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On Maximum Energy of Mechanical Systems Under Shock Disturbances

In this article, we develop a method for estimating the maximum energy of a mechanical system subjected to disturbances whose integral absolute value is bounded. The conditions for this estimate to be exact are found. The method is demonstrated by calculating the maximal displacement in a second-order nonlinear system. © 1993 John Wiley & Sons, Inc.

INTRODUCTION

When designing and developing different engineering systems we often encounter the problem of determining the maximum response of the system under disturbances of some specified class. One of the first problems solved is the well-known Boulgakov's problem of accumulating disturbances in linear systems [Boulgakov, 1946]. In this problem, the absolute values of the external disturbances are supposed to have a bounded integral over time. A number of papers [Balandin, 1989; Bolotnik, 1976; Sevin, 1957] deal with the maximum displacement in systems under disturbances, the integral of whose absolute value is bounded. In Balandin [1989], a broad class of nonlinear second-order systems is found in which the maximum displacement is reached at a single instantaneous impact. In this paper, we consider the problem of estimating the maximal energy of mechanical systems subjected to a disturbance whose absolute value is integrable over time. If such a disturbance is sign-definite, it can be physically interpreted as a force is with a finite impulse.

MECHANICAL SYSTEM AND STATEMENT OF PROBLEM

Let us consider a mechanical system consisting of N point masses, P_ν , connected to a rigid body

(called a base) moving without rotation in an inertial reference system. Denote by S_o a reference system rigidly connected to the base. Let $\mathbf{r}_\nu$ be the position vectors of the point masses in the system S_o . Now for this mechanical system let us introduce the generalized coordinates $q_1, q_2, \dots, q_n$ describing the motion of the system with respect to the base. Because our system is scleronomous, position vectors $\mathbf{r}_\nu$ of points P_ν do not depend on time explicitly and can be expressed as functions of $q_1, q_2, \dots, q_n$:

$$\mathbf{r}_\nu = \mathbf{r}_\nu(q_1, q_2, \dots, q_n).$$

If the mechanical system is holonomic, then its motion can be described by Lagrange equations [Pars, 1968]

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i \quad (1)$$

with initial conditions $q_i(0) = q_i^0, \dot{q}_i(0) = \dot{q}_i^0$, where $L = T - \Pi$ is a Lagrange function, T and Π are the kinetic and potential energies, respectively, and Q_i are the generalized forces. The generalized forces may be described as a sum of dissipative (Q_i^d), gyroscopic (Q_i^g), and inertia (Q_i^m) forces:

$$Q_i = Q_i^d + Q_i^g + Q_i^m.$$

Note that for dissipative and gyroscopic forces the conditions

$$\sum_{i=1}^n Q_i^d \dot{q}_i \leq 0, \sum_{i=1}^n Q_i^g \dot{q}_i = 0 \quad (2)$$

hold.

The virtual work of inertia forces is given by

$$-\sum_{\nu=1}^N m_\nu \delta \mathbf{r}_\nu \mathbf{w}(t) = -M \delta \mathbf{R}_c \mathbf{w}(t) = \sum_{i=1}^n Q_i^m \delta q_i,$$

where m_ν , $\delta \mathbf{r}_\nu$ are the mass and virtual displacement of the point P_ν ; $M = \sum_{\nu=1}^N m_\nu$ is the total mass of the system; $\mathbf{R}_c = 1/M \sum_{\nu=1}^N m_\nu \mathbf{r}_\nu$ is the position vector of the system mass center; $\delta \mathbf{R}_c$ is the virtual displacement of the mass center; $\mathbf{w}(t)$ is the acceleration of the base expressed in the reference system S_o ; δq_i is the virtual displacement of the generalized coordinate q_i . Because our system is scleronomous, the following equation is valid for inertia forces

$$\sum_{i=1}^n Q_i^m \dot{q}_i = -M \dot{\mathbf{R}}_c \mathbf{w}(t). \quad (3)$$

Let us now proceed to the problem statement. Suppose that the vector function $\mathbf{w}(t)$ describing an external disturbance acting on the system is piecewise continuous and belongs to the class of so-called shock disturbances

$$\Theta = \left\{ \mathbf{w}(\cdot): \mathbf{w}(t) \equiv 0 \right. \\ \left. \text{for } t < 0, \int_0^\infty |\mathbf{w}(t)| dt \leq J_o \right\}. \quad (4)$$

For each function of the above class, a mechanical system response will be understood as a maximal (over time) sum of kinetic and potential energies. This sum is obtained by solving system (1) with the specified vector $\mathbf{w}(t)$:

$$W[\mathbf{w}(\cdot)] = \sup_{t \in [0, \infty)} (T + \Pi).$$

Additionally, the potential energy as a function of generalized coordinates

$$\Pi = \Pi(q_1, q_2, \dots, q_n) \geq 0 \quad (5)$$

is assumed nonnegative.

