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Effective Method for Analysis of Electrothermally Coupled Fields

An effective method is presented for solving a nonlinear system of partial differential equations that describe the time-dependent electrothermally coupled fields for passage of constant electric current in a three-dimensional conductive medium. A numerical model of this physical phenomenon was obtained by the finite element method, which takes into account the temperature-dependent characteristics describing the material parameters and conditions of heat transmission outside of the analyzed objects. These characteristics and conditions make the problem strongly nonlinear. The solution uses the Newton–Raphson method with the appropriate procedure for determining the Jacobian matrix elements. The main idea of the proposed method is the use of an automatic time step selection algorithm to solve heat conduction equations. The influence of the assumed accuracy value on the final result of the nonlinear calculation is discussed. The theoretical results were confirmed by the numerical experiments performed with selected physical objects.

INTRODUCTION

The analysis of the 2-D models of electrically and thermally coupled fields has been studied by many authors, for example Bastos et al. (1990), Lavers (1983), and Massé et al. (1985). The primary problem was to find an effective method of solving the nonlinear nonstationary system of partial differential equations. Much attention was paid to this problem in the article written by Massé et al. (1985).

In this article the authors try to solve the problem for the 3-D model, assuming that the direct current flows through a thermal element. When studying the 3-D problem, it is particularly important to choose an effective method of solving the equation that describes the nonstationary thermal field distribution. The new version of "single-step" first-rank method with a variable time step was elaborated on the basis of the methods known from the Massé et al. (1985) and Zienkiewicz et al. (1984) studies.

Numerical computations have confirmed the methods proposed. These computations were arranged in such a way as to avoid nonlinear iterations during calculation.

MATHEMATICAL MODEL OF ANALYZED PROBLEM

It is essential to consider the physical phenomena that occur during the flow of direct or harmonic current through a conductor. When the Kelvin skin effect is neglected, the phenomena

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can be described by the set of partial differential equations:

$$\nabla[\gamma(T)\nabla\varphi] = 0 \tag{1}$$

$$\nabla[\lambda(T)\nabla T] - \rho(T)C(T)\frac{\partial T}{\partial t} = -\gamma(T)(\nabla\varphi)^2 \tag{2}$$

with the appropriate boundary conditions, where $\gamma(T)$ is the conductance, $\lambda(T)$ is the heat conductivity, $\rho(T)$ is the mass density, and $C(T)$ is the specific heat. They are all temperature-dependent coefficients.

The boundary conditions of the electric field depend on the way in which current flow is forced through the conductor. When voltage forcing takes place, the potentials at the load points are assumed to be $\varphi = 0$ and $\varphi = U$, and when the current forcing appears, $\gamma(\partial\varphi/\partial n) = J_n$ is assumed as a nonzero Dirichlet condition. For the remaining part of the conductor condition, $\partial\varphi/\partial n = 0$ should be satisfied. For a thermal field the condition of heat transmission on the surface of the conductor is assumed as follows:

$$\lambda(T)\frac{\partial T}{\partial n} = -\alpha(T)(T - T_0) - \varepsilon(T)(T^4 - T_0^4)$$

or

$$\lambda(T)\frac{\partial T}{\partial n} = -\alpha_z(T)(T - T_0)$$
$$\alpha_z(T) = \alpha(T) + \varepsilon(T)(T + T_0)(T^2 + T_0^2), \tag{3}$$

where $\alpha_z(T)$ is the equivalent coefficient that takes into account heat convection and heat radiation, T_0 is the temperature of a conductor's surroundings, $\alpha(T)$ is the coefficient that accounts for convection heat transfer, and $\varepsilon(T)$ is the temperature-dependent coefficient that accounts for radiation heat transfer.

After the numerical approximation of Eqs. (1) and (2), according to the finite element scheme with the boundary conditions presented above, nonlinear set of algebraic equations is obtained in the folloiwng form

$$\mathbf{H(T)}[\varphi] = \mathbf{G} \tag{4}$$

$$\mathbf{A(T)T} + \mathbf{D(T)\dot{T}} = \mathbf{B}([\varphi], \mathbf{T}), \tag{5}$$

where $\dot{\mathbf{T}}$ is the vector of the temporal temperature derivatives at the discretization nodes of the domain.

