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Limiting Vibroisolation Control of an Oscillating String on a Moving Base

The vibroisolating capability of an elastic object that is assumed to be protected against a class of excitations is studied. It is proposed that this capability be estimated by a quadratic functional. The solution method gives an estimate of the optimal isolation with a criterion of minimum guaranteed quality. A numerical example of the solution technique is presented. © *1995 John Wiley & Sons, Inc.*

INTRODUCTION

The study of limiting vibration isolation capabilities is one of the fundamental problems in the theory of shock and vibration. Until now, studies have been confined to the limiting vibroisolation capabilities of mathematical models of systems in which the external action (input) is assigned as a time function (Gouretski, 1965, 1969; Bolotnik, 1983; Sevin and Pilkey, 1971) or as a stationary random process with declared probability (Kolovski, 1976). However, in real design problems, quite often all that is known of the input is that it belongs to a certain set of functions. In such situations, it is preferable to determine the limiting capabilities for an entire class of inputs rather than for a single input. This study formulates and solves such a problem for the vibroisolation of a viscoelastic string. The problem is shown to be one of optimization with a guaranteed quality of isolation, in which the class of inputs is a set of square-integrable functions.

MATHEMATICAL MODEL AND PROBLEM STATEMENT

Consider a uniform viscoelastic string lying on a base that moves along a straight line according to a prescribed rule. Suppose that one point of the string is fastened to the base through a vibroisolator. The motion of the string is then described by

$$\rho \frac{\partial^2 z}{\partial t^2} = b \frac{\partial^3 z}{\partial x^2 \partial t} + c \frac{\partial^2 z}{\partial x^2} + q(x)u + \rho v. \quad (1)$$

Here $z(x, t)$ is the cross-section shift, with respect to the base, of the point of the string with coordinate x at the instant t; ρ is the density of the string; the positive parameters b and c define the damping and rigidity of the string; $q(x) = \delta(x - x_0)$, where $\delta(x)$ is the delta function, $x_0 \in (0, L]$, and L is the length of the string; u is the force exerted by the vibroisolator; and $v(t)$ is the acceleration of the base. Suppose that the end of

Received February 18, 1993; Accepted October 18, 1993.
Shock and Vibration, Vol. 2, No. 2, pp. 163–171 (1995)

CCC 1070-9622/95/020163-09

the string at $x = 0$ is fixed and the end at $x = L$ is free. Then, the boundary conditions are

$$z(0, t) = 0 \quad z'(L, t) \equiv \left.\frac{\partial z(x, t)}{\partial x}\right|_{x=L} = 0. \quad (2)$$

The function $v(t)$, assumed to be piecewise continuous and absolutely integrable with respect to time over the interval $(-\infty, \infty)$, will be required to satisfy the conditions

$$v(t) \equiv 0 \quad \text{for} \quad t < 0$$

$$\int_0^\infty v^2(t)\, dt \leq s_0^2. \quad (3)$$

The functions $v(t)$ satisfying these conditions will be said to belong to the class S. As for the function $u = u(t)$, it will also be assumed to be piecewise continuous and absolutely integrable with respect to time over the interval $(-\infty, \infty)$. In addition, an *ideal vibroisolator* will be considered to be capable of predicting an input, that is, the control $u(t)$ can be initiated at any instant $t_0 < 0$, and therefore the equality

$$u(t) \equiv 0 \quad \text{for} \quad t < t_0 \quad (4)$$

is true. Such functions $u(t)$ will be said to belong to the class D. The initial state of the system in question will be defined by

$$z(x, t_0) = 0 = \dot{z}(x, t_0). \quad (5)$$

The functional W describing the quality of vibroisolation is defined as

$$W[u(\cdot), v(\cdot)] = \int_{-\infty}^{\infty} w(t)\, dt, \quad (6)$$

where

$$w(t) = \int_0^L [\rho(\dot{z}(x, t))^2 + c(z'(x, t))^2]\, dx.$$

Note that the function $w(t)$ is twice the sum of kinetic and potential energies of the string in its oscillatory motion with respect to the base.

