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Free and Forced Vibrations of Periodic Multispan Beams

In this study, the following two topics are considered for uniform multispan beams of both finite and infinite lengths with rigid transversal and elastic rotational constraints at each support: (a) free vibration and the associated frequencies and mode shapes; (b) forced vibration under a convected harmonic loading. The concept of wave propagation in periodic structures of Brillouin is utilized to investigate the wave motion at periodic supports of a multispan beam. A dispersion equation and its asymptotic form is obtained to determine the natural frequencies. For the special case of zero rotational spring stiffness, an explicit asymptotic expression for the natural frequency is also given. New expressions for the mode shapes are obtained in the complex form for multispan beams of both finite and infinite lengths. The orthogonality conditions of the mode shapes for two cases are formulated. The exact responses of both finite and infinite span beams under a convected harmonic loading are obtained. Thus, the position and the value of each peak in the harmonic response function can be determined precisely, as well as the occurrence of the so-called coincidence phenomenon, when the response is greatly enhanced.

INTRODUCTION

The model of a periodic multispan beam with elastic supports is often utilized in engineering. For example, such a model is a reasonable approximation for a plate-like structure with parallel, regularly spaced stiffeners. The elastic supports may provide both the rotational and transversal restraints to the beam. Krein (1933) and Miles (1956) independently studied an N-span beam by using a finite difference approach, and established that the natural frequencies fell into distinct bands with the same number of natural frequencies in each band as the number of spans. Lin (1962) generalized the finite difference approach to multispan beams with elastic supports. Abramovich and Elishakoff (1987) generalized the Krein and Miles analyses to multispan Timoshenko beams, taking into account shear deformation and rotary inertia. However, use of the finite difference approach may lead to extreme computational efforts and even inaccuracy in the determination of the mode shapes of the system.

Lin and McDaniel (1969) also used a transfer matrix formulation that is more convenient for the imposition of constraints at the supports. However, in practice, numerical difficulty may arise when the number of periodic units in a structure is large. To overcome this difficulty, Yong and Lin (1989) and Cai and Lin (1991) transformed the state vector of displacements and forces into the wave vector of incoming and outgoing waves, and correspondingly transformed the transfer matrix into the wave-scattering matrix. By so doing, the computational efficiency and accuracy are greatly improved, especially when obtaining the dynamic response of a periodic structure due to point excitation because the calculation is channeled in the direction of wave propagation.

Mead (1970) made use of the concept of wave

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propagation in periodic structures originally due to Brillouin (1953) to analyze the free vibration of a multispan beam of infinite length. Sen Gupta (1970) extended the analysis to finite multispan beams and plates on rigid supports. In these studies, the wave propagation band and nonpropagation band were studied in much detail. Sen Gupta (1970) also proposed a graphic method to determine the natural frequencies of the multispan beams with rigid supports. In the framework of wave propagation, *nonharmonic* waves have to be decomposed into an infinite number of *harmonic* components in order to carry out the analysis of the forced vibration. This approach was used by Mead (1971) and Lin, Maekawa, Nijim, and Maestrello (1977) to obtain the response of an infinitely long multispan beam to harmonic excitation, as well as boundary-layer pressure fields. In the actual calculation the infinite sum has to be truncated, and a large system of linear equations have to be solved numerically to determine the unknown coefficients.

It should be noted that in the forced vibration analysis of a periodic multispan beam of finite length, multiple peaks occur in each wave propagation band. The number of peaks in each band is equal to the number of the spans. The larger the number of the span is, the higher the distributed density of peaks will be in the distinct propagation band. However, the approaches mentioned above are still associated with a lot of computational effort, and they may also lead to inaccuracy in the position of the peaks as well as the value of each peak.

In this study, new expressions are proposed for the mode shapes of a periodic multispan beam, based on the wave propagation concept, that can then be used in the forced vibration analysis. Because the transverse displacement within each span of the beam is related uniquely to the displacements at the two ends of the span, we may focus our attention only on the waves that propagate through each periodic support. Once these waves are determined, the motions at all periodic supports and in all span of beams become known. The dispersion equation that establishes the relationship between wave constant and frequency parameter is derived accordingly. The frequency parameters, wave constants, and associated mode shapes for beams of both finite and infinite length can then be determined. The exact responses of multispan beams of both finite and infinite lengths to a convected loading are obtained. Furthermore, the locations of response peaks and their values can be precisely calculated, and the condition for the so-called *coincident phenomenon* can be predicted in exact terms.

FREE VIBRATION ANALYSIS

Basic Equations

Consider an N-span beam with uniformly spaced supports. It is convenient to write the equation of motion in terms of the local nondimensional coordinate ξ as follows:

$$EIw_\beta^{(4)}(\xi, t) + \rho AL^4\ddot{w}_\beta(\xi, t) = 0, \quad (\beta = 1, 2, \ldots, N) \tag{1}$$

where $w_\beta(\xi, t)$ is the transverse displacement in the βth span, and the local coordinate ξ is defined as

$$\xi = x/L - (\beta - 1), \quad (\beta - 1)L \le x \le \beta L, \quad 0 \le \xi \le 1 \tag{2}$$

in which x is global coordinate and L is the span length. Assuming that the motion is harmonic

$$w_\beta(\xi, t) = W_\beta(\xi)e^{i\omega t}, \quad (\beta = 1, 2, \ldots, N) \tag{3}$$

where $W_\beta(\xi)$ is the mode shape function associated with the βth span, Eq. (1) can be reduced to

$$W_\beta^{(4)}(\xi) - \lambda^4 W_\beta(\xi) = 0 \tag{4}$$

where

$$\lambda = \left(\frac{\rho A\omega^2L^4}{EI}\right)^{1/4} \tag{5}$$

is a nondimensional frequency parameter, and ω is the sought angular frequency.

It is assumed that each interior support provides a rigid constraint against transverse motion, as well as an elastic constraint against rotation, with a spring constant k (see Fig. 1). Thus the continuity conditions at each interior support are as follows:

$$\begin{aligned} W_\beta(1) &= W_{\beta+1}(0) = 0 \\ W'_\beta(1) &= W'_{\beta+1}(0) \\ \nu W'_\beta(1) &= W''_{\beta+1}(0) - W''_\beta(1), \\ &(\beta = 1, 2, \ldots, N - 1) \end{aligned} \tag{6}$$

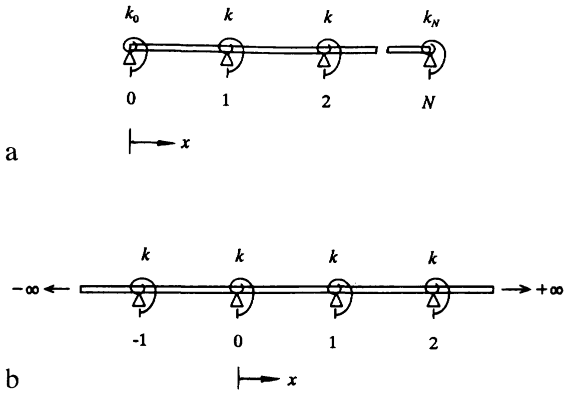

FIGURE 1 Multi-span beams with elastic rotational spring at each support: (a) finite length; (b) infinite length.

where $\nu = kL/EI$. The first two conditions in Eq. (6) represent, respectively, the continuity of vertical and angular displacements. The last condition in Eq. (6) is the requirement of moment equilibrium at each interior support. The conditions at the two end-supports for a multispan beam of finite length will be specified later.

The mode shape that satisfies the first condition in Eq. (6) can be written as

$$W_\beta(\xi) = A_\beta f(\xi, \lambda) + B_\beta f(1 - \xi, \lambda), \quad (\beta = 1, 2, \ldots, N) \tag{7}$$

where A_β and B_β are unknowns. Only the ratio A_β/B_β is of interest in the free vibration case. The function $f(\xi, \lambda)$ is given by

$$f(\xi, \lambda) = \sin(\lambda\xi) - \frac{\sin(\lambda)}{\sinh(\lambda)} \sinh(\lambda\xi). \tag{8}$$

Harmonic Waves and Associated Wave Constants

A simple wave of spatial sinusoidal variation cannot propagate along a multispan beam, due to reflection at each support, giving rise to hyperbolic terms in the expression for the displacement. However, the concept of wave propagation can still be applied in the case of a periodically supported beam, by focusing our attention on the waves that propagate through each support. The motion of a beam segment between two consecutive supports can then be determined from those of the two supports, if so desired. Because all the supports are assumed to be transversely rigid, only the angular displacement at each support needs to be considered. Let $\theta_\beta(t)$ be the angular displacement at the βth support, and be represented in the form of a harmonic wave propagating through the βth support, that is,

$$\begin{aligned} \theta_\beta(t) &= C_\mu e^{i(\omega t - \mu\beta)} = \Theta_\beta e^{i\omega t} \\ \Theta_\beta &= C_\mu e^{-i\mu\beta}, \quad (\beta = 0, 1, \ldots, N) \end{aligned} \tag{9}$$

where the nondimensional parameter μ is known as *wave constant*, and C_μ is the amplitude of the propagating wave associated with the wave constant μ. A positive μ corresponds to a wave propagating in the positive x-direction, whereas a negative μ corresponds to the negative x-direction. The angular displacement function Θ_β is related to the mode shape function $W_\beta(\xi)$ as follows

$$\begin{aligned} W'_\beta(\xi)|_{\xi=0} &= \Theta_{\beta-1} L \\ W'_\beta(\xi)|_{\xi=1} &= \Theta_\beta L, \quad (\beta = 1, 2, \ldots, N) \end{aligned} \tag{10}$$

obtained from the second condition in Eq. (6).

