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Optimum Resolution Bandwidth for Spectral Analysis of Stationary Random Vibration Data

This article presents a methodology for selecting the frequency resolution bandwidth for the spectral analysis of stationary random vibration signals in an optimum manner. Specifically, the resolution bandwidth that will produce power spectral density estimates with a minimum mean square error is determined for any given measurement duration (averaging time), and methods of approximating the optimum bandwidth using practical spectral analysis procedures are detailed. The determination of the optimum resolution bandwidth requires an estimate for the damping ratio of the vibrating structure that produced the measured vibration signal and the analysis averaging time. It is shown that the optimum resolution bandwidth varies approximately with the 0.8 power of the damping ratio and the bandwidth center frequency, and the −0.2 power of the averaging time. Also, any resolution bandwidth within ±50% of the optimum bandwidth will produce power spectral density (PSD) estimates with an error that is no more than 25% above the minimum achievable error. If a damping ratio of about 5% for structural resonances is assumed, a constant percentage resolution bandwidth of 1/12 octave, but no less than 2.5 Hz, will provide a near optimum PSD analysis for an averaging time of 2 seconds over the frequency range from 20 to 2000 Hz. A simple scaling formula allows the determination of appropriate bandwidths for other damping ratios and averaging times.

INTRODUCTION

Random vibration data are commonly reduced for engineering applications into the form of an autospectral density function, also called a power spectral density function or PSD. It is convenient to view the estimation of a PSD in terms of direct frequency domain filtering operations, as illustrated in Fig. 1. Specifically, the random vibration signal $x(t)$ is first passed through an ideal rectangular bandpass filter with a bandwidth of B and a center frequency of f to produce a bandwidth-limited signal $x(f, B, t)$. Next, $x(f, B, t)$ is squared and averaged over the measurement duration T to obtain a bandwidth-limited mean square value estimate $\hat{\Psi}_x^2(f, B)$. Finally, $\hat{\Psi}_x^2(f, B)$ is divided by the bandwidth B to obtain the PSD estimate

$$\hat{G}_{xx}(f) = \frac{\hat{\Psi}_x^2(f, B)}{B} = \frac{1}{BT}\int_0^T x^2(fB, t)\, dt \quad (1)$$

where the hat (^) over $G_{xx}(f)$ denotes "estimate of." It is proven in Bendat and Piersol [1986] that

Shock and Vibration, Vol. 1, No. 1, pp. 33–43 (1993)

CCC 1070-9622/93/01033-11

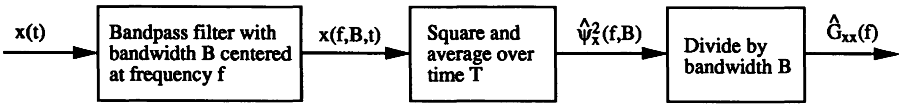

FIGURE 1 Direct frequency domain filtering procedures for PSD computation.

Eq. (1) yields the correct PSD of the random vibration in the limit as $T \to \infty$ and $B \to 0$ such that $BT \to \infty$. Of course, these limiting conditions cannot be achieved in practice, that is, for an actual PSD analysis, BT must be finite and B must be nonzero. The finite value of BT introduces a random (statistical sampling) error in the PSD estimate that decreases as BT becomes large [Bendat and Piersol, 1986], and the nonzero value of B introduces a frequency resolution bias (frequency smoothing) error that decreases as B becomes small [Bendat and Piersol, 1986, 1993; Forlifer, 1964; Schmidt, 1985]. If there is no limit on the total measurement duration T, both the random and bias errors can be made as small as desired by simply increasing T until acceptable values of BT and B are achieved. However, if T is limited, then some compromise between the values of BT and B must be established to minimize the total error in the PSD estimate.

The PSD estimation procedure outlined in Fig. 1 is easily accomplished using digital filters, or can be closely approximated using frequency averaged Fourier transforms of the signal $x(t)$, as will be detailed later. In practice, however, it is common to compute PSD estimates using a fast Fourier transform (FFT) algorithm with ensemble averaging procedures. Specifically, given a stationary signal $x(t)$ representing the time history of a random vibration measurement over the time interval of duration T, the signal duration is divided into n_d contiguous segments, each of duration T_b (called a block of data). The PSD of the random vibration represented by the signal $x(t)$ is then estimated from

$$\hat{G}_{xx}(f) = \frac{2}{n_d T_b} \sum_{i=1}^{n_d} |X_i(f, T_b)|^2; \quad f > 0 \quad (2a)$$

where for each contiguous block of data, $x_i(t)$; $0 \leq t \leq T_b$,

$$X_i(f, T_b) = \int_0^{T_b} x_i(t) e^{-j2\pi ft}\, dt. \quad (2b)$$

Because the Fourier transform in Eq. (2b) has finite limits, it follows that the values of the estimated PSD are computed at discrete frequencies given by

$$f_k = \frac{k}{T_b}; \quad k = 1, 2, 3, \ldots \quad (3)$$

meaning the PSD estimate is computed with an inherent frequency resolution of

$$\Delta f = \frac{1}{T_b}. \quad (4)$$

The exact PSD of the random vibration is given by Eq. (2) in the limit as $n_d \to \infty$ and $T_b \to \infty$ ($\Delta f \to 0$). Again, neither of these conditions can be achieved unless the measurement duration $T = n_d T_b \to \infty$, which is impossible. In this form, the finite value for the number of disjoint averages n_d in Eq. (2a) determines the random error in the estimated PSD, and the finite value for the integration time T_b in Eq. (2b) determines the frequency resolution bias error. As for the estimation procedure in Eq. (1), if the measurement duration T is limited, some compromise between the values of n_d and T_b must be established to minimize the total error in the PSD estimate.

