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Analysis of Vibrating Timoshenko Beams Using the Method of Differential Quadrature

The main advantages of the differential quadrature method are its inherent conceptual simplicity and the fact that easily programmable algorithmic expressions are obtained. It was developed by Bellman in the 1970s but only recently has been applied in the solution of technically important problems. Essentially, it consists of the approximate solution of the differential system by means of a polynomial–collocation approach at a finite number of points selected by the analyst. This article reports some numerical experiments on vibrating Timoshenko beams of nonuniform cross-section. © *1993 John Wiley & Sons, Inc.*

INTRODUCTION

The differential quadrature method was developed by Bellman and Casti [1971] but it has been popularized in recent years by Jang, Bert, and Striz [1989], Striz, Jang, and Bert [1988], and Bert, Jang, and Striz [1989].

A simple explanation of the method is provided and then the technique is applied to the determination of the natural frequencies of Timoshenko beams of nonuniform cross-section.

DESCRIPTION OF THE METHOD

Consider the differential equation

$$M[W(x)] = F(x) \tag{1}$$

subject to certain boundary conditions in the interval $[a, b]$.

One proposes now the polynomial

$$P(x) = a_{N-1}x^{N-1} + a_{N-2}x^{N-2} + \cdots + a_1x + a_0, \tag{2}$$

which is required to satisfy Eq. (1) and the boundary conditions at N points of the interval

$$a = x_1 < x_2 < \cdots < x_i < \cdots < x_N = b. \tag{3}$$

If $N > m$ it is possible to express, at each x_i of expression (3), the derivative of order m of $P(x)$ as a linear combination of the values $P(x_j)$ or in other words:

$$\sum_{j=1}^{N} c_{ij}P(x_j) = P^{(m)}(x_i). \tag{4}$$

Expressing Eq. (4) in the form

$$\sum_{j=1}^{N} c_{ij}(a_{N-1}x_j^{N-1} + a_{n-z}x_j^{N-2} + \cdots + a_1x_j + a_0)$$

$$= \sum_{k=m}^{N-1} k(k-1)\cdots(k-m+1)a_kx_i^{k-m} \tag{5}$$

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leads to the functional relation:

$$\sum_{j=1}^{N}\sum_{k=0}^{N-1} c_{ij}a_k x_j^k = \sum_{k=m}^{N-1} k(k-1)\cdots(k-m+1)a_k x_i^{k-m} \tag{6}$$

and finally to:

$$\sum_{k=0}^{N-1}(c_{i1}x_1^k + c_{i2}x_2^k + \cdots + c_{iN}x_N^k)a_k = \sum_{k=m}^{N-1} k(k-1)\cdots(k-m+1)a_k x_i^{k-m}. \tag{7}$$

The c_{ij}'s are obtained solving the linear system of equations

$$\begin{aligned}
&c_{i1} + c_{i2} + \cdots + c_{iN} = 0\\
&c_{i1}x_1 + c_{i2}x_2 + \cdots + c_{iN}x_N = 0\\
&\vdots\\
&c_{i1}x_1^{m-1} + c_{i2}x_2^{m-1} + \cdots + c_{iN}x_N^{m-1} = 0\\
&c_{i1}x_1^{m} + c_{i2}x_2^{m} + \cdots + c_{iN}x_N^{m} = m(m-1)\cdots 1\\
&c_{i1}x_1^{m+1} + c_{i2}x_2^{m-1} + \cdots + c_{iN}x_N^{m+1} = (m+1)m\cdots 2x_i\\
&\vdots\\
&c_{i1}x^{N-1} + c_{i2}x_2^{N-1} + \cdots + c_{iN}x_N^{N-1} = (N-1)(N-2)\cdots(N-m)x_i^{N-1-m}.
\end{aligned} \tag{8}$$

Accordingly expression (4) is a valid representation of the derivative of order m of $P(x)$.

