**N.J. Fergusson**
**W.D. Pilkey**
*Department of Mechanical and Aerospace Engineering*
*University of Virginia*
*Charlottesville, VA 22901*

# Finite Dynamic Elements and Modal Analysis

*A general modal analysis scheme is derived for forced response that makes use of high accuracy modes computed by the dynamic element method. The new procedure differs from the usual modal analysis in that the modes are obtained from a power series expansion for the dynamic stiffness matrix that includes an extra dynamic correction term in addition to the static stiffness matrix and the consistent mass matrix based on static displacement. A cantilevered beam example is used to demonstrate the relative accuracies of the dynamic element and the traditional finite element methods.* 

## INTRODUCTION

The dynamic element method (DEM) has been employed in the past by Przemieniecki (1968), Downs (1986), and others for computing accurate natural frequencies and mode shapes. The method entails expanding the transcendental dynamic stiffness matrix $\mathbf{d}(\omega)$ in a Taylor series in powers of the frequency-squared, truncating the series to three or more terms. The above authors have demonstrated the highly efficient mode finding capabilities of the DEM in contrast to the traditional finite element method (FEM), in which only two terms, the static stiffness matrix and consistent mass matrix, are retained.

The current paper considers the superposition of modes computed using the DEM, and demonstrates the increased convergence rate as compared to the standard modal analysis based on the FEM.

## REVIEW OF BASIC OPERATORS

First a review of the basic state variables and operators is undertaken. The theory developed here is based on the "mechanics of materials" governing equations for a continuous linear elastic structural member, which can be summarized as

$$\boldsymbol{\epsilon} = \mathbf{D}_u \mathbf{u} \tag{1}$$

$$\mathbf{s} = \mathbf{E}\boldsymbol{\epsilon} \tag{2}$$

$$\mathbf{D}_s^T \mathbf{s} + \bar{\mathbf{p}}_V = \boldsymbol{\rho}\ddot{\mathbf{u}} \tag{3}$$

or the combined (displacement) form

$$\mathbf{D}_s^T \mathbf{E} \mathbf{D}_u \mathbf{u} + \bar{\mathbf{p}}_V = \boldsymbol{\rho}\ddot{\mathbf{u}} \tag{4}$$

in addition to initial and boundary conditions, where $\boldsymbol{\epsilon}$ is the column of generalized strains, which may include slopes, curvatures, etc.; $\mathbf{u}$ is the column of generalized displacements, such as extensions, deflections, or rotations; $\mathbf{s}$ is the column of generalized stresses, or stress-resultants, such as bending moments or shear forces; $\boldsymbol{\rho}$ is the generalized density matrix with components possibly containing radius of gyration terms to account for rotary inertia; finally, $\bar{\mathbf{p}}_V$ represents the applied body forces. The body forces as well as the state variables depend on position $\mathbf{x}$ at time $t$.

The differential operators $\mathbf{D}_u$ and $\mathbf{D}_s^T$ are gen-

Shock and Vibration, Vol. 1, No. 2, pp. 171–176 (1993)

CCC 1070-9622/93/020171-06

eralized forms of the gradient and divergence operators, respectively, as applied in the case of general elasticity. The divergence of the 3-D Cauchy stress tensor **T**, for example, can be written as $\mathbf{D}_s^T\mathbf{s}$ when **s** is a 6 × 1 column containing the six symmetric components of **T**. In light of these facts, the operators $\mathbf{D}_u$ and $\mathbf{D}_s^T$ satisfy a certain formal adjoint relationship

$$\int_V \mathbf{u}^T{}_u\mathbf{D}^T\mathbf{s}dV = \oint_{\partial V} \mathbf{u}^T\mathbf{A}^T\mathbf{s}\, dS - \int_V \mathbf{u}^T\mathbf{D}_s^T\mathbf{s}dV \tag{5}$$

where **u** and **s** are arbitrary vector columns smooth enough for the integrals to make sense, and $\mathbf{A}^T$ is a direction cosine matrix that describes the relationship between the traction **p** on an oblique surface in a body and the generalized stress components

$$\mathbf{p} = \mathbf{A}^T\mathbf{s}. \tag{6}$$

The symbols ${}_u\mathbf{D}^T$ and ${}_s\mathbf{D}$ represent $\mathbf{D}_u$ and $\mathbf{D}_s^T$ transposed, and act on the vector to their left.

