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Dynamic Stiffness Analysis of Curved Thin-Walled Beams

The natural vibration problem of curved thin-walled beams is solved by the dynamic stiffness method. The dynamic stiffness of a curved open thin-walled beam is given. The computed natural frequencies of the beam are compared with those obtained by a completely analytical method to show the high accuracy of the present method. The interaction of in-plane and out-of-plane modes is emphasized. © *1993 John Wiley & Sons, Inc.*

INTRODUCTION

Research on the vibration problems of various structural members including uniform or nonuniform members [Banerjee and Williams, 1985] and straight or curved members [Henrych, 1981; Pearson and Wittrick, 1986] has been intensive. However, comparatively little work has been done on the dynamic problem of curved thin-walled beams. In this article, the natural modes of curved thin-walled beams are studied. The equations of motion are derived by a variational procedure. Warping effects and curvature of the member are considered here to obtain a more rational solution. The dynamic problem is solved by the dynamic stiffness method described in the next section.

Even for the static stability problem, many discrepancies between different works due to different initial assumptions, such as neglecting warping effects and curvature of the beams, were found. Historically, flexural and torsional actions of a beam were considered separately. The common engineering theory of flexure is based on the Bernouilli–Euler–Navier assumption that plane cross-sections remain plane and perpendicular to the deformed locus and suffer no strains in their planes after bending. Torsion was treated by the theory of St. Venant. The effect of warping was first taken into account by Timoshenko for a bisymmetrical I-beam in 1905 [Timoshenko and Gere, 1961]. In Vlasov's theory [1961] for general thin-walled beams, cross-sections are allowed to warp nonuniformly along the beam axis. But even with warping effects being taken into account, disagreements between the known works [Henrych, 1981; Yang and Kuo, 1986] do exist for curved beams, especially when the subtended angles of the members are large. The discrepancies occur due to the fact that analogue generalized strains are adopted for a straight and curved member. For a straight thin-walled beam, the generalized stress–strain relations for compression, flexure, and warping are all uncoupled. Alternatively, we derive the equations of motion for a cylindrically curved thin-walled beam by considering the effects of various deformation modes and the curvature of the member as accurately as possible.

To study the natural vibration problem of a curved thin-walled beam, the dynamic stiffness method is employed herein, and a recently developed program [Leung and Zeng, to appear] with parametric (ω) inverse iteration is used to solve the resulting nonlinear eigenvalue problem.

According to the conventional finite element theory, a subdivision of elements is inevitable for a curved beam and the accuracy is dependent on

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the number of elements. This difficulty can be avoided by using the dynamic stiffness method. The dynamic stiffness matrix is formed by frequency-dependent shape functions, which are exact solutions of the governing differential equations. It eliminates spatial discretization errors and predicts many vibration modes accurately. The method has been applied with success to many dynamic problems including natural vibration [Friberg, 1985; Banerjee and Williams, 1985; Pearson and Wittriek, 1986; Lundon and Akesson, 1983; Leung, 1988, 1990, 1991a,b] and response analysis [Leung, 1987].

AN ANALYTICAL METHOD

In the dynamic stiffness method, analytical solutions of the governing equations are used as shape functions; thus, the stiffness matrices are exact. These matrices are in general parametric in terms of the vibration frequency and the load factor to produce the dynamic stiffness and stability matrices. The whole process including the eigensolutions of the differential governing equations (for shape functions) and stiffness matrix formulation is automated. This is briefly described as follows. (1) When the member is vibrating harmonically, the time variable is excluded from the governing equations, and the frequency appears as a parameter. (2) Expand the equations in the spatial domain by letting the displacement vector $\{u(x)\} = e^{\lambda x}\{\phi\}$ to obtain a characteristic polynomial equation in λ, where λ is the eigenvalue and $\{\phi\}$ is the eigenvector. The polynomial equation for the eigenvalues is solved by a Newtonian algorithm. The ranks are checked and the eigenvectors are found in a standard way. (3) Together with the given natural boundary conditions and the known eigenvalues and eigenvectors, the shape functions are formed. (4) Finally, we find the relation between the generalized boundary forces and the generalized boundary displacements to obtain the dynamic stiffness matrix. Although no explicit expressions for the matrix elements will in general be available, the displacement functions along the element and the numerical values of the stiffness matrix can be found with great accuracy for each frequency considered.

For steady-state harmonic oscillation with excitation frequency ω, the governing equations can be written in a general form,

$$[F(D)]\{u(x)\} = ([A_0] + [A_1]D^{(1)} + \cdots + [A_n]D^{(n)})\{u(x)\} = \{f\} \tag{1}$$

with boundary conditions

$$D^{(i)}\{u(x)\}|_{x=-l/2} = D^{(i)}\{u(-l/2)\} \tag{2a}$$

$$D^{(i)}\{u(x)\}|_{x=l/2} = D^{(i)}\{u(l/2)\}. \tag{2b}$$

Here $D^{(i)}(\ )$ denotes derivatives with respect to the position variable x; l is the length of the element; $\{f(x)\}$ and $\{u(x)\}$ are the excitation and response vectors respectively; $[A_0], [A_1], \ldots [A_n]$ are real square matrices of order m and are ω dependent. Equation (1) is self-adjoint provided that $[A_i]$ is symmetric or skew-symmetric when i is even or odd, respectively.

The homogeneous solution of Eq. (1) can be obtained by letting $\{u(x)\} = e^{\lambda x}\{\phi\}$, which gives

$$[F(\lambda)]\{\phi\} = ([A_0] + \lambda[A_1] + \cdots + \lambda^n[A_n])\{\phi\} = \{0\}. \tag{3}$$

Obviously, this constitutes an eigenproblem for the $n \times m$ nontrivial solutions of eigenvectors $\{\phi_j\}$ and the corresponding $n \times m$ eigenvalues λ_j, $j = 1, 2, \ldots, n \times m$. Solving the matrix polynomial eigenproblem, the frequency dependent shape functions are obtained and then the dynamic stiffness matrix is formed by enforcing the natural boundary conditions.

SHAPE FUNCTIONS AND DYNAMIC STIFFNESS MATRIX

Solving the eigenproblem of Eq. (3), we get the displacements vector

$$\{u(x)\} = \sum_{j=1}^{nm} c_j \exp(\lambda_j)\{\phi_j\} = [\phi]\mathrm{diag}[\exp(\lambda_1 x), \exp(\lambda_2 x), \ldots, \exp(\lambda_{nm} x)]\{C\} \tag{4}$$

where $[\phi]$ is an $m \times nm$ matrix composed of $\{\phi_j\}$ that is the eigenvector corresponding to λ_j; $\{C\}$ is a column containing nm constant coefficients. Substituting Eq. (4) into the boundary conditions (2), one has

$$\{q\} = [H]\{C\}. \tag{5}$$

It is evident that the constants $\{C\}$ can be evaluated in terms of the general nodal displacement vector $\{q\}$ by inverting Eq. (5). Substituting $\{C\}$ into Eq. (4), by definition, we obtain the shape functions relating the distributed displacements and the nodal displacements,

$$\{N(x)\} = [\phi]\text{diag}[\exp(\lambda_1 x), \exp(\lambda_2 x), \ldots, \exp(\lambda_{nm} x)][H]^{-1}. \quad (6)$$

Applying the inner product of Eq. (1) with a virtual displacement vector $\{v(x)\}$, one can integrate by parts and get a series of boundary items. By using Betti's reciprocal theorem, the generalized boundary forces can be expressed in terms of the generalized displacements. Thus, the required dynamic stiffness matrix $[K]$ is obtained. A detailed derivation is given in Leung and Zeng (to appear).

