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Eigensolution Derivatives for Arbitrarily Normalized Modes

Methods for computing eigenvector derivatives for arbitrarily normalized modes are derived. Formulations are presented for modal methods, Nelson's method, and a method for computing eigenvalue and eigenvector derivatives simultaneously. A simple numerical example is provided to illustrate various formulations.

INTRODUCTION

The derivatives of eigenvalue and eigenvectors with respect to design parameters are essential in design sensitivity analysis, structural optimization, and test/analysis correlation studies. Methods of computing eigensolution derivatives have been reported by many researchers. These methods have a common feature: they all use modes normalized to unit generalized mass. In this article, methods for computing eigensolution derivatives for arbitrarily normalized modes are presented. The relationship of eigenvector derivatives computed by various normalizations were studied. We present a simple numerical example to illustrate the various formulations.

The eigenvalue problem for undamped systems in structural dynamics is

$$K\phi = \lambda M\phi \tag{1}$$

where K and M are stiffness mass matrices, λ is an eigenvalue, and ϕ is the associated eigenvector. Assume λ is not a repeated eigenvalue. Let $\tilde{\phi}$ be mass-normalized, and ϕ be arbitrarily normalized. That is

$$\tilde{\phi}^T M\tilde{\phi} = 1 \tag{2}$$

and

$$P^T\phi = c. \tag{3}$$

Note that the normalization scheme presented in Eq. (3) is very general. For example, if the eigenvector is normalized so that the j^{th} component is 1, then

$$P = e_j = \text{a unit vector} \tag{4a}$$

and

$$c = 1. \tag{4b}$$

If the eigenvector is normalized to unit length then

$$P = \phi \tag{5a}$$

and

$$c = 1. \tag{5b}$$

In the literature, methods for eigensolution derivatives are derived based on Eqs. (1) and (2). In this article, formulations are based on Eqs. (1) and (3).

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BASIC EQUATIONS FOR EIGENSOLUTION DERIVATIVES

Differentiating Eqs. (1) and (3) with respect to a design variable yields

$$(K - \lambda M)\phi' + (K' - \lambda' M - \lambda M')\phi = 0 \quad (6)$$

$$P^T\phi' + P'^T\phi = 0 \quad (7)$$

where a prime in the above equations indicates differentiation with respect to a design variable. Equations (6) and (7) are basic equations for computing eigensolution derivatives. The formula for eigenvalue derivative can be derived by premultiplying Eq. (6) by ϕ^T. This yield, after some simplification,

$$\lambda' = \phi^T(K' - \lambda M')\phi/(\phi^T M\phi). \quad (8)$$

This is a well known result.

METHODS OF COMPUTING EIGENVECTOR DERIVATIVES

Several methods for computing eigenvector derivatives of arbitrarily normalized modes include the mode superposition method, improved modal method, Nelson's method, and simultaneous method.

Mode Superposition Method

Following the procedures of Fox and Kapoor (1968), 1, the eigenvector derivative for a system with n degrees of freedom (DOF) can be computed by the following equation.

$$\phi_l' = \sum_{j=1}^{n} c_j\phi_j \quad (9)$$

where

$$c_i = \phi_i^T F_l / G_l(\lambda_i - \lambda_l) \quad \text{for} \quad i = 1, \ldots, n, i \neq l \quad (11)$$

$$F_l = (K' - \lambda_l' M - \lambda_l M')\phi_l \quad (12a)$$

$$G_l = \phi_l^T M\phi_l \quad (12b)$$

and

$$c_l = -\sum_{\substack{j=1 \\ j\neq L}}^{n} c_j(P^T\phi_j)/(P^T\phi_l). \quad (13)$$

Approximate Mode-Superposition Method

If there are only L modes available, $L < n$, then an approximation of ϕ_l' can be computed by changing the summation limit in Eq. (9) from n to L, that is

$$\phi_{la} = \sum_{j=1}^{L} c_j\phi_j \quad (14)$$

where c_j is given by Eq. (11) and c_l is given by Eq. (13) with the upper limit of summation changed to L. The approximate eigenvector derivative may not be accurate enough in application. Its accuracy can be improved by a mode-acceleration approach. Using the procedures of Wang (1991), the following improved approximation to eigenvector derivatives are obtained.