## MODAL ANALYSIS

An initial-boundary value problem with nonhomogeneous boundary conditions can be reduced to one with homogeneous boundary conditions by considering displacement from a reference solution. It thus suffices to consider the following problem as a model for the forced vibration problem of a complex structure.

$$\mathbf{D}_s^T\mathbf{E}\mathbf{D}_u\mathbf{u}(\mathbf{x}, t) - \rho\ddot{\mathbf{u}}(\mathbf{x}, t) = -\rho\hat{\mathbf{p}}_V(\mathbf{x}, t) \quad \mathbf{x} \in V \tag{7}$$

$$\mathbf{B}_u\mathbf{u}(\mathbf{x}, t) = \mathbf{0} \quad \mathbf{x} \in \partial V \tag{8}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \tag{9}$$

$$\dot{\mathbf{u}}(\mathbf{x}, 0) = \dot{\mathbf{u}}_0(\mathbf{x}) \tag{10}$$

where $\mathbf{B}_u$ represents a differential operator (of a lower order than the operator $\mathbf{D}_s^T\mathbf{E}\mathbf{D}_u$) that also evaluates its argument at the boundary; $\mathbf{u}_0$ and $\dot{\mathbf{u}}_0$ represent the prescribed initial displacement and velocity functions; and $\hat{\mathbf{P}}_V$ is given by

$$\hat{\mathbf{p}}_V = \rho^{-1}\bar{\mathbf{p}}_V. \tag{11}$$

According to the classical eigenfunction expansion method for a linear partial differential equation, the solution can be written as an expansion

$$\mathbf{u}(\mathbf{x}, t) = \sum_{j=1}^{\infty}\mathbf{u}_j(\mathbf{x})h_j(t) \tag{12}$$

in terms of the eigenfunctions $\mathbf{u}_j(\mathbf{x})$ obtained by inserting this expansion into the homogeneous form of Eq. (7) and separating the space and time variables. The eigenfunctions thus satisfy the (regular) Sturm–Liouville problem

$$\mathbf{D}_s^T\mathbf{E}\mathbf{D}_u\mathbf{u}_j(\mathbf{x}) + \rho\omega_j^2\mathbf{u}_j(\mathbf{x}) = \mathbf{0} \quad \mathbf{x} \in V \tag{13}$$

$$\mathbf{B}_u\mathbf{u}_j(\mathbf{x}) = \mathbf{0} \quad \mathbf{x} \in \partial V \tag{14}$$

where the eigenpairs $\langle\omega_j^2, \mathbf{u}_j(\mathbf{x})\rangle$ satisfy the usual properties (real unbounded positive monotonicity, orthogonality with respect to $\rho$, Rayleigh quotient property, and completeness) for the regular Sturm–Liouville problem. Because the eigenfunctions are determined only up to scalar multiples, they are assumed to be normalized with respect to $\rho$

$$\int_V \mathbf{u}_i^T\rho\mathbf{u}_j dV = \delta_{ij}. \tag{15}$$

Because the eigenfunctions satisfy the homogeneous problems (13) and (14), they can be determined as eigenpairs $\langle\mathbf{u}, \omega^2\rangle$ each satisfying the principle of virtual work

$$\int_V \delta\mathbf{u}^T{}_u\mathbf{D}^T\mathbf{E}\,\mathbf{D}_u\mathbf{u}dV - \omega^2\int_V \delta\mathbf{u}^T\rho\mathbf{u}\,dV = \int_V \delta\mathbf{u}^T\bar{\mathbf{p}}_V\, dV \tag{16}$$

for arbitrary test functions $\delta\mathbf{u}$ consistent with the geometric constraints.