The ratio A_β/B_β in Eq. (7) will be determined for two special cases: the first case is associated with those mode shapes that are either symmetric or antisymmetric with respect to the midpoint of each span. In such a case, the angular displacements at the two ends of a span are related as

$$\Theta_{\beta-1} = (-1)^s \Theta_\beta, \quad s = \text{integer of } \left[\frac{\lambda}{\pi}\right] \tag{11}$$

where s = odd and s = even correspond to the symmetric and antisymmetric mode shapes, respectively. In view of Eq. (9), the value of the wave constant for this case must be $\mu = m\pi$ implying a *nonpropagating wave* or *standing wave*. The ratio A_β/B_β is obtained by substituting Eqs. (7) and (11) into Eq. (10) to yield

$$\frac{A_\beta}{B_\beta} = (-1)^{s+1}. \tag{12}$$

The second case is associated with those mode shapes that are neither symmetric nor antisymmetric with respect to the midpoint of each span. Therefore, the wave constant μ is not an integer multiple of π, that is, $\mu \neq m\pi$. This implies that there is indeed wave propagation through each periodic support of multispan beam. By applying Eqs. (7), (9), and (10), we find that A_β and B_β are given by

$$A_\beta = L[-f'(0, \lambda)\Theta_{\beta-1} + f'(1, \lambda)\Theta_\beta]/\Delta$$
$$B_\beta = L[-f'(1, \lambda)\Theta_{\beta-1} + f'(0, \lambda)\Theta_\beta]/\Delta \quad (13)$$
$$\Delta = [f'(1, \lambda)]^2 - [f'(0, \lambda)]^2 \neq 0.$$

We note in passing that the first case corresponds precisely to $\Delta = 0$, associated with the same mode shape $W_\beta(\xi)$ as that of a single-span beam with either two simply supported or two fully clamped ends. Return now to the second case, and substitute Eq. (9) into Eq. (13) to obtain

$$\frac{A_\beta}{B_\beta} = -\frac{\eta}{\eta^* e^{-i\mu}}, \quad (14)$$

where the η is given by

$$\eta = f'(1, \lambda)e^{-i\mu} - f'(0, \lambda). \quad (15)$$

It is noted that the ratio A_β/B_β is independent of the span number β for both two cases. This implies that one can choose a mode shape from any span as a reference, then the mode shape for the next span can be obtained by multiplying a phase constant. Thus, a general expression for the mode shape of a multispan beam may be written as follows

$$W_\beta(\xi, \mu, \lambda) = W_1(\xi, \mu, \lambda)e^{-i\mu(\beta-1)} = [af(\xi, \lambda) + bf(1 - \xi, \lambda)]e^{-i\mu(\beta-1)}, \quad (\beta = 1, 2, \ldots, N) \quad (16)$$

which is dependent on the span number β, the local coordinate ξ, the wave constant μ, and frequency parameter λ. Here, a and b are obtained from Eqs. (12) and (14)

$$a = \begin{cases} \eta, & \mu \neq m\pi \\ 1, & \mu = m\pi \end{cases}; \quad b = \begin{cases} -e^{i\mu}\eta^*, & \mu \neq m\pi \\ (-1)^{s+1}, & \mu = m\pi \end{cases}. \quad (17)$$

Coefficients a and b are generally complex and function $f(\cdot)$ is real. Moreover, the span number β in Eq. (16) appears only in the exponential function. It will be shown later that this characteristic of mode shape is very useful, and it will be applied in the analysis of forced vibrations.

Substituting Eq. (16) into the last condition in Eq. (6), the bending moment equilibrium, we obtain a dispersion relationship between μ and λ as follows:

$$\cos(\mu) = F(\lambda), \quad (18)$$

where

$$F(\lambda) = \frac{f'(1, \lambda)}{f'(0, \lambda)} + \nu \frac{f'^2(1, \lambda) - f'^2(0, \lambda)}{2f''(1, \lambda)f'(0, \lambda)}. \quad (19)$$

Equation (18) shows that the values of μ and λ must satisfy a certain relationship for the wave propagation.

To examine the physical meaning of the dispersion equation, function $F(\lambda)$ is plotted in Fig. 2(a,b). It is seen that $F(\lambda)$ has an oscillatory character; thus, each μ value corresponds to multiple values of λ. For the $F(\lambda)$ values between $+1$ and -1, the corresponding wave constants μ are real. This implies that there exists a nonzero phase difference between the motion in adjacent spans, and that the wave is propagating and the wave energy is being transferred from span to span without decay. The associated frequencies are grouped in distinctive bands, called the *propagation bands*. On the other hand, if the absolute values of $F(\lambda)$ are greater than 1, then μ is purely imaginary, indicating an exponential decay of wave motion from span to span. The correspond-

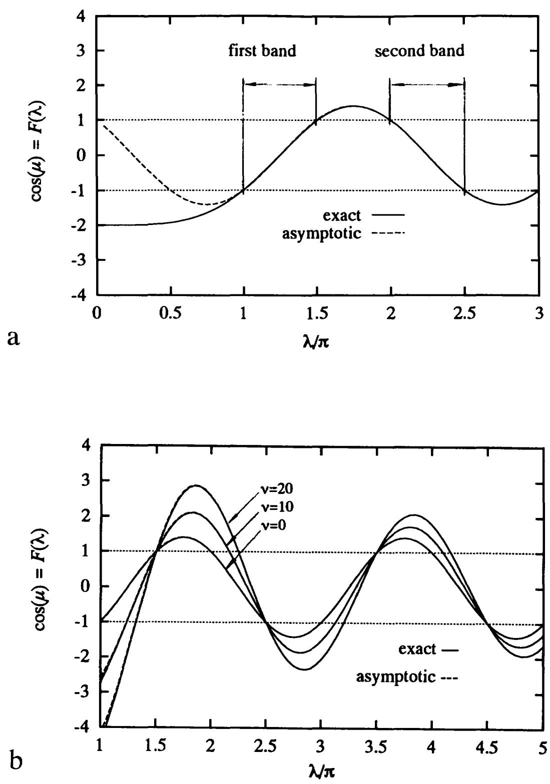

FIGURE 2 Dispersion relationships (a) $\nu = 0$; (b) $\nu = 0, 10, 20$.

ing frequencies are also grouped in distinctive bands, called the *nonpropagation bands*.

As shown in Fig. 2(a,b), the wave constant μ corresponding to the bounding frequencies of a propagation band must be an integer multiples of π. At such a frequency, the motion of a multispan beam reduces to a standing wave the same as that of a single-span beam with symmetric boundary conditions at the ends. The lower bounding frequency of the sth propagation band is the same as the sth natural frequency of a single-span beam with elastic rotational springs at the ends, whereas the upper bounding frequency coincides with the sth natural frequency of a single span with fully clamped ends.

It should be noted that if μ is replaced by

$$\mu_m = \mu + 2\pi m, \quad (m = \pm 1, \pm 2, \ldots) \tag{20}$$

the dispersion equation, Eq. (18), remains unchanged. Thus the state of vibration of the system corresponding to a wave constant μ will be identical to the state corresponding to the other wave constant, namely $\mu + 2\pi m$. Therefore, if we want to have the one-to-one correspondence between the state of vibration of a system and the wave constant μ, the latter must be confined to a range of values of width 2π. The range of μ values satisfying

$$\begin{aligned} &(m-1)\pi < \mu \leq m\pi \\ &-m\pi < \mu \leq -(m-1)\pi, \quad (m = 1, 2, 3, \ldots) \end{aligned} \tag{21}$$

is known as the mth Brillouin (1953) zone. For structural systems, we may restrict to the first Brillouin zone ($m = 1$) without loss of generality, that is,

$$-\pi < \mu \leq \pi. \tag{22}$$

We reiterate that a positive μ corresponds to wave propagation in the positive x-direction and a negative μ corresponds to the negative x-direction.