Next a statement of two problems of optimal vibration reduction for the string's oscillations will be given. The *first problem*, that of limiting vibroisolation capability analysis, is estimating the value

$$W^0 = \sup_{v(\cdot)\in S} \inf_{u(\cdot)\in D} W[u(\cdot), v(\cdot)]. \quad (7)$$

To do this, for any input $v(t)$ in the class S, an optimal vibroisolator $u(t)$ minimizing the functional in Eq. (6) must be found and then the expression obtained must be maximized for all inputs $v(t)$. This problem differs from the traditional statement (Gouretski, 1965, 1969) in that it is to be solved not for only one input but for an entire class of inputs.

To state the *second problem*, that of optimization with a guaranteed quality of vibroisolation, first consider the class D_1 of vibroisolators. This class will involve passive, active, and semiactive vibroisolators. Any vibroisolator will be assumed to become activated concurrently with the initiation of the input at the instant $t_0 = 0$. The D_1-class vibroisolators must satisfy the following condition: at any instant $t_* > 0$, the control u can be determined if the input $v(t)$, the deformation, and the rate of deformation at the point x_0 are known over the time interval $[0, t_*]$. In general terms, the dependence of the control u on $v(t)$, $z(x_0, t)$, and $\dot{z}(x_0, t)$ is expressed by the functional relation

$$u = u_f(z(x_0, t), \dot{z}(x_0, t), v(t)). \quad (8)$$

Thus, if the input $v(t)$ is known, then, upon solving the initial boundary value problem (1), (2), (5) with control of type (8), it is possible to express the control u as a function of time t. It will be assumed that, for any input $v(t)$ in the class S, the control function $u(t)$ will be piecewise continuous and absolutely integrable over the time interval $[0, \infty)$.

Now the second problem can be stated as follows: find the control $u^0(\cdot) \in D_1$ such that

$$\sup_{v(\cdot)\in S} W[u^0(\cdot), v(\cdot)] = \inf_{u(\cdot)\in D_1} \sup_{v(\cdot)\in S} W[u(\cdot), v(\cdot)]. \quad (9)$$

To solve the second problem, it is necessary to find the input maximizing the functional $W[u(\cdot), v(\cdot)]$ for each D_1-class control and then to perform a minimization with respect to all admissible controls.

Next, the relationship between the first and the second problem will be discussed. Note that each D_1-class control can, in principle, be implemented. Principal realizability of control is understood in the sense that to generate a control action at any instant, it is sufficient to have only knowledge of the system's behavior at all previous instants of time, and no knowledge of future values of the input is needed. An optimal

control is generated only for the *worst* case. In contrast, in the first problem, it is assumed that we know a priori what input is applied, and an optimal control is chosen for each of these inputs. In this case, we obtain, so to speak, an *ideal* control that is, in principle, impossible to implement. It is clear from these considerations that a solution of the first problem cannot be improved through any real control. Denote

$$W_*^0 = \inf_{u(\cdot)\in D_1} \sup_{v(\cdot)\in S} W[u(\cdot), v(\cdot)].$$

Then

$$\sup_{v(\cdot)\in S} \inf_{u(\cdot)\in D} W[u(\cdot), v(\cdot)] \le \sup_{v(\cdot)\in S} \inf_{u(\cdot)\in D_1} W[u(\cdot), v(\cdot)].$$

Using the inequality (Fedorov, 1979) known in the theory of antagonistic games, and connecting maximin and minimax, one obtains

$$\sup_{v(\cdot)\in S} \inf_{u(\cdot)\in D_1} W[u(\cdot), v(\cdot)] \le \inf_{u(\cdot)\in D_1} \sup_{v(\cdot)\in S} W[u(\cdot), v(\cdot)].$$

Finally, the last two inequalities yield

$$W^0 \le W_*^0.$$

Thus, the guaranteed vibroisolation quality W_*^0 cannot be smaller than the quality W^0 describing the limiting vibroisolating capability of the input class S.