Substituting now the derivatives that appear in Eq. (1) and in the boundary conditions, by the expressions generated by Eq. (4), one obtains a linear system of equations in the $P(x_i)$'s. These values are approximations to the exact ones, $W(x_i)$. In the case of an eigenvalue problem a homogeneous system of equations results and from the nontriviality condition one obtains a determinantal equation in the characteristic values of the problem under study. As N increases it is reasonable to expect that the approximations will improve (assuming that round-off errors do not come into play). Following the notation used by well-known authors [Jant et al., 1989; Striz et al., 1988; Bert et al., 1989] the coefficients c_{ij} corresponding to first, second, third, and fourth order derivatives are denoted by

$$A_{ij},\ B_{ij},\ C_{ij},\ D_{ij},$$

respectively.

DETERMINATION OF NATURAL FREQUENCIES OF TIMOSHENKO BEAMS OF LINEARLY VARYING THICKNESS

Consider the mechanical system shown in Fig. 1. Making use of Timoshenko's classical theory of vibrating beams one expresses the governing differential equations in the form

$$\begin{cases}
-E\dfrac{\partial}{\partial \bar{x}}\left(I\dfrac{\partial \psi}{\partial \bar{x}}\right) + \dfrac{EA}{\lambda}\left(\psi - \dfrac{\partial v}{\partial \bar{x}}\right) + \rho I\dfrac{\partial^2 \psi}{\partial t^2} = 0\\
\dfrac{E}{\lambda}\dfrac{\partial}{\partial \bar{x}}\left[A\left(\psi - \dfrac{\partial v}{\partial \bar{x}}\right)\right] + \rho A\dfrac{\partial^2 v}{\partial t^2} = 0
\end{cases} \tag{9}$$

where $v(\bar{x}, t)$, transverse displacement; $\psi(x, t)$ angular rotation of the cross-section due to bending; λ, $2(1+\nu)/k$; k, shear factor; ν, Poisson's ratio; $I(\bar{x})$, moment of inertia of the cross-sectional area; $A(\bar{x})$, cross-sectional area; ρ, density of the beam material. In the case of normal modes of vibration one writes

$$\begin{aligned}
v(\bar{x}, t) &= V(\bar{x})\cos \omega t\\
\psi(\bar{x}, t) &= \Psi(\bar{x})\cos \omega t.
\end{aligned} \tag{10}$$

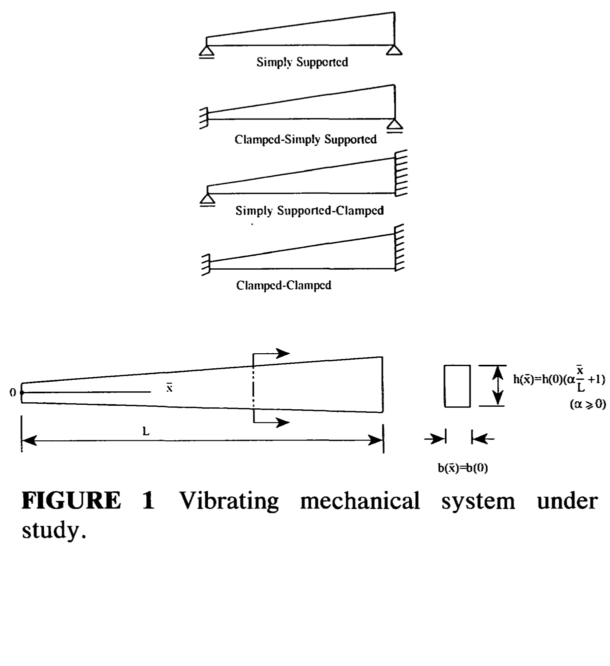

FIGURE 1 Vibrating mechanical system under study.

Introducing the dimensionless variable $x = \bar{x}/L$ and substituting Eq. (10) in (Eq. (9) one obtains

$$\begin{cases} -\lambda\eta_0 \dfrac{d}{dx}(f_1\Phi') + f_2(\Phi - V') - \Omega^2\lambda\eta_0^2 f_1\Phi = 0 \\ \dfrac{d}{dx}[f_2(\Phi - V')] - \Omega^2\lambda\eta_0 f_2 V = 0 \end{cases} \tag{11}$$

where

$$f_1(x) = (\alpha x + 1)^3$$
$$f_2(x) = (\alpha x + 1)$$
$$\eta_0 = \frac{I(0)}{A(0)L^2}$$
$$\Phi(x) = L\Psi(x)$$
$$\Omega^2 = \frac{\rho A(0)L^4\omega^2}{EI(0)}.$$