Asymptotic Dispersion Relations and Natural Frequencies

As seen in Eq. (18), the frequency parameter λ is a multivalued function of μ. Let $\lambda_s(\mu)$ denote the λ value in the sth propagation band. For a large value of $\lambda_s(\mu)$, the following asymptotic approximation is sufficiently accurate

$$F(\lambda) \approx \left(1 + \frac{\nu}{2\lambda}\right)\cos(\lambda) - \sin(\lambda), \quad \text{for} \quad \lambda \geq \pi, \tag{23}$$

which is obtained from Eq. (19) by letting $\tanh(x) \approx 1$ and $\sinh^{-1}(x) \approx 0$. This approximation is compatible with Bolotin's dynamic edge effect method [see Bolotin (1961), Elishakoff (1976)], and is remarkably accurate as shown in Fig. 2(a,b). Moreover, as the rotational spring stiffness ν increases, the position of the lower bounding frequency of each propagation band moves toward the upper bounding frequency that is fixed. This implies that the multispan beam structure becomes more rigid with larger ν, as expected. In the case of $\nu = 0$, the explicit asymptotic expression for the natural frequencies are obtained by combining Eq. (18) and Eq. (23)

$$\lambda_s(\mu) \approx \left(s - \frac{1}{4}\right)\pi + \cos^{-1}\left[(-1)^s \frac{\cos(\mu)}{\sqrt{2}}\right], \quad (s = 1, 2, \ldots, \infty) \tag{24}$$

where s denotes the serial number of the propagation band. Thus, the natural frequencies of a multispan bean can be determined straightforwardly from a given wave constant μ that specific values depends on the exterior boundary conditions of the entire system.

Mode Shapes of a Multispan Beam

It should be recalled that only the boundary conditions at interior supports were used in obtaining an expression for $W_\beta(\xi, \mu, \lambda)$. This implies that the mode shape given in Eq. (16) is valid only for a multispan beam of infinite length. For a finitely long multispan beam, wave reflections occur at two exterior boundaries. Therefore, wave propagating in both positive and negative directions should be included in the analysis. The total angular displacement at the βth support is now given by

$$\begin{aligned} \overline{\Theta}_\beta(\mu) &= \Theta_\beta(\mu) + \Theta_\beta(-\mu) \\ &= C_\mu e^{-i\mu\beta} + C_{-\mu} e^{i\mu\beta}, \\ &\qquad (\beta = 1, 2, \ldots, N) \end{aligned} \tag{25}$$

where the positive and negative subscripts denote two directions of wave propagation. Hence, the associated mode shape for a finite multispan beam becomes

$$\overline{W}_\beta(\xi, \mu, \lambda) = C'_\mu W_\beta(\xi, \mu, \lambda) + C'_{-\mu} W_\beta(\xi, -\mu, \lambda) \quad (26)$$

where the μ value will be chosen in the first Brillouin zone as defined by Eq. (22), and chosen to be positive without loss of generality.

For an infinitely long multispan beam, the wave constant μ varies continuously over the entire zone defined by Eq. (22). The associated frequency parameter λ also varies continuously over the entire propagation band. For a finitely long multispan beam, however, the wave constant μ and the associated frequency parameter λ take on discrete values. The number of the discrete values μ and λ in each propagation band is the same as the number of spans. These discrete wave constants are determined by imposing the boundary conditions at the exterior ends of the entire beam. Referring to Fig. 1, the boundary conditions at the exterior ends of the beam are

$$\begin{aligned} \nu_0 \overline{W}'_1(0, \mu, \lambda) &= \overline{W}''_1(0, \mu, \lambda), \\ \nu_N \overline{W}'_N(1, \mu, \lambda) &= -\overline{W}''_N(1, \mu, \lambda), \\ \nu_0 = \frac{k_0 L}{EI}, \quad \nu_N &= \frac{k_N L}{EI} \end{aligned} \quad (27)$$

where ν_0 and ν_N are the nondimensionalized rotational spring constants at the left and right ends of the N multispan beam, respectively. A vanishing ν corresponds to a simple support, and an infinite ν to a clamped support.

The mode shape of a finitely long multispan beam can be rewritten in abbreviation as follows

$$\begin{aligned} \overline{W}_{\beta,j}(\xi) &= \overline{W}_\beta(\xi, \mu_j, \lambda_j) \\ &= \bar{a}_{\beta,j} f_j(\xi) + \bar{b}_{\beta,j} f_j(1 - \xi), \end{aligned} \quad (28)$$

where

$$\begin{aligned} \bar{a}_{\beta,j} &= \Lambda_j[a_j e^{-i\mu_j(\beta-1)} + \Gamma_j a_j^* e^{i\mu_j(\beta-1)}], \\ \bar{b}_{\beta,j} &= \Lambda_j[b_j e^{-i\mu_j(\beta-1)} + \Gamma_j b_j^* e^{i\mu_j(\beta-1)}], \end{aligned} \quad (29)$$

and where the subscript β denotes the βth span, μ_j is the wave constant corresponding to λ_j, and Λ_j and Γ_j are the unknown constants to be determined by imposing the boundary conditions at the exterior ends.

Examples

The wave constant μ, frequency parameter λ, and mode shapes for an N-span beam will be evaluated in detail for the following three cases.

Case $\nu_0 = \nu_N = \nu/2$. In this particular case, which was first investigated by Lin (1962) using a finite difference approach, the rotational spring constants at both ends of the multispan beam are equal to one-half of that at the interior supports. The boundary conditions at the exterior ends are

$$\begin{aligned} \frac{\nu}{2} \overline{W}'_1(0, \mu, \lambda) &= \overline{W}''_1(0, \mu, \lambda), \\ \frac{\nu}{2} \overline{W}'_N(1, \mu, \lambda) &= -\overline{W}''_N(1, \mu, \lambda). \end{aligned} \quad (30)$$

Using Eqs. (18), (26), and (30), we obtain, after some algebra, an equation for μ as follows:

$$\sin(\mu N) = 0. \quad (31)$$

The possible values of μ in the first Brillouin zone are

$$\mu_j = \frac{j}{N}\pi, \quad (j = 0, 1, 2, \ldots, N). \quad (32)$$

As seen in Fig. 2, the values $\mu = 0$ and $\mu = \pi$ are associated with the bounding frequencies of the propagation bands. In the case of an odd-numbered propagation band, μ is equal to zero at the upper bound and to π at the lower bound. The opposite is true for an even-numbered propagation band. Moreover, the lower and upper bounding frequencies are the same as a single-span beam with elastic supports of rotational spring constant of value $\nu/2$, and with fully clamped supports, respectively. For this case of finite values of ν_0 and ν_N, the value of μ to be either π for the odd-numbered propagation band or zero for the even-numbered propagation band should correspond to the lower bound frequency in each propagation band. Due to the above reasoning, Eq. (32) for μ is modified to read

$$\begin{aligned} \mu_{j=(s-1)N+r} &= \left\{\frac{1}{2}[1 - (-1)^s] + (-1)^s \frac{r-1}{N}\right\} \pi, \\ (s = 1, 2, \ldots, \infty, \quad & r = 1, 2, \ldots, N) \end{aligned} \quad (33)$$

where the subscripts s and r denote, respectively, the sth propagation band and the rth frequency within each band. Then the frequency parameters $\lambda_{j=(s-1)N+r}$ will be numbered in an increasing order of j.