The above noted compromise between random and bias errors in PSD estimates for stationary random vibration data was addressed many years ago by Forlifer [1964], but that article does not translate its results into the resolution bandwidth that provides the minimum mean square error. The compromise was further pursued for the PSD analysis of nonstationary random vibration data in Bendat and Piersol [1993], but that reference does not cover the special case of stationary data. The purpose of this article is to evaluate the optimum frequency resolution bandwidth that will minimize the total mean square error in PSD estimates for a wide class of stationary random vibration data. It should be mentioned that the results herein are developed assuming the PSD is the final data of interest. If the PSD is being estimated as an intermediate step in the computation of some other function, such as a frequency response or coherence function, the criteria for an optimum frequency resolution bandwidth may be quite different [Bendat and Piersol, 1986, 1993; Walker, 1981]. It should also be mentioned that the selection of the frequency resolution bandwidth for a PSD analysis is sometimes dictated by other considerations (e.g., compliance to an established MIL-STD or contractual requirements), which of course take precedence over the recommendations in this article. In particular, when a PSD is computed during a random vibration test to verify that the shaker controller is providing a shaker table motion within specified tolerance limits, the PSD should always be computed with the resolution bandwidth of the shaker controller, whatever that may be. Finally, it should be emphasized that the results herein apply only to random vibration data. The frequency resolution requirements for the accurate analysis of periodic or other types of deterministic vibration data are totally different.

SPECTRAL ESTIMATION ERRORS

To formulate the random and bias errors in PSD estimates for random vibration signals, let the following assumptions apply.

Assumption 1

The measured vibration signal $x(t)$ statistically represents one physical realization of a stationary random process $\{x(t)\}$ with a mean value of $\mu_x = 0$ and a PSD of $G_{xx}(f)$.

Assumption 2

The measured vibration signal $x(t)$ physically represents the response of a structure that behaves in compliance with classical normal mode theory [Bendat and Piersol, 1993], that is, each resonance of the structure responds like a second-order system with a frequency response function given by

$$H(f) = \frac{C_i}{1 - (f/f_i)^2 + j2\zeta_i f/f_i} \tag{5}$$

where f_i is the undamped natural frequency, ζ_i is the damping ratio, and C_i is the reciprocal of stiffness for the ith resonance.

Assumption 3

The random forcing function producing the structural response at each resonance has a PSD, denoted by $G_{yy}(f)$, that is reasonable uniform over the frequency range of the resonance. Noting that $G_{xx}(f) = |H(f)|^2 G_{yy}(f)$ [Bendat and Piersol, 1993], it follows that the PSD for the measured vibration signal in the region of each resonance is

$$G_{xx}(f) = \frac{C_i^2 G_{yy}(f)}{[1 - (f/f_i)^2]^2 + [2\zeta_i f/f_i]^2}. \tag{6}$$

Assumption 4

The damping ratio associated with each structural resonance is $\zeta_i < 0.1$. From Bendat and Piersol [1993], the frequency of the ith resonance (the frequency where the gain factor is a maximum), denoted by f_{ri}, is then approximately the same as the undamped natural frequency f_i, that is,

$$f_{ri} = f_i\sqrt{1 - 2\zeta_i^2} \approx f_i. \tag{7}$$

Furthermore, the frequency bandwidth between the half-power points of the PSD peak at each structural resonance is closely approximated by

$$B_i \approx 2\zeta_i f_i. \tag{8}$$

It is convenient to formulate estimation errors assuming the PSD is computed by the procedure detailed in Fig. 1 and Eq. (1), which approximates an analysis using digital filters or frequency averaged FFT procedures. Modifications required to make the results applicable to the PSD estimates produced by the ensemble averaged FFT computations in Eq. (2) are discussed later.

Random Error

From Bendat and Piersol [1993], the normalized random error (coefficient of variation) for a PSD estimate is given by

$$\varepsilon_r[\hat{G}_{xx}(f)] = \frac{\sigma[\hat{G}_{xx}(f)]}{G_{xx}(f)} \approx \frac{1}{\sqrt{B_s T}} \tag{9}$$

where $\sigma[\hat{G}_{xx}(f)]$ is the standard deviation of the estimate $\hat{G}_{xx}(f)$, and B_s is the statistical bandwidth (sometimes called the effective bandwidth) of the signal passed by the frequency resolution bandpass filter used for the analysis. The statistical bandwidth is defined as [Bendat and Piersol, 1993]

$$B_s(f) = \frac{\left[\int_0^\infty G_F(f)\,df\right]^2}{\int_0^\infty G_F^2(f)\,df} \tag{10}$$

where $G_F(f)$ is the PSD of the signal passed by the frequency resolution bandpass filter (hereafter referred to as the analysis filter). Letting $H_F(f)$ be the frequency response function of the analysis filter,

$$G_F(f) = |H_F(f)|^2 G_{xx}(f). \tag{11}$$

Note if $|H_F(f)| = H; f - B/2 \le f \le f + B/2$, and is zero elsewhere (i.e., the analysis filter has an ideal rectangular bandpass characteristic with a bandwidth B), and $G_{xx}(f) = G; f - B/2 \le f \le f + B/2$ (i.e., the PSD of the vibration signal is uniform over the bandwidth B of the analysis filter), then Eq. (10) yields $B_s = B$.

