose $\Lambda(t)=0.005+0.0035 \cos (2 \pi t / 12)$. Then from Lemma 5, we obtain that $R_{0}=1.8342>1$. From Theorem 9 we have that the infected citrus population $I_{h}(t)$ and the infected psyllid population $I_{v}(t)$ of System (1) are permanence (see Figures 3 and 4).

From the formulae for $R_{0}$, we can predict the general tendency of the epidemic in a long term according to the current situation, which is presented in Figures 1, 2, 3, and 4. From the first two figures we know that the epidemic of


FIGURE 1: Graphs of numerical simulations of (1) showings the tendency of the infected citrus population. (a) $t \in[0,2500]$; (b) $t \in[0,500]$.


Figure 2: It is similar to Figure 1.

Huanglongbing can be rising in a short time but cannot be outbreak with the current prevention and control measures. From Figures 3 and 4, we can see that the epidemic of Huanglongbing dropped heavily after 100 months, while there is still tendency to a stable periodic solution in a long time.

Next, we perform some sensitivity analysis to determine the influence $R_{0}$ on the parameters $\Lambda(t), \beta_{1}(t)$, and $\beta_{2}(t)$.

We choose function $\Lambda(t)=\Lambda_{1}^{0}+\Lambda_{2}^{0} \cos (2 \pi t / 12)$, where $\Lambda_{1}^{0}, \Lambda_{2}^{0}$ denote the average and amplitude values of $\Lambda(t)$, respectively, and $\Lambda_{1}^{0}=(1 / 12) \int_{0}^{12} \Lambda(t) d t$. From Figure 5, we can observe that the blue line is linear relation between $R_{0}$ and $\Lambda_{2}^{0}$, and $R_{0}$ increases as $\Lambda_{2}^{0}$ increases. The red curve reflects the influence of the average value of $\Lambda(t)$ on $R_{0}$. Figure 5 shows that $\Lambda_{1}^{0}$ is more sensitive than $\Lambda_{2}^{0}$ on
the basic reproduction number $R_{0}$. Therefore, in the real world, decreasing the average recruitment rate of citrus is the valuable way to control Huanglongbing.

Now, we consider the combined influence of $\beta_{1}(t)$ and $\beta_{2}(t)$ on $R_{0}$. Set $\Lambda(t)=0.0027+0.00235 \cos (2 \pi t / 12)$, $\beta_{1}(t)=a_{1}+b_{1} \cos (2 \pi t / 12)$ and $\beta_{2}(t)=a_{2}+b_{2} \cos (2 \pi t / 12)$. Moreover, we know that $a_{1}=(1 / 12) \int_{0}^{12} \beta_{1}(t) d t$ and $a_{2}=(1 / 12) \int_{0}^{12} \beta_{2}(t) d t$. Other parameters can be seen in Table 2.

Case (I). We fix $b_{1}=0.003543$ and $b_{2}=0.004838437$, and let $a_{1}$ vary from 0.00001 to 0.015 and $a_{2}$ from 0.00001 to 0.02 . For this case, it is interesting to examine how the average values of adequate contact rate $\beta_{1}(t)$ and $\beta_{2}(t)$ affect the basic reproduction number $R_{0}$. Numerical results shown


Figure 3: The figures show that the infected citrus population is permanence. (a) $t \in[0,2500]$; (b) $t \in[0,500]$.


Figure 4: It is similar to Figure 3.
in Figure 6 imply that the basic reproduction number $R_{0}$ may be less than 1 when $a_{1}$ or $a_{2}$ is small enough. And the results also imply that $R_{0}$ increases as $a_{1}$ and $a_{2}$ increase. Further, we can observe that from Figure 6(i)the smaller the values of $a_{1}$ or $a_{2}$ are, the more sensitive $R_{0}$ is; (ii) increasing $a_{2}$ may be more sensitive for $R_{0}$ when $a_{1}$ is fixed; (iii) increasing $a_{1}$ may be more sensitive for $R_{0}$ when $a_{2}$ is fixed.

Case (II). We fix $a_{1}=0.0042925$ and $a_{2}=0.00877917$, and let $b_{1}$ vary from 0.000001 to 0.005 and $b_{2}$ from 0.000002 to 0.006 . Then we obtain the result of numerical simulation and it is shown in Figure 7. Obviously, Figure 7 shows that $R_{0}$ is linearly related to both $b_{1}$ and $b_{2}$ with the pattern that $R_{0}$ decreases to a relatively small value (less than 1 ) only when $b_{1}$ and $b_{2}$ are very small.

By the above graphs of the basic reproduction number $R_{0}$ on the average values of recruitment rate of citrus $\Lambda(t)$ and adequate contact rate $\beta_{1}(t), \beta_{2}(t)$, we know that the basic reproduction number $R_{0}$ is a monotonic increasing function by the average values. From the sensitivity analysis diagrams, we observe that $R_{0}$ falls to less than 1 by decreasing the values of those parameters.

## 6. Conclusion

In this paper, we have analyzed a HLB transmission model with Logistic growth in periodic environments. It is proved that $R_{0}$ is the threshold for distinguishing the disease extinction or permanence. The disease-free periodic solution is


Figure 5: The graph shows the sensitivity of the basic reproduction number $R_{0}$ to the changes of $\Lambda(t)$.


Figure 6: The graph of $R_{0}$ in terms of $a_{1}$ and $a_{2}$.
globally asymptotically stable and the disease dies out when $R_{0}<1$. When $R_{0}>1$, the disease persists.

The numerical simulations shown in Figure 5 show that there are some parameter ranges of $\Lambda_{1}$ and $\Lambda_{2}$ such that the threshold parameter $R_{0}$ is smaller than 1. It indicates a useful way to eradicate Huanglongbing by limiting the recruitment of citrus, including the average value and amplitude of recruitment function.

The results shown in Figure 6 (Figure 7) show that if the amplitudes of infected functions $b_{1}, b_{2}$ (the average infected rate $a_{1}, a_{2}$ ) are fixed, we can control the infection of citrus and psyllid by limiting the average infected rates $a_{1}, a_{2}$ (the amplitudes of infected functions $b_{1}, b_{2}$ ).

According to the above theoretical analysis and numerical simulations, we can conclude that the recruitment of citrus and the infection of citrus and psyllid have significant effects on Huanglongbing transmission. In order to prevent the epidemic disease from generating endemic, making an appropriate reduction of the recruitment rate of citrus and


Figure 7: The graph of $R_{0}$ in terms of $b_{1}$ and $b_{2}$.
decreasing the contact rate between psyllid and the citrus are effective measures to control Huanglongbing.

## Disclosure

The paper is approved by all authors for publication. The authors would like to declare that the work described was original research that has not been published previously and not under consideration for publication elsewhere.

## Conflict of Interests

No conflict of interests exists in the submission of this paper.

## Acknowledgments

The research has been supported by the Natural Science Foundation of China (11261004), the Natural Science Foundation of Jiangxi Province (20122BAB211010), the Science and Technology Plan Projects of Jiangxi Provincial Education Department (GJJ13646), and the Postgraduate Innovation Fund of Jiangxi Province (YC2012-S121).

## References

[1] P. Christou and R. M. Twyman, "The potential of genetically enhanced plants to address food insecurity," Nutrition Research Reviews, vol. 17, no. 1, pp. 23-42, 2004.
[2] FAO, The State of Food Insecurity in the World (SOFI), FAO, Rome, Italy, 2000, http://www.fao.org/FOCUS/E/SOFI00/ sofi001-e.htm.
[3] C. James, "Global food security," in International Congress of Plant Pathology, Pittsburgh, Pa, USA, August 1998.
[4] J. M. Bové, "Huanglongbing: a destructive, newly-emerging, century-old disease of citrus," Journal of Plant Pathology, vol. 88, no. 1, pp. 7-37, 2006.
[5] S. E. Halbert and K. L. Manjunath, "Asian citrus psyllids (Sternorrhyncha: Psyllidae) and greening disease of citrus: a literature review and assessment of risk in Florida," Florida Entomologist, vol. 87, no. 3, pp. 330-353, 2004.
[6] S. Parnell, T. R. Gottwald, C. A. Gilligan, N. J. Cunniffe, and F. Van Den Bosch, "The effect of landscape pattern on the optimal eradication zone of an invading epidemic," Phytopathology, vol. 100, no. 7, pp. 638-644, 2010.
[7] R. F. Mizell III, C. Tipping, P. C. Andersen, B. V. Brodbeck, W. B. Hunter, and T. Northfield, "Behavioral model for Homalodisca vitripennis (Hemiptera: Cicadellidae): optimization of host plant utilization and management implications," Environmental Entomology, vol. 37, no. 5, pp. 1049-1062, 2008.
[8] G. A. Braga, S. Ternes et al., "Modelagem Matemática da Dinâmica Temporaldo HLB em Citros," in Proceedings of the 8th Congresso Brasileiro de Agroinformática, Bento Goncalves, 2011.
[9] D. G. Hall and M. G. Hentz, "Seasonal flight activity by the Asian citrus psyllid in east central Florida," Entomologia et Applicata, vol. 139, no. 1, pp. 75-85, 2011.
[10] X. Song and A. U. Neumann, "Global stability and periodic solution of the viral dynamics," Journal of Mathematical Analysis and Applications, vol. 329, no. 1, pp. 281-297, 2007.
[11] K. Wang, Z. Teng, and H. Jiang, "On the permanence for n -species non-autonomous Lotka-Volterra competitive system with infinite delays and feedback controls," International Journal of Biomathematics, vol. 1, no. 1, pp. 29-43, 2008.
[12] F. Zhang and X.-Q. Zhao, "A periodic epidemic model in a patchy environment," Journal of Mathematical Analysis and Applications, vol. 325, no. 1, pp. 496-516, 2007.
[13] T. Zhang and Z. Teng, "On a nonautonomous SEIRS model in epidemiology," Bulletin of Mathematical Biology, vol. 69, no. 8, pp. 2537-2559, 2007.
[14] Z. Teng and Z. Li, "Permanence and asymptotic behavior of the tv-species nonautonomous lotka-volterra competitive systems," Computers and Mathematics with Applications, vol. 39, no. 7-8, pp. 107-116, 2000.
[15] W. Wang and X.-Q. Zhao, "Threshold dynamics for compartmental epidemic models in periodic environments," Journal of Dynamics and Differential Equations, vol. 20, no. 3, pp. 699-717, 2008.
[16] Y. Nakata and T. Kuniya, "Global dynamics of a class of SEIRS epidemic models in a periodic environment," Journal of Mathematical Analysis and Applications, vol. 363, no. 1, pp. 230237, 2010.
[17] Y. Nakata, Permanence and Global Asymptotic Stability For Population Models in Mathematical Biology, Waseda University, Tokyo, Japan, 2010.
[18] H. Smith and P. Waltman, The Theory of the Chemostat, Cambridge University Press, Cambridge, Mass, USA, 1995.
[19] X. Zhao, Dynamical Systems in Population Biology, Spring, New York, NY, USA, 2003.
[20] X. M. Deng, "Formming process and basis and technological points of the theory emphasis on control citrus psylla for integrated control Huanglongbing," Chinese Agricultural Science Bulletin, vol. 25, no. 23, pp. 358-363, 2009 (Chinese).
[21] T. Li, C. Z. Cheng et al., "Detection of the bearing rate of liberobacter asiaticum in citrus psylla and its host plant," Acta Agriculturae Universitatis Jiangxiensis, vol. 29, no. 5, pp. 743745, 2007 (Chinese).
[22] G. F. Chen and X. M. Deng, "Dynamic observation adult citrus psyllid quantity live through the winter in spring and winter," South China Fruits, vol. 39, no. 4, pp. 36-38, 2010 (Chinese).
[23] X. M. Deng, G. F. Chen et al., "The newly process of Huanglongbing in citrus," Guangxi Horticulture, vol. 17, no. 3, pp. 49-51, 2006.
[24] R. G. d'A Vilamiu, S. Ternes, B. A. Guilherme et al., "A model for Huanglongbing spread between citrus plants including delay times and human intervention," in Proceedings of the International Conference of Numerical Analysis and Applied Mathematics (ICNAAM '12), vol. 1479, pp. 2315-2319, 2012.

## Research Article

# Geometric Analysis of an Integrated Pest Management Model Including Two State Impulses 

Wencai Zhao, Yulin Liu, Tongqian Zhang, and Xinzhu Meng<br>School of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China<br>Correspondence should be addressed to Wencai Zhao; zhaowencai@sdust.edu.cn

Received 10 December 2013; Accepted 20 January 2014; Published 19 March 2014
Academic Editor: Kaifa Wang
Copyright © 2014 Wencai Zhao et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

According to integrated pest management strategies, we construct and investigate the dynamics of a Holling-Tanner predator-prey system with state dependent impulsive effects by releasing natural enemies and spraying pesticide at different thresholds. Applying the Dulacs criterion, the global stability of the positive equilibrium in the system without impulsive effect is discussed. By using impulsive differential equation geometry theory and the method of successor functions, we prove the existence of periodic solution of the system with state dependent impulsive effects. Furthermore, the stability conditions of periodic solutions are obtained. Some simulations are exerted to illustrate the feasibility of our main results.


## 1. Introduction

In 2012, the main maize area in Northern China embraced the large-scale plant diseases and insect pests [1]. With the good weather condition, Mythimna separata Walker, Ostrinia furnacalis, aphid, and other pests had a mass propagation which brought a large damage to the production of corn. How human beings effectively control the pests has been a significant task. Due to its simple operation and quick effect, spraying pesticide has always been the major way to kill pests for a long time. However, there may be pesticide residues in vegetables and crops threatening people's good health and damaging the environment. With the increasing awareness of the environment, people are paying more attention to developing green agriculture. Releasing natural enemies and artificial capture are becoming significant means of controlling pests.

Different control strategies will be applied due to different behavior features of pests and their damages for crops in different stages. For instance, in the spawning period of Ostrinia furnacalis, artificial releasing of Trichogramma (the natural enemy of Ostrinia furnacalis) will be applied with the control effect of $70 \%-80 \%$; if the hatchability is over $30 \%$, pesticide control will be used instantly [2]. In consideration
of the rapidity of chemical control and nonpollution of biological control, people control pests integrating with biological, physical, and chemical means under the EIL (economic injury level) to realize environmental, economic, and social profits together. Spraying pesticides and releasing natural enemies are instantaneous; these phenomena can be described as the impulsive differential equations. In recent decades, the theoretical research on the impulsive differential equation has represented a significant development and has been widely used in various mathematical ecological models [3-11] and many scholars made a deep analysis of the impulsive differential ecological system at a fixed time and have got some important products [12-26]. However, in the actual process of pest control, relevant measures will be used according to pest quantity and its damage to crops, which is the state impulsive differential system. Tang et al. [27, 28], Zeng et al. [29], Zhao et al. [30], Nie et al. [31, 32], and literatures [33-37] had a further exploration for the system and made great progress. Based on the study of a HollingTanner system, an integrated pest control model with two impulses is established aiming at the specific pest conditions in different threshold.

Let $x=x(t), y=y(t)$ denote the population densities of pest (prey) and natural enemy (predator) at time $t$,
respectively; then the predator-prey system usually can be expressed as [38]

$$
\begin{gather*}
\frac{d x}{d t}=x g(x)-y \Phi(x) \\
\frac{d y}{d t}=y[-q(x)+c \Phi(x)] \tag{1}
\end{gather*}
$$

in which $g(x)$ denotes the relative growth rate of the prey; $\Phi(x)$ is the functional response function of the predator; $q(x)$ is the mortality rate of the prey. The literature [39] studied a class of Holling-Tanner system with functional response function $\Phi(x)=m x /(A+x)$. This system was given by

$$
\begin{align*}
& \frac{d x}{d t}= r x\left(1-\frac{x}{K}\right)-\frac{m x}{A+x} y \\
& \frac{d y}{d t}= y s\left(1-\frac{h y}{x}\right)  \tag{2}\\
& x(0)>0, y(0)>0
\end{align*}
$$

Here, let the prey population be growth in logistic and the environmental capacity is $K$; the intrinsic growth rate of predator is $s$ and the carrying capacity is proportional to the number of prey. Introducing transformation $\tilde{t}=r t, \widetilde{x}(\tilde{t})=$ $x(t) / K, \tilde{y}(\tilde{t})=m y(t) / r K$, and letting $\delta=s / r, \beta=s h / m$, $a=A / K$, the system (2) is changed into a dimensionless form:

$$
\begin{align*}
& \frac{d x}{d t}=x(1-x)-\frac{x}{a+x} y \\
& \frac{d y}{d t}=y\left(\delta-\beta \frac{y}{x}\right)  \tag{3}\\
& x(0)>0, y(0)>0
\end{align*}
$$

In order to carry out integrated control of pests, we adopt strategies as follows.
(1) When the pest density $x(t)$ reached a lower level $x=$ $h_{1}$, we release natural enemies to control pests for low damage of insect pests on crops; that is,

$$
\begin{align*}
& \Delta x(t)=0,  \tag{4}\\
& \Delta y(t)=\lambda,
\end{align*} x=h_{1}
$$

where $\lambda$ is amount of natural enemies $y(t)$ released one time.
(2) If the pest density $x(t)$ reached a higher level $x=h_{2}$, due to the fact that the damage of insect pests on crops is severe at this time, we effectively combine spraying pesticides with releasing natural enemies to control pests; that is,

$$
\begin{gather*}
\Delta x(t)=-p x(t), \quad x=h_{2} .  \tag{5}\\
\Delta y(t)=-q y(t)+\tau,
\end{gather*}
$$

Here, $\tau$ is the amount of natural enemies $y(t)$ released one time, $p, q$ are mortality rates of pests and natural enemies which die from spraying pesticides, and $p, q \in(0,1)$.

Synthesizing systems (3), (4), and (5), the following integrated pest management model is obtained:

$$
\begin{align*}
& \frac{d x(t)}{d t}=x(t)(1-x(t))-\frac{x(t)}{a+x(t)} y(t), \\
& \frac{d y(t)}{d t}=y(t)\left(\delta-\beta \frac{y(t)}{x(t)}\right), \\
& x \neq h_{1}, h_{2}, \quad \text { or } \quad x=h_{1}, \quad y>\tilde{y} \\
& \Delta x(t)=0, \quad x=h_{1}, \quad y \leq \tilde{y}  \tag{6}\\
& \Delta y(t)=\lambda, \\
& \Delta x(t)=-p x(t), \quad x=h_{2} \\
& \Delta y(t)=-q y(t)+\tau, \\
& x(0)>0, \quad y(0)>0,
\end{align*}
$$

where $x(t)$ denotes the population density of pests at time $t$ and $y(t)$ denotes the population density of natural enemies at time $t ; a, \delta, \beta$ are positive numbers; $p, q, \lambda, \tau$ are control parameters and positive numbers; the point $\left(h_{1}, \tilde{y}\right)$ is the intersection of the isoclinic line $y=(1-x)(x+a)$ and the straight line $x=h_{1}$.

## 2. Preliminaries

In order to analyze the dynamics of the system (6), we introduce the basic knowledge of the state impulsive differential equations.

Consider the state impulsive differential equation:

$$
\begin{align*}
& \frac{d x(t)}{d t}=P(x, y), \\
& \frac{d y(t)}{d t}=Q(x, y),  \tag{7}\\
& \Delta x(t)=\alpha(x, y) \neq M(x, y), \quad(x, y) \in M(x, y), \\
& \Delta y(t)=\beta(x, y),
\end{align*}
$$

where $P(x, y)$ and $Q(x, y)$ have order-one continuous partial derivatives.

Definition 1. The dynamic system which is formed by solution mapping of system (7) is called semicontinuous dynamic system, denoted by $(\Omega, f, \varphi, M)$, where $f$ is semicontinuous dynamical system mapping and $f: \Omega \rightarrow \Omega, \varphi(M)=N$, and in which $\varphi$ is pulse mapping. Here, $M(x, y)$ and $N(x, y)$ are straight or curved line in the plane. $M(x, y)$ is called impulsive set, and $N(x, y)$ is called corresponding image set.

In the system (6), let $M_{1}=\left\{(x, y) \mid x=h_{1}, 0<y \leq \tilde{y}\right\}$, and the image set corresponding to impulsive mapping (4) is $N_{1}=\varphi_{1}\left(M_{1}\right)=\left\{(x, y) \mid x=h_{1}, 0<y \leq \tilde{y}+\lambda\right\}$. Let $M_{2}=$ $\left\{(x, y) \mid x=h_{2}, y>0\right\}$, and the image set corresponding to impulsive mapping (5) is $N_{2}=\varphi_{2}\left(M_{2}\right)=\left\{(x, y) \mid x\left(t^{+}\right)=\right.$ $\left.(1-p) h_{2}, y\left(t^{+}\right)=(1-q) y(t)+\tau\right\}$.

Definition 2. Assume that impulsive set $M$ and image set $N$ are straight lines, the orbit $\Pi(A, t)$ of system (7) starting
from point $A$ on $N$ hits $M$ at point $A_{1}$ and then jumps onto point $A_{1}^{+}$, then the function $f(A)=y_{A_{1}^{+}}-y_{A}$ is defined as a successor function about point $A$, and then point $A_{1}^{+}$is called successor point of $A$.

Lemma 3 (see $[10,11])$. Successor function is continuous.
Definition 4. If there exists a point $P_{0}$ on the image set $N$, and a constant $T>0$ such that $\Pi\left(P_{0}, T\right)=P \in M, \varphi(P)=P_{0} \in N$, then the orbit $\Pi\left(P_{0}, t\right)$ starting from $P_{0}$ is called an order-one periodic solution of the system (7).

Lemma 5 (Bendixson theorem of impulsive differential equations $[10,11])$. Assume that $G$ is a Bendixson region of system (7); if $G$ does not contain critical points of system (7), then system (7) contains a closed orbit in $G$.

For the system (6), from Lemma 5, the following conclusion is obtained.

Lemma 6. In system (6), if there exist points $A$ and $B$ on the image set $N$, such that the successor function satisfies $f(A) f(B)<0$, then there must exist an order-one periodic solution in system (6).

Lemma 7 (Analogue of Poincaré Criterion [3, 4]). Assume that $x=\xi(t), y=\eta(t)$ is the T-periodic solution of the following impulsive differential equations:

$$
\begin{align*}
& \frac{d x(t)}{d t}=P(x, y) \\
& \frac{d y(t)}{d t}=Q(x, y)  \tag{8}\\
& \Delta x(t)=\alpha(x, y) \\
& \Delta y(t)=\beta(x, y)
\end{align*}
$$

where $P(x, y)$ and $Q(x, y)$ contain order-one continuous partial derivatives and $\Phi(x, y)$ is a sufficiently smooth function with $\operatorname{grad} \Phi(x, y) \neq 0$.

If the multiplier $\mu$ satisfies the condition $|\mu|<1$, then the periodic solution $(\xi(t), \eta(t))$ of the system (8) is orbitally asymptotically stable, where

$$
\begin{align*}
\mu=\prod_{j=1}^{n} \kappa_{j} \exp \left[\int_{0}^{T}( \right. & \frac{\partial P(\xi(t), \eta(t))}{\partial x}  \tag{9}\\
& \left.\left.+\frac{\partial Q(\xi(t), \eta(t))}{\partial y}\right) d t\right]
\end{align*}
$$

with

$$
\begin{align*}
\kappa_{j}= & \left(\left(\frac{\partial \beta}{\partial y} \cdot \frac{\partial \Phi}{\partial x}-\frac{\partial \beta}{\partial x} \cdot \frac{\partial \Phi}{\partial y}+\frac{\partial \Phi}{\partial x}\right) P_{+}\right. \\
& \left.+\left(\frac{\partial \alpha}{\partial x} \cdot \frac{\partial \Phi}{\partial y}-\frac{\partial \alpha}{\partial y} \cdot \frac{\partial \Phi}{\partial x}+\frac{\partial \Phi}{\partial y}\right) Q_{+}\right)  \tag{10}\\
& \times\left(\frac{\partial \Phi}{\partial x} P+\frac{\partial \Phi}{\partial y} Q\right)^{-1},
\end{align*}
$$

and $P, Q, \partial \alpha / \partial x, \partial \alpha / \partial y, \partial \beta / \partial x, \partial \beta / \partial y, \partial \Phi / \partial x, \partial \Phi / \partial y$ are calculated at the point $\left(\xi\left(\tau_{j}\right), \eta\left(\tau_{j}\right)\right), P_{+}=P\left(\xi\left(\tau_{j}^{+}\right), \eta\left(\tau_{j}^{+}\right)\right)$, $Q_{+}=Q\left(\xi\left(\tau_{j}^{+}\right), \eta\left(\tau_{j}^{+}\right)\right)$, and $\tau_{j}(j \in N)$ is the time of the $j$ th jump.

## 3. The Stability of System (6) without Impulsive Effect

In the system (6), if $p=q=\lambda=\tau=0$, that is, the system without impulsive effect, the following system is obtained:

$$
\begin{align*}
& \frac{d x}{d t}=x(1-x)-\frac{x}{a+x} y \\
& \frac{d y}{d t}=y\left(\delta-\beta \frac{y}{x}\right)  \tag{11}\\
& x(0)>0, y(0)>0 .
\end{align*}
$$

If set $(x(t), y(t))$ is an arbitrary solution of the system (11) satisfying the initial conditions, then the following lemma is obtained.

Lemma 8. The solutions of the system (11) is bounded, which means $\exists T>0$ satisfies $0 \leq x(t) \leq 1$ and $0 \leq y(t) \leq \delta / \beta$ for $t \geq T$.

Obviously, the system (11) exhibits prey isocline $L_{1}: y=$ $(1-x)(x+a)$, predator isocline $L_{2}: y=(\delta / \beta) x$, nontrivial equilibrium points $E_{0}(1,0)$, and $E\left(x^{*}, y^{*}\right)$, and here;

$$
\begin{gather*}
x^{*}=\frac{(\beta-\delta-a \beta)+\sqrt{(\beta-\delta-a \beta)^{2}+4 a \beta^{2}}}{2 \beta}  \tag{12}\\
y^{*}=\frac{\delta}{\beta} x^{*}
\end{gather*}
$$

Calculating the variational matrix of the equilibrium point in the system (11), we get

$$
J\left(E_{0}\right)=\left(\begin{array}{cc}
-1 & -\frac{1}{1+a}  \tag{13}\\
0 & \delta
\end{array}\right)
$$

Obviously, $E_{0}$ is saddle point. At $E$,

$$
J(E)=\left(\begin{array}{cc}
\frac{x^{*}}{x^{*}+a}\left(1-a-2 x^{*}\right) & -\frac{x^{*}}{x^{*}+a}  \tag{14}\\
\frac{\delta^{2}}{\beta} & -\delta
\end{array}\right)
$$

The characteristic equation of $J(E)$ is

$$
\begin{equation*}
f(\lambda)=\lambda^{2}+p \lambda+q=0 \tag{15}
\end{equation*}
$$

in which $p=\delta-\left(x^{*} /\left(x^{*}+a\right)\right)\left(1-a-2 x^{*}\right), q=\left(\delta x^{*} /\left(x^{*}+\right.\right.$ a) ) $\left(\delta / \beta-\left(1-a-2 x^{*}\right)\right)$.

Thus,

$$
\begin{aligned}
\Delta=p^{2}-4 q= & \left(\delta-\frac{x^{*}}{x^{*}+a}\left(1-a-2 x^{*}\right)\right)^{2} \\
& -\frac{4 \delta x^{*}}{x^{*}+a}\left(\frac{\delta}{\beta}-\left(1-a-2 x^{*}\right)\right)
\end{aligned}
$$

$$
\begin{align*}
\lambda_{1} \lambda_{2} & =\frac{\delta x^{*}}{x^{*}+a}\left(\frac{\delta}{\beta}-\left(1-a-2 x^{*}\right)\right) \\
& =\frac{\delta}{x^{*}+a}\left(x^{* 2}+a\right)>0, \\
\lambda_{1}+\lambda_{2} & =-\left(\delta-\frac{x^{*}}{x^{*}+a}\left(1-a-2 x^{*}\right)\right) \\
& =-\frac{2 x^{* 2}+(a-1+\delta) x^{*}+a \delta}{x^{*}+a} . \tag{16}
\end{align*}
$$

Let $P(x)=2 x^{2}+(a-1+\delta) x+a \delta$, and then

$$
\begin{align*}
\Delta & =\frac{P^{2}\left(x^{*}\right)}{\left(x^{*}+a\right)^{2}}-\frac{4 \delta\left(x^{* 2}+a\right)}{x^{*}+a}  \tag{17}\\
& =\frac{1}{\left(x^{*}+a\right)^{2}}\left[P^{2}\left(x^{*}\right)-4 \delta\left(x^{* 2}+a\right)\left(x^{*}+a\right)\right]
\end{align*}
$$

Based on the above analysis, we can get the following conclusion.

Theorem 9. If $P\left(x^{*}\right)>0$, the positive equilibrium point $E\left(x^{*}\right.$, $y^{*}$ ) of the system (11) is locally asymptotically stable. Specially,
(1) if $\left(H_{1}\right): 0<P\left(x^{*}\right)<\left(4 \delta\left(x^{* 2}+a\right)\left(x^{*}+a\right)\right)^{1 / 2}, E\left(x^{*}\right.$, $y^{*}$ ) is a locally asymptotically stable focus,
(2) if $\left(H_{2}\right): P\left(x^{*}\right) \geqslant\left(4 \delta\left(x^{* 2}+a\right)\left(x^{*}+a\right)\right)^{1 / 2}, E\left(x^{*}, y^{*}\right)$ is a locally asymptotically stable node.

Next, we discuss the global stability of $E\left(x^{*}, y^{*}\right)$ about the system (11).

Theorem 10. If $\left(H_{3}\right): a+\delta \geq 1$ or $\left(H_{4}\right): 1-\sqrt{8 a \delta}<a+\delta<1$ is true, then the positive equilibrium $E\left(x^{*}, y^{*}\right)$ of the system (11) is globally asymptotically stable.

Proof. From $\left(H_{3}\right)$ or $\left(H_{4}\right)$, we have $P(x)=2 x^{2}+(a-1+$ $\delta) x+a \delta>0$ for $x>0$, and thus $P\left(x^{*}\right)>0$. Structure a Dulac function as follows:

$$
\begin{equation*}
B(x, y)=\frac{x+a}{x y^{2}}, \quad x>0, y>0 \tag{18}
\end{equation*}
$$

Let $f(x, y)=x(1-x)-(x /(a+x)) y, g(x, y)=y(\delta-\beta(y / x))$, and thus

$$
\begin{align*}
\frac{\partial(f B)}{\partial x}+\frac{\partial(g B)}{\partial y} & =-\frac{1}{x y^{2}}\left[2 x^{2}+(a-1+\delta) x+a \delta\right] \\
& =-\frac{1}{x y^{2}} P(x) \leqslant 0 \tag{19}
\end{align*}
$$

By the Bendixson-Dulac theorem, there does not exist closed orbit of the system (11) around $E$. Based on Lemma 8 and Theorem 9 , the positive equilibrium $E\left(x^{*}, y^{*}\right)$ is globally asymptotically stable.

Remark 11. If $\left(H_{1}\right),\left(H_{3}\right)$ or $\left(H_{1}\right),\left(H_{4}\right)$ are true, then $E\left(x^{*}, y^{*}\right)$ is a globally asymptotically stable focus.


Figure 1: Illustration of vector graph of system (11), where $a=0.05$, $\delta=0.5, \beta=0.7$.

Remark 12. If $\left(H_{2}\right),\left(H_{3}\right)$ or $\left(H_{2}\right),\left(H_{4}\right)$ are true, then $E\left(x^{*}\right.$, $\left.y^{*}\right)$ is a globally asymptotically stable node.

Assume that $E\left(x^{*}, y^{*}\right)$ is globally asymptotically stable focal point of the system (11), and then the illustration of vector graph of the system is as follows (see Figure 1).

## 4. The Geometric Analysis of System (6) with Two State Impulses

In this section, we will discuss the existence and stability of periodic solution of system (6) only at focal point situation. So, we assume that the conditions $\left(H_{1}\right),\left(H_{3}\right)$ or $\left(H_{1}\right),\left(H_{4}\right)$ are true. According to the practical significance of the integrated pest management model, the condition $\left(H_{5}\right): h_{1}<(1-$ p) $h_{2}<h_{2}<x^{*}$ is always given as such. By the analysis of system (6), the curve $L_{1}: y=(1-x)(x+a)$ is $X$-isocline, and the line $L_{2}: y=(\delta / \beta) x$ is $Y$-isocline. Let points $P$, $Q, R$ be the intersection of the curve $L_{1}$ and lines $x=h_{1}$, $x=(1-p) h_{2}, x=h_{2}$, respectively. Obviously $E\left(x^{*}, y^{*}\right)$ is the intersection point of $L_{1}$ and $L_{2}$. From the previous discussion, we know that the first impulsive set is $M_{1}=\{(x, y) \mid x=$ $\left.h_{1}, 0<y \leq \tilde{y}\right\}$, and the image set corresponding to $M_{1}$ is $N_{1}=\left\{(x, y) \mid x=h_{1}, 0<y \leq \tilde{y}+\lambda\right\}$; the second impulsive set is $M_{2}=\left\{(x, y) \mid x=h_{2}, y>0\right\}$, and the image set corresponding to $M_{2}$ is $N_{2}=\left\{(x, y) \mid x\left(t^{+}\right)=\right.$ $\left.(1-p) h_{2}, y\left(t^{+}\right)=(1-q) y(t)+\tau\right\}$. The structure of the system can be shown as in Figure 2.

Using successor function and geometric theory of impulsive differential equations and according to different positions of orbit initial points, the existence and stability of periodic solution of system (6) are discussed as follows.
4.1. The Initial Point on $N_{1}$. Let $C_{0}\left(h_{1}, y_{C_{0}}\right)$ be an initial point of the orbit of the system (6); if $y_{C_{0}}<\tilde{y}$, point $C_{0}$ is below


Figure 2: The structure graph of system (6).
point $P\left(h_{1}, \tilde{y}\right)$, then $C_{0} \in M_{1}$ ( $M_{1}$ is impulsive set), and the image point $C_{0}^{+}$of $C_{0}$ must be above point $P$ with $n$ times impulses; therefore, we only need to discuss the cases of $y_{C_{0}}>$ $\tilde{y}$.

The orbit $\Pi\left(C_{0}, t\right)$ starting from $C_{0}\left(h_{1}, y_{C_{0}}\right)$ hits the impulsive set $M_{1}$ at point $C_{1}\left(h_{1}, y_{C_{1}}\right)$, and then $C_{1}$ jumps to point $C_{11}\left(h_{1}, y_{C_{11}}\right)$. If $y_{C_{11}}<\tilde{y}, C_{11}$ continued to jump to point $C_{12}\left(h_{1}, y_{C_{12}}\right)$, and after $n$ times it reaches the point $C_{1 n}\left(h_{1}, y_{C_{1 n}}\right)$, where $y_{C_{11}}=y_{C_{1}}+\lambda, y_{C_{12}}=y_{C_{1}}+2 \lambda, \ldots$, $y_{C_{1 n}}=y_{C_{1}}+n \lambda$, and $\tilde{y}<y_{C_{1 n}}<\tilde{y}+\lambda$. The situation of point $C_{1 n}$ has three cases as follows.
(a) If $y_{C_{1 n}}=y_{C_{0}}$ (see Figure 3(a)), $C_{1 n}$ is coincident with $C_{0}$, then the curve $C_{0} C_{1} C_{11} \cdots C_{1 n}$ is closed orbit, and the system (6) exhibits a 1-periodic solution.
(b) If $y_{C_{1 n}}>y_{C_{0}}$ (see Figure 3(b)), $C_{1 n}$ is above $C_{0}$; in this time, the successor function of $C_{0}$ satisfies $f\left(C_{0}\right)=y_{C_{1 n}}$ $y_{C_{0}}>0$. In the meantime, choose a point $D_{0}\left(h_{1}, y_{D_{0}}\right)$ on $N_{1}$ satisfying $y_{D_{0}}>\tilde{y}+\lambda$. The orbit $\Pi\left(D_{0}, t\right)$ starting from $D_{0}$ hits the impulsive set at point $D_{1}\left(h_{1}, y_{D_{1}}\right)$, and $D_{1}$ jumps sometimes to point $D_{1 m}\left(h_{1}, y_{D_{1 m}}\right)$, where $\tilde{y}<y_{D_{1 m}}<\tilde{y}+\lambda$. Thus, the successor function of $D_{0}$ satisfies $f\left(D_{0}\right)^{m}=y_{D_{1 m}}-$ $y_{D_{0}}<0$.

According to Lemma 6, the system (6) exhibits a periodic solution, and the initial point of the periodic solution is between $C_{0}$ and $D_{0}$.
(c) If $y_{C_{1 n}}<y_{C_{0}}$ (see Figure 3(c)), $C_{1 n}$ is below $C_{0}$; in this time, the successor function of $C_{0}$ is $f\left(C_{0}\right)=y_{C_{1 n}}-y_{C_{0}}<$ 0 . On the other hand, choose a point $D_{0}\left(h_{1}, y_{D_{0}}\right)$ on $N_{1}$ satisfying $\tilde{y}<y_{D_{0}}<\tilde{y}+\varepsilon$ ( $\varepsilon$ is a sufficiently small positive number). The orbit $\Pi\left(D_{0}, t\right)$ starting from $D_{0}$ hits the impulsive set $M_{1}$ at point $D_{1}\left(h_{1}, y_{D_{1}}\right)$, and $D_{1}$ jumps to point $D_{11}\left(h_{1}, y_{D_{11}}\right)$, where $y_{D_{11}}=y_{D_{1}}+\lambda$. As $D_{0}$ sufficiently closed to $P, D_{1}$ is sufficiently close to $P$, then $y_{D_{11}}=y_{D_{1}}+\lambda>y_{D_{0}}$. Thus, the successor function of $D_{0}$ satisfies $f\left(D_{0}\right)=y_{D_{1}}-$ $y_{D_{0}}>0$.

From Lemma 6, system (6) exhibits a 1-periodic solution. To sum up the above discussed, we get the following.

Theorem 13. If the initial point $C_{0}\left(h_{1}, y_{c_{0}}\right)$ of the orbit of the system (6) is on $N_{1}$ with $y_{C_{0}}>\tilde{y}$, then the system exhibits a 1-periodic solution.

Next, we will discuss the stability of the above periodic solutions.

Theorem 14. Let $(\xi(t), \eta(t))$ be the T-periodic solution of the system (6) with the initial point $C_{0}\left(h_{1}, \eta_{0}\right)$; the closed orbit corresponding to the periodic solution is the curve $C_{0} C_{1} C_{11} \cdots C_{1 n}$, if

$$
\begin{equation*}
|\mu|=\left|\kappa \exp \left\{-\int_{0}^{T}\left(\xi(t)-\frac{\xi(t) \eta(t)}{(\xi(t)+a)^{2}}+\frac{\beta \eta(t)}{\xi(t)}\right) d t\right\}\right|<1 \tag{20}
\end{equation*}
$$

where

$$
\begin{equation*}
\kappa=\frac{\eta_{0}-n \lambda}{\eta_{0}} \prod_{j=1}^{n} \frac{1-h_{1}-\left(1 /\left(h_{1}+a\right)\right)\left(\eta_{0}-(n-j) \lambda\right)}{1-h_{1}-\left(1 /\left(h_{1}+a\right)\right)\left(\eta_{0}-(n-j+1) \lambda\right)}, \tag{21}
\end{equation*}
$$

then the periodic solution $(\xi(t), \eta(t))$ is orbitally asymptotically stable.

Proof. Let the orbit $\Pi\left(C_{0}, t\right)$ with the initial point $C_{0}\left(h_{1}, \eta_{0}\right)$ hit the impulsive set $M_{1}$ at $C_{1}(\xi(T), \eta(T))$, and then $C_{1}$ jumps to the point $C_{11}\left(\xi\left(\tau_{1}\right), \eta\left(\tau_{1}\right)\right)$. The $C_{11}$ continued to jump to point $C_{12}\left(\xi\left(\tau_{2}\right), \eta\left(\tau_{2}\right)\right)$, and at last, the image point $C_{12}$ reaches point $C_{1 n}\left(\xi\left(\tau_{n}\right), \eta\left(\tau_{n}\right)\right)$ with $n$ times pulses. Here, $\eta\left(\tau_{1}\right)=\eta(T)+\lambda, \eta\left(\tau_{2}\right)=\eta(T)+2 \lambda, \ldots, \eta\left(\tau_{n}\right)=\eta(T)+n \lambda$. For the $j$ th time impulse, obviously $\xi\left(\tau_{j}^{+}\right)=\xi\left(\tau_{j+1}\right), \eta\left(\tau_{j}^{+}\right)=$ $\eta\left(\tau_{j+1}\right)$. For the system (6), let $P(x, y)=x(1-x)-(x /(a+x)) y$, $Q(x, y)=y(\delta-\beta(y / x)), \alpha(x, y)=0, \beta(x, y)=\lambda, \Phi(x, y)=$ $x-h_{1}$, and $\xi(T)=h_{1}, \eta(T)=\eta_{0}-n \lambda$; therefore we have

$$
\begin{gather*}
\frac{\partial P}{\partial x}=1-2 x-\frac{a}{(a+x)^{2}} y, \quad \frac{\partial Q}{\partial y}=\delta-\frac{2 \beta y}{x} \\
\frac{\partial \alpha}{\partial x}=\frac{\partial \alpha}{\partial y}=0, \quad \frac{\partial \beta}{\partial x}=\frac{\partial \beta}{\partial y}=0  \tag{22}\\
\frac{\partial \Phi}{\partial x}=1, \quad \frac{\partial \Phi}{\partial y}=0
\end{gather*}
$$

According to Lemma 7, we get

$$
\begin{aligned}
\kappa_{j}= & \left(\left(\frac{\partial \beta}{\partial y} \cdot \frac{\partial \Phi}{\partial x}-\frac{\partial \beta}{\partial x} \cdot \frac{\partial \Phi}{\partial y}+\frac{\partial \Phi}{\partial x}\right) P_{+}\right. \\
& \left.+\left(\frac{\partial \alpha}{\partial x} \cdot \frac{\partial \Phi}{\partial y}-\frac{\partial \alpha}{\partial y} \cdot \frac{\partial \Phi}{\partial x}+\frac{\partial \Phi}{\partial y}\right) Q_{+}\right) \\
& \times\left(\frac{\partial \Phi}{\partial x} P+\frac{\partial \Phi}{\partial y} Q\right)^{-1}
\end{aligned}
$$


(a)

(b)

(c)

Figure 3: The orbit starting from the point $C_{0}$ on $N_{1}$.

$$
\begin{array}{rlrl}
= & \frac{P\left(\xi\left(\tau_{j}^{+}\right), \eta\left(\tau_{j}^{+}\right)\right)}{P\left(\xi\left(\tau_{j}\right), \eta\left(\tau_{j}\right)\right)} & =\prod_{j=1}^{n} \kappa_{j} \exp \left[\int_{0}^{T}\left(1-2 x-\frac{a}{(a+x)^{2}} y+\delta-\frac{2 \beta y}{x}\right) d t\right] \\
=\frac{P\left(h_{1}, \eta(T)+j \lambda\right)}{P\left(h_{1}, \eta(T)+(j-1) \lambda\right)} & =\prod_{j=1}^{n} \kappa_{j} \exp \left\{\int _ { 0 } ^ { T } \left[\left(1-x-\frac{y}{x+a}\right)+\left(\delta-\frac{\beta y}{x}\right)\right.\right. \\
=\frac{1-h_{1}-\left(1 /\left(h_{1}+a\right)\right)\left(\eta_{0}-(n-j) \lambda\right)}{1-h_{1}-\left(1 /\left(h_{1}+a\right)\right)\left(\eta_{0}-(n-j+1) \lambda\right)} & & \left.\left.-x+\frac{x y}{(a+x)^{2}}-\frac{\beta y}{x}\right] d t\right\} \\
=1-\frac{\lambda /\left(h_{1}+a\right)}{1-h_{1}-\left(1 /\left(h_{1}+a\right)\right)\left(\eta_{0}-(n-j+1) \lambda\right)}, \\
\mu=\prod_{j=1}^{n} \kappa_{j} \exp \left[\int_{0}^{T}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right) d t\right] & =\prod_{j=1}^{n} \kappa_{j} \exp \left\{\int_{0}^{T} \frac{d x}{x}+\int_{0}^{T} \frac{d y}{y}\right. \\
& \left.-\int_{0}^{T}\left(x-\frac{x y}{(a+x)^{2}}+\frac{\beta y}{x}\right) d t\right\}
\end{array}
$$

$$
\begin{align*}
= & \prod_{j=1}^{n} \kappa_{j} \frac{\eta_{0}-n \lambda}{\eta_{0}} \exp \left[-\int_{0}^{T}\left(x-\frac{x y}{(a+x)^{2}}+\frac{\beta y}{x}\right) d t\right] \\
= & \frac{\eta_{0}-n \lambda}{\eta_{0}} \\
& \times \prod_{j=1}^{n} \frac{1-h_{1}-\left(1 /\left(h_{1}+a\right)\right)\left(\eta_{0}-(n-j) \lambda\right)}{1-h_{1}-\left(1 /\left(h_{1}+a\right)\right)\left(\eta_{0}-(n-j+1) \lambda\right)} \\
& \times \exp \left[-\int_{0}^{T}\left(x-\frac{x y}{(a+x)^{2}}+\frac{\beta y}{x}\right) d t\right] \\
= & \kappa \exp \left\{-\int_{0}^{T}\left(\xi(t)-\frac{\xi(t) \eta(t)}{(\xi(t)+a)^{2}}+\frac{\beta \eta(t)}{\xi(t)}\right) d t\right\} . \tag{23}
\end{align*}
$$

From Lemma 7, if $|\mu|=\mid \kappa \exp \left\{-\int_{0}^{T}(\xi(t)-(\xi(t) \eta(t) /(\xi(t)+\right.$ $\left.\left.\left.a)^{2}\right)+(\beta \eta(t) / \xi(t))\right) d t\right\} \mid<1$, then the periodic solution of the system (6) is orbitally asymptotically stable. This completes the proof.

Remark 15. If $1-h_{1}-\left(\eta_{0} /\left(h_{1}+a\right)\right)>0$ and $\beta \geq 1$, then the periodic solution with initial point $C_{0}\left(h_{1}, \eta_{0}\right)$ (where $\left.\eta_{0}>\tilde{y}\right)$ is orbitally asymptotically stable.
4.2. The Initial Point on $N_{2}$. Based on existence and uniqueness theorem of differential equations, there exists a unique point $Q_{0}\left((1-p) h_{2}, y_{Q_{0}}\right)$ on $N_{2}$ such that the orbit $\Pi\left(Q_{0}, t\right)$ starting from $Q_{0}$ is tangent to $N_{1}$ at point $P\left(h_{1}, \widetilde{y}\right)$. Assume that $C_{0}\left((1-p) h_{2}, y_{C_{0}}\right)$ is the initial point of the orbit $\Pi\left(C_{0}, t\right)$ of system (6). Next, we will investigate the existence of periodic solution of the system with different positions of $C_{0}$ and $Q_{0}$. Three cases should be discussed.

Case $I\left(y_{C_{0}}=y_{Q_{0}}\right.$; see Figure 4). The initial point $C_{0}$ is exactly $Q_{0}$.

The orbit $\Pi\left(Q_{0}, t\right)$ starting from $Q_{0}$ is tangent to $N_{1}$ at the point $P$, and through $N_{2}$ hit $M_{2}$ at the point $Q_{1}\left(h_{2}, y_{\mathrm{Q}_{1}}\right)$, and then $Q_{1}$ jumps to $Q_{1}^{+}\left(x_{Q_{1}^{+}}, y_{Q_{1}^{+}}\right)$on $N_{2}$. According to (6), the following is obtained:

$$
\begin{equation*}
x_{\mathrm{Q}_{1}^{+}}=(1-p) h_{2}, \quad y_{\mathrm{Q}_{1}^{+}}=(1-q) y_{\mathrm{Q}_{1}}+\tau \tag{24}
\end{equation*}
$$

About the points $Q_{1}^{+}$and $Q_{0}$, there are the following three positional relations.
(a) If $Q_{1}^{+}$coincides with $Q_{0}: y_{Q_{1}^{+}}=y_{Q_{0}}$ (see Figure 4(a)), then the curve $Q_{0} P Q_{1} Q_{1}^{+}$is closed orbit.
(b) If $Q_{1}^{+}$is below $Q_{0}: y_{Q_{1}^{+}}<y_{Q_{0}}$ (see Figure 4(b)), then the successor function of $Q_{0}$ satisfies $f\left(Q_{0}\right)=$ $y_{\mathrm{Q}_{1}^{+}}-y_{\mathrm{Q}_{0}}<0$. In the meantime, take a point $S_{0}((1-$ p) $\left.h_{2}, y_{S_{0}}\right)$ on $N_{2}$ satisfying $0<y_{S_{0}}<\varepsilon(\varepsilon>0$ small enough). The orbit $\Pi\left(S_{0}, t\right)$ starting from $S_{0}$ hits the impulsive $M_{2}$ at the point $S_{1}\left(h_{2}, y_{S_{1}}\right)$, and then $S_{1}$ jumps to the point $S_{1}^{+}\left(x_{S_{1}^{+}}, y_{S_{1}^{+}}\right)$, where $x_{S_{1}^{+}}=(1-p) h_{2}$,
$y_{S_{1}^{+}}=(1-q) y_{S_{1}}+\tau$. Obviously, the successor function of $S_{0}$ is $f\left(S_{0}\right)=y_{S_{1}^{+}}-y_{S_{0}}>0$. From Lemma 6 , the system (6) has an order one periodic solution, where the initial point of the periodic solution is between $Q_{0}$ and $S_{0}$.
(c) If $Q_{1}^{+}$is above $Q_{0}: y_{Q_{1}^{+}}>y_{Q_{0}}$ (see Figure 4(c)), the system (6) does not have closed orbit in the area $\Omega_{1}=$ $\left\{(x, y) \mid h_{1}<x<h_{2}\right\}$ in this time.

Based on the discussion above, we get the following.
Theorem 16. Assume that the orbit $\Pi\left(Q_{0}, t\right)$ starting from $Q_{0}\left((1-p) h_{2}, y_{Q_{0}}\right)$ is tangent to $N_{1}$ at the point $P\left(h_{1}, \tilde{y}\right)$ and hits the impulsive set $M_{2}$ at the point $Q_{1}\left(h_{2}, y_{Q_{1}}\right)$, the image point of $Q_{1}$ is $\mathrm{Q}_{1}^{+}\left(x_{\mathrm{Q}_{1}^{+}}, y_{\mathrm{Q}_{1}^{+}}\right)$on $N_{2}$. If $y_{\mathrm{Q}_{1}^{+}} \leq y_{\mathrm{Q}_{0}}$, the system (6) has 1-periodic solution in the area $\Omega_{1}=\left\{(x, y) \mid h_{1}<x<h_{2}\right\}$.

Case II $\left(y_{C_{0}}<y_{\mathrm{Q}_{0}}\right)$. The initial point $C_{0}$ is below $Q_{0}$.
The isocline $L_{1}: y=(1-x)(x+a)$ intersects with the phase set $N_{2}$ at the point $Q\left((1-p) h_{2}, y_{Q}\right)$, and $Q$ is below $Q_{0}$. In this case, we discuss the existence of periodic solution of the system (6) with the example $y_{C_{0}}=y_{\mathrm{Q}}$ (see Figure 5).

The orbit $\Pi(Q, t)$ starting from $Q$ moves to the point $C_{1}\left(h_{2}, y_{C_{1}}\right)$ on the impulsive set $M_{2}$, and $C_{1}$ jumps onto $C_{1}^{+}\left(x_{C_{1}^{+}}, y_{C_{1}^{+}}\right)$on the image set $N_{2}$, and then

$$
\begin{align*}
& y_{C_{0}}=y_{Q}=\left(1-(1-p) h_{2}\right)\left(a+(1-p) h_{2}\right) \\
& x_{C_{1}^{+}}=(1-p) h_{2}, \quad y_{C_{1}^{+}}=(1-q) y_{C_{1}}+\tau \tag{25}
\end{align*}
$$

(a) If $y_{C_{1}^{+}}=y_{Q}: C_{1}^{+}$coincide with $Q$ (see Figure 5(a)), the curve $Q C_{1} C_{1}^{+}$is the closed orbit of the system (6).
(b) If $y_{\mathrm{Q}}<y_{\mathrm{C}_{1}^{+}} \leq y_{\mathrm{Q}_{0}}: C_{1}^{+}$is between $Q$ and $Q_{0}$ (see Figure 5(b)), in this case the successor function of $Q$ is $f(Q)=y_{C_{1}^{+}}-y_{Q}>0$. On the other hand, consider the orbit $\Pi\left(Q_{0}, t\right)$ starting from $Q_{0}, \Pi\left(Q_{0}, t\right)$ hits the impulsive set $M_{2}$ at $Q_{1}\left(h_{2}, y_{\mathrm{Q}_{1}}\right)$, and $Q_{1}$ jumps onto $Q_{1}^{+}\left(x_{\mathrm{Q}_{1}^{+}}, y_{\mathrm{Q}_{1}^{+}}\right)$on $N_{2}$. Based on the existence and uniqueness theorem of differential equations, $Q_{1}$ must be below $C_{1}$, and $Q_{1}^{+}$must be below $C_{1}^{+}$. Thus $y_{Q_{1}^{+}}<y_{C_{1}^{+}} \leq y_{Q_{0}}$, and the successor function of $Q_{0}$ is $f\left(Q_{0}\right)=y_{Q_{1}^{+}}-y_{Q_{0}}<0$. From Lemma 6, the system (6) has 1-periodic solution, and the initial point of the periodic solution is between $Q_{0}$ and $Q$.
If $y_{C_{1}^{+}}>y_{\mathrm{Q}_{0}}: C_{1}^{+}$is above $Q_{0}$, in this case the successor function of $Q$ is $f(Q)=y_{C_{1}^{+}}-y_{Q}>0$. There are two different cases (case (c) and case (d)).
(c) If $y_{C_{1}^{+}}>y_{Q_{0}}$ and $y_{Q_{1}^{+}} \leq y_{Q_{0}}: Q_{1}^{+}$is below $Q_{0}$ (see Figure 5(c)), $f\left(Q_{0}\right)=y_{Q_{1}^{+}}-y_{Q_{0}} \leq 0$, the system (6) has closed orbit.
(d) If $y_{C_{1}^{+}}>y_{Q_{0}}$ and $y_{Q_{1}^{+}}>y_{Q_{0}}: Q_{1}^{+}$is above $Q_{0}$ (see Figure 5(d)), the system (6) does not have closed orbit in the area $\Omega_{1}=\left\{(x, y) \mid h_{1}<x<h_{2}\right\}$.


Figure 4: The initial point $C_{0}$ on $N_{2}$ (Case I: $y_{C_{0}}=y_{Q_{0}}$ ).
(e) If $y_{C_{1}^{+}}<y_{Q}: C_{1}^{+}$is below $Q$ (see Figure $5(\mathrm{e})$ ), in this case the successor function of $Q$ is $f(Q)=$ $y_{C_{1}^{+}}-y_{\mathrm{Q}}<0$. On the other hand, take a point $D_{0}((1-$ p) $h_{2}, y_{D_{0}}$ ) from $N_{2}$ satisfying that $y_{D_{0}}$ is sufficiently small number, which is $0<y_{D_{0}}<\varepsilon(\varepsilon>0$ small enough). The orbit $\Pi\left(D_{0}, t\right)$ starting from $D_{0}$ moves to $D_{1}\left(h_{2}, y_{D_{1}}\right)$ on the impulsive set $M_{2}$, and $D_{1}$ jumps onto $D_{1}^{+}\left(x_{D_{1}^{+}}, y_{D_{1}^{+}}\right)$on $N_{2}$, where $x_{D_{1}^{+}}=(1-p) h_{2}$, $y_{D_{1}^{+}}=(1-q) y_{D_{1}}+\tau$. Obviously, the successor function
of $D_{0}$ is $f\left(D_{0}\right)=y_{D_{1}^{+}}-y_{D_{0}}>0$. From Lemma 6 , the system (6) has closed orbit.

Based on the discussion above, we get the following.
Theorem 17. Assume that the orbit $\Pi(Q, t)$ starting from $Q\left((1-p) h_{2}, y_{Q}\right)$ hits the impulsive set $M_{2}$ at $C_{1}\left(h_{2}, y_{C_{1}}\right)$, and $C_{1}$ jumps onto $C_{1}^{+}\left(x_{C_{1}^{+}}, y_{C_{1}^{+}}\right)$on $N_{2}$. The orbit $\Pi\left(Q_{0}, t\right)$ starting from the point $Q_{0}\left((1-p) h_{2}, y_{Q_{0}}\right)$ is tangent to $N_{1}$ at $P\left(h_{1}, \tilde{y}\right)$,


Figure 5: The initial point $C_{0}$ on $N_{2}$ (Case II: $y_{C_{0}}<y_{\mathrm{Q}_{0}}$ ).


Figure 6: The initial point $C_{0}$ on $N_{2}$ (Case III: $y_{C_{0}}>y_{Q_{0}}$ ).
and hitting the impulsive set $M_{2}$ at $Q_{1}\left(h_{2}, y_{Q_{1}}\right)$, the image point of $\mathrm{Q}_{1}$ is $\mathrm{Q}_{1}^{+}\left(x_{\mathrm{Q}_{1}^{+}}, y_{\mathrm{Q}_{1}^{+}}\right)$. Then one has the following.
(1) If $y_{C_{1}^{+}} \leq y_{Q_{0}}$, the system (6) has 1-periodic solution.
(2) If $y_{\mathrm{C}_{1}^{+}}>y_{\mathrm{Q}_{0}}$ and $y_{\mathrm{Q}_{1}^{+}} \leq y_{\mathrm{Q}_{0}}$, the system (6) has 1periodic solution.

Case III ( $y_{C_{0}}>y_{\mathrm{Q}_{0}}$; see Figure 6). The initial point $C_{0}$ is above $Q_{0}$.

In this case, the orbit $\Pi\left(C_{0}, t\right)$ starting from $C_{0}$ goes through the isocline $L_{1}$ from the left of the line $x=h_{1}$, hitting the impulsive set $M_{1}$ at $C_{1}\left(h_{1}, y_{C_{1}}\right)$. The same conclusion can be made as in Section 4.1.

Next, we discuss the stability of the periodic solution with the initial point on $N_{2}$.

Theorem 18. Assume that $(\xi(t), \eta(t))$ is the T-periodic solution of the system (6) with initial point $C_{0}\left((1-p) h_{2}, \eta_{0}\right)$; and if

$$
\begin{equation*}
|\mu|=\left|\kappa \exp \left\{-\int_{0}^{T}\left(\xi(t)-\frac{\xi(t) \eta(t)}{(\xi(t)+a)^{2}}+\frac{\beta \eta(t)}{\xi(t)}\right) d t\right\}\right|<1 \tag{26}
\end{equation*}
$$

where

$$
\begin{equation*}
\kappa=\frac{\eta_{0}-\tau}{\eta_{0}} \cdot \frac{1-(1-p) h_{2}-\left(\eta_{0} /\left((1-p) h_{2}+a\right)\right)}{1-h_{2}-\left(\left(\eta_{0}-\tau\right) /\left((1-q)\left(h_{2}+a\right)\right)\right)}, \tag{27}
\end{equation*}
$$

the periodic solution $(\xi(t), \eta(t))$ is orbitally asymptotically stable.

Proof. Assume that the periodic orbit $\Pi\left(C_{0}, t\right)$ starting from the point $C_{0}\left((1-p) h_{2}, \eta_{0}\right)$ moves to the point $C_{1}(\xi(T), \eta(T))$ on impulsive set $M_{2}$, and $C_{1}$ jumps onto the point $C_{1}^{+}\left(\xi\left(T^{+}\right), \eta\left(T^{+}\right)\right)$on $N_{2}$. Therefore, $\Pi\left(C_{0}, T\right)=C_{1}$, $C_{1}^{+}=\varphi_{2}\left(C_{1}\right)=C_{0}, \xi\left(T^{+}\right)=(1-p) \xi(T), \eta\left(T^{+}\right)=(1-q) \eta(T)+$ $\tau$.

Compared with the system (6), we get

$$
\begin{gather*}
P(x, y)=x(1-x)-\frac{x}{a+x} y, \\
Q(x, y)=y\left(\delta-\beta \frac{y}{x}\right), \\
\alpha(x, y)=-p x, \quad \beta(x, y)=-q y+\tau, \\
\Phi(x, y)=x-h_{2}, \quad \xi(T)=h_{2}, \quad \eta(T)=\frac{\eta_{0}-\tau}{1-q}, \\
\frac{\partial P}{\partial x}=1-2 x-\frac{a}{(a+x)^{2}} y, \quad \frac{\partial Q}{\partial y}=\delta-\frac{2 \beta y}{x}, \\
\frac{\partial \alpha}{\partial x}=-p, \quad \frac{\partial \alpha}{\partial y}=0, \quad \frac{\partial \beta}{\partial x}=0, \\
\frac{\partial \beta}{\partial y}=-q, \quad \frac{\partial \Phi}{\partial x}=1, \quad \frac{\partial \Phi}{\partial y}=0 . \tag{28}
\end{gather*}
$$

Thus,

$$
\begin{aligned}
& \kappa_{1}=\left(\left(\frac{\partial \beta}{\partial y} \cdot \frac{\partial \Phi}{\partial x}-\frac{\partial \beta}{\partial x} \cdot \frac{\partial \Phi}{\partial y}+\frac{\partial \Phi}{\partial x}\right) P_{+}\right. \\
& \left.+\left(\frac{\partial \alpha}{\partial x} \cdot \frac{\partial \Phi}{\partial y}-\frac{\partial \alpha}{\partial y} \cdot \frac{\partial \Phi}{\partial x}+\frac{\partial \Phi}{\partial y}\right) Q_{+}\right) \\
& \times\left(\frac{\partial \Phi}{\partial x} P+\frac{\partial \Phi}{\partial y} Q\right)^{-1} \\
& =\frac{(1-q) P\left(\xi\left(T^{+}\right), \eta\left(T^{+}\right)\right)}{P(\xi(T), \eta(T))} \\
& =((1-p)(1-q) \\
& \left.\times\left(1-(1-p) h_{2}-\frac{\eta_{0}}{(1-p) h_{2}+a}\right)\right) \\
& \times\left(1-h_{2}-\frac{\eta_{0}-\tau}{(1-q)\left(h_{2}+a\right)}\right)^{-1}, \\
& \mu=\kappa_{1} \exp \left[\int_{0}^{T}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right) d t\right] \\
& =\kappa_{1} \exp \left[\int_{0}^{T}\left(1-2 x-\frac{a}{(a+x)^{2}} y+\delta-\frac{2 \beta y}{x}\right) d t\right] \\
& =\kappa_{1} \exp \left\{\int _ { 0 } ^ { T } \left[\left(1-x-\frac{y}{x+a}\right)+\left(\delta-\frac{\beta y}{x}\right)\right.\right. \\
& \left.\left.-x+\frac{x y}{(a+x)^{2}}-\frac{\beta y}{x}\right] d t\right\}
\end{aligned}
$$



Figure 7: The periodic solution corresponding to Figure 3.

$$
\begin{align*}
= & \kappa_{1} \exp \left\{\int_{0}^{T} \frac{d x}{x}+\int_{0}^{T} \frac{d y}{y}-\int_{0}^{T}\left(x-\frac{x y}{(a+x)^{2}}+\frac{\beta y}{x}\right) d t\right\} \quad \times \exp \left[-\int_{0}^{T}\left(x-\frac{x y}{(a+x)^{2}}+\frac{\beta y}{x}\right) d t\right] \\
= & \frac{(1-p)(1-q)\left(1-(1-p) h_{2}-\left(\eta_{0} /\left((1-p) h_{2}+a\right)\right)\right)}{1-h_{2}-\left(\left(\eta_{0}-\tau\right) /\left((1-q)\left(h_{2}+a\right)\right)\right)} \quad=\kappa \exp \left[-\int_{0}^{T}\left(x-\frac{x y}{(a+x)^{2}}+\frac{\beta y}{x}\right) d t\right] \\
& \times \exp \left\{\int_{0}^{T} \frac{d x}{x}+\int_{0}^{T} \frac{d y}{y}-\int_{0}^{T}\left(x-\frac{x y}{(a+x)^{2}}+\frac{\beta y}{x}\right) d t\right\}
\end{align*}
$$

$$
=\frac{\eta_{0}-\tau}{\eta_{0}} \cdot \frac{1-(1-p) h_{2}-\left(\eta_{0} /\left((1-p) h_{2}+a\right)\right)}{1-h_{2}-\left(\left(\eta_{0}-\tau\right) /\left((1-q)\left(h_{2}+a\right)\right)\right)}
$$

From Lemma 7, if $|\mu|=\mid \kappa \exp \left\{-\int_{0}^{T}(\xi(t)-(\xi(t) \eta(t) /\right.$ $\left.\left.\left.(\xi(t)+a)^{2}\right)+(\beta \eta(t) / \xi(t))\right) d t\right\} \mid<1$, then the periodic solution


Figure 8: There exists a 1-periodic solution in the area $\Omega_{1}=\left\{(x, y) \mid h_{1}<x<h_{2}\right\}$ corresponding to Figure 4(b).
of system (6) is orbitally asymptotically stable. This completes the proof.

Remark 19. If $\left.\mid\left(\left(1-(1-p) h_{2}\right)\right)-\left(\eta_{0} /\left((1-p) h_{2}+a\right)\right)\right) /(1-$ $\left.h_{2}-\left(\left(\eta_{0}-\tau\right) /\left((1-q)\left(h_{2}+a\right)\right)\right)\right) \mid \leq 1$ and $\beta \geq 1$, the periodic solution of system (6) with initial point $C_{0}\left((1-p) h_{2}, \eta_{0}\right)$ is orbitally asymptotically stable.

## 5. Example and Numerical Simulation

In this part, we use numerical simulation to confirm the conclusion obtained above. Let $a=0.05, \delta=0.5, \beta=0.7$,
$h_{1}=0.24, h_{2}=0.35, \lambda=0.18, p=0.2, q=0.2$. By calculation, we obtain $\tilde{y}=0.22, P(0.24,0.22), Q(0.28,0.2376)$ and $R(0.35,0.26)$. Then, we have an example as follows:

$$
\begin{aligned}
& \frac{d x(t)}{d t}=x(t)(1-x(t))-\frac{x(t)}{0.05+x(t)} y(t), \\
& \frac{d y(t)}{d t}=y(t)\left(0.5-0.7 \frac{y(t)}{x(t)}\right), \\
& x \neq 0.24,0.35 \quad \text { or } \quad x=0.24, y>0.22, \\
& \Delta x(t)=0, \quad x=0.24, \quad y \leq 0.22, \\
& \Delta y(t)=0.18, \quad x=0,
\end{aligned}
$$



Figure 9: 1-periodic solution in the area $\Omega_{1}=\left\{(x, y) \mid h_{1}<x<h_{2}\right\}$ corresponding to Figure 4(c) does not exist.

$$
\begin{gather*}
\Delta x(t)=-0.2 x(t), \quad x=0.35 \\
\Delta y(t)=-0.2 y(t)+\tau, \quad  \tag{30}\\
x(0)>0, y(0)>0
\end{gather*}
$$

Case 1. Let $\tau=0.15$, and the initial point is $(0.24,0.3)$. From Figure 7 corresponding to Figure 3, the system exhibits a 1periodic solution.

Case 2. Let $\tau=0.1$, then $(0.28,0.3056)$ is the initial point. From Figure 8, which corresponds to Figure 4(b), the system
exhibits a 1-periodic solution in the area $\Omega_{1}=\left\{(x, y) \mid h_{1}<\right.$ $\left.x<h_{2}\right\}$.

Case 3. Let $\tau=0.15$, we get the initial point ( $0.28,0.3056$ ). It is easy to find that the system has no 1-periodic solution in the area $\Omega_{1}=\left\{(x, y) \mid h_{1}<x<h_{2}\right\}$ from Figure 9 which corresponds to Figure 4(c).

Case 4. Let $\tau=0.1$, the initial point $(0.28,0.2376)$ is obtained. From Figure 10 corresponding to Figure 5(b), in the area $\Omega_{1}=$ $\left\{(x, y) \mid h_{1}<x<h_{2}\right\}$, the system exhibits a 1-periodic solution.


Figure 10: There exists a 1-periodic solution in the area $\Omega_{1}=\left\{(x, y) \mid h_{1}<x<h_{2}\right\}$ corresponding to Figure 5(b).

Case 5. Let $\tau=0.15$, we can easily get the initial point ( $0.28,0.2376$ ). From Figure 11 corresponding to Figure 5(d), clearly, there is no 1-periodic solution in the area $\Omega_{1}=$ $\left\{(x, y) \mid h_{1}<x<h_{2}\right\}$.

Case 6. Let $\tau=0.15$, and the initial point is $(0.28,0.45)$. Obviously, we can find a 1-periodic solution from Figure 12 which corresponds to Figure 6.

All the simulations above show agreement with the results in Section 4.

## 6. Conclusion

This paper establishes a class of integrated pest management model based on state impulse control. In the initial stage of the occurrence of crop pests, that is, the pest density satisfies $x(t) \leq h_{1}$, we use environment protection measures to control pests, such as releasing natural enemies. Once the pest density reaches a higher level $x(t)=h_{2}$, we will adapt a combination of spraying insecticide and releasing natural enemies to control pests. With a short time to finish spraying insecticide and releasing natural enemies which bring out a


Figure 11: 1-periodic solution in the area $\Omega_{1}=\left\{(x, y) \mid h_{1}<x<h_{2}\right\}$ corresponding to Figure 5(d) does not exist.
sharp change in the number of pests and natural enemies, the state impulsive differential system (6) is obtained. Firstly, let the control parameters $p, q, \lambda, \tau$ be zero, we get HollingTanner ecosystem without impulsive effects. By constructing Dulac function, we discussed the stability of the positive equilibrium point $E\left(x^{*}, y^{*}\right)$, and the globally asymptotically stable conditions are given for focal points and nodal point, respectively. If the control parameters $p, q, \lambda, \tau$ are larger than zero, the system (6) is semicontinuous pulse dynamic system. The existence, uniqueness, and stability of the periodic
solutions are the research difficulties, and we need to consider all the pulse conditions (the value of $h_{1}, h_{2}$ ) and pulse function and its corresponding qualitative properties of the continuous dynamic system. By introducing the successor function, using impulsive differential geometry theory, we have discussed the existence of periodic solutions of the system (6) with a focus. According to the theory of impulsive differential multiplier Analogue of Poincare Criterion, the conditions of periodic solution with orbit asymptotically stable are given. Since (6) is a two-dimensional dynamical


Figure 12: The periodic solution corresponding to Figure 6.
system, geometric method is intuitive and effective. How to study high-dimensional ecological dynamic systems by the geometric theory needs to be resolved in the future.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

Wencai Zhao, Tongqian Zhang, and Xinzhu Meng are financially supported by the National Natural Science Foundation
of China (no. 11371230), the Shandong Provincial Natural Science Foundation, China (no. ZR2012AM012), and a Project of Shandong Province Higher Educational Science and Technology Program of China (no. J13LI05). Xinzhu Meng is financially supported by the SDUST Research Fund (no. 2011KYTD105).

## References

[1] Agricultural Information Network of China, http://www.agri .gov.c/.
[2] C. C. Yang, C. S. Wang, Y. N. Zheng et al., "Sustained effects of Trichogramma dendrolimi on Ostrinia furnacali,"

Journal of Maize Sciences, vol. 19, no. 1, pp. 139-142, 2011 (Chinese).
[3] D. D. Bainov and P. S. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications, CRC Press, Boca Raton, Fla, USA, 1993.
[4] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, Theory of Impulsive Differential Equations, vol. 6 of Series in Modern Applied Mathematics, World Scientific, Singapore, 1989.
[5] P. S. Simeonov and D. D. Baĭnov, "Orbital stability of periodic solutions of autonomous systems with impulse effect," International Journal of Systems Science, vol. 19, no. 12, pp. 2561-2585, 1988.
[6] E. M. Bonotto, "LaSalles theorems in impulsive semidynamical systems," Cadernos de Matematica, vol. 9, pp. 157-168, 2008.
[7] E. M. Bonotto and M. Federson, "Topological conjugation and asymptotic stability in impulsive semidynamical systems," Journal of Mathematical Analysis and Applications, vol. 326, no. 2, pp. 869-881, 2007.
[8] J. J. Nieto and D. O'Regan, "Variational approach to impulsive differential equations," Nonlinear Analysis: Real World Applications, vol. 10, no. 2, pp. 680-690, 2009.
[9] J. J. Nieto, "Periodic boundary value problems for first-order impulsive ordinary differential equations," Nonlinear Analysis: Theory, Methods \& Applications, vol. 51, no. 7, pp. 1223-1232, 2002.
[10] L. S. Chen, "Pest control and geometric theory of semicontinuous dynamical system," Journal of Beihua University (Natural Science), vol. 12, no. 1, pp. 1-9, 2011 (Chinese).
[11] L. S. Chen, "Theory and application of semi-continuous dynamical system," Journal of Yulin Normal University (Natural Science), vol. 34, no. 2, pp. 1-9, 2013 (Chinese).
[12] X. Song, M. Hao, and X. Meng, "A stage-structured predatorprey model with disturbing pulse and time delays," Applied Mathematical Modelling, vol. 33, no. 1, pp. 211-223, 2009.
[13] S. Sun and L. Chen, "Mathematical modelling to control a pest population by infected pests," Applied Mathematical Modelling, vol. 33, no. 6, pp. 2864-2873, 2009.
[14] S. Tang and R. A. Cheke, "Models for integrated pest control and their biological implications," Mathematical Biosciences, vol. 215, no. 1, pp. 115-125, 2008.
[15] S. Tang, Y. Xiao, and R. A. Cheke, "Multiple attractors of hostparasitoid models with integrated pest management strategies: eradication, persistence and outbreak," Theoretical Population Biology, vol. 73, no. 2, pp. 181-197, 2008.
[16] B. Liu, Y. Zhang, and L. Chen, "The dynamical behaviors of a Lotka-Volterra predator-prey model concerning integrated pest management," Nonlinear Analysis: Real World Applications, vol. 6, no. 2, pp. 227-243, 2005.
[17] L. Mailleret and F. Grognard, "Global stability and optimisation of a general impulsive biological control model," Mathematical Biosciences, vol. 221, no. 2, pp. 91-100, 2009.
[18] R. Shi and L. Chen, "The study of a ratio-dependent predatorprey model with stage structure in the prey," Nonlinear Dynamics, vol. 58, no. 1-2, pp. 443-451, 2009.
[19] X. Meng, J. Jiao, and L. Chen, "The dynamics of an age structured predator-prey model with disturbing pulse and time delays," Nonlinear Analysis: Real World Applications, vol. 9, no. 2, pp. 547-561, 2008.
[20] X. Meng, Z. Li, and J. J. Nieto, "Dynamic analysis of MichaelisMenten chemostat-type competition models with time delay and pulse in a polluted environment," Journal of Mathematical Chemistry, vol. 47, no. 1, pp. 123-144, 2010.
[21] J. Jiao, W. Long, and L. Chen, "A single stage-structured population model with mature individuals in a polluted environment and pulse input of environmental toxin," Nonlinear Analysis: Real World Applications, vol. 10, no. 5, pp. 3073-3081, 2009.
[22] T. Zhang, X. Meng, and Y. Song, "The dynamics of a highdimensional delayed pest management model with impulsive pesticide input and harvesting prey at different fixed moments," Nonlinear Dynamics, vol. 64, no. 1-2, pp. 1-12, 2011.
[23] T. Q. Zhang, X. Z. Meng, Y. Song, and T. H. Zhang, "A stagestructured predator-prey SI model with disease in the prey and impulsive effects," Mathematical Modelling and Analysis, vol. 18, no. 4, pp. 505-528, 2013.
[24] H. Zhang, J. Jiao, and L. Chen, "Pest management through continuous and impulsive control strategies," BioSystems, vol. 90, no. 2, pp. 350-361, 2007.
[25] H. Zhang, L. Chen, and J. J. Nieto, "A delayed epidemic model with stage-structure and pulses for pest management strategy," Nonlinear Analysis: Real World Applications, vol. 9, no. 4, pp. 1714-1726, 2008.
[26] J.-J. Jiao, L.-S. Chen, J. J. Nieto, and A. Torres, "Permanence and global attractivity of stage-structured predator-prey model with continuous harvesting on predator and impulsive stocking on prey," Applied Mathematics and Mechanics, vol. 29, no. 5, pp. 653-663, 2008.
[27] S. Tang, Y. Xiao, L. Chen, and R. A. Cheke, "Integrated pest management models and their dynamical behaviour," Bulletin of Mathematical Biology, vol. 67, no. 1, pp. 115-135, 2005.
[28] S. Tang and R. A. Cheke, "State-dependent impulsive models of integrated pest management (IPM) strategies and their dynamic consequences," Journal of Mathematical Biology, vol. 50, no. 3, pp. 257-292, 2005.
[29] G. Zeng, L. Chen, and L. Sun, "Existence of periodic solution of order one of planar impulsive autonomous system," Journal of Computational and Applied Mathematics, vol. 186, no. 2, pp. 466-481, 2006.
[30] L. Zhao, L. Chen, and Q. Zhang, "The geometrical analysis of a predator-prey model with two state impulses," Mathematical Biosciences, vol. 238, no. 2, pp. 55-64, 2012.
[31] L. Nie, Z. Teng, L. Hu, and J. Peng, "Existence and stability of periodic solution of a predator-prey model with statedependent impulsive effects," Mathematics and Computers in Simulation, vol. 79, no. 7, pp. 2122-2134, 2009.
[32] L. Nie, Z. Teng, L. Hu, and J. Peng, "Qualitative analysis of a modified Leslie-Gower and Holling-type II predatorprey model with state dependent impulsive effects," Nonlinear Analysis: Real World Applications, vol. 11, no. 3, pp. 1364-1373, 2010.
[33] B. Liu, Y. Tian, and B. Kang, "Dynamics on a Holling II predator-prey model with state-dependent impulsive control," International Journal of Biomathematics, vol. 5, no. 3, Article ID 1260006, 2012.
[34] C. Dai, M. Zhao, and L. Chen, "Homoclinic bifurcation in semi-continuous dynamic systems," International Journal of Biomathematics, vol. 5, no. 6, Article ID 1250059, 2012.
[35] J. B. Fu and Y. Z. Wang, "The mathematical study of pest management strategy," Discrete Dynamics in Nature and Society, vol. 2012, Article ID 251942, 19 pages, 2012.
[36] C. Wei and L. Chen, "Periodic solution of prey-predator model with Beddington-DeAngelis functional response and impulsive state feedback control," Journal of Applied Mathematics, vol. 2012, Article ID 607105, 17 pages, 2012.
[37] C. Wei and L. Chen, "Heteroclinic bifurcations of a preypredator fishery model with impulsive harvesting," International Journal of Biomathematics, vol. 6, no. 5, Article ID 1350031, 2013.
[38] L. S. Chen, X. Z. Meng, and J. J. Jiao, Biological Dynamics, Science Press, Beijing, China, 2009.
[39] S. B. Hsu and T. W. Huang, "Global stability for a class of predator-prey systems," SIAM Journal on Applied Mathematics, vol. 55, no. 3, pp. 763-783, 1995.

# Analysis of a Patch Model for the Dynamical Transmission of Echinococcosis 

Kai Wang, ${ }^{1,2}$ Xueliang Zhang, ${ }^{2}$ Zhidong Teng, ${ }^{3}$ Lei Wang, ${ }^{2}$ and Liping Zhang ${ }^{2}$<br>${ }^{1}$ Department of Mathematics, North University of China, Taiyuan, Shanxi 030051, China<br>${ }^{2}$ Department of Medical Engineering and Technology, Xinjiang Medical University, Urumqi 830011, China<br>${ }^{3}$ College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, China

Correspondence should be addressed to Kai Wang; wangkaimath@sina.com
Received 18 January 2014; Accepted 4 February 2014; Published 18 March 2014
Academic Editor: Weiming Wang
Copyright © 2014 Kai Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

A patch model for echinococcosis due to dogs migration is proposed to explore the effect of dogs migration among patches on the spread of echinococcosis. We firstly define the basic reproduction number $R_{0}$. The mathematical results show that the dynamics of the model can be completely determined by $R_{0}$. If $R_{0}<1$, the disease-free equilibrium is globally asymptotically stable. When $R_{0}>1$, the model is permanence and endemic equilibrium is globally asymptotically stable. According to the simulations, it is shown that the larger diffusion of dogs from the lower epidemic areas to the higher prevalence areas can intensify the spread of echinococcosis. However, the larger diffusion of dogs from the higher prevalence areas to the lower epidemic areas can reduce the spread and is beneficial for disease control.


## 1. Introduction

Echinococcosis, which is often referred to as hydatid disease, is a parasitic disease that affects both humans and other mammals, such as sheep, dogs, rodents, and horses [1]. The two most clinically relevant species are Echinococcus granulosus and Echinococcus multilocularis, which cause cystic and alveolar echinococcosis respectively. Humans are incidental hosts and, in most cases, do not contribute to continuance of the parasite life cycle, except under unique circumstances [2].

The prevalent scope of echinococcosis in China is approximately 420 square kilometers, accounting for about $41.7 \%$ of the territory. The rate of incidence of echinococcosis has increased in the past decade. The operability of echinococcosis exceeds $10 / 100000$ in each year. High-risk group subject to echinococcosis reaches up to 50 million, and the number of domestic animal amount being faced with the infection of echinococcosis is more than one hundred million, in which the amount of dogs is at least 5 million [3].

Mathematical modeling has become an important tool in analyzing the epidemiological characteristics of infectious disease and can provide useful control measures. Various models have been used to study different aspects of
echinococcosis [4-16]. The models included varied primarily on the basis of six key features that were differentially incorporated in their design [17]. These are (1) the inclusion of a "latent" class (with time delay from host exposure to infectiousness); (2) an age structure for definitive and/or intermediate hosts; (3) the presence of density dependent constraints; (4) accounting for seasonality; (5) stochastic parameters; (6) inclusion of a spatial and risk structures.

In [18], in order to explore effective control and prevention measures authors proposed a deterministic model to study the transmission dynamics of echinococcosis in Xinjiang. The results showed that the dynamics of the model was completely determined by the basic reproductive number $R_{0}$. The model provided an approximate estimate of the basic reproduction number $R_{0}=1.67$.

Many epidemic models with population dispersal among patches have been proposed and studied (see [19-28]). Wang and Zhao [19] proposed an epidemic model to describe the dynamics of disease spread among patches due to population dispersal. The effect of population dispersal among $n$ patches on the spread of a disease was investigated by Jin and Wang in [20]. To understand the effect of transport-related infection on disease spread, an epidemic model for several
regions which are connected by transportation of individuals has been proposed by Cui et al. in [21]. In [23], an SIS patch model with nonconstant transmission coefficients was formulated to investigate the effect of media coverage and human movement on the spread of infectious diseases among patches. Qiu [26] developed a mathematical model to explore the effect of host migration between two patches on the spread of a vector-host disease.

To date, few scholars have researched the echinococcosis transmission models with dogs migration among patches. Considering an increasing number of stray dogs, the dispersal is an essential trait for dogs population. Therefore, we expect to explore the effect of dogs migration among patches on the spread of echinococcosis.

The purpose of this paper is to model the transmission dynamics of echinococcosis spread between two patches due to dogs migration and describe the dynamics of the model. The remaining part of this paper is organized as follows. The model is presented in Section 2. The basic properties on the positivity and boundedness of solutions computing the basic reproduction number are in Section 3. In Section 4, we establish the global stability of the disease-free equilibrium for the model. In Section 5, we will apply the theory of permanence to obtain the permanence of the model. The global stability theorem of endemic equilibrium is stated and proved in Section 6. In Section 7, we give some examples to illustrate how the dogs migration affects the dynamics of echinococcosis. A brief discussion is given in Section 8.

## 2. Model Formulation

In this section, we mainly formulate an epidemic model to describe the transmission dynamics of echinococcosis spread between two discrete patches due to dogs diffusion.

We firstly formulate a model for the spread of echinococcosis in the $i$ th patch. It follows from [18] that the parameters of humans do not affect dynamical behaviors of echinococcosis model. Hence in the paper we only consider dogs, livestock, and Echinococcus eggs in our model. We divide the dogs population in the $i$ th patch into two classes: the susceptible population and the infected population denoted by $S_{D i}(t)$ and $I_{D i}(t)$, respectively. For livestock population, we divide the total livestock population in the $i$ th patch into two classes: susceptible and infectious denoted by $S_{L i}(t)$ and $I_{L i}(t)$, respectively. The density of Echinococcus eggs in the $i$ th patch is denoted by $x_{i}(t)$. Our assumptions on the dynamical transmission of echinococcosis in the $i$ th patch are demonstrated in the flowchart (Figure 1).

If there is no dogs migration among patches, that is, the patches are isolated, we suppose that the echinococcosis dynamics in $i$ th patch is governed by

$$
\begin{gathered}
\dot{S}_{D i}=A_{1 i}-\beta_{1 i} S_{D i} I_{L i}-d_{1 i} S_{D i}+\sigma_{i} I_{D i}, \\
\dot{I}_{D i}=\beta_{1 i} S_{D i} I_{L i}-\left(d_{1 i}+\sigma_{i}\right) I_{D i}, \\
\dot{S}_{L i}=A_{2 i}-\beta_{2 i} S_{L i} x_{i}-d_{2 i} S_{L i}, \\
\dot{I}_{L i}=\beta_{2 i} S_{L i} x_{i}-d_{2 i} I_{L i}, \\
\dot{x}_{i}=a_{i} I_{D i}-d_{i} x_{i} .
\end{gathered}
$$



Figure 1: Transmission diagram for echinococcosis among dogs, livestock.

All parameters are assumed positive. For the dog population in the $i$ th patch, $A_{1 i}$ describes the annual recruitment rate; $d_{1 i}$ is the natural death rate; $\sigma_{i}$ denotes the recovery rate of transition from infected to noninfected dogs, including natural recovery rate and recovery due to anthelmintic treatment; $\beta_{1 i} S_{D i} I_{L i}$ describes the transmission of echinococcosis between susceptible dogs and infectious livestock after the ingestion of cyst-containing organs of the infected livestock. For the livestock population in the $i$ th patch, $A_{2 i}$ is the annual recruitment rate; $d_{2 i}$ is the death rate; $\beta_{2 i} S_{L i} x_{i}$ describes the transmission of echinococcosis to livestock by the ingestion of Echinococcus eggs in the environment. For Echinococcus eggs in the $i$ th patch, $a_{i}$ denotes released rate from infected dogs; $d_{i}$ is the mortality rate of eggs.

When two patches are connected, we assume that susceptible and infected dogs of every patch $i$ leave for patch $j$ at a per capita rate $D_{i}$. Then the dynamics of echinococcosis is governed by the following model:

$$
\begin{gather*}
\dot{S}_{D i}=A_{1 i}-\beta_{1 i} S_{D i} I_{L i}-d_{1 i} S_{D i}+\sigma_{i} I_{D i}-D_{i} S_{D i}+D_{j} S_{D j}, \\
\dot{I}_{D i}=\beta_{1 i} S_{D i} I_{L i}-\left(d_{1 i}+\sigma_{i}\right) I_{D i}-D_{i} I_{D i}+D_{j} I_{D j} \\
\dot{S}_{L i}=A_{2 i}-\beta_{2 i} S_{L i} x_{i}-d_{2 i} S_{L i}, \quad i, j=1,2, i \neq j  \tag{2}\\
\dot{I}_{L i}=\beta_{2 i} S_{L i} x_{i}-d_{2 i} I_{L i} \\
\dot{x}_{i}=a_{i} I_{D i}-d_{i} x_{i}
\end{gather*}
$$

Motivated by biological background of model (2), we always assume that all solutions of model (2) satisfy the following positive initial conditions:

$$
\begin{gather*}
S_{D i}(0)=S_{D i 0}>0, \quad I_{D i}(0)=I_{D i 0}>0, \\
S_{L i}(0)=S_{L i 0}>0, \quad I_{L i}(0)=I_{L i 0}>0,  \tag{3}\\
x_{i}(0)=x_{i 0}>0 .
\end{gather*}
$$

We can easily prove that the solution of model (2) with initial conditions (3) satisfies $S_{D i}(t)>0, I_{D i}(t)>0, S_{L i}(t)>0$, and $I_{L i}(t)>0$ for all $t>0$. Here, we omit the proof.

## 3. Basic Properties and Basic Reproduction Number of the Model

In this section, we mainly present the preliminary results and derive reproduction number for model (2). In order to investigate the dynamics of model (2), we begin with stating some results on model (1). Model (1) has been analyzed in [18]. Model (1) admits a disease-free equilibrium $E_{0 i}=$ $\left(S_{D i}^{0}, 0, S_{L i}^{0}, 0,0\right)$ and a unique positive equilibrium $E_{i}^{*}=$ $\left(S_{D i}^{*}, I_{D i}^{*}, S_{L i}^{*}, I_{L i}^{*}, x_{i}^{*}\right)$, where

$$
\begin{gather*}
S_{D i}^{0}=\frac{A_{1 i}}{d_{1 i}}, \quad S_{L i}^{0}=\frac{A_{2 i}}{d_{2 i}}, \\
S_{D i}^{*}=\frac{d_{2 i}\left(d_{1 i}+\sigma_{i}\right)\left(A_{1 i} \beta_{2 i} a_{i}+d_{1 i} d_{2 i} d_{i}\right)}{a_{i} \beta_{2 i} d_{1 i}\left(\beta_{1 i} A_{2 i}+d_{1 i} d_{2 i}+d_{2 i} \sigma_{i}\right)}, \\
I_{D i}^{*}=\frac{a_{i} \beta_{1 i} \beta_{2 i} A_{1 i} A_{2 i}-\left(d_{1 i}+\sigma_{i}\right) d_{1 i} d_{2 i}^{2} d_{i}}{a_{i} d_{1 i} \beta_{2 i}\left(\beta_{1 i} A_{2 i}+d_{1 i} d_{2 i}+d_{2 i} \sigma_{i}\right)}, \\
S_{L i}^{*}=\frac{d_{i} d_{1 i}\left(\beta_{1 i} A_{2 i}+d_{1 i} d_{2 i}+d_{2 i} \sigma_{i}\right)}{\beta_{1 i}\left(a_{i} A_{1 i} \beta_{2 i}+d_{i} d_{1 i} d_{2 i}\right)},  \tag{4}\\
I_{L i}^{*}=\frac{a_{i} \beta_{1 i} \beta_{2 i} A_{1 i} A_{2 i}-\left(d_{1 i}+\sigma_{i}\right) d_{1 i} d_{2 i}^{2} d_{i}}{d_{2 i} \beta_{1 i}\left(A_{1 i} \beta_{2 i} a_{i}+d_{1 i} d_{2 i} d_{i}\right)}, \\
x_{i}^{*}=\frac{a_{i} \beta_{1 i} \beta_{2 i} A_{1 i} A_{2 i}-\left(d_{1 i}+\sigma_{i}\right) d_{1 i} d_{2 i}^{2} d_{i}}{d_{1 i} \beta_{2 i} d\left(\beta_{1 i} A_{2 i}+d_{1 i} d_{2 i}+d_{2 i} \sigma_{i}\right)} .
\end{gather*}
$$

The reproduction number of model (1) is established in [18], which can be expressed as

$$
\begin{equation*}
R_{0 i}=\sqrt[3]{\frac{\beta_{1 i} \beta_{2 i} A_{1 i} A_{2 i} a_{i}}{\left(d_{1 i}+\sigma_{i}\right) d_{1 i} d_{2 i}^{2} d_{i}}} \tag{5}
\end{equation*}
$$

From Theorems 3 and 5 in [18], we can obtain the following lemma.

Lemma 1. Considering model (1), one has that
(a) if $R_{0 i}<1$, then disease-free equilibrium $E_{0 i}$ is globally asymptotically stable;
(b) if $R_{0 i}>1$, then positive equilibrium $E_{i}^{*}$ is globally asymptotically stable.

In order to obtain our main results, we need the following lemma. Consider the following linear equation:

$$
\begin{align*}
& \widetilde{N}_{D 1}^{\prime}(t)=A_{11}-d_{11} \widetilde{N}_{D 1}(t)-D_{1} \widetilde{N}_{D 1}(t)+D_{2} \widetilde{N}_{D 2}(t), \\
& \widetilde{N}_{D 2}^{\prime}(t)=A_{12}-d_{12} \widetilde{N}_{D 2}(t)-D_{2} \widetilde{N}_{D 2}(t)+D_{1} \widetilde{N}_{D 1}(t) . \tag{6}
\end{align*}
$$

We have the following result on system (6).
Lemma 2. System (6) has a unique equilibrium $N_{D}^{0}\left(N_{D 1}^{0}, N_{D 2}^{0}\right)$ which is globally stable, where

$$
\begin{align*}
& N_{D 1}^{0}=\frac{A_{11}\left(d_{12}+D_{2}\right)+A_{12} D_{2}}{d_{11} d_{12}+d_{11} D_{2}+d_{12} D_{1}} \\
& N_{D 2}^{0}=\frac{A_{12}\left(d_{11}+D_{1}\right)+A_{11} D_{1}}{d_{11} d_{12}+d_{11} D_{2}+d_{12} D_{1}} \tag{7}
\end{align*}
$$

Proof. The Jacobian matrix of (6) at $\left(N_{D 1}^{0}, N_{D 2}^{0}\right)$ is

$$
J\left(N_{D}^{0}\right)=\left(\begin{array}{cc}
-\left(d_{11}+D_{1}\right) & D_{2}  \tag{8}\\
D_{1} & -\left(d_{12}+D_{2}\right)
\end{array}\right) .
$$

By simple calculations, the corresponding characteristic equation is

$$
\begin{equation*}
\Phi(\lambda)=\lambda^{2}+a_{1} \lambda+a_{0}=0 \tag{9}
\end{equation*}
$$

where

$$
\begin{gather*}
a_{1}=d_{11}+d_{12}+D_{1}+D_{2}>0 \\
a_{0}=d_{11} d_{12}+d_{11} D_{2}+d_{12} D_{1}>0 \tag{10}
\end{gather*}
$$

Therefore, all roots of $\Phi(\lambda)$ have negative real parts, and hence $N^{0}\left(N_{D 1}^{0}, N_{D 2}^{0}\right)$ is globally stable.

For any $\varepsilon>0$, we define region $\Gamma_{\varepsilon}$ as follows:

$$
\begin{gather*}
\Gamma_{\varepsilon}=\left\{\left(S_{D 1}, I_{D 1}, S_{L 1}, I_{L 1}, x_{1}, S_{D 2}, I_{D 2}, S_{L 2}, I_{L 2}, x_{2}\right) \in \mathbb{R}_{+}^{10},\right. \\
S_{D i}+I_{D i} \leq N_{D i}^{0}+\varepsilon, S_{L i}+I_{L i} \leq S_{L i}^{0}+\varepsilon, \\
\left.x_{i} \leq \frac{a_{i}}{d_{i}} N_{D i}^{0}+\left(1+\frac{a_{i}}{d_{i}}\right) \varepsilon, i=1,2\right\} . \tag{11}
\end{gather*}
$$

On the ultimate boundedness of solutions for model (2), we have the following result.

Lemma 3. All solutions of model (2) with initial condition (3) ultimately turn into region $\Gamma_{\varepsilon}$ as $t \rightarrow \infty$.

Proof. Let $\quad\left(S_{D 1}(t), I_{D 1}(t), S_{L 1}(t), I_{L 1}(t), x_{1}(t), S_{D 2}(t), I_{D 2}(t)\right.$, $\left.S_{L 2}(t), I_{L 2}(t), x_{2}(t)\right)$ be any solution of model (2) with initial conditions (3) and let $N_{D i}(t)=S_{D i}(t)+I_{D i}(t), i=1,2$. From model (2) we have

$$
\begin{align*}
& \dot{N}_{D 1}(t)=A_{11}-d_{11} N_{D 1}(t)-D_{1} N_{D 1}(t)+D_{2} N_{D 2}(t), \\
& \dot{N}_{D 2}(t)=A_{12}-d_{12} N_{D 2}(t)-D_{2} N_{D 2}(t)+D_{1} N_{D 1}(t), \tag{12}
\end{align*}
$$

and then from Lemma 2 we have $\lim _{t \rightarrow \infty} N_{D i}(t)=N_{D i}^{0}, i=$ 1,2 . Hence, for any $\varepsilon>0$, there is a $t_{1}>0$ such that

$$
\begin{equation*}
S_{D i}(t)+I_{D i}(t) \leq N_{D i}^{0}+\varepsilon, \quad i=1,2, \quad \forall t \geq t_{1} \tag{13}
\end{equation*}
$$

From the third and fourth equations of model (2), we have

$$
\begin{equation*}
\frac{d\left(S_{L i}(t)+I_{L i}(t)\right)}{d t}=A_{2 i}-d_{2 i}\left(S_{L i}(t)+I_{L i}(t)\right) \tag{14}
\end{equation*}
$$

and therefore, there exists a $t_{2}>0$ such that

$$
\begin{equation*}
S_{L i}(t)+I_{L i}(t) \leq S_{L i}^{0}+\varepsilon, \quad i=1,2, \quad \forall t \geq t_{2} \tag{15}
\end{equation*}
$$

Finally, from the fifth equation of model (2), we have

$$
\begin{equation*}
\dot{x}_{i}(t) \leq a_{i}\left(N_{D i}^{0}+\varepsilon\right)-d_{i} x_{i}(t), \quad i=1,2, \quad \forall t \geq t_{1}, \tag{16}
\end{equation*}
$$

and then there is a $t_{3}>t_{1}$ such that

$$
\begin{align*}
x_{i}(t) & \leq \frac{a_{i}}{d_{i}}\left(N_{D i}^{0}+\varepsilon\right)+\varepsilon \\
& =\frac{a_{i}}{d_{i}} N_{D i}^{0}+\left(1+\frac{a_{i}}{d_{i}}\right) \varepsilon, \quad i=1,2, \quad \forall t \geq t_{3} . \tag{17}
\end{align*}
$$

Let $t^{*}=\max \left\{t_{2}, t_{3}\right\}$, and then for all $t>t^{*}$ we have

$$
\begin{align*}
& \left(\left(S_{D 1}(t), I_{D 1}(t), S_{L 1}(t), I_{L 1}(t), x_{1}(t), S_{D 2}(t),\right.\right.  \tag{18}\\
& \left.\left.\quad I_{D 2}(t), S_{L 2}(t), I_{L 2}(t), x_{2}(t)\right)\right) \in \Gamma_{\varepsilon} .
\end{align*}
$$

This completes the proof of Lemma 3.
According to Lemma 3, all feasible solutions of model (2) enter or remain in the region $\Gamma_{\varepsilon}$ as $t$ becomes large enough. In what follows, the dynamics of model (2) can be considered only in $\Gamma_{\varepsilon}$.

Simple algebraic calculation shows that model (2) always has a unique disease-free equilibrium $E_{0}\left(N_{D 1}^{0}, 0, S_{L 1}^{0}, 0,0\right.$, $\left.N_{D 2}^{0}, 0, S_{L 2}^{0}, 0,0\right)$. According to the concepts of next generation matrix and reproduction number presented in [29, 30], we define

$$
\begin{gather*}
\mathscr{F}=\left(\begin{array}{c}
\beta_{11} s_{D 1} I_{L 1} \\
\beta_{21} S_{L 1} x_{1} \\
a_{1} I_{D 1} \\
\beta_{12} s_{D 2} I_{L 2} \\
\beta_{22} S_{L 2} x_{2} \\
a_{2} I_{D 2}
\end{array}\right),  \tag{19}\\
\mathscr{V}=\left(\begin{array}{c}
\left(d_{11}+\sigma_{1}\right) I_{D 1}+D_{1} I_{D 1}-D_{2} I_{D 2} \\
d_{21} I_{L 1} \\
d_{1} x_{1} \\
\left(d_{12}+\sigma_{2}\right) I_{D 2}+D_{2} I_{D 1}-D_{1} I_{D 1} \\
d_{22} I_{L 2} \\
d_{2} x_{2}
\end{array}\right) .
\end{gather*}
$$

Noting that the disease-free equilibrium of model (2) is $E_{0}$, then

$$
F=\left(\begin{array}{cc}
F_{11} & 0  \tag{20}\\
0 & F_{22}
\end{array}\right)
$$

where

$$
\begin{gather*}
F_{11}=\left(\begin{array}{ccc}
0 & \beta_{11} N_{D 1}^{0} & 0 \\
0 & 0 & \beta_{21} S_{L 1}^{0} \\
a_{1} & 0 & 0
\end{array}\right), \\
F_{22}=\left(\begin{array}{ccc}
0 & \beta_{12} N_{D 2}^{0} & 0 \\
0 & 0 & \beta_{22} S_{L 2}^{0} \\
a_{2} & 0 & 0
\end{array}\right), \\
V=\left(\begin{array}{ccccc}
d_{11}+\sigma_{1}+D_{1} & 0 & 0 & -D_{2} & 0 \\
0 \\
0 & d_{21} & 0 & 0 & 0 \\
0 \\
0 & 0 & d_{1} & 0 & 0 \\
0 \\
-D_{1} & 0 & 0 & d_{12}+\sigma_{2}+D_{2} & 0 \\
0 & 0 & 0 & 0 & d_{22} \\
0 & 0 & 0 & 0 & 0 \\
0 & d_{2}
\end{array}\right) . \tag{21}
\end{gather*}
$$

Denote $\Delta=d_{1} d_{2} d_{21} d_{22}\left[d_{11}\left(d_{12}+\sigma_{2}+D_{2}\right)+\sigma_{1}\left(d_{12}+\sigma_{2}+\right.\right.$ $\left.\left.D_{2}\right)+D_{1}\left(d_{12}+\sigma_{2}\right)\right]$. After extensive algebraic calculations, we can obtain

$$
F V^{-1}=\frac{1}{\Delta}\left(\begin{array}{ll}
M_{11} & M_{12}  \tag{22}\\
M_{21} & M_{22}
\end{array}\right)
$$

where

$$
\left.\begin{array}{c}
M_{11}=\left(\begin{array}{ccc}
0 & \frac{\Delta\left(A_{11} d_{12}+A_{11} D_{2}+A_{12} D_{2}\right)}{d_{21}\left(d_{11} d_{12}+d_{11} D_{2}+d_{12} D_{1}\right)} & 0 \\
0 & 0 & \frac{\Delta \beta_{21} A_{21}}{d_{21} d_{1}} \\
d_{1} d_{2} d_{21} d_{22} a_{1}\left(d_{12}+\sigma_{2}+D_{2}\right) & 0 & 0
\end{array}\right), \\
M_{12}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
d_{1} d_{2} d_{21} d_{22} a_{1} D_{2} & 0 & 0
\end{array}\right), \quad M_{21}=\left(\begin{array}{ccc}
0 & 0 \\
0 & 0 & 0 \\
d_{1} d_{2} d_{21} d_{22} a_{2} D_{1} & 0 & 0
\end{array}\right)  \tag{23}\\
0
\end{array} \quad \frac{\Delta \beta_{12}\left(A_{11} D_{1}+A_{12} d_{11}+A_{12} D_{1}\right)}{d_{22}\left(d_{11} d_{12}+d_{11} D_{2}+d_{12} D_{1}\right)} \begin{array}{cc}
0 & \frac{\Delta \beta_{22} A_{22}}{d_{22} d_{2}} \\
0 & 0
\end{array}\right) .
$$

From the proof of Theorem 2 in [30], it follows that

$$
\begin{equation*}
R_{0}<1\left(R_{0}=1, R_{0}>1\right) \Longleftrightarrow s(J)<1(s(J)=0, s(J)>0), \tag{24}
\end{equation*}
$$

$$
J=F-V
$$

$$
=\left(\begin{array}{ccc}
-\left(d_{11}+\sigma_{1}+D_{1}\right) & \beta_{11} N_{D 1}^{0} & 0 \\
0 & -d_{21} & \beta_{21} S_{L 1}^{0} \\
a_{1} & 0 & -d_{1} \\
D_{1} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right.
$$

and $s(J)$ is the maximum real part of the eigenvalues of matrix $J$.

Using Theorem 2 in [30], we can easily obtain the following stability result.

Theorem 4. For model (2), one has that
(a) if $R_{0}<1$, then disease-free equilibrium $E_{0}$ is locally asymptotically stable;
(b) if $R_{0}>1$, then disease-free equilibrium $E_{0}$ is unstable.

## 4. Global Stability of the Disease-Free Equilibrium

We start by considering the global stability of disease-free equilibrium $E_{0}$ when $R_{0}<1$.

Theorem 5. The disease-free equilibrium $E_{0}$ of model (2) is globally asymptotically stable in $\Gamma_{\varepsilon}$ if $R_{0}<1$.

Proof. From Theorem 4 we find that disease-free equilibrium $E_{0}$ is locally asymptotically stable if $R_{0}<1$. In the following we only need to prove the global attractiveness of $E_{0}$. From (24) we can see that if $R_{0}<1$, then $s(J)<0$. Hence, there is a small enough number $\varepsilon>0$ such that $s\left(J_{\varepsilon}\right)<0$, where $J_{\varepsilon}=J+\varepsilon J_{1}$ and

$$
J_{1}=\left(\begin{array}{cccccc}
0 & \beta_{11} & 0 & 0 & 0 & 0  \tag{26}\\
0 & 0 & \beta_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \beta_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & \beta_{22} \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

Let $\left(S_{D 1}(t), I_{D 1}(t), S_{L 1}(t), I_{L 1}(t), x_{1}(t), S_{D 2}(t), I_{D 2}(t), S_{L 2}(t)\right.$, $\left.I_{L 2}(t), x_{2}(t)\right)$ be any solution of model (2) in $\Gamma_{\varepsilon}$, then

$$
\begin{equation*}
S_{D i}(t) \leq N_{D i}^{0}+\varepsilon, \quad S_{L i}(t) \leq S_{L i}^{0}+\varepsilon, \quad i=1,2, \quad \forall t \geq 0 \tag{27}
\end{equation*}
$$

where

From model (2), it follows that

$$
\begin{gather*}
\dot{I}_{D 1} \leq \beta_{11}\left(N_{D 1}^{0}+\varepsilon\right) I_{L 1}-\left(d_{11}+\sigma_{1}\right) I_{D 1}-D_{1} I_{D 1}+D_{2} I_{D 2} \\
\dot{I}_{L 1} \leq \beta_{21}\left(S_{L 1}^{0}+\varepsilon\right) x_{1}-d_{21} I_{L 1} \\
\dot{x}_{1} \leq a_{1} I_{D 1}-d_{1} x_{1} \\
\dot{I}_{D 2} \leq \beta_{12}\left(N_{D 2}^{0}+\varepsilon\right) I_{L 2}-\left(d_{12}+\sigma_{2}\right) I_{D 2}-D_{2} I_{D 2}+D_{1} I_{D 1} \\
\dot{I}_{L 2} \leq \beta_{22}\left(S_{L 2}^{0}+\varepsilon\right) x_{2}-d_{22} I_{L 2} \\
\dot{x}_{2} \leq a_{2} I_{D 2}-d_{2} x_{2} \tag{28}
\end{gather*}
$$

Define an auxiliary linear system:

$$
\begin{gather*}
\bar{I}_{D 1}^{\prime}=\beta_{11}\left(N_{D 1}^{0}+\varepsilon\right) \bar{I}_{L 1}-\left(d_{11}+\sigma_{1}\right) \bar{I}_{D 1}-D_{1} \bar{I}_{D 1}+D_{2} \bar{I}_{D 2} \\
\bar{I}_{L 1}^{\prime}=\beta_{21}\left(S_{L 1}^{0}+\varepsilon\right) \bar{x}_{1}-d_{21} \bar{I}_{L 1}, \\
\bar{x}_{1}^{\prime}=a_{1} \bar{I}_{D 1}-d_{1} \bar{x}_{1}, \\
\bar{I}_{D 2}^{\prime}=\beta_{12}\left(N_{D 2}^{0}+\varepsilon\right) \bar{I}_{L 2}-\left(d_{12}+\sigma_{2}\right) \bar{I}_{D 2}-D_{2} \bar{I}_{D 2}+D_{1} \bar{I}_{D 1}, \\
\bar{I}_{L 2}^{\prime}=\beta_{22}\left(S_{L 2}^{0}+\varepsilon\right) \bar{x}_{2}-d_{22} \bar{I}_{L 2}, \\
\bar{x}_{2}^{\prime}=a_{2} \bar{I}_{D 2}-d_{2} \bar{x}_{2} . \tag{29}
\end{gather*}
$$

Since system (29) is a linear system, the globally stability of origin is determined by the stability of matrix $J_{\varepsilon}$. Since $s\left(J_{\varepsilon}\right)<$ 0 , then all the eigenvalues of matrix $J_{\varepsilon}$ have negative real parts. It then follows that each solution of (29) satisfies

$$
\begin{gather*}
\lim _{t \rightarrow+\infty} \bar{I}_{D i}(t)=0, \quad \lim _{t \rightarrow+\infty} \bar{I}_{L i}(t)=0,  \tag{30}\\
\lim _{t \rightarrow+\infty} \bar{x}_{i}(t)=0, \quad i=1,2
\end{gather*}
$$

By the comparison principle we have

$$
\begin{gather*}
\lim _{t \rightarrow+\infty} I_{D i}(t)=0, \quad \lim _{t \rightarrow+\infty} I_{L i}(t)=0, \\
\lim _{t \rightarrow+\infty} x_{i}(t)=0, \quad i=1,2 \tag{31}
\end{gather*}
$$

Then the limiting system of model (2) is

$$
\begin{gather*}
\dot{S}_{D 1}=A_{11}-d_{11} S_{D 1}-D_{1} S_{D 1}+D_{2} S_{D 2}, \\
\dot{S}_{D 2}=A_{12}-d_{12} S_{D 2}-D_{2} S_{D 2}+D_{1} S_{D 1}, \\
\dot{S}_{L 1}=A_{21}-d_{21} S_{L 1},  \tag{32}\\
\dot{S}_{L 2}=A_{22}-d_{22} S_{L 2} .
\end{gather*}
$$

By Lemma 2 we find that there is a unique equilibrium ( $N_{D 1}^{0}, N_{D 2}^{0}, S_{L 1}^{0}, S_{L 2}^{0}$ ) of system (32), which is globally asymptotically stable. Thus, according to the theory of asymptotic autonomous systems [31], we finally obtain that disease-free equilibrium $E_{0}$ is globally asymptotically stable for model (2) when $R_{0}<1$. This completes the proof of Theorem 5 .

## 5. Permanence

We now turn to the case where $R_{0}>1$. We first establish the permanence for model (2).

Theorem 6. Let $D_{i}>0, i=1,2$. If $R_{0}>1$, then model (2) is permanent. Furthermore, model (2) also has at least one positive equilibrium $E^{*}\left(S_{D 1}^{*}, I_{D 1}^{*}, S_{L 1}^{*}, I_{L 1}^{*}, x_{1}^{*}, S_{D 2}^{*}, I_{D 2}^{*}, S_{L 2}^{*}, I_{L 2}^{*}, x_{2}^{*}\right)$.

Proof. Define

$$
\begin{gather*}
X=\left\{\left(S_{D 1}, I_{D 1}, S_{L 1}, I_{L 1}, x_{1}, S_{D 2}, I_{D 2}, S_{L 2}, I_{L 2}, x_{2}\right):\right. \\
\left.S_{D i} \geq 0, I_{D i} \geq 0, S_{L i} \geq 0, I_{L i} \geq 0, x_{i} \geq 0, i=1,2\right\}, \\
X_{0}=\left\{\left(S_{D 1}, I_{D 1}, S_{L 1}, I_{L 1}, x_{1}, S_{D 2}, I_{D 2}, S_{L 2}, I_{L 2}, x_{2}\right):\right. \\
\left.S_{D i}>0, I_{D i}>0, S_{L i}>0, I_{L i}>0, x_{i}>0, i=1,2\right\}, \\
\partial X_{0}=X \backslash X_{0}, \\
M_{\partial}=\left\{\left(S_{D 1}(0), I_{D 1}(0), S_{L 1}(0), I_{L 1}(0), x_{1}(0),\right.\right. \\
\left.S_{D 2}(0), I_{D 2}(0), S_{L 2}(0), I_{L 2}(0), x_{2}(0)\right): \\
\quad\left(S_{D 1}(t), I_{D 1}(t), S_{L 1}(t), I_{L 1}(t), x_{1}(t),\right. \\
\left.S_{D 2}(t), I_{D 2}(t), S_{L 2}(t), I_{L 2}(t), x_{2}(t)\right) \tag{33}
\end{gather*}
$$

satisfies model (2),

$$
\begin{align*}
& \left(S_{D 1}(t), I_{D 1}(t), S_{L 1}(t), I_{L 1}(t), x_{1}(t), S_{D 2}(t)\right.  \tag{34}\\
& \left.\left.\quad I_{D 2}(t), S_{L 2}(t), I_{L 2}(t), x_{2}(t)\right) \in \partial X_{0}, \forall t \geq 0\right\} .
\end{align*}
$$

In order to prove Theorem 6, it suffices to show that $\partial X_{0}$ repels uniformly the solutions of $X_{0}$.

Firstly, by the form of model (2), it is easy to see that both $X$ and $X_{0}$ are positively invariant. Clearly, $\partial X_{0}$ is relatively closed in $X$. Furthermore, model (2) is point dissipative (see Lemma 3).

We now show that if $D_{i}>0, i=1,2$, then

$$
\begin{align*}
M_{\partial}=\{ & \left(S_{D 1}, 0, S_{L 1}, 0,0, S_{D 2}, 0, S_{L 2}, 0,0\right): \\
& \left.S_{D i} \geq 0, S_{L i} \geq 0, i=1,2\right\} \tag{35}
\end{align*}
$$

Assume

$$
\begin{align*}
& \left(S_{D 1}(0), I_{D 1}(0), S_{L 1}(0), I_{L 1}(0), x_{1}(0),\right.  \tag{36}\\
& \left.\quad S_{D 2}(0), I_{D 2}(0), S_{L 2}(0), I_{L 2}(0), x_{2}(0)\right) \in M_{\partial}
\end{align*}
$$

It suffices to show that

$$
\begin{align*}
I_{D 1}(t) & =I_{L 1}(t)=x_{1}(t)=I_{D 2}(t)  \tag{37}\\
& =I_{L 2}(t)=x_{2}(t)=0, \quad \forall t \geq 0 .
\end{align*}
$$

Suppose not, then there exists a $t_{0} \geq 0$ such that at least one of $I_{D 1}\left(t_{0}\right), I_{L 1}\left(t_{0}\right), x_{1}\left(t_{0}\right), I_{D 2}\left(t_{0}\right), I_{L 2}\left(t_{0}\right)$, or $x_{2}\left(t_{0}\right)$ is greater than zero. Here we only consider the case $I_{D 1}\left(t_{0}\right)>0, I_{D 2}\left(t_{0}\right)=0$, $S_{D i}\left(t_{0}\right)=0, S_{L i}\left(t_{0}\right)=0, I_{L i}\left(t_{0}\right)=0$, and $x_{i}\left(t_{0}\right)=0, i=1,2$. The other case can be deduced in the same way. Since

$$
\begin{gather*}
\dot{S}_{D i}\left(t_{0}\right)=A_{1 i}-\beta_{1 i} S_{D i}\left(t_{0}\right) I_{L i}\left(t_{0}\right)-d_{1 i} S_{D i}\left(t_{0}\right)+\sigma_{i} S_{D i}\left(t_{0}\right) \\
\\
+D_{i} S_{D i}\left(t_{0}\right)-D_{j} S_{D j}\left(t_{0}\right) \geq A_{1 i}>0, \\
\dot{S}_{L i}\left(t_{0}\right)= \\
A_{2 i}-\beta_{2 i} S_{L i}\left(t_{0}\right) x_{i}\left(t_{0}\right)-d_{2 i} S_{L i}\left(t_{0}\right)=A_{2 i}>0,  \tag{38}\\
\dot{x}_{1}\left(t_{0}\right)=a I_{D 1}\left(t_{0}\right)-d_{1} x_{1}\left(t_{0}\right)=a I_{D 1}\left(t_{0}\right)>0, \\
\dot{I}_{D 1}(t) \geq-\left(d_{11}+\sigma_{1}+D_{1}\right) I_{D 1}(t), \quad i=1,2, i \neq j,
\end{gather*}
$$

it follows that there is an $\epsilon_{0}>0$ small enough such that $S_{D i}(t)>0, S_{L i}(t)>0, x_{1}(t)>0$, and $I_{D 1}(t)>0, i=1,2$, for all $t_{0}<t<t_{0}+\epsilon_{0}$. If $I_{L 1}\left(t_{0}+\left(\epsilon_{0} / 2\right)\right)>0$, then we have

$$
\begin{equation*}
\dot{I}_{L 1}(t) \geq-d_{21} I_{L 1}(t) \tag{39}
\end{equation*}
$$

This means that $I_{L 1}(t)>0$ for all $t \geq t_{0}+\left(\epsilon_{0} / 2\right)$. If $I_{L 1}\left(t_{0}+\right.$ $\left.\left(\epsilon_{0} / 2\right)\right)=0$, it then follows from model (2) that

$$
\begin{equation*}
\dot{I}_{L 1}\left(t_{0}+\frac{\epsilon_{0}}{2}\right)=\beta_{21} S_{L 1}\left(t_{0}+\frac{\epsilon_{0}}{2}\right) x_{1}\left(t_{0}+\frac{\epsilon_{0}}{2}\right)>0 . \tag{40}
\end{equation*}
$$

It then follows that there exists an $\epsilon_{1}<\left(\epsilon_{0} / 2\right)$ such that

$$
\begin{equation*}
I_{L 1}(t)>0, \quad \forall t_{0}+\frac{\epsilon_{0}}{2}<t<t_{0}+\frac{\epsilon_{0}}{2}+\epsilon_{1} . \tag{41}
\end{equation*}
$$

By the same way we can obtain that there exists an $\epsilon_{2}<\epsilon_{1}$ such that

$$
\begin{equation*}
I_{D 2}(t)>0, \quad \forall t_{0}+\frac{\epsilon_{0}}{2}<t<t_{0}+\frac{\epsilon_{0}}{2}+\epsilon_{2} . \tag{42}
\end{equation*}
$$

If $x_{2}\left(t_{0}+\left(\epsilon_{0} / 2\right)+\left(\epsilon_{2} / 2\right)\right)>0$, then we have

$$
\begin{equation*}
\dot{x}_{2}(t) \geq d_{2} x_{2}(t) . \tag{43}
\end{equation*}
$$

This means that $x_{2}(t)>0$ for all $t>t_{0}+\left(\epsilon_{0} / 2\right)+\left(\epsilon_{2} / 2\right)$; if $x_{2}\left(t_{0}+\left(\epsilon_{0} / 2\right)+\left(\epsilon_{2} / 2\right)\right)=0$, it then follows from model (2) that

$$
\begin{equation*}
\dot{x}_{2}\left(t_{0}+\frac{\epsilon_{0}}{2}+\frac{\epsilon_{2}}{2}\right)=a_{2} I_{D 2}\left(t_{0}+\frac{\epsilon_{0}}{2}+\frac{\epsilon_{2}}{2}\right)>0 . \tag{44}
\end{equation*}
$$

It then follows that there exists an $\epsilon_{3}<\left(\epsilon_{2} / 2\right)$ such that

$$
\begin{equation*}
x_{2}(t)>0, \quad \forall t_{0}+\frac{\epsilon_{0}}{2}+\frac{\epsilon_{2}}{2}<t<t_{0}+\frac{\epsilon_{0}}{2}+\frac{\epsilon_{2}}{2}+\epsilon_{3} . \tag{45}
\end{equation*}
$$

By the same way we can obtain that there exists an $\epsilon_{4}<\left(\epsilon_{3} / 2\right)$ such that

$$
\begin{gather*}
I_{L 2}(t)>0 \\
\forall t_{0}+\frac{\epsilon_{0}}{2}+\frac{\epsilon_{2}}{2}+\frac{\epsilon_{3}}{2}<t<t_{0}+\frac{\epsilon_{0}}{2}+\frac{\epsilon_{2}}{2}+\frac{\epsilon_{3}}{2}+\epsilon_{4} \tag{46}
\end{gather*}
$$

Thus for all $t \in\left(t_{0}+\left(\epsilon_{0} / 2\right)+\left(\epsilon_{2} / 2\right)+\left(\epsilon_{3} / 2\right), t_{0}+\left(\epsilon_{0} / 2\right)\right.$ $\left.+\left(\epsilon_{2} / 2\right)+\left(\epsilon_{3} / 2\right)+\epsilon_{4}\right)$ we have $S_{D i}(t)>0, I_{D i}(t)>0$,
$S_{L i}(t)>0, I_{L i}(t)>0$, and $x_{i}(t)>0, i=1,2$. This contradicts the assumption that $\left(S_{D 1}(0), I_{D 1}(0), S_{L 1}(0), I_{L 1}(0)\right.$, $\left.x_{1}(0), S_{D 2}(0), I_{D 2}(0), S_{L 2}(0), I_{L 2}(0), x_{2}(0)\right) \in M_{\partial}$. This proves (35).

From (24) we can see that if $R_{0}>1$, then $s(J)>0$. Hence, there is a small enough number $\theta>0$ such that $s\left(J_{\theta}\right)>0$, where $J_{\theta}=J-\theta J_{1}$ and $J_{1}$ is given by (26). Let

$$
\begin{align*}
g(x) & =\left(\begin{array}{l}
g_{1}(x) \\
g_{2}(x) \\
g_{3}(x) \\
g_{4}(x)
\end{array}\right) \\
& =\left(\begin{array}{c}
\frac{A_{11}\left(d_{12}+D_{2}+\beta_{12} x\right)+A_{12} D_{2}}{\left(\beta_{11} x+d_{11}\right)\left(\beta_{12} x+d_{12}\right)+D_{1}\left(\beta_{12} x+d_{12}\right)+D_{2}\left(\beta_{11}+d_{11}\right)} \\
\frac{A_{12}\left(d_{11}+D_{1}+\beta_{11} x\right)+A_{11} D_{1}}{\left(\beta_{11} x+d_{11}\right)\left(\beta_{12} x+d_{12}\right)+D_{1}\left(\beta_{12} x+d_{12}\right)+D_{2}\left(\beta_{11}+d_{11}\right)} \\
\frac{A_{12}}{d_{21}+\beta_{21} x} \\
\frac{A_{22}}{d_{22}+\beta_{22} x}
\end{array}\right), \tag{47}
\end{align*}
$$

and we can see the fact that $\lim _{x \rightarrow 0} g(x)=\left(N_{D 1}^{0}, N_{D 2}^{0}\right.$, $\left.S_{L 1}^{0}, S_{L 2}^{0}\right)^{T}$. Hence we can choose $\delta>0$ small enough such that

$$
\begin{gather*}
g_{1}(\delta)=\left(A_{11}\left(d_{12}+D_{2}+\beta_{12} \delta\right)+A_{12} D_{2}\right) \\
\times\left(\left(\beta_{11} \delta+d_{11}\right)\left(\beta_{12} \delta+d_{12}\right)+D_{1}\left(\beta_{12} \delta+d_{12}\right)\right. \\
\left.+D_{2}\left(\beta_{11}+d_{11}\right)\right)^{-1}>N_{D 1}^{0}-\theta, \\
g_{2}(\delta)=\left(A_{12}\left(d_{11}+D_{1}+\beta_{11} \delta\right)+A_{11} D_{1}\right) \\
\times\left(\left(\beta_{11} \delta+d_{11}\right)\left(\beta_{12} \delta+d_{12}\right)+D_{1}\left(\beta_{12} \delta+d_{12}\right)\right. \\
\left.+D_{2}\left(\beta_{11}+d_{11}\right)\right)^{-1}>N_{D 2}^{0}-\theta, \\
g_{3}(\delta)=\frac{A_{12}}{d_{21}+\beta_{21} \delta}>S_{L 1}^{0}-\theta, \\
g_{4}(\delta)=\frac{A_{22}}{d_{22}+\beta_{22} \delta}>S_{L 2}^{0}-\theta . \tag{48}
\end{gather*}
$$

Suppose $\left(S_{D 1}(t), I_{D 1}(t), S_{L 1}(t), I_{L 1}(t), x_{1}(t), S_{D 2}(t), I_{D 2}(t), S_{L 2}(t)\right.$, $\left.I_{L 2}(t), x_{2}(t)\right)$ is a solution of model (2) with ( $S_{D 1}(0), I_{D 1}(0)$, $\left.S_{L 1}(0), I_{L 1}(0), x_{1}(0), S_{D 2}(0), I_{D 2}(0), S_{L 2}(0), I_{L 2}(0), x_{2}(0)\right) \in X_{0}$. We now claim that

$$
\begin{gather*}
\limsup _{t \rightarrow \infty} \max \left\{I_{D 1}(t), I_{L 1}(t), x_{1}(t), I_{D 2}(t),\right.  \tag{49}\\
\left.I_{L 2}(t), x_{2}(t)\right\}>\delta .
\end{gather*}
$$

Suppose, for the sake of contradiction, that there exists a $T>$ 0 such that $I_{D i} \leq \delta, I_{L i} \leq \delta$, and $x_{i}(t) \leq \delta, i=1,2$, for all $t \geq T$. Then by model (2) we have

$$
\begin{gather*}
\dot{S}_{D 1}(t) \geq A_{11}-\left(\beta_{11} \delta+d_{11}\right) S_{D 1}-D_{1} S_{D 1}+D_{2} S_{D 2} \\
\dot{S}_{D 2}(t) \geq A_{12}-\left(\beta_{12} \delta+d_{12}\right) S_{D 2}-D_{2} S_{D 2}+D_{1} S_{D 1}  \tag{50}\\
\dot{S}_{L 1}(t) \geq A_{21}-\left(\beta_{21} \delta+d_{21}\right) S_{L 1} \\
\dot{S}_{L 2}(t) \geq A_{22}-\left(\beta_{22} \delta+d_{22}\right) S_{L 2}
\end{gather*}
$$

for $t \geq T$. Consider the following auxiliary system:

$$
\begin{align*}
\dot{\tilde{S}}_{D 1}(t)= & A_{11}-\left(\beta_{11} \delta+d_{11}\right) \widetilde{S}_{D 1}-D_{1} \widetilde{S}_{D 1}+D_{2} \widetilde{S}_{D 2} \\
\dot{\tilde{S}}_{D 2}(t)= & A_{12}-\left(\beta_{12} \delta+d_{12}\right) \widetilde{S}_{D 2}-D_{2} \widetilde{S}_{D 2}+D_{1} \widetilde{S}_{D 1}  \tag{51}\\
& \dot{\tilde{S}}_{L 1}(t)=A_{21}-\left(\beta_{21} \delta+d_{21}\right) \widetilde{S}_{L 1} \\
& \dot{\tilde{S}}_{L 2}(t)=A_{22}-\left(\beta_{22} \delta+d_{22}\right) \widetilde{S}_{L 2}
\end{align*}
$$

As in our analysis in Lemma 2, system (51) has a unique positive equilibrium $\left(g_{1}(\delta), g_{2}(\delta), g_{3}(\delta), g_{4}(\delta)\right)$ which is globally stable. By (48) and comparison principle, there is a $\tau>0$ such that $S_{D 1}(t) \geq N_{D 1}^{0}-\theta, S_{D 2}(t) \geq N_{D 2}^{0}-\theta, S_{L 1}(t) \geq S_{L 1}^{0}-\theta$, and $S_{L 2}(t) \geq S_{L 2}^{0}-\theta$ for all $t \geq T+\tau$. Consequently, for $t \geq T+\tau$, we have

$$
\begin{gathered}
\dot{I}_{D 1}(t) \geq \beta_{11}\left(N_{D 1}^{0}-\theta\right) I_{L 1}-\left(d_{11}+\sigma_{1}+D_{1}\right) I_{D 1}+D_{2} I_{D 2} \\
\dot{I}_{L 1}(t) \geq \beta_{21}\left(S_{L 1}^{0}-\theta\right) x_{1}-d_{21} I_{L 1},
\end{gathered}
$$



Figure 2: Time series of echinococcosis disease $I_{L i}, i=1,2$, when the two patches are isolated for the parameters given in Example 8 .


Figure 3: Surface plot of $R_{0}$ as a function of $D_{1}$ and $D_{2}$ for the parameters given in Example 8.

$$
\begin{gather*}
\dot{x}_{1}(t) \geq a_{1} I_{D 1}-d_{1} x_{1}, \\
\dot{I}_{D 2}(t) \geq \beta_{12}\left(N_{D 2}^{0}-\theta\right) I_{L 2}-\left(d_{12}+\sigma_{2}+D_{2}\right) I_{D 2}+D_{1} I_{D 1}, \\
\dot{I}_{L 2}(t) \geq \beta_{22}\left(S_{L 2}^{0}-\theta\right) x_{2}-d_{22} I_{L 2} \\
\dot{x}_{2}(t) \geq a_{2} I_{D 2}-d_{2} x_{2} . \tag{52}
\end{gather*}
$$

Consider an auxiliary system

$$
\begin{gathered}
\dot{\tilde{I}}_{D 1}(t)=\beta_{11}\left(N_{D 1}^{0}-\theta\right) \widetilde{I}_{L 1}-\left(d_{11}+\sigma_{1}+D_{1}\right) \widetilde{I}_{D 1}+D_{2} \widetilde{I}_{D 2} \\
\dot{\tilde{I}}_{L 1}(t)=\beta_{21}\left(S_{L 1}^{0}-\theta\right) \widetilde{x}_{1}-d_{21} \widetilde{I}_{L 1}
\end{gathered}
$$

$$
\begin{gather*}
\dot{\tilde{x}}_{1}(t)=a_{1} \widetilde{I}_{D 1}-d_{1} \tilde{x}_{1} \\
\dot{\tilde{I}}_{D 2}(t)=\beta_{12}\left(N_{D 2}^{0}-\theta\right) \tilde{I}_{L 2}-\left(d_{12}+\sigma_{2}+D_{2}\right) \tilde{I}_{D 2}+D_{1} \widetilde{I}_{D 1} \\
\dot{\tilde{I}}_{L 2}(t)=\beta_{22}\left(S_{L 2}^{0}-\theta\right) \widetilde{x}_{2}-d_{22} \widetilde{I}_{L 2} \\
\dot{\tilde{x}}_{2}(t)=a_{2} \tilde{I}_{D 2}-d_{2} \widetilde{x}_{2} \tag{53}
\end{gather*}
$$

The coefficient matrix of the right hand of (53) is $J_{\theta}$. Since matrix $J_{\theta}$ has a positive eigenvalues $s\left(J_{\theta}\right)$ with a positive eigenvector, it follows from a comparison principle that $I_{D i}(t) \rightarrow \infty, I_{L i}(t) \rightarrow \infty$, and $x_{i}(t) \rightarrow \infty$ as $t \rightarrow \infty, i=$ 1,2 , which leads to a contradiction. This proves (49). Hence $W^{s}\left(E_{0}\right) \cap X_{0}=\emptyset$. Clearly, every forward orbit in $M_{\partial}$ converges to $E_{0}$. By Theorem 4.6 in [32] we are able to conclude that model (2) is uniformly persistent with respect to ( $X_{0}, \partial X_{0}$ ). Thus, by a well-known result in persistence theory in [33] we know that model (2) has at least one positive equilibrium $E^{*}\left(S_{D 1}^{*}, I_{D 1}^{*}, S_{L 1}^{*}, I_{L 1}^{*}, x_{1}^{*}, S_{D 2}^{*}, I_{D 2}^{*}, S_{L 2}^{*}, I_{L 2}^{*}, x_{2}^{*}\right)$. This completes the proof of Theorem 6.

## 6. Global Stability of $E^{*}$

We further have the following result on the stability of the endemic equilibrium.

Theorem 7. If $R_{0}>1$, then model (2) admits a unique equilibrium $E^{*}\left(S_{D 1}^{*}, I_{D 1}^{*}, S_{L 1}^{*}, I_{L 1}^{*}, x_{1}^{*}, S_{D 2}^{*}, I_{D 2}^{*}, S_{L 2}^{*}, I_{L 2}^{*}, x_{2}^{*}\right)$, which is globally asymptotically stable.

Proof. In Lemma 3, we have proved that $S_{D i}(t)+I_{D i}(t) \rightarrow$ $N_{D i}^{0}$ and $S_{L i}(t)+I_{L i}(t) \rightarrow S_{L i}^{0}$ as $t \rightarrow \infty, i=1,2$. Therefore, in model (2) we can represent $S_{D i}$ and $S_{L i}$ by $N_{D i}^{0}-I_{D i}(t)$ and


Figure 4: Time series of echinococcosis disease $I_{L i}, i=1,2$, when the two patches are connected with $D_{1}=0.8, D_{2}=0.2$.
$S_{L i}^{0}-S_{L i}(t), i=1,2$, respectively, and the model (2) will degenerate into the following system with six equations:

$$
\begin{gather*}
\dot{I}_{D 1}(t)=-\left(d_{11}+\sigma_{1}+D_{1}\right) I_{D 1}+\beta_{11}\left(N_{D 1}^{0}-I_{D 1}\right) I_{L 1} \\
+D_{2} I_{D 2} \\
\dot{I}_{L 1}(t)=-d_{21} I_{L 1}+\beta_{21}\left(S_{L 1}^{0}-I_{L 1}\right) x_{1} \\
\dot{x}_{1}(t)=a_{1} I_{D 1}-d_{1} x_{1} \\
\dot{I}_{D 2}(t)=D_{1} I_{D 1}-\left(d_{12}+\sigma_{2}+D_{2}\right) I_{D 2} \\
\dot{I}_{D 1}(t)=+\beta_{12}\left(N_{D 2}^{0}-I_{D 2}\right) I_{L 2} \\
\dot{I}_{L 2}(t)=-d_{22} I_{L 2}+\beta_{22}\left(S_{L 2}^{0}-I_{L 2}\right) x_{2} \\
\dot{x}_{2}(t)=a_{2} I_{D 2}-d_{2} x_{2} \tag{54}
\end{gather*}
$$

By Lemma 3, the dynamics of system (54) can be focused on the following region:

$$
\begin{align*}
\Omega=\{ & \left(I_{D 1}, I_{L 1}, x_{1}, I_{D 2}, I_{L 2}, x_{2}\right): 0 \leq I_{D i} \leq N_{D i}^{0}, \\
& \left.0 \leq I_{L i} \leq S_{L i}^{0}, 0 \leq x_{i} \leq \frac{a_{i}}{d_{i}} N_{D i}^{0}, i=1,2\right\} . \tag{55}
\end{align*}
$$

We will use the theory of cooperate system to prove the global stability of system (54). Therefore, we only verify the assumption in Corollary 3.2 [34] for system (54). Let

$$
f(u)=\left(\begin{array}{l}
f_{1}\left(u_{1}, u_{2}, u_{3}, u_{4}, u_{5}, u_{6}\right) \\
f_{2}\left(u_{1}, u_{2}, u_{3}, u_{4}, u_{5}, u_{6}\right) \\
f_{3}\left(u_{1}, u_{2}, u_{3}, u_{4}, u_{5}, u_{6}\right) \\
f_{4}\left(u_{1}, u_{2}, u_{3}, u_{4}, u_{5}, u_{6}\right) \\
f_{5}\left(u_{1}, u_{2}, u_{3}, u_{4}, u_{5}, u_{6}\right) \\
f_{6}\left(u_{1}, u_{2}, u_{3}, u_{4}, u_{5}, u_{6}\right)
\end{array}\right)
$$

$$
=\left(\begin{array}{c}
-\left(d_{11}+\sigma_{1}+D_{1}\right) u_{1}+\beta_{11}\left(N_{D 1}^{0}-u_{1}\right) u_{2}+D_{2} u_{4}  \tag{56}\\
-d_{21} u_{2}+\beta_{21}\left(S_{L 1}^{0}-u_{2}\right) u_{3} \\
a_{1} u_{1}-d_{1} u_{3} \\
D_{1} u_{1}-\left(d_{12}+\sigma_{2}+D_{2}\right) u_{4}+\beta_{12}\left(N_{D 2}^{0}-u_{4}\right) u_{5} \\
-d_{22} u_{5}+\beta_{22}\left(S_{L 2}^{0}-u_{5}\right) u_{6} \\
a_{2} u_{4}-d_{2} u_{6}
\end{array}\right),
$$

and then $f: \mathbb{R}_{+}^{6} \rightarrow \mathbb{R}_{+}^{6}$ is a continuously differentiable map. Clearly $f(0)=0$ and $f_{i}(u) \geq 0$ for all $u \in \Omega$ with $u_{i}=0$, $i=1,2, \ldots, 6$. Since $\partial f_{i} / \partial u_{j} \geq 0(i \neq j)$ for $u \in \Omega$, we have that $f$ is cooperative on $\Omega$. For every $p \in(0,1)$ and $u \in \Omega$, we have
$f_{1}\left(p u_{1}, p u_{2}, p u_{3}, p u_{4}, p u_{5}, p u_{6}\right)$

$$
\begin{align*}
& =-\left(d_{11}+\sigma_{1}+D_{1}\right) p u_{1}+\beta_{11}\left(N_{D 1}^{0}-p u_{1}\right) p u_{2}+D_{2} p u_{4} \\
& \geq-\left(d_{11}+\sigma_{1}+D_{1}\right) p u_{1}+\beta_{11}\left(N_{D 1}^{0}-u_{1}\right) p u_{2}+D_{2} p u_{4} \\
& =p f_{1}\left(u_{1}, u_{2}, u_{3}, u_{4}, u_{5}, u_{6}\right) . \tag{57}
\end{align*}
$$

Using the same argument, we can show that $f$ is strictly sublinear on $\Omega$. By computing $D f(u)$, we have

$$
\left(\frac{\partial f_{i}}{\partial u_{j}}\right)_{1 \leq i, j \leq 6}=\left(\begin{array}{ll}
f_{11}(u) & f_{12}(u)  \tag{58}\\
f_{21}(u) & f_{22}(u)
\end{array}\right)
$$



Figure 5: Time series of echinococcosis disease $I_{L i}, i=1,2$, when the two patches are connected with $D_{1}=0.2, D_{2}=0.8$.


Figure 6: Time series of echinococcosis disease $I_{L i}, i=1,2$, when the two patches are isolated for the parameters given in Example 9 .
where

$$
\begin{align*}
& f_{11}(u)=\left(\begin{array}{ccc}
-\left(d_{11}+\sigma_{1}+D_{1}\right)-\beta_{11} u_{2} & \beta_{11}\left(N_{D 1}^{0}-u_{1}\right) & 0 \\
0 & -d_{21}-\beta_{21} u_{3} & \beta_{21}\left(s_{L 1}^{0}-u_{3}\right) \\
a_{1} & 0 & -d_{1}
\end{array}\right), \\
& f_{12}(u)=\left(\begin{array}{lll}
D_{2} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right),  \tag{59}\\
& f_{21}(u)=\left(\begin{array}{ccc}
D_{1} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \\
& f_{22}(u)=\left(\begin{array}{ccc}
-\left(d_{12}+\sigma_{2}+D_{2}\right)-\beta_{12} u_{5} & \beta_{12}\left(N_{D 2}^{0}-u_{4}\right) & 0 \\
0 & -d_{22}-\beta_{22} u_{6} & \beta_{22}\left(S_{L 2}^{0}-u_{5}\right) \\
a_{2} & 0 & -d_{2}
\end{array}\right) .
\end{align*}
$$



Figure 7: Surface plot of $R_{0}$ as a function of $D_{1}$ and $D_{2}$ for the parameters given in Example 9.

Clearly, $D f(u)$ is irreducible for $u \in \Omega$. From (24) we can see that if $R_{0}>1$, then $s(J)>0$. Since $D f(0)=J$, we have $s(D f(0))=s(J)>0$. By Corollary 3.2 in [34], one can conclude that system (54) admits a unique positive equilibrium $\left(I_{D 1}^{*}, I_{L 1}^{*}, x_{1}^{*}, I_{D 2}^{*}, I_{L 2}^{*}, x_{2}^{*}\right)$, which is globally asymptotically stable. According to the theory of asymptotic autonomous systems [31], we further obtain that endemic equilibrium $E^{*}\left(S_{D 1}^{*}, I_{D 1}^{*}, S_{L 1}^{*}, I_{L 1}^{*}, x_{1}^{*}, S_{D 2}^{*}, I_{D 2}^{*}, S_{L 2}^{*}, I_{L 2}^{*}, x_{2}^{*}\right)$ is globally attractive for model (2).

## 7. Simulations

To complement the mathematical analysis carried out in the previous section, we now investigate some of the numerical properties of the two-patch model (2).

Example 8. Take parameters in model (2) as follows:
$A_{11}=15, \beta_{11}=0.00065, d_{11}=0.3, \sigma_{1}=0.2, A_{21}=80$, $\beta_{21}=0.004, d_{21}=0.4, a_{1}=150, d_{1}=33, A_{12}=15, \beta_{12}=$ $0.0015, d_{12}=0.3, \sigma_{2}=0.2, A_{22}=80, \beta_{22}=0.004, d_{22}=0.4$, $a_{2}=150$, and $d_{2}=33$. If the two patches are isolated, by simple calculations we have $R_{01}=0.8392, R_{02}=1.1089$.

From Lemma 1 we have that the disease will die out in the first patch and will be endemic in the second patch (see Figure 2). From Figure 3 we can easily see that $R_{0}$ will be larger than 1 under the condition of a larger $D_{1}$ and a smaller $D_{2}$. This means that the larger diffusion of dogs from the lower epidemic areas to the higher prevalence areas can intensify the spread of echinococcosis (see Figure 4). However, when $D_{1}$ is small and $D_{2}$ is large, $R_{0}$ will be smaller than 1 . This indicates that the larger diffusion of dogs from the higher prevalence areas to the lower epidemic areas can reduce the spread and is beneficial for disease control (see Figure 5).

Example 9. We use the parameters given in Example 8 except that $A_{11}=10, \beta_{11}=0.0015, A_{21}=70, A_{12}=20$, and $\beta_{12}=$ 0.0005 . If the two patches are isolated, by simple calculations we have $R_{01}=0.9266, R_{02}=0.8463$.


Figure 8: Time series of echinococcosis disease $I_{L i}, i=1,2$, when the two patches are connected with $D_{1}=0.2, D_{2}=0.6$.

It follows from Lemma 1 that the disease will die out in both two patches when they are isolated (see Figure 6). However, from Figure 7 we can see that $R_{0}$ is not always less than 1. This suggests that dogs diffusion can cause the spread of echinococcosis in two patches (see Figure 8).

## 8. Discussion

In this paper, in order to model the transmission dynamics of echinococcosis spread between two patches due to dogs migration a patch model for echinococcosis is proposed. We define the basic reproduction number $R_{0}$. The mathematical results show that the dynamics of the model is completely determined by $R_{0}$. If $R_{0}<1$, the disease-free equilibrium is globally asymptotically stable. When $R_{0}>$ 1 , the model is permanence and endemic equilibrium is globally asymptotically stable. According to the simulation
we have that the larger diffusion of dogs from a low epidemic area to the high prevalence area can intensify the disease spread. However, the larger diffusion of dogs from the high prevalence area to a low epidemic area can reduce the disease spread and is beneficial to disease control. Additionally, the model presented in this paper can be extended to describe the dynamical transmission of echinococcosis with dogs migration among more than two patches. We leave these in our future work.

## Conflict of Interests

The authors declare that they have no financial and personal relationships with other people or organizations that can inappropriately influence their work and there are no professional or other personal interests of any nature or kind in any product, service, and/or company that could be construed as influencing the position presented in, or the review of, this paper.

## Acknowledgments

This work is State Key Laboratory Basic Medicine Base of Xinjiang Hydatid Disease Research Fund (XJDX0202-2013-2), Cultivation Project Young Scientific Innovative Talents (2013731017), the National Natural Science Foundation of China (11201399, 11271312), the Scientific Research Programmes of Colleges in Xinjiang (XJEDU2012S20), and Academic Discipline Project of Xinjiang Medical University Health Measurements and Health Economics (XYDXK50780308).

## References

[1] S. A. Berger and J. S. Marr, Human Parasitic Diseases Sourcebook, Jones \& Bartlett Learning, Sudbury, Mass, USA, 2006.
[2] C. N. Macpherson, "An active intermediate host role for man in the life cycle of Echinococcus granulosus in Turkana, Kenya," American Journal of Tropical Medicine and Hygiene, vol. 32, no. 2, pp. 397-404, 1983.
[3] W. Zhenghuan, W. Xiaoming, and L. Xiaoqing, "Echinococcosis in China, a review of the epidemiology of Echinococcus spp.," EcoHealth, vol. 5, no. 2, pp. 115-126, 2008.
[4] M. Roberts, J. Lawson, and M. Gemmell, "Population dynamics in Echinococcosis and cysticercosis: mathematical model of the life-cycle of Echinococcus granulosus," Parasitology, vol. 92, no. 3, pp. 621-641, 1986.
[5] P. Torgerson, B. Shaikenov, A. Rysmukhambetova, A. Ussenbayev, A. M. Abdybekova, and K. K. Burtisurnov, "Modelling the transmission dynamics of Echinococcus granulosus in dogs in rural Kazakhstan," Parasitology, vol. 126, no. 5, pp. 417-424, 2003.
[6] P. Torgerson, "Canid immunity to Echinococcus spp.: impact on transmission," Parasite Immunology, vol. 28, no. 7, pp. 295-303, 2006.
[7] P. Torgerson, "The use of mathematical models to simulate control options for echinococcosis," Acta Tropica, vol. 85, no. 2, pp. 211-221, 2003.
[8] D. Heinzmann, A. D. Barbour, and P. R. Torgerson, "A mechanistic individual-based two-host interaction model for the
transmission of a parasitic disease," International Journal of Biomathematics, vol. 4, no. 4, pp. 443-460, 2011.
[9] M. Roberts and M. Aubert, "A model for the control of echinococcus multilocularis in France," Veterinary Parasitology, vol. 56, no. 1-3, pp. 67-74, 1995.
[10] M. Vervaeke, S. Davis, H. Leirs, and R. Verhagen, "Implications of increased susceptibility to predation for managing the sylvatic cycle of Echinococcus multilocularis," Parasitology, vol. 132, no. 6, pp. 893-901, 2006.
[11] H. Ishikawa, Y. Ohga, and R. Doi, "A model for the transmission of Echinococcus multilocularis in Hokkaido, Japan," Parasitology Research, vol. 91, no. 6, pp. 444-451, 2003.
[12] T. Nishina and H. Ishikawa, "A stochastic model of Echinococcus multilocularis transmission in Hokkaido, Japan, focusing on the infection process," Parasitology Research, vol. 102, no. 3, pp. 465-479, 2008.
[13] F. Hansen, K. Tackmann, F. Jeltsch, C. Wissel, and H. Thulke, "Controlling echinococcus multilocularis-ecological implications of field trials," Preventive Veterinary Medicine, vol. 60, no. 1, pp. 91-105, 2003.
[14] K. Takumi and J. Van der Giessen, "Transmission dynamics of Echinococcus multilocularis; its reproduction number, persistence in an area of low rodent prevalence, and effectiveness of control," Parasitology, vol. 131, no. 1, pp. 133-140, 2005.
[15] K. Takumi, A. de Vries, M. L. Chu, J. Mulder, P. Teunis, and J. van der Giessen, "Evidence for an increasing presence of Echinococcus multilocularis in foxes in The Netherlands," International Journal for Parasitology, vol. 38, no. 5, pp. 571-578, 2008.
[16] N. Kato, K. Kotani, S. Ueno, and H. Matsuda, "Optimal risk management of human alveolar echinococcosis with vermifuge," Journal of Theoretical Biology, vol. 267, no. 3, pp. 265271, 2010.
[17] J. A. M. Atkinson, G. M. Williams, L. Yakob et al., "Synthesising 30 years of mathematical modelling of Echinococcus transmission," PLoS Neglected Tropical Diseases, vol. 7, no. 8, p. e2386, 2013.
[18] K. Wang, X. Zhang, Z. Jin, H. Ma, Z. Teng, and L. Wang, "Modeling and analysis of the transmission of Echinococcosis with application to Xinjiang Uygur Autonomous Region of China," Journal of Theoretical Biology, vol. 333, pp. 78-90, 2013.
[19] W. Wang and X.-Q. Zhao, "An epidemic model in a patchy environment," Mathematical Biosciences, vol. 190, no. 1, pp. 97112, 2004.
[20] Y. Jin and W. Wang, "The effect of population dispersal on the spread of a disease," Journal of Mathematical Analysis and Applications, vol. 308, no. 1, pp. 343-364, 2005.
[21] J. Cui, Y. Takeuchi, and Y. Saito, "Spreading disease with transport-related infection," Journal of Theoretical Biology, vol. 239, no. 3, pp. 376-390, 2006.
[22] Y. Yang, J. Wu, J. Li, and Z. Ma, "Global dynamics-convergence to equilibria-of epidemic patch models with immigration," Mathematical and Computer Modelling, vol. 51, no. 5-6, pp. 329337, 2010.
[23] D. Gao and S. Ruan, "An SIS patch model with variable transmission coefficients," Mathematical Biosciences, vol. 232, no. 2, pp. 110-115, 2011.
[24] J. J. Tewa, S. Bowong, and B. Mewoli, "Mathematical analysis of two-patch model for the dynamical transmission of tuberculosis," Applied Mathematical Modelling, vol. 36, no. 6, pp. 24662485, 2012.
[25] R. Lintott, R. Norman, and A. Hoyle, "The impact of increased dispersal in response to disease control in patchy environments," Journal of Theoretical Biology, vol. 323, pp. 57-68, 2013.
[26] Z. Qiu, "Dynamics of an epidemic model with host migration," Applied Mathematics and Computation, vol. 218, no. 8, pp. 46144625, 2011.
[27] J. Liu, "Spread of disease in a patchy environment," Chaos, Solitons and Fractals, vol. 45, no. 7, pp. 942-949, 2012.
[28] D. Gao and S. Ruan, "A multipatch Malaria model with logistic growth populations," SIAM Journal on Applied Mathematics, vol. 72, no. 3, pp. 819-841, 2012.
[29] O. Diekmann, J. A. P. Heesterbeek, and J. A. J. Metz, "On the definition and the computation of the basic reproduction ratio $R_{0}$ in models for infectious diseases in heterogeneous populations," Journal of Mathematical Biology, vol. 28, no. 4, pp. 365-382, 1990.
[30] P. Van den Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," Mathematical Biosciences, vol. 180, pp. 29-48, 2002.
[31] H. R. Thieme, "Convergence results and a Poincaré-Bendixson trichotomy for asymptotically autonomous differential equations," Journal of Mathematical Biology, vol. 30, no. 7, pp. 755763, 1992.
[32] H. R. Thieme, "Persistence under relaxed point-dissipativity (with application to an endemic model)," SIAM Journal on Mathematical Analysis, vol. 24, no. 2, pp. 407-435, 1993.
[33] V. Hutson and K. Schmitt, "Permanence and the dynamics of biological systems," Mathematical Biosciences, vol. 111, no. 1, pp. 1-71, 1992.
[34] X. Q. Zhao and Z. J. Jing, "Global asymptotic behavior in some cooperative systems of functional-differential equations," The Canadian Applied Mathematics Quarterly, vol. 4, no. 4, pp. 421444, 1996.

# Research Article 

# Global Behaviors of a Class of Discrete SIRS Epidemic Models with Nonlinear Incidence Rate 

Lei Wang, ${ }^{1,2}$ Zhidong Teng, ${ }^{2}$ and Long Zhang ${ }^{2}$<br>${ }^{1}$ Department of Medical Engineering and Technology, Xinjiang Medical University, Urumqi 830011, China<br>${ }^{2}$ College of Mathematics and Systems Science, Xinjiang University, Urumqi 830046, China<br>Correspondence should be addressed to Zhidong Teng; zhidong@xju.edu.cn

Received 3 November 2013; Accepted 3 February 2014; Published 17 March 2014
Academic Editor: Yun Kang
Copyright © 2014 Lei Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We study a class of discrete SIRS epidemic models with nonlinear incidence rate $F(S) G(I)$ and disease-induced mortality. By using analytic techniques and constructing discrete Lyapunov functions, the global stability of disease-free equilibrium and endemic equilibrium is obtained. That is, if basic reproduction number $\mathscr{R}_{0}<1$, then the disease-free equilibrium is globally asymptotically stable, and if $\mathscr{R}_{0}>1$, then the model has a unique endemic equilibrium and when some additional conditions hold the endemic equilibrium also is globally asymptotically stable. By using the theory of persistence in dynamical systems, we further obtain that only when $\mathscr{R}_{0}>1$, the disease in the model is permanent. Some special cases of $F(S) G(I)$ are discussed. Particularly, when $F(S) G(I)=\beta S I /(1+\lambda I)$, it is obtained that the endemic equilibrium is globally asymptotically stable if and only if $\mathscr{R}_{0}>1$. Furthermore, the numerical simulations show that for general incidence rate $F(S) G(I)$ the endemic equilibrium may be globally asymptotically stable only as $\mathscr{R}_{0}>1$.


## 1. Introduction

During the past decades, no matter discrete epidemic models or continuous epidemic models, have been widely studied. Many important and interesting results can be found in [1-28] and the references cited therein. The main research subjects are the computation of the threshold value or basic reproduction number which distinguishes whether the infectious disease will persist or die out, the local and global stability of the disease-free equilibrium and endemic equilibrium, the extinction, persistence, and permanence of the disease, and the bifurcations, chaos, and more complex dynamical behaviors of the models.

Among these questions, global stability of equilibria has always been one of the research focuses and difficult problems. Many authors have investigated this question using the second Lyapunov method (see [29]). The most popular types of Lyapunov functions candidate for population biology models are the Volterra-type functions $x-x^{*}-\ln \left(x / x^{*}\right)$ and the quadratic function $(c / 2)\left(x-x^{*}\right)^{2}$. The former has been successfully applied for various disease propagation
models by Korobeinikov and his coworkers (see [7-10] and the references cited therein). In [11], Li et al. presented an algebraic approach to prove the global stability, which can provide the method of constructing a Lyapunov function and prove the negative definiteness of the derivative. Recently, by combining Volterra functions and quadratic functions, Vargas-De-León has studied global stability of classic continuous SIS, SIR, and SIRS epidemic models with constant recruitment, disease-induced death, and standard incidence rate and bilinear incidence rate in [12, 13], respectively. McCluskey in [14-16] introduced the Lyapunov functional formed as $\int_{0}^{\tau}\left(x(t-s) / x^{*}-1-\ln \left(x(t-s) / x^{*}\right)\right) d s$ to investigate global stability of endemic equilibrium of SEIR epidemic model with distributed delay or discrete delay.

It is well known that a crucial role in mathematical models of infectious disease is played by the so-called incidence rate, namely, a function describing the mechanism of transmission of the disease. In most epidemiological models, bilinear incidence rate $\beta S I$ and standard incidence rate $\beta S I / N$ are frequently used, where $N$ is the total number of the population ( $N=S+I+R$ ) and $\beta>0$ is the per capita
contact rate. These incidences imply that the contact number between $S$ and $I$ is proportional to $S I$ or $S I / N$. But the infection probability per contact is likely influenced by the number of infective and susceptible individuals, because more infective individuals can increase the infection risk and susceptible individuals would avoid the contact with infective individuals. Therefore, a number of nonlinear incidence rates are suggested by researchers. After studying the cholera epidemic spread in Bari in 1973, Capasso and Serio [17] introduced the saturated incidence rate $\beta S G(I)$ into epidemic models. To incorporate the effect of the behavioral changes of the susceptible individuals, Liu et al. proposed the general incidence rate $\beta S I^{p} /\left(1+k I^{q}\right)$ in [18], where $p, q>0$ and $k \geq 0$. The special cases when $p$ and $q$ are given different values have been used by many authors (see, e.g., Korobeinikov and Maini [6], Ruan and Wang [19], and Xiao and Ruan [20]).

However, until now, to the best of our knowledge, there are few search results about global stability of equilibria for discrete SIRS model with nonlinear incidence rate. Hu et al. in [28] discussed local stability and complex dynamical behaviors for a class of discrete SIRS epidemic models with general nonlinear incidence rate discretized by the forward Euler scheme. Enatsu et al. in [22] proposed a class of discrete SIR epidemic models with bilinear incidence rate, which are derived from continuous SIR epidemic models with distributed delays by using a variation of the backward Euler method, and obtained that global stability of diseasefree equilibrium and endemic equilibrium. Muroya et al. in [23] discussed global stability and permanence of a discrete epidemic model with bilinear incidence rate and for disease with immunity and latency spreading in a heterogeneous host population, which is also discretized from the continuous case by using the backward Euler method. In [24], Enatsu et al. studied a class of discrete SIR epidemic models with nonlinear incidence rates and distributed delays, which are derived from the corresponding continuous SIR epidemic models by applying a variation of the backward Euler discretization. Using discrete-time analogue of Lyapunov functionals, the global asymptotic stability of the diseasefree equilibrium and endemic equilibrium is fully determined by the basic reproduction number $R_{0}$, when the infection incidence rate has a suitable monotone property.

Motivated by the fact that discrete epidemic models are more appropriate approach to understand disease transmission dynamics and to evaluate eradication policies because they permit arbitrary time step units, preserving the basic features of corresponding continuous models, in this paper, we will extend a discrete-time analogue of Lyapunov techniques proposed in [25-27] to the following discrete SIRS epidemic models with nonlinear incidence rate $F(S) G(I)$, which is established by using the backward Euler scheme (see $[30,31]$ ) to discretize the corresponding continuous SIRS epidemic model:

$$
\begin{aligned}
S(n+1)= & S(n)+\Lambda-F(S(n+1)) G(I(n+1)) \\
& -\mu S(n+1)+\gamma R(n+1)
\end{aligned}
$$

$$
\begin{gather*}
I(n+1)= \\
\quad I(n)+F(S(n+1)) G(I(n+1)) \\
-(k+\mu+\alpha) I(n+1)  \tag{1}\\
R(n+1)=R(n)+k I(n+1)-(\mu+\gamma) R(n+1) .
\end{gather*}
$$

We will investigate the global behaviors of solutions of model (1). By constructing new discrete Lyapunov functions, we will establish some new criteria on the global asymptotic stability of the disease-free equilibrium and endemic equilibrium for model (1). By using the theory of persistence in dynamical systems, we will further obtain the sufficient and necessary conditions for the permanence of the disease for model (1).

The organization of this paper is as follows. In Section 2, the existence of equilibria and positivity of solutions for model (1) are given. In Section 3, the results on the global asymptotic stability of the disease-free equilibrium and endemic equilibrium for model (1) are stated and proved. In Section 4, the results on the permanence of the disease in model (1) are established. In Section 5, the global asymptotic stability of the endemic equilibrium of model (1) for the special case $F(S)=S /(1+\lambda S)$ is discussed. Finally, some examples are given to illustrate the main theoretical results in Section 6.

## 2. Equilibria and Positivity

For model (1), $S(n), I(n)$, and $R(n)$ represent the numbers of susceptible, infectious, and recovered individuals at $n$th generation, respectively. The parameters $\Lambda, \mu, \alpha$, and $k$ are positive constants and $\gamma$ is nonnegative constant in which $\Lambda$ is the recruitment rate into the population, $\mu$ is the natural death rate, $\alpha$ is the disease-induced death rate, $k$ is the recovery rate of the infectious individuals, $\gamma$ is the rate of losing immunity, $\gamma>0$ implies that the recovered individuals would lose the immunity, and $\gamma=0$ implies that the recovered individuals acquire permanent immunity. The spread of disease can be described by general form with incidence rate $F(S) G(I)$; that is, the incidence rate depends on the number of the susceptible individuals and the number of the infectious individuals. This generalizes the bilinear incidence rate (i.e., $F(S) G(I)=\beta S I)$, saturated incidence rate with respect to $S$ (i.e., $F(S) G(I)=\beta S I /(1+\lambda S)$ ), and saturated incidence rate with respect to $I$ (i.e., $F(S) G(I)=\beta S I /(1+\omega I)$ ), where $\beta>0$, $\lambda \geq 0$, and $\omega \geq 0$ are constants, which denotes the contact coefficient and the saturated coefficient, respectively.

The initial condition for model (1) is given by

$$
\begin{equation*}
S(0)>0, \quad I(0)>0, \quad R(0) \geq 0 . \tag{2}
\end{equation*}
$$

In this paper, for functions $F(S)$ and $G(I)$, we firstly introduce the following assumption.
$\left(H_{1}\right) F(S)$ and $G(I)$ are positive, monotonically increasing, and continuous differentiable functions defined for all $S \geq 0$ and $I \geq 0$, the derivative $G^{\prime}(0)$ exists, and $F(0)=$ $G(0)=0$. Furthermore, $G(I) / I$ is nonincreasing for all $I>0$.

Remark 1. Assumption $\left(H_{1}\right)$ is basic for model (1). In fact, for many special cases of $F(S) G(I)$, for example, $F(S) G(I)=\beta S I$, $F(S) G(I)=\beta S I /(1+\lambda S)$, and $F(S) G(I)=\beta S I /(1+\omega I),\left(H_{1}\right)$ is always satisfied.

In order to obtain the existence of disease-free equilibrium and endemic equilibrium of model (1), we introduce a constant

$$
\begin{equation*}
\mathscr{R}_{0}=\frac{F(\Lambda / \mu) G^{\prime}(0)}{k+\mu+\alpha} . \tag{3}
\end{equation*}
$$

We have the following result.
Theorem 2. Assume that $\left(H_{1}\right)$ holds.
(1) When $\mathscr{R}_{0} \leq 1$, then model (1) has only a unique disease-free equilibrium $E^{0}(\Lambda / \mu, 0,0)$.
(2) When $\mathscr{R}_{0}>1$, then model (1) shows a unique endemic equilibrium $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$, except for $E^{0}$, where $S^{*}, I^{*}$, and $R^{*}$ satisfy

$$
\begin{align*}
& \Lambda=F\left(S^{*}\right) G\left(I^{*}\right)+\mu S^{*}-\gamma R^{*} \\
& I^{*}(k+\mu+\alpha)=F\left(S^{*}\right) G\left(I^{*}\right)  \tag{4}\\
& k I^{*}=(\mu+\gamma) R^{*}
\end{align*}
$$

Proof. Obviously, model (1) always has a disease-free equilibrium $E^{0}(\Lambda / \mu, 0,0)$. From (4), we have

$$
\begin{equation*}
R^{*}=\frac{k I^{*}}{\mu+\gamma}, \quad \Lambda=I^{*}(k+\mu+\alpha)+\mu S^{*}-\gamma R^{*} \tag{5}
\end{equation*}
$$

Hence,

$$
\begin{align*}
S^{*} & =\frac{1}{\mu}\left(\Lambda-I^{*}\left(k+\mu+\alpha-\frac{k \gamma}{\mu+\gamma}\right)\right) \\
& =\frac{\Lambda}{\mu}-I^{*} \frac{(\mu+\alpha)(\mu+\gamma)+k \mu}{\mu(\mu+\gamma)} \tag{6}
\end{align*}
$$

and from the second equation of (4) we further have

$$
\begin{equation*}
I^{*}(k+\mu+\alpha)=F\left(\frac{\Lambda}{\mu}-I^{*} \frac{(\mu+\alpha)(\mu+\gamma)+k \mu}{\mu(\mu+\gamma)}\right) G\left(I^{*}\right) . \tag{7}
\end{equation*}
$$

When $I>0$, let

$$
\begin{equation*}
H(I)=k+\mu+\alpha-F\left(\frac{\Lambda}{\mu}-I \frac{(\mu+\alpha)(\mu+\gamma)+k \mu}{\mu(\mu+\gamma)}\right) \frac{G(I)}{I} \tag{8}
\end{equation*}
$$

Then by $\left(H_{1}\right)$ we obtain

$$
\lim _{I \rightarrow 0^{+}} H(I)=k+\mu+\alpha-F\left(\frac{\Lambda}{\mu}\right) G^{\prime}(0) \begin{cases}\geq 0, & \mathscr{R}_{0} \leq 1  \tag{9}\\ <0, & \mathscr{R}_{0}>1\end{cases}
$$

Let $\bar{I}=\Lambda(\mu+\gamma) /((\mu+\gamma)(\mu+\alpha)+k \mu)$; then we obviously have $H(\bar{I})=k+\mu+\alpha>0$. From $\left(H_{1}\right), F((\Lambda / \mu)-I(((\mu+$ $\alpha)(\mu+\gamma)+k \mu) / \mu(\mu+\gamma)))$ is monotonically decreasing for $I \in(0, \bar{I}]$, and hence $H(I)$ is monotonically increasing for $I \in$ ( $0, \bar{I}]$. Thus, from (9), we obtain that when $\mathscr{R}_{0} \leq 1$ equation $H(I)=0$ has not any solution in $(0, \bar{I})$ and when $\mathscr{R}_{0}>1$ equation $H(I)=0$ has a unique positive solution $I^{*}$ in $(0, \bar{I})$. This shows that when $\mathscr{R}_{0} \leq 1$ model (1) does not have any endemic equilibrium. When $\mathscr{R}_{0}>1$, let

$$
\begin{equation*}
R^{*}=\frac{k I^{*}}{\mu+\gamma}, \quad S^{*}=\frac{1}{\mu}\left(\Lambda-I^{*}\left(k+\mu+\alpha-\frac{k \gamma}{\mu+\alpha}\right)\right) \tag{10}
\end{equation*}
$$

and then $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ is a unique endemic equilibrium of model (1). This completes the proof.

From Theorem 2, we can claim that the basic reproduction number of model (1) is $\mathscr{R}_{0}$. On the positivity and ultimate boundedness of solutions of model (1), we obtain the following theorem.

Theorem 3. Assume that $\left(H_{1}\right)$ holds. Let $(S(n), I(n), R(n))$ be the solution of model (1) with initial conditions (2); then $(S(n), I(n), R(n))$ is positive for any $n>0$ and ultimately bounded.

Proof. Let $(S(n), I(n), R(n))$ be any solution of model (1) with initial conditions (2). Further, let $N(n)=S(n)+I(n)+R(n))$; then model (1) is equivalent to the following form:

$$
\begin{gather*}
I(n+1)=\frac{1}{1+k+\mu+\alpha} \\
\quad \times(I(n)+F(N(n+1) \\
\quad-I(n+1)-R(n+1)) \\
\quad \times G(I(n+1)))  \tag{11}\\
R(n+1)=\frac{R(n)+k I(n+1)}{1+\mu+\gamma}, \\
N(n+1)=\frac{N(n)+\Lambda-\alpha I(n+1)}{1+\mu}
\end{gather*}
$$

$$
\begin{align*}
S(n+1)= & S(n)+\Lambda-F(S(n+1)) G(I(n+1)) \\
& -\mu S(n+1)+\gamma R(n+1) . \tag{12}
\end{align*}
$$

In the following, we will use the induction to prove the positivity of $(S(n), I(n), R(n))$. When $n=0$, we have

$$
\begin{align*}
I(1)= & \frac{1}{1+k+\mu+\alpha}  \tag{13}\\
& \times(I(0)+F(N(1)-I(1)-R(1)) G(I(1))), \\
R(1)= & \frac{R(0)+k I(1)}{1+\mu+\gamma}, \quad N(1)=\frac{N(0)+\Lambda-\alpha I(1)}{1+\mu},  \tag{14}\\
S(1)= & S(0)+\Lambda-F(S(1)) G(I(1))-\mu S(1)+\gamma R(1) . \tag{15}
\end{align*}
$$

From (13)-(15) we see that as long as $I(1)$ is confirmed, then $R(1), N(1)$, and $S(1)$ will be whereafter confirmed.

Firstly, we prove that if $I(1)>0$, then $S(1)>0$ and $R(1)>$ 0 . From (14), we directly obtain $R(1)>0$ when $I(1)>0$. Let $x=S(1)$, and from (15) we obtain

$$
\begin{equation*}
\Phi(x) \triangleq(1+\mu) x+F(x) G(I(1))-\gamma R(1)-S(0)-\Lambda=0 . \tag{16}
\end{equation*}
$$

It is obvious that, when $I(1)>0, \Phi(x)$ is monotonically increasing for $x \geq 0$. Obviously, $\Phi(x)$ is a continuous function for $x$. Since $\Phi(0)=-\gamma R(1)-S(0)-\Lambda<0$ and $\lim _{x \rightarrow+\infty} \Phi(x)=+\infty$, we obtain that $\Phi(x)=0$ has a unique positive solution $\bar{x}$. Therefore, we further have $S(1)=\bar{x}>0$. Furthermore, we also have $N(1)=S(1)+I(1)+R(1)>0$.

Let $y=I(1))$; then from (13) we see that $y$ must satisfy the following equation:

$$
\begin{align*}
\Psi(y) \triangleq & y-\frac{1}{1+k+\mu+\alpha}  \tag{17}\\
& \times(I(0)+F(N(1)-y-R(1)) G(y))=0
\end{align*}
$$

where

$$
\begin{equation*}
N(1)=\frac{N(0)+\Lambda-\alpha y}{1+\mu}, \quad R(1)=\frac{R(0)+k y}{1+\mu+\gamma} . \tag{18}
\end{equation*}
$$

Denote

$$
\begin{align*}
& a_{0}=\frac{N(0)+\Lambda}{1+\mu}-\frac{R(0)}{1+\mu+\gamma}, \\
& b_{0}=\frac{\alpha}{1+\mu}+1+\frac{k}{1+\mu+\gamma} . \tag{19}
\end{align*}
$$

Obviously, $a_{0}>0$. Let $y_{0}=a_{0} / b_{0}$; then when $y=y_{0}$ we have $N(1)-y-R(1)=0$. We also have that $N(1)-y-R(1)$ is monotonically decreasing with respect to $y \in\left[0, y_{0}\right]$. Hence, by $\left(H_{1}\right), F(N(1)-y-R(1))$ is also monotonically decreasing with respect to $y \in\left[0, y_{0}\right]$. From the expression of $\Psi(y)$ and $\left(H_{1}\right)$, we obtain that $\Psi(y)$ is monotonically increasing for $y \in\left[0, y_{0}\right]$. Obviously, $\Psi(y)$ is a continuousfunction for $y$.

Since

$$
\begin{gather*}
\Psi(0)=-\frac{I(0)}{1+k+\mu+\alpha}<0, \\
\Psi\left(y_{0}\right)= \\
=y_{0}-\frac{1}{1+k+\mu+\alpha} I(0) \\
=\frac{(\Lambda+N(0))(1+\mu+\gamma)-R(0)(1+\mu)}{\alpha(1+\mu+\gamma)+(1+\mu)(1+\mu+\gamma)+k(1+\mu)} \\
\geq \\
=\frac{1}{1+k+\mu+\alpha} I(0)  \tag{20}\\
1+k+\mu+\alpha \\
1+k+\mu+\alpha
\end{gather*}
$$

there exists a unique $\bar{y} \in\left(0, y_{0}\right)$ such that $\Psi(\bar{y})=0$.
Now, we show that $\bar{y}$ is a unique solution of $\Psi(y)=0$ on $(0, \infty)$. Otherwise, there is a $y^{\prime} \in\left[y_{0}, \infty\right)$ such that $\Psi\left(y^{\prime}\right)=0$. Since $y^{\prime} \geq y_{0}$, we have $N(1)-y-R(1) \leq 0$ when $y=y^{\prime}$. From $\left(H_{1}\right)$, we have $F(S) \geq 0$ for any $S \geq 0$; hence from $\Psi\left(y^{\prime}\right)=0$ we further have $y^{\prime} \leq I(0) /(1+\mu+\alpha+k)$. On the other hand, since $a_{0}>I(0) /(1+\mu)$ and $b_{0}<(1+\mu+\alpha+k) /(1+\mu)$, we obtain $y^{\prime}>I(0) /(1+\mu+\alpha+k)$, which leads to a contradiction.

Therefore, we certainly have $I(1)=\bar{y}>0$. From the above discussions, we finally have $I(1)>0, S(1)>0$, and $R(1)>0$.

When $n=1$, we obtain

$$
\begin{align*}
& I(2)=\frac{1}{1+k+\mu+\alpha} \\
& \times(I(1)+F(N(1)-I(1)-R(1)) G(I(1))), \\
& R(2)=\frac{R(1)+k I(2)}{1+\mu+\gamma},  \tag{21}\\
& N(2)=\frac{N(1)+\Lambda-\alpha I(2)}{1+\mu}, \\
& S(2)=S(1)+\Lambda-F(N(2)-I(2)-R(2)) \\
& \\
& \times G(I(2))-\mu S(2)+\gamma R(2) .
\end{align*}
$$

Obviously, using a similar argument in the above process, we also can obtain $S(2)>0, I(2)>0$, and $R(2)>0$. Lastly, by using the induction, we can finally obtain $S(n)>0, I(n)>0$, and $R(n)>0$ for all $n>0$.

From the third equation of model (11), we have

$$
\begin{equation*}
N(n+1) \leq \frac{1}{1+\mu}(N(n)+\Lambda) . \tag{22}
\end{equation*}
$$

Since comparison equation,

$$
\begin{equation*}
U(n+1)=\frac{1}{1+\mu}(U(n)+\Lambda) \tag{23}
\end{equation*}
$$

has a globally asymptotically stable equilibrium $U^{*}=\Lambda / \mu$, from the comparison principle of difference equations (see [32]), we finally obtain

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} N(n) \leq \frac{\Lambda}{\mu} \tag{24}
\end{equation*}
$$

Therefore, $(S(n), I(n), R(n))$ is also ultimately bound. This completes the proof.

## 3. Global Stability

Now, we are concerned with the global asymptotic stability of disease-free equilibrium $E^{0}$ and endemic equilibrium $E^{*}$ of model (1), respectively.

Theorem 4. Assume that $\left(H_{1}\right)$ holds. Then disease-free equilibrium $E^{0}(\Lambda / \mu, 0,0)$ of model (1) is globally asymptotically stable if $\mathscr{R}_{0}<1$ and is globally attractive if $\mathscr{R}_{0}=1$.

Proof. Calculating the linearization system of model (1) at equilibrium $E^{0}$, we have

$$
\begin{gather*}
u_{n+1}=u_{n}-F\left(\frac{\Lambda}{\mu}\right) G^{\prime}(0) v_{n+1}-\mu u_{n+1}+\gamma w_{n+1} \\
v_{n+1}=v_{n}+F\left(\frac{\Lambda}{\mu}\right) G^{\prime}(0) v_{n+1}-(k+\mu+\alpha) v_{n+1}  \tag{25}\\
w_{n+1}=w_{n}+k v_{n+1}-(\mu+\gamma) w_{n+1}
\end{gather*}
$$

From the second equation of system (25), we have

$$
\begin{equation*}
v_{n+1}=\frac{v_{n}}{1+k+\mu+\alpha-F(\Lambda / \mu) G^{\prime}(0)} . \tag{26}
\end{equation*}
$$

When $\mathscr{R}_{0}<1$, we obtain

$$
\begin{align*}
0 & <\frac{1}{1+k+\mu+\alpha-F(\Lambda / \mu) G^{\prime}(0)} \\
& =\frac{1}{1+(k+\mu+\alpha)\left(1-\mathscr{R}_{0}\right)}<1 \tag{27}
\end{align*}
$$

Therefore, $\lim _{n \rightarrow \infty} v_{n}=0$. By

$$
\begin{equation*}
u_{n+1}=\frac{u_{n}-F(\Lambda / \mu) G^{\prime}(0) v_{n+1}}{1+\mu}, \quad w_{n+1}=\frac{w_{n}+k v_{n+1}}{1+\mu+\gamma} \tag{28}
\end{equation*}
$$

we further obtain $\lim _{n \rightarrow \infty} u_{n}=0$ and $\lim _{n \rightarrow \infty} w_{n}=0$. This shows that $E^{0}$ is locally stable when $\mathscr{R}_{0}<1$. Since the case $\mathscr{R}_{0}=1$ is a critical one for model (1), in the following, we discuss global attractivity of disease-free equilibrium $E^{0}$ when $\mathscr{R}_{0} \leq 1$.