Let us state the following problem: find the estimate from above for the maximum of the total energy, that is, determine the number W^+ so that

$$W[\mathbf{w}(\cdot)] \leq W^+ \quad \forall \mathbf{w}(\cdot) \in \Theta. \quad (6)$$

ANALYSIS OF THE PROBLEM

As it is known [Pars, 1968], the rate of change of the energy $E = T + \Pi$ for a holonomic system is given by

$$\frac{dE}{dt} = \sum_{i=1}^n Q_i \dot{q}_i.$$

Invoking (2) and (3), we rewrite this equation as

$$\frac{dE}{dt} = \sum_{i=1}^n Q_i^d \dot{q}_i - M \dot{\mathbf{R}}_c \mathbf{w}(t).$$

Now, let us take the time segments $\Omega_k = [t_k^-, t_k^+]$, within which we have

$$\frac{dE}{dt} \geq 0,$$

and denote the union of these segments by

$$\Omega_* = \bigcup_k \Omega_k.$$

Taking into account the inequality in (2), we can prove the inequalities

$$\sum_{i=1}^n Q_i^d \dot{q}_i - M \dot{\mathbf{R}}_c \mathbf{w}(t) \geq 0,$$

$$\frac{dE}{dt} \leq |M \dot{\mathbf{R}}_c \mathbf{w}(t)|$$

to hold for all $t \in \Omega_*$. Note that the kinetic energy of a point mass M moving at the same speed as the system mass center does not exceed the kinetic energy of the whole system:

$$\frac{M \dot{\mathbf{R}}_c^2}{2} \leq T = \sum_{\nu=1}^N \frac{m_\nu \dot{\mathbf{r}}_\nu^2}{2}.$$

Besides, because the potential energy is assumed nonnegative (5), we have

$$\frac{M \dot{\mathbf{R}}_c^2}{2} \leq E,$$

and hence

$$|\dot{\mathbf{R}}_c| \leq \sqrt{\frac{2E}{M}}.$$

Therefore,

$$\frac{dE}{dt} \leq \sqrt{2ME} |\mathbf{w}(t)|.$$

By integrating the last inequality over the arbitrary segment Ω_k we obtain

$$\sqrt{E_k^+} - \sqrt{E_k^-} \leq \sqrt{M/2} \int_{t_k^-}^{t_k^+} |\mathbf{w}(t)| dt,$$

where $E_k^- = E(t_k^-)$, $E_k^+ = E(t_k^+)$. Adding all the inequalities obtained for all segments Ω_k and taking Eq. (4) into account, we arrive at

$$\begin{aligned} \sum_{k=1}^L [\sqrt{E_k^+} - \sqrt{E_k^-}] &\leq \sqrt{M/2} \sum_{k=1}^L \int_{t_k^-}^{t_k^+} |\mathbf{w}(t)| dt \\ &\leq \sqrt{M/2} J_o. \end{aligned}$$

According to the definition of the segments Ω_k , we have

$$E_L^+ = \sup_{t \in [0, \infty)} E(t) = W[\mathbf{w}(\cdot)].$$

Note that $E_{k+1}^- \leq E_k^+$, and hence,

$$\sqrt{E_L^+} - \sqrt{E_1^-} \leq \sum_{k=1}^L [\sqrt{E_k^+} - \sqrt{E_k^-}].$$

Finally we obtain

$$\sqrt{E_L^+} \leq \sqrt{E_1^-} + \sqrt{M/2} J_o$$

or

$$E_L^+ = W[\mathbf{w}(\cdot)] \leq \{\sqrt{E_1^-} + \sqrt{M/2} J_o\}^2.$$

Here, E_1^- is determined by initial conditions for Eq. (1) and is just the value of the system energy at the instant when the disturbance starts to act

$$W^+ = \{\sqrt{E_1^-} + \sqrt{M/2} J_o\}^2. \quad (7)$$

DISCUSSION

The above estimate for maximum energy of a system under shock disturbances gives rise to a

quite natural question whether there exist mechanical systems for which this estimate is exact, that is, whether there exist systems for which

$$W^+ = \sup_{\mathbf{w}(\cdot) \in \Theta} W[\mathbf{w}(\cdot)] \quad (8)$$

is valid. Before formulating validity conditions for (8), let us estimate a system response to the instantaneous impact given by