EQUATIONS OF MOTION

We consider a horizontally curved thin-walled beam. Some assumptions that must be made here are: 1) the material is elastic and homogeneous; 2) the length of the beam is very large compared with the cross-sectional dimensions; 3) every cross-section is rigid in its own plane; 4) shearing deformation of the middle surface of the member is negligible; and 5) transverse displacements are much larger than the longitudinal displacement. For a circularly curved member with I-section as shown in Fig. 1, when the effect of curvature is considered, the cross-sectional displacements, say, the displacements at point p (Fig. 1) are derived according to Vlasov's thin-walled beam theory [Yang and Kuo, 1986],

$$u_p = u - yv' - z\left(w' - \frac{u}{R}\right) - \Omega\left(\theta' + \frac{v'}{R}\right) \quad (7a)$$

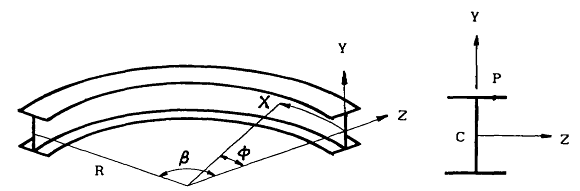

FIGURE 1 Cross-section of a curved thin-walled beam.

$$v_p = v - z\theta \quad (7b)$$

$$w_p = w + y\theta \quad (7c)$$

where R is the radius of curvature, θ is the twist angle, v and w denote the transverse displacements of the centroid, C, from the original position, and u is the longitudinal displacement of C. The displacements u, v, w, θ are the functions of coordinate x that is tangent to the curved axis of the member. In Eq. (7), a prime denotes differentiation with respect to coordinate x, and Ω the normalized sectorial area. Coordinate axes x, y, z form a right-handed frame. The effect of curvature is considered through the expressions of various quantities, such as strains and volumes in terms of the radius of curvature R.

From the finite displacement theory, the first order linear and the second order nonlinear components of the strains can be expressed in terms of the displacements $[u_p, v_p, w_p, \theta]^T$ and hence in terms of the displacements of the centroid C, $[u, v, w, \theta]^T$, from Eq. (7). Therefore we can get a set of strain-displacement relations and hence stress-displacement relations from Hooke's law. According to the principle of virtual displacements, the dynamic stability of a deformed body can be described in a Lagrangian form,

$$\int_v s_{ij}\delta\varepsilon_{ij} d\text{vol} + \delta T = \text{EVW} \quad (8)$$

where $\delta T = \int_{\text{vol}} \rho\{\ddot{u}\}\delta\{u\} d\text{vol}$ is the volume integral of the virtual work done by inertial forces; s_{ij} = the second Piola–Kirchhoff stress tensor; $\delta\varepsilon_{ij}$ = the variation of the Green–Lagrange strain tensor; EVW = the external virtual work; vol denotes the initial volume of the body; and the differential $d$vol equals $(R + Z)/R\, dy\, dz\, dx$ in Cartesian coordinates.

A set of governing differential equations of motion for the curved thin-walled beam can be obtained [Yang and Kuo, 1986] by the following steps applied to Eq. (8): (1) substituting the expression for ε_{ij} and s_{ij} in terms of $[u, v, w, \theta]$; (2) integrating each term by parts to obtain the virtual displacements δu, δv, δw, and $\delta\theta$; (3) admitting the arbitrary nature of virtual displacements; and (4) neglecting the higher order terms than the second order of smallness. Finally, we get a set of equations of motion as follows,

$$\begin{aligned}
&\mathrm{EA}\left(u'' + \frac{w'}{R}\right) + \frac{1}{R}F_x\left(w' - \frac{u}{R}\right) \\
&- \frac{1}{R}\left(M_z + \frac{B}{R}\right)\left(\theta' + \frac{v'}{R}\right) \\
&+ \frac{1}{2R}T'_{\mathrm{SV}}v' - \left(F_y - \frac{M_x}{R}\right)\left(v'' - \frac{\theta}{R}\right) \\
&- F'_z\left(w' - \frac{u}{R}\right) - F_z\left(w'' + \frac{w}{R^2}\right) \\
&- m\left(A + \frac{3I_y}{R^2}\right)\ddot{u} + \frac{2}{R}mI_y\ddot{w}' = 0
\end{aligned} \tag{9a}$$

$$\begin{aligned}
&\mathrm{EI}_y\left(w'''' + 2\frac{w''}{R^2} + \frac{w}{R^4}\right) + \frac{\mathrm{EA}}{R}\left(u' + \frac{w}{R}\right) \\
&- \left[F_x\left(w' - \frac{u}{R}\right)\right]' + \left[\left(M_z + \frac{B}{R}\right)\left(\theta' + \frac{v'}{R}\right)\right]' \\
&- \frac{1}{2}(T'_{sv}v')' - F_y\left(\theta' + \frac{v'}{R}\right) + F_z\left(u'' + \frac{u}{R^2}\right) \\
&+ F'_z\left(u' + \frac{w}{R}\right) - \left[M_x\left(v'' - \frac{\theta}{R}\right)\right]' \\
&+ mA\ddot{w} - mI_y\ddot{w}'' + \frac{2}{R}mI_y\ddot{u}' = 0
\end{aligned} \tag{9b}$$

$$\begin{aligned}
&\mathrm{EI}_z\left(v'''' - \frac{\theta''}{R}\right) - \frac{GJ}{R}\left(\theta'' + \frac{v''}{R}\right) - (F_xv')' \\
&+ \left[M_y\left(\theta' + \frac{v'}{R}\right)\right]' + (F_z\theta)' \\
&+ F_y\left(u'' + \frac{w'}{R}\right) \\
&+ \frac{1}{R}\left[\left(M_z + \frac{B}{R}\right)\left(w' - \frac{u}{R}\right)\right]' \\
&- \frac{r^2}{R}\left[\left(F_x + \frac{M_y}{R}\right)\left(\theta' + \frac{v'}{R}\right)\right]' \\
&+ \left[\left(M_x - \frac{1}{2}T_{\mathrm{SV}}\right)'\left(w' - \frac{u}{R}\right)\right]' \\
&+ \left[M_x\left(w'' - \frac{u'}{R}\right)\right]' \\
&+ mA\ddot{v} - m\left(I_z + \frac{3I_\Omega}{R^2}\right)\ddot{v}'' \\
&- m\frac{I_y}{R}\ddot{\theta} - m\frac{2I_\Omega}{R}\ddot{\theta}'' = 0
\end{aligned} \tag{9c}$$