Of interest here is the minimum value of B_s in Eq. (10) for structural vibration data that occurs when the center frequency of the analysis filter is at a resonance frequency of the structure producing the vibration signal. For this case, assuming an ideal rectangular analysis filter centered at f_i and a uniform excitation spectrum in Eq. (6) of $G_{yyi}(f) = G_{yyi}$ over the analysis filter bandwidth, the PSD passed by the analysis filter centered on a resonance is

$$G_F(f) = \frac{C_i^2 G_{yyi}}{[1 - (f/f_i)^2]^2 + [2\zeta_i f/f_i]^2}; \qquad f_i - \frac{B}{2} < f < f_i + \frac{B}{2}$$

$$= 0; \qquad f < f_i - \frac{B}{2}, \quad f > f_i + \frac{B}{2}. \tag{12}$$

Substituting Eq. (12) into Eq. (10) and solving for B_s versus the ratio B/B_i, where B_i is defined in Eq. (8), yields the results shown in Fig. 2. Note in Fig. 2 that B_s is within 20% of B for values of $B \le 2B_i$. Hence, because B_s appears under a square root sign, Eq. (9) is approximated to within 10% by

$$\varepsilon_r[\hat{G}_{xx}(f)] = \frac{1}{\sqrt{BT}}; \quad B \le 2B_i. \tag{13}$$

For $B > 2B_i$, Eq. (13) will increasingly underestimate the random error at the frequencies of structural resonances.

Frequency Resolution Bias Error

The frequency resolution bias error in a PSD estimate is defined as the expected value of the estimate minus the quantity being estimated, that is,

$$b[\hat{G}_{xx}(f)] = E[\hat{G}_{xx}(f)] - G_{xx}(f) \tag{14}$$

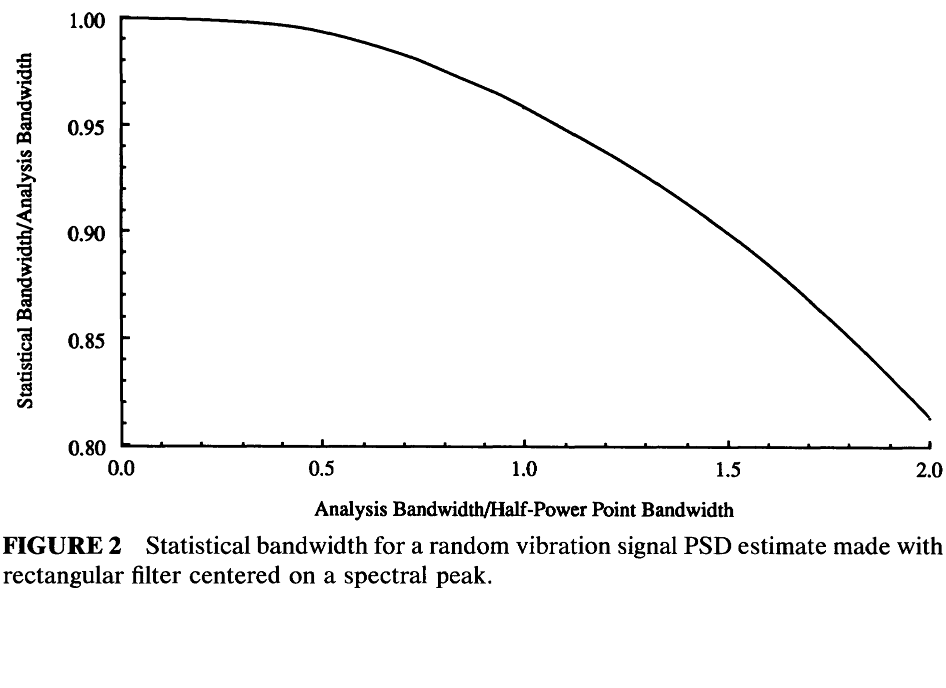

FIGURE 2 Statistical bandwidth for a random vibration signal PSD estimate made with a rectangular filter centered on a spectral peak.

where $E[\ ]$ denotes the expected value of [ ]. For structural vibration signals, this bias error reaches a maximum at those frequencies where the PSD reveals a peak due to a structural resonance. From Forlifer [1964], assuming an ideal rectangular analysis filter, the normalized bias error in the region of a structural resonance is given by

$$\varepsilon_b[\hat{G}_{xx}(f)] = \frac{b[\hat{G}_{xx}(f)]}{G_{xx}(f)} \approx \frac{B_i}{B}\tan^{-1}\left(\frac{B}{B_i}\right) - 1 \quad (15)$$

where B_i is the half-power point bandwidth of the PSD peak due to the ith resonance, as defined in Eq. (8). Equation (15) has been checked against the exact bias error given by a numerical integration of Eq. (12) for various values of B, and has been found to be essentially exact for values of $B/B_i \leq 2$.