$$\mathbf{w}(t) = J_o \mathbf{e} \delta(t), \quad (9)$$

where $\mathbf{e}$ is a constant unit vector ($\mathbf{e}^2 = 1$); J_o is a positive number; and $\delta(t)$ is the Dirac delta-function for which

$$\int_{-\infty}^{\infty} \delta(t) dt = 1.$$

It is known [Pars, 1968] that the instantaneous impact (9) results in an instantaneous change in the velocities $\dot{\mathbf{r}}_v$ of the points P_v by the vectors $\mathbf{c}_v$, with the following inequality holding

$$c_v^2 \leq J_o^2. \quad (10)$$

Let us denote by $T_c(J_o)$ the magnitude by which the kinetic energy of the system mass center is changed under instantaneous impact (9). From (10), we obtain

$$T_c(J_o) \leq \frac{MJ_o^2}{2}.$$

Now let us state the conditions under which the equality (8) holds. Let the initial conditions in (1) be such that

$$\begin{aligned} \dot{q}_1^o = \dot{q}_2^o = \dots = \dot{q}_n^o &= 0, \\ \Pi(q_1^o, q_2^o, \dots, q_n^o) &= 0, \end{aligned} \quad (11)$$

and let the unit vector $\mathbf{e}$ be such that the change in the mass center kinetic energy due to the instantaneous impact (9) is

$$T_c(J_o) = \frac{MJ_o^2}{2}. \quad (12)$$

Then, equality (8) is true. Indeed, for any disturbance $\mathbf{w}(\cdot) \in \Theta$, taking (11) into account, we obtain $E_1^- = 0$, and hence

$$W^+ = \frac{MJ_o^2}{2}.$$

On the other hand, when taking the disturbance

$$\mathbf{w}^*(t) = J_o \mathbf{e}_v(t),$$

where the scalar function $\nu(t)$ is determined by

$$\nu(t) = \begin{cases} 1/\tau, & 0 \leq t \leq \tau \\ 0, & t > \tau \end{cases}, \quad (13)$$

with τ being sufficiently small, we get

$$W[\mathbf{w}^*(\cdot)] = \frac{MJ_o^2}{2} - \mu(\tau),$$

where $\lim_{\tau \rightarrow 0} \mu(\tau) = 0$. Hence, we have

$$\lim_{\tau \rightarrow 0} W[\mathbf{w}^*(\cdot)] = W^+.$$

And equality (8) holds.

Now, let us discuss an example of a mechanical system for which (8) is true. Let us consider a one-dimensional chain of N interconnected point masses connected to the base by elastic-dissipative links. Such a system can be used for modeling the dynamics of different real structures, for example, a motion of railway carriages following a locomotive, or a horizontal displacement of a multistory building under seismic displacement of its base, etc. The equations of motion can be expressed in the form

$$\begin{aligned} m_1(\ddot{x}_1 + w) &= -\varphi_1(x_1) - g_1(\dot{x}_1) + \varphi_2(x_2 - x_1) \\ &\quad + g_2(\dot{x}_2 - \dot{x}_1) \\ &\vdots \\ m_i(\ddot{x}_i + w) &= -\varphi_i(x_i - x_{i-1}) - g_i(\dot{x}_i - \dot{x}_{i-1}) \\ &\quad + \varphi_{i+1}(x_{i+1} - x_i) + g_{i+1}(\dot{x}_{i+1} - \dot{x}_i) \\ &\vdots \\ m_n(\ddot{x}_n + w) &= -\varphi_n(x_n - x_{n-1}) - g_n(\dot{x}_n - \dot{x}_{n-1}) \end{aligned} \quad (14)$$

and initial conditions may be taken as

$$x_i(0) = 0, \quad \dot{x}_i(0) = 0 \quad i \in [1, n].$$

The scalar function $w = w(t)$ is supposed to describe the acceleration of the base and satisfies condition (4); and the functions g_i, φ_i , describing the elastic-dissipative links of the system are continuous and meet the condition

$$g_i(\eta) \operatorname{sign}(\eta) \geq 0, \quad \varphi_i(\eta) \operatorname{sign}(\eta) \geq 0. \quad (15)$$