The mode shape $\overline{W}_{\beta,j}(\xi)$ should be taken as the real part of $W_\beta(\xi, \mu_j, \lambda_j)$. Therefore, the coefficients in Eq. (29) are

$$\Lambda_j = \frac{1}{2}, \quad \Gamma_j = 1. \tag{34}$$

Case $\nu_0 = \nu_N = \infty$. In this case, the boundary conditions at the extreme ends are

$$\begin{aligned} \overline{W}_1'(0, \mu, \lambda) &= \overline{\Theta}_0(\mu)L = 0, \\ \overline{W}_N'(1, \mu, \lambda) &= \overline{\Theta}_N(\mu)L = 0. \end{aligned} \tag{35}$$

Equation (31) remains valid; however, the serialized version now reads

$$\begin{aligned} &\mu_{j=(s-1)N+r} = \left\{\frac{1}{2}\,[1 - (-1)^s] + (-1)^s\,\frac{r}{N}\right\}\pi, \\ &(s = 1, 2, \ldots, \infty, \quad r = 1, 2, \ldots, N). \end{aligned} \tag{36}$$

The wave constant $\mu = 0$ and $\mu = \pi$ correspond to the upper bounds of odd-numbered and even-numbered propagation bands, respectively, contrary to case (a). In this case the mode shape takes the imaginary part of $W_\beta(\xi, \mu_j, \lambda_j)$ and the coefficients in the mode shape Eq. (28) are found to be

$$\begin{aligned} \Lambda_j &= \begin{cases} 1/2, & \mu_j = m\pi \\ -i/2, & \mu_j \neq m\pi, \end{cases} \\ \Gamma_j &= \begin{cases} 1, & \mu_j = m\pi \\ -1, & \mu_j \neq m\pi. \end{cases} \end{aligned} \tag{37}$$

Case $\nu_0 = \nu/2$ and $\nu_N = \infty$. In this case, the left end of the multispan beam is constrained by a rotational spring of stiffness constant $\nu/2$, while the right end is clamped. The corresponding boundary conditions read

$$\begin{aligned} \frac{\nu}{2}\,\overline{W}_1'(0, \mu, \lambda) &= \overline{W}_1''(0, \mu, \lambda), \\ \overline{W}_N'(1, \mu, \lambda) &= \overline{\Theta}_N(\mu)L = 0. \end{aligned} \tag{38}$$

Analogous to cases (a) and (b), the following equation for μ is obtained

$$\cos(\mu N) = 0, \tag{39}$$

from which

$$\begin{aligned} &\mu_{j=(s-1)N+r} = \left\{\frac{1}{2}\,[1 - (-1)^s] + (-1)^s\,\frac{2r-1}{2N}\right\}\pi, \\ &(s = 1, 2, \ldots, \infty, \quad r = 1, 2, \ldots, N). \end{aligned} \tag{40}$$

Note that neither zero nor π is a solution of the above equation; thus a standing wave does not exist. The mode shape for this case is described by the real part of $W_\beta(\xi, \mu_j, \lambda_j)$ with coefficients specified given in Eq. (34). If the left end of the multispan beam is treated as being clamped ($\nu_0 = \infty$), and the right end is treated as being elastically constrained by a rotational spring stiffness of $\nu/2$, then the mode shape is described by the imaginary part of $W_\beta(\xi, \mu_j, \lambda_j)$ with coefficients given in Eq. (37); Eqs. (39) and (40) remain unchanged.

Tables 1–3 list the frequency parameters in the first two bands of a six-span beam evaluated by using both the exact and asymptotic formulas for the three sets of boundary conditions at the extreme ends. It can be seen that the exact and asymptotic solutions differ by less than 0.4% in the first band, and they are almost identical in the higher bands ($s \geq 2$).

The normal modes associated with the frequency parameters in the first band are illustrated in Fig. 3(a,b,c), respectively. The solid and dash lines corresponds to the cases of $\nu = 0$ and $\nu = 10$, respectively. Fig. 3(a,b) portray the mode shapes for the two sets of exterior supports, namely $\nu_0 = \nu_N = \nu/2$ and $\nu_0 = \nu_N = \infty$. In these two cases the mode shapes are either symmetric or antisymmetric with respect to the midpoint of the multispan beam due to symmetric boundary conditions at the two extreme ends. Figure 3(c) illustrates the mode shapes for the case $\nu_0 = \nu/2$ and $\nu_N = \infty$, and they are neither symmetric nor antisymmetric, as expected.

FORCED VIBRATION ANALYSIS

In the preceding section, the frequency parameter λ, wave constant μ, and the associated mode shape have been determined for a multispan beam of finite or infinite length. In this section, the exact analytic harmonic response of such a beam subjected to a convected harmonic loading is obtained using the normal mode approach. Furthermore, both the location and the magnitude of the peak response can be determined in

Table 1. Frequency Parameters of Six-Span Beams with Rotational Spring Parameter ν (Case $\nu_0 = \nu_N = \nu/2$)

	$\nu = kL/EI$					
	0		2		200	
	Exact Eq. (18)	Asymp. Eq. (24)	Exact Eq. (18)	Asymp. Eq. (23)	Exact Eq. (18)	Asymp. Eq. (23)
Frequencies in the first band	π	π	3.398	3.397	4.641	4.624
	3.261	3.267	3.491	3.491	4.647	4.630
	3.556	3.566	3.729	3.730	4.663	4.646
	3.927	3.927	4.042	4.037	4.685	4.668
	4.298	4.288	4.362	4.351	4.707	4.699
	4.601	4.586	4.623	4.607	4.724	4.707
Frequencies in the second band	2π	2π	6.427	6.427	7.710	7.711
	6.410	6.410	6.536	6.536	7.720	7.720
	6.707	6.707	6.802	6.802	7.746	7.746
	7.069	7.069	7.134	7.134	7.781	7.782
	7.430	7.430	7.468	7.468	7.817	7.818
	7.727	7.728	7.740	7.741	7.844	7.844

advance. Thus, the important *coincidence phenomenon* can be investigated in exact terms.

Orthogonality Conditions of Mode Shapes

Multispan Beam of Finite Total Length. Consider two normal modes of a multispan beam satisfying the following equations

$$\begin{aligned} \overline{W}^{(4)}_{\beta,j}(\xi) - \lambda_j^4 \overline{W}_{\beta,j}(\xi) &= 0, \\ \overline{W}^{(4)}_{\beta,k}(\xi) - \lambda_k^4 \overline{W}_{\beta,k}(\xi) &= 0, \end{aligned} \tag{41}$$

where the first subscript β denotes the βth span, and the second subscript, j or k, corresponds to the serial number of natural frequency. Multiplying the first equation in Eq. (41) by $\overline{W}_{\beta,k}(\xi)$ and the second by $\overline{W}_{\beta,j}(\xi)$, and integrating the difference between the two resulting expressions over the total length NL of the N-span beam, we obtain

$$\sum_{\beta=1}^{N} \left\{ \int_0^1 [\overline{W}_{\beta,k}(\xi)\overline{W}^{(4)}_{\beta,j}(\xi) - \overline{W}_{\beta,j}(\xi)\overline{W}^{(4)}_{\beta,k}(\xi)]\, d\xi \right\} = (\lambda_j^4 - \lambda_k^4) \sum_{\beta=1}^{N} \int_0^1 \overline{W}_{\beta,j}(\xi)\overline{W}_{\beta,k}(\xi)\, d\xi. \tag{42}$$

Table 2. Frequency Parameters of Six-Span Beams with Rotational Spring Parameter ν (Case $\nu_0 = \nu_N = \infty$)

	$\nu = kL/EI$					
	0		2		200	
	Exact Eq. (18)	Asymp. Eq. (24)	Exact Eq. (18)	Asymp. Eq. (23)	Exact Eq. (18)	Asymp. Eq. (23)
Frequencies in the first band	3.261	3.267	3.491	3.491	4.647	4.630
	3.556	3.566	3.729	3.730	4.663	4.646
	3.927	3.927	4.042	4.037	4.685	4.668
	4.298	4.288	4.362	4.351	4.724	4.690
	4.601	4.586	4.623	4.607	4.724	4.706
	4.730	4.712	4.730	4.712	4.730	4.713
Frequencies in the second band	6.410	6.410	6.536	6.536	7.720	7.720
	6.708	6.707	6.802	6.802	7.746	7.746
	7.069	7.069	7.134	7.134	7.781	7.782
	7.430	7.430	7.468	7.468	7.817	7.818
	7.727	7.728	7.740	7.741	7.843	7.844
	7.853	7.854	7.853	7.854	7.853	7.854

Table 3. Frequency Parameters of Six-Span Beams with Rotational Spring Parameter ν (Case $\nu_0 = \nu/2$, $\nu_N = \infty$)

	$\nu = kL/EI$					
	0		2		200	
	Exact Eq. (18)	Asymp. Eq. (24)	Exact Eq. (18)	Asymp. Eq. (23)	Exact Eq. (18)	Asymp. Eq. (23)
Frequencies in the first band	3.173	3.175	3.422	3.421	4.643	4.626
	3.393	3.403	3.596	3.598	4.654	4.637
	3.738	3.743	3.881	3.879	4.674	4.656
	4.116	4.111	4.205	4.196	4.697	4.679
	4.463	4.451	4.505	4.491	4.717	4.699
	4.696	4.679	4.702	4.685	4.728	4.711
Frequencies in the second band	6.317	6.317	6.456	6.456	7.713	7.713
	6.545	6.545	6.656	6.656	7.731	7.732
	6.885	6.885	6.964	6.964	7.763	7.763
	7.252	7.253	7.304	7.304	7.800	7.800
	7.592	7.592	7.617	7.618	7.832	7.833
	7.820	7.820	7.823	7.824	7.851	7.852