Let $(S(n), I(n), R(n))$ be any positive solution of model (1) with initial conditions (2). We need to consider the following two cases.

Case 1. $N(n) \geq \Lambda / \mu$ for all $n=1,2, \ldots$.
Case 2. There exists an integer $n_{1}>0$ such that $N\left(n_{1}\right)<\Lambda / \mu$.

For Case 1, from (24), we directly have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} N(n)=\frac{\Lambda}{\mu} \tag{29}
\end{equation*}
$$

From third equation of (11), we further obtain

$$
\begin{align*}
\lim _{n \rightarrow \infty} I(n) & =\lim _{n \rightarrow \infty} \frac{1}{\alpha}[N(n)(1+\mu)-N(n-1)-\Lambda] \\
& =\frac{1}{\alpha}\left[\frac{\Lambda}{\mu}(1+\mu)-\frac{\Lambda}{\mu}-\Lambda\right]=0 \tag{30}
\end{align*}
$$

For Case 2, by using the iterative computations to inequality (22), we can obtain $N(n)<\Lambda / \mu$ for all $n \geq n_{1}$. Hence, $S(n)<\Lambda / \mu$ for all $n \geq n_{1}$. From $\left(H_{1}\right)$, we further obtain

$$
\begin{equation*}
F(S(n+1))<F\left(\frac{\Lambda}{\mu}\right), \quad \forall n \geq n_{1} . \tag{31}
\end{equation*}
$$

Since

$$
\begin{equation*}
\frac{G(I(n+1))}{I(n+1)} \leq \lim _{I \rightarrow 0^{+}} \frac{G(I)}{I}=G^{\prime}(0) \tag{32}
\end{equation*}
$$

from the second equation of model (1), it follows that, for all $n \geq n_{1}$,

$$
I(n+1)-I(n)=F(S(n+1)) G(I(n+1))
$$

$$
-(k+\mu+\alpha) I(n+1)
$$

$$
=I(n+1)(F(S(n+1))
$$

$$
\left.\times \frac{G(I(n+1))}{I(n+1)}-(k+\mu+\alpha)\right)
$$

$$
\leq I(n+1)\left(F\left(\frac{\Lambda}{\mu}\right) G^{\prime}(0)-(k+\mu+\alpha)\right)
$$

$$
\begin{equation*}
=(k+\mu+\alpha) I(n+1)\left(\mathscr{R}_{0}-1\right) . \tag{33}
\end{equation*}
$$

If $\mathscr{R}_{0} \leq 1$, then

$$
\begin{equation*}
I(n+1)-I(n) \leq 0 \quad \text { for all } n \geq n_{1} . \tag{34}
\end{equation*}
$$

Hence, $I(n)$ is nonincreasing for $n \geq n_{1}$. Consequently, $\lim _{n \rightarrow \infty^{\prime}} I(n)=\widehat{I}$ exists and $\widehat{I} \geq 0$.

Suppose $\widehat{I}>0$; then from the second and third equations of model (11), we can obtain that $\lim _{n \rightarrow \infty} R(n)$ and $\lim _{n \rightarrow \infty} N(n)$ exist, and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} R(n)=\frac{k \widehat{I}}{\mu+\gamma}:=\widehat{R}, \quad \lim _{n \rightarrow \infty} N(n)=\frac{\Lambda-\alpha \widehat{I}}{\mu}:=\widehat{N} . \tag{35}
\end{equation*}
$$

From $S(n)=N(n)-I(n)-R(n)$, it follows that $\lim _{n \rightarrow \infty} S(n)=$ $\widehat{S}$ exists. Obviously, we have $\widehat{R}>0, \widehat{N} \geq 0$, and $\widehat{S} \geq 0$.

Taking $n \rightarrow \infty$ from the both sides of model (1), we can obtain the following equations:

$$
\begin{gather*}
\Lambda-F(\widehat{S}) G(\widehat{I})-\mu \widehat{S}+\gamma \widehat{R}=0 \\
\widehat{I}(k+\mu+\alpha)-F(\widehat{S}) G(\widehat{I})=0  \tag{36}\\
k \widehat{I}-(\mu+\gamma) \widehat{R}=0
\end{gather*}
$$

Hence, $(\widehat{S}, \widehat{I}, \widehat{R})$ is an equilibrium of model (1). However, from Theorem 2, we see that when $\mathscr{R}_{0} \leq 1$, (36) only has a unique solution $\widehat{S}=\Lambda / \mu, \widehat{I}=0$, and $\widehat{R}=0$. This leads to a contradiction. Therefore, we have $\widehat{I}=0$.

Therefore, we always have $\lim _{n \rightarrow \infty} I(n)=\widehat{I}=0$. By (35), it follows that $\lim _{n \rightarrow \infty} R(n)=0$ and $\lim _{n \rightarrow \infty} N(n)=\Lambda / \mu$. Consequently, $\lim _{n \rightarrow \infty} S(n)=\Lambda / \mu$. This shows that diseasefree equilibrium $E^{0}=(\Lambda / \mu, 0,0)$ is globally attractive when $\mathscr{R}_{0} \leq 1$. This completes the proof.

In order to obtain the global asymptotic stability of endemic equilibrium $E^{*}$ of model (1), we need the following assumptions.
$\left(H_{2}\right)$ For any $S>0$,

$$
\begin{equation*}
\frac{F(S)}{F(S)-F\left(S^{*}\right)}-\frac{\left(S^{*}\right)^{2}}{F\left(S^{*}\right)\left(S-S^{*}\right)} \geq 0 \tag{37}
\end{equation*}
$$

$\left(H_{3}\right)$ For any $S>0$,

$$
\begin{equation*}
-\frac{\mu\left(S^{*}\right)^{2}}{F\left(S^{*}\right)}+\gamma R^{*}\left(\frac{F(S)}{F(S)-F\left(S^{*}\right)}-\frac{\left(S^{*}\right)^{2}}{F\left(S^{*}\right)\left(S-S^{*}\right)}\right) \leq 0 \tag{38}
\end{equation*}
$$

Theorem 5. Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ hold. If $\mathscr{R}_{0}>1$, then endemic equilibrium $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ of model (1) is globally asymptotically stable.

Proof. We firstly define the auxiliary functions as follows:

$$
\begin{gather*}
V_{1}(S)=S-S^{*}-\int_{S^{*}}^{S} \frac{F\left(S^{*}\right)}{F(\eta)} d \eta \\
V_{2}(I)=I-I^{*}-\int_{I^{*}}^{I} \frac{G\left(I^{*}\right)}{G(\eta)} d \eta  \tag{39}\\
V_{3}(R)=\frac{1}{2}\left(R-R^{*}\right)^{2} \\
V_{4}(N, R)=\frac{1}{2}\left(N-N^{*}+\frac{\alpha}{k}\left(R-R^{*}\right)\right)^{2},
\end{gather*}
$$

where $N=S+I+R$ and $N^{*}=S^{*}+I^{*}+R^{*}$. From $\left(H_{1}\right)$, we easily obtain that when $S \neq S^{*}$

$$
\begin{equation*}
V_{1}(S)>S-S^{*}-\int_{S^{*}}^{S} \frac{F\left(S^{*}\right)}{F\left(S^{*}\right)} d \eta=0 \tag{40}
\end{equation*}
$$

and when $I \neq I^{*}$

$$
\begin{equation*}
V_{2}(I)>I-I^{*}-\int_{I^{*}}^{I} \frac{G\left(I^{*}\right)}{G\left(I^{*}\right)} d \eta=0 . \tag{41}
\end{equation*}
$$

Let $(S(n), I(n), R(n))$ be any positive solution of model (1) with initial condition (2). By computing $\Delta V_{1}(n)=V_{1}(S(n+$ 1)) $-V_{1}(S(n))$, we have

$$
\begin{equation*}
\Delta V_{1}(n)=S(n+1)-S(n)-\int_{S(n)}^{S(n+1)} \frac{F\left(S^{*}\right)}{F(\eta)} d \eta \tag{42}
\end{equation*}
$$

From $\left(H_{1}\right)$, it follows that

$$
\begin{align*}
& -\frac{F\left(S^{*}\right)}{F(\eta)} \leq-\frac{F\left(S^{*}\right)}{F(S(n+1))} \quad \text { if } S(n+1) \geq \eta \geq S(n) \\
& -\frac{F\left(S^{*}\right)}{F(\eta)} \geq-\frac{F\left(S^{*}\right)}{F(S(n+1))} \quad \text { if } S(n+1) \leq \eta \leq S(n) \tag{43}
\end{align*}
$$

Hence,

$$
\begin{aligned}
&-\int_{S(n)}^{S(n+1)} \frac{F\left(S^{*}\right)}{F(\eta)} d \eta \leq-\frac{F\left(S^{*}\right)}{F(S(n+1))}(S(n+1)-S(n)), \\
& \Delta V_{1}(n) \leq S(n+1)-S(n)-\frac{F\left(S^{*}\right)}{F(S(n+1))} \\
& \times(S(n+1)-S(n)) \\
&=\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right) \\
& \times(\Lambda-F(S(n+1)) G(I(n+1)) \\
&-\mu S(n+1)+\gamma R(n+1)) \\
&=\left.1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right) \\
& \times\left(\mu S^{*}-\mu S(n+1)+F\left(S^{*}\right) G\left(I^{*}\right)\right. \\
&-F(S(n+1)) G(I(n+1)) \\
&\left.+\gamma R(n+1)-\gamma R^{*}\right) \\
&=-\mu\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right)\left(S(n+1)-S^{*}\right) \\
&+(k+\mu+\alpha) I^{*} \\
& \times\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right)
\end{aligned}
$$

$$
\begin{align*}
& \times\left(1-\frac{F(S(n+1)) G(I(n+1))}{F\left(S^{*}\right) G\left(I^{*}\right)}\right) \\
& +\gamma\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right)\left(R(n+1)-R^{*}\right) \\
& =-\mu\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right)\left(S(n+1)-S^{*}\right) \\
& +\gamma\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right)\left(R(n+1)-R^{*}\right) \\
& +(k+\mu+\alpha) I^{*} \\
& \times\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right. \\
& \quad-\frac{F(S(n+1)) G(I(n+1))}{F\left(S^{*}\right) G\left(I^{*}\right)} \\
& \left.\quad+\frac{G(I(n+1))}{G\left(I^{*}\right)}\right) \tag{44}
\end{align*}
$$

By computing $\Delta V_{2}(n)=V_{2}(I(n+1))-V_{2}(I(n))$, we also have

$$
\begin{aligned}
\Delta V_{2}(n) \leq & I(n+1)-I(n)-\frac{G\left(I^{*}\right)}{G(n+1)} \\
& \times(I(n+1)-I(n)) \\
= & \left(1-\frac{G\left(I^{*}\right)}{G(I(n+1))}\right) \\
& \times(F(S(n+1)) G(I(n+1)) \\
& -(k+\mu+\alpha) I(n+1)) \\
= & (k+\mu+\alpha) I^{*}\left(1-\frac{G\left(I^{*}\right)}{G(I(n+1))}\right) \\
\times & \left(\frac{F(S(n+1)) G(I(n+1))}{F\left(S^{*}\right) G\left(I^{*}\right)}-\frac{I(n+1)}{I^{*}}\right) \\
= & (k+\mu+\alpha) I^{*} \\
& \times\left(\frac{F(S(n+1)) G(I(n+1))}{F\left(S^{*}\right) G\left(I^{*}\right)}\right. \\
& \quad-\frac{F(S(n+1))}{F\left(S^{*}\right)}-\frac{I(n+1)}{I^{*}} \\
& \left.+\frac{I(n+1)}{I^{*}} \frac{G\left(I^{*}\right)}{G(I(n+1))}\right) .
\end{aligned}
$$

Further, by computing $\Delta V_{3}(n)=V_{3}(R(n+1))-V_{3}(R(n))$, we have

$$
\begin{align*}
\Delta V_{3}(n)= & \frac{1}{2}\left((R(n+1)-R(n))^{2}\right. \\
& +2(R(n+1)-R(n))\left(R(n)-R^{*}\right) \\
& \left.+\left(R(n)-R^{*}\right)^{2}\right) \\
& -\frac{1}{2}\left(R(n)-R^{*}\right)^{2} \\
= & \frac{1}{2}(R(n+1)-R(n))\left(R(n+1)+R(n)-2 R^{*}\right) \\
= & (R(n+1)-R(n))\left(R(n+1)-R^{*}\right) \\
& -\frac{1}{2}(R(n+1)-R(n))^{2} \\
\leq & \left(R(n+1)-R^{*}\right)(R(n+1)-R(n)) \\
= & \left(R(n+1)-R^{*}\right)(k I(n+1)-(\mu+\gamma) R(n+1)) \\
= & \left(R(n+1)-R^{*}\right) \\
\times & \left(k\left(N(n+1)-N^{*}\right)\right. \\
& -k\left(S(n+1)-S^{*}\right) \\
& \left.-(k+\mu+\gamma)\left(R(n+1)-R^{*}\right)\right) \\
= & k\left(R(n+1)-R^{*}\right)\left(N(n+1)-N^{*}\right) \\
& -k\left(R(n+1)-R^{*}\right) \\
\times & \left(S(n+1)-S^{*}\right) \\
\times & (k+\mu+\gamma)\left(R(n+1)-R^{*}\right)^{2} . \tag{46}
\end{align*}
$$

Finally, by computing $\Delta V_{4}(n)=V_{4}(N(n+1), R(n+1))-$ $V_{4}(N(n), R(n))$, we further have

$$
\begin{aligned}
\Delta V_{4}(n)=\frac{1}{2} & {\left[\left(N(n+1)-N(n)+\frac{\alpha}{k}(R(n+1)-R(n))\right)^{2}\right.} \\
& +2\left(N(n+1)-N(n)+\frac{\alpha}{k}(R(n+1)-R(n))\right) \\
& \left.\times\left(N(n)-N^{*}+\frac{\alpha}{k}\left(R(n)-R^{*}\right)\right)\right] \\
= & \frac{1}{2}\left(N(n+1)-N(n)+\frac{\alpha}{k}(R(n+1)-R(n))\right) \\
\times & \left(N(n+1)+N(n)+\frac{\alpha}{k}(R(n+1)+R(n))\right. \\
& \left.-2\left(N^{*}-\frac{\alpha}{k} R^{*}\right)\right)
\end{aligned}
$$

$$
\begin{align*}
& =\left(N(n+1)-N(n)+\frac{\alpha}{k}(R(n+1)-R(n))\right) \\
& \times\left(N(n+1)-N^{*}+\frac{\alpha}{k}\left(R(n+1)-R^{*}\right)\right) \\
& -\frac{1}{2}\left(N(n+1)-N^{*}+\frac{\alpha}{k}\left(R(n+1)-R^{*}\right)\right)^{2} \\
& \leq\left(\left(N(n+1)-N^{*}\right)+\frac{\alpha}{k}\left(R(n+1)-R^{*}\right)\right) \\
& \times\left((N(n+1)-N(n))+\frac{\alpha}{k}(R(n+1)-R(n))\right) \\
& =\left(\left(N(n+1)-N^{*}\right)+\frac{\alpha}{k}\left(R(n+1)-R^{*}\right)\right) \\
& \times((\Lambda-\mu N(n+1)-\alpha I(n+1)) \\
& \left.+\frac{\alpha}{k}(k I(n+1)-(\mu+\gamma) R(n+1))\right) \\
& =\left(\left(N(n+1)-N^{*}\right)+\frac{\alpha}{k}\left(R(n+1)-R^{*}\right)\right) \\
& \times\left(\mu N^{*}-\mu N(n+1)\right. \\
& \left.-\frac{\alpha(\mu+\gamma)}{k}\left(R(n+1)-R^{*}\right)\right) \\
& =-\mu\left(N(n+1)-N^{*}\right)^{2}-\frac{\alpha^{2}}{k^{2}}(\mu+\gamma) \\
& \times\left(R(n+1)-R^{*}\right)^{2} \\
& -\frac{\alpha(2 \mu+\gamma)}{k}\left(N(n+1)-N^{*}\right)\left(R(n+1)-R^{*}\right) . \tag{47}
\end{align*}
$$

Now, we define a Lyapunov function as follows:

$$
\begin{equation*}
V(S, I, R)=V_{1}(S)+V_{2}(I)+\omega_{1} V_{3}(R)+\omega_{2} V_{4}(N, R), \tag{48}
\end{equation*}
$$

where $\omega_{1}$ and $\omega_{2}$ are positive constants which will be chosen in the following. It is obvious that from (40) and (41) $V(S, I, R)>0$ for all $(S, I, R) \neq\left(S^{*}, I^{*}, R^{*}\right)$ and $V(S, I, R)=0$ if and only if $(S, I, R)=\left(S^{*}, I^{*}, R^{*}\right)$. By computing

$$
\begin{align*}
\Delta V(n)= & V(S(n+1), I(n+1), R(n+1)) \\
& -V(S(n),(n), R(n)), \tag{49}
\end{align*}
$$

we have

$$
\begin{align*}
\Delta V(n) \leq & -\mu \omega_{2}\left(N(n+1)-N^{*}\right)^{2} \\
& -\left(\frac{\omega_{2} \alpha^{2}}{k^{2}}(\mu+\gamma)+\omega_{1}(k+\mu+\gamma)\right) \\
& \times\left(R(n+1)-R^{*}\right)^{2} \\
& -\mu\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right)\left(S(n+1)-S^{*}\right) \\
& +\gamma\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right)\left(R(n+1)-R^{*}\right) \\
- & \omega_{1} k\left(R(n+1)-R^{*}\right)\left(S(n+1)-S^{*}\right) \\
& +(k+\mu+\alpha) I^{*}  \tag{50}\\
\times & \left(\left(2-\frac{F\left(S^{*}\right)}{F(S(n+1))}-\frac{F(S(n+1))}{F\left(S^{*}\right)}\right)\right. \\
& +\left(\frac{G(I(n+1))}{G\left(I^{*}\right)}-\frac{I(n+1)}{I^{*}}\right. \\
& \left.\left.+\frac{I(n+1)}{I^{*}} \frac{G\left(I^{*}\right)}{G(I(n+1))}-1\right)\right) \\
+ & \left(\omega_{1} k-\frac{\omega_{2} \alpha(2 \mu+\gamma)}{k}\right) \\
\times & \left(N(n+1)-N^{*}\right)\left(R(n+1)-R^{*}\right)
\end{align*}
$$

Choose constants $\omega_{1}$ and $\omega_{2}$ as follows:

$$
\begin{equation*}
\omega_{1}=\frac{\gamma F\left(S^{*}\right)}{k\left(S^{*}\right)^{2}}, \quad \omega_{2}=\frac{k \gamma F\left(S^{*}\right)}{\alpha(2 \mu+\gamma)\left(S^{*}\right)^{2}} \tag{51}
\end{equation*}
$$

Then we further have

$$
\begin{aligned}
\Delta V(n) \leq & -\frac{\mu k^{2} \gamma F\left(S^{*}\right)}{\alpha(2 \mu+\gamma)\left(S^{*}\right)^{2}}\left(N(n+1)-N^{*}\right)^{2} \\
& -\frac{\gamma F\left(S^{*}\right)}{\left(S^{*}\right)^{2}}\left(\frac{\alpha(\mu+\gamma)}{k(2 \mu+\gamma)}+\frac{\mu+\gamma}{k}+1\right) \\
& \times\left(R(n+1)-R^{*}\right)^{2} \\
& +\frac{F\left(S^{*}\right)}{\left(S^{*}\right)^{2}}\left(S(n+1)-S^{*}\right)\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right)
\end{aligned}
$$

$$
\begin{align*}
& \times\left(-\frac{\mu\left(S^{*}\right)^{2}}{F\left(S^{*}\right)}+\frac{\gamma\left(S^{*}\right)^{2}}{F\left(S^{*}\right)} \frac{R(n+1)-R^{*}}{S(n+1)-S^{*}}\right. \\
& \left.-\gamma\left(R(n+1)-R^{*}\right) \frac{F(S(n+1))}{F(S(n+1))-F\left(S^{*}\right)}\right) \\
& +(k+\mu+\alpha) I^{*} \\
& \times\left(\left(2-\frac{F\left(S^{*}\right)}{F(S(n+1))}-\frac{F(S(n+1))}{F\left(S^{*}\right)}\right)\right. \\
& +\left(\frac{G\left(I^{*}\right)}{G(I(n+1))}-1\right) \\
& \left.\times\left(\frac{I(n+1)}{I^{*}}-\frac{G(I(n+1))}{G\left(I^{*}\right)}\right)\right) \\
& =-\frac{\mu k^{2} \gamma F\left(S^{*}\right)}{\alpha(2 \mu+\gamma)\left(S^{*}\right)^{2}}\left(N(n+1)-N^{*}\right)^{2} \\
& -\frac{\gamma F\left(S^{*}\right)}{\left(S^{*}\right)^{2}}\left(\frac{\alpha(\mu+\gamma)}{k(2 \mu+\gamma)}+\frac{\mu+\gamma}{k}+1\right) \\
& \times\left(R(n+1)-R^{*}\right)^{2} \\
& +\frac{F\left(S^{*}\right)}{\left(S^{*}\right)^{2}}\left(S(n+1)-S^{*}\right)\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right) \\
& \times\left(-\frac{\mu F\left(S^{*}\right)}{\left(S^{*}\right)^{2}}-\gamma\left(R(n+1)-R^{*}\right)\right. \\
& \times\left(\frac{F(S(n+1))}{F(S(n+1))-F\left(S^{*}\right)}\right. \\
& \left.\left.-\frac{\left(S^{*}\right)^{2}}{F\left(S^{*}\right)\left(S(n+1)-S^{*}\right)}\right)\right) \\
& +(k+\mu+\alpha) I^{*} \\
& \times\left(\left(2-\frac{F\left(S^{*}\right)}{F(S(n+1))}-\frac{F(S(n+1))}{F\left(S^{*}\right)}\right)\right. \\
& +\left(\frac{G\left(I^{*}\right)}{G(I(n+1))}-1\right) \\
& \left.\times\left(\frac{I(n+1)}{I^{*}}-\frac{G(I(n+1))}{G\left(I^{*}\right)}\right)\right) . \tag{52}
\end{align*}
$$

Noting that $F(S)>0$, for all $S>0$, then we have

$$
\begin{equation*}
2-\frac{F\left(S^{*}\right)}{F(S(n+1))}-\frac{F(S(n+1))}{F\left(S^{*}\right)} \leq 0 . \tag{53}
\end{equation*}
$$

From $\left(H_{1}\right)$, it follows that $F(S(n+1)) \geq F\left(S^{*}\right)$ when $S(n+1) \geq$ $S^{*}$ and $F(S(n+1)) \leq F\left(S^{*}\right)$ when $S(n+1) \leq S^{*}$. Hence, we have the following inequality:

$$
\begin{equation*}
\left(S(n+1)-S^{*}\right)\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right) \geq 0 \tag{54}
\end{equation*}
$$

Furthermore, from $\left(H_{1}\right)$, we also have the following inequalities:

$$
\begin{align*}
& \frac{G(I(n+1))}{G\left(I^{*}\right)} \geq \frac{I(n+1)}{I^{*}} \quad \text { if } 0<I(n+1) \leq I^{*}, \\
& \frac{G(I(n+1))}{G\left(I^{*}\right)} \leq \frac{I(n+1)}{I^{*}} \quad \text { if } I(n+1) \geq I^{*}, \tag{55}
\end{align*}
$$

which implies that

$$
\begin{equation*}
\left(\frac{G\left(I^{*}\right)}{G(I(n+1))}-1\right)\left(\frac{I(n+1)}{I^{*}}-\frac{G(I(n+1))}{G\left(I^{*}\right)}\right) \leq 0 . \tag{56}
\end{equation*}
$$

From (53), (54), and (56), we further obtain

$$
\begin{align*}
\Delta V(n) \leq & -\frac{\mu k^{2} \gamma F\left(S^{*}\right)}{\alpha(2 \mu+\gamma)\left(S^{*}\right)^{2}}\left(N(n+1)-N^{*}\right)^{2} \\
& -\frac{\gamma F\left(S^{*}\right)}{\left(S^{*}\right)^{2}}\left(\frac{\alpha(\mu+\gamma)}{k(2 \mu+\gamma)}+\frac{\mu+\gamma}{k}+1\right) \\
& \times\left(R(n+1)-R^{*}\right)^{2} \\
+ & \frac{F\left(S^{*}\right)}{\left(S^{*}\right)^{2}}\left(S(n+1)-S^{*}\right)\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right) \\
\times & {\left[-\frac{\mu F\left(S^{*}\right)}{\left(S^{*}\right)^{2}}+\gamma\left(R^{*}-R(n+1)\right)\right.} \\
& \times\left(\frac{F(S(n+1))}{F(S(n+1))-F\left(S^{*}\right)}\right. \\
& \left.\left.\quad-\frac{\left(S^{*}\right)^{2}}{F\left(S^{*}\right)\left(S(n+1)-S^{*}\right)}\right)\right] . \tag{57}
\end{align*}
$$

From $\left(H_{2}\right)$ and $\left(H_{3}\right)$, we finally have $\Delta V(n) \leq 0$ for all $n \geq 0$. Obviously, $\Delta V(n)=0$ if and only if $S(n)=S^{*}, I(n)=I^{*}$, and $R(n)=R^{*}$ for all $n \geq 0$. Therefore, using the theorems of stability of the difference equations (see Theorem 6.3 in [33]), we obtain that $E^{*}$ is globally asymptotically stable. This completes the proof.

As a special case of model (1), we consider the rate of losing immunity $\gamma=0$ in model (1); that is, model (1) degenerates into a SIR epidemic model. Then, in the above
calculation of $\Delta V(n)$, we can directly obtain the following inequality without $\left(\mathrm{H}_{2}\right)$ and $\left(\mathrm{H}_{3}\right)$ :

$$
\begin{align*}
\Delta V(n) \leq & -\mu\left(S(n+1)-S^{*}\right)\left(1-\frac{F\left(S^{*}\right)}{F(S(n+1))}\right) \\
& +(k+\mu+\alpha) I^{*} \\
& \times\left(\left(2-\frac{F\left(S^{*}\right)}{F(S(n+1))}-\frac{F(S(n+1))}{F\left(S^{*}\right)}\right)\right.  \tag{58}\\
& +\left(\frac{G\left(I^{*}\right)}{G(I(n+1))}-1\right) \\
& \left.\times\left(\frac{I(n+1)}{I^{*}}-\frac{G(I(n+1))}{G\left(I^{*}\right)}\right)\right) .
\end{align*}
$$

We have $\Delta V(n) \leq 0$ for all $n \geq 0$ and $\Delta V(n)=0$ if and only if $S(n)=S^{*}$ and $I(n)=I^{*}$ for all $n \geq 0$. Therefore, as a consequence of Theorem 5 , we have the following result.

Corollary 6. Assume that $\left(H_{1}\right)$ holds and $\gamma=0$ in model (1). Then endemic equilibrium $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ is globally asymptotically stable if and only if $\mathscr{R}_{0}>1$.

Remark 7. By comparing the results obtained in [24], then, from Corollary 6, we see that Theorem 5 is a direct extension of the corresponding result given in [24] on the global stability of the endemic equilibrium in the nondelayed case and the recovered individuals are in a position to lose the immunity.

Remark 8. For general model (1), we spontaneously expect that as long as basic reproduction number $\mathscr{R}_{0}>1$, then model (1) shows a unique endemic equilibrium which is globally asymptotically stable. However, it is a pity that, in Theorem 5, in order to obtain the global asymptotic stability of endemic equilibrium $E^{*}$, we need to introduce some additional conditions, that is, $\left(H_{2}\right)$ and $\left(H_{3}\right)$. Furthermore, from the proof of Theorem 5 , we easily see that assumptions $\left(H_{2}\right)$ and $\left(H_{3}\right)$ only are used to ensure $\Delta V(n) \leq 0$ for all $n \geq 0$. Therefore, an important open problem is whether we can directly prove $\Delta V(n) \leq 0$ for all $n \geq 0$ without assumptions $\left(\mathrm{H}_{2}\right)$ and $\left(\mathrm{H}_{3}\right)$ and further obtain the global asymptotic stability of endemic equilibrium $E^{*}$ of model (1) only when basic reproduction number $\mathscr{R}_{0}>1$.

## 4. Permanence of Disease

In this section, we will use the theory of persistence in general discrete dynamical systems to study the permanence of model (1). We will obtain that the disease in model (1) is permanent only when basic reproduction number $\mathscr{R}_{0}>1$ and assumption $\left(H_{1}\right)$ holds.

Let $X$ be a metric space with metric $d$ and let $f: X \rightarrow$ $X$ be a continuous map. For any $x_{0} \in X$, the sequence $\left\{x_{n}\right\}$ defined by $x_{n+1}=f\left(x_{n}\right)$ for any integer $n \geq 0$ is said to be a solution sequence through $x_{0}$, and the omega limit set of $\left\{x_{n}\right\}$ is defined by $\omega\left(x_{0}\right)=\left\{y \in X\right.$ : there is a sequence $n_{k} \rightarrow \infty$ such that $\left.\lim _{k \rightarrow \infty} x_{n_{k}}=y\right\}$. For a nonempty set $M \subset X$, we
further define the stable set of $M$ by $W^{s}(M)=\left\{x_{0} \in X\right.$ : $\left.\lim _{n \rightarrow \infty} d\left(x_{n}, M\right)=0\right\}$.

Let $X_{0}$ be a nonempty open set of $X$. We denote

$$
\begin{equation*}
\partial X_{0}:=X \backslash X_{0}, \quad M_{\partial}:=\left\{x_{0} \in \partial X_{0}: x_{n} \in \partial X_{0} \forall n \geq 0\right\} \tag{59}
\end{equation*}
$$

Lemma 9. Let $f: X \rightarrow X$ be a continuous map. Assume that the following conditions hold.
$\left(C_{1}\right) f$ is compact and point dissipative, and $f\left(X_{0}\right) \subseteq X_{0}$.
$\left(C_{2}\right)$ There exists a finite sequence $\mathscr{M}=\left\{M_{1}, \ldots, M_{k}\right\}$ of compact and isolated invariant sets such that
(a) $M_{i} \cap M_{j}=\emptyset$ for any $i, j=1,2, \ldots, k$ and $i \neq j$;
(b) $\Omega\left(M_{\partial}\right):=\bigcup_{x \in M_{\partial}} \omega(x) \subset \bigcup_{i=1}^{k} M_{i}$;
(c) no subset of $\mathscr{M}$ forms a cycle in $\partial X_{0}$;
(d) $W^{s}\left(M_{i}\right) \bigcap X_{0}=\emptyset$ for each $1 \leq i \leq k$.

Then $f$ is uniformly persistent with respect to $\left(X_{0}, \partial X_{0}\right)$ ); that is, there exists a constant $\eta>0$ such that $\liminf _{n \rightarrow \infty} d\left(x_{n}, \partial X_{0}\right) \geq \eta$ for all $x_{0} \in X_{0}$.

Here, the definitions on the compactness and point dissipativity of map $f$ and the definitions on the compactness, isolated invariance, and the cycle in $\partial X_{0}$ for sequence $\mathscr{M}=$ $\left\{M_{1}, \ldots, M_{k}\right\}$ can be found in [34]. Furthermore, Lemma 9 can be obtained from Theorem 1.1.3, Theorem 1.3.1, Remark 1.3.1, and Theorem 1.3.3 given by Zhao in [34].

On the permanence of the disease for model (1), we have the following result.

Theorem 10. Assume that $\left(H_{1}\right)$ holds. Then, the disease in model (1) is permanent; that is, there are two constants $M>$ $m>0$ such that

$$
\begin{equation*}
m \leq \liminf _{n \rightarrow \infty} I(n) \leq \limsup _{n \rightarrow \infty} I(n) \leq M \tag{60}
\end{equation*}
$$

for any positive solution $(S(n), I(n), R(n))$ of model (1) if and only if $\mathscr{R}_{0}>1$.

Proof. From Theorem 4 we see that the necessity is obvious. Now, we only need to prove the sufficiency. Define two sets as follows:

$$
\begin{align*}
& X=\left\{(S, I, R) \in R^{3}: S>0, I \geq 0, R \geq 0\right\},  \tag{61}\\
& X_{0}=\{(S, I, R) \in X: S>0, I>0, R \geq 0\}
\end{align*}
$$

We have

$$
\begin{equation*}
\partial X_{0}=X \backslash X_{0}=\{(S, I, R): S>0, I=0, R \geq 0\} \tag{62}
\end{equation*}
$$

For any initial point $x_{0}=\left(S_{0}, I_{0}, R_{0}\right) \in X$, let $x_{n}=$ ( $S(n), I(n), R(n)$ ) be the solution of model (1) through $x_{0}$. We define map $f: X \rightarrow X$ by $f\left(x_{0}\right)=x_{1}$.

From the positivity and ultimate boundedness of solutions of model (1), we obtain $f\left(X_{0}\right) \subseteq X_{0}$ and $f$ is also point dissipative.

By observing the proof of Theorem 3, we see that, since $N(1)$ and $R(1)$ are continuous with respect to $N(0)$ and $R(0)$, respectively, $\Psi(y)$ is also continuous with respect to $x_{0}=$ $\left(S_{0}, I_{0}, R_{0}\right)$. Hence, $I(1)$, as the solution of $\Psi(y)=0$, is also continuous for $x_{0}$. Similarly, from the expression of $\Phi(x)$ and the continuity of $I(1)$ with respect to $x_{0}$, we obtain that $\Phi(x)$ is continuous with respect to $x_{0}$. Hence, $S(1)$, as the solution of $\Phi(x)=0$, is also continuous for $x_{0}$. Therefore, we finally obtain that map $f$ is continuous on $X$. From this, we obtain that $f$ also is compact.

In $\partial X_{0}$, we have $I(n) \equiv 0$, and hence $(S(n), R(n))$ satisfies

$$
\begin{align*}
& S(n+1)=S(n)+\Lambda-\mu S(n+1)+\gamma R(n+1), \\
& R(n+1)=R(n)-(\mu+\gamma) R(n+1) . \tag{63}
\end{align*}
$$

Obviously, we can obtain $(S(n), R(n)) \rightarrow(\Lambda / \mu, 0)$ as $n \rightarrow$ $\infty$. This shows that $\omega\left(x_{0}\right)=\left\{E^{0}\right\}$ for any $x_{0} \in M_{\partial}$ and $\Omega\left(M_{\partial}\right)=\bigcup_{x \in M_{\partial}} \omega(x)=\left\{E^{*}\right\}$. Choose $\mathscr{M}=\left\{E^{0}\right\}$; then we easily see that conditions (a)-(c) of Lemma 9 hold.

Now, we prove that condition (d) in Lemma 9 also holds. Otherwise, there is a point $\left(S_{0}, I_{0}, R_{0}\right) \in X_{0}$ such that $(S(n), I(n), R(n)) \rightarrow E^{0}$ as $n \rightarrow \infty$. From $\mathscr{R}_{0}>1$, we can choose a small enough constant $\varepsilon>0$ such that

$$
\begin{equation*}
\left(F\left(\frac{\Lambda}{\mu}\right)-\varepsilon\right)\left(G^{\prime}(0)-\varepsilon\right)-(k+\mu+\alpha)>0 \tag{64}
\end{equation*}
$$

Since $\lim _{n \rightarrow \infty} S(n)=\Lambda / \mu$ and $\lim _{n \rightarrow \infty}(G(I(n)) / I(n))=$ $G^{\prime}(0)$, there exists $N>0$ such that $F(S(n+1))>F(\Lambda / \mu)-\varepsilon$ and $G(I(n+1)) / I(n+1)>G^{\prime}(0)-\varepsilon$ for all $n>N$. Therefore, we have

$$
\begin{align*}
& I(n+1) \\
& \quad=I(n)+F(S(n+1)) G(I(n+1))-(k+\mu+\alpha) I(n+1) \\
& \geq \\
& \quad I(n)+\left[\left(F\left(\frac{\Lambda}{\mu}\right)-\varepsilon\right)\left(G^{\prime}(0)-\varepsilon\right)-(k+\mu+\alpha)\right]  \tag{65}\\
& \quad \times I(n+1),
\end{align*}
$$

for all $n>N$. Consequently,

$$
\begin{align*}
& I(n+1)\left[1+k+\mu+\alpha-\left(F\left(\frac{\Lambda}{\mu}\right)-\varepsilon\right)\left(G^{\prime}(0)-\varepsilon\right)\right]  \tag{66}\\
& \quad \geq I(n)
\end{align*}
$$

for all $n>N$. Since $0 \leq 1+k+\mu+\alpha-(F(\Lambda / \mu)-\varepsilon)\left(G^{\prime}(0)-\varepsilon\right)<1$, we can finally obtain from (66) that $\lim _{n \rightarrow \infty} I(n)=\infty$, which leads to a contradiction. Therefore, condition (d) in Lemma 9 holds. Finally, from Lemma 9 we obtain that the disease in model (1) is permanent. This completes the proof.

Remark 11. From Theorem 10, we directly see that assumptions $\left(H_{2}\right)$ and $\left(H_{3}\right)$ only are used to obtain the global asymptotic stability of endemic equilibrium $E^{*}$.

## 5. Special Case $F(S)=S /(1+\lambda S)$

Now, we especially discuss the special case of model (1): $F(S)=S /(1+\lambda S)$, where $\lambda \geq 0$ is a constant. Firstly, when $F(S)=S /(1+\lambda S)$, the basic reproduction number of model (1) becomes

$$
\begin{equation*}
\mathscr{R}_{0}=\frac{F(\Lambda / \mu) G^{\prime}(0)}{k+\mu+\alpha}=\frac{\Lambda G^{\prime}(0)}{(\mu+\lambda \Lambda)(k+\mu+\alpha)} . \tag{67}
\end{equation*}
$$

Furthermore, by calculating, we obtain that $\left(H_{2}\right)$ naturally holds, and assumption $\left(\mathrm{H}_{3}\right)$ is equivalent to the following simple form:

$$
\begin{equation*}
\gamma R^{*}-\mu S^{*} \leq 0 \tag{68}
\end{equation*}
$$

Therefore, as a direct consequence of Theorem 5, we firstly have the following corollary.

Corollary 12. Assume that $\left(H_{1}\right)$ holds and $F(S)=S /(1+\lambda S)$, where $\lambda \geq 0$ is a constant. If $\mathscr{R}_{0}>1$ and inequality (68) holds, then endemic equilibrium $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ of model (1) is globally asymptotically stable.

Furthermore, in order to validate inequality (68), we have the following result.

Theorem 13. Assume that $\left(H_{1}\right)$ holds and $F(S)=S /(1+\lambda S)$ with $\lambda \geq 0$ is a constant. Then inequality (68) holds if one of the following conditions holds:
(1) $1<\mathscr{R}_{0} \leq(\mu(k+\mu+\alpha)+\Lambda k \lambda) / k(\mu+\lambda \Lambda)$,
(2) $\mathscr{R}_{0}>(\mu(k+\mu+\alpha)+\Lambda k \lambda) / k(\mu+\lambda \Lambda)$ and $0 \leq \gamma \leq$ $\bar{\gamma}:=\mu / L$, where

$$
\begin{equation*}
L=\frac{\mathscr{R}_{0}(\mu+\lambda \Lambda) k}{\mu(k+\mu+\alpha)}-\frac{\mu(k+\mu+\alpha)+\Lambda k \lambda}{\mu(k+\mu+\alpha)} . \tag{69}
\end{equation*}
$$

Proof. When $F(S) G(I)=(S /(1+\lambda S)) G(I)$, endemic equilibrium $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ of model (1) satisfies

$$
\begin{align*}
& \Lambda=\frac{S^{*}}{1+\lambda S^{*}} G\left(I^{*}\right)+\mu S^{*}-\gamma R^{*} \\
& I^{*}(k+\mu+\alpha)=\frac{S^{*}}{1+\lambda S^{*}} G\left(I^{*}\right)  \tag{70}\\
& k I^{*}=(\mu+\gamma) R^{*}
\end{align*}
$$

From the second and third equations of (70), we obtain

$$
\begin{equation*}
S^{*}=\frac{I^{*}(k+\mu+\alpha)}{G\left(I^{*}\right)-\lambda(k+\mu+\alpha) I^{*}}, \quad R^{*}=\frac{k I^{*}}{\mu+\gamma} \tag{71}
\end{equation*}
$$

Putting (71) into the first equation of (70), we have

$$
\begin{align*}
I^{*}= & \left(\Lambda-\frac{\mu(k+\mu+\alpha) I^{*}}{G\left(I^{*}\right)-\lambda(k+\mu+\alpha) I^{*}}\right)  \tag{72}\\
& \times \frac{\mu+\gamma}{\mu(k+\mu+\alpha)+\gamma(\mu+\alpha)}
\end{align*}
$$

Hence, from (71) and (72), we obtain

$$
\begin{align*}
& \mu S^{*}-\gamma R^{*}=\frac{\mu(k+\mu+\alpha) I^{*}}{G\left(I^{*}\right)-\lambda(k+\mu+\alpha) I^{*}}-\gamma R^{*} \\
& =\frac{\mu(k+\mu+\alpha)(\mu+\gamma) R^{*}}{\left(G\left(I^{*}\right)-\lambda(k+\mu+\alpha) I^{*}\right) k}-\gamma R^{*} \\
& =\frac{R^{*}}{k\left(G\left(I^{*}\right)-\lambda(k+\mu+\alpha) I^{*}\right)} \\
& \times[\mu(k+\mu+\alpha)(\mu+\gamma) \\
& \left.-k \gamma\left(\frac{G\left(I^{*}\right)}{I^{*}}-\lambda(k+\mu+\alpha)\right) I^{*}\right] \\
& =\frac{R^{*}}{k\left(G\left(I^{*}\right)-\lambda(k+\mu+\alpha) I^{*}\right)} \\
& \times[\mu(k+\mu+\alpha)(\mu+\gamma) \\
& -k \gamma\left(\frac{G\left(I^{*}\right)}{I^{*}}-\lambda(k+\mu+\alpha)\right)  \tag{73}\\
& \times\left(\Lambda-\frac{\mu(k+\mu+\alpha) I^{*}}{G\left(I^{*}\right)-\lambda(k+\mu+\alpha) I^{*}}\right) \\
& \left.\times \frac{\mu+\gamma}{\mu(k+\mu+\alpha)+\gamma(\mu+\alpha)}\right] \\
& =\frac{R^{*}}{k\left(G\left(I^{*}\right)-\lambda(k+\mu+\alpha) I^{*}\right)} \\
& \times[\mu(k+\mu+\alpha)(\mu+\gamma) \\
& -\frac{\mu+\gamma}{\mu(k+\mu+\alpha)+\gamma(\mu+\alpha)} \\
& \times\left(\Lambda \gamma k\left(\frac{G\left(I^{*}\right)}{I^{*}}-\lambda(k+\mu+\alpha)\right)\right. \\
& -k \gamma \mu(k+\mu+\alpha))] .
\end{align*}
$$

Since $G(I) / I$ is nonincreasing for all $I>0$ in $\left(H_{1}\right)$, we have

$$
\begin{equation*}
\frac{G\left(I^{*}\right)}{I^{*}}-\lambda(k+\mu+\alpha) \leq G^{\prime}(0)-\lambda(k+\mu+\alpha) \tag{74}
\end{equation*}
$$

Therefore, from (73), we further have

$$
\begin{align*}
& \mu S^{*}-\gamma R^{*} \geq \frac{R^{*}}{k\left(G\left(I^{*}\right)-\lambda(k+\mu+\alpha) I^{*}\right)} \\
& \times(\mu(k+\mu+\alpha)(\mu+\gamma) \\
&-\frac{\mu+\gamma}{\mu(k+\mu+\alpha)+\gamma(\mu+\alpha)} \\
& \times\left(\Lambda \gamma k\left(G^{\prime}(0)-\lambda(k+\mu+\alpha)\right)\right. \\
&= \frac{-k \gamma \mu(k+\mu+\alpha)))}{k\left(G^{\prime}\left(I^{*}\right)-\lambda(k+\mu+\alpha) I^{*}\right)} \\
& \times \frac{\mu(\mu+\gamma)(k+\mu+\alpha)^{2}}{\mu(k+\mu+\alpha)+\gamma(\mu+\alpha)} \\
& \times {\left[\frac{1}{k+\mu+\alpha}(\mu(k+\mu+\alpha)+\gamma(\mu+\alpha))\right.} \\
&+\frac{k \gamma}{k+\mu+\alpha} \\
& \times \frac{\mu(\mu+\gamma)(k+\mu+\alpha)^{2}}{\mu(k+\mu+\alpha)+\gamma(\mu+\alpha)} \\
&-\frac{\Lambda k \gamma\left(G^{\prime}(0)-\lambda(k+\mu+\alpha)\right)}{k\left(G^{\prime}\left(I^{*}\right)-\lambda(k+\mu+\alpha) I^{*}\right)} \\
& \frac{\mu(k+\mu+\alpha)^{2}}{k\left(G^{\prime}\left(I^{*}\right)-\lambda(k+\mu+\alpha) I^{*}\right)} \\
& \times \frac{\mu(\mu+\gamma)(k+\mu+\alpha)^{2}}{\mu(k+\mu+\alpha)+\gamma(\mu+\alpha)} \\
& \times\left(\mu+\gamma \frac{\mu(\mu+k+\alpha)+\Lambda k \lambda}{\mu(k+\mu+\alpha)}\right. \\
& \times
\end{align*}
$$

From (75), we obtain that, when the conditions of Theorem 10 hold, $\mu S^{*}-\gamma R^{*} \geq 0$. Therefore, inequality (68) holds. This completesthe proof.

Remark 14. Obviously, from the above discussion for special case $F(S)=S /(1+\lambda S)$ of model (1), we also have an important open problem, that is, whether endemic equilibrium $E^{*}$ of model (1) is globally asymptotically stable as long as basic reproduction number $\mathscr{R}_{0}>1$.

In the following, we will give an affirmative answer for above open problem in allusion to $F(S)=S$ and $G(I)=$ $\beta I /(1+\omega I)$ in model (1), by constructing the other Lyapunov function which is different from the Lyapunov function used in Theorem 5.

Firstly, we see in model (1) when $F(S)=S$ and $G(I)=$ $\beta I /(1+\omega I)$, where $\beta>0$ and $\omega \geq 0$ are two constants, basic reproduction number

$$
\begin{equation*}
\mathscr{R}_{0}=\frac{F(\Lambda / \mu) G^{\prime}(0)}{k+\mu+\alpha}=\frac{\Lambda \beta}{\mu(k+\mu+\alpha)}, \tag{76}
\end{equation*}
$$

and assumption $\left(H_{1}\right)$ naturally holds. Therefore, from Theorem 2, when $\mathscr{R}_{0}>1$, model (1) has a unique endemic equilibrium $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$.

Theorem 15. When $F(S) G(I)=\beta S I /(1+\omega I)$ in model (1), then endemic equilibrium $E^{*}$ is globally asymptotically stable if $\mathscr{R}_{0}>1$.

Proof. We consider the following Lyapunov function:

$$
\begin{align*}
U(S, I, R)= & \frac{\beta}{2 \alpha}\left(N-N^{*}\right)^{2}+\left(I-I^{*}-I^{*} \ln \frac{I}{I^{*}}\right)  \tag{77}\\
& +\frac{\omega}{2}\left(I-I^{*}\right)^{2}+\frac{\beta}{2 k}\left(R-R^{*}\right)^{2} .
\end{align*}
$$

It is clear that $U(S, I, R)>0$ for all $(S, I, R) \neq\left(S^{*}, I^{*}, R^{*}\right)$ and $U(S, I, R)=0$ if and only if $(S, I, R)=\left(S^{*}, I^{*}, R^{*}\right)$.

Let ( $S(n), I(n), R(n))$ be any positive solution of model (1). By computing

$$
\begin{align*}
\Delta U(n)= & U(S(n+1), I(n+1), R(n+1))  \tag{78}\\
& -U(S(n), I(n), R(n)),
\end{align*}
$$

a similar argument as in calculation $\Delta V_{3}(n)$ in Theorem 5, we obtain

$$
\begin{align*}
\Delta U(n) \leq & \frac{\beta}{\alpha}\left(N(n+1)-N^{*}\right)(N(n+1)-N(n)) \\
& +\left(I(n+1)-I(n)-I^{*} \ln \frac{I(n+1)}{I(n)}\right)  \tag{79}\\
& +\lambda\left(I(n+1)-I^{*}\right)(I(n+1)-I(n)) \\
& +\frac{\beta}{k}\left(R(n+1)-R^{*}\right)(R(n+1)-R(n)) .
\end{align*}
$$

By using inequality $\ln (1-x) \leq-x$ for any $x<1$, we obtain

$$
\begin{align*}
& -\ln \frac{I(n+1)}{I(n)} \\
& \quad=\ln 1-\left(1-\frac{I(n)}{I(n+1)}\right) \leq-\left(1-\frac{I(n)}{I(n+1)}\right) \tag{80}
\end{align*}
$$

Hence,

$$
\begin{align*}
\Delta U(n) \leq & \frac{\beta}{\alpha}\left(N(n+1)-N^{*}\right) \\
& \times(\Lambda-\mu N-\alpha I(n+1)) \\
& +\frac{1+\omega I(n+1)}{I(n+1)} \\
& \times\left(I(n+1)-I^{*}\right)(I(n+1)-I(n)) \\
& +\frac{\beta}{k}\left(R(n+1)-R^{*}\right) \\
& \times(k I(n+1)-(\mu+\gamma) R(n+1))  \tag{81}\\
= & \frac{\beta}{\alpha}\left(N(n+1)-N^{*}\right)(\Lambda-\mu N-\alpha I(n+1)) \\
& +\left(I(n+1)-I^{*}\right) \\
& \times(\beta S(n+1)-(1+\omega I(n+1)) \\
& \times(k+\mu+\alpha))+\frac{\beta}{k}\left(R(n+1)-R^{*}\right) \\
& \times(k I(n+1)-(\mu+\gamma) R(n+1)) .
\end{align*}
$$

Since $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ satisfies

$$
\begin{align*}
& \Lambda=\mu N^{*}+\alpha I^{*}=\mu S^{*}+(\alpha+\mu) I^{*}+\mu R^{*} \\
& k+\mu+\alpha=\frac{\beta S^{*}}{1+\omega I^{*}}  \tag{82}\\
& k I^{*}-(\mu+\gamma) R^{*}=0
\end{align*}
$$

then from (81) we further have

$$
\begin{aligned}
\Delta U(n) \leq & -\frac{\mu \beta}{\alpha}\left(N(n+1)-N^{*}\right)^{2} \\
& -\beta\left(N(n+1)-N^{*}\right)\left(I(n+1)-I^{*}\right) \\
& +\beta\left(I(n+1)-I^{*}\right) \\
& \times\left(S(n+1)-\frac{(1+\omega I(n+1)) S^{*}}{1+\omega I^{*}}\right) \\
& +\beta\left(R(n+1)-R^{*}\right)\left(I(n+1)-I^{*}\right) \\
& -\frac{\beta(\mu+\gamma)}{k}\left(R(n+1)-R^{*}\right)^{2}
\end{aligned}
$$

$$
\begin{align*}
= & -\frac{\mu \beta}{\alpha}\left(N(n+1)-N^{*}\right)^{2} \\
& -\beta\left(N(n+1)-N^{*}\right)\left(I(n+1)-I^{*}\right) \\
& +\beta\left(I(n+1)-I^{*}\right) \\
& \times\left(\left(N(n+1)-N^{*}\right)\right. \\
& -\left(I(n+1)-I^{*}\right)-\left(R(n+1)-R^{*}\right) \\
& \left.+S^{*}-\frac{(1+\omega I(n+1)) S^{*}}{1+\omega I^{*}}\right) \\
& +\beta\left(R(n+1)-R^{*}\right)\left(I(n+1)-I^{*}\right) \\
& -\frac{\beta(\mu+\gamma)}{k}\left(R(n+1)-R^{*}\right)^{2} \\
= & -\frac{\mu \beta}{\alpha}\left(N(n+1)-N^{*}\right)^{2} \\
& -\frac{\beta(\mu+\gamma)}{k}\left(R(n+1)-R^{*}\right)^{2} \\
& -\beta\left(I(n+1)-I^{*}\right)^{2}-\beta S^{*} \frac{\omega\left(I(n+1)-I^{*}\right)^{2}}{1+\omega I^{*}} . \tag{83}
\end{align*}
$$

Therefore, we finally get that $\Delta U(n) \leq 0$ for all $n \geq 0$. Obviously, $\Delta U(n)=0$ if and only if $N(n+1)=N^{*}$, $I(n+1)=I^{*}$, and $R(n+1)=R^{*}$. Therefore, from the theorems of stability of difference equations (see Theorem 6.3 in [33]), we obtain that $E^{*}$ is globally asymptotically stable. This completes the proof.

Remark 16. By combining Theorem 4, we can obtain that, when $F(S) G(I)=\beta S I /(1+\omega I)$ in model (1), diseasefree equilibrium $E^{0}$ is globally asymptotically stable if and only if basic reproduction number $\mathscr{R}_{0} \leq 1$ and endemic equilibrium $E^{*}$ is globally asymptotically stable if and only if $\mathscr{R}_{0}>1$.

Remark 17. In [13], the author studied a continuous SIRS epidemic model with bilinear incidence rate and obtained that the disease-free equilibrium is globally stable if basic reproduction number $R_{0} \leq 1$ and the endemic equilibrium is globally stable if $R_{0}>1$. However, in this paper, we established the completely same results for the corresponding backward Euler discretization model with saturation incidence rate. This shows that the results obtained in [13] are extended and improved in the discrete models.

Remark 18. In [25], the following continuous SIRS epidemic model with a class of nonlinear incidence rates and distributed delays is considered:

$$
\begin{gather*}
\dot{S}(t)=B-\mu_{1} S(t)-\beta S(t) \int_{0}^{h} f(\tau) G(I(t-\tau)) d \tau+\delta R(t), \\
\dot{I}(t)=\beta S(t) \int_{0}^{h} f(\tau) G(I(t-\tau)) d \tau-\left(\mu_{2}+\gamma\right) I(t), \\
\dot{R}(t)=\gamma I(t)-\left(\mu_{3}+\delta\right) R(t) . \tag{84}
\end{gather*}
$$

By applying Lyapunov functional techniques, Enatsu et al. obtained that disease-free equilibrium $\left(B / \mu_{1}, 0,0\right)$ of model (84) is globally asymptotically stable if basic reproduction number $\mathscr{R}_{0} \leq 1$ and endemic equilibrium $\left(S^{*}, I^{*}, R^{*}\right)$ of model (84) is globally asymptotically stable if $\mathscr{R}_{0}>1$ and $\mu_{1} S^{*}-\delta R^{*} \geq 0$ hold. Comparing with the results obtained in this paper, we can see that our results are the direct extension of those in [25] for nondelayed discrete SIRS epidemic model with nonlinear incidence rate $F(S) G(I)$.

However, we also see whether the conclusions obtained in [25] can be extended to delayed discrete SIRS epidemic models with more general nonlinear incidence rate $f(S, I)$, which is left to further investigation in our future work.

## 6. Numerical Simulations

In this section, we give the following examples and numerical simulations for model (1).

## Example 1. Consider

$$
\begin{equation*}
F(S) G(I)=\frac{\beta S I}{1+\lambda S} . \tag{85}
\end{equation*}
$$

We choose $\Lambda=3, \alpha=0.2, \lambda=1, \beta=0.8, \mu=0.2, \gamma=0.5$, and $k=0.3$. By calculating, we have the endemic equilibrium $\left.E^{*}=(7,3.2941,1.4118)\right)$ and the basic reproduction number

$$
\begin{equation*}
\mathscr{R}_{0}=\frac{\Lambda \beta}{(k+\mu+\alpha)(\mu+\lambda \Lambda)}=1.0714>1 \tag{86}
\end{equation*}
$$

However, $\gamma R^{*}-\mu S^{*}=-0.6941<0$. Clearly, inequality (56) does not hold. From the numerical simulation (see Figure 1), we obtain that the endemic equilibrium $E^{*}$ is still globally asymptotically stable. Therefore, in our future work, we expect to obtain the corresponding theoretical result for the open problem in Remark 8.

## Example 2. Consider

$$
\begin{equation*}
F(S) G(I)=\frac{\beta S^{2} I}{1+\lambda \sqrt{I}} \tag{87}
\end{equation*}
$$

We choose $\Lambda=4, \alpha=1.2, \lambda=1, \beta=0.4, \mu=1.3$, $\gamma=0.45$, and $k=1$. By calculating, we have the endemic


Figure 1: Time series of $S(n), I(n)$, and $R(n)$.
equilibrium $\left.E^{*}=(3.0637,0.0053,0.0030)\right)$ and the basic reproduction number

$$
\begin{equation*}
\mathscr{R}_{0}=\frac{\Lambda^{2} \beta}{\mu^{2}(k+\mu+\alpha)}=1.082>1 . \tag{88}
\end{equation*}
$$

By further computation, we obtain, when $S=2.7$,

$$
\begin{aligned}
& \frac{F(S)}{F(S)-F\left(S^{*}\right)}-\frac{\left(S^{*}\right)^{2}}{F\left(S^{*}\right)\left(S-S^{*}\right)} \\
& \quad=\frac{S^{2}}{S^{2}-\left(S^{*}\right)^{2}}-\frac{1}{S-S^{*}}=-0.7281<0,
\end{aligned}
$$

and, when $S=3.064$,

$$
\begin{align*}
& -\frac{\mu\left(S^{*}\right)^{2}}{F\left(S^{*}\right)}+\gamma R^{*}\left(\frac{F(S)}{F(S)-F\left(S^{*}\right)}-\frac{\left(S^{*}\right)^{2}}{F\left(S^{*}\right)\left(S-S^{*}\right)}\right) \\
& \quad=-\mu+\gamma R^{*}\left(\frac{S^{2}}{S^{2}-\left(S^{*}\right)^{2}}-\frac{1}{S-S^{*}}\right)=1.0943>0 . \tag{90}
\end{align*}
$$

That is, neither $\left(H_{2}\right)$ nor $\left(H_{3}\right)$ holds. However, from the numerical simulation (see Figure 2), it is clear that the endemic equilibrium $E^{*}$ is still globally asymptotically stable. Therefore, in our future work, we expect to obtain the corresponding theoretical result for the open problem in Remark 8.


Figure 2: Time series of $S(n), I(n)$, and $R(n)$.

## 7. Conclusions

This paper deals with global stability of disease-free equilibrium and endemic equilibrium and the permanence of disease for a class of discrete SIRS epidemic models with nonlinear incidence rate $F(S) G(I)$ and disease-induced mortality. Under the basic assumption $\left(H_{1}\right)$, by applying analytic techniques, we obtain that disease-free equilibrium $E^{0}$ of model (1) is globally asymptotically stable if basic reproduction number $\mathscr{R}_{0} \leq 1$ and disease in the model is permanent if $\mathscr{R}_{0}>1$. Furthermore, motivated by the recent progress of Lyapunov techniques in continuous epidemic models (see, e.g., [25-27]), we construct the corresponding discrete analogue of Lyapunov functions (see Theorem 4) for nonlinear incidence rate $F(S) G(I)$. Under the assumptions burdened on $F(S)$, that is, assumptions $\left(H_{2}\right)$ and $\left(H_{3}\right)$, we prove that the global asymptotic stability for endemic equilibrium $E^{*}$ of model (1) for the case $\mathscr{R}_{0}>1$ is an
extension of SIR-type models with nonlinear incidence rate $F(S) G(I)$ (see, for instance, [7, 26], etc.); that is, when SIRS models degenerate into SIR models, endemic equilibrium of the corresponding SIR models is globally asymptotically stable only if $\mathscr{R}_{0}>1$ and basic assumption $\left(H_{1}\right)$ hold.

From the proof of theorems in this paper, we easily see that discrete Lyapunov functions, such as $V(S, I, R)$ in Theorem 5, also can be applied for advanced models, including the models with delay. We expect to study the global stability of discrete SIRS and SEIRS epidemic models with rather general incidence rate $f(S, I)$ and with discrete or infinite delay, which is left as a future work.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work is supported by the National Natural Science Foundation of China (nos. 11271312 and 11201399), the Postdoctoral Science Foundation of China (Grant no. 20110491750), the Natural Science Foundation of Xinjiang (Grant nos. 2012211B07 and 2011211B08), and the Academic Discipline Project of Xinjiang Medical University Health Measurements and Health Economics (no. XYDXK50780308).

## References

[1] P. van den Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," Mathematical Biosciences, vol. 180, pp. 29-48, 2002.
[2] T. Zhang and Z. Teng, "Global behavior and permanence of SIRS epidemic model with time delay," Nonlinear Analysis, vol. 9, no. 4, pp. 1409-1424, 2008.
[3] T. Zhang and Z. Teng, "On a nonautonomous SEIRS model in epidemiology," Bulletin of Mathematical Biology, vol. 69, no. 8, pp. 2537-2559, 2007.
[4] Z. Teng, Y. Liu, and L. Zhang, "Persistence and extinction of disease in non-autonomous SIRS epidemic models with disease-induced mortality," Nonlinear Analysis, vol. 69, no. 8, pp. 2599-2614, 2008.
[5] Y. Nakata and T. Kuniya, "Global dynamics of a class of SEIRS epidemic models in a periodic environment," Journal of Mathematical Analysis and Applications, vol. 363, no. 1, pp. 230237, 2010.
[6] A. Korobeinikov and P. K. Maini, "A Lyapunov function and global properties for SIR and SEIR epidemiological models with nonlinear incidence," Mathematical Biosciences and Engineering, vol. 1, no. 1, pp. 57-60, 2004.
[7] A. Korobeinikov and P. K. Maini, "Non-linear incidence and stability of infectious disease models," Mathematical Medicine and Biology, vol. 22, no. 2, pp. 113-128, 2005.
[8] A. Korobeinikov, "Lyapunov functions and global stability for SIR and SIRS epidemiological models with non-linear transmission," Bulletin of Mathematical Biology, vol. 68, no. 3, pp. 615-626, 2006.
[9] A. Korobeinikov, "Global properties of infectious disease models with nonlinear incidence," Bulletin of Mathematical Biology, vol. 69, no. 6, pp. 1871-1886, 2007.
[10] A. Korobeinikov, "Global asymptotic properties of virus dynamics models with dose-dependent parasite reproduction and virulence, and nonlinear incidence rate," Mathematical Medicine and Biology, vol. 26, pp. 225-239, 2009.
[11] J. Li, Y. Xiao, F. Zhang, and Y. Yang, "An algebraic approach to proving the global stability of a class of epidemic models," Nonlinear Analysis, vol. 13, no. 5, pp. 2006-2016, 2012.
[12] C. Vargas-De-León, "On the global stability of SIS, SIR and SIRS epidemic models with standard incidence," Chaos, Solitons and Fractals, vol. 44, no. 12, pp. 1106-1110, 2011.
[13] C. Vargas-De-León, "Constructions of Lyapunov functions for classic SIS, SIR and SIRS epidemic models with variable population size," Revista Electrónica de Contenido Matemático, vol. 26, no. 5, pp. 1-12, 2009.
[14] C. C. McCluskey, "Complete global stability for an SIR epidemic model with delay-distributed or discrete," Nonlinear Analysis, vol. 11, no. 1, pp. 55-59, 2010.
[15] C. C. McCluskey, "Global stability for an SEIR epidemiological model with varying infectivity and infinite delay," Mathematical Biosciences and Engineering, vol. 6, no. 3, pp. 603-610, 2009.
[16] C. C. McCluskey, "Global stability for an SIR epidemic model with delay and nonlinear incidence," Nonlinear Analysis, vol. 11, no. 4, pp. 3106-3109, 2010.
[17] V. Capasso and G. Serio, "A generalization of the KermackMcKendrick deterministic epidemic model," Mathematical Biosciences, vol. 42, no. 1-2, pp. 43-61, 1978.
[18] W. M. Liu, S. A. Levin, and Y. Iwasa, "Influence of nonlinear incidence rates upon the behavior of SIRS epidemiological models," Journal of Mathematical Biology, vol. 23, no. 2, pp. 187204, 1986.
[19] S. Ruan and W. Wang, "Dynamical behavior of an epidemic model with a nonlinear incidence rate," Journal of Differential Equations, vol. 188, no. 1, pp. 135-163, 2003.
[20] D. Xiao and S. Ruan, "Global analysis of an epidemic model with nonmonotone incidence rate," Mathematical Biosciences, vol. 208, no. 2, pp. 419-429, 2007.
[21] R. M. Cullen, A. Korobeinikov, and W. J. Walker, "Seasonality and critical community size for infectious diseases," The ANZIAM Journal, vol. 44, no. 4, pp. 501-512, 2003.
[22] Y. Enatsu, Y. Nakata, and Y. Muroya, "Global stability for a class of discrete SIR epidemic models," Mathematical Biosciences and Engineering, vol. 7, no. 2, pp. 347-361, 2010.
[23] Y. Muroya, A. Bellen, Y. Enatsu, and Y. Nakata, "Global stability for a discrete epidemic model for disease with immunity and latency spreading in a heterogeneous host population," Nonlinear Analysis, vol. 13, no. 1, pp. 258-274, 2012.
[24] Y. Enatsu, Y. Nakata, Y. Muroya, G. Izzo, and A. Vecchio, "Global dynamics of difference equations for SIR epidemic models with a class of nonlinear incidence rates," Journal of Difference Equations and Applications, vol. 18, no. 7, pp. 1163-1181, 2012.
[25] Y. Enatsu, Y. Nakata, and Y. Muroya, "Lyapunov functional techniques for the global stability analysis of a delayed SIRS epidemic model," Nonlinear Analysis, vol. 13, no. 5, pp. 21202133, 2012.
[26] G. Huang, Y. Takeuchi, W. Ma, and D. Wei, "Global stability for delay SIR and SEIR epidemic models with nonlinear incidence rate," Bulletin of Mathematical Biology, vol. 72, no. 5, pp. 11921207, 2010.
[27] G. Huang and Y. Takeuchi, "Global analysis on delay epidemiological dynamic models with nonlinear incidence," Journal of Mathematical Biology, vol. 63, no. 1, pp. 125-139, 2011.
[28] Z. Hu, Z. Teng, and H. Jiang, "Stability analysis in a class of discrete SIRS epidemic models," Nonlinear Analysis, vol. 13, no. 5, pp. 2017-2033, 2012.
[29] A. M. Lyapunov, The General Problem of The Stability of Motion, Taylor and Francis, London, UK, 1992.
[30] C. S. Peskin and T. Schlick, "Molecular dynamics by the backward-Euler method," Communications on Pure and Applied Mathematics, vol. 42, no. 7, pp. 1001-1031, 1989.
[31] T. Liu, C. Zhao, Q. Li, and L. Zhang, "An efficient backward Euler time-integration method for nonlinear dynamic analysis
of structures," Computers and Structures, vol. 106, pp. 20-28, 2012.
[32] L. Wang and M. Wang, Ordinary Difference Equation, Xinjiang University Press, Xinjiang, China, 1989 (Chinese).
[33] J. P. LaSalle, The Stability of Dynamical Systems, Society for Industrial and Applied Mathematics, Philadelphia, Pa, USA, 1976.
[34] X. Zhao, Dynamical Systems in Population Biology, Springer, New York, NY, USA, 2003.

Research Article

# Global Property in a Delayed Periodic Predator-Prey Model with Stage-Structure in Prey and Density-Independence in Predator 

Xiaolin Fan, ${ }^{1,2}$ Zhidong Teng, ${ }^{1}$ and Haijun Jiang ${ }^{1}$<br>${ }^{1}$ College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, China<br>${ }^{2}$ Department of Basic Education, Xinjiang Institute of Engineering, Urumqi, Xinjiang 830091, China<br>Correspondence should be addressed to Zhidong Teng; zhidong1960@163.com

Received 14 December 2013; Accepted 7 February 2014; Published 16 March 2014
Academic Editor: Kaifa Wang
Copyright © 2014 Xiaolin Fan et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We study the global property in a delayed periodic predator-prey model with stage-structure in prey and density-independence in predator. The sufficient conditions on the ultimate boundedness of all positive solutions are obtained, and the sufficient conditions of the integrable form for the permanence and extinction are further established, respectively. Some well-known results on the predator density-dependency are improved and extended to the predator density-independent cases. The theoretical results are confirmed by the special examples and the numerical simulations.


## 1. Introduction

There are many different kinds of two-species predatorprey dynamical models in mathematical ecology. Particularly, two-species predator-prey model with stage-structure have been extensively studied by a large number of papers, see [1-5] and the reference cited therein. The main research topics include the persistence, permanence and extinction of species, the existence and the global asymptotic properties of positive periodic solutions in periodic case, and the global stability of models in general nonautonomous cases.

In [2], Cui and Song studied a periodic predator-prey system with stage-structure. They provided a sufficient and necessary condition to guarantee the permanence of species for the system. In [3], Cui and Takeuchi studied a periodic predator-prey system with stage-structure with function response. They provided a sufficient and necessary condition to guarantee the permanence of species for the system with infinite delay. Some known results are extended to the delay case.

So far, from these done works on the predator-prey model with stage-structure, the authors always assume that
the predator is strictly density-dependent, which is much identical with the real biological background. On the other hand, the effect of periodically varying environment plays an important role in the permanence and extinction of species for the system (e.g., seasonal effects of climate, food supply, mating habits, hunting or harvesting seasons, etc.). Thus, the assumptions of periodicity of the parameters and system with time delay are effective ways to characterize and investigate population systems. Owing to many natural and man-made factors such as the low birth rate, high death rate, decreasing habitats, and the hunting of human beings, and the worse ecological system, some predator species become rare and even liable to extinction. For these predator species, we can ignore the effect of density-dependency. Up to now, there are some works on such investigation for the situation of predator density-independence. The authors always assume that the density of predator is proportional to the predation rate, the conversion rate of the immature prey biomass into predator biomass, and the death rate of predator. Predator densityindependece is reasonable to the real ecosystem.

To our knowledge, few scholars consider the delayed periodic predator-prey models with stage-structure in prey
and density-independence in predator. In this paper, we consider the following system:

$$
\begin{align*}
\frac{d x_{1}(t)}{d t}= & a(t) x_{2}(t)-b(t) x_{1}(t)-d(t) x_{1}^{2}(t) \\
& -p(t) \phi\left(x_{1}(t)\right) \int_{-h}^{0} k_{12}(s) y(t+s) d s \\
\frac{d x_{2}(t)}{d t}= & c(t) x_{1}(t)-f(t) x_{2}^{2}(t) \\
\frac{d y(t)}{d t}= & y(t)\left[-g(t)+h(t) \int_{-h}^{0} k_{21}(s) \phi\left(x_{1}(t+s)\right) d s\right] \tag{1}
\end{align*}
$$

Our purpose in this paper is to establish sufficient conditions of integrable form for the permanence and extinction of species for system (1). By using the analysis method, the comparison theorem of cooperative system, and the theory of the persistence of dynamical systems, the integral form criteria on the ultimate boundedness, permanence, and extinction are established. The method used in this paper is motivated by the works on the permanence and extinction for periodic predator-prey systems in patchy environment given by Teng and Chen in [5].

The organization of this paper is as follows. In the next section, the basic assumptions for system (1), some notations, and lemmas which will be used in the later sections are introduced as the preliminaries. In Section 3, the main results of this paper are stated. In Section 4, the proofs of the main theorems are given. In Section 5, the theoretical results are confirmed by some special examples and the numerical simulations. Finally, a conclusion is given in Section 6.

## 2. Preliminaries

In system (1), $x_{i}(t)(i=1,2)$ represent the population density of the infancy prey and maturity prey at time $t$, respectively, and $y(t)$ represents the population density of predator species at time $t$ which only prey on infancy prey $x_{1}(t)$. Functions $a(t), b(t), c(t), d(t), f(t), g(t), h(t)$, and $p(t)$ are periodic and continuous defined on $R_{+}=[0, \infty)$ with common period $\omega>0$, and $a(t), b(t), c(t), d(t)$, and $f(t)$ are also positive, where $a(t), b(t)$, and $d(t)$ denote the birthrate, mortality, and density restriction of infancy prey $x_{1}(t)$ at time $t$, respectively, $c(t)$ denotes the transformation from the infancy prey $x_{1}(t)$ to the maturity prey $x_{2}(t)$ at time $t, f(t)$ denotes the mortality and density restriction of maturity prey $x_{2}(t)$ at time $t, p(t)$ denotes the predation rate in which the predator $y(t)$ captures the infancy prey $x_{1}(t)$ at time $t, g(t)$ is the mortality of predator $y(t)$ at time $t$, and $h(t)$ denotes the transformation from the infancy prey $x_{1}(t)$ to the predator $y(t)$ by the assimilation. Functions $k_{i j}(s)(i, j=1,2)$ are nonnegative and integrable on $[-h, 0]$ and $\int_{-h}^{0} k_{i j}(s) d s=1$. Function $\phi\left(x_{1}\right)$, the number of the prey consumed per predator in unit time, is called the predator functional response. In this paper, we always assume that $\phi\left(x_{1}\right)$ is continuous differentiable function and $\phi(0)=0$.

We define set $C_{+}$as follows:

$$
\begin{align*}
C_{+}=\{ & \psi=\left(\psi_{1}, \psi_{2}, \psi_{3}\right): \psi_{i}(s) \text { is nonnegative } \\
& \text { continous for } \left.s \in[-h, 0], \psi_{i}(0)>0, i=1,2,3\right\} . \tag{2}
\end{align*}
$$

For any $\psi \in C_{+}$, the norm is defined by $\|\psi\|=$ $\sup _{-h \leq \theta \leq 0}|\psi(\theta)|$. Motivated by the biological background of system (1), in this paper, we always assume that the solutions of system (1) satisfy the following initial conditions:

$$
\begin{gather*}
x_{1}(s)=\psi_{1}(s), \quad x_{2}(s)=\psi_{2}(s)  \tag{3}\\
y(s)=\psi_{3}(s), \quad-h \leqslant s \leqslant 0
\end{gather*}
$$

where $\psi=\left(\psi_{1}, \psi_{2}, \psi_{3}\right) \in C_{+}$. It is easy to prove that the right functional of system (1) is continuous and satisfies a local Lipschitz condition with respect to $\psi=\left(\psi_{1}, \psi_{2}, \psi_{3}\right) \in C_{+}$. Therefore, by the fundamental theory of functional differential equations (see [6-8]), for any $\psi=\left(\psi_{1}, \psi_{2}, \psi_{3}\right) \in C_{+}$, system (1) has a unique solution $\left(x_{1}(t, \psi), x_{2}(t, \psi), y(t, \psi)\right)$ satisfying initial condition (3). It is also easy to prove that the solution $\left(x_{1}(t, \psi), x_{2}(t, \psi), y(t, \psi)\right)$ is positive; that is, $x_{i}(t, \psi)>0(i=1,2)$ and $y(t, \psi)>0$ in its maximal interval of existence. In this paper, such a solution of system (1) is called a positive solution.

Let $f(t)$ be a $\omega$-periodic continuous function defined on $R_{+}$, we define

$$
\begin{gather*}
A_{\omega}(f)=\omega^{-1} \int_{0}^{\omega} f(t) d t, \quad f^{m}=\max _{t \in R^{+}} f(t)  \tag{4}\\
f^{l}=\min _{t \in R^{+}} f(t)
\end{gather*}
$$

Consider the following differential equations system:

$$
\begin{align*}
& \frac{d u(t)}{d t}=\alpha(t) v(t)-\beta(t) u(t)-\gamma(t) u^{2}(t)  \tag{5}\\
& \frac{d v(t)}{d t}=\delta(t) u(t)-\eta(t) v^{2}(t)
\end{align*}
$$

where functions $\alpha(t), \beta(t), \gamma(t), \delta(t)$, and $\eta(t)$ are positive periodic and continuous defined on $R_{+}$with common period $\omega>0$. We have the following result.

Lemma 1 (see [9]). System (5) has a positive $\omega$-periodic solution $\left(u^{*}(t), v^{*}(t)\right)$ which is globally asymptotically stable.

Remark 2. Directly from system (5), we can obtain that when we increase coefficients $\alpha(t)$ and $\delta(t)$, or decrease coefficients $\beta(t), \gamma(t)$, and $\eta(t)$, then $u^{*}(t)$ and $v^{*}(t)$ will largen. Otherwise, $u^{*}(t)$ and $v^{*}(t)$ will decrease.

When the predator species $y(t)=0$ in system (1), we obtain the following subsystem of system (1):

$$
\begin{align*}
& \frac{d x_{1}(t)}{d t}=a(t) x_{2}(t)-b(t) x_{1}(t)-d(t) x_{1}^{2}(t)  \tag{6}\\
& \frac{d x_{2}(t)}{d t}=c(t) x_{1}(t)-f(t) x_{2}^{2}(t)
\end{align*}
$$

It is clear that the solution $\left(x_{1}(t), x_{2}(t)\right)$ of system (6) with initial value $\left(x_{1}(0), x_{2}(0)\right)>0$ is positive for all $t>0$. We further have the following result as a corollary of Lemma 1.

Corollary 3. System (6) has a positive w-periodic solution $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)$ which is globally asymptotically stable.

Remark 4. As a direct consequence of Corollary 3, we see that system (1) has a predator extinction periodic solution $\left(x_{1}^{*}(t), x_{2}^{*}(t), 0\right)$.

Remark 5. Obviously, from Remark 2, by increasing coefficients $a(t)$ and $c(t)$, or decreasing coefficients $b(t), d(t)$, and $f(t)$, we can see that $x_{1}^{*}(t)$ and $x_{2}^{*}(t)$ will largen. Otherwise, $x_{1}^{*}(t)$ and $x_{2}^{*}(t)$ will decrease.

For system (1), we introduce the following basic assumptions:

$$
\begin{aligned}
& \left(A_{1}\right) A_{\omega}(g)>0 \\
& \left(A_{2}\right) p^{l}>0 \text { and } h^{l} \geq 0 ; \\
& \left(A_{3}\right) \phi^{\prime}\left(x_{1}\right) \geqslant 0 \text { for all } 0 \leqslant x_{1} \leqslant \max _{t \in(0, \omega]} x_{1}^{*}(t) \text {, and } \phi(0)= \\
& \quad 0 .
\end{aligned}
$$

Let $X$ be a complete metric space with metric $d$. Suppose that $f: X \rightarrow X$ is a continuous map. For any $x \in X$, we denote $f^{n}(x)=f\left(f^{n-1}(x)\right)$ for any integer $n>1$ and $f^{1}(x)=$ $f(x) . f$ is said to be compact in $X$, if for any bounded set $H \subset X$ set $f(H)=\{f(x): x \in H\}$ is precompact in X. $f$ is said to be point dissipative if there is a bounded set $B_{0} \subset X$ such that for any $x \in X$. Consider

$$
\begin{equation*}
\lim _{n \rightarrow \infty} d\left(f^{n}(x), B_{0}\right)=0 \tag{7}
\end{equation*}
$$

For any $x_{0} \in X$, the positive semiorbit through $x_{0}$ is defined by $\gamma^{+}\left(x_{0}\right)=\left\{x_{n}=f^{n}\left(x_{0}\right): n=1,2, \ldots\right\}$, the negative semiorbit through $x_{0}$ is defined as a sequence $\gamma^{-}\left(x_{0}\right)=\left\{x_{k}\right\}$ satisfying $f\left(x_{k-1}\right)=x_{k}$ for integers $k \leqslant 0$, and its $\omega$-limit set is $\omega\left(x_{0}\right)=\left\{y \in X\right.$; there is a time sequence $n_{k} \rightarrow \infty$ such that $\left.\lim _{k \rightarrow \infty} f\left(x_{n_{k}}\right)=y\right\}$ and its $\alpha$-limit set is $\alpha\left(x_{0}\right)=$ $\left\{y \in X\right.$; there is a time sequence $n_{k} \rightarrow-\infty$ such that $\left.\lim _{k \rightarrow \infty} f\left(x_{n_{k}}\right)=y\right\}$.

A nonempty set $A \subset X$ is said to be invariant if $f(A) \subseteq$ A. A nonempty invariant set $M$ of $X$ is called to be isolated in $X$, if it is the maximal invariant set in a neighborhood of itself. For a nonempty set $M$ of $X$, set $W^{s}(M):=\{x \in X$ : $\left.\lim _{n \rightarrow \infty} d\left(f^{n}(x), M\right)=0\right\}$ is called the stable set of $M$.

Let $A$ and $B$ be two isolated invariant sets; set $A$ is said to be chained to set $B$, written as $A \rightarrow B$, if there exists a full orbit though some $x \notin A \cup B$ such that $\omega(x) \subset B$ and $\alpha(x) \subset$ A. A finite sequence $\left\{M_{1}, \ldots, M_{k}\right\}$ of isolated invariant sets is called a chain, if $M_{1} \rightarrow M_{2} \rightarrow \cdots \rightarrow M_{k}$, and if $M_{k}=M_{1}$, the chain is called a cycle.

Let $X^{0}$ and $\partial X^{0}$ be nonempty open set and nonempty closed set of $X$, respectively, and satisfying $X^{0} \cap \partial X^{0}=\emptyset$. We denote

$$
\begin{equation*}
M_{\partial}=\left\{x \in \partial X^{0}: f^{n}(x) \in \partial X^{0}, \forall n \geq 0\right\} \tag{8}
\end{equation*}
$$

Lemma 6. Let $f: X \rightarrow X$ be a continuous map. Assume that the following conditions hold:
$\left(C_{1}\right) f$ is compact and point dissipative, and $f\left(X^{0}\right) \subseteq X^{0}$;
$\left(C_{2}\right)$ there exists a finite sequence $\mathscr{M}=\left\{M_{1}, \ldots, M_{k}\right\}$ of compact and isolated invariant sets such that
(a) $M_{i} \cap M_{j}=\emptyset$ for any $i, j=1,2, \ldots, k$ and $i \neq j$;
(b) $\Omega\left(M_{\partial}\right):=\cup\left\{\omega(x): x \in M_{\partial}\right\} \subset \cup_{i=1}^{k} M_{i}$;
(c) no subset of $\mathscr{M}$ forms a cycle in $\partial X^{0}$;
(d) $W^{s}\left(M_{i}\right) \cap X^{0}=\emptyset$ for each $1 \leq i \leq k$.

Then $f$ is uniformly persistent with respect to $\left(X^{0}, \partial X^{0}\right)$; that is, there exists a constant $\eta>0$ such that $\liminf _{n \rightarrow \infty} d\left(f^{n}(x), \partial X^{0}\right) \geqslant \eta$ for all $x \in X^{0}$.

Lemma 6 can be obtained from Theorem 1.1.3, Theorem 1.3.1, Remark 1.3.1, and Theorem 1.3.3 given by Zhao in [10].

## 3. Main Results

Firstly, concerning the persistence and permanence of species for system (1), we have the following general result.

Theorem 7. Suppose that $\left(A_{1}\right)-\left(A_{3}\right)$ hold. Then there exists a positive constant $M>0$ such that

$$
\begin{gather*}
\lim \sup _{t \rightarrow \infty} x_{i}(t) \leq M \quad(i=1,2) \\
\lim \sup _{t \rightarrow \infty} y(t) \leq M \tag{9}
\end{gather*}
$$

for any positive solution $\left(x_{1}(t), x_{2}(t), y(t)\right)$ of system (1).
Remark 8. Let us see the biological meaning of Theorem 7. In fact, if the predator species is not ultimately bounded, then the population density of predator species will expand unlimitedly. Since the predation rate of predator species for prey species is strictly positive (i.e., $p^{l}>0$ in assumption $\left(A_{2}\right)$ ), the prey species will become extinct because of the massive preying by the predator species. Since the survival of predator is absolutely dependent on the prey species, as an opposite result, the predator species will become extinct too.

However, if the predation rate $p(t)$ of the predator species is not strictly positive, that is, $p^{l}=0$, then it cannot lead to extinction when the population density of predator species expands unlimitedly. Therefore, an important open question is whether we can still obtain the boundedness of predator species which is density-independent when $p^{l}=0$.

Theorem 9. Suppose that $\left(A_{1}\right)-\left(A_{3}\right)$ hold. If

$$
\begin{equation*}
A_{\omega}\left[-g(t)+h(t) \int_{-h}^{0} k_{21}(s) \phi\left(x_{1}^{*}(t+s)\right) d s\right]>0 \tag{10}
\end{equation*}
$$

where $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)$ is the positive $\omega$-periodic solution of system (6), then system (1) is uniformly persistent. That is, there exists a positive constant $\epsilon$, such that any solution $\left(x_{1}(t), x_{2}(t), y(t)\right)$ of system (1) with initial condition (3) satisfies

$$
\begin{equation*}
\liminf _{t \rightarrow \infty}\left(x_{1}(t), x_{2}(t), y(t)\right) \geqslant(\epsilon, \epsilon, \epsilon) \tag{11}
\end{equation*}
$$

Remark 10. Theorem 9 shows that if we guarantee that $\left(A_{1}\right)-\left(A_{3}\right)$ hold, then the prey species must be permanent. In fact, if the prey species $\left(x_{1}(t), x_{2}(t)\right)$ is not permanent, then it may be extinct, as a result the predator species $y(t)$ will be extinct too because its survival is absolutely dependent on $x_{1}(t)$. However, when predator species $y(t)$ become extinct, prey species $\left(x_{1}(t), x_{2}(t)\right)$ will not turn to extinction, because $\left(A_{1}\right)$ shows that $x_{1}(t)$ has a total positive average growth rate.

Remark 11. From Lemma 1, we know that, when there is no predator species $y(t)$, the prey species $\left(x_{1}(t), x_{2}(t)\right)$ will approach a positive periodic solution stable state $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)$. When there is predator species $y(t)$, Theorem 9 shows that if the positive periodic stable state $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)$ of prey species $\left(x_{1}(t), x_{2}(t)\right)$ can guarantee that predator species $y(t)$ obtain a positive total average growth rate, that is, condition (10), then predator species $y(t)$ will be permanent.

Theorem 12. Suppose that $\left(A_{1}\right)-\left(A_{3}\right)$ hold. If

$$
\begin{equation*}
A_{\omega}\left[-g(t)+h(t) \int_{-h}^{0} k_{21}(s) \phi\left(x_{1}^{*}(t+s)\right) d s\right]<0 \tag{12}
\end{equation*}
$$

then for any positive solution $\left(x_{1}(t), x_{2}(t), y(t)\right)$ of system (1), $x_{i}(t) \rightarrow x_{i}^{*}(t)(i=1,2)$ and $y(t) \rightarrow 0$, as $t \rightarrow \infty$.

Remark 13. Theorem 12 shows that when the prey species $\left(x_{1}(t), x_{2}(t)\right)$ approach a positive periodic solution stable state $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)$, the predator species $y(t)$ can only obtain a negative total average growth rate, that is, condition (12), then $y(t)$ will be extinct.

Lastly, from Theorems 9 and 7 given by Teng and Chen in [11] on the existence of positive periodic solutions for general Kolmogorov systems with bounded delays, we have the following result.

Corollary 14. Suppose that $\left(A_{1}\right)-\left(A_{3}\right)$ hold. If

$$
\begin{equation*}
A_{\omega}\left[-g(t)+h(t) \int_{-h}^{0} k_{21}(s) \phi\left(x_{1}^{*}(t+s)\right) d s\right]>0 \tag{13}
\end{equation*}
$$

then system (1) has at least a positive $\omega$-periodic solution.
Remark 15. In this paper we obtain the existence of the positive periodic solutions for system (1) under the assumption that all parameters are with common periodicity. However, considering all parameters fluctuating in time with the same period is unrealistic, because it will be more realistic if we allow time fluctuations with different period or even nonperiod with some almost periodic environment, which will be more identical with the sound ecosystem. Therefore, there is a very important open question that is whether the same result given in Lemma 1 will be true under the assumption that the parameter in system (1) is almost periodic.

Remark 16. From Remark 5 we know that by increasing coefficients $a(t)$ and $c(t)$ or decreasing coefficients $b(t), d(t)$, and $f(t)$, then $x_{1}^{*}(t)$ and $x_{2}^{*}(t)$ will largen. This shows that by
increasing coefficients $a(t)$ and $c(t)$ or decreasing coefficients $b(t), d(t)$, and $f(t)$, we can get that

$$
\begin{equation*}
A_{\omega}\left[-g(t)+h(t) \int_{-h}^{0} k_{21}(s) \phi\left(x_{1}^{*}(t+s)\right) d s\right] \tag{14}
\end{equation*}
$$

increases. Thus, condition (12) can be changed to condition (10). Therefore, from Theorems 9 and 12, we obtain that predator $y(t)$ will become into the permanence from the quondam extinction. This shows that the stage-structure in the prey (i.e., the birthrate, mortality, density restriction of infancy prey, the transformation from the infancy prey to the maturity prey, and the mortality and density restriction of maturity prey) will bring the effect for the permanence and extinction of the predator.

Remark 17. System (1) is a pure delay system with respect to $y(t)$. We cannot use the variable without time delay to control the variable with time delay. This shows that it is very difficult to get the global attractivity of system (1). We will discuss this problem in the future.

Remark 18. An important open question is that what results will be obtained with the condition

$$
\begin{equation*}
A_{\omega}\left[-g(t)+h(t) \int_{-h}^{0} k_{21}(s) \phi\left(x_{1}^{*}(t+s)\right) d s\right]=0 \tag{15}
\end{equation*}
$$

Is it the permanence of system (1) or the extinction of predator $y(t)$ ?

When system (1) degenerates into the nondelayed system of ordinary differential equations, that is, in system (1) $k_{i j} \equiv$ $0(i, j=1,2)$, then we have

$$
\begin{align*}
\frac{d x_{1}(t)}{d t}= & a(t) x_{2}(t)-b(t) x_{1}(t) \\
& -d(t) x_{1}^{2}(t)-p(t) \varphi\left(x_{1}(t)\right) y(t) \\
\frac{d x_{2}(t)}{d t}= & c(t) x_{1}(t)-f(t) x_{2}^{2}(t)  \tag{16}\\
\frac{d y(t)}{d t}= & y(t)\left[-g(t)+h(t) \varphi\left(x_{1}(t)\right)\right]
\end{align*}
$$

we can see that the above assumptions for system (16) will have the following forms:

$$
\begin{aligned}
& \left(A_{1}^{*}\right) A_{\omega}(g)>0 \\
& \left(A_{2}^{*}\right) p^{l}>0 \text { and } h^{l} \geq 0 \\
& \left(A_{3}^{*}\right) \varphi^{\prime}\left(x_{1}\right) \geq 0 \text { and } \varphi(0)=0 \text { for } 0 \leq x_{1} \leq \max _{t \in(0, \omega]} x_{1}^{*}(t)
\end{aligned}
$$

Therefore, as special cases of Theorems 7-12 we have the following results for system (16).

Corollary 19. Suppose that $\left(A_{1}^{*}\right)-\left(A_{3}^{*}\right)$ hold. Then there exists a positive constant $M>0$ such that

$$
\begin{gather*}
\lim \sup _{t \rightarrow \infty} x_{i}(t) \leq M \quad(i=1,2) \\
\lim \sup _{t \rightarrow \infty} y(t) \leq M \tag{17}
\end{gather*}
$$

for any positive solution $\left(x_{1}(t), x_{2}(t), y(t)\right)$ of system (16).

Corollary 20. Suppose that $\left(A_{1}^{*}\right)-\left(A_{3}^{*}\right)$ hold. If

$$
\begin{equation*}
A_{\omega}\left[-g(t)+h(t) \varphi\left(x_{1}^{*}(t)\right)\right]>0, \tag{18}
\end{equation*}
$$

then system (16) is uniformly persistent, where $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)$ is the positive $\omega$-periodic solution of system (6).

Corollary 21. Suppose that $\left(A_{1}^{*}\right)-\left(A_{3}^{*}\right)$ hold. If

$$
\begin{equation*}
A_{\omega}\left[-g(t)+h(t) \varphi\left(x_{1}^{*}(t)\right)\right]<0 \tag{19}
\end{equation*}
$$

then for any positive solution $\left(\left(x_{1}(t), x_{2}(t), y(t)\right)\right.$ of system (16), $x_{i}(t) \rightarrow x_{i}^{*}(t)(i=1,2)$ and $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

## 4. Proof of Theorems

Proof of Theorem 7. For any positive solution $\left(x_{1}(t), x_{2}(t)\right.$, $y(t)$ ) of system (1), we have

$$
\begin{align*}
& \frac{d x_{1}(t)}{d t} \leq a(t) x_{2}(t)-b(t) x_{1}(t)-d(t) x_{1}^{2}(t) \\
& \frac{d x_{2}(t)}{d t}=c(t) x_{1}(t)-f(t) x_{2}^{2}(t) \tag{20}
\end{align*}
$$

By the vector comparison theorem (see [12, 13]) and Corollary 3, we can obtain that for any $\varepsilon>0$ there is a $T_{1}>0$ such that

$$
\begin{equation*}
x_{i}(t) \leq x_{i}^{*}(t)+\varepsilon, \quad i=1,2, \quad \forall t \geq T_{1} \tag{21}
\end{equation*}
$$

This leads to

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} x_{i}(t) \leq M_{1}, \quad i=1,2 \tag{22}
\end{equation*}
$$

where $M_{1}=\max \left\{x_{i}^{*}(t): i=1,2, t \in[0, \omega]\right\}$.
Next, we prove that there exists a positive constant $M_{2}>0$ such that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} y(t) \leq M_{2} \tag{23}
\end{equation*}
$$

And, from $\left(A_{1}\right)$ and $\left(A_{2}\right)$, we can choose positive constants $M_{0}>M_{1}$ and $0<\epsilon_{0}<M_{1}$ such that

$$
\begin{gather*}
M_{1} a^{m}-p^{l} M_{0} \phi\left(\epsilon_{0}\right)<-\epsilon_{0},  \tag{24}\\
A_{\omega}\left[-g(t)+h(t) \phi\left(\epsilon_{0}\right)\right]<-\epsilon_{0} . \tag{25}
\end{gather*}
$$

We firstly prove that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} y(t) \leq M_{0} \tag{26}
\end{equation*}
$$

Otherwise, there exists a positive constant $T_{2}>T_{1}$ such that $y(t)>M_{0}$ for all $t \geq T_{2}$. If $x_{1}(t) \geq \epsilon_{0}$ for all $t \geq T_{2}+h$, then, for any $t \geq T_{2}+h$, we have by (24)

$$
\begin{align*}
\frac{d x_{1}(t)}{d t} & \leq a(t) x_{2}(t)-p(t) \phi\left(x_{1}(t)\right) \int_{-h}^{0} k_{12}(s) y(t+s) d s \\
& <a(t) x_{2}(t)-p(t) \phi\left(x_{1}(t)\right) M_{0} \\
& <M_{1} a^{m}-p^{l} M_{0} \phi\left(\epsilon_{0}\right)<-\epsilon_{0} . \tag{27}
\end{align*}
$$

Integrating (27) from $T_{2}+h$ to $t$ we have

$$
\begin{equation*}
x_{1}(t) \leq x_{1}\left(T_{2}+h\right)-\epsilon_{0}\left(t-T_{2}-h\right) \tag{28}
\end{equation*}
$$

which implies to $x_{1}(t) \rightarrow-\infty$ as $t \rightarrow \infty$. This leads a contradiction. Therefore, there is a $t_{1}>T_{2}+h$ such that $x_{1}\left(t_{1}\right)<$ $\epsilon_{0}$. Now, we prove that $x_{1}(t)<\epsilon_{0}$ for all $t \geq t_{1}$. Otherwise, there exists a $t_{2}>t_{1}$ such that $x_{1}\left(t_{2}\right)=\epsilon_{0}$ and $x_{1}(t)<\epsilon_{0}$ for all $t \in\left(t_{1}, t_{2}\right)$. Then, we have $d x_{1}\left(t_{2}\right) / d t \geq 0$. On the other hand, from the first equation of system (1), a similar calculation as in (27), we have

$$
\begin{equation*}
\frac{d x_{1}\left(t_{2}\right)}{d t} \leq M_{1} a^{m}-p^{l} M_{0} \phi\left(\epsilon_{0}\right) \leq-\epsilon_{0} \tag{29}
\end{equation*}
$$

which leads to a contradiction. Thus, $x_{1}(t)<\epsilon_{0}$ for all $t \geq t_{1}$. For any $t \geq t_{1}+h$, we choose an integer $p_{t}>0$ such that $t \in\left[t_{1}+h+p_{t} \omega, t_{1}+h+\left(p_{t}+1\right) \omega\right]$. Obviously, $p_{t} \rightarrow \infty$ as $t \rightarrow \infty$. From the third equation of system (1) we have

$$
\begin{align*}
y(t)= & y\left(t_{1}+h\right) \\
& \times \exp \int_{t_{1}+h}^{t}\left[-g(s)+h(s) \int_{-h}^{0} k_{21}(\mu) \phi\right. \\
& \left.\times\left(x_{1}(\mu+s)\right) d \mu\right] d s \\
\leq & y\left(t_{1}+h\right) \exp \int_{t_{1}+h}^{t}\left[-g(s)+h(s) \phi\left(\epsilon_{0}\right)\right] d s  \tag{30}\\
\leq & y\left(t_{1}+h\right) \exp \left\{\int_{t_{1}+h}^{t_{1}+h+p_{t} \omega}+\int_{t_{1}+h+p_{t} \omega}^{t}\right\} \\
& \times\left[-g(s)+h(s) \phi\left(\epsilon_{0}\right)\right] d s \\
\leq & y\left(t_{1}+h\right) \exp \left(r^{*} \omega\right) \\
& \times \exp \left\{p_{t} \int_{0}^{\omega}\left[-g(s)+h(s) \phi\left(\epsilon_{0}\right)\right] d s\right\}
\end{align*}
$$

where $r^{*}=\max _{0 \leq t \leq+\infty}\left\{|g(t)|+h(t) \phi\left(\epsilon_{0}\right)\right\}$. Hence, from (25), we obtain $y(t) \rightarrow 0$ as $t \rightarrow \infty$. This leads to a contradiction. Thus, (26) holds.

Now, we prove that (23) is true. Otherwise, there is a sequence of initial functions $\left\{\psi_{n}\right\} \subset C_{+}$for system (1) such that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} y\left(t, \psi_{n}\right)>\left(M_{0}+1\right) n, \quad \forall n=1,2,3, \ldots \tag{31}
\end{equation*}
$$

In view of (26), for each $n$, there are time sequences $\left\{s_{q}^{(n)}\right\}$ and $\left\{t_{q}^{(n)}\right\}$, satisfying $0<s_{1}^{(n)}<t_{1}^{(n)}<s_{2}^{(n)}<t_{2}^{(n)}<\cdots<s_{q}^{(n)}<$ $t_{q}^{(n)}<\cdots$ and $s_{q}^{(n)} \rightarrow \infty$ as $q \rightarrow \infty$, such that

$$
\begin{align*}
& y\left(s_{q}^{(n)}, \psi_{n}\right)=M_{0}, \quad y\left(t_{q}^{(n)}, \psi_{n}\right)=\left(M_{0}+1\right) n  \tag{32}\\
& M_{0}<y\left(t, \psi_{n}\right)<\left(M_{0}+1\right) n, \quad \forall t \in\left(s_{q}^{(n)}, t_{q}^{(n)}\right) \tag{33}
\end{align*}
$$

By the ultimate boundedness of $\left(x_{1}\left(t, \psi_{n}\right), x_{2}\left(t, \psi_{n}\right)\right)$, for each $n$, there is a constant $T^{(n)}>0$ such that $x_{1}\left(t, \psi_{n}\right)<M_{1}$ for
all $t \geq T^{(n)}$. Further, for each $n$ there is $K^{(n)}>0$ such that $s_{q}^{(n)}>T^{(n)}+h$ for all $q \geq K^{(n)}$. Hence, for all $q \geq K^{(n)}$, directly from system (1) we have

$$
\begin{align*}
y\left(t_{q}^{(n)}, \psi_{n}\right)= & y\left(s_{q}^{(n)}, \psi_{n}\right) \\
& \times \exp \int_{s_{q}^{(n)}}^{t_{q}^{(n)}}\left[-g(s)+h(s) \int_{-h}^{0} k_{21}(\mu) \phi\right. \\
& \left.\times\left(x_{1}\left(\mu+s, \psi_{n}\right)\right) d \mu\right] d s \\
\leq & y\left(s_{q}^{(n)}, \psi_{n}\right) \exp \int_{s_{q}^{(n)}}^{t_{q}^{(n)}}\left[-g(s)+h(s) \phi\left(M_{1}\right)\right] d s \\
\leq & y\left(s_{q}^{(n)}, \psi_{n}\right) \exp \left[r_{1}\left(t_{q}^{(n)}-s_{q}^{(n)}\right)\right] \tag{34}
\end{align*}
$$

where $r_{1}=\max _{t \in[0,+\infty)}\left\{|g(t)|+h(t) \phi\left(M_{1}\right)\right\}$. Consequently, by (32) we have

$$
\begin{equation*}
t_{q}^{(n)}-s_{q}^{(n)} \geq \frac{\ln n}{r_{1}}, \quad \forall q \geq K^{(n)} \tag{35}
\end{equation*}
$$

Hence, for any constant $L>0$, there is a $N_{0}>0$ such that $t_{q}^{(n)}>s_{q}^{(n)}+2 L+h$ for all $n \geq N_{0}$ and $q \geq K^{(n)}$. For any fixed $n \geq N_{0}$ and $q \geq K^{(n)}$, we prove that there must be $\tilde{t}_{1} \in$ $\left[s_{q}^{(n)}+h, s_{q}^{(n)}+L+h\right]$ such that $x_{1}\left(\tilde{t}_{1}, \psi_{n}\right)<\epsilon_{0}$. Otherwise, if $x_{1}\left(t, \psi_{n}\right) \geq \epsilon_{0}$ for all $t \in\left[s_{q}^{(n)}+h, s_{q}^{(n)}+L+h\right]$, then, directly from system (1), we have by (24) and (33). Consider

$$
\begin{align*}
\frac{d x_{1}\left(t, \psi_{n}\right)}{d t} \leq & a(t) x_{2}\left(t, \psi_{n}\right) \\
& -p(t) \phi\left(x_{1}\left(t, \psi_{n}\right)\right) \int_{-h}^{0} k_{12}(s) y\left(t+s, \psi_{n}\right) d s \\
\leq & a(t) x_{2}\left(t, \psi_{n}\right)-p(t) \phi\left(x_{1}\left(t, \psi_{n}\right)\right) M_{0} \\
< & M_{1} a^{m}-p^{l} M_{0} \phi\left(\epsilon_{0}\right)<-\epsilon_{0} . \tag{36}
\end{align*}
$$

We can choose enough large $L>0$ such that $M_{1}-L \epsilon_{0}<0$. Integrating this inequality from $s_{q}^{(n)}+h$ to $s_{q}^{(n)}+L+h$, we have

$$
\begin{align*}
x_{1}\left(s_{q}^{(n)}+L+h, \psi_{n}\right)< & x_{1}\left(s_{q}^{(n)}+h, \psi_{n}\right)  \tag{37}\\
& -\epsilon_{0} L \leq M_{1}-L \epsilon_{0}<0 .
\end{align*}
$$

This leads to a contradiction. Next, we prove $x_{1}\left(t, \psi_{n}\right)<\epsilon_{0}$ for all $t \in\left[\tilde{t}_{1}, t_{q}^{(n)}\right]$. Otherwise, there is a $\tilde{t}_{2}>\tilde{t}_{1}$ such that $x_{1}\left(\tilde{t}_{2}, \psi_{n}\right)=\epsilon_{0}$ and $x_{1}\left(t, \psi_{n}\right)<\epsilon_{0}$ for all $t \in\left(\tilde{t}_{1}, \tilde{t}_{2}\right)$. Then, we have $d x_{1}\left(\tilde{t}_{2}, \psi_{n}\right) / d t \geq 0$. On the other hand, a similar calculation as in (36), we have

$$
\begin{equation*}
\frac{d x_{1}\left(\tilde{t}_{2}, \psi_{n}\right)}{d t} \leq M_{1} a^{m}-p^{l} M_{0} \phi\left(\epsilon_{0}\right)<-\epsilon_{0} . \tag{38}
\end{equation*}
$$

This leads to a contradiction. Therefore, $x_{1}(t)<\epsilon_{0}$ for all $t \in$ $\left[s_{q}^{(n)}+L+h, t_{q}^{(n)}\right]$ for all $n \geq N_{0}$ and $q \geq K^{(n)}$. From (32) and (33), we have

$$
\begin{align*}
&\left(M_{0}+1\right) n= y\left(t_{q}^{(n)}, \psi_{n}\right)=y\left(s_{q}^{(n)}+L+h, \psi_{n}\right) \\
& \times \exp \int_{s_{q}^{(n)}+L+h}^{t_{q}^{(n)}}\left[-g(t)+h(t) \int_{-h}^{0} k_{21}\right. \\
&\left.\times(s) \phi\left(x_{1}\left(t+s, \psi_{n}\right)\right) d s\right] d t \\
& \leq y\left(s_{q}^{(n)}+L+h, \psi_{n}\right) \\
& \times \exp \int_{s_{q}^{(n)}+L+h}^{t_{q}^{(n)}}\left[-g(t)+h(t) \phi\left(\epsilon_{0}\right)\right] d t \\
&<\left(M_{0}+1\right) n \exp \int_{s_{q}^{(n)}+L+h}^{t_{q}^{(n)}}[-g(t) \\
&\left.\quad+h(t) \phi\left(\epsilon_{0}\right)\right] d t . \tag{39}
\end{align*}
$$

From (25) we can choose large enough constant $L>0$ such that

$$
\begin{equation*}
\exp \int_{t}^{t+L}\left[-g(t)+h(t) \phi\left(\epsilon_{0}\right)\right] d t<1 \tag{40}
\end{equation*}
$$

for all $t \in R_{+}$. Hence, from (39) we finally obtain a contradiction $\left(M_{0}+1\right) n<\left(M_{0}+1\right) n$. This shows that (23) holds. Choose a constant $M=\max \left\{M_{1}, M_{2}\right\}$. Then we obtain that the conclusion of Theorem 7 is true. This completes the proof.

Proof of Theorem 9. We will use Lemma 6 to prove this theorem. We choose space

$$
\begin{align*}
X=\{\psi & =\left(\psi_{1}, \psi_{2}, \psi_{3}\right): \psi_{i}(\theta)>0, \\
& \left.i=1,2, \psi_{3}(\theta) \geq 0, \forall \theta \in[-h, 0]\right\}, \tag{41}
\end{align*}
$$

and sets $X^{0}$ and $\partial X^{0}$ are defined by

$$
\begin{gather*}
X^{0}=\left\{\psi=\left(\psi_{1}, \psi_{2}, \psi_{3}\right) \in X: \psi_{3}(0)>0\right\} \\
\partial X^{0}=\left\{\psi=\left(\psi_{1}, \psi_{2}, \psi_{3}\right) \in X: \psi_{3}(\theta) \equiv 0, \forall \theta \in[-h, 0]\right\} \tag{42}
\end{gather*}
$$

For any $\psi \in X$, let $x(t, \psi)=\left(x_{1}(t, \psi), x_{2}(t, \psi), y(t, \psi)\right)$ be the solution of system (1) with initial value $\psi$ at $t=0$. We define continuous map $P$ in Lemma 6 as follows:

$$
\begin{equation*}
P(\psi)=x_{\omega}(\psi), \quad \psi \in X \tag{43}
\end{equation*}
$$

where $x_{\omega}(\psi)=x(\omega+s, \psi)$ with $s \in[-h, 0]$.
Now, we verify that all the conditions of Lemma 6 will be satisfied for map $P$. It is easy to see that $X^{0}$ and $\partial X^{0}$ are positively invariant. From the expression of right side functional $f(t, \phi)$ of system (1), we can directly obtain that,
for any bounded set $A \subset X$, there is a constant $M(A)>0$ such that $|f(t, \phi)| \leq M(A)$ for all $t \geq 0$ and $\phi \in A$. By the Ascoli-Arzela theorem, it implies that map $P$ is compact on $X$; that is, for any bounded set $B \subset X$, set $P(B)=\{P(\psi)=$ $\left.x_{\omega}(\psi): \psi \in B\right\}$ is precompact. Moreover, by Theorem 7, we obtain that map $P$ is also point dissipative on $X$.

Further, we define

$$
\begin{equation*}
M_{\partial}=\left\{\psi \in \partial X^{0}: P^{m}(\psi) \in \partial X^{0}, \forall m>0\right\} \tag{44}
\end{equation*}
$$

where $P^{m}=P\left(P^{m-1}\right)$ for all $m>1$ and $P^{1}(\psi)=P(\psi)$. Obviously, we have $M_{\partial}=\partial X^{0}$.

Denote by $\omega(\psi)$ the $\omega$-limit set of solution $x(t, \psi)$ of system (1) starting at $t=0$ with initial value $\psi \in X$. Let

$$
\begin{equation*}
\Omega\left(M_{\partial}\right)=\bigcup\left\{\omega(\psi): \psi \in M_{\partial}\right\} \tag{45}
\end{equation*}
$$

From Remark 2, there is a fixed point of map $P$ in $M_{\partial}$, which is $M_{1}=\left(x_{1}^{*}(0), x_{2}^{*}(0), 0\right)$.

From (10), we can choose a constant $\epsilon_{0}>0$ such that

$$
\begin{equation*}
A_{\omega}\left(-g(t)+h(t) \int_{-h}^{0} k_{21}(s) \phi\left(x_{1}^{*}(t+s)-\epsilon_{0}\right) d s\right)>0 \tag{46}
\end{equation*}
$$

By the continuity of solutions with respect to the initial value, for the above given constant $\epsilon_{0}>0$, there exists $\delta_{0}>0$ such that for all $\psi \in X^{0}$ with $\left\|\psi-M_{1}\right\| \leq \delta_{0}$, it follows that

$$
\begin{equation*}
\left\|x_{t}(\psi)-x_{t}\left(M_{1}\right)\right\|<\epsilon_{0}, \quad \forall t \in[0, \omega] . \tag{47}
\end{equation*}
$$

Now, we prove

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} d\left(P^{m}(\psi), M_{1}\right) \geqslant \delta_{0} \tag{48}
\end{equation*}
$$

Suppose the conclusion is not true, then

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} d\left(P^{m}(\psi), M_{1}\right)<\delta_{0} \tag{49}
\end{equation*}
$$

for some $\psi \in X^{0}$. Without loss of generality, we can assume that

$$
\begin{equation*}
d\left(P^{m}(\psi), M_{1}\right)<\delta_{0}, \quad \forall m \geq 0 \tag{50}
\end{equation*}
$$

Further, from (47) we have

$$
\begin{equation*}
\left\|x_{t}\left(P^{m}(\psi)\right)-x_{t}\left(M_{1}\right)\right\|<\epsilon_{0}, \quad \forall m \geq 0, t \in[0, \omega] \tag{51}
\end{equation*}
$$

For any $t \geq 0$, let $t=m \omega+t^{\prime}$, where $t^{\prime} \in[0, \omega]$ and $m=[t / \omega]$ are the greatest integers less than or equal to $[t / \omega]$, then we can get

$$
\begin{array}{r}
\left\|x_{t}(\psi)-x_{t}\left(M_{1}\right)\right\|=\left\|x_{t^{\prime}}\left(P^{m}(\psi)\right)-x_{t^{\prime}}\left(M_{1}\right)\right\|<\epsilon_{0},  \tag{52}\\
\forall t \geq 0
\end{array}
$$

Since $x_{t}(\psi)=\left(x_{1}(t+s, \psi), x_{2}(t+s, \psi), y(t+s, \psi)\right)$, and $x_{t}\left(M_{1}\right)=\left(x_{1}^{*}(t+s), x_{2}^{*}(t+s), 0\right)$ for all $s \in[-h, 0]$, it follows from (52) that, for all $t \geq-h$,

$$
\begin{gather*}
0 \leq y(t, \psi)<\epsilon_{0}, \quad\left|x_{1}(t, \psi)-x_{1}^{*}(t)\right|<\epsilon_{0} \\
\left|x_{2}(t, \psi)-x_{2}^{*}(t)\right|<\epsilon_{0} \tag{53}
\end{gather*}
$$

Then, by the third equation of system (1), we get, for any $t \geq 0$,

$$
\begin{align*}
\frac{d y(t, \psi)}{d t}= & y(t, \psi) \\
& \times\left(-g(t)+h(t) \int_{-h}^{0} k_{21}(s) \phi\left(x_{1}(t+s, \psi)\right) d s\right) \\
\geq & y(t, \psi) \\
& \times(-g(t) \\
& \left.\quad+h(t) \int_{-h}^{0} k_{21}(s) \phi\left(x_{1}^{*}(t+s)-\epsilon_{0}\right) d s\right) \tag{54}
\end{align*}
$$

Therefore, we further have, for any $t \geq 0$,

$$
\begin{align*}
y(t, \psi) \geq \psi(0) \exp \left(\int_{0}^{t}( \right. & -g(u)+h(u) \int_{-h}^{0} k_{21}(s) \phi \\
& \left.\left.\times\left(x_{1}^{*}(u+s)-\epsilon_{0}\right) d s\right) d u\right) . \tag{55}
\end{align*}
$$

From (46) we can directly obtain that $\lim _{t \rightarrow \infty} y(t, \psi)=\infty$, which leads to a contradiction. Therefore, claim (48) holds. This shows that

$$
\begin{equation*}
W^{s}\left(M_{1}\right) \cap X^{0}=\emptyset \tag{56}
\end{equation*}
$$

From Lemma 1 we can obtain that $\left\{M_{1}\right\}$ is a global attractor of map $P$ in $M_{\partial}$; that is, each orbit of map $P$ in $M_{\partial}$ converges to $\left\{M_{1}\right\}$. Hence, $\left\{M_{1}\right\}$ is isolated in $M_{\partial}$, and, hence, in $X$ by (56). Furthermore, $\left\{M_{1}\right\}$ also is invariant and $\left\{M_{1}\right\}$ does not form a cycle in $M_{\partial}$ and, hence, in $\partial X^{0}$.

Therefore, all the conditions of Lemma 6 are satisfied. By Lemma 6 we finally obtain that map $P$ is uniformly persistent with respect to $\left(X^{0}, \partial X^{0}\right)$. Further, from Theorem 3.1.1 given in [10], we can obtain that all positive solutions of system (1) are uniformly persistent. This completes the proof.

Proof of Theorem 12. From (12), we can choose a constant $0<$ $\epsilon_{0}<1$, such that

$$
\begin{align*}
A_{\omega} & {\left[-g(t)+h(t) \int_{-h}^{0} k_{21}(s) \phi\left(x_{1}^{*}(t+s)\right) d s+h(t) \epsilon_{0}\right] } \\
& \leq-\epsilon_{0} \tag{57}
\end{align*}
$$

We first show that for any positive solution $\left(x_{1}(t), x_{2}(t), y(t)\right)$ of system (1) $\lim _{t \rightarrow \infty} y(t)=0$. Since

$$
\begin{align*}
& \frac{d x_{1}(t)}{d t}<a(t) x_{2}(t)-b(t) x_{1}(t)-d(t) x_{1}^{2}(t)  \tag{58}\\
& \frac{d x_{2}(t)}{d t}=c(t) x_{1}(t)-f(t) x_{2}^{2}(t)
\end{align*}
$$

for all $t>0$. By the comparison theorem and Corollary 3, we obtain that, for any $\epsilon \in\left(0, \epsilon_{0}\right)$, there is a $T_{\epsilon}>0$ such that

$$
\begin{equation*}
x_{i}(t)<x_{i}^{*}(t)+\epsilon, \quad \forall t \geq T_{\epsilon}, \quad i=1,2 . \tag{59}
\end{equation*}
$$

For any $t>T_{\epsilon}+h$, from system (1), we have

$$
\begin{align*}
\frac{d y(t)}{d t}=y(t) & {\left[-g(t)+h(t) \int_{-h}^{0} k_{21}(s) \phi\left(x_{1}(t+s)\right) d s\right] } \\
\leq y(t) & {[-g(t)} \\
& \left.\quad+h(t) \int_{-h}^{0} k_{21}(s) \phi\left(x_{1}^{*}(t+s)+\epsilon_{0}\right) d s\right] \tag{60}
\end{align*}
$$

From (57) and (60), we obtain that $\lim _{t \rightarrow \infty} y(t)=0$.
Consider the following system with a parameter $\alpha$ :

$$
\begin{align*}
& \frac{d x_{1}(t)}{d t}= a(t) x_{2}(t)-b(t) x_{1}(t) \\
&-d(t) x_{1}^{2}(t)-\alpha p(t) x_{1}(t),  \tag{61}\\
& \frac{d x_{2}(t)}{d t}=c(t) x_{1}(t)-f(t) x_{2}^{2}(t)
\end{align*}
$$

From Lemma 1 we obtain that (61) has a unique globally asymptotically stable positive $\omega$-periodic solution $\left(x_{1 \alpha}^{*}(t), x_{2 \omega}^{*}(t)\right)$. By the continuity of solutions with respect to the parameter, we further obtain

$$
\begin{equation*}
\lim _{\alpha \rightarrow 0}\left(x_{1 \alpha}^{*}(t), x_{2 \omega}^{*}(t)\right)=\left(x_{1}^{*}(t), x_{2}^{*}(t)\right) \tag{62}
\end{equation*}
$$

where $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)$ is the globally asymptotically stable positive $\omega$-periodic solution of system (6). Therefore, for any $\epsilon>0$ there is an $\alpha_{0}>0$ such that, for all $t \in R_{+}$,

$$
\begin{equation*}
x_{i \alpha_{0}}^{*}(t)>x_{i}^{*}(t)-\frac{\epsilon}{2}, \quad i=1,2 . \tag{63}
\end{equation*}
$$

Let $\bar{\phi}=\sup \left\{\phi\left(x_{1}\right) / x_{1}: 0 \leq x_{1} \leq M_{1}\right\}$, then, from assumption $\left(A_{3}\right)$, we obtain $0<\bar{\phi}<\infty$, where $M_{1}$ is given in Theorem 7. Since $\lim _{t \rightarrow \infty} y(t)=0$, there is a $T_{0}>0$ such that $y(t)<\alpha_{0} / \bar{\phi}$ for all $t \geq T_{0}$. Hence, for any $t \geq T_{0}+h$, we have

$$
\begin{align*}
\frac{d x_{1}(t)}{d t}= & a(t) x_{2}(t)-b(t) x_{1}(t) \\
& -d(t) x_{1}^{2}(t)-\alpha_{0} p(t) x_{1}(t)  \tag{64}\\
\frac{d x_{2}(t)}{d t} & =c(t) x_{1}(t)-f(t) x_{2}^{2}(t)
\end{align*}
$$

From the comparison theorem and Lemma 1, we can obtain that there is a $T_{1}>T_{0}$ such that, for all $t \geq T_{1}$,

$$
\begin{equation*}
x_{i}(t)>x_{i \alpha_{0}}^{*}(t)-\frac{\epsilon}{2}>x_{i}^{*}(t)-\epsilon, \quad i=1,2 \tag{65}
\end{equation*}
$$

Combining (59), we finally obtain that, for all $t \geq \max \left\{T_{\epsilon}, T_{1}\right\}$,

$$
\begin{equation*}
x_{i}^{*}(t)-\epsilon<x_{i}(t)<x_{i}^{*}(t)+\epsilon, \quad i=1,2 . \tag{66}
\end{equation*}
$$

Therefore, $\lim _{t \rightarrow \infty} x_{i}(t)=x_{i}^{*}(t)(i=1,2)$. This completes the proof.

## 5. Examples and Numerical Simulations

In order to testify the validity of results, we consider the following predator-prey system. The system was obtained by letting $k_{12}(s)=\delta\left(s+\tau_{1}\right)$ and $k_{21}(s)=\delta\left(s+\tau_{2}\right)$ in system (1). Consider

$$
\begin{align*}
\frac{d x_{1}(t)}{d t}= & a(t) x_{2}(t)-b(t) x_{1}(t) \\
& -d(t) x_{1}^{2}(t)-p(t) \phi\left(x_{1}(t)\right) y\left(t-\tau_{1}\right) \\
\frac{d x_{2}(t)}{d t}= & c(t) x_{1}(t)-f(t) x_{2}^{2}(t)  \tag{67}\\
\frac{d y(t)}{d t}= & y(t)\left[-g(t)+h(t) \phi\left(x_{1}\left(t-\tau_{2}\right)\right)\right]
\end{align*}
$$

The corresponding prey subsystem is

$$
\begin{align*}
& \frac{d x_{1}(t)}{d t}=a(t) x_{2}(t)-b(t) x_{1}(t)-d(t) x_{1}^{2}(t) \\
& \frac{d x_{2}(t)}{d t}=c(t) x_{1}(t)-f(t) x_{2}^{2}(t) \tag{68}
\end{align*}
$$

Example 1. In system (67), we let $a(t)=1.2+\sin (2 \pi t)$, $b(t)=0.2, d(t)=0.2, c(t)=0.5 \sin (2 \pi t)+0.75, f(t)=$ $0.8, p(t)=0.4, g(t)=0.28+0.1 \sin (\pi t / 2), h(t)=2.1+$ $2 \sin (\pi t / 3), \tau_{1}=0.2, \tau_{2}=0.3$, and $\phi\left(t, x_{1}(t)\right)=\left(x_{1}(t)\right) /(9+$ $\left.x_{1}^{2}(t)\right)$. We take different initial functions $\psi=\left(\psi_{1}, \psi_{2}, \psi_{3}\right)=$ $(2.5+k, 1+k, 0.3+k), k=0,1,2,3,4,5$ for all $s \in$ $[-0.3,0]$. We easily verify that assumptions $\left(A_{1}\right)-\left(A_{3}\right)$ hold. Therefore, from Lemma 1 , system (68) has a unique globally asymptotically stable positive periodic solution $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)$. By the numerical simulations, we get that the upper and lower bounds of periodic function $x_{1}^{*}(t)$ are 2.87 and 1, respectively.

It is easy to verify that condition (10) in Theorem 9 also holds. Therefore, from Theorem 9 and Corollary 14, we obtain that system (67) is ultimately bounded and permanent and at least has a positive periodic solution. The numerical simulations of the above results can be seen in Figures 1, 2, 3, and 4.

Remark 1. There is an open question: from Figure 3, we see that $y(t)$ of system (1) has more than one periodic solution. So, we cannot get a globally asymptotically stability solution of system (1). Whether we can get a globally asymptotically stability solution of system (1) under some conditions is our future work.

Example 2. In system (67), the coefficients $g(t), h(t), p(t), \tau_{1}$, $\tau_{2}$, and $\phi\left(t, x_{1}(t)\right)$ are given as in Example 1. But, the other coefficients in system (67) are given as the following different values: $a(t)=1+\sin (2 \pi t), c(t)=0.2 \sin (2 \pi t)+0.75$, $b(t)=0.25, d(t)=0.35$, and $f(t)=1$. We see that coefficients $a(t)$ and $c(t)$ are decreased and coefficients $b(t)$, $d(t)$, and $f(t)$ are increased. From Corollary 3, system (68) has a unique globally asymptotically stable positive periodic solution $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)$. Moreover, compared with Example 1, we easily see that $x_{1}^{*}(t)$ and $x_{2}^{*}(t)$ will decrease. Further,


Figure 1


Figure 2


Figure 3


Figure 4


Figure 5
we easily verify that assumptions $\left(A_{1}\right)-\left(A_{3}\right)$ hold. It is easy to verify that condition (10) in Theorem 9 does not hold, but condition (12) in Theorem 12 holds. Therefore, from Theorem 12, we obtain that predator $y(t)$ in system (67) will become into extinction. The numerical simulations of the above results can be seen in Figures 5 and 6 by taking initial function $\psi=\left(\psi_{1}, \psi_{2}, \psi_{3}\right)=(2.5,1,5)$ for all $s \in[-0.3,0]$.

Remark 2. From the numerical simulations given in Examples 1 and 2 , we see that the stage-structure in the prey, specially the birthrate, mortality, density restriction of infancy prey, the transformation from the infancy prey to the maturity prey, and the mortality and density restriction of maturity prey, will bring the very obvious effect for the permanence and extinction of the predator.

Example 3. In system (67), $a(t), b(t), d(t), c(t), f(t), p(t), \tau_{1}$, $\tau_{2}$, and $\phi\left(t, x_{1}(t)\right)$ are given as in Example 1, but $g(t)=0.28+$ $0.3 \sin (\pi t / 2)$ and $h(t)=2.1+2 \sin (\pi t / 7)$. We take initial


Figure 7
function $\psi=\left(\psi_{1}, \psi_{2}, \psi_{3}\right)=(2.5,2,0.1)$ for all $s \in[-0.3,0]$. We easily verify that assumptions $\left(A_{1}\right)-\left(A_{3}\right)$ hold. From Example 1 , system (68) has a unique globally asymptotically stable positive periodic solution $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)$, and the upper and lower bounds of periodic function $x_{1}^{*}(t)$ are 2.87 and 1 , respectively.

It is easy to verify that condition (10) in Theorem 9 does not hold. Therefore, Theorem 9 and Corollary 14 are invalid. Numerical simulations of the above results can be seen in Figures 7 and 8. From Figure 7, we see that the prey species $x$ is permanent; the predator species $y$ is permanent, too.

Example 4. In system (67), $a(t), b(t), d(t), c(t), f(t), p(t)$, $\tau_{1}, \tau_{2}$, and $\phi\left(t, x_{1}(t)\right)$ are given as in Example 1, but $g(t)=$ $0.28+0.1 \sin (t /(2 \pi))$ and $h(t)=1.3+1.2 \sin (t /(5 \pi))$. We take initial function $\psi=\left(\psi_{1}, \psi_{2}, \psi_{3}\right)=(2.5,1,0.3)$ for all $s \in$ $[-0.3,0]$. We easily verify that assumptions $\left(A_{1}\right)-\left(A_{3}\right)$ hold. Therefore, from Lemma 1, system (68) has a unique globally asymptotically stable positive periodic solution $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)$.


Figure 8


Figure 9

By the numerical simulations, we get that the upper and lower bounds of periodic function $x_{1}^{*}(t)$ are 2.87 and 1 , respectively.

Take the upper bounds of periodic function $x_{1}^{*}(t)$ into condition (10), we easily verify that condition (10) in Theorem 9 will hold. But we obtain that predator $y(t)$ in system (67) is extinct. The numerical simulations of the above results can be seen in Figures 9 and 10. From Figure 9, we see that the prey species $x$ is permanent, while the predator species $y$ turns to extinction.

Example 5. In systems (67), $a(t), b(t), d(t), c(t), f(t), p(t)$, $\tau_{1}, \tau_{2}$, and $\phi\left(t, x_{1}(t)\right)$ are given as in Example 1, but $g(t)=$ $0.3+0.1 \sin (\pi t / 2)$ and $h(t)=2.3+0.5 \sin (\pi t / 7)$. We take initial function $\psi=\left(\psi_{1}, \psi_{2}, \psi_{3}\right)=(2.5,1,0.3)$ for all $s \in$ [ $-0.3,0]$. We easily verify that assumptions $\left(A_{1}\right)-\left(A_{3}\right)$ hold. Therefore, from Lemma 1, system (68) has a unique globally asymptotically stable positive periodic solution $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right)$. By the numerical simulations, we get that the upper and lower bounds of periodic function $x_{1}^{*}(t)$ are 2.87 and 1, respectively.


Figure 11

Take the lower bounds of periodic function $x_{1}^{*}(t)$ into condition (12), we easily verify that condition (12) in Theorem 12 will hold. But we obtain that predator $y(t)$ in system (67) is permanent. The numerical simulations of the above results can be seen in Figures 11 and 12. From Figure 12, we see that the prey species $x$ is permanent; the predator species $y$ is permanent, too.

## 6. Conclusions

In the real world, there are many types of interactions between two species. Predator-prey relations are among the most common ecological interactions.

In this paper, we study the global property in a delayed periodic predator-prey model with stage-structure in prey and density-independence in predator. The survival of species in a biological system is one of the most basic and important problems in mathematical biology, and permanence is an important concept when dealing with this problem. Here,


Figure 12
by using the analysis method, the comparison theorem of cooperative system, and the theory of the persistence of dynamical systems, we have established the integral form criteria on the ultimate boundedness, the sufficient integral conditions on the permanence and extinction of species. The method used in this paper is motivated by the works on the permanence and extinction for periodic predator-prey systems in patchy environment given by Teng and Chen in [5]. The results obtained in this paper are different from the predator-prey system given in [4], where the authors studied the necessary and sufficient integral conditions on permanence and extinction of species for nonautonomous predator-prey systems with infinite delays and predator density dependence. However, in our paper, we have considered the effects of general predator functional response on the survival of species. Therefore, we have modeled a general nonautonomous predator-prey system with finite delays and density independence. Some well-known results on the predator density-dependence are improved and extended to the predator density-independent cases.

## Conflict of Interests

The authors declare that they have no financial and personal relationships with other people or organizations that can inappropriately influence their work; there is no professional or other personal interests of any nature or kind in any product, service, and/or company that could be construed as influencing the position presented in, or the review of, this paper.

## Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant nos. 11271312, 11261056, 61164004, and 11301451), the China Postdoctoral Science Foundation (Grant nos. 20110491750 and 2012T50836), the Natural Science Foundation of Xinjiang (Grant nos. 2011211B08 and 2013211B06), and the Research Project at Xinjiang Institute of Engineering (2013XGZ281412).

## References

[1] W. Sokol and J. A. Howell, "Kinetics of phenol oxidation by washed cells," Biotechnology and Bioengineering, vol. 23, no. 9, pp. 2039-2049, 1980.
[2] J. Cui and X. Song, "Permanence of predator-prey system with stage structure," Discrete and Continuous Dynamical Systems B, vol. 4, no. 3, pp. 547-554, 2004.
[3] J. Cui and Y. Takeuchi, "A predator-prey system with a stage structure for the prey," Mathematical and Computer Modelling, vol. 44, no. 11-12, pp. 1126-1132, 2006.
[4] J. Cui and Y. Sun, "Permanence of predator-prey system with infinite delay," Electronic Journal of Differential Equations, vol. 2004, no. 81, pp. 1-12, 2004.
[5] Z. Teng and L. Chen, "Permanence and extinction of periodic predator-prey systems in a patchy environment with delay," Nonlinear Analysis: Real World Applications, vol. 4, no. 2, pp. 335-364, 2003.
[6] J. K. Hale, Theory of Functional Differential Equations, Springer, New York, NY, USA, 1977.
[7] J. K. Hale and J. Kato, "Phase space for retarded equations with infinite delay," Funkcialaj Ekvacioj, vol. 21, no. 1, pp. 11-41, 1978.
[8] Y. Kuang, Delay Differential Equation with Applications in Population Dynamics, Academic Press, New York, NY, USA, 1993.
[9] J. Cui, L. Chen, and W. Wang, "The effect of dispersal on population growth with stage-structure," Computers \& Mathematics with Applications, vol. 39, no. 1-2, pp. 91-102, 2000.
[10] X. Zhao, Dynamical Systems in Population Biology, Springer, New York, NY, USA, 2003.
[11] Z. Teng and L. Chen, "The positive periodic solutions of periodic Kolmogorov type systems with delays," Acta Mathematicae Applicatae Sinica, vol. 22, no. 3, pp. 446-454, 1999.
[12] H. L. Smith, "Cooperative systems of differential equations with concave nonlinearities," Nonlinear Analysis: Theory, Methods \& Applications, vol. 10, no. 10, pp. 1037-1052, 1986.
[13] X. Zhao, "The qualitative analysis of $n$-species Lotka-Volterra periodic competition systems," Mathematical and Computer Modelling, vol. 15, no. 11, pp. 3-8, 1991.

## Research Article

# The Stability of SI Epidemic Model in Complex Networks with Stochastic Perturbation 

Jinqing Zhao, Maoxing Liu, Wanwan Wang, and Panzu Yang<br>Department of Mathematics, North University of China, Taiyuan, Shanxi 030051, China<br>Correspondence should be addressed to Maoxing Liu; liumaoxing@126.com

Received 29 November 2013; Accepted 12 January 2014; Published 13 March 2014
Academic Editor: Kaifa Wang
Copyright © 2014 Jinqing Zhao et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We investigate a stochastic SI epidemic model in the complex networks. We show that this model has a unique global positive solution. Then we consider the asymptotic behavior of the model around the disease-free equilibrium and show that the solution will oscillate around the disease-free equilibrium of deterministic system when $R_{0} \leq 1$. Furthermore, we derive that the disease will be persistent when $R_{0}>1$. Finally, a series of numerical simulations are presented to illustrate our mathematical findings. A new result is given such that, when $R_{0} \leq 1$, with the increase of noise intensity the solution of stochastic system converging to the disease-free equilibrium is faster than that of the deterministic system.


## 1. Introduction

Epidemiology is the science to study the distribution of disease and influencing factors, so as to explore the etiology, clarify the popular rule of the disease, and formulate the countermeasures and measures for preventing, controlling, and eliminating the disease. Many mathematical models of diseases spreading help us to understand the propagation of diseases [1,2]. The transmission of diseases can be influenced by many factors, such as the age and social structure of the population, the contact network among individuals, and the metapopulation characteristics. So it is difficult to establish an accurate epidemic model which is completely consistent with the real world. In recent years, a lot of compartmental epidemic models have been studied by many researchers [35], and complex networks also have been used to study the spread of diseases [6-17].

In this paper we consider an SI model with the birth and death in complex networks. As mentioned in the paper $[6,13]$, the birth and death do not affect the degree of nodes. Suppose $S_{k}(t), I_{k}(t)$ are the number of the healthy and infected nodes with the degree $k$ at time $t$; the mean-field equations can be written as

$$
\frac{d S_{k}}{d t}=b_{k}-\lambda k S_{k} \theta-d S_{k}
$$

$$
\begin{equation*}
\frac{d I_{k}}{d t}=\lambda k S_{k} \theta-(d+\epsilon) I_{k} \tag{1}
\end{equation*}
$$

where $\theta=(1 /\langle k\rangle) \sum_{k=1}^{n} \lambda k P(k)$. For system (1), it can be written as the following form:

$$
\begin{align*}
& d S_{k}(t)=\left(b_{k}-\frac{1}{\langle k\rangle} \sum_{j=1}^{n} \lambda k j P(j) S_{k}(t) I_{j}(t)-d S_{k}(t)\right) d t \\
& d I_{k}(t)=\left(\frac{1}{\langle k\rangle} \sum_{j=1}^{n} \lambda k j P(j) S_{k}(t) I_{j}(t)-(d+\epsilon) I_{k}(t)\right) d t \tag{2}
\end{align*}
$$

We denote $\beta_{k j}=(1 /\langle k\rangle) \lambda k j P(j)$, so we obtain

$$
\begin{align*}
& d S_{k}(t)=\left(b_{k}-\sum_{j=1}^{n} \beta_{k j} S_{k}(t) I_{j}(t)-d S_{k}(t)\right) d t \\
& d I_{k}(t)=\left(\sum_{j=1}^{n} \beta_{k j} S_{k}(t) I_{j}(t)-(d+\epsilon) I_{k}(t)\right) d t \tag{3}
\end{align*}
$$

It always has the disease-free equilibrium $E_{0}=\left(S_{1}^{0}, 0, \ldots\right.$, $\left.S_{n}^{0}, 0\right)$, where $S_{k}^{0}=b_{k} / d, k=1,2, \ldots, n$. If $A=\left(\beta_{k j}\right)_{n \times n}$ is irreducible and $R_{0} \leq 1$, then $E_{0}$ is globally stable in $D$, while
if $R_{0}>1, E_{0}$ is unstable and there is an endemic equilibrium $E^{*}=\left(S_{1}^{*}, I_{1}^{*}, \ldots, S_{n}^{*}, I_{n}^{*}\right)$ belonging to $D$ which is globally asymptotically stable in $D$; here

$$
\begin{align*}
D=\{ & \left(S_{1}, I_{1}, \ldots, S_{n}, I_{n}\right) \in \mathbb{R}_{+}^{2 n}: S_{k}, I_{k} \leq \frac{b_{k}}{d}, S_{k}+I_{k} \leq \frac{b_{k}}{d}, \\
& k=1,2, \ldots, n\}, \\
M_{0}= & M\left(S_{1}^{0}, \ldots, S_{n}^{0}\right)=\left(\frac{\beta_{k j} S_{k}^{0}}{d+\epsilon}\right)_{n \times n}, \quad R_{0}=\rho\left(M_{0}\right), \tag{4}
\end{align*}
$$

and $\rho\left(M_{0}\right)$ denotes the spectral radius of $M_{0}$.
The deterministic models have some limitations in describing the spread of disease. The accident in the process of disease transmission can not be reflected by the deterministic models. This is because of the fact that the deterministic models ignore the effect of the environmental noise. In an ecosystem, the environmental noise is inevitably in the real world; thus stochastic models are more realistic. In the research of stochastic epidemic models, many researchers make a lot of contributions [17-26].

In this paper, we consider the following stochastic system:

$$
\begin{align*}
d S_{k}(t)= & \left(b_{k}-\sum_{j=1}^{n} \beta_{k j} S_{k}(t) I_{j}(t)-d S_{k}(t)\right) d t \\
& +\sigma_{k 1} S_{k}(t) d B_{k 1}(t) \\
d I_{k}(t)= & \left(\sum_{j=1}^{n} \beta_{k j} S_{k}(t) I_{j}(t)-(d+\epsilon) I_{k}(t)\right) d t  \tag{5}\\
& +\sigma_{k 2} I_{k}(t) d B_{k 2}(t)
\end{align*}
$$

where $B_{k i}(t), k=1,2, \ldots, n, i=1,2$, are independent standard Brownian motions with $B_{k i}(0)=0$, and $\sigma_{k i}^{2} \geq 0$, $k=1,2, \ldots, n, i=1,2$, represent the intensities of $B_{k i}(t)$.

The remaining parts of this paper are as follows. In the next section we show the existence and uniqueness of a global positive solution of model (5). In Section 3, we analyze the asymptotic behavior around the disease-free equilibrium. In Section 4, we study the dynamic of system (5) around the endemic of the deterministic model. In Section 5, numerical simulations and conclusions are carried out.

## 2. Global Positive Solution

When we study a dynamical behavior, a global solution is important for the system. In this section we show that the solution of system (5) is global and nonnegative. As we know, for a stochastic differential equation, the coefficients of the equation are generally required to satisfy the linear growth condition and the local Lipschitz condition. It is a sufficient condition for a stochastic differential equation has a unique global (i.e., no explosion in a finite time) solution for any given initial value [27,28]. Although the coefficients of system
(5) satisfy locally Lipschitz continuous, they are not satisfied with the linear growth condition, so the solution of system (5) may explode at a finite time. In this section, Lyapunov analysis method (mentioned in [29]) is used to show that the solution of system (5) is positive and global.

Theorem 1. For any given initial value $\left(S_{1}(0), I_{1}(0), \ldots\right.$, $\left.S_{n}(0), I_{n}(0)\right) \in \mathbb{R}_{+}^{2 n}$, there is a unique positive solution $\left(S_{1}(t)\right.$, $\left.I_{1}(t), \ldots, S_{n}(t), I_{n}(t)\right)$ of model (5) on $t \geq 0$ and the solution will remain in $\mathbb{R}_{+}^{2 n}$ with probability 1 , namely, $\left(S_{1}(t)\right.$, $\left.I_{1}(t), \ldots, S_{n}(t), I_{n}(t)\right) \in \mathbb{R}_{+}^{2 n}$ for all $t \geq 0$ a.s.

Proof. Due to the fact that the coefficients of the system (5) are locally Lipschitz continuous, for any given initial value $\left(S_{1}(0), I_{1}(0), \ldots, S_{n}(0), I_{n}(0)\right) \in \mathbb{R}_{+}^{2 n}$, it has a unique local solution $\left(S_{1}(t), I_{1}(t), \ldots, S_{n}(t), I_{n}(t)\right)$ on $t \in\left[0, \tau_{e}\right)$, where $\tau_{e}$ is the explosion time [30]. If we show that $\tau_{e}=\infty$ a.s., it suggests that this solution is global. Let $l_{0}>0$ be sufficiently large so that $S_{k}(0), I_{k}(0)(k=1,2, \ldots, n)$ all lie within the interval $\left[1 / l_{0}, l_{0}\right]$. For each integer $l \geq l_{0}$, defining the stopping time

$$
\begin{gather*}
\tau_{l}=\inf \left\{t \in\left[0, \tau_{e}\right): \min \left\{S_{k}(t), I_{k}(t), k=1, \ldots, n\right\} \leq \frac{1}{l}\right. \\
\text { or } \left.\max \left\{S_{k}(t), I_{k}(t), k=1, \ldots, n\right\} \geq l\right\} \tag{6}
\end{gather*}
$$

we set $\inf \emptyset=\infty$ (as usual $\emptyset$ denotes the empty set). Obviously, $\tau_{l}$ is increasing as $l \rightarrow \infty$. Set $\tau_{\infty}=\lim _{l \rightarrow \infty} \tau_{l}$; therefore $\tau_{\infty} \leq \tau_{e}$ a.s. If $\tau_{\infty}=\infty$ a.s. is true, then $\tau_{e}=\infty$ a.s. and $\left(S_{1}(t), I_{1}(t), \ldots, S_{n}(t), I_{n}(t)\right) \in \mathbb{R}_{+}^{2 n}$ a.s. for $t \geq 0$. In other words, to complete the proof it is required to show that $\tau_{\infty}=\infty$ a.s. If this statement is false, then there is a pair of constants $T>0$ and $\varepsilon \in(0,1)$ such that $P\left\{\tau_{\infty} \leq T\right\}>\varepsilon$. Thus there is an integer $l_{1} \geq l_{0}$, such that

$$
\begin{equation*}
P\left\{\tau_{l} \leq T\right\} \geq \varepsilon, \quad \forall l \geq l_{1} \tag{7}
\end{equation*}
$$

Define a $C^{2}$-function $V: \mathbb{R}_{+}^{2 n} \rightarrow \mathbb{R}_{+}$as follows:

$$
\begin{align*}
V & \left(S_{1}, I_{1}, \ldots, S_{n}, I_{n}\right) \\
& =\sum_{k=1}^{n}\left[\left(S_{k}-1-\ln S_{k}\right)+\left(I_{k}-1-\ln I_{k}\right)\right] . \tag{8}
\end{align*}
$$

Applying Itô's formula, we obtain

$$
\begin{aligned}
& d\left(S_{1}, I_{1}, \ldots, S_{n}, I_{n}\right) \\
& =\sum_{k=1}^{n}\left[\left(1-\frac{1}{S_{k}}\right) d S_{k}+\frac{1}{2} \frac{1}{S_{k}^{2}}\left(d S_{k}\right)^{2}+\left(1-\frac{1}{I_{k}}\right) d I_{k}\right. \\
& \\
& \left.\quad+\frac{1}{2} \frac{1}{I_{k}^{2}}\left(d I_{k}\right)^{2}\right]
\end{aligned}
$$

$$
\begin{align*}
&=\sum_{k=1}^{n}[ \left(1-\frac{1}{S_{k}}\right)\left[\left(b_{k}-\sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}-d S_{k}\right) d t\right. \\
&\left.+\sigma_{k 1} S_{k}(t) d B_{k 1}(t)\right] \\
&+\frac{1}{2} \sigma_{k 1}^{2} d t+\left(1-\frac{1}{I_{k}}\right) \\
& \times\left[\left(\sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}-(d+\epsilon) I_{k}\right) d t+\sigma_{k 2} I_{k} d B_{k 2}(t)\right] \\
&=\sum_{k=1}^{n}[ b_{k}-\sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}-d S_{k}-\frac{b_{k}}{S_{k}}+\sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}+d \\
&+\frac{1}{2} \sigma_{k 1}^{2}+\sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}-(d+\epsilon) I_{k}-\frac{\sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}}{I_{k}} \\
&\left.+(d+\epsilon)+\frac{1}{2} \sigma_{k 2}^{2}\right] d t \\
&=+\sum_{k=1}^{n}\left[\sigma_{k 1}\left(S_{k}-1\right) d B_{k 1}(t)+\sigma_{k 2}\left(I_{k}-1\right) d B_{k 2}(t)\right] \\
&=\sum_{k=1}^{n}\left(b_{k}+2 d+\epsilon+\sigma_{k 1}\left(S_{k}-1\right) d B_{k 1}(t)+\sigma_{k 2} I_{k} d B_{k 2}(t)\right] \\
&+\sum_{k=1}^{n}\left[\sigma_{k 1}\left(S_{k}-1\right) d B_{k 1}(t)+\frac{\left.\sigma_{k 2} I_{k} d B_{k 2}(t)\right]}{2}\left(\sigma_{k 1}^{2}+\sigma_{k 2}^{2}\right)\right) d t
\end{align*}
$$

We can now integrate both sides of (9) from 0 to $\tau_{l} \wedge T$ and then take the expectations

$$
\begin{align*}
E & {\left[V\left(S_{1}\left(\tau_{l} \wedge T\right), I_{1}\left(\tau_{l} \wedge T\right), \ldots, S_{n}\left(\tau_{l} \wedge T\right), I_{n}\left(\tau_{l} \wedge T\right)\right)\right] } \\
& \leq V\left(S_{1}(0), I_{1}(0), \ldots, S_{n}(0), I_{n}(0)\right)+E\left[\int_{0}^{\tau_{l} \wedge T} K d t\right] \\
& \leq V\left(S_{1}(0), I_{1}(0), \ldots, S_{n}(0), I_{n}(0)\right)+K T . \tag{10}
\end{align*}
$$

Let $\Omega_{l}=\left\{\tau_{l} \leq T\right\}$ for $l \geq l_{1}$ and, by (7), $P\left(\Omega_{l}\right) \geq$ $\varepsilon$. Note that, for every $\omega \in \Omega_{l}$, there is at least one of $S_{k}\left(\tau_{l}, \omega\right)$ and $I_{k}\left(\tau_{l}, \omega\right), k=1,2, \ldots, n$, that equals either $l$ or $1 / l$, and therefore $V\left(S_{1}\left(\tau_{l}, \omega\right), I_{1}\left(\tau_{l}, \omega\right), \ldots, S_{n}\left(\tau_{l}, \omega\right), I_{n}\left(\tau_{l}, \omega\right)\right)$ is not less than either

$$
\begin{equation*}
l-1-\ln l \quad \text { or } \quad \frac{1}{l}-1-\ln \frac{1}{l}=\frac{1}{l}-1+\ln l . \tag{11}
\end{equation*}
$$

Hence,

$$
\begin{align*}
& V\left(S_{1}\left(\tau_{l}, \omega\right), I_{1}\left(\tau_{l}, \omega\right), \ldots, S_{n}\left(\tau_{l}, \omega\right), I_{n}\left(\tau_{l}, \omega\right)\right) \\
& \quad \geq(l-1-\ln l) \wedge\left(\frac{1}{l}-1+\ln l\right) . \tag{12}
\end{align*}
$$

It then follows from (7) and (10) that

$$
\begin{align*}
& V\left(S_{1}(0), I_{1}(0), \ldots, S_{n}(0), I_{n}(0)\right)+K T \\
& \geq E\left[1_{\Omega_{l(\omega)}} V\left(S_{1}\left(\tau_{l}, \omega\right), I_{1}\left(\tau_{l}, \omega\right), \ldots, S_{n}\left(\tau_{l}, \omega\right), I_{n}\left(\tau_{l}, \omega\right)\right)\right] \\
& \geq \varepsilon\left[(l-1-\ln l) \wedge\left(\frac{1}{l}-1+\ln l\right)\right] \tag{13}
\end{align*}
$$

where $1_{\Omega_{l(\omega)}}$ is the indicator function of $\Omega_{l}$. Letting $l \rightarrow \infty$, we have that

$$
\begin{equation*}
\infty>V\left(S_{1}(0), I_{1}(0), \ldots, S_{n}(0), I_{n}(0)\right)+K T \geq \infty \tag{14}
\end{equation*}
$$

is a contradiction. So we must have $\tau_{\infty}=\infty$. Therefore, it implies $S_{k}(t), I_{k}(t), k=1,2, \ldots, n$, will not explode in a finite time with probability one.

## 3. Asymptotic Behavior around the Disease-Free Equilibrium

As mentioned in the Introduction, $E_{0}=\left(b_{1} / d, 0, \ldots, b_{n} / d, 0\right)$ is the disease-free equilibrium of system (3), and when $R_{0} \leq$ $1, E_{0}$ is globally stable, which means that the disease will be extinct in the limited time. In this section, we will study the asymptotic behavior around $E_{0}$ of system (5).

Lemma 2. If $A$ is nonnegative and irreducible, then the spectral radius $\rho(A)$ of $A$ is a simple eigenvalue, and $A$ has a positive eigenvector $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)$ corresponding to $\rho(A)$. Besides, if $0 \leq A \leq B$, then $\rho(A) \leq \rho(B)$. (This lemma can be found in [20].)

Theorem 3. Assume $A=\left(\beta_{k j}\right)_{n \times n}$ is irreducible. If $R_{0} \leq 1$ and the following condition is satisfied:

$$
\begin{equation*}
\sigma_{k 1}^{2} \leq \frac{4}{3} d, \quad \sigma_{k 2}^{2} \leq 2(d+\epsilon) \tag{15}
\end{equation*}
$$

then for any given initial value $\left(S_{1}(0), I_{1}(0), \ldots, S_{n}(0), I_{n}(0)\right) \in$ $\mathbb{R}_{+}^{2 n}$, the solution of system (5) has the property

$$
\limsup _{t \rightarrow \infty} \frac{1}{t} E \int_{0}^{t}\left[\left(S_{k}(r)-\frac{b_{k}}{d}\right)^{2}+I_{k}^{2}(r)\right] d r
$$

$$
\begin{equation*}
\leq \frac{3}{2} \sum_{k=1}^{n} \frac{a_{k} \sigma_{k 1}^{2} b_{k}^{2}}{d^{2} K_{1}} \tag{16}
\end{equation*}
$$

where

$$
\begin{equation*}
K_{1}=\min \left\{\frac{\omega_{k} \beta_{k k}}{(2 d+\epsilon)(d+\epsilon)}\left(d-\frac{3}{4} \sigma_{k 1}^{2}\right), d+\epsilon-\frac{1}{2} \sigma_{k 2}^{2}\right\} . \tag{17}
\end{equation*}
$$

Proof. First change the variables $s_{k}=S_{k}-b_{k} / d, i_{k}=I_{k}$; then $-b_{k} / d \leq s_{k} \leq 0, i_{k} \geq 0$ and system (5) can be written as

$$
\begin{gather*}
d s_{k}=\left(-\sum_{j=1}^{n} \beta_{k j}\left(s_{k}+\frac{b_{k}}{d}\right) i_{j}-d s_{k}\right) d t \\
+\sigma_{k 1}\left(s_{k}+\frac{b_{k}}{d}\right) d B_{k 1}(t) \\
d i_{k}=\left(\sum_{j=1}^{n} \beta_{k j}\left(s_{k}+\frac{b_{k}}{d}\right) i_{j}-(d+\epsilon) i_{k}\right) d t+\sigma_{k 2} i_{k} d B_{k 2}(t) . \tag{18}
\end{gather*}
$$

Let $S^{0}=\left(S_{1}^{0}, S_{2}^{0}, \ldots, S_{n}^{0}\right)$, where $S_{k}^{0}=b_{k} / d, k=1,2, \ldots, n$. Define

$$
M(s)=\left[\begin{array}{cccc}
\frac{\beta_{11} S_{1}^{0}}{d+\epsilon} & \frac{\beta_{12}\left(S_{1}^{0}+s_{1}\right)}{d+\epsilon} & \cdots & \frac{\beta_{1 n}\left(S_{1}^{0}+s_{1}\right)}{d+\epsilon}  \tag{19}\\
\frac{\beta_{21}\left(S_{2}^{0}+s_{2}\right)}{d+\epsilon} & \frac{\beta_{22} S_{2}^{0}}{d+\epsilon} & \cdots & \frac{\beta_{2 n}\left(S_{2}^{0}+s_{2}\right)}{d+\epsilon} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\beta_{n 1}\left(S_{n}^{0}+s_{n}\right)}{d+\epsilon} & \frac{\beta_{n 2}\left(S_{n}^{0}+s_{n}\right)}{d+\epsilon} & \cdots & \frac{\beta_{n n} S_{n}^{0}}{d+\epsilon}
\end{array}\right] ;
$$

then it is nonnegative and irreducible. By Lemma 2, there is a positive eigenvector $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)$ of $M(s)$ corresponding to $\rho(M(s))$, such that

$$
\begin{equation*}
\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right) M(s)=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right) \rho(M(s)) \tag{20}
\end{equation*}
$$

Define a $C^{2}$-function $V: \mathbb{R}_{+}^{2 n} \rightarrow \overline{\mathbb{R}}_{+}$by

$$
\begin{equation*}
V\left(s_{1}, i_{1}, \ldots, s_{n}, i_{n}\right)=\frac{1}{2} \sum_{k=1}^{n} a_{k}\left(s_{k}+i_{k}\right)^{2}+\sum_{k=1}^{n} \frac{\omega_{k}}{d+\epsilon} i_{k}, \tag{21}
\end{equation*}
$$

where $a_{k}, k=1,2, \ldots, n$, are positive constants. Then the function $V$ is positive definite, and

$$
\begin{align*}
& d V=L V d t+\sum_{k=1}^{n} a_{k}\left(s_{k}+i_{k}\right) \\
& \\
& \quad \times\left(\sigma_{k 1}\left(s_{k}+\frac{b_{k}}{d}\right) d B_{k 1}(t)+\sigma_{k 2} i_{k} d B_{k 2}(t)\right)  \tag{22}\\
& +\sum_{k=1}^{n} \frac{\omega_{k}}{d+\epsilon} \sigma_{k 2} i_{k} d B_{k 2}(t),
\end{align*}
$$

where

$$
\begin{align*}
L V= & \sum_{k=1}^{n} a_{k}\left(s_{k}+i_{k}\right)\left[-d s_{k}-(d+\epsilon) i_{k}\right] \\
& +\frac{1}{2} \sum_{k=1}^{n} a_{k}\left[\sigma_{k 1}^{2}\left(s_{k}+\frac{b_{k}}{d}\right)^{2}+\sigma_{k 2}^{2} i_{k}^{2}\right] \\
& +\sum_{k=1}^{n} \frac{\omega_{k}}{d+\epsilon}\left[\sum_{j=1}^{n} \beta_{k j}\left(s_{k}+\frac{b_{k}}{d}\right) i_{j}-(d+\epsilon) i_{k}\right] \\
= & -\sum_{k=1}^{n} a_{k}\left[\left(d-\frac{1}{2} \sigma_{k 1}^{2}\right) s_{k}^{2}+\left(d+\epsilon-\frac{1}{2} \sigma_{k 2}^{2}\right) i_{k}^{2}\right]  \tag{23}\\
& +\sum_{k=1}^{n} \sum_{j \neq k} \frac{\omega_{k}}{d+\epsilon} \beta_{k j} s_{k} i_{k} \\
& -\sum_{k=1}^{n}\left[a_{k}(2 d+\epsilon)-\frac{\omega_{k}}{d+\epsilon} \beta_{k k}\right] s_{k} i_{k} \\
& +\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\omega_{k}}{d+\epsilon} \beta_{k j} \frac{b_{k}}{d} i_{k}-\sum_{k=1}^{n} \omega_{k} i_{k} \\
& +\frac{1}{2} \sum_{k=1}^{n} a_{k} \sigma_{k 1}^{2}\left(2 s_{k} \frac{b_{k}}{d}+\frac{b_{k}^{2}}{d^{2}}\right) .
\end{align*}
$$

Choose $a_{k}=\omega_{k} \beta_{k k} /(2 d+\epsilon)(d+\epsilon), k=1,2, \ldots, n$; then $a_{k}(2 d+\epsilon)-\left(\omega_{k} /(d+\epsilon)\right) \beta_{k k}=0$. And we note that

$$
\begin{align*}
\sum_{k=1}^{n} & \sum_{j \neq k} \frac{\omega_{k}}{d+\epsilon} \beta_{k j} s_{k} i_{k}+\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\omega_{k}}{d+\epsilon} \beta_{k j} \frac{b_{k}}{d} i_{k} \\
& =\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right) \\
& \times\left[\begin{array}{cccc}
\frac{\beta_{11} S_{1}^{0}}{d+\epsilon} & \frac{\beta_{12}\left(S_{1}^{0}+s_{1}\right)}{d+\epsilon} & \cdots & \frac{\beta_{1 n}\left(S_{1}^{0}+s_{1}\right)}{d+\epsilon} \\
\frac{\beta_{21}\left(S_{2}^{0}+s_{2}\right)}{d+\epsilon} & \frac{\beta_{22} S_{2}^{0}}{d+\epsilon} & \cdots & \frac{\beta_{2 n}\left(S_{2}^{0}+s_{2}\right)}{d+\epsilon} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\beta_{n 1}\left(S_{n}^{0}+s_{n}\right)}{d+\epsilon} & \frac{\beta_{n 2}\left(S_{n}^{0}+s_{n}\right)}{d+\epsilon} & \cdots & \frac{\beta_{n n} S_{n}^{0}}{d+\epsilon}
\end{array}\right] \\
& \times\left[\begin{array}{c}
i_{1} \\
i_{2} \\
\vdots \\
i_{n}
\end{array}\right] \\
& =\omega M(s) i . \tag{24}
\end{align*}
$$

## Then

$$
\begin{aligned}
L V= & -\sum_{k=1}^{n} a_{k}\left[\left(d-\frac{1}{2} \sigma_{k 1}^{2}\right) s_{k}^{2}+\left(d+\epsilon-\frac{1}{2} \sigma_{k 2}^{2}\right) i_{k}^{2}\right] \\
& -\omega i+\omega M(s) i+\frac{1}{2} \sum_{k=1}^{n} a_{k} \sigma_{k 1}^{2}\left(2 s_{k} \frac{b_{k}}{d}+\frac{b_{k}^{2}}{d^{2}}\right) \\
\leq & -\sum_{k=1}^{n} a_{k}\left[\left(d-\frac{3}{4} \sigma_{k 1}^{2}\right) s_{k}^{2}+\left(d+\epsilon-\frac{1}{2} \sigma_{k 2}^{2}\right) i_{k}^{2}\right] \\
& +\omega(M(s)-1) i+\frac{3}{2} \sum_{k=1}^{n} a_{k} \sigma_{k 1}^{2} \frac{b_{k}^{2}}{d^{2}} \\
= & -\sum_{k=1}^{n} a_{k}\left[\left(d-\frac{3}{4} \sigma_{k 1}^{2}\right) s_{k}^{2}+\left(d+\epsilon-\frac{1}{2} \sigma_{k 2}^{2}\right) i_{k}^{2}\right] \\
& +\omega(\rho(M(s))-1) i+\frac{3}{2} \sum_{k=1}^{n} a_{k} \sigma_{k 1}^{2} \frac{b_{k}^{2}}{d^{2}}
\end{aligned}
$$

where the last equality is derived from (20). Since $-b_{k} / d \leq$ $s_{k} \leq 0$, then $0 \leq M(s) \leq M\left(S^{0}\right)=\left(S_{k}^{0} \beta_{k j} /(d+\epsilon)\right)_{n \times n}=$ $M_{0}$, and so $\rho(M(s)) \leq \rho\left(M_{0}\right)$ according to Lemma 2. Besides, $R_{0} \leq 1$, and then $\rho(M(s)) \leq 1$. Therefore

$$
\begin{align*}
d V \leq & {\left[-\sum_{k=1}^{n} a_{k}\left[\left(d-\frac{3}{4} \sigma_{k 1}^{2}\right) s_{k}^{2}+\left(d+\epsilon-\frac{1}{2} \sigma_{k 2}^{2}\right) i_{k}^{2}\right]\right.} \\
& \left.+\frac{3}{2} \sum_{k=1}^{n} a_{k} \sigma_{k 1}^{2} \frac{b_{k}^{2}}{d^{2}}\right] d t  \tag{26}\\
& +\sum_{k=1}^{n} a_{k}\left(s_{k}+i_{k}\right) \sigma_{k 1}\left(s_{k}+\frac{b_{k}}{d}\right) d B_{k 1}(t) \\
& +\sum_{k=1}^{n}\left(a_{k}\left(s_{k}+i_{k}\right) \sigma_{k 2}+\frac{\omega_{k}}{d+\epsilon} \sigma_{k 2}\right) i_{k} d B_{k 2}(t)
\end{align*}
$$

Integrating both sides of (26) from 0 to $t$, and taking expectation, yields

$$
\begin{aligned}
& 0 \leq E\left[V\left(s_{1}(t), i_{1}(t), \ldots, s_{n}(t), i_{n}(t)\right)\right] \\
& \leq E\left[V\left(s_{1}(0), i_{1}(0), \ldots, s_{n}(0), i_{n}(0)\right)\right] \\
& \quad+E \int_{0}^{t}\left[-\sum_{k=1}^{n} a_{k}\left[\left(d-\frac{3}{4} \sigma_{k 1}^{2}\right) s_{k}^{2}(r)\right.\right. \\
& \\
& \left.\quad+\left(d+\epsilon-\frac{1}{2} \sigma_{k 2}^{2}\right) i_{k}^{2}(r)\right] \\
& \\
& \left.\quad+\frac{3}{2} \sum_{k=1}^{n} a_{k} \sigma_{k 1}^{2} \frac{b_{k}^{2}}{d^{2}}\right] d r
\end{aligned}
$$

which implies

$$
\begin{align*}
& E \int_{0}^{t}\left[\sum_{k=1}^{n} a_{k}\left[\left(d-\frac{3}{4} \sigma_{k 1}^{2}\right) s_{k}^{2}(r)+\left(d+\epsilon-\frac{1}{2} \sigma_{k 2}^{2}\right) i_{k}^{2}(r)\right]\right] d r \\
& \quad \leq E\left[V\left(s_{1}(0), i_{1}(0), \ldots, s_{n}(0), i_{n}(0)\right)\right]+\frac{3}{2} \sum_{k=1}^{n} a_{k} \sigma_{k 1}^{2} \frac{b_{k}^{2}}{d^{2}} t \tag{28}
\end{align*}
$$

Therefore

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} \frac{1}{t} E \int_{0}^{t}\left[\sum _ { k = 1 } ^ { n } a _ { k } \left[\left(d-\frac{3}{4} \sigma_{k 1}^{2}\right) s_{k}^{2}(r)\right.\right. \\
& \left.\left.\quad+\left(d+\epsilon-\frac{1}{2} \sigma_{k 2}^{2}\right) i_{k}^{2}(r)\right]\right] d r  \tag{29}\\
& \quad \leq \frac{3}{2} \sum_{k=1}^{n} a_{k} \sigma_{k 1}^{2} \frac{b_{k}^{2}}{d^{2}}
\end{align*}
$$

that is,

$$
\begin{aligned}
& \limsup _{t \rightarrow \infty} \frac{1}{t} E \int_{0}^{t}\left[\sum_{k=1}^{n} a_{k}[ \right.\left(d-\frac{3}{4} \sigma_{k 1}^{2}\right)\left(S_{k}(r)-\frac{b_{k}}{d}\right)^{2} \\
&\left.\left.+\left(d+\epsilon-\frac{1}{2} \sigma_{k 2}^{2}\right) I_{k}^{2}(r)\right]\right] d r \\
& \leq \frac{3}{2} \sum_{k=1}^{n} a_{k} \sigma_{k 1}^{2} \frac{b_{k}^{2}}{d^{2}}
\end{aligned}
$$

If we let

$$
\begin{equation*}
K_{1}=\min \left\{a_{k}\left(d-\frac{3}{4} \sigma_{k 1}^{2}\right), d+\epsilon-\frac{1}{2} \sigma_{k 2}^{2}\right\}, \tag{31}
\end{equation*}
$$

then

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} \frac{1}{t} E \int_{0}^{t}\left[\left(S_{k}(r)-\frac{b_{k}}{d}\right)^{2}+I_{k}^{2}(r)\right] d r \\
& \quad \leq \frac{3}{2} \sum_{k=1}^{n} \frac{a_{k} \sigma_{k 1}^{2} b_{k}^{2}}{d^{2} K_{1}} \tag{32}
\end{align*}
$$

as the theorem is proved.
Remark 4. From Theorem 3, we can get the conclusion that the solution of the stochastic system will oscillate around the disease-free equilibrium of the deterministic model; the values of $\sigma_{k 1}$ and $\sigma_{k 2}$ have bearing on the intensity of turbulence. If the stochastic perturbations become small, the solution of system (5) will be close to the disease-free equilibrium of system (3).

Besides, if $\sigma_{k 1}=0$, then $E_{0}$ is also the disease-free equilibrium of system (5). From the proof of Theorem 3, we can obtain

$$
\begin{equation*}
L V \leq-\sum_{k=1}^{n} a_{k}\left[\left(d-\frac{3}{4} \sigma_{k 1}^{2}\right) s_{k}^{2}+\left(d+\epsilon-\frac{1}{2} \sigma_{k 2}^{2}\right) i_{k}^{2}\right] \leq 0 . \tag{33}
\end{equation*}
$$

Therefore, $E_{0}$ is globally asymptotically stable.

## 4. The Dynamic of System (5) around the Endemic of System (3)

In the deterministic model, if $R_{0}>1$, there exists the endemic equilibrium $E^{*}$. But $E^{*}$ is not the endemic equilibrium of stochastic system (5), because there is no endemic equilibrium for the stochastic system (5). In fact, we still want to find the relation between the solution of stochastic system and $E^{*}$.

Given a weighted digraph $(\mathscr{G}, A)$ with $n$ vertices, where $A=\left(a_{k j}\right)_{n n}$ is the weight matrix, whose entry $a_{k j}$ equals the weight of arc $(j, k)$ if it exists, and 0 otherwise, the Laplacian matrix of $A$ is defined as

$$
L_{A}=\left[\begin{array}{cccc}
\sum_{k \neq 1} a_{1 k} & -a_{12} & \cdots & -a_{1 n}  \tag{34}\\
-a_{21} & \sum_{k \neq 2} a_{2 k} & \cdots & -a_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n 1} & -a_{n 2} & \cdots & \sum_{k \neq n} a_{n k}
\end{array}\right] .
$$

Let $c_{k}$ denote the cofactor of the $k$ th diagonal element of $L_{A}$, and we have the following results.

Theorem 5. Assume $A=\left(\beta_{k j}\right)_{n \times n}$ is irreducible and $R_{0}>$ 1. For any given initial value $\left(S_{1}(0), I_{1}(0), \ldots, S_{n}(0), I_{n}(0)\right) \in$ $\mathbb{R}_{+}^{2 n}$, the solution of system (5) has the property

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} \frac{1}{t} \sum_{k=1}^{n} \int_{0}^{t}\left[p_{k} d\left(S_{k}-S_{k}^{*}\right)^{2}+m_{k}(d+\epsilon)\left(I_{k}-I_{k}^{*}\right)^{2}\right] d s \\
& \leq \sum_{k=1}^{n}\left[\left(\frac{a \bar{c}_{k} b_{k}}{d}+\frac{\left(m_{k}+p_{k}\right) b_{k}^{2}}{d^{2}}\right) \sigma_{k 1}^{2}\right. \\
& \left.\quad+\left(\frac{(a+1) \bar{c}_{k} b_{k}}{d}+\frac{m_{k} b_{k}^{2}}{d^{2}}\right) \sigma_{k 2}^{2}\right], \quad \text { a.s. } \tag{35}
\end{align*}
$$

where $E^{*}=\left(S_{1}^{*}, I_{1}^{*}, \ldots, S_{n}^{*}, I_{n}^{*}\right)$ is the endemic equilibrium of system (3) and $\bar{c}_{k}, k=1,2, \ldots, n$, denote the cofactor of the $k$ th diagonal element of $L_{\bar{A}}\left(\bar{A}=\left(\bar{\beta}_{k j}\right)_{n \times n}=\left(\beta_{k j} S_{k}^{*} I_{j}^{*}\right)_{n \times n}\right)$, and $a$, $m_{k}, p_{k}, k=1,2, \ldots, n$, are positive constants defined as in the proof.

Proof. Since $E^{*}$ is the endemic equilibrium of system (3), we have

$$
\begin{equation*}
\sum_{j=1}^{n} \beta_{k j} S_{k}^{*} I_{j}^{*}+d S_{k}^{*}=b_{k}, \quad \sum_{j=1}^{n} \beta_{k j} S_{k}^{*} I_{j}^{*}=(d+\epsilon) I_{k}^{*} \tag{36}
\end{equation*}
$$

Define

$$
\begin{aligned}
& V\left(S_{1}, I_{1}, \ldots, S_{n}, I_{n}\right) \\
& \quad=a \sum_{k=1}^{n} \bar{c}_{k}\left(S_{k}-S_{k}^{*}-S_{k}^{*} \ln \frac{S_{k}}{S_{k}^{*}}+I_{k}-I_{k}^{*}-I_{k}^{*} \ln \frac{I_{k}}{I_{k}^{*}}\right)
\end{aligned}
$$

$$
\begin{align*}
& +\sum_{k=1}^{n} \bar{c}_{k}\left(I_{k}-I_{k}^{*}-I_{k}^{*} \ln \frac{I_{k}}{I_{k}^{*}}\right) \\
& +\frac{1}{2} \sum_{k=1}^{n} m_{k}\left(S_{k}-S_{k}^{*}+I_{k}-I_{k}^{*}\right)^{2}+\frac{1}{2} \sum_{k=1}^{n} p_{k}\left(S_{k}-S_{k}^{*}\right)^{2} \\
:= & a V_{1}+V_{2}+V_{3}+V_{4}, \tag{37}
\end{align*}
$$

where $a, m_{k}, p_{k}, k=1,2, \ldots, n$, are positive constants to be determined later. From the property (1) of Lemma A. 2 (see [20]), we know $\bar{c}_{k}>0, k=1,2, \ldots, n$. Hence $V$ is positive definite. Let $L$ be the generating operator of system (5). Then we get

$$
\begin{align*}
& L V_{1}=\sum_{k=1}^{n} \bar{c}_{k}\left(1-\frac{S_{k}^{*}}{S_{k}}\right)\left(b_{k}-\sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}-d S_{k}\right) \\
& +\sum_{k=1}^{n} \frac{\bar{c}_{k} S_{k}^{*} \sigma_{k 1}^{2}}{2} \\
& +\sum_{k=1}^{n} \bar{c}_{k}\left(1-\frac{I_{k}^{*}}{I_{k}}\right)\left(\sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}-(d+\epsilon) I_{k}\right) \\
& +\sum_{k=1}^{n} \frac{\bar{c}_{k} I_{k}^{*} \sigma_{k 2}^{2}}{2} \\
& =\sum_{k=1}^{n} \bar{c}_{k}\left[b_{k}-d S_{k}-b_{k} \frac{S_{k}^{*}}{S_{k}}+\sum_{j=1}^{n} \beta_{k j} S_{k}^{*} I_{j}+d S_{k}^{*}\right. \\
& -(d+\epsilon) I_{k}+(d+\epsilon) I_{k}^{*}-\sum_{j=1}^{n} \beta_{k j} \frac{S_{k} I_{j} I_{k}^{*}}{I_{k}} \\
& \left.+\frac{1}{2}\left(S_{k}^{*} \sigma_{k 1}^{2}+I_{k}^{*} \sigma_{k 2}^{2}\right)\right] \\
& =\sum_{k=1}^{n} \bar{c}_{k}\left[\sum_{j=1}^{n} \bar{\beta}_{k j}+d S_{k}^{*}-d S_{k}-\sum_{j=1}^{n} \bar{\beta}_{k j} \frac{S_{k}^{*}}{S_{k}}-d S_{k}^{*} \frac{S_{k}^{*}}{S_{k}}\right. \\
& +\sum_{j=1}^{n} \bar{\beta}_{k j} \frac{I_{j}}{I_{j}^{*}}+d S_{k}^{*}-\sum_{j=1}^{n} \bar{\beta}_{k j} \frac{I_{k}}{I_{k}^{*}}-\sum_{j=1}^{n} \bar{\beta}_{k j} \frac{S_{k} I_{j} I_{k}^{*}}{I_{k} S_{k}^{*} I_{j}^{*}} \\
& \left.+\sum_{j=1}^{n} \bar{\beta}_{k j}+\frac{1}{2}\left(S_{k}^{*} \sigma_{k 1}^{2}+I_{k}^{*} \sigma_{k 2}^{2}\right)\right] \\
& =\sum_{k=1}^{n} \bar{c}_{k}\left[-d S_{k}^{*}\left(\frac{S_{k}^{*}}{S_{k}}+\frac{S_{k}}{S_{k}^{*}}-2\right)+\sum_{j=1}^{n} \bar{\beta}_{k j}\left(\frac{I_{j}}{I_{j}^{*}}-\frac{I_{k}}{I_{k}^{*}}\right)\right. \\
& +\sum_{j=1}^{n} \bar{\beta}_{k j}\left(2-\frac{S_{k}^{*}}{S_{k}}-\frac{S_{k} I_{j} I_{k}^{*}}{I_{k} S_{k}^{*} I_{j}^{*}}\right) \\
& \left.+\frac{1}{2}\left(S_{k}^{*} \sigma_{k 1}^{2}+I_{k}^{*} \sigma_{k 2}^{2}\right)\right] \text {. } \tag{38}
\end{align*}
$$



$$
\begin{aligned}
& \quad k=5 \\
& -k=25 \\
& k=48
\end{aligned}
$$

(a)


$$
\begin{aligned}
& k=5 \\
& -k=25 \\
& z=48
\end{aligned}
$$



$$
\begin{aligned}
&- k=5 \\
&- k=25 \\
& k=48
\end{aligned}
$$


(d)

Figure 1: $\lambda=0.025, \sigma_{k 1}=0.005, \sigma_{k 2}=0.03, R_{0} \leq 1$. (a), (b) $P(k)=m^{k} \exp (-m) / k!, m=6, R_{0}=0.4704 \leq 1$; (c), (d) $P(k)=2 m^{2} k^{-3}, m=3$, $R_{0}=0.9071 \leq 1$.

$$
\begin{array}{rlr}
L V_{2}= & \sum_{k=1}^{n} \bar{c}_{k}\left(1-\frac{I_{k}^{*}}{I_{k}}\right)\left(\sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}-(d+\epsilon) I_{k}\right) & =\sum_{k=1}^{n} \bar{c}_{k}\left[\sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}-\sum_{j=1}^{n} \bar{\beta}_{k j} \frac{I_{k}}{I_{k}^{*}}-\sum_{j=1}^{n} \bar{\beta}_{k j} \frac{S_{k} I_{j} I_{k}^{*}}{I_{k} S_{k} I_{j}^{*}}\right. \\
& +\sum_{k=1}^{n} \frac{\bar{c}_{k} I_{k}^{*} \sigma_{k 2}^{2}}{2} & \left.+\sum_{j=1}^{n} \bar{\beta}_{k j}+\frac{I_{k}^{*} \sigma_{k 2}^{2}}{2}\right] .  \tag{39}\\
=\sum_{k=1}^{n} \bar{c}_{k}\left[\sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}-(d+\epsilon) I_{k}-\frac{I_{k}^{*}}{I_{k}} \sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}\right. & L V_{3}=\sum_{k=1}^{n} m_{k}\left(S_{k}-S_{k}^{*}+I_{k}-I_{k}^{*}\right)\left(b_{k}-d S_{k}-(d+\epsilon) I_{k}\right) \\
& \left.\quad+(d+\epsilon) I_{k}^{*}+\frac{I_{k}^{*} \sigma_{k 2}^{2}}{2}\right] & +\sum_{k=1}^{n} m_{k} \frac{\sigma_{k 1}^{2} S_{k}^{2}+\sigma_{k 2}^{2} I_{k}^{2}}{2}
\end{array}
$$

- $k=5$
- $k=48$

(a)



(b)

(d)

Figure 2: $\lambda=0.05, \sigma_{k 1}=0.01, \sigma_{k 2}=0.02$. (a), (b) $P(k)=m^{k} \exp (-m) / k!, m=6, R_{0}=0.9409 \leq 1$; (c), (d) $P(k)=2 m^{2} k^{-3}, m=3$, $R_{0}=1.8142 \geq 1$.

$$
\begin{aligned}
& =\sum_{k=1}^{n} m_{k}\left(S_{k}-S_{k}^{*}+I_{k}-I_{k}^{*}\right) \\
& \times\left(-d\left(S_{k}-S_{k}^{*}\right)-(d+\epsilon)\left(I_{k}-I_{k}^{*}\right)\right) \\
& +\sum_{k=1}^{n} m_{k} \frac{\sigma_{k 1}^{2} S_{k}^{2}+\sigma_{k 2}^{2} I_{k}^{2}}{2} \\
& =\sum_{k=1}^{n} m_{k}\left[-d\left(S_{k}-S_{k}^{*}\right)^{2}-(d+\epsilon)\left(I_{k}-I_{k}^{*}\right)^{2}\right. \\
& -(2 d+\epsilon)\left(S_{k}-S_{k}^{*}\right)\left(I_{k}-I_{k}^{*}\right) \\
& \left.+\frac{\sigma_{k 1}^{2} S_{k}^{2}+\sigma_{k 2}^{2} I_{k}^{2}}{2}\right]
\end{aligned}
$$

$$
\begin{gathered}
\leq \sum_{k=1}^{n} m_{k}\left[-\left(d-\frac{(2 d+\epsilon)^{2}}{2(d+\epsilon)}\right)\left(S_{k}-S_{k}^{*}\right)^{2}\right. \\
\left.-\frac{(d+\epsilon)}{2}\left(I_{k}-I_{k}^{*}\right)^{2}+\frac{\sigma_{k 1}^{2} S_{k}^{2}+\sigma_{k 2}^{2} I_{k}^{2}}{2}\right] \\
=\sum_{k=1}^{n} m_{k}\left[\frac{(d+\epsilon)^{2}+d^{2}}{2(d+\epsilon)}\left(S_{k}-S_{k}^{*}\right)^{2}-\frac{(d+\epsilon)}{2}\left(I_{k}-I_{k}^{*}\right)^{2}\right. \\
\left.+\frac{\sigma_{k 1}^{2} S_{k}^{2}+\sigma_{k 2}^{2} I_{k}^{2}}{2}\right] .
\end{gathered}
$$



(b)


$$
\begin{aligned}
& k=5 \\
& -k=25 \\
& -k=48
\end{aligned}
$$

(a)

(c)

- $k=5$
- $k=25$
- $k=48$
(d)

Figure 3: $\lambda=0.08, \sigma_{k 1}=0.01, \sigma_{k 2}=0.01, R_{0} \geq 1$. (a), (b) $P(k)=m^{k} \exp (-m) / k!, m=6, R_{0}=1.5054 \geq 1$; (c), (d) $P(k)=2 m^{2} k^{-3}, m=3$, $R_{0}=2.9027 \geq 1$.

$$
\begin{align*}
L V_{4}= & \sum_{k=1}^{n} p_{k}\left(S_{k}-S_{k}^{*}\right)\left(b_{k}-\sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}-d S_{k}\right) \\
& +\sum_{k=1}^{n} p_{k} \frac{\sigma_{k 1}^{2} S_{k}^{2}}{2} \\
= & \sum_{k=1}^{n} p_{k}\left(S_{k}-S_{k}^{*}\right) \\
& \quad \times\left(\sum_{j=1}^{n} \beta_{k j}\left(S_{k}^{*} I_{j}^{*}-S_{k} I_{j}\right)-d\left(S_{k}-S_{k}^{*}\right)\right) \tag{40}
\end{align*}
$$

$$
+\sum_{k=1}^{n} p_{k} \frac{\sigma_{k 1}^{2} S_{k}^{2}}{2}
$$

$$
=-\sum_{k=1}^{n} p_{k}\left[\sum_{j=1}^{n} \beta_{k j} S_{k}^{*}\left(S_{k}-S_{k}^{*}\right)\left(I_{j}-I_{j}^{*}\right)\right.
$$

$$
\left.+\sum_{j=1}^{n} \beta_{k j} I_{j}\left(S_{k}-S_{k}^{*}\right)^{2}+d\left(S_{k}-S_{k}^{*}\right)^{2}-\frac{\sigma_{k 1}^{2} S_{k}^{2}}{2}\right]
$$



Figure 4: (a), (b) $\lambda=0.05, \sigma_{k 1}=0.015, \sigma_{k 2}=0.05, P(k)=m^{k} \exp (-m) / k!, m=6, R_{0}=0.94 .9 \leq 1$. (c), (d) $\lambda=0.025, \sigma_{k 1}=0.01, \sigma_{k 2}=0.1$, $P(k)=2 m^{2} k^{-3}, m=3, R_{0}=0.9071 \leq 1$.