According to the familiar procedure for deriving the modal superposition, one uses the continuity of $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{E}\mathbf{D}_u\mathbf{u}(\mathbf{x}, t)$ and the fact that $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{u}_n(\mathbf{x}, t)$ satisfy the same homogeneous boundary conditions, to insert the series (12) in Eq. (7). Taking inner products and invoking orthogonality, one arrives at the familiar solution for the displacement

$$\mathbf{u}(\mathbf{x}, t) = \sum_{j=1}^{\infty}\mathbf{u}_j(\mathbf{x})\left[\frac{1}{\omega_j}\int_0^t \sin\omega_j(t-\tau)\int_V \mathbf{u}_j^T(\mathbf{x})\rho\,\hat{\mathbf{p}}_V(\mathbf{x}, \tau)\, dVd\tau\right.$$

$$+ \cos \omega_j t \int_V \mathbf{u}_j^T(\mathbf{x})\rho \mathbf{u}_0(\mathbf{x})\, dV$$

$$+ \frac{1}{\omega_j} \sin \omega_j t \int_V \mathbf{u}_j^T(\mathbf{x})\rho \dot{\mathbf{u}}_0(\mathbf{x}) dV\Big]. \quad (17)$$

This equation represents the classical eigenfunction expansion method for a nonhomogeneous initial-boundary value problem.

## DISCRETIZATION

Turning to matrix methods, the structure under consideration is subdivided into $M$ elements $\{\Omega^e\}_{e=1}^M$ with a total of $N$ associated nodal points $\{\mathbf{x}_i\}_{i=1}^N$. The nodal values of the eigenfunction $\mathbf{u}_j$, and possibly its derivatives, are stored in the nodal vector $\mathbf{V}_j$.

The eigenfunction $\mathbf{u}$ can be interpolated exactly by the global frequency-dependent shape function $\bar{\mathbf{N}}$

$$\mathbf{u}(\mathbf{x}) = \bar{\mathbf{N}}(\mathbf{x}, \omega)\mathbf{V} \quad (18)$$

where $\mathbf{V}$ signifies the column of global nodal values of the eigenfunction. These global functions can be written in terms of element shape functions $\mathbf{N}(\mathbf{x}, \omega)$ that are found as solutions to the following boundary value problem

$$\mathbf{D}_s^T \mathbf{E} \mathbf{D}_u \mathbf{N} + \rho\omega^2 \mathbf{N} = \mathbf{0} \quad (19)$$

$$\mathbf{B}_u \mathbf{N} = \mathbf{I}. \quad (20)$$

With the introduction of the shape function, the virtual work principle yields the following nonlinear eigenproblem

$$[\mathbf{K}(\omega) - \omega^2 \mathbf{M}(\omega)]\mathbf{V} = \mathbf{0} \quad (21)$$

for the natural frequencies and mode shapes, where $\mathbf{M}$ is the global mass matrix, and $\mathbf{K}$ is the global stiffness matrix. These global matrices are assembled by a summation process of element matrices $\mathbf{m}$ and $\mathbf{k}$ according to the additive property of the Riemann integral. The element matrices are found according to

$$\mathbf{m} = \int_V \mathbf{N}^T \rho \mathbf{N} dV \quad (22)$$

and

$$\mathbf{k} = \int_V \mathbf{N}^T{}_u \mathbf{D}^T \mathbf{E} \mathbf{D}_u\, \mathbf{N} dV. \quad (23)$$