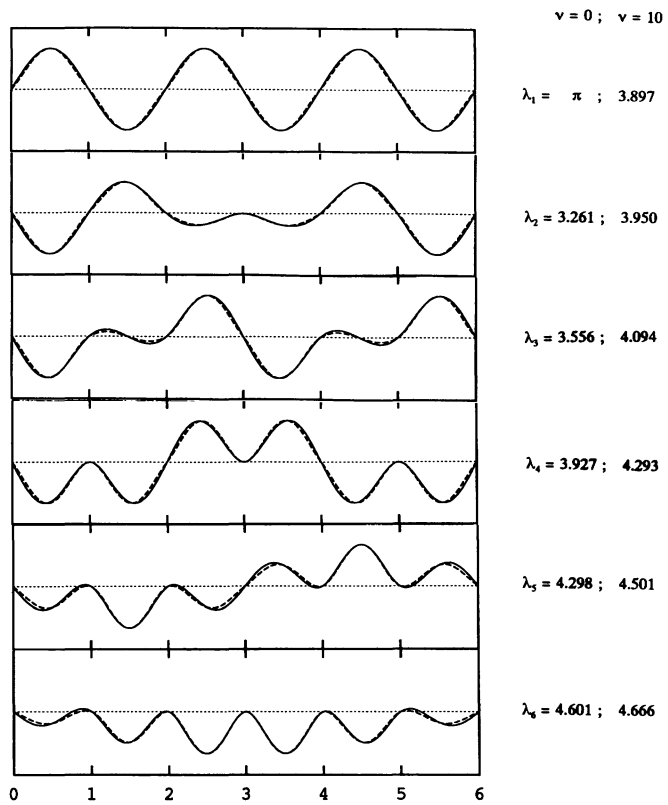

FIGURE 3(a) Normal modes in the first band of a six-span beam (— for $\nu = 0$; --- for $\nu = 10$) (a) $\nu_0 = \nu/2$ and $\nu_N = \nu/2$.

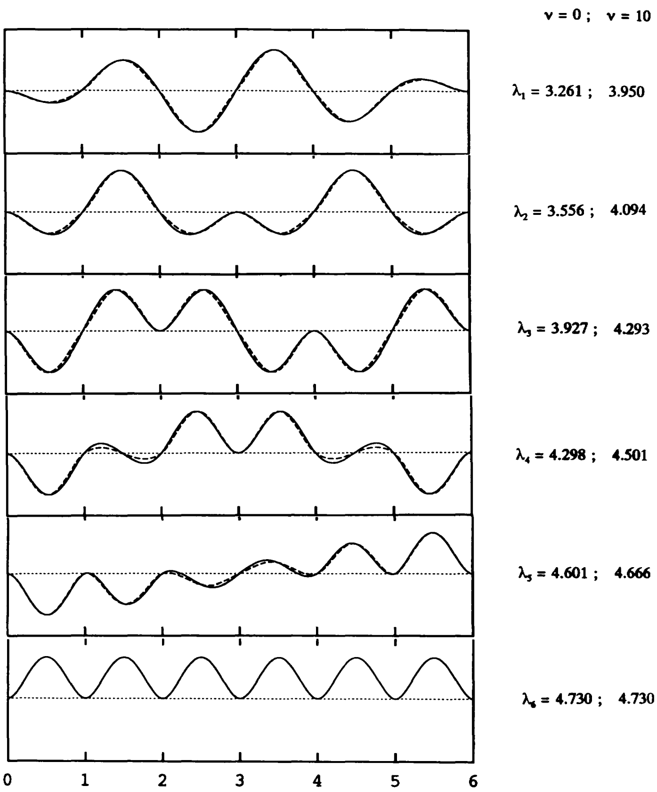

FIGURE 3(b) $\nu_0 = \infty$ and $\nu_N = \infty$.

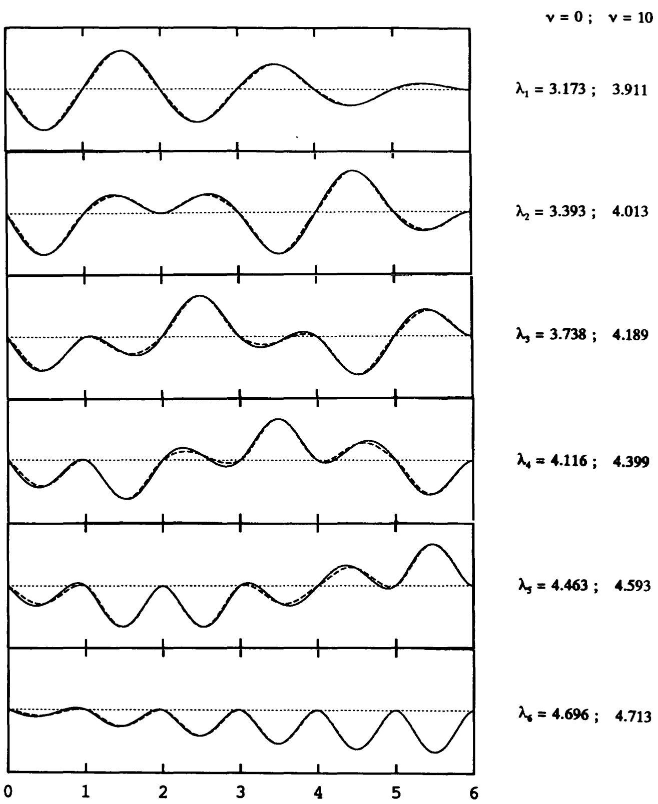

FIGURE 3(c) $\nu_0 = \nu/2$ and $\nu_N = \infty$.

Integrating the product $\overline{W}_{\beta,j}(\xi)\overline{W}^{(4)}_{\beta,j}(\xi)$ by parts yields

$$\begin{aligned}\int_0^1 \overline{W}_{\beta,k}(\xi)\overline{W}^{(4)}_{\beta,j}(\xi)\,d\xi &= [\overline{W}_{\beta,k}(\xi)\overline{W}'''_{\beta,j}(\xi) - \overline{W}'_{\beta,k}(\xi)\overline{W}''_{\beta,j}(\xi) \\ &\quad + \overline{W}''_{\beta,k}(\xi)\overline{W}'_{\beta,j}(\xi) \\ &\quad - \overline{W}'''_{\beta,k}(\xi)\overline{W}_{\beta,j}(\xi)]_0^1 + \int_0^1 \overline{W}_{\beta,j}(\xi)\overline{W}^{(4)}_{\beta,k}(\xi)\,d\xi.\end{aligned} \tag{43}$$

Equations (42) and (43) can be combined to yield

$$\begin{aligned}\sum_{\beta=1}^{N} [\overline{W}_{\beta,k}(\xi)\overline{W}'''_{\beta,j}(\xi) - \overline{W}'_{\beta,k}(\xi)\overline{W}''_{\beta,j}(\xi) \\ + \overline{W}''_{\beta,k}(\xi)\overline{W}'_{\beta,j}(\xi) - \overline{W}'''_{\beta,k}(\xi)\overline{W}_{\beta,j}(\xi)]_0^1 \\ = (\lambda_j^4 - \lambda_k^4)\sum_{\beta=1}^{N}\int_0^1 \overline{W}_{\beta,j}(\xi)\overline{W}_{\beta,k}(\xi)\,d\xi.\end{aligned} \tag{44}$$

Finally, taking into account the continuity conditions Eq. (6) at each interior support, we obtain

$$\begin{aligned}[\overline{W}_{N,k}(1)\overline{W}'''_{N,j}(1) \\ - \overline{W}'_{N,k}(1)\overline{W}''_{N,j}(1) + \overline{W}''_{N,k}(1)\overline{W}'_{N,j}(1) \\ - \overline{W}'''_{N,k}(1)\overline{W}_{N,j}(1)] \\ - [\overline{W}_{1,k}(0)\overline{W}'''_{1,j}(0) - \overline{W}'_{1,k}(0)\overline{W}''_{1,j}(0) \\ + \overline{W}''_{1,k}(0)\overline{W}'_{1,j}(0) - \overline{W}'''_{1,k}(0)\overline{W}_{1,j}(0)] \\ = (\lambda_j^4 - \lambda_k^4)\sum_{\beta=1}^{N}\int_0^1 \overline{W}_{\beta,j}(\xi)\overline{W}_{\beta,k}(\xi)\,d\xi.\end{aligned} \tag{45}$$