By property (2) of Lemma A. 2 (see [20]), we know

$$
\begin{gather*}
\sum_{k=1}^{n} \bar{c}_{k}\left(\sum_{j=1}^{n} \bar{\beta}_{k j} \frac{I_{j}}{I_{j}^{*}}-\sum_{j=1}^{n} \bar{\beta}_{k j} \frac{I_{k}}{I_{k}^{*}}\right)=0 \\
\sum_{k=1}^{n} \bar{c}_{k}\left(\sum_{j=1}^{n} \bar{\beta}_{k j} \ln \frac{I_{j}}{I_{j}^{*}}-\sum_{j=1}^{n} \bar{\beta}_{k j} \ln \frac{I_{k}}{I_{k}^{*}}\right)=0 \tag{41}
\end{gather*}
$$

Besides, note that $x-1-\ln x \geq 0$ for $x>0$; then

$$
\begin{equation*}
\frac{S_{k}^{*}}{S_{k}} \geq 1+\ln \frac{S_{k}^{*}}{S_{k}}, \quad \frac{S_{k} I_{j} I_{k}^{*}}{I_{k} S_{k}^{*} I_{j}^{*}} \geq 1+\ln \frac{S_{k} I_{j} I_{k}^{*}}{I_{k} S_{k}^{*} I_{j}^{*}} . \tag{42}
\end{equation*}
$$

According to (41) and (42), we get

$$
\begin{align*}
& \sum_{k=1}^{n} \bar{c}_{k} \sum_{j=1}^{n} \bar{\beta}_{k j}\left(2-\frac{S_{k}^{*}}{S_{k}}-\frac{S_{k} I_{j} I_{k}^{*}}{I_{k} S_{k}^{*} I_{j}^{*}}\right) \\
& \quad \leq \sum_{k=1}^{n} \bar{c}_{k} \sum_{j=1}^{n} \bar{\beta}_{k j}\left[2-\left(1+\ln \frac{S_{k}^{*}}{S_{k}}\right)-\left(1+\ln \frac{S_{k} I_{j} I_{k}^{*}}{I_{k} S_{k}^{*} I_{j}^{*}}\right)\right] \\
& \quad=\sum_{k=1}^{n} \bar{c}_{k} \sum_{j=1}^{n} \bar{\beta}_{k j}\left(\ln \frac{I_{k}}{I_{k}^{*}}-\ln \frac{I_{j}}{I_{j}^{*}}\right)=0 \tag{43}
\end{align*}
$$

$$
\begin{align*}
\sum_{k=1}^{n} \bar{c}_{k} \sum_{j=1}^{n} \bar{\beta}_{k j} \frac{S_{k} I_{j} I_{k}^{*}}{I_{k} S_{k}^{*} I_{j}^{*}} & \geq \sum_{k=1}^{n} \bar{c}_{k} \sum_{j=1}^{n} \bar{\beta}_{k j}\left(1+\ln \frac{S_{k} I_{j} I_{k}^{*}}{I_{k} S_{k}^{*} I_{j}^{*}}\right) \\
= & \sum_{k=1}^{n} \bar{c}_{k} \sum_{j=1}^{n} \bar{\beta}_{k j}\left(1-\ln \frac{S_{k}^{*}}{S_{k}}\right) \\
& +\sum_{k=1}^{n} \bar{c}_{k} \sum_{j=1}^{n} \bar{\beta}_{k j}\left(\ln \frac{I_{j}}{I_{j}^{*}}-\ln \frac{I_{k}}{I_{k}^{*}}\right)  \tag{44}\\
\geq & \sum_{k=1}^{n} \bar{c}_{k} \sum_{j=1}^{n} \bar{\beta}_{k j}\left(2-\frac{S_{k}^{*}}{S_{k}}\right)
\end{align*}
$$

Substituting (41) and (43) into (38), we get

$$
\begin{align*}
L V_{1} & \leq \sum_{k=1}^{n} \bar{c}_{k}\left[-d S_{k}^{*}\left(\frac{S_{k}^{*}}{S_{k}}+\frac{S_{k}}{S_{k}^{*}}-2\right)+\frac{1}{2}\left(S_{k}^{*} \sigma_{k 1}^{2}+I_{k}^{*} \sigma_{k 2}^{2}\right)\right] \\
& =-\sum_{k=1}^{n} \bar{c}_{k} d \frac{\left(S_{k}-S_{k}^{*}\right)^{2}}{S_{k}}+\sum_{k=1}^{n} \frac{\bar{c}_{k}}{2}\left(S_{k}^{*} \sigma_{k 1}^{2}+I_{k}^{*} \sigma_{k 2}^{2}\right) \tag{45}
\end{align*}
$$

Substituting (44) into (39), we get

$$
\begin{aligned}
& L V_{2} \leq \sum_{k=1}^{n} \bar{c}_{k}[ \sum_{j=1}^{n} \beta_{k j} S_{k} I_{j}-\sum_{j=1}^{n} \bar{\beta}_{k j} \frac{I_{k}}{I_{k}^{*}}-\sum_{j=1}^{n} \bar{\beta}_{k j}\left(2-\frac{S_{k}^{*}}{S_{k}}\right) \\
&\left.+\sum_{j=1}^{n} \bar{\beta}_{k j}+\frac{I_{k}^{*} \sigma_{k 2}^{2}}{2}\right] \\
&=\sum_{k=1}^{n} \bar{c}_{k}\left[\sum_{j=1}^{n} \beta_{k j}\left(S_{k}-S_{k}^{*}\right)\left(I_{j}-I_{j}^{*}\right)\right. \\
& \quad-\sum_{j=1}^{n} \bar{\beta}_{k j}\left(\frac{I_{k}}{I_{k}^{*}}-\frac{I_{j}}{I_{j}^{*}}\right) \\
&\left.\quad-\sum_{j=1}^{n} \bar{\beta}_{k j}\left(2-\frac{S_{k}^{*}}{S_{k}}-\frac{S_{k}}{S_{k}^{*}}\right)+\frac{I_{k}^{*} \sigma_{k 2}^{2}}{2}\right] \\
&=\sum_{k=1}^{n} \bar{c}_{k}[ \sum_{j=1}^{n} \beta_{k j}\left(S_{k}-S_{k}^{*}\right)\left(I_{j}-I_{j}^{*}\right) \\
&\left.+\sum_{j=1}^{n} \beta_{k j} I_{j}^{*} \frac{\left(S_{k}-S_{k}^{*}\right)^{2}}{S_{k}}+\frac{I_{k}^{*} \sigma_{k 2}^{2}}{2}\right] .
\end{aligned}
$$

Therefore,

$$
\begin{align*}
L V & =a L V_{1}+L V_{2}+L V_{3}+L V_{4} \\
& \leq-\sum_{k=1}^{n}\left[p_{k} d-m_{k} \frac{(d+\epsilon)^{2}+d^{2}}{2(d+\epsilon)}\right]\left(S_{k}-S_{k}^{*}\right)^{2} \tag{48}
\end{align*}
$$

Choose $a=\max \left\{\left(\sum_{j=1}^{n} \beta_{k j} I_{j}^{*}\right) / d, k=1,2, \ldots, n\right\}, m_{k}=(d+$ є) $p_{k} d /\left((d+\epsilon)^{2}+d^{2}\right), p_{k}=\bar{c}_{k} / S_{k}^{*}, k=1,2, \ldots, n$; then

$$
\begin{align*}
L V \leq- & \sum_{k=1}^{n} \frac{p_{k} d}{2}\left(S_{k}-S_{k}^{*}\right)^{2}-\frac{1}{2} \sum_{k=1}^{n} m_{k}(d+\epsilon)\left(I_{k}-I_{k}^{*}\right)^{2} \\
+ & \frac{1}{2} \sum_{k=1}^{n}\left[\left(\frac{a \bar{c}_{k} b}{d}+\frac{\left(m_{k}+p_{k}\right) b^{2}}{d^{2}}\right) \sigma_{k 1}^{2}\right.  \tag{46}\\
& \left.+\left(\frac{(a+1) \bar{c}_{k} b}{d}+\frac{m_{k} b^{2}}{d^{2}}\right) \sigma_{k 2}^{2}\right]
\end{align*}
$$

$$
:=F(t)
$$

Therefore,

$$
\begin{align*}
& d V \leq F(t) d t \\
& \qquad \begin{aligned}
+\sum_{k=1}^{n}[ & \\
& \left.=\bar{c}_{k} \sigma_{k 1}\left(S_{k}-S_{k}^{*}\right)+\left(m_{k}+p_{k}\right) \sigma_{k 1}\left(S_{k}-S_{k}^{*}\right) S_{k}\right] \\
& \times d B_{k 1}(t) \\
& +\left[\bar{c}_{k}(a+1) \sigma_{k 2}\left(I_{k}-I_{k}^{*}\right)+m_{k} \sigma_{k 2}\left(I_{k}-I_{k}^{*}\right) I_{k}\right] \\
& \left.\times d B_{k 2}(t)\right] .
\end{aligned}
\end{align*}
$$

Integrating both sides of (49) from 0 to $t$ yields

$$
\begin{align*}
& V(t)-V(0) \\
& \qquad \begin{array}{l}
\leq \int_{0}^{t} F(s) d s \\
\quad+\int_{0}^{t} \sum_{k=1}^{n}\left[a \bar{c}_{k} \sigma_{k 1}\left(S_{k}-S_{k}^{*}\right)+\left(m_{k}+p_{k}\right) \sigma_{k 1}\left(S_{k}-S_{k}^{*}\right) S_{k}\right] \\
\quad \times d B_{k 1}(s) \\
\quad+\int_{0}^{t} \sum_{k=1}^{n}\left[\bar{c}_{k}(a+1) \sigma_{k 2}\left(I_{k}-I_{k}^{*}\right)+m_{k} \sigma_{k 2}\left(I_{k}-I_{k}^{*}\right) I_{k}\right] \\
\quad \times d B_{k 2}(s)
\end{array}
\end{align*}
$$

Let $M_{1}(t):=\int_{0}^{t} \sum_{k=1}^{n}\left[a \bar{c}_{k} \sigma_{k 1}\left(S_{k}-S_{k}^{*}\right)+\left(m_{k}+p_{k}\right) \sigma_{k 1}\left(S_{k}-\right.\right.$ $\left.\left.S_{k}^{*}\right) S_{k}\right] d B_{k 1}(s), M_{2}(t):=\int_{0}^{t} \sum_{k=1}^{n}\left[\bar{c}_{k}(a+1) \sigma_{k 2}\left(I_{k}-I_{k}^{*}\right)+\right.$ $\left.m_{k} \sigma_{k 2}\left(I_{k}-I_{k}^{*}\right) I_{k}\right] d B_{k 2}(s)$, which are local continuous martingale, and $M_{1}(0)=M_{2}(0)=0$. Moreover

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} \frac{\left\langle M_{1}, M_{1}\right\rangle_{t}}{t} \leq 8 \sum_{k=1}^{n} \sigma_{k 1}^{2}\left[a^{2} \bar{c}_{k}^{2}+\left(m_{k}+p_{k}\right)^{2} \frac{b_{k}^{2}}{d^{2}}\right] \frac{b_{k}^{2}}{d^{2}} \\
&<\infty, \\
& \limsup _{t \rightarrow \infty} \frac{\left\langle M_{2}, M_{2}\right\rangle_{t}}{t} \leq 8 \sum_{k=1}^{n} \sigma_{k 2}^{2}\left[\bar{c}_{k}^{2}(a+1)^{2}+m_{k}^{2} \frac{b_{k}^{2}}{d^{2}}\right] \frac{b_{k}^{2}}{d^{2}}<\infty . \tag{51}
\end{align*}
$$

By Lemma A. 4 (see [20]), we obtain

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{M_{1}(t)}{t}=0, \quad \lim _{t \rightarrow \infty} \frac{M_{2}(t)}{t}=0 \quad \text { a.s. } \tag{52}
\end{equation*}
$$

which together with (50) implies

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \frac{\int_{0}^{t} F(s) d s}{t} \geq 0 \quad \text { a.s. } \tag{53}
\end{equation*}
$$

Consequently,

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} \frac{1}{t} \sum_{k=1}^{n} \int_{0}^{t}\left[p_{k} d\left(S_{k}-S_{k}^{*}\right)^{2}+m_{k}(d+\epsilon)\left(I_{k}-I_{k}^{*}\right)^{2}\right] d s \\
& \leq \sum_{k=1}^{n}\left[\left(\frac{a \bar{c}_{k} b_{k}}{d}+\frac{\left(m_{k}+p_{k}\right) b_{k}^{2}}{d^{2}}\right) \sigma_{k 1}^{2}\right. \\
& \left.\quad+\left(\frac{(a+1) \bar{c}_{k} b_{k}}{d}+\frac{m_{k} b_{k}^{2}}{d^{2}}\right) \sigma_{k 2}^{2}\right], \quad \text { a.s. } \tag{54}
\end{align*}
$$

Thus Theorem 5 is proved.
Remark 6. Theorem 5 shows that the solution of system (5) fluctuates around the certain level which is relevant to $E^{*}$ of system (3) and $\sigma_{k 1}^{2}, \sigma_{k 2}^{2}, k=1,2, \ldots, n$. The distance between the solution $X(t)=\left(S_{1}(t), I_{1}(t), \ldots, S_{n}(t), I_{n}(t)\right)$ and $E^{*}$ of system (3) has the following form:

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t}\left\|X(s)-E^{*}\right\|^{2} d s \leq C\|\sigma\|^{2} \tag{55}
\end{equation*}
$$

where $C$ is a positive constant and $\|\sigma\|^{2}=\sum_{k=1}^{n}\left(\sigma_{k 1}^{2}+\sigma_{k 2}^{2}\right)$. Although the solution of system (5) does not have stability as the deterministic system, we can draw a conclusion that system (5) is persistent on the basis of the result of Theorem 5, which also accounts for the fact that the disease is prevalent.

## 5. Simulations and Conclusions

5.1. Numerical Simulations. In order to confirm the results above, we numerically simulate the solution of system (5) with $n=50, b_{k}=0.25, d=0.3, \epsilon=0.01$, and initial value $S_{k}(0)=0.5, I_{k}(0)=0.1, k=1,2, \ldots, 50$. Using Milstein's Higer Order Method [31], we get the discretization equation:

$$
\begin{align*}
S_{k, i+1}= & S_{k, i}+\Delta t\left(b_{k}-\sum_{j=1}^{2} \beta_{k j} S_{k, i} I_{j, i}-d S_{k, i}\right) \\
& +\sigma_{k 1} S_{k, i} \sqrt{\Delta t} \xi_{k 1, i}+\frac{\sigma_{k 1}^{2}}{2} S_{k, i} \Delta t\left(\xi_{k 1, i}^{2}-1\right)  \tag{56}\\
I_{k, i+1}= & I_{k, i}+\Delta t\left(\sum_{j=1}^{2} \beta_{k j} S_{k, i} I_{j, i}-(d+\epsilon) I_{k, i}\right) \\
& +\sigma_{k 2} I_{k, i} \sqrt{\Delta t} \xi_{k 2, i}+\frac{\sigma_{k 2}^{2}}{2} I_{k, i} \Delta t\left(\xi_{k 2, i}^{2}-1\right)
\end{align*}
$$

where $k=1,2, \ldots, n$ and $\xi_{k 1, i}, \xi_{k 2, i}, i=1,2, \ldots, N$, are the independent Gaussian random variables $N(0,1)$.

From Theorem 3 and Remark 4, it is shown that the expectations of $S_{k}(t), I_{k}(t), k=1,2, \ldots, n$, are converging under some conditions, and the solution of system (5) will oscillate around the disease-free equilibrium of system (3). In Figure 1, we choose parameters $\lambda=0.025, \sigma_{k 1}=0.005$, and $\sigma_{k 2}=0.03$, such that $R_{0} \leq 1$, and in Figures 1(a) and 1(b) we choose $P(k)=m^{k} \exp (-m) / k!$, and in Figures $1(c)$ and $1(d)$
we choose $P(k)=2 m^{2} k^{-3}$. From Figure 1, we can see that the disease-free equilibrium $E_{0}$ of system (3) (imaginary lines) is globally asymptotically stable and the curves of system (5) (real lines) always fluctuate around the curves of system (3) (imaginary lines). From Figure 2, we can see that, due to the difference of the degree distribution, the critical value of spread is different.

In Figure 3, parameters $\lambda=0.08, \sigma_{k 1}=0.01$, and $\sigma_{k 2}=0.01$ and others are the same as the previous. From Figure 3, we can see that the position of the equilibrium state is different due to the difference of the degree distribution. From Figures 1(d), 2(b), 4(b), and 4(d), we found that the solution of stochastic system converging to the disease-free equilibrium is faster than that of the deterministic system with the increase of noise intensity.
5.2. Conclusions. The numerical simulations illustrate the mathematical theorems well. Due to the existence of the noise, the solution of the stochastic system goes around the solution of the deterministic system. With intensities decreasing, the turbulence intensity is weaker. From numerical simulations, we have a new discovery. When $R_{0} \leq 1$, with the increase of noise intensity, the solution of stochastic system converging to the disease-free equilibrium is faster than the deterministic system. This is because of the fact that, when $R_{0} \leq 1$, the disease will die out after some time. However in the real world many stochastic factors contributed to the extinction of the disease.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The authors would like to thank the anonymous referees for their useful feedback, helpful comments, and suggestions which have improved the paper. This project is supported by the National Sciences Foundation of China (10901145, 11331009, 11301491), the Top Young Academic Leaders of Higher Learning Institutions of Shanxi, and the Sciences Foundation of Shanxi Province (2012011002-1).

## References

[1] N. T. Bailey, The Mathematical Theory of Infectious Disease, Hafner Press, New York, NY, USA, 2nd edition, 1975.
[2] R. M. Anderson and R. M. May, Infectious Diseases of Humans, Oxford University Press, Oxford, UK, 1992.
[3] O. Diekmann, J. A. P. Heesterbeek, and J. A. J. Metz, "On the definition and the computation of the basic reproduction ratio $R_{0}$ in models for infectious diseases in heterogeneous populations," Journal of Mathematical Biology, vol. 28, no. 4, pp. 365-382, 1990.
[4] O. Diekmann and J. A. P. Heesterbeek, Mathematical Epidemiology of Infectious Diseases, John Wiley \& Sons, Chichester, UK, 2000, Model building, analysis and interpretation.
[5] H. W. Hethcote, "The mathematics of infectious diseases," SIAM Review, vol. 42, no. 4, pp. 599-653, 2000.
[6] J. Wang, M. Liu, and Y. Li, "Analysis of epidemic models with demographics in metapopulation networks," Physica A, vol. 392, no. 7, pp. 1621-1630, 2013.
[7] S. Eubank, H. Guclu, V. S. A. Kumar et al., "Modelling disease outbreaks in realistic urban social networks," Nature, vol. 429, no. 6988, pp. 180-184, 2004.
[8] M. Barthélemy, A. Barrat, R. Pastor-Satorras, and A. Vespignani, "Dynamical patterns of epidemic outbreaks in complex heterogeneous networks," Journal of Theoretical Biology, vol. 235, no. 2, pp. 275-288, 2005.
[9] L. Wang and G.-Z. Dai, "Global stability of virus spreading in complex heterogeneous networks," SIAM Journal on Applied Mathematics, vol. 68, no. 5, pp. 1495-1502, 2008.
[10] L. Mao-Xing and R. Jiong, "Modelling the spread of sexually transmitted diseases on scale-free networks," Chinese Physics B, vol. 18, no. 6, pp. 2115-2120, 2009.
[11] R. Pastor-Satorras and A. Vespignani, "Epidemic spreading in scale-free networks," Physical Review Letters, vol. 86, no. 14, pp. 3200-3203, 2001.
[12] R. Pastor-Satorras and A. Vespignani, "Epidemic dynamics in finite size scale-free networks," Physical Review E, vol. 65, no. 3, Article ID 035108, pp. 035108/1-035108/4, 2002.
[13] J.-p. Zhang and Z. Jin, "The analysis of an epidemic model on networks," Applied Mathematics and Computation, vol. 217, no. 17, pp. 7053-7064, 2011.
[14] J. Z. Liu, Y. F. Tang, and Z. R. Yang, "The spread of disease with birth and death on networks," Journal of Statistical Mechanics, vol. 2004, Article ID P08008, 2004.
[15] X. Fu, M. Small, D. M. Walker, and H. Zhang, "Epidemic dynamics on scale-free networks with piecewise linear infectivity and immunization," Physical Review E, vol. 77, no. 3, pp. 036113/1-036113/8, 2008.
[16] T. Zhou, J.-G. Liu, W.-J. Bai, G. Chen, and B.-H. Wang, "Behaviors of susceptible-infected epidemics on scale-free networks with identical infectivity," Physical Review E, vol. 74, no. 5, pp. 056109/1-056109/6, 2006.
[17] M.-X. Liu and J. Ruan, "A stochastic epidemic model on homogeneous networks," Chinese Physics B, vol. 18, no. 12, pp. 5111-5116, 2009.
[18] C. Q. Xu, S. L. Yuan, and T. H. Zhang, "Asymptotic behavior of a chemostat model with stochastic perturbation on the dilution rate," Abstract and Applied Analysis, vol. 2013, Article ID 423154, 11 pages, 2013.
[19] D. Jiang, J. Yu, C. Ji, and N. Shi, "Asymptotic behavior of global positive solution to a stochastic SIR model," Mathematical and Computer Modelling, vol. 54, no. 1-2, pp. 221-232, 2011.
[20] C. Ji, D. Jiang, and N. Shi, "Multigroup SIR epidemic model with stochastic perturbation," Physica A, vol. 390, no. 10, pp. 17471762, 2011.
[21] C. Yuan, D. Jiang, D. O'Regan, and R. P. Agarwal, "Stochastically asymptotically stability of the multi-group SEIR and SIR models with random perturbation," Communications in Nonlinear Science and Numerical Simulation, vol. 17, no. 6, pp. 2501-2516, 2012.
[22] E. Tornatore, S. M. Buccellato, and P. Vetro, "Stability of a stochastic SIR system," Physica A, vol. 354, no. 1-4, pp. 111-126, 2005.
[23] P. J. Witbooi, "Stability of an SEIR epidemic model with independent stochastic perturbations," Physica A, vol. 392, no. 20, pp. 4928-4936, 2013.
[24] Y. Zhao, D. Jiang, and D. O'Regan, "The extinction and persistence of the stochastic SIS epidemic model with vaccination," Physica A, vol. 392, no. 20, pp. 4916-4927, 2013.
[25] G. Hu, M. Liu, and K. Wang, "The asymptotic behaviours of an epidemic model with two correlated stochastic perturbations," Applied Mathematics and Computation, vol. 218, no. 21, pp. 10520-10532, 2012.
[26] A. Lahrouz and L. Omari, "Extinction and stationary distribution of a stochastic SIRS epidemic model with non-linear incidence," Statistics \& Probability Letters, vol. 83, no. 4, pp. 960968, 2013.
[27] L. Arnold, Stochastic Differential Equations: Theory and Applications, Wiley Press, New York, NY, USA, 1974.
[28] X. Mao, Stochastic Differential Equations and Applications, Horwood Press, Chichester, UK, 1997.
[29] N. Dalal, D. Greenhalgh, and X. Mao, "A stochastic model of AIDS and condom use," Journal of Mathematical Analysis and Applications, vol. 325, no. 1, pp. 36-53, 2007.
[30] X. Mao, G. Marion, and E. Renshaw, "Environmental brownian noise suppresses explosions in population dynamics," Stochastic Processes and their Applications, vol. 97, no. 1, pp. 95-110, 2002.
[31] D. J. Higham, "An algorithmic introduction to numerical simulation of stochastic differential equations," SIAM Review, vol. 43, no. 3, pp. 525-546, 2001.

## Research Article

# Global Stability for a Predator-Prey Model with Dispersal among Patches 

Yang Gao ${ }^{1,2}$ and Shengqiang Liu ${ }^{1}$<br>${ }^{1}$ Academy of Fundamental and Interdisciplinary Science, Harbin Institute of Technology, Harbin 150080, China<br>${ }^{2}$ Department of Mathematics, Daqing Normal University, Daqing, Heilongjiang 163712, China<br>Correspondence should be addressed to Shengqiang Liu; sqliu@hit.edu.cn

Received 19 January 2014; Accepted 6 February 2014; Published 12 March 2014
Academic Editor: Weiming Wang
Copyright © 2014 Y. Gao and S. Liu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We investigate a predator-prey model with dispersal for both predator and prey among $n$ patches; our main purpose is to extend the global stability criteria by Li and Shuai (2010) on a predator-prey model with dispersal for prey among $n$ patches. By using the method of constructing Lyapunov functions based on graph-theoretical approach for coupled systems, we derive sufficient conditions under which the positive coexistence equilibrium of this model is unique and globally asymptotically stable if it exists.


## 1. Introduction

In the literature of predator-prey population systems, both continuous reaction-diffusion systems and discrete patchy models are used to study the spatial heterogeneity [1, 2]; patchy models are often used to describe directed movement of population among niches or migration among habitats. It is naturally interesting problem to consider how the dispersal or migration of predator and prey influences the global dynamics of the interacting ecological system; thus patchy predator-prey model received lots of attentions [1,3-6].

Since the discrete patchy models usually involve highdimensional system, it is rather mathematically challenging to study the uniqueness and stability of the positive equilibrium of the predator-prey patchy models, and the available global dynamics criteria in the literatures mainly focus on the special case of two-patch [3] or on the permanence and existence of periodic solutions [4-6].

Recently, Li and Shuai [7] considered the following predator-prey model with dispersal for prey among $n$-patch:

$$
\begin{gather*}
\dot{x}_{i}=x_{i}\left(r_{i}-b_{i} x_{i}-e_{i} y_{i}\right)+\sum_{j=1}^{n} d_{i j}^{x}\left(x_{j}-\alpha_{i j}^{x} x_{i}\right),  \tag{1}\\
\dot{y}_{i}=y_{i}\left(-\gamma_{i}-\delta_{i} y_{i}+\varepsilon_{i} x_{i}\right), \quad i=1, \ldots, n .
\end{gather*}
$$

Here, $x_{i}, y_{i}$ denote the densities of prey and predators on the patch $i$, respectively. The parameters $r_{i}, b_{i}$ and $\gamma_{i}, \delta_{i}$ in the model are nonnegative constants. What is more, the parameters $e_{i}$ and $\varepsilon_{i}$ in the model are positive constants. Constant $d_{i j}^{x}$ is the dispersal rate of the prey from patch $j$ to patch $i$ and constants $\alpha_{i j}^{x}$ can be selected to represent different boundary conditions in the continuous diffusion case.

In [7], the authors studied the global stability of the coexistence equilibrium of system (1), by considering (1) as a coupled $n$ predator-prey submodels on networks. Using results from graph theory and a developed systematic approach that allows one to construct global Lyapunov functions for largescale coupled systems from building blocks of individual vertex systems, Li and Shuai [7] obtain the following sharp results for (1).

Proposition 1 (see [7, Theorem 6.1]). Assume that $\left(d_{i j}^{x}\right)_{n \times n}$ is irreducible. If there exists $k$ such that $b_{k}>0$ or $\delta_{k}>0$, then, whenever a positive equilibrium $E_{*}$ exists in (1), it is unique and globally asymptotically stable in the positive cone $R_{2 n}^{+}$.

Although well-improved results have been seen in the above work on dispersal predator-prey model, such models are not well studied yet in the sense that model (1) assumes no dispersal for predator, which is not realistic in many cases $[1,3]$. Thus it is interesting for us to consider the global
stability of the positive equilibrium for predator-prey model with dispersal for both predator and prey.

Motivated by the above work in [7], in this paper we generalize model (1) into the following predator-prey model with dispersal for both predator and prey:

$$
\begin{array}{r}
\dot{x}_{i}=x_{i}\left(r_{i}-b_{i} x_{i}-e_{i} y_{i}\right)+\sum_{j=1}^{n} d_{i j}^{x}\left(x_{j}-\alpha_{i j}^{x} x_{i}\right) \\
\dot{y}_{i}=y_{i}\left(-\gamma_{i}-\delta_{i} y_{i}+\varepsilon_{i} x_{i}\right)+\sum_{j=1}^{n} d_{i j}^{y}\left(y_{j}-\alpha_{i j}^{y} y_{i}\right),  \tag{2}\\
i=1, \ldots, n .
\end{array}
$$

Here, the parameters $r_{i}, b_{i}, e_{i}, \gamma_{i}, \delta_{i}$, and $\varepsilon_{i}$ are defined the same as those in (1). The nonnegative constants $d_{i j}^{y}, \alpha_{i j}^{y}$, and $d_{i j}^{y}$ are the dispersal rate of the predators from patch $j$ to patch $i$, and $\alpha_{i j}^{y}$ represents the different boundary conditions in the continuous diffusion case. Clearly, when $d_{i j}^{y}=0$ for all $i, j=1, \ldots, n$, model (2) directly reduces to (1); thus our model (2) directly extends model (1) in [7].

The main purpose of this paper is to obtain the global stability for the coexistence equilibrium of (2). We will engage the techniques of constructing Lyapunov function based on graph-theory which were well developed by Li et al. in [7-9]; we refer to [10-12] for recent applications. Our study seems to be the first attempt in applying the network method for coupled network systems of differential equations to address the predator-prey system with dispersal for both predator and prey among patches. Networked method has been extensively investigated in the several fields. For example, multiagent systems can be seen as complicated network systems. A lot of researchers take their interest in flocking and consensus of the multiagent systems [13-17]. What is more, neural network systems can be seen as complicated network systems. Over the past few decades, various neural network models have been extensively investigated [18-20].

A mathematical description of a network is a directed graph consisting of vertices and directed arcs connecting them. At each vertex, the local dynamics are given by a system of differential equations called the vertex system. The directed arcs indicate interconnections and interactions among vertex systems.

A digraph $G$ with $n$ vertices for the system (2) can be constructed as follows. Each vertex represents a patch and $(j, i) \in E(G)$ if and only if $d_{i j}^{x}, d_{i j}^{y}>0$. At each vertex of $G$, the vertex dynamics is described by a predator-prey system. The coupling among these predator-prey systems is provided by dispersal of predator and prey among patches.

This paper is organized as follows. In the next section, we introduce preliminaries results on graph-theory based on coupled network models. In Section 3, we obtain the main result of system (2). This is followed by a brief conclusion section.

## 2. Preliminaries

In this section, we will list some definitions and Theorems that we will use in the later sections.

A directed graph or digraph $G=(V, E)$ contains a set $V=$ $\{1,2, \ldots, n\}$ of vertices and a set $E$ of $\operatorname{arcs}(i, j)$ leading from initial vertex $i$ to terminal vertex $j$. A subgraph $H$ of $G$ is said to be spanning if $H$ and $G$ have the same vertex set. A digraph $G$ is weighted if each $\operatorname{arc}(j, i)$ is assigned a positive weight. $a_{i j}>0$ if and only if there exists an arc from vertex $j$ to vertex $i$ in $G$.

The weight $w(H)$ of a subgraph $H$ is the product of the weights on all its arcs. A directed path $P$ in $G$ is a subgraph with distinct vertices $i_{1}, i_{2}, \ldots, i_{m}$ such that its set of arcs is $\left\{\left(i_{k}, i_{k+1}\right): k=1,2, \ldots, m\right\}$. If $i_{m}=i_{1}$, we call $P$ a directed cycle.

A connected subgraph $T$ is a tree if it contains no cycles, directed or undirected.

A tree $T$ is rooted at vertex $i$, called the root, if $i$ is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A subgraph $Q$ is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle.

Given a weighted digraph $G$ with $n$ vertices, define the weight matrix $A=\left(a_{i j}\right)_{n \times n}$ whose entry $a_{i j}$ equals the weight of $\operatorname{arc}(j, i)$ if it exists, and 0 otherwise. For our purpose, we denote a weighted digraph as $(G, A)$. A digraph $G$ is strongly connected if for any pair of distinct vertices, there exists a directed path from one to the other. A weighted digraph ( $G, A$ ) is strongly connected if and only if the weight matrix $A$ is irreducible.

The Laplacian matrix of $(G, A)$ is denoted by $L$. Let $c_{i}$ denote the cofactor of the $i$ th diagonal element of $L$. The following results are listed as follows from [7].

Proposition 2 (see [7]). Assume $n \geq 2$. Then

$$
\begin{equation*}
c_{i}=\sum_{\mathbf{T} \in T_{i}} w(\mathbf{T}), \tag{3}
\end{equation*}
$$

where $T_{i}$ is the set of all spanning trees $\mathbf{T}$ of $(G, A)$ that are rooted at vertex $i$, and $w(T)$ is the weight of $T$. In particular, if $(G, A)$ is strongly connected, then $c_{i}>0$ for $1 \leq i \leq n$.

Theorem 3 (see [7]). Assume $n \geq 2$. Let $c_{i}$ be given in Proposition 2. Then the following identity holds:

$$
\begin{equation*}
\sum_{i, j=1}^{n} c_{i} a_{i j} F_{i j}\left(x_{i}, x_{j}\right)=\sum_{\mathrm{Q} \in \mathbf{Q}} w(Q) \sum_{(s, r) \in E\left(C_{\mathrm{Q}}\right)} F_{r s}\left(x_{r}, x_{s}\right), \tag{4}
\end{equation*}
$$

where $F_{i j}\left(x_{i}, x_{j}\right), 1 \leq i, j \leq n$, are arbitrary functions, $\mathbf{Q}$ is the set of all spanning unicyclic graphs of $(G, A), w(Q)$ is the weight of $Q$, and $C_{Q}$ denotes the directed cycle of $Q$.

Given a network represented by digraph $G$ with $n$ vertices, $n \geq 2$, a coupled system can be built on $G$ by assigning each vertex its own internal dynamics and then coupling these vertex dynamics based on directed arcs in $G$. Assume that
each vertex dynamics is described by a system of differential equations

$$
\begin{equation*}
u_{i}^{\prime}=f_{i}\left(t, u_{i}\right) \tag{5}
\end{equation*}
$$

where $u_{i} \in \mathbf{R}^{\mathbf{m}_{\mathbf{i}}}$ and $f_{i}: \mathbf{R} \times \mathbf{R}^{\mathbf{m}_{\mathbf{i}}} \rightarrow \mathbf{R}^{\mathbf{m}_{\mathrm{i}}}$. Let $g_{i j}: \mathbf{R} \times \mathbf{R}^{\mathbf{m}_{\mathbf{i}}} \times$ $\mathbf{R}^{\mathbf{m}_{\mathbf{j}}} \rightarrow \mathbf{R}^{\mathbf{m}_{\mathrm{i}}}$ represent the influence of vertex $j$ on vertex $i$, and let $g_{i j} \equiv 0$ if there exists no arc from $j$ to $i$ in $G$. Then we obtain the following coupled system on graph $G$ :

$$
\begin{equation*}
u_{i}^{\prime}=f_{i}\left(t, u_{i}\right)+\sum_{j=1}^{n} g_{i j}\left(t, u_{i}, u_{j}\right), \quad i=1,2, \ldots, n \tag{6}
\end{equation*}
$$

Here functions $f_{i}, g_{i j}$ are such that initial-value problems have unique solutions.

We assume that each vertex system has a globally stable equilibrium and possesses a global Lyapunov function $V_{i}$.

Theorem 4 (see [7]). Assume that the following assumptions are satisfied.
(1) There exist functions $V_{i}\left(t, u_{i}\right), F_{i j}\left(t, u_{i}, u_{j}\right)$ and constants $a_{i j} \geq 0$ such that
$\dot{V}_{i}\left(t, u_{i}\right) \leq \sum_{i, j=1}^{n} a_{i j} F_{i j}\left(t, u_{i}, u_{j}\right), \quad t>0, u_{i} \in D_{i}$.
(2) Along each directed cycle $C$ of the weighted digraph $(G, A), A=\left(a_{i j}\right)$,

$$
\begin{equation*}
\sum_{(s, r) \in E(C)} F_{r s}\left(t, u_{r}, u_{s}\right) \leq 0 \tag{8}
\end{equation*}
$$

(3) Constants $c_{i}$ are given by the cofactor of the ith diagonal element of $L$.
Then the function $V(t, u)=\sum_{i=1}^{n} c_{i} V_{i}\left(t, u_{i}\right)$ satisfies $\dot{V}(t, u) \leq 0$ for $t>0, u \in D$; namely, $V$ is a Lyapunov function for the system (6).

## 3. Main Results

In this section, the stability for the positive equilibrium of the $n$-patch predator-prey model (2) is considered. We regard (2) as a coupled system on a network. Using a Lyapunov function for the $n$-patch predator-prey model with dispersal and Theorem 4 of Section 2, we will establish that a positive equilibrium of the $n$-patch predator-prey model (2) with dispersal is globally asymptotically stable in $\mathbf{R}_{+}^{2 n}$ as long as it exists.

First of all, we will give a lemma for the system (2).
Lemma 5. The set $\mathbf{R}_{+}^{2 n}$ is the positive invariant set for the system (2).

The next Theorem gives the globally asymptotically stable condition for the positive equilibrium of the system (2).

Theorem 6. Assume that a positive equilibrium $E^{*}=$ $\left(x_{1}^{*}, y_{1}^{*}, x_{2}^{*}, y_{2}^{*}, \ldots, x_{n}^{*}, y_{n}^{*}\right)$ exists for the system (2) and the following assumptions hold.
(1) Dispersal matrixes $\left(d_{i j}^{x}\right)_{n \times n},\left(d_{i j}^{y}\right)_{n \times n}$ are irreducible; moreover there exists $k$ such that $b_{k}>0$ or $\delta_{k}>0$.
(2) There exists nonnegative constant $\lambda$ such that $\lambda$. $d_{i j}^{x} \varepsilon_{i} x_{j}^{*}=d_{i j}^{y} e_{i} y_{j}^{*}$ for $1 \leq i, j \leq n$, or $d_{i j}^{x} \varepsilon_{i} x_{j}^{*}=$ $\lambda \cdot d_{i j}^{y} e_{i} y_{j}^{*}$ for $1 \leq i, j \leq n$.

Then, the positive equilibrium $E^{*}$ is unique and globally asymptotically stable in $R_{+}^{2 n}$.

## Proof. Let

$$
\begin{gather*}
Z_{i}^{1}\left(x_{i}, y_{i}\right)=r_{i}-b_{i} x_{i}-e_{i} y_{i}  \tag{9}\\
Z_{i}^{2}\left(x_{i}, y_{i}\right)=-\gamma_{i}-\delta_{i} y_{i}+\varepsilon_{i} x_{i}
\end{gather*}
$$

In the sequel, we have

$$
\begin{align*}
Z_{i}^{1}\left(x_{i}^{*}, y_{i}^{*}\right) & =-\frac{1}{x_{i}^{*}} \sum_{j=1}^{n} d_{i j}^{x}\left(x_{j}^{*}-\alpha_{i j}^{x} x_{i}^{*}\right), \\
Z_{i}^{2}\left(x_{i}^{*}, y_{i}^{*}\right) & =-\frac{1}{y_{i}^{*}} \sum_{j=1}^{n} d_{i j}^{y}\left(y_{j}^{*}-\alpha_{i j}^{y} y_{i}^{*}\right) . \tag{10}
\end{align*}
$$

Set Lyapunov functions as

$$
\begin{align*}
V_{i}\left(x_{i}, y_{i}\right)= & \varepsilon_{i}\left(x_{i}-x_{i}^{*}-x_{i}^{*} \ln \frac{x_{i}}{x_{i}^{*}}\right) \\
& +e_{i}\left(y_{i}-y_{i}-y_{i}^{*} \ln \frac{y_{i}}{y_{i}^{*}}\right) \tag{11}
\end{align*}
$$

Direct differentiating $V_{i}$ along the system (2), we have

$$
\begin{aligned}
\dot{V}_{i}\left(x_{i}, y_{i}\right)= & \varepsilon_{i}\left(x_{i}-x_{i}^{*}\right)\left[Z_{i}^{1}\left(x_{i}, y_{i}\right)-Z_{i}^{1}\left(x_{i}^{*}, y_{i}^{*}\right)\right] \\
& +e_{i}\left(y_{i}-y_{i}^{*}\right)\left[Z_{i}^{2}\left(x_{i}, y_{i}\right)-Z_{i}^{2}\left(x_{i}^{*}, y_{i}^{*}\right)\right] \\
& +\varepsilon_{i}\left(x_{i}-x_{i}^{*}\right) Z_{i}^{1}\left(x_{i}^{*}, y_{i}^{*}\right) \\
& +\frac{\varepsilon_{i}\left(x_{i}-x_{i}^{*}\right)}{x_{i}^{*}} \sum_{j=1}^{n} d_{i j}^{x}\left(x_{j}-\alpha_{i j}^{x} x_{i}\right) \\
& +e_{i}\left(y_{i}-y_{i}^{*}\right) Z_{i}^{2}\left(x_{i}^{*}, y_{i}^{*}\right) \\
& +\frac{e_{i}\left(y_{i}-y_{i}^{*}\right)}{y_{i}^{*}} \sum_{j=1}^{n} d_{i j}^{y}\left(y_{j}-\alpha_{i j}^{y} y_{i}\right) \\
= & \varepsilon_{i}\left(x_{i}-x_{i}^{*}\right)\left[Z_{i}^{1}\left(x_{i}, y_{i}\right)-Z_{i}^{1}\left(x_{i}^{*}, y_{i}^{*}\right)\right] \\
& +e_{i}\left(y_{i}-y_{i}^{*}\right)\left[Z_{i}^{2}\left(x_{i}, y_{i}\right)-Z_{i}^{2}\left(x_{i}^{*}, y_{i}^{*}\right)\right] \\
& +\sum_{j=1}^{n} d_{i j}^{x} \varepsilon_{i} x_{j}^{*} F_{i j}^{x}\left(x_{i}, x_{j}\right)+\sum_{j=1}^{n} d_{i j}^{y} e_{i} y_{j}^{*} F_{i j}^{y}\left(y_{i}, y_{j}\right)
\end{aligned}
$$

$$
\begin{align*}
= & -\varepsilon_{i} b_{i}\left(x_{i}-x_{i}^{*}\right)^{2}-\varepsilon_{i}\left(x_{i}-x_{i}^{*}\right) e_{i}\left(y_{i}-y_{i}^{*}\right) \\
& -e_{i} \delta_{i}\left(y_{i}-y_{i}^{*}\right)^{2}+e_{i}\left(y_{i}-y_{i}^{*}\right) \varepsilon_{i}\left(x_{i}-x_{i}^{*}\right) \\
& +\sum_{j=1}^{n} d_{i j}^{x} \varepsilon_{i} x_{j}^{*} F_{i j}^{x}\left(x_{i}, x_{j}\right)+\sum_{j=1}^{n} d_{i j}^{y} e_{i} y_{j}^{*} F_{i j}^{y}\left(y_{i}, y_{j}\right) \\
= & -\varepsilon_{i} b_{i}\left(x_{i}-x_{i}^{*}\right)^{2}-e_{i} \delta_{i}\left(y_{i}-y_{i}^{*}\right)^{2} \\
& +\sum_{j=1}^{n} d_{i j}^{x} \varepsilon_{i} x_{j}^{*} F_{i j}^{x}\left(x_{i}, x_{j}\right)+\sum_{j=1}^{n} d_{i j}^{y} e_{i} y_{j}^{*} F_{i j}^{y}\left(y_{i}, y_{j}\right) \tag{12}
\end{align*}
$$

where

$$
\begin{align*}
& F_{i j}^{x}\left(x_{i}, x_{j}\right)=\frac{x_{j}}{x_{j}^{*}}-\frac{x_{i}}{x_{i}^{*}}+1-\frac{x_{i}^{*} x_{j}}{x_{i} x_{j}^{*}}, \\
& F_{i j}^{y}\left(y_{i}, y_{j}\right)=\frac{y_{j}}{y_{j}^{*}}-\frac{y_{i}}{y_{i}^{*}}+1-\frac{y_{i}^{*} y_{j}}{y_{i} y_{j}^{*}} \tag{13}
\end{align*}
$$

Set $a_{i j}^{x}=d_{i j}^{x} \varepsilon_{i} x_{j}^{*}, b_{i j}^{y}=d_{i j}^{y} e_{i} y_{j}^{*}, A=\left(a_{i j}^{x}\right)_{n \times n}$, and $B=\left(b_{i j}^{y}\right)_{n \times n}$. One has

$$
\begin{equation*}
G_{i}^{x}\left(x_{i}\right)=-\frac{x_{i}}{x_{i}^{*}}+\ln \frac{x_{i}}{x_{i}^{*}}, \quad G_{i}^{y}\left(y_{i}\right)=-\frac{y_{i}}{y_{i}^{*}}+\ln \frac{y_{i}}{y_{i}^{*}} . \tag{14}
\end{equation*}
$$

Next, we have two cases to consider.
Case I. $d_{i j}^{x} \varepsilon_{i} x_{j}^{*}=\lambda \cdot d_{i j}^{y} e_{i} y_{j}^{*}$ for $1 \leq i, j \leq n$.
Case II. $\lambda \cdot d_{i j}^{x} \varepsilon_{i} x_{j}^{*}=d_{i j}^{y} e_{i} y_{j}^{*}$ for $1 \leq i, j \leq n$.
For Case I, from the fact that $a_{i j}^{x}=d_{i j}^{x} \varepsilon_{i} x_{j}^{*}$ and $b_{i j}^{y}=$ $d_{i j}^{y} e_{i} y_{j}^{*}$, we obtain that $a_{i j}^{x}=\lambda b_{i j}^{y}$; thus $A=\lambda \cdot B$. Then we obtain that

$$
\begin{aligned}
\dot{V}_{i}\left(x_{i}, y_{i}\right) \leq & -\varepsilon_{i} b_{i}\left(x_{i}-x_{i}^{*}\right)^{2}-e_{i} \delta_{i}\left(y_{i}-y_{i}^{*}\right)^{2} \\
& +\sum_{j=1}^{n} a_{i j}^{x}\left(G_{i}^{x}\left(x_{i}\right)-G_{j}^{x}\left(x_{j}\right)\right) \\
& +\sum_{j=1}^{n} a_{i j}^{x}\left(1-\frac{x_{i}^{*} x_{j}}{x_{i} x_{j}^{*}}+\ln \frac{x_{i}^{*} x_{j}}{x_{i} x_{j}^{*}}\right) \\
& +\sum_{j=1}^{n} b_{i j}^{y}\left(G_{i}^{y}\left(y_{i}\right)-G_{j}^{y}\left(y_{j}\right)\right) \\
& +\sum_{j=1}^{n} b_{i j}^{y}\left(1-\frac{y_{i}^{*} y_{j}}{y_{i} y_{j}^{*}}+\ln \frac{y_{i}^{*} y_{j}}{y_{i} y_{j}^{*}}\right)
\end{aligned}
$$

$$
\begin{align*}
\leq & -\varepsilon_{i} b_{i}\left(x_{i}-x_{i}^{*}\right)^{2}-e_{i} \delta_{i}\left(y_{i}-y_{i}^{*}\right)^{2} \\
& +\lambda \sum_{j=1}^{n} b_{i j}^{y}\left(G_{i}^{x}\left(x_{i}\right)-G_{j}^{x}\left(x_{j}\right)\right) \\
& +\lambda \sum_{j=1}^{n} b_{i j}^{y}\left(1-\frac{x_{i}^{*} x_{j}}{x_{i} x_{j}^{*}}+\ln \frac{x_{i}^{*} x_{j}}{x_{i} x_{j}^{*}}\right) \\
& +\sum_{j=1}^{n} b_{i j}^{y}\left(G_{i}^{y}\left(y_{i}\right)-G_{j}^{y}\left(y_{j}\right)\right) \\
& +\sum_{j=1}^{n} b_{i j}^{y}\left(1-\frac{y_{i}^{*} y_{j}}{y_{i} y_{j}^{*}}+\ln \frac{y_{i}^{*} y_{j}}{y_{i} y_{j}^{*}}\right) . \tag{15}
\end{align*}
$$

Let $c_{i}^{y}$ denote the cofactor of the $i$ th diagonal element of the matrix $B$. From the irreducible character of matrix $B$, we have $c_{i}^{y}>0$.

Furthermore, set Lyapunov functions as

$$
\begin{align*}
V(x, y) & =V\left(x_{1}, y_{1}, \ldots, x_{n}, y_{n}\right) \\
& =\sum_{i=1}^{n} c_{i}^{y} V_{i}^{x}\left(x_{i}\right)+\sum_{i=1}^{n} c_{i}^{y} V_{i}^{y}\left(y_{i}\right) . \tag{16}
\end{align*}
$$

Then differentiating $V$ along the solution of the system (2), we obtain that

$$
\begin{align*}
\dot{V}(x, y) \leq & -\sum_{i=1}^{n} c_{i}^{y} \varepsilon_{i} b_{i}\left(x_{i}-x_{i}^{*}\right)^{2}-\sum_{i=1}^{n} c_{i}^{y} e_{i} \delta_{i}\left(y_{i}-y_{i}^{*}\right)^{2} \\
& +\sum_{i, j=1}^{n} \lambda b_{i j}^{y} c_{i}^{y}\left(G_{i}^{x}\left(x_{i}\right)-G_{j}^{x}\left(x_{j}\right)\right)  \tag{17}\\
& +\sum_{i, j=1}^{n} b_{i j}^{y} c_{i}^{y}\left(G_{i}^{y}\left(y_{i}\right)-G_{j}^{y}\left(y_{j}\right)\right) .
\end{align*}
$$

Let $G$ represent the directed graph associated with matrix $B$. Then $G$ has vertices $1,2, \ldots, n$ with a directed $\operatorname{arc}(k, j)$ from $k$ to $j$ if and only if $b_{k j}^{y} \neq 0$. Then $E(G)$ is the set of all directed arcs of $G$. By Kirchhoff's Matrix-Tree Theorem (see Proposition 2) we know that $v_{k}=C_{k k}$ can be expressed as a sum of weights of all directed spanning subtrees $T$ of $G$ that are rooted at vertex $k$. Thus, each term in $v_{k} a_{k j}$ is the weight $\omega(Q)$ of a unicyclic subgraph $Q$ of $G$ obtained from such a tree $T$ by adding a directed $\operatorname{arc}(k, j)$ from the root $k$ to vertex $j$. Because the $\operatorname{arc}(k, j)$ is a part of the unique cycle $C Q$ of $Q$ and that the same unicyclic graph $Q$ can be formed when each arc of $C Q$ is added to a corresponding rooted tree $T$, then the double sum can be expressed as a sum over all unicyclic subgraphs $Q$ containing vertices $1,2, \ldots, n$.

Therefore, following from the irreducible character of matrix $B$ and Theorem 2.3 in [7], we obtain

$$
\begin{align*}
& \sum_{i, j=1}^{n} \lambda b_{i j}^{y} c_{i}^{y}\left(G_{i}^{x}\left(x_{i}\right)-G_{j}^{x}\left(x_{j}\right)\right)=0 \\
& \sum_{i, j=1}^{n} b_{i j}^{y} c_{i}^{y}\left(G_{i}^{y}\left(y_{i}\right)-G_{j}^{y}\left(y_{j}\right)\right)=0 \tag{18}
\end{align*}
$$

Combining with the fact that $1-a+\ln a \leq 0$, therefore we have

$$
\begin{equation*}
\dot{V}(x, y) \leq 0 \tag{19}
\end{equation*}
$$

When we consider $\dot{V}(x, y)=0$, by condition 1 , there exists $k \in N_{+}$such that

$$
\begin{equation*}
\left(x_{k}-x_{k}^{*}\right)^{2}=0 \quad \text { or } \quad\left(y_{k}-y_{k}^{*}\right)^{2}=0 \tag{20}
\end{equation*}
$$

It means that $x_{k}=x_{k}^{*}$ or $y_{k}=y_{k}^{*}$.
If $i$ and $k$ can be connected with an arc from $k$ to $i$ in $G$, then we have $a_{i k}^{y}>0$ and $b_{i k}^{y}>0$. Furthermore,

$$
\begin{align*}
& 1-\frac{x_{i}^{*} x_{k}}{x_{i} x_{k}^{*}}+\ln \frac{x_{i}^{*} x_{k}}{x_{i} x_{k}^{*}}=0 \\
& 1-\frac{y_{i}^{*} y_{k}}{y_{i} y_{k}^{*}}+\ln \frac{y_{i}^{*} y_{k}}{y_{i} y_{k}^{*}}=0 \tag{21}
\end{align*}
$$

Because of $1-a+\ln a \leq 0$ and $1-a+\ln a=0$, $\Leftrightarrow a=0$. we deduce that

$$
\begin{equation*}
\frac{x_{i}}{x_{i}^{*}}=\frac{x_{k}}{x_{k}^{*}}, \quad \frac{y_{i}}{y_{i}^{*}}=\frac{y_{k}}{y_{k}^{*}} . \tag{22}
\end{equation*}
$$

From $x_{k}=x_{k}^{*}$, or $y_{k}=y_{k}^{*}$, we obtain that $x_{i}=x_{i}^{*}$ and $y_{i} / y_{i}^{*}=$ $y_{k} / y_{k}^{*}$ or $y_{i}=y_{i}^{*}$ and $x_{i} / x_{i}^{*}=x_{k} / x_{k}^{*}$.

By condition 1 and the definition of matrixes $A, B$, we get that $B$ are irreducible. By strong connectivity of $G$, there exists a directed path $P$ from any $i$ to $k$. Then we have that, for any $i=1,2, \ldots, n$, there must be

$$
\begin{equation*}
x_{i}=x_{i}^{*}, \quad \frac{y_{i}}{y_{i}^{*}}=\mu, \quad \mu \geq 0 \tag{23}
\end{equation*}
$$

or for any $i=1,2, \ldots, n$, there must be

$$
\begin{equation*}
y_{i}=y_{i}^{*}, \quad \frac{x_{i}}{x_{i}^{*}}=\mu, \quad \mu \geq 0 \tag{24}
\end{equation*}
$$

Next, we will prove that the largest compact invariant subset of $\{(x, y) \mid \dot{V}(x, y)=0\}$ is the singleton $\left\{E^{*}\right\}$.

We only consider the case that

$$
\begin{equation*}
x_{i}=x_{i}^{*}, \quad \frac{y_{i}}{y_{i}^{*}}=\mu, \quad i=1,2, \ldots, n, \mu \geq 0 \tag{25}
\end{equation*}
$$

The case that

$$
\begin{equation*}
y_{i}=y_{i}^{*}, \quad \frac{x_{i}}{x_{i}^{*}}=\mu, \quad i=1,2, \ldots, n, \mu \geq 0 \tag{26}
\end{equation*}
$$

is similar to this case. So we omit it.

If $\mu=0$, we have $y_{i}=0$ for any $i=1,2, \ldots, n$, and then we have

$$
\begin{equation*}
x_{i}^{*}\left(r_{i}-b_{i} x_{i}^{*}\right)+\sum_{j=1}^{n} d_{i j}^{x}\left(x_{j}^{*}-\alpha_{i j}^{x} x_{i}^{*}\right)=0 \tag{27}
\end{equation*}
$$

which contradicts to the fact that

$$
\begin{equation*}
x_{i}^{*}\left(r_{i}-b_{i} x_{i}^{*}-e_{i} y_{i}^{*}\right)+\sum_{j=1}^{n} d_{i j}^{x}\left(x_{j}^{*}-\alpha_{i j}^{x} x_{i}^{*}\right)=0 \tag{28}
\end{equation*}
$$

If $\mu>0$ and $\mu \neq 1$, we have $y_{i}=\mu y_{i}^{*}$ for any $i=1,2, \ldots, n$, and then we have

$$
\begin{equation*}
x_{i}^{*}\left(r_{i}-b_{i} x_{i}^{*}-e_{i} \mu y_{i}^{*}\right)+\sum_{j=1}^{n} d_{i j}^{x}\left(x_{j}^{*}-\alpha_{i j}^{x} x_{i}^{*}\right)=0 \tag{29}
\end{equation*}
$$

which also contradicts to the fact that

$$
\begin{equation*}
x_{i}^{*}\left(r_{i}-b_{i} x_{i}^{*}-e_{i} y_{i}^{*}\right)+\sum_{j=1}^{n} d_{i j}^{x}\left(x_{j}^{*}-\alpha_{i j}^{x} x_{i}^{*}\right)=0 \tag{30}
\end{equation*}
$$

Therefore, we obtain that $\mu=1$, which means

$$
\begin{equation*}
x_{i}=x_{i}^{*}, \quad y_{i}=y_{i}^{*}, \quad i=1,2, \ldots, n \tag{31}
\end{equation*}
$$

Namely, we get that the largest compact invariant subset of $\{(x, y) \mid \dot{V}(x, y)=0\}$ is the singleton $\left\{E^{*}\right\}$. Therefore, by the LaSalle Invariance Principle ([21]), $E^{*}$ is globally asymptotically stable in $\mathbf{R}_{+}^{2 n}$.

With the similar arguments to the Case I, we can prove that $E^{*}$ is globally asymptotically stable in $\mathbf{R}_{+}^{2 n}$ for Case II. This completes the proof.

Remark 7. Theorem 6 is applicable to model (1): consider model (2) with $d_{i j}^{y}=0, i, j=1, \ldots, n$, and let $\lambda=0$; thus Theorem 6 directly reduces to Proposition 1 by Li and Shuai [7] for (1).

By Theorem 6 and similar arguments to Remark 7, we directly have the following global stability theorem for the predator-prey model with discrete dispersal of predator among patches.

Corollary 8. Consider the model

$$
\begin{array}{r}
\dot{x}_{i}=x_{i}\left(r_{i}-b_{i} x_{i}-e_{i} y_{i}\right) \\
\dot{y}_{i}=y_{i}\left(-\gamma_{i}-\delta_{i} y_{i}+\varepsilon_{i} x_{i}\right)+\sum_{j=1}^{n} d_{i j}^{y}\left(y_{j}-\alpha_{i j}^{y} y_{i}\right),  \tag{32}\\
i=1, \ldots, n
\end{array}
$$

Assume that the matrix $\left(d_{i j}^{y}\right)_{n \times n}$ is irreducible. If there exists $k$ such that $b_{k}>0$ or $\delta_{k}>0$; then, whenever a positive equilibrium $E_{*}$ exists in (32), it is unique and globally asymptotically stable in the positive cone $R_{+}^{2 n}$.

## 4. Discussion

In this paper, we generalize the model of the $n$-patch predator-prey model of [7] to the general model (2) that both the prey and the predator have dispersal among $n$ patches. Based on the network method for coupled systems of differential equations developed in [7-9], we prove that the positive equilibrium of (2) is globally asymptotically stable given some conditions on the coupling (see Theorem 6). Our main theorem generalizes Theorem 6.1 in [7] and our results also cover the other case of (2) in that only the predators disperse among patches.

Biologically, our result of Theorem 6 implies that if predator-prey system is dispersing among strongly connected patches (which is equivalent to the irreducibility of the dispersal matrixes of predator and prey) and if the system is permanent (which guarantees the existence of positive equilibrium), then the numbers of both predators and prey in each patches will eventually be stable at some corresponding positive values given the well-coupled dispersal (condition 2 of Theorem 6).

We remark that our Theorem 6 requires the extra condition 2 for the coupling dispersal coefficients and that the global dynamics for the coexistence equilibrium of (2) without condition 2 of Theorem 6 are still unclear. It remains an interesting future problem for the patchy dispersal predatorprey model.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The first author is supported by the Natural Science Foundation for Doctor of Daqing Normal University (no. 12ZR09). Shengqiang Liu is supported by the NNSF of China (no. 10601042), the Fundamental Research Funds for the Central Universities (no. HIT.NSRIF.2010052), and Program of Excellent Team in Harbin Institute of Technology.

## References

[1] H. I. Freedman and Y. Takeuchi, "Global stability and predator dynamics in a model of prey dispersal in a patchy environment," Nonlinear Analysis: Theory, Methods \& Applications, vol. 13, no. 8, pp. 993-1002, 1989.
[2] J. D. Murry, Mathematical Biology, vol. 1-2, Springer, New York, NY, USA, 2002.
[3] Y. Kuang and Y. Takeuchi, "Predator-prey dynamics in models of prey dispersal in two-patch environments," Mathematical Biosciences, vol. 120, no. 1, pp. 77-98, 1994.
[4] J. Cui, "The effect of dispersal on permanence in a predatorprey population growth model," Computers \& Mathematics with Applications, vol. 44, no. 8-9, pp. 1085-1097, 2002.
[5] R. Xu, M. A. J. Chaplain, and F. A. Davidson, "Periodic solutions for a delayed predator-prey model of prey dispersal
in two-patch environments," Nonlinear Analysis: Real World Applications, vol. 5, no. 1, pp. 183-206, 2004.
[6] L. Zhang and Z. Teng, "Permanence for a delayed periodic predator-prey model with prey dispersal in multi-patches and predator density-independent," Journal of Mathematical Analysis and Applications, vol. 338, no. 1, pp. 175-193, 2008.
[7] M. Y. Li and Z. Shuai, "Global-stability problem for coupled systems of differential equations on networks," Journal of Differential Equations, vol. 248, no. 1, pp. 1-20, 2010.
[8] H. Guo, M. Y. Li, and Z. Shuai, "A graph-theoretic approach to the method of global Lyapunov functions," Proceedings of the American Mathematical Society, vol. 136, no. 8, pp. 2793-2802, 2008.
[9] M. Y. Li and Z. Shuai, "Global stability of an epidemic model in a patchy environment," Canadian Applied Mathematics Quarterly, vol. 17, no. 1, pp. 175-187, 2009.
[10] H. Shu, D. Fan, and J. Wei, "Global stability of multi-group SEIR epidemic models with distributed delays and nonlinear transmission," Nonlinear Analysis: Real World Applications, vol. 13, no. 4, pp. 1581-1592, 2012.
[11] R. Sun and J. Shi, "Global stability of multigroup epidemic model with group mixing and nonlinear incidence rates," Applied Mathematics and Computation, vol. 218, no. 2, pp. 280286, 2011.
[12] J. Wang, J. Zu, X. Liu, G. Huang, and J. Zhang, "Global dynamics of a multi-group epidemic model with general relapse distribution and nonlinear incidence rate," The Journal of Biological Systems, vol. 20, no. 3, pp. 235-258, 2012.
[13] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: algorithms and theory," IEEE Transactions on Automatic Control, vol. 51, no. 3, pp. 401-420, 2006.
[14] N. Moshtagh, A. Jadbabaie, and K. Daniilidis, "Distributed geodesic control laws for flocking of nonholonomic agents," in Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference (CDC-ECC '05), pp. 2835-2840, December 2005.
[15] R. A. Freeman, Y. Peng, and K. M. Lynch, "Distributed estimation and control of swarm formation statistics," in Proceedings of the American Control Conference, pp. 749-755, June 2006.
[16] Y. Hong, L. Gao, D. Cheng, and J. Hu, "Lyapunov-based approach to multiagent systems with switching jointly connected interconnection," IEEE Transactions on Automatic Control, vol. 52, no. 5, pp. 943-948, 2007.
[17] R. Olfati-Saber and J. S. Shamma, "Consensus filters for sensor networks and distributed sensor fusion," in Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference (CDC-ECC '05), pp. 6698-6703, December 2005.
[18] M. T. Hagan, H. B. Demuth, and M. H. Beale, Neural Network Design, China Machine, Beijing, China, 2002.
[19] Z. H. Zhou and C. G. Cao, Neural Network with Applications, Tsinghua University Press, Beijing, China, 2004.
[20] C. Hu, J. Yu, H. Jiang, and Z. Teng, "Exponential stabilization and synchronization of neural networks with time-varying delays via periodically intermittent control," Nonlinearity, vol. 23, no. 10, pp. 2369-2391, 2010.
[21] H. K. Khalil, Nonlinear Systems, Prentice Hall, 3rd edition, 2002.

## Research Article

# Global Exponential Stability of Pseudo Almost Periodic Solutions for SICNNs with Time-Varying Leakage Delays 

Wentao Wang and Bingwen Liu<br>College of Mathematics, Physics and Information Engineering, Jiaxing University, Jiaxing, Zhejiang 314001, China<br>Correspondence should be addressed to Bingwen Liu; liubw007@aliyun.com

Received 13 November 2013; Accepted 6 January 2014; Published 10 March 2014
Academic Editor: Weiming Wang
Copyright © 2014 W. Wang and B. Liu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

This paper is concerned with the shunting inhibitory cellular neural networks (SICNNs) with time-varying delays in the leakage (or forgetting) terms. Under proper conditions, we employ a novel argument to establish a criterion on the global exponential stability of pseudo almost periodic solutions by using Lyapunov functional method and differential inequality techniques. We also provide numerical simulations to support the theoretical result.


## 1. Introduction

In the last three decades, shunting inhibitory cellular neural networks (SICNNs) have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. Hence, they have been the object of intensive analysis by numerous authors in recent years. In particular, there have been extensive results on the problem of the existence and stability of the equilibrium point and periodic and almost periodic solutions of SICNNs with time-varying delays in the literature. We refer the reader to $[1-7]$ and the references cited therein.

It is well known that SICNNs have been introduced as new cellular neural networks (CNNs) in Bouzerdoum et al. in $[1,8,9]$, which can be described by

$$
\begin{aligned}
x_{i j}^{\prime}(t)= & -a_{i j}(t) x_{i j}(t) \\
& -\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(t) f\left(x_{k l}\left(t-\tau_{k l}(t)\right)\right) x_{i j}(t) \\
& -\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(t) \\
& \cdot \int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}(t-u)\right) d u x_{i j}(t)+L_{i j}(t), \\
& i=1,2, \ldots, m, j=1,2, \ldots, n,
\end{aligned}
$$

where $C_{i j}$ denotes the cell at the $(i, j)$ position of the lattice. The $r$-neighborhood $N_{r}(i, j)$ of $C_{i j}$ is given as

$$
\begin{array}{r}
N_{r}(i, j)=\left\{C_{k l}: \max (|k-i|,|l-j|) \leq r\right. \\
1 \leq k \leq m, 1 \leq l \leq n\} \tag{2}
\end{array}
$$

where $N_{q}(i, j)$ is similarly specified, $x_{i j}$ is the activity of the cell $C_{i j}, L_{i j}(t)$ is the external input to $C_{i j}$, the function $a_{i j}(t)>0$ represents the passive decay rate of the cell activity, $C_{i j}^{k l}(t)$ and $B_{i j}^{k l}(t)$ are the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell $C_{i j}$, and the activity functions $f(\cdot)$ and $g(\cdot)$ are continuous functions representing the output or firing rate of the cell $C_{k l}$, and $\tau_{k l}(t) \geq 0$ corresponds to the transmission delay.

Obviously, the first term in each of the right side of (1) corresponds to stabilizing negative feedback of the system which acts instantaneously without time delay; these terms are variously known as "forgettin" or leakage terms (see, for instance, Kosko [10], Haykin [11]). It is known from the literature on population dynamics and neural networks dynamics (see Gopalsamy [12]) that time delays in the stabilizing negative feedback terms will have a tendency to destabilize a system. Therefore, the authors of [13-19] dealt with the existence and stability of equilibrium and periodic solutions for neuron networks model involving
leakage delays. Recently, Liu and Shao [20] considered the following SICNNs with time-varying leakage delays:

$$
\begin{align*}
x_{i j}^{\prime}(t)= & -a_{i j}(t) x_{i j}\left(t-\eta_{i j}(t)\right) \\
& -\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(t) f\left(x_{k l}\left(t-\tau_{k l}(t)\right)\right) x_{i j}(t) \\
& -\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(t)  \tag{3}\\
& \times \int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}(t-u)\right) d u x_{i j}(t)+L_{i j}(t),
\end{align*}
$$

where $i=1,2, \ldots, m, j=1,2, \ldots, n, \eta_{i j},: \mathbb{R} \rightarrow[0+$ $\infty$ ) denotes the leakage delay. By using Lyapunov functional method and differential inequality techniques, in [20], some sufficient conditions have been established to guarantee that all solutions of (1) converge exponentially to the almost periodic solution. Moreover, it is well known that the global exponential convergence behavior of solutions plays a key role in characterizing the behavior of dynamical system since the exponential convergent rate can be unveiled (see [2124]). However, to the best of our knowledge, few authors have considered the exponential convergence on the pseudo almost periodic solution for (1). Motivated by the above discussions, in this paper, we will establish the existence and uniqueness of pseudo almost periodic solution of (1) by using the exponential dichotomy theory and contraction mapping fixed point theorem. Meanwhile, we also will give the conditions to guarantee that all solutions and their derivatives of solutions for (1) converge exponentially to the pseudo almost periodic solution and its derivative, respectively.

For convenience, we denote by $\mathbb{R}^{p}\left(\mathbb{R}=\mathbb{R}^{1}\right)$ the set of all $p$-dimensional real vectors (real numbers). We will use

$$
\begin{align*}
& \left\{x_{i j}(t)\right\}=\left(x_{11}(t), \ldots, x_{1 n}(t), \ldots, x_{i 1}(t), \ldots,\right. \\
& \left.\quad x_{i n}(t), \ldots, x_{m 1}(t), \ldots, x_{m n}(t)\right) \in \mathbb{R}^{m \times n} . \tag{4}
\end{align*}
$$

For any $x(t)=\left\{x_{i j}(t)\right\} \in \mathbb{R}^{m \times n}$, we let $|x|$ denote the absolutevalue vector given by $|x|=\left\{\left|x_{i j}\right|\right\}$ and define $\|x(t)\|=$ $\max _{(i, j)}\left\{\left|x_{i j}(t)\right|\right\}$. A matrix or vector $A \geq 0$ means that all entries of $A$ are greater than or equal to zero. $A>0$ can be defined similarly. For matrices or vectors $A_{1}$ and $A_{2}, A_{1} \geq A_{2}$ (resp. $A_{1}>A_{2}$ ) means that $A_{1}-A_{2} \geq 0$ (resp. $A_{1}-A_{2}>0$ ). For the convenience, we will introduce the notations:

$$
\begin{equation*}
h^{+}=\sup _{t \in \mathbb{R}}|h(t)|, \quad h^{-}=\inf _{t \in \mathbb{R}}|h(t)|, \tag{5}
\end{equation*}
$$

where $h(t)$ is a bounded continuous function.
The initial conditions associated with system (3) are of the form:

$$
\begin{align*}
& x_{i j}(s)=\varphi_{i j}(s), \quad s \in(-\infty, 0]  \tag{6}\\
i j \in J:= & \{11, \ldots, 1 n, 21, \ldots, 2 n, \ldots, m 1, \ldots, m n\}
\end{align*}
$$

where $\varphi_{i j}(\cdot)$ and $\varphi_{i j}^{\prime}(\cdot)$ are real-valued bounded continuous functions defined on $(-\infty, 0]$.

The paper is organized as follows. Section 2 includes some lemmas and definitions, which can be used to check the existence of almost periodic solutions of (3). In Section 3, we present some new sufficient conditions for the existence of the continuously differentiable pseudo almost periodic solution of (3). In Section 4, we establish sufficient conditions on the global exponential stability of pseudo almost periodic solutions of (3). At last, an example and its numerical simulation are given to illustrate the effectiveness of the obtained results.

## 2. Preliminary Results

In this section, we will first recall some basic definitions and lemmas which are used in what follows.

In this paper, $\mathrm{BC}\left(\mathbb{R}, \mathbb{R}^{p}\right)$ denotes the set of bounded continued functions from $\mathbb{R}$ to $\mathbb{R}^{p}$. Note that $\left(\mathrm{BC}\left(\mathbb{R}, \mathbb{R}^{p}\right),\|\cdot\|_{\infty}\right)$ is a Banach space where $\|\cdot\|_{\infty}$ denotes the sup norm $\|f\|_{\infty}:=$ $\sup _{t \in \mathbb{R}}\|f(t)\|$.

Definition 1 (see $[25,26])$. Let $u(t) \in \mathrm{BC}\left(\mathbb{R}, \mathbb{R}^{p}\right) . u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon>0$, the set $T(u, \varepsilon)=$ $\{\delta:\|u(t+\delta)-u(t)\|<\varepsilon$ for all $t \in \mathbb{R}\}$ is relatively dense; that is, for any $\varepsilon>0$, it is possible to find a real number $l=l(\varepsilon)>0$; for any interval with length $l(\varepsilon)$, there exists a number $\delta=$ $\delta(\varepsilon)$ in this interval such that $\|u(t+\delta)-u(t)\|<\varepsilon$, for all $t \in \mathbb{R}$.

We denote by $\operatorname{AP}\left(\mathbb{R}, \mathbb{R}^{n}\right)$ the set of the almost periodic functions from $\mathbb{R}$ to $\mathbb{R}^{n}$. Besides, the concept of pseudo almost periodicity (pap) was introduced by Zhang in the early nineties. It is a natural generalization of the classical almost periodicity. Precisely, define the class of functions $\operatorname{PAP}_{0}(\mathbb{R}, \mathbb{R})$ as follows:

$$
\begin{equation*}
\left\{\left.f \in \mathrm{BC}\left(\mathbb{R}, \mathbb{R}^{n}\right)\left|\lim _{T \rightarrow+\infty} \frac{1}{2 T} \int_{-T}^{T}\right| f(t) \right\rvert\, d t=0\right\} \tag{7}
\end{equation*}
$$

A function $f \in \mathrm{BC}\left(\mathbb{R}, \mathbb{R}^{n}\right)$ is called pseudo almost periodic if it can be expressed as

$$
\begin{equation*}
f=h+\varphi, \tag{8}
\end{equation*}
$$

where $h \in \operatorname{AP}\left(\mathbb{R}, \mathbb{R}^{n}\right)$ and $\varphi \in \operatorname{PAP}_{0}\left(\mathbb{R}, \mathbb{R}^{n}\right)$. The collection of such functions will be denoted by $\operatorname{PAP}\left(\mathbb{R}, \mathbb{R}^{n}\right)$. The functions $h$ and $\varphi$ in the above definition are, respectively, called the almost periodic component and the ergodic perturbation of the pseudo almost periodic function $f$. The decomposition given in definition above is unique. Observe that $\left(\operatorname{PAP}\left(\mathbb{R}, \mathbb{R}^{n}\right),\|\cdot\|_{\infty}\right)$ is a Banach space and $\operatorname{AP}\left(\mathbb{R}, \mathbb{R}^{n}\right)$ is a proper subspace of $\operatorname{PAP}\left(\mathbb{R}, \mathbb{R}^{n}\right)$ since the function $\phi(t)=$ $\cos \pi t+\cos t+e^{-t^{4} \sin ^{2} t}$ is pseudo almost periodic function but not almost periodic. It should be mentioned that pseudo almost periodic functions possess many interesting properties; we shall need only a few of them and for the proofs we shall refer to [25].

Lemma 2 (see [25, page 57]). If $f \in P A P(\mathbb{R}, \mathbb{R})$ and $g$ is its almost periodic component, then we have

$$
\begin{equation*}
g(\mathbb{R}) \subset \overline{f(\mathbb{R})} \tag{9}
\end{equation*}
$$

Therefore $\|f\|_{\infty} \geq\|g\|_{\infty} \geq \inf _{x \in \mathbb{R}}|g(x)| \geq \inf _{x \in \mathbb{R}}|f(x)|$.

Lemma 3 (see [25, page 140]). Suppose that both functions $f$ and its derivative $f^{\prime}$ are in $\operatorname{PAP}(\mathbb{R}, \mathbb{R})$. That is, $f=g+\varphi$ and $f^{\prime}=\alpha+\beta$, where $g, \alpha \in A P(\mathbb{R}, \mathbb{R})$ and $\varphi, \beta \in \operatorname{PAP}_{0}(\mathbb{R}, \mathbb{R})$. Then the functions $g$ and $\varphi$ are continuous differentiable so that

$$
\begin{equation*}
g^{\prime}=\alpha, \quad \varphi^{\prime}=\beta \tag{10}
\end{equation*}
$$

Lemma 4. Let $B^{*}=\left\{f \mid f, f^{\prime} \in \operatorname{PAP}(\mathbb{R}, \mathbb{R})\right\}$ equipped with the induced norm defined by $\|f\|_{B^{*}}=\max \left\{\|f\|_{\infty},\left\|f^{\prime}\right\|_{\infty}\right\}=$ $\max \left\{\sup _{t \in \mathbb{R}}|f(t)|, \sup _{t \in \mathbb{R}}\left|f^{\prime}(t)\right|\right\}$, and then $B^{*}$ is a Banach space.

Proof. Suppose that $\left\{f_{p}\right\}_{p=1}^{+\infty}$ is a Cauchy sequence in $B^{*}$, and then for any $\varepsilon>0$, there exists $N(\varepsilon)>0$, such that

$$
\begin{align*}
& \left\|f_{p}-f_{q}\right\|_{B^{*}} \\
& =\max \left\{\sup _{t \in \mathbb{R}}\left|f_{p}(t)-f_{q}(t)\right|, \sup _{t \in \mathbb{R}}\left|f_{p^{\prime}}(t)-f_{q^{\prime}}(t)\right|\right\}<\varepsilon, \\
& \forall p, q \geq N(\varepsilon) . \tag{11}
\end{align*}
$$

By the definition of pseudo almost periodic function, let

$$
\begin{array}{r}
f_{p}=g_{p}+\varphi_{p}, \quad \text { where } g_{p} \in \operatorname{AP}(\mathbb{R}, \mathbb{R})  \tag{12}\\
\varphi_{p} \in \operatorname{PAP}_{0}(\mathbb{R}, \mathbb{R}), \quad p=1,2, \ldots
\end{array}
$$

From Lemma 3, we obtain

$$
\begin{array}{r}
f_{p}^{\prime}=g_{p}^{\prime}+\varphi_{p}^{\prime}, \quad \text { where } g_{p}^{\prime} \in \mathrm{AP}(\mathbb{R}, \mathbb{R}) \\
\varphi_{p}^{\prime} \in \operatorname{PAP}_{0}(\mathbb{R}, \mathbb{R}), \quad p=1,2, \ldots \tag{13}
\end{array}
$$

On combining (11) with Lemma 2, we deduce that, $\left\{g_{p}\right\}_{p=1}^{+\infty},\left\{g_{p}^{\prime}\right\}_{p=1}^{+\infty} \subset \operatorname{AP}(\mathbb{R}, \mathbb{R})$ are Cauchy sequence, so that $\left\{\varphi_{p}\right\}_{p=1}^{+\infty},\left\{\varphi_{p}^{\prime}\right\}_{p=1}^{+\infty} \subset \operatorname{PAP}_{0}(\mathbb{R}, \mathbb{R})$ are also Cauchy sequence.

Firstly, we show that there exists $g \in A P(\mathbb{R}, \mathbb{R})$ such that $g_{p}$ uniformly converges to $g$, as $p \rightarrow+\infty$.

Note that $\left\{g_{p}\right\}$ is Cauchy sequence in $\operatorname{AP}(\mathbb{R}, \mathbb{R})$. for all $\varepsilon>0, \exists N(\varepsilon)$, such that for all $p, q \geq N(\varepsilon)$

$$
\begin{equation*}
\left|g_{p}(t)-g_{q}(t)\right|<\varepsilon, \quad \forall t \in \mathbb{R} \tag{14}
\end{equation*}
$$

So for fixed $t \in \mathbb{R}$, it is easy to see $\left\{g_{p}(t)\right\}_{p=1}^{+\infty}$ is Cauchy number sequence. Thus, the limits of $g_{p}(t)$ exist as $p \rightarrow+\infty$ and let $g(t)=\lim _{p \rightarrow+\infty} g_{p}(t)$. In (14), let $q \rightarrow+\infty$, and we have

$$
\begin{equation*}
\left|g(t)-g_{p}(t)\right| \leq \varepsilon, \quad \forall t \in \mathbb{R}, p \geq N(\varepsilon) \tag{15}
\end{equation*}
$$

Thus, $g_{n}$ uniformly converges to $g$, as $p \rightarrow+\infty$. Moreover, from the Theorem 1.9 [26, page 5], we obtain $g \in A P(\mathbb{R}, \mathbb{R})$. Similarly, we also obtain that there exist $g^{*} \in \operatorname{AP}(\mathbb{R}, \mathbb{R})$ and $\varphi, \varphi^{*} \in \operatorname{BC}(\mathbb{R}, \mathbb{R})$, such that

$$
\begin{aligned}
& \left|g^{*}(t)-g_{p}^{\prime}(t)\right| \leq \varepsilon \\
& \left|\varphi(t)-\varphi_{p}(t)\right| \leq \varepsilon \\
& \left|\varphi^{*}(t)-\varphi_{p}^{\prime}(t)\right| \leq \varepsilon \\
& \forall t \in \mathbb{R}, p \geq N(\varepsilon)
\end{aligned}
$$

which lead to

$$
\begin{equation*}
g_{p}^{\prime} \Longrightarrow g^{*}, \quad \varphi_{p} \Longrightarrow \varphi, \quad \varphi_{p}^{\prime} \Longrightarrow \varphi^{*} \tag{17}
\end{equation*}
$$

where $p \rightarrow+\infty$ and " $\Rightarrow$ " means uniform convergence.
Next, we claim that $\varphi, \varphi^{*} \in \operatorname{PAP}_{0}(\mathbb{R})$. Together with (16) and the facts that

$$
\begin{align*}
& \lim _{r \rightarrow+\infty} \frac{1}{2 r} \int_{-r}^{r}\left|\varphi_{p}(s)\right| d s=0, \quad \lim _{r \rightarrow+\infty} \frac{1}{2 r} \int_{-r}^{r}\left|\varphi_{p}^{\prime}(s)\right| d s=0 \\
& \quad p=1,2, \ldots \\
& \frac{1}{2 r} \int_{-r}^{r}|\varphi(s)| d s \leq \frac{1}{2 r} \int_{-r}^{r}\left|\varphi(s)-\varphi_{p}(s)\right| d s \\
& \quad+\frac{1}{2 r} \int_{-r}^{r}\left|\varphi_{p}(s)\right| d s, \quad r>0, n=1,2, \ldots \\
& \frac{1}{2 r} \int_{-r}^{r}\left|\varphi^{*}(s)\right| d s \leq \frac{1}{2 r} \int_{-r}^{r}\left|\varphi^{*}(s)-\varphi_{p}^{\prime}(s)\right| d s \\
& \quad+\frac{1}{2 r} \int_{-r}^{r}\left|\varphi_{p}^{\prime}(s)\right| d s, \quad r>0, p=1,2, \ldots \tag{18}
\end{align*}
$$

we have

$$
\begin{equation*}
\lim _{r \rightarrow+\infty} \frac{1}{2 r} \int_{-r}^{r}|\varphi(s)| d s=0, \quad \lim _{r \rightarrow+\infty} \frac{1}{2 r} \int_{-r}^{r}\left|\varphi^{*}(s)\right| d s=0 \tag{19}
\end{equation*}
$$

Hence $\varphi, \varphi^{*} \in \operatorname{PAP}_{0}(\mathbb{R})$. Let $f=\mathrm{g}+\varphi, f^{*}=g^{*}+\varphi^{*}$, then $f=g+\varphi \in \operatorname{PAP}(\mathbb{R}), f^{*}=g^{*}+\varphi^{*} \in \operatorname{PAP}(\mathbb{R})$ and $f_{p} \Rightarrow f$, $f_{p}^{\prime} \Rightarrow f^{*}$ as $p \rightarrow+\infty$.

Finally, we reveal $f^{\prime}=f^{*}$. For $t, \Delta t \in \mathbb{R}$, it follows that

$$
\begin{equation*}
f_{p}(t+\Delta t)-f_{p}(t)=\int_{t}^{t+\Delta t} f_{p}^{\prime}(s) d s \tag{20}
\end{equation*}
$$

In view of the uniform convergence of $f_{p}$ and $f_{p}^{\prime}$, let $p \rightarrow+\infty$ for (20), and we get

$$
\begin{equation*}
f(t+\Delta t)-f(t)=\int_{t}^{t+\Delta t} f^{*}(s) d s \tag{21}
\end{equation*}
$$

which implies that

$$
\begin{align*}
f^{*}(t) & =\lim _{\Delta t \rightarrow 0} \frac{\int_{t}^{t+\Delta t} f^{*}(s) d s}{\Delta t}  \tag{22}\\
& =\lim _{\Delta t \rightarrow 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}=f^{\prime}(t)
\end{align*}
$$

In summary, in view of (15), (16), and (22), we obtain that the Cauchy sequence $\left\{f_{p}\right\}_{p=1}^{+\infty} \subset B^{*}$ satisfies

$$
\begin{equation*}
\left\|f_{p}-f\right\|_{B^{*}} \longrightarrow 0(p \longrightarrow+\infty) \tag{23}
\end{equation*}
$$

and $f \in B^{*}$. This yields that $B^{*}$ is a Banach space. The proof is completed.

Remark 5. Let $B=\left\{f \mid f, f^{\prime} \in \operatorname{PAP}\left(\mathbb{R}, \mathbb{R}^{n \times m}\right)\right\}$ equipped with the induced norm defined by $\|f\|_{B}=$ $\max \left\{\|f\|_{\infty},\left\|f^{\prime}\right\|_{\infty}\right\}=\max \left\{\sup _{t \in \mathbb{R}}\|f(t)\|, \sup _{t \in \mathbb{R}}\left\|f^{\prime}(t)\right\|\right\}$. It follows from Lemma 4 that $B$ is a Banach space.

Definition 6 (see $[19,20]$ ). Let $x \in \mathbb{R}^{p}$ and $Q(t)$ be a $p \times p$ continuous matrix defined on $\mathbb{R}$. The linear system

$$
\begin{equation*}
x^{\prime}(t)=Q(t) x(t) \tag{24}
\end{equation*}
$$

is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist positive constants $k, \alpha$, and projection $P$ and the fundamental solution matrix $X(t)$ of (24) satisfying

$$
\begin{align*}
& \left\|X(t) P X^{-1}(s)\right\| \leq k e^{-\alpha(t-s)}, \quad \text { for } t \geq s \\
& \left\|X(t)(I-P) X^{-1}(s)\right\| \leq k e^{-\alpha(s-t)}, \quad \text { for } t \leq s \tag{25}
\end{align*}
$$

Lemma 7 (see [19]). Assume that $Q(t)$ is an almost periodic matrix function and $g(t) \in P A P\left(\mathbb{R}, \mathbb{R}^{p}\right)$. If the linear system (24) admits an exponential dichotomy, then pseudo almost periodic system

$$
\begin{equation*}
x^{\prime}(t)=Q(t) x(t)+g(t) \tag{26}
\end{equation*}
$$

has a unique pseudo almost periodic solution $x(t)$, and

$$
\begin{align*}
x(t)= & \int_{-\infty}^{t} X(t) P X^{-1}(s) g(s) d s \\
& -\int_{t}^{+\infty} X(t)(I-P) X^{-1}(s) g(s) d s \tag{27}
\end{align*}
$$

Lemma 8 (see [19, 20]). Let $c_{i}(t)$ be an almost periodic function on $\mathbb{R}$ and

$$
\begin{equation*}
M\left[c_{i}\right]=\lim _{T \rightarrow+\infty} \frac{1}{T} \int_{t}^{t+T} c_{i}(s) d s>0, \quad i=1,2, \ldots, p \tag{28}
\end{equation*}
$$

Then the linear system

$$
\begin{equation*}
x^{\prime}(t)=\operatorname{diag}\left(-c_{1}(t),-c_{2}(t), \ldots,-c_{p}(t)\right) x(t) \tag{29}
\end{equation*}
$$

admits an exponential dichotomy on $\mathbb{R}$.

## 3. Existence of Pseudo Almost Periodic Solutions

In this section, we establish sufficient conditions on the existence of pseudo almost periodic solutions of (3).

For $i j, k l \in J, a_{i j}: \mathbb{R} \rightarrow(0,+\infty)$ is an almost periodic function, $\eta_{i j}, \tau_{k l}: \mathbb{R} \rightarrow[0,+\infty)$, and $L_{i j}, C_{i j}^{k l}, B_{i j}^{k l}: \mathbb{R} \rightarrow$ $\mathbb{R}$ are pseudo almost periodic functions. We also make the following assumptions which will be used later.

We also make the following assumptions.
(S1) There exist constants $M_{f}, M_{g}, L_{f}$, and $L_{g}$ such that

$$
\begin{array}{rr}
|f(u)-f(v)| \leq L_{f}|u-v|, & |f(u)| \leq M_{f} \\
|g(u)-g(v)| \leq L_{g}|u-v|, & |g(u)| \leq M_{g} \\
\forall u, v \in \mathbb{R}
\end{array}
$$

(S2) For $i j \in J$, the delay kernels $K_{i j}:[0, \infty) \rightarrow \mathbb{R}$ are continuous, and $\left|K_{i j}(t)\right| e^{\beta t}$ are integrable on $[0, \infty)$ for a certain positive constant $\beta$.
(S3) Let

$$
\begin{align*}
& L=\max \left\{\max _{(i, j)}\left\{\sup _{t \in \mathbb{R}}\left|\int_{-\infty}^{t} e^{-\int_{s}^{t} a_{i j}(u) d u} L_{i j}(s) d s\right|\right\},\right. \\
& \max _{(i, j)}\left\{\sup _{t \in \mathbb{R}} \mid L_{i j}(t)-a_{i j}(t)\right. \\
&\left.\left.\quad \times \int_{-\infty}^{t} e^{-\int_{s}^{t} a_{i j}(u) d u} L_{i j}(s) d s \mid\right\}\right\}>0 . \tag{31}
\end{align*}
$$

Moreover, there exists a constant $\kappa$ such that

$$
\begin{align*}
& 0<\kappa \leq L, \quad \max _{(i, j)}\left\{\frac{1}{a_{i j}^{-}} E_{i j},\left(1+\frac{a_{i j}^{+}}{a_{i j}^{-}}\right) E_{i j}\right\} \leq \kappa,  \tag{32}\\
& \max _{(i, j)}\left\{\frac{1}{a_{i j}^{-}} F_{i j},\left(1+\frac{a_{i j}^{+}}{a_{i j}^{-}}\right) F_{i j}\right\}<1,
\end{align*}
$$

where

$$
\begin{align*}
E_{i j}= & {\left[a_{i j}^{+} \eta_{i j}^{+}+\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(L^{f}(\kappa+L)+|f(0)|\right)\right.} \\
& +\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}} \\
& \left.\times \int_{0}^{\infty}\left|K_{i j}(u)\right| d u\left(L^{g}(\kappa+L)+|g(0)|\right)\right](\kappa+L), \\
F_{i j}= & {\left[a_{i j}^{+} \eta_{i j}^{+}+\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(M_{f}+L^{f}(\kappa+L)\right)\right.} \\
& +\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}}\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M_{g} \in J,\right. \\
& \left.\left.+\int_{0}^{\infty}\left|K_{i j}(u)\right| d u L^{g}(\kappa+L)\right)\right],
\end{align*}
$$

Lemma 9. Assume that assumptions $\left(S_{1}\right)$ and $\left(S_{2}\right)$ hold. Then, for $\varphi(\cdot) \in \operatorname{PAP}(\mathbb{R}, \mathbb{R})$, the function $\int_{0}^{\infty} K_{i j}(u) g(\varphi(t-u)) d u$ belongs to $\operatorname{PAP}(\mathbb{R}, \mathbb{R})$, where $i j \in J$.

Proof. Let $\varphi \in \operatorname{PAP}(\mathbb{R}, \mathbb{R})$. Obviously, $\left(\mathrm{S}_{1}\right)$ implies that $g$ is a uniformly continuous function on $\mathbb{R}$. By using Corollary 5.4 in [25, page 58], we immediately obtain the following:

$$
\begin{equation*}
g(\varphi(t))=\chi_{1}(t)+\chi_{2}(t) \in \operatorname{PAP}(\mathbb{R}, \mathbb{R}) \tag{34}
\end{equation*}
$$

where $\chi_{1} \in \operatorname{AP}(\mathbb{R}, \mathbb{R})$ and $\chi_{2} \in \operatorname{PAP}_{0}(\mathbb{R}, \mathbb{R})$. Then, for any $\varepsilon>0$, it is possible to find a real number $l=l(\varepsilon)>0$; for any interval with length $l$, there exists a number $\tau=\tau(\varepsilon)$ in this interval such that

$$
\begin{gather*}
\left|\chi_{1}(t+\tau)-\chi_{1}(t)\right|<\frac{\varepsilon}{1+\int_{0}^{\infty}\left|K_{\mathrm{i} j}(u)\right| d u}, \quad \forall t \in \mathbb{R}, i j \in J \\
\lim _{r \rightarrow+\infty} \frac{1}{2 r} \int_{-r}^{r}\left|\chi_{2}(v)\right| d v=0 \tag{35}
\end{gather*}
$$

It follows that

$$
\begin{aligned}
& \left|\int_{0}^{\infty} \quad K_{i j}(u) \chi_{1}(t+\tau-u) d u-\int_{0}^{\infty} K_{i j}(u) \chi_{1}(t-u) d u\right| \\
& \quad \leq \int_{0}^{\infty}\left|K_{i j}(u)\right|\left|\chi_{1}(t+\tau-u)-\chi_{1}(t-u)\right| d u \\
& \quad<\int_{0}^{\infty}\left|K_{i j}(u)\right| d u \frac{\varepsilon}{1+\int_{0}^{\infty}\left|K_{i j}(u)\right| d u} \\
& \quad<\varepsilon, \quad \forall t \in \mathbb{R}, i j \in J
\end{aligned}
$$

$$
\lim _{r \rightarrow+\infty} \frac{1}{2 r} \int_{-r}^{r}\left|\int_{0}^{\infty} K_{i j}(u) \chi_{2}(v-u) d u\right| d v
$$

$$
\leq \lim _{r \rightarrow+\infty} \frac{1}{2 r} \int_{-r}^{r} \int_{0}^{\infty}\left|K_{i j}(u)\right|\left|\chi_{2}(v-u)\right| d u d v
$$

$$
=\lim _{r \rightarrow+\infty} \frac{1}{2 r} \int_{0}^{\infty}\left|K_{i j}(u)\right| \int_{-r}^{r}\left|\chi_{2}(v-u)\right| d v d u
$$

$$
=\lim _{r \rightarrow+\infty} \frac{1}{2 r} \int_{0}^{\infty}\left|K_{i j}(u)\right| \int_{-r-u}^{r-u}\left|\chi_{2}(z)\right| d z d u
$$

$$
\leq \lim _{r \rightarrow+\infty} \frac{1}{2 r} \int_{0}^{\infty}\left|K_{i j}(u)\right| \int_{-r-u}^{r+u}\left|\chi_{2}(z)\right| d z d u
$$

$$
\leq \lim _{r \rightarrow+\infty} \int_{0}^{\infty}\left|K_{i j}(u)\right|\left(1+\frac{1}{r} u\right) \frac{1}{2(r+u)}
$$

$$
\times \int_{-r-u}^{r+u}\left|\chi_{2}(z)\right| d z d u
$$

$$
\leq \lim _{r \rightarrow+\infty} \int_{0}^{\infty}\left|K_{i j}(u)\right| e^{(1 / r) u} \frac{1}{2(r+u)} \int_{-r-u}^{r+u}\left|\chi_{2}(z)\right| d z d u
$$

$$
\leq \lim _{r \rightarrow+\infty} \int_{0}^{\infty}\left|K_{i j}(u)\right| e^{\beta u} \frac{1}{2(r+u)} \int_{-r-u}^{r+u}\left|\chi_{2}(z)\right| d z d u
$$

$$
=0, \quad \text { where } r>\frac{1}{\beta}, i j \in J
$$

Thus,

$$
\begin{align*}
& \int_{0}^{\infty} K_{i j}(u) \chi_{1}(t-u) d u \in \operatorname{AP}(\mathbb{R}, \mathbb{R})  \tag{37}\\
& \int_{0}^{\infty} K_{i j}(u) \chi_{2}(t-u) d u \in \operatorname{PAP}_{0}(\mathbb{R}, \mathbb{R})
\end{align*}
$$

which yield

$$
\begin{align*}
\int_{0}^{\infty} & K_{i j}(u) g_{j}(\varphi(t-u)) d u \\
\quad & \int_{0}^{\infty} K_{i j}(u) \chi_{1}(t-u) d u \\
& \quad+\int_{0}^{\infty} K_{i j}(u) \chi_{2}(t-u) d u \in \operatorname{PAP}(\mathbb{R}, \mathbb{R}), \quad i j \in J \tag{38}
\end{align*}
$$

The proof of Lemma 9 is completed.
Theorem 10. Let $\left(S_{1}\right),\left(S_{2}\right)$, and $\left(S_{3}\right)$ hold. Then, there exists at least one continuously differentiable pseudo almost periodic solution of system (3).

Proof. Let $\varphi \in B$. Obviously, the boundedness of $\varphi^{\prime}$ and $\left(\mathrm{S}_{1}\right)$ imply that $f$ and $\varphi_{i j}$ are uniformly continuous functions on $\mathbb{R}$ for $i j \in J$. Set $\widetilde{f}(t, z)=\varphi_{i j}(t-z)(i j \in J)$. By Theorem 5.3 in [25, page 58] and Definition 5.7 in [25, page 59], we can obtain that $\tilde{f} \in \operatorname{PAP}(\mathbb{R} \times \Omega)$ and $\tilde{f}$ is continuous in $z \in K$ and uniformly in $t \in \mathbb{R}$ for all compact subset $K$ of $\Omega$. This, together with $\tau_{i j}, \eta_{i j} \in \operatorname{PAP}(\mathbb{R}, \mathbb{R})$ and Theorem 5.11 in [25, page 60], implies that

$$
\begin{array}{r}
\varphi_{i j}\left(t-\tau_{i j}(t)\right) \in \operatorname{PAP}(\mathbb{R}, \mathbb{R}) \\
\varphi_{i j}\left(t-\eta_{i j}(t)\right) \in \operatorname{PAP}(\mathbb{R}, \mathbb{R})  \tag{39}\\
i j \in J
\end{array}
$$

Again from Corollary 5.4 in [25, page 58], we have

$$
\begin{equation*}
f\left(\varphi_{i j}\left(t-\tau_{i j}(t)\right)\right) \in \operatorname{PAP}(\mathbb{R}, \mathbb{R}), \quad i j \in J \tag{40}
\end{equation*}
$$

which, together with Lemma 9, implies

$$
\begin{aligned}
& a_{i j}(t) \int_{t-\eta_{i j}(t)}^{t} \varphi_{i j}^{\prime}(s) d s \\
& \quad=a_{i j}(t) \varphi_{i j}(t)-a_{i j}(t) \varphi_{i j}\left(t-\eta_{i j}(t)\right) \in \operatorname{PAP}(\mathbb{R}, \mathbb{R}), \\
& \quad i j \in J, \\
& -\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(t) f\left(\varphi_{k l}\left(t-\tau_{k l}(t)\right)\right) \varphi_{i j}(t) \\
& -\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(t) \\
& \times \int_{0}^{\infty} K_{i j}(u) g\left(\varphi_{k l}(t-u)\right) d u \varphi_{i j}(t)+L_{i j}(t) \in \operatorname{PAP}(\mathbb{R}, \mathbb{R}), \\
& \quad i j \in J .
\end{aligned}
$$

For any $\varphi \in B$, we consider the pseudo almost periodic solution $x^{\varphi}(t)$ of nonlinear pseudo almost periodic differential equations

$$
\begin{align*}
& x_{i j}^{\prime}(t)=-a_{i j}(t) x_{i j}(t)+a_{i j}(t) \int_{t-\eta_{i j}(t)}^{t} \varphi_{i j}^{\prime}(s) d s \\
&-\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(t) f\left(\varphi_{k l}\left(t-\tau_{k l}(t)\right)\right) \varphi_{i j}(t) \\
&-\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(t) \\
& \times \int_{0}^{\infty} K_{i j}(u) g\left(\varphi_{k l}(t-u)\right) d u \varphi_{i j}(t)+L_{i j}(t), \\
& \quad i j \in J . \tag{42}
\end{align*}
$$

Then, notice that $M\left[a_{i j}\right]>0, i j \in J$, and it follows from Lemma 8 that the linear system,

$$
\begin{equation*}
x_{i j}^{\prime}(t)=-a_{i j}(t) x_{i j}(t), \quad i j \in J \tag{43}
\end{equation*}
$$

admits an exponential dichotomy on $\mathbb{R}$. Thus, by Lemma 7, we obtain that the system (42) has exactly one pseudo almost periodic solution:

$$
\begin{align*}
& x^{\varphi}(t)=\left\{x_{i j}^{\varphi}(t)\right\} \\
&=\left\{\int_{-\infty}^{t} e^{-\int_{s}^{t} a_{i j}(u) d u}\right. \\
& \times\left[a_{i j}(s) \int_{s-\eta_{i j}(s)}^{s} \varphi_{i j}^{\prime}(u) d u\right. \\
& \quad-\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(s) \\
& \times f\left(\varphi_{k l}\left(s-\tau_{k l}(s)\right)\right) \varphi_{i j}(s) \\
& \quad-\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(s) \\
& \times \int_{0}^{\infty} K_{i j}(u) g\left(\varphi_{k l}(s-u)\right) d u \varphi_{i j}(s)
\end{align*}
$$

From $\left(S_{1}\right),\left(S_{2}\right)$, and the Corollary 5.6 in [25, page 59], we get

$$
\begin{aligned}
\left(x^{\varphi}(t)\right)^{\prime}= & \left\{x_{i j}^{\varphi^{\prime}}(t)\right\} \\
= & \left\{\left[a_{i j}(t) \int_{t-\eta_{i j}(t)}^{t} \varphi_{i j}^{\prime}(s) d s\right.\right. \\
& -\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(t) f\left(\varphi_{k l}\left(t-\tau_{k l}(t)\right)\right) \varphi_{i j}(t)
\end{aligned}
$$

$$
\begin{align*}
& -\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(t) \\
& \left.\times \int_{0}^{\infty} K_{i j}(u) g\left(\varphi_{k l}(t-u)\right) d u \varphi_{i j}(t)+L_{i j}(t)\right] \\
& -a_{i j}(t) \\
& \times \int_{-\infty}^{t} e^{-\int_{s}^{t} a_{i j}(u) d u} \\
& \quad \times\left[a_{i j}(s) \int_{s-\eta_{i j}(s)}^{s} \varphi_{i j}^{\prime}(u) d u\right. \\
& \quad-\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(s) \\
& \quad \times f\left(\varphi_{k l}\left(s-\tau_{k l}(s)\right)\right) \varphi_{i j}(s) \\
& \quad-\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(s) \\
& \quad \times \int_{0}^{\infty} K_{i j}(u) g\left(\varphi_{k l}(s-u)\right) d u \varphi_{i j}(s) \\
& \left.\left.\quad+L_{i j}(s)\right] d s\right\} \tag{45}
\end{align*}
$$

which is a pseudo almost periodic function. Therefore, $x^{\varphi} \in$ B. Let $\varphi^{0}(t)=x^{0}(t)$. Then,

$$
\begin{gather*}
\varphi^{0}(t)=\left\{\varphi_{i j}^{0}(t)\right\}=\left\{\int_{-\infty}^{t} e^{-\int_{s}^{t} a_{i j}(u) d u} L_{i j}(s) d s\right\} \in B  \tag{46}\\
L=\left\|\varphi^{0}\right\|_{B^{\prime}}
\end{gather*}
$$

Set

$$
\begin{equation*}
B^{* *}=\left\{\varphi \mid \varphi \in B,\left\|\varphi-\varphi^{0}\right\|_{B} \leq \kappa\right\} . \tag{47}
\end{equation*}
$$

If $\varphi \in B^{* *}$, then

$$
\begin{equation*}
\|\varphi\|_{B} \leq\left\|\varphi-\varphi^{0}\right\|_{B}+\left\|\varphi^{0}\right\|_{B} \leq \kappa+L \tag{48}
\end{equation*}
$$

Now, we define a mapping $T: B^{* *} \rightarrow B^{* *}$ by setting

$$
\begin{equation*}
T(\varphi)(t)=x^{\varphi}(t), \quad \forall \varphi \in B^{* *} \tag{49}
\end{equation*}
$$

We next prove that the mapping $T$ is a contraction mapping of the $B^{* *}$.

First we show that, for any $\varphi \in B^{* *}, T(\varphi)=x^{\varphi} \in B^{* *}$.

Note that

$$
\begin{aligned}
& \left|T(\varphi)(t)-\varphi^{0}(t)\right| \\
& =\left\{\mid \int_{-\infty}^{t} e^{-\int_{s}^{t} a_{i j}(u) d u}\right. \\
& \quad \times\left[a_{i j}(s) \int_{s-\eta_{i j}(s)}^{s} \varphi_{i j}^{\prime}(u) d u\right. \\
& \quad-\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(s) f\left(\varphi_{k l}\left(s-\tau_{k l}(s)\right)\right) \varphi_{i j}(s) \\
& \quad-\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(s) \\
& \left.\left.\quad \times \int_{0}^{\infty} K_{i j}(u) g\left(\varphi_{k l}(s-u)\right) d u \varphi_{i j}(s)\right] d s \mid\right\}
\end{aligned}
$$

$$
\leq\left\{\int_{-\infty}^{t} e^{-\int_{s}^{t} a_{i j}^{-} d u}\right.
$$

$$
\times\left[a_{i j}^{+} \eta_{i j}^{+}\|\varphi\|_{B}\right.
$$

$$
+\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(\mid f\left(\varphi_{k l}\left(s-\tau_{k l}(s)\right)\right)\right.
$$

$$
-f(0)|+|f(0)|)\|\varphi\|_{B}
$$

$$
+\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}}
$$

$$
\times \int_{0}^{\infty}\left|K_{i j}(u)\right|
$$

$$
\times\left(\left|g\left(\varphi_{k l}(s-u)\right)-g(0)\right|+|g(0)|\right) d u
$$

$$
\left.\left.\times\|\varphi\|_{B}\right] d s\right\}
$$

$$
\leq\left\{\frac { 1 } { a _ { i j } ^ { - } } \left[a_{i j}^{+} \eta_{i j}^{+}+\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(L^{f}\|\varphi\|_{B}+|f(0)|\right)\right.\right.
$$

$$
\left.+\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}} \int_{0}^{\infty}\left|K_{i j}(u)\right| d u\left(L^{g}\|\varphi\|_{B}+|g(0)|\right)\right]
$$

$$
\left.\times\|\varphi\|_{B}\right\}
$$

$$
\leq\left\{\frac { 1 } { a _ { i j } ^ { - } } \left[a_{i j}^{+} \eta_{i j}^{+}+\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(L^{f}(\kappa+L)+|f(0)|\right)\right.\right.
$$

$$
+\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}}
$$

$$
\begin{aligned}
& \left.\left.\times \int_{0}^{\infty}\left|K_{i j}(u)\right| d u\left(L^{g}(\kappa+L)+|g(0)|\right)\right](\kappa+L)\right\}, \\
& \left|\left(T(\varphi)(t)-\varphi^{0}(t)\right)^{\prime}\right| \\
& =\left\{\|\left[a_{i j}(t) \int_{t-\eta_{i j}(t)}^{t} \varphi_{i j}^{\prime}(s) d s\right.\right. \\
& -\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(t) f\left(\varphi_{k l}\left(t-\tau_{k l}(t)\right)\right) \varphi_{i j}(t) \\
& -\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(t) \\
& \left.\times \int_{0}^{\infty} K_{i j}(u) g\left(\varphi_{k l}(t-u)\right) d u \varphi_{i j}(t)\right] \\
& -a_{i j}(t) \int_{-\infty}^{t} e^{-\int_{s}^{t} a_{i j}(u) d u} \\
& \times\left[a_{i j}(s) \int_{s-\eta_{i j}(s)}^{s} \varphi_{i j}^{\prime}(u) d u\right. \\
& -\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(s) f\left(\varphi_{k l}\left(s-\tau_{k l}(s)\right)\right) \varphi_{i j}(s) \\
& -\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(s) \\
& \left.\left.\times \int_{0}^{\infty} K_{i j}(u) g\left(\varphi_{k l}(s-u)\right) d u \varphi_{i j}(s)\right] d s \mid\right\} \\
& \leq\left\{\left(1+\frac{a_{i j}^{+}}{a_{i j}^{-}}\right)\right. \\
& \times\left[a_{i j}^{+} \eta_{i j}^{+}+\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(L^{f}\|\varphi\|_{B}+|f(0)|\right)\right. \\
& \begin{array}{l}
+\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}} \\
\left.\left.\times \int_{0}^{\infty}\left|K_{i j}(u)\right| d u\left(L^{g}\|\varphi\|_{B}+|g(0)|\right)\right]\|\varphi\|_{B}\right\}
\end{array} \\
& \leq\left\{\left(1+\frac{a_{i j}^{+}}{a_{i j}^{-}}\right)\right. \\
& \times\left[a_{i j}^{+} \eta_{i j}^{+}+\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(L^{f}(\kappa+L)+|f(0)|\right)\right. \\
& +\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}}
\end{aligned}
$$

$$
\begin{align*}
& \left.\times \int_{0}^{\infty}\left|K_{i j}(u)\right| d u\left(L^{g}(\kappa+L)+|g(0)|\right)\right] \\
& \times(\kappa+L)\} . \tag{50}
\end{align*}
$$

It follows that

$$
\begin{equation*}
\left\|T(\varphi)-\varphi^{0}\right\|_{B} \leq \max _{(i, j)}\left\{\frac{1}{a_{i j}^{-}} E_{i j},\left(1+\frac{a_{i j}^{+}}{a_{i j}^{-}}\right) E_{i j}\right\} \leq \kappa ; \tag{51}
\end{equation*}
$$

that is, $T(\varphi)=x^{\varphi} \in B^{* *}$.
Second, we show that $T$ is a contract operator.
In fact, in view of (44), (48), $\left(\mathrm{S}_{1}\right),\left(\mathrm{S}_{2}\right)$, and $\left(\mathrm{S}_{3}\right)$, for $\varphi, \psi \in$ $B^{* *}$, we have

$$
\begin{aligned}
& |T(\varphi(t))-T(\psi(t))| \\
& =\left\{\left|(T(\varphi(t))-T(\psi(t)))_{i j}\right|\right\} \\
& =\left\{\mid \int_{-\infty}^{t} e^{-\int_{s}^{t} a_{i j}(u) d u}\right. \\
& \quad \times\left[a_{i j}(s) \int_{s-\eta_{i j}(s)}^{s}\left(\varphi_{i j}^{\prime}(u)-\psi_{i j}^{\prime}(u)\right) d u\right. \\
& \quad-\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(s) \\
& \quad \times\left(f\left(\varphi_{k l}\left(s-\tau_{k l}(s)\right)\right) \varphi_{i j}(s)\right. \\
& \left.\quad-f\left(\psi_{k l}\left(s-\tau_{k l}(s)\right)\right) \psi_{i j}(s)\right) \\
& \quad-\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(s) \\
& \quad \times\left(\int_{0}^{\infty} K_{i j}(u) g\left(\varphi_{k l}(s-u)\right) d u \varphi_{i j}(s)\right. \\
& \quad-\int_{0}^{\infty} K_{i j}(u) g \\
& \left.\left.\left.\quad \times\left(\psi_{k l}(s-u)\right) d u \psi_{i j}(s)\right)\right] d s \mid\right\}
\end{aligned}
$$

$$
\times\left|\varphi_{i j}(s)-\psi_{i j}(s)\right|
$$

$$
\begin{aligned}
& +\mid f\left(\varphi_{k l}\left(s-\tau_{k l}(s)\right)\right) \\
& \left.-f\left(\psi_{k l}\left(s-\tau_{k l}(s)\right)\right)| | \psi_{i j}(s) \mid\right) \\
& +\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}}\left(\int_{0}^{\infty}\left|K_{i j}(u)\right|\right. \\
& \times\left|g\left(\varphi_{k l}(s-u)\right)\right| d u\left|\varphi_{i j}(s)-\psi_{i j}(s)\right| \\
& +\int_{0}^{\infty}\left|K_{i j}(u)\right| \\
& \times\left|g\left(\varphi_{k l}(s-u)\right)-g\left(\psi_{k l}(s-u)\right)\right| d u \\
& \left.\left.\left.\times\left|\psi_{i j}(s)\right|\right)\right] d s\right\} \\
& \leq\left\{\frac { 1 } { a _ { i j } ^ { - } } \left[a_{i j}^{+} \eta_{i j}^{+}\|\varphi-\psi\|_{B}\right.\right. \\
& +\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(M_{f}\|\varphi-\psi\|_{B}\right. \\
& \left.+L^{f}\|\varphi-\psi\|_{B}\|\psi\|_{B}\right) \\
& +\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}} \\
& \times\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M_{g}\|\varphi-\psi\|_{B}\right. \\
& \left.\left.\left.+\int_{0}^{\infty}\left|K_{i j}(u)\right| d u L^{g}\|\varphi-\psi\|_{B}\|\psi\|_{B}\right)\right]\right\} \\
& =\left\{\frac { 1 } { a _ { i j } ^ { - } } \left[a_{i j}^{+} \eta_{i j}^{+}\right.\right. \\
& +\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(M_{f}+L^{f}\|\psi\|_{B}\right) \\
& +\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}} \\
& \times\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M_{g}\right. \\
& \left.\left.\left.+\int_{0}^{\infty}\left|K_{i j}(u)\right| d u L^{g}\|\psi\|_{B}\right)\right]\|\varphi-\psi\|_{B}\right\} \\
& \leq\left\{\frac { 1 } { a _ { i j } ^ { - } } \left[a_{i j}^{+} \eta_{i j}^{+}\right.\right. \\
& +\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(M_{f}+L^{f}(\kappa+L)\right) \\
& +\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}}
\end{aligned}
$$

$$
\begin{aligned}
& \times\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M_{g}\right. \\
& \left.\left.+\int_{0}^{\infty}\left|K_{i j}(u)\right| d u L^{g}(\kappa+L)\right)\right] \\
& \left.\times\|\varphi-\psi\|_{B}\right\}, \\
& \left|(T(\varphi(t))-T(\psi(t)))^{\prime}\right| \\
& =\left\{\left|\left(T^{\prime}(\varphi(t))-T^{\prime}(\psi(t))\right)_{i j}\right|\right\} \\
& =\left\{\mid\left[a_{i j}(t) \int_{t-\eta_{i j}(t)}^{t}\left(\varphi_{i j}^{\prime}(s)-\psi_{i j}^{\prime}(s)\right) d s\right.\right. \\
& -\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(t) \\
& \times\left(f\left(\varphi_{k l}\left(t-\tau_{k l}(t)\right)\right) \varphi_{i j}(t)\right. \\
& \left.-f\left(\psi_{k l}\left(t-\tau_{k l}(t)\right)\right) \psi_{i j}(t)\right) \\
& -\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(t) \\
& \times\left(\int _ { 0 } ^ { \infty } K _ { i j } ( u ) g \left(\varphi_{k l}\right.\right. \\
& \times(t-u)) d u \varphi_{i j}(t) \\
& -\int_{0}^{\infty} K_{i j}(u) g\left(\psi_{k l}(t-u)\right) d u \\
& \left.\left.\times \psi_{i j}(t)\right)\right] \\
& -a_{i j}(t) \int_{-\infty}^{t} e^{-\int_{s}^{t} a_{i j}(u) d u} \\
& \times\left[a_{i j}(s) \int_{s-\eta_{i j}(s)}^{s}\left(\varphi_{i j}^{\prime}(u)-\psi_{i j}^{\prime}(u)\right) d u\right. \\
& -\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(s) \\
& \times\left(f\left(\varphi_{k l}\left(s-\tau_{k l}(s)\right)\right) \varphi_{i j}(s)\right. \\
& -f\left(\psi_{k l}\left(s-\tau_{k l}(s)\right)\right) \\
& \left.\times \psi_{i j}(s)\right) \\
& -\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(s) \\
& \times\left(\int_{0}^{\infty} K_{i j}(u) g\left(\varphi_{k l}(s-u)\right) d u \varphi_{i j}(s)\right. \\
& -\int_{0}^{\infty} K_{i j}(u) g
\end{aligned}
$$

$$
\left.\left.\left.\times\left(\psi_{k l}(s-u)\right) d u \psi_{i j}(s)\right)\right] d s \mid\right\}
$$

$$
\leq\left\{( 1 + \frac { a _ { i j } ^ { + } } { a _ { i j } ^ { - } } ) \left[a_{i j}^{+} \eta_{i j}^{+}\right.\right.
$$

$$
+\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(M_{f}+L^{f}(\kappa+L)\right)
$$

$$
+\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}}\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M_{g}\right.
$$

$$
+\int_{0}^{\infty}\left|K_{i j}(u)\right| d u
$$

$$
\left.\left.\times L^{g}(\kappa+L)\right)\right]
$$

$$
\begin{equation*}
\left.\times\|\varphi-\psi\|_{B}\right\} \tag{52}
\end{equation*}
$$

which yields

$$
\begin{equation*}
\|T(\varphi)-T(\psi)\|_{B} \leq \max _{(i, j)}\left\{\frac{1}{a_{i j}^{-}} F_{i j},\left(1+\frac{a_{i j}^{+}}{a_{i j}^{-}}\right) F_{i j}\right\}\|\varphi-\psi\|_{B}, \tag{53}
\end{equation*}
$$

which implies that the mapping $T: B^{* *} \rightarrow B^{* *}$ is a contraction mapping. Therefore, using Theorem 0.3 .1 of [27], we obtain that the mapping $T$ possesses a unique fixed point

$$
\begin{equation*}
x^{*}=\left\{x_{i j}^{*}(t)\right\} \in B^{* *}, \quad T x^{*}=x^{*} \tag{54}
\end{equation*}
$$

By (42) and (44), $x^{*}$ satisfies (42). So (3) has at least one continuously differentiable pseudo almost periodic solution $x^{*}$. The proof of Theorem 10 is now completed.

## 4. Exponential Stability of the Pseudo Almost Periodic Solution

In this section, we will discuss the exponential stability of the pseudo almost periodic solution of system (3).

Definition 11. Let $x^{*}(t)=\left\{x_{i j}^{*}(t)\right\}$ be the pseudo almost periodic solution of system (3). If there exist constants $\alpha>0$ and $M>1$ such that, for every solution $x(t)=\left\{x_{i j}(t)\right\}$ of system (3) with any initial value $\varphi(t)=\left\{\varphi_{i j}(t)\right\}$ satisfying (6),

$$
\begin{align*}
\| x(t) & -x^{*}(t) \|_{1} \\
& =\max _{(i, j)}\left\{\max \left\{\left|x_{i j}(t)-x_{i j}^{*}(t)\right|,\left|x_{i j}^{\prime}(t)-x_{i j}^{* \prime}(t)\right|\right\}\right\} \\
& \leq M\left\|\varphi-x^{*}\right\|_{0} e^{-\alpha t}, \quad \forall t>0, \tag{55}
\end{align*}
$$

where $\left\|\varphi-x^{*}\right\|_{0}=\max \left\{\sup _{t \leq 0} \max _{(i, j)}\left|\varphi_{i j}(t)-x_{i j}^{*}(t)\right|\right.$, $\left.\sup _{t \leq 0} \max _{(i, j)}\left|\varphi_{i j}^{\prime}(t)-x_{i j}^{* \prime}(t)\right|\right\}$. Then $x^{*}(t)$ is said to be globally exponentially stable.

Theorem 12. Suppose that all conditions in Theorem 10 are satisfied. Then system (3) has at least one pseudo almost periodic solution $x^{*}(t)$. Moreover, $x^{*}(t)$ is globally exponentially stable.

Proof. By Theorem 10, (3) has at least one continuously differentiable pseudo almost periodic solution $x^{*}(t)=\left\{x_{i j}^{*}(t)\right\}$ such that

$$
\begin{equation*}
\left\|x^{*}\right\|_{B} \leq \kappa+L \tag{56}
\end{equation*}
$$

Suppose that $x(t)=\left\{x_{i j}(t)\right\}$ is an arbitrary solution of (1) associated with initial value $\varphi(t)=\left\{\varphi_{i j}(t)\right\}$ satisfying (6). Let $y(t)=\left\{y_{i j}(t)\right\}=\left\{x_{i j}(t)-x_{i j}^{*}(t)\right\}$. Then

$$
\begin{aligned}
& y_{i j}^{\prime}(t)=-a_{i j}(t) y_{i j}\left(t-\eta_{i j}(t)\right) \\
&-\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(t) \\
& \times\left[f\left(x_{k l}\left(t-\tau_{k l}(t)\right)\right) x_{i j}(t)\right. \\
&\left.\quad-f\left(x_{k l}^{*}\left(t-\tau_{k l}(t)\right)\right) x_{i j}^{*}(t)\right] \\
& \quad-\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(t) \\
& \quad \times\left[\int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}(t-u)\right) d u x_{i j}(t)\right. \\
&\left.\quad-\int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}^{*}(t-u)\right) d u x_{i j}^{*}(t)\right]
\end{aligned}
$$

$$
\begin{aligned}
& =-a_{i j}(t) y_{i j}(t) \\
& \quad+a_{i j}(t) \int_{t-\eta_{i j}(t)}^{t} y_{i j}^{\prime}(u) d u \\
& -\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(t)\left[f\left(x_{k l}\left(t-\tau_{k l}(t)\right)\right) x_{i j}(t)\right. \\
& \\
& \quad-f\left(x_{k l}^{*}\left(t-\tau_{k l}(t)\right)\right) \\
& \\
& \left.\times x_{i j}^{*}(t)\right]
\end{aligned}
$$

$$
\begin{align*}
& -\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(t) \\
& \quad \times\left[\int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}(t-u)\right) d u x_{i j}(t)\right. \\
& \left.\quad-\int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}^{*}(t-u)\right) d u x_{i j}^{*}(t)\right] . \tag{57}
\end{align*}
$$

Define continuous functions $\Gamma_{i}(\omega)$ and $\Pi_{i}(\omega)$ by setting

$$
\begin{align*}
& \Gamma_{i j}(\omega)=-a_{i j}^{-}+\omega+a_{i j}^{+} \eta_{i j}^{+} e^{\omega \eta_{i j}^{+}} \\
&+\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(M^{f}+L^{f} e^{\omega \tau_{k l}^{+}}(\kappa+L)\right) \\
&+\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}} \\
& \times\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M^{g}\right. \\
&\left.+\int_{0}^{\infty}\left|K_{i j}(u)\right| L^{g} e^{\omega u} d u(\kappa+L)\right), \\
& \Pi_{i j}(\omega)=\left(1+\frac{a_{i j}^{+}}{a_{i j}^{-}-\omega}\right) \\
& \times {\left[a_{i j}^{+} \eta_{i j}^{+} e^{\omega \eta_{i j}^{+}}\right.} \\
&+\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(M^{f}+L^{f} e^{\omega \tau_{k l}^{+}}(\kappa+L)\right) \\
&+\sum_{C_{k l} \in N_{q}(i, j)}^{B_{i j}^{k l^{+}}\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M^{g}\right.} \\
&+\int_{0}^{\infty}\left|K_{i j}(u)\right| L^{g} e^{\omega u} d u \\
&\times(\kappa+L))] \tag{58}
\end{align*}
$$

where $t>0, \omega \in[0, \beta], i j \in J$. Then, from $\left(\mathrm{S}_{3}\right)$, we have

$$
\begin{aligned}
\Gamma_{i j}(0)= & -a_{i j}^{-}+a_{i j}^{+} \eta_{i j}^{+} \\
& +\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(M^{f}+L^{f}(\kappa+L)\right) \\
& +\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}} \\
& \times\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M^{g}\right. \\
& \left.+\int_{0}^{\infty}\left|K_{i j}(u)\right| L^{g} d u(\kappa+L)\right) \\
= & -a_{i j}^{-}\left(1-\frac{1}{a_{i j}^{-}} F_{i j}\right)<0, \quad i j \in J, \\
\Pi_{i j}(0)= & \left(1+\frac{a_{i j}^{+}}{a_{i j}^{-}}\right)
\end{aligned}
$$

$$
\begin{align*}
\times & \times\left[a_{i j}^{+} \eta_{i j}^{+}+\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(M^{f}+L^{f}(\kappa+L)\right)\right. \\
& +\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}} \\
& \times\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M^{g}\right. \\
& \left.\left.\quad+\int_{0}^{\infty}\left|K_{i j}(u)\right| L^{g} d u(\kappa+L)\right)\right] \\
= & \left(1+\frac{a_{i j}^{+}}{a_{i j}^{-}}\right) F_{i j}<1, \quad i j \in J, \tag{59}
\end{align*}
$$

which, together with the continuity of $\Gamma_{i j}(\omega)$ and $\Pi_{i j}(\omega)$, implies that we can choose a constant $\lambda \in(0, \min \{\beta$, $\left.\left.\min _{(i, j)} a_{i j}^{-}\right\}\right)$such that

$$
\begin{align*}
\Gamma_{i j}(\lambda)= & -a_{i j}^{-}+\lambda+a_{i j}^{+} \eta_{i j}^{+} e^{\lambda \eta_{i j}^{+}} \\
& +\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(M^{f}+L^{f} e^{\lambda \tau_{k l}^{+}}(\kappa+L)\right) \\
& +\sum_{C_{k k} \in N_{q}(i, j)} B_{i j}^{k l^{+}} \\
& \times\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M^{g}\right.  \tag{60}\\
& \left.+\int_{0}^{\infty}\left|K_{i j}(u)\right| L^{g} e^{\lambda u} d u(\kappa+L)\right)  \tag{66}\\
= & \left(a_{i j}^{-}-\lambda\right)\left(\frac{\beta_{i j}}{a_{i j}^{-}-\lambda}-1\right)<0, \\
\Pi_{i j}(\lambda)= & \left(1+\frac{a_{i j}^{+}}{a_{i j}^{-}-\lambda}\right)  \tag{67}\\
& \times\left[a_{i j}^{+} \eta_{i j}^{+} e^{\lambda n_{i j}^{+}+} \sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(M^{f}+L^{f} e^{\lambda \tau_{k l}^{+}}(\kappa+L)\right)\right. \\
& +\sum_{C_{k l} \in N_{q}(i, j)}^{B_{i j}^{k l+}}\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M^{g}\right. \\
= & +\int_{0}^{\infty}\left|K_{i j}(u)\right| L^{g} e^{\lambda u} d u \\
& \left.\times \frac{a_{i j}^{+}}{a_{i j}^{-}-\lambda}\right) \beta_{i j}<1, \\
& \times(\kappa+L))]
\end{align*}
$$

In the following, we will show that

$$
\|y(t)\|_{1}<M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda t}, \quad \forall t>0
$$

Otherwise, there must exist $i j \in J$ and $\theta>0$ such that

$$
\begin{gathered}
\|y(\theta)\|_{1}=\max \left\{\left|y_{i j}(\theta)\right|,\left|y_{i j}^{\prime}(\theta)\right|\right\}=M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda \theta}, \\
\|y(t)\|_{1}<M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda t}, \quad \forall t \in(-\infty, \theta)
\end{gathered}
$$

Note that

$$
\begin{aligned}
y_{i j}^{\prime}(s)+ & a_{i j}(s) y_{i j}(s) \\
= & a_{i j}(s) \int_{s-\eta_{i j}(s)}^{s} y_{i j}^{\prime}(u) d u \\
& -\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(s) \\
\times & {\left[f\left(x_{k l}\left(s-\tau_{k l}(s)\right)\right) x_{i j}(s)\right.} \\
& \left.-f\left(x_{k l}^{*}\left(s-\tau_{k l}(s)\right)\right) x_{i j}^{*}(s)\right] \\
& -\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(s) \\
& \times\left[\int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}(s-u)\right) d u x_{i j}(s)\right.
\end{aligned}
$$

$$
\begin{array}{r}
\left.-\int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}^{*}(s-u)\right) d u x_{i j}^{*}(s)\right] \\
s \in[0, t], t \in[0, \theta] . \tag{68}
\end{array}
$$

Multiplying both sides of (68) by $e^{\int_{0}^{s} a_{i j}(u) d u}$ and integrating on $[0, t]$, we get

$$
\begin{align*}
& y_{i j}(t)= y_{i j}(0) e^{-\int_{0}^{t} a_{i j}(u) d u} \\
&+\int_{0}^{t} e^{-\int_{s}^{t} a_{i j}(u) d u} \\
& \times\left[a_{i j}(s) \int_{s-\eta_{i j}(s)}^{s} y_{i j}^{\prime}(u) d u\right. \\
& \quad-\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(s) \\
& \times\left(f\left(x_{k l}\left(s-\tau_{k l}(s)\right)\right) x_{i j}(s)\right. \\
&\left.\quad-\quad-f\left(x_{k l}^{*}\left(s-\tau_{k l}(s)\right)\right) x_{i j}^{*}(s)\right) \\
& \quad-\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(s) \\
& \quad \times\left(\int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}(s-u)\right) d u x_{i j}(s)\right. \\
&\left.\left.\quad-\int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}^{*}(s-u)\right) d u x_{i j}^{*}(s)\right)\right] d s, \\
& \quad t \in[0, \theta] . \tag{69}
\end{align*}
$$

Thus, with the help of (67), we have

$$
\begin{aligned}
& \left|y_{i j}(\theta)\right| \\
& =\mid y_{i j}(0) e^{-\int_{0}^{\theta} a_{i j}(u) d u} \\
& +\int_{0}^{\theta} e^{-\int_{s}^{\theta} a_{i j}(u) d u} \\
& \times\left[a_{i j}(s) \int_{s-\eta_{i j}(s)}^{s} y_{i j}^{\prime}(u) d u\right. \\
& -\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(s)\left(f\left(x_{k l}\left(s-\tau_{k l}(s)\right)\right) x_{i j}(s)\right. \\
& \left.-f\left(x_{k l}^{*}\left(s-\tau_{k l}(s)\right)\right) x_{i j}^{*}(s)\right) \\
& -\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(s) \\
& \times\left(\int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}(s-u)\right) d u x_{i j}(s)\right. \\
& \left.\left.-\int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}^{*}(s-u)\right) d u x_{i j}^{*}(s)\right)\right] d s \mid \\
& \leq\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-a_{i j}^{-} \theta}+\int_{0}^{\theta} e^{-\int_{s}^{\theta} a_{i j}(u) d u} \\
& \times\left[a_{i j}^{+} \eta_{i j}^{+} M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda\left(s-\eta_{i j}(s)\right)}\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(\left|f\left(x_{k l}\left(s-\tau_{k l}(s)\right)\right)\right|\left|x_{i j}(s)-x_{i j}^{*}(s)\right|\right. \\
& +\mid f\left(x_{k l}\left(s-\tau_{k l}(s)\right)\right) \\
& -f\left(x_{k l}^{*}\left(s-\tau_{k l}(s)\right)\right) \mid \\
& \left.\times\left|x_{i j}^{*}(s)\right|\right)+\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}} \\
& \times\left(\int_{0}^{\infty}\left|K_{i j}(u)\right|\left|g\left(x_{k l}(s-u)\right)\right| d u\left|x_{i j}(s)-x_{i j}^{*}(s)\right|\right. \\
& +\int_{0}^{\infty}\left|K_{i j}(u)\right| \mid g\left(x_{k l}(s-u)\right) \\
& -g\left(x_{k l}^{*}(s-u)\right) \mid d u \\
& \left.\left.\times\left|x_{i j}^{*}(s)\right|\right)\right] d s \\
& \leq\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-a_{i j}^{-} \theta}+\int_{0}^{\theta} e^{-\int_{s}^{\theta} a_{i j}(u) d u} \\
& \times\left[a_{i j}^{+} \eta_{i j}^{+} M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda\left(s-\eta_{i j}(s)\right)}\right. \\
& +\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(M^{f}\left|y_{i j}(s)\right|\right. \\
& \left.+L^{f}\left|y_{k l}\left(s-\tau_{k l}(s)\right)\right|\left|x_{i j}^{*}(s)\right|\right) \\
& +\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}} \\
& \times\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M^{g}\left|y_{i j}(s)\right|\right. \\
& +\int_{0}^{\infty}\left|K_{i j}(u)\right| L^{g}\left|y_{k l}(s-u)\right| d u \\
& \left.\left.\times\left|x_{i j}^{*}(s)\right|\right)\right] d s \\
& \leq\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-a_{i j}^{-\theta}}+\int_{0}^{\theta} e^{-\int_{s}^{\theta} a_{i j}(u) d u} \\
& \times\left[a_{i j}^{+} \eta_{i j}^{+} M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda\left(s-\eta_{i j}(s)\right)}\right. \\
& +\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}} \\
& \times\left(M^{f} M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda s}\right. \\
& \left.+L^{f} M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda\left(s-\tau_{k l}(s)\right)}\left|x_{i j}^{*}(s)\right|\right) \\
& +\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}} \\
& \times\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M^{g} M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda s}\right. \\
& +\int_{0}^{\infty}\left|K_{i j}(u)\right| L^{g} M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) \\
& \left.\left.\times e^{-\lambda(s-u)} d u\left|x_{i j}^{*}(s)\right|\right)\right] d s \\
& \leq M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right)
\end{aligned}
$$

$$
\begin{aligned}
& \times\left\{\frac{e^{-a_{i j}^{-} \theta}}{M}+\int_{0}^{\theta} e^{-\int_{s}^{\theta} a_{i j}(u) d u} e^{-\lambda s}\right. \\
& \times\left[a_{i j}^{+} \eta_{i j}^{+} e^{\lambda \eta_{i j}^{+}}\right. \\
& +\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}\left(M^{f}+L^{f} e^{\lambda \tau_{k l}^{+}}(\kappa+L)\right) \\
& +\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}} \\
& \times\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M^{g}\right. \\
\leq & M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) \\
& \times\left\{\frac{e_{0}^{-a_{i j}^{-} \theta}}{M}+e^{-a_{i j}^{-} \theta} \int_{0}^{\theta} e^{\left(a_{i j}^{-}-\lambda\right) s} d s \beta_{i j}\right\} \\
\leq & \left.\left.\left.M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda \theta} e^{\lambda u} d u(\kappa+L)\right)\right] d s\right\} \\
& \times\left[\frac{e^{\left(\lambda-a_{i j}^{-}\right) \theta}}{M}+\frac{\beta_{i j}}{a_{i j}^{-}-\lambda}\left(1-e^{\left(\lambda-a_{i j}^{-}\right) \theta}\right)\right] \\
= & M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda \theta} \\
& \times\left[\left(\frac{1}{M}-\frac{\beta_{i j}}{a_{i j}^{-}-\lambda}\right) e^{\left(\lambda-a_{i j}^{-}\right) \theta}+\frac{\beta_{i j}}{a_{i j}^{-}-\lambda}\right]
\end{aligned}
$$

which, together with (64) and (67), implies that

From (60), (61) and (67)-(72) yield

$$
\begin{align*}
& \left|y_{i j}^{\prime}(\theta)\right| \\
& \quad \leq a_{i j}(\theta)\left|y_{i j}(\theta)\right| \\
& \quad+\mid a_{i j}(\theta) \int_{\theta-\eta_{i j}(\theta)}^{\theta} y_{i j}^{\prime}(u) d u \\
& \quad-\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l}(\theta) \\
& \quad \times\left[f\left(x_{k l}\left(\theta-\tau_{k l}(\theta)\right)\right) x_{i j}(\theta)\right. \\
& \left.\quad-f\left(x_{k l}^{*}\left(\theta-\tau_{k l}(\theta)\right)\right) x_{i j}^{*}(\theta)\right] \\
& \quad-\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l}(\theta) \tag{73}
\end{align*}
$$

$$
\begin{gather*}
\times\left[\int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}(\theta-u)\right) d u x_{i j}(\theta)\right. \\
\left.-\int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}^{*}(\theta-u)\right) d u x_{i j}^{*}(\theta)\right] \mid \\
\leq a_{i j}^{+}\left|y_{i j}(\theta)\right| \\
+\left[a_{i j}^{+} \eta_{i j}^{+} M\left(\left\|\varphi-x^{*}\right\|_{\xi}+\varepsilon\right) e^{-\lambda\left(\theta-\eta_{i j}(\theta)\right)}\right. \\
+\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}} \\
\times\left(\left|f\left(x_{k l}\left(\theta-\tau_{k l}(\theta)\right)\right)\right|\left|x_{i j}(\theta)-x_{i j}^{*}(\theta)\right|\right. \\
\quad+\left|f\left(x_{k l}\left(\theta-\tau_{k l}(\theta)\right)\right)-f\left(x_{k l}^{*}\left(\theta-\tau_{k l}(\theta)\right)\right)\right| \\
\left.\quad \times\left|x_{i j}^{*}(\theta)\right|\right) \\
+\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}}\left(\int_{0}^{\infty}\left|K_{i j}(u)\right|\left|g\left(x_{k l}(\theta-u)\right)\right| d u\right. \\
\quad \times\left|x_{i j}(\theta)-x_{i j}^{*}(\theta)\right| \\
\quad+\int_{0}^{\infty}\left|K_{i j}(u)\right| \\
\times \mid g\left(x_{k l}(\theta-u)\right) \\
\quad-g\left(x_{k l}^{*}(\theta-u)\right) \mid d u \\
\left.\times\left|x_{i j}^{*}(\theta)\right|\right) \tag{70}
\end{gather*}
$$

$$
\begin{align*}
\leq\left\{a_{i j}^{+}\right. & {\left[\left(\frac{1}{M}-\frac{\beta_{i j}}{a_{i j}^{-}-\lambda}\right) e^{\left(\lambda-a_{i j}^{-}\right) \theta}+\frac{\beta_{i j}}{a_{i j}^{-}-\lambda}\right] }  \tag{71}\\
& +a_{i j}^{+} \eta_{i j}^{+} e^{\lambda \eta_{i j}^{+}}+\sum_{C_{k l} \in N_{r}(i, j)} C_{i j}^{k l^{+}}  \tag{72}\\
& \times\left(M^{f}+L^{f} e^{\lambda \tau_{k l}^{+}}(\kappa+L)\right)+\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l^{+}}  \tag{/2}\\
& \times\left(\int_{0}^{\infty}\left|K_{i j}(u)\right| d u M^{g}\right.
\end{align*}
$$

$$
\left.\left.+\int_{0}^{\infty}\left|K_{i j}(u)\right| L^{g} e^{\lambda u} d u(\kappa+L)\right)\right\} M
$$

$$
\times\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda \theta}
$$

$$
\leq M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda \theta}
$$

$$
\times\left[a_{i j}^{+}\left(\frac{1}{M}-\frac{\beta_{i j}}{a_{i j}^{-}-\lambda}\right) e^{\left(\lambda-a_{i j}^{-}\right) \theta}+\beta_{i j}\left(\frac{a_{i j}^{+}}{a_{i j}^{-}-\lambda}+1\right)\right]
$$

$$
<M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda \theta}
$$

$$
\begin{aligned}
& \left|y_{i j}(\theta)\right|<M\left(\left\|\varphi-x^{*}\right\|_{0}+\varepsilon\right) e^{-\lambda \theta}, \\
& \|y(\theta)\|_{1}=\max \left\{\left|y_{i j}(\theta)\right|,\left|y_{i j}^{\prime}(\theta)\right|\right\} \\
& =\left|y_{i j}^{\prime}(\theta)\right|=M\left(\left\|\varphi-x^{*}\right\|_{\xi}+\varepsilon\right) e^{-\lambda \theta} .
\end{aligned}
$$

which contradicts (72). Hence, (66) holds. Letting $\varepsilon \rightarrow 0^{+}$, we have from (66) that

$$
\begin{equation*}
\|y(t)\|_{1} \leq M\left\|\varphi-x^{*}\right\|_{0} e^{-\lambda t}, \quad \forall t>0 \tag{74}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\left\|x(t)-x^{*}(t)\right\|_{1} \leq M\left\|\varphi-x^{*}\right\|_{0} e^{-\lambda t}, \quad \forall t>0 . \tag{75}
\end{equation*}
$$

This completes the proof.

Set

$$
\begin{gather*}
\kappa=0.7, \quad r=q=1, \quad K_{i j}(u)=|\sin u| e^{-u}, \\
i=1,2,3, j=1,2,3,  \tag{78}\\
f(x)=g(x)=\frac{1}{50}(|x-1|-|x+1|),
\end{gather*}
$$

$$
\sum_{C_{k l} \in N_{1}(1,3)} C_{13}^{k l}=\sum_{C_{k l} \in N_{1}(1,3)} B_{13}^{k l}=0.5
$$

$$
\sum_{C_{k k} \in N_{1}(2,1)} C_{21}^{k l}=\sum_{C_{k k} \in N_{1}(2,1)} B_{21}^{k l}=0.8
$$

clearly,

$$
\sum_{C_{k k} \in N_{1}(2,2)} C_{22}^{k l}=\sum_{C_{k k} \in N_{1}(2,2)} B_{22}^{k l}=1.2
$$

$$
\begin{gathered}
M_{f}=M_{g}=0.04, \quad L_{f}=L_{g}=0.04, \\
\sum_{C_{k l} \in N_{1}(1,1)} C_{11}^{k l}=\sum_{C_{k l} \in N_{1}(1,1)} B_{11}^{k l}=0.5 \\
\sum_{C_{k l} \in N_{1}(1,2)} C_{12}^{k l}=\sum_{C_{k l} \in N_{1}(1,2)} B_{12}^{k l}=0.8
\end{gathered}
$$

$$
\sum_{C_{k k} \in N_{1}(2,3)} C_{23}^{k l}=\sum_{C_{k k} \in N_{1}(2,3)} B_{23}^{k l}=0.8
$$

$$
\sum_{C_{k l} \in N_{1}(3,1)} C_{31}^{k l}=\sum_{C_{k l} \in N_{1}(3,1)} B_{31}^{k l}=0.5
$$

$$
\begin{align*}
& \frac{d x_{i j}}{d t}=-a_{i j}(t) x_{i j}\left(t-\eta_{i j}(t)\right)-\sum_{c_{k l} \in N_{r}(i, j)} C_{i j}^{k l} f\left(x_{k l}\left(t-\sin ^{2} t\right)\right) x_{i j}(t) \\
& -\sum_{C_{k l} \in N_{q}(i, j)} B_{i j}^{k l} \int_{0}^{\infty} K_{i j}(u) g\left(x_{k l}(t-u)\right) d u x_{i j}+L_{i j}(t), \quad i, j=1,2,3,  \tag{76}\\
& {\left[\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right]=\left[\begin{array}{lll}
1 & 1 & 3 \\
3 & 1 & 3 \\
3 & 1 & 3
\end{array}\right]} \\
& {\left[\begin{array}{lll}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{array}\right]=\left[\begin{array}{lll}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{array}\right]=\left[\begin{array}{ccc}
0.1 & 0.2 & 0.1 \\
0.2 & 0 & 0.2 \\
0.1 & 0.2 & 0.1
\end{array}\right],} \\
& {\left[\begin{array}{lll}
\eta_{11} & \eta_{12} & \eta_{13} \\
\eta_{21} & \eta_{22} & \eta_{23} \\
\eta_{31} & \eta_{32} & \eta_{33}
\end{array}\right]=0.01\left[\begin{array}{ccc}
\sin ^{2} \sqrt{3} t+\frac{0.1}{1+t^{2}} & \cos ^{2} \sqrt{3} t+\frac{0.1}{1+t^{2}} & \sin ^{2} 2 t+\frac{0.1}{1+t^{2}} \\
\cos ^{2} \sqrt{5} t+\frac{0.1}{1+t^{2}} & \sin ^{2} \sqrt{5} t+\frac{0.1}{1+t^{2}} & \cos ^{2} 2 t+\frac{0.1}{1+t^{2}} \\
\sin ^{2} 2 t+\frac{0.1}{1+t^{2}} & \cos ^{2} 3 t+\frac{0.1}{1+t^{2}} & \sin ^{2} \sqrt{2} t+\frac{0.1}{1+t^{2}}
\end{array}\right]}  \tag{77}\\
& {\left[\begin{array}{lll}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{array}\right]=\left[\begin{array}{ccc}
0.7+0.24 \sin ^{2} \sqrt{2} t-\frac{1}{1+t^{2}} & 0.41+0.5 \cos ^{2} t & 1 \\
0.61+0.2 \cos ^{2} t-\frac{1}{1+t^{2}} & 0.67+0.2 \sin ^{2} t & 1 \\
0.59+0.4 \cos ^{4} t-\frac{1}{1+t^{2}} & 0.5+0.41 \sin ^{2} t & 1
\end{array}\right] .}
\end{align*}
$$

$$
\begin{align*}
\sum_{C_{k l} \in N_{1}(3,2)} C_{32}^{k l} & =\sum_{C_{k l} \in N_{1}(3,2)} B_{32}^{k l}=0.8, \\
\sum_{C_{k l} \in N_{1}(3,3)} C_{33}^{k l} & =\sum_{C_{k l} \in N_{1}(3,3)} B_{33}^{k l}=0.5, \tag{79}
\end{align*}
$$

where $i j \in J=\{11,12,13,21,22,23,31,32,33\}$. Then,

$$
\begin{align*}
& L=\max \left\{\max _{(i, j)}\left\{\sup _{t \in \mathbb{R}}\left|\int_{-\infty}^{t} e^{-\int_{s}^{t} a_{i j}(u) d u} L_{i j}(s) d s\right|\right\}\right. \\
& \max _{(i, j)}\left\{\sup _{t \in \mathbb{R}} \mid L_{i j}(t)-a_{i j}(t)\right. \\
& \left.\left.\times \int_{-\infty}^{t} e^{-\int_{s}^{t} a_{i j}(u) d u} L_{i j}(s) d s \mid\right\}\right\} \\
& =1>0 \\
& \begin{aligned}
0.7 & =\kappa \leq L=1, \\
\max _{(i, j)} & \left\{\frac{1}{a_{i j}^{-}} E_{i j},\left(1+\frac{a_{i j}^{+}}{a_{i j}^{-}}\right) E_{i j}\right\}=0.6603 \leq \kappa
\end{aligned} \\
& \quad \max _{(i, j)}\left\{\frac{1}{a_{i j}^{-}} F_{i j},\left(1+\frac{a_{i j}^{+}}{a_{i j}^{-}}\right) F_{i j}\right\}=0.5804<1
\end{align*}
$$

It follows that system (56) satisfies all the conditions in Theorems 10 and 12. Hence, system (76) has exactly one pseudo almost periodic solution. Moreover, the pseudo almost periodic solution is globally exponentially stable. The fact is verified by the numerical simulation in Figures 1, 2, and 3 and there are three different initial values which are $\varphi_{11} \equiv 1$, $\varphi_{12} \equiv-3, \varphi_{13} \equiv 4, \varphi_{21} \equiv 2, \varphi_{22} \equiv 5, \varphi_{23} \equiv 3, \varphi_{33} \equiv-1$, $\varphi_{32} \equiv-2, \varphi_{33} \equiv-5 ; \varphi_{11} \equiv 2, \varphi_{12} \equiv-1, \varphi_{13} \equiv 5, \varphi_{21} \equiv 4$, $\varphi_{22} \equiv 2, \varphi_{23} \equiv 1, \varphi_{33} \equiv-3, \varphi_{32} \equiv-4, \varphi_{33} \equiv 3$ and $\varphi_{11} \equiv-2$, $\varphi_{12} \equiv 1, \varphi_{13} \equiv-5, \varphi_{21} \equiv-4, \varphi_{22} \equiv-2, \varphi_{23} \equiv-1, \varphi_{33} \equiv 3$, $\varphi_{32} \equiv 4, \varphi_{33} \equiv-3$, respectively.

Remark 14. By using the inequality analysis technique, in [19, 20], the authors obtained the existence of almost periodic solution of SICNNs with leakage delays, but they did not give the existence and global exponential convergence for the pseudo almost periodic solution. Since [1-9] only dealt with SICNNs without leakage delays, [14-18, 21-24] give no opinions about the problem of pseudo almost periodic solutions for SICNNs with leakage delays. One can observe that all the results in these literatures and the references therein cannot be applicable to prove the existence and exponential stability of pseudo almost periodic solutions for SICNNs (56).

## Conflict of Interests

The authors declare no conflict of interests. They also declare that they have no financial and personal relationships with


Figure 1: Numerical solutions of system (76) for different initial values.


Figure 2: Numerical solutions of system (76) for different initial values.
other people or organizations that can inappropriately influence their work; there is no professional or other personal interest of any nature or kind in any product, service, and/or company that could be construed as influencing the position presented in, or the review of, this present paper.

## Acknowledgments

The authors would like to express the sincere appreciation to the reviewers for their helpful comments in improving the presentation and quality of the paper. This work was


Figure 3: Numerical solutions of system (76) for different initial values.
supported by the National Natural Science Foundation of China (Grant no. 11201184), the Natural Scientific Research Fund of Zhejiang Provincial of P. R. China (Grant no. LY12A01018), and the Natural Scientific Research Fund of Zhejiang Provincial Education Department of P. R. China (Grant no. Z201122436).

## References

[1] A. Bouzerdoum and R. B. Pinter, "Shunting inhibitory cellular neural networks: derivation and stability analysis," IEEE Transactions on Circuits and Systems, vol. 40, no. 3, pp. 215-221, 1993.
[2] A. Chen and J. Cao, "Almost periodic solution of shunting inhibitory CNNs with delays," Physics Letters A, vol. 298, no. 2-3, pp. 161-170, 2002.
[3] F. Chérif, "Existence and global exponential stability of pseudo almost periodic solution for SICNNs with mixed delays," Journal of Applied Mathematics and Computing, vol. 39, no. 12, pp. 235-251, 2012.
[4] M. Cai, H. Zhang, and Z. Yuan, "Positive almost periodic solutions for shunting inhibitory cellular neural networks with time-varying delays," Mathematics and Computers in Simulation, vol. 78, no. 4, pp. 548-558, 2008.
[5] J. Shao, "Anti-periodic solutions for shunting inhibitory cellular neural networks with time-varying delays," Physics Letters A, vol. 372, no. 30, pp. 5011-5016, 2008.
[6] Q. Fan and J. Shao, "Positive almost periodic solutions for shunting inhibitory cellular neural networks with time-varying and continuously distributed delays," Communications in Nonlinear Science and Numerical Simulation, vol. 15, no. 6, pp. 1655-1663, 2010.
[7] C. Zhao, Q. Fan, and W. Wang, "Anti-periodic solutions for shunting inhibitory cellular neural networks with time-varying coefficients," Neural Processing Letters, no. 31, pp. 259-267, 2010.
[8] A. Bouzerdoum, B. Nabet, and R. B. Pinter, "Analysis and analog implementation of directionally sensitive shunting inhibitory
neural networks," in Visual Information Processing: From Neurons to Chips, vol. 1473 of Proceedings of the SPIE, pp. 29-38, April 1991.
[9] A. Bouzerdoum and R. B. Pinter, "Nonlinear lateral inhibition applied to motion detection in the fly visual system," in Nonlinear Vision, R. B. Pinter and B. Nabet, Eds., pp. 423-450, CRC Press, Boca Raton, Fla, USA, 1992.
[10] B. Kosko, Neural Networks and Fuzzy Systems, Prentice Hall, New Delhi, India, 1992.
[11] S. Haykin, Neural Networks, Prentice Hall, New Jersey, NJ, USA, 1999.
[12] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, vol. 74 of Mathematics and its Applications, Kluwer Academic, Dordrecht, The Netherlands, 1992.
[13] K. Gopalsamy, "Leakage delays in BAM," Journal of Mathematical Analysis and Applications, vol. 325, no. 2, pp. 1117-1132, 2007.
[14] H. Zhang and J. Shao, "Almost periodic solutions for cellular neural networks with time-varying delays in leakage terms," Applied Mathematics and Computation, vol. 219, no. 24, pp. 11471-11482, 2013.
[15] X. Li, R. Rakkiyappan, and P. Balasubramaniam, "Existence and global stability analysis of equilibrium of fuzzy cellular neural networks with time delay in the leakage term under impulsive perturbations," Journal of the Franklin Institute, vol. 348, no. 2, pp. 135-155, 2011.
[16] P. Balasubramaniam, V. Vembarasan, and R. Rakkiyappan, "Leakage delays in T-S fuzzy cellular neural networks," Neural Processing Letters, vol. 33, no. 2, pp. 111-136, 2011.
[17] B. Liu, "Global exponential stability for BAM neural networks with time-varying delays in the leakage terms," Nonlinear Analysis: Real World Applications, vol. 14, no. 1, pp. 559-566, 2013.
[18] Z. Chen, "A shunting inhibitory cellular neural network with leakage delays and continuously distributed delays of neutral type," Neural Computing and Applications, vol. 23, no. 7, pp. 2429-2434, 2013.
[19] H. Zhang and M. Yang, "Global exponential stability of almost periodic solutions for SICNNs with continuously distributed leakage delays," Abstract and Applied Analysis, vol. 2013, Article ID 307981, 14 pages, 2013.
[20] B. Liu and J. Shao, "Almost periodic solutions for SICNNs with time-varying delays in the leakage terms," Journal of Inequalities and Applications, vol. 2013, article 494, 2013.
[21] B. Liu, "Global exponential stability of positive periodic solutions for a delayed Nicholson's blowflies model," Journal of Mathematical Analysis and Applications, vol. 412, no. 1, pp. 212221, 2014.
[22] J. Meng, "Global exponential stability of positive pseudo almost periodic solutions for a model of hematopoiesis," Abstract and Applied Analysis, vol. 2013, Article ID 463076, 7 pages, 2013.
[23] C. Ou, "Almost periodic solutions for shunting inhibitory cellular neural networks," Nonlinear Analysis: Real World Applications, vol. 10, no. 5, pp. 2652-2658, 2009.
[24] L. Li, Z. Fang, and Y. Yang, "A shunting inhibitory cellular neural network with continuously distributed delays of neutral type," Nonlinear Analysis: Real World Applications, vol. 13, no. 3, pp. 1186-1196, 2012.
[25] C. Zhang, Almost Periodic Type Functions and Ergodicity, Science Press, Beijing, China, 2003.
[26] A. M. Fink, Almost Periodic Differential Equations, vol. 377 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 1974.
[27] J. K. Hale, Ordinary Differential Equations, Robert E. Krieger, Huntington, NY, USA, 2nd edition, 1980.

## Research Article

# The Dynamics of a Nonautonomous Predator-Prey Model with Infertility Control in the Prey 

Xiaomei Feng, ${ }^{1,2}$ Zhidong Teng, ${ }^{1}$ and Fengqin Zhang ${ }^{2}$<br>${ }^{1}$ College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, China<br>${ }^{2}$ Department of Mathematics, Yuncheng University, Yuncheng 044000, China

Correspondence should be addressed to Zhidong Teng; zhidong@xju.edu.cn
Received 16 December 2013; Accepted 20 January 2014; Published 10 March 2014
Academic Editor: Weiming Wang
Copyright © 2014 Xiaomei Feng et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

A nonautonomous predator-prey model with infertility control in the prey is formulated and investigated. Threshold conditions for the permanence and extinction of fertility prey and infertility prey are established. Some new threshold values of integral form are obtained. For the periodic cases, these threshold conditions act as sharp threshold values for the permanence and extinction of fertility prey and infertility prey. There are also mounting concerns that the quantity of biological sterile drug is obtained in the process of the prevention and control of pest in the grasslands and farmland. Finally, two examples are given to illustrate the main results of this paper. The numerical simulations shown that, when the pest population is permanet, different dynamic behaviors may be found in this model, such as the global attractivity and the chaotic attractor.


## 1. Introduction

Small mammals living in the grasslands, such as the plateau pika, not only burrow, but also accumulate the soil outside the hole, which makes the grass cease growing. More seriously, after a rainstorm, the soil would be washed away which increased soil erosion. So a greater range of damages resulted. And the lack of protective vegetation exacerbated the desertification and degradation of pastures. Besides, pirates of pikas also eat grass, which reduced the carrying capacity. When the number of these small mammals increased sharply, it would cause a lot of trouble and loss to economy, ecology, and people's lives on the grassland. So at this moment, they are referred to as harmful animals.

As the change of the natural environment by the human production activities, agricultural, and the rapid development of cities provide plenty of food resources and good habitat for rodent, rat increases seriously, the management of pest also will be more difficult. Mouse control strategy from the traditional damage caves and machinery catch to fumigation, acute rodenticide, anticoagulant therapy, and the application of many chemical methods has made important progress. At present, the chemical prevention and control play
an important role in the mouse control technology. However, chemical control is effective for short and harmful rat will soon come again and reproduction rapidly leads to the quick rebound in this species. In the fields, the application of acute rodenticide reaches 80 ; the population in the two years can be restored to the original level. In addition, there still exist many problems such as environmental pollution, secondary poisoning, and fungicide resistance in chemical control, which makes chemical Rodenticide restricted to the application of the rodent sustainable control. And integrating multidisciplinary approach and means, the sterility control technology based on ecological security has gradually become the development direction of rodent control. Infertility control technology has both directly and indirectly reduced the rodent population density, and will not lead to sharp fluctuations in ecological system, so it has a very good advantage in the environmental safety and cost-effectiveness.

Now, there are very serious rat in many areas of China, such as Xinjiang, Inner Mongolia, Gansu, Shanxi, and The Tibetan Plateau. In the Inner Mongolia grasslands, it is predicted that pest harm area is about 100 million mu and the serious disaster area is about 50 million mu [1]. Prairie mousehole per hectare is 300 at least and even 900 at most.

In 2006, eleven silver foxes were introduced for the first time at Alxa League in Inner Mongolia grassland in order to control the prairie mice. Those foxes can catch large amounts of prairie gerbil, Meriones unguiculatus, and jerboa [2]. In addition, in 2011, Beowulf Biological antisterility rodenticide was used. The purpose is to test the effect of preventing grassland rat and whether or not achieves these requirements such as restraining the birth rates of harmful rat population, reducing the pest population density, slowing population growth benefiting Environmental Health and Safety [3].

At present, the research about infertility control is at most laboratory studies [4-9] and theoretical analysis even less. Based on the above understanding of the facts and mathematical biology background, the study about a class of predatorprey model with infertility control in the prey (harmful rat) is very meaningful. Moreover, the result indicated that species and quantity are different by vegetation and physiognomy, and change of density is more distinct along with changing season. Therefore, it is a very basilic problem to research this kind of nonautonomous population dynamic systems.

It is interesting to note that rodents living in the North generally have seasonal breeding, such as plateau pika nearby Qinghai lake breed from April to August, Brandt's voles breed from March to September, and Mongolian gerbil in Inner Mongolia breed from April to August. Obviously, this kind of periodic phenomenon, extensively exists in the real world. Therefore, the dynamical behavior of the $\omega$-periodic system is also worthy of being discussed.

Now, we only consider infertility control in the prey (harmful rat) $X(t)$ population. It is composed of two population classes: one is the class of fertility prey, denoted by $F(t)$, and the other is the class of infertility prey, denoted by $S(t)$. Therefore, at any time $t$, the total density of prey population is $X(t)=F(t)+S(t)$. Fertility rodents will become infertile after eating the sterilant. Therefore, $\mu(t)$ is assumed to the rate at which infertility prey contacts occur. In this paper, we study the following nonautonomous predator-prey model with infertility control in the prey:

$$
\begin{gather*}
\frac{d F(t)}{d t}=F(t)\left[b_{1}(t)-a_{11}(t)(F(t)+S(t))\right. \\
\left.-\mu(t)-a_{12}(t) Z(t)\right] \\
\frac{d S(t)}{d t}=\mu(t) F(t)-d_{2}(t) S(t)-a_{11}(t) \\
\times[F(t)+S(t)] S(t)-a_{12}(t) S(t) Z(t) \\
\frac{d Z(t)}{d t}=Z(t)\left[b_{2}(t)+a_{21}(t)(F(t)+S(t))-a_{22}(t) Z(t)\right] \tag{1}
\end{gather*}
$$

with initial conditions

$$
\begin{equation*}
F(0)=F_{0}, \quad S(0)=S_{0}, \quad Z(0)=Z_{0} \tag{2}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(F_{0}, S_{0}, Z_{0}\right) \in R_{+}^{3}=\left\{(F, S, Z) \in R^{3}: F>0, S>0, Z>0\right\} \tag{3}
\end{equation*}
$$

Here, $F(t)$ is the fertility prey population density, $S(t)$ is the sterility prey population density, $Z(t)$ is the predator population density, $b_{1}(t), a_{11}(t)$ are the intrinsic growth rate and density-dependent coefficient of the prey, respectively, $b_{2}(t)$, $a_{22}(t)$ are the intrinsic growth rate and density-dependent coefficient of the predator, respectively, $a_{12}(t)$ is the capturing rate of the predator, and $a_{21}(t)$ is the rate of conversion of nutrients into the reproduction of the predator.

## 2. Preliminaries

For a continuous bounded function $f(t)$ defined on $R_{+}=$ $[0, \infty)$, we denote

$$
\begin{equation*}
f^{m}=\limsup _{t \rightarrow \infty} f(t), \quad f^{l}=\liminf _{t \rightarrow \infty} f(t) \tag{4}
\end{equation*}
$$

If $f$ is $\omega$-periodic, then the average value of on a time interval $[0, \omega]$ can be defined as

$$
\begin{equation*}
\bar{f}=\frac{1}{\omega} \int_{0}^{\omega} f(t) d t \tag{5}
\end{equation*}
$$

For system (1), we introduce the following assumptions.
$\left(\mathrm{H}_{1}\right)$ Functions $a_{11}(t), \mu(t), a_{12}(t), d_{2}(t), a_{21}(t)$, and $a_{22}(t)$ are all negative, continuous, and bounded on $R_{+}$and $b_{i}(t)(i=1,2)$ are continuous and bounded functions.
$\left(\mathrm{H}_{2}\right)$ There exist positive constants $\omega_{j}>0(j=1,2,3$, $4,5,6)$ such that

$$
\begin{array}{ll}
\liminf _{t \rightarrow \infty} \int_{t}^{t+\omega_{i}} b_{i}(\theta) d \theta>0, & \liminf _{t \rightarrow \infty} \int_{t}^{t+\omega_{3}} d_{2}(\theta) d \theta>0 \\
\liminf _{t \rightarrow \infty} \int_{t}^{t+\omega_{3+i}} a_{i i}(\theta) d \theta>0, & \liminf _{t \rightarrow \infty} \int_{t}^{t+\omega_{6}} \mu(\theta) d \theta>0 \tag{6}
\end{array}
$$

In particular, when model (1) degenerates into $\omega$-periodic system, that is, $a_{11}(t), \mu(t), a_{12}(t), d_{2}(t), a_{21}(t), b_{1}(t), b_{2}(t)$, and $a_{22}(t)$ are continuous periodic functions with period $\omega>0$, then assumption $\left(\mathrm{H}_{2}\right)$ is equivalent to the following forms:

$$
\left(\overline{\mathrm{H}_{2}}\right) \overline{b_{i}}>0, \overline{a_{i i}}>0, \overline{d_{2}}>0, \text { and } \bar{\mu}>0(i=1,2)
$$

In the following, we state several lemmas which will be useful in the proof of main results in the paper.

Firstly, we consider the following nonautonomous logistic equation:

$$
\begin{equation*}
\frac{d z(t)}{d t}=z(t)\left(b_{2}(t)-a_{22}(t) z(t)\right) \tag{7}
\end{equation*}
$$

where functions $b_{2}(t)$ and $a_{22}(t)$ are bounded continuous defined on $R_{+}$and $a_{22}(t) \geq 0$ for all $t \geq 0$. We have the following result.

Lemma 1 (see [10]). Suppose that there are constants $\omega_{i}>$ $0(i=1,2)$ such that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \int_{t}^{t+\omega_{1}} b_{2}(\theta) d \theta>0, \quad \liminf _{t \rightarrow \infty} \int_{t}^{t+\omega_{2}} a_{22}(\theta) d \theta>0 \tag{8}
\end{equation*}
$$

Then,
(a) there exist positive constants $m$ and $M$ such that for any positive solution $z(t)$ of (7)

$$
\begin{equation*}
m \leq \liminf _{t \rightarrow \infty} z(t) \leq \limsup _{t \rightarrow \infty} z(t) \leq M ; \tag{9}
\end{equation*}
$$

(b) each fixed positive solution $z^{*}(t)$ of (7) is globally uniformly attractive;
(c) if $a_{22}^{l}>0$, then for any positive solution $z(t)$ of (7)

$$
\begin{equation*}
\left(\frac{b_{2}}{a_{22}}\right)^{l} \leq \liminf _{t \rightarrow \infty} z(t) \leq \limsup _{t \rightarrow \infty} z(t) \leq\left(\frac{b_{2}}{a_{22}}\right)^{m} \tag{10}
\end{equation*}
$$

(d) if (7) is $\omega$-periodic, then condition (8) reduces to $\overline{b_{2}}>0$ and $\overline{a_{22}}>0$; thus (7) has a uniformly attractive positive $\omega$-periodic solution.

Further, we consider the following nonautonomous equation:

$$
\begin{equation*}
\frac{d z(t)}{d t}=z(t)\left[b_{2}(t)-a_{22}(t) z(t)+a_{21}(t)(F(t)+S(t))\right] \tag{11}
\end{equation*}
$$

where $b_{2}(t)$ and $a_{22}(t)$ are defined as in (7) and $a_{21}(t)(F(t)+$ $S(t))$ is continuous and bounded function defined on $R_{+}$.

Let $z\left(t, t_{0}, z_{0}\right)$ be the solution of (11) with initial condition $z\left(t_{0}\right)=z_{0}$ and let $z_{0}(t)$ be some fixed positive solution of (7). We have the following result.

Lemma 2 (see [11]). Suppose that all conditions of Lemma 1 hold. Then for any constants $\varepsilon>0$ and $M>0$ there exist constant $\delta=\delta(\varepsilon)>0$ and $T=T(\varepsilon, M)>0$ such that for any $t_{0} \in R_{+}$and $z_{0} \in\left[M^{-1}, M\right]$, when $\left|a_{21}(t)(F(t)+S(t))\right|<\delta$ for all $t \geq t_{0}$, one has

$$
\begin{equation*}
\left|z\left(t, t_{0}, z_{0}\right)-z_{0}(t)\right|<\varepsilon, \quad \forall t \geq t_{0}+T . \tag{12}
\end{equation*}
$$

Next, we consider the following nonautonomous linear equation:

$$
\begin{equation*}
\frac{d u(t)}{d t}=\mu(t)-d_{2}(t) u(t) \tag{13}
\end{equation*}
$$

where $\mu(t)$ and $d_{2}(t)$ are bounded continuous defined on $R_{+}$ and $\mu(t) \geq 0$ for all $t \geq 0$. We have the following result.

Lemma 3 (see [12]). Suppose that there are constants $\omega_{i}>$ $0(i=1,2)$ such that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \int_{t}^{t+\omega_{1}} \mu(\theta) d \theta>0, \quad \liminf _{t \rightarrow \infty} \int_{t}^{t+\omega_{2}} d_{2}(\theta) d \theta>0 \tag{14}
\end{equation*}
$$

Then,
(a) there exist positive constants $m$ and $M$ such that for any positive solution $u(t)$ of (13)

$$
\begin{equation*}
m \leq \liminf _{t \rightarrow \infty} u(t) \leq \limsup _{t \rightarrow \infty} u(t) \leq M \tag{15}
\end{equation*}
$$

(b) each fixed positive solution $u^{*}(t)$ of (13) is globally uniformly attractive;
(c) if $d_{2}^{l}>0$, then for any positive solution $u(t)$ of (13)

$$
\begin{equation*}
\left(\frac{\mu}{d_{2}}\right)^{l} \leq \liminf _{t \rightarrow \infty} z(t) \leq \limsup _{t \rightarrow \infty} z(t) \leq\left(\frac{\mu}{d_{2}}\right)^{m} ; \tag{16}
\end{equation*}
$$

(d) if (13) is $\omega$-periodic, then the condition (14) reduces to $\bar{\mu}>0$ and $\overline{d_{2}}>0$; thus (13) has a uniformly attractive positive $\omega$-periodic solution.

Further we investigate the following nonautonomous linear equation:

$$
\begin{equation*}
\frac{d u(t)}{d t}=\mu(t)-d_{2}(t) u(t)+e(t) \tag{17}
\end{equation*}
$$

where $\mu(t)$ and $d_{2}(t)$ are defined as in (13) and $e(t)$ is continuous and bounded function defined on $R_{+}$.

Let $u\left(t, t_{0}, u_{0}\right)$ be the solution of (17) with initial condition $u\left(t_{0}\right)=u_{0}$ and let $u_{0}(t)$ be some fixed positive solution of (13). We have the following result.

Lemma 4 (see [13]). Suppose that there exists a constant $\gamma>0$ such that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \int_{t}^{t+\gamma} d_{2}(\theta) d \theta>0 \tag{18}
\end{equation*}
$$

Then for any constants $\varepsilon>0$ and $M>0$ there exist constants $\delta=\delta(\varepsilon)>0$ and $T=T(\varepsilon, M)>0$ such that for any $t_{0} \in R_{+}$ and $u_{0} \in\left[M^{-1}, M\right]$, when $|e(t)|<\delta$ for all $t \geq t_{0}$, one has

$$
\begin{equation*}
\left|u\left(t, t_{0}, u_{0}\right)-u_{0}(t)\right|<\varepsilon, \quad \forall t \geq t_{0}+T . \tag{19}
\end{equation*}
$$

In (17), if function $\mu(t) \equiv 0$, then we can obtain that $u_{0}(t) \equiv 0$. We have the following Corollary 5 of Lemma 4 .

Corollary 5. Suppose that $\mu(t) \equiv 0$ for all $t \in R^{+}$and there exists a constant $\gamma>0$ such that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \int_{t}^{t+\gamma} d_{2}(\theta) d \theta>0 \tag{20}
\end{equation*}
$$

Then for any constants $\varepsilon>0$ and $M>0$ there exist constants $\delta=\delta(\varepsilon)>0$ and $T=T(\varepsilon, M)>0$ such that for any $t_{0} \in R_{+}$ and $u_{0} \in\left[M^{-1}, M\right]$, when $|e(t)|<\delta$ for all $t \geq t_{0}$, one has

$$
\begin{equation*}
\left|u\left(t, t_{0}, z_{0}\right)\right|<\varepsilon, \quad \forall t \geq t_{0}+T . \tag{21}
\end{equation*}
$$

## 3. Main Results

It is obvious that the solution $(F(t), S(t), Z(t))$ of model (1) with initial condition (2) is positive; that is, $F(t)>0, S(t)>0$, $Z(t)>0$ for all $t \geq 0$ in the maximum interval of existence of the solution. On the ultimate boundedness of solutions of system (1), we get the following theorem.

Theorem 6. Suppose that $\left(H_{1}\right)$ and $\left(H_{2}\right)$ hold. Then system (1) is ultimately bounded in the sense that there is a positive constant $M$ such that

$$
\begin{gather*}
\limsup _{t \rightarrow \infty} F(t)<M, \quad \limsup _{t \rightarrow \infty} S(t)<M \\
\limsup _{t \rightarrow \infty} Z(t)<M \tag{22}
\end{gather*}
$$

for any positive solution $(F(t), S(t), Z(t))$ of system (1).
The ecological implication of Theorem 6 is that the fertility prey $F(t)$ is ultimately bounded. The sterility prey $S(t)$, when assumptions $\left(\mathrm{H}_{1}\right)$ and $\left(\mathrm{H}_{2}\right)$ hold, if $S(t)$ is not ultimately bounded, then $S(t)$ will expand unlimitedly. But the conversion of the fertile prey lies on the sterile prey by sterile drugs. So, the prerequisite for the unlimited increase of the sterility prey is that the fertility prey must be expanding unlimitedly. In short, the number of harmful rat will not go on rising forever.