$$\phi_{ll_1}' = \phi_{lL}' + c_{l1}\phi_l + Y_l \quad (15)$$

where

$$\phi_{lL}' = \sum_{\substack{j=1 \\ j\neq l}}^{L} c_j\phi_j \quad (16a)$$

$$Y_l = K^{-1}F_l - \sum_{j=1}^{L} \frac{\phi_j^T F_l}{G_j\lambda_j}\phi_j \quad (16b)$$

$$c_j = \text{Eq. (11)} \quad \text{for} \quad j \neq l$$

$$c_{l1} = -P^T(\phi_{lL}' + Y_l)/(P^T\phi_l). \quad (16c)$$

Equation (15) is the explicit formulation of Wang (1991). Further improvement can be obtained by using implicit formulation from the same reference:

$$\phi_{ll_2}' = c_{l2}\phi_l + \phi_{lL}' + c_y Y \quad (17a)$$

where ϕ_{lL}' is given in Eq. (16a) and

$$c_y = Y^T F_l/(Y^T K Y - \lambda_l Y^T M Y) \quad (17b)$$

$$c_{l2} = -P^T(\phi_{lL}' + c_y Y)/(P^T\phi_l). \quad (17c)$$

Nelson's Method

The modal methods require all modes for an exact solution. Alternatively, the popular Nelson method (1976) requires only the mode itself for computing derivatives. In the traditional Nelson method this is achieved by computing a particular solution of eigenvector derivative by introducing a constraint in Eq. (6). The complete solution can then be calculated using the mass normalization condition. We will derive similar solution procedures for arbitrarily normalized modes. Rewrite Eq. (6) for the lth mode

$$Z_l\phi_l' = F_l \tag{18}$$

where F_l is given by Eq. (11) and

$$Z_l = K_l - \lambda_l M. \tag{19}$$

Next, our development of a Nelson type approach depends on the normalization scheme used

Max-Normalized Modes. For max-normalized modes

$$\phi_{jl}' = 0 \tag{20}$$

because $\phi_{jl} = 1$ by definition.

The eigenvector derivative can then be calculated from equations (18) and (20). Let

$$\phi_l' = T\mathbf{V} \tag{21}$$

where $\mathbf{V}$ is an $(n - 1) \times 1$ vector and T is an $n \times (n - 1)$ matrix constructed by deleting the jth column of an $n \times n$ identity matrix. The vector $\mathbf{V}$ can be computed from

$$Z_l^*\mathbf{V} = F_l^* \tag{22}$$

where

$$Z_l^* = T^T Z_l T \tag{23a}$$

$$F_l^* = T^T F_l \tag{23b}$$

Arbitrarily Normalized Modes. For this case, let

$$\phi_l' = T\omega + c_l\phi_l \tag{24}$$

where ω is an $(n - 1) \times 1$ vector. T is formed by deleting a column in the $n \times n$ identity matrix. Substituting Eq. (24) into Eq. (18), we can compute ω from

$$Z_l^*\omega = F_l^* \tag{25}$$

where Z_l^*, F_l^* have the same form as given in Eqs. (23a, b). Substituting Eq. (24) into (7) yields

$$c_l = P^T T\omega/(P^T\phi_l). \tag{26}$$

In summary, Eqs. (24)–(26) constitute the procedures for computing derivatives of arbitrarily normalized modes when P is a constant matrix. If P is a function of the design variable, then it can be shown that

$$c_l = -(P^T T\omega + P'^T\phi_l)/(P^T\phi_l) \tag{27}$$

Simultaneous Computing of Eigenvalue and Eigenvector Derivatives

Another method for computing ϕ_l', using only ϕ_l, is to compute it with λ_l' at the same time, Murthy and Haftke (1988). This is achieved by putting Eqs. (6) and (7) together and treating ϕ_l' and λ_l' as variables, that is

$$Ax = b \tag{28}$$

where

$$x = \begin{Bmatrix} \phi_l' \\ \lambda_l' \end{Bmatrix} \tag{29a}$$

and

$$A = \begin{bmatrix} Z_l & -M\phi_l \\ P^T & 0 \end{bmatrix} \tag{29b}$$

$$b = \begin{bmatrix} -(K' - \lambda M')\phi_l \\ -P'^T\phi_l \end{bmatrix}. \tag{29c}$$

Note that in the above formulation, we have assumed that P is an explicit function of the design variables. The advantage of this formulation is that ϕ_l' and λ_l' are computed simultaneously.