The element *dynamic stiffness matrix* is then defined as

$$\mathbf{d} = \mathbf{k} - \omega^2 \mathbf{m}. \quad (24)$$

By assuming that the initial displacement and velocity can be interpolated exactly using the static global shape function $\bar{\mathbf{N}}_0(\mathbf{x})$

$$\mathbf{u}_0(\mathbf{x}) = \bar{\mathbf{N}}_0(\mathbf{x})\mathbf{V}_0 \quad (25)$$

$$\dot{\mathbf{u}}_0(\mathbf{x}) = \bar{\mathbf{N}}_0(\mathbf{x})\dot{\mathbf{V}}_0 \quad (26)$$

and that the applied load is of a form

$$\hat{\mathbf{p}}_V(\mathbf{x}, t) = \mathbf{f}(\mathbf{x})\beta(t) \quad (27)$$

with separated variables such that the spatial part can be interpolated exactly using $\bar{\mathbf{N}}_0$

$$\mathbf{f}(\mathbf{x}) = \bar{\mathbf{N}}_0(\mathbf{x})\boldsymbol{\alpha} \quad (28)$$

where $\boldsymbol{\alpha}$ is a vector of nodal values of $\mathbf{f}$, Leung (1983) writes the modal sum of Eq. (17) in the form

$$\mathbf{u}(\mathbf{x}, t) = \sum_{j=1}^{\infty} \bar{N}(\mathbf{x}, \omega_j)\mathbf{V}_j \mathbf{V}_j^T \tilde{\mathbf{M}}(\omega_j)$$

$$\left[\frac{1}{\omega_j} H(\omega_j, t)\boldsymbol{\alpha} + \cos \omega_j t \mathbf{v}_0 + \frac{1}{\omega_j} \sin \omega_j t \dot{\mathbf{v}}_0\right] \quad (29)$$

where

$$H(\omega, t) = \int_0^t \sin \omega(t - \tau)\beta(\tau)d\tau \quad (30)$$

and $\tilde{\mathbf{M}}(\omega)$ is the global quasi-static mass matrix formed by summing element matrices defined by

$$\tilde{\mathbf{m}} = \int_V \mathbf{N}^T \rho \mathbf{N}_0 dV. \quad (31)$$

The eigenvectors $\mathbf{V}_j$ are assumed to have been normalized in accordance with Eq. (15), this being accomplished by dividing each computed eigenvector $\bar{\mathbf{V}}_j$ by the exact modal mass

$$M_j = [\bar{\mathbf{V}}_j^T \mathbf{M}(\omega_j)\bar{\mathbf{V}}_j]^{1/2}. \quad (32)$$

The validity of the dynamic equivalent modal analysis rests on the orthogonality of the computed (normalized) eigenvectors with respect to the global form of the mixed mass matrix defined by Leung (1983)

$$\mathbf{M}(\omega_1, \omega_2) = \int_V \bar{\mathbf{N}}^T(\omega_1)\rho\bar{\mathbf{N}}(\omega_2)dV. \quad (33)$$

These orthogonality relations appear as

$$\mathbf{V}_i^T\mathbf{M}(\omega_i, \omega_j)\mathbf{V}_j = 0 \quad (34)$$

for distinct eigenpairs $i \neq j$.

Another form of modal analysis is now derived as a simple consequence of Eq. (29) by replacing the frequency-dependent quantities $\bar{\mathbf{N}}$ and $\tilde{\mathbf{M}}$ by their corresponding Taylor series expressions

$$\bar{\mathbf{N}} = \sum_{n=0}^{\infty} \bar{\mathbf{N}}_n\, \omega^{2n} \quad (35)$$

$$\tilde{\mathbf{M}} = \sum_{n=0}^{\infty} \tilde{\mathbf{M}}_n\, \omega^{2n}. \quad (36)$$

This method is analogous to the finite dynamic element method used by Przemieniecki (1968), Downs (1986), and others in the analysis of free vibrations. The terms $\bar{\mathbf{N}}_n$ are defined in terms of element shape function terms, calculated according to Eq. (19) by solving the recursive system of differential equations

$$\mathbf{D}_s^T\mathbf{E}\mathbf{D}_u\mathbf{N}_0 = \mathbf{0} \quad (37)$$

$$\mathbf{D}_s^T\mathbf{E}\mathbf{D}_u\mathbf{N}_n + \rho\mathbf{N}_{n-1} = \mathbf{0}, \quad n \geq 1. \quad (38)$$