THE FIRST PROBLEM

Let the dimensionless variables $\tilde{z}$, $\tilde{x}$, and $\tilde{t}$ be defined by

$$z = L\tilde{z} \quad x = L\tilde{x} \quad t = (\rho/c)^{1/2}L\tilde{t}.$$

In terms of these dimensionless variables, Eq. (1) becomes

$$\frac{\partial^2\tilde{z}}{\partial\tilde{t}^2} = \beta\frac{\partial^3\tilde{z}}{\partial\tilde{x}^2\partial\tilde{t}} + \frac{\partial^2\tilde{z}}{\partial\tilde{x}^2} + \tilde{q}(\tilde{x})\tilde{u} + \tilde{v}(\tilde{t}),$$

where

$$\beta = b(\rho/c)^{-1/2}L^{-1} \qquad \tilde{q}(\tilde{x}) = \delta(\tilde{x} - \tilde{x}_0)$$
$$\tilde{x}_0 = x_0/L \qquad \tilde{u} = uL/c$$
$$\tilde{v}(\tilde{t}) = v(t)\rho L/c \qquad \tilde{z}(\tilde{x}, \tilde{t})|_{\tilde{x}=0} = 0$$

and

$$\left.\frac{\partial\tilde{z}(\tilde{x}, \tilde{t})}{\partial\tilde{x}}\right|_{\tilde{x}=1} = 0.$$

The functional defined in Eq. (6) is also written in the dimensionless form

$$W[u(\cdot), v(\cdot)] = \chi_0 \int_{-\infty}^{\infty} \tilde{w}(\tilde{t})\, d\tilde{t}$$

in which

$$\tilde{w}(\tilde{t}) = \int_0^1 \left[\left(\frac{\partial\tilde{z}(\tilde{x}, \tilde{t})}{\partial\tilde{t}}\right)^2 + \left(\frac{\partial\tilde{z}(\tilde{x}, \tilde{t})}{\partial\tilde{x}}\right)^2\right] d\tilde{x}$$

and

$$\chi_0 = L^2(\rho c)^{1/2}.$$

Then Eq. (3) becomes

$$\int_0^{\infty} (\tilde{v}(\tilde{t}))^2\, d\tilde{t} \le \eta_0^2 \quad \eta_0^2 = s_0^2 L(\rho/c)^{3/2}.$$

In the rest of the article these problems will always be expressed in terms of dimensionless variables, and, for brevity, the *tilde* over the symbols will be omitted whenever this omission is unlikely to cause confusion. A dot over a symbol will stand for differentiation with respect to $\tilde{t}$ and a prime for differentiation with respect to $\tilde{x}$.

In terms of the new variables, then, the first problem is stated as follows. The initial boundary value problem is given as

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= \beta\frac{\partial^3 z}{\partial x^2\partial t} + \frac{\partial^2 z}{\partial x^2} + q(x)u + v(t) \\ z(0, t) &= 0 = z'(1, t) \\ z(x, t_0) &= 0 = \dot{z}(x, t_0), \end{aligned} \tag{10}$$

where $\beta > 0$, $q(x) = \delta(x - x_0)$, $x_0 \in (0, 1]$ and $v(t)$ is a piecewise continuous, absolutely integrable function, satisfying, according to Eq. (3), the conditions

$$v(t) \equiv 0 \quad \text{for} \quad t < 0$$
$$\int_0^\infty v^2(t)\, dt \le \eta_0^2. \tag{11}$$

Here $u = u(t)$ is a piecewise continuous, absolutely integrable function, satisfying, according to Eq. (4), the condition

$$u(t) \equiv 0 \quad \text{for} \quad t < t_0. \tag{12}$$

In other words, if $v(t)$ and $u(t)$ satisfy Eqs. (11) and (12), then $v(\cdot) \in S$ and $u(\cdot) \in D$. Next, find the value of W^0 defined in Eq. (7), where

$$W[u(\cdot), v(\cdot)] = \chi_0 \int_{-\infty}^{\infty} w(t)\, dt$$
$$w(t) = \int_0^1 [\dot{z}(x, t)^2 + z'(x, t)^2]\, dx. \tag{13}$$

The initial boundary value problem of Eq. (10) is solved using a series expansion in terms of the eigenfunctions of the associated boundary value problem. To find the eigenfunctions, consider the homogeneous equation

$$\frac{\partial^2 z}{\partial t^2} = \beta \frac{\partial^3 z}{\partial x^2 \partial t} + \frac{\partial^2 z}{\partial x^2}$$

with boundary conditions $z(0, t) = 0 = z'(1, t)$. If this equation is solved by the method of separation of variables, by representing $z(x, t)$ as the product $X(x)T(t)$, the following boundary value problem for finding the eigenfunctions is obtained

$$X'' + \lambda^2 X = 0 \quad X(0) = 0 = X'(1).$$

The eigenvalues λ_n and the eigenfunctions $X_n(x)$ are found in the usual way (Tikhonov and Samarski, 1966)

$$X_n(x) = \sqrt{2} \sin \lambda_n x \quad \lambda_n = \frac{(2n-1)\pi}{2} \quad n \ge 1.$$