Equation (15) involves a transcendental function that makes it difficult to work with. However, from Bendat and Piersol [1993], for $B/B_i < 0.4$, Eq. (15) is closely approximated by a much simpler algebraic function, namely,

$$\varepsilon_b[\hat{G}_{xx}(f)] \approx -\frac{1}{3}\left(\frac{B}{B_i}\right)^2; \quad B < 0.4\,B_i. \quad (16)$$

A comparison of the bias error approximations given by Eqs. (15) and (16) for values of $B/B_i \leq 2$ is shown in Fig. 3. Note that the Eq. (16) approximation is within 10% of the actual error in Eq. (15) only for values of $B/B_i \leq 0.4$. For $B/B_i > 0.4$, Eq. (16) increasingly overestimates the frequency resolution bias error.

OPTIMUM ANALYSIS BANDWIDTH

Let an optimum analysis bandwidth be defined as that value of B that will minimize the total mean square error of the estimate, that is, the sum of the squares of the random and bias errors. Using the random error in Eq. (13) and the approximation for the frequency resolution bias error in Eq. (16), the total normalized mean square error in a PSD estimate is approximated by

$$\varepsilon^2 = \varepsilon_r^2[\hat{G}_{xx}(f)] + \varepsilon_b^2[\hat{G}_{xx}(f)] \approx \frac{B^4}{9B_i^4} + \frac{1}{BT} \quad (17)$$

Taking the derivative of Eq. (17) with respect to B, equating to zero, and solving for B gives the optimum analysis bandwidth as

$$B_0 \approx \left[\frac{9B_i^4}{4T}\right]^{1/5} \approx 2\,\frac{(\zeta_i f_i)^{4/5}}{T^{1/5}} \quad (18)$$

where the second expression is obtained using Eq. (8). Note in Eq. (18) that the optimum analysis bandwidth is inversely proportional to the 0.2 power of the averaging time, meaning the analysis bandwidth is not very sensitive to this parameter, for example, B_0 for $T = 1$ second is less than

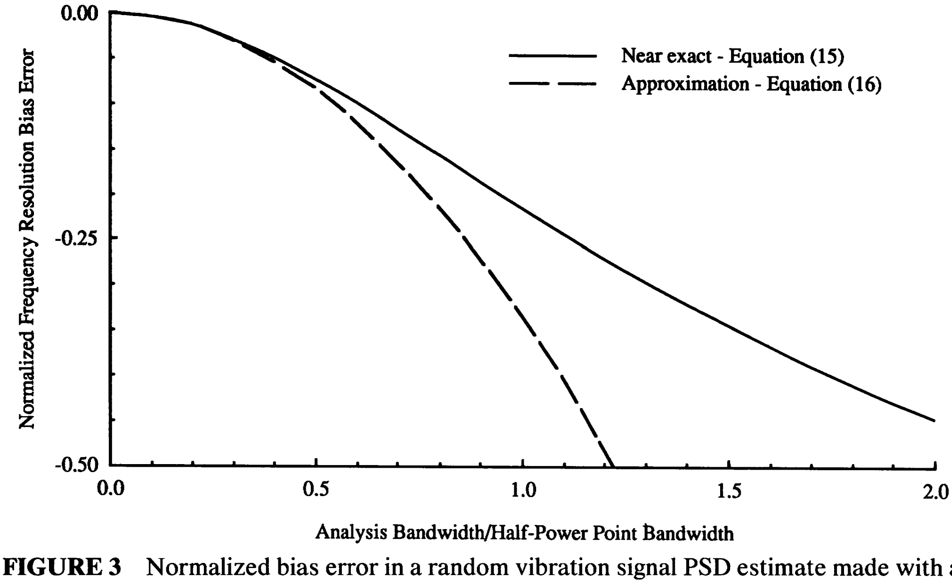

FIGURE 3 Normalized bias error in a random vibration signal PSD estimate made with a rectangular filter centered on a spectral peak.

60% greater than B_0 for $T = 10$ seconds. On the other hand, the optimum analysis bandwidth is proportional to the 0.8 power of both the damping ratio and the frequency of the various structural resonances.

Equation (18) helps clarify parametric relationships, but for many practical applications it violates the restriction of $B/B_i \leq 0.4$ for the bias error approximation in Eq. (16). Specifically, dividing Eq. (18) by $B_i = 2\zeta_i f_i$ and solving for f_i when $B/B_i \leq 0.4$ yields

$$f_i \geq \frac{110}{\zeta_i T}. \qquad (19)$$

For example, if $T = 10$ seconds and $\zeta_i = 0.05$, Eq. (18) is fully applicable only for resonance frequencies of $f_i \geq 220$ Hz. If $T = 1$ second, the lowest resonance frequency for reasonable accuracy is above 2000 Hz, which is often the highest frequency of interest in the analysis of random vibration data.