Proof. Let $(F(t), S(t), Z(t))$ be any positive solution of system (1). From the first equation of system (1) we have

$$
\begin{equation*}
\frac{d F(t)}{d t} \leq F(t)\left(b_{1}(t)-a_{11}(t) F(t)\right) \tag{23}
\end{equation*}
$$

From $\left(\mathrm{H}_{2}\right)$, it is easy to verify that the comparison equation

$$
\begin{equation*}
\frac{d x(t)}{d t}=x(t)\left[b_{1}(t)-a_{11}(t) x(t)\right] \tag{24}
\end{equation*}
$$

satisfies all conditions of Lemma 1. So, the comparison theorem and Lemma 1 imply that we obtain there is a constant $M_{1}$ such that for any positive solution $(F(t), S(t), Z(t))$ of system (1), there is a $T_{1}>0$ such that we have $F(t)<M_{1}$ for all $t \geq T_{1}$. Further, from the second equation of system (1) we have

$$
\begin{equation*}
\frac{d S(t)}{d t} \leq \mu(t) M_{1}-d_{2}(t) S(t) \tag{25}
\end{equation*}
$$

for all $t \geq T_{2}$. From Lemma 3 it can be obtained that under assumption $\left(\mathrm{H}_{2}\right)$ any positive solution $x(t)$ of the following nonautonomous linear equation:

$$
\begin{equation*}
\frac{d x(t)}{d t}=\mu(t) M_{1}-d_{2}(t) S(t) \tag{26}
\end{equation*}
$$

is ultimately bounded. Hence, using the comparison theorem, we further can obtain that there is a constant $M_{2}>0$ such that for any positive solution $(F(t), S(t), Z(t))$ of system (1), there is a $T_{2} \geq T_{1}$ such that $S(t)<M_{2}$ for all $t \geq T_{1}$. Lastly, from the third equation of equation of system (1) we have

$$
\begin{equation*}
\frac{d Z(t)}{d t} \leq Z(t)\left(b_{2}(t)+a_{21}(t)\left(M_{1}+M_{2}\right)-a_{22}(t) Z(t)\right) \tag{27}
\end{equation*}
$$

for all $t \geq T_{2}$. Consider the following nonautonomous equation:

$$
\begin{equation*}
\frac{d x(t)}{d t}=x(t)\left(b_{2}(t)+a_{21}(t)\left(M_{1}+M_{2}\right)-a_{22}(t) x(t)\right) \tag{28}
\end{equation*}
$$

the comparison theorem and Lemma 1 imply that there is a constant $M_{3}$ such that for any positive solution ( $F(t), S(t), Z(t))$ of system (2), there is a $T_{3}>0$ such that $Z(t)<M_{3}$ for all $t \geq T_{3}$.

Now, let $M=\max \left\{M_{1}, M_{2}, M_{3}\right\}$; then from the above proofs, we have

$$
\begin{gather*}
\limsup _{t \rightarrow \infty} F(t)<M, \quad \limsup _{t \rightarrow \infty} S(t)<M, \\
\underset{t \rightarrow \infty}{\limsup Z(t)<M .} \tag{29}
\end{gather*}
$$

Therefore, solution $(F(t), S(t), Z(t))$ is ultimately bounded. This completes the proof.

Remark 7. Applying the comparison theorem and combining conclusion (c) of Lemmas 1 and 3, we can obtain that if assumptions $\left(\mathrm{H}_{1}\right)$ and $\left(\mathrm{H}_{2}\right)$ hold in system (1), $a_{11}^{l}>0, d_{2}^{l}>0$, and $a_{22}^{l}>0$, then constants $M_{i}(i=1,2,3)$ given above can be chosen by

$$
\begin{gather*}
M_{1}=\left(\frac{b_{1}}{a_{11}}\right)^{m}, \quad M_{2}=\left(\frac{\mu M_{1}}{d_{2}}\right)^{m} \\
M_{3}=\left(\frac{b_{2}+a_{21}\left(M_{1}+M_{2}\right)}{a_{22}}\right)^{m} \tag{30}
\end{gather*}
$$

Next, we discuss the permanence and extinction of fertility prey $F(t)$ and infertility prey $S(t)$.

Let $Z_{0}(t)$ be some fixed positive solution of the following nonautonomous logistic equation:

$$
\begin{equation*}
\frac{d Z(t)}{d t}=Z(t)\left(b_{2}(t)-a_{22}(t) Z(t)\right) \tag{31}
\end{equation*}
$$

Particularly, if $a_{22}^{l}>0$, using conclusion (c) of Lemma 1, we can obtain

$$
\begin{equation*}
\left(\frac{b_{2}}{a_{22}}\right)^{l} \leq \liminf _{t \rightarrow \infty} Z_{0}(t) \leq \limsup _{t \rightarrow \infty} Z_{0}(t) \leq\left(\frac{b_{2}}{a_{22}}\right)^{m} \tag{32}
\end{equation*}
$$

Theorem 8. Suppose that $\left(H_{1}\right)$ and $\left(H_{2}\right)$ hold and there exists a constant $\lambda>0$ such that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \int_{t}^{t+\lambda}\left(b_{1}(\theta)-\mu(\theta)-a_{12}(\theta) Z_{0}((\theta))\right) d \theta>0 \tag{33}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} F(t)>m, \quad \liminf _{t \rightarrow \infty} S(t)>m \tag{34}
\end{equation*}
$$

for any positive solution $(F(t), S(t), Z(t))$ of system (1).

Theorem 8 shows that if we guarantee that assumptions $\left(\mathrm{H}_{1}\right),\left(\mathrm{H}_{2}\right)$ and (35) hold, then the prey species must be permanent. In the ecological system, each component part, including the animal, plant and microorganism, plays its own role, and they are indispensable and irreplaceable. Every creature may deviate from its original trajectory, which lead to the outbreak of this population and the negative effect on human beings, such as harmful rat. Even if it happens, this species should not be extinct through the human activity. What we should do is to control the rat population to such a degree that will not be harmful to human beings. Therefore, the permanence of harmful rat given by Theorem 8 is very necessary.

Proof. Let $(F(t), S(t), Z(t))$ be any positive solution of system (1). From condition (17) there are positive constants $\varepsilon_{0}, \eta$ and $T^{*}$ such that for all $t \geq T^{*}$

$$
\begin{gather*}
\int_{t}^{t+\lambda}\left(b_{1}(\theta)-\mu(\theta)-2 \varepsilon_{0} a_{11}(\theta)-a_{12}(\theta)\right.  \tag{35}\\
\left.\quad \times\left(Z_{0}(\theta)+\varepsilon_{0}\right)\right) d \theta>\eta
\end{gather*}
$$

According to Theorem 6, there exists a constant $T^{* *} \geq T^{*}$ such that

$$
\begin{equation*}
F(t)<M, \quad S(t)<M, \quad Z(t)<M \tag{36}
\end{equation*}
$$

for all $t \geq T^{* *}$. Consider (11), that is,

$$
\begin{equation*}
\frac{d z(t)}{d t}=z(t)\left(b_{2}(t)-a_{22}(t) z(t)+a_{21}(t)(F(t)+S(t))\right) \tag{37}
\end{equation*}
$$

from Lemma 2, for $\varepsilon_{0}$ and $M$ given in above there exist constants $\delta_{0}=\delta_{0}\left(\varepsilon_{0}\right)>0$ and $T_{0}=T_{0}\left(\varepsilon_{0}, M\right)>0$ such that for any $t_{0} \in R_{+}$and $Z_{0} \in[0, M]$, when $\left|a_{21}(t)(F(t)+S(t))\right|<$ $\delta_{0}$ for all $t \geq t_{0}$, we have

$$
\begin{equation*}
\left|z\left(t, t_{0}, z_{0}\right)-z_{0}(t)\right|<\varepsilon_{0}, \quad \forall t \geq t_{0}+T_{0} \tag{38}
\end{equation*}
$$

where $z\left(t, t_{0}, z_{0}\right)$ is the solution of (11) with initial condition $z\left(t_{0}\right)=z_{0}$.

Choose constant $\alpha_{0}$ as follows:

$$
\begin{equation*}
0<\alpha_{0} \leq \min \left\{\varepsilon_{0}, \frac{\delta_{0}}{a_{21}^{m}\left(F^{m}+S^{m}\right)+1}\right\} . \tag{39}
\end{equation*}
$$

Consider the following nonautonomous linear equation:

$$
\begin{equation*}
\frac{d u(t)}{d t}=\mu(t) F(t)-d_{2}(t) u(t) \tag{40}
\end{equation*}
$$

From Corollary 5, for $\alpha_{0}$ and $M$ given in above there exist constants $\delta_{1}=\delta_{1}\left(\alpha_{0}\right)<\alpha_{0}$ and $T_{1}=T_{1}\left(\alpha_{0}, M\right)>0$ such that for any $t_{0} \in R_{+}$and $u_{0} \in[0, M]$, when $|\mu(t) F(t)|<\delta_{1}$ for all $t \geq t_{0}$, we have

$$
\begin{equation*}
\left|u\left(t, t_{0}, u_{0}\right)\right|<\alpha_{0}, \quad \forall t \geq t_{0}+T_{1} \tag{41}
\end{equation*}
$$

Let $\alpha_{1}=\min \left\{\varepsilon_{0}, \delta_{1} /\left(\mu^{m}+1\right)\right\}$, we will discuss the following three cases.
Case 1. There exists a constant $T^{\prime} \geq T_{0}$ such that $F(t) \leq \alpha_{1}$ for all $t \geq T^{\prime}$.
Case 2. There exists a constant $T^{\prime} \geq T_{0}$ such that $F(t) \geq \alpha_{1}$ for all $t \geq T^{\prime}$.
Case 3. There exists a time sequence $\left\{\left[s_{k}, t_{k}\right]\right\}$ satisfying $T_{0} \leq$ $s_{1}<t_{1}<s_{2}<t_{2}<\cdots<s_{k}<t_{k}<\cdots$, and $\lim _{k \rightarrow \infty} s_{k}=\infty$ such that

$$
\begin{align*}
& F(t) \leq \alpha_{1}, \quad \forall t \in \bigcup_{k=1}^{\infty}\left[s_{k}, t_{k}\right]  \tag{42}\\
& F(t)>\alpha_{1}, \quad \forall t \notin \bigcup_{k=1}^{\infty}\left[s_{k}, t_{k}\right]
\end{align*}
$$

If Case 1 appears, we have

$$
\begin{equation*}
\frac{d S(t)}{d t} \leq \mu(t) F(t)-d_{2}(t) S(t) \tag{43}
\end{equation*}
$$

for all $t \geq T^{\prime}$. Considering the auxiliary system

$$
\begin{equation*}
\frac{d u(t)}{d t}=\mu(t) F(t)-d_{2}(t) u(t) \tag{44}
\end{equation*}
$$

Let $S(t)$ be the solution of the above equation satisfying initial condition $S\left(T^{\prime}\right)=u\left(T^{\prime}\right)$, by the comparison theorem, we have $S(t) \leq u(t)$ for all $t \geq T^{\prime}$. Since $F(t) \leq \alpha_{1}$ for all $t \geq T^{\prime}$. Hence, $|\mu(t) F(t)|<\delta_{1}$ for all $t \geq T^{\prime}$ and $u\left(T^{\prime}\right) \leq M$. By (41), we have $u(t)=u\left(t, T^{\prime}, u_{0}\left(T^{\prime}\right)\right)<\alpha_{0}$ for all $t \geq T^{\prime}+T_{1}$. Then, we obtain $S(t)<\alpha_{0}$ for all $t \geq T^{\prime}+T_{1}$. So,

$$
\begin{equation*}
F(t) \leq \alpha_{1}, \quad S(t)<\alpha_{0}, \quad Z(t)<M, \quad \forall t \geq T^{\prime}+T_{1} \tag{45}
\end{equation*}
$$

Hence, $a_{21}(t)(F(t)+S(t))<\delta_{0}$ for all $t \geq T^{\prime}+T_{1}$. In (38), choosing $t_{0}=T^{\prime}+T_{1}, Z_{0}=Z\left(T^{\prime}+T_{1}\right)$ and $Z(t)=$ $Z\left(t, t_{0}, Z\left(T^{\prime}+T_{1}\right)\right)$, by (38), we can get

$$
\begin{equation*}
Z\left(t, t_{0}, Z\left(T^{\prime}+T_{1}\right)\right)<Z_{0}(t)+\varepsilon_{0}, \quad \forall t \geq T^{\prime}+T_{1}+T_{0} \tag{46}
\end{equation*}
$$

Then,

$$
\begin{align*}
& F(t) \leq \alpha_{1}<\varepsilon_{0}, \quad S(t)<\alpha_{0}<\varepsilon_{0}, \quad Z(t)<Z_{0}(t)+\varepsilon_{0} \\
& \forall t \geq T^{\prime}+T_{1}+T_{0} \tag{47}
\end{align*}
$$

For any $t \geq T^{\prime}+T_{1}+T_{0}$, we have

$$
\begin{align*}
& \frac{d F(t)}{d t} \\
& =F(t)\left(b_{1}(t)-a_{11}(t)(F(t)+S(t))-\mu(t)-a_{12}(t) Z(t)\right) \\
& \geq F(t)\left(b_{1}(t)-2 \varepsilon_{0} a_{11}(t)-\mu(t)-a_{12}(t)\left(Z_{0}(t)+\varepsilon_{0}\right)\right) . \tag{48}
\end{align*}
$$

Integrating the above inequality from $T^{\prime}+T_{1}+T_{0}$ to $t>T^{\prime}+$ $T_{1}+T_{0}$, we can obtain

$$
\begin{align*}
& F(t) \geq F\left(T^{\prime}+T_{1}+T_{0}\right) \\
& \quad \times \exp \int_{T^{\prime}+T_{1}+T_{0}}^{t}\left(b_{1}(\theta)-2 \varepsilon_{0} a_{11}(\theta)\right. \\
& \tag{49}
\end{align*}
$$

From this and (35), it follows $\lim _{t \rightarrow \infty} F(t)=\infty$ which leads to a contradiction.

If Case 2 appears, then obviously $F(t)$ is permanent.
If Case 3 appears, for any $\left\{\left[s_{k}, t_{k}\right]\right\}$ we have $F\left(s_{k}\right)=$ $F\left(t_{k}\right)=\alpha_{1}$ and $F(t) \leq \alpha_{1}$ for all $t \in\left[s_{k}, t_{k}\right]$. If $t_{k}-s_{k} \leq T_{1}+T_{0}$, choosing constant

$$
\begin{equation*}
h=\sup _{t \geq 0}\left\{b_{1}(t)+a_{11}(t)\left(\varepsilon_{0}+M\right)+\mu(t)+a_{12}(t) M\right\}, \tag{50}
\end{equation*}
$$

integrating the first equation of model (1) in interval $\left[s_{k}, t_{k}\right]$, we get

$$
\begin{align*}
& \frac{d F(t)}{d t}= F\left(s_{k}\right) \exp \int_{s_{k}}^{t}\left(b_{1}(\theta)-a_{11}(\theta)(F(\theta)+S(\theta))\right. \\
&\left.\quad-\mu(\theta)-a_{12}(\theta) Z(\theta)\right) d \theta \\
& \geq F\left(s_{k}\right) \exp \int_{s_{k}}^{t}\left(b_{1}(\theta)-a_{11}(\theta)\left(\varepsilon_{0}+M\right)\right.  \tag{51}\\
&\left.\quad-\mu(\theta)-a_{12}(\theta) M\right) d \theta \\
& \geq \alpha_{1} \exp \left\{-h\left(T_{1}+T_{0}\right)\right\} .
\end{align*}
$$

If $t_{k}-s_{k}>T_{1}+T_{0}$, because $F(t) \leq \alpha_{1}$ for all $t \in\left[s_{k}, t_{k}\right]$, we have $|\mu(t) F(t)|<\delta_{1}$ for all $t \in\left[s_{k}, t_{k}\right]$ and $u\left(s_{k}\right)=S\left(s_{k}\right) \leq M$. Hence, we have $u(t)<\alpha_{0}$ for all $t \in\left[s_{k}+T_{1}, t_{k}\right]$. Then, we obtain $S(t)<\alpha_{0}$ for all $t \in\left[s_{k}+T_{1}, t_{k}\right]$. So,

$$
\begin{array}{r}
F(t) \leq \alpha_{1}, \quad S(t)<\alpha_{0}, \quad Z(t)<M  \tag{52}\\
\forall t \in\left[s_{k}+T_{1}, t_{k}\right]
\end{array}
$$

Hence, $a_{21}(t)(F(t)+S(t))<\delta_{0}$ for any $t \in\left[s_{k}+T_{1}, t_{k}\right]$. In (38), choosing $t_{0}=s_{k}+T_{1}, Z_{0}=Z\left(s_{k}+T_{1}\right)$ and $Z(t)=$ $Z\left(t, t_{0}, Z\left(s_{k}+T_{1}\right)\right)$, by (38), we can get

$$
\begin{equation*}
Z\left(t, t_{0}, Z\left(s_{k}+T_{1}\right)\right)<Z_{0}(t)+\varepsilon_{0}, \quad \forall t \in\left[s_{k}+T_{1}+T_{0}, t_{k}\right] \tag{53}
\end{equation*}
$$

Then,

$$
\begin{array}{r}
F(t) \leq \alpha_{1}<\varepsilon_{0}, \quad S(t)<\alpha_{0}<\varepsilon_{0}, \quad Z(t)<Z_{0}(t)+\varepsilon_{0} \\
\forall t \in\left[s_{k}+T_{1}+T_{0}, t_{k}\right] . \tag{54}
\end{array}
$$

For any $t \in\left[s_{k}, t_{k}\right]$, when $t \leq s_{k}+T_{1}+T_{0}$, we can obtain from the above discussion on the case $t_{k}-s_{k} \leq T_{1}+T_{0}$,

$$
\begin{equation*}
F(t) \geq \alpha_{1} \exp \left\{-h\left(T_{1}+T_{0}\right)\right\} \tag{55}
\end{equation*}
$$

In particular, we have $F\left(s_{k}+T_{1}+T_{0}\right) \geq \alpha_{1} \exp \left\{-h\left(T_{1}+T_{0}\right)\right\}$. When $t>s_{k}+T_{1}+T_{0}$, then we choose an integer $p>0$ such
that $t \in\left[s_{k}+T_{1}+T_{0}+p \lambda, s_{k}+T_{1}+T_{0}+(p+1) \lambda\right]$; integrating the first equation of system (1) from $s_{k}+T_{1}+T_{0}$ to $t>s_{k}+T_{1}+T_{0}$ we can obtain

$$
\begin{align*}
& \frac{d F(t)}{d t} \\
& =F\left(s_{k}+T_{1}+T_{0}\right) \\
& \times \exp \int_{s_{k}+T_{1}+T_{0}}^{t}\left(b_{1}(\theta)-a_{11}(\theta)(F(\theta)+S(\theta))\right. \\
& \left.-\mu(\theta)-a_{12}(\theta) Z(\theta)\right) d \theta \\
& \geq \alpha_{1} \exp \left\{-h\left(T_{1}+T_{0}\right)\right\} \\
& \times \exp \int_{s_{k}+T_{1}+T_{0}}^{t}\left(b_{1}(\theta)-2 \varepsilon_{0} a_{11}(\theta)-\mu(\theta)\right. \\
& \left.-a_{12}(\theta)\left(Z_{0}(\theta)+\varepsilon_{0}\right)\right) d \theta \\
& =\alpha_{1} \exp \left\{-h\left(T_{1}+T_{0}\right)\right\}  \tag{56}\\
& \times \exp \left(\int_{s_{k}+T_{1}+T_{0}}^{s_{k}+T_{1}+T_{0}+p \lambda}+\int_{s_{k}+T_{1}+T_{0}+p \lambda}^{t}\right) \\
& \times\left(b_{1}(\theta)-2 \varepsilon_{0} a_{11}(\theta)\right. \\
& \left.-\mu(\theta)-a_{12}(\theta)\left(Z_{0}(\theta)+\varepsilon_{0}\right)\right) d \theta \\
& \geq \alpha_{1} \exp \left\{-h\left(T_{1}+T_{0}\right)\right\} \\
& \times \exp \int_{s_{k}+T_{1}+T_{0}+p \lambda}^{t}\left(b_{1}(\theta)-2 \varepsilon_{0} a_{11}(\theta)-\mu(\theta)\right. \\
& \left.-a_{12}(\theta)\left(Z_{0}(\theta)+\varepsilon_{0}\right)\right) d \theta \\
& \geq \alpha_{1} \exp \left\{-h\left(T_{1}+T_{0}\right)\right\} \exp \left\{-h_{1} \lambda\right\},
\end{align*}
$$

where $h_{1}=\sup _{t \geq 0}\left\{b_{1}(t)+2 \varepsilon_{0} a_{11}(t)+\mu(t)+a_{12}(t)\left(Z_{0}(\theta)+\varepsilon_{0}\right)\right\}$. Choose

$$
\begin{equation*}
m_{1}=\alpha_{1} \exp \left\{-\left(h\left(T_{1}+T_{0}\right)+h_{1} \lambda\right)\right\} ; \tag{57}
\end{equation*}
$$

then from above discussion we finally obtain

$$
\begin{equation*}
F(t) \geq m_{1}, \quad \forall t \in \bigcup_{k=1}^{\infty}\left[s_{k}, t_{k}\right] \tag{58}
\end{equation*}
$$

In addition, we have $F(t)>\alpha_{1}$ for all $t \notin \bigcup_{k=1}^{\infty}\left[s_{k}, t_{k}\right]$. Then, we finally obtain

$$
\begin{equation*}
F(t) \geq m_{1}, \quad \forall t \geq T^{\prime} \tag{59}
\end{equation*}
$$

Considering the second equation of system (1), according to Theorem 6, we have

$$
\begin{align*}
\frac{d S(t)}{d t}= & \mu(t) F(t)-d_{2}(t) S(t)-a_{11}(t) \\
& \times(F(t)+S(t)) S(t)-a_{12}(t) S(t) Z(t) \\
\geq & \mu(t) m_{1}-\left(d_{2}(t)+2 M a_{11}(t)+a_{12}(t) M\right) S(t) \tag{60}
\end{align*}
$$

for all $t \geq T^{\prime}$.
Considering the auxiliary equation

$$
\begin{equation*}
\frac{d u(t)}{d t}=\mu(t) m_{1}-\left(d_{2}(t)+2 M a_{11}(t)+a_{12}(t) M\right) u(t) \tag{61}
\end{equation*}
$$

According to Lemma 3, there exists a constant $m_{2}$ such that $\liminf _{t \rightarrow \infty} u(t) \geq m_{2}$ for any positive solution of (61). By the comparison theorem and (60), we have

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} S(t) \geq \liminf _{t \rightarrow \infty} u(t) \geq m_{2} \tag{62}
\end{equation*}
$$

Let $m=\min \left\{m_{1}, m_{2}\right\}$; from (59) and (62) we obtain

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} F(t) \geq m, \quad \liminf _{t \rightarrow \infty} S(t) \geq m \tag{63}
\end{equation*}
$$

This completes the proof.
Theorem 9. Suppose that $\left(H_{1}\right)$ and $\left(H_{2}\right)$ hold and there exists a constant $\lambda>0$ such that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int_{t}^{t+\lambda}\left(b_{1}(\theta)-\mu(\theta)-a_{12}(\theta) Z_{0}(\theta)\right) d \theta \leq 0 \tag{64}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} F(t)=0, \quad \lim _{t \rightarrow \infty} S(t)=0 \tag{65}
\end{equation*}
$$

for any positive solution $(F(t), S(t), Z(t))$ of system (1).
The biological meaning of Theorem 9 is that if $\left(\mathrm{H}_{1}\right),\left(\mathrm{H}_{2}\right)$, and (64) hold, the prey species will be extinct. Form the viewpoint of the Nature Conservancy and Human Health, the best way for our human beings is to keep the existence of the species, and, meanwhile, guarantee such existence do no harm to us. Thus, the condition of making harmful rat extinct for management expert is very important. Therefore, it is a critical threshold value.

Proof. By (64), we have for any $0<\varepsilon<1$, there are positive constants $\varepsilon_{1}<\varepsilon$ and $\varepsilon_{0}$ and $T_{0}>0$ such that

$$
\begin{equation*}
\int_{t}^{t+\lambda}\left(b_{1}(\theta)-\mu(\theta)-a_{11}(\theta) \varepsilon-a_{12}(\theta)\left(Z_{0}(\theta)-\varepsilon_{1}\right)\right) d \theta<-\varepsilon_{0} \tag{66}
\end{equation*}
$$

for all $t \geq T_{0}$. From the third equation of system (1) we have

$$
\begin{equation*}
\frac{d Z(t)}{d t} \geq Z(t)\left(b_{2}(t)-a_{22}(t) Z(t)\right), \quad \forall t \geq T_{0} \tag{67}
\end{equation*}
$$

applying the comparison theorem and conclusion (b) of Lemma 1, there exists a constant $T_{1} \geq T_{0}$ such that $Z(t) \geq$ $Z_{0}(t)-\varepsilon_{1}$ for all $t \geq T_{1}$. For any $t \geq T_{1}$, we have

$$
\begin{align*}
& \frac{d F(t)}{d t} \\
& =F(t)\left(b_{1}(t)-a_{11}(t)(F(t)+S(t))-\mu(t)-a_{12}(t) Z(t)\right) \\
& \leq F(t)\left(b_{1}(t)-a_{11}(t) F(t)-\mu(t)-a_{12}(t) Z(t)\right) . \tag{68}
\end{align*}
$$

For any $0<\varepsilon<1$, if $F(t) \geq \varepsilon$ for all $t \geq T_{1}$, integrating (68) from $T_{1}$ to $t$, we obtain

$$
\begin{align*}
F(t) \leq F\left(T_{1}\right) \exp \int_{T_{1}}^{t} & \left(b_{1}(\theta)-a_{11}(\theta) \varepsilon-\mu(\theta)\right.  \tag{69}\\
& \left.-a_{12}(\theta)\left(Z_{0}(\theta)-\varepsilon_{1}\right)\right) d \theta .
\end{align*}
$$

From (66), it follows that $F(t) \rightarrow 0$ as $t \rightarrow \infty$ which leads to a contradiction. Hence, there exists a $t_{1} \geq T_{1}$ such that $F\left(t_{1}\right)<\varepsilon$. Let

$$
\begin{equation*}
h=\sup _{t \geq T_{1}}\left\{b_{1}(t)+a_{11}(t)+\mu(t)+a_{12}(t)\left(Z_{0}(\theta)-\varepsilon_{1}\right)\right\} \tag{70}
\end{equation*}
$$

we prove

$$
\begin{equation*}
F(t) \leq \varepsilon \exp \{h \lambda\}, \quad \forall t \geq t_{1} . \tag{71}
\end{equation*}
$$

If (71) is not true, then there exists a $t_{2}>t_{1}$ such that $F\left(t_{2}\right)>$ $\varepsilon \exp \{h \lambda\}$. From $F\left(t_{1}\right)<\varepsilon$, there exists a $t_{3} \in\left(t_{1}, t_{2}\right)$ such that $F\left(t_{3}\right)=\varepsilon$ and $F(t)>\varepsilon$ for all $t \in\left(t_{3}, t_{2}\right)$. Let $p \geq 0$ be an integer such that $t_{2} \in\left(t_{3}+p \lambda, t_{3}+(p+1) \lambda\right]$, integrating (68) from $t_{3}$ to $t_{2}$,

$$
\begin{align*}
& \varepsilon \exp \{h \lambda\}< F\left(t_{2}\right) \leq F\left(t_{3}\right) \\
& \times \exp \int_{t_{3}}^{t_{2}}\left(b_{1}(\theta)-a_{11}(\theta) \varepsilon\right. \\
&\left.\quad-\mu(\theta)-a_{12}(\theta)\left(Z_{0}(\theta)-\varepsilon_{1}\right)\right) d \theta \\
& \leq \varepsilon \exp \{h \lambda\}, \tag{72}
\end{align*}
$$

which leads to a contradiction. Hence, (71) holds. From the arbitrariness of $\varepsilon$, we finally obtain $F(t) \rightarrow 0$ as $t \rightarrow \infty$. Considering the second equation

$$
\begin{align*}
& \frac{d S(t)}{d t} \\
& =\mu(t) F(t)-d_{2}(t) S(t)-a_{11}(t) \\
& \quad \times(F(t)+S(t)) S(t)-a_{12}(t) S(t) Z(t) \\
& \leq \mu(t) F(t)-d_{2}(t) S(t) \leq \varepsilon \exp \{h \lambda\} \mu(t)-d_{2}(t) S(t) \tag{73}
\end{align*}
$$

for all $t \geq t_{1}$. Using Corollary 5 , we can easily obtain $S(t) \rightarrow 0$ as $t \rightarrow \infty$. This completes the proof.

Further, from conclusion (c) of Lemma 1, as consequence of Theorems 8 and 9 , we also have the following corollaries.

Corollary 10. Suppose that $\left(H_{1}\right)$ and $\left(H_{2}\right)$ hold, $a_{22}^{l}>0$ and there exists a constant $\lambda>0$ such that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \int_{t}^{t+\lambda}\left(b_{1}(\theta)-\mu(\theta)-a_{12}(\theta)\left(\frac{b_{2}}{a_{22}}\right)^{m}\right) d \theta>0 \tag{74}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} F(t)>m, \quad \liminf _{t \rightarrow \infty} S(t)>m \tag{75}
\end{equation*}
$$

for any positive solution $(F(t), S(t), Z(t))$ of system (1).


Figure 1: Permanence of system (1) with parameters in Example 1.

Corollary 11. Suppose that $\left(H_{1}\right)$ and $\left(H_{2}\right)$ hold, $a_{22}^{l}>0$ and there exists a constant $\lambda>0$ such that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int_{t}^{t+\lambda}\left(b_{1}(\theta)-\mu(\theta)-a_{12}(\theta)\left(\frac{b_{2}}{a_{22}}\right)^{l}\right) d \theta \leq 0 \tag{76}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} F(t)=0, \quad \lim _{t \rightarrow \infty} S(t)=0 \tag{77}
\end{equation*}
$$

for any positive solution $(F(t), S(t), Z(t))$ of system (1).
As consequences of Theorems 8 and 9, we have the following corollaries. Firstly, from Lemmas 1 we obtain that if $\left(\mathrm{H}_{1}\right)$ and $\left(\overline{\mathrm{H}_{2}}\right)$ hold, then (31) have the globally uniformly attractive nonnegative $\omega$-periodic solutions $Z_{0}(t)$.

Corollary 12. Suppose that system (1) is $\omega$-periodic and $\left(H_{1}\right)$ and $\left(\overline{H_{2}}\right)$ hold. Then the fertility prey $F(t)$ and infertility prey $S(t)$ in model (1) are permanent if and only if

$$
\begin{equation*}
\bar{\mu}<\overline{b_{1}-a_{12} Z_{0}} . \tag{78}
\end{equation*}
$$

Corollary 13. Suppose that system $(1)$ is $\omega$-periodic and $\left(H_{1}\right)$ and $\left(\overline{H_{2}}\right)$ hold. Then the fertility prey $F(t)$ and infertility prey $S(t)$ in model (1) are extinct if and only if

$$
\begin{equation*}
\bar{\mu} \geq \overline{b_{1}-a_{12} Z_{0}} . \tag{79}
\end{equation*}
$$

Remark 14. In the process of the prevention and control of rat in the grasslands and farmland, we are concerned about how many biological sterile drug should be put in a period in order to make the population of the harmful rat reduce to a very low level. From Corollaries 12 and 13, we can easily obtain that $\bar{\mu}=\overline{b_{1}-a_{12} Z_{0}}$ is a critical value. If $\bar{\mu} \geq \overline{b_{1}-a_{12} Z_{0}}$, we can control the population of rat at a very low level. The results are very meaningful and significant.

When system (1) is simplified into the corresponding autonomous system, that is,

$$
\begin{align*}
\frac{d F(t)}{d t}= & F(t)\left(b_{1}-a_{11}(F(t)+S(t))-\mu-a_{12} Z(t)\right), \\
\frac{d S(t)}{d t}= & \mu F(t)-d_{2} S(t)-a_{11}(F(t)+S(t)) S(t)  \tag{80}\\
& -a_{12} S(t) Z(t) \\
\frac{d Z(t)}{d t}= & Z(t)\left(b_{2}+a_{21}(F(t)+S(t))-a_{22} Z(t)\right) .
\end{align*}
$$

Remark 15. For system (80), we know that $\mu=b_{1}-a_{12} b_{2} / a_{22}$ is a critical value. Then, we can obtain that the quantity of biological sterile drug should be $\mu \geq b_{1}-a_{12} b_{2} / a_{22}$.

## 4. Example and Numerical Simulation

In this section, we give some examples and numerical simulations to the above theoretical analysis.


Figure 2: Chaotic behavior of system (1) with parameters in Example 2.

Example 1. Take $b_{1}(t)=10+\sin (6 t / \pi), a_{11}(t)=0.09+0.001 \times$ $\sin (6 t / \pi), \mu(t)=9 \times(0.35+0.2 \times \cos (4 t / \pi)+0.01 \times \sin (4 t / \pi))$, $a_{12}(t)=1.2+\cos (6 t / \pi), d_{2}(t)=2+0.005 \times \sin (t \pi / 6), b_{2}(t)=$ $1.5+\cos (6 t / \pi), a_{21}(t)=0.7+0.3 \times \cos (6 t / \pi)$ and $a_{22}(t)=5+$ $3 \times \sin (6 t / \pi)$ in system (1). We easily verify that assumptions $\left(\mathrm{H}_{1}\right),\left(\mathrm{H}_{2}\right)$ hold. From Lemma 1, some fixed positive solution $Z_{0}(t)$ of system (32) satisfies $0.00625 \leq Z_{0}(t) \leq 1.25$. Moreover, condition (35) $\liminf _{t \rightarrow \infty} \int_{t}^{t+\lambda}\left(b_{1}(\theta)-\mu(\theta)-\right.$ $\left.a_{12}(\theta) Z_{0}((\theta))\right) d \theta>0>9-9 \times(0.35+0.21)-2.2 \times 1.25=$ $1.21>0$ holds, therefore, by Theorem 8, system (1) with these parameters is permanent. The corresponding numerical simulations are given in Figure 1, and this figure illustrates that the solutions will tend towards periodic oscillation along with time passing. It means that there exists a periodic solution, and it is seemed that this periodic solution is globally attractive.

Example 2. Take $b_{1}(t)=10+\sin (6 t / \pi), a_{11}(t)=0.09+0.001 \times$ $\sin (6 t / \pi), \mu(t)=9 \times(0.35+0.1 \times \cos (t / 4)+0.1 \times \sin (t / 4))$, $a_{12}(t)=1.2+\cos (t / 6), d_{2}(t)=2+0.005 \times \sin (6 t / \pi)$, $b_{2}(t)=1.5+\cos (t / 6), a_{21}(t)=0.7+0.3 \times \cos (6 t / \pi)$ and $a_{22}(t)=5+3 \times \sin (t / 6)$ in system (1). By similar calculation, we can obtain that $\left(\mathrm{H}_{1}\right),\left(\mathrm{H}_{2}\right)$ and (35) hold. Therefore, by

Theorem 8, system (1) with these parameters is permanent, as shown in numerical simulations of Figures 2(a)-2(d), which not only illustrate the validity of the proposed results, but also display the interesting complex dynamic behaviors; that is, there is not periodic oscillation along with time passing as like Figure 1, and from (a)-(d) in Figure 2, it can be obviously seen that there is a strange chaotic attractor, which may contribute to a better understanding of the complex chaotic behaviors which can be a high risk of the uncertain number of the population due to the unpredictability.

## 5. Conclusion

Based on the mouse rampant phenomenon in some areas, a predator-prey model with infertility control in rat species is established in the situation where all coefficients depend on time. For the nonautonomous system threshold conditions for the permanence and the extinction of fertility prey and infertility prey are established. The condition for permanence has the form of a lim inf condition for some time-dependent sterility conversion rate $(\mu(t))$ while the condition for extinction assumes the form of a lim sup condition. Hence, in
the general case the main results are not threshold criteria in a strict sense. However, in the periodic cases, the conditions merge into a sharp threshold criterion and sterile drug dosage can be obtained. Two numerical examples are carried out to support theoretical results, and the second simulation result suggests that there may be interesting dynamic behaviors in this model-a strange chaotic attractor. Furthermore, chaos may cause the number of pests approaching to the uncontrollable state due to the unpredictability. Thus, how to control chaos in the population model is very important, which needs further investigation.

## Conflict of Interests

The authors declare that they have no financial and personal relationships with other people or organizations that can inappropriately influence their work, there is no professional or other personal interests of any nature or kind in any product, service, and/or company that could be construed as influencing the position presented in, or the review of, this paper.

## Acknowledgments

This work was supported by The National Natural Science Foundation of China [11271312,11371313,11241005], The Natural Science Foundation of Shanxi Province [20130110025], and the Research Project at Yuncheng University [XK2012001,XK2012007].

## References

[1] Hainan Online News Center, "Inner Mongolia steppe: starving rat injury by oneself and alse eat live sheep," 2005, http://news.hainan.net/newshtml/2005w6r9/23094f1.htm.
[2] Modern Express, "Inner Mongolia steppe introduced eleven Yinhu in order to destroy rat," 2006, http://kb.dsqq.cn/old/ $\mathrm{html} / 2006-11 / 23 /$ content_49946823.htm.
[3] "Inner Mongolia DongWuZhMuQinQi farming and animal husbandry information network," 2011, http://www.nmgdwzmqq.agri.gov.cn/sites/MainSite/.
[4] C. M. Hardy, L. A. Hinds, P. J. Kerr et al., "Biological control of vertebrate pests using virally vectored immunocontraception," Journal of Reproductive Immunology, vol. 71, no. 2, pp. 102-111, 2006.
[5] A. D. Arthur, R. P. Pech, and G. R. Singleton, "Cross-strain protection reduces effectiveness of virally vectored fertility control: results from individual-based multistrain models," Journal of Applied Ecology, vol. 44, no. 6, pp. 1252-1262, 2007.
[6] A. T. Rutberg, R. E. Naugle, L. A. Thiele, and I. K. M. Liu, "Effects of immunocontraception on a suburban population of whitetailed deer Odocoileus virginianus," Biological Conservation, vol. 116, no. 2, pp. 243-250, 2004.
[7] C. K. Williams, C. C. Davey, R. J. Moore et al., "Population responses to sterility imposed on female European rabbits," Journal of Applied Ecology, vol. 44, no. 2, pp. 291-301, 2007.
[8] J. Jacob, Rahmini, and Sudarmaji, "The impact of imposed female sterility on field populations of ricefield rats (Rattus argentiventer)," Agriculture, Ecosystems \& Environment, vol. 115, no. 1-4, pp. 281-284, 2006.
[9] Z. Zhang, "Mathematical models of wildlife management by contraception," Ecological Modelling, vol. 132, no. 1-2, pp. 105113, 2000.
[10] Z. Teng and Z. Li, "Permanence and asymptotic behavior of the $N$-species nonautonomous lotka-volterra competitive systems," Computers \& Mathematics with Applications, vol. 39, no. 7-8, pp. 107-116, 2000.
[11] X. Niu, T. Zhang, and Z. Teng, "The asymptotic behavior of a nonautonomous eco-epidemic model with disease in the prey," Applied Mathematical Modelling, vol. 35, no. 1, pp. 457-470, 2011.
[12] T. Zhang and Z. Teng, "On a nonautonomous SEIRS model in epidemiology," Bulletin of Mathematical Biology, vol. 69, no. 8, pp. 2537-2559, 2007.
[13] X. Feng, Z. Teng, and L. Zhang, "Permanence for nonautonomous n -species Lotka-Volterra competitive systems with feedback controls," Rocky Mountain Journal of Mathematics, vol. 38, no. 5, pp. 1355-1376, 2008.

## Research Article

# Mathematical Model of Schistosomiasis under Flood in Anhui Province 

Longxing Qi, ${ }^{1}$ Jing-an Cui, ${ }^{2}$ Tingting Huang, ${ }^{1}$ Fengli Ye, ${ }^{3}$ and Longzhi Jiang ${ }^{4}$<br>${ }^{1}$ School of Mathematical Sciences, Anhui University, Hefei 230601, China<br>${ }^{2}$ College of Science, Beijing University of Civil Engineering and Architecture, Beijing 100044, China<br>${ }^{3}$ Tongcheng Health Bureau, Tongcheng 231400, China<br>${ }^{4}$ Tongcheng Schistosomiasis Control Station, Tongcheng 231400, China<br>Correspondence should be addressed to Longxing Qi; qilx@ahu.edu.cn

Received 11 January 2014; Accepted 1 February 2014; Published 6 March 2014
Academic Editor: Weiming Wang
Copyright © 2014 Longxing Qi et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

Based on the real observation data in Tongcheng city, this paper established a mathematical model of schistosomiasis transmission under flood in Anhui province. The delay of schistosomiasis outbreak under flood was considered. Analysis of this model shows that the disease free equilibrium is locally asymptotically stable if the basic reproduction number is less than one. The stability of the unique endemic equilibrium may be changed under some conditions even if the basic reproduction number is larger than one. The impact of flood on the stability of the endemic equilibrium is studied and the results imply that flood can destabilize the system and periodic solutions can arise by Hopf bifurcation. Finally, numerical simulations are performed to support these mathematical results and the results are in accord with the observation data from Tongcheng Schistosomiasis Control Station.


## 1. Introduction

As we know, schistosomiasis is a serious water-borne disease. It is not easy to control because of many reasons such as flood. Many reports have shown that flood leads to a serious outbreak of schistosomiasis [1-3]. During the flood period there are a lot of people that come into contact with contaminated water, which may lead to the fact that a lot of people are infected by schistosome [1-3]. In China, Anhui province often encounters floods; in particular in 1998 the flood was one of the most serious flood [1]. Based on the observation data from Tongcheng Schistosomiasis Control Station in Anhui province (Figure 1), we can see that the number of patients and the area of snails increase by a big margin after 1998 in Tongcheng city in Anhui province. Although people know the phenomenon that schistosomiasis will be serious after flood, people do not know the reason and there are only some live reports. Hence, it is necessary to investigate theoretically the effect of flood on the schistosomiasis transmission.

After flood the infected human by cercaria will have an incubation period to become an infectious human. In fact, it is about five weeks from the time of cercaria penetration through skins of human to the time when eggs are discharged [4]. Adult schistosomes in human are capable of producing eggs for a number of years [5]. This leads to breakout of schistosomiasis in many places after flood. For example, the catastrophic flood in 1998 brought a serious impact on the prevalence of schistosomiasis in Anhui province from 1998 to 2000 [1]. Furthermore, the data from Tongcheng Schistosomiasis Control Station (Figure 1) and the report of Ge et al. [1] both show that schistosomiasis is more serious in three years after flood than in the flood year. This phenomenon is called the delayed effect of flood [1]. In this paper we want to investigate how flood affects the dynamical behavior of schistosomiasis.

Many schistosomiasis models have involved many aspects such as drug-resistant, age-structure, incubation period of snail, and chemotherapy [6-10]. Their results imply that many factors affect the transmission of schistosomiasis. However,


Figure 1: The observation data in Tongcheng city from Tongcheng Schistosomiasis Control Station in 1998-2007.
there are few mathematical models considering the effect of flood in previous papers.

To study the delayed effect of flood, we use a time delay to reflect the incubation period in the infected human. We modified the model in [11]. Distribute human into susceptible $x_{s}(t)$ and infectious $x_{i}(t)$ and snails into susceptible $y_{s}(t)$, preshedding $y_{e}(t)$, and infectious $y_{i}(t)$. The model in [11] reads

$$
\begin{align*}
& \frac{d x_{s}}{d t}=A_{x}-\mu_{x} x_{s}-\beta_{x} x_{s} y_{i}, \\
& \frac{d x_{i}}{d t}=\beta_{x} x_{s} y_{i}-\left(\mu_{x}+\alpha_{x}\right) x_{i}, \\
& \frac{d y_{s}}{d t}=A_{y}-\mu_{y} y_{s}-\beta_{y} x_{i} y_{s},  \tag{1}\\
& \frac{d y_{e}}{d t}=\beta_{y} x_{i} y_{s}-\left(\mu_{y}+\theta\right) y_{e}, \\
& \frac{d y_{i}}{d t}=\theta y_{e}-\left(\mu_{y}+\alpha_{y}\right) y_{i},
\end{align*}
$$

where $A_{x}$ is the recruitment rate of human, $\mu_{x}$ is the death rate of human, $\alpha_{x}$ is the disease-induced death rate of human, $\beta_{x}$ is the transmission rate from infectious snails to susceptible human, $A_{y}$ is the recruitment rate of snail host, $\mu_{y}$ is the death rate of snail host, $\alpha_{y}$ is the disease-induced death rate of snail host, $\beta_{y}$ is the transmission rate from infectious human to susceptible snails, and $\theta$ is the translate rate from infected and preshedding snails to shedding snails. In the model, we have studied the stability of equilibria and preferable control strategies.

The goal of this paper is to study the impact of flood on the basic reproduction number and the dynamics of the schistosomiasis transmission. This paper is organized as follows. In Section 2 we establish a schistosomiasis model with a time delay and define the basic reproduction number $R_{0}$. The stability of the disease free equilibrium is obtained in Section 3. We devote Section 4 to the Hopf bifurcation analysis. Section 5 examines mathematical results by numerical simulations.

## 2. The Delayed Model

By incorporating a time delay in human, we have the following model:

$$
\begin{gather*}
\frac{d x_{s}}{d t}=A_{x}-\mu_{x} x_{s}(t)-\beta_{x} x_{s}(t-\tau) y_{i}(t-\tau) e^{-\mu_{x} \tau} \\
\frac{d x_{i}}{d t}=\beta_{x} x_{s}(t-\tau) y_{i}(t-\tau) e^{-\mu_{x} \tau}-\left(\mu_{x}+\alpha_{x}\right) x_{i}(t), \\
\frac{d y_{s}}{d t}=A_{y}-\mu_{y} y_{s}(t)-\beta_{y} x_{i}(t) y_{s}(t)  \tag{2}\\
\frac{d y_{e}}{d t}=\beta_{y} x_{i}(t) y_{s}(t)-\left(\mu_{y}+\theta\right) y_{e}(t) \\
\frac{d y_{i}}{d t}=\theta y_{e}(t)-\left(\mu_{y}+\alpha_{y}\right) y_{i}(t)
\end{gather*}
$$

where $\tau$ is the incubation period in the infected human, that is, the time from cercaria penetration through skins to the time when eggs are discharged.

Define the basic reproduction number according to biological meanings:

$$
\begin{equation*}
R_{0}=\frac{A_{x} A_{y} \theta \beta_{x} e^{-\mu_{x} \tau} \beta_{y}}{\mu_{x} \mu_{y}\left(\mu_{x}+\alpha_{x}\right)\left(\mu_{y}+\alpha_{y}\right)\left(\mu_{y}+\theta\right)} \tag{3}
\end{equation*}
$$

These quantities have a clear biological interpretation. Consider the case when an infectious snail is introduced into a purely susceptible people population with size $A_{x} / \mu_{x}$. The size of susceptible people who become infectious people per unit time is $\beta_{x}\left(A_{x} / \mu_{x}\right) .1 /\left(\mu_{x}+\alpha_{x}\right)$ is the mean infective period of the infectious people and $e^{-\mu_{x} \tau}$ represents the survived rate of people during his infection. On the other hand, infectious people can infect $\beta_{y}\left(A_{y} / \mu_{y}\right)$ susceptible snails which should get through the latent time where the rate of transmission is $\theta$ and then infective period is $1 /\left(\mu_{y}+\alpha_{y}\right)\left(\mu_{y}+\theta\right)$. Thus, $R_{0}$ gives the total number of secondary infectious snails produced by a typical infected snail during its entire period of infectiousness in a completely susceptible population. The following section shows that the
basic reproduction number $R_{0}$ provides a threshold condition for parasite extinction.

Theorem 1. There exist at most two equilibria:
(i) if $R_{0} \leq 1$, system (2) has a disease free equilibrium $E_{0}=$ $\left(A_{x} / \mu_{x}, 0, A_{y} / \mu_{y}, 0,0\right)$;
(ii) if $R_{0}>1$, system (2) has two equilibria, the disease free equilibrium $E_{0}$ and the unique endemic equilibrium $E=\left(x_{s}^{*}, x_{i}^{*}, y_{s}^{*}, y_{e}^{*}, y_{i}^{*}\right)$, where

$$
\begin{gather*}
x_{s}^{*}=\frac{A_{x}\left(A_{x} \beta_{y}+\mu_{y}\left(\mu_{x}+\alpha_{x}\right)\right)}{\mu_{x}\left(A_{x} \beta_{y}+\mu_{y} R_{0}\left(\mu_{x}+\alpha_{x}\right)\right)}, \\
x_{i}^{*}=\frac{A_{x} \mu_{y}\left(R_{0}-1\right)}{A_{x} \beta_{y}+\mu_{y} R_{0}\left(\mu_{x}+\alpha_{x}\right)}, \\
y_{s}^{*}=\frac{A_{y}\left(A_{x} \beta_{y}+\mu_{y} R_{0}\left(\mu_{x}+\alpha_{x}\right)\right)}{\mu_{y} R_{0}\left(A_{x} \beta_{y}+\mu_{y}\left(\mu_{x}+\alpha_{x}\right)\right)},  \tag{4}\\
y_{e}^{*}=\frac{\mu_{x} \mu_{y}\left(\mu_{x}+\alpha_{x}\right)\left(\mu_{y}+\alpha_{y}\right)\left(R_{0}-1\right)}{\theta \beta_{x} e^{-\mu_{x} \tau}\left(A_{x} \beta_{y}+\mu_{y}\left(\mu_{x}+\alpha_{x}\right)\right)}, \\
y_{i}^{*}=\frac{\mu_{x} \mu_{y}\left(\mu_{x}+\alpha_{x}\right)\left(R_{0}-1\right)}{\beta_{x} e^{-\mu_{x} \tau}\left(A_{x} \beta_{y}+\mu_{y}\left(\mu_{x}+\alpha_{x}\right)\right)} .
\end{gather*}
$$

Next we will discuss the stabilities of $E_{0}$ and $E$ in system (2).

## 3. Stability Analysis of $E_{0}$

In this section, we will analyze the stability of the disease free equilibrium $E_{0}$ of the delayed model (2) in the two cases: $R_{0}<$ 1 and $R_{0}>1$.

Theorem 2. The disease free equilibrium $E_{0}$ of the system (2) is locally asymptotically stable if $R_{0}<1$ and unstable if $R_{0}>1$.

Proof. Denote $b=\mu_{x}+\alpha_{x}, c=\mu_{y}+\alpha_{y}, d=\mu_{y}+$ $\theta$. By linearizing the system (2) around $E_{0}$ we can obtain the characteristic roots that are $-\mu_{x},-\mu_{y}$ and roots of the following equation:

$$
\begin{align*}
& \lambda^{3}+(b+c+d) \lambda^{2}+(b c+b d+c d) \lambda \\
& \quad+b c d-\frac{A_{x} A_{y} \theta \beta_{x} e^{-\mu_{x} \tau} \beta_{y}}{\mu_{x} \mu_{y}} e^{-\lambda \tau}=0 . \tag{5}
\end{align*}
$$

Denote the left-hand side of (5) as $F(\lambda, \tau)$. It is easy to see that

$$
\begin{aligned}
F(0, \tau) & =b c d-\frac{A_{x} A_{y} \theta \beta_{x} e^{-\mu_{x} \tau} \beta_{y}}{\mu_{x} \mu_{y}} e^{-\lambda \tau} \\
& =\operatorname{bcd}\left(1-R_{0}\right)
\end{aligned}
$$

$$
\begin{align*}
F_{\lambda}^{\prime}(\lambda, \tau)= & 3 \lambda^{2}+2(b+c+d) \lambda+(b c+b d+c d) \\
& +\tau \frac{A_{x} A_{y} \theta \beta_{x} e^{-\mu_{x} \tau} \beta_{y}}{\mu_{x} \mu_{y}} e^{-\lambda \tau} \tag{6}
\end{align*}
$$

(i) If $R_{0}>1, F(0, \tau)<0, F_{\lambda}^{\prime}(\lambda, \tau)>0$ for $\lambda \geq 0$ and $\tau>0$. Thus, (5) has a positive real solution for $\tau>0$ and the disease free equilibrium $E_{0}$ is unstable.
(ii) If $R_{0}<1, F(0, \tau)>0$. Since $F_{\lambda}^{\prime}(\lambda, \tau)>0$ for $\lambda \geq 0$ and $\tau>0$, (5) does not have nonnegative real roots for $\tau>0$. Hence, if (5) has roots with nonnegative real parts they must be complex roots. Moreover these complex roots should be obtained from a pair of complex conjugate roots crossing the imaginary axis. Thus, (5) must have a pair of purely imaginary roots $\lambda= \pm \omega i$ for some $\tau>0$. Without loss of generality we assume that $\omega>0$. Then $\omega$ must be a positive solution of the following equation:

$$
\begin{align*}
& -\omega^{3} i-(b+c+d) \omega^{2}+(b c+b d+c d) \omega i+b c d \\
& -\frac{A_{x} A_{y} \theta \beta_{x} e^{-\mu_{x} \tau} \beta_{y}}{\mu_{x} \mu_{y}}(\cos (\omega \tau)-i \sin (\omega \tau))=0 \tag{7}
\end{align*}
$$

which is equivalent to

$$
\begin{align*}
& -\omega^{3}+(b c+b d+c d) \omega \\
& \quad+\frac{A_{x} A_{y} \theta \beta_{x} e^{-\mu_{x} \tau} \beta_{y}}{\mu_{x} \mu_{y}} \sin (\omega \tau)=0 \\
& -(b+c+d) \omega^{2}+b c d  \tag{8}\\
& -\frac{A_{x} A_{y} \theta \beta_{x} e^{-\mu_{x} \tau} \beta_{y}}{\mu_{x} \mu_{y}} \cos (\omega \tau)=0
\end{align*}
$$

Let $A_{x} A_{y} \theta \beta_{x} e^{-\mu_{x} \tau} \beta_{y} / \mu_{x} \mu_{y}=e$. Hence,

$$
\begin{align*}
\omega^{6}+ & \left(b^{2}+c^{2}+d^{2}\right) \omega^{4}+\left(b^{2} c^{2}+b^{2} d^{2}+c^{2} d^{2}\right) \omega^{2} \\
& +\left(b^{2} c^{2} d^{2}-e^{2}\right)=0 \tag{9}
\end{align*}
$$

Assuming $z=\omega^{2}$, we can obtain

$$
\begin{equation*}
z^{3}+\alpha z^{2}+\beta z+\gamma=0 \tag{10}
\end{equation*}
$$

where $\alpha=b^{2}+c^{2}+d^{2}>0, \beta=b^{2} c^{2}+b^{2} d^{2}+c^{2} d^{2}>0, \gamma=$ $b^{2} c^{2} d^{2}-e^{2}>0$ as $R_{0}<1$.

From [12, Lemma 3.31], if $\alpha \geq 0, \beta>0, \gamma \geq 0$, then (10) has no positive real roots. This implies that (7) does not have positive solution $\omega$ since $R_{0}<1$. Therefore, (5) does not have purely imaginary roots. Consequently, the real parts of all eigenvalues of $E_{0}$ are negative for all positive $\tau$. This indicates that the disease free equilibrium $E_{0}$ is locally asymptotically stable if $R_{0}<1$.

## 4. Hopf Bifurcation Analysis

In this section, we turn to the study of the stability of the endemic equilibrium $E$ when $R_{0}>1$. Notice that $R_{0}>1$ is equivalent to

$$
\begin{equation*}
\tau<\tau^{*}=\frac{1}{\mu_{x}} \ln \frac{A_{x} A_{y} \theta \beta_{x} \beta_{y}}{\mu_{x} \mu_{y}\left(\mu_{x}+\alpha_{x}\right)\left(\mu_{y}+\alpha_{y}\right)\left(\mu_{y}+\theta\right)} \tag{11}
\end{equation*}
$$

The characteristic equation of $E$ is

$$
\begin{align*}
& \lambda^{5}+a_{1} \lambda^{4}+a_{2} \lambda^{3}+a_{3} \lambda^{2}+a_{4} \lambda+a_{5} \\
&=e^{-\lambda \tau}\left(b_{1} \lambda^{4}+b_{2} \lambda^{3}+b_{3} \lambda^{2}+b_{4} \lambda+b_{5}\right) \tag{12}
\end{align*}
$$

where

$$
\begin{gather*}
a_{1}=\mu_{x}+\mu_{y}+b+c+d>0, \\
a_{2}=b c+b d+b \mu_{x}+b \mu_{y}+c d+c \mu_{x}+c \mu_{y} \\
+d \mu_{x}+d \mu_{y}+\mu_{x} \mu_{y}>0 \\
a_{3}=b c d+b c \mu_{x}+b c \mu_{y}+c d \mu_{x} \\
+c d \mu_{y}+d \mu_{x} \mu_{y} \\
a_{4}=b c d \mu_{x}+b c d \mu_{y}+c d \mu_{x} \mu_{y} \\
a_{5}=b c d \mu_{x} \mu_{y} \\
b_{1}=-\beta_{x} y_{i}^{*} e^{-\mu_{x} \tau}<0, \\
b_{3}=-\beta_{x} y_{i}^{*} e^{-\mu_{x} \tau}\left(b \mu_{y}+b d+d \mu_{y}+b c+c d+c \mu_{y}\right)+b c d, \\
b_{4}=-\beta_{x} y_{i}^{*} e^{-\mu_{x} \tau}\left(b d \mu_{y}+b c \mu_{y}+c d \mu_{y}+b c d\right) \\
+b c d\left(\mu_{x}+\mu_{y}\right), \\
b_{5}=-\beta_{x} y_{i}^{*} e^{-\mu_{x} \tau} b c d \mu_{y}+b c d \mu_{x} \mu_{y} .
\end{gather*}
$$

In the following, it can be shown that (12) does not have nonnegative real roots for $\tau>0$. Let

$$
\begin{gather*}
\tilde{a}_{3}=a_{3}-b c d e^{-\lambda \tau}, \\
\widetilde{a}_{4}=a_{4}-b c d\left(\mu_{x}+\mu_{y}\right) e^{-\lambda \tau}, \\
\widetilde{a}_{5}=a_{5}-b c d \mu_{x} \mu_{y} e^{-\lambda \tau} \\
\widetilde{b}_{3}=-\beta_{x} y_{i}^{*} e^{-\mu_{x} \tau}\left(b \mu_{y}+b d+d \mu_{y}+b c+c d+c \mu_{y}\right)<0, \\
\widetilde{b}_{4}=-\beta_{x} y_{i}^{*} e^{-\mu_{x} \tau}\left(b d \mu_{y}+b c \mu_{y}+c d \mu_{y}+b c d\right)<0, \\
\widetilde{b}_{5}=-\beta_{x} y_{i}^{*} e^{-\mu_{x} \tau} b c d \mu_{y}<0 . \tag{14}
\end{gather*}
$$

Note that $\widetilde{a}_{3}>0, \widetilde{a}_{4}>0, \widetilde{a}_{5}>0$ for all $\lambda \geq 0$ and $\tau>0$. We rewrite (12) in the following form:

$$
\begin{align*}
& \lambda^{5}+a_{1} \lambda^{4}+a_{2} \lambda^{3}+\tilde{a}_{3} \lambda^{2}+\tilde{a}_{4} \lambda+\tilde{a}_{5} \\
&=e^{-\lambda \tau}\left(b_{1} \lambda^{4}+b_{2} \lambda^{3}+\widetilde{b}_{3} \lambda^{2}+\widetilde{b}_{4} \lambda+\widetilde{b}_{5}\right) \tag{15}
\end{align*}
$$

It is easy to see that the left-hand side in (15) is positive while the right-hand side is negative for all $\lambda \geq 0$. Then (12) does not have nonnegative real solutions. Now we consider whether or not (12) has purely imaginary solutions.

Suppose $\lambda=\omega i, \omega>0$ for some $\tau>0$, is a root of (12). Then we have

$$
\begin{align*}
\omega^{5} i+ & a_{1} \omega^{4}-a_{2} \omega^{3} i-a_{3} \omega^{2}+a_{4} \omega i+a_{5} \\
= & {[\cos (\omega \tau)-i \sin (\omega \tau)] }  \tag{16}\\
& \times\left(b_{1} \omega^{4}-b_{2} \omega^{3} i-b_{3} \omega^{2}+b_{4} \omega i+b_{5}\right)
\end{align*}
$$

Therefore

$$
\begin{align*}
\omega^{5}-a_{2} \omega^{3}+a_{4} \omega= & \cos (\omega \tau)\left(-b_{2} \omega^{3}+b_{4} \omega\right) \\
& -\sin (\omega \tau)\left(b_{1} \omega^{4}-b_{3} \omega^{2}+b_{5}\right) \\
a_{1} \omega^{4}-a_{3} \omega^{2}+a_{5}= & \cos (\omega \tau)\left(b_{1} \omega^{4}-b_{3} \omega^{2}+b_{5}\right)  \tag{17}\\
& +\sin (\omega \tau)\left(-b_{2} \omega^{3}+b_{4} \omega\right)
\end{align*}
$$

From (17), we obtain

$$
\begin{align*}
\left(a_{1} \omega^{4}\right. & \left.-a_{3} \omega^{2}+a_{5}\right)^{2}+\left(\omega^{5}-a_{2} \omega^{3}+a_{4} \omega\right)^{2}  \tag{18}\\
& =\left(b_{1} \omega^{4}-b_{3} \omega^{2}+b_{5}\right)^{2}+\left(-b_{2} \omega^{3}+b_{4} \omega\right)^{2}
\end{align*}
$$

that is,

$$
\begin{align*}
& \omega^{10}+\left(a_{1}^{2}-2 a_{2}-b_{1}^{2}\right) \omega^{8}+\left(a_{2}^{2}+2 a_{4}-2 a_{1} a_{3}\right. \\
& \left.\quad-b_{2}^{2}+2 b_{1} b_{3}\right) \omega^{6} \\
& +\left(a_{3}^{2}-2 a_{2} a_{4}+2 a_{1} a_{5}+2 b_{2} b_{4}-b_{3}^{2}-2 b_{1} b_{5}\right) \omega^{4} \\
& +\left(a_{4}^{2}-2 a_{3} a_{5}-b_{4}^{2}+2 b_{3} b_{5}\right) \omega^{2}+\left(a_{5}^{2}-b_{5}^{2}\right)=0 \tag{19}
\end{align*}
$$

Let $z=\omega^{2}$ again; we obtain

$$
\begin{equation*}
z^{5}+c_{1} z^{4}+c_{2} z^{3}+c_{3} z^{2}+c_{4} z+c_{5}=0 \tag{20}
\end{equation*}
$$

where

$$
\begin{gather*}
c_{1}=a_{1}^{2}-2 a_{2}-b_{1}^{2} \\
c_{2}=a_{2}^{2}+2 a_{4}-2 a_{1} a_{3}-b_{2}^{2}+2 b_{1} b_{3} \\
c_{3}=a_{3}^{2}-2 a_{2} a_{4}+2 a_{1} a_{5}+2 b_{2} b_{4}-b_{3}^{2}-2 b_{1} b_{5}  \tag{21}\\
c_{4}=a_{4}^{2}-2 a_{3} a_{5}-b_{4}^{2}+2 b_{3} b_{5} \\
c_{5}=a_{5}^{2}-b_{5}^{2}
\end{gather*}
$$

Because (20) is very complex, the roots cannot easily be found. However, we know there are positive roots in some conditions. For example, if $c_{5}<0$, then (20) has at least a positive root, say $z_{0}$, and (19) has at least a positive root $\omega_{0}=\sqrt{z_{0}}$. Consequently, the endemic equilibrium $E$ may lose stability and lead to oscillations because the time delay $\tau>0$. In this case, we will do bifurcation analysis by $\tau$ as bifurcation parameter in the following.

Let $\lambda(\tau)=\xi(\tau)+i \omega(\tau)$ be a root of (12) such that $\xi\left(\tau_{0}\right)=0, \omega\left(\tau_{0}\right)=\omega_{0}\left(\omega_{0}>0\right)$ for some initial value of the bifurcation parameter $\tau_{0}$. From (17) we can obtain

$$
\begin{align*}
& \tau_{j} \\
& \begin{aligned}
=\frac{1}{\omega_{0}} \arccos ( & \left(\left(a_{1} b_{1}-b_{2}\right) \omega_{0}^{8}+\left(b_{4}+a_{2} b_{2}-a_{1} b_{3}-a_{3} b_{1}\right) \omega_{0}^{6}\right. \\
& \left.+\left(-a_{2} b_{4}-a_{4} b_{2}+a_{1} b_{5}+a_{3} b_{3}+a_{5} b_{1}\right) \omega_{0}^{4}\right) \\
& \times\left(\left(b_{1} \omega_{0}^{4}-b_{3} \omega_{0}^{2}+b_{5}\right)^{2}+\left(-b_{2} \omega_{0}^{3}+b_{4} \omega_{0}\right)^{2}\right)^{-1} \\
& +\frac{\left(a_{4} b_{4}-a_{3} b_{5}-a_{5} b_{3}\right) \omega_{0}^{2}+a_{5} b_{5}}{\left.\left(b_{1} \omega_{0}^{4}-b_{3} \omega_{0}^{2}+b_{5}\right)^{2}+\left(-b_{2} \omega_{0}^{3}+b_{4} \omega_{0}\right)^{2}\right)} \\
+\frac{2 j \pi}{\omega_{0}}, \quad j= & 0,1,2, \ldots
\end{aligned}
\end{align*}
$$

Now we can show the transversal condition $\left.(d \operatorname{Re} \lambda(\tau) / d \tau)\right|_{\tau=\tau_{0}} \neq 0$.

Differentiating (12) with respect to $\tau$ yields

$$
\begin{align*}
\left(5 \lambda^{4}+\right. & 4 a_{1} \lambda^{3}+3 a_{2} \lambda^{2}+2 a_{3} \lambda+a_{4}+\tau e^{-\lambda \tau} \\
& \times\left(b_{1} \lambda^{4}+b_{2} \lambda^{3}+b_{3} \lambda^{2}+b_{4} \lambda+b_{5}\right) \\
& \left.-e^{-\lambda \tau}\left(4 b_{1} \lambda^{3}+3 b_{2} \lambda^{2}+2 b_{3}+b_{4}\right)\right) \frac{d \lambda}{d \tau}  \tag{23}\\
= & -\lambda e^{-\lambda \tau}\left(b_{1} \lambda^{4}+b_{2} \lambda^{3}+b_{3} \lambda^{2}+b_{4} \lambda+b_{5}\right)
\end{align*}
$$

Using (12), we obtain

$$
\begin{align*}
\left(\frac{d \lambda}{d \tau}\right)^{-1}= & \left(5 \lambda^{4}+4 a_{1} \lambda^{3}+3 a_{2} \lambda^{2}+2 a_{3} \lambda+a_{4}\right. \\
& +\tau e^{-\lambda \tau}\left(b_{1} \lambda^{4}+b_{2} \lambda^{3}+b_{3} \lambda^{2}+b_{4} \lambda+b_{5}\right) \\
& \left.-e^{-\lambda \tau}\left(4 b_{1} \lambda^{3}+3 b_{2} \lambda^{2}+2 b_{3}+b_{4}\right)\right) \\
\times & \left(-\lambda e^{-\lambda \tau}\left(b_{1} \lambda^{4}+b_{2} \lambda^{3}+b_{3} \lambda^{2}+b_{4} \lambda+b_{5}\right)\right)^{-1} \\
= & \frac{5 \lambda^{4}+4 a_{1} \lambda^{3}+3 a_{2} \lambda^{2}+2 a_{3} \lambda+a_{4}}{-\lambda\left(\lambda^{5}+a_{1} \lambda^{4}+a_{2} \lambda^{3}+a_{3} \lambda^{2}+a_{4} \lambda+a_{5}\right)} \\
& +\frac{4 b_{1} \lambda^{3}+3 b_{2} \lambda^{2}+2 b_{3}+b_{4}}{\lambda\left(b_{1} \lambda^{4}+b_{2} \lambda^{3}+b_{3} \lambda^{2}+b_{4} \lambda+b_{5}\right)}-\frac{\tau}{\lambda} \tag{24}
\end{align*}
$$

Then,

$$
\begin{align*}
& \operatorname{sign}\left\{\frac{d \operatorname{Re} \lambda}{d \tau}\right\}_{\lambda=i \omega_{0}} \\
& =\operatorname{sign}\left\{\operatorname{Re}\left(\frac{d \lambda}{d \tau}\right)^{-1}\right\}_{\lambda=i \omega_{0}} \\
& =\operatorname{sign}\left\{\operatorname{Re}\left[\frac{5 \lambda^{4}+4 a_{1} \lambda^{3}+3 a_{2} \lambda^{2}+2 a_{3} \lambda+a_{4}}{-\lambda\left(\lambda^{5}+a_{1} \lambda^{4}+a_{2} \lambda^{3}+a_{3} \lambda^{2}+a_{4} \lambda+a_{5}\right)}\right]_{\lambda=i \omega_{0}}\right. \\
& \left.+\operatorname{Re}\left[\frac{4 b_{1} \lambda^{3}+3 b_{2} \lambda^{2}+2 b_{3}+b_{4}}{\lambda\left(b_{1} \lambda^{4}+b_{2} \lambda^{3}+b_{3} \lambda^{2}+b_{4} \lambda+b_{5}\right)}\right]_{\lambda=i \omega_{0}}\right\} \\
& =\operatorname{sign}\left\{\operatorname { R e } \left[\left(\left(-2 a_{3} \omega_{0}+4 a_{1} \omega_{0}^{3}\right)\right.\right.\right. \\
& \left.+\left(-3 a_{2} \omega_{0}^{2}+5 \omega_{0}^{4}+a_{4}\right) i\right) \\
& \times\left(\omega _ { 0 } \left(\left(a_{1} \omega_{0}^{4}-a_{3} \omega_{0}^{2}+a_{5}\right)\right.\right. \\
& \left.\left.\left.+\left(\omega_{0}^{5}-a_{2} \omega_{0}^{3}+a_{4} \omega_{0}\right) i\right)\right)^{-1}\right] \\
& \left.+\operatorname{Re}\left[\frac{\left(2 b_{3} \omega_{0}-4 b_{1} \omega_{0}^{3}\right)+\left(3 b_{2} \omega_{0}^{2}-b_{4}\right) i}{\omega_{0}\left(\left(b_{1} \omega_{0}^{4}+b_{5}-b_{3} \omega_{0}^{2}\right)+\left(b_{4} \omega_{0}-b_{2} \omega_{0}^{3}\right) i\right)}\right]\right\} \\
& =\operatorname{sign}\left\{\left(5 \omega_{0}^{8}+\left(4 a_{1}^{2}-8 a_{2}-4 b_{1}^{2}\right) \omega_{0}^{6}\right.\right. \\
& +\left(3 a_{2}^{2}-6 a_{1} a_{3}+6 a_{4}+6 b_{1} b_{3}-3 b_{2}^{2}\right) \omega_{0}^{4} \\
& +\left(2 a_{3}^{2}-4 a_{2} a_{4}+4 a_{1} a_{5}+4 b_{2} b_{4}-2 b_{3}^{2}-4 b_{1} b_{5}\right) \omega_{0}^{2} \\
& \left.+\left(a_{4}^{2}-2 a_{3} a_{5}+2 b_{3} b_{5}-b_{4}^{2}\right)\right) \\
& \times\left(\left(a_{1} \omega_{0}^{4}-a_{3} \omega_{0}^{2}+a_{5}\right)^{2}\right. \\
& \left.\left.+\left(\omega_{0}^{5}-a_{2} \omega_{0}^{3}+a_{4} \omega_{0}\right)^{2}\right)^{-1}\right\} . \tag{25}
\end{align*}
$$

If we denote $z_{0}=\omega_{0}^{2}$, we get

$$
\begin{align*}
\operatorname{sign} & \left\{\frac{d \operatorname{Re} \lambda}{d \tau}\right\}_{\lambda=i \omega_{0}} \\
& =\operatorname{sign}\left\{\frac{5 z_{0}^{4}+4 c_{1} z_{0}^{3}+3 c_{2} z_{0}^{2}+2 c_{3} z_{0}+c_{4}}{\left(a_{1} \omega_{0}^{4}-a_{3} \omega_{0}^{2}+a_{5}\right)^{2}+\left(\omega_{0}^{5}-a_{2} \omega_{0}^{3}+a_{4} \omega_{0}\right)^{2}}\right\} \tag{26}
\end{align*}
$$

Denote $f(z)=z^{5}+c_{1} z^{4}+c_{2} z^{3}+c_{3} z^{2}+c_{4} z+c_{5}$. Suppose $\omega_{0}$ is the largest positive simple root of (19); from [12, Lemma 3.32 and Theorem 3.32], we have

$$
\begin{equation*}
\left.\frac{d f(z)}{d z}\right|_{z=z_{0}}=5 z_{0}^{4}+4 c_{1} z_{0}^{3}+3 c_{2} z_{0}^{2}+2 c_{3} z_{0}+c_{4}>0 \tag{27}
\end{equation*}
$$



Figure 2: The trajectories of $x_{i}$ and $y_{i}$ occur oscillations when $\tau_{0}=3$.

Thus,

$$
\begin{align*}
& \operatorname{sign}\left\{\frac{d \operatorname{Re} \lambda}{d \tau}\right\}_{\lambda=i \omega_{0}} \\
& =\operatorname{sign}\left\{\left(\left.\frac{d f(z)}{d z}\right|_{z=z_{0}}\right)\right. \\
& \\
& \quad \times\left(\left(a_{1} \omega_{0}^{4}-a_{3} \omega_{0}^{2}+a_{5}\right)^{2}\right.  \tag{28}\\
& \left.\left.\quad+\left(\omega_{0}^{5}-a_{2} \omega_{0}^{3}+a_{4} \omega_{0}\right)^{2}\right)^{-1}\right\}=+1
\end{align*}
$$

Summarizing the above analysis, we have the following results.

Theorem 3. If $R_{0}>1, c_{5}<0$ and $\omega_{0}$ is the largest positive simple root of (19), a Hopf bifurcation occurs around the endemic equilibrium $E$ of the delayed model (2).

## 5. Numerical Simulations

Based on the observation data from the investigation of Tongcheng Schistosomiasis Control Station in Anhui province, we estimated transmission rates in our model. Also according to the previous papers $[7-9,11,13]$, we choose the parameter values in Table 1. Thus, $R_{0}>1, \tau^{*}=327$, and $c_{5}<0$ when $\tau=0.1$.

Note that the bifurcation parameter $\tau_{0}=3$ at this time. We performed some simulations and obtained Figure 2. From Figure 2 we can see that Hopf bifurcation can occur when $\tau_{0}=3$. This implies that schistosomiasis will break out in about three years after flood. It is also in accord with the investigation of Tongcheng Schistosomiasis Control Station. This phenomenon is also in accord with the report of the whole Anhui province [1]. From our theoretical results and

Table 1: Values of parameters.

| Parameters | Values (per capita per year) | References |
| :--- | :---: | :---: |
| $A_{x}$ | 6 | $[8,9]$ |
| $\mu_{x}$ | 0.014 | $[8,9,11]$ |
| $\alpha_{x}$ | $10^{-5}$ | $[8,9]$ |
| $\beta_{x}$ | 0.003 | Estimated |
| $A_{y}$ | 100 | $[8,9]$ |
| $\mu_{y}$ | 0.3 | $[8,9,11]$ |
| $\alpha_{y}$ | 0.01 | $[9]$ |
| $\beta_{y}$ | 0.001 | Estimated |
| $\theta$ | 9.125 | $[13]$ |

the reports we can see that schistosomiasis will break out in about the third year after a flood. Hence, we can get the result that the delayed effect of flood may be caused by the incubation period of schistosome in the infected human.

## 6. Discussion

In this paper, based on the observation data in Tongcheng Schistosomiasis Control Station in Anhui province we have modified our previous model by including a time delay that describes the incubation period of schistosome within infected human. We define the basic reproduction number $R_{0}$ according to biological meanings and give the existence of the disease free equilibrium and the endemic equilibrium. We find that, if $R_{0}<1$, then the disease free equilibrium is locally asymptotically stable. However, the stability of the unique endemic equilibrium may be changed under some condition even if the basic reproduction number is larger than one. The results imply that the time delay can destabilize the system and periodic solutions can arise by Hopf bifurcation.

Numerical simulations imply that schistosomiasis will break out in about three years after flood. Furthermore the
observation data show that schistosomiasis will be the most serious in about the third year after flood. From Figure 1, we can see that the number of patients and snails did not greatly change in 1998 and 1999. However, in 2001 the number of patients became about 5 times that of 1998, and the area of snails became about two times that of 1998. In our simulations, there is a little difference. Our results are higher than the observation data. We think the reason may be that after flood the government dispatched a large number of manpower and material resources to control the spread of the disease. In summary, our theoretical results are in accord with the investigation of Tongcheng Schistosomiasis Control Station and the report of Anhui Province Institute of Schistosomiasis for the whole Anhui province. Hence, we can obtain the result that after flood the delayed effect of flood may be caused by the incubation period of schistosome in the definitive host. Furthermore, the period of outbreak is about three years after flood in Anhui province.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This research is supported by National Natural Science Foundation of China (11126177 and 11071011), Natural Science Foundation of Anhui Province (1208085QA15), and the Foundation for Young Talents in College of Anhui Province (2012SQRL021), as well as National Scholarship Foundation of China, Funding Project for Academic Human Resources Development in Institutions of Higher Learning under the Jurisdiction of Beijing Municipality (PHR201107123), Doctoral Fund of Ministry of Education of China (20113401110001, 20103401120002), the Key Natural Science Foundation of the Anhui Higher Education Institutions of China (KJ2009A49), and Students Research Training Program of Anhui University. The authors would like to thank anonymous reviewers for very helpful suggestions which improved greatly this paper.

## References

[1] J. H. Ge, S. Q. Zhang, T. P. Wang et al., "Efects of flood on the prevalence of schistosomiasis in Anhui province in 1998," Journal of Tropical Diseases and Parasitology, vol. 2, pp. 131-134, 2004.
[2] S. B. Mao, Biology of Schistosome and Control of Schistosomiasis, People's Health Press, Beijing, China, 1990.
[3] X. N. Zhou, J. G. Guo, X. H. Wu et al., "Epedemiology of schistosomiasis in the people's republic of China," Emerging Infectious Diseases, vol. 13, no. 10, pp. 1470-1476, 2004.
[4] C. Castillo-Chavez, Z. Feng, and D. Xu, "A schistosomiasis model with mating structure and time delay," Mathematical Biosciences, vol. 211, no. 2, pp. 333-341, 2008.
[5] A. D. Barbour, "Modeling the transmission of schistosomiasis: an introductory view," The American Journal of Tropical Medicine and Hygiene, vol. 55, no. 5, pp. 135-143, 1996.
[6] D. Cioli, "Chemotherapy of schistosomiasis: an update," Parasitology Today, vol. 14, no. 10, pp. 418-422, 1998.
[7] Z. Feng, J. Curtis, and D. J. Minchella, "The influence of drug treatment on the maintenance of schistosome genetic diversity," Journal of Mathematical Biology, vol. 43, no. 1, pp. 52-68, 2001.
[8] Z. Feng, A. Eppert, F. A. Milner, and D. J. Minchella, "Estimation of parameters governing the transmission dynamics of schistosomes," Applied Mathematics Letters, vol. 17, no. 10, pp. 1105-1112, 2004.
[9] Z. Feng, C.-C. Li, and F. A. Milner, "Schistosomiasis models with two migrating human groups," Mathematical and Computer Modelling, vol. 41, no. 11-12, pp. 1213-1230, 2005.
[10] S. Liang, D. Maszle, and R. C. Spear, "A quantitative framework for a multi-group model of schistosomiasis japonicum transmission dynamics and control in Sichuan, China," Acta Tropica, vol. 82, no. 2, pp. 263-277, 2002.
[11] L. Qi, J.-A. Cui, Y. Gao, and H. Zhu, "Modeling the schistosomiasis on the islets in Nanjing," International Journal of Biomathematics, vol. 5, no. 4, Article ID 1250037, 17 pages, 2012.
[12] H.-M. Wei, X.-Z. Li, and M. Martcheva, "An epidemic model of a vector-borne disease with direct transmission and time delay," Journal of Mathematical Analysis and Applications, vol. 342, no. 2, pp. 895-908, 2008.
[13] E. J. Allen and H. D. Victory Jr., "Modelling and simulation of a schistosomiasis infection with biological control," Acta Tropica, vol. 87, no. 2, pp. 251-267, 2003.

## Research Article

# Estimation of Hospital Potential Capacity and Basic Reproduction Number 

Fei Wang, ${ }^{1}$ Linhua Wang, ${ }^{2}$ and Peng Wang ${ }^{1}$<br>${ }^{1}$ Information Department, Southwest Hospital, Third Military Medical University, Chongqing 400038, China<br>${ }^{2}$ Information Department, Daping Hospital, Third Military Medical University, Chongqing 400042, China

Correspondence should be addressed to Fei Wang; wangfxnyy@sina.com
Received 7 January 2014; Accepted 5 February 2014; Published 6 March 2014
Academic Editor: Weiming Wang
Copyright © 2014 Fei Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

In order to reflect the population covered by institutional medical services, the concept of hospital potential capacity is proposed and a formula for its estimation is developed based on a population dynamic model. Using the collected data on hospital outpatient and inpatient services and the demographical information on Chongqing as an example, the demand for medical resource allocation in Chongqing is dynamically estimated. Moreover, the proposed formula is also useful in the estimation of the basic reproduction number in epidemiology. The results can be contributed to the improvement of decision-making in the allocation of medical resources and the evaluation of the interventions and control efforts of the infectious disease.


## 1. Introduction

Hospital potential capacity reflects the population covered by the medical institutes, which is a crucial indicator to evaluate publicly served medical resource allocation. As the patient flow of a hospital is a complicated dynamic process, it is rather difficult to estimate the hospital potential capacity effectively by a simple statistical analysis of hospital outpatient and inpatient numbers.

Currently, there is no particular prototype for the estimation of hospital potential capacity, and most people are more concerned about the bed number or the bed capacity. However, bed planning and management are carried out under uncertainty and verification of environmental assessments and limited resources, and the spreadsheet calculations determined by simple planning and management capacity often underestimate the true demand for beds [1,2]. On the other hand, the dynamic capacity of hospitals can be determined to a certain extent by studying the patients' length of stay in hospitals, simulating relevant activities for patient flow in hospitals, and employing mathematical methods such as mixed exponential distributions, compartmental modelling, and simulation modelling to conduct integrated data analysis [3]. For example, the demand for beds and the waiting time
for bed appointments can be estimated by analyzing a heart surgery patient's length of stay in an intensive care unit [4].

A new planning paradigm for improving the hospital capacity is to improve hospital resource allocation through the simulation of patient flow so as to avoid possible bottlenecks, which is also the new direction of research, and on which a growing number of research studies are focusing currently [5]. If the entry or the exit of patients under special medical conditions is considered as the patient flow and all the in-between activities or services require medical resources, this type of patient flow can be described as a resource network with basic network features. Hospital administrators can predict and assess the queuing model by determining the network features and, therefore, can improve the control of patient in-flows and enhance the resource utilization rate [6].

From a point of view of epidemiology, the basic reproduction number $R_{0}$ is commonly used to characterize disease transmissibility during an epidemic, which means the mean number of secondary cases of disease caused by a typical infected individual in a totally susceptible population in its lifetime without any control policies [7-9]. Now, many different methods have been proposed to estimate the basic reproduction number $R_{0}$ from the surveillance data [9-11].

For example, using the final epidemic size, [10] introduced the following calculation formula:

$$
\begin{equation*}
R_{0}=\frac{N-1}{C} \sum_{i=S_{f}+1}^{S_{0}} \frac{1}{i}, \tag{1}
\end{equation*}
$$

where $N$ is the population size, $C$ is the total number of cases in the given disease, and $S_{0}$ and $S_{f}$ are the numbers of susceptible individuals in the population at the start and end of the given disease, respectively. In addition, let $u_{0}=$ $S_{0} / N$ and $u_{\infty}=S_{f} / N$. Reference [12] obtained the following formula to estimate the basic reproduction number $R_{0}$ of the disease:

$$
\begin{equation*}
R_{0}=\frac{1}{u_{0}-u_{\infty}}\left[\ln u_{0}-\ln u_{\infty}\right] \tag{2}
\end{equation*}
$$

Note that the parameter $N$ in (1) is equal to the hospital potential capacity to a certain extent. The estimation of the hospital potential capacity has very definite value in epidemiology.

In this study, by use of the surveillance data from Southwest Hospital and the demographic data of Chongqing city, we proposed a formula to estimate the hospital potential capacity. Chongqing is located in the southeast of inland China, on the middle-upper reaches of the Yangtze River, with a total population of more than 30 million. As a pilot zone for the national urban and rural comprehensive reform, Chongqing has a distinctive feature of large urban and rural areas with unbalanced economic and social developments, especially uneven distribution of medical resources. According to the statistics in 2012, there are 18 top-rated hospitals, among which six are large general hospitals with more than 1500 treatment tables, such as Southwest Hospital, the First Affiliated Hospital of Chongqing Medical University, Three Gorges Central Hospital, Xinqiao Hospital, Daping Hospital, and the Second Affiliated Hospital of Chongqing Medical University, mostly situated in the main city areas [13]. Southwest Hospital is the largest integrated medical and health institution, which provides medical treatment, education, and research facilities, accommodates more than 3000 beds with an average of over 10,000 outpatients per day and over 100,000 inpatients per year. Therefore, the study of population covered by the medical services offered by Southwest Hospital is regionally representative, and it provides important guidance to the allocation of medical resources in Chongqing.

This paper first constructs a population dynamics model, which reflects the hospital potential capacity on base of compartmental modeling, then utilizes the dynamic feature of the model and the data of the outpatients and inpatient numbers in Southwest Hospital from 2000 to 2012 to dynamically estimate the potential capacity for the same period. Based on the estimated potential capacity together with the surveillance data from the hospital on two common infectious diseases, viral hepatitis and brothers mouth disease, the corresponding basic reproduction number of these disease is dynamically estimated. By combining the annual demographic data in Chongqing for this time span, we dynamically estimate the annual demand of medical resource allocation in Chongqing


Figure 1: Flow chart of the population dynamic model.
from the estimated potential capacity. Considering the fact that Southwest Hospital is the largest medical institution in Chongqing (including staff, equipment, etc.), we collect further statistical data from another big general hospitalDaping Hospital-in order to adjust the data of Southwest Hospital. In this way, we can estimate the demand of annual medical resource allocation more reasonably and provide suggestions on the medical resource allocation in Chongqing, which enhance the decision-making in the allocation of medical resources to achieve the balance of demands-resource.

## 2. Methods and Data

2.1. Model Description. Intuitively, we divide the population into three types: healthy persons, outpatients, and inpatients. They are called the three compartments and are denoted by $x, y, z$, respectively. Generally, the population in the hospital service areas is relatively stable; therefore, the influences of birth and death in such areas are negligible. In other words, we can assume that the input and output of the population in this area remain the same. Hence, the population flow can be described as the following block diagram (Figure 1). Here $a_{1}$ represents the disease rate of healthy persons, $a_{2}$, the cure rate of outpatients, $a_{3}$, the hospitalizing rate of outpatients, and $a_{4}$, the cure rate of inpatients. According to the biological significance of the parameters, all $a_{i}, i=1,2,3,4$, are positive and $a_{2}+a_{3}=1$.

From Figure 1, we have the following differential equation model:

$$
\begin{gather*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=-a_{1} x+a_{2} y+a_{4} z \\
\frac{\mathrm{~d} y}{\mathrm{~d} t}=a_{1} x-\left(a_{2}+a_{3}\right) y  \tag{3}\\
\frac{\mathrm{~d} z}{\mathrm{~d} t}=a_{3} y-a_{4} z
\end{gather*}
$$

Clearly, the population in a hospital service area is $N=x+$ $y+z$. According to model (3), we have $d N / d t=0$, which means that the hospital potential capacity remains constant. Note that $x=N-y-z$. Model (3) can be simplified as

$$
\begin{gather*}
\frac{\mathrm{d} y}{\mathrm{~d} t}=a_{1} N-\left(1+a_{1}\right) y-a_{1} z \triangleq F_{1}(y, z),  \tag{4}\\
\frac{\mathrm{d} z}{\mathrm{~d} t}=a_{3} y-a_{4} z \triangleq F_{2}(y, z) .
\end{gather*}
$$

Here, we used $a_{2}+a_{3}=1$.
2.2. Data. The demographic data of Chongqing is taken from the Chongqing Statistical Yearbook 2012 [14]. The hospital outpatient and inpatient data are extracted from the statistical reports of medical services offered by Southwest Hospital and Daping Hospital from 2000 to 2012. The data of viral hepatitis and hand-foot-and-mouth disease are extracted from Southwest Hospital between the years of 2003 and 2012. The sources of these data are valid and reliable, including the overall numbers of outpatients and inpatients. Among them, the data for Southwest Hospital are used for the dynamic estimation of potential capacity $N$ and the basic reproduction number $R_{0}$, whereas the data for Daping Hospital are mainly used for the estimation of percentages of outpatients from other hospitals in comparison to Southwest Hospital for more reasonable results in estimated demand of allocation of medical resources in Chongqing.

## 3. Results

3.1. Dynamics of Model (4). Let $F_{1}=0$ and $F_{2}=0$. It is easy to obtain a unique positive equilibrium $\left(y^{*}, z^{*}\right)$ in model (4), where

$$
\begin{equation*}
y^{*}=\frac{a_{1} a_{4} N}{\left(1+a_{1}\right) a_{4}+a_{1} a_{3}}, \quad z^{*}=\frac{a_{1} a_{3} N}{\left(1+a_{1}\right) a_{4}+a_{1} a_{3}} \tag{5}
\end{equation*}
$$

The Jacobian matrix $J$ of model (4) at the equilibrium is

$$
J=\left[\begin{array}{cc}
-\left(1+a_{1}\right) & -a_{1}  \tag{6}\\
a_{3} & -a_{4}
\end{array}\right]
$$

After a simple calculation, we find that the determinant of the Jacobian matrix $J$ is positive, and its trace is negative. Thus, the unique equilibrium $\left(y^{*}, z^{*}\right)$ is globally asymptotically stable based on the Routh-Hurwitz criterion and linearity of the model [15].

### 3.2. Estimation of the Hospital Potential Capacity and the Basic

 Reproduction Number. According to the dynamics of model (4), the hospital outpatients and inpatients tend towards the positive equilibrium (5). From the collected data for the monthly outpatients in Chongqing Southwest Hospital from 2000 to 2012, we see that the monthly outpatient numbers are relatively stable. Therefore, we use the number of annual outpatients for the estimation of $y^{*}$ at the equilibrium. We then assume that everyone gets sick at least once per year; that is, $a_{1}=1$. The annual parameters $a_{3}$ and $a_{4}$ can be estimated from the collected data. From the expression of $y^{*}$ in (5), we can determine the annual hospital potential capacity$$
\begin{equation*}
N=\frac{y^{*}\left(\left(1+a_{1}\right) a_{4}+a_{1} a_{3}\right)}{a_{1} a_{4}} \tag{7}
\end{equation*}
$$

Using formula (7) and the data from Southwest Hospital, we have the dynamical estimation of the potential capacity $N$ (corresponding to the no adjust term in Figure 2(a)). Based on formula (2) and the surveillance data of the viral hepatitis and hand-foot-and-mouth disease, the corresponding basic reproduction number $R_{0}$ of the diseases is further obtained
(Figure 3). From Figure 3, we can see that almost all basic reproduction numbers are greater than unity.

Furthermore, from the estimated annual potential capacity results and the annual demographic statistics of Chongqing, we can estimate the numbers of medical institutions required to meet the medical and health demand of population in Chongqing (corresponding to the no adjust term in Figure 2(b)). As seen from Figure 2, the population covered by the medical services of Southwest Hospital increases annually, and this rate of increase is much faster than the rate of increase in the population of Chongqing. Thus, the number of medical institutions required decreases each year.
3.3. Adjustment of the Number of Hospitals Required. In 2012, the medical and health demand of the whole population in Chongqing could be satisfied with six medical institutions of similar scale to Southwest Hospital (Figure 2(b)). However, there are 18 top-rated hospitals in the city [13]. Does it indicate that the present number of medical institutions is already saturated? Because the Southwest Hospital is the largest one in the region, no matter in the software or hardware facilities, but also the number of outpatients and inpatients compared with medical institutions. The collected data of the other large general hospital-Daping Hospital-suggest that the increasing rate of outpatients is still much slower than that of Southwest Hospital (Figure 4), with the average number of outpatients constituting only $45 \%$ that of Southwest Hospital. Thus, in this section, considering the imbalance of resources and service ability between different medical institutions, we adjust the annual number of outpatients of Southwest Hospital in order to obtain a better estimation of the annual demand of medical resources in Chongqing.

With respect to the statistical data of Daping Hospital, we consider $20 \%$ and $40 \%$ as correction coefficients in order to adjust the number of outpatients in Southwest Hospital while keeping the other parameters unchanged. Thus, we determine the hospital potential capacity and the number of medical institutions required using these coefficients, and these results are presented in Figure 2 (corresponding to the $20 \%$ and $40 \%$ adjust term in Figure 2).

## 4. Discussion

Because Southwest Hospital is the largest individual general medical institution in Chongqing, it has the largest annual outpatient and inpatient numbers. The results, presented in this paper and calculated from the dynamic analysis of data from Southwest Hospital, indicate the number of medical resource allocations for the same scale as that of Southwest Hospital. Under the same conditions, these numbers could be taken as the maximum number of allocations for medical resources. However, the data obtained from other large medical institutions mentioned in this paper cannot reach such level. We can assume that this sequence of data is in a descending order. Thus, in order to obtain realistic results, we adjust the data of Southwest Hospital using the correction coefficients. For instance, in 2012, the number of medical institutions required for a coefficient of $40 \%$ for Southwest


Figure 2: Illustration of the potential capacity and the number of hospital required under different correction coefficients. (a) Potential capacity, and (b) number of hospital required.


Figure 3: Illustration of the basic reproduction number for viral hepatitis and brothers mouth disease between the years of 2003 and 2012.


Figure 4: Comparison of outpatients in Chongqing Southwest Hospital and Chongqing Daping Hospital.
conclude that the existing medical institutions in Chongqing still cannot meet the demand of the whole population.

Regarding the estimation of the basic reproduction number $R_{0}$, such as formula (1) or (2), we can use the proposed formula (7) to estimate the requisite population size. Thus, it is a meaningful study to combine the formulae (2), (7) with clinical surveillance data during an epidemic to obtain the estimation of the basic reproduction number. For two
common infectious diseases, such as viral hepatitis and hand-foot-and-mouth disease, the estimated results indicate that almost all $R_{0}$ are greater than unity. Note that $R_{0}>1$ means that the disease is endemic in the area. Therefore, the incidence will not be significant in the near future with the current status. This will help us to understand the trend of the epidemic and has potential benefits in evaluating different interventions and control efforts [16, 17]. Our conclusion shows $R_{0}$ is generally larger as the infectious disease is caused by the enteric viruses, and happens to children. More stricter and effective control efforts are necessary in control of the epidemic. In order to control the epidemic, we need to take stricter and more effective control efforts.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

This work is partially supported by the National High-tech Research and Development Program (863 Program) (no. 2012AA02A616).

## References

[1] J. Appleby, "The hospital bed: on its way out?" British Medical Journal, vol. 346, no. 11, p. f1563, 2013.
[2] P. R. Harper and A. K. Shahani, "Modelling for the planning and management of bed capacities in hospitals," Journal of the Operational Research Society, vol. 53, no. 1, pp. 11-18, 2002.
[3] A. Marshall, C. Vasilakis, and E. El-Darzi, "Length of staybased patient flow models: recent developments and future directions," Health Care Management Science, vol. 8, no. 3, pp. 213-220, 2005.
[4] S. Gallivan, M. Utley, T. Treasure, and O. Valencia, "Booked inpatient admissions and hospital capacity: mathematical modelling study," British Medical Journal, vol. 324, no. 5, pp. 280-282, 2002.
[5] B. Rechel, S. Wright, J. Barlow, and M. McKee, "Hospital capacity planning: from measuring stocks to modelling flows," Bulletin of the World Health Organization, vol. 88, no. 8, pp. 632636, 2010.
[6] M. J. Côté, "Understanding patient flow," Decision Line, vol. 31, no. 1, pp. 8-10, 2000.
[7] O. Diekmann, J. A. Heesterbeek, and J. A. Metz, "On the definition and the computation of the basic reproduction ratio R0 in models for infectious diseases in heterogeneous populations," Journal of Mathematical Biology, vol. 28, no. 4, pp. 365-382, 1990.
[8] P. van den Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," Mathematical Biosciences, vol. 180, pp. 29-48, 2002.
[9] A. Cori, N. M. Ferguson, C. Fraser, and S. Cauchemez, "A new framework and software to estimate time-varying reproduction numbers during epidemics," The American Journal of Epidemiology, vol. 178, no. 9, pp. 1505-1512, 2013.
[10] E. Vynnycky, A. Trindall, and P. Mangtani, "Estimates of the reproduction numbers of Spanish influenza using morbidity data," International Journal of Epidemiology, vol. 36, no. 4, pp. 881-889, 2007.
[11] H. F. Huo, S. J. Dang, and Y. N. Li, "Stability of a two-strain tuberculosis model with general contact rate," Abstract and Applied Analysis, vol. 2010, Article ID 293747, 31 pages, 2010.
[12] S. Liao, W. Yang, and X. Chen, "The basic reproduction number for the cholera outbreak," Journal on Numerical Methods and Computer Applications, vol. 33, no. 3, pp. 189-197, 2012.
[13] Health and Family Planning Commission of Chongqing, http:// www.cqwsj.gov.cn/jyzn/yyxx/2012-8/10653.html.
[14] Chongqing Municipal Bureau of Statistics, http://www.cqtj.gov .cn/tjnj/2012/indexch.htm.
[15] J. Y. Zhang and B. Y. Feng, Geometric Theory of Ordinary Differential Equations and Bifurcation Problems, Peking University Press, Beijing, China, 2000 (Chinese).
[16] C. Fraser, S. Riley, R. M. Anderson, and N. M. Ferguson, "Factors that make an infectious disease outbreak controllable," Proceedings of the National Academy of Sciences of the United States of America, vol. 101, no. 16, pp. 6146-6151, 2004.
[17] N. M. Ferguson, D. A. T. Cummings, C. Fraser, J. C. Cajka, P. C. Cooley, and D. S. Burke, "Strategies for mitigating an influenza pandemic," Nature, vol. 442, no. 7101, pp. 448-452, 2006.

## Research Article

# Hopf Bifurcation Analysis in a Modified Price Differential Equation Model with Two Delays 

Yanhui Zhai, Ying Xiong, and Xiaona Ma<br>School of Science, Tianjin Polytechnic University, Tianjin 300387, China<br>Correspondence should be addressed to Ying Xiong; 542790840@qq.com

Received 14 November 2013; Accepted 13 January 2014; Published 3 March 2014
Academic Editor: Weiming Wang
Copyright © 2014 Yanhui Zhai et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

The paper investigates the behavior of price differential equation model based on economic theory with two delays. The primary aim of this thesis is to provide a research method to explore the undeveloped areas of the price model with two delays. Firstly, we modify the traditional price model by considering demand function as a downward opening quadratic function, and supply and demand functions both depending on the price of the past and the present. Then the price model with two delays is established. Secondly, by considering the price model with one delay, we get the stable interval. Regarding another delay as a parameter, we studied the linear stability and local Hopf bifurcation. In addition, we pay attention to the direction and stability of the bifurcating periodic solutions which are derived by using the normal form theory and center manifold method. Afterwards, the study turns to simulate the results through numerical analysis, which shows that the provided method is valid.


## 1. Introduction

Recent developments in mathematical economics and in problems of business administration have led to extensive use of differential equation model. Bifurcations and chaos always show in the contemporary literature of economics as basic concepts. The paper of Shuhe [1] was the pioneering work in studying price differential equation model which provides a dynamic system to investigate sectoral dynamics of an economy phenomenon. Since demand depends on the past price, further study on price model with delay can be found in [2]. Reference [3] focused on the phenomenon of bifurcation which forms an integral part of qualitative approach to study dynamical systems. Similarly, [4-6], which investigated the local Hopf bifurcation and the existence of periodic solutions of price model, had important consequences for theoretical and empirical model building in economics. Reference [7] provided a brief survey of the literature of bifurcation detections in economic models. The existence of different bifurcate parameters leads to chaos and the causes of complicated phenomenon were argued in [8]. In short, few people studied price model with delay; what is more, no results involving price model with two delays have occurred.

In order to illustrate the economic phenomena with price varying accurately, a reasonable mathematical model of price is needed. Thus, we introduce a traditional price differential equation model in [1]:

$$
\begin{equation*}
\frac{d^{2} P(t)}{d t^{2}}=\mu \delta(P(t)) \frac{d P(t)}{d t}-\mu b_{0} P(t)+\mu d_{0}-\mu g_{0} \tag{1}
\end{equation*}
$$

The meaning of parameters refers to [1]. We modify the traditional price differential equation model by considering the following factors.

Firstly, we denote the correlation coefficient between demand and price rising rate by $\delta(P(t))$. We consider demand function as a downward opening quadratic function and supply function as a linear function:

$$
\begin{equation*}
\delta(P(t))=b(P(t)+\beta)^{2}+C_{0}, \quad b<0, \beta<0, C_{0}<0 \tag{2}
\end{equation*}
$$

Secondly, according to the cobweb theory, since manufacturers need a production cycle time from obtaining market information to adjust the production line, the role of price adjustment lags on the supply function of time. The purchase of the consumers also depends on the price change and
decision whether to buy or not. Then, supply and demand functions both depend on the price of the past and the present. We introduce two delays $\tau_{1}, \tau_{2}$ to denote supply and demand functions, which depend on the price of the past, respectively. A differential system with two delays for price differential equation model is transformed into the following form:

$$
\begin{align*}
\frac{d^{2} P(t)}{d t^{2}}= & \mu\left[b(P+\beta)^{2}+C_{0}\right] \frac{d P\left(t-\tau_{2}\right)}{d t}  \tag{3}\\
& -\mu b_{0} P(t)-\mu a P\left(t-\tau_{1}\right)+\mu\left(d_{0}-g_{0}\right) .
\end{align*}
$$

Let $P(t)=x(t), d P(t) / d t=y(t)$, and notice that supply and demand functions both depend on the price of the past and the present; then the model is described by the following autonomous system:

$$
\begin{gather*}
\dot{x}(t)=d y(t)+k y\left(t-\tau_{2}\right), \\
\dot{y}(t)=\mu\left[b(x(t)+\beta)^{2}+C_{0}\right] y\left(t-\tau_{2}\right)-\mu b_{0} x(t)  \tag{4}\\
-\mu a x\left(t-\tau_{1}\right)+\mu\left(d_{0}-g_{0}\right),
\end{gather*}
$$

where $d, k$ refers to [2]. $x(t)$ is the price at time $t$, and $y(t)$ is the amount of supply at time $t ; \mu>0, b_{0}>0, a>0, b<0$, $\beta<0, C_{0}<0$.

Different from the previous work in [1, 2, 4], the purpose of this paper is to investigate the stability of the Hopf bifurcation and the direction of bifurcation periodic solution of a price differential equation model with two delays. The structure of the paper is as follows. In Section 2, linear stability and local Hopf bifurcations are studied by using qualitative methods. In Section 3, we regard $\tau_{2}$ as bifurcation parameter and consider (4) with $\tau_{1}$ in its stable interval; the direction of Hopf bifurcation and the stability of the bifurcation periodic solutions are derived by using the normal form theory and center manifold method. Afterwards, the presented numerical simulations in Section 4 have demonstrated the theoretical analysis.

## 2. Local Stability and Hopf Bifurcation

Obviously, system (4) always has an equilibrium $E^{*}=$ $\left(x^{*}, y^{*}\right)=\left(\left(d_{0}-g_{0}\right) /\left(b_{0}+a\right), 0\right)$. In the following, we will investigate the effect of the delay $\tau_{1}, \tau_{2}$ on the dynamics of system (4).

Let $u(t)=x(t)-x^{*}, v(t)=y(t)-y^{*}$; the linearization of system (4) at zero steady state is

$$
\begin{gather*}
\dot{u}(t)=d v(t)+k v\left(t-\tau_{2}\right), \\
\dot{v}(t)=\mu\left[b\left(P_{0}+\beta\right)^{2}+C_{0}\right] v\left(t-\tau_{2}\right)  \tag{5}\\
-\mu b_{0} u(t)-\mu a u\left(t-\tau_{1}\right) .
\end{gather*}
$$

For convenience, as in literature of [5], we denote $A=$ $\left.-\mu\left[b\left(P_{0}+\beta\right)^{2}+C_{0}\right)\right]>0, B=\mu b_{0}>0, C=\mu a>0$. Then (5) become

$$
\begin{gather*}
\dot{u}(t)=d v(t)+k v\left(t-\tau_{2}\right), \\
\dot{v}(t)=-A v\left(t-\tau_{2}\right)-B u(t)-C u\left(t-\tau_{1}\right) . \tag{6}
\end{gather*}
$$

The corresponding characteristic equation is

$$
\begin{equation*}
\lambda^{2}+A e^{-\lambda \tau_{2}} \lambda+B k e^{-\lambda \tau_{2}}+C d e^{-\lambda \tau_{1}}+C k e^{-\lambda\left(\tau_{1}+\tau_{2}\right)}=0 . \tag{7}
\end{equation*}
$$

To study the stability of equilibrium $E^{*}$ of (4) and Hopf bifurcation, it is sufficient to analyze the distribution of the roots of (7). It is stable if all roots of (7) have negative real parts and unstable if one root has positive real part. In order to study the characteristic (7) with two delays, we first consider (7) with one delay. Without loss of generality, we choose $\tau_{1}$ as a parameter and employ Rouche's theorem in Cooke and Grossman [9]; we will find the stable interval for $\tau_{1}$. Then we consider (7) with $\tau_{1}$ in its stable intervals. Using Rouche's theorem again, regard $\tau_{2}$ as a parameter; we will find the stable interval (depends on $\tau_{1}$ ) for $\tau_{2}$. Then we obtain the stable interval for system (4).

Now we analyse the case when $\tau_{2}=0$; (7) becomes

$$
\begin{equation*}
\lambda^{2}+A \lambda+B k+C d e^{-\lambda \tau_{1}}+C k e^{-\lambda \tau_{1}}=0 . \tag{8}
\end{equation*}
$$

Using a procedure similar to [10], we make some hypotheses as follows:
(H1) $A^{2} \leq 4[B k+C d+C k]$,
(H2) $A^{2}>4[B k+C d+C k]$,
(H3) $B^{2} k^{2}<C^{2}(d+k)^{2}$,
(H4) $B^{2} k^{2}>C^{2}(d+k)^{2}, 2 B k>A^{2}$, and $\left(-2 B k+A^{2}\right)^{2}>$ $4\left[B^{2} k^{2}-C^{2}(d+k)^{2}\right]$,
(H5) neither (H3) nor (H4).
Obviously, when $\tau_{1}=\tau_{2}=0$, (4) becomes a system of ODE. Under the hypothesis (H1), all roots of (7) have negative real parts if and only if $A>0$; under the hypothesis (H2), all roots of (8) have negative real parts if and only if $A>0$ and $B k+$ $C d+C k>0$. Above all, either (H1) or (H2) holds for $\tau_{1}=$ $\tau_{2}=0$, and all roots of (7) have negative real parts.

Applying the lemma in [9] again, we obtain the following results.

Lemma 1. For (8), one has the following:
(i) if(H3) holds and $\tau_{1}=\tau_{1, n}^{(1)}$, then (8) has a pair of purely imaginary roots $\pm i \omega_{+}$;
(ii) if(H4) holds and $\tau_{1}=\tau_{1, n}^{(1)}\left(\right.$ res. $\left.\tau_{1}=\tau_{1, n}^{(2)}\right)$ then (8) has a pair of imaginary roots of $\pm i \omega_{+}$(res. $\pm i \omega_{-}$);
(iii) if (H5) holds and $\tau_{2}>0$, then (8) has no purely imaginary root, where

$$
\begin{align*}
\omega_{ \pm}^{2}= & \frac{2 B k-A^{2}}{2}  \tag{9}\\
& \pm\left[\frac{1}{4}\left(-2 B k+A^{2}\right)^{2}-B^{2} k^{2}+C^{2}(d+k)^{2}\right]^{1 / 2} \\
& \tau_{1, n}^{(1)}=\frac{1}{\omega_{+}} \arccos \left\{\frac{\left(\omega_{+}^{2}-B k\right)}{C d+C k}\right\}+\frac{2 n \pi}{\omega_{+}} \\
\tau_{1, n}^{(2)}= & \frac{1}{\omega_{-}} \arccos \left\{\frac{\left(\omega_{-}^{2}-B k\right)}{C d+C k}\right\}+\frac{2 n \pi}{\omega_{-}}, \quad(n=0,1, \ldots) \tag{10}
\end{align*}
$$

Proof. Let $\lambda=i \omega$ be the root of (8); we thus have

$$
\begin{equation*}
-\omega^{2}+A \omega i+B k+C(d+k) e^{-\lambda \tau_{1}}=0 \tag{11}
\end{equation*}
$$

Separating the real and imaginary parts

$$
\begin{gather*}
-\omega^{2}+B k+C(d+k) \cos \omega \tau_{1}=0 \\
A \omega-C(d+k) \sin \omega \tau_{1}=0 \tag{12}
\end{gather*}
$$

Eliminating the harmonic terms gives

$$
\begin{equation*}
\omega^{4}+\left(-2 B k+A^{2}\right) \omega^{2}+B^{2} k^{2}-C^{2}(d+k)^{2}=0 \tag{13}
\end{equation*}
$$

Obviously

$$
\begin{equation*}
\omega_{1}^{2} \cdot \omega_{2}^{2}=B^{2} k^{2}-C^{2}(d+k)^{2}, \quad \omega_{1}^{2}+\omega_{2}^{2}=2 B k-A^{2} \tag{14}
\end{equation*}
$$

From (12) and by direct computation, we obtain $\omega_{ \pm}^{2}, \tau_{1, n}^{(1)}, \tau_{1, n}^{(2)}$. The result is straightforward.

Denote the minimum value of $\tau_{1, n}$ by $\tau_{1}^{0}$; that is, $\min \left(\tau_{1, n}\right)=\tau_{1}^{0}$, and

$$
\begin{align*}
\lambda_{k, n} & =\alpha_{k, n}\left(\tau_{1}\right)+i \omega_{k, n}\left(\tau_{1}\right), \\
\alpha_{1, n}\left(\tau_{1, n}^{(1)}\right) & =0, \quad \omega_{1, n}\left(\tau_{1, n}^{(1)}\right)=\omega_{+},  \tag{15}\\
\alpha_{2, n}\left(\tau_{1, n}^{(2)}\right) & =0, \quad \omega_{2, n}\left(\tau_{1, n}^{(2)}\right)=\omega_{-} .
\end{align*}
$$

To see if $\tau_{1, n}^{(1)}$ and $\tau_{1, n}^{(2)}$ are bifurcation values, we need to verify if the transversality conditions hold.

Lemma 2. The following transversality conditions

$$
\begin{equation*}
\frac{d \operatorname{Re} \lambda_{1, n}\left(\tau_{1, n}^{(1)}\right)}{d \tau_{1}}>0, \quad \frac{d \operatorname{Re} \lambda_{1, n}\left(\tau_{1, n}^{(2)}\right)}{d \tau_{1}}>0 \tag{16}
\end{equation*}
$$

hold.
Proof. Differentiating (8) with respect to $\tau_{1}$ yields

$$
\begin{gather*}
2 \lambda \frac{d \lambda}{d \tau_{1}}+A \frac{d \lambda}{d \tau_{1}}+C(k+d) e^{-\lambda \tau_{1}}\left(-\tau_{1} \frac{d \lambda}{d \tau_{1}}-\lambda\right)=0 \\
\frac{d \lambda}{d \tau_{1}}=\frac{C(k+d) \lambda e^{-\lambda \tau_{1}}}{2 \lambda+A-C(k+d) \tau_{1} e^{-\lambda \tau_{1}}} \tag{17}
\end{gather*}
$$

then we have

$$
\begin{align*}
\operatorname{Re} \frac{d \lambda}{d \tau_{1}} & =\operatorname{Re}\left\{\frac{2 \lambda+A}{C(k+d) \lambda e^{-\lambda \tau_{1}}}-\frac{\tau_{1}}{\lambda}\right\}=\frac{2 \lambda+A}{C(k+d) \lambda e^{-\lambda \tau_{1}}} \\
& =\frac{1}{C(d+k)} \cdot \frac{2 i \omega+A}{i \omega\left(\cos \omega \tau_{1}-i \sin \omega \tau_{1}\right)} \\
& =\frac{1}{C(d+k)} \cdot \frac{A \omega \sin \omega \tau_{1}+2 \omega^{2} \cos \omega \tau_{1}}{\omega^{2}}, \tag{18}
\end{align*}
$$

which satisfied $C>0$ and $d+k>0$; then for $\lambda_{1, n}\left(\tau_{1, n}^{(1)}\right), d \operatorname{Re} \lambda_{1, n}\left(\tau_{1, n}^{(1)}\right) / d \tau_{1}>0$, and for $\lambda_{1, n}\left(\tau_{1, n}^{(2)}\right)$, $d \operatorname{Re} \lambda_{1, n}\left(\tau_{1, n}^{(2)}\right) / d \tau_{1}>0$. We complete the proof.

Thus, we get the distribution of the characteristic roots of (8).

Lemma 3. For (8), one has the following:
(i) if (H3) and either (1) (H1) $A>0$ or (2) (H2) $A>0$ and $B k+C d+C k>0$ hold, then when $\tau_{1} \in\left[0, \tau_{1,0}^{(1)}\right)$, all roots of (8) have negative real parts, and when $\tau_{1}>\tau_{1,0}^{(1)}$, (8) has at least one root with positive real part;
(ii) if (H4) and either (H1) or (H2) hold, then there are $k$ switches from stability to instability; that is, when $\tau_{1} \in$ $\left(\tau_{1, n}^{(2)}, \tau_{1, n+1}^{(1)}\right), n=-1,0,1, \ldots, k-1$, all roots of (8) have negative real parts, where $\tau_{2,-1}^{(2)}=0, \tau_{1} \in\left[\tau_{1, n}^{(1)}, \tau_{1, n+1}^{(2)}\right)$ and $\tau_{1}>\tau_{1, k}^{(1)}, n=0,1, \ldots, k-1$, and (8) has at least one root with positive real part.

That is to say, under those conditions when $\tau_{2}=0, \tau_{1} \in\left[0, \tau_{1}^{0}\right)$, system (4) is asymptotically stable, and system (4) undergoes a Hopf bifurcation when $\tau_{1}=\tau_{1}^{0}$.

Then, we consider stable interval for $\tau_{1}$ in which all roots of (8) have negative real parts, regarding $\tau_{2}>0$ as a parameter.

Lemma 4. If all roots of (8) have negative real parts, then there exists a $\tau_{2}^{0}\left(\tau_{1}\right)>0$, such that when $\tau_{2} \in\left[0, \tau_{2}^{0}\left(\tau_{1}\right)\right)$ all roots of (7) have negative real parts.

Proof. All roots of (8) have negative real parts which means that system (4) is stable when $\tau_{2}=0$. In what follows, we consider (4) with fixed $\tau_{1}$ in its stable interval, regarding $\tau_{2}$ as a parameter. Let $i v(v>0)$ be a root of (4); then we obtain

$$
\begin{align*}
& -v^{2}+A\left(\cos v \tau_{2}-i \sin v \tau_{2}\right) v i+B k\left(\cos v \tau_{2}-i \sin v \tau_{2}\right) \\
& \quad+C d\left(\cos v \tau_{1}-i \sin v \tau_{1}\right) \\
& \quad+C k\left(\cos v\left(\tau_{1}+\tau_{2}\right)-i \sin v\left(\tau_{1}+\tau_{2}\right)\right)=0 \tag{19}
\end{align*}
$$

Suppose that $F(v)=v^{2}-A v i \cos v \tau_{2}-A v \sin v \tau_{2}-B k \cos v \tau_{2}+$ $B k i \sin v \tau_{2}-C d \cos v \tau_{2}+C d i \sin v \tau_{2}-C k \cos v\left(\tau_{1}+\tau_{2}\right)+$ $C k i \sin v\left(\tau_{1}+\tau_{2}\right)$. Since $F(0)=-(B k+C d+C k)<0$ and $F(+\infty)=+\infty$, then (19) has at least one positive root. Without loss of generality, the roots of (19) are defined by $v_{1}, v_{2}, \ldots, v_{k}$. For every $v_{i}(i=1,2, \ldots, k)$, there exists a sequence $\left\{\tau_{2 i}^{(j)} \mid j=1,2, \ldots\right\}$, such that (19) holds. The expression of $\tau_{2 i}^{(j)}$ and $v_{i}$ can be derived by (19) for fixed $\tau_{1}$; we will calculate them directly by the use of Mathematica software in Section 4 ; here we omit them. Let $\tau_{2}^{0}=\left\{\min \tau_{2 i}^{j} \mid\right.$ $i=1,2, \ldots, k, j=1,2, \ldots\}$ and let $v_{0}$ be the positive and simple root of (19) when $\tau_{2}=\tau_{2}^{0}$. When $\tau_{2}=\tau_{2}^{0}$, (19) has a pair of purely imaginary roots $\pm i v^{*}$ for $\tau_{1} \in\left[0, \tau_{1}^{0}\right)$. Then, as $\tau_{2}$ varies, the sum of the multiplicities of zeros in the open right half-plane can change only if a zero appears on or crosses the imaginary axis. In what follows, we assume that (H6) $\left[d \operatorname{Re} \lambda\left(\tau_{2}\right) / d \tau_{2}\right]_{\tau_{2}=\tau_{2 i}^{j}} \neq 0$. Therefore, by the general Hopf bifurcation theorem for FDEs in Wei and Ruan [10],
we get that when $\tau_{2} \in\left[0, \tau_{2}^{0}\right)$ all roots of (7) have negative real parts.

Applying the discussion above and noticing that all roots of (8) have negative real parts, we know that there exist $\tau_{2}^{0}>0$ such that all roots of (7) with $\tau_{2} \in\left[0, \tau_{2}^{0}\right)$ have negative real parts. The proof is complete.

Summarizing the above lemmas and literature of Hale [11], we obtain the following sufficient conditions for all characteristic roots of (8) to have negative real parts.

Theorem 5. Suppose that (H6) holds and either (H1) or (H2) is satisfied.
(i) If (H3) holds, then for any $\tau_{1} \in\left[0, \tau_{1,0}^{(1)}\right)$ there exists a $\tau_{2}^{0}\left(\tau_{1}\right)>0$ such that when $\tau_{2}^{0} \in\left[0, \tau_{2}\left(\tau_{1}\right)\right)$, all roots of (7) have negative real parts.
(ii) If (H4) holds, then for any $\tau_{1} \in \cup_{n=-1}^{k-1}\left(\tau_{1, n}^{(2)}, \tau_{1, n+1}^{(1)}\right)$ there exists a $\tau_{2}\left(\tau_{1}\right)>0$, such that when $\tau_{1} \in\left[0, \tau_{1,0}^{(1)}\right)$ all roots of (7) have negative real parts, where $\tau_{1, j}^{(1)}$ and $\tau_{1, j}^{(2)}$ are defined by (10).
(iii) If(H5) holds, then for any $\tau_{1} \geq 0$, there exists a $\tau_{2}\left(\tau_{1}\right)>$ 0 , such that when $\tau_{2} \in\left[0, \tau_{2}\left(\tau_{1}\right)\right)$ all roots of (7) have negative real parts.

That is to say, under the conditions that $\tau_{1}$ is stable interval, there exist a $\tau_{2}$ (depend on $\tau_{1}$ ) such that when $\tau_{2} \in\left[0, \tau_{2}^{0}\right.$ ), system (4) is asymptotically stable, and system (4) undergoes a Hopf bifurcation when $\tau_{2}=\tau_{2 i}^{(j)}, i=1,2, \ldots, k ; j=1,2 \ldots$.

## 3. Direction and Stability of the Bifurcating Periodic Solutions

In the previous section, we obtain the conditions under which a family of periodic solutions bifurcate from the steady state and the equilibrium loses its stability when $\tau_{2}=\tau_{2 i}^{(j)}, i=$ $1,2, \ldots, k ; j=1,2, \ldots$ for fixed $\tau_{1}$, and the relationship between $\tau_{1}$ and $\tau_{2}$ can be derived by (19). Throughout this section, by using techniques of the normal form and center manifold theory due to Hale [11], we derive the algorithm for determining the direction of the Hopf bifurcations and the stability of the bifurcating periodic solutions at critical values on the center manifold.

Since the analysis is local, we regard $\tau_{2}=\tau_{2}^{0}+\gamma, \gamma \in$ $R$ as bifurcation parameter. Choosing the space as $C=$ $C\left(\left[-\tau_{2}^{0}, 0\right], R^{2}\right)$ and $u_{t}=u(t+\theta) \in C$ for $\theta \in\left[-\tau_{2}^{0}, 0\right]$, system (3) is transformed into FDE as

$$
\begin{equation*}
\dot{u}(t)=L_{\gamma}(\phi)+F(\phi, \gamma), \tag{20}
\end{equation*}
$$

with

$$
\begin{gather*}
L_{\gamma}(\phi)=B_{1} \phi(0)+B_{2} \phi\left(-\tau_{1}\right)+B_{3} \phi\left(-\tau_{2}\right), \\
F(\phi, \gamma)=\binom{0}{\mu b \phi_{1}^{2}(0) \phi_{2}\left(-\tau_{2}\right)+2 \mu\left(P_{0}+\beta\right) \phi_{1}(0) \phi_{2}(0)}, \tag{21}
\end{gather*}
$$

where $B_{1}=\left(\begin{array}{cc}0 & d \\ -\mu b_{0} & 0\end{array}\right), B_{2}=\left(\begin{array}{cc}0 & 0 \\ -\mu & 0\end{array}\right)$, and $B_{3}=\left(\begin{array}{cc}0 & k \\ 0 & 2 \mu\left(P_{0}+\beta\right)^{2}+C_{0}\end{array}\right)$. Obviously, $L_{\gamma}(\phi)$ is continuous linear function mapping $C\left(\left[-\tau_{2}^{0}, 0\right], R^{2}\right)$ into $R^{2}$. By the Riesz representation theorem, there exists a matrix whose elements are bounded variation functions $\eta(\theta, \gamma)$ in $\theta \in\left[-\tau_{2}^{0}, 0\right]$ such that

$$
\begin{equation*}
L_{\gamma} \phi=\int_{-\tau_{2}^{0}}^{0} d \eta(\theta, \gamma) \phi(\theta), \quad \text { for } \phi \in C \tag{22}
\end{equation*}
$$

In fact, we choose

$$
\begin{equation*}
\eta(\theta, \gamma)=B_{1} \delta(\theta)+B_{2} \delta\left(\theta+\tau_{1}\right)+B_{3} \delta\left(\theta+\tau_{2}\right) \tag{23}
\end{equation*}
$$

where $\delta(\theta)$ is a delta function.
For $\phi \in C^{\prime}\left(\left[-\tau_{2}^{0}, 0\right], R^{2}\right)$ the operators $A$ and $R$ are defined as

$$
\begin{gather*}
A(\mu) \phi(\theta)= \begin{cases}\frac{d \phi(\theta)}{d \theta}, & \theta \in\left[-\tau_{2}^{0}, 0\right), \\
\int_{-\tau_{2}^{0}}^{0} d(\eta(t, \mu) \phi(t)), & \theta=0\end{cases}  \tag{24}\\
R(\mu) \phi(\theta)= \begin{cases}0, & \theta \in\left[-\tau_{2}^{0}, 0\right), \\
f(\gamma, \theta), & \theta=0 .\end{cases}
\end{gather*}
$$

Let $\psi \in C^{\prime}\left[0, \tau_{2}^{0}\right]$; the adjoint operator $A^{*}(0)$ corresponding to $A(0)$ is defined as follows:

$$
A^{*} \psi(s)= \begin{cases}-\frac{d \psi(s)}{d s}, & s \in\left(0, \tau_{2}^{0}\right]  \tag{25}\\ \int_{-\tau_{2}^{0}}^{0} d\left(\eta^{T}(t, 0) \psi(-t)\right), & s=0\end{cases}
$$

Then system (20) can be written in the following form: $\dot{u}_{t}=$ $A(\alpha) u_{t}+R(\alpha) u_{t}$, where $u_{t}=u(t+\theta)$ for $\theta \in[-1,0)$.

For $\phi \in C^{\prime}\left(\left[-\tau_{2}^{0}, 0\right], R^{2}\right)$ and $\psi \in C^{\prime}\left[0, \tau_{2}^{0}\right]$, define the adjoint bilinear:

$$
\begin{equation*}
\langle\psi, \phi\rangle=\bar{\psi}(0) \phi(0)-\int_{-\tau_{2}^{0}}^{0} \int_{\varepsilon=0}^{\theta} \bar{\psi}(\xi-\theta) d \eta(\theta) \phi(\xi) d \xi, \tag{26}
\end{equation*}
$$

where $\eta(\theta)=\eta(\theta, 0)$.
Proposition 6. $\operatorname{Let} q(\theta)$ and $q^{*}(s)$ be eigenvectors of $A$ and $A^{*}$ corresponding to $i \omega_{0}$ and $-i \omega_{0}$, respectively, satisfying $\left\langle q^{*}, q\right\rangle=$ 1 and $\left\langle q^{*}, \bar{q}\right\rangle=0$. Then

$$
\begin{gather*}
q(\theta)=\left(q_{1}, q_{2}\right)^{T} e^{i \omega_{0} \theta}=\left(d+k e^{-i \omega_{0} \tau_{2}}, i \omega\right)^{T} e^{i \omega_{0} \theta},  \tag{27}\\
q^{*}(s)=\bar{D}\left(q_{1}^{*}, q_{2}^{*}\right) e^{-i \omega_{0} s}=\left(\mu b_{0}+\mu a e^{i \omega_{0} \tau_{1}}, i \omega\right) e^{-i \omega_{0} s},
\end{gather*}
$$

where

$$
\begin{align*}
\bar{D}= & {\left[\left(q_{1} \bar{q}_{1}^{*}+q_{2} \bar{q}_{2}^{*}\right)+\tau_{1}^{0}\left[-\mu a e^{-i \omega_{0} \tau_{1}^{0}} q_{1} \bar{q}_{2}^{*}\right]\right.} \\
& \left.+\left[k q_{2} \bar{q}_{1}^{*}+\left[2 \mu b\left(P_{0}+\beta\right)^{2}+C_{0}\right] \bar{q}_{2}^{*} q_{2}\right] \tau_{2}^{0} e^{-i \omega_{0} \tau_{2}^{0}}\right]^{-1} \tag{28}
\end{align*}
$$

Proof. We assume that $q(\theta)=\left(q_{1}, q_{2}\right)^{T} e^{i \omega_{0} \theta}$ is the eigenvector of $A(0)$ corresponding to $i \omega$ and $q^{*}(s)=\bar{D}\left(q_{1}^{*}, q_{2}^{*}\right)^{T} e^{-i \omega_{0} s}$ is the eigenvector of $A^{*}(0)$ corresponding to $-i \omega$. It follows from the definition of $A(0), A^{*}(0),(22)$, and (23) that we have $A q(0)=\int_{-\tau_{2}^{0}}^{0} d(\eta(t, \mu) \phi(t))=i \omega_{0} q(0)$ and $A^{*} q(0)=$ $\int_{-\tau_{2}^{0}}^{0} d\left(\eta^{T}(t, \mu) \phi(-t)\right)=-i \omega_{0} q(0)$; we have

$$
\begin{align*}
& {\left[i \omega_{0} I-\left(B_{1}+B_{2} e^{-i \omega_{0} \tau_{1}}+B_{3} e^{-i \omega_{0} \tau_{2}}\right)\right] q(0)=0}  \tag{29}\\
& {\left[-i \omega_{0} I-\left(B_{1}+B_{2} e^{i \omega_{0} \tau_{1}}+B_{3} e^{i \omega_{0} \tau_{2}}\right)\right] q^{*}(0)=0}
\end{align*}
$$

where $I$ is identity matrix; that is,

$$
\begin{align*}
& \left(\begin{array}{cc}
i \omega_{0} & -\left(d+k e^{-i \omega_{0} \tau_{2}}\right) \\
\mu\left(b_{0}+a e^{-i \omega_{0} \tau_{1}}\right) & i \omega_{0}-\mu\left[b\left(P_{0}+\beta\right)+C_{0}\right] e^{-i \omega_{0} \tau_{2}}
\end{array}\right) \\
& \times\binom{ q_{1}}{q_{2}}=0, \\
& \left(\begin{array}{cc}
-i \omega_{0} & \mu\left(b_{0}+a e^{i \omega_{0} \tau_{1}}\right) \\
-\left(d+k e^{i \omega_{0} \tau_{2}}\right) & -i \omega_{0}-\mu\left[b\left(P_{0}+\beta\right)+C_{0}\right] e^{i \omega_{0} \tau_{2}}
\end{array}\right)  \tag{30}\\
& \times\binom{ q_{1}^{*}}{q_{2}^{*}}=0 .
\end{align*}
$$

By direct computation and considering $q(\theta)=q(0) e^{i \omega \theta}$, $q^{*}(s)=q^{*}(0) e^{-i \omega s}$, we obtain $q(\theta)$ and $q^{*}(s)$. Now, we calculate $\left\langle q^{*}, q\right\rangle$ as follows:

$$
\begin{align*}
& \left\langle q(s)^{*}, q(\theta)\right\rangle \\
& \begin{aligned}
&= \bar{D}\left\{\left(\bar{q}_{1}^{*}, \bar{q}_{2}^{*}\right)\binom{q_{1}}{q_{2}}\right. \\
&\left.\quad-\int_{-\tau_{2}^{0}}^{0} \int_{\varepsilon=0}^{\theta}\binom{\bar{q}_{1}^{*}}{\bar{q}_{2}^{*}}^{T} e^{-i \omega_{0}(\varepsilon-\theta)} d \eta(\theta)\binom{q_{1}}{q_{2}} e^{i \omega_{0} \varepsilon} d \varepsilon\right\} \\
&=\bar{D}\left\{\left(q_{1} \bar{q}_{1}^{*}+q_{2} \bar{q}_{2}^{*}\right)-\int_{-\tau_{2}^{0}}^{0}\binom{\bar{q}_{1}^{*}}{\bar{q}_{2}^{*}}^{T} \theta e^{i \omega_{0} \theta} d \eta(\theta)\binom{q_{1}}{q_{2}}\right\} \\
&=\bar{D}\left\{\left(q_{1} \bar{q}_{1}^{*}+q_{2} \bar{q}_{2}^{*}\right)+\tau_{1}^{0}\left[-\mu a e^{-i \omega_{0} \tau_{1}^{0}} q_{1} \bar{q}_{2}^{*}\right]\right. \\
&=\left.+\left[k q_{2} \bar{q}_{1}^{*}+\left[2 \mu b\left(P_{0}+\beta\right)^{2}+C_{0}\right] \bar{q}_{2}^{*} q_{2}\right] \tau_{2}^{0} e^{-i \omega_{0} \tau_{2}^{0}}\right\} \\
&=
\end{aligned} \\
& \\
&
\end{align*}
$$

Since $\langle\psi, A \phi\rangle=\left\langle A^{*} \psi, \phi\right\rangle$, we get

$$
\begin{equation*}
i \omega\left\langle q^{*}, \bar{q}\right\rangle=\left\langle A^{*} q^{*}, \bar{q}\right\rangle=\left\langle-i \omega_{0} q^{*}, \bar{q}\right\rangle=i \omega_{0}\left\langle q^{*}, \bar{q}\right\rangle . \tag{32}
\end{equation*}
$$

Therefore, $\left\langle q^{*}, \bar{q}\right\rangle=0$. This completes the proof.
Then, we construct the coordinates of the center manifold $C_{0}$ at $\gamma=0$. Let

$$
\begin{equation*}
z(t)=\left\langle q^{*}, u_{t}\right\rangle, \quad W(t, \theta)=u_{t}(\theta)-2 \operatorname{Re}\{z(t) q(\theta)\} \tag{33}
\end{equation*}
$$

On the center manifold $C_{0}$, we have

$$
\begin{equation*}
W(t, \theta)=W(z(t), \bar{z}(t), \theta), \tag{34}
\end{equation*}
$$

where

$$
\begin{align*}
W(z, \bar{z}, \theta)= & W_{20}(\theta) \frac{z^{2}}{2}+W_{11}(\theta) z \bar{z}  \tag{35}\\
& +W_{02} \frac{\bar{z}^{2}}{2}+W_{30} \frac{z^{3}}{6}+\cdots
\end{align*}
$$

and $z$ and $\bar{z}$ are local coordinates for the center manifold $C_{0}$ in the direction of $q$ and $q^{*}$, respectively. Since $\gamma=0$, we have

$$
\begin{align*}
z^{\prime}(t) & =i \omega_{0} z(t)+\left\langle q^{*}(\theta), f(W+2 \operatorname{Re}\{z(t) q(\theta)\})\right\rangle \\
& =i \omega_{0} z(t)+\bar{q}^{*}(0) f(W(z, \bar{z}, 0)+2 \operatorname{Re}\{z(t) q(0)\}) \\
& \triangleq i \omega_{0} z(t)+\bar{q}^{*}(0) f_{0}(z, \bar{z}) \tag{36}
\end{align*}
$$

where

$$
\begin{equation*}
f_{0}(z, \bar{z})=f_{z^{2}} \frac{z^{2}}{2}+f_{\bar{z}^{2}} \frac{\bar{z}^{2}}{2}+f_{z \bar{z}} z \bar{z}+\cdots \tag{37}
\end{equation*}
$$

We rewrite in abbreviated form as

$$
\begin{equation*}
z^{\prime}(t)=i \omega_{0} z+g(z, \bar{z}) \tag{38}
\end{equation*}
$$

where

$$
\begin{align*}
g(z, \bar{z}) & =\bar{q}^{*}(0) f_{0}(z, \bar{z}) \\
& =g_{20} \frac{z^{2}}{2}+g_{11} z \bar{z}+g_{02} \frac{\bar{z}^{2}}{2}+g_{21} \frac{z^{2} \bar{z}}{2}+\cdots \tag{39}
\end{align*}
$$

By (20) and (38), we obtain

$$
\begin{align*}
\dot{W} & =\dot{u}_{t}-\dot{z} q-\dot{\bar{z}} \bar{q} \\
& = \begin{cases}A W-2 \operatorname{Re} \bar{q}^{*}(0) f_{0} q(\theta), & \theta \in\left[-\tau_{2}^{0}, 0\right] \\
A W-2 \operatorname{Re} \bar{q}^{*}(0) f_{0} q(\theta)+f_{0}, & \theta=0\end{cases}  \tag{40}\\
& \triangleq A W+H(z, \bar{z}, \theta),
\end{align*}
$$

where

$$
\begin{equation*}
H(z, \bar{z}, \theta)=H_{20}(\theta) \frac{z^{2}}{2}+H_{11}(\theta) z \bar{z}+H_{02}(\theta) \frac{\bar{z}^{2}}{2}+\cdots \tag{41}
\end{equation*}
$$

Substituting (26) and (38) into $\dot{W}=W_{z} \dot{z}+W_{\bar{z}} \dot{\bar{z}}$ on the center manifold $C_{0}$ and comparing the coefficients we get

$$
\begin{gather*}
\left(A-2 i \omega_{0} I\right) W_{20}(\theta)=-H_{20}(\theta), \quad A W_{11}(\theta)=-H_{11}(\theta), \\
\left(A+2 i \omega_{0} I\right) W_{02}(\theta)=-H_{02}(\theta) \tag{42}
\end{gather*}
$$

Comparing the coefficients with (41) gives that

$$
\begin{align*}
& H_{20}(\theta)=-g_{20} q(\theta)-\bar{g}_{02} \bar{q}(\theta), \\
& H_{11}(\theta)=-g_{11} q(\theta)-\bar{g}_{11} \bar{q}(\theta) . \tag{43}
\end{align*}
$$

From (42), (43), and the definition of $A$, we can derive the following equation:

$$
\begin{gather*}
\dot{W}_{20}(\theta)=2 i \omega_{0} W_{20}(\theta)+g_{20} q(\theta)+\bar{g}_{20} \bar{q}(\theta), \\
\dot{W}_{11}(\theta)=g_{11} q(\theta)+\bar{g}_{11} \bar{q}(\theta) \tag{44}
\end{gather*}
$$

Solving for $W_{20}(\theta)$ and $W_{11}(\theta)$, we get

$$
\begin{gather*}
W_{20}(\theta)=\frac{i g_{20}}{\omega_{0}} q(0) e^{i \omega_{0} \theta}-\frac{\bar{g}_{02}}{3 i \tau_{0} \omega_{0}} \bar{q}(0) e^{-i \omega_{0} \theta}+E_{1} e^{2 i \omega_{0} \theta}, \\
W_{11}(\theta)=-\frac{i g_{11}}{\omega_{0}} q(0) e^{i \omega_{0} \theta}+\frac{i \bar{g}_{11}}{\omega_{0}} \bar{q}(0) e^{-i \omega_{0} \theta}+E_{2}, \tag{45}
\end{gather*}
$$

where $E_{1}=\left(E_{1}^{(1)}, E_{1}^{2}\right) \in R^{2}, E_{2}=\left(E_{2}^{(1)}, E_{2}^{2}\right) \in R^{2}$ are two constant vectors, which can be determined by setting $\theta=0$ in $H$.
$\operatorname{By}(33), \phi_{t}(\theta)=\left[u_{1}\left(t-\tau_{1}\right), u_{2}\left(t-\tau_{2}\right)\right]=W(t, \theta)+z q(\theta)+$ $\bar{z} \bar{q}(\theta)$, and noticing $q(\theta)=\left(q_{1}, q_{2}\right)^{T} e^{i \omega_{0} \theta}$ we have

$$
\begin{align*}
\phi_{t}(\theta)= & z\binom{d+k e^{-i \omega_{0} \tau_{2}^{0}}}{i \omega} e^{-i \omega_{0} \theta}  \tag{46}\\
& +\bar{z}\binom{d+k e^{i \omega_{0} \tau_{2}^{0}}}{-i \omega} e^{-i \omega_{0} \theta}+W(t, \theta)
\end{align*}
$$

Then it is easy to obtain

$$
\begin{align*}
\phi_{1}(0) & =z\left(d+k e^{-i \omega_{0} \tau_{2}^{0}}\right)+\bar{z}\left(d+k e^{i \omega_{0} \tau_{2}^{o}}\right)+W^{(1)}(0), \\
\phi_{2}(0) & =z \omega i+\bar{z}(-i \omega)+W^{(2)}(0), \\
\phi_{2}\left(t-\tau_{2}^{0}\right) & =z \omega i e^{-i \omega_{0} \tau_{2}^{0}}-\bar{z} e^{i \omega_{0} \tau_{2}^{0}}+W^{2}\left(t-\tau_{2}^{0}\right), \tag{47}
\end{align*}
$$

where

$$
\begin{align*}
& W^{(1)}(0)= W_{20}^{(1)}(0) \frac{z^{2}}{2}+W_{11}^{(1)}(0) z \bar{z} \\
&+W_{02}^{(1)}(0) \frac{\bar{z}}{2}+o\left(|z, \bar{z}|^{3}\right), \\
& W^{(2)}(0)= W_{20}^{(2)}(0) \frac{z^{2}}{2}+W_{11}^{(2)}(0) z \bar{z} \\
&+W_{02}^{(2)}(0) \frac{\bar{z}}{2}+o\left(|z, \bar{z}|^{3}\right), \\
& W^{(2)}\left(t-\tau_{2}\right)= W_{20}^{(2)}\left(-\tau_{2}^{0}\right) \frac{z^{2}}{2}+W_{11}^{(2)}\left(-\tau_{2}^{0}\right) z \bar{z} \\
&+W_{02}^{(2)}\left(-\tau_{2}^{0}\right) \frac{\bar{z}}{2}+o\left(|z, \bar{z}|^{3}\right), \\
& f_{0}(z, \bar{z})=( 0  \tag{48}\\
&\left.\mu b \phi_{1}^{2}(0) \phi_{2}\left(-\tau_{2}^{0}\right)+2 \mu\left(P_{0}+\beta\right) \phi_{1}(0) \phi_{2}(0)\right) .
\end{align*}
$$

Thus, from (39), it follows that

$$
g(z, \bar{z})=\bar{q}^{*}(0) f_{0}(z, \bar{z})=\bar{D}\left(\bar{q}_{1}^{*} \quad \bar{q}_{2}^{*}\right)
$$

$$
\begin{aligned}
& \times\binom{ 0}{\mu b \phi_{1}^{2}(0) \phi_{2}\left(-\tau_{2}^{0}\right)+2 \mu\left(P_{0}+\beta\right) \phi_{1}(0) \phi_{2}(0)} \\
= & \bar{D} \bar{q}_{2}^{*}\left[\mu \phi_{1}^{2}(0) \phi_{2}\left(-\tau_{2}^{0}\right)+2 \mu\left(P_{0}+\beta\right) \phi_{1}(0) \phi_{2}(0)\right] \\
= & \bar{D}\left\{2 \mu\left(P_{0}+\beta\right) \bar{q}_{2}^{*} q_{1} q_{2} z^{2}+2 \mu\left(P_{0}+\beta\right)\right.
\end{aligned}
$$

$$
\times \bar{q}_{2}^{*}\left(q_{1} \bar{q}_{2}+\bar{q}_{1} q_{2}\right) z \bar{z}+2 \mu\left(P_{0}+\beta\right) \bar{q}_{2}^{*} \bar{q}_{1} \bar{q}_{2} \bar{z}^{2}
$$

$$
+\left[\mu b \bar{q}_{2}^{*}\left(q_{1}^{2} \bar{q}_{2} e^{i \omega_{0} \tau_{2}^{0}}+2 q_{1} \bar{q}_{1} q_{2} e^{-i \omega_{0} \tau_{2}^{0}}\right)\right.
$$

$$
+2 \mu\left(P_{0}+\beta\right) \bar{q}_{2}^{*}
$$

$$
\times\left(q_{2} W_{11}^{(1)}(0)+q_{1} W_{11}^{(2)}(0)\right.
$$

$$
\left.\left.\left.+\frac{\bar{q}_{2} W_{20}^{(1)}(0)+\bar{q}_{1} W_{20}^{(2)}(0)}{2}\right)\right] z^{2} \bar{z}\right\}
$$

$$
\begin{equation*}
+o\left(\phi^{4}\right) \tag{49}
\end{equation*}
$$

Comparing the coefficient with (39), we have

$$
\begin{gather*}
g_{20}=4 \bar{D} \mu\left(P_{0}+\beta\right) \bar{q}_{2}^{*} q_{1} q_{2} \\
g_{11}=4 \bar{D} \mu\left(P_{0}+\beta\right) \bar{q}_{2}^{*} \operatorname{Re}\left(q_{1} \bar{q}_{2}\right) \\
g_{02}=4 \bar{D} \mu\left(P_{0}+\beta\right) \bar{q}_{2}^{*} \bar{q}_{1} \bar{q}_{2} \\
g_{21}=2 \bar{D}\left[\mu b \bar{q}_{2}^{*}\left(q_{1}^{2} \bar{q}_{2} e^{i \omega_{0} \tau_{2}^{0}}+2 q_{1} \bar{q}_{1} q_{2} e^{-i \omega_{0} \tau_{2}^{0}}\right)\right.  \tag{50}\\
+2 \mu\left(P_{0}+\beta\right) \bar{q}_{2}^{*} \\
\times\left(q_{2} W_{11}^{(1)}(0)+q_{1} W_{11}^{(2)}(0)\right. \\
\left.\left.\quad+\frac{\bar{q}_{2} W_{20}^{(1)}(0)+\bar{q}_{1} W_{20}^{(2)}(0)}{2}\right)\right]
\end{gather*}
$$

From (40), we get that $H(z, \bar{z}, 0)=-2 \operatorname{Re} \bar{q}^{*}(0) f_{0} q(0)+f_{0}=$ $-2 \operatorname{Re}(g q(0))+f_{0}=-g q(0)-\bar{g} \bar{q}(0)+f_{0}$; that is,

$$
\begin{gather*}
H_{20}(0)=-g_{20} q(0)-\bar{g}_{20} \bar{q}(0)+\binom{0}{4 \mu\left(P_{0}+\beta\right) q_{1} q_{2}}, \\
H_{11}(0)=-g_{11} q(0)-\bar{g}_{11} \bar{q}(0)+\binom{0}{2\left(P_{0}+\beta\right)\left(q_{1} \bar{q}_{2}+\bar{q}_{1} q_{2}\right)} . \tag{51}
\end{gather*}
$$

By the definition of $A$ and (42) we have

$$
\begin{gather*}
\int_{-\tau_{2}^{0}}^{0} d \eta(\theta) W_{20}(\theta)=2 i \omega_{0} W_{20}-H_{20}(0) \\
\int_{-\tau_{2}^{0}}^{0} d \eta(\theta) W_{11}(\theta)=-H_{11}(0) \tag{52}
\end{gather*}
$$

Notice that

$$
\begin{aligned}
& \left(i \omega_{0} I-\int_{-\tau_{2}^{0}}^{0} d \eta(\theta) e^{i \omega_{0} \theta}\right) q(0)=0 \\
& \left(-i \omega_{0} I-\int_{-\tau_{2}^{0}}^{0} d \eta(\theta) e^{i \omega_{0} \theta}\right) \bar{q}(0)=0
\end{aligned}
$$

Substituting (45) and (52) into (51), we obtain

$$
\begin{align*}
& \left(\begin{array}{cc}
2 i \omega_{0} & -d-k e^{2 i \omega_{0} \tau_{2}^{0}} \\
\mu b_{0}+\mu a e^{-2 i \omega_{0} \tau_{1}^{0}} & 2 i \omega_{0}-\mu\left[b\left(P_{0}+\beta\right)^{2}+C_{0}\right] e^{2 i \omega_{0} \tau_{2}^{0}}
\end{array}\right) E_{1} \\
& =\left(\begin{array}{cc}
0 & \\
4 \mu\left(P_{0}+\beta\right) & q_{1} q_{2}
\end{array}\right) \\
& \left(\begin{array}{cc}
0 & -d-k \\
\mu\left(b_{0}+a\right)-\mu\left[b\left(P_{0}+\beta\right)^{2}+C_{0}\right]
\end{array}\right) E_{2} \\
& =\binom{0}{2\left(P_{0}+\beta\right)\left(q_{1} \bar{q}_{2}+\bar{q}_{1} q_{2}\right)} \tag{54}
\end{align*}
$$

By direct computation, we obtain

$$
\begin{gather*}
E_{1}^{(1)}=\frac{\left[4 \mu\left(P_{0}+\beta\right) q_{1} q_{2}\right]\left(d+k e^{-2 i \omega_{0} \tau_{2}^{0}}\right)}{\left[\mu b_{0}+\mu a e^{-2 i \omega_{0} \tau_{1}^{0}}\right]\left(d+k e^{-2 i \omega_{0} \tau_{2}^{0}}\right)-2 i \omega_{0}\left[\mu\left[b\left(P_{0}+\beta\right)^{2}+C_{0}\right] e^{-2 i \omega_{0} \tau_{2}^{0}}-2 i \omega_{0}\right]}, \\
E_{1}^{(2)}=\frac{-6 i \omega_{0} \mu\left(P_{0}+\beta\right) q_{1} q_{2}}{-\left[\mu b_{0}+\mu a e^{-2 i \omega_{0} \tau_{1}^{0}}\right]\left(d+k e^{-2 i \omega_{0} \tau_{2}^{0}}\right)+2 i \omega_{0}\left[\mu\left[b\left(P_{0}+\beta\right)^{2}+C_{0}\right] e^{-2 i \omega_{0} \tau_{2}^{0}}-2 i \omega_{0}\right]},  \tag{55}\\
E_{2}^{(1)}=\frac{2\left(P_{0}+\beta\right)\left(q_{1} \bar{q}_{2}+\bar{q}_{1} q_{2}\right)}{\mu\left(b_{0}\right)-a}, \quad E_{2}^{(2)}=0 .
\end{gather*}
$$

Thus, we can compute the following values which determine the properties of bifurcating periodic solutions at the critical value $\tau_{2}^{0}$ :

$$
\begin{gather*}
C_{1}(0)=\frac{i}{2 \tau_{2}^{0} \omega_{0}}\left(g_{20} g_{11}-2\left|g_{11}\right|-\frac{1}{3}\left|g_{02}\right|^{2}\right)+\frac{g_{21}}{2} \\
\mu_{2}=-\frac{\operatorname{Re}\left\{C_{1}(0)\right\}}{\operatorname{Re}\left\{\lambda^{\prime}\left(\tau_{2}^{0}\right)\right\}}  \tag{56}\\
\beta_{2}=2 \operatorname{Re}\left\{C_{1}(0)\right\} \\
T_{2}=-\frac{\operatorname{Im}\left\{C_{1}(0)\right\}+\mu_{2}\left(\operatorname{Im}\left\{\lambda^{\prime}\left(\tau_{2}^{0}\right)\right\}\right)}{\omega_{0}}
\end{gather*}
$$

More specifically (see Hassard et al. [12]), $\mu_{2}$ determines the direction of the Hopf bifurcation: if $\mu_{2}>0\left(\mu_{2}<0\right)$, then the Hopf bifurcation is forward (backward) and the bifurcating periodic solutions exist for $\tau>\tau_{2}^{0}\left(\tau<\tau_{2}^{0}\right)$. $\beta_{2}$ determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_{2}<0\left(\beta_{2}>0\right) . T_{2}$ determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_{2}>0\left(T_{2}<0\right)$.

## 4. Numerical Simulation

In order to validate the theoretical analysis, we will present some numerical simulations. We take the following coefficients as an example: $\mu=0.13, b_{0}=10, a=12, d_{0}=60$,
$g_{0}=20, \beta=-2, b=1, k=1$, and $d=1$, it is easy to obtain $E^{*}=(1.82,0), \omega_{0} \approx 0.89$, and $\tau_{1}^{0} \approx 0.72$. We choose $\tau_{1}^{0}=0.7$; then from (19) and by means of Mathematica software, we get $v_{0} \approx 2$ and $\tau_{2}^{0} \approx 0.86$. Taking $(x(0), y(0)=(2,2))$ as the initial conditions. By Theorem 5 and the above results, we know equilibrium $E^{*}$ is locally asymptotically stable when $\tau_{2}=0.8<\tau_{2}^{0}$ as is illustrated in Figure 1. When $\tau_{2}>\tau_{2}^{0}$, $E^{*}$ is unstable and periodic solutions occur from $E^{*}$; we take $\tau_{2}=0.9$, the corresponding phase plots are shown in Figure 2.

Finally, the numerical simulation shows that it is a complex transformation process for the system changes from stable equilibrium to chaos.

## 5. Conclusion

Different from the previous work in $[1,2,4]$, the main contribution of this paper lies in the following aspects. Firstly, we modify the traditional price differential equation model by considering demand function that is settled as a downward opening quadratic function and considering supply and demand functions that are both depending on the price of the past and the present. Then the price differential equation model with two delays is established. Secondly, to study the stability and Hopf bifurcation of system (4), we consider (4) with one delay $\tau_{1}$ and find the stable interval for $\tau_{1}$. In the following, regarding $\tau_{2}$ as a parameter, we obtain the stable interval for $\tau_{2}$. Then we get the stable interval for (4). In addition, we derive the algorithm for determining


Figure 1: Equilibrium $E^{*}$ is locally asymptotically stable when $\tau_{1}=$ $0.7, \tau_{2}=0.8<\tau_{2}^{0}$.


Figure 2: Equilibrium $E^{*}$ is unstable when $\tau_{1}=0.7, \tau_{2}=0.9>\tau_{2}^{0}$.
the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions by using techniques of normal form theory and center manifold method. Lastly, a numerical analysis confirms the effectiveness of our research results. The paper provided the preparative work for further discussion. For instance, we consider that supply cannot increase with the price without limit; the production capacity of enterprises and social resources are limited; then we can modify the supply function as a fractional linear function of price. The results in the paper enrich the toolbox for the qualitative analysis of mathematical economics and business administration.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

The authors are grateful to the referees for their helpful comments and constructive suggestions.

## References

[1] W. Shuhe, "Differential equation model and chaos," Journal of China Science and Technology University, pp. 312-324, 1999.
[2] Z. Xi-fan, C. Xia, and C. Yun-qing, "A qualitative analysis of price model in differential equations of price," Journal of Shenyang Institute of Aeronautical Engineering, vol. 21, no. 1, pp. 83-86, 2004.
[3] S. Banerjee and W. A. Barnett, "Bifurcation analysis of Zellner's marshallain macroeconomic model," Journal of Economic Dynamics and Control, vol. 35, no. 9, pp. 1577-1585.
[4] L. Tanghong and L. Zhenwen, "Hopf bifurcation of price Reyleigh equation with time delay," Journal of Jilin University, vol. 47, no. 3, 2009.
[5] W. Yong and Z. Yanhui, "Stability and Hopf bifurcation of differential equation model of price with time delay," Highlights of Sciencepaper Online, vol. 4, no. 1, 2011.
[6] Y. Zhai, H. Bai, Y. Xiong, and X. Ma, "Hopf bifurcation analysis for the modified Rayleigh price model with time delay," Abstract and Applied Analysis, vol. 2013, Article ID 290497, 6 pages, 2013.
[7] O. I. Adeyemi and L. C. Hunt, "Modelling OECD industrial energy demand: asymmetric price responses and energy saving technical change," Energy Economics, vol. 29, no. 4, pp. 693-709, 2007.
[8] L. Tanghong and Z. Linhua, "Hopf and codimension two bifurcation for the price Rayleigh equation with two time delays," Journal of Jilin University, vol. 50, no. 3, pp. 409-416, 2012.
[9] K. L. Cooke and Z. Grossman, "Discrete delay, distributed delay and stability switches," Journal of Mathematical Analysis and Applications, vol. 86, no. 2, pp. 592-627, 1982.
[10] J. Wei and S. Ruan, "Stability and bifurcation in a neural network model with two delays," Physica D, vol. 130, no. 3-4, pp. 255-272, 1999.
[11] J. Hale, Theory of Functional Differential Equations, Springer, New York, NY, USA, 2nd edition, 1977.
[12] D. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, Theory and Applications of Hopf Bifurcation, Cambridge University Press, Cambridge, UK, 1981.

## Research Article

# Optimal Control Strategies in an Alcoholism Model 

Xun-Yang Wang, ${ }^{1,2}$ Hai-Feng Huo, ${ }^{1,2}$ Qing-Kai Kong, ${ }^{3}$ and Wei-Xuan Shi ${ }^{2}$<br>${ }^{1}$ College of Electrical and Information Engineering, Lanzhou University of Technology, Lanzhou, Gansu 730050, China<br>${ }^{2}$ Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, China<br>${ }^{3}$ School of Electrical and Information Engineering, Anhui Industrial University, Maanshan, Anhui 243032, China<br>Correspondence should be addressed to Hai-Feng Huo; huohf1970@gmail.com

Received 10 December 2013; Accepted 9 January 2014; Published 2 March 2014
Academic Editor: Weiming Wang
Copyright © 2014 Xun-Yang Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

This paper presents a deterministic SATQ-type mathematical model (including susceptible, alcoholism, treating, and quitting compartments) for the spread of alcoholism with two control strategies to gain insights into this increasingly concerned about health and social phenomenon. Some properties of the solutions to the model including positivity, existence and stability are analyzed. The optimal control strategies are derived by proposing an objective functional and using Pontryagin's Maximum Principle. Numerical simulations are also conducted in the analytic results.


## 1. Introduction

Alcoholism, also known as alcohol dependence, is a disease that includes the desire for alcohol and continuing to drink it despite its negative effect on individual's health, relationships, and social status [1]. Similar to all other drug addictions, alcoholism can be regarded as a treatable disease. The World Health Organization estimates that about 140 million people throughout the world suffer from alcohol dependence with related problems, such as being sick, losing a job, among a host of other things [2]. Particularly, young people's alcoholism problem is a major concern to public health. US surveys indicate that approximately $90 \%$ of college students have consumed alcohol at least once [3], and more than $40 \%$ of college students have engaged in binge drinking [4, 5]. Unfortunately, the biological mechanisms underpinning alcoholism are not known; however, risk factors include social environment, stress, mental health, genetic sensitivity, age, ethnic group, and sex [6, 7]. Long-term alcohol abuse will produce negative changes in the brain such as tolerance and physical dependence. The subtle changes make it difficult for the alcoholics to stop drinking and result in alcohol withdrawal symptoms upon discontinuation of alcohol consumption. Alcohol damages almost all parts of the body and contribute to a number of human diseases including but not limited to liver cirrhosis, pancreatitis, heart disease, and
sexual dysfunction and can eventually be deadly [8]. Damage to the central and peripheral nervous systems can take place from sustained alcohol consumption [9-13].

Although alcoholism is becoming more and more dangerous and serious as well as a widespread social phenomenon, only much less work has been done in the mathematical modelling of alcoholism as a growing health problem, including a few studies which offered some mathematical approaches to understand the growing burden of alcoholism [10, 14-19]. In [10], a SIR-type model was proposed; the authors used standard contact rate between susceptibles and alcoholism, getting alcoholism reproductive number and discussing the existence and stability of two equilibria. In [14], a framework where drinking was modeled as a socially contagious process in low- and high-risk connected environments was introduced; they found that high levels of social interaction between light and moderate drinkers in low-risk environments can diminish the importance of the distribution of relative drinking times on the prevalence of heavy drinking. In [15], neurophysiological examinations of 100 long-term alcohol dependent patients, who were having neuropsychiatric treatment, showed symptoms of polytopic damage of the peripheral and central nervous system. The results showed that for recognition of the damage an extensive diagnostic programme must be used. In [16], the authors considered a kind of binge drinking model with two equal
infectivity drunk states; mathematical analyses established that the global dynamics of the model were determined by the basic reproduction number. In [17], the authors modified the model from [16]; that is, they considered different infectivity of two drunk states, and a SEIR-type model of alcoholism was thus presented, in which two alcohol related states were involved, namely, no alcohol dependent consumers $D(t)$ and alcohol dependent consumers $A(t)$. In [18], the authors formulated a deterministic model for evaluating the impact of heavy alcohol drinking on the reemerging gonorrhea epidemic, and both analytical and numerical results were provided to ascertain whether heavy alcohol drinking had an impact on the transmission dynamics of gonorrhea. The approach of the literature [18] was very meaningful, since it provided a new direction of thinking when the crossinfection between alcoholism and other pathological diseases occurs. In recent monograph [19], the authors also proposed a SIR-type model to investigate alcohol abuse phenomenon and generated some useful insights; for example, the basic reproductive number was not always the key to controlling drinking within the population. For other papers that study the model of giving up smoking or quitting drinking, please see $[20,21]$ and references cited therein.

As living standard and health awareness get improved, more and more people who fall into binge drinking state are actively seeking the quitting alcoholism measures and treatment methods [1, 11, 22]. In [22], treatment strategy was introduced into a simple SIR-type alcoholics quitting model, in which the authors used the bilinear incidence to depict the "infection" between the occasional drinkers $S$ and problem drinkers $D$. Motivated by some aforementioned documents [ $10,19,22$ ], in this paper, we will formulate a more reasonable alcoholics quitting model. The fact that our model is more reasonable is embodied from the following three aspects.
(1) Taking into account that alcoholism is a widespread social phenomenon, so the standard incidence is superior to bilinear incidence when we portray the relationship between the alcoholism and the susceptibles during the course of infection. While in [22], the authors adopted bilinear incidence, we will adopt standard incidence in this paper.
(2) Since alcohol is harmful to health, moreover, as we all know, alcoholism is treatable if we can take approximate measure in time, for example, artificial isolation from alcoholisms, medications, persuasion, and education programing on alcoholism. So it is necessary to take effective measures to avoid alcohol or to treat after alcoholism. Documents [10, 19] have not considered these aspects.
(3) Since there is effective prevention and treatment in describing the phenomenon of alcoholism, there are some people who will never drink due to successful prevention or some people who no longer drink after successful treatment. Therefore, when we formulate the model in this paper, it's reasonable to introduce a new compartment $Q$, the people in which will never drink for ever. Obviously, the models of $[10,19,22]$ are not involving the quitting compartment $Q$.

Based on the above considerations, we will premeditate two treating methods, namely, prevention of susceptibles from alcoholism and treatment on alcoholism as control variables; hence, we will derive a SATQ-type model. We note the fact that many authors are interested in solving optimal control problems, such as cost minimization and optimal control of various disease, especially with biological background and various mathematical models [22-24]. In this paper, we will propose an objective functional which considers not only alcohol quitting effects but also the cost of controlling alcohol. Then, we consider a range of issues related to the optimal control with the method of Pontryagin's Maximum Principle, including optimal control existence, uniqueness, and characterization.

The organization of this paper is as follows. In the next section, the alcoholism model with prevention for the susceptibles and treatment for alcoholism is formulated. In Section 3, the basic reproduction number and the existence of equilibria are investigated. The stability of the disease free and endemic equilibria is proved in Section 4. Optimal control strategies by the classic method of PMP (Pontryagin's Maximum Principle) are discussed in Section 5. In Section 6, we give some numerical simulations. We give some discussions and conclusions in the last section.

## 2. The Model Formulation and Some Fundamental Properties

In this section, we introduce a mathematical model with prevention and treatment for the alcoholism and then study some important properties such as the boundness and positivity of its solutions.
2.1. Model Formulation and Parameter Explanation. The total population is partitioned into four compartments: the susceptible compartment $S$ which refers to the persons who never drink or drink moderately without affecting the physical health, the alcoholism compartment $A$ which refers to the persons who binge drink and affect the physical health seriously, the treatment compartment $T$ which refers to the persons who have been receiving treatments by taking pills or other medical interventions after alcoholism, and the quitting compartment $Q$ which refers to the persons who recover from alcoholism after treatment and stay off alcohol hereafter. In this paper, we focus on a closed environment, such as a community, a university, or a village. So the total number of population to be considered is a constant; we denote it as $N$. The population flow among those compartments is shown in the following diagram (Figure 1).

The schematic diagram leads to the following system of ordinary differential equations:

$$
\begin{gathered}
S^{\prime}=\mu N-\left(1-u_{1}\right) \frac{\beta S A}{N}-\mu S \\
A^{\prime}=\left(1-u_{1}\right) \frac{\beta S A}{N}+\xi T-\left(u_{2}+\mu\right) A
\end{gathered}
$$



Figure 1: Transfer diagram for the dynamics of alcoholism model.

$$
\begin{gather*}
T^{\prime}=u_{2} A-(\mu+\xi+\delta) T \\
Q^{\prime}=\delta T-\mu Q \tag{1}
\end{gather*}
$$

Here, $\mu N$ is the birth number of the population; $\mu$ is the natural death rate of the population; $u_{1}$ is the fraction of the susceptible individuals who successfully avoid to stay off the alcoholism; $u_{2}$ is the fraction of the alcoholics who take part in treatments; here, $0 \leq u_{i} \leq 1, i=1,2$, and they will be considered as two control variables in Section 5; $\beta$ is the transmission coefficient of the "infection" for the susceptible individuals from the alcoholic individuals; $\xi$ is the rate coefficient of the person who fail to be treated and return to the alcoholism compartment mostly due to their own weak will; $\delta$ is the rate coefficient of the person who have received effective treatment and recovered from alcoholism forever.
2.2. Boundedness of Solutions to System and Positively Invariant Region. It is important to show positivity and boundedness for the system (1) as they represent populations. Firstly, we present the positivity of the solutions. System (1) can be put into the matrix form

$$
\begin{equation*}
X^{\prime}=G(X) \tag{2}
\end{equation*}
$$

where $X=(S, A, T, Q)^{T} \in R^{4}$ and $G(X)$ is given by

$$
\begin{align*}
G(X) & =\left(\begin{array}{l}
G_{1}(X) \\
G_{2}(X) \\
G_{3}(X) \\
G_{4}(X)
\end{array}\right) \\
& =\left(\begin{array}{c}
\mu N-\left(1-u_{1}\right) \frac{\beta S A}{N}-\mu S \\
\left(1-u_{1}\right) \frac{\beta S A}{N}+\xi T-\left(u_{2}+\mu\right) A \\
u_{2} A-(\mu+\xi+\delta) T \\
\delta T-\mu Q
\end{array}\right) \tag{3}
\end{align*}
$$

It is easy to check that

$$
\begin{equation*}
\left.G_{i}(X)\right|_{X_{i}(t)=0, X_{t} \in C_{+}} \geq 0, \quad i=1,2,3,4 . \tag{4}
\end{equation*}
$$

Due to Lemma 2 in [25], any solution of (1) is $X(t) \in R_{+}^{4}$ for all $t \geq 0$.

We denote $N(t)=S(t)+A(t)+T(t)+Q(t)$; summing equations in (1) yields

$$
\begin{equation*}
\frac{d N(t)}{d t}=0 \tag{5}
\end{equation*}
$$

so $N(t)=S(t)+A(t)+T(t)+Q(t)=$ constant (denoted as $N$ ), and the set

$$
\begin{equation*}
\Omega=\left\{(S, A, T, Q) \in R_{+}^{4}: S+A+T+Q \leq N\right\} \tag{6}
\end{equation*}
$$

is a positively invariant region for (1). Therefore, we will consider the global stability of (1) on the set $\Omega$.

## 3. The Basic Reproduction Number and Existence of Alcoholism Equilibria

3.1. The Basic Reproduction Number $R_{0}$. In epidemiology, the basic reproduction number (sometimes called basic reproductive rate or basic reproductive ratio) of an infection is the number of infectious cases that one infectious case generates on average over the course of its infectious period. While in this context, it means the number of persons that an alcoholic will "infect" during his "infectious" period in the pure susceptible environment so that the infected persons will enter the alcoholism compartment. It is easy to see that the model has an alcohol free equilibrium $E_{0}=\left(S_{0}, 0,0,0\right)=$ ( $N, 0,0,0$ ). In the following, the basic reproduction number of system (1) will be obtained by the next generation matrix method formulated in [26].

Let $x=(A, T, Q, S)^{T}$, then system (1) can be written as

$$
\begin{equation*}
\frac{d x}{d t}=\mathscr{F}(x)-\mathscr{V}(x) \tag{7}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathscr{F}(x)=\left(\begin{array}{c}
\left(1-u_{1}\right) \frac{\beta S A}{N} \\
0 \\
0 \\
0
\end{array}\right), \\
\mathscr{V}(x)=\left(\begin{array}{c}
\left(u_{2}+\mu\right) A-\xi T \\
(\mu+\xi+\delta) T-u_{2} A \\
\mu Q-\delta T \\
\left(1-u_{1}\right) \frac{\beta S A}{N}+\mu S-\mu N
\end{array}\right) . \tag{8}
\end{gather*}
$$

The Jacobian matrices of $\mathscr{F}(x)$ and $\mathscr{V}(x)$ at the alcohol free equilibrium $E_{0}$ are, respectively,

$$
\left.\begin{array}{c}
D \mathscr{F}\left(E_{0}\right)=\left(\begin{array}{cc}
F_{3 \times 3} & 0 \\
0 & 0
\end{array}\right), \\
D \mathscr{V}\left(E_{0}\right)=\left(\begin{array}{ccc}
V_{3 \times 3} & 0 \\
\left(1-u_{1}\right) \beta & 0 & 0
\end{array}\right) \tag{9}
\end{array}\right), ~ \$
$$

where

$$
F=\left(\begin{array}{ccc}
\left(1-u_{1}\right) \beta & 0 & 0  \tag{10}\\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad V=\left(\begin{array}{ccc}
u_{2}+\mu & -\xi & 0 \\
-u_{2} & \mu+\xi+\delta & 0 \\
0 & -\delta & \mu
\end{array}\right) .
$$

The basic reproduction number, denoted by $R_{0}$, is thus given by

$$
\begin{equation*}
R_{0}=\rho\left(F V^{-1}\right)=\frac{\beta\left(1-u_{1}\right)(\mu+\xi+\delta)}{u_{2}(\mu+\delta)+\mu(\mu+\xi+\delta)} \tag{11}
\end{equation*}
$$

It is easy to see that both of the control parameters contributed to reducing the alcoholism. From this point, the control measures are meaningful.
3.2. Existence of Alcoholism Equilibrium. The endemic equilibrium $E^{*}\left(S^{*}, A^{*}, T^{*}, Q^{*}\right)$ of system (1) is determined by equations

$$
\begin{gather*}
\mu N-\left(1-u_{1}\right) \frac{\beta S A}{N}-\mu S=0 \\
\left(1-u_{1}\right) \frac{\beta S A}{N}+\xi T-\left(u_{2}+\mu\right) A=0  \tag{12}\\
u_{2} A-(\mu+\xi+\delta) T=0  \tag{21}\\
\delta T-\mu Q=0
\end{gather*}
$$

The third equation in (12) leads to

$$
\begin{equation*}
A=\frac{\mu+\xi+\delta}{u_{2}} T \tag{13}
\end{equation*}
$$

From the last equation in (12), we have

$$
\begin{equation*}
Q=\frac{\delta T}{\mu} \tag{14}
\end{equation*}
$$

From the first equation of (12), and together with (13), we can get

$$
\begin{align*}
S & =\frac{\mu N}{\mu+\left(\left(\left(1-u_{1}\right) \beta\right) / N\right) A}  \tag{22}\\
& =\frac{\mu N^{2} u_{2}}{\mu N u_{2}+\left(1-u_{1}\right) \beta T(\mu+\xi+\delta)} . \tag{15}
\end{align*}
$$

Substituting (13)-(15) into the second equation of (12) gives

$$
\begin{align*}
\mu N & -\frac{\mu^{2} N^{2} u_{2}}{\mu N u_{2}+\left(1-u_{1}\right) \beta T(\mu+\xi+\delta)}  \tag{16}\\
& +\xi T-\left(\mu+u_{2}\right) \frac{\mu+\xi+\delta}{u_{2}} T=0
\end{align*}
$$

By simplifying (16), we can get

$$
\begin{align*}
& T\left\{\left[u_{2} \xi\left(1-u_{1}\right) \beta(\mu+\xi+\delta)\right.\right.  \tag{17}\\
& \left.\left.\quad-\left(1-u_{1}\right) \beta(\mu+\xi+\delta)^{2}\left(\mu+u_{2}\right)\right] T+\sigma\right\}=0 \tag{23}
\end{align*}
$$

where

$$
\begin{align*}
\sigma= & u_{2} \mu N\left(1-u_{1}\right) \beta(\mu+\xi+\delta)  \tag{18}\\
& +\left(u_{2}\right)^{2} \mu N \xi-\mu N u_{2}\left(\mu+u_{2}\right)(\mu+\xi+\delta)
\end{align*}
$$

Hence, we get two explicit solutions to (17); one is $T_{0}=0$, which is corresponding to the alcohol free equilibria, and the other is

$$
\begin{align*}
T^{*}= & (\sigma)\left(\left(1-u_{1}\right) \beta(\mu+\xi+\delta)^{2}\left(\mu+u_{2}\right)\right. \\
& \left.\quad-u_{2} \xi\left(1-u_{1}\right) \beta(\mu+\xi+\delta)\right)^{-1}  \tag{19}\\
& =\frac{\sigma}{\left(1-u_{1}\right) \beta(\mu+\xi+\delta)\left[\mu \xi+(\delta+\mu)\left(\mu+u_{2}\right)\right]} \tag{24}
\end{align*}
$$

which should be corresponding to the alcoholism equilibria on condition that $T^{*}>0$; otherwise, the alcoholism equilibria are nonexistent. It is enough to show the positivity of $\sigma$ to make sure the existence of alcoholism equilibria on the condition $R_{0} \geq 1$. By some simple calculations, we simplify the expression of $\sigma$ to be

$$
\begin{align*}
& \sigma=\mu N u_{2}\left\{(\mu+\xi+\delta)\left(1-u_{1}\right) \beta\right.  \tag{20}\\
&\left.-\left[\left(\mu+u_{2}\right)(\delta+\mu)+\mu \xi\right]\right\}
\end{align*}
$$

Since $R_{0}>1$ is equivalent to

$$
\beta\left(1-u_{1}\right)(\mu+\xi+\delta)>u_{2}(\mu+\delta)+\mu(\mu+\xi+\delta)
$$

the right side of this inequality is exactly equal to $\left(\mu+u_{2}\right)(\delta+$ $\mu)+\mu \xi$. Hence, we have proved the existence of $T^{*}>0$, so are the alcoholism equilibria. We summarize this result in Theorem 1.

Theorem 1. For system (1), there is always an alcohol free equilibrium $E_{0}=(N, 0,0,0)$. When $R_{0}>1$, besides alcohol free equilibrium $E_{0}$, system (1) also has a unique alcoholism equilibrium $E^{*}\left(S^{*}, A^{*}, T^{*}, Q^{*}\right)$, where

$$
\begin{gathered}
S^{*}=\frac{\mu N^{2} u_{2}}{\mu N u_{2}+\left(1-u_{1}\right) \beta T^{*}(\mu+\xi+\delta)}, \\
A^{*}=\frac{\mu+\xi+\delta}{u_{2}} T^{*} \\
Q^{*}=\frac{\delta T^{*}}{\mu} \\
T^{*}=\frac{\sigma}{\left(1-u_{1}\right) \beta(\mu+\xi+\delta)\left[\mu \xi+(\delta+\mu)\left(\mu+u_{2}\right)\right]}
\end{gathered}
$$

## 4. Stability Analysis of Equilibria

For the convenience of subsequent proof, we denote a vector $X=(A, T, Q, S)^{T}$ and

$$
f(X)=\left(\begin{array}{c}
\left(1-u_{1}\right) \frac{\beta S A}{N}+\xi T-\left(u_{2}+\mu\right) A \\
u_{2} A-(\mu+\xi+\delta) T \\
\delta T-\mu Q \\
\mu N-\left(1-u_{1}\right) \frac{\beta S A}{N}-\mu S
\end{array}\right)
$$

So the Jacobian matrix of $f(x)$ about vector $X$ is as the following:

$$
\begin{aligned}
J & =\frac{\partial f(X)}{\partial X} \\
& =\left(\begin{array}{cccc}
\frac{\left(1-u_{1}\right) \beta S}{N}-\left(\mu+u_{2}\right) & \xi & 0 & \frac{\left(1-u_{1}\right) \beta A}{N} \\
u_{2} & -(\mu+\xi+\delta) & 0 & 0 \\
0 & \delta & -\mu & 0 \\
-\frac{\left(1-u_{1}\right) \beta S}{N} & 0 & 0 & -\mu-\frac{\left(1-u_{1}\right) \beta A}{N}
\end{array}\right) .
\end{aligned}
$$

Theorem 2. For system (1), the alcohol free equilibrium $E_{0}$ is locally asymptotically stable if $R_{0}<1$.