The solution of the problem stated in Eq. (10) is represented as (Tikhonov and Samarski, 1966)

$$z(x, t) = \sum_{n=1}^{\infty} T_n(t) X_n(x), \tag{14}$$

where the functions $T_n(t)$ are determined in accordance with the initial conditions. By substituting the series in Eq. (14) into Eq. (10) one obtains

$$\sum_{n=1}^{\infty} [\ddot{T}_n X_n - \beta \dot{T}_n X_n'' - T_n X_n''] = q(x)u + v(t).$$

Now multiply this equality by $X_k(x)$ and integrate it with respect to x over the interval [0, 1] using the equalities

$$X_k'' = -\lambda_k^2 X_k$$
$$\int_0^1 X_k X_n\, dx = \begin{cases} 1 & k = n \\ 0 & k \ne n \end{cases}$$

to find

$$\ddot{T}_k + \beta\lambda_k^2 \dot{T}_k + \lambda_k^2 T_k = q_k u + r_k v(t),$$

where

$$q_k = \sqrt{2} \int_0^1 \delta(x - x_0) \sin \lambda_k x\, dx = \sqrt{2} \sin \lambda_k x_0$$

$$r_k = \sqrt{2} \int_0^1 \sin \lambda_k x\, dx = \frac{\sqrt{2}}{\lambda_k}.$$

Thus

$$\begin{aligned} \ddot{T}_1 + \beta\lambda_1^2 \dot{T}_1 + \lambda_1^2 T_1 &= q_1 u + r_1 v(t) \\ \ddot{T}_2 + \beta\lambda_2^2 \dot{T}_2 + \lambda_2^2 T_2 &= q_2 u + r_2 v(t) \\ &\cdots\cdots \\ \ddot{T}_n + \beta\lambda_n^2 \dot{T}_n + \lambda_n^2 T_n &= q_n u + r_n v(t) \\ &\cdots\cdots \end{aligned} \tag{15}$$

with initial conditions

$$T_1(t_0) = \dot{T}_1(t_0) = \cdots = T_n(t_0) = \dot{T}_n(t_0) = 0.$$

The function $w(t)$ defined in Eq. (13) will be transformed, using Eq. (14), into

$$w(t) = \sum_{n=1}^{\infty} [\dot{T}_n^2 + \lambda_n^2 T_n^2]. \tag{16}$$

To solve the first problem, each equation in (15) will be multiplied by the factor $e^{-i\omega t}$ (here i is the imaginary unit), and the result obtained will be integrated with respect to t over the infinite interval $(-\infty, \infty)$. This operation is valid because the system (15) is asymptotically stable; and hence, because the functions $u(t)$ and $v(t)$ are absolutely integrable, so are the functions $T_n(t)$. In terms of Fourier transforms, Eq. (15) is written as

$$(-\omega^2 + i\beta\omega\lambda_n^2 + \lambda_n^2)\Theta_n(\omega) = q_n U(\omega) + r_n V(\omega) \quad n \geq 1, \quad (17)$$

where $\Theta_n(\omega)$, $U(\omega)$, and $V(\omega)$ stand for the Fourier transforms of $T_n(t)$, $u(t)$, and $v(t)$. From Eq. (17)

$$\Theta_n(\omega) = \frac{q_n U(\omega) + r_n V(\omega)}{-\omega^2 + i\beta\omega\lambda_n^2 + \lambda_n^2}.$$

Let U^* and V^* be complex conjugates of the functions U and V. Employing Parseval's equality (Korn and Korn, 1974) and invoking Eq. (16), one obtains

$$\int_{-\infty}^{\infty} w(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, d\omega$$

where

$$G(\omega) = \sum_{n=1}^{\infty} \Delta_n(\omega)[q_n^2 UU^* + q_n r_n(UV^* + U^*V) + r_n^2 VV^*]$$

with

$$\Delta_n(\omega) = \frac{\omega^2 + \lambda_n^2}{(\omega^2 - \lambda_n^2)^2 + (\beta\omega\lambda_n^2)^2}.$$