To obtain an optimum averaging time with a wider range of application, the more accurate bias error expression in Eq. (15) is used to define the total normalized mean square error as

$$\varepsilon^2 \approx \left[\frac{B_i}{B}\tan^{-1}\left(\frac{B}{B_i}\right) - 1\right]^2 + \frac{1}{BT}. \qquad (20)$$

The analysis bandwidth that will minimize the total mean square error in Eq. (20) is computed by numerical procedures over the frequency range from 20 to 2000 Hz (a common range for random vibration data analysis) with a structural damping ratio of $\zeta = 0.05$ and averaging times of $T = 1$ and $T = 10$ seconds in Fig. 4. Also shown in Fig. 4 is the analysis bandwidth predicted by minimizing Eq. (17) for the same values of ζ and T. It is seen in this figure that the more accurate bias error expression used in Eq. (20) gives a somewhat higher optimum analysis bandwidth, particularly at the lower resonance frequencies. Also, the discrepancy between the optimum analysis bandwidths determined using Eqs. (17) and (20) is greater for the shorter averaging time, as predicted by Eq. (19).

As a final point of evaluation, the normalized random error, the bias error, and the total rms error (the square root of the total mean square error) are plotted against the analysis bandwidth in Fig. 5. The errors in Fig. 5 are computed by numerically minimizing Eq. (20) with $f_i = 500$ Hz, $\zeta_i = 0.05$, and $T = 1$ second. Note that the rms error reaches a minimum at $B_0 = 30$ Hz, in agreement with the results in Fig. 4 at $f = 500$ Hz with $T = 1$ second. However, the rms error is no more than 25% above the minimum error for all values of B within ±50% of B_0. Similar results occur for essential all values of f_i, ζ_i, and T. It follows that a precise selection of the analysis bandwidth is not necessary, that is, any analysis bandwidth within ±50% of the optimum bandwidth computed by minimizing Eq. (20) will produce acceptably accurate PSD estimates.

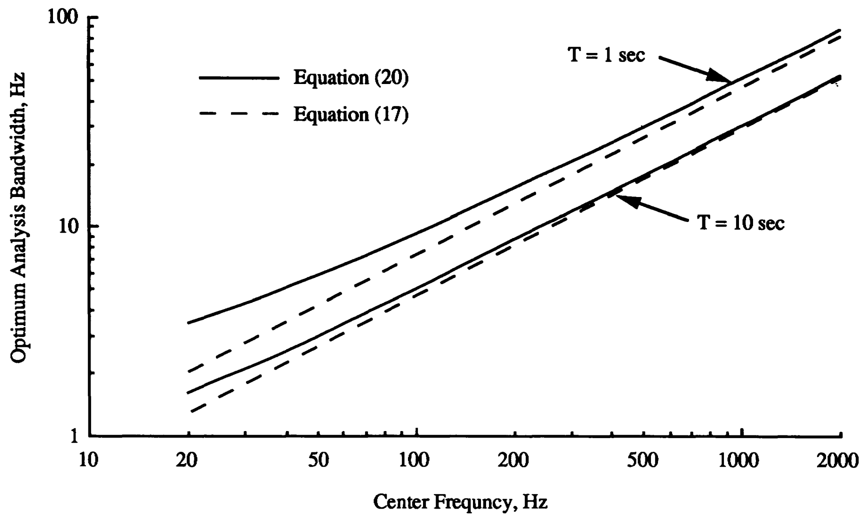

FIGURE 4 Optimum analysis bandwidth for the computation of a random vibration signal PSD ($\zeta = 0.05$).

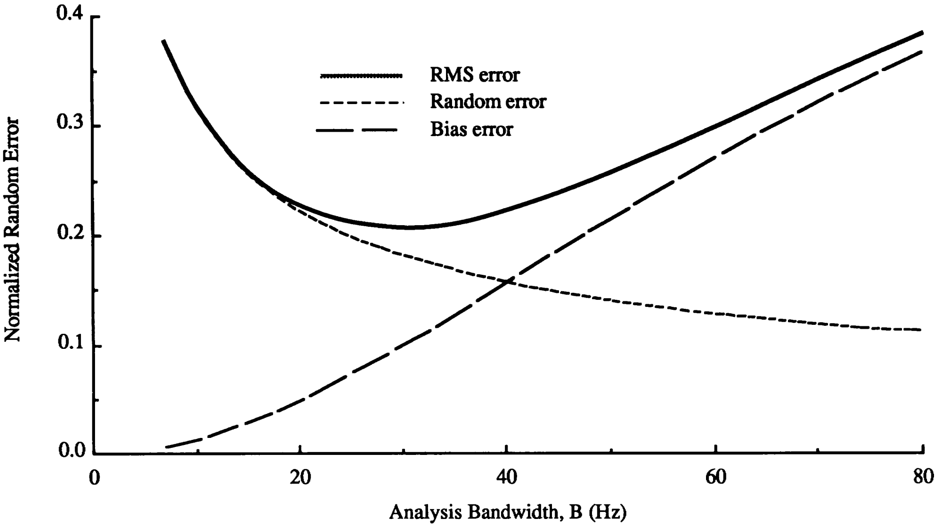

FIGURE 5 Errors in PDS estimates of random vibration signal (f = 500 Hz, ζ = 0.05, T = 2 seconds).