If the beam is hinged at both ends the boundary conditions are

$$\begin{aligned} V(0) &= \Phi'(0) = 0 \\ V(1) &= \Phi'(1) = 0 \end{aligned} \tag{12}$$

and if they are clamped

$$\begin{aligned} V(0) &= \Phi(0) = 0 \\ V(1) &= \Phi(1) = 0. \end{aligned} \tag{13}$$

The interval [0, 1] is now subdivided and N nodes are adopted. In correspondence with each node one has two unknowns: $\Phi_k = \Phi(x_k)$ and $V_k = V(x_k)$ and two equations are expressed. The unknowns will now be defined in the form

$$U_1 = \Phi_1, \ldots, U_N = \Phi_N, \quad U_{N+1} = V_1, \ldots, U_{2N} = V_N. \tag{14}$$

Substituting the polynomial expression (4) in the governing differential system and using the notation defined in earlier, one obtains the following system of equations for the case of a hinged–hinged beam

$$\sum_{k=1}^{N} A_{1k}U_k = 0$$

$$-\lambda\eta_0 \sum_{k=1}^{N} (f_1' A_{ik} + f_1 B_{ik})U_k + f_2 U_i - \sum_{k=2}^{N-1} f_2 A_{ik} U_{k+N} - \Omega^2\lambda\eta_0^2 f_1 U_i = 0$$
$$(i = 2, \ldots, N-1)$$

$$f_2 \sum_{k=1}^{N} A_{(i-N)k}U_k + f_2' U_{i-N} - \sum_{k=2}^{N-1} [f_2' A_{(i-N)k} + f_2 B_{(i-N)k}]U_{k+N} - \Omega^2\lambda\eta_0 f_2 U_i = 0 \quad (i = 2 + N, \ldots, 2N-1)$$

$$\sum_{k=1}^{N} A_{Nk}U_k = 0 \tag{15}$$

Analogous procedures are followed for other combinations of boundary conditions.

NUMERICAL RESULTS

Fundamental frequency coefficients were obtained for the following situations (Fig. 1): simply supported; clamped–simply supported; simply supported–clamped; clamped–clamped. In order to ascertain the relative accuracy of the results obtained by means of the differential quadrature method, they were compared with values obtained using the finite element algorithmic procedure [Gutierrez, Laura, and Rossi, 1991]. Results are presented for several values of η_0 and α and Poisson's ratio is equal to 0.30 and $k = 0.833$. Table 1 depicts numerical results for the case of a simply supported beam.

The comparison with the results obtained by means of the finite elements method, (Table 2) indicates very good relative accuracy. Excellent agreement is also achieved when the results are compared with the exact fundamental eigenvalues (Table 1), for $\alpha = 0$.

The cases of: clamped–simply supported, simply supported–clamped and clamped–clamped ends are dealt with in Tables 3–7. Excellent agreement with the finite element predictions are observed for the cases considered in Tables 3, 4, 6, and 7 (no finite elements results are available for the situation posed in Table 5).

CONCLUSIONS

Present numerical experiments indicate that the method of differential quadrature may be advan-

Table 1. Fundamental Frequency Coefficients Ω_1 in the Case of a Simply Supported Beam of Linearly Varying Thickness

	0		0.05	0.10	0.15	0.20
η_0/α	(A)	(B)	(A)	(B)	(A)	(B)
0.0009	9.694	9.695	9.925	10.153	10.376	10.597
0.0016	9.565	9.567	9.788	10.007	10.220	10.429
0.0025	9.409	9.411	9.622	9.829	10.030	10.228
0.0036	9.231	9.232	9.431	9.626	9.815	9.999
0.0049	9.034	9.036	9.222	9.403	9.581	9.753
0.0064	8.825	8.827	9.001	9.171	9.333	9.491

See Fig. 1. (A) Determined by means of the differential quadrature method (n = 9). (B) Exact results.