In this case, the power of the dissipative forces is equal to

$$-\sum_{i=1}^{n-1} g_{i+1}(\dot{x}_i - \dot{x}_{i+1})(\dot{x}_i - \dot{x}_{i+1}) - g_1(\dot{x}_1)\dot{x}_1 \leq 0,$$

and the potential energy is

$$\Pi = \sum_{i=1}^{n-1} \int_0^{(x_{i+1} - x_i)} \varphi_{i+1}(\eta) d\eta + \int_0^{x_1} \varphi_1(\eta) d\eta \geq 0,$$

with Π vanishing ($\Pi = 0$) when all $x_i = 0$. Thus, conditions (2), (5), and (11) are satisfied. If the base is subjected to an instantaneous impact so that

$$w(t) = J_o \delta(t),$$

then from the equations of motion (14) we immediately obtain the instantaneous increment in the velocity $\dot{x}_i(0)$:

$$\dot{x}_i(0) = -J_o \quad i \in [1, n]$$

Hence, condition (12) is also valid.

Thus, the disturbance that maximizes the sum of the kinetic and potential energies of the given mechanical system is an instantaneous impact of intensity J_o [equality (8) being true], and the desired value of the maximum response is given by

$$W^+ = \frac{MJ_o^2}{2} \left(M = \sum_{i=1}^n m_i \right).$$

Note here that the value W^+ also enables us to estimate a maximal displacement in system (14) described by

$$\sigma_o[w(\cdot)] = \max\{\sigma_1, \sigma_2, \dots, \sigma_n\},$$

where $\sigma_1 = \sup_{t \in [0, \infty)} |x_1[t, w(\cdot)]|$,

$$\sigma_{i+1} = \sup_{t \in [0, \infty)} |x_i[t, w(\cdot)] - x_{i+1}[t, w(\cdot)]|$$

$$i \in [1, n-1],$$

and $\{x_1[t, w(\cdot)], \dots, x_i[t, w(\cdot)], \dots, x_n[t, w(\cdot)]\}$ is a solution of system (14) corresponding to the disturbance $w(t)$.

Indeed, from $\Pi \leq W^+$, taking into account the

second relationship in (15), we arrive at the inequality

$$\sigma_i \leq |s_i^*|, \quad i \in [1, n],$$

where s_i^* is the maximum (in absolute value) root of the equation

$$\int_0^{s_i^*} \varphi_i(\eta) d\eta = W^+.$$

This means that the desired estimate of the maximal displacement is given by

$$\sigma_o[w(\cdot)] \leq \max(|s_1^*|, |s_2^*|, \dots, |s_n^*|) = S^+.$$

It is worth mentioning that the estimate may appear to be significantly overstated in the sense that the value

$$\sigma^+ = \sup_{w(\cdot) \in \Theta} \sigma_o[w(\cdot)], \quad (16)$$

will be drastically less than S^+ . However, if (14) is a single-mass system, then the value of σ^+ is the same as the estimate S^+ , provided the dissipative link is absent.

APPLICATION OF RESULTS

Now let us discuss an example of results above to estimate exactly the maximum displacement of a second-order nonlinear system with damping.

Let us consider the following system

$$\begin{aligned} \dot{x} &= y, \quad \dot{y} = -g(x, y) - \varphi(x) + u(t), \\ x(0) &= y(0) = 0. \end{aligned} \quad (17)$$

For the continuous functions g and φ , it is supposed that

$$\begin{aligned} g(x, y) \operatorname{sign}(y) &\geq 0, \\ \frac{g(x, y_1)}{y_1} &\geq \frac{g(x, y_2)}{y_2} \quad \text{at } |y_2| > |y_1| > 0, \end{aligned} \quad (18)$$

$$\begin{aligned} \varphi(x) \operatorname{sign}(x) &\geq 0, \quad \varphi(x) \neq 0, \\ \varphi(x_1) &\geq \varphi(x_2) \quad \text{at } x_1 > x_2. \end{aligned} \quad (19)$$

Depending upon the situation, by the external disturbance $u(t)$, we understand $w(t)$ from the class Θ , or some instantaneous impact

$$\Delta^+(t) = J_o \delta(t) \quad [\text{or} \quad \Delta^-(t) = -J_o \delta(t)]. \quad (20)$$

Note that if $u(t) = w(t)$, then, with (18) and (19) taken into consideration, the following estimate is valid

$$W[w(\cdot)] \leq J_o^2/2, \quad (21)$$

where $W[w(\cdot)]$ is maximum of the sum of kinetic and potential energy

$$E = \frac{y^2}{2} + \int_0^x \varphi(x) dx$$

calculated for the solution $\{x[t, w(\cdot)], y[t, w(\cdot)]\}$ of system (17).