It may be noted that the left-hand side of Eq. (45) vanishes for any set of homogeneous boundary conditions of the form

$$a\overline{W}(x) + b\overline{W}'''(x) = 0 \tag{46}$$

or

$$c\overline{W}'(x) + d\overline{W}''(x) = 0 \tag{47}$$

at the two ends, where a, b, c, d are constants. The idealized boundary conditions, such as clamped-free, simply–simply supports, and so on, are examples. Equation (46) corresponds to a transverse elastic support, and Eq. (47) to a rotational elastic support. Thus, we obtain orthogonality condition for normal modes of an N-span beam as follows

$$\sum_{\beta=1}^{N}\int_0^1 \overline{W}_{\beta,j}(\xi)\overline{W}_{\beta,k}(\xi)\,d\xi = \gamma_j^2\delta_{jk}, \tag{48}$$

where δ_{jk} denotes the Kronecker delta and γ_j^2 is defined as follows:

$$\gamma_j^2 = \sum_{\beta=1}^{N}\int_0^1 \overline{W}^2_{\beta,j}(\xi)\,d\xi. \tag{49}$$

Note that the span serial number β and the local coordinate ξ are separable in the expression for the mode shape given in Eqs. (28) and (29). Indeed, β appears only in the exponential functions not involving ξ. Therefore, integration over the entire length of a multispan beam can be carried out with respect to the local coordinate ξ, and then summed over the span serial number β. It will be shown later that these properties can be used to reduce the computational effort when evaluating the dynamic response of the system.

Equation (49) may be rewritten as follows:

$$\gamma_j^2 = \sum_{\beta=1}^{N}\int_0^1 \overline{W}^2_{\beta,j}(\xi)\,d\xi = C^a_{jj}I_1(\lambda_j) + C^b_{jj}I_2(\lambda_j), \tag{50}$$

where integrals $I_1(\cdot)$ and $I_2(\cdot)$ are defined as

$$\begin{aligned}I_1(\lambda) &= \int_0^1 f^2(\xi)\,d\xi \\ &= \frac{1}{2}\left\{1 + \frac{1}{2\lambda}\sin(2\lambda) - \frac{\sin^2(\lambda)}{\sinh^2(\lambda)}\left[1 + \frac{1}{2\lambda}\sinh(2\lambda)\right]\right\},\end{aligned} \tag{51}$$

$$\begin{aligned}I_2(\lambda) &= \int_0^1 f(\xi)f(1-\xi)\,d\xi \\ &= \frac{1}{2}\left\{-\cos(\lambda) - \frac{1}{\lambda}\sin(\lambda) + \frac{\sin^2(\lambda)}{\sinh^2(\lambda)}\left[\cosh(\lambda) + \frac{1}{\lambda}\sinh(\lambda)\right]\right\},\end{aligned} \tag{52}$$

and where C^a_{jj} and C^b_{jj} can be obtained from the following more general expressions

$$C^a_{jk} = \sum_{\beta=1}^{N} (\bar{a}_{\beta,j}\bar{a}_{\beta,k} + \bar{b}_{\beta,j}\bar{b}_{\beta,k})$$
$$= \Lambda_j\Lambda_k\{(a_ja_k + b_jb_k)S_N(\mu_j + \mu_k) + \Gamma_j\Gamma_k(a_j^*a_k^* + b_j^*b_k^*)S_N^*(\mu_j + \mu_k) + \Gamma_j(a_j^*a_k + b_j^*b_k)S_N^*(\mu_j - \mu_k) + \Gamma_k(a_ja_k^* + b_jb_k^*)S_N(\mu_j - \mu_k)\}, \tag{53}$$

$$C^b_{jk} = \sum_{\beta=1}^{N} (\bar{a}_{\beta,j}\bar{b}_{\beta,k} + \bar{a}_{\beta,k}\bar{b}_{\beta,j})$$
$$= \Lambda_j\Lambda_k\{(a_jb_k + a_kb_j)S_N(\mu_j + \mu_k) + \Gamma_j\Gamma_k(a_j^*b_k^* + a_k^*b_j^*)S_N^*(\mu_j + \mu_k) + \Gamma_k(a_jb_k^* + a_kb_j^*)S_N(\mu_j - \mu_k) + \Gamma_j(a_j^*b_k + a_k^*b_j)S_N^*(\mu_j - \mu_k)\}. \tag{54}$$

In Eqs. (53) and (54), an asterisk denotes the complex conjugate. Γ and Λ are defined by

$$\Lambda = \frac{1}{2}, \Gamma = 1, \quad \text{if } \overline{W}_\beta(\xi) = \mathrm{Re}\{W_\beta(\xi, \mu, \lambda)\} \text{ or } \mu = m\pi;$$
$$\Lambda = \frac{1}{2i}, \Gamma = -1, \quad \text{if } \overline{W}_\beta(\xi) = \mathrm{Im}\{W_\beta(\xi, \mu, \lambda)\} \text{ and } \mu \neq m\pi. \tag{55}$$

and function $S_N(\mu)$ is given by

$$S_N(\mu) = \sum_{\beta=1}^{N} e^{-i\mu(\beta-1)} = \begin{cases} \dfrac{1 - e^{-i\mu N}}{1 - e^{-i\mu}} & \mu \neq 2m\pi \\ N & \mu = 2m\pi \end{cases}. \tag{56}$$

Mutispan Beam of Infinite Length. In the case of the beam of infinite length, the orthogonality condition of normal modes can also be derived by using a similar procedure. However, it is no longer necessary to impose any boundary conditions at the exterior supports. The mode shape $W_\beta[\xi, \mu, \lambda_s(\mu)]$ given in Eq. (16) is now applicable throughout the entire length. Equation (41) through (44) still holds, except that the finite sum is replaced by an infinite sum. By taking into account the continuity conditions, Eq. (6), at the interior supports, it is easy to show that the left-hand side of Eq. (42) vanishes, that is,

$$2\pi\delta(\mu + \mu')\{(e^{i\mu} - e^{-i\mu'}) W_1''[0, \mu, \lambda_s(\mu)]W_1''[1, \mu', \lambda_{s'}(\mu')] + (e^{-i\mu} - e^{i\mu'})W_1''[1, \mu, \lambda_s(\mu)] W_1''[0, \mu', \lambda_{s'}(\mu')] \equiv 0, \tag{57}$$

where $\delta(\cdot)$ is Dirac's delta function, and where use has been made of the identity

$$\sum_{\beta=-\infty}^{\infty} e^{-i\mu(\beta-1)} = 2\pi\delta(\mu). \tag{58}$$

The orthogonality condition of the mode shapes for an infinitely long multispan beam read

$$\sum_{\beta=-\infty}^{\infty} \int_0^1 W_\beta[\xi, \mu, \lambda_s(\mu)]W_\beta[\xi, \mu', \lambda_{s'}(\mu')]\, d\xi = \gamma_0^2[\mu, \lambda_s(\mu)]\, \delta(\mu + \mu')\, \delta_{ss'}, \tag{59}$$

where $\delta_{ss'}$ is the Kronecker's delta, and $\gamma_0^2[\cdot]$ is defined as follows:

$$\gamma_0^2[\mu, \lambda_s(\mu)] = \int_0^1 |W_1[\xi, \mu, \lambda_s(\mu)]|^2\, d\xi = (|a|^2 + |b|^2)I_1[\lambda_s(\mu)] + 2\mathrm{Re}\{ab^*\}I_2[\lambda_s(\mu)], \tag{60}$$

in which integrals $I_1(\cdot)$ and $I_2(\cdot)$ are given in Eqs. (51) and (52), respectively, and a, b are defined in Eq. (17).

Response of Multispan Beam Under Convected Loading

Let us consider the damped forced vibration of a multispan beam. The equation of motion in the local coordinate system is

$$EIL^{-4}y_\beta^{(4)}(\xi, t) + c\dot{y}_\beta(\xi, t) + \rho A\ddot{y}_\beta(\xi, t) = p_\beta(\xi, t),$$
$$(\beta = N_-, \ldots, 1, 2, \ldots, N_+; \quad 0 \leq \xi \leq 1) \tag{61}$$

where c = damping coefficient, $p_\beta(\xi, t)$ = transverse pressure per unit length, $N = 1$ and $N_+ = N$ for a finite span beam, whereas $N_- = -\infty$ and $N_+ = \infty$ for an infinite span beam. Assuming that the excitation and response are harmonic in time, we have

$$p_\beta(\xi, t) = P_\beta(\xi)e^{i\omega t},$$
$$y_\beta(\xi, t) = Y_\beta(\xi)e^{i\omega t}. \tag{62}$$

Function $Y_\beta(\xi)$ will be referred to as the harmonic response function. For a harmonic loading convected over the beam at the velocity $\omega L/\mu_f$

$$P_\beta(\xi) = P_0 e^{-i\mu_f(\xi+\beta-1)}, \quad (\beta = 1, 2, \ldots, N) \tag{63}$$

where P_0 is the amplitude, and μ_f is the wave constant of the loading.