During the remainder of the discussion, it will be assumed that the eigenfunctions are adequately represented by a form that is quadratic in the frequency-squared, and thus each eigenfunction $\mathbf{u}_j(\mathbf{x})$ can be interpolated between the nodal values, using the first two terms in the Taylor expansion for the shape function $\bar{\mathbf{N}}(\mathbf{x}, \omega)$ by

$$\mathbf{u}_j = (\bar{\mathbf{N}}_0 + \omega_j^2\bar{\mathbf{N}}_1)\mathbf{V}_j \quad (39)$$

where $\langle\omega_j^2, \mathbf{V}_j\rangle$ is the $j$th eigenpair of the quadratic eigenproblem

$$\{\mathbf{K}_0 - \omega^2\mathbf{M}_0 - \omega^4\mathbf{M}_1/2\}\ \mathbf{V} = \mathbf{0} \quad (40)$$

the global matrices $\mathbf{K}_0$, $\mathbf{M}_0$, and $\mathbf{M}_1$ having been assembled from their element counterparts whose Taylor series terms are given by Pilkey and Fergusson (1990) as

$$\mathbf{m}_n = (n + 1) \int_V \mathbf{N}_j^T\rho\mathbf{N}_{n-j}dV, \quad n \geq 0 \quad (41)$$

where $j$ is any integer between 0 and $n$ inclusive, and

$$\mathbf{k}_n = (n - 1) \int_V \mathbf{N}_{iu}^T\mathbf{D}^T\mathbf{E}\ \mathbf{D}_u\mathbf{N}_{n-i}dV, \quad n \geq 2 \quad (42)$$

where $i$ is any integer between 1 and $n - 1$ inclusive. The term $\mathbf{k}_0$ is expressed using the appropriate equation for the static stiffness matrix, and the term $\mathbf{k}_1$ vanishes identically.

The eigenvectors $\mathbf{V}_j$ are assumed to have been normalized in accordance with Eq. (15), this being accomplished by dividing each computed eigenvector $\bar{\mathbf{V}}_j$ by the approximation to the $j$th modal mass given as

$$M_j = [\bar{\mathbf{V}}_j^T(\mathbf{M}_0 + \omega_j^2\mathbf{M}_1)\ \bar{\mathbf{V}}_j]^{1/2}. \quad (43)$$

The insertion of the modes extracted from the quadratic eigenproblem along with the appropriate Taylor series for $\bar{\mathbf{N}}$ and $\tilde{\mathbf{M}}$ yields the modal sum for the displacement

$$\mathbf{u}(\mathbf{x}, t) = \sum_{j=1}^{\infty} (\bar{\mathbf{N}}_0 + \omega_j^2\bar{\mathbf{N}}_1)\mathbf{V}_j^T\mathbf{V}_j(\tilde{\mathbf{M}}_0 + \omega_j^2\tilde{\mathbf{M}}_1 + \omega_j^4\tilde{\mathbf{M}}_2) \left[\frac{1}{\omega_j} H(\omega_j, t)\ \boldsymbol{\alpha} + \cos \omega_j t\ \mathbf{V}_0 + \frac{1}{\omega_j} \sin \omega_j t\ \dot{\mathbf{V}}_0\right]. \quad (44)$$

The fact that three $\tilde{\mathbf{M}}_i$ terms can be obtained from only two $\bar{\mathbf{N}}_i$ terms is evidenced by the above mentioned results given in Eq. (41) and (42), along with the proportionality relationship given by Fergusson and Pilkey (1992) that states that for any $n \geq 1$, the Taylor series terms for the stiffness, mass, quasi-static mass, and dynamic stiffness matrices are related according to

$$(n + 1)\mathbf{k}_{n+1} = n\mathbf{m}_n = n(n + 1)\tilde{\mathbf{m}}_n = -n(n + 1)\mathbf{d}_{n+1}. \quad (45)$$