Let us state the following problem: find the value σ^+ determined in accordance with (16), provided the maximum displacement given by

$$\sigma_o[w(\cdot)] = \sup_{t \in [0, \infty)} |x[t, w(\cdot)]|.$$

Let us first consider the motion of system (17) under instantaneous impact (20). This motion is described as

$$\begin{aligned} \dot{x} &= y, \quad \dot{y} = -g(x, y) - \varphi(x) \\ x(0) &= 0, \quad y(0) = J_o \\ [y(0) &= -J_o, \quad \text{if } u(t) = \Delta^-(t)]. \end{aligned} \quad (22)$$

Note here some features of system (22). Let us separate the initial segments of the system trajectories corresponding to the disturbances $\Delta^+(t)$ and $\Delta^-(t)$ and lying in the first and third quadrants of the phase plane $\{x, y\}$, respectively. Denote these segments by F^+ and F^- (Fig. 1).

The curve F^+ is described as follows

$$y = y_o^+(x),$$

and the curve F^- is given by

$$y = y_o^-(x).$$

Now using the assumed properties of the functions g, φ it is easy to show that a) unique roots of the equations

$$y_o^+(x) = 0 \quad \text{and} \quad y_o^-(x) = 0$$

denoted by $\lambda^+ > 0, \lambda^- < 0$ assume finite values; and b)

FIGURE 1

$$\max\{\sigma_o[\Delta^+(\cdot)], \sigma_o[\Delta^-(\cdot)]\} \equiv \max\{\lambda^+, |\lambda^-|\}.$$

On the phase plane, let us determine a set of points μ^* containing the point $\{0,0\}$ and bounded by the curves F^+ and F^- within the first and the third quadrants, and by rectangles with sides $\{|\lambda^-|, J_o\}$ and $\{\lambda^+, J_o\}$, within the second and the fourth (Fig. 1). Let us show that a phase trajectory of system (17), corresponding to an arbitrary disturbance $w(t)$ of the class Θ , will not leave the set μ^* .

Now, let us have a look at the phase plane $\{x, y\}$ of system (17). According to (17) and with (18) and (19), the phase trajectory [that corresponds to an arbitrary disturbance $w(t)$] in the first quadrant connects a point on the y-axis (or x-axis) to a point on the x-axis. Now, let us separate the phase trajectory segments fully lying within the first quadrant and denote them by F_i . The segments are numbered according to the time they pass the first quadrant. Let t_i^- and t_i^+ be the instants at which motion starts and ends along the segment F_i ; and let $x = x[t, w(\cdot)]$, $y = y[t, w(\cdot)]$ for $t \in [t_i^-, t_i^+]$ be a parametric equation for the segment F_i . We denote $x_i^- = x[t_i^-, w(\cdot)]$, $x_i^+ = x[t_i^+, w(\cdot)]$, $y_i^- = y[t_i^-, w(\cdot)]$, $y_i^+ = y[t_i^+, w(\cdot)]$. According to the definition of segments F_i , we have $y_i^+ = 0$. Because $y = y[t, w(\cdot)] > 0$ for $t \in (t_i^-, t_i^+)$, y may be expressed through x : $y = y_i(x)$ at $x \in [x_i^-, x_i^+]$. Among all F_i , we then choose those starting at the points lying on the y-axis. Denote a subset of all such F_i through Γ_o , that is,

$F_i \in \Gamma_o$, if $y_i^- \geq 0$, $x_i^- = 0$. Note that Γ_o is non-empty if the set of all F_i is non-empty, because under zero initial conditions in system (17) F_i [with allowance for (18), (19)] necessarily belongs to Γ_o . Consider two segments, F_j and F_k , so that these segments belong to Γ_o , where F_i whose indices range from $j+1$ through $k-1$ do not belong to Γ_o . Now let us make some auxiliary constructions. We consider the following two possible variants for mutual position of segments F_{j+1} and F_j : 1) F_{j+1} and F_j have at least one common point; 2) F_{j+1} and F_j have no common points.