Proof. Since

$$
\begin{align*}
& J\left(E_{0}\right) \\
& \qquad=\left(\begin{array}{cccc}
\left(1-u_{1}\right) \beta-\left(\mu+u_{2}\right) & \xi & 0 & 0 \\
u_{2} & -(\mu+\xi+\delta) & 0 & 0 \\
0 & \delta & -\mu & 0 \\
-\left(1-u_{1}\right) \beta & 0 & 0 & -\mu
\end{array}\right), \tag{25}
\end{align*}
$$

we can easily get that two of the eigenvalues are $\lambda_{1}=\lambda_{2}=$ $-\mu<0$, while $\lambda_{3}, \lambda_{4}$ satisfy

$$
\begin{align*}
\lambda^{2}+ & {\left[2 \mu+\xi+\delta+u_{2}-\left(1-u_{1}\right) \beta\right] \lambda }  \tag{26}\\
& +(\mu+\xi+\delta)\left(\mu+u_{2}-\left(1-u_{1}\right) \beta\right)-u_{2} \xi=0
\end{align*}
$$

Thus,

$$
\begin{equation*}
\lambda_{3}+\lambda_{4}=\left(1-u_{1}\right) \beta-(\mu+\xi+\delta)-\left(\mu+u_{2}\right) . \tag{27}
\end{equation*}
$$

Since $R_{0}<1$ is equivalent to

$$
\begin{align*}
\beta\left(1-u_{1}\right)(\mu+\xi+\delta) & <u_{2}(\mu+\delta)+\mu(\mu+\xi+\delta) \\
& <\left(\mu+u_{2}\right)(\mu+\xi+\delta), \tag{28}
\end{align*}
$$

so

$$
\begin{equation*}
\beta\left(1-u_{1}\right)<\mu+u_{2}, \tag{29}
\end{equation*}
$$

and then

$$
\begin{equation*}
\lambda_{3}+\lambda_{4}<-(\mu+\xi+\delta)<0 \tag{30}
\end{equation*}
$$

while

$$
\begin{equation*}
\lambda_{3} \lambda_{4}=(\mu+\xi+\delta)\left(\mu+u_{2}-\left(1-u_{1}\right) \beta\right)-u_{2} \xi \tag{31}
\end{equation*}
$$

Similarly from $R_{0}<1$, we can derive the inequality

$$
\begin{equation*}
-\beta\left(1-u_{1}\right)(\mu+\xi+\delta)>-u_{2}(\mu+\delta)-\mu(\mu+\xi+\delta) \tag{32}
\end{equation*}
$$

so

$$
\begin{align*}
\lambda_{3} \lambda_{4}> & \left(\mu+u_{2}\right)(\mu+\xi+\delta)  \tag{33}\\
& -u_{2} \xi-u_{2}(\mu+\delta)-\mu(\mu+\xi+\delta)
\end{align*}
$$

It reduces to

$$
\begin{equation*}
\lambda_{3} \lambda_{4}>0 \tag{34}
\end{equation*}
$$

Hence, $\operatorname{Re} \lambda_{3}<0, \operatorname{Re} \lambda_{4}<0$. The proof is complete.
Next, we will turn to investigate the global stability of $E_{0}$.
Theorem 3. For system (1), the alcohol free equilibrium $E_{0}$ is globally asymptotically stable if $R_{0}<1$.

Proof. Consider the subsystem of (1) as follows:

$$
\begin{gather*}
A^{\prime}=\left(1-u_{1}\right) \frac{\beta S A}{N}+\xi T-\left(u_{2}+\mu\right) A, \\
T^{\prime}=u_{2} A-(\mu+\xi+\delta) T  \tag{35}\\
Q^{\prime}=\delta T-\mu Q .
\end{gather*}
$$

Equation (35) can be rewritten as

$$
\begin{align*}
\left(\begin{array}{c}
\dot{A} \\
\dot{T} \\
\dot{Q}
\end{array}\right)= & (F-V)\left(\begin{array}{c}
A \\
T \\
Q
\end{array}\right)-\left(1-\frac{S}{N}\right)  \tag{36}\\
& \times\left(\begin{array}{ccc}
\beta\left(1-u_{1}\right) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)\left(\begin{array}{c}
A \\
T \\
Q
\end{array}\right) .
\end{align*}
$$

Since $S \leq N$ and $0 \leq u_{1} \leq 1$, then for all $t>0$, we can get

$$
\left(\begin{array}{c}
\dot{A}  \tag{37}\\
\dot{T} \\
\dot{Q}
\end{array}\right) \leq(F-V)\left(\begin{array}{c}
A \\
T \\
Q
\end{array}\right)
$$

According to Lemma 1 in [26], all the eigenvalues of matrix $F-V$ have negative real parts, so the solutions of this subsystem are stable whenever $R_{0}<1$. So $(A(t), T(t), Q(t)) \rightarrow$ $(0,0,0)$ as $t \rightarrow \infty$. By the comparison theorem [27], and based on the fact that the total population is constant $N$, it follows that $(A(t), T(t), Q(t)) \rightarrow(0,0,0)$ and $S(t) \rightarrow N$ as $t \rightarrow \infty$. So the alcohol free equilibrium $E_{0}$ is globally asymptotically stable; the proof is complete.

Theorem 4. For system (1), the alcoholism equilibrium $E^{*}\left(S^{*}, A^{*}, T^{*}, Q^{*}\right)$ is globally asymptotically stable if $R_{0}>1$.

Proof. Since the total population in model (1) is a constant number $N$, in order to prove the global stability of system (1), it is sufficed to prove the corresponding stability of subsystem (38):

$$
\begin{gather*}
S^{\prime}=\mu N-\left(1-u_{1}\right) \frac{\beta S A}{N}-\mu S \\
A^{\prime}=\left(1-u_{1}\right) \frac{\beta S A}{N}+\xi T-\left(u_{2}+\mu\right) A  \tag{38}\\
T^{\prime}=u_{2} A-(\mu+\xi+\delta) T
\end{gather*}
$$

We make normalization transform and still use the same symbols $S, A, T$ to denote the variables; then (38) can be transformed into

$$
\begin{gather*}
S^{\prime}=\mu-\left(1-u_{1}\right) \beta S A-\mu S \\
s A^{\prime}=\left(1-u_{1}\right) \beta S A+\xi T-\left(u_{2}+\mu\right) A,  \tag{39}\\
T^{\prime}=u_{2} A-(\mu+\xi+\delta) T
\end{gather*}
$$

From (39), we can easily know that the equilibria ( $S^{*}, A^{*}, T^{*}$ ) satisfy the following three equalities to be used later:

$$
\begin{gather*}
\mu=\left(1-u_{1}\right) \beta S^{*} A^{*}-\mu S^{*} \\
\left(1-u_{1}\right) \beta S^{*} A^{*}+\xi T^{*}=\left(u_{2}+\mu\right) A^{*}  \tag{40}\\
u_{2} A^{*}=(\mu+\xi+\delta) T^{*}
\end{gather*}
$$

Let $V=x_{1}\left(S-S^{*}-S^{*} \ln \left(S / S^{*}\right)\right)+x_{2}\left(A-A^{*}-\right.$ $\left.A^{*} \ln \left(A / A^{*}\right)\right)+x_{3}\left(T-T^{*}-T^{*} \ln \left(T / T^{*}\right)\right)$; then

$$
\begin{align*}
& \left.V^{\prime}\right|_{(39)}=x_{1}\left[\mu-\left(1-u_{1}\right) \beta S A-\mu S\right. \\
& \left.-\frac{S^{*}}{S} \mu+\left(1-u_{1}\right) \beta S^{*} A+\mu S^{*}\right] \\
& +x_{2}\left[\left(1-u_{1}\right) \beta S A+\xi T-\left(u_{2}+\mu\right) A\right. \\
& \left.-\left(1-u_{1}\right) \beta S A^{*}-\frac{A^{*}}{A} \xi T+\left(u_{2}+\mu\right) A^{*}\right] \\
& +x_{3}\left[u_{2} A-(\mu+\xi+\delta) T\right. \\
& \left.-\frac{T^{*}}{T} u_{2} A+(\mu+\xi+\delta) T^{*}\right] \\
& =x_{1}\left[\left(1-u_{1}\right) \beta S^{*} A^{*}+\mu S^{*}-\left(1-u_{1}\right) \beta S A-\mu S\right. \\
& -\frac{S^{*}}{S}\left(\left(1-u_{1}\right) \beta S^{*} A^{*}+\mu S^{*}\right) \\
& \left.+\left(1-u_{1}\right) \beta S^{*} A+\mu S^{*}\right] \\
& +x_{2}\left[\left(1-u_{1}\right) \beta S A+\xi T-\left(u_{2}+\mu\right) A\right. \\
& \left.-\left(1-u_{1}\right) \beta S A^{*}-\frac{A^{*}}{A} \xi T+\left(u_{2}+\mu\right) A^{*}\right] \\
& +x_{3}\left[u_{2} A-(\mu+\xi+\delta) T\right. \\
& \left.-\frac{T^{*}}{T} u_{2} A+(\mu+\xi+\delta) T^{*}\right] \\
& =x_{1} \mu S^{*}\left(2-\frac{S}{S^{*}}-\frac{S^{*}}{S}\right) \\
& +\left[x_{1}\left(1-u_{1}\right) \beta S^{*} A^{*}+x_{2}\left(u_{2}+\mu\right) A^{*}\right. \\
& \left.+x_{3}(\mu+\xi+\delta) T^{*}\right] \\
& -\left[x_{1} \frac{\left(S^{*}\right)^{2}\left(1-u_{1}\right) \beta A^{*}}{S}+x_{2}\left(1-u_{1}\right) \beta S A^{*}\right. \\
& \left.+x_{2} \frac{A^{*} \xi T}{A}+x_{3} \frac{T^{*}}{T} u_{2} A\right] \\
& +S A\left[-x_{1}\left(1-u_{1}\right) \beta+x_{2}\left(1-u_{1}\right) \beta\right] \\
& +A\left[-\left(u_{2}+\mu\right) x_{2}+x_{1}\left(1-u_{1}\right) \beta S^{*}+u_{2} x_{3}\right] \\
& +T\left[\xi x_{2}-(\mu+\xi+\delta) x_{3}\right] . \tag{41}
\end{align*}
$$

To eliminate the cross-term $S A$ and two single-variable terms $A$ and $T$, we let

$$
\begin{gather*}
-x_{1}\left(1-u_{1}\right) \beta+x_{2}\left(1-u_{1}\right) \beta=0, \\
-\left(u_{2}+\mu\right) x_{2}+x_{1}\left(1-u_{1}\right) \beta S^{*}+u_{2} x_{3}=0,  \tag{42}\\
\xi x_{2}-(\mu+\xi+\delta) x_{3}=0 .
\end{gather*}
$$

By solving them, we can get

$$
\begin{gather*}
x_{1}=1, \quad x_{2}=1, \\
x_{3}=\frac{\xi}{\mu+\xi+\delta}=\frac{u_{2}+\mu-\left(1-u_{1}\right) \beta S^{*}}{u_{2}} . \tag{43}
\end{gather*}
$$

Next, we let

$$
\begin{aligned}
V_{1}^{\prime}= & x_{1}\left(1-u_{1}\right) \beta S^{*} A^{*}+x_{2}\left(u_{2}+\mu\right) A^{*} \\
& +x_{3}(\mu+\xi+\delta) T^{*} \\
V_{2}^{\prime}= & -\left[x_{1} \frac{\left(S^{*}\right)^{2}\left(1-u_{1}\right) \beta A^{*}}{S}\right.
\end{aligned}
$$

$$
\begin{equation*}
\left.+x_{2}\left(1-u_{1}\right) \beta S A^{*}+x_{2} \frac{A^{*} \xi T}{A}+x_{3} \frac{T^{*}}{T} u_{2} A\right] \tag{44}
\end{equation*}
$$

and then

$$
\begin{equation*}
V^{\prime}=x_{1} \mu S^{*}\left(2-\frac{S}{S^{*}}-\frac{S^{*}}{S}\right)+V_{1}^{\prime}+V_{2}^{\prime} \tag{45}
\end{equation*}
$$

Due to

$$
\begin{gather*}
\left(u_{2}+\mu\right) A^{*}=\left(1-u_{1}\right) \beta S^{*} A^{*}+\xi T^{*} \\
x_{3}(\mu+\xi+\delta) T^{*}=\xi T^{*} x_{2}=\xi T^{*} \tag{46}
\end{gather*}
$$

so

$$
\begin{align*}
& V_{1}^{\prime}=2\left(1-u_{1}\right) \beta S^{*} A^{*}+2 \xi T^{*}, \\
V_{2}^{\prime}= & -\left[\frac{\left(S^{*}\right)^{2}\left(1-u_{1}\right) \beta A^{*}}{S}\right. \\
& \left.+\left(1-u_{1}\right) \beta S A^{*}+\frac{A^{*} \xi T}{A}+x_{3} \frac{T^{*}}{T} u_{2} A\right] \\
\leq & -2\left[\frac{\left(S^{*}\right)^{2}\left(1-u_{1}\right) \beta A^{*}}{S} \cdot\left(1-u_{1}\right) \beta S A^{*}\right]^{1 / 2} \\
= & -2\left(1-u_{1}\right) \beta S^{*} A^{*}-2\left(x_{3} A^{*} \xi u_{2} T^{*}\right)^{1 / 2}  \tag{47}\\
= & -2\left(1-u_{1}\right) \beta S^{*} A^{*}-2\left[\xi T^{*}\left(x_{3} u_{2} A^{*}\right)\right]^{1 / 2} \\
= & -2\left(1-u_{1}\right) \beta S^{*} A^{*} \\
& -2\left\{\xi T^{*}\left[\left(u_{2}+\mu\right) A^{*}-\left(1-u_{1}\right) \beta S^{*} A^{*}\right]\right\}^{1 / 2} \\
= & -2\left(1-u_{1}\right) \beta S^{*} A^{*}-2\left(\xi T^{*} \xi T^{*}\right)^{1 / 2} \\
= & -2\left(1-u_{1}\right) \beta S^{*} A^{*}-2 \xi T^{*} .
\end{align*}
$$

Hence,

$$
\begin{align*}
V^{\prime}= & \mu S^{*}\left(2-\frac{S}{S^{*}}-\frac{S^{*}}{S}\right)+V_{1}^{\prime}+V_{2}^{\prime} \\
\leq & \mu S^{*}\left(2-\frac{S}{S^{*}}-\frac{S^{*}}{S}\right)+2\left(1-u_{1}\right) \beta S^{*} A^{*}  \tag{48}\\
& +2 \xi T^{*}-2\left(1-u_{1}\right) \beta S^{*} A^{*}-2 \xi T^{*} \\
= & \mu S^{*}\left(2-\frac{S}{S^{*}}-\frac{S^{*}}{S}\right) \leq 0,
\end{align*}
$$

$V^{\prime}=0$ if and only if $S=S^{*}, A=A^{*}, T=T^{*}$. According to LaSalle's invariance principle [28], we can derive the conclusion that the alcoholism equilibria $E^{*}\left(S^{*}, A^{*}, T^{*}, Q^{*}\right)$ are globally asymptotically stable; the proof is complete.

## 5. Optimal Control Problem

5.1. The Existence of Optimal Control. In order to investigate an effective campaign to control alcoholism in a community which pursue the goals of the minimized alcoholisms and more recovered individuals, we will reconsider the system (1) and use two control variables to reduce the numbers of alcoholics. The difference is that we will change the parameters $u_{1}, u_{2}$ into control variable $u_{1}(t), u_{2}(t)$. Their aforementioned definitions allow us to do so. $u_{1}(t)$ is used to limit the proportion of the susceptible individual to contact with alcoholism, usually by propaganda and education, so that the susceptible individual can stay off alcoholism consciously and be free of "infection," we can understand the effect of $u_{1}(t)$
is to prevent the the susceptible from contacting with the alcoholism. The control variable $u_{2}(t)$ is used to control the alcoholism to take appropriate treatment measures, such as taking pills or seeking other medical help. However, just as a coin has two sides, there will be a lot of costs generated during the control process. So it is advisable to balance between the costs and the alcohol effects. In view of this, our optimal control problem to minimize the objective functional is given by

$$
\begin{equation*}
J\left(u_{1}, u_{2}\right)=\int_{0}^{t_{f}}\left[A(t)+\frac{c_{1}}{2} u_{1}^{2}(t)+\frac{c_{2}}{2} u_{2}^{2}(t)\right] d t \tag{49}
\end{equation*}
$$

which subjects to system

$$
\begin{gather*}
S^{\prime}=\mu N-\left(1-u_{1}(t)\right) \frac{\beta S A}{N}-\mu S, \\
A^{\prime}=\left(1-u_{1}(t)\right) \frac{\beta S A}{N}+\xi T-\left(u_{2}(t)+\mu\right) A,  \tag{50}\\
T^{\prime}=u_{2}(t) A-(\mu+\xi+\delta) T, \\
Q^{\prime}=\delta T-\mu \mathrm{Q},
\end{gather*}
$$

with initial conditions

$$
\begin{array}{ll}
S(0)=S_{0}, & A(0)=A_{0} \\
T(0)=T_{0}, & Q(0)=Q_{0} . \tag{51}
\end{array}
$$

Here, $u_{i}(t) \in U \triangleq\left\{\left(u_{1}, u_{2}\right) \mid u_{i}(t)\right.$ is measurable and $0 \leq$ $u_{i}(t) \leq 1$, for all $\left.t \in\left[0, t_{f}\right]\right\}, t_{f}$ is the end time to be controlled, $U$ is an admissible control set, $c_{i}$, and $i=1,2$, are weight factors (positive constants) that adjust the intensity of two different control measures.

Next, we will investigate the existence of the optimal control of the above-mentioned problem.

Theorem 5. There exists an optimal control pair $u^{*}=$ $\left(u_{1}^{*}, u_{2}^{*}\right) \in U$ such that

$$
\begin{equation*}
J\left(u_{1}^{*}, u_{2}^{*}\right)=\min J\left(u_{1}, u_{2}\right), \quad u_{1}(t), u_{2}(t) \in U \tag{52}
\end{equation*}
$$

subjects to the control system (1) with initial conditions (50).
Proof. To prove the existence of an optimal control, according to the classic literature [29], we have to show the following.
(1) The control and state variables are nonnegative values.
(2) The control set $U$ is convex and closed.
(3) The right side of the state system is bounded by linear function in the state and control variables.
(4) The integrand of the objective functional is concave on $U$.
(5) There exist constants $d_{1}, d_{2}>0$ and $\alpha>1$ such that the integrand $L\left(t ; u_{1} ; u_{2}\right) \triangleq A(t)+\left(c_{1} / 2\right) u_{1}^{2}(t)+$ $\left(c_{2} / 2\right) u_{2}^{2}(t)$ of the objective functional satisfies

$$
\begin{equation*}
L\left(t ; u_{1} ; u_{2}\right) \geq d_{1}\left(\left|u_{1}\right|^{2}+\left|u_{2}\right|^{2}\right)^{\alpha / 2}-d_{2} \tag{53}
\end{equation*}
$$

statements (1), (2) and (3) are obvious satisfied, we only need to test and verify the latter two ones. Since the four state variables have been all proved to be up bounded by $N$, we will get the following equalities:

$$
\begin{gather*}
S^{\prime} \leq \mu N, \quad A^{\prime} \leq\left(1-u_{1}(t)\right) \beta S+\xi T \\
T^{\prime} \leq u_{2}(t) A, \quad Q^{\prime} \leq \delta T \tag{54}
\end{gather*}
$$

so the fourth condition is set up. As for the last condition,

$$
\begin{equation*}
L\left(t ; u_{1} ; u_{2}\right) \geq d_{1}\left(\left|u_{1}\right|^{2}+\left|u_{2}\right|^{2}\right)^{\alpha / 2}-d_{2} \tag{55}
\end{equation*}
$$

is also true, when we choose $d_{1}=\min \left\{c_{1} / 2, c_{2} / 2\right\}$, and for all $d_{2} \in R^{+}, \alpha=2$. The proof is complete.

We next come to the core of this section.
5.2. The Characterization of the Optimal Control. With the existence of the optimal control pairs established, we now present the optimality system and use a result from [30]; we can easily know the existence of the solutions to the optimality system (71) which will be gotten later. Firstly, we come to discuss the theorem that relates to the characterization of the optimal control. The optimality system can be used to compute candidates for optimal control pairs. To do this, we begin by defining an augmented Hamiltonian $H$ with penalty terms for the control constraints as follows:

$$
\begin{align*}
H= & A(t)+\frac{c_{1}}{2} u_{1}^{2}(t)+\frac{c_{2}}{2} u_{2}^{2}(t) \\
& +\lambda_{1}\left[\mu N-\left(1-u_{1}(t)\right) \frac{\beta S A}{N}-\mu S\right] \\
& +\lambda_{2}\left[\left(1-u_{1}(t)\right) \frac{\beta S A}{N}+\xi T-\left(u_{2}(t)+\mu\right) A\right]  \tag{56}\\
& +\lambda_{3}\left[u_{2}(t) A-(\mu+\xi+\delta) T\right]+\lambda_{4}(\delta T-\mu Q) \\
& -w_{11} u_{1}(t)-w_{12}\left(1-u_{1}(t)\right) \\
& -w_{21} u_{2}(t)-w_{22}\left(1-u_{2}(t)\right)
\end{align*}
$$

where $w_{i j}(t) \geq 0$ are the penalty multipliers satisfying

$$
\begin{aligned}
w_{11}(t) u_{1}(t) & =w_{12}(t)\left(1-u_{1}(t)\right) \\
& =0 \text { at optimal control } u_{1}^{*} \\
w_{21}(t) u_{2}(t) & =w_{22}(t)\left(1-u_{2}(t)\right) \\
& =0 \text { at optimal control } u_{2}^{*}
\end{aligned}
$$

Theorem 6. Given optimal control pairs $\left(u_{1}^{*}, u_{2}^{*}\right)$ and solutions $S(t), A(t), T(t), Q(t)$ of the corresponding state system (50), there exist adjoint variables $\lambda_{i}, i=1,2,3,4$, satisfying

$$
\begin{gather*}
\lambda_{1}^{\prime}=\lambda_{1}\left(1-u_{1}(t)\right) \frac{\beta A}{N}+\mu \lambda_{1}-\lambda_{2}\left(1-u_{1}(t)\right) \frac{\beta A}{N} \\
\lambda_{2}^{\prime}=-1+\lambda_{1}\left(1-u_{1}(t)\right) \frac{\beta S}{N}-\lambda_{2}\left(1-u_{1}(t)\right) \frac{\beta S}{N} \\
+\lambda_{2}\left(\mu+u_{2}(t)\right)-\lambda_{3} u_{2}(t)  \tag{58}\\
\lambda_{3}^{\prime}=-\lambda_{2} \xi+\lambda_{3}(\mu+\xi+\delta)-\lambda_{4} \delta \\
\lambda_{4}^{\prime}=\mu \lambda_{4}
\end{gather*}
$$

with the terminal conditions

$$
\begin{equation*}
\lambda_{i}\left(t_{f}\right)=0, \quad i=1,2,3,4 . \tag{59}
\end{equation*}
$$

Furthermore, $\left(u_{1}^{*}, u_{2}^{*}\right)$ are represented by

$$
\begin{align*}
& u_{1}^{*}=\min \left(1, \max \left(0, \frac{\beta S A\left(\lambda_{2}-\lambda_{1}\right)}{c_{1} N}\right)\right),  \tag{60}\\
& u_{2}^{*}=\min \left(1, \max \left(0, \frac{A\left(\lambda_{2}-\lambda_{3}\right)}{c_{2}}\right)\right)
\end{align*}
$$

Proof. According to Pontryagin Maximum Principle [2931], we first differentiate the Hamiltonian operator $H$, with respect to states. Then the adjoint system can be written as

$$
\begin{array}{ll}
\lambda_{1}^{\prime}=-\frac{\partial H}{\partial S}, & \lambda_{2}^{\prime}=-\frac{\partial H}{\partial A} \\
\lambda_{3}^{\prime}=-\frac{\partial H}{\partial T}, & \lambda_{4}^{\prime}=-\frac{\partial H}{\partial Q} . \tag{61}
\end{array}
$$

The terminal condition (56) of adjoint equations is given by $\lambda_{i}\left(t_{f}\right)=0, i=1,2,3,4$.

To obtain the necessary conditions of optimality (59), we also differentiate the Hamiltonian operator $H$, with respect to $U=\left(u_{1}, u_{2}\right)$ and set them equal to zero; then

$$
\begin{gather*}
\frac{\partial H}{\partial u_{1}}=c_{1} u_{1}(t)+\lambda_{1} \frac{\beta S A}{N}-\lambda_{2} \frac{\beta S A}{N}-w_{11}+w_{12}=0, \\
\frac{\partial H}{\partial u_{2}}=c_{2} u_{2}(t)-\lambda_{2} A+\lambda_{3} A-w_{21}+w_{22}=0 . \tag{62}
\end{gather*}
$$

By solving the optimal control, we obtain

$$
\begin{equation*}
u_{1}^{*}=\frac{1}{c_{1}}\left[\left(\lambda_{2}-\lambda_{1}\right) \frac{\beta S A}{N}+w_{11}-w_{12}\right] . \tag{63}
\end{equation*}
$$

To determine an explicit expression for the optimal control without $w_{11}$ and $w_{12}$, a standard optimality technique is utilized [29]. We consider the following three cases.
(i) On the set $\left\{t \mid 0<u_{1}^{*}(t)<1\right\}$, we have $w_{11}(t)=$
$w_{12}(t)=0$. Hence, the optimal control is

$$
\begin{equation*}
u_{1}^{*}=\frac{1}{c_{1}}\left[\left(\lambda_{2}-\lambda_{1}\right) \frac{\beta S A}{N}\right] . \tag{64}
\end{equation*}
$$

(ii) On the set $\left\{t \mid u_{1}^{*}(t)=1\right\}$, we have $w_{11}(t)=0$. Hence,

$$
\begin{equation*}
1=u_{1}^{*}(t)=\frac{1}{c_{1}}\left[\left(\lambda_{2}-\lambda_{1}\right) \frac{\beta S A}{N}-w_{12}\right] . \tag{65}
\end{equation*}
$$

This implies that

$$
\begin{equation*}
\frac{1}{c_{1}}\left(\lambda_{2}-\lambda_{1}\right) \frac{\beta S A}{N} \geq 1 \quad \text { since } w_{12}(t) \geq 0 \tag{66}
\end{equation*}
$$

(iii) On the set $\left\{t \mid u_{1}^{*}(t)=0\right\}$, we have $w_{12}(t)=0$. Hence,

$$
\begin{equation*}
0=u_{1}^{*}(t)=\frac{1}{c_{1}}\left[\left(\lambda_{2}-\lambda_{1}\right) \frac{\beta S A}{N}+w_{11}\right] . \tag{67}
\end{equation*}
$$

This implies that

$$
\begin{equation*}
\frac{1}{c_{1}}\left(\lambda_{2}-\lambda_{1}\right) \frac{\beta S A}{N} \leq 0 \quad \text { since } w_{11}(t) \geq 0 . \tag{68}
\end{equation*}
$$

Combining these results, the optimal control $u_{1}^{*}(t)$ is characterized as

$$
\begin{equation*}
u_{1}^{*}=\min \left\{1, \max \left\{0, \frac{1}{c_{1}}\left[\left(\lambda_{2}-\lambda_{1}\right) \frac{\beta S A}{N}\right]\right\}\right\} \tag{69}
\end{equation*}
$$

Using the similar arguments, we can also obtain the other optimal control function

$$
\begin{equation*}
u_{2}^{*}=\min \left\{1, \max \left\{0, \frac{\left(\lambda_{2}-\lambda_{3}\right) A}{c_{2}}\right\}\right\} \tag{70}
\end{equation*}
$$

The proof is complete.

We point out that the optimality system consists of the state system (50) with the initial conditions $S(0), A(0), T(0), Q(0)$, the adjoint (or costate) system (58) with the terminal conditions (59), and the optimality condition (60). Any optimal control pairs must satisfy this optimality system. For the convenience of subsequent
numerical simulation in Section 6, we give the optimality system as follows:

$$
\begin{align*}
& S^{\prime}=\mu N-\left(1-\min \left\{1, \max \left\{0, \frac{1}{c_{1}}\left[\left(\lambda_{2}-\lambda_{1}\right) \frac{\beta S A}{N}\right]\right\}\right\}(t)\right) \\
& \times \frac{\beta S A}{N}-\mu S, \\
& A^{\prime}=\left(1-\min \left\{1, \max \left\{0, \frac{1}{c_{1}}\left[\left(\lambda_{2}-\lambda_{1}\right) \frac{\beta S A}{N}\right]\right\}\right\}(t)\right) \frac{\beta S A}{N} \\
& +\xi T-\left(\min \left\{1, \max \left\{0, \frac{\left(\lambda_{2}-\lambda_{3}\right) A}{c_{2}}\right\}\right\}(t)+\mu\right) A, \\
& T^{\prime}=\min \left\{1, \max \left\{0, \frac{\left(\lambda_{2}-\lambda_{3}\right) A}{c_{2}}\right\}\right\}(t) A \\
& -(\mu+\xi+\delta) T, \\
& Q^{\prime}=\delta T-\mu Q, \\
& \lambda_{1}^{\prime}=\lambda_{1}\left(1-\min \left\{1, \max \left\{0, \frac{1}{c_{1}}\left[\left(\lambda_{2}-\lambda_{1}\right) \frac{\beta S A}{N}\right]\right\}\right\}(t)\right) \\
& \times \frac{\beta A}{N}+\mu \lambda_{1} \\
& -\lambda_{2}\left(1-\min \left\{1, \max \left\{0, \frac{1}{c_{1}}\left[\left(\lambda_{2}-\lambda_{1}\right) \frac{\beta S A}{N}\right]\right\}\right\}(t)\right) \\
& \times \frac{\beta A}{N}, \\
& \lambda_{2}^{\prime}=-1 \\
& +\lambda_{1}\left(1-\min \left\{1, \max \left\{0, \frac{1}{c_{1}}\left[\left(\lambda_{2}-\lambda_{1}\right) \frac{\beta S A}{N}\right]\right\}\right\}(t)\right) \\
& \times \frac{\beta S}{N}-\lambda_{3} \cdot \min \left\{1, \max \left\{0, \frac{\left(\lambda_{2}-\lambda_{3}\right) A}{c_{2}}\right\}\right\}(t) \\
& -\lambda_{2}\left(1-\min \left\{1, \max \left\{0, \frac{1}{c_{1}}\left[\left(\lambda_{2}-\lambda_{1}\right) \frac{\beta S A}{N}\right]\right\}\right\}\right) \\
& \times \frac{\beta S}{N}+\lambda_{2}\left(\mu+\min \left\{1, \max \left\{0, \frac{\left(\lambda_{2}-\lambda_{3}\right) A}{c_{2}}\right\}\right\}(t)\right), \\
& \lambda_{3}^{\prime}=-\lambda_{2} \xi+\lambda_{3}(\mu+\xi+\delta)-\lambda_{4} \delta, \\
& \lambda_{4}^{\prime}=\mu \lambda_{4}, \\
& \begin{array}{lll}
S(0)=S_{0}, & A(0)=A_{0}, & T(0)=T_{0}, \\
Q(0)=Q_{0}, & \lambda_{i}\left(t_{f}\right)=0, & i=1,2,3,4 .
\end{array} \tag{71}
\end{align*}
$$

5.3. The Uniqueness of Optimal Control. Due to the a priori boundedness of the state, adjoint functions, and the resulting Lipschitz structure of the ODEs, we can obtain the uniqueness of the optimal control.

Lemma 7 (see [23]). The function $u^{*}(s)=\min (b, \max (s, a))$ is Lipschitz continuous in $s$, where $a<b$ are some fixed positive constants.

Theorem 8. For all $t \in\left[0, t_{f}\right]$, the solution to the optimality system (71) is unique.

Proof. Suppose $\quad\left(S, A, T, Q, \lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right) \quad$ and $\left(\bar{S}, \bar{A}, \bar{T}, \bar{Q}, \bar{\lambda}_{1}, \bar{\lambda}_{2}, \bar{\lambda}_{3}, \bar{\lambda}_{4}\right)$ are two different solutions of our optimality system (71). Let

$$
\begin{gathered}
S=e^{\lambda t} m, \quad A=e^{\lambda t} n, \quad T=e^{\lambda t} p, \\
Q=e^{\lambda t} q, \quad \lambda_{1}=e^{-\lambda t} r, \quad \lambda_{2}=e^{-\lambda t} s, \\
\lambda_{3}=e^{-\lambda t} w, \quad \lambda_{4}=e^{-\lambda t} v, \\
\bar{S}=e^{\lambda t} \bar{m}, \quad \bar{A}=e^{\lambda t} \bar{n}, \quad \bar{T}=e^{\lambda t} \bar{p}, \\
\bar{Q}=e^{\lambda t} \bar{q}, \quad \bar{\lambda}_{1}=e^{-\lambda t} \bar{r}, \quad \bar{\lambda}_{2}=e^{-\lambda t} \bar{s}, \\
\bar{\lambda}_{3}=e^{-\lambda t} \bar{w}, \quad \bar{\lambda}_{4}=e^{-\lambda t} \bar{v}
\end{gathered}
$$

where $\lambda>0$ is to be chosen.
Accordingly, we have

$$
\begin{gather*}
u_{1}^{*}(t)=\min \left\{1, \max \left\{0, \frac{\beta m n(s-r) e^{\lambda t}}{c_{1} N}\right\}\right\}, \\
u_{2}^{*}(t)=\min \left\{1, \max \left\{0, \frac{(s-w) n}{c_{2}}\right\}\right\},  \tag{73}\\
\bar{u}_{1}^{*}(t)=\min \left\{1, \max \left\{0, \frac{\beta \overline{m n}(\bar{s}-\bar{r}) e^{\lambda t}}{c_{1} N}\right\}\right\}, \\
\bar{u}_{2}^{*}(t)=\min \left\{1, \max \left\{0, \frac{(\bar{s}-\bar{w}) \bar{n}}{c_{2}}\right\}\right\} .
\end{gather*}
$$

Now we substitute $S=e^{\lambda t} m, A=e^{\lambda t} n, T=e^{\lambda t} p, Q=e^{\lambda t} q$, $\lambda_{1}=e^{-\lambda t} r, \lambda_{2}=e^{-\lambda t} s, \lambda_{3}=e^{-\lambda t} w, \lambda_{4}=e^{-\lambda t} v$ and $\bar{S}=$ $e^{\lambda t} \bar{m}, \bar{A}=e^{\lambda t} \bar{n}, \bar{T}=e^{\lambda t} \bar{p}, \bar{Q}=e^{\lambda t} \bar{q}, \overline{\lambda_{1}}=e^{-\lambda t} \bar{r}, \overline{\lambda_{2}}=$ $e^{-\lambda t} \bar{s}, \overline{\lambda_{3}}=e^{-\lambda t} \bar{w}, \overline{\lambda_{4}}=e^{-\lambda t} \bar{v}$ into the first ODE of (71), respectively; then we can obtain

$$
\begin{aligned}
\dot{m}+\lambda m= & \mu N e^{-\lambda t} \\
& -\left(1-\min \left\{1, \max \left\{0, \frac{\beta m n(s-r) e^{\lambda t}}{c_{1} N}\right\}\right\}\right) \\
& \times \frac{\beta m n e^{\lambda t}}{N}-\mu m,
\end{aligned}
$$

for $m$ and $\bar{m}$, respectively. Similarly, we can derive

$$
\begin{align*}
\dot{n}+ & \lambda n \\
= & \left(1-\min \left\{1, \max \left\{1, \frac{\beta m n(s-r) e^{\lambda t}}{c_{1} N}\right\}\right\}\right) \frac{\beta m n e^{\lambda t}}{N} \\
& +\xi p-\left(\min \left\{1, \max \left\{0, \frac{n(s-w)}{c_{2}}\right\}\right\}+\mu\right) n, \tag{75}
\end{align*}
$$

for $n$ and $\bar{n}$, respectively;

$$
\begin{align*}
\dot{p}+\lambda p= & \min \left\{1, \max \left\{0, \frac{n(s-w)}{c_{2}}\right\}\right\} n  \tag{76}\\
& -(\mu+\xi+\delta) p
\end{align*}
$$

for $p$ and $\bar{p}$, respectively;

$$
\begin{equation*}
\dot{q}+\lambda q=\delta p-\mu q \tag{77}
\end{equation*}
$$

for $q$ and $\bar{q}$, respectively;

$$
\begin{align*}
\dot{r}-\lambda r= & r\left(1-\min \left\{1, \max \left\{0, \frac{\beta m n(s-r) e^{\lambda t}}{c_{1} N}\right\}\right\}\right) \frac{\beta n e^{\lambda t}}{N} \\
& +\mu r-s\left(1-\min \left\{1, \max \left\{0, \frac{\beta m n(s-r) e^{\lambda t}}{c_{1} N}\right\}\right\}\right) \\
& \times \frac{\beta n e^{\lambda t}}{N} \tag{78}
\end{align*}
$$

for $r$ and $\bar{r}$, respectively;

$$
\begin{align*}
\dot{s}-\lambda s= & -e^{\lambda t} \\
& +r\left(1-\min \left\{1, \max \left\{0, \frac{\beta m n(s-r) e^{\lambda t}}{c_{1} N}\right\}\right\}\right) \frac{\beta m e^{\lambda t}}{N} \\
& -s\left(1-\min \left\{1, \max \left\{0, \frac{\beta m n(s-r) e^{\lambda t}}{c_{1} N}\right\}\right\}\right) \\
& \times \frac{\beta m e^{\lambda t}}{N}+s\left(\min \left\{1, \max \left\{0, \frac{n(s-w)}{c_{2}}\right\}\right\}+\mu\right) \\
& -w\left(\min \left\{1, \max \left\{0, \frac{n(s-w)}{c_{2}}\right\}\right\}\right), \tag{79}
\end{align*}
$$

for $s$ and $\bar{s}$, respectively;

$$
\begin{equation*}
\dot{w}-\lambda w=-s \xi+(\mu+\xi+\delta) w-\delta v \tag{80}
\end{equation*}
$$

for $w$ and $\bar{w}$, respectively;

$$
\begin{equation*}
\dot{v}-\lambda v=\mu v \tag{81}
\end{equation*}
$$

for $v$ and $\bar{v}$, respectively.

$$
\begin{align*}
& \text { By Lemma 7, we can obtain } \\
& \qquad \begin{array}{l}
\left|u_{1}^{*}(t)-\bar{u}_{1}^{*}(t)\right| \leq \frac{\beta e^{\lambda t}}{c_{1} N}|m n(s-r)-\overline{m n}(\bar{s}-\bar{r})|, \\
\quad\left|u_{2}^{*}(t)-\bar{u}_{2}^{*}(t)\right| \leq \frac{1}{c_{2}}|n(s-w)-\bar{n}(\bar{s}-\bar{w})|
\end{array} \tag{82}
\end{align*}
$$

The equations for $m, n, p, q, r, s, w, v$ and the equations for $\bar{m}, \bar{n}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{w}, \bar{v}$ are subtracted, respectively; then we multiply each equation by appropriate difference of functions and integrate from 0 to $t_{f}$. Next, we add all eight integral equations and some inequality techniques to obtain uniqueness. The following calculation is similar; for the sake of simplicity, we only take $m$ and $\bar{m}$ for an example:

$$
\begin{aligned}
& \dot{m}-\dot{\bar{m}}+(\mu+\lambda)(m-\bar{m}) \\
&=\frac{\beta e^{\lambda t}}{N}[ -\left(1-\min \left\{1, \max \left\{0, \frac{\beta m n(s-r) e^{\lambda t}}{c_{1} N}\right\}\right\}\right) m n \\
&\left.+\left(1-\min \left\{1, \max \left\{0, \frac{\beta \overline{m n}(\bar{s}-\bar{r}) e^{\lambda t}}{c_{1} N}\right\}\right\}\right) \overline{m n}\right] .
\end{aligned}
$$

Multiplying both sides of (83) by $(m-\bar{m})$ and integrating from 0 to $t_{f}$ gives

$$
\begin{aligned}
& \begin{array}{l}
\frac{1}{2}(m-\bar{m})^{2}\left(t_{f}\right)+(\mu+\lambda) \int_{0}^{t_{f}}(m-\bar{m})^{2} d t \\
= \\
\quad \int_{0}^{t_{f}}(m-\bar{m}) \frac{\beta e^{\lambda t}}{N} \\
\quad \times\left[-\left(1-u_{1}^{*}\right) m n+\left(1-\bar{u}_{1}^{*}\right) \overline{m n}\right] d t \\
=\int_{0}^{t_{f}}(m-\bar{m}) \frac{\beta e^{\lambda t}}{N}\left[\left(\bar{u}_{1}^{*}-1\right)(m n-m \bar{n}+m \bar{n}-\overline{m n})\right. \\
\left.\quad+m n\left(u_{1}^{*}-\bar{u}_{1}^{*}\right)\right] d t \\
\leq \frac{\beta e^{\lambda t_{f}}}{N} \int_{0}^{t_{f}}(m-\bar{m}) \\
\quad \times\left[\bar{u}_{1}^{*}-1\left|(m|n-\bar{n}|+|m-\bar{m}| \bar{n})+m n \frac{\beta e^{\lambda t}}{c_{1} N}\right|\right. \\
\quad \times m n(s-r)-\overline{m n}(\bar{s}-\bar{r})] d t \\
\leq \frac{\beta e^{\lambda t_{f}}}{N} \int_{0}^{t_{f}}(m-\bar{m}) \\
\quad \times\left[\left|\bar{u}_{1}^{*}-1\right|(m|n-\bar{n}|+|m-\bar{m}| \bar{n})+m n \frac{\beta e^{\lambda t}}{c_{1} N}\right.
\end{array}
\end{aligned}
$$

$$
\begin{align*}
& \mid m n(s-\bar{s})+(m n-\overline{m n}) \bar{s} \\
& -(m n(r-\bar{r})+(m n-\overline{m n}) \bar{r}) \mid] d t \\
& \leq \frac{\beta e^{\lambda t_{f}}}{N} \int_{0}^{t_{f}}\left[\left|\bar{u}_{1}^{*}-1\right||m||m-\bar{m}||n-\bar{n}|\right. \\
& +|m-\bar{m}|^{2}|\bar{n}|+|B||\bar{s}||\bar{n}||m-\bar{m}|^{2} \\
& +|B||m n||m-\bar{m}||r-\bar{r}| \\
& +|B||m n||m-\bar{m}||s-\bar{s}| \\
& +|B||\bar{s}||m||m-\bar{m}||n-\bar{n}| \\
& +|B||\bar{r}||m||m-\bar{m}||n-\bar{n}| \\
& \left.+|B||\bar{r}||\bar{n}||m-\bar{m}|^{2}\right] d t \\
& \leq \frac{\beta e^{\lambda t_{f}}}{N} \int_{0}^{t_{f}}\left[\frac{\left|\bar{u}^{*}-1\right|}{2}|m|\left((m-\bar{m})^{2}+(n-\bar{n})^{2}\right)\right. \\
& +|\bar{n}|(m-\bar{m})^{2} \\
& +\frac{|B||m n|}{2}\left((m-\bar{m})^{2}+(s-\bar{s})^{2}\right) \\
& +\frac{|B||\bar{s}||m|}{2}\left((m-\bar{m})^{2}+(n-\bar{n})^{2}\right) \\
& +|B||\bar{s}||\bar{n}|(m-\bar{m})^{2} \\
& +\frac{|B||m n|}{2}\left((m-\bar{m})^{2}+(r-\bar{r})^{2}\right) \\
& +\frac{|B||m||\bar{r}|}{2}\left((m-\bar{m})^{2}+(n-\bar{n})^{2}\right) \\
& \left.+|B||\bar{r}||\bar{n}|(m-\bar{m})^{2}\right] d t . \tag{84}
\end{align*}
$$

In the above derivation, we use many scaling techniques for inequality or absolute inequality. Particularly, what should be noted is that to get the first inequality of above derivation, we use the estimation of $\left|u_{1}^{*}(t)-\bar{u}_{1}^{*}(t)\right|$ which has been given before; besides, for the sake of convenience, we note $B=$ $m n\left(\beta e^{\lambda t_{f}} / N\right)$. Furthermore, we notice that the coefficients of all the eight terms in the last formula: $(m-\bar{m})^{2}+(n-\bar{n})^{2}$, $(m-\bar{m})^{2},(m-\bar{m})^{2}+(s-\bar{s})^{2},(m-\bar{m})^{2}+(n-\bar{n})^{2},(m-\bar{m})^{2}$, $(m-\bar{m})^{2}+(r-\bar{r})^{2},(m-\bar{m})^{2}+(n-\bar{n})^{2},(m-\bar{m})^{2}$, namely, $\left(\mid \bar{u}_{1}^{*}-\right.$ $1 \mid / 2)|m|,|\bar{n}|,(|B||m n|) / 2,(|B||\bar{s}||m|) / 2,|B||\bar{s}| \bar{n},(|B||m n|) / 2$, $(|B||m \||\bar{r}|) / 2,|B||\bar{r}||\bar{n}|$ are nonnegative and bounded. So there exists a positive constant $c_{2}$ such that

$$
\frac{1}{2}(m-\bar{m})^{2}\left(t_{f}\right)+(\mu+\lambda) \int_{0}^{t_{f}}(m-\bar{m})^{2} d t
$$

$$
\begin{align*}
\leq c_{2} \frac{\beta e^{\lambda t_{f}}}{N} \int_{0}^{t_{f}} & \left((m-\bar{m})^{2}+(n-\bar{n})^{2}\right.  \tag{85}\\
& \left.+(s-\bar{s})^{2}+(r-\bar{r})^{2}\right) d t
\end{align*}
$$

Combining eight of these inequalities gives

$$
\begin{align*}
& \frac{1}{2}(m-\bar{m})\left(t_{f}\right)+\frac{1}{2}(n-\bar{n})\left(t_{f}\right)+\frac{1}{2}(p-\bar{p})\left(t_{f}\right) \\
& \quad+\frac{1}{2}(q-\bar{q})\left(t_{f}\right)+\frac{1}{2}(r-\bar{r})(0)+\frac{1}{2}(s-\bar{s})(0) \\
& \quad+\frac{1}{2}(w-\bar{w})(0)+\frac{1}{2}(v-\bar{v})(0)+(\mu+\lambda) \\
& \times \int_{0}^{t_{f}}\left\{(m-\bar{m})^{2}+(n-\bar{n})^{2}+(p-\bar{p})^{2}\right. \\
&  \tag{86}\\
& \quad+(q-\bar{q})^{2}+(s-\bar{s})^{2} \\
& \left.\left.\quad+(r-\bar{r})^{2}+(w-\bar{w})^{2}+(v-\bar{v})^{2}\right)\right\} d t \\
& \leq \bar{B} \int_{0}^{t_{f}}\left\{(m-\bar{m})^{2}+(n-\bar{n})^{2}+(p-\bar{p})^{2}\right. \\
& \\
& \quad+(q-\bar{q})^{2}+(r-\bar{r})^{2}+(s-\bar{s})^{2} \\
& \\
& \left.\quad+(w-\bar{w})^{2}+(v-\bar{v})^{2}\right\} d t
\end{align*}
$$

Thus, from the above inequality we can conclude that

$$
\begin{align*}
&(\mu+\lambda-\bar{B}) \int_{0}^{t_{f}}\left\{(m-\bar{m})^{2}+(n-\bar{n})^{2}\right. \\
&+(p-\bar{p})^{2}+(q-\bar{q})^{2}+(r-\bar{r})^{2} \\
&\left.+(s-\bar{s})^{2}+(w-\bar{w})^{2}+(v-\bar{v})^{2}\right\} d t \leq 0 \tag{87}
\end{align*}
$$

where $\bar{B}$ depends on the coefficients and the bounds depend on $m, n, p, q, r, s, w, v$. If we choose $\lambda$ such that $\mu+\lambda>\bar{B}$, then $m=\bar{m}, n=\bar{n}, p=\bar{p}, q=\bar{q}, r=\bar{r}, s=\bar{s}, w=\bar{w}$, and $v=\bar{v}$. Hence, the solution to the optimality system is unique. The proof is complete.

## 6. Numerical Simulation

6.1. The Simulation of State System (1) without Control Parameters. For the sake of simplicity but without loss of generality, we will perform the numerical simulation of state system (1) with parameters $u_{1}=0, u_{2}=0$. Before illustrating the analytic properties of the alcoholism model (1), we will target the populations in the environment of a community or a university, for example, the school of material science and engineering in our university, that is, Lan zhou University of Technology (LUT for short), owing to the accurate and available information we can obtain. Referring to the information provided by the admissions office of LUT, this school will enroll almost 1200 undergraduates and almost 300 various postgraduates at the beginning of
fall semester; at the same time, there will be almost 1500 various students graduated and left this school, so the scale of students in school remained almost 6000; we can take the total population $N=6000$. In this simulation, we will take September as the initial time and units in one week, period in one year. According to the investigations of the student union implemented in September every year, we can take initial values as $S(0)=4500, A(0)=1000, T(0)=300$, and $Q(0)=200$. It seems that the alcoholism is a little bit more, but it is rather natural because many freshmen feel confused when they are faced with the new environment and a new lifestyle; many of them have no better choice but gather together to drink in small groups to mediate the anxiety and get to know each other; over time, some of them develop the habit of drinking. To a certain extent, for example, the frequent drinking badly affects their study; we can classify them into the alcoholism compartment. Other initial values seem more reasonable, so we need no more explanation. As we know, alcoholism death is seldom happen within one year, so we omit mortality from alcoholism; then how to understand the recruitment rate as well as natural death rate $\mu$ ? We can treat the freshmen admission as the recruitment population and graduation students as the natural "death" parts. So we can take $\mu=1500 / 6000=0.25$, which is exactly consistent with the value in [16]. As for the infection rate $\beta$ and recovery rate $\delta$, we will let them be variables, since the drinking behaviors are related to many factors such as the season and the pressure.

According to the data we get from the student union, we choose $\xi=0.4$. To summarize, we list the values of the parameters in Table 1. Using the values of parameters in Table 1, we can plot Figures 2 and 3 which are on condition $R_{0}<1$ and $R_{0} \geq 1$, respectively. From Figure 2, we easily know when $R_{0}<1$ holds; the solution of system (1) tends to the alcohol free equilibrium $E_{0}$ and verifies the global stability of $E_{0}$. While seen from Figure 3, we also know that if $R_{0} \geq$ 1 holds, the solution of system (1) tends to the alcoholism equilibrium $E^{*}$ and verifies the global stability of $E^{*}$.
6.2. The Sensitivity of $R_{0}$ about Two Control Parameters. Although from the expression of the model reproduction number $R_{0}$, we can easily find out the fact that the two control variables, that is, $u_{1}$ and $u_{2}$, attribute to reducing the severity of alcoholism; we will still depict the graph between $R_{0}$ and the two control variables to see more intuitiveness see Figure 4. It seems from the figure that $R_{0}$ is a monotonically decreasing function about two control parameters, so it is advisable to take two approaches simultaneously to control the alcoholics.
6.3. The Simulation of Optimality System (71). In this subsection, we will investigate numerically the optimal solution to optimality system (71) by numerical method from [32]; the optimality system is solved with a fourth-order Runge-Kutta scheme. Beginning with a guess for the control variables, the state system is solved forward in time and then those values of state are used to solve the adjoint equations backward in time. The controls are updated at the end of each iteration using


Figure 2: When $R_{0}<1$, the alcohol free equilibrium $E_{0}$ is globally asymptotically stable ( $\beta=0.2, \delta=0.3$, and $R_{0}=0.71698$ ).


Figure 3: When $R_{0} \geq 1$, the alcoholism equilibrium $E^{*}=$ $(4466,476,32,26)$ corresponding to the given parameters is globally asymptotically stable ( $\beta=0.3, \delta=0.2$, and $R_{0}=1.08511$ ).
the values of optimal controls obtained lastly. The iterations continue until convergence takes place.

In the simulations, we choose the available variable values as Table 1 shows; besides, $\beta=0.3, \delta=0.2$. The initial value of model (1) is assumed to be $S(0)=4500, A(0)=1000$, $T(0)=300$, and $Q(0)=200$ as before.

The ideal weights in objective functional are very difficult to obtain in reality; it needs much work on data mining and fitting. Hence, the acquisition of appropriate practical weights

Table 1: The parameters description of model (1).

| Parameter | Description | Values |
| :--- | :--- | :---: |
| $\mu$ | Natural birth rate or death rate | 0.25 |
| $\beta$ | Transmission coefficient between <br> alcoholism and susceptibles | Variables |
| $N$ | The total populations to be considered | 6000 |
| $\delta$ | The rate of populations quitting from <br> alcoholism permanently after treatment | Variables |
| $\xi$ | The rate of populations failed in <br> treatment and returned to be alcoholic | 0.4 |



Figure 4: The relationship between $R_{0}$ and two control variables $u_{1}$ and $u_{2}$.
is still a difficult problem and remains for further investigations. The cost associated with $A(t)$ and $u_{1}(t)$ mainly includes the cost of dangerous behaviours during the alcoholism time and educating the public, while the cost associated with $u_{2}(t)$ mainly comes from health professional and the medical resource including medicines and nursing care. In view of this and taking the expressions of $u_{1}, u_{2}$ into account, after many numerical simulations, we finally give weighting coefficients as $c_{1}=10^{2} ; c_{2}=10^{4}$. It should be pointed out that the weights here are of only theoretical interest to reveal the control strategies proposed in this paper. Another point to note is that the maximum control is very difficult to achieve in reality, so we will omit the situation of the maximum control during the series of simulations.

Next, we will make some necessary instructions and explanations to the above simulation graphs. Figures 5, 6, 7 , and 8 depict the number of four compartments under different control levels when we choose the weight coefficients in objective function to be $c_{1}=10^{2} ; c_{2}=10^{4}$. From the four simulation graphs, we can observe the following simple facts, in reducing the total number of alcoholisms and increasing the number of susceptibles; the effectiveness of various control measures is as follows: optimal control is evidently better than middle control, and middle control is better than single control $u_{2}$, single control $u_{2}$ is better than single control $u_{1}$, while single control $u_{1}$ is much better than no control.

Figures 9 and 10 depict the optimal control law of $u_{1}, u_{2}$, respectively. In the beginning of the simulation, the control


Figure 5: Number of the susceptibles when we choose the weights in objective function are $c_{1}=10^{2} ; c_{2}=10^{4}$ and under different optimal control strategies, that is, (1) without control: $u_{1}=u_{2}=0$; (2) single control: $u_{1}=0.5, u_{2}=0$; (3) single control: $u_{1}=0, u_{2}=0.5$; (4) middle control: $u_{1}=u_{2}=0.5$; (5) optimal control: $u_{1}=u_{1}^{*}, u_{2}=u_{2}^{*}$.


Figure 6: Number of the alcoholics when we choose the weights in objective function are $c_{1}=10^{2} ; c_{2}=10^{4}$ and under different optimal control strategies, that is, (1) without control: $u_{1}=u_{2}=0$; (2) single control: $u_{1}=0.5, u_{2}=0$; (3) single control: $u_{1}=0, u_{2}=0.5$; (4) middle control: $u_{1}=u_{2}=0.5$; (5) optimal control: $u_{1}=u_{1}^{*}, u_{2}=u_{2}^{*}$.


Figure 7: Number of the persons in treatment when we choose the weights in objective function are $c_{1}=10^{2} ; c_{2}=10^{4}$ and under different optimal control strategies, that is, (1) without control: $u_{1}=$ $u_{2}=0$; (2) single control: $u_{1}=0.5, u_{2}=0$; (3) single control: $u_{1}=0$, $u_{2}=0.5$; (4) middle control: $u_{1}=u_{2}=0.5$; (5) optimal control: $u_{1}=u_{1}^{*}, u_{2}=u_{2}^{*}$.


Figure 8: Number of people in quitting compartment when we choose the weights in objective function are $c_{1}=10^{2} ; c_{2}=10^{4}$ and under different optimal control strategies, that is, (1) without control: $u_{1}=u_{2}=0$; (2) single control: $u_{1}=0.5, u_{2}=0$; (3) single control: $u_{1}=0, u_{2}=0.5$; (4) middle control: $u_{1}=u_{2}=0.5$; (5) optimal control: $u_{1}=u_{1}^{*}, u_{2}=u_{2}^{*}$.


Figure 9: Figures of the optimal control $u_{1}$ when we choose the weight in objective function are $c_{1}=10^{2} ; c_{2}=10^{4}$.


Figure 10: Figures of the optimal control $u_{2}$ when we choose the weight in objective function are $c_{1}=10^{2} ; c_{2}=10^{4}$.
effort of $u_{1}$ should be decreased from 0.65 to 0.5 within the first month, and over the next week, it should be increased to the maximum control until 50 weeks, then rapidly decreased to 0 at the end of the simulation. As for the control $u_{2}$, it should start from around 0.5 due to the initial alcoholics then increase to 0.55 within one week since the rapid infection and next decrease to almost 0 since the effectiveness of treatment in three weeks, but with the infection going on, the control effort of $u_{2}$ should gradually increase to the maximum and maintain this level until the tenth week for the purpose of consolidation therapy and preventing rebound; hereafter, it should be gradually decreased to the level of almost 0.43 until the fifty weeks then quickly decreased to 0 in the end.


Figure 11: Number of the susceptibles when we choose the weights in objective function are $c_{1}=10^{4}, c_{2}=10^{2}$ and under different optimal control strategies, that is, (1) without control: $u_{1}=u_{2}=0$; (2) single control: $u_{1}=0.5, u_{2}=0$; (3) single control: $u_{1}=0$, $u_{2}=0.5$; (4) middle control: $u_{1}=u_{2}=0.5$; (5) optimal control: $u_{1}=u_{1}^{*}, u_{2}=u_{2}^{*}$.

In order to investigate the influence of different weight coefficients in the objective functional on the effect of controlling, at the same time, for a better comparison, we will change the weight coefficients in objective function into $c_{2}=$ $10^{4} ; c_{1}=10^{2}$, and we will list the corresponding numerical simulation results as Figures 11-16 show.

When we change the weight coefficients in objective function into $c_{1}=10^{4}, c_{2}=10^{2}$, we find that the results of simulations derived from the graphs are very similar to the ones before. We speculate that the most likely reasons of this result are due to three respects; one is that the weight coefficients are not too sensitive in the numerical simulation, and another possible reason is that both of the two controls are important, in some sense, equivalently important. The last but not the most unlikely reason is that we have not found the most appropriate weight coefficients in the simulation, which is very difficult to find as previously mentioned.

## 7. Conclusions

In this paper, we formulate an alcoholics quitting model and firstly investigate the variation discipline of various populations from the perspective of global stability; then we propose an objective functional to examine two different control measures (i.e., prevention and treatment) on the effect of alcohol. The basic reproduction number of the model was derived and the global stability of the two equilibria is given. From the expression of the basic reproduction number $R_{0}$ and


Figure 12: Number of the alcoholics when we choose the weights in objective function are $c_{1}=10^{4}, c_{2}=10^{2}$ and under different optimal control strategies, that is, (1) without control: $u_{1}=u_{2}=0$; (2) single control: $u_{1}=0.5, u_{2}=0$; (3) single control: $u_{1}=0, u_{2}=0.5$; (4) middle control: $u_{1}=u_{2}=0.5$; (5) optimal control: $u_{1}=u_{1}^{*}, u_{2}=u_{2}^{*}$.


Figure 13: Number of the persons in treatment when we choose the weights in objective function are $c_{1}=10^{4}, c_{2}=10^{2}$ and under different optimal control strategies, that is, (1) without control: $u_{1}=$ $u_{2}=0$; (2) single control: $u_{1}=0.5, u_{2}=0$; (3) single control: $u_{1}=0$, $u_{2}=0.5$; (4) middle control: $u_{1}=u_{2}=0.5$; (5) optimal control: $u_{1}=u_{1}^{*}, u_{2}=u_{2}^{*}$.


Figure 14: Number of people in quitting compartment when we choose the weights in objective function are $c_{1}=10^{4}, c_{2}=10^{2}$ and under different optimal control strategies, that is, (1) without control: $u_{1}=u_{2}=0$; (2) single control: $u_{1}=0.5, u_{2}=0$; (3) single control: $u_{1}=0, u_{2}=0.5$; (4) middle control: $u_{1}=u_{2}=0.5$; (5) optimal control: $u_{1}=u_{1}^{*}, u_{2}=u_{2}^{*}$.


Figure 15: Figures of the optimal control $u_{1}$ when we choose the weight in objective function are $c_{1}=10^{4}, c_{2}=10^{2}$.
related numerical simulation, we can easily see that the two control strategies are effective in the alcoholics process.

Using Pontryagin's Maximum Principle, we firstly determine the necessary conditions for existence of optimal control pairs. The uniqueness of the solution to the optimality system (71) is derived by the classical method of contradiction. Numerical simulations of the model suggest that the two different groups of weights in the objective function have


Figure 16: Figures of the optimal control $u_{2}$ when we choose the weight in objective function are $c_{1}=10^{4}, c_{2}=10^{2}$.
much similar effects on the transmission of the alcoholism; from this point, the two control measures are almost equally important in controlling the alcoholism, although they will probably have great influences on the cost of the objective function. From the simulation figures, it seems that the effect of optimal control, which is measured by the reduction in the number of alcoholics and the increase in the number of susceptibles, is much better than other control strategies as noted earlier in the simulation section. According to the realtime curve of two optimal controls, we point out the specific implementation methods of optimal control which can be achieved in practice.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work was partially supported by the NSF of Gansu Province of China (2013GS09485), the NNSF of China (10961018), the NSF of Gansu Province of China (1107RJZA088), the NSF for Distinguished Young Scholars of Gansu Province of China (1111RJDA003), the Special Fund for the Basic Requirements in the Research of University of Gansu Province of China and the Development Program for Hong Liu Distinguished Young Scholars in Lanzhou University of Technology.

## References

[1] R. Room, T. Babor, and J. Rehm, "Alcohol and public health," The Lancet, vol. 365, no. 9458, pp. 519-530, 2005.
[2] J. B. Saunders, O. G. Aasland, A. Amundsen, and M. Grant, "Alcohol consumption and related problems among primary
health care patients: WHO collaborative project on early detection of persons with harmful alcohol consumption I," Addiction, vol. 88, no. 3, pp. 349-362, 1993.
[3] L. D. Johnston, P. M. OMalley, and J. G. Bachman, National Survey Results on Drug Use From the Monitoring the Future Study, 1975-1992, National Institute on Drug Abuse, Rockville, Md, USA, 1993.
[4] H. Wechsler, J. E. Lee, M. Kuo, and H. Lee, "College binge drinking in the 1990s: a continuing problem. Results of the Harvard School of Public Health 1999 College Alcohol Study," Journal of American College Health, vol. 48, no. 5, pp. 199-210, 2000.
[5] P. M. O'Malley and L. D. Johnston, "Epidemiology of alcohol and other drug use among American college students," Journal of Studies on Alcohol, vol. 63, no. 14, pp. 23-39, 2002.
[6] K. Agarwal-Kozlowski and D. P. Agarwal, "Genetic predisposition to alcoholism," Therapeutische Umschau, vol. 57, no. 4, pp. 179-184, 2000.
[7] C. Y. Chen, C. L. Storr, and J. C. Anthony, "Early-onset drug use and risk for drug dependence problems," Addictive Behaviors, vol. 34, no. 3, pp. 319-322, 2009.
[8] M. Glavas and J. Weinberg, "Stress, alcohol consumption and the hypothalamic-pituitary adrenal axis," in Nutrients, Stress and Medical Disorders, pp. 165-183, Humana Press, New York, NY, USA, 2006.
[9] L. N. Blum, N. H. Nielsen, J. A. Riggs, and L. B. Bresolin, "Alcoholism and alcohol abuse among women: report of the Council on Scientific Affairs," Journal of Women's Health, vol. 7, no. 7, pp. 861-871, 1998.
[10] B. Benedict, "Modeling alcoholism as a contagious disease: how "infected" drinking buddies spread problem drinking," SIAM News, vol. 40, no. 3, 2007.
[11] H. Walter, K. Gutierrez, K. Ramskogler, I. Hertling, A. Dvorak, and O. M. Lesch, "Gender-specific differences in alcoholism: Implications for treatment," Archives of Women's Mental Health, vol. 6, no. 4, pp. 253-258, 2003.
[12] D. Müller, R. D. Koch, H. von Specht, w. Völker, and E. M. Münch, "Neurophisiologic findings in chronic alcohol abuse," Psychiatrie, Neurologie, und Medizinische Psychologie, vol. 37, no. 3, pp. 129-132, 1985.
[13] G. Testino, "Alcoholic diseases in hepato-gastroenterology: a point of view," Hepato-Gastroenterology, vol. 55, no. 82-83, pp. 371-377, 2008.
[14] A. Mubayi, P. E. Greenwood, C. Castillo-Chávez, P. J. Gruenewald, and D. M. Gorman, "The impact of relative residence times on the distribution of heavy drinkers in highly distinct environments," Socio-Economic Planning Sciences, vol. 44, no. 1, pp. 45-56, 2010.
[15] D. Müller, R. D. Koch, H. von Specht, W. Völker, and E. M. Münch, "Neurophysiologic findings in chronic alcohol abuse," Psychiatrie, Neurologie, Und Medizinische Psychologie, vol. 37, no. 3, pp. 129-132, 1985, (Leipz).
[16] G. Mulone and B. Straughan, "Modeling binge drinking," International Journal of Biomathematics, vol. 5, no. 1, Article ID 1250005, 14 pages, 2012.
[17] H. F. Huo and N. N. Song, "Global stability for a binge drinking model with two stages," Discrete Dynamics in Nature and Society, vol. 2012, Article ID 829386, 15 pages, 2012.
[18] S. S. Mushayabasa and C. P. Bhunu, "Modelling the effects of heavy alcohol consumption on the transmission dynamics of gonorrhea," Nonlinear Dynamics, vol. 66, no. 4, pp. 695-706, 2011.
[19] F. S. Sáncheznchez, X. Wang, C. Castillo-Chvez, D. M. Gorman, and P. J. Gruenewald, "Drinking as an epidemic: a simple mathematical model with recovery and relapse," in Therapists Guide to Evidence-Based Relapse Prevention, K. Witkiewitz and G. Marlatt, Eds., Academic Press, New York, NY, USA, 2007.
[20] H.-F. Huo and C.-C. Zhu, "Influence of relapse in a giving up smoking model," Abstract and Applied Analysis, vol. 2013, Article ID 525461, 12 pages, 2013.
[21] H. F. Huo and Q. Wang, "The effects of awareness programs by 17 media on the drinking dynamics," Submitted.
[22] S. Lee, E. Jung, and C. Castillo-Chavez, "Optimal control intervention strategies in low- and high-risk problem drinking populations," Socio-Economic Planning Sciences, vol. 44, no. 4, pp. 258-265, 2010.
[23] K. R. Fister, S. Lenhart, and J. S. McNally, "Optimizing chemotherapy in an HIV model," Electronic Journal of Differential Equations, vol. 1998, no. 32, pp. 1-12, 1998.
[24] S. Lenhart and J. T. Workman, Optimal Control Applied to Biological Models, Chapman \& Hall/CRC Mathematical and Computational Biology Series, Chapman \& Hall/CRC, Boca Raton, Fla, USA, 2007.
[25] X. Yang, L. Chen, and J. Chen, "Permanence and positive periodic solution for the single-species nonautonomous delay diffusive models," Computers \& Mathematics with Applications. An International Journal, vol. 32, no. 4, pp. 109-116, 1996.
[26] P. van den Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," Mathematical Biosciences, vol. 180, pp. 29-48, 2002.
[27] V. Lakshmikantham, S. Leela, and A. A. Martynyuk, Stability Analysis of Nonlinear Systems, vol. 125, Marcel Dekker, New York, NY, USA, 1989.
[28] J. P. LaSalle, The Stability of Dynamical Systems, vol. 25 of Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics, 1976.
[29] W. H. Fleming and R. W. Rishel, Deterministic and Stochastic Optimal Control, vol. 1 of Applications of Mathematics, Springer, Berlin, Germany, 1975.
[30] D. L. Lukes, Differential Equations: Classical to Controlled, Mathematics in Science and Engineering, Academic Press, New York, NY, USA, 1982.
[31] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, The Mathematical Theory of Optimal Processes, John Wiley \& Sons, New Jersey, NJ, USA, 1962.
[32] W. Hackbusch, "A numerical method for solving parabolic equations with opposite orientations," Computing, vol. 20, no. 3, pp. 229-240, 1978.

## Research Article

# Dynamics Analysis of a Stochastic SIR Epidemic Model 

Feng Rao<br>College of Sciences, Nanjing University of Technology, Nanjing 211816, China<br>Correspondence should be addressed to Feng Rao; raofeng2002@163.com

Received 2 January 2014; Accepted 27 January 2014; Published 2 March 2014
Academic Editor: Weiming Wang
Copyright © 2014 Feng Rao. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We investigate an SIR epidemic model with stochastic perturbations. We assume that stochastic perturbations are of a white noise type which is directly proportional to the distances of three variables from the steady-state values, respectively. By constructing suitable Lyapunov functions and applying Itô's formula, some qualitative properties are obtained, such as the existence of global positive solutions, stochastic boundedness, and permanence. A series of numerical simulations to illustrate these mathematical findings are presented.


## 1. Introduction

Almost all mathematical models for the transmission of infectious diseases descend from the classical susceptible-infective-removed (SIR) model of Kermack and McKendrick [1]. The dynamic behavior of different epidemic models and a lot of their extensions is well investigated by a number of scholars; see [2-11]. The basic and important research subjects for recent studies are the existence of the threshold values which distinguish whether the disease dies out, the stability of the disease-free and the endemic equilibria, permanence, and extinction [12]. During the last few decades, a number of realistic transmission functions have become the focus of considerable attention, and many authors are interested in the formulation of nonlinear incidence rate (see [13-17]). A nonlinear incidence rate can arise from saturation effects that if the proportion of the infection in a population is very high, so that exposure to the disease agent is virtually certain, then the transmission rate may respond more slowly than linear to the increase in the number of infection [18]. For example, Capasso and Serio [19] introduced a saturated transmission rate $f(S, I)=k S I /(1+\alpha I)$, where $k I$ measures the infection force of the disease and $1 /(1+\alpha I)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. To be biologically feasible, the function of the incidence rate $f(S, I)=k S I /(1+\alpha I)$
is a positive continuous and differentiable and satisfies the conditions

$$
\begin{gather*}
f(S, 0)=0=f(0, I) \\
\frac{\partial f(S, I)}{\partial S}=\frac{k I}{1+\alpha I}>0, \quad \frac{\partial f(S, I)}{\partial I}=\frac{k S}{(1+\alpha I)^{2}}>0 \tag{1}
\end{gather*}
$$

for all $S, I>0$. It is easy to know that the function $f(S, I)$ is concave with respect to the variable $I$; that is,

$$
\begin{equation*}
\frac{\partial^{2} f(S, I)}{\partial I^{2}}=-\frac{2 \alpha k S}{(1+\alpha I)^{3}}<0 \quad \text { for } S, I>0 \tag{2}
\end{equation*}
$$

which implies that when the number of infections is very high that the exposure to the disease agent is virtually certain, the incidence rate will respond more slowly than linearly to the disease in $I$.

In the real world, population dynamics is inevitably subjected to environmental noise, which is an important component in an ecosystem. Most natural phenomena do not follow strictly deterministic laws but rather oscillate randomly about some average values, so that the population density never attains a fixed value with the advancement of time [20, 21]. Recent advances in stochastic differential equations enable a lot of authors to introduce randomness into deterministic model of physical phenomena to reveal the effect of environmental variability, whether it is a random
noise in the system of differential equations or environmental fluctuations in parameters; see [12, 13, 22-30]. Of them, Tuckwell and Williams [28] investigated the properties of a simple discrete time stochastic epidemic model. A classical model of an SIRS epidemic in an open population was considered by El Maroufy et al. [12]. They established the global stability of disease-free and endemic equilibrium points for both the deterministic and stochastic models. Based on the theory of stochastic differential equation, Cai et al. [13] studied the dynamics of an SIRS epidemic model with a ratiodependent incidence rate. In [29], the authors extended the classical SIRS epidemic model incorporating media coverage from a deterministic framework to a stochastic differential equation and focused on how environmental fluctuations of the contact coefficient affect the extinction of the disease.

To the best of our knowledge, a small amount of work has been done with stochastic perturbation on an SIR epidemic model with a saturated transmission rate $k S I /(1+\alpha I)$. The purpose of this paper is to study that the stochastic factor has a significant effect on the dynamics of SIR epidemic model with a saturated incidence rate. The organization of this paper is as follows. In the next section, we present the formulation of mathematical model with environmental noise. We give some properties about deterministic model (4) and carry out the analysis of the dynamical properties of stochastic model (3), respectively. Finally, we give a concluding section.

## 2. Model and Dynamics Analysis

Let $S(t)$ be the number of susceptible individuals, $I(t)$ the number of infective individuals, and $R(t)$ the number of removed individuals at time $t$, respectively. Motivated by [31], we assume that stochastic perturbations are of white noise type, which are directly proportional to distances $S(t), I(t), R(t)$ from the steady-state values of $S^{*}, I^{*}, R^{*}$ and influence on $\mathrm{d} S(t) / \mathrm{d} t, \mathrm{~d} I(t) / \mathrm{d} t, \mathrm{~d} R(t) / \mathrm{d} t$, respectively. In this way, an SIR epidemic model with a saturated transmission rate and stochastic fluctuations will be reduced to the following form:

$$
\begin{gather*}
\mathrm{d} S=\left(b-d S-\frac{k S I}{1+\alpha I}+\gamma R\right) \mathrm{d} t+\sigma_{1}\left(S-S^{*}\right) \mathrm{d} B(t), \\
\mathrm{d} I=\left(\frac{k S I}{1+\alpha I}-(d+\mu) I\right) \mathrm{d} t+\sigma_{2}\left(I-I^{*}\right) \mathrm{d} B(t),  \tag{3}\\
\mathrm{d} R=(\mu I-(d+\gamma) R) \mathrm{d} t+\sigma_{3}\left(R-R^{*}\right) \mathrm{d} B(t) .
\end{gather*}
$$

All parameters are positive constants, $b$ is the recruitment rate of the population, $d$ is the natural death rate of the population, $k$ is the proportionality constant, $\alpha$ is the parameter that measures the psychological or inhibitory effect, $\gamma$ is the rate at which recovered individuals lose immunity and return to the susceptible class, and $\mu$ is the natural recovery rate of the infective individuals. Note that $\sigma_{1}, \sigma_{2}$, and $\sigma_{3}$ are real constants and known as the intensity of the stochastic environment and $B(t)$ is standard Brownian motion.
2.1. Dynamics of the Deterministic Model. In this subsection, when $\sigma_{1}=\sigma_{2}=\sigma_{3}=0$, we consider the deterministic SIR epidemic model:

$$
\begin{gather*}
\frac{\mathrm{d} S}{\mathrm{~d} t}=b-d S-\frac{k S I}{1+\alpha I}+\gamma R, \\
\frac{\mathrm{~d} I}{\mathrm{~d} t}=\frac{k S I}{1+\alpha I}-(d+\mu) I,  \tag{4}\\
\frac{\mathrm{~d} R}{\mathrm{~d} t}=\mu I-(d+\gamma) R .
\end{gather*}
$$

Because of the biological meaning of the components ( $S(t), I(t), R(t)$ ), we focus on the model in the first quadrant $\mathbb{R}_{+}^{3}=\left\{(S, I, R) \in \mathbb{R}^{3}: S \geq 0, I \geq 0, R \geq 0\right\}$. Model (4) always has a disease-free equilibrium $E_{0}=(b / d, 0,0)$, which corresponds to the extinction of the disease.

Define the basic reproduction number as

$$
\begin{equation*}
R_{0}=\frac{b k}{d(d+\mu)} \tag{5}
\end{equation*}
$$

which denotes the number of individuals infected by a single infected individual placed in a totally susceptible population.

## Theorem 1. From model (4), it follows that

(i) if $R_{0} \leq 1$, there is no positive equilibrium;
(ii) if $R_{0}>1$, there is a unique endemic equilibrium $E^{*}=$ ( $S^{*}, I^{*}, R^{*}$ ), which corresponds to the coexistence of $S$, $I$, and $R$ and is given by

$$
\begin{gather*}
S^{*}=\frac{(d+\mu)(d(d+\mu+\gamma)+\alpha b(d+\gamma))}{\alpha d(d+\mu)(d+\gamma)+d k(d+\mu+\gamma)} \\
I^{*}=\frac{(d+\gamma)(b k-d(d+\mu))}{(d+\mu)(d(d+\mu+\gamma)+\alpha b(d+\gamma))} S^{*}  \tag{6}\\
R^{*}=\frac{\mu}{d+\gamma} I^{*}
\end{gather*}
$$

In other words, when $R_{0}>1$, the disease can invade a totally susceptible population and the number of cases will increase, whereas when $R_{0} \leq 1$, the disease will always fail to spread.

Lemma 2. The plane $S+I+R=b / d$ is a manifold of model (4), which is attracting in the first octant.

Proof. Summing up the three equations in (4) and denoting $N(t)=S(t)+I(t)+R(t)$, we obtain

$$
\begin{equation*}
\frac{\mathrm{d} N}{\mathrm{~d} t}=b-d N \tag{7}
\end{equation*}
$$

It is clear that $N(t)=b / d$ is a solution of (7) and for any $N\left(t_{0}\right) \geq 0$, the general solution of (7) is

$$
\begin{equation*}
N(t)=\frac{1}{d}\left(b-\left(b-d N\left(t_{0}\right)\right) e^{-d\left(t-t_{0}\right)}\right) \tag{8}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} N(t)=\frac{b}{d} \tag{9}
\end{equation*}
$$

which implies the conclusion.

Theorem 3. The endemic equilibrium point $E^{*}=\left(S^{*}, I^{*}, R^{*}\right)$ is globally asymptotically stable in $\mathbb{R}_{+}^{3}$.

Proof. The Jacobian matrix $\mathbf{J}(S, I, R)$ at equilibrium point $E^{*}=\left(S^{*}, I^{*}, R^{*}\right)$ is given by

$$
\mathbf{J}\left(E^{*}\right)=\left(\begin{array}{lll}
J_{11} & J_{12} & J_{13}  \tag{10}\\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{array}\right)
$$

where

$$
\begin{gather*}
J_{11}=-d-\frac{k I^{*}}{1+\alpha I^{*}}, \quad J_{12}=-\frac{k S^{*}}{\left(1+\alpha I^{*}\right)^{2}}, \quad J_{13}=\gamma \\
J_{21}=\frac{k I^{*}}{1+\alpha I^{*}}, \quad J_{22}=\frac{k S^{*}}{\left(1+\alpha I^{*}\right)^{2}}-d-\mu, \quad J_{23}=0 \\
J_{31}=0, \quad J_{32}=\mu, \quad J_{33}=-d-\gamma . \tag{11}
\end{gather*}
$$

The characteristic equation at the interior equilibrium point $E^{*}$ is

$$
\begin{equation*}
\lambda^{3}+Q_{1} \lambda^{2}+Q_{2} \lambda+Q_{3}=0 \tag{12}
\end{equation*}
$$

where

$$
\begin{align*}
& Q_{1}=-\left(J_{11}+J_{22}+J_{33}\right) \\
& Q_{2}=J_{11} J_{22}-J_{12} J_{21}+\left(J_{11}+J_{22}\right) J_{33}  \tag{13}\\
& Q_{3}=J_{12} J_{21} J_{33}-J_{21} J_{13} J_{32}-J_{11} J_{22} J_{33}
\end{align*}
$$

It is clear that

$$
\begin{gather*}
J_{11}<0, \quad J_{12}<0, \quad J_{13}>0, \quad J_{21}>0, \\
J_{22}<0, \quad J_{32}>0, \quad J_{33}<0 . \tag{14}
\end{gather*}
$$

Here $Q_{1}>0, Q_{2}>0$ and $Q_{3}>0$.
Now $Q_{1} Q_{2}-Q_{3}=\left(J_{11}+J_{22}\right) J_{12} J_{21}+J_{21} J_{13} J_{32}-J_{11}^{2}\left(J_{22}+\right.$ $\left.J_{33}\right)-J_{22}^{2}\left(J_{11}+J_{33}\right)-J_{33}^{2}\left(J_{11}+J_{22}\right)-2 J_{11} J_{22} J_{33}>0$. Therefore, model (4) is globally stable at the equilibrium $E^{*}=$ $\left(S^{*}, I^{*}, R^{*}\right)$.
2.2. Dynamics of the Stochastic Model. Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \in \mathbb{R}_{+}}$satisfying the usual conditions; that is, it is right continuous and increasing while $\mathscr{F}_{0}$ contains all P-null sets. Denote

$$
\begin{equation*}
X(t)=(S(t), I(t), R(t)) \triangleq\left(x_{1}(t), x_{2}(t), x_{3}(t)\right) \tag{15}
\end{equation*}
$$

and the norm $|X(t)|=\sqrt{S^{2}(t)+I^{2}(t)+R^{2}(t)}$. And denote $C^{2,1}\left(\mathbb{R}^{3} \times(0, \infty) ; \mathbb{R}_{+}\right)$as the family of all nonnegative functions $V(X, t)$ defined on $\mathbb{R}^{3} \times(0, \infty)$ such that they are continuously twice differentiable in $X$ and once in $t$.

We define the differential operator $\mathbf{L}$ associated with three-dimensional stochastic differential equation

$$
\begin{equation*}
\mathrm{d} X(t)=f(X, t) \mathrm{d} t+g(X, t) \mathrm{d} B(t), \tag{16}
\end{equation*}
$$

as

$$
\begin{equation*}
\mathbf{L}=\frac{\partial}{\partial t}+\sum_{i=1}^{3} f_{i}(X, t) \frac{\partial}{\partial x_{i}}+\frac{1}{2} \sum_{i, j=1}^{3}\left(g^{T}(X, t) g(X, t)\right)_{i j} \frac{\partial^{2}}{\partial x_{i} \partial x_{j}} \tag{17}
\end{equation*}
$$

where

$$
\begin{gather*}
f=\left(\begin{array}{c}
b-d S-\frac{k S I}{1+\alpha I}+\gamma R \\
\frac{k S I}{1+\alpha I}-(d+\mu) I \\
\mu I-(d+\gamma) R
\end{array}\right)  \tag{18}\\
g=\operatorname{diag}\left(\sigma_{1}\left(S-S^{*}\right), \sigma_{2}\left(I-I^{*}\right), \sigma_{3}\left(R-R^{*}\right)\right)
\end{gather*}
$$

If $L$ acts on a function $V \in C^{2,1}\left(\mathbb{R}^{3} \times(0, \infty) ; \mathbb{R}_{+}\right)$, then we denote

$$
\begin{align*}
\mathbf{L} V(X, t)= & V_{t}(X, t)+V_{X}(X, t) f(X, t) \\
& +\frac{1}{2} \operatorname{trace}\left(g^{T}(X, t) V_{X X}(X, t) g(X, t)\right) \tag{19}
\end{align*}
$$

where $T$ means transposition.
In this subsection, we first show the existence of a unique positive global solution of the stochastic model (3).

Theorem 4. For model (3) and any given initial value ( $S(0)$, $I(0), R(0)) \in \mathbb{R}_{+}^{3}$, there is a unique solution $(S(t), I(t), R(t))$ on $t \geq 0$ and will remain in $\mathbb{R}_{+}^{3}$ with probability one.

Proof. Since the coefficients of model (3) satisfy the local Lipschitz condition, there is a unique local solution on $\left[0, \tau_{e}\right)$, where $\tau_{e}$ is the explosion time. Therefore, by Itô's formula, the unique local solution of model (3) is positive. Next, let us show that this solution is global; that is, $\tau_{e}=\infty$ a.s.

Let $n_{0}>0$ be sufficiently large for $S(0), I(0)$, and $R(0)$ lying with the interval $\left[1 / n_{0}, n_{0}\right]$. For each integer $n \geq n_{0}$, define a sequence of stopping times by

$$
\begin{align*}
& \tau_{n}=\inf \left\{t \in\left[0, \tau_{e}\right]: S(t) \notin\left(\frac{1}{n}, n\right)\right. \\
& \left.\quad \text { or } I(t) \notin\left(\frac{1}{n}, n\right) \text { or } R(t) \notin\left(\frac{1}{n}, n\right)\right\}, \tag{20}
\end{align*}
$$

where we set $\inf \emptyset=\infty$ ( $\emptyset$ represents the empty set) in this paper. Since $\tau_{n}$ is nondecreasing as $n \rightarrow \infty$, there exists the limit

$$
\begin{equation*}
\tau_{\infty}=\lim _{n \rightarrow \infty} \tau_{n} . \tag{21}
\end{equation*}
$$

Then $\tau_{\infty} \leq \tau_{e}$ a.s. Now, we need to show $\tau_{\infty}=\infty$ a.s. If this statement is violated, then there exist $T>0$ and $\varepsilon \in(0,1)$ such that

$$
\begin{equation*}
\mathbf{P}\left\{\tau_{\infty} \leq T\right\}>\varepsilon . \tag{22}
\end{equation*}
$$

Thus, there is an integer $n_{1} \geq n_{0}$ such that

$$
\begin{equation*}
\mathbf{P}\left\{\tau_{n} \leq T\right\} \geq \varepsilon \quad \forall n \geq n_{1} \tag{23}
\end{equation*}
$$

Define a $C^{3}$-function $V: \mathbb{R}_{+}^{3} \rightarrow \mathbb{R}_{+}$by
$V(S, I, R)=(S-1-\ln S)+(I-1-\ln I)+(R-1-\ln R)$,
which is a nonnegative function. If $(S(t), I(t), R(t)) \in \mathbb{R}_{+}^{3}$, by using Itô's formula, we compute

$$
\begin{aligned}
& \mathrm{d} V(S, I, R)=\left(\left(1-\frac{1}{S}\right)\left(b-d S-\frac{k S I}{1+\alpha I}+\gamma R\right)\right. \\
& +\left(1-\frac{1}{I}\right)\left(\frac{k S I}{1+\alpha I}-(d+\mu) I\right) \\
& +\left(1-\frac{1}{R}\right)(\mu I-(d+\gamma) R)+\frac{\sigma_{1}^{2}}{2}\left(1-\frac{S^{*}}{S}\right)^{2} \\
& \left.+\frac{\sigma_{2}^{2}}{2}\left(1-\frac{I^{*}}{I}\right)^{2}+\frac{\sigma_{3}^{2}}{2}\left(1-\frac{R^{*}}{R}\right)^{2}\right) \mathrm{d} t \\
& +\left(\sigma_{1}\left(1-\frac{1}{S}\right)\left(S-S^{*}\right)+\sigma_{2}\left(1-\frac{1}{I}\right)\left(I-I^{*}\right)\right. \\
& \left.+\sigma_{3}\left(1-\frac{1}{R}\right)\left(R-R^{*}\right)\right) \mathrm{d} B(t) \\
& =\left(b+3 d+\mu+\gamma+\frac{\sigma_{1}^{2}}{2}\left(1-\frac{S^{*}}{S}\right)^{2}\right. \\
& +\frac{\sigma_{2}^{2}}{2}\left(1-\frac{I^{*}}{I}\right)^{2}+\frac{\sigma_{3}^{2}}{2}\left(1-\frac{R^{*}}{R}\right)^{2} \\
& +\frac{k I}{1+\alpha I}-\frac{k S}{1+\alpha I}-d(S+I+R) \\
& \left.-\frac{b}{S}-\frac{\gamma R}{S}-\frac{\mu I}{R}\right) \mathrm{d} t \\
& +\left(\sigma_{1}\left(1-\frac{1}{S}\right)\left(S-S^{*}\right)+\sigma_{2}\left(1-\frac{1}{I}\right)\left(I-I^{*}\right)\right. \\
& \left.+\sigma_{3}\left(1-\frac{1}{R}\right)\left(R-R^{*}\right)\right) \mathrm{d} B(t) \\
& \leq\left(b+3 d+\mu+\gamma+\frac{\sigma_{1}^{2}}{2}\left(1-\frac{S^{*}}{S}\right)^{2}\right. \\
& \left.+\frac{\sigma_{2}^{2}}{2}\left(1-\frac{I^{*}}{I}\right)^{2}+\frac{\sigma_{3}^{2}}{2}\left(1-\frac{R^{*}}{R}\right)^{2}+\frac{k}{\alpha}\right) \mathrm{d} t \\
& +\left(\sigma_{1}\left(1-\frac{1}{S}\right)\left(S-S^{*}\right)+\sigma_{2}\left(1-\frac{1}{I}\right)\left(I-I^{*}\right)\right. \\
& \left.+\sigma_{3}\left(1-\frac{1}{R}\right)\left(R-R^{*}\right)\right) \mathrm{d} B(t)
\end{aligned}
$$

$$
\leq M \mathrm{~d} t
$$

$$
\begin{align*}
& +\left(\sigma_{1}\left(1-\frac{1}{S}\right)\left(S-S^{*}\right)+\sigma_{2}\left(1-\frac{1}{I}\right)\left(I-I^{*}\right)\right. \\
& \left.\quad+\sigma_{3}\left(1-\frac{1}{R}\right)\left(R-R^{*}\right)\right) \mathrm{d} B(t) \tag{25}
\end{align*}
$$

where $M$ is a positive constant. Integrating both sides of the above inequality from 0 to $\tau_{n} \wedge T$, we get

$$
\begin{align*}
& \int_{0}^{\tau_{n} \wedge T} \mathrm{~d} V(S(s), I(s), R(s)) \\
& \leq \int_{0}^{\tau_{n} \wedge T} M \mathrm{~d} s \\
& \quad+\int_{0}^{\tau_{n} \wedge T}\left(\sigma_{1}\left(1-\frac{1}{S}\right)\left(S-S^{*}\right)+\sigma_{2}\left(1-\frac{1}{I}\right)\left(I-I^{*}\right)\right. \\
& \left.\quad+\sigma_{3}\left(1-\frac{1}{R}\right)\left(R-R^{*}\right)\right) \mathrm{d} B(s), \tag{26}
\end{align*}
$$

where $\tau_{n} \wedge T=\min \left\{\tau_{n}, T\right\}$. Then taking the expectations leads to

$$
\begin{align*}
& \mathbf{E} V\left(S\left(\tau_{n} \wedge T\right), I\left(\tau_{n} \wedge T\right), R\left(\tau_{n} \wedge T\right)\right)  \tag{27}\\
& \quad \leq V(S(0), I(0), R(0))+M T
\end{align*}
$$

Set $\Omega_{n}=\left\{\tau_{n} \leq T\right\}$ for $n \geq n_{1}$ and from (23), we have $\mathbf{P}\left(\Omega_{n}\right) \geq \varepsilon$. For every $v \in \Omega_{n}$, there are some $i$ such that $x_{i}\left(\tau_{n}, \nu\right)$ equals either $n$ or $1 / n$ for $i=1,2,3$; hence $V\left(S\left(\tau_{n}, \nu\right), I\left(\tau_{n}, \nu\right), R\left(\tau_{n}, \nu\right)\right)$ is no less than $\min \{n-1-$ $\ln n, 1 / n-1-\ln (1 / n)\}$. Then we obtain

$$
\begin{align*}
& V(S(0), I(0), R(0))+M T \\
& \quad \geq \mathbf{E}\left(1_{\Omega_{n}(v)} V\left(S\left(\tau_{n}\right), I\left(\tau_{n}\right), R\left(\tau_{n}\right)\right)\right)  \tag{28}\\
& \quad \geq \varepsilon \min \left\{n-1-\ln n, \frac{1}{n}-1-\ln \frac{1}{n}\right\},
\end{align*}
$$

where $1_{\Omega_{n}(v)}$ is the indicator function of $\Omega_{n}$. Letting $n \rightarrow \infty$ leads to the contradiction $\infty=V(S(0), I(0), R(0))+M T<$ $\infty$. This completes the proof.

Theorem 4 shows that the solution to model (3) will remain in $\mathbb{R}_{+}^{3}$. The property makes us continue to discuss how the solution varies in $\mathbb{R}_{+}^{3}$ in more detail. Here, we present that the definition of stochastic ultimate boundedness [32] is one of the important topics in population dynamics and is defined as follows.

Definition 5. The solutions $X(t)=(S(t), I(t), R(t))$ of model (3) are said to be stochastically ultimately bounded, if for any $\varepsilon \in(0,1)$, there is a positive constant $\delta=\delta(\varepsilon)$, such that for any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_{+}^{3}$, the solution $X(t)$ to model (3) has the property that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \mathbf{P}\{|X(t)|>\delta\}<\varepsilon \tag{29}
\end{equation*}
$$

Theorem 6. The solutions of model (3) are stochastically ultimately bounded for any initial value $(S(0), I(0), R(0)) \in$ $\mathbb{R}_{+}^{3}$.

Proof. From Theorem 4, the solution will remain in $\mathbb{R}_{+}^{3}$ for all $t \geq 0$ almost surely. Define a function

$$
\begin{equation*}
V(S, I, R)=e^{t}\left(S^{\theta}+I^{\theta}+R^{\theta}\right) \tag{30}
\end{equation*}
$$

for $(S, I, R) \in \mathbb{R}_{+}^{3}$ and $\theta>1$. By Itô's formula we obtain $\mathrm{d} V(S, I, R)$

$$
\begin{align*}
& =e^{t}\left(\theta S^{\theta-1}\left(b-d S-\frac{k S I}{1+\alpha I}+\gamma R\right)\right. \\
& +\theta I^{\theta-1}\left(\frac{k S I}{1+\alpha I}-(d+\mu) I\right) \\
& +\theta R^{\theta-1}(\mu I-(d+\gamma) R) \\
& +\frac{\theta(\theta-1)}{2}\left(\sigma_{1}^{2} S^{\theta}\left(1-\frac{S^{*}}{S}\right)^{2}+\sigma_{2}^{2} I^{\theta}\left(1-\frac{I^{*}}{I}\right)^{2}\right. \\
& \\
& \left.\left.+\sigma_{3}^{2} R^{\theta}\left(1-\frac{R^{*}}{R}\right)^{2}\right)\right) \mathrm{d} t \\
& +e^{t} \theta\left(\sigma_{1} S^{\theta}\left(1-\frac{S^{*}}{S}\right)+\sigma_{2} I^{\theta}\left(1-\frac{I^{*}}{I}\right)\right. \\
& \left.\quad+\sigma_{3} R^{\theta}\left(1-\frac{R^{*}}{R}\right)\right) \mathrm{d} B(t) \\
& \leq C e^{t} \mathrm{~d} t+e^{t} \theta\left(\sigma_{1} S^{\theta}\left(1-\frac{S^{*}}{S}\right)+\sigma_{2} I^{\theta}\left(1-\frac{I^{*}}{I}\right)\right.  \tag{31}\\
& \left.\quad+\sigma_{3} R^{\theta}\left(1-\frac{R^{*}}{R}\right)\right) \mathrm{d} B(t),
\end{align*}
$$

where $C>0$ is a suitable constant.
Based on Theorem 4 and from (31), we have

$$
\begin{align*}
& \mathbf{E}\left(e^{t \wedge \tau_{n}} V\left(S\left(t \wedge \tau_{n}\right), I\left(t \wedge \tau_{n}\right), R\left(t \wedge \tau_{n}\right)\right)\right) \\
& \quad \leq V(S(0), I(0), R(0))+C \mathbf{E} \int_{0}^{t \wedge \tau_{n}} e^{s} \mathrm{~d} s \tag{32}
\end{align*}
$$

Letting $n \rightarrow \infty$ yields

$$
\begin{equation*}
e^{t} \mathbf{E} V(S(t), I(t), R(t)) \leq V(S(0), I(0), R(0))+C\left(e^{t}-1\right), \tag{33}
\end{equation*}
$$

which implies
$\mathbf{E} V(S(t), I(t), R(t)) \leq e^{-t} V(S(0), I(0), R(0))+C$.
Note that

$$
\begin{aligned}
|X(t)|^{\theta} & =\left(S^{2}(t)+I^{2}(t)+R^{2}(t)\right)^{\theta / 2} \\
& \leq 3^{\theta / 2} \max \left\{S^{\theta}(t), I^{\theta}(t), R^{\theta}(t)\right\} \\
& \leq 3^{\theta / 2}\left(S^{\theta}+I^{\theta}+R^{\theta}\right) .
\end{aligned}
$$

Then we get

$$
\begin{equation*}
\mathbf{E}|X(t)|^{\theta} \leq 3^{\theta / 2}\left(e^{-t} V(S(0), I(0), R(0))+C\right), \tag{36}
\end{equation*}
$$

which means

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \mathbf{E}|X(t)|^{\theta} \leq 3^{\theta / 2} C<\infty \tag{37}
\end{equation*}
$$

Therefore, there exists a positive constant $\delta_{1}$ such that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \mathbf{E}|\sqrt{X(t)}|<\delta_{1} . \tag{38}
\end{equation*}
$$

For any $\varepsilon>0$, set $\delta=\delta_{1}^{2} / \varepsilon^{2}$, then by Chebyshev's inequality,

$$
\begin{equation*}
\mathbf{P}\{|X(t)>\delta|\} \leq \frac{\mathbf{E}|\sqrt{X(t)}|}{\sqrt{\delta}} \tag{39}
\end{equation*}
$$

Thus, we obtain

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \mathbf{P}\{|X(t)>\delta|\} \leq \frac{\delta_{1}}{\sqrt{\delta}}=\varepsilon \tag{40}
\end{equation*}
$$

which yields the required assertion.
Generally speaking, the nonexplosion property, the existence, and the uniqueness of the solution are not enough but the property of permanence is more desirable since it means the long time survival in a population dynamics. Now, the definition of stochastic permanence [33] will be given below.

Definition 7. The solutions $X(t)=(S(t), I(t), R(t))$ of model (3) are said to be stochastically permanent, if for any $\varepsilon \in(0,1)$, there exists a pair of positive constants $\delta=\delta(\varepsilon)$ and $\chi=\chi(\varepsilon)$ such that for any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_{+}^{3}$, the solution $X(t)$ to model (3) has the properties

$$
\begin{align*}
& \liminf _{t \rightarrow \infty} \mathbf{P}\{|X(t)| \leq \delta\} \geq 1-\varepsilon, \\
& \liminf _{t \rightarrow \infty} \mathbf{P}\{|X(t)| \geq \chi\} \geq 1-\varepsilon . \tag{41}
\end{align*}
$$

Theorem 8. Assume $d<b$ and for any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_{+}^{3}$, the solution $(S(t), I(t), R(t))$ satisfies

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \mathrm{E}\left(|X(t)|^{-9}\right) \leq Q \tag{42}
\end{equation*}
$$

where $\vartheta$ is an arbitrary positive constant satisfying

$$
\begin{align*}
& \frac{\vartheta+1}{2} \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\}<b-d  \tag{43}\\
& Q= \frac{3^{9}\left(4 \omega C_{1}+C_{2}\right)}{4 \omega C_{1}} \\
& \times \max \left\{1,\left(\frac{2 C_{1}+C_{2}+\sqrt{C_{2}^{2}+4 C_{1} C_{2}}}{2 C_{1}}\right)^{9-2}\right\} \tag{44}
\end{align*}
$$

in which $\omega$ is an arbitrary positive constant satisfying

$$
\begin{gather*}
\omega<b-d-\frac{\vartheta+1}{2} \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\},  \tag{45}\\
C_{1}=b-d-\frac{\vartheta+1}{2} \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\}-\omega,  \tag{46}\\
C_{2}=d+\max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\}+2 \omega .
\end{gather*}
$$

Proof. Define a function

$$
\begin{equation*}
V(S, I, R)=\frac{1}{S+I+R} \tag{47}
\end{equation*}
$$

for $(S(t), I(t), R(t)) \in \mathbb{R}_{+}^{3}$; using Itô's formula, we get
$\mathrm{d} V(S, I, R)$

$$
\begin{align*}
= & -V^{2}(b-d(S+I+R)) \mathrm{d} t \\
& +V^{3}\left(\sigma_{1}^{2}\left(S-S^{*}\right)^{2}+\sigma_{2}^{2}\left(I-I^{*}\right)^{2}+\sigma_{3}^{2}\left(R-R^{*}\right)^{2}\right) \mathrm{d} t \\
& -V^{2}\left(\sigma_{1}\left(S-S^{*}\right)+\sigma_{2}\left(I-I^{*}\right)+\sigma_{3}\left(R-R^{*}\right)\right) \mathrm{d} B(t) . \tag{48}
\end{align*}
$$

Choosing a positive constant $\vartheta$ that satisfies (43) and applying Itô's formula, we obtain

$$
\begin{align*}
& \mathbf{L}(1+V)^{9} \\
& =\vartheta(1+V)^{9-1} \\
& \times\left(-V^{2}(b-d(S+I+R))\right. \\
& \left.+V^{3}\left(\sigma_{1}^{2}\left(S-S^{*}\right)^{2}+\sigma_{2}^{2}\left(I-I^{*}\right)^{2}+\sigma_{3}^{2}\left(R-R^{*}\right)^{2}\right)\right) \\
& +\frac{\mathcal{Y}(\mathcal{\vartheta}-1)}{2} V^{4}(1+V)^{9-2} \\
& \times\left(\sigma_{1}^{2}\left(S-S^{*}\right)^{2}+\sigma_{2}^{2}\left(I-I^{*}\right)^{2}+\sigma_{3}^{2}\left(R-R^{*}\right)^{2}\right) \\
& =\vartheta(1+V)^{9-2} \\
& \times\left(-V^{2}(b-d(S+I+R))-V^{3}(b-d(S+I+R))\right. \\
& +V^{3}\left(\sigma_{1}^{2}\left(S-S^{*}\right)^{2}+\sigma_{2}^{2}\left(I-I^{*}\right)^{2}+\sigma_{3}^{2}\left(R-R^{*}\right)^{2}\right) \\
& \left.+\frac{\vartheta+1}{2} V^{4}\left(\sigma_{1}^{2}\left(S-S^{*}\right)^{2}+\sigma_{2}^{2}\left(I-I^{*}\right)^{2}+\sigma_{3}^{2}\left(R-R^{*}\right)^{2}\right)\right) \\
& =\vartheta(1+V)^{9-2} W \text {, } \tag{49}
\end{align*}
$$

where

$$
\begin{align*}
W= & -V^{2}(b-d(S+I+R))-V^{3}(b-d(S+I+R)) \\
& +V^{3}\left(\sigma_{1}^{2}\left(S-S^{*}\right)^{2}+\sigma_{2}^{2}\left(I-I^{*}\right)^{2}+\sigma_{3}^{2}\left(R-R^{*}\right)^{2}\right) \\
& +\frac{\vartheta+1}{2} V^{4}\left(\sigma_{1}^{2}\left(S-S^{*}\right)^{2}+\sigma_{2}^{2}\left(I-I^{*}\right)^{2}+\sigma_{3}^{2}\left(R-R^{*}\right)^{2}\right) \\
\leq & d V-(b-d) V^{2} \\
& +V^{3}\left(\sigma_{1}^{2}\left(S-S^{*}\right)^{2}+\sigma_{2}^{2}\left(I-I^{*}\right)^{2}+\sigma_{3}^{2}\left(R-R^{*}\right)^{2}\right) \\
& +\frac{\vartheta+1}{2} V^{4}\left(\sigma_{1}^{2}\left(S-S^{*}\right)^{2}+\sigma_{2}^{2}\left(I-I^{*}\right)^{2}+\sigma_{3}^{2}\left(R-R^{*}\right)^{2}\right) . \tag{50}
\end{align*}
$$

Using the facts that

$$
\begin{align*}
V^{3} & \left(\sigma_{1}^{2}\left(S-S^{*}\right)^{2}+\sigma_{2}^{2}\left(I-I^{*}\right)^{2}+\sigma_{3}^{2}\left(R-R^{*}\right)^{2}\right) \\
& <\max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\} V \\
V^{4} & \left(\sigma_{1}^{2}\left(S-S^{*}\right)^{2}+\sigma_{2}^{2}\left(I-I^{*}\right)^{2}+\sigma_{3}^{2}\left(R-R^{*}\right)^{2}\right)  \tag{51}\\
& <\max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\} V^{2}
\end{align*}
$$

then,

$$
\begin{align*}
W \leq & \left(d+\max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\}\right) V \\
& -\left(b-d-\frac{\vartheta+1}{2} \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\}\right) V^{2} \tag{52}
\end{align*}
$$

Let $\omega>0$ be sufficiently small such that it satisfies (45), by Itô's formula; then

$$
\begin{align*}
\mathbf{L}\left(e^{\omega t}(1+V)^{9}\right) & =\omega e^{\omega t}(1+V)^{9}+e^{\omega t} \mathbf{L}(1+V)^{9} \\
& =e^{\omega t}(1+V)^{9-2}\left(\omega(1+V)^{2}+W\right) \\
& \leq e^{\omega t}(1+V)^{9-2}\left(\omega-C_{1} V^{2}+C_{2} V\right)  \tag{53}\\
& \leq Q_{0} e^{\omega t}
\end{align*}
$$

where $Q_{0}=\left(4 \omega C_{1}+C_{2}\right) /\left(4 C_{1}\right) \max \left\{1,\left(\left(2 C_{1}+C_{2}+\right.\right.\right.$ $\left.\left.\left.\sqrt{C_{2}^{2}+4 C_{1} C_{2}}\right) /\left(2 C_{1}\right)\right)^{9-2}\right\}$ and $C_{1}, C_{2}$ have been defined in the statement of the theorem. Thus,

$$
\begin{equation*}
\mathbf{E}\left(e^{\omega t}(1+V)^{\vartheta}\right) \leq(1+V(0))^{\vartheta}+\frac{Q_{0}}{\omega} e^{\omega t} \tag{54}
\end{equation*}
$$

Therefore we obtain

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \mathbf{E}\left(V(t)^{\vartheta}\right) \leq \limsup _{t \rightarrow \infty} \mathbf{E}(1+V)^{\vartheta} \leq \frac{Q_{0}}{\omega} \tag{55}
\end{equation*}
$$

For $(S, I, R) \in \mathbb{R}_{+}^{3}$, we know that $(S+I+R)^{9} \leq 3^{9}\left(S^{2}+\right.$ $\left.I^{2}+R^{2}\right)^{9 / 2} \leq 3^{9}|X(t)|^{9}$; consequently,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \mathrm{E}\left(\frac{1}{|X(t)|^{9}}\right) \leq 3^{9} \limsup _{t \rightarrow \infty} \mathrm{E}\left(V(t)^{9}\right) \leq \frac{3^{9} Q_{0}}{\omega}=Q \tag{56}
\end{equation*}
$$

which completes the proof.


FIGure 1: Solutions of model (3) with different noise. Other parameters and initial condition are given in text. (a) $\sigma_{1}=0.05, \sigma_{2}=0.01$, and $\sigma_{3}=0.03$ and (b) $\sigma_{1}=0.1, \sigma_{2}=0.06$, and $\sigma_{3}=0.12$.

Considering Chebyshev inequality, Theorems 6 and 8, we immediately obtain the following result.

Theorem 9. Assume $\max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\}<2(b-d)$; then the solutions of model (3) are stochastically permanent.

Proof. From Theorem 6, we have $\mathbf{P}\{|X(t)|>\delta\} \leq \varepsilon$ which implies

$$
\begin{equation*}
\mathbf{P}\{|X(t)| \leq \delta\} \geq 1-\varepsilon . \tag{57}
\end{equation*}
$$

This follows that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \mathbf{P}\{|X(t)| \leq \delta\} \geq 1-\varepsilon \tag{58}
\end{equation*}
$$

By Theorem 8, we get

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \mathbf{E}\left(\frac{1}{|X(t)|^{9}}\right) \leq Q . \tag{59}
\end{equation*}
$$

For any $\varepsilon>0$, let $\chi=\varepsilon^{9} / Q^{9}$; then

$$
\begin{equation*}
\mathbf{P}\{|X(t)|<\chi\}=\mathbf{P}\left\{\frac{1}{|X(t)|}>\frac{1}{\chi}\right\} \leq \chi^{1 / 9} \mathbf{E}\left(|X(t)|^{-9}\right) \tag{60}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \mathbf{P}\{|X(t)|<\chi\} \leq \chi^{1 / 9} \mathrm{Q}=\varepsilon \tag{61}
\end{equation*}
$$

which follows that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \mathbf{P}\{|X(t)| \geq \chi\} \geq 1-\varepsilon \tag{62}
\end{equation*}
$$

The proof is complete.

## 3. Conclusions

In this paper, we propose an SIR epidemic model with a nonlinear incidence rate of the form $k S I /(1+\alpha I)$. We extend to consider and analyze the epidemic model with stochastic perturbations. The value of this study lies in two aspects. First, it presents existence and global stability analysis of the endemic equilibrium for the deterministic model (4). Second, it verifies some relevant properties of the corresponding stochastic model (3) and reveals the effect of environmental noise on the epidemic model.

To study the effect of environmental noise on the deterministic model (4), we stochastically perturb model (4) with respect to white noise around its endemic equilibrium. By constructing suitable Lyapunov functions and applying Itô's formula, we obtain that there is a unique positive solution to model (3) for any positive initial value and derive that the solution is stochastically bounded and permanent under some conditions. These conditions depend on the intensities of noise $\sigma_{1}, \sigma_{2}$, and $\sigma_{3}$. When the intensities of noise satisfy some conditions and are not sufficiently large, the population of the stochastic model may be stochastically permanent.

As an example, we perform some numerical simulations to illustrate the analytical results of stochastic model (3) by referring to the method mentioned in Higham [34]. Then model (3) can be rewritten as the following discretization equations:

$$
\begin{aligned}
S_{i+1}= & S_{i}+\left(b-d S_{i}-\frac{k S_{i} I_{i}}{1+\alpha I_{i}}+\gamma R_{i}\right) \Delta t \\
& +\sigma_{1}\left(S_{i}-S^{*}\right) \sqrt{\Delta t} \xi_{i}+\frac{\sigma_{1}^{2}}{2}\left(S_{i}-S^{*}\right)^{2}\left(\xi_{i}^{2}-1\right) \Delta t
\end{aligned}
$$



Figure 2: Solutions of model (3) with different noise. Other parameters and initial condition are the same as Figure 1. (a) $\sigma_{1}=0, \sigma_{2}=0.06$, and $\sigma_{3}=0$ and (b) $\sigma_{1}=1.7$ and $\sigma_{2}=\sigma_{3}=0$.

$$
\begin{align*}
I_{i+1}= & I_{i}+\left(\frac{k S_{i} I_{i}}{1+\alpha I_{i}}-(d+\mu) I_{i}\right)+\sigma_{2}\left(I_{i}-I^{*}\right) \sqrt{\Delta t} \xi_{i} \\
& +\frac{\sigma_{2}^{2}}{2}\left(I_{i}-I^{*}\right)^{2}\left(\xi_{i}^{2}-1\right) \Delta t, \\
R_{i+1}= & R_{i}+\left(\mu I_{i}-(d+\gamma) R_{i}\right) \Delta t+\sigma_{3}\left(R_{i}-R^{*}\right) \sqrt{\Delta t} \xi_{i} \\
& +\frac{\sigma_{3}^{2}}{2}\left(R_{i}-R^{*}\right)^{2}\left(\xi_{i}^{2}-1\right) \Delta t, \tag{63}
\end{align*}
$$

where $\xi_{i}(i=1,2, \ldots, n)$ is the Gaussian random variables $N(0,1)$.

Figure 1 shows time-series plots for model (3) with and without stochastic perturbations. The parameters are taken as $b=1, d=0.2, k=1, \alpha=0.5, \gamma=$ 0.25 , and $\mu=0.3$ and initial value $(S(0), I(0), R(0))=$ (1.35, $0.9,0.45$ ). In this case, model (4) has the endemic point $E^{*}=(1.087,2.3478,1.5652)$. The only difference between conditions of Figures $1(\mathrm{a})$ and $1(\mathrm{~b})$ is that the values of environmental noise intensities $\sigma_{1}, \sigma_{2}$, and $\sigma_{3}$ are different. In Figure 1(a), with $\sigma_{1}=0.05, \sigma_{2}=0.01$, and $\sigma_{3}=0.03$ and in Figure 1(b), with $\sigma_{1}=0.1, \sigma_{2}=0.06$, and $\sigma_{3}=0.12$, the condition of Theorem 9 is satisfied. That is, the solutions of model (3) are stochastically permanent. From Figures 1(a) and 1(b), one can see that with increasing the noise intensities, the solutions of model (3) will be oscillating strongly around the endemic point $E^{*}$ of model (4).

To study the effect of noise in model (3) further, in Figure 2(a), we choose $\sigma_{1}=\sigma_{3}=0, \sigma_{2}=0.06$ which satisfies the condition of Theorem 9, while in Figure 2(b), $\sigma_{1}=1.7, \sigma_{2}=\sigma_{3}=0$ that does not satisfy the condition of

Theorem 9. From Figure 2(a), one can see that the infective population $I$ will be oscillating slightly around $I^{*}=2.3478$, and both the susceptible $S$ and the removed $R$ population will be affected by the noise but the effect is very small. From Figure 2(b), when the condition of Theorem 9 is not satisfied, the noise can force the population to become largely fluctuating. In this case, the solution of model (3) is not stochastically permanent.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The author thanks the editor and the anonymous referees for very helpful suggestions and comments which led to the improvement of the original paper.

## References

[1] W. O. Kermack and A. G. McKendrick, "Contribution to mathematical theory of epidemics," Proceedings of the Royal Society of London A, vol. 115, no. 5, pp. 700-721, 1927.
[2] A. Abta, A. Kaddar, and H. T. Alaoui, "Global stability for delay SIR and SEIR epidemic models with saturated incidence rates," Electronic Journal of Differential Equations, vol. 2012, no. 23, pp. 1-13, 2012.
[3] E. Beretta and Y. Takeuchi, "Global stability of an SIR epidemic model with time delays," Journal of Mathematical Biology, vol. 33, no. 3, pp. 250-260, 1995.
[4] Y. Kang and C. Castillo-Chavez, "A simple epidemiological model for populations in the wild with Allee effects and diseasemodified fitness," Discrete and Continuous Dynamical Systems B, vol. 19, no. 1, pp. 89-130, 2014.
[5] M. Liu and Y. H. Xiao, "Modeling and analysis of epidemic diffusion within small-world network," Journal of Applied Mathematics, vol. 2012, Article ID 841531, 14 pages, 2012.
[6] Y. Nakata, Y. Enatsu, and Y. Muroya, "On the global stability of an SIRS epidemic model with distributed delays," Discrete and Continuous Dynamical Systems A, vol. 2011, pp. 1119-1128, 2011.
[7] G. Ujjainkar, V. K. Gupta, B. Singh, R. Khandelwal, and N. Trivedi, "An epidemic model with modified non-monotonic incidence rate under treatment," Applied Mathematical Sciences, vol. 6, no. 21-24, pp. 1159-1171, 2012.
[8] D. M. Xiao and S. G. Ruan, "Global analysis of an epidemic model with nonmonotone incidence rate," Mathematical Biosciences, vol. 208, no. 2, pp. 419-429, 2007.
[9] F.-F. Zhang, G. Huo, Q.-X. Liu, G.-Q. Sun, and Z. Jin, "Existence of travelling waves in nonlinear SI epidemic models," Journal of Biological Systems, vol. 17, no. 4, pp. 643-657, 2009.
[10] J.-Z. Zhang, Z. Jin, Q.-X. Liu, and Z.-Y. Zhang, "Analysis of a delayed SIR model with nonlinear incidence rate," Discrete Dynamics in Nature and Society, vol. 2008, Article ID 636153, 16 pages, 2008.
[11] T. Zhang and Z. D. Teng, "Permanence and extinction for a nonautonomous SIRS epidemic model with time delay," Applied Mathematical Modelling, vol. 33, no. 2, pp. 1058-1071, 2009.
[12] H. El Maroufy, A. Lahrouz, and P. G. L. Leach, "Qualitative behaviour of a model of an SIRS epidemic: stability and permanence," Applied Mathematics é Information Sciences, vol. 5, no. 2, pp. 220-238, 2011.
[13] Y. L. Cai, X. X. Wang, W. M. Wang, and M. Zhao, "Stochastic dynamics of an SIRS epidemic model with ratio-dependent incidence rate," Abstract and Applied Analysis, vol. 2013, Article ID 172631, 11 pages, 2013.
[14] A. Kaddar, "On the dynamics of a delayed SIR epidemic model with a modified saturated incidence rate," Electronic Journal of Differential Equations, vol. 2009, no. 133, pp. 1-7, 2009.
[15] A. Kaddar, "Stability analysis in a delayed SIR epidemic model with a saturated incidence rate," Nonlinear Analysis: Modelling and Control, vol. 15, no. 3, pp. 299-306, 2010.
[16] S. Pathak, A. Maiti, and G. P. Samanta, "Rich dynamics of an SIR epidemic model," Nonlinear Analysis: Modelling and Control, vol. 15, no. 1, pp. 71-81, 2010.
[17] F. A. Rihan and M. N. Anwar, "Qualitative analysis of delayed SIR epidemic model with a saturated incidence rate," International Journal of Differential Equations, vol. 2012, Article ID 408637, 13 pages, 2012.
[18] A. Korobeinikov and P. K. Maini, "Non-linear incidence and stability of infectious disease models," Mathematical Medicine and Biology, vol. 22, no. 2, pp. 113-128, 2005.
[19] V. Capasso and G. Serio, "A generalization of the KermackMcKendrick deterministic epidemic model," Mathematical Biosciences, vol. 42, no. 1-2, pp. 43-61, 1978.
[20] T. C. Gard, "Persistence in stochastic food web models," Bulletin of Mathematical Biology, vol. 46, no. 3, pp. 357-370, 1984.
[21] T. C. Gard, "Stability for multispecies population models in random environments," Nonlinear Analysis: Theory, Methods \& Applications, vol. 10, no. 12, pp. 1411-1419, 1986.
[22] P. E. Greenwood and L. F. Gordillo, "Stochastic epidemic modeling," in Mathematical and Statistical Estimation Approaches
in Epidemiology, pp. 31-52, Springer, Amsterdam, The Netherlands, 2009.
[23] Y. Kang and D. Armbruster, "Noise and seasonal effects on the dynamics of plant-herbivore models with monotonic plant growth functions," International Journal of Biomathematics, vol. 4, no. 3, pp. 255-274, 2011.
[24] F. Rao, "Dynamical analysis of a stochastic predator-prey model with an Allee effect," Abstract and Applied Analysis, vol. 2013, Article ID 340980, 10 pages, 2013.
[25] F. Rao, "The complex dynamics of a stochastic toxic-phytoplankton-zooplankton model," Advances in Difference Equations, vol. 2014, no. 1, article 22, 2014.
[26] F. Rao, S. J. Jiang, Y. Q. Li, and H. Liu, "Stochastic analysis of a Hassell-Varley type predation model," Abstract and Applied Analysis, vol. 2013, Article ID 738342, 10 pages, 2013.
[27] F. Rao, W. M. Wang, and Z. B. Li, "Stability analysis of an epidemic model with diffusion and stochastic perturbation," Communications in Nonlinear Science and Numerical Simulation, vol. 17, no. 6, pp. 2551-2563, 2012.
[28] H. C. Tuckwell and R. J. Williams, "Some properties of a simple stochastic epidemic model of SIR type," Mathematical Biosciences, vol. 208, no. 1, pp. 76-97, 2007.
[29] L. Y. Wang, H. L. Huang, A. C. Xu, and W. M. Wang, "Stochastic extinction in an SIRS epidemic model incorporating media coverage," Abstract and Applied Analysis, vol. 2013, Article ID 891765, 8 pages, 2013.
[30] X. X. Wang, H. L. Huang, Y. L. Cai, and W. M. Wang, "The complex dynamics of a stochastic predator-prey model," Abstract and Applied Analysis, vol. 2012, Article ID 401031, 24 pages, 2012.
[31] E. Beretta, V. Kolmanovskii, and L. Shaikhet, "Stability of epidemic model with time delays influenced by stochastic perturbations," Mathematics and Computers in Simulation, vol. 45, no. 3-4, pp. 269-277, 1998.
[32] A. Bahar and X. Mao, "Stochastic delay Lotka-Volterra model," Journal of Mathematical Analysis and Applications, vol. 292, no. 2, pp. 364-380, 2004.
[33] D. Q. Jiang, N. Shi, and X. Y. Li, "Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation," Journal of Mathematical Analysis and Applications, vol. 340, no. 1, pp. 588-597, 2008.
[34] D. J. Higham, "An algorithmic introduction to numerical simulation of stochastic differential equations," SIAM Review, vol. 43, no. 3, pp. 525-546, 2001.

## Research Article

# Stability of a Mathematical Model of Malaria Transmission with Relapse 

Hai-Feng Huo and Guang-Ming Qiu<br>Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, China<br>Correspondence should be addressed to Hai-Feng Huo; huohf1970@gmail.com

Received 6 December 2013; Accepted 10 January 2014; Published 27 February 2014
Academic Editor: Weiming Wang
Copyright © 2014 H.-F. Huo and G.-M. Qiu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

A more realistic mathematical model of malaria is introduced, in which we not only consider the recovered humans return to the susceptible class, but also consider the recovered humans return to the infectious class. The basic reproduction number $R_{0}$ is calculated by next generation matrix method. It is shown that the disease-free equilibrium is globally asymptotically stable if $R_{0} \leq 1$, and the system is uniformly persistence if $R_{0}>1$. Some numerical simulations are also given to explain our analytical results. Our results show that to control and eradicate the malaria, it is very necessary for the government to decrease the relapse rate and increase the recovery rate.


## 1. Introduction

Malaria is caused by a parasite called Plasmodium, which is transmitted via the bites of infected mosquitoes. Approximately half of the world's population is at risk of malaria. Most malaria cases and deaths occur in Sub-Saharan Africa. In 2011, 99 countries and territories had ongoing malaria transmission [1]. Recently, the incidence of malaria has been rising due to drug resistance. Various control strategies have been taken to reduce malaria transmissions.

Many epidemic models have been analyzed mathematically and applied to specific diseases [2,3]. Since the first mathematical model of malaria transmission is introduced by Ross [4], quite a few mathematical models have been formulated to investigate the transmission dynamics of malaria [5-12]. Ngwa and Shu [5] analyze a deterministic differential equation model for endemic malaria involving variable human and mosquito populations. Ngwa [6] also analyzes a mathematical model for endemic malaria involving variable human and mosquito populations and uses a perturbation analysis to approximate the endemic equilibrium in the important case where the disease related death rate is nonzero, small but significant. Furthermore, in quasistationarity, the stochastic process undergoes oscillations about a mean population whose size can be approximated by
the stable endemic deterministic equilibrium. Chitnis et al. [ 7,8 ] study a model that both human and vector species follow a logistic population, and human have immigration and disease-induced death. They present a bifurcation analysis and analyze a periodically-forced difference equation model for malaria in mosquitoes that captures the effects of seasonality and allows the mosquitoes to feed on a heterogeneous population of hosts. Chamchod and Britton [9] incorporate a vector-bias term into a malaria transmission model to account for the greater attractiveness of infectious humans to mosquitoes in terms of differing probabilities that a mosquito arriving at a human depending on whether he is infectious or susceptible. To take account of the incubation periods of parasites within the human and the mosquito, a delayed Ross-Macdonald model is taken by Ruan et al. [10]. Further, Xiao and Zou [11] use mathematical models to explored a natural concern of possible epidemics caused by multiple species of malaria parasites in one region. They find that epidemics involving both species in a single region are possible. Li [12] provides a basic analysis for the stage-structured malaria model and shows that both the baseline and the stage-structured malaria models undergo backward bifurcations.

Recently, Li et al. [13] consider a fast and slow dynamics of malaria model with relapse, and analyse the global dynamics


Figure 1: Transfer diagram of the model (1).
by using the geometric singular perturbation theory. They find that a treatment should be given to symptomatic patients completely and adequately rather than asymptomatic infection. On the other hand, for the asymptomatic patients, their results strongly suggest that to control and eradicate the malaria, it is very necessary for the government to control the relapse rate strictly. Nadjm and Behrens [14] state that relapse is when symptoms reappear after the parasites had been eliminated from blood but persist as dormant hypnozoites in liver cells. This commonly occurs between 8-24 weeks and is commonly seen with $P$. vivax and $P$. ovale infections. Other papers also consider the inluence of relapse in giving up smoking or quitting drinking, please see $[15,16]$ and references cited therein.

Chitnis et al. [7] assume that the recovered humans have some immunity to the disease and do not get clinically ill, but they still harbor low levels of parasite in their blood streams and can pass the infection to mosquitoes. After some period of time, they lose their immunity and return to the susceptible class. Unfortunately, they do not consider that the recovered humans will return to their infectious state because of incomplete treatment. Li et al. [13] consider the relapse but not that the recovered humans may return to the susceptible class.

Motivated by these works, in this paper, we propose a more realistic mathematical model of malaria, in which we assume that the recovered humans return to the susceptible class and relapse. The basic reproductive number $R_{0}$ is calculated and the persistence theory is used to analyze the uniformly persistence of the system.

The organization of this paper is as follows. In the next section, a mathematical model of malaria with relapse is formulated. In Section 3, the basic reproduction number and the stability of disease-free equilibria are investigated. The existence of endemic equilibrium and uniformly persistence are proved in Section 4, and some numerical simulations are given in Section 5. In the last section, we give some brief discussions.

## 2. The Model

2.1. System Description. In this section, we introduce a mathematical model of malaria with relapse. Because hosts might
get repeatedly infected due to not acquiring complete immunity so the population is assumed to be described by the SIRS model. Mosquitoes are assumed not to recover from the parasites so the mosquito population can be described by the SI model. The total number of population at time $t$ is given by $N=S_{h}(t)+I_{h}(t)+R_{h}(t)$ and $M=S_{m}(t)+I_{m}(t)$. The structure of model is shown in Figure 1. The transfer diagram leads to the following system of ordinary differential equations:

$$
\begin{gather*}
\frac{d S_{h}(t)}{d t}=\mu N-\frac{\beta S_{h} I_{m}}{N}+\rho_{1} R_{h}-\mu S_{h}, \\
\frac{d I_{h}(t)}{d t}=\frac{\beta S_{h} I_{m}}{N}+\rho_{2} R_{h}-(\gamma+\mu) I_{h}, \\
\frac{d R_{h}(t)}{d t}=\gamma I_{h}-\left(\rho_{1}+\rho_{2}+\mu\right) R_{h},  \tag{1}\\
\frac{d S_{m}(t)}{d t}=\eta M-\frac{\alpha_{1} S_{m} I_{h}}{N}-\frac{\alpha_{2} S_{m} R_{h}}{N}-\eta S_{m}, \\
\frac{d I_{m}(t)}{d t}=\frac{\alpha_{1} S_{m} I_{h}}{N}+\frac{\alpha_{2} S_{m} R_{h}}{N}-\eta I_{m},
\end{gather*}
$$

where $S_{h}, I_{h}, R_{h}, S_{m}, I_{m}, N$, and $M$ represent the number of susceptible humans, infectious humans, recovered humans, susceptible mosquitoes, infectious mosquitoes, the total size of the human population, and the total size of the mosquitoes population, respectively. $\mu$ is the natural birth and death rate of humans, $\eta$ is the natural birth and death rate of mosquitoes, $\beta$ is from an infectious mosquito to a susceptible human transmission rate in humans, $\alpha_{1}$ and $\alpha_{2}$ represent both infectious and recovered human to a susceptible mosquito transmission rate in mosquitoes, $\gamma$ is treatment rate, $\rho_{1}$ is recovery rate (individuals from recovered class could back to susceptible class again because they had a very small amount of parasites, which would be cleared quickly by their own immune system), $\rho_{2}$ is relapse rate, and $q$ is the number of mosquitoes per individual. All the parameters can be found in Table 1. In the model, $N$ and $M$ are constant, so we introduce the new variables in terms of proportion as follows:

$$
\begin{gather*}
s_{h}=\frac{S_{h}}{N}, \quad x_{1}=\frac{I_{h}}{N}, \quad x_{2}=\frac{R_{h}}{N}, \\
s_{m}=\frac{S_{m}}{M}, \quad y=\frac{I_{m}}{M}, \tag{2}
\end{gather*}
$$

Table 1: The parameters description of malaria model.

| $\alpha_{1}$ | From an infectious human to a susceptible mosquito, <br> transmission rate in mosquitoes |
| :--- | :--- |
| $\alpha_{2}$ | From a recovered human to a susceptible mosquito, <br> transmission rate in mosquitoes. |
| $\beta$ | From an infectious mosquito to a susceptible human, <br> transmission rate in humans |
| $N$ | The total size of human population |
| $M$ | The total size of mosquito population |
| $\mu$ | Natural birth and death rate of humans |
| $\gamma$ | Treatment rate |
| $\rho_{1}$ | Recovery rate |
| $\rho_{2}$ | Relapse rate |
| $\eta$ | Natural birth and death rate of mosquitoes |
| $q$ | The number of mosquitoes per individual |

with $s_{h}+x_{1}+x_{2}=1, s_{m}+y=1$. Then the system (1) becomes

$$
\begin{gather*}
\frac{d x_{1}(t)}{d t}=q \beta\left(1-x_{1}-x_{2}\right) y+\rho_{2} x_{2}-(\gamma+\mu) x_{1} \\
\frac{d x_{2}(t)}{d t}=\gamma x_{1}-\left(\rho_{1}+\rho_{2}+\mu\right) x_{2}  \tag{3}\\
\frac{d y(t)}{d t}=\alpha_{1}(1-y) x_{1}+\alpha_{2}(1-y) x_{2}-\eta y
\end{gather*}
$$

### 2.2. Basic Properties

2.2.1. Invariant Region. Notice that from (1) we have

$$
\begin{equation*}
\frac{d N(t)}{d t}=0, \quad \frac{d M(t)}{d t}=0 \tag{4}
\end{equation*}
$$

Thus, the total human population $N$ and mosquitoes' population $M$ are constant. Since the system (3) monitor human population, it is plausible to assume that all its state variables and parameters are nonnegative for all $t \geq 0$. Further, it can be shown that the region

$$
\begin{align*}
\Omega=\{ & \left(x_{1}(t), x_{2}(t), y(t)\right) \in R_{+}^{3}: \\
& \left.x_{1}(t)+x_{2}(t) \leq 1\right\} \tag{5}
\end{align*}
$$

is positively-invariant. Thus, each solution of the system (3), with initial conditions in $\Omega$, remains there for $t \geq 0$. Therefore, the $\omega$-limit sets of solutions of the system (3), are contained in $\Omega$. Furthermore, in $\Omega$, the usual existence, uniqueness, and continuation results hold for the system, so that the system (3), is well-posed mathematically and epidemiologically. So we consider dynamics of system (3) on the set $\Omega$ in this paper.
2.2.2. Positivity of Solutions. For system (3), to ensure the solutions of the system with positive initial conditions remain positive for all $t>0$, it is necessary to prove that all the state variables are nonnegative, so we have the following lemma.

Lemma 1. If $x_{1}(0)>0, x_{2}(0)>0, y(0)>0$, the solutions $x_{1}(t), x_{2}(t)$, and $y(t)$ of system (3) are positive for all $t \geq 0$.

Proof. Under the given initial conditions, it is easy to prove that the solutions of the system (3) are positive; if not, we assume a contradiction: that there exists a first time $t_{1}$ such that

$$
\begin{gather*}
x_{1}\left(t_{1}\right)=0, \quad x_{1}^{\prime}\left(t_{1}\right) \leq 0, \quad x_{2}\left(t_{1}\right) \geq 0 \\
y\left(t_{1}\right) \geq 0, \quad x_{2}\left(t_{1}\right)+y\left(t_{1}\right)>0  \tag{6}\\
x_{2}(t)>0, \quad y(t)>0 \\
t \in\left(0, t_{1}\right)
\end{gather*}
$$

there exists a $t_{2}$, such that

$$
\begin{gather*}
x_{2}\left(t_{2}\right)=0, \quad x_{2}^{\prime}\left(t_{2}\right) \leq 0, \quad x_{1}\left(t_{2}\right) \geq 0 \\
y\left(t_{2}\right) \geq 0, \quad x_{1}\left(t_{2}\right)+y\left(t_{2}\right)>0  \tag{7}\\
x_{1}(t)>0, \quad y(t)>0 \\
t \in\left(0, t_{2}\right)
\end{gather*}
$$

there exists a $t_{3}$, such that

$$
\begin{gather*}
y\left(t_{3}\right)=0, \quad y^{\prime}\left(t_{3}\right) \leq 0, \quad x_{1}\left(t_{3}\right) \geq 0 \\
x_{2}\left(t_{3}\right) \geq 0, \quad x_{1}\left(t_{3}\right)+x_{2}\left(t_{3}\right)>0 \\
x_{1}(t)>0, \quad x_{2}(t)>0  \tag{8}\\
t \in\left(0, t_{3}\right)
\end{gather*}
$$

In the first case, we have

$$
\begin{equation*}
x_{1}^{\prime}\left(t_{1}\right)=q \beta\left(1-x_{2}\right) y+\rho_{2} x_{2}>0 \tag{9}
\end{equation*}
$$

which is a contradiction meaning that $x_{1}(t)>0, t \geq 0$.
In the second case, we have

$$
\begin{equation*}
x_{2}^{\prime}\left(t_{2}\right)=\gamma x_{1}>0 \tag{10}
\end{equation*}
$$

which is a contradiction meaning that $x_{2}(t)>0, t \geq 0$.
In the third case, we have

$$
\begin{equation*}
y^{\prime}\left(t_{3}\right)=\alpha_{1} x_{1}+\alpha_{2} x_{2}>0 \tag{11}
\end{equation*}
$$

which is a contradiction meaning that $y(t)>0, t \geq 0$. Thus, the solutions $x_{1}(t), x_{2}(t)$, and $y(t)$ of system (3) remain positive for all $t>0$.

## 3. Analysis of the Model

The model (3) has one disease-free equilibrium $E_{0}$ and one endemic equilibrium $E^{*}$.
3.1. Disease-Free Equilibrium and the Basic Reproduction Number. The model has a disease-free equilibrium given by

$$
\begin{equation*}
E_{0}=(0,0,0) \tag{12}
\end{equation*}
$$

In the following, the basic reproduction number of system (3) will be obtained by the next generation matrix method formulated in [17].

Let $X=\left(x_{1}, x_{2}, y\right)^{T}$, then system (3) can be written as

$$
\begin{equation*}
\frac{d X}{d t}=\mathscr{F}(X)-\mathscr{V}(X) \tag{13}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathscr{F}(X)=\left(\begin{array}{c}
q \beta\left(1-x_{1}-x_{2}\right) y \\
0 \\
\alpha_{1}(1-y) x_{1}+\alpha_{2}(1-y) x_{2}
\end{array}\right), \\
\mathscr{V}(X)=\left(\begin{array}{c}
-\rho_{2} x_{2}+(\gamma+\mu) x_{1} \\
-\gamma x_{1}+\left(\rho_{1}+\rho_{2}+\mu\right) x_{2} \\
\eta y
\end{array}\right) . \tag{14}
\end{gather*}
$$

The Jacobian matrices of $\mathscr{F}(X)$ and $\mathscr{V}(X)$ at the disease-free equilibrium $E_{0}$ are, respectively,

$$
\begin{gather*}
D \mathscr{F}\left(E_{0}\right)=\left(\begin{array}{ccc}
0 & 0 & q \beta \\
0 & 0 & 0 \\
\alpha_{1} & \alpha_{2} & 0
\end{array}\right),  \tag{15}\\
D \mathscr{V}\left(E_{0}\right)=\left(\begin{array}{ccc}
\gamma+\mu & -\rho_{2} & 0 \\
-\gamma & \rho_{1}+\rho_{2}+\mu & 0 \\
0 & 0 & -\eta
\end{array}\right) .
\end{gather*}
$$

The model reproduction number denoted by $R_{0}$ is thus given by $R_{0}=$ $\sqrt{q \beta\left[\alpha_{1}\left(\rho_{1}+\rho_{2}+\mu\right)+\alpha_{2} \gamma\right] / \eta\left[(\mu+\gamma)\left(\rho_{1}+\rho_{2}+\mu\right)-\gamma \rho_{2}\right]}$. Here $R_{0}$ is associated with disease transmission by infected humans as well as the infection of susceptible humans by infected mosquitoes. Susceptible mosquitoes acquire malaria infection from infected humans in two ways, namely, by infected or recoveries. Susceptible humans acquire infection following effective contacts with infected mosquitoes.

### 3.2. Global Stability of $E_{0}$

Theorem 2. For system (3), the disease-free equilibrium $E_{0}$ is locally asymptotically stable if $R_{0}<1$.

Proof. The linearised system (3) at the disease-free equilibrium is given by

$$
\begin{gather*}
\frac{d x_{1}}{d t}=-(\gamma+\mu) x_{1}+\rho_{2} x_{2}+q \beta y \\
\frac{d x_{2}}{d t}=\gamma x_{1}-\left(\rho_{1}+\rho_{2}+\mu\right) x_{2}  \tag{16}\\
\frac{d y}{d t}=\alpha_{1} x_{1}+\alpha_{2} x_{2}-\eta y
\end{gather*}
$$

Therefore, the characteristic equation is

$$
\begin{equation*}
\lambda^{3}+A_{1} \lambda^{2}+A_{2} \lambda+A_{3}\left(1-R_{0}^{2}\right)=0 \tag{17}
\end{equation*}
$$

with $A_{1}=\gamma+2 \mu+\rho_{1}+\rho_{2}+\eta, A_{2}=\left[(\gamma+\mu)\left(\rho_{1}+\rho_{2}+\mu\right)(1+\right.$ $\left.\eta)-q \beta \alpha_{1}\right]$, and $A_{3}=\left[(\gamma+\mu)\left(\rho_{1}+\rho_{2}+\mu\right)-\gamma \rho_{2}\right] \eta$. We use
the Routh-Hurwitz criterion [18] to prove that when $R_{0}<1$, all roots of (17) have negative real part. From (17), we see that $H_{1}=\gamma+2 \mu+\rho_{1}+\rho_{2}+\eta>0$ and

$$
\begin{align*}
H_{2}= & A_{1} A_{2}-A_{3}\left(1-R_{0}^{2}\right) \\
= & \left(\gamma+2 \mu+\rho_{1}+\rho_{2}+\eta\right) \\
& \times\left[(\gamma+\mu)\left(\rho_{1}+\rho_{2}+\mu\right)-\gamma \rho_{2}\right](1+\eta)  \tag{18}\\
& -q \beta \alpha_{1}-\eta(\gamma+\mu)\left(\rho_{1}+\rho_{2}+\mu\right) \\
& +\eta \gamma \rho_{2}+q \beta\left[\alpha_{1}\left(\rho_{1}+\rho_{2}+\mu\right)+\alpha_{2} \gamma\right] .
\end{align*}
$$

For ease of notation, we introduce $B_{1}=\gamma+\mu$ and $B_{2}=\rho_{1}+$ $\rho_{2}+\mu$, so that

$$
\begin{align*}
H_{2}= & \left(B_{1}+B_{2}\right)\left(B_{1} B_{2}-\gamma \rho_{2}\right)+\eta\left(B_{1}+B_{2}+\eta\right) \\
& \times\left[1-\frac{\left(B_{1}+\eta\right) q \beta \alpha_{1}}{\eta\left(B_{1}+B_{2}\right)\left(B_{1}+B_{2}+\eta\right)}\right]+q \beta \alpha_{2} \gamma \\
\geq & \left(B_{1}+B_{2}\right)\left(B_{1} B_{2}-\gamma \rho_{2}\right)+\eta\left(B_{1}+B_{2}+\eta\right) \\
& \times\left[1-\frac{q \beta \alpha_{1}}{\eta\left(B_{1}+B_{2}\right)}\right]+q \beta \alpha_{2} \gamma \\
\geq & \left(B_{1}+B_{2}\right)\left(B_{1} B_{2}-\gamma \rho_{2}\right)+\eta\left(B_{1}+B_{2}+\eta\right) \\
& \times\left[1-\frac{q \beta \alpha_{1} B_{2}+q \beta \alpha_{2} \gamma}{\eta\left(B_{1} B_{2}+B_{2}^{2}\right)}\right]+q \beta \alpha_{2} \gamma  \tag{19}\\
\geq & \left(B_{1}+B_{2}\right)\left(B_{1} B_{2}-\gamma \rho_{2}\right)+\eta\left(B_{1}+B_{2}+\eta\right) \\
& \times\left[1-\frac{q \beta \alpha_{1} B_{2}+q \beta \alpha_{2} \gamma}{\eta\left(B_{1} B_{2}-\gamma \rho_{2}\right)}\right]+q \beta \alpha_{2} \gamma \\
= & \left(B_{1}+B_{2}\right)\left(B_{1} B_{2}-\gamma \rho_{2}\right)+\eta\left(B_{1}+B_{2}+\eta\right) \\
& \times\left[1-R_{0}^{2}\right]+q \beta \alpha_{2} \gamma .
\end{align*}
$$

Thus, for $R_{0}<1, H_{2}>0$. Lastly, $H_{3}=H_{2} A_{3}\left(1-R_{0}^{2}\right)$. Thus, for $R_{0}<1$, all roots of (17) have negative real parts. The dis-ease-free equilibrium point $E_{0}$, is locally asymptotically stable if $R_{0}<1$.

In the following, we prove that when $R_{0} \leq 1, E_{0}$ is globally asymptotically stable in $\Omega$.

Theorem 3. For system (3), the disease-free equilibrium $E_{0}$ is globally asymptotically stable if $R_{0} \leq 1$.

Proof. We introduce the following Lyapunov function [19, 20]:

$$
\begin{equation*}
V=a x_{1}+b x_{2}+c y \tag{20}
\end{equation*}
$$

where $a=\alpha_{1}\left(\rho_{1}+\rho_{2}+\mu\right)+\alpha_{2} \gamma, b=\alpha_{2}(\gamma+\mu)+\alpha_{1} \rho_{2}$, and $c=$ $(\gamma+\mu)\left(\rho_{1}+\rho_{2}+\mu\right)-\gamma \rho_{2}$. It is easy to see that $a, b$, and $c$ are all positive. The derivative of $V$ is given by

$$
\begin{align*}
\dot{V}= & a \dot{x}_{1}+b \dot{x}_{2}+c \dot{y} \\
= & a\left[q \beta\left(1-x_{1}-x_{2}\right) y+\rho_{2} x_{2}-(\gamma+\mu) x_{1}\right] \\
& +b\left[\gamma x_{1}-\left(\rho_{1}+\rho_{2}+\mu\right) x_{2}\right] \\
& +c\left[\alpha_{1}(1-y) x_{1}+\alpha_{2}(1-y) x_{2}-\eta y\right] \\
= & (a q \beta-c \eta) y-\left[a(\gamma+\mu)-b \gamma-c \alpha_{1}\right] x_{1}  \tag{21}\\
& +\left[a \rho_{2}-b\left(\rho_{1}+\rho_{2}+\mu\right)+c \alpha_{2}\right] x_{2} \\
& -\left(a q \beta+c \alpha_{1}\right) x_{1} y-\left(a q \beta+c \alpha_{2}\right) x_{2} y \\
= & -\left(a q \beta+c \alpha_{1}\right) x_{1} y-\left(a q \beta+c \alpha_{2}\right) x_{2} y \\
& +\frac{\left(R_{0}^{2}-1\right) y}{\left[(\gamma+\mu)\left(\rho_{1}+\rho_{2}+\mu\right)-\gamma \rho_{2}\right] \eta} .
\end{align*}
$$

If $R_{0} \leq 1$, then $\left(R_{0}^{2}-1\right) /\left[(\gamma+\mu)\left(\rho_{1}+\rho_{2}+\mu\right)-\gamma \rho_{2}\right] \eta \leq 0$. As we know that $-\left(a q \beta+c \alpha_{1}\right)<0$ and $-\left(a q \beta+c \alpha_{2}\right)<0$, so we obtain $\dot{V} \leq 0$. Furthermore, $\dot{V}=0$ only if $y=0$ or $R_{0}=1$. The maximum invariant set in $\left\{\left(x_{1}, x_{2}, y\right): \dot{V}=0\right\}$ is the singleton $E_{0}$. By LaSalle's Invariance Principle [21], $E_{0}$ is globally asymptotically stable in $\Omega$.

### 3.3. Endemic Equilibrium

### 3.3.1. Existence of the Endemic Equilibrium

Theorem 4. If $R_{0}>1$, system (3) has a unique endemic equilibrium $E^{*}=\left(x_{1}^{*}, x_{2}^{*}, y^{*}\right)$, where

$$
\begin{gather*}
x_{1}^{*}=\left(\left(\rho_{1}+\rho_{2}+\mu\right)\left[(\gamma+\mu)\left(\rho_{1}+\rho_{2}+\mu\right)-\gamma \rho_{2}\right]\right. \\
\left.\times \eta\left(R_{0}^{2}-1\right)\right) \\
\times\left(\left[(\gamma+\mu)\left(\rho_{1}+\rho_{2}+\mu\right)-\gamma \rho_{2}\right]\right. \\
\times\left[\alpha_{1}\left(\rho_{1}+\rho_{2}+\mu\right)+\alpha_{2} \gamma\right] \\
\left.+\left(\rho_{1}+\rho_{2}+\mu+\gamma\right)\right)^{-1}  \tag{22}\\
x_{2}^{*}=\frac{\gamma}{\rho_{1}+\rho_{2}+\mu} x_{1} \\
y^{*}=\frac{\left[\alpha_{1}\left(\rho_{1}+\rho_{2}+\mu\right)+\alpha_{2} \gamma\right] x_{1}}{\left[\alpha_{1}\left(\rho_{1}+\rho_{2}+\mu\right)+\alpha_{2} \gamma\right] x_{1}+\eta\left(\rho_{1}+\rho_{2}+\mu\right)}
\end{gather*}
$$

Proof. It follows from system (3) that

$$
\begin{gather*}
q \beta\left(1-x_{1}^{*}-x_{2}^{*}\right) y^{*}+\rho_{2} x_{2}^{*}-(\gamma+\mu) x_{1}^{*}=0 \\
\gamma x_{1}^{*}-\left(\rho_{1}+\rho_{2}+\mu\right) x_{2}^{*}=0  \tag{23}\\
\alpha_{1}\left(1-y^{*}\right) x_{1}^{*}+\alpha_{2}\left(1-y^{*}\right) x_{2}^{*}-\eta y=0
\end{gather*}
$$

From the second equation of (23), we obtain

$$
\begin{equation*}
x_{2}^{*}=\frac{\gamma}{\rho_{1}+\rho_{2}+\mu} x_{1}^{*} \tag{24}
\end{equation*}
$$

Substituting $x_{2}^{*}$ into the third equation of (23), we have

$$
\begin{equation*}
y^{*}=\frac{\left[\alpha_{1}\left(\rho_{1}+\rho_{2}+\mu\right)+\alpha_{2} \gamma\right] x_{1}^{*}}{\left[\alpha_{1}\left(\rho_{1}+\rho_{2}+\mu\right)+\alpha_{2} \gamma\right] x_{1}^{*}+\eta\left(\rho_{1}+\rho_{2}+\mu\right)} \tag{25}
\end{equation*}
$$

Then substituting (24) and (25) into first equation of (23), we get

$$
\begin{gather*}
x_{1}^{*}=\left(\left(\rho_{1}+\rho_{2}+\mu\right)\left[(\gamma+\mu)\left(\rho_{1}+\rho_{2}+\mu\right)-\gamma \rho_{2}\right]\right. \\
\left.\times \eta\left(R_{0}^{2}-1\right)\right) \\
\times\left(\left[(\gamma+\mu)\left(\rho_{1}+\rho_{2}+\mu\right)-\gamma \rho_{2}\right]\right.  \tag{26}\\
\times\left[\alpha_{1}\left(\rho_{1}+\rho_{2}+\mu\right)+\alpha_{2} \gamma\right] \\
\\
\left.+\left(\rho_{1}+\rho_{2}+\mu+\gamma\right)\right)^{-1}
\end{gather*}
$$

Hence, if $R_{0} \leq 1$, there is no positive root of (26), while if $R_{0}>1$ there is one positive root.
3.3.2. Uniform Persistence of the Disease. We using the persistence theory of dynamical system to show the uniform persistence of the disease when $R_{0}>1$. Let $E$ be a closed positively invariant subset of $\Omega$, on which a continuous flow $\mathscr{F}$ is defined. We denote the restriction $\mathscr{F}$ to $\partial E$ by $\partial \mathscr{F}$ and note that $\partial E$ is in general not positively invariant. Let $N$ be the maximal invariant set of $\partial \mathscr{F}$ on $\partial E$. Suppose $N$ is a closed invariant set and there exists a cover $\left\{N_{\alpha}\right\}_{\alpha \in A}$ of $N$, where $A$ is a nonempty index set. $N_{\alpha} \subset \partial E, N \subset \bigcup_{\alpha \in A} N_{\alpha}$, and $\left\{N_{\alpha}\right\}(\alpha \in$ $A)$ are pairwise disjoint closed invariant sets. Furthermore, we propose the following hypothesis and Lemma. $\left(H_{1}\right)$ All $N_{\alpha}$ are isolated invariant sets of the flow $\mathscr{F} .\left(H_{2}\right),\left\{N_{\alpha}\right\}_{\alpha \in A}$ is acyclic; that is, any finite subset of $\left\{N_{\alpha}\right\}_{\alpha \in A}$ does not form a cycle. $\left(H_{3}\right)$ Any compact subset of $\partial E$ contains, at most, finitely many sets of $\left\{N_{\alpha}\right\}_{\alpha \in A}$ [22].

Lemma 5 (see [22, Theorem 4.3]). Let E be a closed positively invariant subset of $\Omega$ on which a continuous flow $\mathscr{F}$ is defined. Suppose there is a constant $\varepsilon>0$ such that $\mathscr{F}$ is point dissipative on $S[\partial E, \varepsilon] \cap E^{0}$ and the assumption $\left(H_{1}-H_{3}\right)$ holds. Then the flow $\mathscr{F}$ is uniformly persistent, if and only if $W^{+}\left(N_{\alpha}\right) \bigcap S[\partial E, \alpha] \bigcap E^{0}=\phi$. For any $\alpha \in A$, where $W^{+}\left(N_{\alpha}\right)=\left\{y \in \Omega, \omega(y) \subset N_{\alpha}\right\}, S[\partial E, \varepsilon]=\{x: x \in$ $\Omega, d(x, \partial E) \leq \varepsilon\}$, and $E^{0}$ is interior of set $E$.

By this lemma, we can show the uniform persistence of disease when $R_{0}>1$, and similar to the proof of Theorem 2.3 in [13], we have the following.

Theorem 6. In system (3), assume that $R_{0}>1$, and the disease is initially present, then the disease is uniformly persistent; that is, there is a constant $k>0$ such that ${\lim \inf _{t \rightarrow+\infty}}^{x_{1}}(t) \geq k$, $\lim \inf _{t \rightarrow+\infty} x_{2}(t) \geq k$, and $\liminf _{t \rightarrow+\infty} y(t) \geq k$.

Proof. We set $E=\left\{\left(x_{1}, x_{2}, y\right) \in \mathbf{R}_{+}^{3} \mid 0 \leq x_{1}+x_{2} \leq 1,0 \leq\right.$ $y \leq 1\}, \partial E=\left\{\left(x_{1}, x_{2}, y\right) \in E \mid x_{1}=0\right\}$; we will prove below that the conditions of Lemma 5 are satisfied. Clearly $N_{\alpha}=E_{0}=(0,0,0)$ is isolated. Hence, the covering is simply $N=E_{0}$, which is acyclic. Thus, the condition $\left(H_{1}-H_{3}\right)$ holds.


Figure 2: $R_{0}<1$, the disease-free equilibrium, $E_{0}$, is globally asymptotically stable.

We also can obtain $\mathscr{F}$ is point dissipative by Lemma 1. Now we show that $W^{+}\left(E_{0}\right) \bigcap E^{0}=\phi$; suppose this is not true, then there exists a solution $\left(x_{1}(t), x_{2}(t), y(t)\right) \in E^{0}$ such that: $\lim _{t \rightarrow+\infty} x_{1}(t)=0, \lim _{t \rightarrow+\infty} x_{2}(t)=0, \lim _{t \rightarrow+\infty} y(t)=0$. For any sufficiently small constant $\varepsilon>0$, there exists a positive constant $T=T(\varepsilon)$ such that $x_{1}(t)<\varepsilon, x_{2}(t)<$ $\varepsilon, y(t)<\varepsilon$, for all $t \geq T$.

Noting that

$$
\begin{gather*}
\frac{d x_{1}}{d t} \geq-(\gamma+\mu) x_{1}+\rho_{2} x_{2}+q \beta y \\
\frac{d x_{2}}{d t}=\gamma x_{1}-\left(\rho_{+} \rho_{2}+\mu\right) x_{2}  \tag{27}\\
\frac{d y}{d t} \geq \alpha_{1} x_{1}+\alpha_{2} x_{2}-\eta y
\end{gather*}
$$

Therefore, if $x_{1}, x_{2}, y \rightarrow 0$, as $t \rightarrow \infty$, then by a standard comparison argument and the nonnegativity, the solution $x_{1}, x_{2}, y$ of

$$
\begin{gathered}
\frac{d x_{1}}{d t}=-(\gamma+\mu) x_{1}+\rho_{2} x_{2}+q \beta y \\
\frac{d x_{2}}{d t}=\gamma x_{1}-\left(\rho_{+} \rho_{2}+\mu\right) x_{2} \\
\frac{d y}{d t}=\alpha_{1} x_{1}+\alpha_{2} x_{2}-\eta y
\end{gathered}
$$

with initial data $x_{1}(T)=x_{1}(T), x_{2}(T)=x_{2}(T), y(T)=y(T)$, converges to $(0,0,0)$ as well. Thus $\lim W(t)=0$, where $W(t)>0$, is defined by

$$
\begin{align*}
\frac{d W}{d t}= & k_{1}\left[-(\gamma+\mu) x_{1}+\rho_{2} x_{2}+q \beta y\right] \\
& +k_{2}\left[\gamma x_{1}-\left(\rho_{+} \rho_{2}+\mu\right) x_{2}\right]  \tag{29}\\
& +k_{3}\left[\alpha_{1} x_{1}+\alpha_{2} x_{2}-\eta y\right]
\end{align*}
$$

Here, $k_{1}=\alpha_{1}\left(\rho_{1}+\rho_{2}+\mu\right)+\alpha_{2} \gamma, k_{2}=\alpha_{2}(\gamma+\mu)+\rho_{2} \rho, k_{3}=$ $(\gamma+\mu)\left(\rho_{1}+\rho_{2}+\mu\right)-\gamma \rho_{2}$. The derivative of $W(t)$ is given by

$$
\begin{equation*}
\frac{d W}{d t}=\left[(\gamma+\mu)\left(\rho_{1}+\rho_{2}+\mu\right)-\gamma \rho_{2}\right]\left(R_{0}^{2}-1\right) y \geq 0 \tag{30}
\end{equation*}
$$

Therefore, $W(t)$ goes to either infinity or some positive number as $t \rightarrow \infty$, which is a contradiction to $\lim _{t \rightarrow+\infty} W(t)=$ 0 . Thus, we have $W^{+}\left(E_{0}\right) \bigcap E^{0}=\phi$. Then, we obtain $\lim \inf _{t \rightarrow+\infty} x_{1}(t) \geq k_{1}$, for some constant $k_{1}>0$. By the second and third equations of (3) and the use of Lemma 1, we have $k_{2}=\gamma k_{1} /\left(\rho_{1}+\rho_{2}+\mu\right), k_{3}=\left(\alpha_{1} k_{1}+\alpha_{2} k_{2}\right) / \eta$, such that $\liminf _{t \rightarrow+\infty} x_{2}(t) \geq k_{2}, \liminf _{t \rightarrow+\infty} y(t) \geq k_{3}$. Denote $k=$ $\min \left\{k_{1}, k_{2}, k_{3}\right\}, \liminf \operatorname{in+\infty }_{t \rightarrow+\infty} x_{1}(t) \geq k, \liminf _{t \rightarrow+\infty} x_{2}(t) \geq$ $k, \liminf _{t \rightarrow+\infty} y(t) \geq k$. Then the proof of Theorem 6 is completed.

## 4. Numerical Simulation

To illustrate the analytical results obtained above, we give some simulations using the parameter values in Table 2. Numerical results are displayed in Figures 2-5. First, we

Table 2: The parameters values of malaria model.

| $\alpha_{1}$ | From an infectious human to a susceptible mosquito, transmission rate in mosquitoes | $0.8333\left(\right.$ day $\left.^{-1}\right)$ | $[7]$ |
| :--- | :---: | :---: | :---: |
| $\alpha_{2}$ | From a recovered human to a susceptible mosquito, transmission rate in mosquitoes | $8.333 * 10^{-2}\left(\right.$ day $\left.^{-1}\right)$ | $[7]$ |
| $\beta$ | From an infectious mosquito to a susceptible human, transmission rate in humans | $2.000 * 10^{-2}\left(\right.$ day $\left.^{-1}\right)$ | $[7]$ |
| $N$ | The total size of human population | Estimated |  |
| $M$ | The total size of mosquito population | $q N$ | $[9]$ |
| $\mu$ | Natural birth and death rate of humans | $1 / 70\left(\right.$ year $\left.^{-1}\right)$ | $[9]$ |
| $\gamma$ | Treatment rate | $3.704 * 10^{-3}\left(\right.$ day $\left.^{-1}\right)$ | $[7]$ |
| $\rho_{1}$ | Recovery rate | Estimated |  |
| $\rho_{2}$ | Relapse rate | Estimated |  |
| $\eta$ | Natural birth and death rate of mosquitoes | $0.1429\left(\right.$ day $\left.^{-1}\right)$ | $[7]$ |
| $q$ | The number of mosquitoes per individual | $1-2$ | $[9]$ |



Figure 3: $R_{0}>1$, the disease is uniformly persistent.
choose $\rho_{2}=0.004, \rho_{1}=0.0146$, and $q=1.5$, numerical simulation gives $R_{0}=0.6940<1$, then the disease-free equilibrium $E_{0}$ is globally asymptotically stable (Figure 2). Second, we choose $\rho_{2}=0.04, \rho_{1}=0.0146$, and $q=1.5$, numerical simulation gives $R_{0}=1.1254>1$, the disease is uniformly persistent (Figure 3).

Finally, for showing the effect of relapse and recover rate to the basic reproduction number, we give the relation between $R_{0}$ and $\rho_{2}$ (Figure 4), and the relation between $R_{0}$ and $\rho_{1}$ (Figure 5) in the numerical simulation. From Figures 4 and 5, we know that $R_{0}$ is increasing with respect to the relapse rate, while it is decreasing with respect to the recovery rate.

## 5. Discussion

An ordinary differential equation for the transmission of malaria is formulated in this paper. The model exhibits two equilibria, that is, the disease-free equilibrium and endemic equilibrium. By constructing Lyapunov function and persistence theory of dynamical system, it is shown that if $R_{0} \leq 1$, then the disease-free equilibrium point $E_{0}$ is globally stable, and if $R_{0}>1$, the disease is uniformly persistent. Some numerical simulations for $R_{0}$ in terms of relapse rate and recover rate are performed. $R_{0}$ is increasing with respect to the relapse rate while it is decreasing with respect to the recovery rate.


Figure 4: The relationship between $R_{0}$ and $\rho_{2}$.


Figure 5: The relationship between $R_{0}$ and $\rho_{1}$.

Our results strongly suggest that to control and eradicate the malaria, it is very necessary for the government to decrease the relapse rate and increase the recovery rate.

## Conflict of Interests

On behalf of all the authors, Hai-Feng Huo declares that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work was partially supported by the NNSF of China (10961018), the NSF of Gansu Province of China (1107RJZA088), the NSF for Distinguished Young Scholars of Gansu Province of China (1111RJDA003), the Special Fund for the Basic Requirements in the Research of University of Gansu Province of China, and the Development Program
for Hong Liu Distinguished Young Scholars in Lanzhou University of Technology.

## References

[1] WHO, World malaria, 2012, http://www.who.int/en/.
[2] S. Gupta, J. Swinton, and R. M. Anderson, "Theoretical studies of the effects of heterogeneity in the parasite population on the transmission dynamics of malaria," Proceedings of the Royal Society B, vol. 256, no. 1347, pp. 231-238, 1994.
[3] H. W. Hethcote, "The mathematics of infectious diseases," SIAM Review, vol. 42, no. 4, pp. 599-653, 2000.
[4] R. Ross, "An application of the theory of probabilities to the study of a priori pathometry," Proceedings of the Royal Society A, vol. 92, pp. 204-230, 1916.
[5] G. A. Ngwa and W. S. Shu, "A mathematical model for endemic malaria with variable human and mosquito populations," Mathematical and Computer Modelling, vol. 32, no. 7-8, pp. 747-763, 2000.
[6] G. A. Ngwa, "Modelling the dynamics of endemic malaria in growing populations," Discrete and Continuous Dynamical Systems $B$, vol. 4, no. 4, pp. 1173-1202, 2004.
[7] N. Chitnis, J. M. Cushing, and J. M. Hyman, "Bifurcation analysis of a mathematical model for malaria transmission," SIAM Journal on Applied Mathematics, vol. 67, no. 1, pp. 24-45, 2006.
[8] N. Chitnis, D. Hardy, and T. Smith, "A periodically-forced mathematical model for the seasonal dynamics of malaria in mosquitoes," Bulletin of Mathematical Biology, vol. 74, no. 5, pp. 1098-1124, 2012.
[9] F. Chamchod and N. F. Britton, "Analysis of a vector-bias model on malaria transmission," Bulletin of Mathematical Biology, vol. 73, no. 3, pp. 639-657, 2011.
[10] S. Ruan, D. Xiao, and J. C. Beier, "On the delayed Ross-Macdonald model for malaria transmission," Bulletin of Mathematical Biology, vol. 70, no. 4, pp. 1098-1114, 2008.
[11] Y. Xiao and X. Zou, "Can multiple malaria species co-persist?" SIAM Journal on Applied Mathematics, vol. 73, no. 1, pp. 351-373, 2013.
[12] J. Li, "Malaria model with stage-structured mosquitoes," Mathematical Biosciences and Engineering, vol. 8, no. 3, pp. 753-768, 2011.
[13] J. Li, Y. Zhao, and S. Li, "Fast and slow dynamics of Malaria model with relapse," Mathematical Biosciences, vol. 246, no. 1, pp. 94-104, 2013.
[14] B. Nadjm and R. H. Behrens, "Malaria: an update for physicians," Infectious Disease Clinics of North America, vol. 26, pp. 243-259, 2012.
[15] H. F. Huo and C. C. Zhu, "Stability of a quit drinking model with relapse," Journal of Biomathematics. In press.
[16] H.-F. Huo and C.-C. Zhu, "Influence of relapse in a giving up smoking model," Abstract and Applied Analysis, vol. 2013, Article ID 525461, 12 pages, 2013.
[17] P. van den Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," Mathematical Biosciences, vol. 180, no. 1-2, pp. 29-48, 2002.
[18] G. A. Korn and T. M. Korn, Mathematical Handbook for Scientists and Engineers: Definitions, Theorems, and Formulas for Reference and Review, Dover, Mineola, NY, USA, 2000.
[19] M. Y. Li and H. Shu, "Global dynamics of an in-host viral model with intracellular delay," Bulletin of Mathematical Biology, vol. 72, no. 6, pp. 1492-1505, 2010.
[20] M. Y. Li and J. S. Muldowney, "Global stability for the SEIR model in epidemiology," Mathematical Biosciences, vol. 125, no. 2, pp. 155-164, 1995.
[21] J. P. LaSalle, The Stability of Dynamical Systems, Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, Pa, USA, 1976.
[22] H. I. Freedman, S. G. Ruan, and M. X. Tang, "Uniform persistence and flows near a closed positively invariant set," Journal of Dynamics and Differential Equations, vol. 6, no. 4, pp. 583-600, 1994.

## Research Article

# An SIRS Model for Assessing Impact of Media Coverage 

Jing'an Cui and Zhanmin Wu<br>School of Science, Beijing University of Civil Engineering and Architecture, Beijing 100044, China<br>Correspondence should be addressed to Jing'an Cui; cuijingan@bucea.edu.cn

Received 15 November 2013; Accepted 9 January 2014; Published 25 February 2014
Academic Editor: Weiming Wang
Copyright © 2014 J. Cui and Z. Wu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

An SIRS model incorporating a general nonlinear contact function is formulated and analyzed. When the basic reproduction number $\mathscr{R}_{0}<1$, the disease-free equilibrium is locally asymptotically stable. There is a unique endemic equilibrium that is locally asymptotically stable if $\mathscr{R}_{0}>1$. Under some conditions, the endemic equilibrium is globally asymptotically stable. At last, we conduct numerical simulations to illustrate some results which shed light on the media report that may be the very effective method for infectious disease control.

## 1. Introduction

Media coverage has an enormous impact on the spread and control of infectious diseases [1-6]. The paper [7] considered that the evidence shows that, faced with lethal or novel pathogens, people will change their behavior to try to reduce their risk.

In [8], the authors studied the effect of media coverage on the spreading of disease by using the following model:

$$
\begin{align*}
& \frac{d S(t)}{d t}=\Lambda-\mu S-\frac{\left(\beta_{1}-\beta_{2} f(I)\right) S I}{(S+I)}+\gamma I,  \tag{1}\\
& \frac{d I(t)}{d t}=\frac{\left(\beta_{1}-\beta_{2} f(I)\right) S I}{(S+I)}-(\mu+\alpha+\gamma) I,
\end{align*}
$$

where the authors proposed an SIS model with the general nonlinear contact function $\beta(I)=\beta_{1}-\beta_{2} f(I)$ and $\beta_{1}$ and $\beta_{2}$ are positive constants. Here, $\beta_{1}$ is the usual contact rate without considering the infective individuals and $\beta_{2}$ is the maximum reduced contact rate due to the presence of the infected individuals. Everyone cannot avoid contact with others in every case so it is assumed $\beta_{1}>\beta_{2}$. When infective individuals appear in a region, people reduce their contact with others to avoid being infected when they are aware of the potential danger of being infected, and the more infective individuals being reported, the less contact the susceptible will make with others. Therefore, it is assumed that $f^{\prime}(I) \geq 0$. The limited power of the infection due to
contact is reflected by the saturating function $\lim _{I \rightarrow \infty} f(I)=$ 1. In summary, the functional $f(I)$ satisfies $f(0)=0, f^{\prime}(I) \geq$ $0, \lim _{I \rightarrow \infty} f(I)=1$.

In this paper, using the same contact function as [8], we study an SIRS model with media coverage. Let $S(t), I(t)$, and $R(t)$ denote the number of susceptible individuals, infected individuals, and recovered individuals at time $t$, respectively. The ordinary differential equation with nonnegative initial conditions is as follows:

$$
\begin{align*}
& \frac{d S(t)}{d t}=\Lambda-\mu S-\left(\beta_{1}-\beta_{2} f(I)\right) S I+\sigma R \\
& \frac{d I(t)}{d t}=\left(\beta_{1}-\beta_{2} f(I)\right) S I-(\alpha+\mu+\lambda) I  \tag{2}\\
& \frac{d R(t)}{d t}=\lambda I-(\mu+\sigma) R
\end{align*}
$$

Here, all the variables and parameters of the model are nonnegative. $\Lambda$ is the recruitment rate, $\mu$ represents the natural death rate, $\sigma$ is the loss of constant immunity rate, $\alpha$ is the diseases induced constant death rate, and $\lambda$ is constant recovery rate.

We have $d S /\left.d t\right|_{S=0, R \geq 0}>0, d I /\left.d t\right|_{I=0}=0, d R /\left.d t\right|_{R=0, I \geq 0} \geq$ 0 , and $d(S+I+R) /\left.d t\right|_{S+I+R=\Lambda / \mu} \leq 0$. So,

$$
\begin{equation*}
\Omega=\left\{(S, I, R) \in \mathbb{R}_{+}^{3}: S+I+R \leq \frac{\Lambda}{\mu}\right\} \tag{3}
\end{equation*}
$$

is a positive invariant set of (2).

## 2. The Existence of the Equilibria

It is easy to see that model (2) always has a disease-free equilibrium $E_{0}=\left(S_{0}, 0,0\right)$, where $S_{0}=\Lambda / \mu$. Let $x=$ $(I, S, R)^{\top}$. Then model (2) can be written as

$$
\begin{equation*}
\frac{d x}{d t}=\mathscr{F}(x)-\mathscr{V}(x) \tag{4}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathscr{F}(x)=\left(\begin{array}{c}
\left(\beta_{1}-\beta_{2} f(I)\right) S I \\
0 \\
0
\end{array}\right) \\
\mathscr{V}(x)=\left(\begin{array}{c}
(\alpha+\mu+\lambda) I \\
-\Lambda+\mu S+\left(\beta_{1}+\beta_{2} f(I)\right) S I-\sigma R \\
-\lambda I+(\mu+\sigma) R
\end{array}\right) \tag{5}
\end{gather*}
$$

According to Theorem 2 in [9], the basic reproduction number of model (2) is

$$
\begin{equation*}
\mathscr{R}_{0}=\frac{\beta_{1} S_{0}}{\alpha+\mu+\lambda}=\frac{\beta_{1} \Lambda}{\mu(\alpha+\mu+\lambda)} \tag{6}
\end{equation*}
$$

In the following, the existence and uniqueness of the endemic equilibrium is established when $\mathscr{R}_{0}>1$. The components of the endemic equilibrium $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ satisfy

$$
\begin{gather*}
\Lambda-\mu S^{*}-\left(\beta_{1}-\beta_{2} f\left(I^{*}\right)\right) S^{*} I^{*}+\sigma R^{*}=0 \\
\left(\beta_{1}-\beta_{2} f\left(I^{*}\right)\right) S^{*}-(\alpha+\mu+\lambda)=0  \tag{7}\\
\lambda I^{*}-(\mu+\sigma) R^{*}=0
\end{gather*}
$$

which gives

$$
\begin{gather*}
R^{*}=\frac{\lambda I^{*}}{\mu+\sigma}  \tag{8}\\
S^{*}=\frac{\alpha+\mu+\lambda}{\beta_{1}-\beta_{2} f\left(I^{*}\right)},  \tag{9}\\
\Lambda-\mu R^{*}-\mu S^{*}-(\mu+\alpha) I^{*}=0 . \tag{10}
\end{gather*}
$$

Substituting (8) and (9) into (10), we get $\phi\left(I^{*}\right)=0$, where

$$
\begin{equation*}
\phi(I)=\Lambda-\frac{\mu \lambda I}{\mu+\sigma}-\frac{\mu(\alpha+\mu+\lambda)}{\beta_{1}-\beta_{2} f(I)}-(\alpha+\mu) I . \tag{11}
\end{equation*}
$$

Hence, if an endemic equilibrium exists, its coordinate must be a root of $\phi(I)=0$ in the interval $I \in(0, \Lambda / \mu)$.

Note that

$$
\begin{equation*}
\phi^{\prime}(I)=-\frac{\mu \lambda}{\mu+\sigma}-\frac{\beta_{2} \mu(\alpha+\mu+\lambda) f^{\prime}(I)}{\left(\beta_{1}-\beta_{2} f(I)\right)^{2}}-\alpha-\mu<0 \tag{12}
\end{equation*}
$$

Hence, $\phi(I)$ is monotonically decreasing for $I>0$.
Besides,

$$
\begin{gather*}
\phi\left(\frac{\Lambda}{\mu}\right)=-\frac{\lambda \Lambda}{\mu+\sigma}-\frac{\mu(\alpha+\mu+\lambda)}{\beta_{1}-\beta_{2} f(\Lambda / \mu)}-\frac{(\alpha+\mu) \Lambda}{\mu}<0 \\
\phi(0)=\frac{\mu(\alpha+\mu+\lambda)\left(\mathscr{R}_{0}-1\right)}{\beta_{1}} . \tag{13}
\end{gather*}
$$

Therefore, when $\mathscr{R}_{0}>1, \phi(0)>0, \phi(I)$ has unique positive root $I^{*}$ in the interval $I \in(0, \Lambda / \mu)$. $S^{*}$ and $R^{*}$ are uniquely determined by $I^{*}$. Therefore, model (2) has a unique endemic equilibrium $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ if $\mathscr{R}_{0}>1$. Otherwise, there is no endemic equilibrium.

## 3. Stability of the Disease-Free Equilibrium

Theorem 1. The disease-free equilibrium $E_{0}$ is locally asymptotically stable for $\mathscr{R}_{0}<1$ and unstable for $\mathscr{R}_{0}>1$.

Proof. The Jacobian matrix of system (2) at $X=E_{0}$ is

$$
J\left(E_{0}\right)=\left(\begin{array}{ccc}
-\mu & \frac{\beta_{1} \Lambda}{\mu} & \sigma  \tag{14}\\
0 & \frac{\beta_{1} \Lambda}{\mu}-(\alpha+\mu+\lambda) & 0 \\
0 & \lambda & -(\mu+\sigma)
\end{array}\right)
$$

The eigenvalues of the matrix $J\left(E_{0}\right)$ are given by

$$
\begin{equation*}
\xi_{1}=-\mu, \quad \xi_{2}=-(\mu+\sigma), \quad \xi_{3}=(\alpha+\mu+\lambda)\left(\mathscr{R}_{0}-1\right) \tag{15}
\end{equation*}
$$

If $\mathscr{R}_{0}<1$, then $\xi_{3}<0$. Thus, using the Routh-Hurwitz criterion, all eigenvalues of $J\left(E_{0}\right)$ have negative real parts, and $E_{0}$ is locally asymptotically stable for system (2).

## 4. Stability of the Endemic Equilibrium

Theorem 2. If $\mathscr{R}_{0}>1, E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ is locally asymptotically stable.

Proof. Let

$$
\begin{gather*}
A=\left(\beta_{1}-\beta_{2} f\left(I^{*}\right)\right) I^{*}>0 \\
B=\beta_{2} f^{\prime}\left(I^{*}\right) S^{*} I^{*}>0 \tag{16}
\end{gather*}
$$

The Jacobian matrix at $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ is

$$
J\left(E^{*}\right)=\left(\begin{array}{ccc}
-\mu-A & B-(\alpha+\mu+\lambda) & \sigma  \tag{17}\\
A & -B & 0 \\
0 & \lambda & -(\mu+\sigma)
\end{array}\right)
$$

The characteristic polynomial of the matrix $J\left(E^{*}\right)$ is given by

$$
\begin{equation*}
\operatorname{det}\left(\delta I-J\left(E^{*}\right)\right)=a_{0} \delta^{3}+a_{1} \delta^{2}+a_{2} \delta+a_{3} \tag{18}
\end{equation*}
$$

where

$$
\begin{gathered}
a_{0}=1, \\
a_{1}=A+B+\sigma+2 \mu>0, \\
a_{2}=2 B \mu+\mu \sigma+\mu^{2}+B \sigma+A \sigma+A \alpha+A \lambda \\
+2 A \mu>0,
\end{gathered}
$$

$$
\begin{align*}
a_{3}=A \alpha \sigma & +B \mu^{2}+B \mu \sigma+A \mu(\mu+\sigma+\alpha+\lambda)>0, \\
a_{1} a_{2}-a_{3}= & \sigma(A+B)^{2}+A \lambda \sigma+5 A \mu \sigma+A \mu \lambda \\
& +4 B \mu \sigma+5 A B \mu+A \mu \alpha+4 B \mu^{2}+6 A \mu^{2}+2 \\
& \cdot B^{2} \mu+3 A^{2} \mu+\mu \sigma^{2}+3 \sigma \mu^{2}+B \sigma^{2}+A \sigma^{2} \\
& +2 \mu^{3}+A B \alpha+A B \lambda+\Phi^{2} \alpha+A^{2} \lambda>0 . \tag{19}
\end{align*}
$$

Thus, using Routh-Hurwitz criterion, all eigenvalues of $J\left(E^{*}\right)$ have negative real parts which means $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ is locally asymptotically stable.

Theorem 3. If $\mathscr{R}_{0}>1, E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ is globally asymptotically stable, provided that inequalities $\mu>\sigma$ and $\mu>\lambda$ hold true.

In order to study the global stability of $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$, we use the geometrical approach which is developed in the papers of Smith [10] and Li and Muldowney [11]. We obtain simple sufficient conditions that $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ is globally asymptotically stable when $\mathscr{R}_{0}>1$. At first, we give a brief outline of this geometrical approach.

Let $x \mapsto f(x) \in R^{n}$ be a $C^{1}$ function for $x$ in an open set $D \in R^{n}$. Consider the differential equation

$$
\begin{equation*}
x^{\prime}=f(x) \tag{20}
\end{equation*}
$$

Denote by $x\left(t, x_{0}\right)$ the solution to (20) such that $x\left(0, x_{0}\right)$. We make the following two assumptions.
(i) There exists a compact absorbing set $K \subset D$.
(ii) Equation (20) has a unique equilibrium $\bar{x}$ in $D$.

The equilibrium $\bar{x}$ is said to be globally stable in $D$ if it is locally stable and all trajectories in $D$ converge to $\bar{x}$.