APPLICATIONS

For the PSD analysis of random vibration data in practice, the frequencies of spectral peaks due to structural resonances usually cannot be anticipated in advance. It follows that the analysis must be performed with an analysis bandwidth that will minimize the error in a spectral peak at all frequencies. Of course, this means that the error at most frequencies will be much less than predicted by Eq. (20). Also, the damping ratios of the structural resonances producing spectral peaks cannot be precisely anticipated, so a single common value of damping for all resonances must be assumed. A value of $\zeta = 0.05$ is recommended, although a different value can be used if it is considered more appropriate. Finally, a wide range of averaging times might be used for a PSD analysis, but an averaging time of $T \approx 2$ seconds is common. Referring to Eq. (18), the optimum bandwidth is relatively insensitive to averaging time, so $T = 2$ seconds should produce acceptable analysis bandwidth selections for averaging times in the range $1 \leq T \leq 4$ seconds. These parameters ($\zeta = 0.05$ and $T = 2$ seconds) are assumed in the analysis bandwidth selections to follow. The bandwidth selections can be modified for different values of ζ and T by scaling in accordance with Eq. (18).

PSD Analysis Using Digital Filters

The PSD analysis of random vibration data is sometimes performed using digital filter algorithms, either in a special purpose spectral analysis instrument, or with appropriate software in a personal or mainframe computer. The computational procedure is as detailed in Eq. (1) and Fig. 1. Digital filters provide the near-rectangular bandpass characteristic assumed to derive Eqs. (17)–(20). Although not essential, it is common to design digital filter algorithms with a bandwidth that is a constant percentage of the center frequency, for example, 1/3, 1/6, or 1/12 octave bandwidth. It turns out that a 1/12 octave bandwidth filter ($B \approx 0.058 f$) falls within the $\pm 50\%$ bounds on the optimum analysis bandwidth determined by minimizing Eq. (20) with $\zeta = 0.05$ and $T = 2$ seconds over most of the frequency range from 20 to 2000. This is illustrated in Fig. 6. It is seen in Fig. 6 that the largest discrepancies between the optimum and 1/12 octave bandwidths occur at the frequency extremes, namely 20 and 2000 Hz. The discrepancy at 2000 Hz is not important because the rms error at this frequency is very small even with the discrepancy. However, the discrepancy at 20 Hz is important because this is where the rms error of the analysis is largest. Hence, the results of a 1/12 octave band analysis can be enhanced by breaking away from the 1/12 octave bandwidth at frequencies below 43.1 Hz to a fixed bandwidth of 2.5 Hz (Fig. 6). The rms errors for the optimum bandwidth analysis and the 1/12 octave bandwidth analysis, with and without the constant 2.5-Hz bandwidth below 43.1 Hz, are shown in Fig. 7. Note in Fig. 7 that the rms error for the 1/12

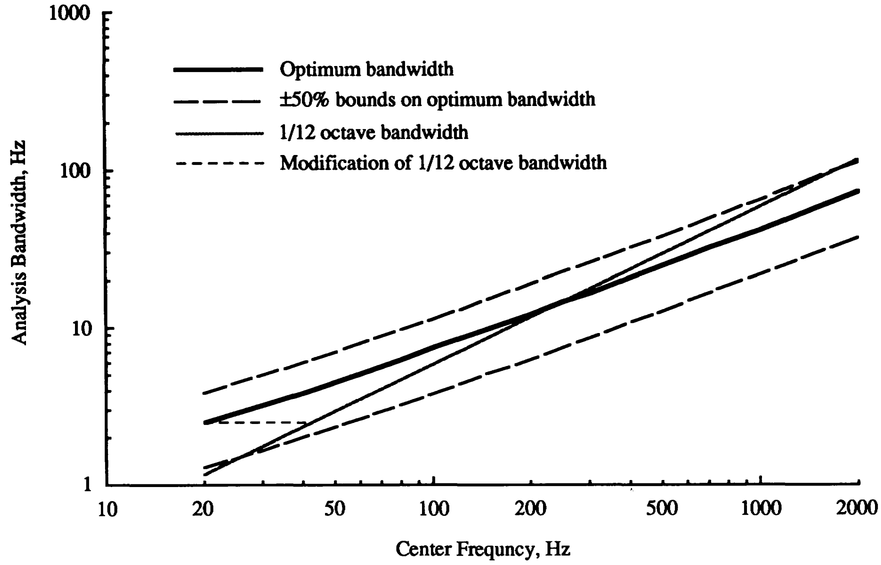

FIGURE 6 Analysis bandwidths for optimum and 1/12 octave PSD estimates ($\zeta = 0.05$, $T = 2$ seconds).

octave bandwidth analysis is within 25% of the minimum rms error for the optimum bandwidth analysis at most frequencies (Fig. 5). Also, with the modification of the 1/12 octave bandwidth to a constant 2.5 Hz bandwidth below 43.1 Hz, the large rms error values at the low frequencies are reduced. It follows that 1/12 octave bandwidth filters truncated at 43.1 Hz to a fixed bandwidth of 2.5 Hz will provide a near optimum PSD analysis of random vibration data under the stated assumptions ($\zeta = 0.05$, $T = 2$ seconds).