$$\begin{aligned}
&\mathrm{EI}_\Omega\left(\theta'''' + \frac{2\theta''}{R^2} + \frac{\theta}{R^4}\right) - \frac{\mathrm{EI}_z}{R}\left(v'' - \frac{\theta}{R}\right) \\
&- GJ\left(\theta'' + \frac{v''}{R}\right) + M_y\left(v'' - \frac{\theta}{R}\right) \\
&+ \left[\left(M_z + \frac{B}{R}\right)\left(w' - \frac{u}{R}\right)\right]' \\
&- r^2\left[\left(F_x + \frac{M_y}{R}\right)\left(\theta' + \frac{v'}{R}\right)\right]' \\
&+ F_y\left(w' - \frac{u}{R}\right) - \frac{M_x}{R}\left(w' - \frac{u}{R}\right) \\
&+ m(I_y + I_z + r^2A)\ddot{\theta} - mI_\Omega\ddot{\theta}'' \\
&- \frac{1}{R}mI_y\ddot{v} - \frac{2}{R}mI_\Omega\ddot{v}'' = 0
\end{aligned} \tag{9d}$$

where $F_x, F_y, F_z, M_x, M_y, M_z$ are the stress resultants on a cross-section of a member; T_{SV} is the St. Venant torque; B is the bimoment; I_y and I_z are the moments of inertia about the y and z axes, respectively; I_Ω is the warping constant; J is the torsional constant; m is the mass of unit volume; r is the polar radius of gyration; and A is the area of the cross section.

Equations (9a), (9b), (9c), and (9d) represent the dynamic stability of a curved thin-walled beam. The different deformation modes, such as tensile, flexural, and torsional modes (including possible warping), are coupled as the result of including the effect of curvature.

EXAMPLE

For the present purpose, a horizontally curved member of an I-section as shown in Fig. 1 will be considered. The circular beam was subjected to a constant bending moment M_y with different boundary conditions in the present studies. Figure 2 is an in-plain diagram of the beam. The section properties adopted from Yang and Kuo [1986] are: $A = 92.9$ cm^2; $I_z = 11360$ cm^4; $I_y = 3870$ cm^4; $I_\omega = 55590$ cm^6, $J = 58.9$ cm^6, $r = 12.81$ cm; and $L = 1024$ cm. The moduli of elasticity are $E = 200$ GPa, $G = 77.2$ GPa.

If the condition of inextensibility, $u' + w/R =$

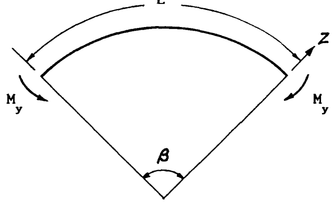

FIGURE 2 Curved beam with in-plane bending moments.

0, and the harmonic conditions are assumed, u is eliminated from Eq. (9a) and Eqs. (9b), (9c), and (9d) become

$$\mathrm{EI}_y\left(w'''' + \frac{2w''}{R^2} + \frac{w}{R^4}\right) - \frac{M_y}{R}\left(w'' + \frac{w}{R^2}\right) - Am\omega^2 w + I_y m\omega^2 w'' + \frac{2I_y}{R^2} m\omega^2 w = 0 \tag{10a}$$

$$\begin{aligned} &\mathrm{EI}_z\left(v'''' - \frac{\theta''}{R}\right) - \frac{\mathrm{GJ}}{R}\left(\theta'' + \frac{v''}{R}\right) \\ &+ M_y\left(1 - \frac{r^2}{R^2}\right)\left(\theta'' + \frac{v''}{R}\right) \\ &- Am\omega^2 v + \left(I_z + \frac{3I_\Omega}{R^2}\right) m\omega^2 v'' \\ &+ \frac{I_y}{R} m\omega^2\theta + \frac{2I_\Omega}{R} m\omega^2\theta'' = 0 \end{aligned} \tag{10b}$$

$$\begin{aligned} &\mathrm{EI}_\Omega\left(\theta'''' + \frac{2\theta''}{R^2} + \frac{\theta}{R^4}\right) - \frac{\mathrm{EI}_z}{R}\left(v'' - \frac{\theta}{R}\right) \\ &- \mathrm{GJ}\left(\theta'' + \frac{v''}{R}\right) + M_y v'' - \frac{r^2}{R} M_y\left(\theta'' + \frac{v''}{R}\right) \\ &- (I_y + I_z + r^2 A)m\omega^2\theta + I_\Omega m\omega^2\theta'' + \frac{I_y}{R} m\omega^2 v \\ &+ \frac{2I_\Omega}{R} m\omega^2 v'' = 0. \end{aligned} \tag{10c}$$

The corresponding natural boundary conditions are,

$$[Q_1\delta w + Q_2\delta v + Q_3\delta\theta + Q_4\delta w' + Q_5\delta v' + Q_6\delta\theta']_0^L = 0 \tag{11}$$

where the generalized forces are,

$$\begin{aligned} Q_1(x) &= -\mathrm{EI}_y\left(w''' + \frac{w'}{R^2}\right) + \rho w^2 I_y w' \\ Q_2(x) &= -\mathrm{EI}_z\left(v''' - \frac{\theta'}{R}\right) + \frac{\mathrm{GJ}}{R}\left(\theta' + \frac{v'}{R}\right) \\ &\quad - M_y\left(1 - \frac{r^2}{R^2}\right)\left(\theta' + \frac{v'}{R}\right) \\ &\quad - m\omega^2\left(I_z + \frac{3I_\Omega}{R^2}\right) v' - 2m\omega^2\frac{I_\Omega}{R}\theta \\ Q_3(x) &= -\mathrm{EI}_\Omega\left(\theta''' + \frac{\theta'}{R^2}\right) + \mathrm{GJ}\left(\theta' + \frac{v'}{R}\right) \\ &\quad + M_y\frac{r^2}{R}\theta' - M_y\left(1 - \frac{r^2}{R^2}\right) v' \\ &\quad - m\omega^2 I_\Omega\left(\theta' + \frac{2v'}{R}\right) \\ Q_4(x) &= \mathrm{EI}_y\left(w'' + \frac{w}{R^2}\right) + M_y \\ Q_5(x) &= \mathrm{EI}_z\left(v'' - \frac{\theta}{R}\right) \\ Q_6(x) &= \mathrm{EI}_\Omega\left(\theta'' + \frac{\theta}{R}\right). \end{aligned} \tag{12}$$