Because the dynamic shape functions satisfy Boolean type boundary conditions at the nodes, they are not dependent on $\omega_i$ there. The nodal solutions $\mathbf{V}(t)$ can thus be extracted from the expression given in Eq. (44) for the modal sum. The three forms of such a solution, corresponding to the three types of modal analysis (usual modal analysis based on FEM; current modal analysis based on DEM; and dynamic equivalent modal analysis given by Leung, 1983), can be written, respectively, as

$$\mathbf{V}(t) = \sum_{j=1}^{\infty} \mathbf{V}_j V_j^T(\tilde{\mathbf{M}}_0) \left[\frac{1}{\omega_j} H(\omega_j, t)\boldsymbol{\alpha} + \cos \omega_j t \mathbf{V}_0 + \frac{1}{\omega_j} \sin \omega_j t \dot{\mathbf{V}}_0\right] \quad (46)$$

$$\mathbf{V}(t) = \sum_{j=1}^{\infty} \mathbf{V}_j V_j^T(\tilde{\mathbf{M}}_0 + \omega_j^2\tilde{\mathbf{M}}_1 + \omega_j^4\tilde{\mathbf{M}}_2) \left[\frac{1}{\omega_j} H(\omega_j, t)\boldsymbol{\alpha} + \cos \omega_j t \mathbf{V}_0 + \frac{1}{\omega_j} \sin \omega_j t \dot{\mathbf{V}}_0\right] \quad (47)$$

$$\mathbf{V}(t) = \sum_{j=1}^{\infty} \mathbf{V}_j V_j^T(\tilde{\mathbf{M}}(\omega_j)) \left[\frac{1}{\omega_j} H(\omega_j, t)\boldsymbol{\alpha} + \cos \omega_j t \mathbf{V}_0 + \frac{1}{\omega_j} \sin \omega_j t \dot{\mathbf{V}}_0\right] \quad (48)$$

Of course the three types of analysis differ also in the fact that the modes are computed using 1 linear eigenproblem, quadratic eigenproblem, and highly nonlinear eigenproblem, respectively.

## EXAMPLE

Consider as an example, a cantilevered beam with unit material parameters and length, subject to a transverse driving force at the free end of magnitude $1000(t^4 - t^5)$, and zero initial conditions. This is the same beam studied by Leung (1983).

The quantity $\sum \tilde{\mathbf{M}}_i\omega_j^{2i}\boldsymbol{\alpha}$ appearing in Eq. (44) and (47) can be replaced by a column of zeros with a "1000" as the second to last component. The various matrices and state vectors are given by

$$\mathbf{u} = [w] \quad (49)$$

$$\boldsymbol{\epsilon} = [\kappa] \quad (50)$$

$$\mathbf{s} = [M] \quad (51)$$

$$\mathbf{D}_u = [-\partial_x^2] \quad (52)$$

$$\mathbf{D}_s^T = [\partial_x^2] \quad (53)$$

$$\mathbf{E} = [EI] \quad (54)$$

$$\boldsymbol{\rho} = [\rho] \quad (55)$$

with $EI$ and $\rho$ each set equal to unity. The symbols $w$, $\kappa$, and $M$ represent the transverse deflection, curvature, and bending moment, respectively. The element nodal vector takes the form

$$\mathbf{v} = [w(0) \quad w'(0) \quad w(1) \quad w'(1)]^T. \quad (56)$$

The present method using one dynamic correction term to the shape function and two dynamic corrections to the mass matrix has been implemented for the beam example, and the transverse response $w_{\text{calc}}$ of the free end at time $t = 1.0$ s computed. The results are compared to a similar analysis made using the more traditional static equivalent form (FEM) of modal analysis using mass and stiffness matrices derived with static shape functions. For the traditional method only half as many modes as the number of degrees of freedom are considered accurate, whereas all the modes computed by the dynamic element are of sufficient accuracy to be used in the modal sum.