Therefore,

$$G(\omega) = a_1 UU^* + a_2(UV^* + U^*V) + a_3 VV^* \quad (18)$$

in which

$$a_1 = a_1(\omega) = \sum_{n=1}^{\infty} \Delta_n(\omega) q_n^2$$

$$a_2 = a_2(\omega) = \sum_{n=1}^{\infty} \Delta_n(\omega) q_n r_n$$

$$a_3 = a_3(\omega) = \sum_{n=1}^{\infty} \Delta_n(\omega) r_n^2.$$

The partial derivatives of G with respect to U and U^* are given by

$$\frac{\partial G}{\partial U} = a_1 U^* + a_2 V^*$$

$$\frac{\partial G}{\partial U^*} = a_1 U + a_2 V.$$

These partial derivatives are zero if

$$U^* = -\frac{a_2}{a_1} V^* \quad U = -\frac{a_2}{a_1} V. \quad (19)$$

Thus, for any ω, the expression (18) assumes an extreme value if U is determined from Eq. (19). To determine the nature of this extremum, calculate the second derivatives

$$\frac{\partial^2 G}{\partial U^2} = \frac{\partial^2 G}{(\partial U^*)^2} = 0 \quad \frac{\partial^2 G}{\partial U \partial U^*} = a_1$$

Because $a_1 > 0$, this extremum is a minimum. Let $Q(\omega) = -a_2/a_1$. It is not difficult to show that the function $Q(\omega)$ is continuous and bounded in the interval $(-\infty, \infty)$. Hence, the complex function $U(\omega) = Q(\omega)V(\omega)$ is the Fourier transform of some function $u_0(t)$. Substitution of Eq. (19) into Eq. (18) gives

$$G(\omega) = G_0(\omega) VV^*$$

where

$$G_0(\omega) = a_3 - \frac{a_2^2}{a_1}.$$

Thus, the inequality

$$\inf_{u(\cdot)\in D} W[u(\cdot), v(\cdot)] \geq \frac{\chi_0}{2\pi}\int_{-\infty}^{\infty} G_0(\omega) VV^*\, d\omega \quad (20)$$

holds. Note that if the function $u_0(t)$, whose Fourier transform is given by Eq. (19), belongs to the class D, then (20) becomes an equality. However, a function

$$u_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(\omega)V(\omega)e^{i\omega t}\, d\omega$$

may, in general, not be identically zero for all $t < t_0$ for any finite value t_0, and, therefore, $u_0(t)$ may not belong to the class D. In this case, construct a minimizing sequence of functions

$$u_0^\varepsilon(t) = \begin{cases} u_0(t) & t \geq t_0^\varepsilon \\ 0 & t < t^\varepsilon{}_0 \end{cases}$$

where t_0^ε is found from

$$\int_{-\infty}^{t_0^\varepsilon} |u_0(t)|\, dt = \varepsilon.$$

The Fourier transform of the function $u_0^\varepsilon(t)$ is

$$U_\varepsilon(\omega) = U(\omega) + \Omega_\varepsilon(\omega) = Q(\omega)V(\omega) + \Omega_\varepsilon(\omega)$$

in which

$$\Omega_\varepsilon(\omega) = -\int_{-\infty}^{t_0^\varepsilon} u_0(t)e^{-i\omega t}\, dt.$$

It follows from the definition of t_0^ε that $|\Omega_\varepsilon(\omega)| \leq \varepsilon$. Substituting $U_\varepsilon(\omega)$ into Eq. (18), one obtains

$$G(\omega) = G_0(\omega)VV^* + \gamma_\varepsilon(\omega),$$

where

$$\gamma_\varepsilon(\omega) = a_1|\Omega_\varepsilon(\omega)|^2.$$

Thus,

$$W[u_0^\varepsilon(\cdot), v(\cdot)] = \frac{\chi_0}{2\pi}\int_{-\infty}^{\infty} G_0(\omega)VV^*\, d\omega + \frac{\chi_0}{2\pi}\int_{-\infty}^{\infty} \gamma_\varepsilon(\omega)\, d\omega.$$