Let us consider the first variant, and denote by $\{x_{j+1}^j, y_{j+1}^j\}$ the first (after the moment t_{j+1}^-) common point. Let us construct a curve composed from a piece of the segment F_j (from the start up to the intersection point) and a piece of the segment F_{j+1} (from the intersection point up to the end) (Fig. 2). This curve, denoted through H_j^j , may be written as

$$y = y_j^j(x) = \begin{cases} y_j(x), & x \in [0, x_{j+1}^j] \\ y_{j+1}(x), & x \in [x_{j+1}^j, x_{j+1}^+] \end{cases} \quad (23)$$

Denote by ξ_{j+1}^j and ζ_{j+1}^j the instants when the phase trajectory segments F_{j+1} and F_j pass through the point $\{x_{j+1}^j, y_{j+1}^j\}$:

$$\begin{aligned} \xi_{j+1}^j &\in [t_j^-, t_j^+], \quad \zeta_{j+1}^j \in [t_{j+1}^-, t_{j+1}^+], \\ x[\xi_{j+1}^j, w(\cdot)] &= x[\zeta_{j+1}^j, w(\cdot)] = x_{j+1}^j, \\ y[\xi_{j+1}^j, w(\cdot)] &= y[\zeta_{j+1}^j, w(\cdot)] = y_{j+1}^j. \end{aligned}$$

Now let the second variant take place, that is, F_{j+1} and F_j do not intersect. Then, according to (17) and with (18) and (19), the segments F_{j+1} lie between F_j and the x-axis:

FIGURE 2

$$0 < y_{j+1}(x) < y_j(x) \quad \text{at } x \in (x_{j+1}^-, x_{j+1}^+).$$

In this case, the curve H_j^i is assumed to coincide with F_j :

$$y_j^i(x) = y_j(x). \quad (24)$$

Suppose that the curve H_j^r , where $r = l - j$ and $l < k - 1$, has been built, and now we will show how to construct the curve H_j^{r+1} . As before we have two variants of mutual positions of H_j^r and F_{l+1} : 1) H_j^r and F_{l+1} intersect; 2) H_j^r and F_{l+1} have no common points. If the first variant takes place, then denote by $\{x_{l+1}^n, y_{l+1}^n\}$ the first (after t_{l+1}^-) intersection point of H_j^r and F_{l+1} . Here the superscript index n denotes the number of the segment F_n that belongs to the curve H_j^r , and is intersected by the segment F_{l+1} ; in this case $n \in [j, l]$. If an intersection point falls into the "joint" of the segments F_m and F_n , then the smaller of the numbers m and n is taken as a superscript. Let us construct the curve H_j^{r+1} composed of the piece of the curve H_j^r (from the start up to the intersection point) and the piece of segment F_{l+1} (from the intersection point up to the end). The curve H_j^{r+1} is given by

$$y = y_j^{r+1}(x) = \begin{cases} y_j^r(x), & x \in [0, x_{l+1}^n] \\ y_{l+1}(x), & x \in [x_{l+1}^n, x_{l+1}^+] \end{cases} \quad (25)$$

Denote also by $\xi_{l+1}^n, \zeta_{l+1}^n$ the instants when the curve segments H_j^r and F_{l+1} pass through point $\{x_{l+1}^n, y_{l+1}^n\}$; $\xi_{l+1}^n \in [t_n^-, t_n^+]$, $\zeta_{l+1}^n \in [t_{l+1}^-, t_{l+1}^+]$. If the second variant takes place, that is, H_j^r and F_{l+1} do not intersect, then the segment F_{l+1} lies below the curve H_j^r . In this case H_j^{r+1} is assumed to coincide with H_j^r :

$$y_j^{r+1}(x) = y_j^r(x). \quad (26)$$

Expression (25) combined with (23), (24), and (26), constitutes a recursive formula to determine the curve H_j^p for $p \in [1, k - 1]$. To all curves H_j^p let us add the curves F_j from the set Γ_o denoting them by H_j^p : $y_j^p(x) = y_j(x)$. The set of curves thus obtained is denoted by G_o . For each $H_j^p \in G_o$, we also determine a set of points within the phase plane; this set contains the point $\{0, 0\}$ and is bounded by the coordinate axes and the curve H_j^p . Denote this set by θ_j^p (Fig. 3). According to the definition of the set G_o , for every segment of the phase trajectory F_i there can be found a curve H_j^p such that F_i does not leave the set θ_j^p .

FIGURE 3

Thus, to prove that the system phase trajectory (17) lies within the first quadrant of the phase plane and does not leave the set μ^* , it is sufficient to prove that the curves $H_j^p \in G_o$ do not intersect F^+ and do not leave the set μ^* . Let us examine the curve H_j^p . For simplicity, assume that this curve consists of only two pieces of the segments F_j and F_n . Let the coordinates of the intersection point of F_j and F_n be $\{x_n^j, y_n^j\}$; and the instants when the segments F_j and F_n pass through this point be denoted by ξ_n^j, ζ_n^j ($\xi_n^j < \zeta_n^j$), respectively. Consider an external disturbance in the form

$$w^*(t) = \begin{cases} w(t), & t \in [0, \xi_n^j] \\ w(t - \xi_n^j + \zeta_n^j), & t \geq \xi_n^j \end{cases}.$$