RELATIONSHIP OF EIGENVECTOR DERIVATIVES

Let $\tilde{\phi}_l$ and ϕ_l be a mass-normalized mode and an arbitrarily normalized mode, respectively. Then by definition

$$\tilde{\phi}_l = \phi_l/\sqrt{G_l} \tag{30}$$

Table 1. First Eigenvector With Different Normalization

Mass	Max	Length
0.4248	0.8261	0.5207
0.4697	0.9134	0.5757
0.5142	1.000	0.6304

where

$$G_l = \phi_l^T M \phi_l. \tag{31}$$

It follows that

$$\tilde{\phi}_l' = (G_l \phi_l' - G_l' \phi_l/2)/G_l^{3/2} \tag{32}$$

where

$$G_l' = 2\phi_l^T M \phi_l' + \phi_l^T M' \phi_l. \tag{33}$$

Thus, given ϕ_l', $\tilde{\phi}_l'$ can be computed from Eqs. (31) and (33). Alternatively,

$$\phi_l = \tilde{\phi}_l / r \tag{34}$$

where

$$\begin{aligned} r &= \max(\phi_{jl}) \\ &= \text{maximum elements in } \tilde{\phi}_l. \end{aligned} \tag{35}$$

Assume $r > 0$. Then

$$\phi_l' = (r\tilde{\phi}_l' - \tilde{\phi}_l r')/r^2 \tag{36}$$

where

$$r' = \tilde{\phi}_{jl}'. \tag{37}$$

Equation (36) relates ϕ_l' to $\tilde{\phi}_l$.

Numerical Example

A simple 3-DOF system is used to illustrate the above formulation. Consider a 3-DOF spring-mass system with the following mass and stiffness matrices

$$M = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.9 + 0.1\alpha \end{bmatrix}$$

$$K = \begin{bmatrix} 5+\alpha & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix}.$$

The eigensolution sensitivities at $\alpha = 0$ are to be computed. For this case,

$$M' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

$$K' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues for the system are: 0.1925, 2.0000, and 7.6964. The corresponding mass-normalized eigenvectors are:

$$\tilde{\phi} = \begin{bmatrix} 0.4248 & 0.3393 & 0.1943 \\ 0.4697 & -0.0848 & -8787 \\ 0.5142 & -0.8482 & 0.3567 \end{bmatrix}.$$

Table 1 summarizes the first mode with three different normalization methods. The corresponding eigenvector sensitivities are given in Table 2. Using the relationship between $\tilde{\phi}_l'$ and ϕ_l, and the same $\tilde{\phi}_l'$ can be computed from $\phi_{l\max}$ and $\phi_{l,\text{length}}'$.

Note that Table 2 shows very different eigenvector derivatives for modes normalized by different methods. However, when these derivatives are used to study the change of eigenvector

Table 2. Derivatives of First Eigenvector With Different Normalization

Mass	Max	Length
−0.0363	−0.1686	−0.0607
0.0102	−0.0885	−0.0054
0.0610	0	0.0551

Table 3. Comparison of Eigenvectors for $\alpha - 0.1$

Mass	Max	Length
0.4212	0.8093	0.5147
0.4707	0.9045	0.5752
0.5203	1.0000	0.6359

Values were predicted by eigenvector derivatives using different modes.

Table 4. Comparison of Eigenvectors for α – 0.1

Exact	Mass	Max	Length
0.8093	0.8095	0.8093	0.8094
0.9045	0.9046	0.9045	0.9046
1.0000	1.0000	1.0000	1.0000

Values were predicted by different eigenvector derivatives.

with respect to design change the results are independent of normalization schemes. For example, let us predict the new eigenvector for the system when $\alpha = 0.1$. The predict eigenvectors for various normalizations are given in Table 3. After max-normalization, these modes become those shown in Table 4. Also shown in Table 4 are the exact mode shape of the modified system. It is clear that all eigenvector derivatives provided the same results.

CONCLUSIONS

Methods for computing eigensolution derivatives for arbitrarily normalized modes are similar to the conventional eigenvector derivatives based on mass-normalized modes. The relationship between these derivatives are presented. Because in practice the computed modes are often max-normalized, the proposed formulation can be used to compute the required derivatives. Additionally, from their relationships, we can convert eigenvector derivatives for modes normalized by different methods. It should be noted that in Haftka, Gurdal, and Kamat (1989), methods for computing eigenvector derivatives are presented for simultaneous methods for mode shapes normalized with a general symmetric weight matrix W such that $\phi^T W \phi = 1$. The contribution of this study is to present a direct formulation of computing eigenvector derivatives for various methods, including an improved modal method, for arbitrarily normalized modes.

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