Multispan Beam of Finite Length. First, let us consider the case of an N-span beam. We expand $Y_\beta(\xi)$ and $P_\beta(\xi)$ in terms of the normal modes of the system

$$Y_\beta(\xi) = \sum_{j=1}^{\infty} c_j \overline{W}_{\beta,j}(\xi),$$
$$P_\beta(\xi) = \sum_{j=1}^{\infty} d_j \overline{W}_{\beta,j}(\xi), \quad (\beta = 1, 2, \ldots, N) \tag{64}$$

where c_j and d_j are the coefficients. The relationship between c_j and d_j can be found by substituting Eqs. (62) and (64) into Eq. (61) and using Eq. (41) to obtain

$$\sum_{j=1}^{\infty} (\lambda_j^4 - \lambda_f^4 + i\zeta\lambda_f^2) c_j \overline{W}_{\beta,j}(\xi) = \frac{L^4}{EI} \sum_{j=1}^{\infty} d_j \overline{W}_{\beta,j}(\xi), \tag{65}$$

where $\lambda_f^4 = \rho A L^4 \omega^2 / EI$ is a nondimensional loading frequency parameter, $\zeta = cL^2/(\rho AEI)^{1/2}$ is a nondimensional damping parameter. Comparison of coefficients on the two sides of Eq. (65) yields

$$c_j = H(\lambda_j, \lambda_f) d_j, \tag{66}$$

where $H(\lambda_j, \lambda_f)$ is the frequency response function given by

$$H(\lambda_j, \lambda_f) = \frac{L^4}{EI} (\lambda_j^4 - \lambda_f^4 + i\zeta\lambda_f^2)^{-1}. \tag{67}$$

Hence, the harmonic response function is obtained in the following form

$$Y_\beta(\xi) = \sum_{j=1}^{\infty} d_j H(\lambda_j, \lambda_f) \overline{W}_{\beta,j}(\xi), \quad (\beta = 1, 2, \ldots, N). \tag{68}$$

The coefficients d_j are obtained by applying the orthogonality condition of mode shapes, namely Eq. (48), to the second equation in Eq. (64) to yield

$$d_j = \frac{1}{\gamma_j^2} \sum_{\beta=1}^{N} \int_0^1 \overline{W}_{\beta,j}(\xi) P_\beta(\xi)\, d\xi = \frac{P_0}{\gamma_j^2} [E_j^a g(\lambda_j, \mu_f) + E_j^b e^{-i\mu_f} g^*(\lambda_j, \mu_f)], \tag{69}$$

where

$$E_j^a = \Lambda_j [a_j S_N(\mu_j + \mu_f) + \Gamma_j a_j^* S_N(\mu_f - \mu_j)], \tag{70}$$

$$E_j^b = \Lambda_j [b_j S_N(\mu_j + \mu_f) + \Gamma_j b_j^* S_N(\mu_f - \mu_j)], \tag{71}$$

$$g(\lambda, \mu_f) = \int_0^1 f(\xi) e^{-i\mu_f \xi}\, d\xi = u - \frac{\sin(\lambda)}{\sinh(\lambda)} \nu, \tag{72}$$

$$u = \begin{cases} \dfrac{1}{2(\mu_f^2 - \lambda^2)} [(\lambda + \mu_f) e^{i(\lambda-\mu_f)} + (\lambda - \mu_f) e^{-i(\lambda+\mu_f)} - 2\lambda], & |\mu_f| \neq \lambda \\ -\dfrac{i}{2} \operatorname{sgn}(\mu_f) + \dfrac{1}{2(\lambda + |\mu_f|)} \{1 - e^{-i[\lambda \operatorname{sgn}(\mu_f) + \mu_f]}\}, & |\mu_f| = \lambda \end{cases} \tag{73}$$

$$\nu = \frac{1}{2(\mu_f^2 + \lambda^2)} [(\lambda - i\mu_f) e^{-(\lambda + i\mu_f)} + (\lambda + i\mu_f) e^{\lambda - i\mu_f} - 2\lambda], \tag{74}$$

in which sgn(·) denotes the sign function.

Multispan Beam of Infinite Length. The dynamic response of an infinitely long multispan beam subjected to a convected harmonic loading can also be evaluated in a similar way. Let us expand both the harmonic response and the loading function in the mode shapes of such a beam

$$Y_\beta(\xi) = \sum_{s=1}^{\infty} c_s(\mu) W_\beta[\xi, \mu, \lambda_s(\mu)]\, d\mu,$$
$$P_\beta(\xi) = \sum_{s=1}^{\infty} d_s(\mu) W_\beta[\xi, \mu, \lambda_s(\mu)]\, d\mu, \tag{75}$$
$$(\beta = 0, \pm 1, \pm 2, \ldots, \pm\infty)$$

where s denotes the serial number of a propagation band. Multiplying both sides of the second equation in Eq. (75) by $W_\beta^*[\xi, \mu', \lambda_{s'}(\mu')]$ and performing integration over the length of the entire beam, we obtain upon applying the orthogonality condition, Eq. (59),

$$d_s(\mu) = \frac{1}{2\pi\gamma_0^2[\mu, \lambda_s(\mu)]} \times \sum_{\beta=-\infty}^{\infty} \int_0^1 P_\beta(\xi) W_\beta^*[\xi, \mu, \lambda_s(\mu)]\, d\xi. \quad (76)$$

For an excitation in the form of Eq. (63), the numerator in Eq. (76) may be simplified to

$$\sum_{\beta=-\infty}^{\infty} \int_0^1 P_\beta(\xi) W_\beta^*[\xi, \mu, \lambda_s(\mu)]\, d\xi = 2\pi\delta(\mu_f - \mu) P_0 D[\mu, \lambda_s(\mu), \mu_f],$$
$$D[\mu, \lambda_s(\mu), \mu_f] = a^*[\mu, \lambda_s(\mu)] g[\lambda_s(\mu), \mu_f] + b^*[\mu, \lambda_s(\mu)] e^{-i\mu_f} g^*[\lambda_s(\mu), \mu_f], \quad (77)$$

in which $a(\cdot)$ and $b(\cdot)$ are defined in Eq. (17), and $g(\cdot)$ is given by Eq. (72). Equation (66) is still valid for $c_s(\mu)$ and $d_s(\mu)$. Hence, $c_s(\mu)$ may be expressed as follows

$$c_s(\mu) = \frac{\delta(\mu_f - \mu) P_0}{\gamma_0^2[\mu, \lambda_s(\mu)]} D[\mu, \lambda_s(\mu), \mu_f] H[\lambda_s(\mu), \mu_f]. \quad (78)$$

The harmonic response function for the infinitely long multispan beam can be obtained by substituting Eq. (78) into the first equation in Eq. (75) to yield

$$Y_\beta(\xi) = \sum_{s=1}^{\infty} \int_{\mu\in R} \frac{\delta(\mu_f - \mu) P_0}{\gamma_0^2[\mu, \lambda_s(\mu)]} \times D[\mu, \lambda_s(\mu), \mu_f] H[\lambda_s(\mu), \lambda_f] W_\beta[\xi, \mu, \lambda_s(\mu)]\, d\mu,$$
$$(\beta = 0, \pm 1, \pm 2, \ldots, \pm\infty). \quad (79)$$

Here, the integration range R must be chosen from the particular *Brillouin zone* defined in Eq. (21) that includes the loading wave constant μ_f. This is always possible because the union of all Brillouin zones constitutes the entire one-dimensional space. Note the presence of a Dirac's delta function in Eq. (79) with an argument $\mu_f - \mu$. It implies that only a group of propagating waves associated with μ_f contributes to the response. Carrying out the integration in Eq. (79), we obtain

$$Y_\beta(\xi) = \sum_{s=1}^{\infty} \frac{P_0}{\gamma_0^2[\mu, \lambda_s(\mu)]} D[\mu_f, \lambda_s(\mu_f), \mu_f] H[\lambda_s(\mu_f), \lambda_f] W_\beta[\xi, \mu_f, \lambda_s(\mu_f)],$$
$$(\beta = 0, \pm 1, \pm 2, \ldots, \pm\infty) \quad (80)$$

where $\gamma_0^2(\cdot)$, $D(\cdot)$, $H(\cdot)$, and $W_\beta(\cdot)$ are given by Eqs. (60), (77), (67), and (16), respectively.