The following general global stability principle is established in [11].

Let $x \mapsto P(x)$ be an $\binom{n}{2} \times\binom{ n}{2}$ matrix-valued function that is $C^{1}$ for $x \in D$. Assume that $P^{-1}(x)$ exists and is continuous for $x \in K$, the compact absorbing set. A quantity $q$ is defined as

$$
\begin{equation*}
q=\limsup _{t \rightarrow \infty} \sup _{x \in K} \frac{1}{t} \int_{0}^{t} \bar{\mu}\left(Q\left(x\left(s, x_{0}\right)\right)\right) d s \tag{21}
\end{equation*}
$$

where

$$
\begin{equation*}
Q=P_{f} P^{-1}+P J^{[2]} P^{-1} \tag{22}
\end{equation*}
$$

and $J^{[2]}$ is the second additive compound matrix of the Jacobian matrix $J$. The matrix $P_{f}$ is obtained by replacing each entry $p_{i j}$ of $P$ by its derivative in the direction of $f, p_{i j} f$, and $\bar{\mu}(Q)$ is the Lozinskiĭ measure of $Q$ with respect to a vector norm $|\cdot|$ in $R^{N}\left(\right.$ where $\left.N=\binom{n}{2}\right)$ defined by [12]

$$
\begin{equation*}
\bar{\mu}(Q)=\lim _{h \rightarrow 0^{+}} \frac{|I+h Q|-1}{h} \tag{23}
\end{equation*}
$$

It is shown in [11] that, if $D$ is simply connected, the condition $q<0$ rules out the presence of any orbit that gives rise to a simple closed rectifiable curve that is invariant for (20), such as periodic orbits, homoclinic orbits, and heteroclinic cycles. As a consequence, the following global stability result is proved in Theorem 3.5 of [11].

Lemma 4. Assume that $D$ is simply connected and that the assumptions (i) and (ii) hold. Then, the unique equilibrium $\bar{x}$ of (20) is globally asymptotically stable in $D$ if $q<0$.

We now apply Lemma 4 to prove Theorem 3.

Proof. The paper [13] showed that the existence of a compact set which is absorbing in the interior of $\Omega$ is equivalent to proving that (2) is uniformly persistent, which means that there exits $c>0$ such that every solution $(S(t), I(t), R(t))$ of (2) with $(S(0), I(0), R(0))$ in the interior $\Omega$ satisfies

$$
\begin{equation*}
\liminf _{t \rightarrow \infty}|(S(t), I(t), R(t))| \geq c \tag{24}
\end{equation*}
$$

In fact, when $\mathscr{R}_{0}>1$, then $E_{0}$ is unstable. The instability of $E_{0}$, together with $E_{0} \in \partial \Omega$, implies the uniform persistence [14]. Thus, (i) is verified. Moreover, as previously shown, $E^{*}$ is the only equilibrium in the interior of $\Omega$, so that (ii) is verified, too. Let $x=(S, I, R)$ and $f(x)$ denote the vector field of (2). The Jacobian matrix $J=\partial f / \partial x$ associated with a general solution $x(t)$ of (2) is

$$
J=\left(\begin{array}{ccc}
-\mu-\Phi & \Psi-(\alpha+\mu+\lambda) & \sigma  \tag{25}\\
\Phi & -\Psi & 0 \\
0 & \lambda & -(\mu+\sigma)
\end{array}\right)
$$

where

$$
\begin{gather*}
\Phi=\left(\beta_{1}-\beta_{2} f(I)\right) I>0 \\
\Psi=\beta_{2} f^{\prime}(I) S I>0 \tag{26}
\end{gather*}
$$

and its second additive compound matrix $J^{[2]}$ is

$$
J^{[2]}=\left(\begin{array}{ccc}
-\mu-\Phi-\Psi & 0 & -\sigma  \tag{27}\\
\lambda & -\Phi-2 \mu-\sigma & \Psi-(\alpha+\mu+\lambda) \\
0 & \Phi & -\Psi-\mu-\sigma
\end{array}\right)
$$

Set the function $P(x)=P(S, I, R)=\operatorname{diag}\{I / R, I / R, I / R\}$; then

$$
\begin{equation*}
P_{f} P^{-1}=\operatorname{diag}\left\{\frac{I^{\prime}}{I}-\frac{R^{\prime}}{R}, \frac{I^{\prime}}{I}-\frac{R^{\prime}}{R}, \frac{I^{\prime}}{I}-\frac{R^{\prime}}{R}\right\} \tag{28}
\end{equation*}
$$

and the matrix $Q=P_{f} P^{-1}+P J^{[2]} P^{-1}$ can be written in block form

$$
Q=\left(\begin{array}{ll}
Q_{11} & Q_{12}  \tag{29}\\
Q_{21} & Q_{22}
\end{array}\right)
$$



Figure 1: The tendency of the infected population varies. The solid line represents the case when $\beta_{2}=0.0018$, and the dashed line represents the case when $\beta_{2}=0$.


Figure 2: Variation of the number of infected under different $\Lambda$. The solid line represents the case when $\Lambda=5$, and the dashed line represents the case when $\Lambda=2$.
where

$$
\begin{gather*}
Q_{11}=-\frac{R^{\prime}}{R}-\mu-\Phi-\Psi \\
Q_{12}=(0,-\sigma), \\
Q_{21}=\binom{\lambda}{0}, \\
Q_{22}=\left(\begin{array}{cc}
\frac{I^{\prime}}{I}-\frac{R^{\prime}}{R}-\Phi-2 \mu-\sigma & \Psi-\alpha-\mu-\lambda \\
\Phi & \frac{I^{\prime}}{I}-\frac{R^{\prime}}{R}-\Psi-\mu-\sigma
\end{array}\right) \tag{30}
\end{gather*}
$$

The vector norm $|\cdot|$ in $R^{3} \cong R^{\left(\frac{3}{2}\right)}$ is chosen as $|(u, v, w)|=$ $\sup \{|u|,|v|+|w|\}$ and let $\mu(\cdot)$ be the Lozinskiĭ measure with respect to this norm. Following the method in [15], we have

$$
\begin{equation*}
\bar{\mu}(Q) \leq \sup \left\{g_{1}, g_{2}\right\} \tag{31}
\end{equation*}
$$

where

$$
\begin{align*}
& g_{1}=\bar{\mu}_{1}\left(Q_{11}\right)+\left|Q_{12}\right|,  \tag{32}\\
& g_{2}=\bar{\mu}_{1}\left(Q_{22}\right)+\left|Q_{21}\right| .
\end{align*}
$$

$\left|Q_{12}\right|$ and $\left|Q_{21}\right|$ being the matrix norm with respect to the $l_{1}$ vector norm. More specifically,

$$
\begin{gather*}
\bar{\mu}_{1}\left(Q_{11}\right)=-\frac{R^{\prime}}{R}-\mu-\Phi-\Psi, \\
\left|Q_{12}\right|=\sigma,  \tag{33}\\
\left|Q_{21}\right|=\lambda .
\end{gather*}
$$

To calculate $\bar{\mu}_{1}\left(Q_{22}\right)$, add the absolute value of the offdiagonal elements to the diagonal one in each column of $Q_{22}$ and then take the maximum of two sums. We thus obtain

$$
\begin{equation*}
\bar{\mu}_{1}\left(Q_{22}\right)=\frac{I^{\prime}}{I}-\frac{R^{\prime}}{R}-2 \mu-\sigma \tag{34}
\end{equation*}
$$

Therefore, we have

$$
\begin{align*}
& g_{1}=\bar{\mu}_{1}\left(Q_{11}\right)+\left|Q_{12}\right|=\sigma-\frac{R^{\prime}}{R}-\mu-\Phi-\Psi \\
& g_{2}=\bar{\mu}_{1}\left(Q_{22}\right)+\left|Q_{21}\right|=\lambda+\frac{I^{\prime}}{I}-\frac{R^{\prime}}{R}-2 \mu-\sigma \tag{35}
\end{align*}
$$

This leads to

$$
\begin{equation*}
\bar{\mu}(Q) \leq \frac{I^{\prime}}{I}-\mu+\max \{\sigma, \lambda\} \tag{36}
\end{equation*}
$$

Table 1: Parameters for the simulation.

| Figure | Parameter values |  |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\Lambda$ | $\mu$ | $\beta_{1}$ | $\beta_{2}$ | $\alpha$ | $\lambda$ | $\sigma$ |
| Figure 1(a) | 5 | 0.02 | 0.002 | $0.0018,0$ | 0.1 | 0.05 | 0.01 |
| Figure 1(b) | 5 | 0.2 | 0.002 | $0.0018,0$ | 0.1 | 0.05 | 0.01 |
| Figure 2 | 5,2 | 0.02 | 0.002 | 0.0018 | 0.1 | 0.05 | 0.01 |
| Figure 3 | 5 | 0.02 | 0.002 | 0.0018 | 0.1 | $0.05,0.5$ | 0.01 |

We can deduce that if

$$
\begin{align*}
& \mu>\sigma  \tag{37}\\
& \mu>\lambda
\end{align*}
$$

hold, then

$$
\begin{equation*}
\bar{\mu}(Q) \leq \frac{I^{\prime}}{I}-d \tag{38}
\end{equation*}
$$

where

$$
\begin{equation*}
d=\min \{\mu-\sigma, \mu-\lambda\}>0 \tag{39}
\end{equation*}
$$

Along each solution ( $S(t), I(t), R(t)$ ) of system (2) for which $(S(0), I(0), R(0)) \in \Omega$, we have

$$
\begin{align*}
q & =\limsup _{t \rightarrow \infty} \sup _{x_{0} \in \Omega} \frac{1}{t} \int_{0}^{t} \bar{\mu}\left(Q\left(x\left(s, x_{0}\right)\right)\right) d s  \tag{40}\\
& \leq-\frac{d}{2}<0 .
\end{align*}
$$

According to Lemma 4 , if $\mathscr{R}_{0}>1$, then the endemic equilibrium $E^{*}\left(S^{*}, I^{*}, R^{*}\right)$ of system (2) is globally asymptotically stable in $\Omega$.

## 5. Simulation Study and Discussion

To complement the mathematical analysis carried out in the previous section, using the Runge-Kutta method, we now investigate some numerical properties of (2). Choose $f(I)=$ $I /(b+I), b>0$, and $b$ reflects the reactive velocity of people and media coverage to the disease. Related parameter values are listed in Table 1.

Figure 1(a) shows that, when $\mathscr{R}_{0}=2.941>1$, the number of infected individuals is asymptotically stable, and the media coverage is beneficial to decrease the number of infected individuals. Figure 1(b) shows that, when $\mathscr{R}_{0}=0.029<1$, the number of infected individuals tends to zero point, and the media coverage can quicken the extinction of infectious disease.

Furthermore, the analysis of the impact of related parameters on the infectious disease progression is fairly important. From the definition of $\mathscr{R}_{0}$, it can be seen that

$$
\begin{align*}
\frac{\partial \mathscr{R}_{0}}{\partial \Lambda} & =\frac{\beta_{1}}{\mu(\alpha+\mu+\lambda)}>0, \\
\frac{\partial \mathscr{R}_{0}}{\partial \lambda} & =-\frac{\beta_{1} \Lambda}{\mu(\alpha+\mu+\lambda)^{2}}<0 . \tag{41}
\end{align*}
$$



Figure 3: Variation of the number of infected under different $\lambda$. The solid line represents the case when $\lambda=0.05$, and the dashed line represents the case when $\lambda=0.5$.

Hence, $\mathscr{R}_{0}$ is an increasing function of $\Lambda$ and is a decreasing function of $\lambda$. The mathematical results show that the basic reproduction number $\mathscr{R}_{0}$ satisfies a threshold property. When $\mathscr{R}_{0}<1$, it has been proved that the diseasefree equilibrium $E_{0}$ is locally asymptotically stable, and the diseases will be eliminated from the community. And, when $\mathscr{R}_{0}>1$, the unique endemic equilibrium $E^{*}$ is globally asymptotically stable, and the diseases persist. This shows that $\mathscr{R}_{0}$ reduces to a value less than unity by reducing $\Lambda$ or increasing $\lambda$, so as to control the spread of infectious diseases.

From Figure 2, we can find that the number of infected individuals decreases as the recruitment rate ( $\Lambda$ ) decreases. Organized measures such as limitation of travel, closure of public places, or isolation are beneficial to lessen the recruitment rate to control the spreading of infectious diseases. Figure 3 reveals that the number of infected individuals decreases as the recovery rate $(\lambda)$ increases. So timely and effective treatment is regarded as one good method in managing infectious diseases.

Based on the obtained results, we can get that media coverage has an effective impact on the control and spread of infectious diseases. It is hoped that these control strategies we considered may offer some useful suggestions for authorities. In addition, we can generalize the current model by incorporating some control methods, such as isolation and treatment strategies. A more realistic model deserves to be considered.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work is supported by the National Natural Science Foundation of China (no. 11371048), Funding Project for Academic

Human Resources Development in Institutions of Higher Learning Under the Jurisdiction of Beijing Municipality (no. PHR201107123). The authors wish to express their thanks for the financial support.

## References

[1] S. J. Etuk and E. I. Ekanem, "Impact of mass media campaigns on the knowledge and attitudes of pregnant Nigerian woman towards HIV/AIDS," Tropical Doctor, vol. 35, no. 2, pp. 101-102, 2005.
[2] M. S. Rahman and M. L. Rahman, "Media and education play a tremendous role in mounting AIDS awareness among married couples in Banladesh," AIDS Research and Therapy, vol. 4, no. 1, pp. 1-7, 2007.
[3] C. Sun, W. Yang, J. Arino, and K. Khan, "Effect of mediainduced social distancing on disease transmission in a two patch setting," Mathematical Biosciences, vol. 230, no. 2, pp. 87-95, 2011.
[4] S. Funk, E. Gilad, and V. A. A. Jansen, "Endemic disease, awareness, and local behavioural response," Journal of Theoretical Biology, vol. 264, no. 2, pp. 501-509, 2010.
[5] J. A. Cui, Y. H. Sun, and H. P. Zhu, "The impact of media on the control of infectious diseases," Journal of Dynamics and Differential Equations, vol. 20, no. 1, pp. 31-53, 2008.
[6] Y. P. Liu and J.-A. Cui, "The impact of media coverage on the dynamics of infectious disease," International Journal of Biomathematics, vol. 1, no. 1, pp. 65-74, 2008.
[7] N. Ferguson, "Capturing human behaviour," Nature, vol. 446, no. 7137, article 733, 2007.
[8] J.-A. Cui, X. Tao, and H. P. Zhu, "An SIS infection model incorporating media coverage," The Rocky Mountain Journal of Mathematics, vol. 38, no. 5, pp. 1323-1334, 2008.
[9] P. van den Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," Mathematical Biosciences, vol. 180, pp. 29-48, 2002.
[10] R. A. Smith, "Some applications of Hausdorff dimension inequalities for ordinary differential equations," Proceedings of the Royal Society of Edinburgh A, vol. 104, no. 3-4, pp. 235-259, 1986.
[11] M. Y. Li and J. S. Muldowney, "A geometric approach to globalstability problems," SIAM Journal on Mathematical Analysis, vol. 27, no. 4, pp. 1070-1083, 1996.
[12] M. Fan, M. Y. Li, and K. Wang, "Global stability of an SEIS epidemic model with recruitment and a varying total population size," Mathematical Biosciences, vol. 170, no. 2, pp. 199-208, 2001.
[13] G. Butler and P. Waltman, "Persistence in dynamical systems," Journal of Differential Equations, vol. 63, no. 2, pp. 255-263, 1986.
[14] H. I. Freedman, S. G. Ruan, and M. X. Tang, "Uniform persistence and flows near a closed positively invariant set," Journal of Dynamics and Differential Equations, vol. 6, no. 4, pp. 583-600, 1994.
[15] R. H. Martin Jr., "Logarithmic norms and projections applied to linear differential systems," Journal of Mathematical Analysis and Applications, vol. 45, pp. 432-454, 1974.

## Research Article

# Antimicrobial Resistance within Host: A Population Dynamics View 

Chunji Huang ${ }^{1}$ and Aijun Fan ${ }^{2}$<br>${ }^{1}$ Xinqiao Hospital, Third Military Medical University, Chongqing 400037, China<br>${ }^{2}$ Chongqing Research Center for Information and Automation Technology, Chongqing Academy of Science e Technology, Chongqing 401123, China<br>Correspondence should be addressed to Aijun Fan; aijun71@163.com

Received 27 December 2013; Accepted 19 January 2014; Published 23 February 2014
Academic Editor: Weiming Wang
Copyright © 2014 C. Huang and A. Fan. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

To study the relationship between antimicrobial resistance and the concentration of antibiotics, a competitive population dynamical model is proposed between the susceptible strain and the resistant strain with antibiotic exposure. The strict mathematical analysis is performed, and the results indicate that long-term high strength antibiotic treatment and prevention can induce the extinction of susceptible strain. Thus, the prescribed dose of antibiotics must be strictly controlled during the treatment and prevention of the infections in clinics.


## 1. Introduction

It was thought that the war against infectious diseases has been won in the initial stages of the discovery of antibiotics and their widespread introduction [1]. However, during the multiplication process of bacteria, there are high degrees of individuality or phenotypic heterogeneity in populations of genetically identical cells [2-5]. As a result of the cell-to-cell variation, a high probability of the selection of antimicrobial resistance is particularly prone to occur. Thus, the overuse of antibiotic therapy may result in the prevalence of antibioticresistant bacteria and an apparently inexorable advent of a postantibiotic era or a super wicked challenge [6-9]. In fact, antimicrobial resistance has now become an unfolding catastrophe [1] and the new strategy and action plan has been proposed by the Department of Health in the United Kingdom [9].

To extend the life of existing antibiotics, it is necessary to analyze the molecular mechanism of antibiotic resistance and strategize about slowdown and avoid antibiotic resistance during anti-infective therapy. In the process numerous research articles have highlighted that both molecular biology and computational biology, including mathematical modeling, are vitally important methods [5, 10-15]. Especially, recent biological study has confirmed that the signaling
nucleotide (p)ppGpp can control bacterial persistence by stochastic induction of toxin-antitoxin activity, and there is a special resistant strain, which can switch into slow growth through the changes of (p)ppGpp level in high antibiotic concentration [5]. However, under different concentrations of antibiotic, the long-term competitive ending between the susceptible strain and the resistant strain remains unknown.

In this paper, based on the above mentioned mechanism of bacterial antibiotic resistance within the host, a competitive population dynamical model is proposed to explore the competitive interactions between the susceptible strain and the resistant strain with antibiotic exposure. The focus is the relationship between antibiotic resistance and the concentration of antibiotics, which may be added to the host by injection, orally, or by transfusion. The organization of this paper is as follows. In the next section, the proposed model is described and the global dynamics is obtained. In Section 3, some numerical simulations are performed. Finally, a brief discussion is given to conclude this work.

## 2. Model and Its Dynamical Behaviors

2.1. Description of the Model. According to the pharmacokinetic, we know that the concentration of the drug within-host
will tend to be approximately constant after multiple dosing. Thus, it is reasonable to assume that the plasma concentration of the antibiotics is a constant, which is denoted as $S_{0}$. In addition, let $x(t)$ be the number of susceptible strains, and let $y(t)$ be the number of resistant strains at time $t$, respectively. The following differential equations can be used to describe the basic dynamics of the interaction between $x(t)$ and $y(t)$ :

$$
\begin{align*}
& \frac{\mathrm{d} x(t)}{\mathrm{d} t}=x(t)\left(r_{1}-\delta_{11} x(t)-\delta_{12} y(t)-\beta S_{0}\right) \triangleq F_{1}(x, y), \\
& \frac{\mathrm{d} y(t)}{\mathrm{d} t}=y(t)\left(r_{2} e^{-\mu S_{0}}-\delta_{21} x(t)-\delta_{22} y(t)\right) \triangleq F_{2}(x, y), \tag{1}
\end{align*}
$$

where the natural growth rates and death rates of susceptible strain and resistant strain are $r_{1}, r_{2}, \delta_{11} x(t)$, and $\delta_{22} y(t)$, respectively. Parameter $\beta$ is the coefficient of the effect of destroying susceptible bacteria by antibiotics, and function $e^{-\mu S_{0}}$ denotes the decline of growth rate of resistant strain by the signaling nucleotide (p)ppGpp. For biological consistency, all parameters are positive constants, $r_{1}>\beta S_{0}$ and the initial values of system (1) are $x(0)>0$ and $y(0)>0$.
2.2. Mathematical Analysis. Because of the biological meaning of the components $(x(t), y(t))$, we focus on the model in the first octant of $\mathbb{R}^{2}$. To study the dynamics of system (1), we first show that that model (1) is biologically well behaved and dissipative; that is, all solutions of model (1) in $\mathbb{R}_{+}^{2}$ are ultimately bounded and the solutions with positive initial values are positive.

Theorem 1. Under the given initial conditions, all solutions of system (1) are positive and system (1) is dissipative.

This theorem is clear to be seen, thus, the detailed proof is omitted for the sake of simplicity.

In order to obtain the global dynamics of system (1), we first have the following result regarding the nonexistence of periodic orbits in system (1).

Theorem 2. System (1) does not have nontrivial periodic orbits.

Proof. Consider system (1) for $x>0$ and $y>0$. Take a Dulac function:

$$
\begin{equation*}
D(x, y)=\frac{1}{x y} \tag{2}
\end{equation*}
$$

We have

$$
\begin{equation*}
\frac{\partial D(x, y) F_{1}(x, y)}{\partial x}+\frac{\partial D(x, y) F_{2}(x, y)}{\partial y}=-\frac{\delta_{11}}{y}-\frac{\delta_{22}}{x}<0 . \tag{3}
\end{equation*}
$$

The conclusion follows from Dulac criterion [16, 17].
We now consider the existence of equilibria of system (1). Let $F_{1}(x, y)=0$ and let $F_{2}(x, y)=0$. Clearly, when the plasma concentration of the antibiotics $S_{0}<r_{1} / \beta$, model (1) always
has three equilibria: one is $E_{0}=(0,0)$, meaning that both bacteria become extinct and the others are $E_{1}=\left(x_{1}, 0\right)$ and $E_{2}=\left(0, y_{2}\right)$, in which

$$
\begin{equation*}
x_{1}=\frac{r_{1}-\beta S_{0}}{\delta_{11}}, \quad y_{2}=\frac{r_{2} e^{-\mu S_{0}}}{\delta_{22}} \tag{4}
\end{equation*}
$$

which are corresponding to the extinction of resistant strain and susceptible strain, respectively. Furthermore, we have the positive equilibrium $E_{+}=\left(x^{*}, y^{*}\right)$, corresponding to coexistence of susceptible strain and resistant strain, that is given by intersections of the zero growth isoclines:

$$
\begin{gather*}
l_{1}: r_{1}-\delta_{11} x(t)-\delta_{12} y(t)-\beta S_{0}=0 \\
l_{2}: r_{2} e^{-\mu S_{0}}-\delta_{21} x(t)-\delta_{22} y(t)=0 \tag{5}
\end{gather*}
$$

Apparently, the isoclines $l_{1}$ and $l_{2}$ pass through the points $E_{1},\left(0, y_{1}\right)$ and $\left(x_{2}, 0\right), E_{2}$, respectively. Here

$$
\begin{equation*}
y_{1}=\frac{r_{1}-\beta S_{0}}{\delta_{11}}, \quad x_{2}=\frac{r_{2} e^{-\mu S_{0}}}{\delta_{21}} \tag{6}
\end{equation*}
$$

According to the position relation between $l_{1}$ and $l_{2}$, we know that there are four cases (Figure 1) depending on the size of parameters $A_{1}, A_{2}, A_{3}$, and $A_{4}$, in which

$$
\begin{array}{ll}
A_{1}=r_{2} \delta_{12} e^{-\mu S_{0}}+\beta \delta_{22} S_{0}, & A_{2}=r_{1} \delta_{22} \\
A_{3}=r_{2} \delta_{11} e^{-\mu S_{0}}+\beta \delta_{21} S_{0}, & A_{4}=r_{1} \delta_{21} \tag{7}
\end{array}
$$

The object of the next analysis is to study the asymptotical stabilizability of the equilibria. Since $l_{1}$ and $l_{2}$ are the isoclines of system (1), $l_{1}$ and $l_{2}$ divide the first octant into several subregions, and the derivative of $x$ and $y$ keeps a fixed sign in each subregion as indicated in Figure 1. By the combination of Theorem 1, Theorem 2, and the Poincaré-Bendixson theorem, with the help of the fixed sign in each subregion (Figure 1), we have the complete dynamical behaviors of system (1), which is summarized in Table 1.

## 3. Simulations

From Table 1, we know that equilibrium $E_{2}$ is globally asymptotically stable in the case of (IV); that is, the susceptible strain will extinct and the resistant strain will persist, which means that antimicrobial resistance occurs. What is the relationship between the concentration of antibiotics and the phenomenon of antimicrobial resistance? In this section, we will give some qualitative analyses from a numerical simulation standpoint.

Let

$$
\begin{array}{llr}
r_{1}=1.5, & r_{2}=3.5, & \delta_{11}=2.0 \\
\delta_{22}=12.0, & \beta=1.0, & \mu=0.5 \tag{8}
\end{array}
$$

When $\delta_{12}=4.0$ and $\delta_{21}=3.0$, if there is no antibiotics, that is, $S_{0}=0$, after a simple calculation, we have $A_{1}<A_{2}$ and $A_{3}>A_{4}$. Thus, Case (I) occurs (Figure 2(a)). Increasing

Table 1: Dynamical behaviors of system (1).

| Case | Conditions | Dynamics |
| :--- | :--- | :--- |
| I | $A_{1}<A_{2}$ and $A_{3}>A_{4}$ | $E_{0}, E_{1}$, and $E_{2}$ are unstable, and $E_{+}$is globally asymptotically stable |
| II | $A_{1}>A_{2}$ and $A_{3}<A_{4}$ | $E_{0}$ and $E_{+}$are unstable; $E_{1}$ and $E_{2}$ are locally stable dependent on the |
| III | $A_{1}<A_{2}$ and $A_{3}<A_{4}$ | $E_{0}$ and $E_{2}$ are unstable, and $E_{1}$ is globally asymptotically stable |
| IV | $A_{1}>A_{2}$ and $A_{3}>A_{4}$ | $E_{0}$ and $E_{1}$ are unstable, and $E_{2}$ is globally asymptotically stable |



FIgure 1: Illustration of the equilibria and the vector field of system (1). (I) $A_{1}<A_{2}$ and $A_{3}>A_{4}$; (II) $A_{1}>A_{2}$ and $A_{3}<A_{4}$; (III) $A_{1}<A_{2}$ and $A_{3}<A_{4}$; (IV) $A_{1}>A_{2}$ and $A_{3}>A_{4}$. Each equilibrium is represented by a closed cycle ( $\cdot$ ), and the sign of the derivative of $x, y$ in each subregion is denoted by $(a, b)$, where $a$ is the sign of the derivative of $x$ and $b$ is the sign of the derivative of $y$. Note that the expressions of $x_{1}, x_{2}, y_{1}, y_{2}, A_{1}, A_{2}, A_{3}$, and $A_{4}$ are shown in (4), (6), and (7).
the concentration of antibiotics, $S_{0}=0.1$, the inequalities remain valid. However, when the concentration of antibiotics increase to $S_{0}=0.9$, we find that the inequalities become $A_{1}>A_{2}, A_{3}>A_{4}$, and Case (IV) occurs, which is also shown in Figure 2(a). Thereby long-term high strength antibiotic treatment and prevention can induce the extinction of susceptible strain and accelerate the phenomenon of antimicrobial resistance.

By changing the parameter $\delta_{21}$ to 6.0, because $A_{1}<$ $A_{2}$ and $A_{3}<A_{4}$ are valid, we can obtain the extinction of resistant strain and persistence of susceptible strain if there is no antibiotics or low strength antibiotic treatment (Figure 2(b), Case (III) in Table 1). Similarly, when there is a high strength antibiotic treatment, $S_{0}=0.9$, the inequalities change to $A_{1}>A_{2}$ and $A_{3}>A_{4}$ (Case (IV) in Table 1) and the simulated time series is shown in Figure 2(b). Thus, the serious consequences of the abuse of antibiotic were proved afresh during the treatment and prevention of the infections.

Holding $\delta_{21}=6.0$ and changing the parameter $\delta_{12}$ to 5.5, the inequalities $A_{1}>A_{2}$ and $A_{3}<A_{4}$ are valid if $S_{0}=0.0$ or $S_{0}=0.1$ (Case (II) in Table 1). Thus, both extinction and persistence of the resistant strain may happen in course of
the competition because $E_{1}$ and $E_{2}$ are locally stable dependent on the initial conditions (Figures 2(c) and 2(d)). However, when the concentration of antibiotics increase to $S_{0}=$ 0.9 , the resistant strain is survived since the inequalities change to $A_{1}>A_{2}$ and $A_{3}>A_{4}$ (Case (IV) in Table 1) and the equilibrium $E_{2}$ is globally asymptotically stable (Figures 2(c) and 2(d)), which also means that it is necessary to control the dose of antibiotics. Otherwise, antimicrobial resistance will occur.

## 4. Discussion

According to the latest mechanism of bacterial antibiotic resistance within the host [5], a competitive population model (1) between the susceptible strain and resistant strain is proposed under the circumstance of antibiotic exposure. Based on the global dynamics of system (1), the relationship is explored between antimicrobial resistance and the concentration of antibiotics by numerical simulations. The results indicate that the resistant strain will ultimately survive along with the long-term high strength antibiotic treatment and

(a)

(c)


$$
\begin{array}{lll}
- & x(t), S_{0}=0.0 & --y(t), S_{0}=0.1 \\
-- & y(t), S_{0}=0.0 & - \\
- & x(t), S_{0}=0.1 & --y(t), S_{0}=0.9 \\
- & y(t), S_{0}=0.9
\end{array}
$$

(b)

(d)

Figure 2: Time series of the susceptible strain $(x(t))$ and resistant strain $(y(t))$ within host as predicted by the model (1). $\delta_{12}=4.0$ and $\delta_{21}=3.0$ in (a); $\delta_{12}=4.0$ and $\delta_{21}=6.0$ in (b); $\delta_{12}=5.5$ and $\delta_{21}=6.0$ in (c) and (d). The initial condition is $(x(0), y(0))=(0.8,0.2)$ in (a), (b), and (c), and $(x(0), y(0))=(0.2,0.8)$ in (d). Other parameters are shown in (8).
prevention, which has been found in many recurrent and chronic infections [18-20].

Note that the assumption that infections can be prevented or treated has become the backbone of the whole modern healthcare [1]. Thus, resistance is not just an infectious disease issue, it is also a surgical issue, a cancer issue, and a health system issue [1]. Antimicrobial prescribing needs to be more evidence based and more efficiently targeted [9]. In particular, in order to inhibit or decelerate resistance to antibiotics, the prescribed dose of antibiotics must be strictly controlled during the treatment and prevention of the infections in
clinics. Otherwise, a postantibiotic era or a super wicked challenge is likely to occur [6-9]. Though the risk-benefit balance for antibiotic prescribing is becoming even more complex [9], mathematical modeling may be a useful research tool because it can involve and integrate a wide range of subjects, including biology, medicine, and economics.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

This work is supported by the Research Institutes Innovation Capacity-Building Program of CQ (no. cstc2012ptkyys40002).

## References

[1] F. Godlee, "Antimicrobial resistance-an unfolding catastrophe," British Medical Journal, vol. 346, article f1663, 2013.
[2] D. Dubnau and R. Losick, "Bistability in bacteria," Molecular Microbiology, vol. 61, no. 3, pp. 564-572, 2006.
[3] A. Eldar and M. B. Elowitz, "Functional roles for noise in genetic circuits," Nature, vol. 467, no. 7312, pp. 167-173, 2010.
[4] M. E. Lidstrom and M. C. Konopka, "The role of physiological heterogeneity in microbial population behavior," Nature Chemical Biology, vol. 6, no. 10, pp. 705-712, 2010.
[5] E. Maisonneuve, M. Castro-Camargo, and K. Gerdes, "(p)ppGpp controls bacterial persistence by stochastic induction of toxin-antitoxin activity," Cell, vol. 154, no. 5, pp. 1140-1150, 2013.
[6] F. E. Berkowitz, "Antibiotic resistance in bacteria," Southern Medical Journal, vol. 88, no. 8, pp. 797-804, 1995.
[7] F. C. Tenover and J. M. Hughes, "The challenges of emerging infectious diseases: development and spread of multiplyresistant bacterial pathogens," Journal of the American Medical Association, vol. 275, no. 4, pp. 300-304, 1996.
[8] A. M. Garber, "Antibiotic exposure and resistance in mixed bacterial populations," Theoretical Population Biology, vol. 32, no. 3, pp. 326-346, 1987.
[9] A. S. Kessel and M. Sharland, "The new UK antimicrobial resistance strategy and action plan," British Medical Journal, vol. 346, article f1601, 2013.
[10] M. Lipsitch, C. T. Bergstrom, and B. R. Levin, "The epidemiology of antibiotic resistance in hospitals: paradoxes and prescriptions," Proceedings of the National Academy of Sciences of the United States of America, vol. 97, no. 4, pp. 1938-1943, 2000.
[11] M. J. M. Bonten, D. J. Austin, and M. Lipsitch, "Understanding the spread of antibiotic resistant pathogens in hospitals: mathematical models as tools for control," Clinical Infectious Diseases, vol. 33, no. 10, pp. 1739-1746, 2001.
[12] N. Jumbe, A. Louie, R. Leary et al., "Application of a mathematical model to prevent in vivo amplification of antibiotic-resistant bacterial populations during therapy," Journal of Clinical Investigation, vol. 112, no. 2, pp. 275-285, 2003.
[13] A. Sotto and J. P. Lavigne, "A mathematical model to guide antibiotic treatment strategies," BMC Medicine, vol. 10, article 90, 2012.
[14] P. Ankomah and B. R. Levin, "Two-drug antimicrobial chemotherapy: a mathematical model and experiments with Mycobacterium marinum," PLoS Pathogens, vol. 8, no. 1, Article ID e1002487, 2012.
[15] I. H. Spicknall, B. Foxman, C. F. Marrs, and J. N. S. Eisenberg, "A modeling framework for the evolution and spread of antibiotic resistance: literature review and model categorization," American Journal of Epidemiology, vol. 178, no. 4, pp. 508-520, 2013.
[16] C. C. McCluskey and J. S. Muldowney, "Bendixson-Dulac criteria for difference equations," Journal of Dynamics and Differential Equations, vol. 10, no. 4, pp. 567-575, 1998.
[17] O. Osuna and G. Villasenor, "On the Dulac functions," Qualitative Theory of Dynamical Systems, vol. 10, no. 1, pp. 43-49, 2011.
[18] K. Lewis, "Persister cells, dormancy and infectious disease," Nature Reviews Microbiology, vol. 5, no. 1, pp. 48-56, 2007.
[19] M. D. LaFleur, Q. Qi, and K. Lewis, "Patients with longterm oral carriage harbor high-persister mutants of Candida albicans," Antimicrobial Agents and Chemotherapy, vol. 54, no. 1, pp. 39-44, 2010.
[20] K. R. Allison, M. P. Brynildsen, and J. J. Collins, "Metaboliteenabled eradication of bacterial persisters by aminoglycosides," Nature, vol. 473, no. 7346, pp. 216-220, 2011.

## Research Article

# An SIR Epidemic Model with Time Delay and General Nonlinear Incidence Rate 

Mingming Li and Xianning Liu<br>Key Laboratory of Eco-Environments in Three Gorges Reservoir Region (Ministry of Education), School of Mathematics and Statistics, Southwest University, Chongqing 400715, China

Correspondence should be addressed to Xianning Liu; liuxn@swu.edu.cn
Received 26 December 2013; Accepted 6 January 2014; Published 20 February 2014
Academic Editor: Weiming Wang
Copyright © $2014 \mathrm{M} . \operatorname{Li}$ and X. Liu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

An SIR epidemic model with nonlinear incidence rate and time delay is investigated. The disease transmission function and the rate that infected individuals recovered from the infected compartment are assumed to be governed by general functions $F(S, I)$ and $G(I)$, respectively. By constructing Lyapunov functionals and using the Lyapunov-LaSalle invariance principle, the global asymptotic stability of the disease-free equilibrium and the endemic equilibrium is obtained. It is shown that the global properties of the system depend on both the properties of these general functions and the basic reproductive number $R_{0}$.


## 1. Introduction

The mechanism of transmission is usually qualitatively known for most diseases from epidemiological point of view. For modeling the spread process of infectious diseases mathematically and quantitatively, many classical epidemic models have been proposed and studied, such as SIR, SIS, SEIR, and SIRS models. Recently, considerable attention have been paid to study the dynamics of epidemic models with epidemiological meaningful time delays.

The fundamental assumption in epidemic models is that the population can be divided into distinct groups. The most common groups are the susceptible ( $S$ ) which contains individuals that may be infected by the disease; the infected $(I)$ which contains individuals that are already infected and can spread the disease to susceptible individuals; the removed $(R)$ which contains individuals that have the immunity and cannot be infected. Therefore such models are referred to SIR models. The simplest forms of these models are ordinary differential equations (ODEs).

It is well known that the disease transmission progress plays an important role in the epidemic dynamics; that is, applying different incidence rates can potentially change the behaviors of the system. In many epidemic models, following incidence functions with delay or without delay are widely used in different epidemiological backgrounds.
(1) The bilinear incidence rate $\beta S I$ (e.g., [1-8]), where $\beta$ is the average number of contacts per infected individual per day.
(2) The standard incidence rate $\beta S I / N$ ([9-12]), where $N=S+I+R$.
(3) The Holling type incidence rate of the form $\beta S I /(1+$ $\left.\alpha_{1} S\right)([13-15])$, where $\alpha_{1}$ is a positive constant.
(4) The saturated incidence rate of the form $\beta S I /\left(1+\alpha_{2} I\right)$ ([16-21]), where $\alpha_{2}$ is a positive constant.
(5) The saturated incidence rate of the form $\beta S I /\left(1+\alpha_{1} S+\right.$ $\left.\alpha_{2} I\right)([22-25])$, where $\alpha_{1}, \alpha_{2}$ is a positive constant.
The bilinear incidence rate in (1) is based on the law of mass action, which is more appropriate for communicable diseases, such as influenza, but not suitable for sexually transmitted diseases. It has been pointed out that standard incidence rate in (2) may be a good approximation when the number of available partners is large enough and everybody could not make more contacts than that is practically feasible. In fact, the infection probability per contact is likely influenced by the number of infective individuals, because more infective individuals can increase the infection risk.

In the incidence rates in (3) and (4), $\beta S I$ measures the infection force of the disease and $1 /\left(1+\alpha_{1} S\right), 1 /\left(1+\alpha_{2} I\right)$ measure the inhibition effect from the behavioral changes of
the susceptible individuals when their number increases or from the crowding effect of the infective individuals. In these incidence rates, the number of effective contacts between infectived and susceptible individuals may saturate at high infective levels. These incidence rates seem more reasonable than the bilinear incidence rate $\beta S I$, because they include the behavioral changes of susceptible individuals and crowding effect of the infective individuals which prevent the unboundedness of the contact rate by choosing suitable parameters.

Obviously, the incidence rate in (5) includes the former three incidence rates: the bilinear incidence rate $\beta S I$ (when $\left.\alpha_{1}=0, \alpha_{2}=0\right)$, the Holling type incidence rate $\beta S I /\left(1+\alpha_{1} S\right)$ (when $\left.\alpha_{2}=0\right)$, and the saturated incidence rate $\beta S I /\left(1+\alpha_{2} I\right)$ (when $\alpha_{1}=0$ ).

The incidence rate can also be modeled by many other kinds of more general functions. It is interesting that whether the functional form of the incidence rate can change the epidemic dynamics or not. Korobeinikov studied the global properties for epidemiological models with various nonlinear incidence rates, such as $f(s) g(i)$ in [26], $f(s, i)$ in [27-29]. By constructing Lyapunov functions, [27, 28] established the global stability for ordinary differential equations models of epidemiological dynamics with nonlinear incidence rate $f(s, i)$ under some conditions.

These models have not included time delays, which are usually used to model the fact that an individual may not be infectious until some time after becoming infected. In the context of epidemiology, delays can be caused by a variety of factors. The most common reasons for a delay are (i) the latency of the infection in a vector and (ii) the latency of the infection in an infected host [30]. In these cases, some time should elapse before the level of infection in the infected host or the vector reaching a sufficiently high level to transmit the infection further.

Motivated by all the above, we present a model described by delay differential equations (DDEs) with two general nonlinear terms as follows:

$$
\begin{gather*}
\frac{d S(t)}{d t}=\lambda-\mu S(t)-F(S(t), I(t)) \\
\frac{d I(t)}{d t}=e^{-\mu \tau} F(S(t-\tau), I(t-\tau))-(\mu+\alpha) I(t)-G(I(t)), \\
\frac{d R(t)}{d t}=G(I(t))-\mu R(t) \tag{1}
\end{gather*}
$$

where $S(t), I(t)$, and $R(t)$, as mentioned above, represent the population of the susceptible, the infected, and the removed at time $t$, respectively. The parameters in the equations are explained as below. The positive $\lambda$ is the recruitment rate of the population, $\mu$ is the natural death rate of the population, $\alpha$ is the death rate due to disease, all $\tau \geq 0$ is the latent period. The term $0 \leq e^{-\mu \tau}<1$ represents the survival rate of population and the time they take to become infectious is $\tau$. We assume that the force of infection at any time $t$ is given by the general function $F(S(t), I(t))$, and the recovered infected individuals at any time $t$ is given by the function $G(I(t))$.

Since $R(t)$ does not appear in equations for $d S(t) / d t$ and $d I(t) / d t$, it is sufficient to analyze the behaviors of solutions of (1) by the following system of DDEs:

$$
\begin{gather*}
\frac{d S(t)}{d t}=\lambda-\mu S(t)-F(S(t), I(t)) \\
\frac{d I(t)}{d t}=e^{-\mu \tau} F(S(t-\tau), I(t-\tau))-(\mu+\alpha) I(t)-G(I(t)) \tag{2}
\end{gather*}
$$

The initial conditions for system (2) take the form

$$
\begin{gather*}
S(\theta)=\phi_{1}(\theta), \quad I(\theta)=\phi_{2}(\theta), \\
\phi_{1}(\theta) \geq 0, \quad \phi_{2}(\theta) \geq 0, \quad \theta \in[-\tau, 0],  \tag{3}\\
\phi_{1}(0)>0, \quad \phi_{2}(0)>0,
\end{gather*}
$$

where $\phi=\left(\phi_{1}(\theta), \phi_{2}(\theta)\right) \in C^{+} \times C^{+}$. Here $C$ denotes the Banach space $C=C([-\tau, 0], \mathbb{R})$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}$, equipped with the supremum norm. The nonnegative cone of $C$ is defined as $C^{+}=C\left([-\tau, 0], \mathbb{R}^{+}\right)$.

The organization of this paper is as follows. In Section 2, we study the existence of a positive equilibrium. In Section 3, we show that the global asymptotic stability of the diseasefree equilibrium and the endemic equilibrium of model (2) depend only on the basic reproductive number under some hypotheses. A brief discussion is given in the last section to conclude this paper.

## 2. The Existence of Positive Equilibrium

In this section, we prove the existence of a positive equilibrium. We assume that $F(S, I)$ and $G(I)$ are always positive, continuously differentiable, and monotonically increasing for all $S>0$ and $I>0$. That is, they satisfy the following conditions:
(H1) $F(S, I)>0, F_{S}^{\prime}(S, I)>0, F_{I}^{\prime}(S, I)>0$ for $S>0$ and $I>0$.
(H2) $F(S, 0)=F(0, I)=0, F_{S}^{\prime}(S, 0)=0, F_{I}^{\prime}(S, 0)>0$ for $S>0$ and $I>0$.
(H3) $G(0)=0 . G^{\prime}(I)>0$ for $I \geq 0$.
Global behaviors of system (2) may depend on the basic reproduction number $R_{0}$, which is the average number of secondary cases produced by a single infective individual introduced into an entirely susceptible population. The basic reproductive number for system (2) can be computed as

$$
\begin{equation*}
R_{0}=\frac{e^{-\mu \tau} F_{I}^{\prime}\left(S_{0}, 0\right)}{\mu+\alpha+G^{\prime}(0)} \tag{4}
\end{equation*}
$$

where $S_{0}=\lambda / \mu$. Usually, $R_{0}<1$ implies that the number of infected individuals will tend to zero and $R_{0}>1$ implies that the number will increase.

The epidemiologically natural condition $F(S, 0)=0$ ensures that system (2) always has a disease-free equilibrium $E_{0}=\left(S_{0}, 0\right)$. And it may also admit an endemic equilibrium
$E^{*}=\left(S^{*}, I^{*}\right)$ which depends on $R_{0}$. And $S^{*}, I^{*}$ satisfy the following equations:

$$
\begin{gather*}
\lambda-\mu S^{*}=F\left(S^{*}, I^{*}\right), \\
F\left(S^{*}, I^{*}\right)=e^{\mu \tau}\left[(\mu+\alpha) I^{*}+G\left(I^{*}\right)\right] . \tag{5}
\end{gather*}
$$

We will show that under certain epidemiologically reasonable conditions, the existence of the positive equilibrium state $E^{*}$ is ensured. We have the following theorem.

Theorem 1. Assume that $F(S, I)$ satisfies (H1) and (H2), and $G(I)$ satisfies (H3). Then system (2) has a positive equilibrium state $E^{*}=\left(S^{*}, I^{*}\right)$ if $R_{0}>1$.

Proof. Let the right-hand sides of the three equations in system (2) equal zero; we have that

$$
\begin{equation*}
\lambda-\mu S=F(S, I)=e^{\mu \tau}[(\mu+\alpha) I+G(I)] \tag{6}
\end{equation*}
$$

Substituting the expression of $S$ by $I$, we obtain the following equation for $I$ :

$$
\begin{align*}
H(I)= & F\left(\frac{\lambda-e^{\mu \tau}[(\mu+\alpha) I+G(I)]}{\mu}, I\right)  \tag{7}\\
& -e^{\mu \tau}[(\mu+\alpha) I+G(I)] .
\end{align*}
$$

It is obvious that $H(0)=0$, and we can compute that there exists a positive $I_{0}$ such that $\lambda=e^{\mu \tau}\left[(\mu+\alpha) I_{0}+G\left(I_{0}\right)\right]$. Hence

$$
\begin{equation*}
H\left(I_{0}\right)=F\left(0, I_{0}\right)-\lambda=-\lambda<0 . \tag{8}
\end{equation*}
$$

And when $I \geq 0$, since $H(I)$ is continuously differentiable, we have

$$
\begin{align*}
H^{\prime}(0)= & \lim _{I \rightarrow 0^{+}} \frac{H(I)-H(0)}{I-0} \\
= & F_{I}^{\prime}\left(S_{0}, 0\right)-e^{\mu \tau}\left[\mu+\alpha+G^{\prime}(0)\right] F_{S}^{\prime}\left(S_{0}, 0\right) \\
& -e^{\mu \tau}\left[\mu+\alpha+G^{\prime}(0)\right]  \tag{9}\\
= & F_{I}^{\prime}\left(S_{0}, 0\right)-e^{\mu \tau}\left[\mu+\alpha+G^{\prime}(0)\right] \\
= & e^{\mu \tau}\left[\mu+\alpha+G^{\prime}(0)\right]\left(R_{0}-1\right)
\end{align*}
$$

Thus, $R_{0}>1$ ensures that $H^{\prime}(0)>0$. And $H(I)$ is continuous on $\left[0, I_{0}\right]$; then there exists some $I^{*} \in\left(0, I_{0}\right)$ such that $H\left(I^{*}\right)=0$. Since $G(I)$ is strictly monotonically increasing, we have $e^{\mu \tau}\left[(\mu+\alpha) I^{*}+G\left(I^{*}\right)\right]<e^{\mu \tau}\left[(\mu+\alpha) I_{0}+G\left(I_{0}\right)\right]$. Therefore $S^{*}=\left(\lambda-e^{\mu \tau}\left[(\mu+\alpha) I^{*}+G\left(I^{*}\right)\right]\right) / \mu>0$, and we have proved the existence of the endemic equilibrium $E^{*}=\left(S^{*}, I^{*}\right)$ for system (2) under condition $R_{0}>1$. This completes the proof.

## 3. Global Dynamics of the Model

In this section, we will analyze the global dynamics of system (2) and show the global asymptotic stability of the disease-free equilibrium and the endemic equilibrium.
3.1. Stability of the Disease-Free Equilibrium. In this subsection, we will study the global stability of the disease-free equilibrium $E_{0}=\left(S_{0}, 0\right)$ of system (2). We propose the following conditions:
(H4) $F_{I}^{\prime}(S, 0)$ is increasing with respect to $S>0$.
(H5) $F(S, I) \leq I \cdot(\partial F(S, 0) / \partial I)$ with respect to $I>0$.
(H6) $G^{\prime}(0) \leq G(I) / I$ with respect to $I>0$.
By ( H 4 ), the following inequalities hold true:

$$
\begin{align*}
& \frac{F_{I}^{\prime}\left(S_{0}, 0\right)}{F_{I}^{\prime}(S, 0)}>1 \quad \text { for } S \in\left(0, S_{0}\right) \\
& \frac{F_{I}^{\prime}\left(S_{0}, 0\right)}{F_{I}^{\prime}(S, 0)}<1 \quad \text { for } S>S_{0} \tag{10}
\end{align*}
$$

Under these conditions, we have the following theorems.
Theorem 2. Suppose that conditions (H1)-(H3) are satisfied. Then the disease-free equilibrium $E_{0}=\left(S_{0}, 0\right)$ of system (2) is locally asymptotically stable for any $\tau>0$ if $R_{0}<1 ; E_{0}=$ $\left(S_{0}, 0\right)$ is unstable if $R_{0}>1$.

Proof. The characteristic equation of system (2) at $E_{0}=$ $\left(S_{0}, 0\right)$ is

$$
\begin{equation*}
(\lambda+u)\left(\lambda-e^{-\tau(\lambda+\mu)} F_{I}^{\prime}\left(S_{0}, 0\right)+\mu+\alpha+G^{\prime}(0)\right)=0 \tag{11}
\end{equation*}
$$

It has a negative real root $\lambda_{1}=-\mu$. Moreover, it has a root of

$$
\begin{equation*}
\lambda-e^{-\tau(\lambda+\mu)} F_{I}^{\prime}\left(S_{0}, 0\right)+\mu+\alpha+G^{\prime}(0)=0 \tag{12}
\end{equation*}
$$

In (12), if $\tau=0, R_{0}<1$ becomes $R_{01}=F_{I}^{\prime}\left(S_{0}, 0\right) /[\mu+\alpha+$ $\left.G^{\prime}(0)\right]<1$; one can see that $\lambda_{2}=\left[\mu+\alpha+G^{\prime}(0)\right]\left(R_{01}-1\right)<0$. Hence the $E_{0}=\left(S_{0}, 0\right)$ is locally asymptotically stable. In (12), if $\tau>0, x>0, y \neq 0, \lambda=x+i \cdot y$ is a root of (12), then we have

$$
\begin{gather*}
x+\mu+\alpha+G^{\prime}(0)=e^{-\tau(x+\mu)} F_{I}^{\prime}\left(S_{0}, 0\right) \cos (y \tau) \\
y=-e^{-\tau(x+\mu)} F_{I}^{\prime}\left(S_{0}, 0\right) \sin (y \tau) \tag{13}
\end{gather*}
$$

Further we have

$$
\begin{align*}
{\left[x+\mu+\alpha+G^{\prime}(0)\right]^{2}+y^{2} } & =\left[e^{-\tau(x+\mu)} F_{I}^{\prime}\left(S_{0}, 0\right)\right]^{2} \\
& \leq\left[e^{-\mu \tau} F_{I}^{\prime}\left(S_{0}, 0\right)\right]^{2} \tag{14}
\end{align*}
$$

which is a contradiction. Hence the $E_{0}=\left(S_{0}, 0\right)$ is locally asymptotically stable.

If $R_{0}>1$, let $h(\lambda)=\lambda-e^{-\tau(\lambda+\mu)} F_{I}^{\prime}\left(S_{0}, 0\right)+\mu+\alpha+G^{\prime}(0)$; then we have

$$
\begin{gather*}
h(0)=\left[\mu+\alpha+G^{\prime}(0)\right]\left(1-R_{0}\right)<0, \\
\lim _{\lambda \rightarrow+\infty} h(\lambda)=+\infty . \tag{15}
\end{gather*}
$$

And when $\lambda \geq 0$, since $h(\lambda)$ is continuously, then $h(\lambda)=0$ has at least one positive root. Hence $E_{0}=\left(S_{0}, 0\right)$ is unstable. This completes the proof.

Theorem 3. Suppose that conditions (H1)-(H6) are satisfied. Then the disease-free equilibrium $E_{0}=\left(S_{0}, 0\right)$ of system (2) is globally asymptotically stable for any $\tau>0$ if $R_{0} \leq 1$.

Proof. Define a Lyapunov functional

$$
\begin{equation*}
V_{1}(t)=U_{1}(t)+U_{2}(t), \tag{16}
\end{equation*}
$$

where

$$
\begin{gather*}
U_{1}(t)=S(t)-S_{0}-\int_{S_{0}}^{S(t)} \lim _{I \rightarrow 0^{+}} \frac{F\left(S_{0}, I(t)\right)}{F(\theta, I(t))} d \theta+e^{\mu \tau} I(t) \\
U_{2}(t)=\int_{0}^{\tau} F(S(t-\eta), I(t-\eta)) d \eta \tag{17}
\end{gather*}
$$

By ( H 1$)-(\mathrm{H} 6)$, it is obvious that $V_{1}$ is defined and continuously differentiable for all $S(t), I(t)>0$, and $V_{1}=0$ at $E_{0}=$ $\left(S_{0}, 0\right)$. The system (2) at $E_{0}=\left(S_{0}, 0\right)$ has $\lambda=\mu S_{0}$. The time derivative of $U_{1}$ along the solutions of system (2) is given by

$$
\begin{align*}
& \frac{d U_{1}(t)}{d t} \\
&= S^{\prime}(t)-\lim _{I \rightarrow 0^{+}} \frac{F\left(S_{0}, I(t)\right)}{F(S(t), I(t))} \cdot S^{\prime}(t)+e^{\mu \tau} I^{\prime}(t) \\
&=\left(1-\lim _{I \rightarrow 0^{+}} \frac{F\left(S_{0}, I(t)\right)}{F(S(t), I(t))}\right)[\lambda-\mu S(t)-F(S(t), I(t))] \\
&+F(S(t-\tau), I(t-\tau))-e^{\mu \tau}[(\mu+\alpha) I(t)+G(I(t))] \\
&= \mu S(t)\left(\frac{S_{0}}{S(t)}-1\right)\left(1-\lim _{I \rightarrow 0^{+}} \frac{F\left(S_{0}, I(t)\right)}{F(S(t), I(t))}\right) \\
&-F(S(t), I(t))+F(S(t), I(t)) \cdot \lim _{I \rightarrow 0^{+}} \frac{F\left(S_{0}, I(t)\right)}{F(S(t), I(t))} \\
&+F(S(t-\tau), I(t-\tau))-e^{\mu \tau}[(\mu+\alpha) I(t)+G(I(t))] . \tag{18}
\end{align*}
$$

Further, we have

$$
\begin{align*}
\frac{d U_{2}(t)}{d t} & =\frac{d}{d t} \int_{0}^{\tau} F(S(t-\eta), I(t-\eta)) d \eta  \tag{19}\\
& =F(S(t), I(t))-F(S(t-\tau), I(t-\tau))
\end{align*}
$$

Thus

$$
\begin{align*}
\frac{d V_{1}(t)}{d t}= & \frac{d U_{1}(t)}{d t}+\frac{d U_{2}(t)}{d t} \\
= & \mu S(t)\left(\frac{S_{0}}{S(t)}-1\right)\left(1-\lim _{I \rightarrow 0^{+}} \frac{F\left(S_{0}, I(t)\right)}{F(S(t), I(t))}\right) \\
& +F(S(t), I(t)) \cdot \lim _{I \rightarrow 0^{+}} \frac{F\left(S_{0}, I(t)\right)}{F(S(t), I(t))} \\
& -e^{\mu \tau}[(\mu+\alpha) I(t)+G(I(t))] \tag{20}
\end{align*}
$$

By (10), we have

$$
\begin{align*}
& \left(\frac{S_{0}}{S(t)}-1\right)\left(1-\lim _{I \rightarrow 0^{+}} \frac{F\left(S_{0}, I(t)\right)}{F(S(t), I(t))}\right)  \tag{21}\\
& \quad=\left(\frac{S_{0}}{S(t)}-1\right)\left(1-\frac{F_{I}^{\prime}\left(S_{0}, 0\right)}{F_{I}^{\prime}(S(t), 0)}\right) \leq 0 .
\end{align*}
$$

Note that $F_{I}^{\prime}\left(S_{0}, 0\right) / F_{I}^{\prime}(S, 0) \neq 1$, for $S \neq S_{0}, S>0$, and by $F_{I}^{\prime}(S, 0)>0$ and (H4), the equality in (21) holds if and only if $S=S_{0}$. Furthermore, (H5) and (H6) imply that

$$
\begin{align*}
& F(S(t), I(t)) \cdot \lim _{I \rightarrow 0^{+}} \frac{F\left(S_{0}, I(t)\right)}{F(S(t), I(t))} \\
&-e^{\mu \tau}[(\mu+\alpha) I(t)+G(I(t))] \\
&= F(S(t), I(t)) \cdot \frac{F_{I}^{\prime}\left(S_{0}, 0\right)}{F_{I}^{\prime}(S(t), 0)}  \tag{22}\\
&-e^{\mu \tau}[(\mu+\alpha) I(t)+G(I(t))] \\
& \leq I(t) \cdot\left[F_{I}^{\prime}\left(S_{0}, 0\right)-e^{\mu \tau}\left(\mu+\alpha+G^{\prime}(0)\right)\right] \\
&= e^{\mu \tau}\left(\mu+\alpha+G^{\prime}(0)\right) I(t)\left(R_{0}-1\right)
\end{align*}
$$

Therefore, $R_{0} \leq 1$ ensures that $d V_{1} / d t \leq 0$ for all $S(t) \geq 0$, $I(t) \geq 0$, where $d V_{1} / d t=0$ holds only for $S=S_{0}$. It is easy to verify that the disease-free equilibrium $E_{0}$ is the only fixed point of the systems on the plane $S=S_{0}$ and hence it is easy to show that the largest invariant set in $\left\{(S(t), I(t)) \mid d V_{1} / d t=\right.$ $0\}$ is the singleton $\left\{E_{0}\right\}$. By the Lyapunov-LaSalle asymptotic stability theorem in [31], $E_{0}$ is globally asymptotically stable for any $\tau>0$. This completes the proof.
3.2. Global Stability of the Endemic Equilibrium. In this subsection, we will study the global stability of the endemic equilibrium $E^{*}=\left(S^{*}, I^{*}\right)$ of system (2) by the Lyapunov direct method. We propose the following hypotheses:
(H7) $I / I^{*} \leq F(S, I) / F\left(S, I^{*}\right)$ for $I \in\left(0, I^{*}\right), F(S, I) /$ $F\left(S, I^{*}\right) \leq I / I^{*}$ for $I \geq I^{*}$.
(H8) $G(I) / G\left(I^{*}\right) \leq I / I^{*}$ for $I \in\left(0, I^{*}\right), I / I^{*} \leq G(I) / G\left(I^{*}\right)$ for $I \geq I^{*}$.

Based on these, we have the following theorem.
Theorem 4. Suppose that conditions (H1)-(H3) and (H7)(H8) are satisfied. Then the endemic equilibrium $E^{*}=\left(S^{*}, I^{*}\right)$ of system (2) is globally asymptotically stable for any $\tau>0$ if $R_{0}>1$.

Proof. Define a Lyapunov functional

$$
\begin{equation*}
V_{2}(t)=W_{1}(t)+W_{2}(t) \tag{23}
\end{equation*}
$$

where

$$
\begin{aligned}
W_{1}(t)= & S(t)-S^{*}-\int_{S^{*}}^{S(t)} \frac{F\left(S^{*}, I^{*}\right)}{F\left(\varphi, I^{*}\right)} d \varphi \\
& +e^{\mu \tau}\left(I(t)-I^{*}-I^{*} \ln \frac{I(t)}{I^{*}}\right) .
\end{aligned}
$$

$$
\begin{align*}
W_{2}(t)=F\left(S^{*}, I^{*}\right) \int_{0}^{\tau}( & \frac{F(S(t-\xi), I(t-\xi))}{F\left(S^{*}, I^{*}\right)}-1 \\
& \left.-\ln \frac{F(S(t-\xi), I(t-\xi))}{F\left(S^{*}, I^{*}\right)}\right) d \xi \tag{24}
\end{align*}
$$

By (H1)-(H6), $V_{2}(t)=W_{1}(t)+W_{2}(t)$ is defined and continuously differentiable for all $S(t), I(t)>0$. And $V_{2}(0)=$ 0 at $E^{*}=\left(S^{*}, I^{*}\right)$. At $E^{*}=\left(S^{*}, I^{*}\right)$, system (2) has

$$
\begin{gather*}
\lambda-\mu S^{*}=F\left(S^{*}, I^{*}\right) \\
F\left(S^{*}, I^{*}\right)=e^{\mu \tau}\left[(\mu+\alpha) I^{*}+G\left(I^{*}\right)\right] . \tag{25}
\end{gather*}
$$

The time derivative of $W_{1}$ along the solutions of system (2) is given by

$$
\begin{align*}
& \frac{d W_{1}(t)}{d t} \\
&= S^{\prime}(t)-\frac{F\left(S^{*}, I^{*}\right)}{F\left(S(t), I^{*}\right)} S^{\prime}(t)+e^{\mu \tau} I^{\prime}(t)\left(1-\frac{I^{*}}{I(t)}\right) \\
&=\left(1-\frac{F\left(S^{*}, I^{*}\right)}{F\left(S(t), I^{*}\right)}\right)\left[\mu S^{*}-\mu S(t)+F\left(S^{*}, I^{*}\right)\right. \\
&-F(S(t), I(t))] \\
&+\left(1-\frac{I^{*}}{I(t)}\right)\left[F(S(t-\tau), I(t-\tau))-F\left(S^{*}, I^{*}\right) \frac{I(t)}{I^{*}}\right. \\
&= \mu S^{*}\left(1-\frac{F\left(S^{*}, I^{*}\right)}{F\left(S(t), I^{*}\right)}\right)\left(1-\frac{S(t)}{S^{*}}\right) \\
&\left.+e^{\mu \tau} G\left(I^{*}\right) \frac{I(t)}{I^{*}}-e^{\mu \tau} G(I(t))\right] \\
&+F\left(S^{*}, I^{*}\right)\left(1-\frac{F\left(S^{*}, I^{*}\right)}{F\left(S(t), I^{*}\right)}+\frac{F(S(t), I(t))}{F\left(S(t), I^{*}\right)}\right) \\
&+e^{\mu \tau} G\left(I^{*}\right)\left[\frac{I(t)}{I^{*}}-\frac{I^{*}}{I(t)} \cdot \frac{F(S(t-\tau), I(t-\tau))}{F\left(S^{*}, I^{*}\right)}\right) \\
&-F(S(t), I(t))+F(S(t-\tau), I(t-\tau)) .
\end{align*}
$$

Further, we have

$$
\begin{aligned}
& \frac{d W_{2}(t)}{d t} \\
& \quad=F\left(S^{*}, I^{*}\right) \cdot \frac{d}{d t} \int_{0}^{\tau}\left(\frac{F(S(t-\xi), I(t-\xi))}{F\left(S^{*}, I^{*}\right)}-1\right. \\
& \\
& \left.\quad-\ln \frac{F(S(t-\xi), I(t-\xi))}{F\left(S^{*}, I^{*}\right)}\right) d \xi
\end{aligned}
$$

$$
\begin{align*}
= & F(S(t), I(t))-F(S(t-\tau), I(t-\tau)) \\
& +F\left(S^{*}, I^{*}\right) \ln \frac{F(S(t-\tau), I(t-\tau))}{F(S(t), I(t))} . \tag{27}
\end{align*}
$$

Then we have

$$
\begin{align*}
\frac{d V_{2}(t)}{d t}= & \frac{d W_{1}(t)}{d t}+\frac{d W_{2}(t)}{d t} \\
= & \mu S^{*}\left(1-\frac{F\left(S^{*}, I^{*}\right)}{F\left(S(t), I^{*}\right)}\right)\left(1-\frac{S(t)}{S^{*}}\right) \\
& +F\left(S^{*}, I^{*}\right)\left(1-\frac{F\left(S^{*}, I^{*}\right)}{F\left(S(t), I^{*}\right)}+\frac{F(S(t), I(t))}{F\left(S(t), I^{*}\right)}\right) \\
& +F\left(S^{*}, I^{*}\right)\left(1-\frac{I(t)}{I^{*}}-\frac{I^{*}}{I(t)}\right. \\
& \cdot \frac{F(S(t-\tau), I(t-\tau))}{F\left(S^{*}, I^{*}\right)} \\
& +e^{\mu \tau} G\left(I^{*}\right)\left[\frac{I(t)}{I^{*}}-1-\frac{F(S(t-\tau), I(t-\tau))}{F(S(t), I(t))}\right) \\
& \left.+\frac{G(I(t))}{G\left(I^{*}\right)} \cdot \frac{I^{*}}{I(t)}\right] \\
& +e^{\mu \tau} G\left(I^{*}\right)\left(\frac{G(I(t))}{G\left(I^{*}\right)}-\frac{I(t)}{I^{*}}\right)\left(\frac{I^{*}}{I(t)}-1\right) \\
= & \mu S^{*}\left(1-\frac{F\left(S^{*}, I^{*}\right)}{F\left(S(t), I^{*}\right)}\right)\left(1-\frac{S(t)}{S^{*}}\right) \\
& +F\left(S^{*}, I^{*}\right)\left(\frac{I(t)}{I^{*}}-\frac{F(S(t), I(t))}{F\left(S(t), I^{*}\right)}\right) \\
& +F\left(S^{*}, I^{*}\right)\left(1-\frac{F\left(S^{*}, I^{*}\right)}{F\left(S(t), I^{*}\right)}+\ln \frac{F\left(S^{*}, I^{*}\right)}{F\left(S(t), I^{*}\right)}\right) \\
+ & \left.\left.+\frac{I(t))}{I^{*}}\right) \cdot \frac{F\left(S(t), I^{*}\right)}{F(S(t), I(t))}\right) \\
+ & F\left(I^{*}, I^{*}\right)\left(1-\frac{I^{*}}{I(t)} \cdot \frac{F(S(t-\tau), I(t-\tau))}{F\left(S^{*}, I^{*}\right)}\right. \\
&
\end{align*}
$$

The function $F(S, I)$ is monotonically increasing for any $S>$ 0 ; hence the following inequality holds:

$$
\begin{equation*}
\left(1-\frac{F\left(S^{*}, I^{*}\right)}{F\left(S(t), I^{*}\right)}\right)\left(1-\frac{S(t)}{S^{*}}\right) \leq 0 . \tag{29}
\end{equation*}
$$

And by the properties of the function $g(x)=1-x+\ln x$, $(x>0)$, we note that $g(x)$ has its global maximum $g(1)=0$. Hence $g(x) \leq 0$ when $x>0$ and the following inequalities hold true:

$$
\begin{gather*}
1-\frac{F\left(S^{*}, I^{*}\right)}{F\left(S(t), I^{*}\right)}+\ln \frac{F\left(S^{*}, I^{*}\right)}{F\left(S(t), I^{*}\right)} \leq 0, \\
1-\frac{I^{*}}{I(t)} \cdot \frac{F(S(t-\tau), I(t-\tau))}{F\left(S^{*}, I^{*}\right)} \\
+\ln \frac{I^{*}}{I(t)} \cdot \frac{F(S(t-\tau), I(t-\tau))}{F\left(S^{*}, I^{*}\right)} \leq 0, \\
1-\frac{I(t)}{I^{*}} \cdot \frac{F\left(S(t), I^{*}\right)}{F(S(t), I(t))}+\ln \frac{I(t)}{I^{*}} \cdot \frac{F\left(S(t), I^{*}\right)}{F(S(t), I(t))} \leq 0 . \tag{30}
\end{gather*}
$$

Furthermore, by (H7) the following inequality holds:

$$
\begin{equation*}
\left(\frac{I(t)}{I^{*}}-\frac{F(S(t), I(t))}{F\left(S(t), I^{*}\right)}\right)\left(\frac{F\left(S(t), I^{*}\right)}{F(S(t), I(t))}-1\right) \leq 0 . \tag{31}
\end{equation*}
$$

And by (H8) we have the following inequality:

$$
\begin{equation*}
\left(\frac{G(I(t))}{G\left(I^{*}\right)}-\frac{I(t)}{I^{*}}\right)\left(\frac{I^{*}}{I(t)}-1\right) \leq 0 . \tag{32}
\end{equation*}
$$

By (29)-(32), we see that $d V_{2} / d t \leq 0$ for all $S(t) \geq 0$, $I(t) \geq 0$. It is easy to verify that the largest invariant set in $\left\{(S(t), I(t)) \mid d V_{2} / d t=0\right\}$ is the singleton $\left\{E^{*}\right\}$. By the Lyapunov-LaSalle asymptotic stability theorem in [31], $E^{*}$ is globally asymptotically stable for any $\tau>0$. This completes the proof.

## 4. Discussion and Conclusion

In this paper, we formulated an SIR epidemic model with delay and two general functions, one is $F(S, I)$ which represents the incidence rate, and the other is $G(I)$ which represents the recovered infected individuals from the infected compartment. We studied the global asymptotic stability of disease-free equilibrium and endemic equilibrium of system (2), respectively. We showed that in Theorem 2 the diseasefree equilibrium $E_{0}=\left(S_{0}, 0\right)$ is locally asymptotically for any $\tau>0$ if the basic reproduction number $R_{0}<1$ and $E_{0}=\left(S_{0}, 0\right)$ are unstable if $R_{0}>1$; in Theorem 3 the diseasefree equilibrium $E_{0}=\left(S_{0}, 0\right)$ is globally asymptotically for any $\tau>0$ if $R_{0} \leq 1$, while in Theorem 4, the endemic equilibrium $E^{*}=\left(S^{*}, I^{*}\right)$ is globally asymptotically for any $\tau>0$ if $R_{0}>1$.

In order to obtain the global properties of the system (2), we proposed assumptions (H1)-(H8) for functions of $F(S, I)$ and $G(I)$. Conditions (H1)-(H3) are some basic assumptions; for example, (H1) implies that the function $F(S, I)$ is a nonnegative differentiable function on nonnegative quadrant and
is positive if and only if both arguments are positive. We used (H4)-(H6), (H7)-(H8) to establish the global asymptotic stability of disease-free equilibrium and endemic equilibrium of system (2), respectively. These hypotheses seem to be mathematical techniques; however, they may be obviously true for many concrete forms of the functions of $F(S, I)$ and $G(I)$ in previous studies.

A special case of system (2) is that when $G(I)=\gamma I, \gamma$ is the recovery rate of the infective individuals. System (2) becomes the following DDEs:

$$
\begin{gather*}
\frac{d S(t)}{d t}=\lambda-\mu S(t)-F(S(t), I(t))  \tag{33}\\
\frac{d I(t)}{d t}=e^{-\mu \tau} F(S(t-\tau), I(t-\tau))-(\mu+\alpha+\gamma) I(t) .
\end{gather*}
$$

The basic reproductive number for system (33) is presented as

$$
\begin{equation*}
R_{0}^{\prime}=\frac{e^{-\mu \tau} F_{I}^{\prime}\left(S_{0}, 0\right)}{\mu+\alpha+\gamma} . \tag{34}
\end{equation*}
$$

Using Theorems 3 and 4, we can easily obtain the global asymptotic stability of the disease-free equilibrium and the endemic equilibrium of system (33). Regarding to system (33), we now give examples of incidence function $F(S, I)$ that satisfies the required hypotheses obviously.

Example 5. Without delay: let $\tau=0, \lambda=\mu$ and $\delta=\mu+\alpha+\gamma$. Then system (33) becomes to the SIR model studied in [27].

Example 6. Holling type II incidence rate: let $F(S, I)=$ $v_{m} S I /\left(C_{h}+S\right)$ for some constant $v_{m}, C_{h}>0$. Then the hypotheses on $F(S, I)$ are satisfied and the global properties are determined by the basic reproductive number. This model was introduced by $[32,33]$ for considering delays in the standard bacterial growth model in a chemostat. Its global dynamics were first proved by [32] by the fluctuation lemma and a different proof was given in [34, Theorem 5.16] by comparison.

Example 7. Saturate incidence rate: let $F(S, I)=\beta S I /\left(1+\alpha_{1} S+\right.$ $\left.\alpha_{2} I\right)$ for some constant $\alpha_{1}>0, \alpha_{2}>0$. Then hypotheses about $F(S, I)$ are also satisfied and the global properties are determined by the basic reproductive number. The behaviors of this model were previously studied in [23, 24]. In [23], the local stability of disease-free equilibrium and endemic equilibrium was obtained. And in [24], the global stability of disease-free equilibrium and endemic equilibrium was studied.

From these Examples 5-7, we can see that system (2) is reasonably established and it can contain many classical epidemic models and imply their global dynamics as special cases.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work was supported by the National Nature Science Foundation of China (11271303 and 10971168) and the Fundamental Research Funds for the Center Universities (XDJK2009B012).

## References

[1] M. Gabriela, M. Gomes, L. J. White, and G. F. Medley, "The reinfection threshold," Journal of Theoretical Biology, vol. 236, no. 1, pp. 111-113, 2005.
[2] Y. Zhou and H. Liu, "Stability of periodic solutions for an SIS model with pulse vaccination," Mathematical and Computer Modelling, vol. 38, no. 3-4, pp. 299-308, 2003.
[3] A. Gray, D. Greenhalgh, L. Hu, X. Mao, and J. Pan, "A stochastic differential equation SIS epidemic model," SIAM Journal on Applied Mathematics, vol. 71, no. 3, pp. 876-902, 2011.
[4] M. Song, W. Ma, and Y. Takeuchi, "Permanence of a delayed SIR epidemic model with density dependent birth rate," Journal of Computational and Applied Mathematics, vol. 201, no. 2, pp. 389-394, 2007.
[5] W. Ma, M. Song, and Y. Takeuchi, "Global stability of an SIR epidemic model with time delay," Applied Mathematics Letters, vol. 17, no. 10, pp. 1141-1145, 2004.
[6] F. Zhang, Z. Li, and F. Zhang, "Global stability of an SIR epidemic model with constant infectious period," Applied Mathematics and Computation, vol. 199, no. 1, pp. 285-291, 2008.
[7] X. Liu, Y. Takeuchi, and S. Iwami, "SVIR epidemic models with vaccination strategies," Journal of Theoretical Biology, vol. 253, no. 1, pp. 1-11, 2008.
[8] J. Wang, J. Zhang, and Z. Jin, "Analysis of an SIR model with bilinear incidence rate," Nonlinear Analysis, vol. 11, no. 4, pp. 2390-2402, 2010.
[9] A. Korobeinikov and G. C. Wake, "Lyapunov functions and global stability for SIR, SIRS, and SIS epidemiological models," Applied Mathematics Letters, vol. 15, no. 8, pp. 955-960, 2002.
[10] S. M. O’Regan, T. C. Kelly, A. Korobeinikov, M. J. A. O'Callaghan, and A. V. Pokrovskii, "Lyapunov functions for SIR and SIRS epidemic models," Applied Mathematics Letters, vol. 23, no. 4, pp. 446-448, 2010.
[11] L. Liu, X. Q. Zhao, and Y. Zhou, "A tuberculosis model with seasonality," Bulletin of Mathematical Biology, vol. 72, no. 4, pp. 931-952, 2010.
[12] N. Yoshida and T. Hara, "Global stability of a delayed SIR epidemic model with density dependent birth and death rates," Journal of Computational and Applied Mathematics, vol. 201, no. 2, pp. 339-347, 2007.
[13] R. M. Anderson and R. M. May, "Regulation and stability of host-parasite population interactions. I. Regulatory processes," Journal of Animal Ecology, vol. 47, no. 1, pp. 219-267, 1978.
[14] C. Wei and L. Chen, "A delayed epidemic model with pulse vaccination," Discrete Dynamics in Nature and Society, vol. 2008, Article ID 746951, 12 pages, 2008.
[15] J. Zhang, Z. Jin, Q. Liu, and Z. Zhang, "Analysis of a delayed SIR model with nonlinear incidence rate," Discrete Dynamics in Nature and Society, vol. 2008, Article ID 636153, 16 pages, 2008.
[16] Z. Jiang and J. Wei, "Stability and bifurcation analysis in a delayed SIR model," Chaos, Solitons and Fractals, vol. 35, no. 3, pp. 609-619, 2008.
[17] R. Xu and Z. Ma, "Stability of a delayed SIRS epidemic model with a nonlinear incidence rate," Chaos, Solitons and Fractals, vol. 41, no. 5, pp. 2319-2325, 2009.
[18] R. Xu, Z. Ma, and Z. Wang, "Global stability of a delayed SIRS epidemic model with saturation incidence and temporary immunity," Computers \& Mathematics with Applications, vol. 59, no. 9, pp. 3211-3221, 2010.
[19] R. Xu and Z. Ma, "Global stability of a SIR epidemic model with nonlinear incidence rate and time delay," Nonlinear Analysis, vol. 10, no. 5, pp. 3175-3189, 2009.
[20] X. Zhang and X. Liu, "Backward bifurcation of an epidemic model with saturated treatment function," Journal of Mathematical Analysis and Applications, vol. 348, no. 1, pp. 433-443, 2008.
[21] C. C. McCluskey, "Global stability for an SIR epidemic model with delay and nonlinear incidence," Nonlinear Analysis, vol. 11, no. 4, pp. 3106-3109, 2010.
[22] A. Kaddar, "On the dynamics of a delayed SIR epidemic model with a modified saturated incidence rate," Electronic Journal of Differential Equations, vol. 2009, no. 133, pp. 1-7, 2009.
[23] A. Kaddar, A. Abta, and H. T. Alaoui, "A comparison of delayed SIR and SEIR epidemic models," Nonlinear Analysis, vol. 16, no. 2, pp. 181-190, 2011.
[24] A. Abta, A. Kaddar, and H. T. Alaoui, "Global stability for delay SIR and SEIR epidemic models with saturated incidence rates," Electronic Journal of Differential Equations, vol. 2012, no. 23, pp. 1-13, 2012.
[25] Z. Liu, "Dynamics of positive solutions to SIR and SEIR epidemic models with saturated incidence rates," Nonlinear Analysis, vol. 14, no. 3, pp. 1286-1299, 2013.
[26] A. Korobeinikov and K. Philip Maini, "Nonliear incidence and stability of infectious diseasemodels," Mathematical Medicine and Biology, vol. 22, pp. 113-128, 2005.
[27] A. Korobeinikov, "Lyapunov functions and global stability for SIR and SIRS epidemiological models with non-linear transmission," Bulletin of Mathematical Biology, vol. 68, no. 3, pp. 615-626, 2006.
[28] A. Korobeinikov, "Global properties of infectious disease models with nonlinear incidence," Bulletin of Mathematical Biology, vol. 69, no. 6, pp. 1871-1886, 2007.
[29] A. Korobeinikov, "Stability of ecosystem: global properties of a general predator-prey model," Mathematical Medicine and Biology, vol. 26, no. 4, pp. 309-321, 2009.
[30] G. Huang, Y. Takeuchi, W. Ma, and D. Wei, "Global stability for delay SIR and SEIR epidemic models with nonlinear incidence rate," Bulletin of Mathematical Biology, vol. 72, no. 5, pp. 11921207, 2010.
[31] J. K. Hale and S. M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer, New York, NY, USA, 1993.
[32] S. F. Ellermeyer, "Competition in the chemostat: global asymptotic behavior of a model with delayed response in growth," SIAM Journal on Applied Mathematics, vol. 54, no. 2, pp. 456465, 1994.
[33] S. F. Ellermeyer, J. Hendrix, and N. Ghoochan, "A theoretical and empirical investigation of delayed growth response in the continuous culture of bacteria," Journal of Theoretical Biology, vol. 222, no. 4, pp. 485-494, 2003.
[34] H. Smith, An Introduction to Delay Differential Equations with Applications to the Life Sciences, Springer, New York, NY, USA, 2011.

## Research Article

# Dynamics of a Viral Infection Model with General Contact Rate between Susceptible Cells and Virus Particles 

Chenxi Dai, ${ }^{1,2}$ Cui Ma, ${ }^{1}$ Lijuan Song, ${ }^{1}$ and Kaifa Wang ${ }^{1}$<br>${ }^{1}$ Department of Mathematics, School of Biomedical Engineering, Third Military Medical University, Chongqing 400038, China<br>${ }^{2}$ Nineteen Student Battalion, School of Biomedical Engineering, Third Military Medical University, Chongqing 400038, China

Correspondence should be addressed to Kaifa Wang; kfwang001@gmail.com
Received 23 December 2013; Accepted 9 January 2014; Published 13 February 2014
Academic Editor: Weiming Wang
Copyright © 2014 Chenxi Dai et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

This paper investigates the dynamic behavior of a viral infection model with general contact rate between susceptible host cells and free virus particles. If the basic reproduction number of the virus is less than unity, by LaSalle's invariance principle, the diseasefree equilibrium is globally asymptotically stable. If the basic reproduction number of the virus is greater than unity, then the virus persists in the host and the endemic equilibrium is locally asymptotically stable.


## 1. Introduction

Viral infection within-host, such as hepatitis B virus (HBV), hepatitis C virus (HCV), and human immunodeficiency virus (HIV) infections, is a complicated kinetic process, and mathematical model is always important, which can give a hand to understand the complexity between the responses of the body and variant conditions [1-6].

The basic viral infection model contains three variables, susceptible host cells $(x)$, infected host cells $(y)$, and free virus particles $(v)$, which can be formulated by the following differential equations [7, 8]:

$$
\begin{gather*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=r-m x-\beta x v \\
\frac{\mathrm{~d} y}{\mathrm{~d} t}=\beta x v-a y  \tag{1}\\
\frac{\mathrm{~d} v}{\mathrm{~d} t}=k y-u v
\end{gather*}
$$

in which susceptible host cells are produced at a constant rate, $r$, die at the rate of $m x$, and become infected with the rate of $\beta x v$. Infected host cells are produced at the rate of $\beta x v$ and die at the rate of $a y$. Free virus particles are released from infected host cells at the rate of $k y$ and die at the rate of $u v$. It is assumed that parameters $r, m, \beta, a, k$, and $u$ are all positive constants.

Note that there is an assumption that the infection term is based on the mass-action principle, which means that there is a constant contact rate $(\beta)$ between susceptible host cells and virus particles in (1). However, many experiments of microparasitic infections suggest the infection rate may be a nonlinear relationship [3, 9-11], such as dose-dependent infection rate. Thus, to meet more biological practice, we replace the constant contact rate $(\beta)$ with a general contact rate $(f(v))$ between susceptible cells and virus particles and obtain the following modified viral infection model:

$$
\begin{gather*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=r-m x-f(v) x v \\
\frac{\mathrm{~d} y}{\mathrm{~d} t}=f(v) x v-a y  \tag{2}\\
\frac{\mathrm{~d} v}{\mathrm{~d} t}=k y-u v
\end{gather*}
$$

where the contact rate function $f(v)$ satisfy the following assumption (H1):
(H1) $f(v): \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$, continuous and differentiable, $f(0)=\beta, f^{\prime}(v)<0$ and $f(\infty)=0$.
The primary goal of this paper is to carry out a mathematical analysis of system (2) and predict whether the infection
disappears or survives. The organization of this paper is as follows. In the next section, some preliminary results are given, including the dissipativity of system (2), the definition of basic reproduction number of the virus, and the existence of the disease-free equilibrium and endemic equilibrium. In Section 3, by analyzing the corresponding characteristic equations, we study the local stability of the equilibria. In Section 4, by using suitable Lyapunov function and LaSalle's invariance principle [12], we first prove that if the basic reproduction number is less than unity, the disease-free equilibrium is globally asymptotically stable. Then using Theorem 4.6 in [13], we obtain the uniform persistence of (2) if the basic reproduction number is greater than unity. A brief discussion is given in Section 5 to conclude this work.

## 2. Preliminary Results

In this section, we first show that all solutions of system (2) are positive and ultimately bounded. Then the existence of feasible equilibria is given under the condition of basic reproduction number of the virus.

Because of the biological meaning of the components $(x(t), y(t), v(t))$, we focus on the model in the first octant of $\mathbb{R}^{3}$ and consider system (2) with initial conditions

$$
\begin{equation*}
x(0)>0, \quad y(0)>0, \quad v(0)>0 . \tag{3}
\end{equation*}
$$

The following result shows that system (2) is dissipative.
Theorem 1. Under the initial conditions (3), all solutions of system (2) are positive for $t>0$ and there exists a constant $M>0$, such that all solutions satisfy $x(t)<M, y(t)<M$, and $v(t)<M$ for all sufficiently large $t$.

Proof. Note that $\left.x^{\prime}\right|_{x=0}=r>0,\left.y^{\prime}\right|_{y=0}=f(v) x v$ and $\left.V^{\prime}\right|_{\nu=0}=k y$. This implies that $(x(t), y(t), v(t)) \in \mathbb{R}_{+}^{3}$ for all $t>0$, provided that $(x(0), y(0), v(0)) \in \mathbb{R}_{+}^{3}$. Suppose that $x(t)$ is not always positive. Let $\tau>0$ be the first time such that $x(\tau)=0$. By the first equation of (2) we have $x^{\prime}(\tau)=r>0$, which implies $x(t)<0$ for $t \in(\tau-\varepsilon, \tau)$ for sufficiently small $\varepsilon>0$, a contradiction. Thus, $x(t)$ is positive for all $t>0$. In addition, by the second and third equations of (2), we have

$$
\begin{align*}
y(t) & =\left(y(0)+\int_{0}^{t} f(v(s)) x(s) v(s) e^{a s} d s\right) e^{-a t} \\
& \geq y(0) e^{-a t}>0  \tag{4}\\
v(t) & =\left(v(0)+\int_{0}^{t} k y(s) e^{u s} d s\right) e^{-u t} \geq v(0) e^{-u t}>0
\end{align*}
$$

for all $t>0$. Therefore, it is easy to see that $y(t)$ and $v(t)$ are positive with initial conditions (3).

Next, we sketch the arguments for ultimate boundedness of solution of (2). Let $N_{1}(t)=x(t)+y(t), N_{2}(t)=x(t)+y(t)+$ $(a v(t) / k), d_{1}=\min \{m, a\}$, and $d_{2}=\min \{m, a, u\}$. Since all solutions of (2) are positive, we have

$$
\begin{gather*}
N_{1}^{\prime}=r-m x-a y<r-d_{1} N_{1} \\
N_{2}^{\prime}=r-m x-\frac{a u}{k} v<2 r-d_{2} N_{2} \tag{5}
\end{gather*}
$$

Therefore, $N_{1}(t)<2 r / d_{1}$ and $N_{2}(t)<3 r / d_{2}$ for all sufficiently large $t$, and hence, $x(t), y(t)$, and $v(t)$ are ultimately bounded by some positive constant $M$.

Note that a free virus particle has an average lifetime of $1 / u$ and parameter $k$ is the burst size, which means the total number of virions produced by an infected cell during its life span. Thus, at the beginning of the infectious process, the average number of newly virus particles generated from one virus particle, which is the basic reproduction number of virus by $[14,15]$, can be defined as

$$
\begin{equation*}
R_{0}=\frac{r k f(0)}{a u m}=\frac{r k \beta}{a u m} . \tag{6}
\end{equation*}
$$

Now, we begin to find the equilibria of model (2) by the following algebraic system

$$
\begin{gather*}
r-m x-f(v) x v=0 \\
f(v) x v-a y=0  \tag{7}\\
k y-u v=0
\end{gather*}
$$

Solving the third algebraic equation of (7), we can obtain $y=u v / k$. By combining this equality with the second equation of (7), we have $x=a u / k f(v)$ or $v=0$. When $v=0$, it is easy to have $y=0$ and $x=r / m$ by the third and first equations of (7); that is, system (2) always has a diseasefree equilibrium state, denoted as $E_{0}=(r / m, 0,0)$. If $v \neq 0$, substituting $x=a u / k f(v)$ in the first equation of (7), we have

$$
\begin{equation*}
\varphi_{1}(v) \equiv r-\frac{a u}{k} v=\frac{a m u}{k f(v)} \equiv \varphi_{2}(v) \tag{8}
\end{equation*}
$$

Note that

$$
\begin{gather*}
\varphi_{1}(0)=r, \quad \varphi_{2}(0)=\frac{a u m}{k \beta}, \quad \varphi_{1}^{\prime}(v)=-\frac{a u}{k}<0, \\
\varphi_{2}^{\prime}(v)=-\frac{a u m k f^{\prime}(v)}{(k f(v))^{2}}>0 . \tag{9}
\end{gather*}
$$

Thus, if $\varphi_{1}(0)>\varphi_{2}(0)$, that is, $R_{0}>1$, there is a unique positive root for (8).

We summarize the above analyses in the following result.
Proposition 2. For system (2), the disease-free equilibrium $E_{0}=(r / m, 0,0)$ always exists. Furthermore, the unique endemic equilibrium $E_{1}=\left(x^{*}, y^{*}, v^{*}\right)$ exists only if $R_{0}>1$; here $x^{*}=\operatorname{au} / k f\left(v^{*}\right), y^{*}=u v^{*} / k$, and $v^{*}$ is the unique positive root of (8).

## 3. Local Stability

In this section, we study the local stability of each of feasible equilibria of system (2) by analyzing the corresponding characteristic equations, respectively.

The Jacobian matrix $J$ of (2) at $(x, y, v)$ is

$$
J=\left[\begin{array}{ccc}
-m-v f(v) & 0 & -x f(v)-x v f^{\prime}(v)  \tag{10}\\
f(v) v & -a & x f(v)+x v f^{\prime}(v) \\
0 & k & -u
\end{array}\right]
$$

At disease-free equilibrium $E_{0}$,

$$
J_{E_{0}}=\left[\begin{array}{ccc}
-m & 0 & -\frac{\beta r}{m}  \tag{11}\\
0 & -a & \frac{\beta r}{m} \\
0 & k & -u
\end{array}\right]
$$

Clearly, the determinant of the lower right-hand $2 \times 2$ matrix is positive and its trace is negative only if $R_{0}<1$, so its eigenvalues have negative real parts in this case. Thus, $E_{0}$ is locally asymptotically stable if and only if $R_{0}<1$.

When $R_{0}>1$, the endemic equilibrium $E_{1}$ exists, and the Jacobian matrix at $E_{1}$ is

$$
J_{E_{1}}=\left[\begin{array}{ccc}
-m-v^{*} f\left(v^{*}\right) & 0 & -x^{*} f\left(v^{*}\right)-x^{*} v^{*} f^{\prime}\left(v^{*}\right)  \tag{12}\\
f\left(v^{*}\right) v^{*} & -a & x^{*} f\left(v^{*}\right)+x^{*} v^{*} f^{\prime}\left(v^{*}\right) \\
0 & k & -u
\end{array}\right]
$$

The characteristic equation of (12) is given by

$$
\begin{equation*}
\lambda^{3}+A_{1} \lambda^{2}+A_{2} \lambda+A_{3}=0 \tag{13}
\end{equation*}
$$

in which

$$
\begin{gather*}
A_{1}=a+u+m+v^{*} f\left(v^{*}\right)>0, \\
A_{2}=a u+(a+u)\left(m+v^{*} f\left(v^{*}\right)\right) \\
-k\left(x^{*} f\left(v^{*}\right)+x^{*} v^{*} f^{\prime}\left(v^{*}\right)\right) \\
=(a+u)\left(m+v^{*} f\left(v^{*}\right)\right)-k x^{*} v^{*} f^{\prime}\left(v^{*}\right)>0, \\
A_{3}=a u\left(m+v^{*} f\left(v^{*}\right)\right)-m k\left(x^{*} f\left(v^{*}\right)+x^{*} v^{*} f^{\prime}\left(v^{*}\right)\right) \\
=a u v^{*} f\left(v^{*}\right)-m k x^{*} v^{*} f^{\prime}\left(v^{*}\right)>0 . \tag{14}
\end{gather*}
$$

Here, we used $x^{*} f\left(v^{*}\right)=a u / k$ and the assumption (H1); that is, $f^{\prime}(v)<0$.

Because $A_{1}$ and $A_{3}$ are both positive, by Routh-Hurwitz criterion, $E_{1}$ is locally asymptotically stable if and only if
$A_{1} A_{2}-A_{3}>0$. After a simple algebraic calculation, we have that

$$
\begin{align*}
& A_{1} A_{2}-A_{3} \\
& =a u m-a k x^{*} v^{*} f^{\prime}\left(v^{*}\right)+\left(m+v^{*} f\left(v^{*}\right)\right)\left(a^{2}+a u+m u\right) \\
& \quad+\left(u+v^{*} f\left(v^{*}\right)\right)\left((a+u)\left(m+v^{*} f\left(v^{*}\right)\right)-k x^{*} v^{*} f^{\prime}\left(v^{*}\right)\right) \tag{15}
\end{align*}
$$

is positive because $f^{\prime}(v)<0$. Thus, $E_{1}$ is locally asymptotically stable if and only if $R_{0}>1$.

We summarize the above results and Proposition 2 in the following theorem.

Theorem 3. If $R_{0}<1$, then only disease-free equilibrium $E_{0}$ exists and is locally asymptotically stable. When $R_{0}>1, E_{0}$ is unstable and the endemic equilibrium $E_{1}$ appears and is locally asymptotically stable.

## 4. Global Stability and Disease Persistence

For the global stability of the equilibria, we first have the following.

Theorem 4. The disease-free equilibrium $E_{0}$ is globally asymptotically stable if only $E_{0}$ exists; that is, $R_{0}<1$.

Proof. Define a Lyapunov function

$$
\begin{equation*}
V=x-\frac{r}{m}-\ln \frac{m x}{r}+y+\frac{a}{k} v . \tag{16}
\end{equation*}
$$

Along the trajectories of system (2), we have

$$
\begin{align*}
\left.V^{\prime}\right|_{(2)}= & \left(1-\frac{r}{m x}\right) x^{\prime}+y^{\prime}+\frac{a}{k} v^{\prime} \\
= & \left(1-\frac{r}{m x}\right)(r-m x-f(v) x v) \\
& +f(v) x v-a y+a y-\frac{a u}{k} v  \tag{17}\\
= & -\frac{m}{x}\left(x-\frac{r}{m}\right)^{2}-\frac{a u}{k}\left(1-\frac{k r f(v)}{a u m}\right) v .
\end{align*}
$$

Based on Theorem 1, we know that all solutions of system (2) are positive for $t>0$. Taking $\varphi(v)=1-k r f(v) / a u m$, we have $\varphi(0)=1-R_{0}, \varphi^{\prime}(v)=-k r f^{\prime}(v) /$ aum $>0$; that is, $\varphi(v)$ is a monotone increasing function. Thus, $\varphi(v)>0$ is always valid if $R_{0}<1$. Consequently, all terms of the right hand side of (17) are nonpositive when $R_{0}<1$, which implies that $\left.V^{\prime}\right|_{(2)} \leq$ 0 and $\left.V^{\prime}\right|_{(2)}=0$ if and only if $x=r / m$ and $v=0$. As a result, the maximal invariant set in $\left\{(x, y, v):\left.V^{\prime}\right|_{(2)}=0\right\}$ is the singleton $\left\{E_{0}\right\}$. According to the results in Theorem 3 and LaSalle's invariance principle [12], we have that $E_{0}$ is globally asymptotically stable if $R_{0}<1$.

Next, we investigate the uniform persistence of (2) and have the following result.

Theorem 5. If $R_{0}>1$, then system (2) is uniformly persistent; that is, there exists $\varepsilon>0$ (independent of initial conditions), such that $\liminf _{t \rightarrow+\infty} x(t)>\varepsilon, \liminf _{t \rightarrow+\infty} y(t)>\varepsilon$, and $\liminf _{t \rightarrow+\infty} v(t)>\varepsilon$ for all solutions of (2) with initial conditions (3).

Proof. The result follows from an application of Theorem 4.6 in [13], with $X_{1}=\operatorname{int}\left(\mathbb{R}_{+}^{3}\right)$ and $X_{2}=\operatorname{bd}\left(\mathbb{R}_{+}^{3}\right)$. Since the proof is similar to that of Lemma 3.5 in [16], here we only sketch the modifications that $E_{0}$ is a weak repeller for $X_{1}$.

Since $R_{0}>1$, that is, au $<r k f(0) / m$, together with the continuity of the function $f(v)$, there exists a sufficiently small constant $\epsilon>0$ such that $a u<k(r / m-\epsilon) f(\epsilon)$ is valid. Suppose that there exists a solution $(x(t), y(t), v(t))$ such that $(x(t), y(t), v(t)) \rightarrow(r / m, 0,0)$. Thus, when $t$ is sufficiently large, we have

$$
\begin{equation*}
\frac{r}{m}-\epsilon<x(t)<\frac{r}{m}+\epsilon, \quad y(t) \leq \epsilon, \quad v(t) \leq \epsilon \tag{18}
\end{equation*}
$$

By the second equation of (2), we have

$$
\begin{equation*}
\frac{\mathrm{d} y}{\mathrm{~d} t}=f(v) x v-a y \geq f(\epsilon)\left(\frac{r}{m}-\epsilon\right) v-a y . \tag{19}
\end{equation*}
$$

Take an auxiliary system of (2) as

$$
\begin{gather*}
\frac{\mathrm{d} y}{\mathrm{~d} t}=f(\epsilon)\left(\frac{r}{m}-\epsilon\right) v-a y  \tag{20}\\
\frac{\mathrm{~d} v}{\mathrm{~d} t}=k y-u v
\end{gather*}
$$

Clearly, $(0,0)$ is the unique equilibrium of (20) and the Jacobian matrix $J$ of (20) is given by

$$
J=\left[\begin{array}{cc}
-a & f(\epsilon)\left(\frac{r}{m}-\epsilon\right)  \tag{21}\\
k & -u
\end{array}\right]
$$

After a simple calculation, we have that the determinant of matrix (21)

$$
\begin{equation*}
\operatorname{det}(J)=a u-k\left(\frac{r}{m}-\epsilon\right) f(\epsilon)<0 \tag{22}
\end{equation*}
$$

is valid for some sufficiently small constant $\epsilon>0$ if $R_{0}>1$. Thus, $(0,0)$ is unstable in this case. This is a contradiction to that $(y(t), v(t)) \rightarrow(0,0)$. As a result, $E_{0}$ is a weak repeller for $X_{1}$ 。

## 5. Discussion

Considering the biological practice during viral or microparasitic infection [3,9-11], we proposed a viral infection model with general contact rate between susceptible cells and virus particles, which is a generalization of the basic viral infection model $[7,8]$. The biological meaning of the assumption (H1) is that the accumulation of free virus particles can affect the contact rate between susceptible cells and virus particles, and the contact function is gradually weaker along with the increasing of free virus particles.


Figure 1: Phase diagram of system (2) under different initial conditions. Here $f(v)=\beta /(1+b v)$ and $r=10.0, m=0.01$, $\beta=3.60 \times 10^{-6}, a=0.02, b=0.01, k=50$, and $u=0.67$.

Though the rigorous analysis of stability of equilibria is obtained in [17] for the basic model, it is usually very complicated [18] and we cannot obtain the global stability of the endemic equilibrium $E_{1}$. However, we have the conditions of globally asymptotic stability of the diseasefree equilibrium and persistence of virus. In addition, the phase diagram of system (2) indicates that all solutions tend to the unique disease steady state $E_{1}$ under different initial conditions (Figure 1). Thus, we conjecture that $E_{1}$ is globally asymptotically stable only if it exists even though the rigorous mathematical proof remains open.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The authors are very grateful to the anonymous reviewers for their helpful comments and suggestions. This work is supported by the National Natural Science Fund of China (nos. 11271369 and 11201434).

## References

[1] H. Dahari, J. E. Layden-Almer, E. Kallwitz et al., "A mathematical model of hepatitis C virus dynamics in patients with high baseline viral loads or advanced liver disease," Gastroenterology, vol. 136, no. 4, pp. 1402-1409, 2009.
[2] L. Rong and A. S. Perelson, "Modeling latently infected cell activation: viral and latent reservoir persistence, and viral blips in

HIV-infected patients on potent therapy," PLoS Computational Biology, vol. 5, no. 10, Article ID el000533, 2009.
[3] K. Wang, W. Tan, Y. Tang, and G. Deng, "Numerical diagnoses of superinfection in chronic hepatitis B viral dynamics," Intervirology, vol. 54, no. 6, pp. 349-356, 2011.
[4] J. Pang, J.-A. Cui, and J. Hui, "The importance of immune responses in a model of hepatitis B virus," Nonlinear Dynamics, vol. 67, no. 1, pp. 723-734, 2012.
[5] Q. Li, F. Lu, and K. Wang, "Modeling of HIV-1 infection: insights to the role of monocytes/macrophages, latently infected T4 cells, and HAART regimes," PLoS ONE, vol. 7, no. 9, Article ID e46026, 2012.
[6] G. Huang, A. Fan, and K. Wang, "Dynamics behavior of mutation during reproduction on HIV-1 drug resistance," International Journal of Biomathematics, vol. 6, no. 3, Article ID 1350018, 2013.
[7] M. A. Nowak, S. Bonhoeffer, A. M. Hill, R. Boehme, H. C. Thomas, and H. Mcdade, "Viral dynamics in hepatitis B virus infection," Proceedings of the National Academy of Sciences of the United States of America, vol. 93, no. 9, pp. 4398-4402, 1996.
[8] S. Bonhoeffer, R. M. May, G. M. Shaw, and M. A. Nowak, "Virus dynamics and drug therapy," Proceedings of the National Academy of Sciences of the United States of America, vol. 94, no. 13, pp. 6971-6976, 1997.
[9] D. Ebert, C. D. Zschokke-Rohringer, and H. J. Carius, "Dose effects and density-dependent regulation of two microparasites of Daphnia magna," Oecologia, vol. 122, no. 2, pp. 200-209, 2000.
[10] R. R. Regoes, D. Ebert, and S. Bonhoeffer, "Dose-dependent infection rates of parasites produce the Allee effect in epidemiology," Proceedings of the Royal Society B, vol. 269, no. 1488, pp. 271-279, 2002.
[11] K. Wang and Y. Kuang, "Novel dynamics of a simple Daphniamicroparasite model with dose-dependent infection," Discrete and Continuous Dynamical Systems S, vol. 4, no. 6, pp. 15991610, 2011.
[12] J. P. LaSalle, The Stability of Dynamical Systems, Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, Pa, USA, 1976.
[13] H. R. Thieme, "Persistence under relaxed point-dissipativity (with application to an endemic model)," SIAM Journal on Mathematical Analysis, vol. 24, no. 2, pp. 407-435, 1993.
[14] O. Diekmann, J. A. P. Heesterbeek, and J. A. J. Metz, "On the definition and the computation of the basic reproduction ratio $R_{0}$ in models for infectious diseases in heterogeneous populations," Journal of Mathematical Biology, vol. 28, no. 4, pp. 365-382, 1990.
[15] P. van den Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," Mathematical Biosciences, vol. 180, no. 1-2, pp. 29-48, 2002.
[16] P. de Leenheer and H. L. Smith, "Virus dynamics: a global analysis," SIAM Journal on Applied Mathematics, vol. 63, no. 4, pp. 1313-1327, 2003.
[17] A. Korobeinikov, "Global properties of basic virus dynamics models," Bulletin of Mathematical Biology, vol. 66, no. 4, pp. 879-883, 2004.
[18] A. Murase, T. Sasaki, and T. Kajiwara, "Stability analysis of pathogen-immune interaction dynamics", Journal of Mathematical Biology, vol. 51, no. 3, pp. 247-267, 2005.

## Research Article

# Stochastic Permanence, Stationary Distribution and Extinction of a Single-Species Nonlinear Diffusion System with Random Perturbation 

Li Zu, ${ }^{1,2}$ Daqing Jiang, ${ }^{2}$ and Donal O'Regan ${ }^{3}$<br>${ }^{1}$ School of Science, Changchun University, Changchun 130022, China<br>${ }^{2}$ School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China<br>${ }^{3}$ School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland<br>Correspondence should be addressed to Daqing Jiang; daqingjiang2010@hotmail.com

Received 5 October 2013; Accepted 22 November 2013; Published 22 January 2014
Academic Editor: Weiming Wang
Copyright © 2014 Li Zu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We analyze the influence of stochastic perturbations on a single-species logistic model with the population's nonlinear diffusion among $n$ patches. First, we show that this system has a unique positive solution. Then we obtain sufficient conditions for stochastic permanence and persistence in mean, stationary distribution, and extinction. Finally, we illustrate our conclusions through numerical simulation.


## 1. Introduction

Spatial factors which play a fundamental role in persistence and evolution of species can be modeled by a diffusion process. We have two typical equations to model the diffusion process. One is semilinear parabolic equations, that is, reaction-diffusion systems, where the populations are continuously spread out in space. The other is discrete diffusion systems, where several species are distributed over an interconnected network of multiple patches and there are population migrations among patches. Allen [1] studied and investigated the logistic nonlinear directed diffusion model

$$
\begin{equation*}
\dot{x}_{i}=x_{i}\left(a_{i}-b_{i} x_{i}\right)+\sum_{j=1, j \neq i}^{n} d_{i j}\left(x_{j}^{2}-\alpha_{i j} x_{i}^{2}\right), \quad i=1,2, \ldots, n, \tag{1}
\end{equation*}
$$

where $x_{i}$ denotes the density dependent growth rate in patch $i$. The constants $d_{i j}(i, j=1,2, \ldots, n, j \neq i)$ are the dispersal
rate from the $j$ th patch to the $i$ th patch, and the nonnegative constant $\alpha_{i j}$ can be selected to represent different boundary conditions [2]. Allen proved that the system (1) has a unique positive solution on a maximal interval (see [3]) and is strongly persistent and the population size can increase without bound or bounded under reversed conditions (see [1]). The fundamental tools to prove these results are the cooperative system theory and the cooperative matrix [1-4]. For system (1), Lu and Takeuchi [2, Theorem 3] extended Allen's results and obtained the following necessary and sufficient conditions.
(i) The system (1) possesses a globally stable positive equilibrium point $\left(x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}\right)$, if the largest eigenvalue of the cooperative negative matrix $A$ is less than 0.
(ii) Every solution of the system is unbounded, if the above condition is not satisfied.

Here

$$
A=\left(\begin{array}{cccc}
b_{1}+\sum_{j=1, j \neq 1}^{n} d_{1 j} \alpha_{1 j} & -d_{12} & \cdots & -d_{1 n}  \tag{2}\\
-d_{21} & b_{2}+\sum_{j=1, j \neq 2}^{n} d_{2 j} \alpha_{2 j} & \cdots & -d_{2 n} \\
\vdots & \vdots & \ddots & \cdots \\
-d_{n 1} & -d_{n 2} & \cdots & b_{n}+\sum_{j=1, j \neq n}^{n} d_{n j} \alpha_{n j}
\end{array}\right)
$$

Deterministic models are often subject to stochastic perturbations, and it is useful to reveal how the noise affects the population system. There are many papers which study differential equations with stochastic perturbations (see [510] and the references therein). Li et al. [7] studied the stochastic logistic populations system under regime switching and analyzed the asymptotic properties of their model. Jiang et al. $[8,9]$ investigated a logistic equation with random perturbation and obtained many results such as global stability and stochastic permanence. More investigations and improvements of these stochastic models can be found in [11, 12]. There is very little known on the dynamic behavior in the single-species dispersal system with stochastic perturbation.

Now we introduce randomly perturbation into the intrinsic growth rate $a_{i}$ and assume that parameters $a_{i}$ are disturbed to

$$
\begin{equation*}
a_{i}+\sigma_{i} \dot{B}_{i}(t), \quad i=1,2, \ldots, n \tag{3}
\end{equation*}
$$

where $B_{i}(t)$ is mutually independent Brownian motion and $\sigma_{i}$ is a positive constant representing the intensity of the white noise. Then the stochastic system takes the form
$d x_{i}=\left[x_{i}\left(a_{i}-b_{i} x_{i}\right)+\sum_{j=1, j \neq i}^{n} d_{i j}\left(x_{j}^{2}-\alpha_{i j} x_{i}^{2}\right)\right] d t+\sigma_{i} x_{i} d B_{i}(t)$,
$i=1,2, \ldots, n$

For convenience, let $\bar{b}_{i}=b_{i}+\sum_{j=1, j \neq i}^{n} d_{i j} \alpha_{i j}$ and $d_{i i}=0$. Thus, the equation is rewritten as

$$
\begin{array}{r}
d x_{i}=\left[x_{i}\left(a_{i}-\bar{b}_{i} x_{i}\right)+\sum_{j=1}^{n} d_{i j} x_{j}^{2}\right] d t+\sigma_{i} x_{i} d B_{i}(t)  \tag{5}\\
i=1,2, \ldots, n
\end{array}
$$

In this paper, we assume that $d_{i j}$ and $\alpha_{i j}$ are nonnegative constants, the parameters $a_{i}, b_{i}$ are positive constants, and so $\bar{b}_{i}>0$.

The rest of the paper is arranged as follows. We will show that there exists a unique positive global solution with any initial positive value in Section 2. In Section 3, we will investigate sufficient conditions for stochastic permanence
and persistence in mean which are important in an ecological system. In a deterministic system, the global attractivity of the positive equilibrium is studied, but it is impossible to expect system (5) to tend to a steady state. We investigate the stationary distribution of this system by the Lyapunov functional technique. This can be considered as weak stability, which appears as the solution is fluctuating in a neighborhood of the point. In Section 4, we show that if the white noise is small, there is a stationary distribution of (5) and it has an ergodic property. Results on dynamic in a patchy environment have largely been restricted to extinction analysis which means that the population system will survive or die out in the future. In Section 5, we give sufficient conditions for extinction. In Sections 6 and 7, we make numerical simulation to confirm the effect of white noise intensity and the diffusion coefficient on the species and give a conclusion. Finally, for the completeness of the paper, we give an Appendix containing some results which will be used in other sections.

The key method used in this paper is the analysis of Lyapunov functions [5, 8-10, 12].

Throughout this paper, unless otherwise specified, let $\left(\Omega,\left\{\mathscr{F}_{t}\right\}_{t \geq 0}, P\right)$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_{0}$ contains all $P$-null sets). Let $R_{+}^{n}$ is the positive cone of $R^{n}$, namely, $R_{+}^{n}=\left\{\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in R^{n}\right.$ : $\left.x_{i}>0, i=1,2, \ldots, n\right\}$. For convenience and simplicity in the following discussion, denote $x(t)=\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)$ and $x_{i}=x_{i}(t)$. If $A$ is a vector or matrix, its transpose is denoted by $A^{T}$. By $A \gg 0$ we mean all elements of $A$ are positive. If $A$ is a matrix, its trace norm is denoted by $|A|=$ $\sqrt{\operatorname{trace}\left(A^{T} A\right)}$ whilst its operator norm is denoted by $\|A\|=$ $\sup \{|A x|:|x|=1\}$. We impose the following assumptions.

Assumption 1. $\bar{b}_{i}>\sum_{j=1}^{n} d_{j i}, i=1,2, \ldots, n$.
Assumption 2. $a_{i}>\sigma_{i}^{2} / 2, i=1,2, \ldots, n$.

## 2. Positive and Global Solutions

As the solution of $\operatorname{SDE}$ (5) has biological significance, it should be nonnegative. Moreover, in order for a stochastic differential equation to have a unique global (i.e., no explosion in a finite time) solution for any given initial
value, the coefficients of the equation are generally required to satisfy a linear growth condition and a local Lipschitz condition (cf. Mao [13]). However, the coefficients of SDE (5) do not satisfy a linear growth condition, though they are locally Lipschitz continuous. In this section, we will use a method similar to Mao et al. [5, Theorem 2.1] to prove that the solution of $\operatorname{SDE}$ (5) is nonnegative and global.

Theorem 3. Let Assumption 1 hold. For any given initial value $x(0) \in R_{+}^{n}$, there is a unique positive solution $x(t)$ of system (5), and the solution will remain in $R_{+}^{n}$ with probability 1.

Proof. Define a $C^{2}$-function $V: R_{+}^{n} \rightarrow R_{+}$by

$$
\begin{equation*}
V(x)=\sum_{i=1}^{n}\left(x_{i}-1-\log x_{i}\right) \tag{6}
\end{equation*}
$$

The nonnegativity of this function can be observed from $a-$ $1-\log a \geq 0$ on $a>0$ with equality holding if and only if $a=1$. For $x \in R_{+}^{n}$, applying Itô's formula, we have

$$
\begin{align*}
& d V(x) \\
& \begin{aligned}
= & \sum_{i=1}^{n}\left[d x_{i}-\frac{1}{x_{i}} d x_{i}+\frac{1}{2 x_{i}^{2}}\left(d x_{i}\right)^{2}\right] \\
= & \sum_{i=1}^{n}\left[-\bar{b}_{i} x_{i}^{2}+\sum_{j=1}^{n} d_{i j} x_{j}^{2}+\left(a_{i}+\bar{b}_{i}\right) x_{i}-\sum_{j=1}^{n} d_{i j} \frac{x_{j}^{2}}{x_{i}}-a_{i}+\frac{\sigma_{i}^{2}}{2}\right] d t \\
& +\sum_{i=1}^{n} \sigma_{i}\left(x_{i}-1\right) d B_{i}(t) \\
\leq & \sum_{i=1}^{n}\left[\left(-\bar{b}_{i}+\sum_{j=1}^{n} d_{j i}\right) x_{i}^{2}+\left(a_{i}+\bar{b}_{i}\right) x_{i}-a_{i}+\frac{\sigma_{i}^{2}}{2}\right] d t \\
& +\sum_{i=1}^{n} \sigma_{i}\left(x_{i}-1\right) d B_{i}(t) \\
= & L V d t+\sum_{i=1}^{n} \sigma_{i}\left(x_{i}-1\right) d B_{i}(t),
\end{aligned}
\end{align*}
$$

where

$$
\begin{equation*}
L V=\sum_{i=1}^{n}\left[\left(-\bar{b}_{i}+\sum_{j=1}^{n} d_{j i}\right) x_{i}^{2}+\left(a_{i}+\bar{b}_{i}\right) x_{i}-a_{i}+\frac{\sigma_{i}^{2}}{2}\right] \tag{8}
\end{equation*}
$$

and by Assumption 1, we know that there exists a positive constant number $K$ satisfying

$$
\begin{equation*}
L V \leq K \tag{9}
\end{equation*}
$$

and $K$ is independent of $x_{i}$ and $t$. By a proof similar to Mao et al. [5, Theorem 2.1], we obtain the desired assertion.

## 3. Stochastic Permanence and Persistence in Mean

In this section, we will investigate the persistence under two different meanings: stochastic permanence and persistence in mean.
3.1. Stochastic Permanence. Theorem 3 shows that the solution of SDE (5) will remain in the positive cone $R_{+}^{n}$ with probability 1 . We now further discuss how the solution varies in $R_{+}^{n}$ in detail. We will first give the definitions of stochastically ultimate boundedness and stochastic permanence.

Definition 4. The $\operatorname{SDE}$ (5) is said to be stochastically ultimately bounded, if for any $\epsilon \in(0,1)$, there exist positive constants $\chi_{i}\left(=\chi_{i}(\epsilon)\right)(i=1,2, \ldots, n)$ such that for any initial value $x(0) \in R_{+}^{n}$, the solution of $\operatorname{SDE}$ (5) has the property that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} P\left\{x_{i}(t)>\chi_{i}\right\}<\epsilon, \quad i=1,2, \ldots, n \tag{10}
\end{equation*}
$$

where $\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right)$ is the solution of SDE (5) with any initial value $x(0) \in R_{+}^{n}$.

Definition 5. The $\operatorname{SDE}$ (5) is said to be stochastically permanent, if for any $\epsilon \in(0,1)$, there are positive constants $\chi_{i}\left(=\chi_{i}(\epsilon)\right)$ and $\delta_{i}\left(=\delta_{i}(\epsilon)\right)(i=1,2, \ldots, n)$ such that

$$
\begin{array}{r}
\liminf _{t \rightarrow \infty} P\left\{x_{i}(t) \leq \chi_{i}\right\} \geq 1-\epsilon \\
\liminf _{t \rightarrow \infty} P\left\{x_{i}(t) \geq \delta_{i}\right\} \geq 1-\epsilon  \tag{11}\\
i=1,2, \ldots, n
\end{array}
$$

It is clear that if the system is stochastically permanent, it must be stochastically ultimately bounded.

Lemma 6. Under Assumption 1, for any given initial value $x(0) \in R_{+}^{n}$, there exists a positive constant $\kappa(p)$ such that the solution $x(t)$ of SDE (5) has the following property:

$$
\begin{equation*}
E\left(\sum_{i=1}^{n} x_{i}(t)\right)^{p} \leq \kappa(p), \quad t \geq 0, p>1 \tag{12}
\end{equation*}
$$

Proof. By Theorem 3, we know that the solution $x(t)$ with initial value $x(0) \in R_{+}^{n}$ will remain in $R_{+}^{n}$ with probability 1 . For any given positive constant $p>1$, define

$$
\begin{equation*}
V(x(t))=\left(\sum_{i=1}^{n} x_{i}(t)\right)^{p} \tag{13}
\end{equation*}
$$

By Itô's formula, we find that

$$
\begin{align*}
& d V=p\left(\sum_{i=1}^{n} x_{i}\right)^{p-1} d\left(\sum_{i=1}^{n} x_{i}\right) \\
& +\frac{p(p-1)}{2}\left(\sum_{i=1}^{n} x_{i}\right)^{p-2} d\left(\sum_{i=1}^{n} x_{i}\right)^{2} \\
& =\left[p\left(\sum_{i=1}^{n} x_{i}\right)^{p-1} \sum_{i=1}^{n}\left[x_{i}\left(a_{i}-\bar{b}_{i} x_{i}\right)+\sum_{j=1}^{n} d_{i j} x_{j}^{2}\right]\right. \\
& \left.+\frac{p(p-1)}{2}\left(\sum_{i=1}^{n} x_{i}\right)^{p-2} \sum_{i=1}^{n} \sigma_{i}^{2} x_{i}^{2}\right] d t \\
& +p\left(\sum_{i=1}^{n} x_{i}\right)^{p-1} \sum_{i=1}^{n} \sigma_{i} x_{i} d B_{i}(t) \\
& =p\left[\left(\sum_{i=1}^{n} x_{i}\right)^{p-1} \sum_{i=1}^{n}\left(-\bar{b}_{i}+\sum_{j=1}^{n} d_{j i}\right) x_{i}^{2}\right. \\
& +p\left(\sum_{i=1}^{n} x_{i}\right)^{p-1} \sum_{i=1}^{n}\left(a_{i} x_{i}\right) \\
& \left.+\frac{p(p-1)}{2}\left(\sum_{i=1}^{n} x_{i}\right)^{p-2} \sum_{i=1}^{n} \sigma_{i}^{2} x_{i}^{2}\right] d t \\
& +p\left(\sum_{i=1}^{n} x_{i}\right)^{p-1} \sum_{i=1}^{n} \sigma_{i} x_{i} d B_{i}(t) \\
& \leq\left[-\min _{1 \leq i \leq n}\left\{\frac{p\left(\bar{b}_{i}-\sum_{j=1}^{n} d_{j i}\right)}{n}\right\}\left(\sum_{i=1}^{n} x_{i}\right)^{p+1}\right. \\
& \left.+\max _{1 \leq i \leq n}\left\{p a_{i}+\frac{p(p-1)}{2} \sigma_{i}^{2}\right\}\left(\sum_{i=1}^{n} x_{i}\right)^{p}\right] d t \\
& +p\left(\sum_{i=1}^{n} x_{i}\right)^{p-1} \sum_{i=1}^{n} \sigma_{i} x_{i} d B_{i}(t) \\
& =:\left[-\hat{\beta}\left(\sum_{i=1}^{n} x_{i}\right)^{p+1}+\tilde{\alpha}\left(\sum_{i=1}^{n} x_{i}\right)^{p}\right] d t \\
& +p\left(\sum_{i=1}^{n} x_{i}\right)^{p-1} \sum_{i=1}^{n} \sigma_{i} x_{i} d B_{i}(t), \tag{14}
\end{align*}
$$

where $\widehat{\beta}=\min _{1 \leq i \leq n}\left\{p\left(\bar{b}_{i}-\sum_{j=1}^{n} d_{j i}\right) / n\right\}$, and $\check{\alpha}=\max _{1 \leq i \leq n}\left\{p a_{i}+\right.$ $\left.(p(p-1) / 2) \sigma_{i}^{2}\right\}$. It's clear that $\check{\alpha}>0$ and $\widehat{\beta}>0$. Hence we get

$$
\begin{align*}
& \frac{d E\left(\sum_{i=1}^{n} x_{i}(t)\right)^{p}}{d t} \\
& \quad \leq \check{\alpha} E\left(\sum_{i=1}^{n} x_{i}(t)\right)^{p}-\widehat{\beta}\left[E\left(\sum_{i=1}^{n} x_{i}(t)\right)^{p}\right]^{(p+1) / p}  \tag{15}\\
& \quad=E\left(\sum_{i=1}^{n} x_{i}(t)\right)^{p}\left\{\check{\alpha}-\widehat{\beta}\left[E\left(\sum_{i=1}^{n} x_{i}(t)\right)^{p}\right]^{1 / p}\right\}
\end{align*}
$$

Therefore, letting $z(t)=E\left(\sum_{i=1}^{n} x_{i}(t)\right)^{p}$, we have

$$
\begin{equation*}
\frac{d z(t)}{d t} \leq z(t)\left[\check{\alpha}-\widehat{\beta} z^{1 / p}(t)\right] \tag{16}
\end{equation*}
$$

Notice that the solution of equation

$$
\begin{equation*}
\frac{d \bar{z}(t)}{d t}=\bar{z}(t)\left[\check{\alpha}-\hat{\beta} \bar{z}^{1 / p}(t)\right] \tag{17}
\end{equation*}
$$

obeys

$$
\begin{equation*}
\bar{z}(t) \longrightarrow\left(\frac{\check{\alpha}}{\widehat{\beta}}\right)^{p}, \quad \text { as } t \longrightarrow \infty \tag{18}
\end{equation*}
$$

Thus by the comparison argument we get

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} z(t) \leq\left(\frac{\check{\alpha}}{\widehat{\widehat{\beta}}}\right)^{p} \tag{19}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} E\left(\sum_{i=1}^{n} x_{i}(t)\right)^{p} \leq\left(\frac{\check{\alpha}}{\hat{\beta}}\right)^{p}=: L(p) \tag{20}
\end{equation*}
$$

which implies that there exists a $T>0$, such that

$$
\begin{equation*}
E\left(\sum_{i=1}^{n} x_{i}(t)\right)^{p} \leq 2 L(p), \quad t>T \tag{21}
\end{equation*}
$$

In addition, $E\left(\sum_{i=1}^{n} x_{i}(t)\right)^{p}$ is continuous, so we have

$$
\begin{equation*}
E\left(\sum_{i=1}^{n} x_{i}(t)\right)^{p} \leq C(p), \quad t \in[0, T] \tag{22}
\end{equation*}
$$

Let $\kappa(p)=\max \{2 L(p), C(p)\}$, and therefore

$$
\begin{equation*}
E\left(\sum_{i=1}^{n} x_{i}(t)\right)^{p} \leq \kappa(p), \quad t \geq 0, p>1 \tag{23}
\end{equation*}
$$

This completes the proof.
Theorem 7. Under Assumption 1, solutions of SDE (5) are stochastically ultimately bounded.

The proof of Theorem 7 is a simple application of the Chebyshev inequality and Lemma 6.

Since the solution of SDE (5) is positive, by the classical comparison theorem of stochastic differential equations [14], we can obtain the lemma.

Lemma 8. Let Assumptions 1 and 2 hold, and $x(t) \in R_{+}^{n}$ is the solution of SDE (5) with initial value $x(0) \in R_{+}^{n}$. Then $x(t)$ has the property that

$$
\begin{equation*}
x_{i}(t) \geq \phi_{i}(t) \tag{24}
\end{equation*}
$$

where $\phi_{i}(t)(i=1,2, \ldots, n)$ are the solutions of the following equations:

$$
\begin{array}{r}
d \phi_{i}(t)=\phi_{i}(t)\left[\left(a_{i}-\bar{b}_{i} \phi_{i}(t)\right) d t+\sigma_{i} d B_{i}(t)\right] \\
\phi_{i}(0)=x_{i}(0)  \tag{25}\\
i=1,2, \ldots, n
\end{array}
$$

In view of Li et al. [7, Lemma 3.6], one sees that, if Assumption 2 holds, there exist positive constants $H_{i}$ and $\theta$ such that $a_{i}-((\theta+1) / 2) \sigma_{i}^{2}>0(i=1,2, \ldots, n)$ satisfying the following inequalities:

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} E\left[\frac{1}{\left(\phi_{i}(t)\right)^{\theta}}\right] \leq H_{i}, \quad i=1,2, \ldots, n  \tag{26}\\
& \liminf _{t \rightarrow \infty} \frac{\log \phi_{i}(t)}{\log t} \geq-\frac{1}{\theta} \quad \text { a.s. } i=1,2, \ldots, n
\end{align*}
$$

These, together with Lemma 8, then we have.
Lemma 9. Under Assumptions 1 and 2, the solution $x(t)$ of $S D E(5)$ with any initial value $x(0) \in R_{+}^{n}$ has the property that

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} E\left[\frac{1}{\left(x_{i}(t)\right)^{\theta}}\right] \leq H_{i}, \quad i=1,2, \ldots, n  \tag{27}\\
& \liminf _{t \rightarrow \infty} \frac{\log x_{i}(t)}{\log t} \geq-\frac{1}{\theta} \quad \text { a.s. } i=1,2, \ldots, n \tag{28}
\end{align*}
$$

where $H_{i}$ are positive constants and $\theta>0$ such that $a_{i}-((\theta+$ 1)/2) $\sigma_{i}^{2}>0, i=1,2, \ldots, n$.

Theorem 10. Under Assumptions 1 and 2, SDE (5) is stochastically permanent.

Proof. Let $x(t)$ be the solution of $\operatorname{SDE}$ (5) with any given positive initial value $x(0) \in R_{+}^{n}$. By Lemma 9, we have

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} E\left[\frac{1}{\left(x_{i}(t)\right)^{\theta}}\right] \leq H_{i}, \quad i=1,2, \ldots, n \tag{29}
\end{equation*}
$$

For $x(t) \in R_{+}^{n}$ and for any $\epsilon>0$, let $\delta_{i}=\left(\epsilon / H_{i}\right)^{1 / \theta}$, we get the following:

$$
\begin{align*}
P\left\{x_{i}(t)<\delta_{i}\right\} & =P\left\{\frac{1}{\left(x_{i}(t)\right)^{\theta}}>\frac{1}{\delta_{i}^{\theta}}\right\} \\
& \leq \frac{E\left[1 /\left(x_{i}(t)\right)^{\theta}\right]}{1 / \delta_{i}^{\theta}}  \tag{30}\\
& \leq \delta_{i}^{\theta} H_{i}=\epsilon, \quad i=1,2, \ldots, n .
\end{align*}
$$

Hence

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} P\left\{x_{i}(t)<\delta_{i}\right\} \leq \epsilon, \quad i=1,2, \ldots, n \tag{31}
\end{equation*}
$$

and this implies

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} P\left\{x_{i}(t) \geq \delta_{i}\right\} \geq 1-\epsilon, \quad i=1,2, \ldots, n \tag{32}
\end{equation*}
$$

The other part of Definition 5 follows from Theorem 7.
3.2. Persistence in Mean. In this section, we will investigate persistence in mean. First we introduce one definition.

Definition 11. SDE (5) is said to be persistent in mean, if there exist positive constants $m_{i}, M_{i}(i=1,2, \ldots, n)$ such that the solution $x(t)$ of SDE (5) has the following property:

$$
\begin{array}{ll}
\limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{i}(s) d s \leq M_{i} & \text { a.s. } i=1,2, \ldots, n \\
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{i}(s) d s \geq m_{i} & \text { a.s. } i=1,2, \ldots, n \tag{33}
\end{array}
$$

From the result in [12], we know that

$$
\begin{align*}
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \phi_{i}(s) d s=\frac{a_{i}-\sigma_{i}^{2} / 2}{\bar{b}_{i}}, & \lim _{t \rightarrow \infty} \frac{\log \phi_{i}(t)}{t}=0 \\
& \text { a.s. } i=1,2, \ldots, n \tag{34}
\end{align*}
$$

Using the above conclusions, we get the following lemmas.
Lemma 12. Suppose that Assumptions 1 and 2 are satisfied. Then the solution $x(t)$ of SDE (5) with any initial value $x(0) \in$ $R_{+}^{n}$ has the following property:

$$
\begin{array}{r}
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{i}(s) d s \geq \frac{a_{i}-\sigma_{i}^{2} / 2}{\bar{b}_{i}}, \quad \liminf _{t \rightarrow \infty} \frac{\log x_{i}(t)}{t} \geq 0 \\
\text { a.s. } i=1,2, \ldots, n . \tag{35}
\end{array}
$$

Lemma 13. Let Assumption 1 hold. For any given initial value $x(0) \in R_{+}^{n}$, the solution $x(t)$ of $S D E(5)$ has the property that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{\log \left[\sum_{i=1}^{n} x_{i}(t)\right]}{\log t} \leq 1 \quad \text { a.s. } i=1,2, \ldots, n . \tag{36}
\end{equation*}
$$

Proof. Define $V: R_{+}^{n} \rightarrow R_{+}$by

$$
\begin{equation*}
V(x(t))=\sum_{i=1}^{n} x_{i}(t) \tag{37}
\end{equation*}
$$

and applying the Itô's formula, one can see that

$$
\begin{align*}
& E\left[\sup _{t \leq u \leq t+1} V(x(u))\right] \\
& \quad \leq E[V(x(t))]+\check{a} \int_{t}^{t+1} E\left[\sum_{i=1}^{n} x_{i}(s)\right] d s \\
& \quad+\check{b} \int_{t}^{t+1} E\left[\sum_{i=1}^{n} x_{i}^{2}(s)\right] d s  \tag{38}\\
& \quad+E\left[\sup _{t \leq u \leq t+1} \int_{t}^{u} \sum_{i=1}^{n} \sigma_{i} x_{i}(s) d B_{i}(s)\right]
\end{align*}
$$

here $\check{a}=\max _{1 \leq i \leq n}\left\{a_{i}\right\}, \check{b}=\max _{1 \leq i \leq n}\left\{\left|-\bar{b}_{i}+\sum_{j=1}^{n} d_{j i}\right|\right\}$. From (12) of Lemma 6, we have

$$
\begin{gather*}
\limsup _{t \rightarrow \infty} E[V(x(t))]=\limsup _{t \rightarrow \infty} E\left[\sum_{i=1}^{n} x_{i}(t)\right] \leq[\kappa(2)]^{1 / 2}, \\
\limsup _{t \rightarrow \infty} \int_{t}^{t+1} E\left[\sum_{i=1}^{n} x_{i}^{2}(s)\right] d s \leq \kappa(2) \tag{39}
\end{gather*}
$$

An application of the Burkholder-Davis-Gundy inequality (see [12, 14]) and the Hölder inequality (see [12]) yields

$$
\begin{align*}
& E\left[\sup _{t \leq u \leq t+1} \int_{t}^{u} \sum_{i=1}^{n} \sigma_{i} x_{i}(s) d B_{i}(s)\right] \\
& \quad \leq 3 \max _{1 \leq i \leq n}\left\{\sigma_{i}\right\} E\left(\int_{t}^{t+1}\left[\sum_{i=1}^{n} x_{i}^{2}(s)\right] d s\right)^{1 / 2}  \tag{40}\\
& \quad \leq 3 \check{\sigma}[\kappa(2)]^{1 / 2}
\end{align*}
$$

where $\check{\sigma}=\max _{1 \leq i \leq n}\left\{\sigma_{i}\right\}$. This together with (39) yields

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} E\left[\sup _{t \leq u \leq t+1} V(x(u))\right]  \tag{41}\\
& \quad \leq(1+\check{a}+3 \check{\sigma})[\kappa(2)]^{1 / 2}+\check{b} \kappa(2) .
\end{align*}
$$

We observe from (41) that there is a positive constant $K^{*}$ such that

$$
\begin{equation*}
E\left(\sup _{t \leq u \leq t+1}\left[\sum_{i=1}^{n} x_{i}(u)\right]\right) \leq K^{*} \tag{42}
\end{equation*}
$$

Let $\epsilon>0$ be arbitrary. Then, by the well-known Chebyshev inequality, we have

$$
\begin{equation*}
P\left\{\sup _{t \leq u \leq t+1}\left[\sum_{i=1}^{n} x_{i}(u)\right]>m^{1+\epsilon}\right\} \leq \frac{K^{*}}{m^{1+\epsilon}}, \quad m=1,2, \ldots . \tag{43}
\end{equation*}
$$

Applying the Borel-Cantelli lemma (see [12]), for almost all $\omega \in \Omega$, we obtain that

$$
\begin{equation*}
\sup _{t \leq u \leq t+1}\left[\sum_{i=1}^{n} x_{i}(u)\right] \leq m^{1+\epsilon} \tag{44}
\end{equation*}
$$

holds for all but finitely many $m$. Hence, we have a $m_{0}(\omega)$ such that (44) holds whenever $m \geq m_{0}$, for almost all $\omega \in \Omega$. Consequently, for almost all $\omega \in \Omega$, if $m \geq m_{0}$ and $m \leq t \leq$ $m+1$, we have

$$
\begin{equation*}
\frac{\log \left[\sum_{i=1}^{n} x_{i}(t)\right]}{\log t} \leq \frac{(1+\epsilon) \log m}{\log m}=1+\epsilon \tag{45}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{\log \left[\sum_{i=1}^{n} x_{i}(t)\right]}{\log t} \leq 1+\epsilon \quad \text { a.s. } \tag{46}
\end{equation*}
$$

Letting $\epsilon \rightarrow 0$ we obtain the desired assertion (36).
Theorem 14. Under Assumptions 1 and 2, for any given initial value $x(0) \in R_{+}^{n}$, the solution $x(t)$ of $\operatorname{SDE}(5)$ is persistent in mean.

Proof. Assume that $V: R_{+}^{n} \rightarrow R_{+}$is defined as in (37). From the inequality (28) of Lemma 9 and (36) of Lemma 13, one can derive that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{\log V(x(t))}{t}=0 \quad \text { a.s. } \tag{47}
\end{equation*}
$$

By virtue of the Itô's formula and the Cauchy inequality, we have

$$
\begin{align*}
d \log V= & \frac{1}{V} \sum_{i=1}^{n}\left[a_{i} x_{i}-\left(\bar{b}_{i}-\sum_{j=1}^{n} d_{j i}\right) x_{i}^{2}\right] d t \\
& -\frac{1}{2 V^{2}} \sum_{i=1}^{n} \sigma_{i}^{2} x_{i}^{2} d t+\frac{1}{V} \sum_{i=1}^{n} \sigma_{i} x_{i} d B_{i}(t) \\
= & {\left[\frac{1}{V} \sum_{i=1}^{n} a_{i} x_{i}-\frac{1}{2 V^{2}} \sum_{i=1}^{n} \sigma_{i}^{2} x_{i}^{2}\right] d t } \\
& -\frac{1}{V} \sum_{i=1}^{n}\left(\bar{b}_{i}-\sum_{j=1}^{n} d_{j i}\right) x_{i}^{2} d t+\frac{1}{V} \sum_{i=1}^{n} \sigma_{i} x_{i} d B_{i}(t) \\
\leq & {\left[\max _{1 \leq i \leq n}\left\{a_{i}\right\}-\frac{\sum_{i=1}^{n} \sigma_{i}^{2} x_{i}^{2}}{2\left(\sum_{i=1}^{n}\left(1 / \sigma_{i}^{2}\right)\right)\left(\sum_{i=1}^{n} \sigma_{i}^{2} x_{i}^{2}\right)}\right] d t } \\
& -\frac{1}{n} \min _{1 \leq i \leq n}\left\{\bar{b}_{i}-\sum_{j=1}^{n} d_{j i}\right\} \frac{\left(\sum_{i=1}^{n} x_{i}\right)^{2}}{\sum_{i=1}^{n} x_{i}} d t \\
& +\frac{1}{V} \sum_{i=1}^{n} \sigma_{i} x_{i} d B_{i}(t) \\
= & \left(\check{a}-\frac{\widehat{\sigma}^{2}}{2}\right) d t-\frac{\hat{b}^{2}}{n} \sum_{i=1}^{n} x_{i} d t+\frac{1}{V} \sum_{i=1}^{n} \sigma_{i} x_{i} d B_{i}(t) . \tag{48}
\end{align*}
$$

Here $\check{a}=\max _{1 \leq i \leq n}\left\{a_{i}\right\}, \widehat{\sigma}^{2} / 2=1 /\left(2\left(\sum_{i=1}^{n}\left(1 / \sigma_{i}^{2}\right)\right)\right)$ and $\widehat{b}=$ $\min _{1 \leq i \leq n}\left\{\bar{b}_{i}-\sum_{j=1}^{n} d_{j i}\right\}$. Integrating both sides of the above inequality (48) from 0 to $t$ gives

$$
\begin{align*}
\log V & (x(t))+\frac{\widehat{b}}{n} \int_{0}^{t} \sum_{i=1}^{n} x_{i}(s) d s  \tag{49}\\
& \leq \log V(x(0))+\int_{0}^{t}\left(\check{a}-\frac{\widehat{\sigma}^{2}}{2}\right) d s+M(t)
\end{align*}
$$

where $M(t)$ is a martingale defined by

$$
\begin{equation*}
M(t)=\int_{0}^{t} \frac{\sum_{i=1}^{n} \sigma_{i} x_{i}(s) d B_{i}(s)}{\sum_{i=1}^{n} x_{i}(s)} \tag{50}
\end{equation*}
$$

with $M(0)=0$. The quadratic variation of this martingale is

$$
\begin{equation*}
\langle M, M\rangle_{t}=\int_{0}^{t} \frac{\sum_{i=1}^{n} \sigma_{i}^{2} x_{i}^{2}(s)}{\left(\sum_{i=1}^{n} x_{i}(s)\right)^{2}} d s \leq \max _{1 \leq i \leq n}\left\{\sigma_{i}^{2}\right\} t \tag{51}
\end{equation*}
$$

By the strong law of large numbers for martingales (see [11]), we have

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{M(t)}{t}=0 \quad \text { a.s. } \tag{52}
\end{equation*}
$$

It finally follows from (49) by dividing by $t$ on both sides and then letting $t \rightarrow \infty$; that is,

$$
\begin{equation*}
\frac{\widehat{b}}{n} \limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \sum_{i=1}^{n} x_{i}(s) d s \leq \check{a}-\frac{\widehat{\sigma}^{2}}{2} \quad \text { a.s. } \tag{53}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \sum_{i=1}^{n} x_{i}(s) d s \leq \frac{n}{\hat{b}}\left(\check{a}-\frac{\widehat{\sigma}^{2}}{2}\right) \quad \text { a.s. } \tag{54}
\end{equation*}
$$

On the other hand, from Lemma 12, we know that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{i}(s) d s \geq \frac{1}{\bar{b}_{i}}\left(a_{i}-\frac{\sigma_{i}^{2}}{2}\right) \quad \text { a.s. } i=1,2, \ldots, n . \tag{55}
\end{equation*}
$$

Let $M_{i}=(n / \widehat{b})\left(\check{a}-\left(\widehat{\sigma}^{2} / 2\right)\right), m_{i}=\left(1 / \bar{b}_{i}\right)\left(a_{i}-\left(\sigma_{i}^{2} / 2\right)\right)(i=$ $1,2, \ldots, n)$, then we have

$$
\begin{array}{r}
\limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{i}(s) d s \leq M_{i}, \quad \liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{i}(s) d s \geq m_{i} \\
\text { a.s. } i=1,2, \ldots, n . \tag{56}
\end{array}
$$

Thus the required assertion follows.

## 4. Stationary Distribution

In this section, we investigate that there is a stationary distribution for SDE (5) instead of asymptotically stable equilibria. In order to ensure that system (1) has a globally stable positive equilibrium point $x^{*}=\left(x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}\right)$, we need to introduce the following lemmas.

Lemma 15 (Mao and Yuan [11, Lemma 5.3]). If $B=\left(b_{i j}\right) \in$ $Z^{n \times n}$ has all of its row sums positive,

$$
\begin{equation*}
\sum_{j=1}^{n} b_{i j}>0 \quad \forall 1 \leq i \leq n \tag{57}
\end{equation*}
$$

then $\operatorname{det} B>0$, where $Z^{n \times n}=\left\{B=\left(b_{i j}\right)_{n \times n}: b_{i j} \leq 0, i \neq j\right\}$.
Lemma 16 (Mao and Yuan [11, Theorem 2.10]). If $B \in Z^{n \times n}$, then the following statements are equivalent:
(a) $B$ is a nonsingular M-matrix;
(b) for any $y \gg 0$ in $R^{n}$, the linear equation $B x=y$ has a unique solution $x \gg 0$;
(c) all of the principal minors of $B$ are positive; that is,
$L_{B}=\left|\begin{array}{cccc}b_{11} & b_{12} & \ldots & b_{1 k} \\ b_{21} & b_{22} & \ldots & b_{2 k} \\ \vdots & \vdots & \ddots & \ldots \\ b_{k 1} & b_{k 2} & \ldots & b_{k k}\end{array}\right|>0 \quad$ for every $k=1,2, \ldots, n ;$
(d) $B$ is positive stable; that is, the real part of each eigenvalue of $B$ is positive.

Let the matrix $A$ be defined as in Section 1 which can be simply written as

$$
A=\left(\begin{array}{cccc}
\bar{b}_{1} & -d_{12} & \ldots & -d_{1 n}  \tag{59}\\
-d_{21} & \bar{b}_{2} & \ldots & -d_{2 n} \\
\vdots & \vdots & \ddots & \ldots \\
-d_{n 1} & d_{n 2} & \ldots & \bar{b}_{n}
\end{array}\right)
$$

Lemma 17. If Assumption 1 holds, then both $A$ and $A^{T}$ are nonsingular M-matrices.

Proof. We can obtain by Lemma 15 that if $\bar{b}_{i}>\sum_{j=1}^{n} d_{j i}(i=$ $1,2, \ldots, n)$, then all of the principal minors of $A^{T}$ are positive, and from (a) and (c) of Lemma 16, we know that $A^{T}$ is a nonsingular M-matrix.

Since all of the principal minors of $A$ and $A^{T}$ are the same, so $A$ is also a nonsingular $M$-matrix.

From Lemma 17, we know that if $A$ is a nonsingular $M$-matrix, then the real part of each eigenvalue of $A$ is positive based on (a) and (d) of Lemma 16, and we can also deduce that the maximum of the real part eigenvalues of negative matrix $A$ is less than 0 . This together with Theorem 3 of [2] stated in Section 1, we know that system (1) possesses a globally stable positive equilibrium point $x^{*}=$ $\left(x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}\right)$ satisfying the equations

$$
\begin{equation*}
a_{i} x_{i}^{*}-\bar{b}_{i} x_{i}^{* 2}+\sum_{j=1}^{n} d_{i j} x_{j}^{* 2}=0, \quad i=1,2, \ldots, n \tag{60}
\end{equation*}
$$

where $x_{i}^{*}$ are positive constants.

Theorem 18. Let Assumption 1 hold. Let $\delta_{2}=(1 / 2) \sum_{i=1}^{n} \sigma_{i}^{2} x_{i}^{*}$ and $\delta_{2}<\min _{1 \leq i \leq n}\left\{\left(\bar{b}_{i}-\sum_{i=1}^{n} d_{j i}\right) x_{i}^{* 2}\right\}$. Then there is a stationary distribution $\mu(\cdot)$ for SDE (5) and it has the ergodic property.

Proof. Define $V: E_{l}=R_{+}^{n} \rightarrow R_{+}$by

$$
\begin{equation*}
V(x)=\sum_{i=1}^{n}\left(x_{i}-x_{i}^{*}-x_{i}^{*} \log \frac{x_{i}}{x_{i}^{*}}\right) . \tag{61}
\end{equation*}
$$

By Itô's formula, we have

$$
\begin{equation*}
L V=\sum_{i=1}^{n}\left[\left(x_{i}-x_{i}^{*}\right)\left(a_{i}-\bar{b}_{i} x_{i}+\sum_{j=1}^{n} d_{i j} \frac{x_{j}^{2}}{x_{i}}\right)+\frac{1}{2} \sigma_{i}^{2} x_{i}^{*}\right] . \tag{62}
\end{equation*}
$$

From (60), we know that

$$
\begin{equation*}
a_{i}=\bar{b}_{i} x_{i}^{*}-\sum_{j=1}^{n} d_{i j} \frac{x_{j}^{* 2}}{x_{i}^{*}} . \tag{63}
\end{equation*}
$$

Substituting (63) into (62) one sees that

$$
\begin{align*}
L V=\sum_{i=1}^{n}( & \left(x_{i}-x_{i}^{*}\right) \\
& \times\left[\bar{b}_{i}\left(x_{i}^{*}-x_{i}\right)+\sum_{j=1}^{n} d_{i j}\left(\frac{x_{j}^{2}}{x_{i}}-\frac{x_{j}^{* 2}}{x_{i}^{*}}\right)\right] \\
& \left.+\frac{1}{2} \sigma_{i}^{2} x_{i}^{*}\right) \\
=\sum_{i=1}^{n}[ & -\bar{b}_{i}\left(x_{i}-x_{i}^{*}\right)^{2}  \tag{64}\\
& +\sum_{j=1}^{n} d_{i j}\left(x_{j}^{2}-\frac{x_{i} x_{j}^{* 2}}{x_{i}^{*}}-\frac{x_{i}^{*} x_{j}^{2}}{x_{i}}+x_{j}^{* 2}\right) \\
& \left.+\frac{1}{2} \sigma_{i}^{2} x_{i}^{*}\right] .
\end{align*}
$$

Using the inequality $a^{2}+b^{2} \geq 2 a b$, we compute $-\left(x_{i} x_{j}^{* 2} / x_{i}^{*}\right)-$ $\left(x_{i}^{*} x_{j}^{2} / x_{i}\right) \leq-2 x_{j} x_{j}^{*}$, and from the above inequality, we have

$$
\begin{aligned}
L V \leq \sum_{i=1}^{n}[ & -\bar{b}_{i}\left(x_{i}-x_{i}^{*}\right)^{2} \\
& \left.+\sum_{j=1}^{n} d_{i j}\left(x_{j}^{2}-2 x_{j} x_{j}^{*}+x_{j}^{* 2}\right)+\frac{1}{2} \sigma_{i}^{2} x_{i}^{*}\right]
\end{aligned}
$$

$$
\begin{align*}
& =\sum_{i=1}^{n}\left[-\bar{b}_{i}\left(x_{i}-x_{i}^{*}\right)^{2}+\sum_{j=1}^{n} d_{j i}\left(x_{i}-x_{i}^{*}\right)^{2}+\frac{1}{2} \sigma_{i}^{2} x_{i}^{*}\right] \\
& =-\sum_{i=1}^{n}\left(\bar{b}_{i}-\sum_{j=1}^{n} d_{j i}\right)\left(x_{i}-x_{i}^{*}\right)^{2}+\sum_{i=1}^{n} \frac{1}{2} \sigma_{i}^{2} x_{i}^{*} \\
& =:-\sum_{i=1}^{n}\left(\bar{b}_{i}-\sum_{j=1}^{n} d_{j i}\right)\left(x_{i}-x_{i}^{*}\right)^{2}+\delta_{2}, \tag{65}
\end{align*}
$$

where $\delta_{2}=\sum_{i=1}^{n}(1 / 2) \sigma_{i}^{2} x_{i}^{*}$. By Assumption 1, we know that the quadratic coefficients are less than zero. The following proof of ergodicity is similar to Theorem 3.2 in [10]. Note that $\delta_{2}<\min _{1 \leq i \leq n}\left\{\left(\bar{b}_{i}-\sum_{i=1}^{n} d_{j i}\right) x_{i}^{* 2}\right\}$; then the ellipse

$$
\begin{equation*}
-\sum_{i=1}^{n}\left(\bar{b}_{i}-\sum_{j=1}^{n} d_{j i}\right)\left(x_{i}-x_{i}^{*}\right)^{2}+\delta_{2}=0 \tag{66}
\end{equation*}
$$

lies entirely in $R_{+}^{n}$.
We can take $U$ to be a neighborhood of the ellipsoid with $\bar{U} \subset E_{l}=R_{+}^{n}$, so for $x \in E_{l} \backslash U, L V \leq-N$ ( $N$ is a positive constant), which implies that the condition (B2) in Assumption A. 1 (see the Appendix) is satisfied. By Remark A. 3 and Lemma A. 4 and using the similar method as [10], we can prove that (A1) is also satisfied (see page 349 of [15]). Therefore, the stochastic system (5) has a stable stationary distribution $\mu(\cdot)$ and it is ergodic.

## 5. Extinction

We know that, if Assumption 1 holds, the solution of ODE (1) converges to a positive equilibrium point or is unbounded, so the population will not become extinct, and by Theorem 10, we note that if the condition $a_{i}>\sigma_{i}^{2} / 2(i=1,2, \ldots, n)$ is also satisfied, that is, the white noise intensity is smaller, then the species will be stochastically permanent and persistent in mean. We will show in this section that if the noise is sufficiently large, the solution to the associated SDE (5) will become extinct with probability 1.

Theorem 19. Let Assumption 1 hold. Let $\check{a}=\max _{1 \leq i \leq n}\left\{a_{i}\right\}$ and $\widehat{\sigma}^{2} / 2=1 /\left(2\left(\sum_{i=1}^{n}\left(1 / \sigma_{i}^{2}\right)\right)\right)$. For any given initial value $x(0) \in$ $R_{+}^{n}$, the solution of the SDE (5) has the property that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{\log \left(\sum_{i=1}^{n} x_{i}(t)\right)}{t} \leq \check{a}-\frac{\widehat{\sigma}^{2}}{2} \quad \text { a.s. } \tag{67}
\end{equation*}
$$

Particularly, if $\check{a}-\left(\hat{\sigma}^{2} / 2\right)<0$, then $\lim _{t \rightarrow \infty} x(t)=0$ a.s.
Proof. Define $V: R_{+}^{n} \rightarrow R_{+}$as in (37). Using Itô's formula, one can derive that

$$
\begin{equation*}
d V=\sum_{i=1}^{n}\left[a_{i} x_{i}-\left(\bar{b}_{i}-\sum_{j=1}^{n} d_{j i}\right) x_{i}^{2}\right] d t+\sum_{i=1}^{n} \sigma_{i} x_{i} d B_{i}(t) \tag{68}
\end{equation*}
$$



Figure 1: The pictures on the left are the solutions of stochastic system (73) and the corresponding undisturbed system, and the blue lines and the black lines represent them, respectively. The middle of the subgraphs is the histogram of stochastic system (73) and the subgraphs on the right are normal quantile-quantile plots of the values of the paths $x_{1}(t)$ and $x_{2}(t)$. The stochastic system is stochastically permanent and has a stationary distribution. $\sigma_{1}=0.1, \sigma_{2}=0.09$.

Following the scaling method of (48) and applying the Cauchy inequality and Assumption 1, we find

$$
\begin{align*}
d \log V= & \frac{1}{V} \sum_{i=1}^{n}\left[a_{i} x_{i}-\left(\bar{b}_{i}-\sum_{j=1}^{n} d_{j i}\right) x_{i}^{2}\right] d t \\
& -\frac{1}{2 V^{2}} \sum_{i=1}^{n} \sigma_{i}^{2} x_{i}^{2} d t+\frac{1}{V} \sum_{i=1}^{n} \sigma_{i} x_{i} d B_{i}(t) \\
\leq & {\left[\frac{1}{V} \sum_{i=1}^{n} a_{i} x_{i}-\frac{1}{2 V^{2}} \sum_{i=1}^{n} \sigma_{i}^{2} x_{i}^{2}\right] d t }  \tag{69}\\
& +\frac{1}{V} \sum_{i=1}^{n} \sigma_{i} x_{i} d B_{i}(t) \\
= & \left(\check{a}-\frac{\widehat{\sigma}^{2}}{2}\right) d t+\frac{1}{V} \sum_{i=1}^{n} \sigma_{i} x_{i} d B_{i}(t)
\end{align*}
$$

Integrating both sides of the above inequality (69) from 0 to $t$ gives

$$
\begin{equation*}
\log V(x(t)) \leq \log V(x(0))+\int_{0}^{t}\left(\check{a}-\frac{\widehat{\sigma}^{2}}{2}\right) d s+M(t) \tag{70}
\end{equation*}
$$

where $M(t)$ is a martingale defined in the proof of Theorem 14. By the strong law of large numbers for martingales (see [11]), we have

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{M(t)}{t}=0 \quad \text { a.s. } \tag{71}
\end{equation*}
$$

It finally follows from (70) by dividing by $t$ on both sides and then letting $t \rightarrow \infty$; that is,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{\log V}{t} \leq \limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t}\left(\check{a}-\frac{\widehat{\sigma}^{2}}{2}\right) d s=\check{a}-\frac{\widehat{\sigma}^{2}}{2} \quad \text { a.s. } \tag{72}
\end{equation*}
$$

Thus the required assertion follows.

## 6. Numerical Simulation

For the purpose of discussing the results, we consider the single-species nonlinear dispersal system with $n=2$. Assume that $\alpha_{i j}=1(i, j=1,2)$ and then $\bar{b}_{1}=b_{1}+d_{12}, \bar{b}_{2}=b_{2}+d_{21}$, so the SDE (5) becomes

$$
\begin{align*}
& d x_{1}=\left[x_{1}\left(a_{1}-\bar{b}_{1} x_{1}\right)+d_{12} x_{2}^{2}\right] d t+\sigma_{1} x_{1} d B_{1}(t) \\
& d x_{2}=\left[x_{2}\left(a_{2}-\bar{b}_{2} x_{2}\right)+d_{21} x_{1}^{2}\right] d t+\sigma_{2} x_{2} d B_{2}(t) \tag{73}
\end{align*}
$$



FIgure 2: The subgraphs are defined as in Figure 1. $\sigma_{1}=0.2, \sigma_{2}=0.3$. The stochastic system is stochastically permanent and persistent in mean and has a stationary distribution.

We numerically simulate the solution of (73). By the method mentioned in [16], we consider the discretized equation

$$
\begin{align*}
x_{1, k+1}= & x_{1, k}+\left[x_{1, k}\left(a_{1}-\bar{b}_{1} x_{1, k}\right)+d_{12} x_{2, k}^{2}\right] h \\
& +\sigma_{1} x_{1, k} \sqrt{h} \xi_{1, k}+\frac{1}{2} \sigma_{1}^{2} x_{1, k}\left(h \xi_{1, k}^{2}-h\right), \\
x_{2, k+1}= & x_{2, k}+\left[x_{2, k}\left(a_{2}-\bar{b}_{2} x_{2, k}\right)+d_{21} x_{1, k}^{2}\right] h  \tag{74}\\
& +\sigma_{2} x_{2, k} \sqrt{h} \xi_{2, k}+\frac{1}{2} \sigma_{2}^{2} x_{2, k}\left(h \xi_{2, k}^{2}-h\right) .
\end{align*}
$$

We will use the numerical simulation method and the help of Matlab software to illustrate our results. Choose $a_{1}=$ $0.3, a_{2}=0.4, b_{1}=b_{2}=0.6$. Assume that $d_{12}=0.6, d_{21}=0.5, \bar{b}_{1}=$ 1.2, $\bar{b}_{2}=1.1$ in Figures 1, 2, 3, 4, 6(a), and 7 except in Figures 5 and 6(b) (in Figures 5 and 6(b), we choose $d_{12}=0.01, d_{21}=0$ for the purpose of illustrating the impact of different diffusion coefficients on population), the initial value $\left(x_{1}(0), x_{2}(0)\right)=$ $(0.58,0.60)$, and time step $h=0.01$. Then Assumption 1 is satisfied, so the corresponding deterministic model has
a globally stable positive equilibrium point $x^{*}=\left(x_{1}^{*}, x_{2}^{*}\right) \doteq$ ( $0.5734,0.6090$ ). Obviously, by Theorem 3, system (73) has a unique positive solution. The following discussion will be divided into two cases.

Case 1. The effect of different white noise intensity on the population.

In Figure 1, we choose $\sigma_{1}=0.1, \sigma_{2}=0.09$. Obviously Assumption 2 holds and the $\operatorname{SDE}$ (73) is stochastically permanent and persistent in mean. We compute $\delta_{2}=(1 / 2) \sigma_{1}^{2} x_{1}^{*}+$ $(1 / 2) \sigma_{2}^{2} x_{2}^{*} \doteq 5.3334 * 10^{-3}$, and $\min \left\{\left(\bar{b}_{1}-d_{21}\right)\left(x_{1}^{*}\right)^{2},\left(\bar{b}_{2}-\right.\right.$ $\left.\left.d_{12}\right)\left(x_{2}^{*}\right)^{2}\right\} \doteq 0.18544$, so the condition $\delta_{2}<\min \left\{\left(\bar{b}_{1}-\right.\right.$ $\left.\left.d_{21}\right)\left(x_{1}^{*}\right)^{2},\left(\bar{b}_{2}-d_{12}\right)\left(x_{2}^{*}\right)^{2}\right\}$ is also satisfied. Therefore, by Theorem 18, there is a stationary distribution (see the middle histogram in Figure 1). The left pictures in Figure 1 show that the stochastic system imitate the deterministic system. The right subgraphs are the normal quantile-quantile plots of the values of the paths $x_{1}(t)$ and $x_{2}(t)$, and they are similar to the straight lines. This means that the distribution is approximately standard normal distribution. The scatter plot of $x_{1}(t)$ and $x_{2}(t)$ is Figure 3(a); we find that almost all


FIGURE 3: Population distribution of stochastic system (73) around the deterministic model's positive equilibrium $x^{*} \doteq(0.5734,0.6090)$. $\sigma_{1}=0.1, \sigma_{2}=0.09$ in left subgraph (a) and $\sigma_{1}=0.2, \sigma_{2}=0.3$ in the right subgraph (b).


Figure 4: The pictures on the left are the solutions of stochastic system (73) and the corresponding undisturbed system, and the blue lines and the black lines represent them, respectively. The right subgraphs are the histogram of stochastic system (73). $\sigma_{1}=0.2, \sigma_{2}=0.8 . d_{12}=0.6$, $d_{21}=0.5$.
population distribution lies in a small neighborhood, which can be imagined as a circular or elliptic region centered at $\left(x_{1}^{*}, x_{2}^{*}\right)$. Hence, although there is no equilibrium of the stochastic system (73) as the deterministic system, it is stochastically permanent, persistent in mean and has the ergodic property by Theorems 10, 14, and 18.

In Figure 2, we choose $\sigma_{1}=0.2, \sigma_{2}=0.3$. The populations of $x_{1}$ and $x_{2}$ suffer relatively large white noise. By comparing Figure 1, we can see that in Figure 2 the left curves fluctuations are more violent, the histogram distribute in relatively large regions and the curves of QQ plots slightly deviate from the straight line. Comparing with Figure 3(a),


Figure 5: The subgraphs are defined as in Figure 4. Because there is no diffusion, $x_{2}$ is isolated and will die out; $\sigma_{1}=0.2, \sigma_{2}=0.8 ; d_{12}=0.01$, $d_{21}=0$.


Figure 6: Population distribution of stochastic system (73); $\sigma_{1}=0.2, \sigma_{2}=0.8, d_{12}=0.6$, and $d_{21}=0.5$ in the left subgraph (a) and $\sigma_{1}=0.2$, $\sigma_{2}=0.8, d_{12}=0.01$, and $d_{21}=0$ in the right subgraph (b).
its scatter distributes in a larger area (see the scatter picture in Figure 3(b)), but we can find an ellipse to meet the condition $\delta_{2} \doteq 3.8873 \times 10^{-2}<\min \left\{\left(\bar{b}_{1}-d_{21}\right)\left(x_{1}^{*}\right)^{2},\left(\bar{b}_{2}-d_{12}\right)\left(x_{2}^{*}\right)^{2}\right\} \doteq$ 0.18544 ; by Theorems 10, 14, and 18, we know that SDE (73) is stochastically permanent, persistent in mean and has stationary distribution.

Comparing with small white noise as in Figures 1 and 2, we choose $\sigma_{1}=0.9, \sigma_{2}=1.0$ in Figure 7. Both $x_{1}$ and $x_{2}$ suffer large white noise. We find that $a_{i}<(1 / 2) \sigma_{i}^{2}(i=1,2), \delta_{2} \doteq$ $0.5367>\min \left\{\left(\bar{b}_{1}-d_{21}\right)\left(x_{1}^{*}\right)^{2},\left(\bar{b}_{2}-d_{12}\right)\left(x_{2}^{*}\right)^{2}\right\} \doteq 0.18544$, so the conditions of Theorems 10, 14, and 18 are not satisfied and the extinction conditions in Theorem 19 are satisfied, that is,


Figure 7: The subgraphs are defined as in Figure 4; $\sigma_{1}=0.9, \sigma_{2}=1$. The populations of $x_{1}$ and $x_{2}$ will become extinct.
$\check{a}-\left(\widehat{\sigma}^{2} / 2\right) \doteq-4.75 \times 10^{-2}<0$; as the case in Theorem 19 expected, the species $x_{1}$ and $x_{2}$ will become extinct although the deterministic system is globally asymptotic stable.

Case 2. The effect of different diffusion coefficient on the population.

In Figure 4, we select $\sigma_{1}=0.2, \sigma_{2}=0.8$. The conditions of Theorems 10,14 , and 18 are satisfied. $x_{2}$ suffers relatively large white noise. From the left pictures of $x_{1}(t)$ in Figures 2 and 4 , we see that the fluctuations of the two curves are different and the reason is that larger white noise of $x_{2}$ impacts $x_{1}$ in Figure 4. In other words, due to the presence of diffusion, the relatively big white noise intensity in the individual patches will be evenly distributed to the other patches, which reduces the risk of extinction of the population. Therefore, system (73) is stochastically permanent and has a stationary distribution.

In Figure 5, we choose $d_{12}=0.01, d_{21}=0, \sigma_{1}=0.2$, and $\sigma_{2}=0.8$. Figures 4 and 5 have the same white noise intensity but have different diffusion coefficients. Because there is no diffusion effects, we can see that $x_{2}$ will die out from Figure 5 and the scatter plot Figure 6(b), that is to say, the isolated patches may become extinct if the white noise is large.

## 7. Conclusion

In this paper, we study the stochastic logistic single-species model with nonlinear directed diffusion among $n$ patches.

First, we divide the white noise intensity into small, medium, and large three cases, and through numerical simulation, we can more clearly understand the important role played by the white noise in biological populations. From these figures, we find that when the white noise is small, system (73) imitates its deterministic system and it is
stochastically permanent and persistent in mean and has a stationary distribution (see Figures 1 and 3(a)). When the white noise is relatively large in some groups, it will bring relatively large deviation (see Figures 2, 3(b), 4, and 6(a)) but will not bring the species extinction due to the presence of diffusion. But, when the noise is sufficiently large in all the groups (see Figure 7), the species will become extinct even if diffusion exists. We also study the effect of different diffusion coefficient on the species and we find that isolated plaque affected by big white noise may become extinct if the diffusion coefficient is very small or equals zero (see Figures 5 and 6(b)).

In the real world, the large white noise may be bad weather, serious epidemic, which can be considered as the decisive factor responsible for the extinction of populations. Diffusion phenomena, however, play a crucial role in the development of biological populations, and human activities without control will affect the biological diffusion process which is likely to cause fatal consequences. Therefore, our research and analysis on population have great practical significance.

## Appendix

In this section, we list some results about the stationary distribution (see [15, 17]) which will be used in the previous sections.

Let $X(t)$ be a homogeneous Markov process in $E_{l}\left(E_{l}\right.$ denotes $l$-space) described by the stochastic equation

$$
\begin{equation*}
d X(t)=b(X) d t+\sum_{r=1}^{k} g_{r}(X) d B_{r}(t) \tag{A.1}
\end{equation*}
$$

The diffusion matrix is

$$
\begin{equation*}
\Lambda(x)=\left(\lambda_{i j}(x)\right), \quad \lambda_{i j}(x)=\sum_{r=1}^{k} g_{r}^{i}(x) g_{r}^{j}(x) \tag{A.2}
\end{equation*}
$$

Assumption A.1. There exists a bounded domain $U \subset E_{l}$ with regular boundary $\Gamma$, having the following properties.
(A1) In the domain $U$ and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.
(B2) If $x \in E_{l} \backslash U$, the mean time $\tau$ at which a path starting from $x$ reaches the set $U$ is finite, and $\sup _{x \in K} E_{x} \tau<\infty$ for every compact subset $K \subset E_{l}$.

Lemma A. 2 (see [17]). If Assumption A. 1 holds, then the Markov process $X(t)$ has a stationary distribution $\mu(A)$. Let $f(\cdot)$ be a function integrable with respect to the measure $\mu$. Then

$$
\begin{equation*}
P_{x}\left\{\lim _{T \rightarrow \infty} \frac{1}{T} \int_{0}^{T} f(X(t)) d t=\int_{E_{l}} f(x) \mu(d x)\right\}=1 \tag{A.3}
\end{equation*}
$$

for all $x \in E_{l}$.
Remark A.3. Theorem 3 shows that there exists a unique positive solution $x(t)$ of $\operatorname{SDE}$ (5). Also from the proof of Theorem 3, we obtain

$$
\begin{equation*}
L V \leq K \tag{A.4}
\end{equation*}
$$

Now define $\bar{V}=V+K$; then

$$
\begin{equation*}
L \bar{V} \leq \bar{V} \tag{A.5}
\end{equation*}
$$

and we can get

$$
\begin{equation*}
\bar{V}_{R}=\inf _{x \in R_{+}^{n} \backslash D_{m}} \bar{V}(x) \longrightarrow \infty \quad \text { as } m \longrightarrow \infty \tag{A.6}
\end{equation*}
$$

where $D_{m}=(1 / m, m) \times(1 / m, m) \times \cdots \times(1 / m, m)$. By [17], we can obtain that the solution $x(t)$ is a homogeneous Markov process in $R_{+}^{n}$.

Lemma A. 4 (see [17]). Let $x(t)$ be a regular temporally homogeneous Markov process in $E_{l}$. If $x(t)$ is recurrent relative to some bounded domain $U$, then it is recurrent relative to any nonempty domain in $E_{l}$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The work was supported by the NSFC (no. 10971021 and no. 11001032), the Ministry of Education of China (no. 109051), the Ph.D. Programs Foundation of Ministry of China (no. 200918), and the Program for Changjiang Scholars and Innovative Research Team in University (PCSIRT).

## References

[1] L. J. S. Allen, "Persistence and extinction in single-species reac-tion-diffusion models," Bulletin of Mathematical Biology, vol. 45, no. 2, pp. 209-227, 1983.
[2] Z. Y. Lu and Y. Takeuchi, "Global asymptotic behavior in singlespecies discrete diffusion systems," Journal of Mathematical Biology, vol. 32, no. 1, pp. 67-77, 1993.
[3] L. Allen, "Persistence, extinction, and critical patch number for island populations," Bulletin of Mathematical Biology, vol. 65, pp. 1-12, 1987.
[4] H. I. Freedman and Y. Takeuchi, "Global stability and predator dynamics in a model of prey dispersal in a patchy environment," Nonlinear Analysis: Theory, Methods \& Applications, vol. 13, no. 8, pp. 993-1002, 1989.
[5] X. R. Mao, C. G. Yuan, and J. Zou, "Stochastic differential delay equations of population dynamics," Journal of Mathematical Analysis and Applications, vol. 304, no. 1, pp. 296-320, 2005.
[6] M. Liu and K. Wang, "Persistence and extinction in stochastic non-autonomous logistic systems," Journal of Mathematical Analysis and Applications, vol. 375, no. 2, pp. 443-457, 2011.
[7] X. Y. Li, A. Gray, D. Q. Jiang, and X. R. Mao, "Sufficient and necessary conditions of stochastic permanence and extinction for stochastic logistic populations under regime switching," Journal of Mathematical Analysis and Applications, vol. 376, no. 1, pp. 11-28, 2011.
[8] D. Q. Jiang and N. Z. Shi, "A note on nonautonomous logistic equation with random perturbation," Journal of Mathematical Analysis and Applications, vol. 303, no. 1, pp. 164-172, 2005.
[9] D. Q. Jiang, N. Z. Shi, and X. Y. Li, "Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation," Journal of Mathematical Analysis and Applications, vol. 340, no. 1, pp. 588-597, 2008.
[10] C. Y. Ji, D. Q. Jiang, H. Liu, and Q. S. Yang, "Existence, uniqueness and ergodicity of positive solution of mutualism system with stochastic perturbation," Mathematical Problems in Engineering, vol. 2010, Article ID 684926, 18 pages, 2010.
[11] X. R. Mao and C. G. Yuan, Stochastic Differential Equations with Markovian Switching, Imperial College Press, London, UK, 2006.
[12] C. Ji, D. Jiang, and N. Shi, "Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation," Journal of Mathematical Analysis and Applications, vol. 359, no. 2, pp. 482-498, 2009.
[13] X. R. Mao, Stochastic Differential Equations and Applications, Horwood, New York, NY, USA, 1997.
[14] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, vol. 24, North-Holland, Amsterdam, The Netherlands; Kodansha Ltd., Tokyo, Japan, 2nd edition, 1989.
[15] G. Strang, Linear Algebra and Its Applications, Thomson Learning Inc., 1988.
[16] D. J. Higham, "An algorithmic introduction to numerical simulation of stochastic differential equations," SIAM Review, vol. 43, no. 3, pp. 525-546, 2001.
[17] R. Z. Hasminskii, Stochastic Stability of Differential Equations, vol. 7, Sijthoff \& Noordhoff, Alphen aan den Rijn, The Netherlands, 1980.

## Research Article

# Bifurcation Analysis of an SIR Epidemic Model with the Contact Transmission Function 

Guihua $\mathrm{Li}^{1}$ and Gaofeng Li ${ }^{2}$<br>${ }^{1}$ School of Science, North University of China, Taiyuan, Shanxi 030051, China<br>${ }^{2}$ Xinjiang Agriculture Second Division Korla Hospital, Korla, Xinjiang 841000, China

Correspondence should be addressed to Guihua Li; ttl1013@163.com
Received 8 December 2013; Accepted 23 December 2013; Published 21 January 2014
Academic Editor: Kaifa Wang
Copyright © 2014 G. Li and G. Li. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We consider an SIR endemic model in which the contact transmission function is related to the number of infected population. By theoretical analysis, it is shown that the model exhibits the bistability and undergoes saddle-node bifurcation, the Hopf bifurcation, and the Bogdanov-Takens bifurcation. Furthermore, we find that the threshold value of disease spreading will be increased, when the half-saturation coefficient is more than zero, which means that it is an effective intervention policy adopted for disease spreading. However, when the endemic equilibria exist, we find that the disease can be controlled as long as we let the initial values lie in the certain range by intervention policy. This will provide a theoretical basis for the prevention and control of disease.


## 1. Introduction

The classical SIR model for disease transmission has been widely studied. It is one of the most important issues that the dynamical behaviors are changed by the different incidence rate in epidemic system. For the incidence rate, we divided into two categories: one is that Capasso and Serio [1] proposed the infection force which is a saturated curve, described "crowding effect" or "protection measures;" the other is the infection force that describes the effect of "intervention policy," for example, closing schools and restaurants and postponing conferences (see Figure 1). For the model with the saturated infection force, $a I^{2} /\left(b+I^{2}\right)$, which is one of the typical infection forces, the rich dynamical behaviors were founded by Ruan and Wang [2] and Tang et al. [3]. The model with the incidence rate can be suited for many infectious diseases, including measles, mumps, rubella, chickenpox, and influenza. For more research literatures about nonlinear infection rate see [4-9]. However, for some parasite-host models, by observing macro- and microparasitic infections, one finds that the infection rate is an increasing function of the parasite dose, usually sigmoidal in shape $[10,11]$. So we will build a model with sigmoidal incidence rate which is taken into account "crowding effect" and "saturated effect."

According to the parasite-host model which is proposed by Anderson and May (1979) [12, 13], the model is as follows:

$$
\begin{align*}
\frac{d S}{d t} & =A-d S-\beta(I) S \\
\frac{d I}{d t} & =\beta(I) S-(d+\gamma+\epsilon) I  \tag{1}\\
\frac{d R}{d t} & =\gamma I-d R
\end{align*}
$$

where $S, I, R$ are susceptible hosts, infected hosts, and removed hosts, respectively. $A$ is the birth rate of susceptible host, $d$ is the natural death rate of a population, $\gamma$ is the removal rate, and $\epsilon$ is the per capita infection-related death rate. If we denote infection force $\beta(I)=g(I) I, g(I)$ can be explained as the rate of valid contact. At the beginning of disease, most people have poor awareness of prevention, then the rate of valid contact $g(I)$ can be first increasing then tends to a certain value. As the time flies, people are gradually aware of the seriousness and take measures to prevent and control development of the disease and will reduce to be contact with infected, so the rate of contact $g(I)$ is first increasing then


Figure 1: The plotting for the contact transmission function $\beta(I)$.
decreasing. In short time, it does not tend to zero, but tends to a nonzero constant. To simplify the study, we take

$$
\begin{equation*}
g(I)=\frac{\beta I^{2}}{b+a I+I^{2}}, \tag{2}
\end{equation*}
$$

where $b>0$ and $-2 \sqrt{b}<a$. If $a \geq 0, g(I)$ is increasing monotonically and tends to $\beta$. If $-2 \sqrt{b}<a<0, g(I)$ is first increasing then decreasing and tends to $\beta$ (see Figure 1).

Then model (1) becomes

$$
\begin{align*}
& \frac{d S}{d t}=A-d S-\frac{\beta S I^{3}}{b+a I+I^{2}}, \\
& \frac{d I}{d t}=\frac{\beta S I^{3}}{b+a I+I^{2}}-(d+\gamma+\epsilon) I,  \tag{3}\\
& \frac{d R}{d t}=\gamma I-d R,
\end{align*}
$$

where $\beta$ is the valid contact coefficient.
When $a=0$ and $b=0$, model (3) becomes

$$
\begin{gather*}
\frac{d S}{d t}=A-d S-\beta S I \\
\frac{d I}{d t}=\beta S I-\mu I  \tag{4}\\
\frac{d R}{d t}=\gamma I-d R
\end{gather*}
$$

where $\mu=d+\gamma+\epsilon$.
We know that $R_{0}=\beta A / d \mu$ is the basic reproduction number of (4). It is easy to see that there is a unique positive equilibrium $I^{*}$ in system (4) when $R_{0}>1$ and there is no positive equilibrium when $R_{0} \leq 1$. In the next sections, we will study that parameters $a$ and $b$ would have any effect on the dynamic behaviors of model (3).

The organization of this paper is as follows. In the next section, we analyze the existence and stability of the endemic equilibria for model (3). Then we discuss conditions for the Hopf bifurcation and the Bogdanov-Takens bifurcation in Sections 3. Section 4 presents numerical simulations to indicate dynamical behaviors and bifurcation structures, and gives with a brief discussion.