Equations (10a,b,c) can be written in the form of Eq. (1), and the matrices of Eq. (1) are,

$$A_0 = \begin{bmatrix} \frac{\mathrm{EI}_y}{R^4} - \frac{M_y}{R^3} - m\omega^2\left(A - \frac{2I_y}{R^2}\right) & 0 & 0 \\ & -m\omega^2 A & m\omega^2\frac{I_y}{R} \\ \text{(Sym)} & & \frac{\mathrm{EI}_\Omega}{R^4} + \frac{\mathrm{EI}_z}{R^2} - m\omega^2(I_y + I_z + r^2A) \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \frac{2\mathrm{EI}_y}{R^4} - \frac{M_y}{R} + m\omega^2 I_y & 0 & 0 \\ & -\frac{GJ}{R} + \frac{M_y}{R}\left(1 - \frac{r^2}{R^2}\right) + m\omega^2\left(I_z + \frac{3I_\Omega}{R^2}\right) & -\frac{\mathrm{EI}_z}{R} - \frac{\mathrm{GJ}}{R} + M_y\left(1 - \frac{r^2}{R^2}\right) + m\omega^2\frac{2I_\Omega}{R} \\ \text{(Sym)} & & \frac{2EI_\Omega}{R^2} - \mathrm{GJ} - \frac{r^2}{R}M_y + m\omega^2 I_\Omega \end{bmatrix}$$

$$A_4 = \begin{bmatrix} \mathrm{EI}_y & 0 & 0 \\ & \mathrm{EI}_z & 0 \\ & & \mathrm{EI}_\Omega \end{bmatrix}$$

and $[A_1]$, $[A_3]$ are zero matrices in this case. Thus, the dynamic stiffness matrix can be formed explicitly by the following procedure with the help of a microcomputer:

(a) solve for the eigenvalues λ_j and the corresponding eigenvectors $\{\phi_j\}, j = 1, 2, \ldots,$ nm from the eigenproblem (3);
(b) calculate the matrix $[H]$ and then invert it to obtain the shape function;
(c) forming the dynamic stiffness matrix $[K]$.

The natural frequencies of the system are determined by equating the determinant of the dynamic stiffness of the structure to zero,

$$\det[D(\omega)] = 0. \tag{14}$$

Here $[D(\omega)]$ is the system dynamic stiffness matrix obtained from the element matrices $[K(\omega)]$. $[K(\omega)]$ is a real square matrix of order 12. $[D(\omega)]$ is one of order 12, 6, 6, 3 when the natural boundary conditions are free–free, clamped–free, pinned–pinned, and clamped–pinned, respectively.

The six lowest natural frequencies are plotted in Fig. 4(a–d) for various boundary conditions with different end bending moment M_y against the subtended angles. Figure 5(a,b,c,d) represent the relations between frequencies and bending moment M_y for different boundary conditions and subtended angles. We plot Fig. 3, which represents the variation of frequencies against subtended angle, by adopting the boundary conditions as pinned–pinned and using the analytical solution of Eq. (10) as,

$$\begin{aligned} w &= a_1 \sin\lambda x \\ v &= a_2 \sin\lambda x \\ \theta &= a_3 \sin\lambda x \end{aligned} \tag{15}$$

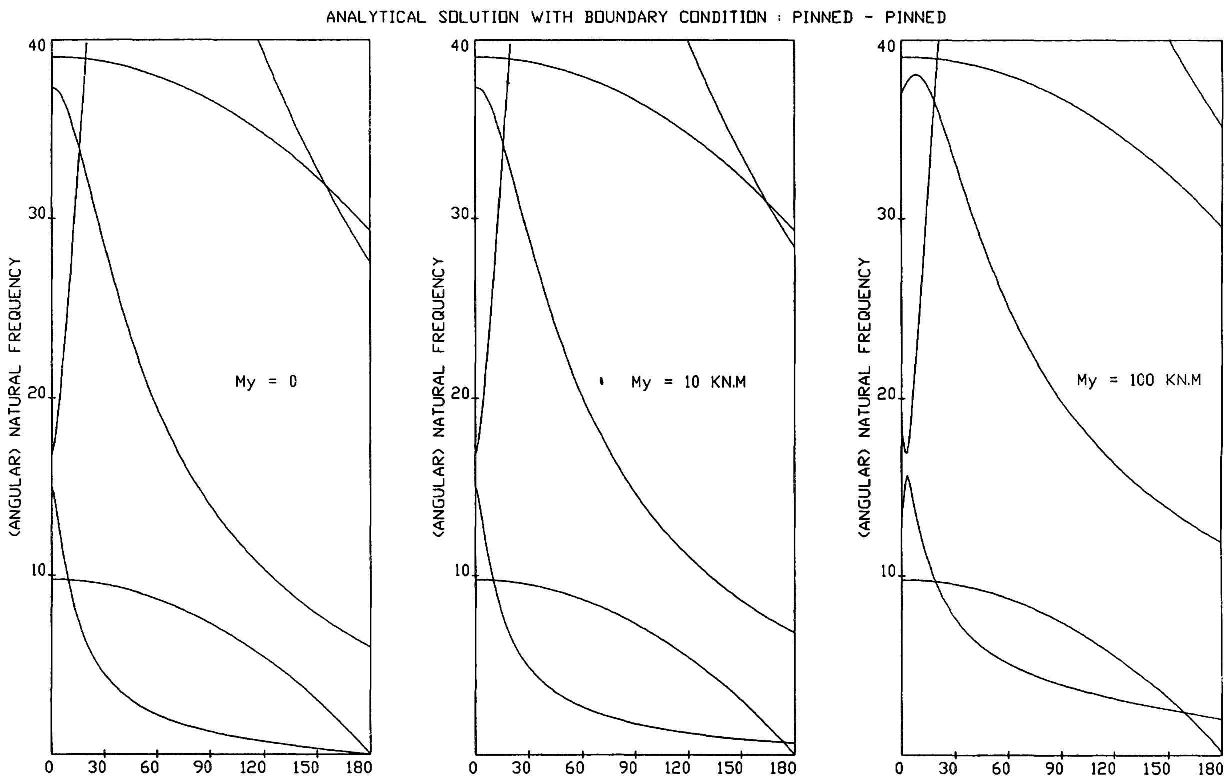

FIGURE 3 Frequency diagrams of a pinned–pinned curved thin-walled beam using analytical solutions.