Because

$$\int_{-\infty}^{\infty} \gamma_\varepsilon(\omega)\, d\omega \leq \varepsilon^2 \int_{-\infty}^{\infty} a_1(\omega)\, d\omega$$

and it can be easily shown that the integral on the right-hand side of this inequality converges, the limit as ε goes to zero becomes

$$\lim_{\varepsilon\to 0} W[u_0^\varepsilon(\cdot), v(\cdot)] = \frac{\chi_0}{2\pi}\int_{-\infty}^{\infty} G_0(\omega)VV^*\, d\omega.$$

Thus, for any input $v(t)$ in the class S, the equality

$$\inf_{u(\cdot)\in D} W[u(\cdot), v(\cdot)] = \frac{\chi_0}{2\pi}\int_{-\infty}^{\infty} G_0(\omega)VV^*\, d\omega$$

will hold. The required magnitude of W^0 is determined as

$$W^0 = \frac{\chi_0}{2\pi}\sup_{v(\cdot)\in S}\int_{-\infty}^{\infty} G_0(\omega)VV^*\, d\omega. \qquad (21)$$

Now consider the input

$$v^0(t) = \begin{cases} \dfrac{\eta_0}{\sqrt{\sigma_+}}\sin\omega_0 t & 0 \leq t \leq t_k \\ 0 & t_k < t < 0 \end{cases}$$

where

$$\sigma_+ = \int_0^{t_k} \sin^2 \omega t\, dt$$

and ω_0 is a constant. By calculating the Fourier transform $V_0(\omega)$ of the input $v^0(t)$ and then letting t_k approach infinity, one obtains (Besekerski and Popov, 1966)

$$\lim_{t_k\to\infty} V_0V_0^* = \pi\eta_0^2[\delta(\omega - \omega_0) + \delta(\omega - \omega_0)], \qquad (22)$$

where $\delta(\omega)$ is the Dirac delta function. It follows from Eq. (22) and the Parseval equality that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} VV^*\, d\omega \leq \eta_0^2.$$

Denote by ω_+ the value of ω at which the function $G_0(\omega)$ assumes its maximum value

$$G_0(\omega_+) = \max_{\omega\in(-\infty,\infty)} G_0(\omega).$$

With the relationships Eqs. (21) and (22), the required magnitude $W^0 = \eta_0^2\chi_0 G_0(\omega_+)$ is now found. Using the explicit expression for $G_0(\omega)$, one obtains

$$W^0 = \eta_0^2\chi_0\left[\sum_{n=1}^{\infty}\Delta_n(\omega_+)r_n^2 - \frac{\left(\sum_{n=1}^{\infty}\Delta_n(\omega_+)q_nr_n\right)^2}{\sum_{n=1}^{\infty}\Delta_n(\omega_+)q_n^2}\right],$$

and taking into account the explicit expressions for the parameters q_n and r_n

$$W^0 = 2\eta_0^2\chi_0 \max_{\omega\in(-\infty,\infty)}\left\{\sum_{n=1}^{\infty}\frac{\Delta_n(\omega)}{\lambda_n^2} - \frac{\left(\sum_{n=1}^{\infty}(\sin\lambda_n x_0\,\Delta_n(\omega))\right)^2}{\sum_{n=1}^{\infty}\sin^2\lambda_n x_0\,\Delta_n(\omega)}\right\}. \qquad (23)$$

THE SECOND PROBLEM

Unfortunately, it is difficult to find a general solution technique for the second problem. Consider, therefore, only a specific case: a simple linear vibroisolator with a damper and an elastic element connected in parallel. Let the coupling function of Eq. (8) be

$$u = -B\dot{z}(x_0, t) - Cz(x_0, t). \tag{24}$$

Then, passing from the boundary value problem to Eq. (15) in dimensionless variables, one has

$$\ddot{T}_n + \beta\lambda_n^2\dot{T}_n + \lambda_n^2 T_n = -q_n\left[\sum_{k=1}^{\infty} q_k(\gamma\dot{T}_k + \sigma T_k)\right] + r_n v(t) \quad n \geq 1, \tag{25}$$

where $\gamma = BL(\rho c)^{-1/2}$ and $\sigma = CL^2/c$. Thus, in this case, the second problem may be stated as follows: determine the parameters γ^0 and σ^0 such that

$$\sup_{v(\cdot)\in S} W[\gamma^0, \sigma^0, v(\cdot)] = \inf_{\substack{\sigma\geq 0\\ \sigma\geq 0}} \sup_{v(\cdot)\in S} W[\gamma, \sigma, v(\cdot)].$$