## 2. Existence and Stability of Equilibria

We consider the positive equilibria of (3). Setting the right hand sides of system (3) to zero, we find that the first and second equations of system (3) do not include $R$, so we only consider

$$
\begin{gather*}
A-d S-\frac{\beta S I^{3}}{b+a I+I^{2}}=0 \\
\frac{\beta S I^{3}}{b+a I+I^{2}}-\mu I=0 \tag{5}
\end{gather*}
$$

From the above two equations, except for the disease-free equilibrium (DFE) at ( $A / d, 0$ ), any endemic equilibrium (EE), if exists, is the intersection of the following two curves in the positive quadrant

$$
\begin{align*}
& S=\frac{A-\mu I}{d} \\
& S=\frac{\mu\left(b+a I+I^{2}\right)}{\beta I^{2}} \tag{6}
\end{align*}
$$

From (6), I must satisfy the following equation:

$$
\begin{equation*}
H(I):=\beta I^{3}+d\left(1-R_{0}\right) I^{2}+d a I+d b=0 . \tag{7}
\end{equation*}
$$

Thus the intersection of two curves (6) is transformed into the positive root of (7).

The derivative of $H^{\prime}(I)$ is

$$
\begin{equation*}
H^{\prime}(I):=3 \beta I^{2}+2 d\left(1-R_{0}\right) I+d a . \tag{8}
\end{equation*}
$$

In the following, we consider three cases according to the sign of $a$. By calculation, we have the following three theorems.

Set

$$
\begin{gather*}
a_{1}=d^{2}\left(1-R_{0}\right)^{2}-3 \beta a d  \tag{9}\\
a_{2}=27 \beta^{2} d b-d b\left(1-R_{0}\right)\left(3 \beta a d-2 a_{1}\right) .
\end{gather*}
$$

Theorem 1. Suppose $a>0$. Then we have the following.
(a) If $R_{0} \leq 1+\sqrt{3 \beta a / d}$, then system (3) has no endemic equilibrium.
(b) If $R_{0}>1+\sqrt{3 \beta a / d}$, then we have the following.
(i) When $2 a_{1}^{3 / 2}<a_{2}$, system (3) has no endemic equilibrium.
(ii) When $2 a_{1}^{3 / 2}=a_{2}$, system (3) has a unique endemic equilibrium.
(iii) When $2 a_{1}^{3 / 2}>a_{2}$, system (3) has two endemic equilibria $E_{1}\left(S_{1}, I_{1}\right), E_{2}\left(S_{2}, I_{2}\right)$.

Theorem 2. Suppose $a=0$. Then we have the following.
(a) If $R_{0}<1+\sqrt[3]{27 \beta^{2} b / 4 d^{2}}$, then system (3) has no endemic equilibrium.
(b) If $R_{0}=1+\sqrt[3]{27 \beta^{2} b / 4 d^{2}}$, then system (3) has a unique endemic equilibrium.
(c) If $R_{0}>1+\sqrt[3]{27 \beta^{2} b / 4 d^{2}}$, then system (3) has two endemic equilibria $E_{1}\left(S_{1}, I_{1}\right), E_{2}\left(S_{2}, I_{2}\right)$.

Theorem 3. Suppose $-2 \sqrt{b}<a<0$. Then we have the following.
(a) If $2 a_{1}^{3 / 2}<a_{2}$, then system (3) has no endemic equilibrium.
(b) If $2 a_{1}^{3 / 2}=a_{2}$, then system (3) has a unique endemic equilibrium $E^{*}\left(S^{*}, I^{*}\right)$.
(c) If $2 a_{1}^{3 / 2}>a_{2}$, then system (3) has two endemic equilibria $E_{1}\left(S_{1}, I_{1}\right), E_{2}\left(S_{2}, I_{2}\right)$, where $I_{1}<I^{*}<I_{2}$.

Remark 4. From Theorems 1 and 2, we can find that the basic reproduction number for the model (3) is less than that of the standard model. It means that the disease will spread more easily. For Theorem 3, it is obvious that the disease can exist if $R_{0}<1$.

For disease-free equilibrium (DFE), it is easy to calculate that the Jacobian matrix of system (3) at DFE has eigenvalues $\lambda_{1}=-d$ and $\lambda_{2}=-\mu$. Hence, DFE is always stable.

In the following, the stability of the endemic equilibrium in system (3) will be studied. Firstly, evaluating the Jacobian matrix of system (3) at $E(S, I)$ gives

$$
J=\left.\left(\begin{array}{ll}
j_{11} & j_{12}  \tag{10}\\
j_{21} & j_{22}
\end{array}\right)\right|_{(S, I)}
$$

where

$$
\begin{array}{cc}
j_{11}=-d-\frac{\beta I^{3}}{b+a I+I^{2}}, & j_{12}=-\mu-\frac{\mu(2 b+a I)}{b+a I+I^{2}}  \tag{11}\\
j_{21}=\frac{\beta I^{3}}{b+a I+I^{2}}, & j_{22}=\frac{\mu(2 b+a I)}{b+a I+I^{2}}
\end{array}
$$

Its characteristic equation is

$$
\begin{equation*}
P(\lambda)=\lambda^{2}-\operatorname{tr}(J) \lambda+\operatorname{det}(J)=0 \tag{12}
\end{equation*}
$$

where

$$
\begin{align*}
& \operatorname{det}(J)=\frac{\beta \mu I^{3}-d a \mu I-2 d b \mu}{b+a I+I^{2}}  \tag{13}\\
& \operatorname{tr}(J)=\frac{-\beta A I^{2}+a \mu^{2} I+2 b \mu^{2}}{\mu\left(b+a I+I^{2}\right)} \tag{14}
\end{align*}
$$

It is easy to calculate

$$
\begin{equation*}
\beta \mu I^{* 3}-d a \mu I^{*}-2 d b \mu=\mu\left[I^{*} H^{\prime}\left(I^{*}\right)-2 H\left(I^{*}\right)\right]=0 \tag{15}
\end{equation*}
$$

that is, $\left.\operatorname{det}(J)\right|_{I=I^{*}}=0$.
Now suppose that the model has two endemic equilibria $E_{1}\left(S_{1}, I_{1}\right), E_{2}\left(S_{2}, I_{2}\right)$, with $I_{1}<I^{*}<I_{2}<A / \mu$; that is, in Theorem 1, the item (b) (iii) holds or Theorems 2 and 3, the item $c$ holds. If $J_{i}(i=1,2)$ is the Jacobian matrix at $\left(S_{i}, I_{i}\right)$, then (13) gives

$$
\begin{equation*}
\operatorname{det}\left(J_{i}\right)=\left.\frac{\beta \mu I^{3}-d a \mu I-2 d b \mu}{b+a I+I^{2}}\right|_{I=I_{i}} \tag{16}
\end{equation*}
$$

Thus, it is easily obtained that $\operatorname{det}\left(J_{1}\right)$ is negative and $\operatorname{det}\left(J_{2}\right)$ is positive. We can immediately conclude that the endemic equilibrium $E_{1}$ with low number of infected individuals is always a saddle, and that the endemic equilibrium $E_{2}$ with high number of infected individuals is a node or focus but the stability of $E_{2}$ is determined by $\operatorname{tr}\left(J_{2}\right)$. From (14), we notice that the sign of the trace of $J_{2}$ is determined by

$$
\begin{equation*}
\operatorname{tr} 1:=-\beta A I^{2}+a \mu^{2} I+2 b \mu^{2} \tag{17}
\end{equation*}
$$

Set

$$
\begin{gather*}
b_{0}:=\frac{a \mu^{2}+\mu \sqrt{a^{2} \mu^{2}+8 \beta A b}}{2 \beta A}, \\
b_{1}:=\frac{-d\left(1-R_{0}\right)+\sqrt{a_{1}}}{3 \beta \mu},  \tag{18}\\
r_{0}:=\beta A\left(a b_{0}(A+1)+A b\right), \\
r_{1}:=d\left(R_{0}-1\right)(2+A)\left(a+2 b_{0}\left(1-R_{0}\right)\right)
\end{gather*}
$$

Theorem 5. Assume that (3) has two endemic equilibria. Then $E_{2}$ is asymptotically stable if one of the following is satisfied.
(a) $b_{0}<b_{1}$;
(b) $b_{0}>b_{1}$ and $r_{0}>r_{1}$.

Further, $E_{2}$ is unstable if $b_{0}>b_{1}$ and $r_{0}<r_{1}$.
Proof. If $b_{0}<b_{1}$, then $-\beta A I^{* 2}+a \mu^{2} I^{*}+2 b \mu^{2}<0$. It follows from $I_{2}>I^{*}$ that $\operatorname{tr} 1<0$. Hence, $E_{2}$ is asymptotically stable in this case. If $b_{0}>b_{1}$, we have $b_{0}>I^{*}$. By direct calculations we see that $r_{0}>r_{1}$ implies $I_{2}>b_{0}$, which leads to $\operatorname{tr} 1<0$. Therefore, $E_{2}$ is asymptotically stable if condition (b) holds. Similarly, if $b_{0}<b_{1}$ and $r_{0}<r_{1}$, we have $I_{2}>b_{0}$, which leads to $\operatorname{tr} 1>0$. It follows that $E_{2}$ is unstable.


Figure 2: The bifurcation curves the palne of $\left(I, R_{0}\right)$.

## 3. Bifurcation of the System

3.1. Hopf Bifurcation. When the condition (b) (ii) in Theorem 1 and the condition (b) in Theorems 2 and 3 hold and $r_{0}=$ $r_{1}$, there are a pair of purely imaginary eigenvalues (Figure 2). Thus for suitable parameter values a Hopf bifurcation may occur, which means that there is a periodic solution around the larger nontrivial equilibrium. In order to determine the type of the Hopf bifurcation, we set

$$
\begin{gather*}
q_{1}:=\frac{\rho p^{3}\left(3 \beta b A+a^{2}-4 b\right)}{(2 b \mu p+a \rho A)^{2}}, \\
q_{2}:=\frac{-2 \beta S_{2}\left[\left(a^{2}-b\right) b I_{2}^{4}+4 a b I_{2}^{3}+6 b^{2} I_{2}^{2}-b^{3}\right]}{\left(b+a I_{2}+I_{2}\right)^{4}} . \tag{19}
\end{gather*}
$$

Then we consider the transformation $X=S-S_{2}, Y=I-I_{2}$ to move $\left(S_{2}, I_{2}\right)$ to the origin of $(X, Y)$. After some manipulations, the model can be transformed into the following equations:

$$
\begin{align*}
& \frac{d X}{d t}=a_{11} X+a_{12} Y-C(X, Y)  \tag{20}\\
& \frac{d Y}{d t}=b_{11} X+b_{12} Y+C(X, Y)
\end{align*}
$$

where $C(X, Y)$ represents the higher order terms and

$$
\left.\begin{array}{rl}
a_{11}=-d-\frac{\beta I_{2}^{3}}{b+a I_{2}+I_{2}^{2}}, & a_{12}=-\mu-\frac{\mu\left(2 b+a I_{2}\right)}{b+a I_{2}+I_{2}^{2}}, \\
b_{11}=\frac{\beta I_{2}^{3}}{b+a I_{2}+I_{2}^{2}}, & b_{12} \tag{21}
\end{array}\right) \frac{\mu\left(2 b+a I_{2}\right)}{b+a I_{2}+I_{2}^{2}} .
$$

Suppose $r_{0}=r_{1}$. Then

$$
\begin{equation*}
\operatorname{tr}(J)=-d-\frac{\beta I_{2}^{3}}{b+a I_{2}+I_{2}^{2}}+\frac{\mu\left(2 b+a I_{2}\right)}{b+a I_{2}+I_{2}^{2}}=0 \tag{22}
\end{equation*}
$$

By defining $\rho=\beta I_{2}^{3} /\left(b+a I_{2}+I_{2}^{2}\right)$ and $p=\mu\left(2 b+a I_{2}\right) /(b+$ $\left.a I_{2}+I_{2}^{2}\right)$, it can be seen that

$$
\begin{gather*}
a_{11}=-d-\rho, \quad a_{12}=-\mu-p, \quad b_{11}=\rho  \tag{23}\\
b_{12}=p, \quad d=p-\rho
\end{gather*}
$$

Set

$$
\begin{equation*}
\omega=\sqrt{\operatorname{det}\left(J_{2}\right)}=\sqrt{\mu \rho-d p} . \tag{24}
\end{equation*}
$$

Then the eigenvalues of $J_{2}$ are $\lambda_{1}=\omega i$ and $\lambda_{2}=-\omega i$.
Now, using the transformation $u=X, v=-(1 / \omega)\left(a_{11} X+\right.$ $a_{12} Y$ ) to (20), we obtain

$$
\begin{align*}
& \frac{d u}{d t}=-\omega v+F_{1}(u, v) \\
& \frac{d v}{d t}=\omega u+F_{2}(u, v) \tag{25}
\end{align*}
$$

where

$$
\begin{align*}
& F_{1}(u, v)=-C\left(u, \frac{\omega v-p u}{\mu+p}\right), \\
& F_{2}(u, v)=\frac{\mu}{\omega} C\left(u, \frac{\omega v-p u}{\mu+p}\right) . \tag{26}
\end{align*}
$$

If

$$
\begin{align*}
\sigma= & \frac{1}{16}\left[\frac{\partial^{3} F_{1}}{\partial u^{3}}+\frac{\partial^{3} F_{2}}{\partial u \partial v^{2}}+\frac{\partial^{3} F_{2}}{\partial u^{2} \partial v}+\frac{\partial^{3} F_{2}}{\partial v^{3}}\right] \\
& +\frac{1}{16 \omega}\left[\frac{\partial^{2} F_{1}}{\partial u \partial v}\left(\frac{\partial^{2} F_{1}}{\partial u^{2}}+\frac{\partial^{2} F_{1}}{\partial v^{2}}\right)-\frac{\partial^{2} F_{2}}{\partial u \partial v}\left(\frac{\partial^{2} F_{2}}{\partial u^{2}}+\frac{\partial^{2} F_{2}}{\partial v^{2}}\right)\right. \\
& \left.\quad-\frac{\partial^{2} F_{1}}{\partial u^{2}} \frac{\partial^{2} F_{2}}{\partial u^{2}}+\frac{\partial^{2} F_{1}}{\partial v^{2}} \frac{\partial^{2} F_{2}}{\partial v^{2}}\right]\left.\right|_{u=0, v=0}, \tag{27}
\end{align*}
$$

by some tedious calculations, we see that the sign of $\sigma$ is determined by $\xi$, where

$$
\begin{align*}
\xi= & A^{2} q_{1}\left(2 d \rho q_{1}+\mu p^{2}\left(2 d p-\mu \rho-4 p^{2}\right)\right) \\
& +p^{2}\left(p^{3}(\mu+d)(\mu+p)^{2}-3 q_{2} \rho A^{2} \omega^{2}\right) \tag{28}
\end{align*}
$$

By the results in [14], the direction of the Hopf bifurcation is determined by the sign of $\sigma$. Therefore, we have the following result.

Theorem 6. Suppose one condition of (c) in Theorem (6) holds and $r_{0}=r_{1}$. If $\xi \neq 0$, then a curve of periodic solutions bifurcates from the endemic equilibrium $E_{2}$ such that
(i) for $\xi<0$, system (3) undergoes a supercritical Hopf bifurcation;
(ii) for $\xi>0$, system (3) undergoes a subcritical Hopf bifurcation.

Remark 7. Theorems 5 and 6 imply the occurrence of the Allee effect because endemic equilibrium $E_{2}$ and the diseasefree equilibrium can be stable at the same time, or a stable limit cycle and the disease-free equilibrium can be stable at the same time.
3.2. Bogdanov-Takens Bifurcations. The purpose of this subsection is to study the Bogdanov-Takens bifurcation of (3) when there is a unique degenerate positive equilibrium. Assume that
(1) $a>0, R_{0}>1+\sqrt{3 \beta a / d}$ and $2 a_{1}^{3 / 2}=a_{2}$;
(2) $a=0$ and $R_{0}=1+\sqrt[3]{27 \beta^{2} b / 4 d^{2}}$;
(3) $a<0$ and $2 a_{1}^{3 / 2}=a_{2}$.

Then system (3) admits a unique positive equilibrium $\left(S^{*}, I^{*}\right)$ if one of (H1) is satisfied.

The Jacobian matrix of (3) at this point is

$$
J=\left(\begin{array}{cc}
-d-\frac{\beta I^{* 3}}{b+a I^{*}+I^{* 2}} & -\mu-\frac{\mu\left(2 b+a I^{*}\right)}{b+a I^{*}+I^{* 2}}  \tag{29}\\
\frac{\beta I^{* 3}}{b+a I^{*}+I^{* 2}} & \frac{\mu\left(2 b+a I^{*}\right)}{b+a I^{*}+I^{* 2}}
\end{array}\right)
$$

Since we are interested in codimension 2 bifurcations, we assume further
(H2) $r_{0}=r_{1}$
By (15), we have

$$
\begin{equation*}
\operatorname{det}(J)=\frac{\beta \mu I^{* 3}-d a \mu I^{*}-2 d b \mu}{b+a I^{*}+I^{* 2}}=0 \tag{30}
\end{equation*}
$$

Furthermore, (H2) implies that

$$
\begin{equation*}
\operatorname{tr}(J)=\frac{-\beta A I^{* 2}+a \mu^{2} I^{*}+2 b \mu^{2}}{\mu\left(b+a I^{*}+I^{* 2}\right)}=0 \tag{31}
\end{equation*}
$$

Thus, (H1) and (H2) imply that the Jacobian matrix has a zero eigenvalue with multiplicity 2 . This suggests that (3) may admit a Bogdanov-Takens bifurcation. The next theorem will confirm this.

Theorem 8. Suppose that (H1) and (H2) hold. Then the equilibrium $\left(S^{*}, I^{*}\right)$ of (3) is a cusp of codimension 2; that is, it is a Bogdanov-Takens singularity.

Proof. In order to translate the interior equilibrium $\left(S^{*}, I^{*}\right)$ to the origin, we set $x=S-S^{*}, y=I-I^{*}$. Expanding the
right-hand side of the system (3) in a Taylor series about the origin, we obtain

$$
\begin{align*}
& \frac{d x}{d t}=a_{11} x+a_{12} y+a_{21} x y+a_{22} y^{2}+P_{1}(x, y) \\
& \frac{d y}{d t}=-\frac{a_{11}^{2}}{a_{12}} x-a_{11} y-a_{21} x y-a_{22} y^{2}+P_{2}(x, y) \tag{32}
\end{align*}
$$

where $P_{i}(x, y)$ is a smooth function in $(x, y)$ at least of order three and

$$
\begin{gather*}
a_{11}=-d-\frac{\beta I^{* 3}}{b+a I^{*}+I^{* 2}}<0, \\
a_{12}=-\mu-\frac{\mu\left(2 b+a I^{*}\right)}{b+a I^{*}+I_{2}^{* 2}}<0, \\
a_{21}=\frac{-\beta I^{* 2}\left(3 b+2 a I^{*}+I^{* 2}\right)}{\left(b+a I^{*}+I^{* 2}\right)^{2}}<0,  \tag{33}\\
a_{22}=\frac{-2 \beta S^{*} I^{*}\left[3 b^{2}+3 a b I^{*}+\left(a^{2}-b\right) I^{* 2}\right]}{\left(b+a I^{*}+I^{* 2}\right)^{3}} .
\end{gather*}
$$

Set $X=x, Y=a_{11} x+a_{12} y$. Then (32) is transformed into

$$
\begin{align*}
& \frac{d X}{d t}=Y+c_{1} X^{2}+c_{2} X Y+c_{3} Y^{2}+Q_{1}(X, Y)  \tag{34}\\
& \frac{d Y}{d t}=-d_{1} X^{2}+d_{2} X Y+d_{3} Y^{2}+Q_{2}(X, Y)
\end{align*}
$$

where $Q_{i}$ are smooth functions in $(X, Y)$ at least of order three and

$$
\begin{gather*}
c_{1}=\frac{a_{11}\left(a_{11} a_{22}-a_{12} a_{21}\right)}{a_{12}^{2}}, \\
d_{1}=\frac{a_{11}\left(a_{12} a_{21}-a_{11} a_{22}\right)\left(a_{11}-a_{12}\right)}{a_{12}^{2}}, \\
c_{2}=\frac{a_{12} a_{21}-2 a_{11} a_{22}}{a_{12}^{2}},  \tag{35}\\
d_{2}=\frac{\left(a_{12} a_{21}-2 a_{11} a_{22}\right)\left(a_{11}-a_{12}\right)}{a_{12}^{2}}, \\
c_{3}=\frac{a_{22}}{a_{12}^{2}}, \\
d_{3}=\frac{a_{22}\left(a_{11}-a_{12}\right)}{a_{12}^{2}}
\end{gather*}
$$

Change the variables one more time by letting $X=X, P=$ $Y+c_{3} Y^{2}$; we have

$$
\begin{gather*}
\frac{d X}{d t}=P+c_{1} X^{2}+c_{2} X Y+Q_{3}(X, P)  \tag{36}\\
\frac{d P}{d t}=-d_{1} X^{2}+d_{2} X P+d_{3} P^{2}+Q_{4}(X, P)
\end{gather*}
$$

Let $X=X, Z=P-d_{3} X P$. Then system (36) becomes

$$
\begin{gather*}
\frac{d X}{d t}=Z+c_{1} X^{2}+\left(c_{2}+d_{3}\right) X Z+Q_{5}(X, Z)  \tag{37}\\
\frac{d Z}{d t}=-d_{1} X^{2}+d_{2} X Z+Q_{6}(X, Z)
\end{gather*}
$$

In order to obtain the canonical normal forms, we perform the transformation of variables by

$$
\begin{equation*}
u=X-\frac{c_{2}+d_{3}}{2} X^{2}, \quad v=Z+c_{1} X^{2} \tag{38}
\end{equation*}
$$

Then, we obtain

$$
\begin{gather*}
\frac{d u}{d t}=v+R_{1}(u, v)  \tag{39}\\
\frac{d v}{d t}=-d_{1} u^{2}+\left(d_{2}+2 c_{1}\right) u v+R_{2}(u, v)
\end{gather*}
$$

where $R_{i}$ are smooth functions in $(u, v)$ at least of the third order.

Note that $d_{1}>0$ and

$$
\begin{equation*}
d_{2}+2 c_{1}=\frac{-a_{11} a_{21}-a_{21} a_{12}+2 a_{22} a_{11}}{a_{12}} \tag{40}
\end{equation*}
$$

In addition, by (30) and (31), it is obtained that

$$
\begin{gather*}
a_{11}=-\frac{A}{S^{*}}, \quad a_{12}=-\mu-\frac{A}{S^{*}}, \\
a_{21}=\frac{\mu I^{*}}{S^{*}}, \quad a_{22}=\frac{A}{S^{*}},  \tag{41}\\
\frac{A^{2}}{S^{* 2}}=\frac{\mu I^{*}}{S^{*}}\left(\mu+\frac{A}{S^{*}}\right) .
\end{gather*}
$$

So

$$
\begin{align*}
d_{2}+2 c_{1} & =\frac{-a_{11} a_{21}-a_{21} a_{12}+2 a_{22} a_{11}}{a_{12}} \\
& =\frac{1}{a_{12}} \frac{A\left(\mu I^{*}-A\right)}{S^{* 2}}>0 \tag{42}
\end{align*}
$$

It follows that (3) admits that a Bogdanov-Takens bifurcation from [15, 16] or [17].

## 4. Simulations and Conclusions

In the following, we use numerical simulations, based upon the MatCont package [18], to reveal how parameters $a$ induce bifurcations and limit cycles in system (3). Firstly, by fixing $A=2, d=0.1, \beta=0.8, b=2.4, \epsilon=0.6, \gamma=0.2$, we plot a 2D-plot of variable $I$ versus parameter $a$ shown in Figure 3. We find a Hopf bifurcation at $a=-0.090429$, a limit point (fold) bifurcation at $a=2.378982$. The Lyapunov coefficient is $1.68171 \times 10^{-2}$, which means that the periodic orbits are unstable. Furthermore, $a$ is fixed -0.13 ; we observe the orbits of system (3) is how to vary with $t$. From Figure 4, we can find


Figure 3: Bifurcation curves in ( $a, I$ ) plane by fixed $A=2, d=$ $0.1, \beta=0.8, b=2.4, \epsilon=0.6, \gamma=0.2$, where H denotes the Hopf bifurcation, LP is the limit point (flod) bifurcation.


Figure 4: Phase trajectory in system (3) by fixed $A=2, d=0.1, \beta=$ $0.8, b=2.4, \epsilon=0.6, \gamma=0.2, a=-0.13$.
that the periodic orbits will occur, but the disease will die out when $t \rightarrow+\infty$, though there exist the positive equilibria for system (3). Furthermore, we take $R_{0}$ and $a$ as bifurcation parameters; from Figure 5, we can show that the system has no positive equilibrium when $R_{0}$ and $a$ lie in the left side of red curve and two endemic equilibria when they are in the right side of red curve. If parameters $R_{0}$ and $a$ are between red and green curves, we find that system will undergo Hopf bifurcation.

In the paper, we built a model with contact transmission function and obtained the dynamical behaviors. From the analysis, we find that the threshold value of disease spreading will be larger. It means that it is an effective intervention policy adopted for disease spreading. For the disease-free equilibrium is always locally stable and when a positive equilibrium exist and is stable, we can control the disease as long as we let the initial values be in the certain range by intervention policy. If the positive equilibrium is unstable, the


Figure 5: Bifurcation figure when $R_{0}$ and $a$ are taken as bifurcation parameters in system (3) by fixed $A=2, d=0.1, \beta=0.8, b=2.4, \epsilon=$ $0.6, \gamma=0.2$.
disease will die out. This will provide a theoretical basis for the prevention and control of disease.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work is supported by the National Science Foundation of China $(11201434,11271369)$ and Shanxi Scholarship Council of China (2013-087).

## References

[1] V. Capasso and G. Serio, "A generalization of the KermackMcKendrick deterministic epidemic model," Mathematical Biosciences, vol. 42, no. 1-2, pp. 43-61, 1978.
[2] S. Ruan and W. Wang, "Dynamical behavior of an epidemic model with a nonlinear incidence rate," Journal of Differential Equations, vol. 188, no. 1, pp. 135-163, 2003.
[3] Y. Tang, D. Huang, S. Ruan, and W. Zhang, "Coexistence of limit cycles and homoclinic loops in a SIRS model with a nonlinear incidence rate," SIAM Journal on Applied Mathematics, vol. 69, no. 2, pp. 621-639, 2008.
[4] D. Xiao and H. Zhu, "Multiple focus and Hopf bifurcations in a predator-prey system with nonmonotonic functional response," SIAM Journal on Applied Mathematics, vol. 66, no. 3, pp. 802819, 2006.
[5] W. Wang, "Epidemic models with nonlinear infection forces," Mathematical Biosciences and Engineering, vol. 3, no. 1, pp. 267279, 2006.
[6] D. Xiao and S. Ruan, "Global analysis of an epidemic model with nonmonotone incidence rate," Mathematical Biosciences, vol. 208, no. 2, pp. 419-429, 2007.
[7] A. B. Gumel and S. M. Moghadas, "A qualitative study of a vaccination model with non-linear incidence," Applied Mathematics and Computation, vol. 143, no. 2-3, pp. 409-419, 2003.
[8] W. M. Liu, H. W. Hethcote, and S. A. Levin, "Dynamical behavior of epidemiological models with nonlinear incidence rates," Journal of Mathematical Biology, vol. 25, no. 4, pp. 359380, 1987.
[9] W. M. Liu, S. A. Levin, and Y. Iwasa, "Influence of nonlinear incidence rates upon the behavior of SIRS epidemiological models," Journal of Mathematical Biology, vol. 23, no. 2, pp. 187204, 1986.
[10] R. R. Regoes, D. Ebert, and S. Bonhoeffer, "Dose-dependent infection rates of parasites produce the Allee effect in epidemiology," Proceedings of the Royal Society B, vol. 269, no. 1488, pp. 271-279, 2002.
[11] G. Li and W. Wang, "Bifurcation analysis of an epidemic model with nonlinear incidence," Applied Mathematics and Computation, vol. 214, no. 2, pp. 411-423, 2009.
[12] R. M. Anderson and R. M. May, "Population biology of infectious diseases: Part I," Nature, vol. 280, no. 5721, pp. 361367, 1979.
[13] R. M. May and R. M. Anderson, "Population biology of infectious diseases: Part II," Nature, vol. 280, no. 5722, pp. 455461, 1979.
[14] L. Perko, Differential Equations and Dynamical Systems, vol. 7, Springer, New York, NY, USA, 2nd edition, 1996.
[15] R. Bogdanov, "Bifurcations of a limit cycle for a family of vector fields on the plan," Selecta Mathematica, vol. 1, pp. 373-388, 1981.
[16] R. Bogdanov, "Versal deformations of a singular point on the plan in the case of zero eigenvalues," Selecta Mathematica, vol. 1, pp. 389-421, 1981.
[17] F. Takens, "Forced oscillations and bifurcations," in Applications of Global Analysis I, pp. 1-59, Rijksuniversitat Utrecht, 1974.
[18] A. Dhooge, W. Govaerts, and Y. A. Kuznetsov, "Limit cycles and their bifucations in MatCont," http://www.matcont.ugent.be/ Tutorial2.pdf.

## Research Article

# Modelling the Drugs Therapy for HIV Infection with Discrete-Time Delay 

Xueyong Zhou ${ }^{1}$ and Xiangyun Shi ${ }^{1,2}$<br>${ }^{1}$ College of Mathematics and Information Science, Xinyang Normal University, Xinyang, Henan 464000, China<br>${ }^{2}$ College of Forest, Beijing Forest University, Beijing 100083, China

Correspondence should be addressed to Xueyong Zhou; xueyongzhou@126.com
Received 2 October 2013; Accepted 3 November 2013; Published 16 January 2014
Academic Editor: Weiming Wang
Copyright © 2014 X. Zhou and X. Shi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A discrete-time-delay differential mathematical model that described HIV infection of $\mathrm{CD} 4{ }^{+} \mathrm{T}$ cells with drugs therapy is analyzed. The stability of the two equilibria and the existence of Hopf bifurcation at the positive equilibrium are investigated. Using the normal form theory and center manifold argument, the explicit formulas which determine the stability, the direction, and the period of bifurcating periodic solutions are derived. Numerical simulations are carried out to explain the mathematical conclusions.

## 1. Introduction

Recently there has been a substantial effort in the mathematical modelling of virus dynamics [1-8]. These models focus on uninfected target cells, infected cells that are producing virus, and virus. A basic mathematical model describing HIV infection dynamic model is of the following form which has been studied in [5, 9]:

$$
\begin{align*}
& \frac{d T(t)}{d t}=s-\mu_{1} T(t)-k T(t) V(t) \\
& \frac{d I(t)}{d t}=k T(t) V(t)-\delta I(t)  \tag{1}\\
& \frac{d V(t)}{d t}=N \delta I(t)-c V(t)
\end{align*}
$$

In system (1), the following variables are includes: $T(t)$ uninfected cells at time $t$ (unit is cells $\mathrm{mm}^{-3}$ ), $I(t)$ infected cells at time $t$ (unit is cells $\mathrm{mm}^{-3}$ ), and $V(t)$ virus at time $t$ (unit is virions $\mathrm{mm}^{-3}$ ). Parameters $\mu_{1}, \delta$, and $c$ are the death rates of the uninfected $T$ cells, the infected $T$ cells, and the virus particles, respectively. $k$ is the contact rate between uninfected $T$ cells and the virus particles. $N$ is the average number of virus particles produced by an infected $T$ cell.

Reverse transcriptase inhibitors (RTIs) are a class of antiretroviral drugs used to treat HIV infection. RTIs
inhibitors work by inhibiting the action of reverse transcriptase. RTIs inhibit the activity of reverse transcriptase, a viral DNA polymerase enzyme that retroviruses need to reproduce. In [10], Srivastava et al. developed a mathematical model for primary infection with RTIs. They subdivided the infected cells class in two subclasses: pre-RT (denoted by $I_{1}(t)$ ) and post-RT (denoted by $I_{2}(t)$ ). They assumed that a virus enters a resting $\mathrm{CD} 4^{+} \mathrm{T}$ cell, the viral RNA may not be completely reverse transcribed into DNA, the unintegrated virus may decay with time and partial DNA transcripts are labile and degrade quickly [11, 12]. And they also assumed that a fraction of cells $\eta a I_{1}(t)$ in pre-RT class reverts back to uninfected class and the remaining $(1-\eta) a I_{1}(t)$ proceeds to post-RT class and becomes productively infected due to presence of RT inhibitors. The model of Srivastava et al. is as follows

$$
\begin{align*}
\frac{d T(t)}{d t} & =s-\mu_{1} T(t)-k T(t) V(t)+(\eta a+b) I_{1}(t) \\
\frac{d I_{1}(t)}{d t} & =k T(t) V(t)-(d+a+b) I_{1}(t) \\
\frac{d I_{2}(t)}{d t} & =(1-\eta) a I_{1}(t)-\delta I_{2}(t) \\
\frac{d V(t)}{d t} & =N \delta I_{2}(t)-c V(t) \tag{2}
\end{align*}
$$

where $0<\eta<1$ is the efficacy of reverse transcriptase inhibitors (RTIs), $a$ is the transition rate from pre-RT (i.e., $\left.I_{1}(t)\right)$ infected $\mathrm{CD} 4^{+} \mathrm{T}$ cells class to productively post-RT (i.e., $I_{2}(t)$ ) which is a productively infected class, and $b$ is the reverting rate of infected cells to uninfected class due to noncompletion of reverse transcription [11, 12].

Protease inhibitors (PIs) are a class of drugs used to treat or prevent infection by viruses, including HIV and hepatitis C. PIs prevent viral replication by inhibiting the activity of HIV-1 protease, an enzyme used by the viruses to cleave nascent proteins for final assembly of new virus. The new virous are noninfectious. Virions that were created prior to drug treatment remain infectious. Thus, in the presence of a protease inhibitor, two types of virus particles (i.e., infectious virions and noninfectious virions) should be considered [5]. We need the drug to be highly effective if we use single drug to treat. Hence, combination anti-HIV therapy is now the standard of care for people with HIV. So far as we know, there are few mathematical models about the effects of combination anti-HIV therapy [7,13]. Therefore, considering the effects of both RTIs and PIs, model (2) can be modified to

$$
\begin{align*}
& \frac{d T(t)}{d t}=s-\mu_{1} T(t)-k T(t) V_{1}(t)+(\eta a+b) I_{1}(t) \\
& \frac{d I_{1}(t)}{d t}=k T(t) V_{1}(t)-(d+a+b) I_{1}(t) \\
& \frac{d I_{2}(t)}{d t}=(1-\eta) a I_{1}(t)-\delta I_{2}(t)  \tag{3}\\
& \frac{d V_{1}(t)}{d t}=(1-p) N \delta I_{2}(t)-c V_{1}(t) \\
& \frac{d V_{2}(t)}{d t}=p N \delta I_{2}(t)-c V_{2}(t)
\end{align*}
$$

where variables $V_{1}(t)$ and $V_{2}(t)$ denote infectious and noninfectious virus at time $t$, respectively. And $V(t)=V_{1}(t)+$ $V_{2}(t)$ is the total virus concentration at time $t$. Parameter $p \in$ $[0,1]$ denotes the effectiveness of PIs with $p=1$ meaning that the therapy with PIs is $100 \%$ effective and no newly infectious virus particles will be produced [5].

In the real situation, there may be a delay between the time target cells which are contacted by the virus particles and the time the contacted cells become actively affected meaning that the contacting virions enter cells. Hence, time delays of one type or another have been incorporated into viral dynamical models by many authors. The first model that included this type "intracellular" delay was developed by Herz et al. [14] and assumed that cells became productively infected time units after HIV initial infection. Nelson et al. [15] extend the development of delay models of HIV infection and treatment to the general case of combination antiviral therapy that is less than completely efficacious. Recently, in studying the viral clearance rates, Perelson et al. [9] assumed that there are two types of delays that occur between the administration of drug and the observed decline in viral load: a pharmacological delay that occurs between the ingestion of drug and its appearance within cells and an intracellular delay that is between initial infection of a cell by HIV and the release
of new virions. Furthermore, the growth of $\mathrm{CD} 4^{+} \mathrm{T}$ cells in humans is not well understood.

Recently, studies in various fields such as biology, control, economy, chemistry, and electrodynamics have shown that delay differential equations play an important role in explaining many different phenomena [16-20]. Srivastava et al. [10] proposed and analyzed a mathematical model for the effect of RTIs on the dynamics of HIV. In [21], Culshaw and Ruan have considered that the basic model of HIV infection in host was extended to incorporate logistic growth and an intracellular delay. However, none of these models have incorporated antiretroviral therapy, logistic growth of the $\mathrm{CD} 4^{+} \mathrm{T}$ cell, and intracellular delay. Here, we build on the basic model of HIV pathogenesis in host, adding the effects of antiretroviral therapy, logistic growth of the $\mathrm{CD} 4^{+} \mathrm{T}$ cell, and intracellular delay. Hence, we can obtain the following model:

$$
\begin{align*}
\frac{d T(t)}{d t}= & s+r T(t)\left(1-\frac{T(t)}{T_{\max }}\right)-\mu_{1} T(t) \\
& -k T(t) V_{1}(t)+(\eta a+b) I_{1}(t) \\
\frac{d I_{1}(t)}{d t}= & k T(t-\tau) V_{1}(t-\tau)-(d+a+b) I_{1}(t) \\
\frac{d I_{2}(t)}{d t}= & (1-\eta) a I_{1}(t)-\delta I_{2}(t)  \tag{4}\\
\frac{d V_{1}(t)}{d t}= & (1-p) N \delta I_{2}(t)-c V_{1}(t) \\
\frac{d V_{2}(t)}{d t}= & p N \delta I_{2}(t)-c V_{2}(t)
\end{align*}
$$

In model (4), $T(t), I_{1}(t), I_{2}(t), V_{1}(t)$, and $V_{2}(t)$ represent the density of susceptible $\mathrm{CD} 4^{+} \mathrm{T}$ cells, infected $\mathrm{CD} 4^{+} \mathrm{T}$ cells before reverse transcription (i.e., those infected cells which are in pre-RT class), infected $\mathrm{CD} 4^{+} \mathrm{T}$ cells in which reverse transcription is completed (post-RT class), infectious virus, and noninfectious virus at time $t$, respectively. The meaning of the parameters are as follows: $s$ is the source term for uninfected $\mathrm{CD} 4^{+} \mathrm{T}$ cell, $k$ is the rate at which $\mathrm{CD} 4^{+} \mathrm{T}$ cell becomes infected with virus, $\mu_{1}$ is the death rate of healthy $\mathrm{CD} 4^{+} \mathrm{T}$ cell, $\eta$ is the efficacy of RTIs, $a$ is the transition rate from pre-RT infected $\mathrm{CD} 4^{+} \mathrm{T}$ cells to productively post-RT, $b$ is the reverting rate of infected cells to uninfected class, $d$ is the death rate of infected $T$ cells, $\delta$ is the death rate of actively infected $T$ cells $I_{2}, N$ is the number of virions produced by infected $\mathrm{CD} 4^{+} \mathrm{T}$ cells, $c$ is the clearance rate of virus, $r$ is the maximum proliferation rate, $T_{\max }$ is the $T$ cell population density at which proliferation shuts off, $p$ is the efficacy of protease inhibitor, and $\tau$ is the "intracellular" delay.

Note that the non-infectious HIV virus $V_{2}(t)$ does not appear in the first four equations of system (4). Thus, we can consider the following subsystem of system (4):

$$
\begin{aligned}
\frac{d T(t)}{d t}= & s+r T(t)\left(1-\frac{T(t)}{T_{\max }}\right)-\mu_{1} T(t)-k T(t) V_{1}(t) \\
& +(\eta a+b) I_{1}(t)
\end{aligned}
$$

$$
\begin{gather*}
\frac{d I_{1}(t)}{d t}=k T(t-\tau) V_{1}(t-\tau)-(d+a+b) I_{1}(t) \\
\frac{d I_{2}(t)}{d t}=(1-\eta) a I_{1}(t)-\delta I_{2}(t) \\
\frac{d V_{1}(t)}{d t}=  \tag{5}\\
(1-p) N \delta I_{2}(t)-c V_{1}(t)
\end{gather*}
$$

In this paper, we will discuss the dynamics of model (5). This paper is organized as follows. In Section 2, we present some preliminaries about system (5), for example, the positivity of solutions and the expression of equilibria. We discuss the local stability of the uninfected equilibrium in Section 3. In Section 4, we discuss the local stability and Hopf bifurcation at the infected equilibrium. In Section 5, the direction and stability of the local Hopf bifurcation are established. In Section 6, some numerical simulations are performed to illustrate the analytical results found. A brief discussion is presented in the last section.

## 2. Preliminaries

System (5) is a system of delay differential equations. For such a system, initial functions need to be specified and wellposedness needs to be addressed. We denote by $\mathbb{C}$ the Banach space of continuous functions $\varphi:[-\tau, 0] \rightarrow \mathbb{R}^{4}$ with norm

$$
\begin{equation*}
\|\varphi\|=\sup _{-\tau \leq \varsigma \leq 0}\left\{\left|\varphi_{1}(\varsigma)\right|,\left|\varphi_{2}(\varsigma)\right|,\left|\varphi_{3}(\varsigma)\right|,\left|\varphi_{4}(\varsigma)\right|\right\} \tag{6}
\end{equation*}
$$

where $\varphi=\left(\varphi_{1}, \varphi_{2}, \varphi_{3}, \varphi_{4}\right)$. As usual, the initial condition of (5) is given as

$$
\begin{gather*}
T(\varsigma)=\varphi_{1}(\varsigma), \quad I_{1}(0)=\varphi_{2}(0), \quad I_{2}(0)=\varphi_{3}(0) \\
V_{1}(\varsigma)=\varphi_{4}(\varsigma), \quad \varsigma \in[-\tau, 0] \tag{7}
\end{gather*}
$$

where the initial function $\varphi=\left(\varphi_{1}, \varphi_{2}, \varphi_{3}, \varphi_{4}\right)$ belongs to the Banach space $\mathbb{C}=\mathbb{C}\left([-\tau, 0], \mathbb{R}^{4}\right)$ of continuous functions mapping the initial $[-\tau, 0]$ into $\mathbb{R}^{4}$. For biological reasons, the initial functions are assumed as

$$
\begin{gather*}
T(\varsigma)=\varphi_{1}(\varsigma) \geq 0 \\
\varsigma \in[-\tau, 0], \quad \varphi_{1}(0)>0 ; \\
I_{1}(0)=\varphi_{2}(0)>0 ;  \tag{8}\\
I_{2}(0)=\varphi_{3}(0)>0 ; \\
V_{1}(\varsigma)=\varphi_{4}(\varsigma) \geq 0, \quad \varsigma \in[-\tau, 0], \quad \varphi_{4}(0)>0 .
\end{gather*}
$$

In this paper, we will discuss the dynamical behavior of system (5) with the initial conditions in (8). By the fundamental theory of functional differential equations [22], we know that there is a unique solution $\left(T(t), I_{1}(t), I_{2}(t), V_{1}(t)\right)$ to system (5) with initial conditions (8).

Firstly, we present the positivity of the solutions. System (5) can be put into the matrix form

$$
\begin{equation*}
\dot{X}(t)=G(X(t)), \tag{9}
\end{equation*}
$$

where $X(t)=\left(T(t), I_{1}(t), I_{2}(t), V_{1}(t)\right)^{\top} \in \mathbb{R}^{4}$ and $G(X(t))$ is given by

$$
\begin{align*}
& G(X(t)) \\
& =\left(\begin{array}{c}
G_{1}(X(t)) \\
G_{2}(X(t)) \\
G_{3}(X(t)) \\
G_{4}(X(t))
\end{array}\right) \\
& =\left(\begin{array}{c}
s+r T(t)\left(1-\frac{T(t)}{T_{\max }}\right)-\mu_{1} T(t)-k T(t) V_{1}(t)+(\eta a+b) I_{1}(t) \\
k T(t-\tau) V_{1}(t-\tau)-(d+a+b) I_{1}(t) \\
(1-\eta) a I_{1}(t)-\delta I_{2}(t) \\
(1-p) N \delta I_{2}(t)-c V_{1}(t)
\end{array}\right) . \tag{10}
\end{align*}
$$

Let $\mathbb{R}_{+}^{4}=[0,+\infty) \times[0,+\infty) \times[0,+\infty) \times[0,+\infty)$ be the nonnegative octant in $\mathbb{R}^{4} ; G: \mathbb{R}_{+}^{4+1} \rightarrow \mathbb{R}^{4}, G \in \mathbb{C}^{\infty}\left(\mathbb{R}^{4}\right)$ (where $G$ is a function of the variable $X(t) \in \mathbb{R}_{+}^{4}$ ) is locally Lipschitz and satisfies the condition

$$
\begin{equation*}
\left.G_{i}(X(t))\right|_{x_{i}(t)=0, X(t) \in \mathbb{R}_{+}^{4}} \geq 0 \tag{11}
\end{equation*}
$$

where $x_{1}(t)=T(t), x_{2}(t)=I_{1}(t), x_{3}(t)=I_{2}(t)$, and $x_{4}(t)=$ $V_{1}(t)$.

Due to lemma in [23] any solution of (9) with $X(\varsigma) \in \mathbb{C}_{+}$, say $X(t)=X(t, X(\varsigma))$, is such that $X(t) \in \mathbb{R}_{+}^{4}$ for all $t \geq 0$.

System (5) has an uninfected (boundary) equilibrium and an infected (positive) steady state. The uninfected equilibrium is $E_{0}\left(T_{0}, 0,0,0\right)$, where

$$
\begin{equation*}
T_{0}=\frac{T_{\max }}{2 r}\left[r-\mu_{1}+\sqrt{\left(r-\mu_{1}\right)^{2}+\frac{4 r s}{T_{\max }}}\right] \tag{12}
\end{equation*}
$$

The infected equilibrium is $E^{*}\left(T^{*}, I_{1}^{*}, I_{2}^{*}, V_{1}^{*}\right)$, where

$$
\begin{gather*}
T^{*}=\frac{c(d+a+b)}{(1-p)(1-\eta) k N a}, \\
I_{1}^{*}=\frac{1}{d+(1-\eta) a}\left[s-d T^{*}+r T^{*}\left(1-\frac{T^{*}}{T_{\max }}\right)\right], \\
I_{2}^{*}=\frac{(1-\eta) a}{\delta[d+(1-\eta) a]}\left[s-d T^{*}+r T^{*}\left(1-\frac{T^{*}}{T_{\max }}\right)\right], \\
V_{1}^{*}=\frac{(1-p)(1-\eta) N a}{c[d+(1-\eta) a]}\left[s-d T^{*}+r T^{*}\left(1-\frac{T^{*}}{T_{\max }}\right)\right] . \tag{13}
\end{gather*}
$$

The basic reproductive number is given as $\mathscr{R}_{0}=T_{0} / T^{*}$. The basic reproductive number $\mathscr{R}_{0}$ measures the average number virus-producing target cells produced by an single virus-producing target cell during its entire infectious period in an entirely uninfected targeT cell population [24, 25]. It is easy to see that $\mathscr{R}_{0}>1$ ensures the existence of the infected equilibrium $E^{*}$.

## 3. Stability of Uninfected Equilibrium $E_{0}$

In this section, we will discuss the stability of the uninfected equilibrium $E_{0}\left(T_{0}, 0,0,0\right)$.

Let $\bar{E}\left(\bar{T}, \bar{I}_{1}, \bar{I}_{2}, \bar{V}_{1}\right)$ be any arbitrary equilibrium. To study the stability of the steady state $\bar{E}$, let us define

$$
\begin{array}{cc}
x(t)=T(t)-\bar{T}, & y_{1}(t)=I_{1}(t)-\bar{I}_{1}  \tag{14}\\
y_{2}(t)=I_{2}(t)-\bar{I}_{2}, & z(t)=V_{1}(t)-\bar{V}_{1}
\end{array}
$$

Then, the linearized system of (5) around the equilibrium $\bar{E}$ is given by

$$
\frac{d}{d t}\left(\begin{array}{c}
x(t)  \tag{15}\\
y_{1}(t) \\
y_{2}(t) \\
z(t)
\end{array}\right)=A_{1}\left(\begin{array}{c}
x(t) \\
y_{1}(t) \\
y_{2}(t) \\
z(t)
\end{array}\right)+A_{2}\left(\begin{array}{c}
x(t-\tau) \\
y_{1}(t-\tau) \\
y_{2}(t-\tau) \\
z(t-\tau)
\end{array}\right)
$$

where $A_{1}$ and $A_{2}$ are $4 \times 4$ matrices given by

$$
\begin{gather*}
A_{1}=\left(\begin{array}{cccc}
-\mu_{1}+r-\frac{2 r \bar{T}}{T_{\max }}-k \bar{V}_{1} & \eta a+b & 0 & -k \bar{T} \\
0 & -a-b-d & 0 & 0 \\
0 & & (1-\eta) a & -\delta \\
0 & 0 & (1-p) N \delta & -c
\end{array}\right), \\
A_{2}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
k \bar{V}_{1} & 0 & 0 & k \bar{T} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) . \tag{16}
\end{gather*}
$$

Hence, the characteristic equation of system (5) at $\bar{E}$ is given by

$$
\begin{equation*}
\operatorname{det}\left(A+B e^{-\lambda \tau}-\lambda \mathscr{I}\right)=0 \tag{17}
\end{equation*}
$$

where $\mathscr{F}$ is a $4 \times 4$ identity matrix that is,

$$
\left|\begin{array}{cccc}
-\mu_{1}+r-\frac{2 r \bar{T}}{T_{\max }}-k \bar{V}_{1}-\lambda & \eta a+b & 0 & -k \bar{T}  \tag{18}\\
k \bar{V}_{1} e^{-\lambda \tau} & -a-b-d-\lambda & 0 & k \bar{T} e^{-\lambda \tau} \\
0 & (1-\eta) a & -\delta-\lambda & 0 \\
0 & 0 & (1-p) N \delta & -c-\lambda
\end{array}\right|=0 .
$$

Theorem 1. (1) If $\mathscr{R}_{0}<1, E_{0}$ is locally asymptotically stable for any time delay $\tau \geq 0$. (2) If $\mathscr{R}_{0}>1, E_{0}$ is unstable for any time delay $\tau \geq 0$. (3) If $\mathscr{R}_{0}=1$, it is a critical case.

Proof. For uninfected equilibrium $E_{0}$, (18) reduces to

$$
\begin{equation*}
\left(r-\mu_{1}-\frac{2 r T_{0}}{T_{\max }}-\lambda\right)\left[\lambda^{3}+b_{1} \lambda^{2}+b_{2} \lambda+b_{3}+c_{3} e^{-\lambda \tau}\right]=0 \tag{19}
\end{equation*}
$$

where

$$
\begin{align*}
& b_{1}=a+b+d+\delta+c \\
& b_{2}=(a+b+d)(c+\delta)+c \delta \\
& b_{3}=(a+b+d) c \delta  \tag{20}\\
& c_{3}=-(1-p)(1-\eta) a k N \delta T_{0}
\end{align*}
$$

It is clear that (19) has the characteristic root $\lambda_{1}=r-\mu_{1}-$ $\left(2 r T_{0} / T_{\max }\right)=-\sqrt{\left(r-\mu_{1}\right)^{2}+4 r s / T_{\max }}<0$.

Next, we will consider the transcendental polynomial

$$
\begin{equation*}
\lambda^{3}+b_{1} \lambda^{2}+b_{2} \lambda+b_{3}+c_{3} e^{-\lambda \tau}=0 \tag{21}
\end{equation*}
$$

For $\tau=0$, we have that

$$
\begin{equation*}
\lambda^{3}+b_{1} \lambda^{2}+b_{2} \lambda+b_{3}+c_{3}=0 \tag{22}
\end{equation*}
$$

Obviously, $b_{1}>0, b_{2}>0$, and $b_{3}+c_{3}>0$ since $\mathscr{R}_{0}<1$. We also get

$$
\begin{align*}
b_{1} b_{2}-\left(b_{3}+c_{3}\right)= & (a+b+d)^{2}(c+\delta) \\
& +(a+b+d)(c+\delta)^{2}+c \delta(c+\delta)  \tag{23}\\
& +(1-p)(1-\eta) a k N \delta T_{0}>0
\end{align*}
$$

This shows that all the roots of (22) have negative real parts for $\tau=0$ by using Routh-Hurwitz theorem.

In the following, we investigate the existence of purely imaginary roots $\lambda=i \omega, \omega>0$, of (21). If $\tau>0$ and $\lambda=i \omega$ with $\omega>0$ is a solution of (21), then separating the real and imaginary parts gives

$$
\begin{align*}
& \omega^{3}-b_{2} \omega=-c_{3} \sin (\omega \tau)  \tag{24}\\
& b_{1} \omega^{2}-b_{3}=c_{3} \cos (\omega \tau)
\end{align*}
$$

Squaring and adding both equations of (24) yields

$$
\begin{equation*}
f(\omega, \tau)=\omega^{6}+m_{1} \omega^{4}+m_{2} \omega^{2}+b_{3}^{2}-c_{3}^{2}=0 \tag{25}
\end{equation*}
$$

where

$$
\begin{align*}
& m_{1}=(a+b+d)^{2}+c^{2}+\delta^{2}>0, \\
m_{2}= & (a+b+d)^{2}\left(c^{2}+\delta^{2}\right)+(a+b+d)^{2} c \delta  \tag{26}\\
& +(c \delta)^{2}+c \delta(c+\delta)(a+b+d)>0 .
\end{align*}
$$

Letting $y=\omega^{2}$ yields

$$
\begin{equation*}
y^{3}+m_{1} y^{2}+m_{2} y+b_{3}^{2}-c_{3}^{2}=0 \tag{27}
\end{equation*}
$$

If $\mathscr{R}_{0}<1$, then $b_{3}^{2}-c_{3}^{2}>0$. Therefore, by claim 1 in [21], it is evident that (27) has no positive real roots. This shows that (21) cannot have a purely imaginary root for any $\tau>0$. Therefore, the uninfected equilibrium $E_{0}$ is locally asymptotically stable for any $\tau \geq 0$ provided that $\mathscr{R}_{0}<1$.

If $\mathscr{R}_{0}=1$, the transcendental polynomial (21) becomes

$$
\begin{equation*}
\lambda^{3}+b_{1} \lambda^{2}+b_{2} \lambda+b_{3}-b_{3} e^{-\lambda \tau}=0 \tag{28}
\end{equation*}
$$

It is clear that $\lambda=0$ is a simple root of (28). We further show that any root of (28) must have negative real part except $\lambda=0$.

In fact, if (28) has imaginary root $u \pm i \omega$ for some $u \geq 0$, $\omega \geq 0$, and $\tau \geq 0$, from (28) we have

$$
\begin{gather*}
u^{3}-3 u \omega^{2}+b_{1} u^{2}-b_{1} \omega^{2}+b_{2} u+b_{3}=b_{3} e^{-u \tau} \cos (\omega \tau) \\
-\omega^{3}+3 u^{2} \omega+b_{2} \omega+2 b_{1} u \omega=-b_{3} e^{-u \tau} \sin (\omega \tau) \tag{29}
\end{gather*}
$$

which, together with $u \geq 0$, implies that

$$
\begin{align*}
& {\left[u^{3}-3 u \omega^{2}+b_{1} u^{2}-b_{1} \omega^{2}+b_{2} u+b_{3}\right]^{2}} \\
& \quad+\left[-\omega^{3}+3 u^{2} \omega+b_{2} \omega+2 b_{1} u \omega\right]^{2}  \tag{30}\\
& \quad=b_{3}^{2} e^{-2 u \tau} \leq b_{3}^{2} .
\end{align*}
$$

However, it is easy to check that the previous inequality is not true. Hence, it shows that any root of (28) has negative real part except $\lambda=0$, which implies that the trivial solution of (5) is stable for any time delay $\tau \geq 0$.

If $\mathscr{R}_{0}>1$, let

$$
\begin{equation*}
f(\lambda)=\lambda^{3}+b_{1} \lambda^{2}+b_{2} \lambda+b_{3}+c_{3} e^{-\lambda \tau}=0 \tag{31}
\end{equation*}
$$

Note that $f(0)=b_{3}+c_{3}<0$ since $R_{0}>1$ and $\lim _{\lambda \rightarrow+\infty} f(\lambda)=$ $+\infty$. It follows from the continuity of the function $f(\lambda)$ on $(-\infty,+\infty)$ that equation $f(\lambda)=0$ has at least one positive root. Hence, characteristic equation (19) has at least one positive. Thus, $E_{0}$ is unstable. Therefore, our results in this theorem are proved.

## 4. Dynamical Behavior of Endemic Equilibrium $E^{*}$

In general, the nonlinear delay system will undergo a Hopf bifurcation when the delay passes through a critical value of the delay, for which the stability of the existing equilibrium changes from stable status to unstable status and a self-excited limit cycle emerges at this moment. Under certain conditions, the existence of a Hopf bifurcation can be determined from linear stability analysis; it requires that at the bifurcation point, the characteristic function has exactly one pair of conjugate roots on the imaginary axis, and as the delay passes through the bifurcation point, this pair of characteristic roots cross from the left-half complex plane to the right-half complex plane or vice verse [19, 26]. The crossing direction is the same as that mentioned previously in linear stability analysis. Thus, the determination of the crossing direction is very important for both stability analysis and Hopf bifurcation. In this section, we will consider the dynamical behavior of endemic equilibrium $E^{*}$. Some conditions for Hopf bifurcation around equilibrium $E^{*}$ to occur are obtained by using the time delay $\tau$ as a bifurcation parameter.

For endemic equilibrium $E^{*}\left(T^{*}, I_{1}^{*}, I_{2}^{*}, V_{1}^{*}\right),(18)$ reduces to

$$
\left|\begin{array}{cccc}
-\mu_{1}+r-\frac{2 r T^{*}}{T_{\max }}-k V_{1}^{*}-\lambda & \eta a+b & 0 & -k T^{*}  \tag{32}\\
k V_{1}^{1} e^{-\lambda \tau} & -a-b-d-\lambda & 0 & k T^{*} e^{-\lambda \tau} \\
0 & a(1-\eta) & -\delta-\lambda & 0 \\
0 & 0 & (1-p) N \delta & -c-\lambda
\end{array}\right|=0 ;
$$

that is,

$$
\begin{equation*}
\lambda^{4}+p_{1} \lambda^{3}+p_{2} \lambda^{2}+p_{3} \lambda+p_{4}-\left(q_{2} \lambda^{2}+q_{3} \lambda+q_{4}\right) e^{-\lambda \tau}=0 \tag{33}
\end{equation*}
$$

where

$$
\begin{gather*}
p_{1}=a+b+d+c+\delta-\Omega \\
p_{2}=-\Omega(a+b+d)+(c+\delta)(a+b+d-\Omega)+c \delta, \\
p_{3}=-(c+\delta) \Omega(a+b+d)+c \delta(a+b+d-\Omega), \\
p_{4}=-c \delta \Omega(a+b+d), \\
q_{2}=(\eta a+b) k V_{1}^{*}, \\
q_{3}=(c+\delta)(\eta a+b) k V_{1}^{*}+(1-p)(1-\eta) N \delta a k T^{*}, \\
q_{4}=c \delta(\eta a+b) k V_{1}^{*}-(1-p)(1-\eta) N \delta a k T^{*}\left(\Omega+k V_{1}^{*}\right), \\
\Omega=-\mu_{1}+r-\frac{2 r T^{*}}{T_{\max }}-k V_{1}^{*}<0 . \tag{34}
\end{gather*}
$$

Obviously, $p_{1}>0$. In addition, in view of Routh-Hurwitz criteria, we can easily know that all roots of (33) with $\tau=0$ have negative real parts if the following condition holds:

$$
\begin{gather*}
(\mathrm{H}): p_{4}-q_{4}>0, \quad p_{3}-q_{3}>0 \\
p_{1}\left[\left(p_{3}-q_{3}\right)\left(p_{2}-q_{2}\right)-p_{1}\left(p_{4}-q_{4}\right)\right]>\left(p_{1}-q_{3}\right)^{2} \tag{35}
\end{gather*}
$$

Let us consider $\tau \neq 0$ and assume $\lambda(\tau)=\phi(\tau)+i \psi(\tau)$, where $\phi(\tau), \psi(\tau) \in R$. Substituting $\lambda(\tau)=\phi(\tau)+i \psi(\tau)$ and rewriting (33) in terms of its real and imaginary parts, we obtain

$$
\begin{gather*}
v^{4}+\omega^{4}-6 v^{2} \omega^{2}+p_{1}\left(v^{3}-3 v \omega^{2}\right)+p_{2}\left(v^{2}-\omega^{2}\right)+p_{3} v+p_{4} \\
=e^{-\tau v}\left\{q_{2}\left[\left(v^{2}-\omega^{2}\right) \cos (\tau \omega)+2 v \omega \sin (\tau \omega)\right]\right. \\
 \tag{36a}\\
\left.+q_{3}[v \cos (\tau \omega)+\omega \sin (\tau \omega)]+q_{4} \cos (\tau \omega)\right\},
\end{gather*}
$$

$$
\begin{align*}
& 4 v \omega\left(v^{2}-\omega^{2}\right)+p_{1}\left(3 v^{2} \omega-\omega^{3}\right)+p_{2}(2 v \omega)+p_{3} \omega \\
& =e^{-\tau v}\left\{q_{2}\left[\left(-v^{2}+\omega^{2}\right) \sin (\tau \omega)+2 v \omega \cos (\tau \omega)\right]\right. \\
&  \tag{36b}\\
& \left.\quad+q_{3}[-v \sin (\tau \omega)+\omega \cos (\tau \omega)]+q_{4}[-\sin (\tau \omega)]\right\} .
\end{align*}
$$

Let $\tau_{1}^{*}$ be such that $v\left(\tau_{1}^{*}\right)=0$ and $\omega\left(\tau_{1}^{*}\right)=\omega\left(\tau^{*}\right)$; then (36a) and (36b) reduce to

$$
\begin{align*}
\omega_{1}^{* 4}-p_{2} \omega_{1}^{* 2}+p_{4}= & \left(-q_{2} \omega_{1}^{* 2}+q_{4}\right) \cos \left(\tau_{1}^{*} \omega_{1}^{*}\right)  \tag{37a}\\
& +q_{3} \omega_{1}^{*} \sin \left(\tau_{1}^{*} \omega_{1}^{*}\right) \\
-p_{1} \omega_{1}^{* 3}+p_{3} \omega_{1}^{*}= & q_{3} \omega_{1}^{*} \cos \left(\tau_{1}^{*} \omega_{1}^{*}\right) \\
& +\left(q_{2} \omega_{1}^{* 2}-q_{4}\right) \sin \left(\tau_{1}^{*} \omega_{1}^{*}\right) \tag{37b}
\end{align*}
$$

Eliminating $\tau$, we have

$$
\begin{align*}
\omega_{1}^{* 8} & +\left(p_{1}^{2}-2 p_{2}\right) \omega_{1}^{* 6}+\left(p_{2}^{2}+2 p_{4}-2 p_{1} p_{3}-q_{2}^{2}\right) \omega_{1}^{* 4} \\
& +\left(p_{3}^{2}-2 p_{2} p_{4}+2 q_{2} q_{4}-q_{3}^{2}\right) \omega_{1}^{* 2}+\left(p_{4}^{2}-q_{4}^{2}\right)=0 \tag{38}
\end{align*}
$$

Suppose that $\omega_{1}^{*}$ is the last positive simple root of (38). We will now show that, with this value of $\omega_{1}^{*}$, there is a $\tau_{1}^{*}$ such that $v\left(\tau_{1}^{*}\right)=0$ and $\omega\left(\tau_{1}^{*}\right)=\omega_{1}^{*}$. Given $\omega_{1}^{*},(37 \mathrm{a})$ and (37b) can be written as

$$
\begin{align*}
& U=\Phi \cos \left(\tau_{1}^{*} \omega_{1}^{*}\right)+\Psi \sin \left(\tau_{1}^{*} \omega_{1}^{*}\right)  \tag{39a}\\
& V=\Psi \cos \left(\tau_{1}^{*} \omega_{1}^{*}\right)-\Phi \sin \left(\tau_{1}^{*} \omega_{1}^{*}\right) \tag{39b}
\end{align*}
$$

where

$$
\begin{gather*}
\Phi=-q_{2} \omega_{1}^{* 2}+q_{4}, \quad \Psi=q_{3} \omega_{1}^{*}, \\
U=\omega_{1}^{* 4}-p_{2} \omega_{1}^{* 2}, \quad V=-p_{1} \omega_{1}^{* 3}+p_{3} \omega_{1}^{*},  \tag{40}\\
\Phi^{2}+\Psi^{2}=U^{2}+V^{2}=H^{2},
\end{gather*}
$$

where $H>0$.
Equations

$$
\begin{equation*}
\Phi=H \cos \theta, \quad \Psi=H \sin \theta \tag{41}
\end{equation*}
$$

determine a unique $\theta \in[0,2 \pi)$. With this value of $\theta$,

$$
\begin{align*}
& H \cos \left(\tau_{1}^{*} \omega_{1}^{*}\right) \cos \theta+H \sin \left(\tau_{1}^{*} \omega_{1}^{*}\right) \sin \theta=U  \tag{42a}\\
& H \cos \left(\tau_{1}^{*} \omega_{1}^{*}\right) \sin \theta-H \sin \left(\tau_{1}^{*} \omega_{1}^{*}\right) \cos \theta=V \tag{42b}
\end{align*}
$$

Hence,

$$
\begin{align*}
& H \cos \left(\tau_{1}^{*} \omega_{1}^{*}-\theta\right)=U  \tag{43a}\\
& H \sin \left(\tau_{1}^{*} \omega_{1}^{*}-\theta\right)=-V \tag{43b}
\end{align*}
$$

Equations (43a) and (43b) determine $\tau_{1}^{*} \omega_{1}^{*}-\theta$ uniquely in $[0,2 \pi)$ and hence $\tau_{1}^{*}$ uniquely in $\left[\theta / \omega_{1}^{*},(\theta+2 \pi) / \omega_{1}^{*}\right)$. To apply the Hopf bifurcation theorem as stated in [27], we state the following lemma.

Lemma 2 (see [28]). Suppose (38) has at least one simple positive root and $\omega_{1}^{*}$ is the last such root. Then, $i \omega\left(\tau_{1}^{*}\right)=i \omega_{1}^{*}$ is a simple root of $(33)$ and $v(\tau)+i \omega(\tau)$ is differentiable with respect to $\tau$ in a neighbourhood of $\tau=\tau_{1}^{*}$.

Next, to establish Hopf bifurcation at $\tau_{1}=\tau_{1}^{*}$, we need to verify the transversality condition

$$
\begin{equation*}
\left.\frac{d v}{d \tau}\right|_{\tau=\tau_{1}^{*}} \neq 0 \tag{44}
\end{equation*}
$$

Differentiating equations (36a) and (36b) with respect to $\tau$, setting $v=0$ and $\omega=\omega_{1}^{*}$, solving for $d v /\left.d \tau\right|_{\tau=\tau_{1}^{*}}$ and $d \omega /\left.d \tau\right|_{\tau=\tau_{1}^{*}}$, and using (37a) and (37b), we obtain

$$
\begin{align*}
\left.\frac{d v}{d \tau}\right|_{\tau=\tau_{1}^{*}}=\frac{1}{\Gamma_{1}^{2}+\Gamma_{2}^{2}}\left\{\omega_{1}^{* 2}[ \right. & 4 \omega_{1}^{* 6}+3 \omega_{1}^{* 4}\left(p_{1}^{2}-2 p_{2}\right) \\
& +2 \omega_{1}^{* 2}\left(p_{2}^{2}+2 p_{4}-2 p_{1} p_{3}-q_{2}^{2}\right) \\
& \left.\left.+p_{3}^{2}-q_{3}^{2}-2 p_{2} p_{4}+2 q_{2} q_{4}\right]\right\} . \tag{45}
\end{align*}
$$

Here,

$$
\begin{gather*}
\Gamma_{1}=-4 \omega_{1}^{* 3}+2 p_{2} \omega_{1}^{*}+\tau_{1}^{*}\left(-p_{1} \omega_{1}^{* 3}+p_{3} \omega_{1}^{*}\right) \\
+q_{3} \sin \left(\tau_{1}^{*} \omega_{1}^{*}\right)-2 q_{2} \omega_{1}^{*} \cos \left(\tau_{1}^{*} \omega_{1}^{*}\right), \\
\Gamma_{2}=-3 p_{1} \omega_{1}^{* 2}+p_{3}+\tau_{1}^{*}\left(\omega_{1}^{* 4}-p_{2} \omega_{1}^{* 2}+p_{4}\right)  \tag{46}\\
+\left(-2 q_{2} \omega_{1}^{*}\right) \sin \left(\tau_{1}^{*} \omega_{1}^{*}\right)-q_{3} \cos \left(\tau_{1}^{*} \omega_{1}^{*}\right), \\
\Gamma_{1}^{2}+\Gamma_{2}^{2}>0
\end{gather*}
$$

as i $\omega\left(\tau_{1}^{*}\right)$ is a simple root of (33). Let $\varsigma=\omega_{1}^{* 2}$; then (38) reduces to $\nu(\varsigma)=0$, where

$$
\begin{align*}
\nu(\varsigma)= & \varsigma^{4}+\left(p_{1}^{2}-2 p_{2}\right) \varsigma^{3}+\left(p_{2}^{2}+2 p_{4}-2 p_{1} p_{3}-q_{2}^{2}\right) \varsigma^{2} \\
& +\left(p_{3}^{2}-2 p_{2} p_{4}+2 q_{2} q_{4}-q_{3}^{2}\right) \varsigma+\left(p_{4}^{2}-q_{4}^{2}\right) . \tag{47}
\end{align*}
$$

Hence,

$$
\begin{align*}
\frac{d v}{d \varsigma}= & 4 \varsigma^{3}+3 \varsigma^{2}\left(p_{1}^{2}-2 p_{2}\right)+2 \varsigma\left(p_{2}^{2}+2 p_{4}-2 p_{1} p_{3}-q_{2}^{2}\right) \\
& +\left(p_{3}^{2}-q_{3}^{2}-2 p_{2} p_{4}+2 q_{2} q_{4}\right) \tag{48}
\end{align*}
$$

If $\omega_{1}^{* 2}$ is the first positive simple root of (38), then

$$
\begin{equation*}
\left.\frac{d v}{d \varsigma}\right|_{\varsigma=\omega_{1}^{* 2}}>0 \tag{49}
\end{equation*}
$$

Hence, using (45) and (48) we deduce that

$$
\begin{equation*}
\left.\frac{d v}{d \tau}\right|_{\tau=\tau_{1}^{*}}>0 \tag{50}
\end{equation*}
$$

Theorem 3. Suppose that (38) has at least one simple positive root and $\omega_{1}^{*}$ is the last such root. Then, there is a Hopf bifurcation for the system (5) as $\tau$ passes upwards through $\tau_{1}^{*}$ leading to a periodic solution that bifurcates from $E^{*}$.

Next, we will give the sensible conditions that the Hopf bifurcation occurs around equilibrium $E^{*}$. Firstly, we need the following important lemma.

Define $f_{1}=p_{1}^{2}-2 p_{2}, f_{2}=p_{2}^{2}+2 p_{4}-2 p_{1} p_{3}-q_{2}^{2}, f_{3}=$ $p_{3}^{2}-2 p_{2} p_{4}+2 q_{2} q_{4}-q_{3}^{2}, f_{4}=p_{4}^{2}-q_{4}^{2}$, and $\varsigma=\omega_{1}^{* 2}$, then (38) becomes

$$
\begin{equation*}
\varsigma^{4}+f_{1} \varsigma^{3}+f_{2} \varsigma^{2}+f_{3} \varsigma+f_{4}=0 \tag{51}
\end{equation*}
$$

Lemma 4 (see [28]). If $f_{4}<0$, then the quartic equation

$$
\begin{equation*}
\nu(\varsigma)=\varsigma^{4}+f_{1} \varsigma^{3}+f_{2} \varsigma^{2}+f_{3} \varsigma+f_{4}=0 \tag{52}
\end{equation*}
$$

has a strictly positive triple root $k_{1}$ if and only if
(1) $3 f_{1}^{2} \geq 8 f_{2}$;
(2) $f_{1}<0$ or $f_{2}<0$;
(3) $\beta_{1}$ satisfies the equation $6 \beta^{2}+3 f_{1} \beta+f_{2}=0$;
(4) $f_{3}=\beta_{1}^{2}\left(3 f_{1}+8 \beta_{1}\right)$;
(5) $f_{4}=\beta_{1}^{3}\left(-f_{1}-3 \beta_{1}\right)$.

We also need the following mild condition.
Condition 1. Either
(i) $8 f_{2}>3 f_{1}^{2}$;
(ii) $f_{1} \geq 0$ and $f_{2} \geq 0$;
(iii) or if $3 f_{1}^{2} \geq 8 f_{2}$ and also either $f_{1}<0$ or $f_{2}<0$, then if $\beta_{1}$ is a strictly positive root of the quadratic equation, $6 \beta^{2}+3 f_{1} \beta+f_{2}=0$; either $f_{3} \neq \beta_{1}^{2}\left(3 f_{1}+8 \beta_{1}\right)$ or $f_{4} \neq \beta_{1}^{3}\left(-f_{1}-3 \beta_{1}\right)$.

Equation (38) has at least one positive real root for $\omega_{1}^{* 2}$ if $\left|p_{4}\right|<\left|q_{4}\right|$. By Lemma 2, this is a simple root if Condition 1 is satisfied. Thus, from Lemma 2 and Theorem 3, we can get the following theorem.

Theorem 5. Suppose that
(i) $\mathscr{R}_{0}>1$ and the unique endemic equilibrium $E^{*}$ exists; and
(ii) Condition 1 holds and $\left|p_{4}\right|<\left|q_{4}\right|$ so $f_{4}<0$.

Then, there is a Hopf bifurcation for the system (5) as $\tau$ passes upwards through $\tau_{1}^{*}$ leading to a periodic solution that bifurcates from $E^{*}$.

Remark 6. If (38) has a positive root $\omega_{1}^{*}$, from (37a) and (37b) we can obtain

$$
\begin{align*}
& \tau_{j}^{*}=\frac{1}{\omega_{1}^{*}} \arcsin \left[\left(\left(-q_{3} \omega_{1}^{*}\left(\omega_{1}^{* 4}-p_{2} \omega_{1}^{* 2}+p_{4}\right)\right.\right.\right. \\
&\left.+\left(p_{1} \omega_{1}^{* 3}-p_{3} \omega_{1}^{*}\right)\left(q_{2} \omega_{1}^{* 2}-q_{4}\right)\right) \\
&\left.\times\left(\left(q_{3}^{* 4} \omega_{1}^{*}\right)^{2}+\left(q_{2} \omega_{1}^{* 2}-q_{4}\right)^{2}\right)^{-1}\right) \\
&+2 j \pi], \quad j=0,1,2, \ldots \tag{53}
\end{align*}
$$

## 5. Direction and Stability of the Hopf Bifurcation

In the previous section, we obtain the conditions under which a family of periodic solutions bifurcates from the positive equilibrium $E^{*}$ at the critical value of $\tau_{1}^{*}$. As pointed out in Hassard et al. [29], it is interesting to determine the direction, stability, and period of the periodic solutions bifurcating from the positive equilibrium $E^{*}$. Following the ideas of Hassard et al., we derive the explicit formulas for determining the properties of the Hopf bifurcation at the critical value of $\tau_{1}^{*}$ by using the normal form and the center manifold theory. Throughout this section, we always assume that system (5) undergoes Hopf bifurcation at the positive equilibrium $E^{*}$
for $\tau=\tau_{1}^{*}$, and then $\pm i \omega_{1}^{*}$ is corresponding purely imaginary roots of the characteristic equation at the positive equilibrium $E^{*}$. In the this section, for convenience, we use $\tau^{*}$ and $\omega^{*}$ instead of $\tau_{1}^{*}$ and $\omega_{1}^{*}$, respectively.

Let $x_{1}(t)=T(t)-T^{*}, x_{2}(t)=I_{1}(t)-I_{1}^{*}, x_{3}(t)=I_{2}(t)-I_{2}^{*}$, $x_{4}(t)=V_{1}(t)-V_{1}^{*}, \bar{x}_{i}(t)=x_{i}(\tau t),(i=1,2,3,4)$, and $\tau=$ $\tau^{*}+\mu$; system (5) is transformed into an functional differential equation (FDE) in $\mathbb{C}=\mathbb{C}\left([-1,0], \mathbb{R}^{4}\right)$ as

$$
\begin{equation*}
\frac{d x}{d t}=L_{\mu}\left(x_{t}\right)+f\left(\mu, x_{t}\right) \tag{54}
\end{equation*}
$$

where $x(t)=\left(x_{1}(t), x_{2}(t), x_{3}(t), x_{4}(t)\right)^{\top} \in \mathbb{R}^{4}$ and $L \mu: \mathbb{C} \rightarrow$ $\mathbb{R}, f: \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}$ are given, respectively, by

$$
\begin{align*}
L_{\mu}(\phi)= & \left(\tau^{*}+\mu\right) \\
& \times\left(\begin{array}{cccc}
\Omega & \eta a+b & 0 & -k T^{*} \\
0 & -a-b-d & 0 & 0 \\
0 & a(1-\eta) & -\delta & 0 \\
0 & 0 & (1-p) N \delta & -c
\end{array}\right) \\
& \times\left(\begin{array}{l}
\phi_{1}(0) \\
\phi_{2}(0) \\
\phi_{3}(0) \\
\phi_{4}(0)
\end{array}\right) \\
& +\left(\tau^{*}+\mu\right)\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
k V_{1}^{*} & 0 & 0 & k T^{*} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)\left(\begin{array}{l}
\phi_{1}(-1) \\
\phi_{2}(-1) \\
\phi_{3}(-1) \\
\phi_{4}(-1)
\end{array}\right)  \tag{55}\\
f(\mu, \phi)= & \left(\tau^{*}+\mu\right)\left(\begin{array}{c}
-\frac{r}{T_{\max }} \phi_{1}^{2}(0)-k \phi_{1}(0) \phi_{4}(0) \\
k \phi_{1}(-1) \phi_{4}(-1) \\
0 \\
0
\end{array}\right) \tag{56}
\end{align*}
$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in[-1,0]$, such that

$$
\begin{equation*}
L_{\mu} \phi=\int_{-1}^{0} d \eta(\theta, 0) \phi(\theta) \tag{57}
\end{equation*}
$$

for $\phi \in \mathbb{C}$.
In fact, we can choose

$$
\begin{aligned}
& \eta(\theta, \mu) \\
& =\left(\tau^{*}+\mu\right)\left(\begin{array}{cccc}
\Omega & \eta a+b & 0 & -k T^{*} \\
0 & -a-b-d & 0 & 0 \\
0 & a(1-\eta) & -\delta & 0 \\
0 & 0 & (1-p) N \delta & -c
\end{array}\right) \delta(\theta)
\end{aligned}
$$

$$
-\left(\tau^{*}+\mu\right)\left(\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{58}\\
k V_{1}^{*} & 0 & 0 & k T^{*} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) \delta(\theta+1)
$$

where $\delta$ is the Dirac delta function. For $\phi \in \mathbb{C}^{\prime}\left([-1,0], \mathbb{R}^{4}\right)$, define

$$
\begin{gather*}
A(\mu) \phi= \begin{cases}\frac{d \phi(\theta)}{d \theta}, & \theta \in[-1,0), \\
\int_{-1}^{0} d \eta(\mu, s) \phi(s), & \theta=0\end{cases}  \tag{59}\\
R(\mu) \phi= \begin{cases}0, & \theta \in[-1,0) \\
f(\mu, \phi), & \theta=0 .\end{cases}
\end{gather*}
$$

Then, system (54) is equivalent to

$$
\begin{equation*}
\dot{x}_{t}=A(\mu) x_{t}+R(\mu) x_{t}, \tag{60}
\end{equation*}
$$

where $x_{t}(\theta)=x(t+\theta)$ for $\theta \in[-1,0]$.
For $\psi \in \mathbb{C}^{1}\left([-1,0],\left(\mathbb{R}^{4}\right)^{*}\right)$, define

$$
A^{*} \psi(s)= \begin{cases}-\frac{d \psi(s)}{d s}, & s \in(0,1]  \tag{61}\\ \int_{-1}^{0} d \eta^{T}(t, 0) \psi(-t), & s=0\end{cases}
$$

and a bilinear inner product

$$
\begin{align*}
\langle\psi(s), \phi(\theta)\rangle= & \bar{\psi}(0) \phi(0) \\
& -\int_{-1}^{0} \int_{\zeta-\theta}^{\theta} \bar{\psi}(\zeta-\theta) d \eta(\theta) \phi(\zeta) d \zeta \tag{62}
\end{align*}
$$

where $\eta(\theta)=\eta(\theta, 0)$. Then, $A(0)$ and $A^{*}$ are adjoint operators. By the discussion in Section 4 , we know that $\pm i \omega^{*} \tau^{*}$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of $A^{*}$. We first need to compute the eigenvector of $A(0)$ and $A^{*}$ corresponding to $+i \omega^{*} \tau^{*}$ and $-i \omega^{*} \tau^{*}$, respectively.

Suppose that $q(\theta)=(1, a, \beta, \gamma)^{\top} e^{i \omega^{*} \tau^{*} \theta}$ is the eigenvector of $A(0)$ corresponding to $+i \omega^{*} \tau^{*}$; then $A(0) q(\theta)=$ $i \omega^{*} \tau^{*} q(\theta)$. It follows from the definition of $A(0)$ and (55), (57), and (58) that

$$
\begin{align*}
& \tau^{*}\left(\begin{array}{cccc}
i \omega^{*}-\Omega & -\eta a-b & 0 & k T^{*} \\
-k V_{1}^{*} e^{-i \omega^{*} \tau^{*}} & i \omega^{*}+a+b+d & 0 & -k T^{*} e^{-i \omega^{*} \tau^{*}} \\
0 & -a(1-\eta) & i \omega^{*}+\delta & 0 \\
0 & 0 & -(1-p) N \delta & i \omega^{*}+c
\end{array}\right) \\
& \times q(0)=\left(\begin{array}{l}
0 \\
0 \\
0 \\
0
\end{array}\right) . \tag{63}
\end{align*}
$$

Thus, we can easily obtain $q(0)=(1, \alpha, \beta, \gamma)^{\top}$, where

$$
\begin{gather*}
\alpha=\frac{\left(i \omega^{*}-\Omega-k V_{1}^{*}\right) e^{-i \omega^{*} \tau^{*}}}{(\eta a+b) e^{-i \omega^{*} \tau^{*}}-\left(i \omega^{*}+a+b+d\right)}, \\
\gamma=\frac{1}{k T^{*}}\left[(\eta a+b) \alpha-i \omega^{*}+\Omega\right], \quad \beta=\frac{\left(i \omega^{*}+c\right) \gamma}{(1-p) N \delta} . \tag{64}
\end{gather*}
$$

Similarly, let $q^{*}(s)=D\left(1, \alpha^{*}, \beta^{*}, \gamma^{*}\right) e^{i \omega^{*} \tau^{*} s}$ be the eigenvector of $A^{*}$ corresponding to $-i \omega^{*} \tau^{*}$. By the definition of $A^{*}$ and (55)-(57), we can compute

$$
\begin{gather*}
\alpha^{*}=\frac{i \omega^{*}-\Omega}{k V_{1}^{*} e^{-i \omega^{*} \tau^{*}}} \\
\beta^{*}=\frac{-\eta a-b+\left(i \omega^{*}+a+b+d\right) \alpha^{*}}{a(1-\eta)}  \tag{65}\\
\gamma^{*}=\frac{\alpha^{*} k T^{*} e^{-i \omega^{*} \tau^{*}}-k T^{*}}{i \omega^{*}+c}
\end{gather*}
$$

In order to assure $\left\langle q^{*}(s), q(\theta)\right\rangle=1$, we need to determine the value of $D$. From (62), we have

$$
\begin{align*}
& \left\langle q^{*}(s), q(\theta)\right\rangle \\
& =\bar{D}\left(1, \overline{\alpha^{*}}, \overline{\beta^{*}}, \overline{\gamma^{*}}\right)(1, \alpha, \beta, \gamma)^{\top} \\
& \quad-\int_{-1}^{0} \int_{\zeta=0}^{\theta} \bar{D}\left(1, \overline{\alpha^{*}}, \overline{\beta^{*}}, \overline{\gamma^{*}}\right) e^{-i \omega^{*} \tau^{*}(\zeta-\theta)} d \eta(\theta)(1, \alpha, \beta, \gamma)^{\top} e^{i \omega^{*} \tau^{*} \zeta} d \zeta \\
& = \\
& \bar{D}\left\{1+\alpha \overline{\alpha^{*}}+\beta \overline{\beta \beta^{*}}+\gamma \overline{\gamma^{*}}\right. \\
&  \tag{66}\\
& \left.\quad-\int_{-1}^{0}\left(1, \overline{\alpha^{*}}, \overline{\beta^{*}}, \overline{\gamma^{*}}\right) \theta e^{i \omega^{*} \tau^{*} \theta} d \eta(\theta)(1, \alpha, \beta, \gamma)^{\top}\right\} \\
& = \\
& \bar{D}\left\{1+\alpha \overline{\alpha^{*}}+\beta \overline{\beta \beta^{*}}+\gamma \overline{\gamma \gamma^{*}}+\tau^{*} e^{-i \omega^{*} \tau^{*}}\left(\alpha^{*} k V_{1}^{*}+\alpha^{*} k T^{*} \bar{\gamma}\right)\right\} .
\end{align*}
$$

Thus, we can choose $D$ as

$$
\begin{equation*}
D=\frac{1}{1+\alpha \overline{\alpha^{*}}+\beta \overline{\beta^{*}}+\gamma \overline{\gamma^{*}}+\tau^{*} e^{i \omega^{*} \tau^{*}}\left(k V_{1}^{*} \alpha^{*}+k T^{*} \alpha^{*} \bar{\gamma}\right)} . \tag{67}
\end{equation*}
$$

In the remainder of this section, we use the same notations as in [29]; we first compute the coordinates to describe the center manifold $\mathbf{C}_{0}$ at $\mu=0$. Let $x_{t}$ be the solution of (60) when $\mu=0$. Define

$$
\begin{equation*}
Z(t)=\left\langle q^{*}, x_{t}\right\rangle, \quad W(t, \theta)=x_{t}(\theta)-2 \operatorname{Re}\{z(t) q(\theta)\} \tag{68}
\end{equation*}
$$

On the center manifold $\mathrm{C}_{0}$, we have

$$
\begin{equation*}
W(t, \theta)=W(z(t), \bar{z}(t), \theta), \tag{69}
\end{equation*}
$$

where

$$
\begin{align*}
W(z, \bar{z}, \theta)= & W_{20}(\theta) \frac{z^{2}}{2}+W_{11}(\theta) z \bar{z}+W_{02}(\theta) \frac{\bar{z}^{2}}{2}  \tag{70}\\
& +W_{30}(\theta) \frac{z^{3}}{6}+\cdots,
\end{align*}
$$

$z$ and $\bar{z}$ are local coordinates for center manifold $\mathbf{C}_{0}$ in the direction of $q^{*}$ and $\overline{q^{*}}$. Note that $W$ is real if $x_{t}$ is real. We only consider real solutions. For solution $x_{t} \in \mathrm{C}_{0}$ of (60), since $\mu=0$, we have

$$
\begin{align*}
\dot{z}(t) & =i \omega^{*} \tau^{*} z+\overline{q^{*}}(0) f(0, W(z, \bar{z}, \theta)+2 \operatorname{Re}\{z q(\theta)\}) \\
& =i \omega^{*} \tau^{*} z+\overline{q^{*}}(0) f_{0}(z, \bar{z}) \tag{71}
\end{align*}
$$

We rewrite this equation as

$$
\begin{equation*}
\dot{z}(t)=i \omega^{*} \tau^{*} z(t)+g(z, \bar{z}), \tag{72}
\end{equation*}
$$

where

$$
\begin{align*}
g(z, \bar{z}) & =\overline{q^{*}}(0) f_{0}(z, \bar{z}) \\
& =g_{20} \frac{z^{2}}{2}+g_{11} z \bar{z}+g_{02} \frac{\bar{z}^{2}}{2}+g_{21} \frac{z^{2} \bar{z}}{2}+\cdots \tag{73}
\end{align*}
$$

It follows from (68) and (70) that

$$
\begin{align*}
& x_{t}(\theta) \\
& \qquad=W(t, \theta)-2 \operatorname{Re}\{z(t) q(t)\} \\
& =W_{20}(\theta) \frac{z^{2}}{2}+W_{11}(\theta) z \bar{z}+W_{02}(\theta) \frac{\bar{z}^{2}}{2} \\
& \quad+(1, \alpha, \beta, \gamma)^{T} e^{i \omega^{*} \tau^{*} \theta} z+(1, \bar{\alpha}, \bar{\beta}, \bar{\gamma})^{T} e^{-i \omega^{*} \tau^{*} \theta} \bar{z}+\cdots \tag{74}
\end{align*}
$$

It follows together with (56) that

$$
\begin{aligned}
& g(z, \bar{z}) \\
& =\overline{q^{*}}(0) f_{0}(z, \bar{z}) \\
& =\overline{q^{*}}(0) f\left(0, x_{t}\right) \\
& =\tau_{k} \bar{D}\left(1, \overline{\alpha^{*}}, \overline{\beta^{*}}, \overline{\gamma^{*}}\right)\left(\begin{array}{c}
-\frac{r}{T_{\max }} \phi_{1}^{2}(0)-k \phi_{1}(0) \phi_{4}(0) \\
k \phi_{1}(-1) \phi_{4}(-1) \\
0
\end{array}\right) \\
& =-\tau^{*} \bar{D} \frac{r}{T_{\max }}\left[z+\bar{z}+W_{20}^{(1)}(0) \frac{z^{2}}{2}\right. \\
& \left.\quad+W_{11}^{(1)}(0) z \bar{z}+W_{02}^{(1)}(0) \frac{\bar{z}^{2}}{2}+o\left(\left|(z, \bar{z})^{3}\right|\right)\right]^{2} \\
& \quad-\tau^{*} \bar{D} k\left[z+\bar{z}+W_{20}^{(1)}(0) \frac{z^{2}}{2}+W_{11}^{(1)}(0) z \bar{z}\right. \\
& \\
& \left.\quad+W_{02}^{(1)}(0) \frac{\bar{z}^{2}}{2}+o\left(\left|(z, \bar{z})^{3}\right|\right)\right] \\
& \quad \times\left[\gamma z+\bar{\gamma} \bar{z}+W_{20}^{(4)}(0) \frac{z^{2}}{2}\right. \\
& \left.\quad+W_{11}^{(4)}(0) z \bar{z}+W_{02}^{(4)}(0) \frac{\bar{z}^{2}}{2}+o\left(|z, \bar{z}|^{3}\right)\right] \\
& \quad+\tau^{*} \bar{D} \bar{\alpha}^{*} k\left[e^{-i \omega^{*} \tau^{*}} z+e^{i \omega^{*} \tau^{*}} \bar{z}+W_{20}^{(1)}(-1) \frac{z^{2}}{2}\right.
\end{aligned}
$$

$$
\begin{gather*}
\left.+W_{11}^{(1)}(-1) z \bar{z}+W_{02}^{(1)}(-1) \frac{\bar{z}^{2}}{2}+o\left(|z, \bar{z}|^{3}\right)\right] \\
\times\left[\gamma e^{-i \omega^{*} \tau^{*}} z+\bar{\gamma} e^{i \omega^{*} \tau^{*}} \bar{z}+W_{20}^{(4)}(-1) \frac{z^{2}}{2}\right. \\
\left.+W_{11}^{(4)}(-1) z \bar{z}+W_{02}^{(4)}(-1) \frac{\bar{z}^{2}}{2}+o\left(|z, \bar{z}|^{3}\right)\right] \tag{75}
\end{gather*}
$$

Comparing the coefficients with (73), we have

$$
\begin{gather*}
g_{20}=2 \tau^{*} \bar{D}\left[-\frac{r}{T_{\max }}-k \gamma+k \bar{\alpha}^{*} \gamma e^{-2 i \omega^{*} \tau^{*}}\right] \\
g_{11}=2 \tau^{*} \bar{D}\left[-\frac{r}{T_{\max }}-k \operatorname{Re} \gamma+k \bar{\alpha}^{*} \operatorname{Re} \gamma\right] \\
g_{02}=2 \tau^{*} \bar{D}\left[-\frac{r}{T_{\max }}-k \bar{\gamma}+k \bar{\alpha}^{*} \bar{\gamma} e^{2 i \omega^{*} \tau^{*}}\right] \\
g_{21} \\
=\tau^{*} \bar{D} k \bar{\alpha}^{*} \\
\times\left[-\frac{r}{T_{\max }}\left(4 W_{11}^{(1)}(0)+2 W_{20}^{(1)}(0)\right)\right. \\
\quad-k\left(2 W_{11}^{(4)}(0)+W_{20}^{(4)}(0)+\bar{\gamma} W_{20}^{(1)}(0)+2 \gamma W_{11}^{(1)}(0)\right) \\
+k \bar{\alpha}^{*}\left(2 e^{-i \omega^{*} \tau^{*}} W_{11}^{(4)}(-1)+e^{i \omega^{*} \tau^{*}} W_{20}^{(4)}(-1)\right. \\
\left.\left.+\bar{\gamma} e^{i \omega^{*} \tau^{*}} W_{20}^{(1)}(-1)+2 \gamma e^{-i \omega^{*} \tau^{*}} W_{11}^{(1)}(-1)\right)\right] . \tag{76}
\end{gather*}
$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in $g_{21}$, we still need to compute them. From (60) and (68), we have

$$
\begin{align*}
\dot{W} & =\dot{x}_{t}-\dot{z} q-\dot{\bar{z}} \bar{q} \\
& = \begin{cases}A W-2 \operatorname{Re}\left\{\overline{q^{*}}(0) f_{0} q(\theta)\right\}, & \theta \in[-1,0), \\
A W-2 \operatorname{Re}\left\{\overline{q^{*}}(0) f_{0} q(\theta)\right\}+f_{0}, & \theta=0,\end{cases} \\
& \triangleq A W+H(z, \bar{z}, \theta), \tag{77}
\end{align*}
$$

where

$$
\begin{equation*}
H(z, \bar{z}, \theta)=H_{20}(\theta) \frac{z^{2}}{2}+H_{11}(\theta) z \bar{z}+H_{02}(\theta) \frac{\bar{z}^{2}}{2}+\cdots \tag{78}
\end{equation*}
$$

Substituting the corresponding series into (77) and comparing the coefficients, we obtain

$$
\begin{equation*}
\left(A-2 i \omega^{*} \tau^{*}\right) W_{20}(\theta)=-H_{20}, \quad A W_{11}(\theta)=-H_{11}, \ldots \tag{79}
\end{equation*}
$$

From (77), we know that for $\theta \in[-1,0)$,

$$
\begin{align*}
H(z, \bar{z}, \theta) & =-\overline{q^{*}}(0) f_{0} q(\theta)-q^{*}(0) \bar{f}_{0} \bar{q}(\theta)  \tag{80}\\
& =-g(z, \bar{z}) q(\theta)-g(z, \bar{z}) \bar{q}(\theta)
\end{align*}
$$

Comparing the coefficients with (78) gives

$$
\begin{align*}
& H_{20}(\theta)=-g_{20} q(\theta)-\bar{g}_{02} \bar{q}(\theta),  \tag{81}\\
& H_{11}(\theta)=-g_{11} q(\theta)-\bar{g}_{11} \bar{q}(\theta) . \tag{82}
\end{align*}
$$

From (79), (81), and the definition of $A$, it follows that

$$
\begin{equation*}
\dot{W}_{20}=2 i \omega^{*} \tau^{*} W_{20}(\theta)+g_{20} q(\theta)+\bar{g}_{02} \bar{q}(\theta) . \tag{83}
\end{equation*}
$$

Notice that $q(\theta)=(1, \alpha, \beta, \gamma)^{\top} e^{i \omega^{*} \tau^{*} \theta}$; hence,

$$
\begin{align*}
W_{20}(\theta)= & \frac{i g_{20}}{\omega^{*} \tau^{*}} q(0) e^{i \omega^{*} \tau^{*} \theta}+\frac{i \bar{g}_{02}}{3 \omega^{*} \tau^{*}} \bar{q}(0) e^{-i \omega^{*} \tau^{*} \theta}  \tag{84}\\
& +E_{1} e^{2 i \omega^{*} \tau^{*} \theta}
\end{align*}
$$

where $E_{1}=\left(E_{1}^{(1)}, E_{1}^{(2)}, E_{1}^{(3)}, E_{1}^{(4)}\right) \in \mathbb{R}^{4}$ is a constant vector. Similarly, from (79) and (82), we obtain

$$
\begin{equation*}
W_{11}(\theta)=-\frac{i g_{11}}{\omega^{*} \tau^{*}} q(0) e^{i \omega^{*} \tau^{*} \theta}+\frac{i \bar{g}_{11}}{\omega^{*} \tau^{*}} \bar{q}(0) e^{-i \omega^{*} \tau^{*} \theta}+E_{2} \tag{85}
\end{equation*}
$$

where $E_{2}=\left(E_{2}^{(1)}, E_{2}^{(2)}, E_{2}^{(3)}, E_{2}^{(4)}\right) \in \mathbb{R}^{4}$ is also a constant vector.

In what follows, we will seek appropriate $E_{1}$ and $E_{2}$. From the definition of $A$ and (79), we obtain

$$
\begin{gather*}
\int_{-1}^{0} d \eta(\theta) W_{20}(\theta)=2 i \omega^{*} \tau^{*} W_{20}(\theta)-H_{20}(\theta),  \tag{86}\\
\int_{-1}^{0} d \eta(\theta) W_{11}(\theta)=-H_{11}(\theta), \tag{87}
\end{gather*}
$$

where $\eta(\theta)=\eta(0, \theta)$. By (77), we have

$$
\begin{align*}
& H_{20}(\theta)=-g_{20} q(0)-\bar{g}_{20} \bar{q}(0)+2 \tau^{*}\left(\begin{array}{c}
-\frac{r}{T_{\max }}-k \gamma \\
k \gamma e^{-2 i \omega^{*} \tau^{*}} \\
0 \\
0
\end{array}\right),  \tag{88}\\
& H_{11}(\theta)=-g_{11} q(0)-\bar{g}_{11} \bar{q}(0)+2 \tau^{*}\left(\begin{array}{c}
-\frac{r}{T_{\max }}-k \operatorname{Re} \gamma \\
k \operatorname{Re} \gamma \\
0 \\
0
\end{array}\right) \tag{89}
\end{align*}
$$

Substituting (83) and (88) into (86), we obtain

$$
\left(2 \omega^{*} \tau^{*} I-\int_{-1}^{0} e^{2 i \omega^{*} \tau^{*} \theta} d \eta(\theta)\right) E_{1}=2 \tau^{*}\left(\begin{array}{c}
-\frac{r}{T_{\max }}-k \gamma  \tag{90}\\
k \gamma e^{-2 i \omega^{*} \tau^{*}} \\
0 \\
0
\end{array}\right)
$$

which leads to

$$
\begin{gather*}
\left(\begin{array}{cccc}
2 i \omega^{*}-\Omega & -\eta a-b & 0 & k T^{*} \\
-k V_{1}^{*} e^{-2 i \omega^{*} \tau^{*}} & 2 i \omega^{*}+a+b+d & 0 & -k T^{*} e^{-2 i \omega^{*} \tau^{*}} \\
0 & -a(1-\eta) & 2 i \omega^{*}+\delta & 0 \\
0 & 0 & -(1-p) N \delta & 2 i \omega^{*}+c
\end{array}\right) \\
\times E_{1}=2\left(\begin{array}{c}
-\frac{r}{T_{\max }}-k \gamma \\
k \gamma e^{-2 i \omega^{*} \tau^{*}} \\
0 \\
0
\end{array}\right) . \tag{91}
\end{gather*}
$$

It follows that

$$
E_{1}^{(1)}
$$

$$
=\frac{2}{\Delta}\left|\begin{array}{cccc}
-\frac{r}{T_{\max }}-k \gamma & -\eta a-b & 0 & k T^{*} \\
k \gamma e^{-2 i \omega^{*} \tau^{*}} & 2 i \omega^{*}+a+b+d & 0 & -k T^{*} e^{-2 i \omega^{*} \tau^{*}} \\
0 & -a(1-\eta) & 2 i \omega^{*}+\delta & 0 \\
0 & 0 & -(1-p) N \delta & 2 i \omega^{*}+c
\end{array}\right| \text {, }
$$

$$
E_{1}^{(2)}
$$

$$
=\frac{2}{\Delta}\left|\begin{array}{cccc}
2 i \omega^{*}-\Omega & -\frac{r}{T_{\max }}-k \gamma & 0 & k T^{*} \\
-k V_{1}^{*} e^{-2 i \omega^{*} \tau^{*}} & k \gamma e^{-2 i \omega^{*} \tau^{*}} & 0 & -k T^{*} e^{-2 i \omega^{*} \tau^{*}} \\
0 & -a(1-\eta) & 2 i \omega^{*}+\delta & 0 \\
0 & 0 & -(1-p) N \delta & 2 i \omega^{*}+c
\end{array}\right|,
$$

$$
E_{1}^{(3)}
$$

$$
=\frac{2}{\Delta}\left|\begin{array}{cccc}
2 i \omega^{*}-\Omega & -\eta a-b & -\frac{r}{T_{\max }}-k \gamma & k T^{*} \\
-k V_{1}^{*} e^{-2 i \omega^{*} \tau^{*}} & 2 i \omega^{*}+a+b+d & k \gamma e^{-2 i \omega^{*} \tau^{*}} & -k T^{*} e^{-2 i \omega^{*} \tau^{*}} \\
0 & 0 & \delta & 0 \\
0 & 0 & 0 & 2 i \omega^{*}+c
\end{array}\right|
$$

$$
\begin{align*}
& E_{1}^{(4)}  \tag{92}\\
& =\frac{2}{\Delta}\left|\begin{array}{cccc}
2 i \omega^{*}-\Omega & -\eta a-b & 0 & -\frac{r}{T_{\max }}-k \gamma \\
-k V_{1}^{*} e^{-2 i \omega^{*} \tau^{*}} & 2 i \omega^{*}+a+b+d & 0 & k \gamma e^{-2 i \omega^{*} \tau^{*}} \\
0 & -a(1-\eta) & 2 i \omega^{*}+\delta & 0 \\
0 & 0 & -(1-p) N \delta & 0
\end{array}\right|
\end{align*}
$$

where

$$
\Delta=\left|\begin{array}{cccc}
2 i \omega^{*}-\Omega & -\eta a-b & 0 & k T^{*}  \tag{93}\\
-k V_{1}^{*} e^{-2 i \omega^{*} \tau^{*}} & 2 i \omega^{*}+a+b+d & 0 & -k T^{*} e^{-2 i \omega^{*} \tau^{*}} \\
0 & -a(1-\eta) & 2 i \omega^{*}+\delta & 0 \\
0 & 0 & -(1-p) N \delta & 2 i \omega^{*}+c
\end{array}\right|
$$

Similarly, substituting (85) and (89) into (87), we can get

$$
\begin{gather*}
\left(\begin{array}{cccc}
-\Omega & -\eta a-b & 0 & k T^{*} \\
-k V_{1}^{*} & a+b+d & 0 & -k T^{*} \\
0 & -a(1-\eta) & \delta & 0 \\
0 & 0 & -(1-p) N \delta & c
\end{array}\right) E_{2} \\
\quad=\left(\begin{array}{c}
-\frac{r}{T_{\max }}-k \operatorname{Re} \gamma \\
k \operatorname{Re} \gamma \\
0 \\
0
\end{array}\right) \tag{94}
\end{gather*}
$$



Figure 1: When $\tau=0.8<\tau^{*}$, the positive equilibrium $E^{*}$ is stable.
and hence

$$
\begin{aligned}
E_{2}^{(1)} & =\frac{2}{\Delta_{1}} \\
& \times\left|\begin{array}{cccc}
-\frac{r}{T_{\max }}-k \operatorname{Re} \gamma & -\eta a-b & 0 & k T^{*} \\
k \operatorname{Re} \gamma & a+b+d & 0 & -k T^{*} \\
0 & -a(1-\eta) & \delta & 0 \\
0 & 0 & -(1-p) N \delta & c
\end{array}\right|, \\
E_{2}^{(2)}= & \frac{2}{\Delta_{1}}\left|\begin{array}{cccc}
-\Omega & -\frac{r}{T_{\max }}-k \operatorname{Re} \gamma & 0 & k T^{*} \\
-k V_{1}^{*} & k \operatorname{Re} \gamma & 0 & -k T^{*} \\
0 & 0 & -(1-p) N \delta & c
\end{array}\right| \\
E_{2}^{(3)}= & \frac{2}{\Delta_{1}}\left|\begin{array}{cccc} 
& 0 & r & 0 \\
-\Omega & -\eta a-b & -\frac{r}{T_{\max }}-k \operatorname{Re} \gamma & k T^{*} \\
-k V_{1}^{*} & a+b+d & k \operatorname{Re} \gamma & -k T^{*} \\
0 & -a(1-\eta) & 0 & 0 \\
0 & 0 & 0 & c
\end{array}\right|
\end{aligned}
$$

$$
E_{2}^{(4)}=\frac{2}{\Delta_{1}}\left|\begin{array}{cccc}
-\Omega & -\eta a-b & 0 & -\frac{r}{T_{\max }}-k \operatorname{Re} \gamma  \tag{95}\\
-k V_{1}^{*} & a+b+d & 0 & k \operatorname{Re} \gamma \\
0 & -a(1-\eta) & \delta & 0 \\
0 & 0 & -(1-p) N \delta & 0
\end{array}\right|,
$$

where

$$
\Delta_{1}=\left|\begin{array}{cccc}
-\Omega & -\eta a-b & 0 & k T^{*}  \tag{96}\\
-k V_{1}^{*} & a+b+d & 0 & -k T^{*} \\
0 & -a(1-\eta) & \delta & 0 \\
0 & 0 & -(1-p) N \delta & c
\end{array}\right|
$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (83) and (85). Furthermore, $g_{21}$ in (75) can be expressed by the parameters and delay. Then, we can compute the following


Figure 2: When $\tau=2.4>\tau^{*}$, the positive equilibrium $E^{*}$ losses its stability and periodic solution occurs.
values:

$$
\begin{gather*}
c_{1}(0)=\frac{i}{2 \omega^{*} \tau^{*}}\left(g_{20} g_{11}-2\left|g_{11}\right|^{2}-\frac{\left|g_{02}\right|^{2}}{3}\right)+\frac{g_{21}}{2}, \\
\mu_{2}=-\frac{\operatorname{Re}\left\{c_{1}(0)\right\}}{\operatorname{Re}\left\{\lambda^{\prime}\left(\tau^{*}\right)\right\}}  \tag{97}\\
\beta_{2}=2 \operatorname{Re}\left\{c_{1}(0)\right\} \\
T_{2}=-\frac{\operatorname{Im}\left\{c_{1}(0)\right\}+\mu_{2} \operatorname{Im}\left\{\lambda^{\prime}\left(\tau^{*}\right)\right\}}{\omega^{*} \tau^{*}} .
\end{gather*}
$$

By the result of Hassard et al. [29], we have the following.
Theorem 7. In (97), the sign of $\mu_{2}$ determined the direction of Hopfbifurcation: if $\mu_{2}>0\left(\mu_{2}<0\right)$, then the Hopfbifurcation is supercritical (subcritical) and the bifurcating periodic solution
exists for $\tau>\tau^{*}\left(\tau<\tau^{*}\right)$. $\beta_{2}$ determines the stability of the bifurcating periodic solution: the bifurcating periodic solution is stable (unstable) if $\beta_{2}<0\left(\beta_{2}>0\right)$, and $T_{2}$ determines the period of the bifurcating periodic solution: the period increases (decreases) if $T_{2}>0\left(T_{2}<0\right)$.

## 6. Numerical Simulation

In the previous sections, we introduced the analytical tools proposed and used for a qualitative analysis of the system obtaining some results about the dynamics of the system. In this section, we perform a numerical analysis of the model based on the previous results.

Our model involves 13 parameters, including the delay $\tau$. In the following, we choose a set of parameters in Table 1. Correspondingly, $\mathscr{R}_{0}=17.68614487>1$ and $E^{*}(72.94117647,167.1814866,94.73617573,21315.63954)$.

Table 1: Variables and parameters for viral spread.

|  | Variables and meaning parameters | Values |
| :---: | :---: | :---: |
| T | Uninfected CD4 ${ }^{+}$T-cell population size | $1000 \mathrm{~mm}^{-3}$ |
| $I_{1}$ | Pre-RT | $0 \mathrm{~mm}^{-3}$ |
| $I_{2}$ | Post-RT | $0 \mathrm{~mm}^{-3}$ |
| $V_{1}$ | Infectious virus | $10^{-3} \mathrm{~mm}^{-3}$ |
| $s$ | Source term for uninfected CD4 ${ }^{+}$ T-cell | 5 day $^{-1} \mathrm{~mm}^{-3}$ |
| $k$ | Rate at which CD4 ${ }^{+}$T-cell becomes infected with virus | $0.00005 \mathrm{~mm}^{3} \mathrm{day}^{-1}$ |
| $\mu_{1}$ | Death rate of healthy CD4 ${ }^{+}$T-cell | 0.01 day $^{-1}$ |
| $\eta$ | Efficacy of RTIs | 0.15 |
| $a$ | Transition rate from pre-RT infected $\mathrm{CD} 4^{+} \mathrm{T}$ cells to productively post-RT | 0.4 day $^{-1}$ |
| $b$ | Reverting rate of infected cells to uninfected class | 0.05 day $^{-1}$ |
| d | Death rate of infected T-cells | 0.015 day $^{-1}$ |
| $\delta$ | Death rate of actively infected T-cells $I_{2}$ | 0.6 day $^{-1}$ |
| $N$ | Number of virions produced by infected CD4 ${ }^{+}$T-cells | 1000 virions cell ${ }^{-1}$ |
| c | Clearance rate of virus | 2.4 day $^{-1}$ |
| $r$ | Growth rate of T-cells | $0.8 \mathrm{day}^{-1}$ |
| $T_{\text {max }}$ | Carrying capacity of T-cells | $1300 \mathrm{~mm}^{3}$ |
| $p$ | Protease inhibitor efficacy | 0.1 |

We can compute that $\omega^{*}=0.3205862054$, and $\tau^{*}=$ 1.201514430. By Theorem 5, equilibrium $E^{*}$ is locally asymptotically stable when $\tau<\tau^{*}$ (see Figure 1), and Hopf bifurcation occurs at $\tau=\tau^{*}$; a periodic solution exists when $\tau>\tau^{*}$ (see Figure 2). Furthermore, we compute $c_{1}(0)=-13.62874664-2.28357223 i$. Therefore, $\operatorname{Re}\left(c_{1}(0)\right)<0$. By Theorem 7, we know that the Hopf bifurcation is supercritical: the bifurcating periodic solutions exist for $\tau>\tau^{*}$ and they are orbitally asymptotically stable.

The ranges of time delay $\tau$ are reported in [30,31], are between 0 and 2 days. By the theory of Hopf bifurcation, we have shown that sustained oscillations are possible in the realistic parameter space. This shows that our model is reasonable.

## 7. Discussion

We have considered a mathematical model for drugs therapy to the infection of $\mathrm{CD} 4^{+} \mathrm{T}$ cells in vivo by HIV. The model incorporates the effects of antiretroviral therapy, logistic growth of the $\mathrm{CD} 4^{+} \mathrm{T}$ cell, and intracellular delay. We have carried out a rigorous mathematical analysis of global dynamics of the model and have shown that the time delay can destabilize the positive equilibrium and lead to periodic solutions through Hopf bifurcation.

If we cannot consider the effect of "intracellular" delay, the viral oscillation will not occur [10]. Intracellular delay can induce rich dynamics in the viral system. Moreover, in system (5), we used a logistic term to model the generation
and death of target cells. In fact, we can find the logistic term to model the generation by using simulation. And Li and Su have studied that both the "intracellular" delay and target cell can proliferate on virus dynamics [32]. All in all, based on the analytic and simulation results, we can conclude that both the "intracellular" delay and logistic term may give rise to the viral oscillation in the host. Hence, the oscillation behaviors of virus population can be understood in these ways. We will discuss the effect of the "intracellular" delay and logistic term in theory in the future.

It is well to know that current treatment regimens cannot eradicate the virus. And the single drug may be highly effective. From the expression of the basic reproductive number
$\mathscr{R}_{0}=T_{0} / T^{*}=\left(T_{\max }\left[r-\mu_{1}+\sqrt{\left(r-\mu_{1}\right)^{2}+4 r s / T_{\max }}\right] / 2 r\right)((1-$ $p)(1-\eta) k N a / c(d+a+b))$, we can find that $\mathscr{R}_{0}$ is a decreasing function for $p$. The value of $\mathscr{R}_{0}$ is smaller for a larger $p$. That is to say, PIs are positive for the treatment of HIV. Hence, our results show that we need a combination therapy to obtain the better results of drug therapy.

Finally, if we assume a constant death rate $m$ for infected but not yet virus-producing cells, the probability of surviving from time $t-\tau$ to time $t$ is just $e^{-m \tau}$ [14]. Thus the refined model can be rewritten as (4). Hence, we have the following system:

$$
\begin{gather*}
\frac{d T}{d t}=s+r T\left(1-\frac{T}{T_{\max }}\right)-\mu_{1} T-k T V_{1}+(\eta a+b) I_{1}, \\
\frac{d I_{1}}{d t}=k e^{-m \tau} T(t-\tau) V_{1}(t-\tau)-(d+a+b) I_{1}, \\
\frac{d I_{2}}{d t}=(1-\eta) a I_{1}-\delta I_{2},  \tag{98}\\
\frac{d V_{1}}{d t}=(1-p) N \delta I_{2}-c V_{1}, \\
\frac{d V_{2}}{d t}=p N \delta I_{2}-c V_{2} .
\end{gather*}
$$

It is easy to obtain that the characteristic equation about the positive equilibrium of model (98) is delay dependent coefficients. We can deduce that the stability switches around the positive equilibrium may occur. We leave it in the future.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work is supported by the National Natural Science Foundation of China (nos. 11171284 and 11371306), Basic and Frontier Technology Research Program of Henan Province (nos. 132300410025 and 132300410364), and the Key Project for the Education Department of Henan Province (no. 13A110771).

## References

[1] P. de Leenheer and H. L. Smith, "Virus dynamics: a global analysis," SIAM Journal on Applied Mathematics, vol. 63, no. 4, pp. 1313-1327, 2003.
[2] D. Li and W. Ma, "Asymptotic properties of a HIV-1 infection model with time delay," Journal of Mathematical Analysis and Applications, vol. 335, no. 1, pp. 683-691, 2007.
[3] S. Liu and L. Wang, "Global stability of an HIV-1 model with distributed intracellular delays and a combination therapy," Mathematical Biosciences and Engineering, vol. 7, no. 3, pp. 675685, 2010.
[4] K. A. Pawelek, S. Liu, F. Pahlevani, and L. Rong, "A model of HIV-1 infection with two time delays: mathematical analysis and comparison with patient data," Mathematical Biosciences, vol. 235, no. 1, pp. 98-109, 2012.
[5] A. S. Perelson and P. W. Nelson, "Mathematical analysis of HIV1 dynamics in vivo," SIAM Review, vol. 41, no. 1, pp. 3-44, 1999.
[6] X. Zhou, X. Song, and X. Shi, "Analysis of stability and Hopf bifurcation for an HIV infection model with time delay," Applied Mathematics and Computation, vol. 199, no. 1, pp. 23-38, 2008.
[7] M. von Kleist, S. Menz, and W. Huisinga, "Drug-class specific impact of antivirals on the reproductive capacity of HIV," PLoS Computational Biology, vol. 6, no. 3, Article ID e1000720, 2010.
[8] E. A. Hernandez-Vargas and R. H. Middleton, "Modeling the three stages in HIV infection," Journal of Theoretical Biology, vol. 320, pp. 33-40, 2013.
[9] A. S. Perelson, A. U. Neumann, and M. Markowitz, "HIV-1 dynamics in vivo: virion clearance rate, infected cell life-span, and viral generation time," Science, vol. 271, no. 5255, pp. 15821586, 1996.
[10] P. K. Srivastava, M. Banerjee, and P. Chandra, "Modeling the drug therapy for HIV infection," Journal of Biological Systems, vol. 17, no. 2, pp. 213-223, 2009.
[11] J. A. Zack, S. J. Arrigo, S. R. Weitsman, A. S. Go, A. Haislip, and I. S. Y. Chen, "HIV-1 entry into quiescent primary lymphocytes: molecular analysis reveals a labile, latent viral structure," Cell, vol. 61, no. 2, pp. 213-222, 1990.
[12] J. A. Zack, A. M. Haislip, P. Krogstad, and I. S. Y. Chen, "Incompletely reverse-transcribed human immunodeficiency virus type 1 genomes in quiescent cells can function as intermediates in the retroviral life cycle," Journal of Virology, vol. 66, no. 3, pp. 1717-1725, 1992.
[13] M. von Kleist, Combining pharmacology and mutational dynamics to understand and combat drug resistance in HIV, [PhD Dissertation Thesis], National University of Ireland Maynooth, 2010.
[14] A. V. M. Herz, S. Bonhoeffer, R. M. Anderson, R. M. May, and M. A. Nowak, "Viral dynamics in vivo: limitations on estimates of intracellular delay and virus decay," Proceedings of the National Academy of Sciences of the United States of America, vol. 93, no. 14, pp. 7247-7251, 1996.
[15] P. W. Nelson, J. D. Murray, and A. S. Perelson, "A model of HIV-1 pathogenesis that includes an intracellular delay," Mathematical Biosciences, vol. 163, no. 2, pp. 201-215, 2000.
[16] S. M. Lee and J. H. Park, "Delay-dependent criteria for absolute stability of uncertain time-delayed Lur'e dynamical systems," Journal of the Franklin Institute, vol. 347, no. 1, pp. 146-153, 2010.
[17] C. Liu, Q. Zhang, and X. Duan, "Dynamical behavior in a harvested differential-algebraic prey-predator model with discrete time delay and stage structure," Journal of the Franklin Institute, vol. 346, no. 10, pp. 1038-1059, 2009.
[18] Q. Wang and B. Dai, "Existence of positive periodic solutions for neutral population model with delays," International Journal of Biomathematics, vol. 1, no. 1, pp. 107-120, 2008.
[19] X. Xu, H. Y. Hu, and H. L. Wang, "Stability switches, Hopf bifurcation and chaos of a neuro model with delay-dependent parameter," Physics Letters A, vol. 354, no. 1-2, pp. 126-136, 2006.
[20] X. Zhou and J. Cui, "Stability and Hopf bifurcation analysis of an eco-epidemiological model with delay," Journal of the Franklin Institute, vol. 347, no. 9, pp. 1654-1680, 2010.
[21] R. V. Culshaw and S. Ruan, "A delay-differential equation model of HIV infection of CD4 ${ }^{+}$T-cells," Mathematical Biosciences, vol. 165, no. 1, pp. 27-39, 2000.
[22] J. Hale, Theory of Functional Differential Equations, Springer, New York, NY, USA, 2nd edition, 1977.
[23] X. Yang, L. Chen, and J. Chen, "Permanence and positive periodic solution for the single-species nonautonomous delay diffusive models," Computers \& Mathematics with Applications, vol. 32, no. 4, pp. 109-116, 1996.
[24] M. Y. Li and H. Shu, "Global dynamics of an in-host viral model with intracellular delay," Bulletin of Mathematical Biology, vol. 72, no. 6, pp. 1492-1505, 2010.
[25] A. S. Perelson, D. E. Kirschner, and R. de Boer, "Dynamics of HIV infection of CD4 $4^{+}$T cells," Mathematical Biosciences, vol. 114, no. 1, pp. 81-125, 1993.
[26] Z. H. Wang and H. Y. Hu, "Pseudo-oscillator analysis of scalar nonlinear time-delay systems near a Hopf bifurcation," International Journal of Bifurcation and Chaos in Applied Sciences and Engineering, vol. 17, no. 8, pp. 2805-2814, 2007.
[27] J. E. Marsden and M. McCracken, The Hopf Bifurcation and Its Applications, Springer, New York, NY, USA, 1976.
[28] W. Liu and H. I. Freedman, "A mathematical model of vascular tumor treatment by chemotherapy," Mathematical and Computer Modelling, vol. 42, no. 9-10, pp. 1089-1112, 2005.
[29] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, Theory and Applications of Hopf Bifurcation, vol. 41 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, UK, 1981.
[30] D. M. Bortz and P. W. Nelson, "Sensitivity analysis of a nonlinear lumped parameter model of HIV infection dynamics," Bulletin of Mathematical Biology, vol. 66, no. 5, pp. 1009-1026, 2004.
[31] N. M. Dixit and A. S. Perelson, "Complex patterns of viral load decay under antiretroviral therapy: influence of pharmacokinetics and intracellular delay," Journal of Theoretical Biology, vol. 226, no. 1, pp. 95-109, 2004.
[32] M. Y. Li and H. Shu, "Joint effects of mitosis and intracellular delay on viral dynamics: two-parameter bifurcation analysis," Journal of Mathematical Biology, vol. 64, no. 6, pp. 1005-1020, 2012.

## Research Article

# Nonlinear Dynamics of a Nutrient-Plankton Model 

Yapei Wang, ${ }^{1,2}$ Min Zhao, ${ }^{2,3}$ Chuanjun Dai, ${ }^{2,3}$ and Xinhong Pan ${ }^{1,2}$<br>${ }^{1}$ School of Mathematics and Information Science, Wenzhou University, Wenzhou, Zhejiang 325035, China<br>${ }^{2}$ Zhejiang Provincial Key Laboratory for Water Environment and Marine Biological Resources Protection, Wenzhou University, Wenzhou, Zhejiang 325035, China<br>${ }^{3}$ School of Life and Environmental Science, Wenzhou University, Wenzhou, Zhejiang 325035, China

Correspondence should be addressed to Min Zhao; zmen@tom.com
Received 11 October 2013; Revised 3 December 2013; Accepted 4 December 2013; Published 16 January 2014
Academic Editor: Malay Banerjee
Copyright © 2014 Yapei Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We investigated a nonlinear model of the interaction between nutrients and plankton, which was addressed using a pair of reaction-advection-diffusion equations. Based on numerical analysis, we studied a model without diffusion and sinking terms, and we found that the phytoplankton density (a stable state) increased with the increase of nutrient density. We analyzed the model using a linear analysis technique and found that the sinking of phytoplankton could affect the system. If the sinking velocity exceeded a certain critical value, the stable state became unstable and the wavelength of phytoplankton increased with the increase of sinking velocity. Furthermore, band patterns were also produced by our model, which was affected by the diffusion and sinking of phytoplankton. Thus, the change in the diffusion and sinking of phytoplankton led to different spatial distributions of phytoplankton. All of these results are expected to be useful in the study of plankton dynamics in aquatic ecosystems.


## 1. Introduction

Plankton play an important role in the ecology of the ocean and climate because of their participation in the global carbon cycle at the base of the food chain [1]. In certain environmental conditions, lakes, reservoir, and marine waters may experience plankton or algal blooms [2,3]. However, the local and global impacts of plankton blooms on water quality, carbon cycling, and climate may be damaging. If nutrient source is abundant, and some conditions are satisfied, blooms may become long-term events that affect ecosystems. Plankton blooms can change the types of species present at the base of the aquatic food web and affect human health. Thus, the study of plankton dynamics is currently of major interest.

In the past years, there were many researches on the model between nutrient and phytoplankton and zooplankton [4-6]. A larger number of researchers have attempted to model the relationship between nutrient and phytoplankton and zooplankton, to investigate the dynamics in plankton model. Truscott and Brindley [7] presented a model for the evolution of phytoplankton and zooplankton populations which resembles models for the behavior of excitable media.

Luo [8] investigated phytoplankton-zooplankton dynamics in periodic environments, where eutrophication was considered. El Saadi and Bah [9] modeled phytoplankton aggregation using numerical treatment and explored the asymptotic behavior of the model. Banerjee and Venturino [10] studied a phytoplankton-toxic phytoplankton-zooplankton model and found that the toxic phytoplankton does not drive the zooplankton population towards extinction under a certain mechanism. The result is very important for study on plankton. These works make contributions for the study on plankton.

In recent years, many ecologists have paid increasing attention to spatial processes in a wide variety of practical contexts [11]. For example, theoretical community ecologists have explored ecosystems, including vegetation systems [12] and phytoplankton systems [13]. In particular, the modeling of plankton systems is becoming increasingly important because of their roles in carbon cycling and temperature control, particularly their major impacts on global climate change [14]. These modeling strategies are focused on two areas: (i) studies of large and complex systems, which are eventually used to fit field data or to forecast future changes;
and (ii) studies of skeleton models for various mechanisms, which can provide insights or stimulate new experiments [14].

The present study belongs to the latter area. We propose a model on phytoplankton using a pair of reaction-advectiondiffusion equations, which allow spatial phenomena, such as sinking, and turbulence to be described directly, thereby enabling spatial structures to be studied. It is known that the sinking and mixing of phytoplankton have pronounced effects on the tendency of different phytoplankton to increase. Experimental studies indicated that most fresh water diatoms and other phytoplankton sink in undisturbed water [15]. Theoretical results also demonstrated the importance of sinking, mixing, and diffusion [16, 17]. Theoretical models predicted that a process with reduced vertical mixing may induce oscillations and chaos in the phytoplankton of the deep chlorophyll maxima, which leads to differences in the timescale between the sinking flux of phytoplankton and the upward flux of nutrients [18]. A remarkable finding was the survival of a sinking phytoplankton population even when the diffusivity in the deep layers could not prevent population washout [19-21]. Mellard et al. demonstrated how externally imposed heterogeneity in the form of resource gradients and mixing interacted with internally generated heterogeneity in the form of competition, population dynamics, and movement to determine the spatial distribution of phytoplankton [22]. Ryabov et al. showed that the upper mixed layer was an important factor that had the potential to shape the spatial distribution and species composition of phytoplankton, but it also provided insights with general ecological importance [23]. van de Koppel et al. studied selforganized spatial patterning in an algae-mussel model, where regular spatial patterns were formed in young mussel beds on soft sediments in the Wadden Sea [24]. Self-organized spatial patterns are of considerable interest to theoretical biology [25-30] because of the basic paper by Turing [31] on the role of nonequilibrium reaction-diffusion prepatterns in biomorphogenesis. Furthermore, recent modeling studies of plankton support the self-organized spatial patterns, such as patchiness $[32,33]$ and bands $[25,34,35]$.

The rest of this paper is organized as follows. In the next section, we present a model based on the theoretical ecology and partial differential equations, which is addressed using a pair of reaction-advection-diffusion equations. In Section 3, we analyze stable behavior of the nonspatial system firstly. What is more, the stable behavior of the spatial system is analyzed. And we obtain the condition under which the steady state becomes unstable. Finally, a series of simulations are given. Using simulation, we investigate the effect of critical factor on the system. In Section 4, discussion and conclusion are presented.

## 2. The Model

Natural ecosystems of plankton exhibit great variability in space and time. The growth of phytoplankton is dependent mainly on nutrients and light. After the mortality of phytoplankton, nutrients are returned to the system over short time scales with minimal losses [39] through
microbial decomposition. In addition, biological factors such as higher predation and physical factors such as the sinking of phytoplankton into the water column also affect the ecosystem, which has been examined previously [40]. Turbulence also affects these systems [3, 19, 41]. Vertical mixing brings nutrients from the lower layers of the ocean into the mixed layer. Based on the previous analysis, the following general structure is obtained:

$$
\begin{gather*}
\frac{\partial N}{\partial t}=\text { input }- \text { uptake }+ \text { recycling }+ \text { mixing }  \tag{1a}\\
\frac{\partial P}{\partial t}=\text { growth }- \text { mortality }- \text { predation }- \text { sinking }+ \text { mixing } \tag{lb}
\end{gather*}
$$

where $N$ is the nutrient density and $P$ is the phytoplankton population density.

Dugdale proposed the use of Michaelis-Menten enzyme kinetics to describe nutrient-phytoplankton interactions [42]. The Michaelis-Menten equations have the same form as the well-known Monod equations [43], which are used in the Droop equations, and they have formed the basis of a number of modeling studies that aimed to simulate phytoplankton blooms [44]. Thus, we employed MichaelisMenten kinetics in terms of "uptake." Furthermore, a Holling type II functional response has been used widely to describe zooplankton predation in various theoretical studies [45, 46]. It has also been reported that the Holling type II functional response shows good concordance with experimental data [33, 47, 48]. Hence, in the present paper, Holling II functional response is adopted to describe zooplankton grazing on phytoplankton. Therefore, a pair of specific models is defined as follows:

$$
\begin{align*}
\frac{\partial N}{\partial t}= & f(N, P)+d_{N} \Delta N=k\left(N_{0}-N\right) \\
& -\alpha \beta \frac{N}{H_{N}+N} P+\varepsilon m P+d_{N} \Delta N  \tag{2a}\\
\frac{\partial P}{\partial t}= & g(N, P)-v \frac{\partial P}{\partial z}+d_{P} \Delta P=\beta \frac{N}{H_{N}+N} P  \tag{2b}\\
& -m P-f_{P} \frac{P}{H_{P}+P}-v \frac{\partial P}{\partial z}+d_{P} \Delta P
\end{align*}
$$

where a vertical water column is considered. Let $z$ indicate the depth in the water column; $x$ is the width in the water column. For vertical mixing, we assume that $N_{0}$ is a constant concentration, which includes the nutrient input flowing into the system and the nutrient from below the mixed layer, $k$ is the rate of exchange between the lower and upper layers, $\alpha$ is the nutrient content of phytoplankton, $\beta$ denotes the maximum growth rate of phytoplankton, $H_{N}$ is the half-saturation constant for nutrients, $H_{P}$ is the half-saturation constant for phytoplankton, $f_{P}$ denotes the maximum predation rate of zooplankton on phytoplankton, $m$ is the mortality of phytoplankton, $\varepsilon$ is the proportion of nutrients in dead phytoplankton that is recycled, $v$ is the sinking velocity of phytoplankton, and $d_{N}$ and $d_{P}$ are the
diffusion rates of nutrients and phytoplankton, respectively, which are caused by mixing and turbulence. In addition, $\Delta$ is the Laplacian operator. Table 1 provides the parameter values used and their units, which is obtained from published studies [19, 36-38].

## 3. Results

3.1. Stable Behavior of the Nonspatial System. In the nonspatial system (i.e., system (2a), (2b) without spatial derivatives), according to $f(x, y)=0$ and $g(x, y)=0$, vertical isocline and horizontal isocline can be obtained, respectively, as follows. Vertical isocline $l_{1}: P=\left(k\left(N_{0}-N\right)\left(H_{N}+N\right)\right) /((\alpha \beta-\varepsilon m) N-$ $\left.\varepsilon m H_{N}\right)$. Horizontal isocline $l_{2}: P=\left(f_{P}\left(H_{N}+N\right) /((\beta-\right.$ $\left.\left.m) N-m H_{N}\right)\right)-H_{P}$. Obviously, the line, $N=\varepsilon m H_{N} /$ $(\alpha \beta-\varepsilon m)$, is asymptote of vertical isocline $l_{1}$, and the line, $N=\left(m H_{N}\right) /(\beta-m)$, is asymptote of horizontal isocline $l_{2}$. In the following discussion, it is assumed that the condition $\beta>m$ always holds; otherwise phytoplankton become extinct eventually. For vertical isocline $l_{1}, P^{\prime}=\left((\varepsilon m-\alpha \beta) N^{2}+\right.$ $\left.2 \varepsilon m H_{N} N+\left(\varepsilon m H_{N}-\alpha \beta N_{0}\right) H_{N}\right) /\left((\varepsilon m-\alpha \beta) N+\varepsilon m H_{N}\right)^{2}$, which is derivative. There are two roots in $P^{\prime}=0$ when the condition $\varepsilon m>\alpha \beta$ holds, root is $N=$ $\left(\varepsilon m H_{N} \pm \sqrt{\alpha \beta H_{N}\left(\varepsilon m H_{N}+(\varepsilon m-\alpha \beta) N_{0}\right)}\right) /(\alpha \beta-\varepsilon m)$. Then, it is obvious that the asymptote, $N=\varepsilon m H_{N} /(\alpha \beta-\varepsilon m)$, is on the left of line $N=0$. Therefore, the vertical isocline $l_{1}$ is continuous when $N>0$. And $P>0$ holds when $N>N_{0}$; $P<0$ holds when $0<N<N_{0}$. From the horizontal isocline $l_{2}$, if $f_{P}>H_{P}(\beta-m)$, then there is a positive equilibrium in the nonspatial system at least; if $f_{P}<H_{P}(\beta-m)$, then there is a positive equilibrium in the nonspatial system at least when the condition $m H_{N} /(\beta-m)>N_{0}$ holds.

When the condition $\varepsilon m<\alpha \beta$ holds, there is no root in $P^{\prime}=0$ if $N_{0}>\left(\varepsilon m H_{N} /(\alpha \beta-\varepsilon m)\right)$. From $P^{\prime}$, vertical isocline $l_{1}$ is monotone decreasing when $N>\left(\varepsilon m H_{N} /(\alpha \beta-\varepsilon m)\right)$, and $P<0$ holds when $N \in\left(0,\left(\varepsilon m H_{N} /(\alpha \beta-\varepsilon m)\right)\right) ; P>0$ holds when $N \in\left(\left(\varepsilon m H_{N} /(\alpha \beta-\varepsilon m)\right), N_{0}\right)$. According to horizontal isocline $l_{2}$, if $f_{P}>H_{P}(\beta-m)$, then there is a positive equilibrium in the nonspatial system at least when the condition $(m / \beta) \varepsilon<\alpha<\varepsilon$ holds; if $f_{P}<H_{P}(\beta-m)$, then there is no positive equilibrium in the nonspatial system when the condition $\left(m H_{N} /(\beta-m)\right)>N_{0}$ holds.

When the condition $\varepsilon m<\alpha \beta$ holds, there are two roots in $P^{\prime}=0$ if $N_{0}<\left(\varepsilon m H_{N} /(\alpha \beta-\varepsilon m)\right)$. From $P^{\prime}$, vertical isocline $l_{1}$ is monotone increasing, when $N \in\left(N_{0},\left(\varepsilon m H_{N} /(\alpha \beta-\varepsilon m)\right)\right)$. And $P<0$ holds, when $N \in\left(0, N_{0}\right) \cup\left(\left(\varepsilon m H_{N} /(\alpha \beta-\varepsilon m)\right)\right.$, $+\infty) ; P>0$ holds, when $N \in\left(N_{0},\left(\varepsilon m H_{N} /(\alpha \beta-\varepsilon m)\right)\right)$. According to horizontal isocline $l_{2}$, if $f_{P}>H_{P}(\beta-m)$, then there is a positive equilibrium state in the nonspatial system at least when the condition $(m / \beta) \varepsilon<\alpha<\varepsilon$ holds; if $f_{P}<H_{P}(\beta-m)$, then there is no positive equilibrium state in the nonspatial system when the condition $\alpha>\varepsilon$ holds.

It is noted that these conditions only confirm that there exists positive equilibrium state in the nonspatial system when these conditions are satisfied. It does not mean that there must be no positive equilibrium state in the nonspatial system when these conditions are not satisfied. In addition, it is not difficult to find that there is always a trivial steady state,

Table 1: Parameter values used.

| Symbol | Value | Unit |
| :--- | :---: | :---: |
| $N_{0}$ | 0.5 | $\mathrm{~g} \cdot \mathrm{~m}^{-3}$ |
| $k$ | 0.08 | day $^{-1}$ |
| $\alpha$ | 0.02 | dimensionless |
| $\beta$ | 0.5 | day $^{-1}$ |
| $\varepsilon$ | 0.01 | dimensionless $^{m}$ |
| $m$ | 0.24 | day $^{-1}$ |
| $f_{P}$ | 2 | $\mathrm{~g} \cdot \mathrm{~m}^{-2} \cdot \mathrm{day}^{-1}$ |
| $H_{N}$ | 0.005 | $\mathrm{~g} \cdot \mathrm{~m}^{-2}$ |
| $H_{P}$ | 4 | $\mathrm{~g} \cdot \mathrm{~m}^{-2}$ |
| $v$ | 1.008 | $\mathrm{~m} \cdot \mathrm{day}^{-1}$ |
| $d_{N}$ | 1.038 | $\mathrm{~m} \cdot \mathrm{day}^{-1}$ |
| $d_{P}$ | 1.038 | $\mathrm{~m}^{2} \cdot \mathrm{day}^{-1}$ |

Note: parameter value $\varepsilon$ was estimated, parameter value $H_{P}$ was estimated based on de Angelis et al. [36], and the other parameter values were derived from previous studies [19, 36-38].
$E_{0}=\left(N_{0}, 0\right)$, consisting of bare nutrients without phytoplankton in the nonspatial model. The Jacobian matrix of nonspatial system at the equilibrium $E_{0}=\left(N_{0}, 0\right)$ is

$$
A=\left(\begin{array}{cc}
-k & \varepsilon m-\frac{\alpha \beta N_{0}}{H_{N}+N_{0}}  \tag{3}\\
0 & \frac{(\beta-m) N_{0}-m H_{N}}{H_{N}+N_{0}}-\frac{f_{P}}{H_{P}}
\end{array}\right)
$$

It is obvious that the index of equilibrium $E_{0}$ is +1 , when the condition $\beta N_{0} H_{P}<\left(m H_{P}+f_{P}\right)\left(N_{0}+H_{N}\right)$ holds, which is stable. In particular, when the conditions $\varepsilon m<\alpha \beta$ and $f_{P}<$ $H_{P}(\beta-m)$ hold, if $\left(m H_{N} /(\beta-m)\right)>N_{0}>\left(\varepsilon m H_{N} /(\alpha \beta-\varepsilon m)\right)$ or $N_{0}<\left(\varepsilon m H_{N} /(\alpha \beta-\varepsilon m)\right)$ and $\alpha>\varepsilon$, then there is no positive equilibrium in nonspatial system. Under these conditions, equilibrium $E_{0}$ is locally asymptotically stable. Furthermore, equilibrium $E_{0}$ is globally asymptotically stable in $\Omega=[0,+\infty) \times[0,+\infty]$. Because first quadrant is a positive invariant set according to $f(x, y)$ and $g(x, y)$, there is no limit cycle because there is no equilibrium in first quadrant.

Based on previous discussion, there exists positive equilibrium in the nonspatial model under some conditions, which is defined by $E_{*}=\left(N_{*}, P_{*}\right)$. The Jacobian matrix of nonspatial model at the equilibrium $E_{*}=\left(N_{*}, P_{*}\right)$ is

$$
T=\left(\begin{array}{cc}
-k-\frac{\alpha \beta H_{N} P_{*}}{\left(H_{N}+N_{*}\right)^{2}} & \frac{k\left(N_{*}-N_{0}\right)}{P_{*}}  \tag{4}\\
\frac{\beta H_{N} P_{*}}{\left(H_{N}+N_{*}\right)^{2}} & \frac{f_{P} P_{*}}{\left(H_{P}+P_{*}\right)^{2}}
\end{array}\right)
$$

From $T$, it is easy to find that the equilibrium $E_{*}=\left(N_{*}\right.$, $P_{*}$ ) is unstable when $N_{*}>N_{0}$, which is saddle. When $N_{*}<$ $N_{0}$, the index of equilibrium $E_{*}$ is +1 when the condition $k \beta H_{N}\left(N_{0}-N_{*}\right)\left(H_{P}+P_{*}\right)^{2}>\left(k\left(H_{N}+N_{*}\right)^{2}+\alpha \beta H_{N} P_{*}\right) f_{P} P_{*}$ holds, and it is locally asymptotically stable using RothHurwitz criterion when the condition $\left(f_{P} P_{*} /\left(H_{P}+P_{*}\right)^{2}\right)<$ $k+\left(\alpha \beta H_{N} P_{*} /\left(H_{N}+N_{*}\right)^{2}\right)$ holds.


FIGURE 1: The existence and stability of an equilibrium versus the parameter $N_{0}(\mathrm{a}, \mathrm{b})$ and parameter $\varepsilon(\mathrm{c}, \mathrm{d})$. The other parameters are given in Table 1.

Although the expression of equilibrium $E_{*}$ can hardly be obtained, the stable behavior of the nonspatial system is determined when some parameters are given in Table 1. In the present paper, our interest is how some factors, such as the nutrient concentration $N_{0}$ and nutrient cycling effect $\varepsilon$, affect the system. Hence, the stable behavior of the nonspatial system is analyzed using the graph (see Figure 1). In Figure 1, when the nutrient concentration $N_{0}$ or nutrient cycling effect $\varepsilon$ increases, there is always a trivial equilibrium consisting of bare nutrients without phytoplankton. In Figures 1(a) and 1 (b), when the nutrient concentration is $0 \leq N_{0}<0.489$, there is only a trivial steady state in the nonspatial system. When the nutrient concentration $0.489<N_{0}<0.609$, there were two other steady states: one is always unstable (green dashed, saddle), while the other is stable (red solid line, focus). When the nutrient concentration is $0.609<N_{0}<2$, the focus disappears, and a node emerges (blue line, node). In Figures 1(c) and 1(d), a similar analysis is obtained, and the difference among Figures 1(a), 1(b), 1(c), and 1(d) is indicated by the grey zone. The trivial steady state and saddle coexist in the grey zone. In the following discussion, the nontrivial homogeneous steady state (focus or node) is defined by $E^{*}=$ $\left(N^{*}, P^{*}\right)$.
3.2. Stable Behavior of the Spatial System. In this section, we consider the sensitivity of the system (2a), (2b) to change in the parameter values. A linear analysis technique is employed to focus on the parameters essential for the system behavior
[49]. Our interest is how the nutrient concentration $N_{0}$, nutrient cycling effect $\varepsilon$, sinking velocity $v$, and the diffusion rate of phytoplankton $d_{P}$ affect the system. Symmetry breaking occurred when the stationary homogeneous solution, $E^{*}=$ ( $N^{*}, P^{*}$ ), is linearly unstable to small spatial perturbations in the presence of diffusion and advection, but that is linearly stable to perturbations in the absence of the diffusion and advection terms. To analyze the spatial system and determine how a small heterogeneous perturbation of the homogeneous steady state developed within a large time period, the following perturbation is considered [41]:

$$
\begin{equation*}
\binom{N}{P}=\binom{N^{*}}{P^{*}}+\delta\binom{N_{0}}{P_{0}} \exp (\lambda t+i k z)+c . c .+O\left(\varepsilon^{2}\right) \tag{5}
\end{equation*}
$$

where $\lambda$ is the perturbation growth rate, $k$ is the wavenumber, and $i$ is an imaginary unit $\left(i^{2}=-1\right)$. Substituting expression (5) into (2a), (2b) and neglecting all nonlinear terms in $N$ and $P$, the following characteristic equation is obtained for the eigenvalues $\lambda$ :

$$
\left|\begin{array}{cc}
a_{11}-k^{2} d_{N}-\lambda & a_{12}  \tag{6}\\
a_{21} & a_{22}-k^{2} d_{P}-i v k-\lambda
\end{array}\right|=0
$$



Figure 2: (a) An illustration of $a_{22}>0$ versus $\varepsilon$ and $N_{0}$. The other parameters are given in Table 1. (b) An illustration of the sign of $\Delta_{k}$ versus $d_{P}$ and $N_{0}, v=0.6\left(\mathrm{~m} \cdot \mathrm{day}^{-1}\right)$.
where the elements of the Jacobian determinant of the nonspatial system are taken at the stationary homogeneous solution $E^{*}=\left(N^{*}, P^{*}\right)$, as follows:

$$
J=\left(\begin{array}{cc}
-k-\frac{\alpha \beta H_{N} P^{*}}{\left(H_{N}+N^{*}\right)^{2}} & \frac{k\left(N^{*}-N_{0}\right)}{P^{*}}  \tag{7}\\
\frac{\beta H_{N} P^{*}}{\left(H_{N}+N^{*}\right)^{2}} & \frac{f_{P} P^{*}}{\left(H_{P}+P^{*}\right)^{2}}
\end{array}\right)=\left(\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right)
$$

By $J$, it is not difficult to find that $a_{11}<0, a_{21}>0$, and $a_{22}>0$, when $E^{*}$ is positive equilibrium. When $N^{*}<N_{0}$, $a_{12}<0$. The characteristic equation (8) can be described as

$$
\begin{equation*}
\lambda^{2}-\left(\operatorname{tr}_{k}-i v k\right) \lambda+\Delta_{k}+i v k\left(k^{2} d_{N}-a_{11}\right)=0 \tag{8}
\end{equation*}
$$

where $\operatorname{tr}_{k}=\left(a_{11}+a_{22}\right)-\left(d_{N}+d_{P}\right) k^{2}$ and $\Delta_{k}=a_{11} a_{22}-$ $a_{21} a_{12}-k^{2}\left(a_{11} d_{P}+a_{22} d_{N}\right)+k^{4} d_{N} d_{P}$. In the previous analysis, the parameters $N_{0}, \varepsilon, v$, and $d_{P}$ are allowed to vary, but the other parameters are fixed in Table 1. In Figure 2(a), the value of $a_{22}$ is given when the parameters $N_{0}$ and $\varepsilon$ are changed. In addition, in Figure 2(b), in zone III, $a_{11} d_{P}+a_{22} d_{N}<0$, so $\Delta_{k}>0$ for $k>0$; in zone II, $a_{11} d_{P}+a_{22} d_{N}>0$, but $\min \left(\Delta_{k}\right)>0$ for $k>0$, so $\Delta_{k}>0$; in zone I, $a_{11} d_{P}+a_{22} d_{N}>0$, and $\min \left(\Delta_{k}\right)<0$ for $k>0$. In zone I, to determine the sign of $\Delta_{k}$ for different values of $N_{0}$ and $d_{P}$, we need to analyze $\Delta_{k}$ further because of the $\min \left(\Delta_{k}\right)<0$ for $k>0$.

From expression (8), we can obtain

$$
\begin{equation*}
\lambda=\frac{1}{2}\left[\operatorname{tr}_{k}-i v k \pm \sqrt{\Phi+i \Theta}\right] \tag{9}
\end{equation*}
$$

where $\Phi=\operatorname{tr}_{k}^{2}-v^{2} k^{2}-4 \Delta_{k}$ and $\Theta=-2\left(\operatorname{tr}_{k}+2\left(k^{2} d_{N}-a_{11}\right)\right) v k$.

To analyze the spatial system, the real and imaginary parts of the eigenvalues must be obtained, which are described as follows:

$$
\begin{gather*}
\operatorname{Re}(\lambda)=\frac{1}{2}\left[\operatorname{tr}_{k}+j \sqrt{\frac{1}{2}\left(\sqrt{\Phi^{2}+\Theta^{2}}+\Phi\right)}\right],  \tag{10a}\\
\operatorname{Im}(\lambda)=\frac{1}{2}\left[-v k+j \operatorname{sign}(\Phi) \sqrt{\frac{1}{2}\left(\sqrt{\Phi^{2}+\Theta^{2}}-\Phi\right)}\right], \tag{10b}
\end{gather*}
$$

where $j= \pm 1$. The solution is stable when the real parts of all eigenvalues are less than zero; that is, $\operatorname{Re}(\lambda)<0$. The solution is unstable when one of the real parts with a finite wave number $k>0$ is greater than zero at least. The critical point is got when $\operatorname{Re}(\lambda)=0$. However, the analytical expression for the critical point is difficult to be obtained. Indeed, we only need to consider the maximum value of $\operatorname{Re}(\lambda)$. Thus, the critical condition can be obtained using $\operatorname{Re}(\lambda)=0$, as follows:

$$
\begin{equation*}
v^{2}=\frac{\operatorname{tr}_{k}^{2} \Delta_{k}}{\left(k^{2} d_{N}-a_{11}\right)\left(a_{22}-d_{P} k^{2}\right) k^{2}} \tag{11}
\end{equation*}
$$

In expression (11), the solution is unstable if the righthand side of the equal sign is always less than zero. Otherwise, a necessary condition for expression (11) to hold is that $\Delta_{k}\left(a_{22}-d_{p} k^{2}\right)>0$. We consider the following case: the nutrient concentration $N_{0}$ is allowed to vary, but the values of other parameters are in Table 1. Then, the sinking velocity $v$ is a function related to the nutrient concentration $N_{0}$ and the wave number $k$. By Figure 2(b), if the diffusion rate of phytoplankton, $d_{P}$, is larger than $0.2 \mathrm{~cm}^{2} \cdot \mathrm{~s}^{-1}$, then $\Delta_{k}>0$. Thus, the right-hand side of expression (11) is positive within $0<k<\sqrt{a_{22} / d_{P}}$, and the sinking velocity $v$ has a minimum $v_{c}$ at the point $k=k_{c}$. Figure 3(a) confirms expression (11). In Figure 3(a), the neutral curve is convex with a unique minimum in the range $0<k<\sqrt{a_{22} / d_{P}}$.


Figure 3: (a) A typical neutral curve $v$, defined using expression (11) for different values of $N_{0}$. (b) Numerical calculation of the stability on $\left(v, N_{0}\right)$ space. (c) An illustration of the dispersion relation $(\operatorname{Re}(\lambda), \operatorname{Im}(\lambda)$ versus the wave number $k)$, where $d_{p}=0.3\left(\mathrm{~m}^{2} \cdot\right.$ day $\left.^{-1}\right)$ and the other parameters are given in Table 1. Blue line: $v=0.6>v_{c} \approx 0.358\left(\mathrm{~m} \cdot\right.$ day $\left.{ }^{-1}\right)$; red line: $v=v_{c} \approx 0.358\left(\mathrm{~m} \cdot\right.$ day $\left.{ }^{-1}\right)$; green line: $v=0.1<v_{c} \approx$ $0.358\left(\mathrm{~m} \cdot \mathrm{day}^{-1}\right)$.

The effects of sinking velocity $v$ and nutrient concentration $N_{0}$ on the behavior of system (2a), (2b) are shown in Figure 3(b), which shows the transition from a no phytoplankton state through a banded phytoplankton state to a homogeneous phytoplankton state when the nutrient concentration $N_{0}$ increases and the sinking velocity $v$ is fixed. However, when the nutrient concentration $N_{0}$ is fixed, a homogeneous phytoplankton state becomes a banded phytoplankton state with the increase of sinking velocity $v$, that is, a banded self-organized spatial pattern emerges because of the sinking velocity $v$.

In zone I of Figure 3(b), there is only a trivial steady state, that is, no phytoplankton. In zone II, the steady state is stable, while the steady state becomes unstable in zone III
because of the effect of sinking velocity of phytoplankton, which is further confirmed by Figure 3(c), where $v_{c} \approx$ $0.358\left(\mathrm{~m} \cdot\right.$ day $\left.^{-1}\right)$ and $k_{c} \approx 0.283$. It is obvious that the maximal real part of $\lambda$ is larger than zero when the sinking velocity of phytoplankton is larger than the critical value of the sinking velocity of phytoplankton; that is, the instability of the steady state will occur. The imaginary value of $\lambda$ is not equal to zero. In Figure 3(b), the red line represents the critical value of the sinking velocity of phytoplankton. The critical value of the sinking velocity of phytoplankton increases with the increase of nutrient concentration $N_{0}$; that is, when the nutrient concentration $N_{0}$ increases, the sinking velocity $v$ cannot affect the stability of the stable state.

### 3.3. Effects of the Parameters on the Wavelength of the

 Banded Pattern. Section 3.2 has described how a banded phytoplankton state emerges when the sinking velocity $v$ has reached a certain critical value. The determination of the wavelength of the banded pattern is a key issue. In particular, how do the parameters, such as the nutrient concentration $N_{0}$, and the sinking velocity $v$, affect the change of wavelength? The relationships among the wavelength, the nutrient concentration $N_{0}$, and the sinking velocity $v$ are shown in Figure 4, which shows that the wavelength increases when the sinking velocity $v$ exceeds the critical value $v_{c}$, but the wavelength decreases when the nutrient concentration $N_{0}$ increases.3.4. The Simulation. In the previous sections, we discussed the effects of parameters, including the nutrient concentration $N_{0}$ and the sinking velocity $v$, on the system (2a), (2b). In this section, we discuss the numerical solution of the system (2a), (2b) in one-dimensional and two-dimensional spaces. In a one-dimensional space, a periodic boundary condition is employed, and system (2a), (2b) is solved on a rectangular spatial grid of $1 \times 200$ points. In two-dimensional space, system (2a), (2b) is studied in a horizontal $(x, z)$ plane with zero-flux boundary conditions (left and right) and periodic boundary conditions (top and bottom), which is solved on a rectangular spatial grid of $100 \times 300$ points. The initial conditions comprise a homogeneous state which is randomly perturbed. Furthermore, we assume that the diffusion rate of phytoplankton is larger or smaller than that of nutrients because of the viscosity and living of phytoplankton. Of course, it is also feasible that the diffusion rate of phytoplankton is equal to that of nutrients.

Firstly, the one-dimensional solution of system (2a), (2b) is shown in Figure 5. In Figure 5, we consider a vertical water column, where the depth of the water column is 120 m and the time is 600 days. We found that oscillations did occur; that is, the stable state became unstable because of spatial effects. Figure 6 shows the analysis of relationship between nutrients and phytoplankton further. In Figure 6(a), the relationship between the spatial distributions of nutrients and phytoplankton is given at the 600th day, which shows that nutrient concentration reaches the minimal value when the density of phytoplankton reaches the maximal value. The nutrient concentration affects the density of phytoplankton, and the density of phytoplankton increases with the increases of nutrient concentration. Thus, eutrophication may explain phytoplankton blooms. Furthermore, the effects of phytoplankton sinking on the relative maxima for nutrients and phytoplankton are shown in Figure 6(b). The relative maxima of phytoplankton increase with the increase of sinking velocity, whereas the relative maxima of nutrients decrease with the increase of sinking velocity. Therefore, the sinking flux has an important role in the increase of the density of phytoplankton.

To further analyze the dynamic behavior of system (2a), (2b), we consider the solution of system (2a), (2b) in two-dimensional space. The band patterns are observed in the field, as shown in Figure 7. Figures 7(a), 7(b), and


Figure 4: An illustration of the variation in the pattern wavelength with $v$ and $N_{0}$, where the symbol ( $\square$ ) represents the critical value of the sinking velocity $v, d_{P}=0.3\left(\mathrm{~m}^{2} \cdot\right.$ day $\left.^{-1}\right)$, and the other parameters are given in Table 1.


Figure 5: Model simulations based on depth and time: (a) nutrient density (unit $\mathrm{g} \cdot \mathrm{m}^{-3}$ ); (b) phytoplankton density (unit $\mathrm{g} \cdot \mathrm{m}^{-3}$ ), where $d_{P}=1.2\left(\mathrm{~m}^{2} \cdot \mathrm{day}^{-1}\right), v=1.2\left(\mathrm{~m} \cdot \mathrm{day}^{-1}\right)$, and the other parameters are given in Table 1.


Figure 6: (a) Spatial distributions of the nutrient density and phytoplankton density on the 600th day, where the left vertical axis denotes the nutrient density, the right vertical axis denotes the phytoplankton density, $d_{P}=0.3\left(\mathrm{~m}^{2} \cdot \mathrm{day}^{-1}\right), v=1.2\left(\mathrm{~m} \cdot\right.$ day $\left.{ }^{-1}\right)$, and the other parameters are given in Table 1. (b) Relationship between nutrient density, phytoplankton density, and the sinking velocity $v$, where the right vertical axis denotes the maximal nutrient density, and the left vertical axis denotes the maximal phytoplankton density. The purple field shows that the equilibrium state is stable, where $N_{0}=0.5\left(\mathrm{~g} \cdot \mathrm{~m}^{-3}\right)$ and $d_{P}=0.3\left(\mathrm{~m}^{2} \cdot \mathrm{day}^{-1}\right)$.


FIgure 7: Simulation in two-dimensional space. The figures show the density levels of the phytoplankton on the 1000th day, where the width is 100 m and the depth is 300 m . (a) $d_{P}=0.3\left(\mathrm{~m}^{2} \cdot\right.$ day $\left.^{-1}\right)$ and $v=0.9\left(\mathrm{~m} \cdot\right.$ day $\left.{ }^{-1}\right)$. (b) $d_{P}=1.038\left(\mathrm{~m}^{2} \cdot\right.$ day $\left.^{-1}\right)$ and $v=1.2\left(\mathrm{~m} \cdot \mathrm{day}^{-1}\right)$. (c) $d_{P}=1.038\left(\mathrm{~m}^{2} \cdot \mathrm{day}^{-1}\right)$ and $v=0.9+0.25 * \sin (4 * \pi * x / 100)$. The other parameters are given in Table 1.

7(c) show the patterns of phytoplankton at the 1000th day in the two-dimensional space. As discussed in Section 3.2, our numerical results confirm the predictions of the linear analysis that a band pattern of phytoplankton occurs if the nutrient concentration $N_{0}$ and the sinking velocity of phytoplankton satisfy some conditions. Figure 7(a) shows the emergence of parallel and crossed patterns, which indicate that band patterns with different speeds coexist in the system
(2a), (2b) where the wavelength of patterns is different. By contrast, the patterns in Figure 7(b) are much more regular and almost parallel. In the real world, the sinking velocity of phytoplankton varies at different spatial points. Thus, to add more realism to the system, we forced the model to undergo periodic changes in the sinking velocity of phytoplankton, that is, $v=v_{*}+A \times \sin (4 \pi \times x / L)$, where $L$ denotes the width of water column. The results are shown in Figure 7(c), where
the values of the parameters are the same as those used in Figures 7(a) and 7(b), except the sinking velocity $v$. However, it is obvious that the patterns in Figure 7(c) are very different from those in Figures 7(a) and 7(b). Thus, the sinking flux of phytoplankton has an important role in the system.

## 4. Discussion and Conclusion

Banded patterns have been described in several resourcelimited ecosystems around the world. In the real world, numerous population patterns have been observed, including banded vegetation, patches, and spiral waves, which can be regular or irregular. Physical factors may cause these types of pattern, such as wind, water flow, and turbulence. Internal factors in populations also force these patterns to occur.

In the present study, we used a nutrient-plankton model with both diffusion and advection to investigate the interaction between nutrient and plankton. Our model was simple because it was only an abstraction of real-world phenomena but the model reproduced many features of real-world phenomena. Our explanation focuses on a predator-prey interaction between phytoplankton and their nutrient source. In particular, how do the sinking of phytoplankton and the input of nutrients affect the interaction? Our analytical results showed that the homogeneous steady state became unstable because of the sinking of phytoplankton. The critical value of the sinking phytoplankton led to an instability in the homogeneous steady state, which depended on the input of nutrients. Our numerical results showed that the homogeneous steady state was unstable against small spatially heterogeneous perturbations.

Figure 1(b) shows that when the nutrient concentration $N_{0}$ increased beyond a critical value, the increase in the concentration of phytoplankton was stable; that is, the concentration of phytoplankton tended toward a certain stable state. Spatial effects did not influence the stable state when the sinking flux was below a critical value, as shown in Figure 3(b). Thus, an abundance of nutrient inputs flowed into the system, which led to the high-level reproduction of phytoplankton, which may trigger phytoplankton blooms.

Figures 5 and 7 show that oscillation could occur because of the sinking flux. In particular, Figure 5 shows that both spatial and temporal oscillations arose in the nutrients and the phytoplankton. The sinking of phytoplankton can also lead to the increase in the phytoplankton density and wavelength when the input of nutrients is fixed. It is possible that the phytoplankton sinks from the surface of water until it reaches a depth where the nutrient conditions are suitable for growth. Figure 6(a) shows that the relationship between phytoplankton and nutrients is mutually constrained. Thus, abundant nutrition leads to the mass propagation of phytoplankton, which consumes large amounts of nutrient, thereby depleting the nutrient levels. Thus, the sinking of phytoplankton and the input of nutrients can change the spatial distribution of phytoplankton under these conditions, which may promote the increase of phytoplankton density. In particular, eutrophication may promote phytoplankton blooms.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work was supported by the Key Program of Zhejiang Provincial Natural Science Foundation of China (Grant no. LZ12C03001), the National Natural Science Foundation of China (Grant no. 31170338), and the National Key Basic Research Program of China (973 Program, Grant no. 2012CB426510).

## References

[1] R. Reigada, R. M. Hillary, M. A. Bees, J. M. Sancho, and F. Sagués, "Plankton blooms induced by turbulent flows," Proceedings of the Royal Society B, vol. 270, no. 1517, pp. 875-880, 2003.
[2] A. Huppert, B. Blasius, R. Olinky, and L. Stone, "A model for seasonal phytoplankton blooms," Journal of Theoretical Biology, vol. 236, no. 3, pp. 276-290, 2005.
[3] M. Sandulescu, C. López, and U. Feudel, "Plankton blooms in vortices: the role of biological and hydrodynamic timescales," Nonlinear Processes in Geophysics, vol. 14, no. 4, pp. 443-454, 2007.
[4] C. J. Dai, M. Zhao, and L. S. Chen, "Bifurcations and periodic solutions for an algae-fish semicontinuous system," Abstract and Applied Analysis, vol. 2013, Article ID 619721, 11 pages, 2013.
[5] S. Abbas, M. Banerjee, and N. Hungerbühler, "Existence, uniqueness and stability analysis of allelopathic stimulatory phytoplankton model," Journal of Mathematical Analysis and Applications, vol. 367, no. 1, pp. 249-259, 2010.
[6] J. F. Zhang and D. Zhang, "Hopf-Pitchfork bifurcation in a phytoplankton-zooplankton model with delays," Abstract and Applied Analysis, vol. 2013, Article ID 340174, 5 pages, 2013.
[7] J. E. Truscott and J. Brindley, "Ocean plankton populations as excitable media," Bulletin of Mathematical Biology, vol. 56, no. 5, pp. 981-998, 1994.
[8] J. H. Luo, "Phytoplankton-zooplankton dynamics in periodic environments taking into account eutrophication," Mathematical Biosciences, vol. 245, pp. 126-136, 2013.
[9] N. El Saadi and A. Bah, "Numerical treatment of a nonlocal model for phytoplankton aggregation," Applied Mathematics and Computation, vol. 218, no. 17, pp. 8279-8287, 2012.
[10] M. Banerjee and E. Venturino, "A phytoplankton-toxic phytoplankton-zooplankton model," Ecological Complexity, vol. 8, no. 3, pp. 239-248, 2011.
[11] E. E. Holmes, M. A. Lewis, J. E. Banks, and R. R. Veit, "Partial differential equations in ecology: spatial interactions and population dynamics," Ecology, vol. 75, no. 1, pp.17-29, 1994.
[12] C. A. Klausmeier, "Regular and irregular patterns in semiarid vegetation," Science, vol. 284, no. 5421, pp. 1826-1828, 1999.
[13] A. D. Barton, S. Dutkiewicz, G. Flierl, J. Bragg, and M. J. Follows, "Patterns of diversity in marine phytoplankton," Science, vol. 327, no. 5972, pp. 1509-1511, 2010.
[14] H. Malchow, "Spatio-temporal pattern formation in nonlinear non-equilibrium plankton dynamics," Proceedings of the Royal Society B, vol. 251, no. 1331, pp. 103-109, 1993.
[15] D. A. Bella, "Simulating the effect of sinking and vertical mixing on algal population dynamics," Journal of the Water Pollution Control Federation, vol. 42, no. 5, pp. 140-152, 1970.
[16] O. Kerimoglu, D. Straile, and F. Peeters, "Role of phytoplankton cell size on the competition for nutrients and light in incompletely mixed systems," Journal of Theoretical Biology, vol. 300, pp. 330-343, 2012.
[17] R. K. Upadhyay, N. Kumari, and V. Rai, "Wave of chaos in a diffusive system: generating realistic patterns of patchiness in plankton-fish dynamics," Chaos, Solitons and Fractals, vol. 40, no. 1, pp. 262-276, 2009.
[18] J. Huisman, N. N. Pham Thi, D. M. Karl, and B. Sommeijer, "Reduced mixing generates oscillations and chaos in the oceanic deep chlorophyll maximum," Nature, vol. 439, no. 7074, pp. 322-325, 2006.
[19] N. Shigesada and A. Okubo, "Effects of competition and shading in planktonic communities," Journal of Mathematical Biology, vol. 12, no. 3, pp. 311-326, 1981.
[20] D. C. Speirs and W. S. C. Gurney, "Population persistence in rivers and estuaries," Ecology, vol. 82, no. 5, pp. 1219-1237, 2001.
[21] A. V. Straube and A. Pikovsky, "Mixing-induced global modes in open active flow," Physical Review Letters, vol. 99, no. 18, Article ID 184503, 2007.
[22] J. P. Mellard, K. Yoshiyama, E. Litchman, and C. A. Klausmeier, "The vertical distribution of phytoplankton in stratified water columns," Journal of Theoretical Biology, vol. 269, no. 1, pp. 1630, 2011.
[23] A. B. Ryabov, L. Rudolf, and B. Blasius, "Vertical distribution and composition of phytoplankton under the influence of an upper mixed layer," Journal of Theoretical Biology, vol. 263, no. 1, pp. 120-133, 2010.
[24] J. van de Koppel, M. Rietkerk, N. Dankers, and P. M. J. Herman, "Scale-dependent feedback and regular spatial patterns in young mussel beds," The American Naturalist, vol. 165, no. 3, pp. 66-77, 2005.
[25] W. M. Wang, Z. G. Guo, R. K. Upadhyay, and Y. Z. Lin, "Pattern formation in a cross-diffusive Holling-Tanner model," Discrete Dynamics in Nature and Society, vol. 2012, Article ID 828219, 12 pages, 2012.
[26] M. P. Hassell, H. N. Comins, and R. M. May, "Spatial structure and chaos in insect population dynamics," Nature, vol. 353, no. 6341, pp. 255-258, 1991.
[27] R. K. Upadhyay, N. K. Thakur, and V. Rai, "Diffusiondriven instabilities and spatio-temporal patterns in an aquatic predator-prey system with Beddington-Deangelis type functional response," International Journal of Bifurcation and Chaos, vol. 21, no. 3, pp. 663-684, 2011.
[28] J. von Hardenberg, E. Meron, M. Shachak, and Y. Zarmi, "Diversity of vegetation patterns and desertification," Physical Review Letters, vol. 87, no. 19, Article ID 198101, 4 pages, 2001.
[29] M. Rietkerk, S. C. Dekker, P. C. De Ruiter, and J. Van De Koppel, "Self-organized patchiness and catastrophic shifts in ecosystems," Science, vol. 305, no. 5692, pp. 1926-1929, 2004.
[30] A. I. Borthagaray, M. A. Fuentes, and P. A. Marquet, "Vegetation pattern formation in a fog-dependent ecosystem," Journal of Theoretical Biology, vol. 265, no. 1, pp. 18-26, 2010.
[31] A. M. Turing, "The chemical basis of morphogenesis," Philosophical Transactions of the Royal Society B, vol. 237, pp. 37-72, 1952.
[32] R. Reigada, R. M. Hillary, M. A. Bees, J. M. Sancho, and F. Sagués, "Plankton blooms induced by turbulent flows,"

Proceedings of the Royal Society B, vol. 270, no. 1517, pp. 875-880, 2003.
[33] H. Serizawa, T. Amemiya, and K. Itoh, "Patchiness in a minimal nutrient—phytoplankton model," Journal of Biosciences, vol. 33, no. 3, pp. 391-403, 2008.
[34] R.-H. Wang, Q.-X. Liu, G.-Q. Sun, Z. Jin, and J. Van De Koppel, "Nonlinear dynamic and pattern bifurcations in a model for spatial patterns in young mussel beds," Journal of the Royal Society Interface, vol. 6, no. 37, pp. 705-718, 2009.
[35] Q. X. Liu, E. J. Weerman, P. M. J. Herman et al., "Alternative mechanisms alter the emergent properties of self-organization in mussel beds," Proceedings of the Royal Society B, vol. 279, no. 1739, pp. 2744-2753, 2012.
[36] D. L. de Angelis, S. M. Bartell, and A. L. Brenkert, "Effects of nutrient recycling and food-chain length on resilience," American Naturalist, vol. 134, pp. 778-805, 1989.
[37] A. M. Edwards, "Adding detritus to a nutrient-phytoplanktonzooplankton model: a dynamical-systems approach," Journal of Plankton Research, vol. 23, no. 4, pp. 389-413, 2001.
[38] G. L. Bowie and W. B. Mills, Rates, Constants, and Kinetics Formulations in Surface Water Quality Modeling, US Envitonmental Protection Agency, Athens, Greece, 2nd edition, 1985.
[39] M. Ramin, G. Perhar, Y. Shimoda, and G. B. Arhonditsis, "Examination of the effects of nutrient regeneration mechanisms on plankton dynamics using aquatic biogeochemical modeling," Ecological Modelling, vol. 240, pp. 139-155, 2012.
[40] A. Huppert, B. Blasius, and L. Stone, "A model of phytoplankton blooms," American Naturalist, vol. 159, no. 2, pp. 156-171, 2002.
[41] M. Lévy, D. Iovino, L. Resplandy et al., "Large-scale impacts of submesoscale dynamics on phytoplankton: local and remote effects," Ocean Modelling, vol. 43-44, pp. 77-93, 2012.
[42] R. C. Dugdale, "Nutrient limitation in the sea: dynamics, identification, and significance," Limnology and Oceanography, vol. 12, pp. 685-695, 1967.
[43] D. L. de Angelis, Dynamics of Nutrient Cycling and Food Webs, vol. 9, Springer, 1992.
[44] W. J. O'Brien, "The dynamics of nutrient limitation of phytoplankton algae: a model reconsidered," Ecology, vol. 55, no. 1, pp. 135-141, 1974.
[45] C. S. Holling, "Resilience and stability of ecological systems," Annual Review of Ecology, Evolution, and Systematics, vol. 4, pp. 1-23, 1973.
[46] M. Scheffer, "Fish and nutrients interplay determines algal biomass: a minimal model," Oikos, vol. 62, no. 3, pp. 271-282, 1991.
[47] W. Gentleman, A. Leising, B. Frost, S. Strom, and J. Murray, "Functional responses for zooplankton feeding on multiple resources: a review of assumptions and biological dynamics," Deep-Sea Research Part II, vol. 50, no. 22-26, pp. 2847-2875, 2003.
[48] M. M. Mullin, E. F. Stewart, and F. J. Fuglister, "Ingestion by planktonic grazers as a function of food concentration," Limnology and Oceanography, vol. 20, no. 2, pp. 259-262, 1975.
[49] J. D. Murray, "Mathematical biology," in Biomathematics, vol. 19, Springer, Berlin, Germany, 2nd edition, 1993.

## Research Article

# Multiple Periodic Solutions of a Nonautonomous Plant-Hare Model 

Yongfei Gao, ${ }^{1}$ P. J. Y. Wong, ${ }^{2}$ Y. H. Xia, ${ }^{1}$ and Xiaoqing Yuan ${ }^{1}$<br>${ }^{1}$ Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China<br>${ }^{2}$ School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798

Correspondence should be addressed to Y. H. Xia; yhxia@zjnu.cn
Received 27 November 2013; Accepted 24 December 2013; Published 9 January 2014
Academic Editor: Weiming Wang
Copyright © 2014 Yongfei Gao et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Based on Mawhin's coincidence degree theory, sufficient conditions are obtained for the existence of at least two positive periodic solutions for a plant-hare model with toxin-determined functional response (nonmonotone). Some new technique is used in this paper, because standard arguments in the literature are not applicable.

## 1. Introduction

In the past few decades, the classical predator-prey model has been well studied. Such classical predator-prey model has, however, been questioned by several biologists (e.g., see [1, 2]). Based on experimental data, Holling [3] has proposed several types of monotone functional responses $g(x)=$ $c(t) x, c(t) x /(m+x), c(t) x^{2} /\left(m+x^{2}\right), c(t) x /\left(m+a x+x^{2}\right)$ for these and other models. However, this will not be appropriate if we explore the impact of plant toxicity on the dynamics of plant-hare interactions [4]. Recently, Gao and Xia [5] considered a nonautonomous plant-hare dynamical system with a toxin-determined functional response given by

$$
\begin{gather*}
\dot{N}(t)=r(t) N(t)\left[1-\frac{N(t)}{K}\right]-C(N(t)) P(t)  \tag{1}\\
\dot{P}(t)=B(t) C(N(t)) P(t)-d(t) P(t)
\end{gather*}
$$

where

$$
\begin{gather*}
C(N(t))=f(N(t))\left[1-\frac{f(N(t))}{4 G}\right] \\
f(N(t))=\frac{e \delta N(t)}{1+h e \delta N(t)} \tag{2}
\end{gather*}
$$

Here, $N(t)$ denotes the density of plant at time $t, P(t)$ denotes the herbivore biomass at time $t, r(t)$ is the plant
intrinsic growth rate at time $t, d(t)$ is the per capita rate of herbivore death unrelated to plant toxicity at time $t, B(t)$ is the conversion rate at time $t, e$ is the encounter rate per unit plant, $\delta$ is the fraction of food items encountered that the herbivore ingests, $K$ is the carrying capacity of plant, $G$ measures the toxicity level, and $h$ is the time for handing one unit of plant. The functions $r(t), d(t)$, and $B(t)$ are continuous, positive, and periodic with period $\omega$, and $e, \delta, K, G$, and $h$ are positive real constants. For any continuous $\omega$-periodic function $F$, we let

$$
\begin{equation*}
\bar{F}=\frac{1}{\omega} \int_{0}^{\omega} F(t) \mathrm{d} t . \tag{3}
\end{equation*}
$$

The topological degree of a mapping has long been known to be a useful tool for establishing the existence of fixed points of nonlinear mappings. In particular, a powerful tool to study the existence of periodic solution of nonlinear differential equations is the coincidence degree theory (see [6]). Many papers study the existence of periodic solutions of biological systems by employing the topological degree theory; see, for example, [7-12] and references cited therein. However, most of them investigated the classical predator-prey model or the models with Holling functional responses; see [710]. There is no paper studying the functional responses in model (1) except for [5]. Gao and Xia [5] have obtained some sufficient conditions for the existence of at least one positive periodic solution for the system (1). Unlike the traditional Holling Type II functional response, systems with
nonmonotone functional responses are capable of supporting multiple interior equilibria and bistable attractors. Thus, for nonautonomous system (1), it is possible to find two periodic solutions of (1). However, to date there is no work done on the existence of multiple periodic solutions of (1). Therefore, in this paper we will establish the existence of at least two positive periodic solutions of (1). We will be using the continuation theorem of Mawhin's coincidence degree theory; to this end some novel estimation technique will be employed to obtain a priori bounds of unknown solutions to some operator equation, as the standard estimation techniques used in the literature are not applicable to the system (1) due to the term $C(N(t))$. We will elaborate this in Remark 3.

## 2. Existence of Multiple Positive Periodic Solutions

In this section, we will establish sufficient conditions for the existence of at least two positive periodic solutions of (1). We will first summarize in the following a few concepts and results from [6] that will be required later.

Let $X, Y$ be normed vector spaces, $L: \operatorname{Dom} L \subset X \rightarrow Y$ a linear mapping, and $N: X \rightarrow Y$ a continuous mapping. The mapping $L$ is called a Fredholm mapping of index zero if $\operatorname{dim} \operatorname{Ker} L=$ codim $\operatorname{Im} L<+\infty$ and $\operatorname{Im} L$ is closed in $Y$. If $L$ is a Fredholm mapping of index zero, there exist continuous projectors $P: X \rightarrow X$ and $Q: Y \rightarrow Y$ such that $\operatorname{Im} P=\operatorname{Ker} L$ and $\operatorname{Ker} Q=\operatorname{Im} L=\operatorname{Im}(I-Q)$. It follows that $L \mid \operatorname{dom} L \cap \operatorname{Ker} P:(I-P) X \rightarrow \operatorname{Im} L$ is invertible. We denote the inverse of that map by $K_{p}$. If $\Omega$ is an open bounded subset of $X$, then the mapping $N$ will be called $L$-compact on $\bar{\Omega}$ if $Q N(\bar{\Omega})$ is bounded and $K_{p}(I-Q) N: \bar{\Omega} \rightarrow X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J: \operatorname{Im} Q \rightarrow \operatorname{Ker} L$.