Note that if $w(t)$ belongs to the class Θ , the function $w^*(t)$ will also belong to it. Moreover, the phase trajectory $\{x[t, w(\cdot)], y[t, w(\cdot)]\}$ corresponding to the disturbance $w(t)$, differs from the phase trajectory $\{x[t, w^*(\cdot)], y[t, w^*(\cdot)]\}$ [corresponding to the disturbance $w^*(t)$] only within the interval (ξ_n^j, ζ_n^j) . Within the time interval $[t_j^-, t_n^+ + \xi_n^j - \zeta_n^j]$, the phase trajectory $\{x[t, w^*(\cdot)], y[t, w^*(\cdot)]\}$ will coincide with the curve H_j^p . Because $y[t, w^*(\cdot)] > 0$ in the interval $(t_j^-, t_n^+ + \xi_n^j - \zeta_n^j)$, y and t can be expressed as a function of x :

$$y = y_j^p(x), \quad t = t_j^p(x) \quad \text{at } x \in [x_j^-, x_n^+].$$

We can also find the function $w_j^p(x) = w^*[t_j^p(x)]$ for $x \in [x_j^-, x_n^+]$. The function $y_j^p(x)$ for $x \in [x_j^-, x_n^+]$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{-g(x, y) - \varphi(x) + w_j^p(x)}{y}, \quad y(x_j^-) = y_j^- \quad (27)$$

derived from system (17) by dividing the second equation by the first one. Let us write Eq. (27) in the integral form:

$$y_j^p(x) = y_j^- - \int_{x_j^-}^x R[x, y_j^p(x)] dx + \int_{x_j^-}^x \frac{w_j^p(x)}{y_j^p(x)} dx, \quad (28)$$

where

$$R[x, y_j^p(x)] = \frac{g[x, y_j^p(x)] + \varphi(x)}{y_j^p(x)}.$$

Note that according to the definition of the curve H_j^p , we obtain $x_j^- = 0$ and

$$\int_0^x \frac{w_j^p(x)}{y_j^p(x)} dx = \int_{t_j^-}^{t_j^p(x)} w^*(t) dt \leq \int_{t_j^-}^\infty |w(t)| dt. \quad (29)$$

The function $y = y_o^+(x)$ describing the curve F^+ satisfies the differential equation

$$\frac{dy}{dx} = \frac{-g(x, y) - \varphi(x)}{y}, \quad y(0) = J_o$$

for $x \in [0, \lambda^+]$. We can write this equation in the integral form:

$$y_o^+(x) = J_o - \int_0^x R[x, y_o^+(x)] dx.$$

Subtracting expression (28) from both sides of this equation we obtain:

$$y_o^+(x) - y_j^p(x) = J_o - y_j^- - \int_0^x \frac{w_j^p(x)}{y_j^p(x)} dx - \int_0^x \eta[x, y_j^p(x)] dx, \quad (30)$$

where $\eta[x, y_j^p(x)] = R[x, y_o^+(x)] - R[x, y_j^p(x)]$. Let us show that the curve H_j^p does not intersect the curve F^+ and hence does not leave the set μ^* . This is proved by contradiction. Suppose that the curve H_j^p intersects the curve F^+ at some point. In other words, there exists x^* such that

$$y_o^+(x^*) = y_j^p(x^*), \quad y_o^+(x) > y_j^p(x) \quad \text{at } x \in [0, x^*].$$

Taking into account the second inequality (18), we arrive at

$$\eta[x, y_j^p(x)] < 0 \quad \text{at } x \in [0, x^*]. \quad (31)$$

Now consider the difference

$$K_j^p(x) = J_o - y_j^- - \int_0^x \frac{w_j^p(x)}{y_j^p(x)} dx$$

and use estimate (21) for the maximal energy of system (17). Let

$$I_o \int_0^{t_j^-} |w(t)| dt.$$

If we consider disturbances $w'(t)$ of the form

$$w'(t) = \begin{cases} w(t), & t \in [0, t_j^-], \\ 0, & t \geq t_j^- \end{cases}$$

then according to estimate (21) we have

$$W[w'(\cdot)] \leq I_o^2/2$$

and, given the expression for the energy E , we get

$$y_j^- \leq I_o.$$

Thus, taking (29), into account, we have

$$\begin{aligned} K_j^p(x) &\geq J_o - I_o - \int_0^{x^*} \frac{w_j^p(x)}{y_j^p(x)} dx \\ &\geq J_o - I_o - \int_{t_j^-}^\infty |w(t)| dt \geq 0. \end{aligned}$$