An approximation technique will be used to solve this problem. If only a finite number N of Eq. (25) are retained, then

$$\ddot{T} + F\dot{T} + \Phi T = rv(t) \quad T(0) = 0 = \dot{T}(0), \tag{26}$$

where T is a column vector with components $\{T_1, T_2, \ldots, T_N\}$, r is a column vector with components $\{r_1, r_2, \ldots, r_N\}$, and F and Φ are the $N \times N$ symmetric matrices

$$F = \begin{bmatrix} \beta\lambda_1^2 + q_1^2\gamma & q_1q_2\gamma & \cdots & q_1q_N\gamma \\ q_1q_2\gamma & \beta\lambda_2^2 + q_2^2\gamma & \cdots & q_2q_N\gamma \\ \vdots & \vdots & \vdots & \vdots \\ q_1q_N\gamma & q_2q_N\gamma & \cdots & \beta\lambda_N^2 + q_N^2\gamma \end{bmatrix}$$

$$\Phi = \begin{bmatrix} \lambda_1^2 + q_1^2\sigma & q_1q_2\sigma & \cdots & q_1q_N\sigma \\ q_1q_2\sigma & \lambda_2^2 + q_2^2\sigma & \cdots & q_2q_N\sigma \\ \vdots & \vdots & \vdots & \vdots \\ q_1q_N\sigma & q_2q_N\sigma & \cdots & \beta\lambda_N^2 + q_N^2\sigma \end{bmatrix}$$

The Fourier transform of Eq. (26) and the reasoning given earlier lead to

$$\sup_{v(\cdot)\in S} W[\gamma, \sigma, v(\cdot)] = \chi_0\eta_0^2 \max_{\omega} r^T A(\omega) r,$$

where

$$A(\omega) = \Gamma^*(\omega)(\omega^2 E + L_0)\Gamma(\omega)$$

with

$$\Gamma(\omega) = (-\omega^2 E + i\omega F + \Phi)^{-1}.$$

The matrices $\Gamma^T(\omega)$ and $\Gamma^*(\omega)$ are the transpose and the complex conjugate, respectively, of $\Gamma(\omega)$; L_0 is a diagonal matrix with elements $\{\lambda_1^2, \lambda_2^2, \ldots, \lambda_N^2\}$ and E is the identity matrix. Thus, for $\gamma, \sigma \geq 0$, it is possible to evaluate

$$W_S[\gamma, \sigma] = \chi_0\eta_0^2 \max_{\omega} r^T A(\omega) r.$$

Then, minimizing the function $W_S[\gamma, \sigma]$ with respect to γ and σ will yield the optimal values γ^0 and σ^0. Thus,

$$W_S^0 = W_S[\gamma^0, \sigma^0] = \inf_{\gamma,\sigma\geq 0} W_S[\gamma, \sigma].$$

Finally, the ratio α defined as

$$\alpha = \frac{W_S^0}{W^0} \geq 1$$

will show how close the vibroisolating quality of a passive linear vibroisolator is to the limiting possible quality.

Note that the values W^0 and W_S^0 obtained in this section for the first and second problems depend on the parameter x_0 that assigns a location to the vibroisolator. Thus, along with the problems formulated above, the problem of optimal vibroisolator location may be solved. This problem, if it is combined with the first problem, may be stated as follows: find x_0^0 such that

$$W^0(x_0^0) = \min_{x_0\in(0,1]} W^0(x_0). \tag{27}$$

NUMERICAL EXAMPLE

Suppose, without loss of generality, that $\eta_0^2\chi_0 = 1$. To solve the first problem, use Eq. (23). It can be shown that the infinite series in this equation