RESULTS AND DISCUSSION

For a multispan beam under harmonic excitation, large response is likely to occur if the excitation frequency is within a propagation frequency band. When the excitation is convected along the beam, the response can be further amplified due to the so-called *coincidence effect*. For an N-span beam the magnitude of the harmonic response function may have as many as N peaks in each propagation band. The possible location of a peak in propagation band for $|Y_\beta(\xi)|$ can be determined from the condition

$$\lambda_f = \left(\lambda_j^4 - \frac{\zeta^2}{2}\right)^{1/4}, \quad (j = 1, 2, \ldots), \quad (81)$$

that maximizes $|H(\lambda_j, \lambda_f)|$, whereas the magnitude of harmonic response function $Y_\beta(\xi)$ is dominated by the term with $|H(\lambda_j, \lambda_f)|$. For an infinitely long multispan beam, however, there is only one group of propagating waves, whose wave constant coincides with μ_f, contributing to the response, as it can be seen from Eq. (80). Therefore, there is only one peak at

$$\lambda_f = \left[\lambda_s(\mu_f)^4 - \frac{\zeta^2}{2}\right]^{1/4}, \quad (s = 1, 2, \ldots) \quad (82)$$

appearing in each propagation band.

Figure 4(a) portrays the harmonic response at the midpoint of the second span of a four multispan beam. It shows that there are four peaks in the first propagation band. In contrast, there is only one peak in the first band for the infinite multispan beam shown in Fig. 4(b), and the response is considerably magnified due to the coincidence effect.

The effects of damping coefficient to the harmonic responses are also shown in Fig. 5(a,b) for

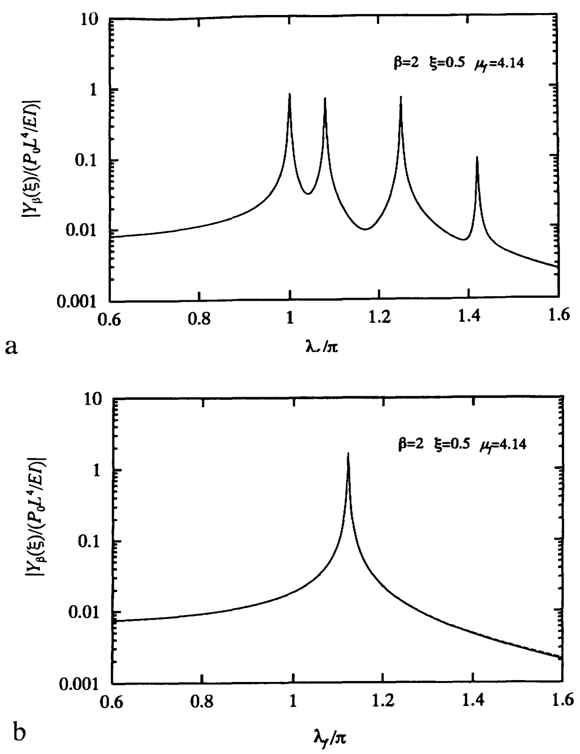

FIGURE 4(a) Nondimensional harmonic response for a four-span beam vs loading frequency parameter λ_f (— Using normal modes of the first band; --- Using normal modes of the first two bands).

FIGURE 4(b) Nondimensional harmonic response of an infinitely long multi-span beams vs loading frequency parameter λ_f (— Using normal modes of the first band; --- Using normal modes of the first two bands).

a four-span beam and an infinite long multispan beam, respectively. It can be seen that the values of the peaks in each propagation band will be reduced with a larger damping coefficient. Hence, the profile of each peak becomes flatter as the damping coefficient increases.

The results obtained from the present approach and Mead's approach (1971) are compared in Fig. 6(a,b) for an infinitely long beam with an evenly spaced hinge supports ($\nu = 0$). The dash line represents a 100-term approximation in Mead's formulation, whereas the solid curve represents a one-term approximation by the present approach in the first propagation band. The results are seen to be very close. The two results become indistinguishable in the first three propagation bands, when a 20-term was used in the present approach, as shown in Fig. 6(b). Most significantly, the present approach has the advantage in that the location of each possible peak in each propagation band can be determined; thus, the value of each peak can be evaluated precisely from Eq. (80).

CONCLUSION

The free and forced vibrations of the periodically supported multispan beam of both finite and infinite length were studied. The wave propagation concept was applied in the analysis of free vibration of the beam systems. The dispersion equation and its asymptotic form were derived from which the natural frequencies can be determined from a given wave constant. An explicit asymp-

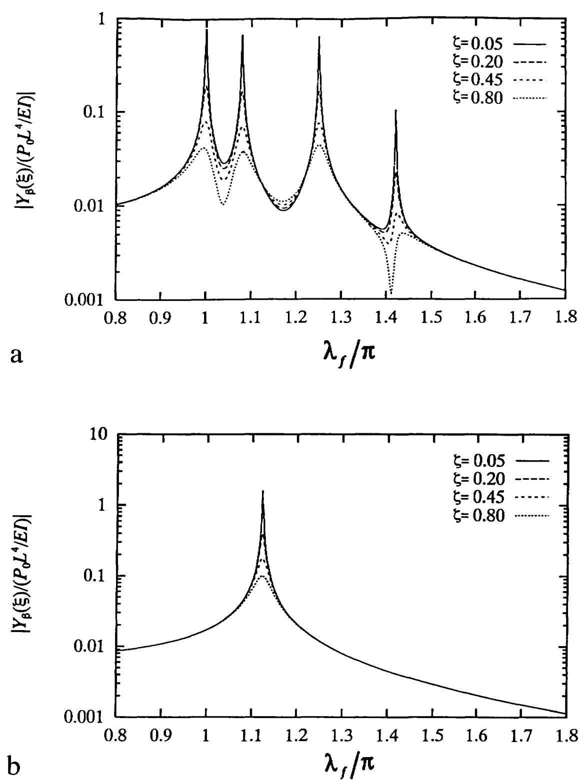

FIGURE 5(a) Nondimensional harmonic response of a four-span beam vs loading frequency parameter λ_f with different damping coefficients ($\beta = 2$, $\xi = 0.5$, $\mu_f = 4.14$).

FIGURE 5(b) Nondimensional harmonic response of an infinite long multi-span beam vs loading frequency parameter λ_f with different damping coefficients ($\beta = 2$, $\xi = 0.5$, $\mu_f = 4.14$).

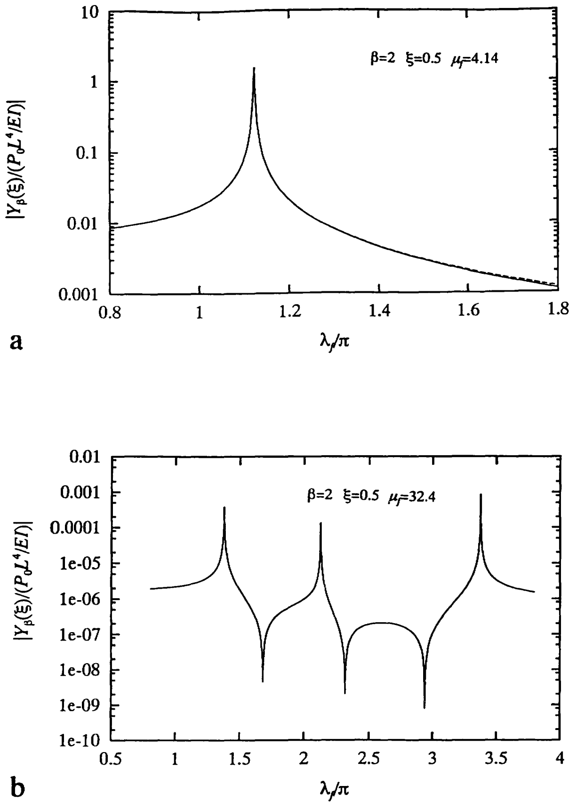

FIGURE 6(a) Nondimensional harmonic response of an infinitely long multi-span beam in the first band (— by present approach; --- by Mead's approach)
FIGURE 6(b) Nondimensional harmonic response of an infinitely long multi-span beam in the first three bands (— by present approach; --- by Mead's approach).

totic expression for the natural frequency was also proposed for the specific case of zero rotational spring stiffness. It is shown that the agreement between the asymptotic and the exact frequencies is excellent. The mode shapes of free vibration are obtained in the complex form. In these mode shapes the span serial number and the local spatial coordinate are separable; thus, an integration over the entire length was reduced to one within a single span, and a summation over the serial numbers of the spans. It was shown that use of these mode shapes can greatly reduce the computational efforts in the forced vibration analysis.

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