Table 2. Fundamental Frequency Coefficients Ω_1 in the Case of a Simply Supported Beam of Linearly Varying Thickness

η_0/α	0.05	0.10	0.15	0.20
0.0009	9.927	10.154	10.377	10.597
0.0016	9.790	10.007	10.221	10.430
0.0025	9.623	9.830	10.031	10.229
0.0036	9.433	9.627	9.816	10.001
0.0049	9.224	9.406	9.582	9.754
0.0064	9.003	9.172	9.336	9.494

See Fig. 1. Obtained by means of the finite element method [Gutierrez et al., 1991].

Table 3. Fundamental Frequency Coefficient Ω_1 in the Case of a Clamped–Simply Supported Beam of Linearly Varying Thickness

	0		0.05	0.10	0.15	0.20
η_0/α	(A)	(B)	(A)	(A)	(A)	(A)
0.0009	14.792	14.793	15.032	15.267	15.497	15.724
0.0016	14.358	14.358	14.575	14.786	14.991	15.192
0.0025	13.854	13.856	14.046	14.231	14.413	14.587
0.0036	13.310	13.311	13.478	13.638	13.792	13.940
0.0049	12.745	12.746	12.888	13.024	13.155	13.280
0.0064	12.179	12.178	12.298	12.413	12.522	12.626

(A) determined by means of the differential quadrature method (n = 10). (B) Exact results.

Table 4. Fundamental Frequency Coefficient Ω_1 in the Case of a Clamped–Simply Supported Beam of Linearly Varying Thickness

η_0/α	0.05	0.10	0.15	0.20
0.0009	15.035	15.271	15.502	15.728
0.0016	14.578	14.789	14.996	15.197
0.0025	14.050	14.236	14.417	14.592
0.0036	13.480	13.641	13.796	13.285
0.0049	12.892	13.029	13.160	13.285
0.0064	12.303	12.418	12.527	12.632

Obtained by means of the finite element method [Gutierrez et al., 1991].

Table 5. Fundamental Frequency Coefficient Ω_1 in the Case of a Simply Supported–Clamped Beam of Linearly Varying Thickness

η_0/α	0	0.05	0.10	0.15	0.20
0.0009	14.792	15.226	15.653	16.075	16.491
0.0016	14.358	14.757	15.149	15.532	15.909
0.0025	13.854	14.218	14.569	14.915	15.251
0.0036	13.310	13.634	13.948	14.255	14.550
0.0049	12.745	13.054	13.312	13.581	13.841
0.0064	12.179	12.433	12.679	12.913	13.140

Results obtained using the method of differential quadrature, (n = 10).

Table 6. Fundamental Frequency Coefficient Ω_1 in the Case of a Clamped–Clamped Beam of Linearly Varying Thickness

	0		0.05	0.10	0.15	0.20
η_0/α	(A)	(B)	(A)	(A)	(A)	(A)
0.0009	20.872	20.872	21.321	21.763	22.194	22.616
0.0016	19.901	19.901	20.290	20.669	21.038	21.397
0.0025	18.837	18.837	19.167	19.845	19.794	20.093
0.0036	17.749	17.749	18.024	18.290	18.544	18.790
0.0049	16.683	16.682	16.911	17.130	17.339	17.540
0.0064	15.665	15.666	15.856	16.036	16.208	16.372

(A) Differential quadrature method (n = 11). (B) Exact results.

Table 7. Fundamental Frequency Coefficient Ω_1 in the Case of a Clamped–Clamped Beam of Linearly Varying Thickness

η_0/α	0.05	0.10	0.15	0.20
0.0009	21.325	21.765	22.197	22.621
0.0016	20.294	20.673	21.043	21.403
0.0025	19.173	19.492	19.801	20.101
0.0036	18.031	18.297	18.552	18.799
0.0049	16.919	17.138	17.348	17.549
0.0064	15.864	16.045	16.217	16.381

Finite elements method [Gutierrez et al., 1991].

tageous when dealing with vibrating Timoshenko beams. The methodology is also applicable in the case of forced vibration situations.

It also appears at this moment that the technique can be conveniently used when dealing with vibrating Timoshenko–Mindlin plates.

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