From relationship (30), with (31), we immediately reach the contradiction

$$y_o^+(x^*) - y_j^p(x^*) > 0.$$

Hence, the curve H_j^p does not intersect F^+ and does not leave the set μ^* . A similar proof may be applied to any curve H_j^p from the set G_o . So we have shown that the system phase trajectory (17) lying within the first quadrant of the phase plane does not leave the set μ^* . This means that for all x_i^+ , the inequality $x_i^+ < \lambda^+$ is valid. Then, according to (17), (18), and (19), for any trajectory segment within the fourth quadrant, the inequality $x[t, w(\cdot)] < \lambda^+$ holds. In addition, because of estimate (21) in the fourth quadrant, the inequality $|y[t, w(\cdot)]| \leq J_o$ holds. Hence, the phase trajectory lying within the fourth quadrant does not leave the set μ^* .

The behavior of the phase trajectory within the third and second quadrants is analyzed, with al-

lowance made for the properties of the functions g and φ [see (18), (19)], in a similar way, if in Eqs. (17) the change of variables $x' = -x$, $y' = -y$ is made.

Thus, the phase trajectory corresponding to any disturbance $w(t)$ of the class Θ , does not leave the set μ^* . This means that for the maximal displacement in (17) the estimate

$$\sigma_o[w(\cdot)] \leq \max\{\lambda^+, |\lambda^-|\} \quad \forall w(\cdot) \in \Theta$$

holds. To prove the equality

$$\sigma^+ = \max\{\lambda^+, |\lambda^-|\}$$

it is sufficient to consider the disturbances

$$w_+^*(t) = +J_o\nu(t), \quad w_-^*(t) = -J_o\nu(t),$$

where $\nu(t)$ is the function of the kind (13). As τ tends to zero, we obtain

$$\begin{aligned} \max\{\lim_{\tau \rightarrow 0} \sigma_o[w_+^*(\cdot)], \\ \lim_{\tau \rightarrow 0} \sigma_o[w_-^*(\cdot)]\} = \max\{\lambda^+, |\lambda^-|\}. \end{aligned}$$

Hence, the instantaneous impact of intensity J_o or $-J_o$ results in the maximum value σ^+ of the maximal displacement $\sigma_o[w(\cdot)]$ in (17). This result generalizes the results obtained by Balandin [1989].

EXAMPLE

As an example let us consider a nonlinear oscillator described by the equation

$$\ddot{x} = -q\dot{x} + k|x|x + w(t), \quad \dot{x}(0) = x(0) = 0.$$

We assume that $q = 0.528$ and $k = 0.441$. Having solved this differential equation numerically, we can compare the above estimate σ^+ with the values of the maximal displacement $\sigma_o[w(\cdot)]$ under various disturbances $w(t)$. Let us consider shock disturbances of two kinds

$$\begin{aligned} w_1(t) &= \begin{cases} 1/T, & 0 \leq t \leq T \\ 0, & t > T \end{cases} \\ w_2(t) &= \begin{cases} 6t(T-t)/T^3, & 0 \leq t \leq T \\ 0, & t > T \end{cases} \end{aligned}$$

Table 1. Maximal Displacement Values

T	0.20	0.50	1.00	1.50	1.80
σ_1	0.998	0.997	0.996	0.990	0.985
σ_2	0.999	0.999	0.998	0.995	0.992

for which the equality

$$\int_0^\infty w_1(t) dt = \int_0^\infty w_2(t) dt = 1$$

holds. If we let Θ_o be the class of disturbances consisting of $w_1(t)$ and $w_2(t)$, then for this class the greatest value of the maximal displacement σ^+ will be equal to 1.000. Table 1 contains maximal displacement values for the oscillator influenced by the disturbances from the class Θ_o . The displacements depend on the duration of the disturbance. The values of σ_1 and σ_2 differ from σ^+ at most by 1.5%. Hence, the magnitude σ^+ can be considered a sufficiently good estimate for the maximal displacement of the given system under forces belonging to the class Θ_o .

SUMMARY

In this paper, an estimate for the maximal energy is obtained for mechanical systems under disturbances the integral of whose absolute value is below a given value J_o . The conditions for this estimate to be exact are formulated. An example is given that shows how the results obtained can be used for determining the maximal displacement in a second-order nonlinear system. These results may also be used for estimating a guaranteed quality of various mechanical systems, in particular shock isolation systems.

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