BOUNDARY CONDITIONS : PINNED – PINNED

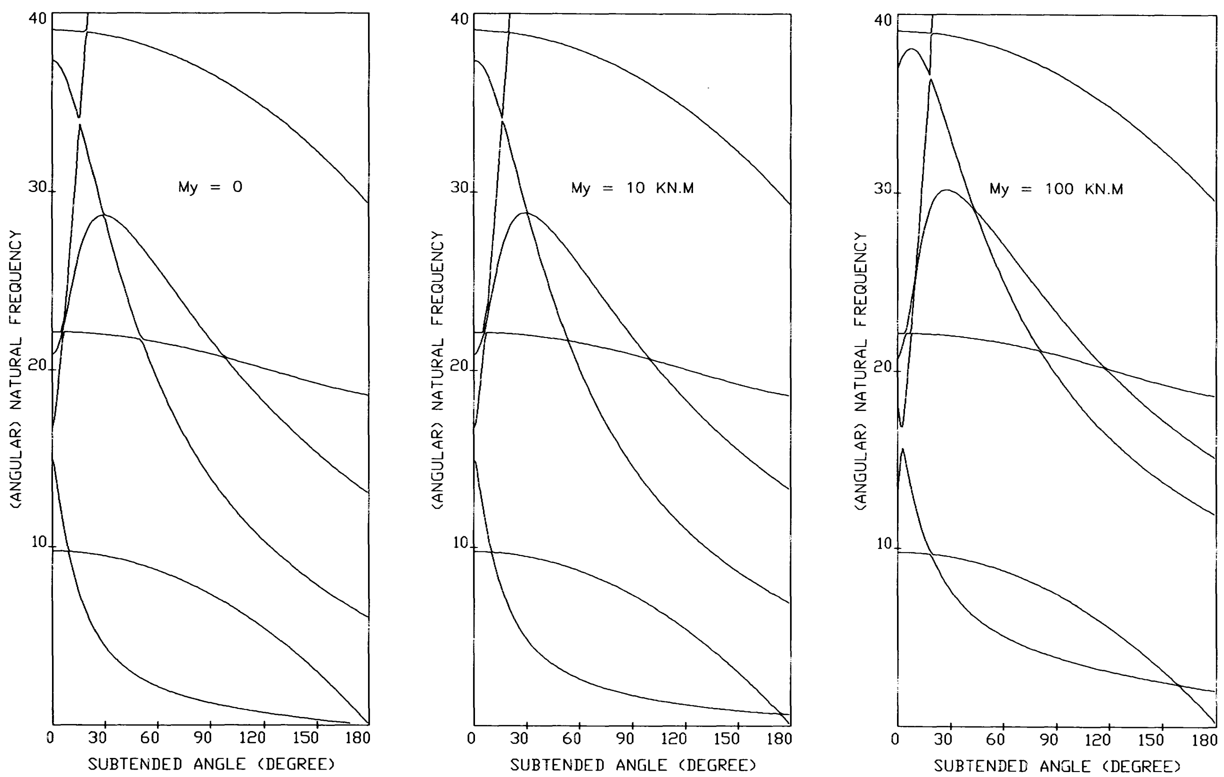

FIGURE 4(a) Frequency diagrams of a pinned–pinned curve thin-walled beam against subtended angle.

BOUNDARY CONITIONS : CLAMPED – PINNED

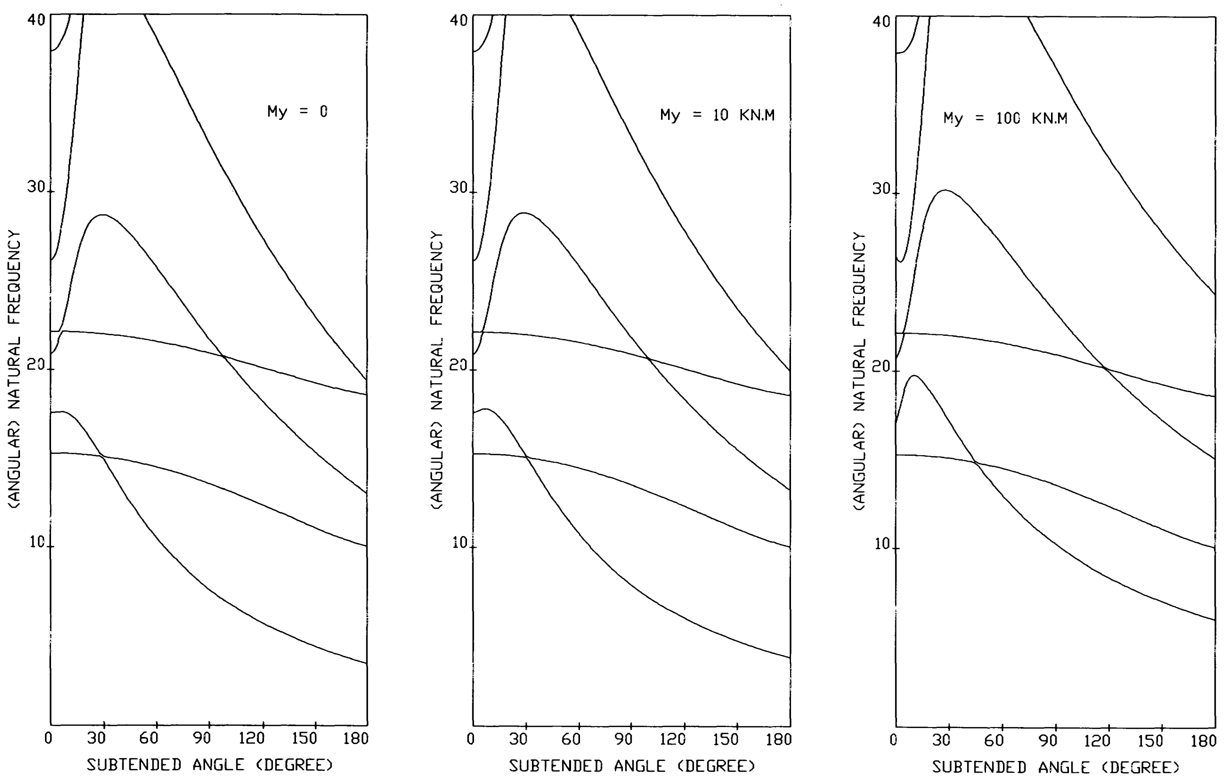

FIGURE 4(b) Frequency diagrams of a clamped–pinned curved thin-walled beam against subtended angle.