Although a simple iteration procedure was used to calculate the modes, the solution effort can be decreased by employing the methods used by Gupta (1973) for solving the quadratic eigenproblem. Gupta's method enables solution of the quadratic eigenproblem with not much more effort than is required in solving the analogous linear eigenproblem associated with the FEM using the same number of elements.

The percentage relative error in the response

$$e_{\text{rel}} = \frac{\omega_{\text{calc}} - \omega_{\text{exact}}}{\omega_{\text{exact}}} 100 \quad (57)$$

where $\omega_{\text{exact}}$ is the exact deflection of the free end of the beam, is displayed graphically in Fig. 1 for the two methods.

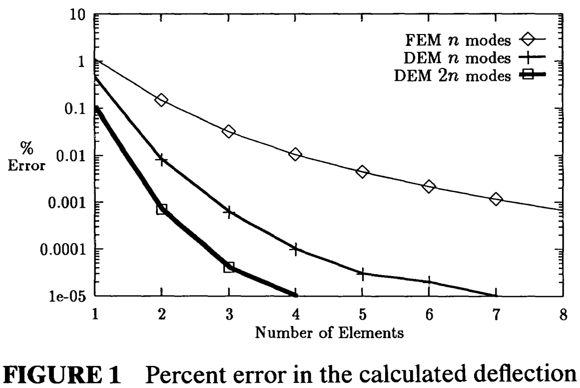


**FIGURE 1** Percent error in the calculated deflection of the free end at time $t = 1.0$ s using available modes.

For the FEM, half of the available modes were included; for the DEM half the modes and all the modes were included, resulting in three curves corresponding to

1. FEM using $n$ modes in the modal sum;
2. DEM using $n$ modes in the modal sum;
3. DEM using $2n$ modes in the modal sum;

where $n$ is the number of elements.

The advantage of the new method is readily apparent upon examination of the graph shown in the figure. To achieve an accuracy of three decimal places in the response, for example, a percentage error of 0.0018 or better is needed. This would require seven elements using the traditional FEM, but only two or three elements using the DEM, depending on whether $n$ or $2n$ modes are used. The results are based on an exact value of $w_{\text{exact}} = 27.137519088$ computed using exact eigenfunctions.

## CONCLUSIONS

The dynamic element method, involving the solution of a quadratic eigenproblem for the (high accuracy) modes of a vibrating structure, has been extended to the calculation of the response via modal analysis. The new method differs from the conventional modal analysis in that the modes employed are found using the high accuracy dynamic element method. In addition, the associated modal masses used to normalize the computed modes are evaluated using a dynamic correction term in the mass matrix.

The results for the transverse vibration of a cantilevered beam show that a much greater accuracy is obtained for the response when the dynamic element is used instead of the usual finite element method.

This work was supported under Grant NAG1745 from the NASA Langley Research Center.

## REFERENCES

Downs, B., 1986, "An Equivalent Inertia Matrix for a Segment of a Uniform Timoshenko Beam of Any Geometry and Material," *Journal of Sound and Vibration,* Vol. 105, pp. 109–120.

Fergusson, N.J., and Pilkey, W.D., 1992, "Frequency-Dependent Element Mass Matrices," *Journal of Applied Mechanics,* Vol. 59, pp. 136–139.

Gupta, K.K., 1973, "Solution of Quadratic Matrix Equations for Free Vibration Analysis of Structures," *International Journal of Numerical Methods,* Vol. 6, pp. 129–135.

Leung, A.Y.T., 1983, "Fast Convergence Modal Analysis for Continuous Systems," *Journal of Sound and Vibration,* Vol. 87, pp. 449–467.

Pilkey, W.D., and Fergusson, N.J., 1990, "Some Characteristics of Frequency-Dependent Structural Matrices," *Computers and Structures,* Vol. 35, pp. 413–416.

Przemieniecki, J.S., 1968, *Theory of Matrix Structural Analysis,* McGraw–Hill, New York.