Table 1 Limiting Possible Quality

x_0	0.01	0.1	0.2	0.4	0.6	0.8	1.0
W^0	9.23	1.12	0.475	0.123	0.0466	0.0975	0.146

converges. Hence, when determining the numerical value of W^0, the sum of the series can be calculated to any desired accuracy by keeping a finite number of terms. Table 1 shows a set of values of W^0, depending on x_0, with the assumption that $\beta = 0.1$. Further, for convenience, the magnitude W_+ is introduced so that it determines a maximal S-input value of the functional (6) for the case $u(t) = 0$

$$W_+ = \min_{v(\cdot)\in S} W[0, v(\cdot)].$$

For these values of the initial parameters, one finds $W_+ = 26.64$. Hence, the limiting efficiency of vibroisolation may be stated as a percentage

$$\xi = 100\,\frac{W_+ - W^0}{W^+}.$$

The values of ξ are shown as a function of x_0 in Table 2. The optimal location x_0 of the isolator, determined by the condition (27), is found from a one-dimensional minimization of $W^0(x_0)$ with respect to x_0 to be $x_0^0 = 0.61$, and one obtains $W^0(x_0^0) = 0.0444$ and $\xi = 99.83\%$. Note here that $W^0(x_0^0)$ indicates the lowest possible value; a smaller value of guaranteed vibroisolation quality cannot be obtained by any real vibroisolator.

Next consider the numerical solution of the first problem for the linear vibroisolator given in Eq. (24). The two-parameter optimization at $x_0 = x_0^0 = 0.61$ yields the optimal values $\gamma^0 = 3.85$, $\sigma^0 = 0.878$, and the corresponding optimal value of the indicator of guaranteed quality is

$$W_S^0 = W_S[\gamma^0, \sigma^0] = 0.163.$$

For this example, the ratio α of the guaranteed quality indicator S_S^0 to the limiting possible quality is equal to 3.67.

DISCUSSION OF RESULTS

A physical interpretation of the results obtained above will be suggested in this section. Consider a finite chain of solids sequentially interconnected by linear elastic–dissipative links. Let this mechanical system be connected, at one of its ends, by a linear elastic–dissipative link to a massive body called the base. Suppose that the base vibrates under the action of external forces. Then, all the elements of the chain will start vibrating with respect to the base. In particular, this mechanical system can be considered as a model of oscillations that a multistory building performs under seismic action. If the elements in the chain are homogeneous and there is a sufficiently large number of them, each having a mass much smaller than the mass of the entire system, then it is possible to replace the finite-dimensional model with a distributed one described by a partial differential equation like Eq. (1).

The results obtained will be interpreted by considering the base-fixed chain as a model for a uniform multistory building under seismic excitation. Suppose that a seismoisolation system needs to be designed for the building. The designers do not know when an earthquake may strike or how severe it is likely to be. What they have is only the data collected for the earthquakes that have already occurred in this region. One may say that designers have an impression about an entire set of possible seismic excitations, but do not know anything about the next earthquake to come. Hence, they have to create the seismoisolation design for the *worst* case, to ensure the best protection of the building against the most dangerous seismic excitation. In other words, design follows the principle of optimizing with a guaranteed result. Upon obtaining guaranteed seismoisolating quality data, derived through a mathematical model analysis, it becomes possi-

Table 2 Limiting Efficiency of Vibroisolation

x_0	0.01	0.1	0.2	0.4	0.6	0.8	1.0
ξ (%)	65.36	95.80	98.22	99.54	99.82	99.63	99.45

ble to compare various seismoisolation designs. Here arises the question of limits on what can be achieved by modifying the proposed seismoisolation designs. In essence, this article is devoted to answering this question. By solving the *first problem*, concerning limiting vibroisolation capabilities, one can only obtain the theoretically attainable limit on the guaranteed quality. No real isolator can exceed this limit. Note that the isolator discussed here couples only one of the stories to the base. It is certainly possible to consider more complicated structures with several stories coupled to the base, each through its own isolator. In this case, limiting seismoisolation capabilities may also be determined. These limiting values would be lower than those for the single-isolator case. The numerical example given in this article demonstrates a simple linear isolator that ensures a protecting quality that differs from the limiting quality by as much as a factor of 3.67. This approach may be applied to the problem of deriving objective numerical estimates in the design of vibroisolation devices for a variety of systems.

CONCLUSION

The optimal control problems of string oscillations studied in this article have natural generalizations to other mathematical models. The proposed technique makes it possible to study the control of elastic oscillations of a bar, rod, membrane, or plate with various boundary conditions, and to investigate control problems for elastic bodies coupled through linear elastic–dissipative elements. In addition, instead of a scalar control applied to one of the points of the elastic body, there appears to be the possibility of studying both a vector control applied to several points of the body and a distributed control acting within a region of the body.

This research was supported by a grant from the Russian Fund for Fundamental Studies, Project No. 93-013-16282.

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