BOUNDARY CONDITIONS : CLAMPED – FREE

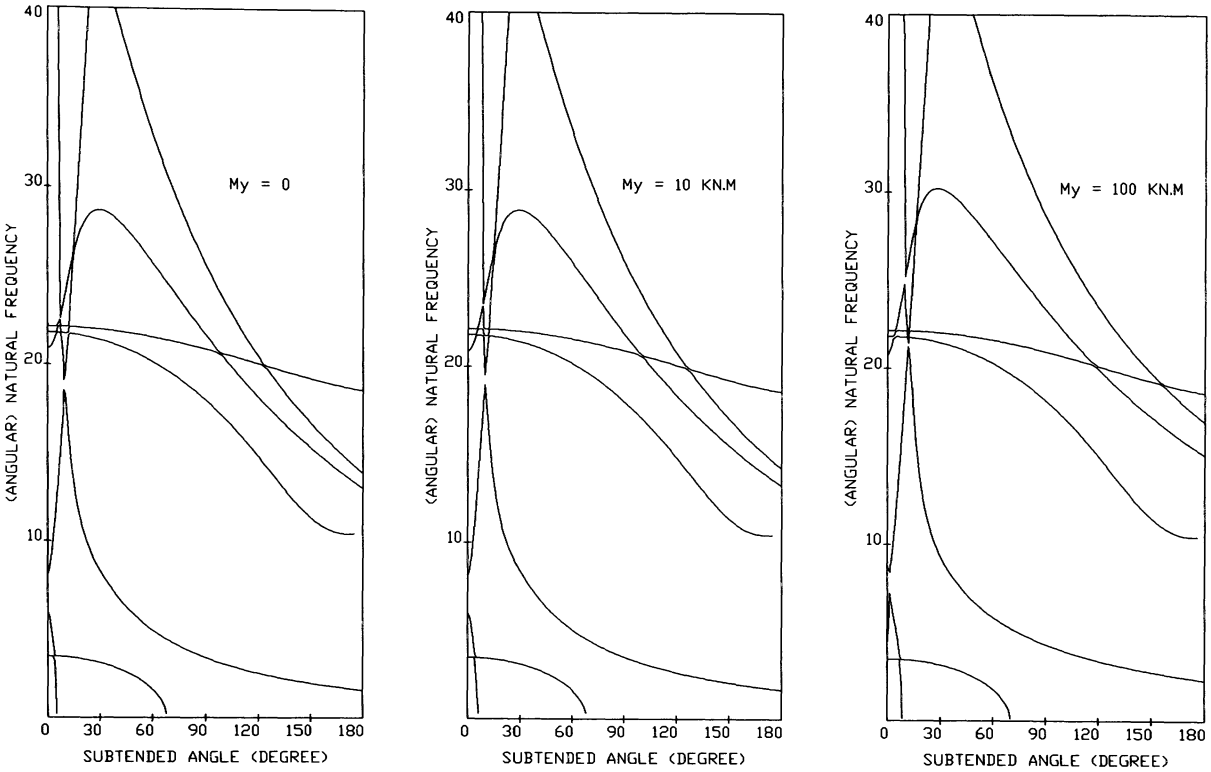

FIGURE 4(c) Frequency diagrams of a clamped–free curved thin-walled beam against subtended angle.

BOUNDARY CONDITIONS : FREE – FREE

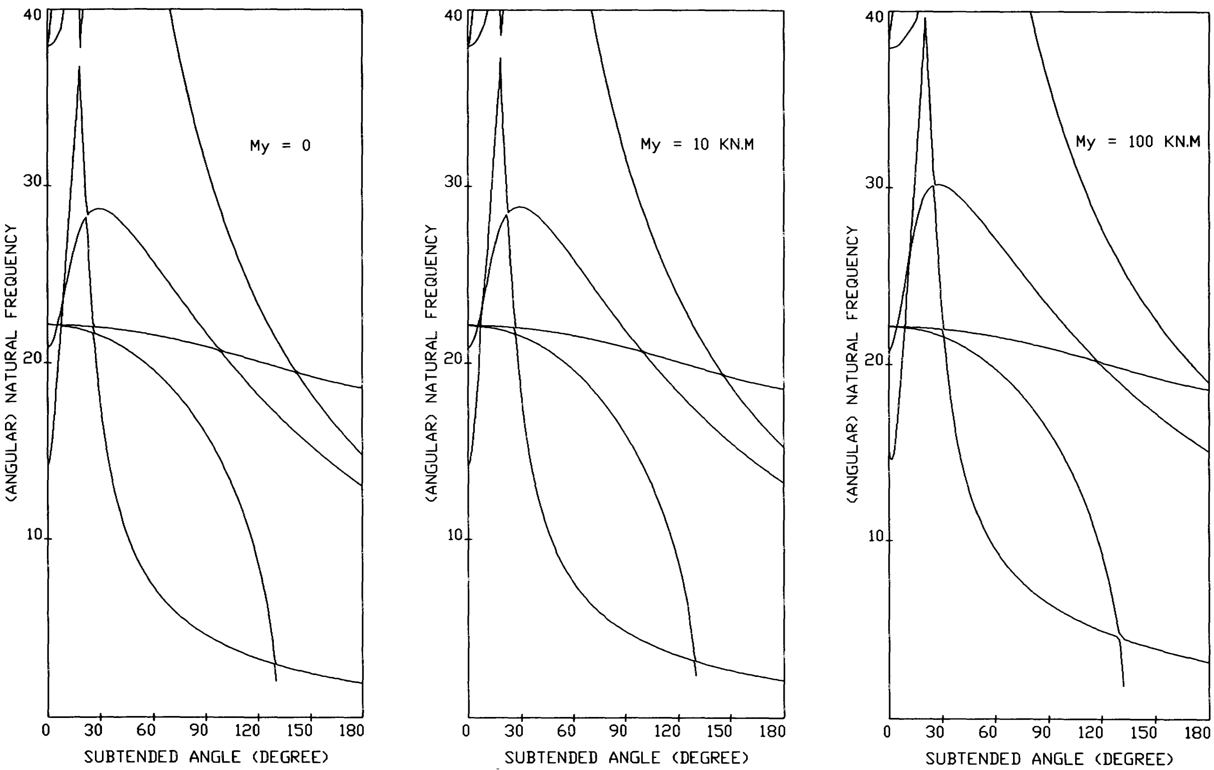

FIGURE 4(d) Frequency diagrams of a free–free curved thin-walled beam against subtended angle.