Spectral Analysis Using FFT Algorithms with Frequency Averaging

The PSD analysis of random vibration data is commonly performed using an FFT algorithm that efficiently computes the finite (discrete) Fourier transform defined in Eq. (2b). If a single FFT of the random vibration signal $x(t)$ is computed over the entire measurement duration T, a "raw" PSD is computed with a basic frequency resolution of $\Delta f = 1/T$, as follows:

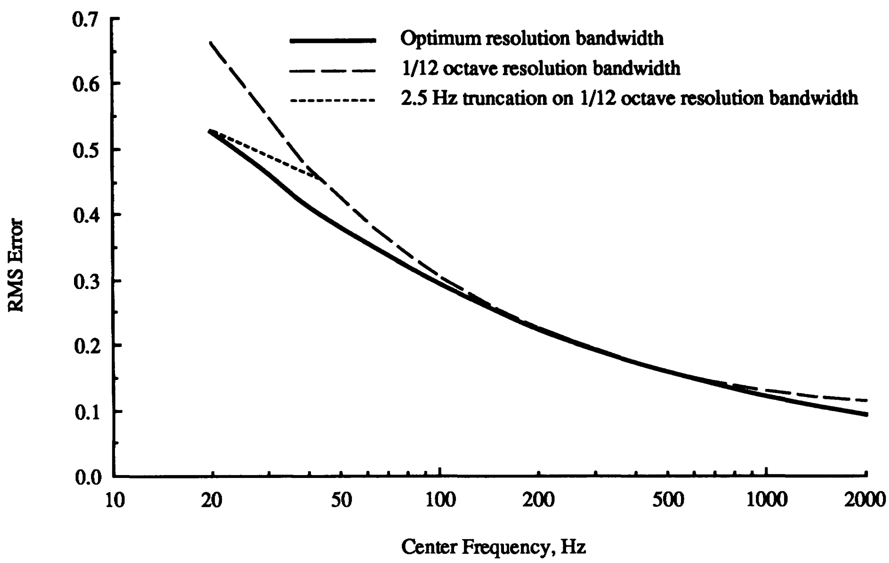

FIGURE 7 RMS errors for optimum and 1/12 octave PSD estimates ($\zeta = 0.05$, $T = 2$ seconds).

$$\hat{G}_{xx}(f) = \frac{2}{T}\left|\int_0^T x_i(t)\, e^{-j2\pi ft}\, dt\right|^2. \quad (21)$$

The basic resolution for the raw PSD estimate in Eq. (21) is fixed at all center frequencies to $\Delta f = 1/T$ (e.g., if $T = 2$ seconds, $\Delta f = 0.5$ Hz). Furthermore, the basic spectral window at each center frequency $f_k = k\Delta f$, $k = 1, 2, 3, \ldots$, is a (sin x)/x function, rather than a rectangular function [Bendat and Piersol, 1986]. However, a variable frequency resolution bandwidth and a near rectangular shape can be achieved by frequency averaging the raw PSD estimate [Otnes and Enochson, 1978]. Specifically, for each center frequency f_k, average n_k contiguous raw PSD values centered on f_k to obtain the final PSD estimate at f_k with the desired spectral bandwidth of $B_k = n_k \Delta f$.

Using the above frequency averaging procedure, a PSD with any desired frequency resolution can be achieved. For example, a 1/12 octave bandwidth resolution (simulating the previous analysis with digital filters) with $T = 2$ seconds is given by

$$n_k \approx 0.058\, f_k T = 0.12\, f_k. \quad (22)$$

Rounding off n_k to the next highest integer value, $n_k = 3$ at $f_k = 20$ Hz ($n_k = 5$ at $f_k = 20$ Hz if the 2.5-Hz constant bandwidth below 43.1 Hz is used), $n_k = 30$ at $f_k = 250$ Hz, and $n_k = 240$ at $f_k = 2000$ Hz. The rms error for the frequency averaged PSD estimate will be as shown in Fig. 7. Of course, there is no need to restrict the analysis bandwidth to a constant percentage of center frequency, that is, n_k could be selected to more closely match the optimum analysis bandwidth shown in Fig. 6.

Spectral Analysis Using FFT Algorithms with Ensemble Averaging

A common procedure for estimating the PSD of a random vibration signal is to use the ensemble averaged FFT computation detailed in Eq. (2). This procedure inherently produces estimates with a fixed analysis bandwidth and, hence, is not desirable to obtain optimum PSD estimates in the minimum mean square error sense. Nevertheless, some discussion of the PSD estimates produced by an ensemble averaged FFT analysis is warranted because the procedure is so widely used.

Because of the ±50% range for acceptable values of the analysis bandwidth B indicated in Fig. 6, a PSD could be estimated using ensemble averaged FFT estimates by repeatedly computing the PSD with different resolutions $\Delta f = 1/T_b$ over several different frequency ranges. For example, with $\zeta = 0.05$ and $T = 2$ seconds, the following analysis bandwidth selections would provide results within ±50% of the optimum bandwidth at most frequencies between 20 and 2000 Hz.

$$\begin{aligned} f &= 20\text{–}100\ \text{Hz}: B = 3\ \text{Hz} \\ f &= 100\text{–}400\ \text{Hz}: B = 10\ \text{Hz} \quad (23) \\ f &= 400\text{–}2000\ \text{Hz}: B = 30\ \text{Hz}. \end{aligned}$$