Lemma 1 (see [6]). Let $\Omega \subset X$ be an open bounded set. Let $\bar{L}$ be a Fredholm mapping of index zero and N L-compact on $\bar{\Omega}$. Assume
(a) for each $\lambda \in(0,1), x \in \partial \Omega \cap \operatorname{Dom} L, L x \neq \lambda N x$;
(b) for each $x \in \partial \Omega \cap \operatorname{Ker} L, Q N x \neq 0$;
(c) $\operatorname{deg}\{J Q N, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then $L x=N x$ has at least one solution in $\bar{\Omega} \cap \operatorname{Dom} L$.
To proceed, we note that (1) is equivalent to

$$
\begin{align*}
\dot{N}(t)=N(t) & {\left[r(t)\left(1-\frac{N(t)}{K}\right)\right.} \\
& \left.-\frac{4 G e \delta P(t)+(4 G h-1) e^{2} \delta^{2} N(t) P(t)}{4 G(1+h e \delta N(t))^{2}}\right], \\
\dot{P}(t)=P(t)[ & \frac{4 G e \delta B(t) N(t)+(4 G h-1) e^{2} \delta^{2} B(t) N^{2}(t)}{4 G(1+h e \delta N(t))^{2}} \\
& -d(t)] . \tag{4}
\end{align*}
$$

Throughout, we assume the following:

$$
\begin{aligned}
& \left(A_{1}\right) 1 / 4 h<G<1 / 3 h \\
& \left(A_{2}\right) 4 h \bar{d} \exp (2 \bar{r} \omega)<\bar{B}<4 G \bar{d} h^{2} /(4 G h-1)
\end{aligned}
$$

We further introduce six positive numbers which will be used later as follows:

$$
\begin{gather*}
h_{ \pm}=\frac{(e \delta \bar{B} \exp (-2 \bar{r} \omega)-2 h e \delta \bar{d}) \pm \sqrt{\Delta_{1}}}{2 \bar{d} h^{2} e^{2} \delta^{2}}, \\
l_{ \pm}=\frac{\left[4 G h^{2} e \delta \bar{B} \exp (2 \bar{r} \omega)-2 h e \delta\left(4 G h^{2} \bar{d}-(4 G h-1) \bar{B}\right)\right] \pm \sqrt{\Delta_{2}}}{2 h^{2} e^{2} \delta^{2}\left[4 G h^{2} \bar{d}-(4 G h-1) \bar{B}\right]}, \\
u_{ \pm}=\frac{(4 G e \delta \bar{B}-8 G h e \delta \bar{d}) \pm \sqrt{\Delta_{3}}}{2\left[4 G \bar{d} h^{2} e^{2} \delta^{2}-(4 G h-1) e^{2} \delta^{2} \bar{B}\right]} \tag{5}
\end{gather*}
$$

where

$$
\begin{gather*}
\Delta_{1}=[e \delta \bar{B} \exp (-2 \bar{r} \omega)-2 h e \delta \bar{d}]^{2}-4 \bar{d}^{2} h^{2} e^{2} \delta^{2}, \\
\Delta_{2}=\left[4 G h^{2} e \delta \bar{B} \exp (2 \bar{r} \omega)-2 h e \delta\left(4 G h^{2} \bar{d}-(4 G h-1) \bar{B}\right)\right]^{2} \\
-4 h^{2} e^{2} \delta^{2}\left[4 G h^{2} \bar{d}-(4 G h-1) \bar{B}\right]^{2}, \\
\Delta_{3}=(4 G e \delta \bar{B}-8 G h e \delta \bar{d})^{2} \\
\quad-16 G \bar{d}\left[4 G \bar{d} h^{2} e^{2} \delta^{2}-(4 G h-1) e^{2} \delta^{2} \bar{B}\right] \tag{6}
\end{gather*}
$$

Under assumptions $\left(A_{1}\right)$ and $\left(A_{2}\right)$, it is not difficult to show that

$$
\begin{equation*}
l_{-}<u_{-}<h_{-}<h_{+}<u_{+}<l_{+} \tag{7}
\end{equation*}
$$

Theorem 2. In addition to $\left(A_{1}\right)$ and $\left(A_{2}\right)$, suppose that

$$
\left(A_{3}\right) 1-(1 / K) \exp \left(\ln l_{+}+2 \bar{r} \omega\right)>0 .
$$

Then system (4) has at least two positive $\omega$-periodic solutions.
Proof. Since we are concerned with positive solutions of system (4), we make use of the change of variables

$$
\begin{equation*}
N(t)=\exp \left(u_{1}(t)\right), \quad P(t)=\exp \left(u_{2}(t)\right) . \tag{8}
\end{equation*}
$$

Then, system (4) can be rewritten as

$$
\begin{align*}
\dot{u}_{1}(t)= & r(t)-\frac{r(t)}{K} \exp \left(u_{1}(t)\right) \\
& -\frac{4 G e \delta \exp \left(u_{2}(t)\right)+(4 G h-1) e^{2} \delta^{2} \exp \left(u_{1}(t)+u_{2}(t)\right)}{4 G\left(1+h e \delta \exp \left(u_{1}(t)\right)\right)^{2}}, \\
\dot{u}_{2}(t)= & -d(t) \\
& +\frac{4 G e \delta B(t) \exp \left(u_{1}(t)\right)+(4 G h-1) e^{2} \delta^{2} B(t) \exp \left(2 u_{1}(t)\right)}{4 G\left(1+h e \delta \exp \left(u_{1}(t)\right)\right)^{2}} \tag{9}
\end{align*}
$$

Take

$$
\begin{equation*}
X=Y=\left\{x=\left(u_{1}, u_{2}\right)^{T} \in C\left(\mathbb{R}, \mathbb{R}^{2}\right) \mid x(t+\omega)=x(t)\right\} \tag{10}
\end{equation*}
$$

and define

$$
\begin{gather*}
\|x\|=\max _{t \in[0, \omega]}\left|u_{1}(t)\right|+\max _{t \in[0, \omega]}\left|u_{2}(t)\right|,  \tag{11}\\
x=\left(u_{1}, u_{2}\right)^{T} \in X \text { or } Y .
\end{gather*}
$$

Here $|\cdot|$ denotes the Euclidean norm. Then $X$ and $Y$ are Banach spaces with the norm $\|\cdot\|$. For any $x=\left(u_{1}, u_{2}\right)^{T} \in X$, by means of the periodicity assumption, we can easily check that

$$
\begin{align*}
& r(t)- \frac{r(t)}{K} \exp \left(u_{1}(t)\right) \\
&-\frac{4 G e \delta \exp \left(u_{2}(t)\right)+(4 G h-1) e^{2} \delta^{2} \exp \left(u_{1}(t)+u_{2}(t)\right)}{4 G\left(1+h e \delta \exp \left(u_{1}(t)\right)\right)^{2}} \\
&:=f_{1}(t) \in C(\mathbb{R}, \mathbb{R}), \\
&-d(t)+ \frac{4 G e \delta B(t) \exp \left(u_{1}(t)\right)+(4 G h-1) e^{2} \delta^{2} B(t) \exp \left(2 u_{1}(t)\right)}{4 G\left(1+h e \delta \exp \left(u_{1}(t)\right)\right)^{2}} \\
&:=f_{2}(t) \in C(\mathbb{R}, \mathbb{R}) \tag{12}
\end{align*}
$$

are $\omega$-periodic.
Set
$L: \operatorname{Dom} L \cap X, \quad L\left(u_{1}(t), u_{2}(t)\right)^{T}=\left(\frac{\mathrm{d} u_{1}(t)}{\mathrm{d} t}, \frac{\mathrm{~d} u_{2}(t)}{\mathrm{d} t}\right)^{T}$,
where $\operatorname{Dom} L=\left\{\left(u_{1}(t), u_{2}(t)\right)^{T} \in C^{1}\left(\mathbb{R}, \mathbb{R}^{2}\right)\right\}$. Further, $N$ : $X \rightarrow X$ is defined by

$$
\begin{equation*}
N\binom{u_{1}}{u_{2}}=\binom{f_{1}(t)}{f_{2}(t)} \tag{14}
\end{equation*}
$$

Define

$$
\begin{align*}
P\binom{u_{1}}{u_{2}} & =\binom{u_{1}}{u_{2}} \\
& =\binom{\frac{1}{\omega} \int_{0}^{\omega} u_{1}(t) \mathrm{d} t}{\frac{1}{\omega} \int_{0}^{\omega} u_{2}(t) \mathrm{d} t}, \quad\binom{u_{1}}{u_{2}} \in X=Y \tag{15}
\end{align*}
$$

It is not difficult to show that

$$
\operatorname{Ker} L=\left\{x \mid x \in X, x=C_{0}, C_{0} \in \mathbb{R}^{2}\right\}
$$

$\operatorname{Im} L=\left\{y \mid y \in Y, \int_{0}^{\omega} y(t) \mathrm{d} t=0\right\}$ is closed in $Y$,

$$
\operatorname{dim} \operatorname{Ker} L=\operatorname{codimIm} L=2,
$$

and $P$ and $Q$ are continuous projectors such that

$$
\begin{equation*}
\operatorname{Im} P=\operatorname{Ker} L, \quad \operatorname{Ker} Q=\operatorname{Im} L=\operatorname{Im}(I-Q) \tag{17}
\end{equation*}
$$

It follows that $L$ is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to $L$ ) $K_{p}: \operatorname{Im} L \rightarrow$ Dom $L \cap \operatorname{Ker} P$ exists and is given by

$$
\begin{equation*}
K_{p}(y)=\int_{0}^{t} y(s) \mathrm{d} s-\frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} y(s) \mathrm{d} s \mathrm{~d} t \tag{18}
\end{equation*}
$$

Then $Q N: X \rightarrow Y$ and $K_{p}(I-Q) N: X \rightarrow X$ are, respectively, defined by

$$
\begin{align*}
& \mathrm{QN} x=\left(\frac{1}{\omega} \int_{0}^{\omega} f_{1}(t) \mathrm{d} t, \frac{1}{\omega} \int_{0}^{\omega} f_{2}(t) \mathrm{d} t\right)^{T} \\
& K_{p}(I-Q) N x \\
& =\int_{0}^{t} N x(s) \mathrm{d} s \\
& \quad-\frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} N x(s) \mathrm{d} s \mathrm{~d} t-\left(\frac{t}{\omega}-\frac{1}{2}\right) \int_{0}^{\omega} N x(s) \mathrm{d} s \tag{19}
\end{align*}
$$

Clearly, QN and $K_{p}(I-Q) N$ are continuous. By using the Arzelà-Ascoli Theorem, it is not difficult to prove that $\overline{K_{p}(I-Q) N(\bar{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $\mathrm{QN}(\bar{\Omega})$ is bounded. Therefore, $N$ is $L$-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset X$.

Now, we will search for two appropriate open bounded subsets in order to apply the continuation theorem.

Corresponding to the operator equation $L x=\lambda N x, \lambda \in$ $(0,1)$, we have

$$
\begin{aligned}
& \dot{u}_{1}(t) \\
& =\lambda\left[r(t)-\frac{r(t)}{K} \exp \left(u_{1}(t)\right)\right. \\
& \left.\quad-\frac{4 G e \delta \exp \left(u_{2}(t)\right)+(4 G h-1) e^{2} \delta^{2} \exp \left(u_{1}(t)+u_{2}(t)\right)}{4 G\left(1+h e \delta \exp \left(u_{1}(t)\right)\right)^{2}}\right]
\end{aligned}
$$

$$
\begin{align*}
& \dot{u}_{2}(t)  \tag{20}\\
& =\lambda[-d(t) \\
& \left.\quad+\frac{4 G e \delta B(t) \exp \left(u_{1}(t)\right)+(4 G h-1) e^{2} \delta^{2} B(t) \exp \left(2 u_{1}(t)\right)}{4 G\left(1+h e \delta \exp \left(u_{1}(t)\right)\right)^{2}}\right] . \tag{21}
\end{align*}
$$

Suppose $x=\left(u_{1}(t), u_{2}(t)\right)^{T} \in X$ is a solution of (20) and (21) for a certain $\lambda \in(0,1)$. Integrating (20), (21) over the interval $[0, \omega]$, we obtain

$$
\begin{align*}
& \int_{0}^{\omega} \frac{r(t)}{K} \exp \left(u_{1}(t)\right) \mathrm{d} t \\
& \quad+\int_{0}^{\omega} \frac{4 G e \delta \exp \left(u_{2}(t)\right)+(4 G h-1) e^{2} \delta^{2} \exp \left(u_{1}(t)+u_{2}(t)\right)}{4 G\left(1+h e \delta \exp \left(u_{1}(t)\right)\right)^{2}} \mathrm{~d} t \\
& \quad=\bar{r} \omega, \tag{22}
\end{align*}
$$

$$
\begin{align*}
\int_{0}^{\omega} & \frac{4 G e \delta B(t) \exp \left(u_{1}(t)\right)+(4 G h-1) e^{2} \delta^{2} B(t) \exp \left(2 u_{1}(t)\right)}{4 G\left(1+h e \delta \exp \left(u_{1}(t)\right)\right)^{2}} \mathrm{~d} t \\
& =\bar{d} \omega . \tag{23}
\end{align*}
$$

It follows from $\left(A_{1}\right),(20)$, and (22) that

$$
\begin{align*}
& \int_{0}^{\omega}\left|\dot{u}_{1}(t)\right| \mathrm{d} t \\
& =\lambda \int_{0}^{\omega} \left\lvert\,\left[r(t)-\frac{r(t)}{K} \exp \left(u_{1}(t)\right)\right.\right. \\
& \left.\quad-\frac{4 G e \delta \exp \left(u_{2}(t)\right)+(4 G h-1) e^{2} \delta^{2} \exp \left(u_{1}(t)+u_{2}(t)\right)}{4 G\left(1+h e \delta \exp \left(u_{1}(t)\right)\right)^{2}}\right] \mathrm{d} t \\
& <\int_{0}^{\omega} r(t) \mathrm{d} t+\int_{0}^{\omega} \frac{r(t)}{K} \exp \left(u_{1}(t)\right) \mathrm{d} t \\
& \quad+\int_{0}^{\omega} \frac{4 G e \delta \exp \left(u_{2}(t)\right)+(4 G h-1) e^{2} \delta^{2} \exp \left(u_{1}(t)+u_{2}(t)\right)}{4 G\left(1+h e \delta \exp \left(u_{1}(t)\right)\right)^{2}} \mathrm{~d} t \\
& =\int_{0}^{\omega} r(t) \mathrm{d} t+\bar{r} \omega=2 \bar{r} \omega ; \tag{24}
\end{align*}
$$

that is,

$$
\begin{equation*}
\int_{0}^{\omega}\left|\dot{u}_{1}(t)\right| \mathrm{d} t<2 \bar{r} \omega . \tag{25}
\end{equation*}
$$

Similarly, it follows from $\left(A_{1}\right)$, (21), and (23) that

$$
\begin{equation*}
\int_{0}^{\omega}\left|\dot{u}_{2}(t)\right| \mathrm{d} t<2 \bar{d} \omega \tag{26}
\end{equation*}
$$

Since $\left(u_{1}(t), u_{2}(t)\right)^{T} \in X$, there exist $\xi_{i}, \eta_{i} \in[0, \omega]$ such that

$$
\begin{equation*}
u_{i}\left(\xi_{i}\right)=\min _{t \in[0, \omega]} u_{i}(t), \quad u_{i}\left(\eta_{i}\right)=\max _{t \in[0, \omega]} u_{i}(t), \quad i=1,2 \tag{27}
\end{equation*}
$$

From $\left(A_{1}\right)$ and (23), we see that

$$
\begin{align*}
\bar{d} \omega \leq & \int_{0}^{\omega} \frac{4 G e \delta B(t) \exp \left(u_{1}(t)\right)}{4 G\left(1+h e \delta \exp \left(u_{1}(t)\right)\right)^{2}} \mathrm{~d} t  \tag{28}\\
& +\int_{0}^{\omega} \frac{(4 G h-1) e^{2} \delta^{2} B(t) \exp \left(2 u_{1}(t)\right)}{4 G h^{2} e^{2} \delta^{2} \exp \left(2 u_{1}(t)\right)} \mathrm{d} t
\end{align*}
$$

which implies

$$
\begin{equation*}
\bar{d} \leq \frac{e \delta \bar{B} \exp \left(u_{1}\left(\eta_{1}\right)\right)}{\left(1+h e \delta \exp \left(u_{1}\left(\xi_{1}\right)\right)\right)^{2}}+\frac{(4 G h-1) \bar{B}}{4 G h^{2}} \tag{29}
\end{equation*}
$$

So

$$
\begin{equation*}
u_{1}\left(\eta_{1}\right) \geq \ln \frac{\left[4 G h^{2} \bar{d}-(4 G h-1) \bar{B}\right]\left(1+h e \delta \exp \left(u_{1}\left(\xi_{1}\right)\right)\right)^{2}}{4 G h^{2} e \delta \bar{B}} \tag{30}
\end{equation*}
$$

This, combined with (25), gives

$$
\begin{align*}
u_{1}(t) \geq & u_{1}\left(\eta_{1}\right)-\int_{0}^{\omega}\left|\dot{u}_{1}(t)\right| \mathrm{d} t \\
& >\ln \frac{\left[4 G h^{2} \bar{d}-(4 G h-1) \bar{B}\right]\left(1+h e \delta \exp \left(u_{1}\left(\xi_{1}\right)\right)\right)^{2}}{4 G h^{2} e \delta \bar{B}} \\
& -2 \bar{r} \omega . \tag{31}
\end{align*}
$$

In particular, we have

$$
\begin{align*}
u_{1}\left(\xi_{1}\right)> & \ln \frac{\left[4 G h^{2} \bar{d}-(4 G h-1) \bar{B}\right]\left(1+h e \delta \exp \left(u_{1}\left(\xi_{1}\right)\right)\right)^{2}}{4 G h^{2} e \delta \bar{B}} \\
& -2 \bar{r} \omega, \tag{32}
\end{align*}
$$

or

$$
\begin{align*}
& {\left[4 G h^{2} \bar{d}-(4 G h-1) \bar{B}\right] h^{2} e^{2} \delta^{2} \exp \left(2 u_{1}\left(\xi_{1}\right)\right)} \\
& \quad-\left[4 G h^{2} e \delta \bar{B} \exp (2 \bar{r} \omega)\right. \\
& \left.\quad-2 h e \delta\left(4 G h^{2} \bar{d}-(4 G h-1) \bar{B}\right)\right] \exp \left(u_{1}\left(\xi_{1}\right)\right)  \tag{33}\\
& \quad+\left[4 G h^{2} \bar{d}-(4 G h-1) \bar{B}\right]<0
\end{align*}
$$

In view of $\left(A_{2}\right)$, we have

$$
\begin{equation*}
\ln l_{-}<u_{1}\left(\xi_{1}\right)<\ln l_{+} \tag{34}
\end{equation*}
$$

Similarly, it follows from $\left(A_{1}\right)$ and (23) that

$$
\begin{equation*}
\bar{d} \omega \geq \int_{0}^{\omega} \frac{4 G e \delta B(t) \exp \left(u_{1}(t)\right)}{4 G\left(1+h e \delta \exp \left(u_{1}(t)\right)\right)^{2}} \mathrm{~d} t \tag{35}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\bar{d} \geq \frac{e \delta \bar{B} \exp \left(u_{1}\left(\xi_{1}\right)\right)}{\left(1+h e \delta \exp \left(u_{1}\left(\eta_{1}\right)\right)\right)^{2}} \tag{36}
\end{equation*}
$$

So

$$
\begin{equation*}
u_{1}\left(\xi_{1}\right) \leq \ln \frac{\bar{d}\left(1+h e \delta \exp \left(u_{1}\left(\eta_{1}\right)\right)\right)^{2}}{e \delta \bar{B}} \tag{37}
\end{equation*}
$$

This, combined with (25), gives

$$
\begin{align*}
u_{1}(t) & \leq u_{1}\left(\xi_{1}\right)+\int_{0}^{\omega}\left|\dot{u}_{1}(t)\right| \mathrm{d} t \\
& <\ln \frac{\bar{d}\left(1+h e \delta \exp \left(u_{1}\left(\eta_{1}\right)\right)\right)^{2}}{e \delta \bar{B}}+2 \bar{r} \omega . \tag{38}
\end{align*}
$$

In particular, we have

$$
\begin{equation*}
u_{1}\left(\eta_{1}\right)<\ln \frac{\bar{d}\left(1+h e \delta \exp \left(u_{1}\left(\eta_{1}\right)\right)\right)^{2}}{e \delta \bar{B}}+2 \bar{r} \omega, \tag{39}
\end{equation*}
$$

or

$$
\begin{align*}
& \bar{d} h^{2} e^{2} \delta^{2} \exp \left(2 u_{1}\left(\eta_{1}\right)\right) \\
& \quad-(e \delta \bar{B} \exp (-2 \bar{r} \omega)-2 h e \delta \bar{d}) \exp \left(u_{1}\left(\eta_{1}\right)\right)+\bar{d}>0 \tag{40}
\end{align*}
$$

It follows from $\left(A_{2}\right)$ that

$$
\begin{equation*}
u_{1}\left(\eta_{1}\right)<\ln h_{-} \quad \text { or } \quad u_{1}\left(\eta_{1}\right)>\ln h_{+} . \tag{41}
\end{equation*}
$$

From (25) and (34), we find

$$
\begin{equation*}
u_{1}(t) \leq u_{1}\left(\xi_{1}\right)+\int_{0}^{\omega}\left|\dot{u}_{1}(t)\right| \mathrm{d} t<\ln l_{+}+2 \bar{r} \omega \triangleq H_{11} . \tag{42}
\end{equation*}
$$

On the other hand, it follows from $\left(A_{1}\right),(22)$, and (42) that

$$
\begin{gather*}
\bar{r} \omega \geq \int_{0}^{\omega} \frac{4 G e \delta \exp \left(u_{2}\left(\xi_{2}\right)\right)}{4 G\left(1+h e \delta \exp \left(\ln l_{+}+2 \bar{r} \omega\right)\right)^{2}} \mathrm{~d} t  \tag{43}\\
\bar{r} \omega \leq \int_{0}^{\omega} \frac{r(t)}{K} \exp \left(\ln l_{+}+2 \bar{r} \omega\right) \mathrm{d} t \\
\quad+\int_{0}^{\omega} e \delta \exp \left(u_{2}\left(\eta_{2}\right)\right) \mathrm{d} t  \tag{44}\\
\quad+\int_{0}^{\omega} \frac{e \delta \exp \left(u_{2}\left(\eta_{2}\right)\right)}{2} \mathrm{~d} t
\end{gather*}
$$

It follows from (43) that

$$
\begin{equation*}
u_{2}\left(\xi_{2}\right) \leq \ln \frac{\bar{r}\left(1+h e \delta \exp \left(\ln l_{+}+2 \bar{r} \omega\right)\right)^{2}}{e \delta} \tag{45}
\end{equation*}
$$

This, combined with (26), gives

$$
\begin{align*}
u_{2}(t) & \leq u_{2}\left(\xi_{2}\right)+\int_{0}^{\omega}\left|\dot{u}_{2}(t)\right| \mathrm{d} t \\
& <\ln \frac{\bar{r}\left(1+h e \delta \exp \left(\ln l_{+}+2 \bar{r} \omega\right)\right)^{2}}{e \delta}+2 \bar{d} \omega \triangleq H_{21} . \tag{46}
\end{align*}
$$

Moreover, because of $\left(A_{3}\right)$, it follows from (44) that

$$
\begin{equation*}
u_{2}\left(\eta_{2}\right) \geq \ln \frac{2 \bar{r}\left(1-(1 / K) \exp \left(\ln l_{+}+2 \bar{r} \omega\right)\right)}{3 e \delta} \tag{47}
\end{equation*}
$$

This, combined with (26) again, gives

$$
\begin{align*}
u_{2}(t) & \geq u_{2}\left(\eta_{2}\right)-\int_{0}^{\omega}\left|\dot{u}_{2}(t)\right| \mathrm{d} t \\
& >\ln \frac{2 \bar{r}\left(1-(1 / K) \exp \left(\ln l_{+}+2 \bar{r} \omega\right)\right)}{3 e \delta}-2 \bar{d} \omega \triangleq H_{22} . \tag{48}
\end{align*}
$$

It follows from (46) and (48) that

$$
\begin{equation*}
\max _{t \in[0, \omega]} u_{2}(t)<\max \left\{\left|H_{21}\right|,\left|H_{22}\right|\right\} \triangleq H_{2} \tag{49}
\end{equation*}
$$

Now, let us consider $Q N x$ with $x=\left(u_{1}, u_{2}\right)^{T} \in \mathbb{R}^{2}$. Note that

$$
\begin{align*}
& Q N\left(u_{1}, u_{2}\right)^{T} \\
& \begin{array}{l}
=\left(\bar{r}-\frac{\bar{r}}{K} \exp \left(u_{1}\right)\right. \\
\quad-\frac{4 G e \delta \exp \left(u_{2}\right)+(4 G h-1) e^{2} \delta^{2} \exp \left(u_{1}+u_{2}\right)}{4 G\left(1+h e \delta \exp \left(u_{1}\right)\right)^{2}}, \\
\left.\quad-\bar{d}+\frac{4 G e \delta \bar{B} \exp \left(u_{1}\right)+(4 G h-1) e^{2} \delta^{2} \bar{B} \exp \left(2 u_{1}\right)}{4 G\left(1+h e \delta \exp \left(u_{1}\right)\right)^{2}}\right)^{T} .
\end{array}
\end{align*}
$$

Noting $\left(A_{1}\right),\left(A_{2}\right)$, and $\left(A_{3}\right)$, we can show that the equation QN $\left(u_{1}, u_{2}\right)^{T}=0$ has two distinct solutions:

$$
\begin{align*}
& \widetilde{u}=\left(\ln u_{-}, \ln \frac{4 G\left(\bar{r}-(\bar{r} / K) u_{-}\right)\left(1+h e \delta u_{-}\right)^{2}}{4 G e \delta+(4 G h-1) e^{2} \delta^{2} u_{-}}\right) \\
& \widehat{u}=\left(\ln u_{+}, \ln \frac{4 G\left(\bar{r}-(\bar{r} / K) u_{+}\right)\left(1+h e \delta u_{+}\right)^{2}}{4 G e \delta+(4 G h-1) e^{2} \delta^{2} u_{+}}\right) \tag{51}
\end{align*}
$$

Choose $C>0$ such that

$$
\begin{gather*}
C>\max \left\{\left|\ln \frac{4 G\left(\bar{r}-(\bar{r} / K) u_{-}\right)\left(1+h e \delta u_{-}\right)^{2}}{4 G e \delta+(4 G h-1) e^{2} \delta^{2} u_{-}}\right|,\right. \\
\left\{\left.\ln \frac{4 G\left(\bar{r}-(\bar{r} / K) u_{+}\right)\left(1+h e \delta u_{+}\right)^{2}}{4 G e \delta+(4 G h-1) e^{2} \delta^{2} u_{+}} \right\rvert\,\right\} . \tag{52}
\end{gather*}
$$

We are now ready to define two open bounded subsets in order to apply the continuation theorem. Let

$$
\begin{align*}
\Omega_{1}= & \left\{x=\left(u_{1}, u_{2}\right)^{T} \in X \mid u_{1}(t) \in\left(\ln l_{-}, \ln h_{-}\right),\right. \\
& \left.\max _{t \in[0, \omega]}\left|u_{2}(t)\right|<H_{2}+C\right\}, \\
\Omega_{2}= & \left\{x=\left(u_{1}, u_{2}\right)^{T} \in X \mid \min _{t \in[0, \omega]} u_{1}(t) \in\left(\ln l_{-}, \ln l_{+}\right),\right. \\
& \left.\max _{t \in[0, \omega]} u_{1}(t) \in\left(\ln h_{+}, H_{11}\right), \max _{t \in[0, \omega]}\left|u_{2}(t)\right|<H_{2}+C\right\} . \tag{53}
\end{align*}
$$

Then both $\Omega_{1}$ and $\Omega_{2}$ are bounded open subsets of $X$. It follows from (4) and (52) that $\widetilde{\mathcal{u}} \in \Omega_{1}$ and $\widehat{\mathcal{u}} \in \Omega_{2}$. With the help of (4), (34), (41), (42), (49), and (52), it is easy to see that $\Omega_{1} \cap \Omega_{2}=\phi$ and $\Omega_{i}$ satisfies the requirement (a) in Lemma 1 for $i=1,2$. Moreover, $Q N x \neq 0$ for $x \in \partial \Omega \cap \operatorname{Ker} L=\partial \Omega \cap \mathbb{R}^{2}$. A direct computation gives $\operatorname{deg}\left\{J Q N, \Omega_{i} \cap \operatorname{Ker} L, 0\right\} \neq 0$. Here, $J$ is taken as the identity mapping since $\operatorname{Im} Q=\operatorname{Ker} L$. So far we have proved that $\Omega_{i}$ satisfies all the assumptions in Lemma 1. Hence, (4) has at least two $\omega$-periodic solutions. This completes the proof of Theorem 2.

Remark 3. In the proof of Theorem 2, we have employed some new technique to obtain a priori bounds for $u_{1}$. Here, the standard arguments in the literature (see, e.g., [7-12]) do not work. Indeed, from (23) in the proof it follows that

$$
\begin{equation*}
\bar{d} \omega \leq \frac{4 G e \delta \bar{B} \omega \exp \left(u_{1}\left(\eta_{1}\right)\right)+4 G h e^{2} \delta^{2} \bar{B} \omega \exp \left(2 u_{1}\left(\eta_{1}\right)\right)}{4 G\left(1+h e \delta \exp \left(u_{1}\left(\xi_{1}\right)\right)\right)^{2}} . \tag{54}
\end{equation*}
$$

If we were to use the standard arguments in the literature, then we have

$$
\begin{align*}
& {\left[4 \bar{d} h^{3} e^{2} \delta^{2} \bar{B}-4 h^{2} e^{2} \delta^{2} \bar{B}^{2} \exp (4 \bar{r} \omega)\right] \exp \left(2 u_{1}\left(\xi_{1}\right)\right)} \\
& \quad+\left[8 \bar{d} h^{2} e \delta \bar{B}-4 h e \delta \bar{B}^{2} \exp (2 \bar{r} \omega)\right] \exp \left(u_{1}\left(\xi_{1}\right)\right)  \tag{55}\\
& \quad+4 \bar{d} h \bar{B}<0
\end{align*}
$$

where $u_{1}\left(\xi_{1}\right)=\min _{t \in[0, \omega]} u_{1}(t)$ and $u_{1}\left(\eta_{1}\right)=\max _{t \in[0, \omega]} u_{1}(t)$. It follows from (55) that

$$
\begin{equation*}
\tilde{l}_{-}<\exp \left(u_{1}\left(\xi_{1}\right)\right)<\tilde{l}_{+}, \tag{56}
\end{equation*}
$$

where $\tilde{l}_{-}$and $\tilde{l}_{+}$are the roots of the following equation in $x$ :

$$
\begin{align*}
& {\left[4 \bar{d} h^{3} e^{2} \delta^{2} \bar{B}-4 h^{2} e^{2} \delta^{2} \bar{B}^{2} \exp (4 \bar{r} \omega)\right] x^{2}} \\
& \quad+\left[8 \bar{d} h^{2} e \delta \bar{B}-4 h e \delta \bar{B}^{2} \exp (2 \bar{r} \omega)\right] x  \tag{57}\\
& \quad+4 \bar{d} h \bar{B}=0 .
\end{align*}
$$

We claim that (57) has at least a negative root; that is, at least one of $\widetilde{l}_{-}, \tilde{l}_{+}$is negative. Otherwise, if both $\tilde{l}_{-}$and $\tilde{l}_{+}$are positive, then from (57) we see that

$$
\begin{equation*}
\tilde{l}_{+} \cdot \tilde{l}_{-}=\frac{4 \bar{d} h \bar{B}}{4 \bar{d} h^{3} e^{2} \delta^{2} \bar{B}-4 h^{2} e^{2} \delta^{2} \bar{B}^{2} \exp (4 \bar{r} \omega)}>0 \tag{58}
\end{equation*}
$$

which implies

$$
\begin{equation*}
h \bar{d}>\bar{B} \exp (4 \bar{r} \omega) \tag{59}
\end{equation*}
$$

On the other hand, it follows form (57) and (58) that

$$
\begin{equation*}
\tilde{l}_{+}+\widetilde{l}_{-}=-\frac{8 \bar{d} h^{2} e \delta \bar{B}-4 h e \delta \bar{B}^{2} \exp (2 \bar{r} \omega)}{4 \bar{d} h^{3} e^{2} \delta^{2} \bar{B}-4 h^{2} e^{2} \delta^{2} \bar{B}^{2} \exp (4 \bar{r} \omega)}<0, \tag{60}
\end{equation*}
$$

which contradicts the positivity of $\tilde{l}_{-}$and $\tilde{l}_{+}$. Therefore, at least one of $\tilde{l}_{-}, \tilde{l}_{+}$is negative. However, to use the standard arguments in the literature we need both $\tilde{l}_{-}$and $\tilde{l}_{+}$to be positive. Hence, we have illustrated that standard arguments in the literature are not applicable to the system (4) and some new technique should be used. To see how this problem is handled, the reader may refer to (27)-(34) in the proof of Theorem 2.

## Conflict of Interest

No conflict of interests exists in the submission of this paper, and the paper is approved by all authors for publication. The authors would like to declare that the work described was original research that has not been published previously and not under consideration for publication elsewhere.

## Funding

This work is supported by NNSFC 11271333 and 11171090.

## References

[1] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, vol. 191, Academic Press, Boston, Mass, USA, 1993.
[2] H. I. Freedman, Deterministic Mathematical Models in Population Ecology, Marcel Dekker, New York, NY, USA, 1980.
[3] C. S. Holling, "The functional response of predator to prey density and its role in mimicry and population regulation," Memoirs of the Entomological Society of Canada, vol. 45, pp. 160, 1965.
[4] R. S. Liu, Z. L. Feng, H. Zhu, and D. L. de Angelis, "Bifurcation analysis of a plant-herbivore model with toxin-determined functional response," Journal of Differential Equations, vol. 245, no. 2, pp. 442-467, 2008.
[5] Y.-F. Gao and Y.-H. Xia, "Periodic solutions of a nonautonomous plant-hare model," Journal of Zhejiang University, vol. 39, no. 5, pp. 507-511, 2012.
[6] R. E. Gains and J. L. Mawhin, Coincidence Degree and Nonlinear Differential Equations, Springer, Berlin, Germany, 1977.
[7] Z. Q. Zhang, Z. Hou, and L. Wang, "Multiplicity of positive periodic solutions to a generalized delayed predator-prey system with stocking," Nonlinear Analysis. Theory, Methods \& Applications, vol. 68, no. 9, pp. 2608-2622, 2008.
[8] Y. H. Xia, J. Cao, and S. S. Cheng, "Multiple periodic solutions of a delayed stage-structured predator-prey model with nonmonotone functional responses," Applied Mathematical Modelling, vol. 31, pp. 1947-1959, 2007.
[9] F. Y. Wei, "Existence of multiple positive periodic solutions to a periodic predator-prey system with harvesting terms and Holling III type functional response," Communications in Nonlinear Science and Numerical Simulation, vol. 16, no. 4, pp. 2130-2138, 2011.
[10] Q. Wang, B. X. Dai, and Y. Chen, "Multiple periodic solutions of an impulsive predator-prey model with Holling-type IV functional response," Mathematical and Computer Modelling, vol. 49, no. 9-10, pp. 1829-1836, 2009.
[11] N. H. Zhao, Y. Xia, W. Liu, P. Wong, and R. T. Wang, "Existence of almost periodic solutions of a nonlinear system," The Journal of Applied Analysis and Computation, vol. 3, pp. 301-306, 2013.
[12] T. Zhang, J. Liu, and Z. Teng, "Existence of positive periodic solutions of an SEIR model with periodic coefficients," Applications of Mathematics, vol. 57, no. 6, pp. 601-616, 2012.

## Research Article

# Stochastic Analysis of a Hassell-Varley Type Predation Model 

Feng Rao, Shunjun Jiang, Yanqiu Li, and Hao Liu<br>College of Sciences, Nanjing University of Technology, Nanjing 211816, China<br>Correspondence should be addressed to Feng Rao; raofeng2002@163.com

Received 14 November 2013; Accepted 3 December 2013
Academic Editor: Weiming Wang
Copyright © 2013 Feng Rao et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We investigate a Hassell-Varley type predator-prey model with stochastic perturbations. By perturbing the growth rate of prey population and death rate of predator population with white noise terms, we construct a stochastic differential equation model to discuss the effects of the environmental noise on the dynamical behaviors. Applying the comparison theorem of stochastic equations and Itô's formula, the unique positive global solution to the model for any positive initial value is obtained. We find out some sufficient conditions for stochastically asymptotically boundedness, permanence, persistence in mean and extinction of the solution. Furthermore, a series of numerical simulations to illustrate our mathematical findings are presented. The results indicate that the stochastic perturbations do not cause drastic changes of the dynamics in the deterministic model when the noise intensity is small under some conditions, but while the noise intensity is sufficiently large, the species may die out, which does not happen in the deterministic model.


## 1. Introduction

It is well known that predator-prey interaction is one of basic interspecies relations for ecosystems, and it is also the basic block of more complicated food chain, food web, and biophysical network structure [1]. Because of the universal existence of predator and prey and their importance in ecology, the dynamical relationship between them has long been and will continue to be one of the dominant themes $[2,3]$.

The classical predator-prey model has received extensive attentions from mathematicians as well as ecologists [4-7], and it can be expressed by a model of nonlinear ordinary differential equations as follows:

$$
\begin{align*}
\frac{d N}{d t} & =f(N) N-b g(N, P) P \\
\frac{d P}{d t} & =P(c g(N, P)-d) \tag{1}
\end{align*}
$$

where $N=N(t)$ and $P=P(t)$ denote the density of prey and predator population at time $t$, respectively. Parameters $b$, $c$, and $d$ are positive constants. $b$ stands for capturing rate of prey by predator, $c$ is conversion rate of prey into predator, and $d$ is the natural death rate of the predator. The function $f(N)$ represents the density-dependent specific growth rate
of prey in absence of predator. The amount of prey biomass consumed by each predator per unit of time is described by the functional response $g(N, P)$.

In this paper, we consider the usual logistic form of the growth function for prey in the absence of predator as

$$
\begin{equation*}
f(N)=r\left(1-\frac{N}{K}\right) \tag{2}
\end{equation*}
$$

where $r(>0)$ is the natural growth rate of prey and $K(>0)$ is the environmental carrying capacity. The functional response $g(N, P)$ is taken as

$$
\begin{equation*}
g(N, P)=\frac{N}{N+m P^{\alpha}} \tag{3}
\end{equation*}
$$

which is called the Hassell-Varley type functional response and $\alpha \in(0,1)$ is the Hassell-Varley constant [8] and $m(>0)$ stands for half capturing saturation constant. The predatorprey model with Hassell-Varley type functional response has been studied in the ecological literature [6, 9-11].

For more biological motivation in population dynamics, we take into account the density-dependence of predator
population. And the corresponding Hassell-Varley type pred-ator-prey model is described by the following form:

$$
\begin{align*}
\frac{d N}{d t} & =r N\left(1-\frac{N}{K}\right)-\frac{b N P}{N+m P^{\alpha}} \\
\frac{d P}{d t} & =P\left(\frac{c N}{N+m P^{\alpha}}-d-h P\right)  \tag{4}\\
N(0) & =N_{0}>0, \quad P(0)=P_{0}>0,
\end{align*}
$$

where $h P$ stands for the density-dependence of the predator population and $h>0$.

On the other hand, most natural phenomena do not follow strictly deterministic laws but rather oscillate randomly about some average. So that the population density never attains a fixed value with the advancement of time but rather exhibits continuous oscillation around some average values [12, 13]. In fact, there are many benefits to be gained by using stochastic models because real life is full of random fluctuations (i.e., the effects of noise), which undeniably arise from either environmental variability or internal species. The basic mechanism and factors of population growth like resources and vital rates-birth, death, immigration, and emigrationchange nondeterministically due to continuous fluctuations in the environment (e.g., variation in intensity of sunlight, temperature, water level, etc.) [2, 3, 14]. Recent advances in stochastic differential equations enable a lot of authors to introduce noise into the model of physical phenomena, whether it is a random noise in the system of differential equations or environmental fluctuations in parameters [1531]. So far as our knowledge is concerned, the work of a modified Hassell-Varley type predator-prey model with stochastic perturbations seems rare. Motivated by these, we attempt to study the stochastic behaviors of the modified Hassell-Varley type predation model in a random fluctuating environment.

The organization of this paper is as follows. In Section 2, we present a stochastic model corresponding to the deterministic model (4) and discuss it in detail. In Section 3, we use numerical simulations to reveal the influence of noise on the dynamical behaviors of the model. A brief discussion is given in Section 4.

## 2. The Stochastic Model and Analysis

In this section, we investigate the effects of fluctuating environments on the dynamical behaviors of model (4). Assuming that random fluctuations in the environment would display themselves as fluctuations in the growth rate of prey population $N$ and in the death rate of predator population $P$, then the parameters $r$ and $d$ in model (4) can be replaced by

$$
\begin{equation*}
r \longrightarrow r+\sigma_{1} \dot{B}_{1}(t), \quad-d \longrightarrow-d+\sigma_{2} \dot{B}_{2}(t) \tag{5}
\end{equation*}
$$

In this way, model (4) will be reduced to the following form:

$$
\begin{gather*}
d N=N\left(r-\frac{r}{K} N-\frac{b P}{N+m P^{\alpha}}\right) d t+\sigma_{1} N d B_{1}(t)  \tag{6}\\
d P=P\left(\frac{c N}{N+m P^{\alpha}}-d-h P\right) d t+\sigma_{2} P d B_{2}(t)
\end{gather*}
$$

where $\sigma_{1}^{2}$ and $\sigma_{2}^{2}$ are known as the intensities of environmental noise and $\dot{B}_{i}(t)(i=1,2)$ is a standard white noise; that is, $B_{i}(t)(i=1,2)$ is a Brownian motion defined in a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$ with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \in R_{+}}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathscr{F}_{0}$ contains all $\mathbf{P}$-null sets) [14].
2.1. Positive and Global Solution. For model (6), there is a positive local solution.

Lemma 1. There is a unique local solution $(N(t), P(t))$ for $t \in$ $\left[0, \tau_{e}\right)$ to model (6) almost surely for initial value $\left(N_{0}, P_{0}\right) \in R_{+}^{2}$, where $\tau_{e}$ is the explosion time.

The proof of this lemma is rather standard and hence is omitted.

Lemma 1 only tells us that there is a unique positive local solution to model (6). Next, we show that this solution is global which is more interesting.

In particular, let us consider the one-dimensional stochastic population model

$$
\begin{gather*}
d \bar{N}(t)=r \bar{N}(t)\left(1-\frac{\bar{N}(t)}{K}\right) d t+\sigma_{1} \bar{N}(t) d B_{1}(t), \quad t \geq 0 \\
\bar{N}(0)=\bar{N}_{0} \tag{7}
\end{gather*}
$$

there is an explicit solution

$$
\begin{align*}
& \bar{N}(t)=\exp \left\{\left(r-\frac{\sigma_{1}^{2}}{2}\right) t+\sigma_{1} B_{1}(t)\right\} \\
& \times\left(\frac{1}{\bar{N}_{0}}+\frac{r}{K} \int_{0}^{t} \exp \left\{\left(r-\frac{\sigma_{1}^{2}}{2}\right) s\right.\right.  \tag{8}\\
& \\
& \left.\left.\quad+\sigma_{1} B_{1}(s)\right\} d s\right)^{-1}
\end{align*}
$$

From model (6), we have

$$
\begin{equation*}
d N(t) \leq r N(t)\left(1-\frac{N(t)}{K}\right) d t+\sigma_{1} N(t) d B_{1}(t) \tag{9}
\end{equation*}
$$

By the comparison theorem of stochastic equations [14], we have $N(t) \leq \bar{N}(t)$ a.s. $t \in\left[0, \tau_{e}\right)$.

Besides, for the following equation

$$
\begin{align*}
d \underline{N}(t)= & \underline{N}(t)\left(r-\frac{b}{m}-\frac{r}{K} \underline{N}(t)\right) d t  \tag{10}\\
& +\sigma_{1} \underline{N}(t) d B_{1}(t), \quad \underline{N}(0)=\underline{N}_{0}
\end{align*}
$$

there is a unique solution as

$$
\begin{align*}
& \underline{N}(t)=\exp \left\{\left(r-\frac{b}{m}-\frac{\sigma_{1}^{2}}{2}\right) t+\sigma_{1} B_{1}(t)\right\} \\
& \times\left(\frac{1}{\underline{N}_{0}}+\frac{r}{K} \int_{0}^{t} \exp \left\{\left(r-\frac{b}{m}-\frac{\sigma_{1}^{2}}{2}\right) s\right.\right.  \tag{11}\\
& \left.\left.\quad+\sigma_{1} B_{1}(s)\right\}\right)^{-1}
\end{align*}
$$

In model (6), for $\alpha \in(0,1)$, we can get

$$
\begin{align*}
d N(t) \geq & N(t)\left(r-\frac{b}{m}-\frac{r}{K} N(t)\right) d t  \tag{12}\\
& +\sigma_{1} N(t) d B_{1}(t)
\end{align*}
$$

then $N(t) \geq \underline{N}(t)$ a.s. $t \in\left[0, \tau_{e}\right)$.
Consequently, we obtain

$$
\begin{equation*}
\underline{N}(t) \leq N(t) \leq \bar{N}(t) \quad \text { a.s. } t \in\left[0, \tau_{e}\right) . \tag{13}
\end{equation*}
$$

On the other hand, the equation

$$
\begin{array}{r}
d \underline{P}(t)=\underline{P}(t)(c-d-c m-h \underline{P}(t)) d t+\sigma_{2} \underline{P}(t) d B_{2}(t), \\
\underline{P}(0)=\underline{P}_{0}, \tag{14}
\end{array}
$$

has a unique solution as follows:

$$
\begin{array}{r}
\underline{P}(t)=\exp \left\{\left(c-d-c m-\frac{\sigma_{2}^{2}}{2}\right) t+\sigma_{2} B_{2}(t)\right\} \\
\times\left(\frac{1}{\underline{P}_{0}}+h \int_{0}^{t} \exp \left\{\left(c-d-c m-\frac{\sigma_{2}^{2}}{2}\right) s\right.\right.  \tag{15}\\
\left.\left.+\sigma_{2} B_{2}(s)\right\} d s\right)^{-1}
\end{array}
$$

Considering the predator population $P(t)$ in model (6), we have

$$
\begin{align*}
& d P(t) \leq P(t)(c-d) d t+\sigma_{2} P(t) d B_{2}(t) \\
& d P(t)= P(t)\left(c-d-\frac{c m P^{\alpha}(t)}{N(t)+m P^{\alpha}(t)}-h P(t)\right) d t \\
&+\sigma_{2} P(t) d B_{2}(t) \\
& \geq P(t)(c-d-c m-h P(t)) d t+\sigma_{2} P(t) d B_{2}(t) . \tag{16}
\end{align*}
$$

By the comparison theorem, we obtain $P(t) \geq \underline{P}(t)$ a.s. $t \in$ $\left[0, \tau_{e}\right)$; then

$$
\begin{align*}
\underline{P}(t) & \leq P(t) \leq P_{0} \exp \left\{\left(c-d-\frac{\sigma_{2}^{2}}{2}\right) t+\sigma_{2} B_{2}(t)\right\}  \tag{17}\\
& =\bar{P}(t) \quad \text { a.s. } t \in\left[0, \tau_{e}\right)
\end{align*}
$$

From the representation of solutions $\bar{N}(t), \underline{N}(t), \underline{P}(t)$, and $\bar{P}(t)$, we can see that they are all existence for $t \in[0, \infty)$; that is, $\tau_{e}=\infty$. Therefore, we have the following theorem to show that the positive solution of model (6) is global, which is essential for a population system.

Theorem 2. There is a unique positive solution $(N(t), P(t))$ of model (6) almost surely for any initial value $\left(N_{0}, P_{0}\right) \in$ $R_{+}^{2}$. Moreover there exist functions $\bar{N}(t), \underline{N}(t), \underline{P}(t)$, and $\bar{P}(t)$ defined as (8), (11), (15), and (17) such that

$$
\begin{gather*}
\underline{N}(t) \leq N(t) \leq \bar{N}(t), \\
\underline{P}(t) \leq P(t) \leq \bar{P}(t) \quad \text { a.s. } t \geq 0 . \tag{18}
\end{gather*}
$$

2.2. Stochastic Boundedness. In this subsection, we show that the solution $(N(t), P(t))$ of model (6) with any positive initial value is uniformly bounded in mean.

Theorem 3. The solution $(N(t), P(t))$ of model (6) with any positive initial value has the property that

$$
\begin{gather*}
\limsup _{t \rightarrow \infty} \mathbf{E}[N(t)] \leq K \\
\limsup _{t \rightarrow \infty} \mathbf{E}[P(t)] \leq \frac{c K(r+d)^{2}}{4 r b d} \tag{19}
\end{gather*}
$$

Proof. From (7), we obtain

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \mathrm{E}[\bar{N}(t)] \leq K \tag{20}
\end{equation*}
$$

combining $N(t) \leq \bar{N}(t)$, then

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \sup [N(t)] \leq K \tag{21}
\end{equation*}
$$

Set

$$
\begin{equation*}
M(t)=N(t)+\frac{b}{c} P(t) \tag{22}
\end{equation*}
$$

then

$$
\begin{align*}
d M(t)= & \left(r N(t)\left(1-\frac{N(t)}{K}\right)-\frac{b}{c} P(t)(d+h P(t))\right) d t \\
& +\sigma_{1} N(t) d B_{1}(t)+\frac{\sigma_{2} b}{c} P(t) d B_{2}(t) \\
= & \left((r+d) N(t)-\frac{r}{K} N^{2}(t)-\frac{b h}{c} P^{2}(t)-d M(t)\right) d t \\
& +\sigma_{1} N(t) d B_{1}(t)+\frac{\sigma_{2} b}{c} P(t) d B_{2}(t) . \tag{23}
\end{align*}
$$

Integrating the above equation from 0 to $t$, we obtain

$$
\begin{align*}
M(t)= & M(0)+\int_{0}^{t}((r+d) N(s) \\
& \left.-\frac{r}{K} N^{2}(s)-\frac{b h}{c} P^{2}(s)-d M(s)\right) d s \\
& +\sigma_{1} \int_{0}^{t} N(s) d B_{1}(s)+\frac{\sigma_{2} b}{c} \int_{0}^{t} P(s) d B_{2}(s) \tag{24}
\end{align*}
$$

and taking expectations leads to

$$
\begin{align*}
\mathbf{E}[M(t)]=M(0)+\int_{0}^{t} \mathbf{E}[ & (r d) N(s)-\frac{r}{K} N^{2}(s)  \tag{25}\\
& \left.-\frac{b h}{c} P^{2}(s)-d M(s)\right] d s
\end{align*}
$$

then

$$
\begin{align*}
\frac{d \mathbf{E}[M(t)]}{d t}= & (r+d) \mathbf{E}[N(t)] \\
& -\frac{r}{K} \mathbf{E}\left[N^{2}(t)\right]-\frac{b h}{c} \mathbf{E}\left[P^{2}(t)\right]-d \mathbf{E}[M(t)] \\
\leq & (r+d) \mathbf{E}[N(t)]-\frac{r}{K}(\mathbf{E}[N(t)])^{2} \\
& -\frac{b h}{c}(\mathbf{E}[P(t)])^{2}-d \mathbf{E}[M(t)] \\
\leq & \frac{K(r+d)^{2}}{4 r}-d \mathbf{E}[M(t)] \tag{26}
\end{align*}
$$

By the comparison theorem, we can get

$$
\begin{align*}
& \lim _{t \rightarrow \infty} \sup \mathbf{E}[M(t)] \\
& =\underset{t \rightarrow \infty}{\lim \sup }\left(\mathbf{E}[N(t)]+\frac{b}{c} \mathbf{E}[P(t)]\right) \leq \frac{K(r+d)^{2}}{4 r d} . \tag{27}
\end{align*}
$$

Therefore, we obtain

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \sup E[P(t)] \leq \frac{c K(r+d)^{2}}{4 r b d} \tag{28}
\end{equation*}
$$

This completes the proof.
2.3. The Long Time Behavior. It is well known that the property of permanence is more desirable since it means the long time survival in a population dynamics. Now, the definition of stochastic permanence will be given below [32, 33].

Definition 4. The solution $(N(t), P(t))$ of model (6) is said to be stochastically permanent, if, for any $\varepsilon \in(0,1)$, there exists a pair of positive constants $\delta=\delta(\varepsilon)$ and $\chi=\chi(\varepsilon)$ such that, for any initial value $\left(N_{0}, P_{0}\right) \in R_{+}^{2}$, the solution $(N(t), P(t))$ to model (6) has the properties that

$$
\begin{align*}
& \liminf _{t \rightarrow \infty} \mathbf{P}\{|N(t), P(t)| \geq \delta\} \geq 1-\varepsilon \\
& \liminf _{t \rightarrow \infty} \mathbf{P}\{|N(t), P(t)| \leq \chi\} \geq 1-\varepsilon \tag{29}
\end{align*}
$$

Lemma 5. For any initial value $\left(N_{0}, P_{0}\right) \in R_{+}^{2}$, the solution $(N(t), P(t))$ satisfies that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \mathrm{E}\left[\left(N^{2}+P^{2}\right)^{-\theta / 2}\right] \leq \frac{C}{k} \tag{30}
\end{equation*}
$$

where $C=C(\theta)$ is a positive constant and $\theta, k$ are arbitrary positive constants satisfying

$$
\begin{equation*}
\theta \min \left\{r-\frac{b}{m}, c-d\right\}>\frac{\theta(\theta+1)}{2} \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}\right\}+k \tag{31}
\end{equation*}
$$

Proof. Set a function

$$
\begin{equation*}
V(N, P)=\frac{1}{N+P} \tag{32}
\end{equation*}
$$

for $(N(t), P(t)) \in R_{+}^{2}$; using Itô's formula, we have

$$
\begin{align*}
d V= & -V^{2}\left[N\left(r-\frac{r}{K} N-\frac{b P}{N+m P^{\alpha}}\right)\right. \\
& \left.+P\left(\frac{c N}{N+m P^{\alpha}}-d-h P\right)\right] d t \\
& +V^{3}\left[\sigma_{1}^{2} N^{2}+\sigma_{2}^{2} P^{2}\right] d t-V^{2}\left[\sigma_{1} N d B_{1}+\sigma_{2} P d B_{2}\right] \tag{33}
\end{align*}
$$

Choosing a positive constant $\theta$ and by Itô's formula, we get

$$
\begin{align*}
\mathbf{L}(1+V)^{\theta}=\theta(1+V)^{\theta-1}\left\{-V^{2}[ \right. & N\left(r-\frac{r}{K} N-\frac{b P}{N+m P^{\alpha}}\right) \\
& \left.+P\left(\frac{c N}{N+m P^{\alpha}}-d-h P\right)\right] \\
+ & \left.V^{3}\left[\sigma_{1}^{2} N^{2}+\sigma_{2}^{2} P^{2}\right]\right\} \\
+ & \frac{\theta(\theta-1)}{2} V^{4}(1+V)^{\theta-2}\left[\sigma_{1}^{2} N^{2}+\sigma_{2}^{2} P^{2}\right] \tag{34}
\end{align*}
$$

Let $k>0$ be sufficiently small such that it satisfies (31); by Itô's formula, then

$$
\begin{align*}
& \mathrm{Le}^{k t}(1+V)^{\theta} \\
& \begin{aligned}
=\mathrm{e}^{k t}(1+V)^{\theta-2}\{ & \left\{k(1+V)^{2}-\theta V^{2}\right. \\
\times & {\left[N\left(r-\frac{r}{K} N-\frac{b P}{N+m P^{\alpha}}\right)\right.} \\
& \left.+P\left(\frac{c N}{N+m P^{\alpha}}-d-h P\right)\right] \\
& -\theta V^{3}\left[N\left(r-\frac{r}{K} N-\frac{b P}{N+m P^{\alpha}}\right)\right. \\
& \left.+P\left(\frac{c N}{N+m P^{\alpha}}-d-h P\right)\right] \\
& +\theta V^{3}\left[\sigma_{1}^{2} N^{2}+\sigma_{2}^{2} P^{2}\right] \\
& \left.+\frac{\theta(\theta-1)}{2} V^{4}\left[\sigma_{1}^{2} N^{2}+\sigma_{2}^{2} P^{2}\right]\right\}
\end{aligned}
\end{align*}
$$

Based on the following inequality,

$$
\begin{equation*}
V^{3}\left(\sigma_{1}^{2} N^{2}+\sigma_{2}^{2} P^{2}\right) \leq \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}\right\} V \tag{36}
\end{equation*}
$$

## Therefore, we obtain

$$
\begin{aligned}
& \mathbf{L e}^{k t}(1+V)^{\theta} \\
& \leq \mathrm{e}^{k t}(1+V)^{\theta-2}\left[k(1+V)^{2}\right. \\
& -\theta V^{2}\left(N\left(r-\frac{r}{K} N-\frac{b P}{N+m P^{\alpha}}\right)\right. \\
& \left.+P\left(\frac{c N}{N+m P^{\alpha}}-d-h P\right)\right) \\
& -\theta V^{3}\left(N\left(r-\frac{r}{K} N-\frac{b P}{N+m P^{\alpha}}\right)\right. \\
& \left.+P\left(\frac{c N}{N+m P^{\alpha}}-d-h P\right)\right) \\
& +\theta V \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}\right\} \\
& \left.+\frac{\theta(\theta-1)}{2} V^{2} \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}\right\}\right] \\
& =\mathrm{e}^{k t}(1+V)^{\theta-2}\left[k+\left(2 k-\theta V^{2}\right.\right. \\
& \times\left(r N-\frac{r}{K} N^{2}-\frac{b N P}{N+m P^{\alpha}}\right. \\
& \left.+\frac{c N P}{N+m P^{\alpha}}-d P-h P^{2}\right) \\
& \left.+\theta \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}\right\}\right) V \\
& -\left(\theta \left(r N-\frac{r}{K} N^{2}-\frac{b N P}{N+m P^{\alpha}}\right.\right. \\
& \left.+\frac{c N P}{N+m P^{\alpha}}-d P-h P^{2}\right) \\
& \left.\left.-\frac{\theta(\theta-1)}{2} \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}\right\}\right) V^{2}\right] \\
& \leq \mathrm{e}^{k t}(1+V)^{\theta-2} \\
& \times\left[k+\theta V^{2}\left(\frac{r}{K} N^{2}+h P^{2}\right)\right. \\
& +\left(2 k+\theta V^{2}\left(\frac{r}{K} N^{2}+h P^{2}\right)-\theta V^{2}\right. \\
& \times\left(r N-\frac{b N P}{N+m P^{\alpha}}+\frac{c N P}{N+m P^{\alpha}}-d P\right) \\
& \left.+\theta \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}\right\}\right) V \\
& -\left(\theta\left(r N-\frac{b N P}{N+m P^{\alpha}}+\frac{c N P}{N+m P^{\alpha}}-d P\right)\right. \\
& \left.\left.-\frac{\theta(\theta-1)}{2} \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}\right\}-k\right) V^{2}\right]
\end{aligned}
$$

$$
\begin{align*}
\leq \mathrm{e}^{k t}(1+V)^{\theta-2}[ & \left(k+\theta \max \left\{\frac{r}{K}, h\right\}\right) \\
& +\left(2 k+\theta \max \left\{\frac{r}{K}, h\right\}\right. \\
& -\theta \min \left\{r-\frac{b}{m}, c-d\right\} \\
& \left.+\theta \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}\right\}\right) V \\
- & \left(\theta \min \left\{r-\frac{b}{m}, c-d\right\}\right. \\
& \left.\left.\quad \frac{\theta(\theta-1)}{2} \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}\right\}-k\right) V^{2}\right] . \tag{37}
\end{align*}
$$

There exists a positive constant $C_{0}$ such that $\mathbf{L e}^{k t}(1+V)^{\theta} \leq$ $C_{0} \mathrm{e}^{k t}$; then

$$
\begin{equation*}
\mathbf{E}\left[\mathrm{e}^{k t}(1+V)^{\theta}\right] \leq(1+V(0))^{\theta}+\frac{C_{0}}{k} \mathrm{e}^{k t} \tag{38}
\end{equation*}
$$

So, we can get

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \mathrm{E} V^{\theta}(t) \leq \limsup _{t \rightarrow \infty} \mathrm{E}(1+V(t))^{\theta} \leq \frac{C_{0}}{k} \tag{39}
\end{equation*}
$$

In addition, we know that $(N+P)^{\theta} \leq 2^{\theta}\left(N^{2}+P^{2}\right)^{\theta / 2}$; consequently,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} E\left[\left(N^{2}+P^{2}\right)^{-\theta / 2}\right] \leq 2^{\theta} \limsup _{t \rightarrow \infty} E V^{\theta}(t) \leq \frac{2^{\theta} C_{0}}{k} \triangleq \frac{C}{k} \tag{40}
\end{equation*}
$$

The proof is complete.
Based on the results of Theorem 3, Lemma 5, and the Chebyshev inequality [14], we can obtain the following theorem.

Theorem 6. Assume that $\max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}\right\}<2 \min \{r-b / m, c-d\}$; the solution of model (6) is stochastically permanent.

In a view of ecology, the coexistence of species may be a good situation. In the following, we consider the stochastic persistence (i.e., stochastic persistence in mean) of the species.

Theorem 7. Assume that $r-\sigma_{1}^{2} / 2>b / m$ holds, for any initial value $N_{0}>0$; then the solution $N(t)$ to model (6) has the property

$$
\begin{align*}
& \frac{K\left(r-b / m-\sigma_{1}^{2} / 2\right)}{r} \\
& \quad \leq \liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} N(s) d s  \tag{41}\\
& \quad \leq \limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} N(s) d s \leq \frac{K\left(r-\sigma_{1}^{2} / 2\right)}{r} \quad \text { a.s. }
\end{align*}
$$

Proof. Denoting $V(N)=\ln N$ and by Itô's formula, we obtain

$$
\begin{align*}
d V= & \left(r-\frac{r}{K} N(t)-\frac{b P(t)}{N(t)+m P^{\alpha}(t)}-\frac{\sigma_{1}^{2}}{2}\right) d t  \tag{42}\\
& +\sigma_{1} d B_{1}(t) .
\end{align*}
$$

Then, we have

$$
\begin{align*}
\ln N(t)= & \ln N_{0}+\left(r-\frac{\sigma_{1}^{2}}{2}\right) t-\frac{r}{K} \int_{0}^{t} N(s) d s \\
& -b \int_{0}^{t} \frac{P(s)}{N(s)+m P^{\alpha}(s)} d s+\sigma_{1} B_{1}(t) \tag{43}
\end{align*}
$$

And we get

$$
\begin{equation*}
\frac{r}{K} \int_{0}^{t} N(s) d s \leq-\ln N(t)+\ln N_{0}+\left(r-\frac{\sigma_{1}^{2}}{2}\right) t+\sigma_{1} B_{1}(t) \tag{44}
\end{equation*}
$$

Dividing $t$ on both sides of the previously mentioned inequality yields

$$
\begin{align*}
\frac{r}{K} \frac{1}{t} \int_{0}^{t} N(s) d s \leq & -\frac{\ln N(t)}{t} \\
& +\frac{\ln N_{0}}{t}+\left(r-\frac{\sigma_{1}^{2}}{2}\right)+\frac{\sigma_{1} B_{1}(t)}{t} \tag{45}
\end{align*}
$$

Letting $t \rightarrow \infty$, we know that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{\ln N(t)}{t}=0 \quad \text { a.s.; } \tag{46}
\end{equation*}
$$

Then we obtain

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} N(s) d s \leq \frac{K\left(r-\sigma_{1}^{2} / 2\right)}{r} \quad \text { a.s. } \tag{47}
\end{equation*}
$$

On the other hand,

$$
\begin{align*}
\frac{r}{K} \int_{0}^{t} N(s) d s \geq & -\ln N(t)+\ln N_{0} \\
& +\left(r-\frac{\sigma_{1}^{2}}{2}\right) t-\frac{b}{m} t+\sigma_{1} B_{1}(t) \tag{48}
\end{align*}
$$

dividing $t$ on both sides and letting $t \rightarrow \infty$, then

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} N(s) d s \geq \frac{K\left(r-b / m-\sigma_{1}^{2} / 2\right)}{r} \tag{49}
\end{equation*}
$$

From the above results, inequality (41) holds.
Theorem 8. Assume that $c-d-\sigma_{2}^{2} / 2>0$ holds and that $(N(t), P(t))$ is the solution of model (6) for any initial value $\left(N_{0}, P_{0}\right) \in R_{+}^{2}$; then

$$
\begin{gather*}
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \frac{N(s)}{N(s)+m P^{\alpha}(s)} d s \geq \frac{d+\sigma_{2}^{2} / 2}{c} \\
\limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \frac{P^{\alpha}(s)}{N(s)+m P^{\alpha}(s)} d s \leq \frac{c-d-\sigma_{2}^{2} / 2}{c m}, \tag{50}
\end{gather*}
$$

Proof. Denoting $V(P)=\ln P$ and by Itô's formula, we have

$$
\begin{equation*}
d V=\left(\frac{c N(t)}{N(t)+m P^{\alpha}(t)}-d-h P(t)-\frac{\sigma_{2}^{2}}{2}\right) d t+\sigma_{2} d B_{2}(t) \tag{51}
\end{equation*}
$$

Then,

$$
\begin{align*}
& c \int_{0}^{t} \frac{N(s)}{N(s)+m P^{\alpha}(s)} d s \\
& =\ln P(t)-\ln P_{0}+\left(d+\frac{\sigma_{2}^{2}}{2}\right) t+h \int_{0}^{t} P(s) d s-\sigma_{2} B_{2}(t) \\
& \geq \ln P(t)-\ln P_{0}+\left(d+\frac{\sigma_{2}^{2}}{2}\right) t-\sigma_{2} B_{2}(t) \tag{52}
\end{align*}
$$

Dividing $t$ on both sides yields

$$
\begin{align*}
\frac{c}{t} \int_{0}^{t} & \frac{N(s)}{N(s)+m P^{\alpha}(s)} d s  \tag{53}\\
& \geq \frac{\ln P(t)}{t}-\frac{\ln P_{0}}{t}+d+\frac{\sigma_{2}^{2}}{2}-\frac{\sigma_{2} B_{2}(t)}{t}
\end{align*}
$$

we obtain

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \frac{N(s)}{N(s)+m P^{\alpha}(s)} d s \geq \frac{d+\sigma_{2}^{2} / 2}{c} \tag{54}
\end{equation*}
$$

Furthermore,

$$
\begin{align*}
d V= & \left(c-\frac{c m P^{\alpha}(t)}{N(t)+m P^{\alpha}(t)}-d-h P(t)-\frac{\sigma_{2}^{2}}{2}\right) d t  \tag{55}\\
& +\sigma_{2} d B_{2}(t)
\end{align*}
$$

then we can get

$$
\begin{align*}
& c m \int_{0}^{t} \frac{P^{\alpha}(s)}{N(s)+m P^{\alpha}(s)} d s \\
& =-\ln P(t)+\ln P_{0}+\left(c-d-\frac{\sigma_{2}^{2}}{2}\right) t  \tag{56}\\
& \quad-h \int_{0}^{t} P(s) d s+\sigma_{2} B_{2}(t) \\
& \leq-\ln P(t)+\ln P_{0}+\left(c-d-\frac{\sigma_{2}^{2}}{2}\right) t+\sigma_{2} B_{2}(t)
\end{align*}
$$

Dividing $t$ on both sides, we obtain

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \frac{P^{\alpha}(s)}{N(s)+m P^{\alpha}(s)} d s \leq \frac{c-d-\left(\sigma_{2}^{2} / 2\right)}{c m} \tag{57}
\end{equation*}
$$

which are stable in time average.
2.4. Extinction. From (17), if $c-d-\sigma_{2}^{2} / 2<0$, then $\lim _{t \rightarrow \infty} P(t)=0$ a.s. Moreover, from Theorem 7 and (49), we know that if $r-b / m-\sigma_{1}^{2} / 2>0$ holds, then

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} N(s) d s \geq \frac{K\left(r-b / m-\sigma_{1}^{2} / 2\right)}{r} \quad \text { a.s., } \tag{58}
\end{equation*}
$$

which implies that there are a $T_{0}>0$ and a positive constant $n_{0}$ such that $N(t)>n_{0}$ a.s. for $t \geq T_{0}$. Besides, for all $\varepsilon>0$, there are $T>T_{0}$ and $\Omega_{\varepsilon}$ such that $\mathbf{P}\left(\Omega_{\varepsilon}\right) \geq 1-\varepsilon$ and $b P(t) / N(t) \leq \varepsilon$ for $t \geq T$. Then we obtain

$$
\begin{align*}
d N(t)= & N(t)\left(r-\frac{r}{K} N(t)-\frac{b P(t)}{N(t)+m P^{\alpha}(t)}\right) d t \\
& +\sigma_{1} N(t) d B_{1}(t) \\
\geq & N(t)\left(r-\frac{r}{K} N(t)-\frac{b P(t)}{N(t)}\right) d t+\sigma_{1} N(t) d B_{1}(t) \\
\geq & N(t)\left(r-\frac{r}{K} N(t)-\varepsilon\right) d t+\sigma_{1} N(t) d B_{1}(t) \tag{59}
\end{align*}
$$

which implies that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} N(s) d s \geq \frac{K\left(r-\varepsilon-\sigma_{1}^{2} / 2\right)}{r}>0 \tag{60}
\end{equation*}
$$

From Theorem 7 and (47), we have

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \sup \frac{1}{t} \int_{0}^{t} N(s) d s \leq \frac{K\left(r-\sigma_{1}^{2} / 2\right)}{r} \text { a.s. } \tag{61}
\end{equation*}
$$

Therefore, by the arbitrary of $\varepsilon$, we get

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} N(s) d s=\frac{K\left(r-\sigma_{1}^{2} / 2\right)}{r} \quad \text { a.s. } \tag{62}
\end{equation*}
$$

Combining the above arguments, we can get the theorem as follows.

Theorem 9. Let $(N(t), P(t))$ be the solution of model (6) with any initial value $\left(N_{0}, P_{0}\right) \in R_{+}^{2}$. If $r-b / m-\sigma_{1}^{2} / 2>0$ and $c-d-\sigma_{2}^{2} / 2<0$, then

$$
\begin{gather*}
\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} N(s) d s=\frac{K\left(r-\sigma_{1}^{2} / 2\right)}{r}  \tag{63}\\
\lim _{t \rightarrow \infty} P(t)=0
\end{gather*}
$$

Furthermore, set $u(t)=\ln N(t)$ and $v(t)=\ln P(t)$; for the first equation of model (6) we have

$$
\begin{align*}
d u(t)= & \left(r-\frac{r}{K} \mathrm{e}^{u(t)}-\frac{b \mathrm{e}^{v(t)}}{\mathrm{e}^{u(t)}+m \mathrm{e}^{\alpha v(t)}}-\frac{\sigma_{1}^{2}}{2}\right) d t  \tag{64}\\
& +\sigma_{1} d B_{1}(t) \leq\left(r-\frac{\sigma_{1}^{2}}{2}\right) d t+\sigma_{1} d B_{1}(t)
\end{align*}
$$

Taking the comparison theorem of stochastic equations and the theory of diffusion processes [14], then $\lim _{t \rightarrow \infty} u(t)=$ $-\infty$ a.s.; that is,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} N(t)=0 \quad \text { a.s. } \tag{65}
\end{equation*}
$$

Similarly, we obtain

$$
\begin{equation*}
\lim _{t \rightarrow \infty} P(t)=0 \quad \text { a.s. } \tag{66}
\end{equation*}
$$

If not, then there is a positive constant $H$ such that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} P(t)=H>0 \quad \text { a.s. } \tag{67}
\end{equation*}
$$

Hence, for any given $\varepsilon>0$, there exist $t_{0}$ and a set $\Omega_{\varepsilon}$ such that $\mathbf{P}\left(\Omega_{\varepsilon}\right) \geq 1-\varepsilon$ and $c N(t) /\left(N(t)+m P^{\alpha}(t)\right) \leq \varepsilon$ for $t \geq t_{0}$. Therefore,

$$
\begin{align*}
& -P(t)(d+h P(t)) d t+\sigma_{2} P(t) d B_{2}(t) \\
& \quad \leq d P(t) \leq P(t)(-d+\varepsilon) d t+\sigma_{2} P(t) d B_{2}(t) \\
& -\left(d+h P(t)+\frac{\sigma_{2}^{2}}{2}\right) d t+\sigma_{2} d B_{2}(t)  \tag{68}\\
& \quad \leq d v(t) \leq\left(-d+\varepsilon-\frac{\sigma_{2}^{2}}{2}\right) d t+\sigma_{2} d B_{2}(t)
\end{align*}
$$

By the same reasoning as previously stated, we can get $\lim _{t \rightarrow \infty} v(t)=-\infty$ a.s.; that is,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} P(t)=0 \quad \text { a.s. } \tag{69}
\end{equation*}
$$

There is a contradiction; hence (66) is true.
Based on the above, we obtain the following theorem which means that if the noise satisfies some conditions, then both species $N$ and $P$ of model (6) will die out.

Theorem 10. Let $(N(t), P(t))$ be the solution of model (6) with any initial value $\left(N_{0}, P_{0}\right) \in R_{+}^{2}$. If $r-b / m-\sigma_{1}^{2} / 2<0$, then

$$
\begin{equation*}
\lim _{t \rightarrow \infty} N(t)=0, \quad \lim _{t \rightarrow \infty} P(t)=0 \tag{70}
\end{equation*}
$$

## 3. Numerical Simulations

In this section, we perform some numerical simulations for model (6) with environmental noise to illustrate the previously mentioned analytical findings by referring to the


Figure 1: Phase portrait of model (4). Other parameters are taken as $r=2, K=0.15, b=0.8, c=0.5, d=0.1, m=0.1, \alpha=0.2$, and $h=0.1$. The horizontal axis is prey population $N$ and the vertical axis is predator population $P . E_{0}=(0,0)$ and $E_{1}=(0.15,0)$ are two saddle points; $E^{*}=(0.023,0.2025)$ is stable.
method mentioned in Higham [34]. Next, we consider the discretization equations

$$
\begin{align*}
N_{i+1}= & N_{i}+N_{i}\left(r-\frac{r}{K} N_{i}-\frac{b P_{i}}{N_{i}+m P_{i}^{\alpha}}\right) \Delta t \\
& +\alpha N_{i} \sqrt{\Delta t} \xi_{1 i}+\frac{\alpha^{2}}{2} N_{i}^{2}\left(\xi_{1 i}^{2}-1\right) \Delta t  \tag{71}\\
P_{i+1}= & P_{i}+P_{i}\left(\frac{c N_{i}}{N_{i}+m P_{i}^{\alpha}}-d-h P_{i}\right) \Delta t \\
& +\beta P_{i} \sqrt{\Delta t} \xi_{2 i}+\frac{\beta^{2}}{2} P_{i}^{2}\left(\xi_{2 i}^{2}-1\right) \Delta t
\end{align*}
$$

where $\xi_{1 i}$ and $\xi_{2 i}(i=1,2, \ldots, n)$ are the Gaussian random variables $\mathbf{N}(0,1)$.

When choosing the values of parameters $r=2, K=$ $0.15, b=0.8, c=0.5, d=0.1, m=0.1, \alpha=0.2$, and $h=0.1$ for model (4), which has three equilibria in the positive quadrant, where $E_{0}=(0,0)$ (total extinct) and $E_{1}=(0.15,0)$ (extinct of the predator or prey only) are saddle points, $E^{*}=(0.023,0.2025)$ (coexistence of the prey and predator) is globally asymptotically stable. The trajectory of the prey and predator population of model (4) is shown in Figure 1.

Figure 2 shows the time-series plots of model (6) with different noise intensities $\sigma_{1}^{2}$ and $\sigma_{2}^{2}$. When choosing $\sigma_{1}=\sigma_{2}=$ 0.045 (Figure 2(a)) and $\sigma_{1}=0.12, \sigma_{2}=0.3$ (Figure 2(b)), from Theorem 6, we know that the positive solution of model (6) is stochastically permanent, which means that stochastic perturbations do not change the permanence of the deterministic model (4). Moreover, from Theorem 7, the model will be stochastically persistent in mean. In other words, we can use the deterministic model (4) describing the dynamics of the stochastic model when the noise intensities $\sigma_{1}^{2}$ and $\sigma_{2}^{2}$ are small. From Figure 2(b), we can observe
that the violent fluctuations appear as the noise intensities further increase. It means that noise has strong destabilizing effects on the model and the amplitude of the fluctuations in population density of prey and predator species increases obviously, implying instability of the coexisting equilibrium point in the stochastic environment.

In Figure 3, we continue to choose different noise intensities $\sigma_{1}^{2}$ and $\sigma_{2}^{2}$ to consider the effects of noise to model (6). When choosing $\sigma_{1}=0.15$ and $\sigma_{2}=0.9$ (Figure 3(a)), the conditions of Theorem 9 are satisfied; then we can find that prey population $N(t)$ of model (6) is permanent and predator population $P(t)$ will die out. Choosing $\sigma_{1}=0.918$ and $\sigma_{2}=0.6$ in Figure 3(b), which satisfies the conditions of Theorem 10, both species $N$ and $P$ in model (6) become extinct. That is to say, big noise can make the two species die out. From the above numerical results and by Theorems 3, 6, 7,9 , and 10 , we conclude that for some noise intensities $\sigma_{1}^{2}$ and $\sigma_{2}^{2}$ the dynamical behaviors of stochastically ultimately boundedness, permanence, persistence in mean, and extinction can be observed in model (6).

## 4. Conclusions and Remarks

In this paper, we consider a stochastic Hassell-Varley type predator-prey model. The value of this study lies in twofolds. First, it verifies some relevant properties of the corresponding stochastic model (6), which shows the global existence, boundedness and stochastic permanence, persistence in mean, and extinction of the positive solution. Second, it illustrates the dynamics of the model via numerical simulations, which shows that if the noise is not large and satisfies some conditions, the stochastic perturbations do not cause drastic changes of the dynamics in the deterministic model (4), while if the noise is sufficiently large and satisfies some conditions, it will force two species in the model to die out.

In order to study the stochastic model (6), we perturb the deterministic model (4) by incorporating white noise terms in the growth rate of prey population and in the death rate of predator population. Establishing a Lyapunov function, there is a unique positive solution to the model for any positive initial value. Applying Itô's formula, we derive that, under some conditions, the solution of model (6) is stochastically bounded, permanent, and stochastic persistent in mean and extinct. For the fixed parameters $r, K, b, c, d, m, \alpha$, and $h$, some conditions depend on the intensities of noise $\sigma_{1}^{2}$ and noise $\sigma_{2}^{2}$. When the noise intensities satisfy some conditions of Theorem 9, we can find that prey population $N(t)$ of model (6) is permanent and predator population $P(t)$ will die out (see Figure 3(a)), while with the noise intensities increasing which satisfy the conditions of Theorem 10, from Figure 3(b), two species $N$ and $P$ will die out. In other words, when the noise satisfies some conditions of Theorems 6 and 7 and is not sufficiently large, the populations $N$ and $P$ may be stochastically permanent and persistent in mean, while large noise satisfying the conditions of Theorems 9 and 10 will force the population to become extinct. Our complete analysis of the stochastic model will give some suggestions to the studies of the population dynamics.

$\qquad$
(a)

$\qquad$
(b)

FIgure 2: Solutions of model (6) with different noise and other parameters are the same as those of Figure 1 and initial condition $\left(N_{0}, P_{0}\right)=$ $(0.05,0.23)$. (a) $\sigma_{1}=\sigma_{2}=0.045$ and (b) $\sigma_{1}=0.12, \sigma_{2}=0.3$.

$\qquad$

$\qquad$
(a)
(b)

Figure 3: Solutions of model (6) with different noise and other parameters are the same as those of Figure 1. (a) $\sigma_{1}=0.15, \sigma_{2}=0.9$ and (b) $\sigma_{1}=0.918, \sigma_{2}=0.6$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The authors thank the editor and the anonymous referees for the very helpful suggestions and comments which led to improvement of their original paper. This research was supported by the National Science Foundation of China (no. 11301263).

## References

[1] A. A. Berryman, "The origins and evolution of predator-prey theory," Ecology, vol. 73, no. 5, pp. 1530-1535, 1992.
[2] R. M. May, Stability and Complexity in Model Ecosystems, Princeton University Press, Princeton, NJ, USA, 2001.
[3] J. D. Murray, Mathematical Biology II: Spatial Models and Biomedical Applications, vol. 18, Springer, New York, NY, USA, 3rd edition, 2003.
[4] E. Beretta and Y. Kuang, "Global analyses in some delayed ratiodependent predator-prey systems," Nonlinear Analysis. Theory, Methods \& Applications, vol. 32, no. 3, pp. 381-408, 1998.
[5] K. Mischaikow and G. Wolkowicz, "A predator-prey system involving group defense: a connection matrix approach," Nonlinear Analysis. Theory, Methods \& Applications, vol. 14, no. 11, pp. 955-969, 1990.
[6] C. Cosner, D. L. Deangelis, J. S. Ault, and D. B. Olson, "Effects of spatial grouping on the functional response of predators," Theoretical Population Biology, vol. 56, no. 1, pp. 65-75, 1999.
[7] M. Bandyopadhyay and J. Chattopadhyay, "Ratio-dependent predator-prey model: effect of environmental fluctuation and stability," Nonlinearity, vol. 18, no. 2, pp. 913-936, 2005.
[8] M. P. Hassell and G. C. Varley, "New inductive population model for insect parasites and its bearing on biological control", Nature, vol. 223, no. 5211, pp. 1133-1137, 1969.
[9] P. A. Abrams and L. R. Ginzburg, "The nature of predation: prey dependent, ratio dependent or neither?" Trends in Ecology and Evolution, vol. 15, no. 8, pp. 337-341, 2000.
[10] G. T. Skalski and J. F. Gilliam, "Functional responses with predator interference: viable alternatives to the Holling type II model," Ecology, vol. 82, no. 11, pp. 3083-3092, 2001.
[11] W. J. Sutherland, "Aggregation and the "ideal free" distribution," The Journal of Animal Ecology, vol. 52, no. 3, pp. 821-828, 1983.
[12] T. C. Gard, "Persistence in stochastic food web models," Bulletin of Mathematical Biology, vol. 46, no. 3, pp. 357-370, 1984.
[13] T. C. Gard, "Stability for multispecies population models in random environments," Nonlinear Analysis. Theory, Methods \& Applications, vol. 10, no. 12, pp. 1411-1419, 1986.
[14] B. K. Øksendal, Stochastic Differential Equations: An Introduction with Applications, Springer, New York, NY, USA, 4th edition, 1995.
[15] K. Liu and X. Mao, "Exponential stability of non-linear stochastic evolution equations," Stochastic Processes and Their Applications, vol. 78, no. 2, pp. 173-193, 1998.
[16] R. R. Sarkar, "A stochastic model for autotroph-herbivore system with nutrient reclycing," Ecological Modelling, vol. 178, no. 3-4, pp. 429-440, 2004.
[17] A. Bahar and X. Mao, "Stochastic delay Lotka-Volterra model," Journal of Mathematical Analysis and Applications, vol. 292, no. 2, pp. 364-380, 2004.
[18] E. Tornatore, S. M. Buccellato, and P. Vetro, "Stability of a stochastic SIR system," Physica A, vol. 354, no. 1-4, pp. 111-126, 2005.
[19] D. Q. Jiang and N. Z. Shi, "A note on nonautonomous logistic equation with random perturbation," Journal of Mathematical Analysis and Applications, vol. 303, no. 1, pp. 164-172, 2005.
[20] S. A. L. M. Kooijman, J. Grasman, and B. W. Kooi, "A new class of non-linear stochastic population models with mass conservation," Mathematical Biosciences, vol. 210, no. 2, pp. 378394, 2007.
[21] N. Dalal, D. Greenhalgh, and X. R. Mao, "A stochastic model for internal HIV dynamics," Journal of Mathematical Analysis and Applications, vol. 341, no. 2, pp. 1084-1101, 2008.
[22] S. L. Pang, F. Q. Deng, and X. R. Mao, "Asymptotic properties of stochastic population dynamics," Dynamics of Continuous, Discrete \& Impulsive Systems A, vol. 15, no. 5, pp. 603-620, 2008.
[23] F. Rao, W. M. Wang, and Z. Q. Li, "Spatiotemporal complexity of a predator-prey system with the effect of noise and external forcing," Chaos, Solitons \& Fractals, vol. 41, no. 4, pp. 1634-1644, 2009.
[24] X.-Z. Meng, "Stability of a novel stochastic epidemic model with double epidemic hypothesis," Applied Mathematics and Computation, vol. 217, no. 2, pp. 506-515, 2010.
[25] J. Lv and K. Wang, "Asymptotic properties of a stochastic predator-prey system with Holling II functional response," Communications in Nonlinear Science and Numerical Simulation, vol. 16, no. 10, pp. 4037-4048, 2011.
[26] C. Y. Ji, D. Q. Jiang, and X. Y. Li, "Qualitative analysis of a stochastic ratio-dependent predator-prey system," Journal of Computational and Applied Mathematics, vol. 235, no. 5, pp. 1326-1341, 2011.
[27] X. X. Wang, H. L. Huang, Y. L. Cai, and W. M. Wang, "The complex dynamics of a stochastic predatorprey model," Abstract and Applied Analysis, vol. 2012, Article ID 401031, 24 pages, 2012.
[28] M. Liu and K. Wang, "Dynamics of a Leslie-Gower Holling-type II predator-prey system with Lévy jumps," Nonlinear Analysis. Theory, Methods \& Applications, vol. 85, pp. 204-213, 2013.
[29] D. Jiang, C. Ji, X. Li, and D. O'Regan, "Analysis of autonomous Lotka-Volterra competition systems with random perturbation," Journal of Mathematical Analysis and Applications, vol. 390, no. 2, pp. 582-595, 2012.
[30] M. Liu and K. Wang, "Analysis of a stochastic autonomous mutualism model," Journal of Mathematical Analysis and Applications, vol. 402, no. 1, pp. 392-403, 2013.
[31] F. Rao, "Dynamical analysis of a stochastic predator-prey model with an Allee effect," Abstract and Applied Analysis, vol. 2013, Article ID 340980, 10 pages, 2013.
[32] D. Q. Jiang, N. Z. Shi, and X. Y. Li, "Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation," Journal of Mathematical Analysis and Applications, vol. 340, no. 1, pp. 588-597, 2008.
[33] X. Y. Li and X. R. Mao, "Population dynamical behavior of nonautonomous Lotka-Volterra competitive system with random perturbation," Discrete and Continuous Dynamical Systems A, vol. 24, no. 2, pp. 523-545, 2009.
[34] D. J. Higham, "An algorithmic introduction to numerical simulation of stochastic differential equations," SIAM Review, vol. 43, no. 3, pp. 525-546, 2001.

## Research Article

# Stochastic Extinction in an SIRS Epidemic Model Incorporating Media Coverage 

Liyan Wang, Huilin Huang, Ancha Xu, and Weiming Wang<br>College of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, China<br>Correspondence should be addressed to Weiming Wang; weimingwang2003@163.com

Received 2 December 2013; Accepted 12 December 2013
Academic Editor: Kaifa Wang
Copyright © 2013 Liyan Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

We extend the classical SIRS epidemic model incorporating media coverage from a deterministic framework to a stochastic differential equation (SDE) and focus on how environmental fluctuations of the contact coefficient affect the extinction of the disease. We give the conditions of existence of unique positive solution and the stochastic extinction of the SDE model and discuss the exponential $p$-stability and global stability of the SDE model. One of the most interesting findings is that if the intensity of noise is large, then the disease is prone to extinction, which can provide us with some useful control strategies to regulate disease dynamics.


## 1. Introduction

Recent years, a number of mathematical models have been formulated to describe the impact of media coverage on the dynamics of infectious diseases [1-10]. Mass media (television, radio, newspapers, billboards, and booklets) has been used as a way of delivering preventive health messages as it has the potential to influence people's behavior and deter them from risky behavior or from taking precautionary measures in relation to a disease outbreak [7, 11, 12]. Hence, media coverage has an enormous impact on the spread and control of infectious diseases $[2,3,9]$.

On the other hand, for human disease, the nature of epidemic growth and spread is inherently random due to the unpredictability of person-to-person contacts [13], and population is subject to a continuous spectrum of disturbances [14, 15]. In epidemic dynamics, stochastic differential equation (SDE) models could be the more appropriate way of modeling epidemics in many circumstances and many realistic stochastic epidemic models can be derived based on their deterministic formulations [16-28].

In [10], Liu investigated an SIRS epidemic model incorporating media coverage with random perturbation. He assumed that stochastic perturbations were of white noise type, which were directly proportional to distance susceptible $S(t)$, infectious $I(t)$, and recover $R(t)$ from values of endemic
equilibrium point $\left(S^{*}, I^{*}, R^{*}\right)$, influence on the $d S(t) / d t$, $d I(t) / d t, d R(t) / d t$, respectively. In fact, besides the possible equilibrium approach in [10], there are different possible approaches to introduce random effects in the epidemic models affected by environmental white noise from biological significance and mathematical perspective [28-30]. Some scholars [17, 28, 30, 31] demonstrated that one or more system parameter(s) can be perturbed stochastically with white noise term to derive environmentally perturbed system.

In [10], the author proved that the endemic equilibrium of the stochastic model is asymptotically stable in the large. Therefore, it is natural to ask how environmental fluctuations of the contact coefficient affect the extinction of the disease.

In this paper, we will focus on the effects of environmental fluctuations on the disease's extinction through studying the stochastic dynamics of an SIRS model incorporating media coverage. The rest of this paper is organized as follows. In Section 2, based on the results of Cui et al. [2] and [10], we derive the stochastic differential SIRS model incorporating media coverage. In Section 3, we give the conditions of existence of unique positive solution and the stochastic extinction of the SDE model. In Section 4, we provide some examples to support our research results. In the last section, we provide a brief discussion and the summary of main results.

## 2. Model Derivation and Related Definitions

2.1. Model Derivation. Let $S(t)$ be the number of susceptible individuals, $I(t)$ the number of infective individuals, and $R(t)$ the number of removed individuals at time $t$, respectively. Based on the work of Cui et al. [2] and [10], we consider the SIRS epidemic model incorporating media coverage as follows:

$$
\begin{gather*}
\frac{d S}{d t}=\Lambda-\mu S-\left(\beta_{1}-\frac{\beta_{2} I}{b+I}\right) S I+\eta R \\
\frac{d I}{d t}=\left(\beta_{1}-\frac{\beta_{2} I}{b+I}\right) S I-(\mu+\alpha+\lambda) I  \tag{1}\\
\frac{d R}{d t}=\lambda I-(\mu+\eta) R
\end{gather*}
$$

where $\Lambda$ is the recruitment rate, $\mu$ represents the natural death rate, $\eta$ is the loss of constant immunity rate, $\alpha$ is the diseases induced constant death rate, and $\lambda$ is constant recovery rate. $\beta_{1}$ is the usual contact rate without considering the infective individuals and $\beta_{2}$ is the maximum reduced contact rate due to the presence of the infected individuals. No one can avoid contacting with others in every case, so it is assumed that $\beta_{1}>\beta_{2}$. The half-saturation constant $b>0$ reflects the impact of media coverage on the contact transmission. The function $I /(b+I)$ is a continuous bounded function which takes into account disease saturation or psychological effects.

For model (1), the basic reproduction number

$$
\begin{equation*}
R_{0}=\frac{\Lambda \beta_{1}}{\mu(\mu+\alpha+\lambda)} \tag{2}
\end{equation*}
$$

is the threshold of the system for an epidemic to occur. Model (1) has a disease-free equilibrium $E_{0}=(\Lambda / \mu, 0,0)$ and the endemic equilibrium if $R_{0}>1$. The disease-free equilibrium is globally asymptotically stable if $R_{0} \leq 1$ and unstable if $R_{0}>$ 1. The endemic equilibrium is globally asymptotically stable if $R_{0}>1$. These results of model (1) were studied in [10].

If we replace the contact rate $\beta_{1}$ in model (1) by $\beta_{1}+$ $\sigma(d B / d t)$, where $d B / d t$ is a white noise (i.e., $B(t)$ is a Brownian motion), model (1) becomes as follows:

$$
\begin{gather*}
d S=\left[\Lambda-\mu S-\left(\beta_{1}-\frac{\beta_{2} I}{b+I}\right) S I+\eta R\right] d t+\sigma S I d B(t) \\
d I=\left[\left(\beta_{1}-\frac{\beta_{2} I}{b+I}\right) S I-(\mu+\alpha+\lambda) I\right] d t+\sigma S I d B(t) \\
d R=(\lambda I-(\mu+\eta) R) d t \tag{3}
\end{gather*}
$$

Obviously, the stochastic model (3) has the same diseasefree equilibrium $E_{0}=(\Lambda / \mu, 0,0)$ as model (1).

Throughout this paper, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \in \mathbb{R}_{+}}$satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathscr{F}_{0}$ contains all $\mathbb{P}$-null sets). Define a bounded set $\Gamma$ as follows:

$$
\begin{equation*}
\Gamma=\left\{(S, I, R) \in \mathbb{R}_{+}^{3}: 0<S+I+R<\frac{\Lambda}{\mu}\right\} \subset \mathbb{R}_{+}^{3} \tag{4}
\end{equation*}
$$

2.2. Related Definitions. Consider the general $n$-dimensional stochastic differential equation

$$
\begin{equation*}
d x(t)=f(x(t), t) d t+\varphi(x(t), t) d B(t) \tag{5}
\end{equation*}
$$

on $t \geq 0$ with initial value $x(0)=x_{0}$, the solution is denoted by $x\left(t, x_{0}\right)$. Assume that $f(0, t)=0$ and $\varphi(0, t)=0$ for all $t \geq 0$, so (5) has the solution $x(t)=0$, which is called the trivial solution.

Let us first recall a few definitions.
Definition 1 (see [32]). The trivial solution $x(t)=0$ of (5) is said to be
(i) stable in probability if, for all $\varepsilon>0$,

$$
\begin{equation*}
\lim _{x_{0} \rightarrow 0} \mathbb{P}\left(\sup _{t \geq 0}\left|x\left(t, x_{0}\right)\right| \geq \varepsilon\right)=0 \tag{6}
\end{equation*}
$$

(ii) asymptotically stable if it is stable in probability and moreover if

$$
\begin{equation*}
\lim _{x_{0} \rightarrow 0} \mathbb{P}\left(\lim _{t \rightarrow \infty} x\left(t, x_{0}\right)=0\right)=1 \tag{7}
\end{equation*}
$$

(iii) globally asymptotically stable if it is stable in probability and moreover if, for all $x_{0} \in \mathbb{R}^{n}$

$$
\begin{equation*}
\mathbb{P}\left(\lim _{t \rightarrow \infty} x\left(t, x_{0}\right)=0\right)=1 \tag{8}
\end{equation*}
$$

(iv) almost surely exponentially stable if for all $x_{0} \in \mathbb{R}^{n}$,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log \left|x\left(t, x_{0}\right)\right|<0 \quad \text { a.s.; } \tag{9}
\end{equation*}
$$

(v) exponentially $p$-stable if there is a pair of positive constants $C_{1}$ and $C_{2}$ such that for all $x_{0} \in \mathbb{R}^{n}$,

$$
\begin{equation*}
\mathbb{E}\left(\left|x\left(t, x_{0}\right)\right|^{p}\right) \leq C_{1}\left|x_{0}\right|^{p} e^{-C_{2} t} \quad \text { on } t \geq 0 \tag{10}
\end{equation*}
$$

## 3. Dynamics of the SDE Model (3)

In what follows, we first use the method of Lyapunov functions to find conditions of existence of unique positive solution of model (3).
3.1. Existence of Unique Positive Solution of Model (3). In this subsection, we show the existence of the unique positive global solution of SDE model (3).

Theorem 2. Consider model (3), for any given initial value $(S(0), I(0), R(0)) \in \Gamma$; then there is a unique solution $(S(t)$, $I(t), R(t))$ on $t \geq 0$ and it will remain in $\mathbb{R}_{+}^{3}$ with probability one.

Proof. The proof is almost identical to Theorem 2 of [33], but for completeness we repeat it here. Let $(S(0), I(0), R(0)) \in \Gamma$. Summing up the three equations in (3) and denoting $N(t)=$ $S(t)+I(t)+R(t)$, we have

$$
\begin{equation*}
d N(t)=(\Lambda-\mu N(t)-\alpha I(t)) d t \tag{11}
\end{equation*}
$$

Then, if $(S(s), I(s), R(s)) \in \mathbb{R}_{+}^{3}$ for all $0 \leq s \leq t$ almost surely (briefly a.s.), we get

$$
\begin{equation*}
(\Lambda-(\mu+\alpha) N(s)) d s \leq d N(s) \leq(\Lambda-\mu N(s)) d s \quad \text { a.s. } \tag{12}
\end{equation*}
$$

Hence, by integration, we check

$$
\begin{align*}
\frac{\Lambda}{\mu+\alpha} & +\left(N(0)-\frac{\Lambda}{\mu+\alpha}\right) e^{-(\mu+\alpha) s}  \tag{13}\\
& \leq N(s) \leq \frac{\Lambda}{\mu}+\left(N(0)-\frac{\Lambda}{\mu}\right) e^{-\mu s}
\end{align*}
$$

Then, $0<\Lambda /(\mu+\alpha)<N(s)<\Lambda / \mu$ a.s., so,

$$
\begin{equation*}
(S(s), I(s), R(s)) \in\left(0, \frac{\Lambda}{\mu}\right)^{3} \quad \text { for all } s \in[0, t] \text { a.s. } \tag{14}
\end{equation*}
$$

Since the coefficients of model (3) satisfy the local Lipschitz condition, there is a unique local solution on $\left[0, \tau_{e}\right)$, where $\tau_{e}$ is the explosion time. Therefore, the unique local solution to model (3) is positive by the Itô's formula. Now, let us show that this solution is global; that is, $\tau_{e}=\infty$ a.s.

Let $\epsilon_{0}>0$ such that $S(0), I(0), R(0)>\epsilon_{0}$. For $\epsilon \leq \epsilon_{0}$, define the stop-times

$$
\begin{equation*}
\tau_{\epsilon}=\inf \left\{t \in\left[0, \tau_{e}\right]: S(t) \leq \epsilon \text { or } I(t) \leq \epsilon \text { or } R(t) \leq \epsilon\right\} \tag{15}
\end{equation*}
$$

Then

$$
\begin{align*}
\tau & =\lim _{\epsilon \rightarrow 0} \tau_{\epsilon} \\
& =\inf \left\{t \in\left[0, \tau_{e}\right]: S(t) \leq 0 \text { or } I(t) \leq 0 \text { or } R(t) \leq 0\right\} . \tag{16}
\end{align*}
$$

Define a $C^{2}$-function $V: \mathbb{R}_{+}^{3} \rightarrow \mathbb{R}_{+}$by

$$
\begin{equation*}
V(S, I, R)=-\log \left(\frac{\mu S}{\Lambda}\right)-\log \left(\frac{\mu I}{\Lambda}\right)-\log \left(\frac{\mu R}{\Lambda}\right) \tag{17}
\end{equation*}
$$

By the Itô's formula, for all $t \geq 0, s \in\left[0, t \wedge \tau_{\epsilon}\right]$, we obtain

$$
\begin{align*}
d V= & -\frac{1}{S(s)} d S+\frac{1}{2 S(s)^{2}} d S d S-\frac{1}{I(s)} d I \\
& +\frac{1}{2 I(s)^{2}} d I d I-\frac{1}{R(s)} d R+\frac{1}{2 R(s)^{2}} d R d R  \tag{18}\\
\triangleq & L V d s+\sigma(I(s)-S(s)) d B(s)
\end{align*}
$$

where

$$
\begin{aligned}
L V= & 3 \mu+2 \lambda+\alpha+\beta_{1} I+\frac{\beta_{2} S I}{b+I}+\frac{\sigma^{2}}{2}\left(S^{2}+I^{2}\right) \\
& -\frac{\beta_{2} I^{2}}{b+I}-\frac{\eta R}{S}-\beta_{1} S-\frac{\Lambda}{S}-\frac{\alpha I}{R} \\
\leq & 3 \mu+2 \lambda+\alpha+\beta_{1} I+\beta_{2} S+\frac{\sigma^{2}}{2}\left(S^{2}+I^{2}\right)
\end{aligned}
$$

By (14) we assert that $(S(s), I(s), R(s)) \in(0, \Lambda / \mu)$ for all $s \in$ $\left[0, t \wedge \tau_{\epsilon}\right]$ a.s. Hence

$$
\begin{equation*}
L V \leq 3 \mu+2 \lambda+\alpha+\frac{\Lambda}{\mu}\left(\beta_{1}+\beta_{2}+\frac{\sigma^{2} \Lambda}{\mu}\right):=M \tag{20}
\end{equation*}
$$

Substituting this inequality into (18), we see that

$$
\begin{equation*}
d V(S, I, R) \leq M d s+\sigma(I-S) d B(s) \tag{21}
\end{equation*}
$$

which implies that

$$
\begin{align*}
& \int_{0}^{t \wedge \tau_{e}} d V(S(s), I(s), R(s)) \\
& \quad \leq \int_{0}^{t \wedge \tau_{\epsilon}} M d s+\sigma \int_{0}^{t \wedge \tau_{\varepsilon}}(I(s)-S(s)) d B(s) \tag{22}
\end{align*}
$$

Taking the expectations of the above inequality leads to

$$
\begin{align*}
& \mathbb{E} V\left(S\left(t \wedge \tau_{\epsilon}\right), I\left(t \wedge \tau_{\epsilon}\right), R\left(t \wedge \tau_{\epsilon}\right)\right) \\
& \quad \leq V(S(0), I(0), R(0))+M t \tag{23}
\end{align*}
$$

On the other hand, in view of (14), we have $V(S(t \wedge$ $\left.\left.\tau_{\epsilon}\right), I\left(t \wedge \tau_{\epsilon}\right), R\left(t \wedge \tau_{\epsilon}\right)\right)>0$. It then follows that

$$
\begin{align*}
\mathbb{E} V & \left(S\left(t \wedge \tau_{\epsilon}\right), I\left(t \wedge \tau_{\epsilon}\right), R\left(t \wedge \tau_{\epsilon}\right)\right) \\
= & \mathbb{E}\left[\square_{\left(\tau_{\epsilon} \leq t\right)} V\left(S\left(t \wedge \tau_{\epsilon}\right), I\left(t \wedge \tau_{\epsilon}\right), R\left(t \wedge \tau_{\epsilon}\right)\right)\right] \\
& +\mathbb{E}\left[\square_{\left(\tau_{\epsilon}>t\right)} V\left(S\left(t \wedge \tau_{\epsilon}\right), I\left(t \wedge \tau_{\epsilon}\right), R\left(t \wedge \tau_{\epsilon}\right)\right)\right]  \tag{24}\\
\geq & \mathbb{E}\left[\square_{\left(\tau_{\epsilon} \leq t\right)} V\left(S\left(\tau_{\epsilon}\right), I\left(\tau_{\epsilon}\right), R\left(\tau_{\epsilon}\right)\right)\right],
\end{align*}
$$

where $\square_{A}$ is the indicator function of $A$. Note that there is some component of $\left(S\left(\tau_{\epsilon}\right), I\left(\tau_{\epsilon}\right), R\left(\tau_{\epsilon}\right)\right)$ equal to $\epsilon$; therefore, $V\left(S\left(\tau_{\epsilon}\right), I\left(\tau_{\epsilon}\right), R\left(\tau_{\epsilon}\right)\right) \geq-\log (\mu \epsilon / \Lambda)>0$. Thereby

$$
\begin{equation*}
\mathbb{E} V\left(S\left(t \wedge \tau_{\epsilon}\right), I\left(t \wedge \tau_{\epsilon}\right), R\left(t \wedge \tau_{\epsilon}\right)\right) \geq-\log \left(\frac{\mu \epsilon}{\Lambda}\right) \mathbb{P}(\tau \leq t) \tag{25}
\end{equation*}
$$

Combining (23) with (25) gives, for all $t \geq 0$,

$$
\begin{equation*}
\mathbb{P}(\tau \leq t) \leq-\frac{V(S(0), I(0), R(0))+M t}{\log (\mu \epsilon / \Lambda)} \tag{26}
\end{equation*}
$$

Let $\epsilon \rightarrow 0$; we obtain, for all $t \geq 0, \mathbb{P}(\tau \leq t)=0$. Hence, $\mathbb{P}(\tau=\infty)=1$. As $\tau_{e} \geq \tau$, then $\tau_{e}=\tau=\infty$ a.s. which completes the proof of the theorem.

From Theorem 2 and (14), we can conclude the following corollary.

Corollary 3. The set $\Gamma$ is almost surely positive invariant of model (3); that is, if $(S(0), I(0), R(0)) \in \Gamma$, then $\mathbb{P}((S(t), I(t)$, $R(t)) \in \Gamma)=1$ for all $t \geq 0$.
3.2. Stochastic Extinction of Model (3). In this subsection, we investigate stochastic stability of the disease-free equilibrium $E_{0}=(\Lambda / \mu, 0,0)$ in almost sure exponential and exponential $p$ stability by using the suitable Lyapunov function and other techniques of stochastic analysis.

The following theorem gives a sufficient condition for the almost surely exponential stability of the disease-free equilibrium $E_{0}=(\Lambda / \mu, 0,0)$ of model (3).

Theorem 4 (almost sure exponential stability). If $\sigma^{2}>\beta_{1}^{2} / 2 \mu$, then disease-free $E_{0}=(\Lambda / \mu, 0,0)$ of model (3) is almost surely exponentially stable in $\Gamma$.

Proof. Define a function $V$ by

$$
\begin{equation*}
V(S, I, R)=\log \left(\frac{\Lambda}{\mu}-S+I+R\right) \tag{27}
\end{equation*}
$$

Using the Itô's formula, we have

$$
\begin{align*}
& d V=\frac{\partial V}{\partial S} d S+\frac{\partial V}{\partial I} d I+\frac{\partial V}{\partial R} d R \\
& +\frac{1}{2}\left(\frac{\partial^{2} V}{\partial S^{2}} d S d S+\frac{\partial^{2} V}{\partial I^{2}} d I d I+\frac{\partial^{2} V}{\partial R^{2}} d R d R\right) \\
& +\frac{\partial^{2} V}{\partial S \partial I} d S d I+\frac{\partial^{2} V}{\partial S \partial R} d S d R+\frac{\partial^{2} V}{\partial I \partial R} d I d R \\
& =\frac{1}{(\Lambda / \mu)-S+I+R}(-d S+d I+d R) \\
& -\frac{1}{2((\Lambda / \mu)-S+I+R)^{2}}(d S d S+d I d I) \\
& +\frac{1}{((\Lambda / \mu)-S+I+R)^{2}} d S d I \\
& =\frac{1}{(\Lambda / \mu)-S+I+R} \\
& \times\left(-\Lambda+\mu S+2 \beta_{1} S I-\frac{2 \beta_{2} S I^{2}}{b+I}\right. \\
& -(\mu+\alpha) I-(\mu+2 \eta) R) d t \\
& -\frac{2 \sigma^{2} S^{2} I^{2}}{((\Lambda / \mu)-S+I+R)^{2}} d t+\frac{2 \sigma S I}{(\Lambda / \mu)-S+I+R} d B \\
& =\left(2 \beta_{1} Z-2 \sigma^{2} Z-\mu\right) d t+2 \sigma d B Z \\
& -\frac{1}{(\Lambda / \mu)-S+I+R}\left(\frac{2 \beta_{2} S I^{2}}{b+I}+\alpha I+2 \eta R\right) d t \\
& \leq\left(2 \beta_{1} Z-2 \sigma^{2} Z-\mu\right) d t+2 \sigma Z d B, \tag{28}
\end{align*}
$$

where $Z(S, I, R)=S I /((\Lambda / \mu)-S+I+R)$. Since $2 \beta_{1} Z-2 \sigma^{2} Z-$ $\mu=-2 \sigma^{2}\left(Z-\left(\beta_{1} / 2 \sigma^{2}\right)\right)+\left(\beta_{1}^{2}-2 \sigma^{2} \mu\right) / 2 \sigma^{2}$, we obtain

$$
\begin{equation*}
d V \leq \frac{\beta_{1}^{2}-2 \sigma^{2} \mu}{2 \sigma^{2}} d t+2 \sigma Z d B \tag{29}
\end{equation*}
$$

Hence,

$$
\begin{align*}
& \log \left(\frac{\Lambda}{\mu}-S(t)+I(t)+R(t)\right) \\
& \quad \leq \log \left(\frac{\Lambda}{\mu}-S(0)+I(0)+R(0)\right)+\frac{\beta_{1}^{2}-2 \sigma^{2} \mu}{2 \sigma^{2}} t+G(t) \tag{30}
\end{align*}
$$

where $G(t)$ is a martingale defined by $G(t)=2 \sigma \int_{0}^{t} Z d B(s)$. In virtue of Corollary 3, the solution of model (3) remains in $\Gamma$. It then follows that

$$
\begin{equation*}
\langle G, G\rangle_{t}=4 \sigma^{2} \int_{0}^{t} Z^{2} d s \leq C t \tag{31}
\end{equation*}
$$

where $C$ is a positive constant which is dependent on $\Lambda, \mu$. By the strong law of large numbers for martingales [16], we have $\lim \sup _{t \rightarrow \infty} G(t) / t=0$ a.s. It finally follows from (30) by dividing $t$ on the both sides and then letting $t \rightarrow \infty$ that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \log \left(\frac{\Lambda}{\mu}-S+I+R\right) \leq \frac{\beta_{1}^{2}-2 \sigma^{2} \mu}{2 \sigma^{2}}<0 \quad \text { a.s. } \tag{32}
\end{equation*}
$$

which is the required assertion.
We now consider the concept of exponential $p$-stability. The following lemma gives sufficient conditions for exponential $p$-stability of stochastic systems in terms of the Lyapunov functions (see [32]).

Lemma 5 (see [32]). Suppose that there exists a function $V(z, t) \in C^{2}(\Omega)$ satisfying the following inequalities:

$$
\begin{gather*}
K_{1}|z|^{p} \leq V(z, t) \leq K_{2}|z|^{p},  \tag{33}\\
L V(z, t) \leq-K_{3}|z|^{p}, \tag{34}
\end{gather*}
$$

where $p>0$ and $K_{i}(i=1,2,3)$ is positive constant. Then the equilibrium of mode (3) is exponentially $p$-stable for $t \geq 0$. When $p=2$, it is usually said to be exponentially stable in mean square and the the equilibrium is globally asymptotically stable.

From the above Lemma, we obtain the following theorem.
Theorem 6 (exponential $p$-stability). Let $p \geq 2$. If the conditions $R_{0}=\beta_{1} \Lambda / \mu(\mu+\alpha+\lambda)<1$ and $R_{0}^{s}:=R_{0}+((p-$ 1) $\left.\Lambda^{2} \sigma^{2} / 2 \mu^{2}(\mu+\alpha+\lambda)\right)<1$ hold, the disease-free equilibrium $E_{0}=(\Lambda / \mu, 0,0)$ of model (3) is pth moment exponentially stable in $\Gamma$.

Proof. Let $p \geq 2$ and $(S(0), I(0), R(0)) \in \Gamma$; in view of Corollary 3, the solution of model (3) remains in $\Gamma$. We define the Lyapunov function $V(S, I, R)$ as follows:

$$
\begin{equation*}
V=c_{1}\left(\frac{\Lambda}{\mu}-S\right)^{p}+\frac{1}{p} I^{p}+c_{2} R^{p} \tag{35}
\end{equation*}
$$

where $c_{1}>0$ and $c_{2}>0$ are real positive constants that are to be chosen later. It is easy to check that inequalities (33) are true.

Furthermore, by the Itô's formula, it follows from $S, I, R \in$ ( $0, \Lambda / \mu$ ) that

$$
\begin{align*}
L V= & -c_{1} p\left(\frac{\Lambda}{\mu}-S\right)^{p-1}\left(\Lambda-\mu S-\beta_{1} S I+\frac{\beta_{2} S I^{2}}{b+I}+\eta R\right) \\
& +\frac{1}{2} p(p-1) c_{1} \sigma^{2} S^{2} I^{2}\left(\frac{\Lambda}{\mu}-S\right)^{p-2} \\
& +I^{p-1}\left(\beta_{1} S I-\frac{\beta_{2} S I^{2}}{b+I}-(\mu+\alpha+\lambda) I\right) \\
& +\frac{1}{2}(p-1) \sigma^{2} S^{2} I^{p}+c_{2} p R^{p-1}(\lambda I-(\mu+\eta) R) \\
\leq & -c_{1} \mu p\left(\frac{\Lambda}{\mu}-S\right)^{p}+\frac{c_{1} \beta_{1} p \Lambda}{\mu}\left(\frac{\Lambda}{\mu}-S\right)^{p-1} I \\
& +\frac{1}{2 \mu^{2}} p(p-1) c_{1} \sigma^{2} \Lambda^{2}\left(\frac{\Lambda}{\mu}-S\right)^{p-2} I^{2} \\
& -\left(\mu+\alpha+\lambda-\frac{\beta_{1} \Lambda}{\mu}-\frac{1}{2 \mu^{2}}(p-1) \sigma^{2} \Lambda^{2}\right) I^{p} \\
& -c_{2} p(\mu+\eta) R^{p}+c_{2} p \lambda I R^{p-1} \tag{36}
\end{align*}
$$

Using the fact that

$$
\begin{align*}
\left(\frac{\Lambda}{\mu}-S\right)^{p-1} I & \leq \frac{p-1}{p} \varepsilon\left(\frac{\Lambda}{\mu}-S\right)^{p}+\frac{1}{p} \varepsilon^{1-p} I^{p} \\
\left(\frac{\Lambda}{\mu}-S\right)^{p-2} I^{2} & \leq \frac{p-2}{p} \varepsilon\left(\frac{\Lambda}{\mu}-S\right)^{p}+\frac{2}{p} \varepsilon^{(2-p) / 2} I^{p}  \tag{37}\\
R^{p-1} I & \leq \frac{p-1}{p} \varepsilon R^{p}+\frac{1}{p} \varepsilon^{1-p} I^{p}
\end{align*}
$$

we get

$$
\begin{equation*}
L V \leq-A_{1}\left(\frac{\Lambda}{\mu}-S\right)^{p}-A_{2} I^{p}-A_{3} R^{p} \tag{38}
\end{equation*}
$$

where

$$
\begin{gather*}
A_{1}=\left(\mu p-\left(\frac{\beta_{1} \Lambda(p-1)}{\mu}+\frac{\sigma^{2} \Lambda^{2}(p-1)}{2 \mu^{2}}\right) \varepsilon\right) c_{1}, \\
A_{2}=\mu+\alpha+\lambda-\frac{\beta_{1} \Lambda}{\mu}-\frac{1}{2 \mu^{2}}(p-1) \sigma^{2} \Lambda^{2} \\
-\left(\frac{\beta_{1} \Lambda}{\mu} \varepsilon^{1-p}+\frac{\sigma^{2} \Lambda^{2}(p-1)}{\mu^{2}} \varepsilon^{(2-p) / 2}\right) c_{1}-c_{2} \lambda \varepsilon^{1-p} \\
A_{3}=c_{2}(p(\mu+\eta)-\lambda(p-1) \varepsilon) . \tag{39}
\end{gather*}
$$

In view of $R_{0}+\left((p-1) \Lambda^{2} \sigma^{2} / 2 \mu^{2}(\mu+\alpha+\lambda)\right)<1$, we have $\mu+\alpha+\lambda-\left(\beta_{1} \Lambda / \mu\right)-\left(1 / 2 \mu^{2}\right)(p-1) \sigma^{2} \Lambda^{2}>0$. Hence, we chose $\varepsilon$ sufficiently small and $c_{1}, c_{2}$ are positive such that $A_{1}, A_{2}, A_{3}>$ 0 . According to Lemma 5 the proof is completed.

Under Lemma 5 and Theorems 6, we have in the case $p=$ 2 the following corollary.

Corollary 7 (globally asymptotically stable). If the conditions $R_{0}=\beta_{1} \Lambda / \mu(\mu+\alpha+\lambda)<1$ and $R_{0}^{s}:=R_{0}+\left(\Lambda^{2} \sigma^{2} / 2 \mu^{2}(\mu+\right.$ $\alpha+\lambda))<1$ hold, the disease-free equilibrium $E_{0}=(\Lambda / \mu, 0,0)$ of model (3) is globally asymptotically stable in $\Gamma$.

## 4. Numerical Simulations and Dynamics Comparison

In this section, as an example, we give some numerical simulations to show different dynamic outcomes of the deterministic model (1) versus its stochastic version (3) with the same set of parameter values by using the Milstein method mentioned in Higham [34]. In this way, model (3) can be rewritten as the following discretization equations:

$$
\begin{align*}
S_{k+1}= & S_{k}+\left(\Lambda-\mu S_{k}-\beta_{1} S_{k} I_{k}+\frac{\beta_{2} S_{k} I_{k}^{2}}{b+I_{k}}+\eta R_{k}\right) \Delta t \\
& +\sigma S_{k} I_{k} \sqrt{\Delta t} \xi_{k}+\frac{\sigma^{2}}{2} S_{k} I_{k}\left(\xi_{k}^{2}-1\right) \Delta t \\
I_{k+1}= & I_{k}+\left(\beta_{1} S_{k} I_{k}-\frac{\beta_{2} S_{k} I_{k}^{2}}{b+I_{k}}-(\mu+\alpha+\lambda) I_{k}\right) \Delta t  \tag{40}\\
& +\sigma S_{k} I_{k} \sqrt{\Delta t} \xi_{k}+\frac{\sigma^{2}}{2} S_{k} I_{k}\left(\xi_{k}^{2}-1\right) \Delta t \\
& R_{k+1}=R_{k}+\left(\lambda I_{k}-(\mu+\eta) R_{k}\right) \Delta t
\end{align*}
$$

where $\xi_{k}, k=1,2, \ldots, n$, are the Gaussian random variables $N(0,1)$.

For the deterministic model (1) and its stochastic model (3), the parameters are taken as follows:

$$
\begin{gather*}
\Lambda=1, \quad \mu=0.03, \quad \beta_{1}=0.02, \quad \beta_{2}=0.018 \\
\eta=0.01, \quad \alpha=0.1, \quad \lambda=0.05, \quad b=10 \tag{41}
\end{gather*}
$$



Figure 1: The paths of $S(t), I(t)$, and $R(t)$ for the deterministic model (1) with initial values $(S(0), I(0), R(0))=(9,1,0)$. The parameters are taken as (41) ( $R_{0}=7.407$ ).

(b) $\sigma=0.02$

FIGURE 2: The paths of $S(t), I(t)$, and $R(t)$ for the stochastic model (3) with initial values $(S(0), I(0), R(0))=(9,1,0)$. The parameters are taken as (41) $\left(R_{0}=7.407\right)$.
(1) The Endemic Dynamics of the Deterministic Model (1). For the deterministic model (1), $R_{0}=\beta_{1} \Lambda / \mu(\mu+\alpha+\lambda)=$ $7.407>1$; thus, it admits a unique endemic equilibrium $E^{*}=(8.1035,9.7664,12.2080)$ which is globally stable for any initial values $(S(0), I(0), R(0)) \in \Gamma$ according to [10] (see, Figure 1).
(2) The Stochastic Dynamics of Model (3). For the corresponding stochastic model (3), we choose $\sigma=0.1$; then, we have $0.01=\sigma^{2}>\beta_{1}^{2} / 2 \mu=0.007$. Thus, from Theorem 4, we can conclude that for any initial value $(S(0), I(0), R(0)) \in \Gamma$, disease-free $E_{0}=(\Lambda / \mu, 0,0)$ of model (3) is almost surely exponentially stable in $\Gamma$ (see Figure 2(a)).

To see the disease dynamics of model (3) more, we decrease the noise intensity $\sigma$ to be 0.02 and keep the other parameters unchanged. Then, we have $0.0004=\sigma^{2}<$ $\beta_{1}^{2} / 2 \mu=0.007$. Therefore, the condition of Theorem 4 is not satisfied. In this case, our simulations suggest that model (3) is stochastically persistent (see Figure 2(b)).

## 5. Concluding Remarks

In this paper, we propose an SIRS epidemic model with media coverage and environment fluctuations to describe disease transmission. It is shown that the magnitude of environmental fluctuations will have an effective impact on the control and spread of infectious diseases. In a nutshell, we summarize our main findings as well as their related biological implications as follows.

Theorem 4 and [10] combined with numerical simulations (see Figures 1 and 2) provide us with a full picture on the dynamics of the deterministic model (1) and stochastic model (3). In [10], the authors showed that the deterministic model (1) admits a unique endemic equilibrium $E^{*}$ which is globally asymptotically stable if its basic reproduction number $R_{0}>1$ (see Figure 1). If the magnitude of the intensity of noise $\sigma$ is large, that is, $\sigma^{2}>\beta_{1}^{2} / 2 \mu$, the extinction of disease in the stochastic model (3) occurs whether $R_{0}$ is greater than 1 or less than 1 (see Figure 2(a)). While the magnitude of the intensity of noise $\sigma$ is small, one of our most interesting findings is that disease may persist if $R_{0}>1$, (see Figure 2(b)).

Needless to say, both equilibrium possible approach and parameter possible approach in the present paper have their important roles to play. Obviously, our results in the present paper may be a useful supplement for [10].

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This research was supported by the National Science Foundation of China (61373005, 11201344, and 11201345) and Zhejiang Provincial Natural Science Foundation (LY12A01014).

## References

[1] D. Xiao and S. Ruan, "Global analysis of an epidemic model with nonmonotone incidence rate," Mathematical Biosciences, vol. 208, no. 2, pp. 419-429, 2007.
[2] J.-A. Cui, X. Tao, and H. Zhu, "An SIS infection model incorporating media coverage," The Rocky Mountain Journal of Mathematics, vol. 38, no. 5, pp. 1323-1334, 2008.
[3] J. Cui, Y. Sun, and H. Zhu, "The impact of media on the control of infectious diseases," Journal of Dynamics and Differential Equations, vol. 20, no. 1, pp. 31-53, 2008.
[4] Y. Liu and J.-A. Cui, "The impact of media coverage on the dynamics of infectious disease," International Journal of Biomathematics, vol. 1, no. 1, pp. 65-74, 2008.
[5] Y. Li and J. Cui, "The effect of constant and pulse vaccination on SIS epidemic models incorporating media coverage," Communications in Nonlinear Science and Numerical Simulation, vol. 14, no. 5, pp. 2353-2365, 2009.
[6] J. Pang and J.-A. Cui, "An SIRS epidemiological model with nonlinear incidence rate incorporating media coverage," in Proceedings of the 2nd International Conference on Information and Computing Science (ICIC '09), pp. 116-119, May 2009.
[7] M. P. Brinn, K. V. Carson, A. J. Esterman, A. B. Chang, and B. J. Smith, "Mass media interventions for preventing smoking in young people," Cochrane Database of Systematic Reviews, vol. 11, Article ID CD001006, 2010.
[8] S. Funk, M. Salathé, and V. A. A. Jansen, "Modelling the influence of human behaviour on the spread of infectious diseases: a review," Journal of the Royal Society Interface, vol. 7, no. 50, pp. 1247-1256, 2010.
[9] Y. Xiao, T. Zhao, and S. Tang, "Dynamics of an infectious diseases with media/psychology induced non-smooth incidence," Mathematical Biosciences and Engineering. MBE, vol. 10, no. 2, pp. 445-461, 2013.
[10] W. Liu, "A SIRS epidemic model incorporating media coverage with random perturbation," Abstract and Applied Analysis, vol. 2013, Article ID 792308, 9 pages, 2013.
[11] M. E. Young, G. R. Norman, and K. R. Humphreys, "Medicine in the popular press: the influence of the media on perceptions of disease," PLoS ONE, vol. 3, no. 10, Article ID e3552, 2008.
[12] J. M. Tchuenche and C. T. Bauch, "Dynamics of an infectious disease where media coverage influences transmission," ISRN Biomathematics, vol. 2012, Article ID 581274, 10 pages, 2012.
[13] S. Spencer, Stochastic epidemic models for emerging diseases [Ph.D. thesis], University of Nottingham, 2008.
[14] J. R. Beddington and R. M. May, "Harvesting natural populations in a randomly fluctuating environment," Science, vol. 197, no. 4302, pp. 463-465, 1977.
[15] L. J. S. Allen, "An introduction to stochastic epidemic models," in Mathematical Epidemiology, vol. 1945 of Lecture Notes in Math., pp. 81-130, Springer, Berlin, Germany, 2008.
[16] X. Mao, Stochastic Differential Equations and Their Applications, Horwood Publishing Series in Mathematics \& Applications, Horwood, Chichester, UK, 1997.
[17] X. Mao, G. Marion, and E. Renshaw, "Environmental Brownian noise suppresses explosions in population dynamics," Stochastic Processes and their Applications, vol. 97, no. 1, pp. 95-110, 2002.
[18] J. E. Truscott and C. A. Gilligan, "Response of a deterministic epidemiological system to a stochastically varying environment," Proceedings of the National Academy of Sciences of the United States of America, vol. 100, no. 15, pp. 9067-9072, 2003.
[19] D. Jiang and N. Shi, "A note on nonautonomous logistic equation with random perturbation," Journal of Mathematical Analysis and Applications, vol. 303, no. 1, pp. 164-172, 2005.
[20] X. Li and X. Mao, "Population dynamical behavior of nonautonomous Lotka-Volterra competitive system with random perturbation," Discrete and Continuous Dynamical Systems. Series A, vol. 24, no. 2, pp. 523-545, 2009.
[21] M. Liu and K. Wang, "Survival analysis of stochastic singlespecies population models in polluted environments," Ecological Modelling, vol. 220, no. 9-10, pp. 1347-1357, 2009.
[22] O. Ovaskainen and B. Meerson, "Stochastic models of population extinction," Trends in Ecology and Evolution, vol. 25, no. 11, pp. 643-652, 2010.
[23] W. Wang, Y. Cai, M. Wu, K. Wang, and Z. Li, "Complex dynamics of a reaction-diffusion epidemic model," Nonlinear Analysis: Real World Applications, vol. 13, no. 5, pp. 2240-2258, 2012.
[24] F. Ball and P. Neal, "A general model for stochastic SIR epidemics with two levels of mixing," Mathematical Biosciences, vol. 180, pp. 73-102, 2002, John A. Jacquez memorial volume.
[25] H. C. Tuckwell and R. J. Williams, "Some properties of a simple stochastic epidemic model of SIR type," Mathematical Biosciences, vol. 208, no. 1, pp. 76-97, 2007.
[26] T. Britton, "Stochastic epidemic models: a survey," Mathematical Biosciences, vol. 225, no. 1, pp. 24-35, 2010.
[27] A. Gray, D. Greenhalgh, L. Hu, X. Mao, and J. Pan, "A stochastic differential equation SIS epidemic model," SIAM Journal on Applied Mathematics, vol. 71, no. 3, pp. 876-902, 2011.
[28] Q. Yang, D. Jiang, N. Shi, and C. Ji, "The ergodicity and extinction of stochastically perturbed SIR and SEIR epidemic models with saturated incidence," Journal of Mathematical Analysis and Applications, vol. 388, no. 1, pp. 248-271, 2012.
[29] L. Imhof and S. Walcher, "Exclusion and persistence in deterministic and stochastic chemostat models," Journal of Differential Equations, vol. 217, no. 1, pp. 26-53, 2005.
[30] Z. Liu, "Dynamics of positive solutions to SIR and SEIR epidemic models with saturated incidence rates," Nonlinear Analysis: Real World Applications, vol. 14, no. 3, pp. 1286-1299, 2013.
[31] P. S. Mandal and M. Banerjee, "Stochastic persistence and stationary distribution in a HollingTanner type preypredator model," Physica A, vol. 391, no. 4, pp. 1216-1233, 2012.
[32] R. Khasminskii, Stochastic Stability of Differential Equations, vol. 66 of Stochastic Modelling and Applied Probability, Springer, Heidelberg, Germany, 2nd edition, 2012.
[33] A. Lahrouz, L. Omari, and D. Kiouach, "Global analysis of a deterministic and stochastic nonlinear SIRS epidemic model," Nonlinear Analysis: Modelling and Control, vol. 16, no. 1, pp. 5976, 2011.
[34] D. J. Higham, "An algorithmic introduction to numerical simulation of stochastic differential equations," SIAM Review, vol. 43, no. 3, pp. 525-546, 2001.

## Research Article

# The Existence of Positive Nonconstant Steady States in a Reaction: Diffusion Epidemic Model 

Yuan Yuan, ${ }^{1}$ Hailing Wang, ${ }^{2}$ and Weiming Wang ${ }^{1}$<br>${ }^{1}$ College of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, China<br>${ }^{2}$ Department of Mathematics, Hubei Minzu University, Enshi, Hubei 445000, China<br>Correspondence should be addressed to Weiming Wang; weimingwang2003@163.com

Received 19 November 2013; Accepted 1 December 2013
Academic Editor: Kaifa Wang
Copyright © 2013 Yuan Yuan et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate the disease's dynamics of a reaction-diffusion epidemic model. We first give a priori estimates of upper and lower bounds for positive solutions to model and then give the conditions of the existence and nonexistence of the positive nonconstant steady states, which guarantees the existence of the stationary patterns.

## 1. Introduction

Infectious diseases are the second leading cause of death worldwide, after heart disease, and are responsible for more deaths annually than cancer [1]. Since the pioneer work of Kermark and McKendrick [2], mathematical models have been contributing to improve our understanding of infectious disease dynamics and help us develop preventive measures to control infection spread qualitatively and quantitatively.

Many studies indicate that spatial epidemiology with selfdiffusion has become a principal scientific discipline aiming at understanding the causes and consequences of spatial heterogeneity in disease transmission [3]. In these studies, reaction-diffusion equations have been intensively used to describe spatiotemporal dynamics. In particular, the spatial spread of infections has been studied by analyzing traveling wave solutions and calculating spread rates [4-10].

Besides, there has been some research on pattern formation in the spatial epidemic model, starting with Turing's seminal paper [11]. Turing's revolutionary idea was that the passive diffusion could interact with chemical reaction in such a way that even if the reaction by itself has no symmetrybreaking capabilities, diffusion can destabilize the symmetric solutions with the result that the system with diffusion has them [12]. In these studies [3, 13-20], via standard linear analysis, the authors obtained the conditions of Turing instability, and, via numerical simulation, they showed the pattern
formation induced by self-diffusion or cross-diffusion and found that model dynamics exhibits a diffusion controlled formation growth to stripes, spots, and coexistence or chaos pattern replication.

Recently, the researchers are interested in research on the stationary patterns due to the existence and nonexistence nonconstant solutions of the reaction-diffusion model [2129]. But the research on the existence and nonexistence nonconstant solutions of reaction-diffusion epidemic model, seems rare [3].

In this paper, we will focus on the disease's dynamics through studying the existence of the constant and nonconstant steady states of a simple reaction-diffusion epidemic model.

The rest of this paper is organized as follows. In Section 2, we derive a reaction-diffusion epidemic model. In Section 3, we give a priori estimates of upper and lower bounds for positive solutions to model. In Section 4, we give the main results on the existence and nonexistence of positive nonconstant steady states of the model. The paper ends with a brief discussion in Section 5.

## 2. Basic Model

In [30], Berezovsky and coworkers introduced a simple epidemic model through the incorporation of variable population, disease induced mortality, and emigration into the
classic model of Kermark and McKendrick [2]. The total population $(N)$ is divided into two groups susceptible $(S)$ and infectious $(I)$; that is, $N=S+I$. The model describing the relations between the state variables is

$$
\begin{gather*}
\frac{d S}{d t}=r N\left(1-\frac{N}{K}\right)-\beta \frac{S I}{N}-(\mu+\theta) S  \tag{1}\\
\frac{d I}{d t}=\beta \frac{S I}{N}-(\mu+d) I
\end{gather*}
$$

where the birth process incorporates density dependent effects via a logistic equation with the intrinsic growth rate $r$ and the carrying capacity $K ; S(t), I(t)$ represent population densities of susceptible and infected population, respectively; $\beta$ denotes the transmission rate (the infection rate constant); $\mu$ is the natural mortality; $d$ denotes the disease-induced mortality; $\theta$ is the per-capita emigration rate of noninfective.

For model (1), the epidemic threshold of basic reproduction number $R_{0}$ is then computed as

$$
\begin{equation*}
R_{0}=\frac{\beta}{\mu+d} \tag{2}
\end{equation*}
$$

The basic demographic reproductive number $R_{d}$ is given by

$$
\begin{equation*}
R_{d}=\frac{r}{\mu+\theta} \tag{3}
\end{equation*}
$$

For simplicity, rescalling the model (1) by letting $S \rightarrow$ $S / K, I \rightarrow I / K$, and $t \rightarrow t /(\mu+d)$ leads to the following model:

$$
\begin{gather*}
\frac{d S}{d t}=\nu R_{d}(S+I)(1-(S+I))-R_{0} \frac{S I}{S+I}-\nu S,  \tag{4}\\
\frac{d I}{d t}=R_{0} \frac{S I}{S+I}-I,
\end{gather*}
$$

where $\nu=(\mu+\theta) /(\mu+d)$ defined by the ratio of the average life-span of susceptibles to that of infections and $S+I \leq 1$.

See [30] for more details.
Assume that the habitat $\Omega \subset \mathbb{R}^{m}(m \geq 1)$ is a bounded domain with smooth boundary $\partial \Omega$ (when $m>1$ ), and $\mathbf{n}$ is the outward unit normal vector on $\partial \Omega$. We consider the following reaction-diffusion SI epidemic model:

$$
\begin{align*}
& \frac{\partial S}{\partial t}-d_{S} \Delta S= \\
& \quad \nu R_{d}(S+I)(1-(S+I)) \\
&  \tag{5}\\
& -R_{0} \frac{S I}{S+I}-\nu S, \quad x \in \Omega, t>0, \\
& \frac{\partial I}{\partial t}-d_{I} \Delta I=R_{0} \frac{S I}{S+I}-I, \quad x \in \Omega, t>0, \\
& S(x, 0)=S_{0}(x)>0, \quad I(x, 0)=I_{0}(x) \geq 0, \quad x \in \Omega, \\
& \frac{\partial S}{\partial \mathbf{n}}=\frac{\partial I}{\partial \mathbf{n}}=0, \quad x \in \partial \Omega, t>0,
\end{align*}
$$

where the nonnegative constants $d_{S}$ and $d_{I}$ are the diffusion coefficients of $S$ and $I$, respectively. The symbol $\Delta$ is
the Laplacian operator. The homogeneous Neumann boundary condition implies that the above model is self-contained and there is no infection across the boundary.

The corresponding kinetic model (5) with $m=2$ has been investigated by Wang et al. [20].

In this paper, we concentrated on the steady states of model (5) which satisfy

$$
\begin{gather*}
-d_{S} \Delta S=\nu R_{d}(S+I)(1-(S+I))-R_{0} \frac{S I}{S+I}-\nu S, \quad x \in \Omega \\
-d_{I} \Delta I=R_{0} \frac{S I}{S+I}-I, \quad x \in \Omega \\
\frac{\partial S}{\partial \mathbf{n}}=\frac{\partial I}{\partial \mathbf{n}}=0, \quad x \in \partial \Omega . \tag{6}
\end{gather*}
$$

Throughout this paper, the positive solution $(S, I)$ satisfying model (6) refers to a classical one with $S>0, I>0$ on $\bar{\Omega}$. Clearly, model (6) has a unique positive constant solution (endemic equilibrium) $E^{*}=\left(S^{*}, I^{*}\right)$ if $R_{d}>\left(\nu+R_{0}-1\right) / R_{0} v$ and $R_{0}>1$, where

$$
\begin{equation*}
S^{*}=\frac{\nu R_{0} R_{d}-R_{0}+1-v}{\nu R_{0}^{2} R_{d}}, \quad I^{*}=\left(R_{0}-1\right) S^{*} \tag{7}
\end{equation*}
$$

## 3. A Priori Estimates for Positive Solutions to Model (6)

The main purpose of this section is to give a priori upper and lower positive bounds for positive solution of model (6). To this aim, we first cite two known results. The first is due to Lin et al. [31] and the second to Lou and Ni [32]. In the following, let us denote the constants $\nu, R_{d}$, and $R_{0}$ collectively by $\Lambda$. The positive constants $C, \underline{C}, \bar{C}, C^{*}$, and so forth will depend only on the domains $\Omega$ and $\Lambda$.

Lemma 1 (Harnack inequality [31]). Let $w \in C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$ be a positive solution to $\Delta w(x)+c(x) w(x)=0$, where $c \in C(\bar{\Omega})$, satisfying the homogeneous Neumann boundary conditions. Then there exists a positive constant $C^{*}=C^{*}\left(\|c\|_{\infty}, \Omega\right)$, such that

$$
\begin{equation*}
\max _{\bar{\Omega}} w \leq C^{*} \min _{\bar{\Omega}} w \tag{8}
\end{equation*}
$$

Lemma 2 (maximum principle [32]). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^{m}$ and $g \in C(\bar{\Omega} \times \mathbb{R})$.
(a) Assume that $w \in C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$ and satisfies

$$
\begin{gather*}
\Delta w(x)+g(x, w(x)) \geq 0, \quad x \in \Omega \\
\frac{\partial w}{\partial \mathbf{n}} \leq 0, \quad x \in \partial \Omega \tag{9}
\end{gather*}
$$

$$
\text { If } w\left(x_{0}\right)=\max _{\bar{\Omega}} w(x), \text { then } g\left(x_{0}, w\left(x_{0}\right)\right) \geq 0
$$

(b) Assume that $w \in C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$ and satisfies

$$
\begin{gather*}
\Delta w(x)+g(x, w(x)) \leq 0, \quad x \in \Omega, \\
\frac{\partial w}{\partial \mathbf{n}} \geq 0, \quad x \in \partial \Omega .  \tag{10}\\
\text { If } w\left(x_{1}\right)=\min _{\bar{\Omega}} w(x), \text { then } g\left(x_{1}, w\left(x_{1}\right)\right) \leq 0 .
\end{gather*}
$$

Theorem 3. If $R_{0}>1$, then the positive solution $(S(x), I(x))$ of model (6) satisfies

$$
\begin{equation*}
\max _{\bar{\Omega}} S(x)<\frac{1}{4} R_{d}, \quad \max _{\bar{\Omega}} I(x)<\frac{1}{4} R_{d}\left(R_{0}-1\right) \tag{11}
\end{equation*}
$$

Proof. Assume that $(S(x), I(x))$ is a positive solution of model (6). We set

$$
\begin{equation*}
S\left(x_{1}\right)=\max _{\bar{\Omega}} S(x), \quad I\left(x_{2}\right)=\max _{\bar{\Omega}} I(x) . \tag{12}
\end{equation*}
$$

By applying Lemma 2, we have

$$
\begin{align*}
& \frac{1}{4} R_{d}-v S\left(x_{1}\right) \\
& \quad \geq \nu R_{d}\left(S\left(x_{1}\right)+I\left(x_{1}\right)\right)\left(1-\left(S\left(x_{1}\right)+I\left(x_{1}\right)\right)\right)-\nu S\left(x_{1}\right) \\
& \quad \geq \frac{R_{0} S\left(x_{1}\right) I\left(x_{1}\right)}{S\left(x_{1}\right)+I\left(x_{1}\right)}>0, \tag{13}
\end{align*}
$$

and $R_{0} S\left(x_{2}\right) I\left(x_{2}\right) /\left(S\left(x_{2}\right)+I\left(x_{2}\right)\right) \geq I\left(x_{2}\right)$. This clearly gives

$$
\begin{equation*}
S\left(x_{1}\right)<\frac{1}{4} R_{d}, \quad I\left(x_{2}\right) \leq\left(R_{0}-1\right) S\left(x_{2}\right)<\frac{1}{4} R_{d}\left(R_{0}-1\right) . \tag{14}
\end{equation*}
$$

Theorem 4. Assume that $R_{d}>1$ and $R_{0}>1$. Let $d$ and $D$ be fixed positive constants. Then there exists a positive constant $\underline{C}=\underline{C}(\Lambda, d)$ such that, if $d_{S}, d_{I}>d$, every positive solution $(S(x), I(x))$ of model (6) satisfies

$$
\begin{equation*}
\min _{\bar{\Omega}} S(x)>\underline{C}, \quad \min _{\bar{\Omega}} I(x)>\underline{C} . \tag{15}
\end{equation*}
$$

Proof. Let

$$
\begin{gather*}
c_{1}(x)=\frac{1}{d_{S}}\left(\nu R_{d}\left(1+\frac{I}{S}\right)(1-(S+I))-\frac{R_{0} I}{S+I}-v\right) \\
c_{2}(x)=\frac{1}{d_{I}}\left(\frac{R_{0} S}{S+I}-1\right) . \tag{16}
\end{gather*}
$$

In view of Theorem 3, there exists a positive constant $C=$ $C(\Lambda)$ such that $\left\|c_{1}(x)\right\|_{\infty} \leq C,\left\|c_{2}(x)\right\|_{\infty} \leq C$ provided that $d_{S}, d_{I}>d$. As $S$ and $I$ satisfy

$$
\begin{align*}
& \Delta S(x)+c_{1}(x) S=0, \quad x \in \Omega, \\
& \Delta I(x)+c_{2}(x) I=0, \quad x \in \Omega,  \tag{17}\\
& \frac{\partial S}{\partial \mathbf{n}}=\frac{\partial I}{\partial \mathbf{n}}=0, \quad x \in \partial \Omega .
\end{align*}
$$

It follows from Lemma 1 that there exists a positive constant $C^{*}=C^{*}(\Lambda, d)$ such that

$$
\begin{equation*}
\max _{\bar{\Omega}} S \leq C^{*} \min _{\bar{\Omega}} S, \quad \max _{\bar{\Omega}} I \leq C^{*} \min _{\bar{\Omega}} I \tag{18}
\end{equation*}
$$

for $d_{S}, d_{I} \geq d$.
Now, on the contrary, suppose that (15) is not true, then there exist sequences $\left\{d_{S, i}\right\}_{i=1}^{\infty},\left\{d_{I, i}\right\}_{i=1}^{\infty}$ with $\left(d_{S, i}, d_{I, i}\right) \in$ $[d, \infty) \times[d, \infty)$ and the positive solution $\left(S_{i}, I_{i}\right)$ of model (6) corresponding to $\left(d_{S}, d_{I}\right)=\left(d_{S, i}, d_{I, i}\right)$, such that

$$
\begin{equation*}
\min _{\bar{\Omega}} S_{i}(x) \longrightarrow 0 \quad \text { or } \min _{\bar{\Omega}} I_{i}(x) \longrightarrow 0 \quad \text { as } i \longrightarrow \infty . \tag{19}
\end{equation*}
$$

It follows from Lemma 1 that

$$
\begin{align*}
& S_{i}(x) \longrightarrow 0 \text { or } I_{i}(x) \longrightarrow 0 \\
& \text { uniformly on } \bar{\Omega} \text { as } i \longrightarrow \infty . \tag{20}
\end{align*}
$$

$\left(S_{i}, I_{i}\right)$ satisfies

$$
\begin{align*}
-d_{S, i} \Delta S_{i}= & \nu R_{d}\left(S_{i}+I_{i}\right)\left(1-\left(S_{i}+I_{i}\right)\right) \\
& -\frac{R_{0} S_{i} I_{i}}{S_{i}+I_{i}}-v S_{i}, \quad x \in \Omega \\
-d_{I, i} \Delta I_{i}= & \frac{R_{0} S_{i} I_{i}}{S_{i}+I_{i}}-I_{i}, \quad x \in \Omega  \tag{21}\\
\frac{\partial S_{i}}{\partial \mathbf{n}}=\frac{\partial I_{i}}{\partial \mathbf{n}}= & 0, \quad x \in \partial \Omega
\end{align*}
$$

Integrating by parts, we obtain that, for $i=1,2, \ldots$,

$$
\begin{gather*}
\int_{\Omega}\left(\nu R_{d}\left(S_{i}+I_{i}\right)\left(1-\left(S_{i}+I_{i}\right)\right)-\frac{R_{0} S_{i} I_{i}}{S_{i}+I_{i}}-\nu S_{i}\right) d x=0 \\
\int_{\Omega} I_{i}\left(\frac{R_{0} S_{i}}{S_{i}+I_{i}}-1\right) d x=0 \tag{22}
\end{gather*}
$$

By the regularity theory for elliptic equations [33], we see that there exist a subsequence of $\left\{\left(S_{i}, I_{i}\right)\right\}_{i}^{\infty}$, which we will still denote by $\left\{\left(S_{i}, I_{i}\right)\right\}_{i}^{\infty}$, and two nonnegative functions $\widetilde{S}, \widetilde{I} \in$ $C^{2}(\Omega)$, such that $\left(S_{i}, I_{i}\right) \rightarrow(\widetilde{S}, \widetilde{I})$ in $\left[C^{2}(\Omega)\right]^{2}$ as $i \rightarrow \infty$. By (20), we have that $\widetilde{S} \equiv 0$ or $\widetilde{I} \equiv 0$.

Letting $i \rightarrow \infty$ in (22) we obtain that

$$
\begin{gather*}
\int_{\Omega}\left(\nu R_{d}(\widetilde{S}+\widetilde{I})(1-(\widetilde{S}+\widetilde{I}))-\frac{R_{0} \widetilde{S} \widetilde{I}}{\widetilde{S}+\widetilde{I}}-v \widetilde{S}\right) d x=0  \tag{23}\\
\int_{\Omega} \widetilde{I}\left(\frac{R_{0} \widetilde{S}}{\widetilde{S}+\widetilde{I}}-1\right) d x=0
\end{gather*}
$$

Case 1 ( $\widetilde{S} \equiv 0, \widetilde{I} \not \equiv 0$ or $\widetilde{S} \equiv 0, \widetilde{I} \equiv 0)$. Since $I_{i}$ satisfies the second inequality of (18), $I_{i}>0$ on $\bar{\Omega}$. Therefore, $R_{0} S_{i} /\left(S_{i}+\right.$ $\left.I_{i}\right)-1 \rightarrow-1<0$ on $\bar{\Omega}$ as $i \rightarrow \infty$. Hence, $\int_{\Omega} I_{i}\left(R_{0} S_{i} /\left(S_{i}+\right.\right.$ $\left.\left.I_{i}\right)-1\right) d x<0$ for sufficiently large $i$ which contradicts the second integral identity of (22).

Case $2(\widetilde{I} \equiv 0, \widetilde{S} \not \equiv 0)$. As above, $\widetilde{S}>0$ on $\bar{\Omega}$. It follows from the first integral identity of (23) that

$$
\begin{equation*}
\int_{\Omega} \widetilde{S}\left(\nu R_{d}(1-\widetilde{S})-v\right) d x=0 \tag{24}
\end{equation*}
$$

This fact combines with $0<\widetilde{S} \leq(1 / 4) R_{d}$ yielding to $\widetilde{S}=$ $1-\underline{1} / R_{d}$, which implies that $R_{0} S_{i} /\left(S_{i}+I_{i}\right) \rightarrow R_{0}$ uniformly on $\bar{\Omega}$ as $i \rightarrow \infty$, since $I_{i} \rightarrow 0$ uniformly on $\bar{\Omega}$. As $R_{0}>1$, this contradicts the second integral identity of (23) and the fact that $I_{i}>0$. This completes the proof.

## 4. Existence and Nonexistence of Positive Nonconstant Steady States

In this section, we provide some sufficient conditions for the existence and nonexistence of nonconstant positive solution of model (6) by using the Leray-Schauder degree theory [34]. From now on, we denote by

$$
\begin{equation*}
0=\mu_{0}<\mu_{1}<\mu_{2}<\mu_{3}<\cdots \tag{25}
\end{equation*}
$$

the eigenvalues of the operator $-\Delta$ on $\Omega$ with the zero-flux boundary conditions.
4.1. Nonexistence for Positive Nonconstant Steady States to Model (6). This section is devoted to the consideration of the nonexistence for the nonconstant positive solutions of model (6), and, in the following results, the diffusion coefficients do play a significant role.

Theorem 5. Assume that $R_{0}>1$. Let $D_{2}$ be a fixed positive constant with $D_{2}>\left(R_{0}-1\right) / \mu_{1}$. Then there exists a positive constant $D_{1}\left(\Lambda, D_{2}\right)$ such that model (6) has no positive nonconstant solution provided that $d_{S} \geq D_{1}$ and $d_{I} \geq D_{2}$.

Proof. Let $(S(x), I(x))$ be any positive solution of model (6) and denote $\bar{g}=|\Omega|^{-1} \int_{\Omega} g d x$. Then, multiplying the first equation of model (6) by $(S-\bar{S})$, integrating over $\Omega$, by virtue of Theorem 3, we have that

$$
\begin{aligned}
& d_{S} \int_{\Omega}|\nabla S|^{2} d x \\
&= \int_{\Omega}(S-\bar{S}) \\
& \times\left(\nu R_{d}(S+I)(1-(S+I))-\frac{R_{0} S I}{S+I}-v S\right) d x \\
&= \int_{\Omega}(S-\bar{S})^{2} \\
& \times\left(\nu\left(R_{d}-1\right)-\nu R_{d}(S+\bar{S})+2 v R_{d} I-\frac{R_{0} I \bar{I}}{(\bar{S}+\bar{I})(S+I)}\right) d x
\end{aligned}
$$

$$
\begin{align*}
& +\int_{\Omega}\left(\nu R_{d}(1-(I+\bar{I})+2 \bar{S})-\frac{R_{0} s \bar{S}}{(\bar{S}+\bar{I})(S+I)}\right) \\
& \times(S-\bar{S})(I-\bar{I}) d x \\
& \leq C_{1} \int_{\Omega}(S-\bar{S})^{2}+C_{2} \int_{\Omega}|S-\bar{S}||I-\bar{I}| d x, \tag{26}
\end{align*}
$$

where $C_{1}, C_{2}$ depend only on $\Lambda$. In a similar manner, we multiply the second equation in model (6) by $(I-\bar{I})$ to have

$$
\begin{align*}
& d_{I} \int_{\Omega}|\nabla I|^{2} d x \\
& \quad=\int_{\Omega}(I-\bar{I})\left(\frac{R_{0} S I}{S+I}-I\right) d x \\
& =\int_{\Omega}(I-\bar{I})^{2}\left(-1+\frac{R_{0} \bar{S} S}{(\bar{S}+\bar{I})(S+I)}\right) d x  \tag{27}\\
& \quad+\int_{\Omega} \frac{R_{0} \bar{I} I}{(\bar{S}+\bar{I})(S+I)}(S-\bar{S})(I-\bar{I}) d x \\
& \leq \\
& \quad\left(R_{0}-1\right) \int_{\Omega}(I-\bar{I})^{2} d x \\
& \quad+R_{0} \int_{\Omega}|S-\bar{S}||I-\bar{I}| d x .
\end{align*}
$$

It follows from (26), (27) and the $\varepsilon$-Young inequality that

$$
\begin{align*}
& \int_{\Omega}\left(d_{S}|\nabla S|^{2}+d_{I}|\nabla I|^{2}\right) d x \\
& \leq \int_{\Omega}\left(\left(C_{1}+\frac{C}{2 \varepsilon}\right)(S-\bar{S})^{2}+\left(R_{0}-1+\frac{\varepsilon C}{2}\right)(I-\bar{I})^{2}\right) d x \tag{28}
\end{align*}
$$

where $C=C_{2}+R_{0}$. It follows from the well-known Poincaré inequality that

$$
\begin{align*}
\int_{\Omega} & \left(d_{S}|\nabla S|^{2}+d_{I}|\nabla I|^{2}\right) d x \\
\leq & \frac{1}{\mu_{1}}\left(C_{1}+\frac{C}{2 \varepsilon}\right) \int_{\Omega}|\nabla S|^{2} d x  \tag{29}\\
& +\frac{1}{\mu_{1}}\left(R_{0}-1+\frac{\varepsilon C}{2}\right) \int_{\Omega}|\nabla I|^{2} d x
\end{align*}
$$

Since $d_{I} \mu_{1}>R_{0}-1$ from the assumption, we can find a sufficiently small $\varepsilon_{0}$ such that $d_{I} \mu_{1} \geq R_{0}-1+\varepsilon C / 2$. Finally, by taking $D_{1}:=\left(1 / \mu_{1}\right)\left(C_{1}+C / 2 \varepsilon_{0}\right)$ one can conclude that $S=\bar{S}$ and $I=\bar{I}$, which asserts our results.
4.2. Existence for Positive Nonconstant Steady States to Model (6). In this section, we discuss the global existence of nonconstant positive classical solutions to model (6), which guarantees the existence of the stationary patterns [21, 24, 26, 27].

Unless otherwise specified, in this section, we always require that $R_{d}>\left(\nu+R_{0}-1\right) / R_{0} \nu$ and $R_{0}>1$, which guarantees that model (6) has one positive constant steady state $E^{*}$. From now on, let us denote

$$
\begin{gather*}
\mathbf{u}=(S, I) \\
\mathbf{u}^{*}=\left(S^{*}, I^{*}\right)=\left(\frac{\nu R_{0} R_{d}-R_{0}+1-v}{\nu R_{0}^{2} R_{d}},\left(R_{0}-1\right) S^{*}\right) . \tag{30}
\end{gather*}
$$

Let $\mathbf{X}=\left\{\mathbf{u} \in\left[C^{2}(\Omega)\right]^{2} \mid \partial \mathbf{u} / \partial \mathbf{n}=0, x \in \partial \Omega\right\}$ and $\mathbf{X}^{+}=$ $\{\mathbf{u} \in \mathbf{X} \mid S, I>0, x \in \bar{\Omega}\}$. Then we write model (6) in the form

$$
\begin{align*}
-\Delta \mathbf{u} & =\mathbf{G}(\mathbf{u}), \quad x \in \Omega  \tag{35}\\
\frac{\partial \mathbf{u}}{\partial \mathbf{n}} & =0, \quad x \in \partial \Omega \tag{31}
\end{align*}
$$

$$
\begin{align*}
& \mathbf{A}=\left(\begin{array}{cc}
\frac{4 R_{0}+2 \nu-R_{0}^{2}-\nu R_{0} R_{d}-\nu R_{0}-3}{d_{S} R_{0}} & -\frac{\nu R_{0} R_{d}+3-2 R_{0}-2 v}{d_{S} R_{0}} \\
\frac{\left(R_{0}-1\right)^{2}}{d_{I} R_{0}} & -\frac{R_{0}-1}{d_{I} R_{0}}
\end{array}\right)  \tag{36}\\
&:=\left(\begin{array}{cc}
d_{S}^{-1} a_{1} & -d_{S}^{-1} a_{2} \\
d_{I}^{-1} a_{3} & -d_{I}^{-1} a_{4}
\end{array}\right)
\end{align*}
$$

$\lambda$ is an eigenvalue of (35) if and only if $\lambda$ is an eigenvalue of the matrix $\left(\mu_{i}+1\right)^{-1}\left(\mu_{i} \mathbf{I}-\mathbf{A}\right)$ for any $i \geq 0$. Therefore, $\mathbf{I}-\mathscr{F}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)$ is invertible if and only if, for any $i \geq 0$, the matrix

$$
M_{i}:=\mu_{i} \mathbf{I}-\mathbf{A}=\left(\begin{array}{cc}
\mu_{i}-d_{S}^{-1} a_{1} & d_{S}^{-1} a_{2}  \tag{37}\\
-d_{I}^{-1} a_{3} & \mu_{i}+d_{I}^{-1} a_{4}
\end{array}\right)
$$

is invertible. A straightforward computation yields

$$
\begin{align*}
\operatorname{det}\left(M_{i}\right)= & d_{S}^{-1} d_{I}^{-1}  \tag{40}\\
& \times\left(d_{S} d_{I} \mu_{i}^{2}+\left(d_{S} a_{4}-d_{I} a_{1}\right) \mu_{i}+\rho\right) \tag{38}
\end{align*}
$$

where $\rho=\left(1 / R_{0}\right)\left(\nu R_{0} R_{d}-R_{0}+1-v\right)\left(R_{0}-1\right)>0$. For the sake of convenience, we denote

$$
\begin{equation*}
H\left(d_{S}, d_{I}, \mu\right)=d_{S} d_{2} \mu_{i}^{2}+\left(d_{S} a_{4}-d_{I} a_{1}\right) \mu_{i}+\rho \tag{39}
\end{equation*}
$$

Then $H\left(d_{S}, d_{I}, \mu\right)=d_{S} d_{2} \operatorname{det}\left(M_{i}\right)$.
where

$$
\begin{equation*}
\mathbf{G}(\mathbf{u})=\binom{\frac{1}{d_{S}}\left(\nu R_{d}(S+I)(1-(S+I))-\frac{R_{0} S I}{S+I}-\nu S\right)}{\frac{1}{d_{I}}\left(\frac{R_{0} S I}{S+I}-I\right)} . \tag{32}
\end{equation*}
$$

Define a compact operator $\mathscr{F}: \mathbf{X}^{+} \rightarrow \mathbf{X}^{+}$by

$$
\mathscr{F}(\mathbf{u}):=(\mathbf{I}-\Delta)^{-1}\{\mathbf{G}(\mathbf{u})+\mathbf{u}\}
$$

where $(\mathbf{I}-\Delta)^{-1}$ is the inverse operator of $\mathbf{I}-\Delta$ subject to the zero-flux boundary condition. Then $\mathbf{u}$ is a positive solution of model (31) if and only if $\mathbf{u}$ satisfies

$$
\begin{equation*}
(\mathbf{I}-\mathscr{F}) \mathbf{u}=0, \quad x \in \Omega \tag{34}
\end{equation*}
$$

To apply the index theory, we investigate the eigenvalue of the problem

$$
-\left(\mathbf{I}-\mathscr{F}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)\right) \Psi=\lambda \Psi, \quad \Psi \neq \mathbf{0}
$$

where $\Psi=\left(\Psi_{1}, \Psi_{2}\right)^{T}$ and $\mathscr{F}_{\mathbf{u}}\left(\mathbf{u}^{*}\right)=(\mathbf{I}-\Delta)^{-1}(\mathbf{I}+\mathbf{A})$ with

If $\left(d_{S} a_{4}-d_{I} a_{1}\right)^{2}>4 d_{S} d_{2} \rho$, then $H\left(d_{S}, d_{I}, \mu\right)=0$ has two real roots $\mu^{ \pm}$given by

$$
\begin{aligned}
& \mu^{+}\left(d_{S}, d_{I}\right) \\
& \quad=\frac{1}{2 d_{S} d_{2}}\left(d_{I} a_{1}-d_{S} a_{4}+\sqrt{\left(d_{S} a_{4}-d_{I} a_{1}\right)^{2}-4 d_{S} d_{2} \rho}\right) \\
& \mu^{-}\left(d_{S}, d_{I}\right) \\
& \quad=\frac{1}{2 d_{S} d_{2}}\left(d_{I} a_{1}-d_{S} a_{4}-\sqrt{\left(d_{S} a_{4}-d_{I} a_{1}\right)^{2}-4 d_{S} d_{2} \rho}\right)
\end{aligned}
$$

Set $B:=B\left(d_{S}, d_{I}\right)=\left\{\mu: \mu \geq 0, \mu^{-}\left(d_{S}, d_{I}\right)<\mu<\right.$ $\left.\mu^{+}\left(d_{S}, d_{I}\right)\right\}, S_{p}=\left\{\mu_{0}, \mu_{1}, \mu_{2}, \ldots\right\}$, and $m\left(\mu_{i}\right)$ the multiplicity of $\mu_{i}$.

To compute index $\left(\mathbf{I}-\mathscr{F}, \mathbf{u}^{*}\right)$, we can assert the following conclusion by Pang and Wang [22].

Lemma 6 (see [22]). Suppose $H\left(d_{S}, d_{I}, \mu_{i}\right) \neq 0$ for all $\mu_{i} \in S_{p}$. Then

$$
\begin{equation*}
\text { index }\left(\mathbf{I}-\mathscr{F}, \mathbf{u}^{*}\right)=(-1)^{\sigma} \tag{41}
\end{equation*}
$$

where

$$
\sigma= \begin{cases}\sum_{\mu \in B \cap S_{j}} m\left(u_{i}\right), & \text { if } B \cap S_{p} \neq \emptyset  \tag{42}\\ 0, & \text { if } B \cap S_{p}=\emptyset\end{cases}
$$

In particular, if $H\left(d_{S}, d_{I}, \mu\right)>0$ for all $\mu \geq 0$, then $\sigma=0$.
From Lemma 6, we see that to calculate the index of index $\left(\mathbf{I}-\mathscr{F}, \mathbf{u}^{*}\right)$, the key step is to determine the range of $\mu$ for which $H\left(d_{S}, d_{I}, \mu\right)<0$.

Theorem 7. Assume that $R_{d}>\max \left\{1,\left(\nu+R_{0}-1\right) / R_{0} v\right\}$. If $4 R_{0}+2 \nu-R_{0}^{2}-\nu R_{0} R_{d}-\nu R_{0}-3>0,\left(4 R_{0}+2 v-R_{0}^{2}-\nu R_{0} R_{d}-\right.$ $\left.\nu R_{0}-3\right) / d_{S} R_{0} \in\left(\mu_{j}, \mu_{j+1}\right)$ for some $j \geq 1$, and $\sum_{i=1}^{j} m\left(\mu_{i}\right)$ is odd, then there exists a positive constant $d^{*}$ such that model (6) has at least one nonconstant solution if $d_{I}>d^{*}$.

Proof. Since $4 R_{0}+2 v-R_{0}^{2}-\nu R_{0} R_{d}-\nu R_{0}-3>0$, equivalently, $a_{1}>0$, it follows that if $d_{I}$ is large enough, then $\left(d_{S} a_{4}-d_{I} a_{1}\right)^{2}>4 d_{S} d_{I} \rho$ and $0<\mu^{-}\left(d_{S}, d_{I}\right)<\mu^{+}\left(d_{S}, d_{I}\right)$. Furthermore,

$$
\begin{equation*}
\mu^{-}\left(d_{S}, d_{I}\right) \longrightarrow 0, \quad \mu^{+}\left(d_{S}, d_{I}\right) \longrightarrow \frac{a_{1}}{d_{S}}, \quad \text { as } d_{I} \longrightarrow \infty \tag{43}
\end{equation*}
$$

Since $a_{1} / d_{S} \in\left(\mu_{j}, \mu_{j+1}\right)$ for some $j \geq 1$, there exists $d_{0} \gg 1$ such that

$$
\begin{array}{r}
\mu^{+}\left(d_{S}, d_{I}\right) \in\left(\mu_{j}, \mu_{j+1}\right), \quad 0<\mu^{-}\left(d_{S}, d_{I}\right)<\mu_{1}  \tag{44}\\
\forall d_{I} \geq d_{0}
\end{array}
$$

By Theorem 5, we know that there exists $d>d_{0}$ such that model (6) with diffusion coefficients $d_{S}=d$ and $d_{I} \geq d$ has no nonconstant solutions. Moreover, we can choose $d$ so large that $a_{1} / d<\mu_{1}$. It follows that there exists $d^{*}>d$ such that

$$
\begin{equation*}
0<\mu^{-}\left(d_{S}, d_{I}\right)<\mu^{+}\left(d_{S}, d_{I}\right)<\mu_{1}, \quad \forall d_{I} \geq d^{*} \tag{45}
\end{equation*}
$$

We shall prove that, for any $d_{I} \geq d^{*}$, model (6) has at least one nonconstant positive solution. On the contrary, suppose that this assertion is not true for some $d_{I}^{*}>d^{*}$. In the following, we will derive a contradiction by using a homotopy argument.

By virtue of Theorems 3 and 4, there exists a positive constant $C=C\left(\Lambda, d, d_{S}, d^{*}, d_{I}{ }^{*}\right)$ such that the positive solution $(S(x), I(x))$ of model (6) satisfies $C^{-1}<S, I<C$.

Set

$$
\begin{equation*}
\mathscr{M}=\left\{(S, I) \in C(\bar{\Omega}) \times C(\bar{\Omega}): C^{-1}<S, I<C, x \in \bar{\Omega}\right\} \tag{46}
\end{equation*}
$$

and define

$$
\begin{equation*}
\Phi: \mathscr{M} \times[0,1] \longrightarrow C(\bar{\Omega}) \times C(\bar{\Omega}) \tag{47}
\end{equation*}
$$

by

$$
\begin{equation*}
\Phi(\mathbf{u}, \theta)=(\mathbf{I}-\Delta)^{-1}\{\mathbf{G}(\mathbf{u}, \theta)+\mathbf{u}\} \tag{48}
\end{equation*}
$$

where

$$
\mathbf{G}(\mathbf{u}, \theta)
$$

$$
\begin{equation*}
=\binom{\left(\theta d_{S}+(1-\theta) d\right)^{-1}\left(\nu R_{d}(S+I)(1-(S+I))-\frac{R_{0} S I}{S+I}-\nu S\right)}{\left(\theta d_{I}+(1-\theta) d^{*}\right)^{-1}\left(\frac{R_{0} S I}{S+I}-I\right)} . \tag{49}
\end{equation*}
$$

It is clear that finding the positive solution of model (31) is equivalent to finding the fixed point of $\Phi(\mathbf{u}, 1)$ in $\mathscr{M}$. Further, by virtue of the definition of $\mathscr{M}$, we have that $\Phi(\mathbf{u}, \theta)=0$ has no fixed point in $\partial \mathscr{M}$ for all $0 \leq \theta \leq 1$.

Since $\Phi(\mathbf{u}, t)$ is compact, the Leray-Schauder topological degree $\operatorname{deg}(\mathbf{I}-\Phi(\mathbf{u}, \theta), \mathscr{M}, 0)$ is well defined. From the invariance of Leray-Schauder degree at the homotopy, we deduce

$$
\begin{equation*}
\operatorname{deg}(\mathbf{I}-\Phi(\mathbf{u}, 1), \mathscr{M}, 0)=\operatorname{deg}(\mathbf{I}-\Phi(\mathbf{u}, 0), \mathscr{M}, 0) \tag{50}
\end{equation*}
$$

In view of $\mu^{-} \in\left(\mu_{i}, \mu_{i+1}\right)$ and $\mu^{+} \in\left(\mu_{j}, \mu_{j+1}\right)$, we have $B\left(d_{S}, d_{I}\right) \cap S_{j}=\left\{\mu_{i+1}, \mu_{i+2}, \ldots, \mu_{p}\right\}$. Clearly, $\mathbf{I}-\Phi(\mathbf{u}, 1)=\mathbf{I}-\mathscr{F}$. Thus, if model (6) has no other solutions except the constant one $\mathbf{u}^{*}$, then Lemma 6 shows that

$$
\begin{align*}
& \operatorname{deg}(\mathbf{I}-\Phi(\mathbf{u}, 1), \mathscr{M}, 0) \\
& \quad=\operatorname{index}\left(\mathbf{I}-\mathscr{F}, \mathbf{u}^{*}\right)=(-1)^{\sum_{i=1}^{j} m\left(u_{i}\right)}=-1 \tag{51}
\end{align*}
$$

On the contrary, by the choice of $d$ and $d^{*}$, we have that $B\left(\bar{d}_{1}, \bar{d}_{2}\right) \cap S_{p}=\emptyset$ and $\mathbf{u}^{*}$ is the only fixed point of $\Phi(\mathbf{u}, 0)$. It therefore follows from Lemma 6 that

$$
\begin{align*}
& \operatorname{deg}(\mathbf{I}-\Phi(\mathbf{u}, 0), \mathscr{M}, 0) \\
& \quad=\operatorname{index}\left(\mathbf{I}-\mathscr{F}, \mathbf{u}^{*}\right)=(-1)^{0}=1 \tag{52}
\end{align*}
$$

From (50)-(52), we get a contradiction. Therefore, there exists a nonconstant solution of model (6). The proof is completed.

## 5. Discussion

In this paper, we investigate the disease's dynamics through studying the existence and nonexistence positive constant steady states of a reaction-diffusion epidemic model. We give a priori estimates for positive solutions to model and show that the nonconstant positive steady states exist due to the emergence of diffusion, which demonstrates that stationary patterns can be found as a result of diffusion. The numerical results about the stationary patterns for model (5) can be found in [20].

On the other hand, there are plenty of papers which focus on the pattern formation of reaction-diffusion population models via standard linear analysis method and numerical simulations. But there is little literature analytically concerning the existence of a stationary patterns via theory and methods of partial differential equations infrequently. The methods and results in the present paper may enrich the research of pattern formation in the spatial epidemic model.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The authors thank the anonymous referee for very helpful suggestions and comments which led to improvement of our original paper. This research was supported by the National Science Foundation of China (61373005) and Zhejiang Provincial Natural Science Foundation (LY12A01014).

## References

[1] A. Lahrouz and A. Settati, "Asymptotic properties of switching diffusion epidemic model with varying population size," Applied Mathematics and Computation, vol. 219, no. 24, pp. 11134-11148, 2013.
[2] M. D. Kermark and A. G. Mckendrick, "Contributions to the mathematical theory of epidemics," Proceedings of the Royal Society of London A, vol. 115, pp. 700-721, 1927.
[3] Y. Cai, D. Chi, W. Liu, and W. Wang, "Stationary patterns of a cross-diffusion epidemic model," Abstract and Applied Analysis, vol. 2013, Article ID 852698, 10 pages, 2013.
[4] E. E. Holmes, M. A. Lewis, J. E. Banks, and R. R. Veit, "Partial differential equations in ecology: spatial interactions and population dynamics," Ecology, vol. 75, no. 1, pp. 17-29, 1994.
[5] Y. Hosono and B. Ilyas, "Traveling waves for a simple diffusive epidemic model," Mathematical Models and Methods in Applied Sciences, vol. 5, no. 7, pp. 935-966, 1995.
[6] B. T. Grenfell, O. N. Bjørnstad, and J. Kappey, "Travelling waves and spatial hierarchies in measles epidemics," Nature, vol. 414, no. 6865, pp. 716-723, 2001.
[7] A. L. Lloyd and V. A. A. Jansen, "Spatiotemporal dynamics of epidemics: synchrony in metapopulation models," Mathematical Biosciences, vol. 188, no. 1-2, pp. 1-16, 2004.
[8] K. Wang, W. Wang, and S. Song, "Dynamics of an HBV model with diffusion and delay," Journal of Theoretical Biology, vol. 253, no. 1, pp. 36-44, 2008.
[9] R. Xu and Z. Ma, "An HBV model with diffusion and time delay", Journal of Theoretical Biology, vol. 257, no. 3, pp. 499-509, 2009.
[10] S. Wang, W. Liu, Z. Guo, and W. Wang, "Traveling wave solutions in a reaction-diffusion epidemic model," Abstract and Applied Analysis, vol. 2013, Article ID 216913, 13 pages, 2013.
[11] A. M. Turing, "The chemical basis of morphogenesis," Philosophical Transactions of the Royal Society of London B, vol. 237, no. 641, pp. 37-72, 1952.
[12] N. F. Britton, Essential Mathematical Biology, Springer, 2003.
[13] G. Sun, Z. Jin, Q.-X. Liu, and L. Li, "Pattern formation in a spatial S-I model with non-linear incidence rates," Journal of Statistical Mechanics, vol. 2007, no. 11, Article ID P11011, 2007.
[14] M. Bendahmane and M. Langlais, "A reaction-diffusion system with cross-diffusion modeling the spread of an epidemic disease," Journal of Evolution Equations, vol. 10, no. 4, pp. 883-904, 2010.
[15] Y. Cai and W. Wang, "Spatiotemporal dynamics of a reactiondiffusion epidemic model with nonlinear incidence rate," Journal of Statistical Mechanics, vol. 2011, no. 2, Article ID P02025, 2011.
[16] W. Wang, Y. Lin, H. Wang, H. Liu, and Y. Tan, "Pattern selection in an epidemic model with self and cross diffusion," Journal of Biological Systems, vol. 19, no. 1, pp. 19-31, 2011.
[17] W.-M. Wang, H.-Y. Liu, Y.-L. Cai, and Z.-Q. Li, "Turing pattern selection in a reaction-diffusion epidemic model," Chinese Physics B, vol. 20, no. 7, Article ID 074702, 2011.
[18] S. Berres and R. Ruiz-Baier, "A fully adaptive numerical approximation for a two-dimensional epidemic model with nonlinear cross-diffusion," Nonlinear Analysis, vol. 12, no. 5, pp. 28882903, 2011.
[19] F. Rao, W. Wang, and Z. Li, "Stability analysis of an epidemic model with diffusion and stochastic perturbation," Communications in Nonlinear Science and Numerical Simulation, vol. 17, no. 6, pp. 2551-2563, 2012.
[20] W. Wang, Y. Cai, M. Wu, K. Wang, and Z. Li, "Complex dynamics of a reaction-diffusion epidemic model," Nonlinear Analysis, vol. 13, no. 5, pp. 2240-2258, 2012.
[21] M. Wang, "Non-constant positive steady states of the Sel'kov model," Journal of Differential Equations, vol. 190, no. 2, pp. 600620, 2003.
[22] P. Y. H. Pang and M. Wang, "Qualitative analysis of a ratiodependent predator-prey system with diffusion," Royal Society of Edinburgh A, vol. 133, no. 4, pp. 919-942, 2003.
[23] P. Y. H. Pang and M. Wang, "Strategy and stationary pattern in a three-species predator-prey model," Journal of Differential Equations, vol. 200, no. 2, pp. 245-273, 2004.
[24] M. Wang, "Stationary patterns for a prey-predator model with prey-dependent and ratio-dependent functional responses and diffusion," Physica D, vol. 196, no. 1-2, pp. 172-192, 2004.
[25] M. Wang, "Stationary patterns caused by cross-diffusion for a three-species prey-predator model," Computers and Mathematics with Applications, vol. 52, no. 5, pp. 707-720, 2006.
[26] R. Peng, J. Shi, and M. Wang, "Stationary pattern of a ratiodependent food chain model with diffusion," SIAM Journal on Applied Mathematics, vol. 67, no. 5, pp. 1479-1503, 2007.
[27] R. Peng, M. Wang, and G. Yang, "Stationary patterns of the holling-tanner prey-predator model with diffusion and crossdiffusion," Applied Mathematics and Computation, vol. 196, no. 2, pp. 570-577, 2008.
[28] R. Peng and J. Shi, "Non-existence of non-constant positive steady states of two Holling type-II predator-prey systems: strong interaction case," Journal of Differential Equations, vol. 247, no. 3, pp. 866-886, 2009.
[29] C. Bianca, "Existence of stationary solutions in kinetic models with gaussian thermostats," Mathematical Methods in the Applied Sciences, vol. 36, no. 13, pp. 1768-1775, 2013.
[30] F. Berezovsky, G. Karev, B. Song, and C. Castillo-Chavez, "A simple epidemic model with surprising dynamics," Mathematical Biosciences and Engineering, vol. 2, no. 1, pp. 133-152, 2005.
[31] C.-S. Lin, W.-M. Ni, and I. Takagi, "Large amplitude stationary solutions to a chemotaxis system," Journal of Differential Equations, vol. 72, no. 1, pp. 1-27, 1988.
[32] Y. Lou and W.-M. Ni, "Diffusion, self-diffusion and crossdiffusion," Journal of Differential Equations, vol. 131, no. 1, pp. 79-131, 1996.
[33] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, 1983.
[34] L. Nirenberg, Topics in Nonlinear Functional Analysis, AMS Bookstore, 2001.