BOUNDARY CONDITIONS : PINNED - PINNED

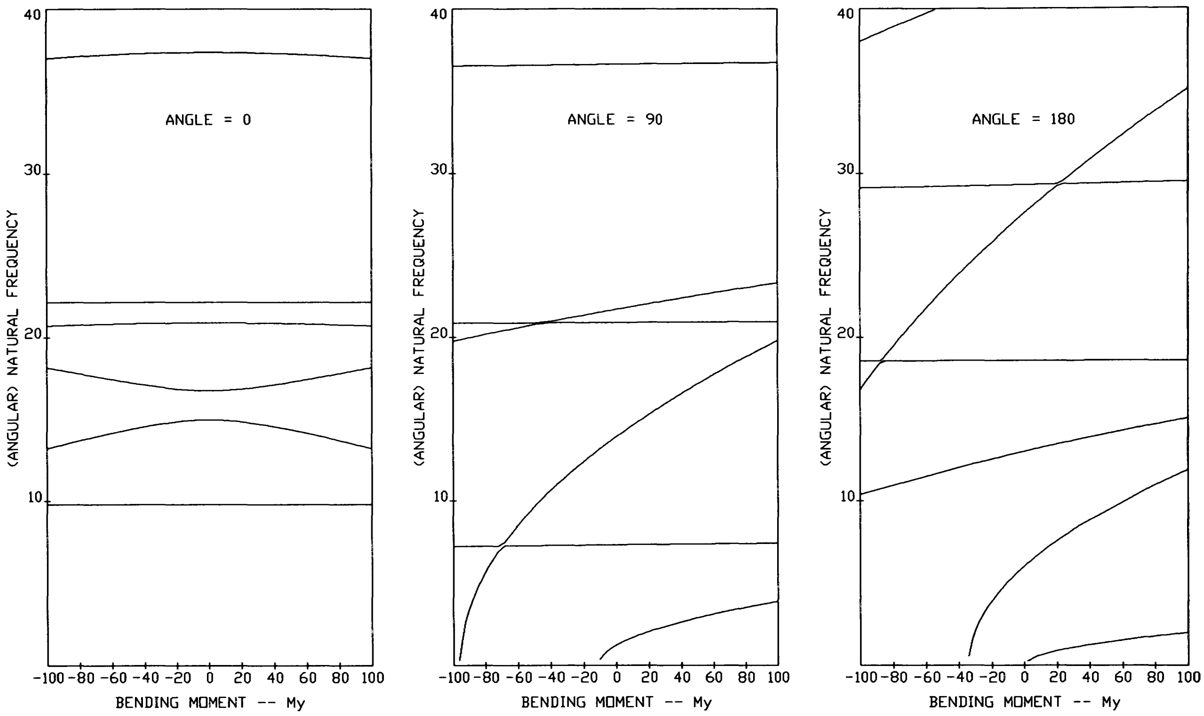

FIGURE 5(a) Frequency diagrams of a pinned–pinned curved thin-walled beam against bending moment, M_y.

BOUNDARY CONDITIONS : CLAMPED - PINNED

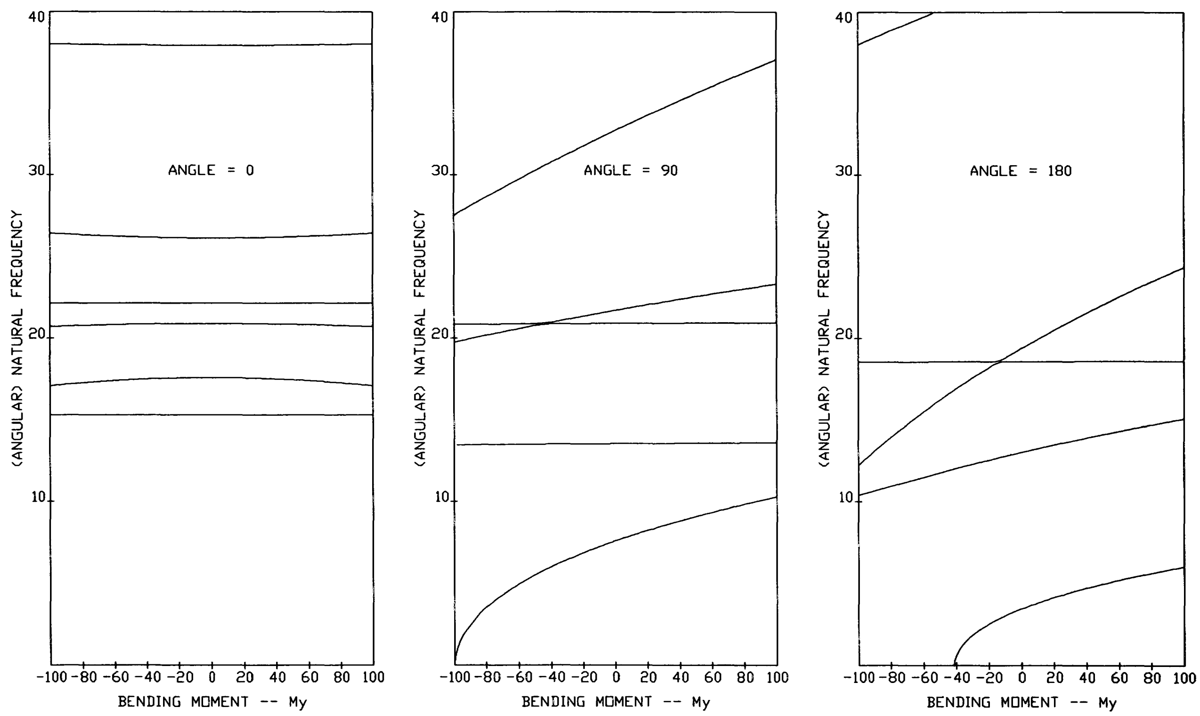

FIGURE 5(b) Frequency diagrams of a clamped–pinned curved thin-walled beam against bending moment, M_y.

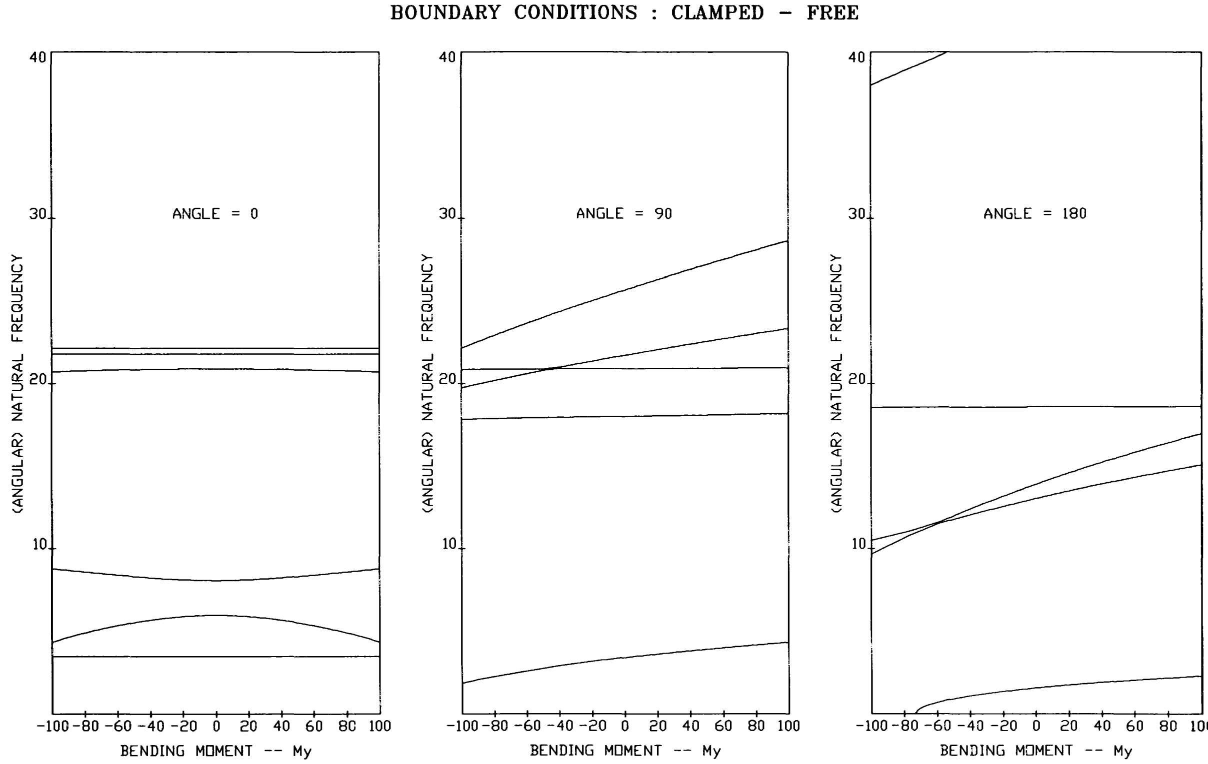

FIGURE 5(c) Frequency diagrams of a clamped–free curved thin-walled beam against bending moment, M_y.