The problem is to establish a relationship between the ideal rectangular bandwidth B assumed for the derivation of the optimum analysis bandwidth in Fig. 6, and the basic frequency resolution $\Delta f = 1/T_b$ for the ensemble averaged FFT analysis. Specifically, the equivalent bandpass filter (the spectral window) for the ensemble averaged FFT estimates has a nonrectangular shape that is heavily dependent on the time history tapering operation (the time window) used to suppress sidelobe leakage [Bendat and Piersol, 1986]. This fact dramatically influences the value of the statistical bandwidth B_s defined in Eq. (10) relative to the basic frequency resolution Δf. For example, if no tapering is used (i.e., the analysis is performed using a rectangular time window producing a sin x/x type spectral window [Bendat and Piersol, 1986]), $B_s \approx 0.75\ \Delta f$, but if a cosine squared taper is used (referred to as a "Hanning" window [Bendat and Piersol, 1986]), $B_s \approx 1.35\ \Delta f$. Also, the frequency resolution bias error is heavily influenced by the tapering operation, as detailed in Schmidt [1985] and illustrated in Fig. 8. If the bandwidth B for a rectangular bandpass filter is equated to Δf, it is seen in Fig. 8 that the resolution bias error is more severe for an ensemble averaged FFT estimate, with either a rectangular or Hanning time window, than for the rectangular frequency window simulated by the digital filter or frequency averaged FFT procedures discussed earlier. Furthermore, the factor that relates the rectangular frequency window B to the frequency resolution Δf in the random and bias error expressions in Eq. (13) and (15), respectively, are generally different.

There is at least one case where the recommendations in Eq. (23) might provide acceptable results, namely, when the ensemble averaged FFT analysis is performed with a Hanning win-

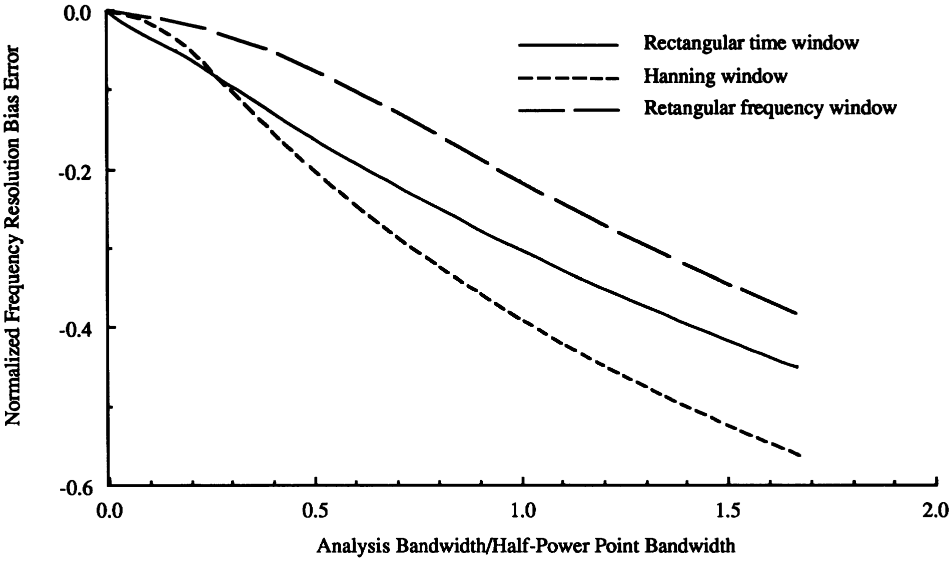

FIGURE 8 Frequency resolution bias errors for ensemble averaged FFT computation of random vibration signal PSD estimates.

dow. For this case, assuming $B/B_i < 1$, $B \approx 1.5\ \Delta f$ will coarsely relate the random and bias errors for the ensemble averaged FFT analysis to those given for a rectangular frequency window analysis in Eq. (13) and (15). Again assuming $\zeta = 0.05$ and $T = 2$ seconds, Eq. (23) becomes

$$\begin{aligned} f &= 20\text{–}100\ \text{Hz}: \Delta f = 2\ \text{Hz} \\ f &= 100\text{–}400\ \text{Hz}: \Delta f = 7\ \text{Hz} \\ f &= 400\text{–}2000\ \text{Hz}: \Delta f = 20\ \text{Hz}. \end{aligned} \tag{24}$$

CONCLUSIONS

This study of the optimum frequency resolution (analysis) bandwidth that will minimize the total mean square error in the computation of a PSD function of a random vibration signal leads to the following primary conclusions:

1. The optimum analysis bandwidth is dependent on the damping ratio of the structure on which the vibration signal is measured, the center frequency of the analysis bandwidth, and the analysis duration (averaging time). Specifically, the optimum bandwidth varies approximately with the 0.8 power of the damping ratio and center frequency, and the -0.2 power of the averaging time.
2. For a given set of values for damping ratio, center frequency, and averaging time, any analysis bandwidth within $\pm 50\%$ of the computed optimum bandwidth produces an estimated PSD with an error no more than 25% greater than the minimum error.
3. For an assumed damping ratio of 5% and an averaging time of 2 seconds, the optimum analysis bandwidth over the frequency range from 20 to 2000 Hz can be acceptably approximated by a 1/12 octave bandwidth that is bounded at the lower frequencies to be no less than 2.5 Hz.
4. For other damping ratios and averaging times, the appropriate optimum analysis bandwidth can be determined by scaling in accordance with the functional relationships stated in Conclusion 1. However, because of the weak dependence on averaging time, the analysis bandwidth in Conclusion 3 will generally provide acceptable results in the frequency range from 20 to 2000 Hz for all averaging times between 1 and 4 seconds (assuming a damping ratio of 5%).

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