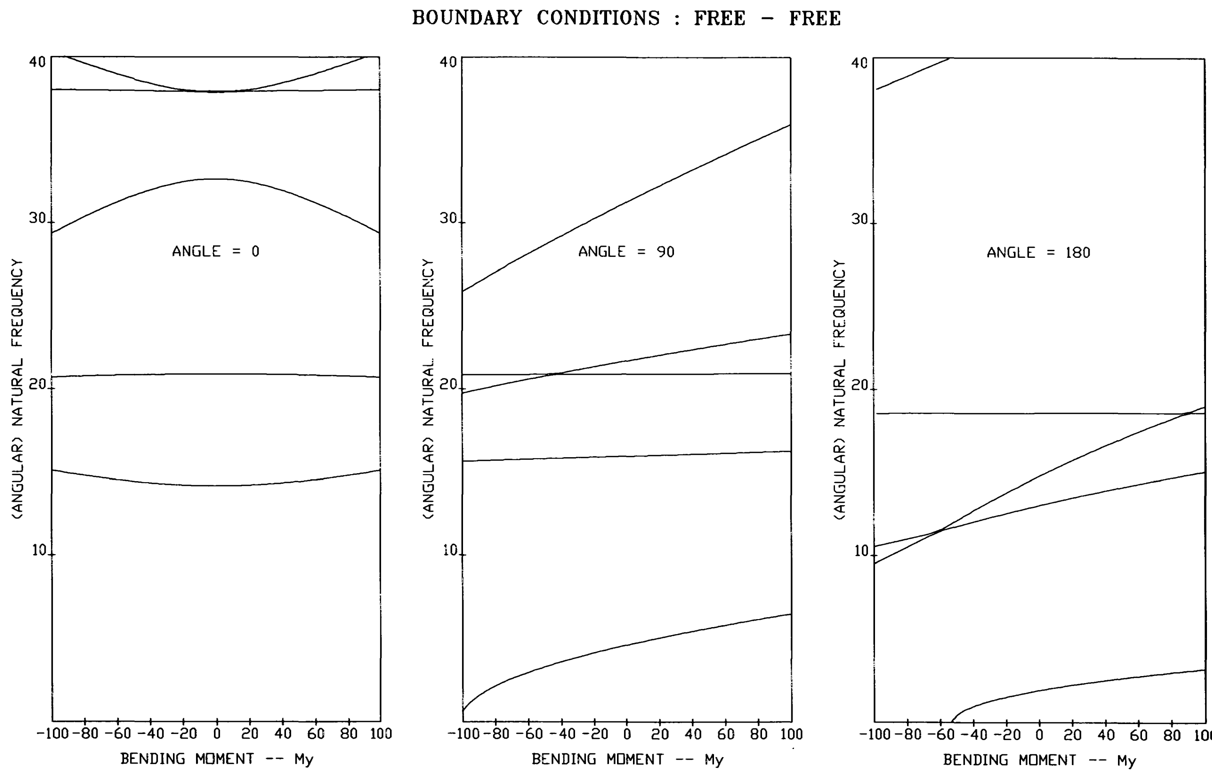

FIGURE 5(d) Frequency diagrams of a free–free curved thin-walled beam against bending moment, M_y.

where $\lambda = n\pi/L$, $n = 1, 2, 3, \ldots$. Substitution of Eq. (15) in Eq. (10) generates a characteristic problem

$$([A] - \omega^2[B])\{\phi\} = \{0\} \quad \text{or} \quad \det[A - \omega^2 B] = 0 \tag{16}$$

here

$$[A] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & d \end{bmatrix}; \quad [B] = \begin{bmatrix} e & 0 & 0 \\ 0 & f & g \\ 0 & g & h \end{bmatrix};$$

$$\{\phi\} = \begin{Bmatrix} w \\ v \\ \theta \end{Bmatrix}$$

$$a = \mathrm{EI}_y \left(\lambda^4 - \frac{2}{R^2}\lambda^2 + \frac{1}{R^4}\right) + \frac{M_y}{R}\left(\lambda^2 - \frac{1}{R^2}\right)$$

$$b = \mathrm{EI}_z\lambda^4 + \frac{\mathrm{GJ}}{R^2}\lambda^2 - \frac{M_y}{R}\left(1 - \frac{r^2}{R^2}\right)\lambda^2$$

$$c = \frac{\mathrm{EI}_z}{R}\lambda^2 + \frac{GJ}{R}\lambda^2 - M_y\left(1 - \frac{r^2}{R^2}\right)\lambda^2$$

$$d = \mathrm{EI}_\Omega\left(\lambda^4 - \frac{2}{R^2}\lambda^2 + \frac{1}{R^4}\right) + \frac{\mathrm{EI}_z}{R^2} + \mathrm{GJ}\lambda^2 + M_y\frac{r^2}{R}\lambda^2 \tag{17}$$

$$e = mA - 2m\frac{I_y}{R^2} + mI_y\lambda^2$$

$$f = mA + m\left(I_z + \frac{3I_\Omega}{R^2}\right)\lambda^2$$

$$g = -m\frac{I_y}{R} + m\frac{2I_\Omega}{R}\lambda^2$$

$$h = m(I_y + I_z + r^2A) + mI_\Omega\lambda^2.$$

Equation (16) gives a relation between frequency ω and subtended angle β. Comparing Fig. 3 and Fig. 4(a), we can see that they fit completely. From Fig. 4(a) we find two extra curves that represent the two lowest frequencies with the clamped–clamped boundary condition where the value of the determinant crosses ∞ and $-\infty$. For the clamped–clamped boundary condition, the order of $[D(\omega)]$ in Eq. (12) is zero, and the determinant of $[D(\omega)]$ tends to be infinite. The same phenomenon is found in other cases with various boundary conditions.

Figure 4 shows that increasing the subtended angles of the beam softens the flexural modes including in-plane and out-of-plane flexural modes but quickly hardens the torsional mode, simultaneously. We can still find that whatever acts against the subtended angles or against the bending moments at the beam ends, the frequencies do not vary in a monotonic way due to the changing of modes between flexure and torsion. Avoided crossing, or frequency veering, occurs if two or more frequencies approach each other, but then veer off without becoming equal.

The natural frequencies found, the actual dynamic stiffness matrices can be constructed without difficulty. The global stiffness matrix can be formed by the standard assembly procedure of the finite element method.

CONCLUSION

The dynamic stiffness method is used to give a better approximation of the model. It is in fact a kind of continuum element compared with the finite element. Using the dynamic stiffness method, we have efficiently studied the dynamic characteristics of a curved thin-walled beam with open sections subjected to the in-plane bending moment. A structure made from curved thin-walled members with various boundary conditions can be analyzed without difficulty.

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