The elements of the matrices in Eqs. (4) and (5) for the eth discretization element of the domain of analysis are expressed as follows:

$$H_{ij}^{(e)} = \int_{V^{(e)}} \gamma(T)^{(e)}(\nabla N_i)^T \nabla N_j \, dv \tag{6}$$

$$A_{ij}^{(e)} = \int_{V^{(e)}} \lambda(T)^{(e)}(\nabla N_i)^T \nabla N_j \, dv + \int_{S^{(e)}} \alpha_z(T)^{(e)} N_i N_j \, ds \tag{7}$$

$$D_{ij}^{(e)} = \int_{V^{(e)}} \rho(T)^{(e)} C(T)^{(e)} N_i N_j \, dv \tag{8}$$

$$B_i^{(e)} = \int_{V^{(e)}} \gamma(T)^{(e)}(\nabla\varphi)^2 N_i \, dv + \int_{S^{(e)}} \alpha_z(T)^{(e)} T_0 N_i \, ds, \tag{9}$$

where $V^{(e)}$ is the volume of the eth element, $S^{(e)}$ is the surface of the eth element that forms a part of the flank of the examined conductor, and N_i is the shape functions over the element.

METHOD OF SOLVING HEAT EQUATION

From the numerical point of view two serious problems must be considered. The first is to solve the set of nonlinear equations, and the second is to choose the proper way of approximating the temporal temperature derivative in order to solve heat Eq. (5).

To solve the second problem, the single-step method was described by Zienkiewicz et al. (1984). Additionally, an algorithm for the automatic calculation of the time step is presented below. The temperature for the interval $\Delta t_n = t_{n+1} - t_n$ is approximated by the pth order polynomial:

$$T = T_n + \dot{T}_n t + \ddot{T}_n \frac{1}{2} t^2 + \cdots + \alpha_n^{(p)} t^p \frac{1}{p!}, \tag{10}$$

where $0 \le t \le \Delta t_n$. In order to find the temperature distribution for the nth interval, the coefficient $\alpha_n^{(p)}$ has to be found. For the assumption $p = 1$, the single-step algorithm becomes the linear algorithm, described by the equations:

$$\mathbf{\tilde{T}_{n+1}} = \mathbf{T_n}$$
$$(\mathbf{D} + \delta\Delta t_n \mathbf{A})[\alpha_n] = \mathbf{B} - \mathbf{AT_n} \tag{11}$$

$$\mathbf{T}_{n+1} = \mathbf{T}_n + [\alpha_n]\Delta t_n, \tag{12}$$

where $\delta \in \langle 0.5; 1\rangle$ and $\tilde{\mathbf{T}}_{n+1}$ is an assumed approximate value of $\mathbf{T}_{n+1}$.

Equation (11) is nonlinear because the elements of the matrices **D**, **A**, and **B** depend both on the temperature and on $[\alpha_n]$. Therefore, the following calculations at each instant of time should be performed:

$$\mathbf{H}(\mathbf{T})[\varphi_n] = \mathbf{G} \tag{13}$$

$$(\mathbf{D}(\mathbf{T}) + \delta\Delta t_n \mathbf{A}(\mathbf{T}))[\alpha_n] = \mathbf{B}([\varphi_n], \mathbf{T}) - \mathbf{A}(\mathbf{T})\mathbf{T}_n \tag{14}$$

$$\mathbf{T}_{n+1} = \mathbf{T}_n + [\alpha_n]\Delta t_n. \tag{15}$$

AUTOMATIC CALCULATION OF TIME STEP

For the single-step method (10), we assume the temperature in the $(n + 1)$st instant of time

$$T^*_{n+1} = T_n + \alpha_n \Delta t_n. \tag{16}$$

At the end of the nth instant of time the temperature can be expanded in a Taylor series:

$$T_{n+1} = T_n + \Delta T_n \dot{T}_n + \frac{\Delta t_n^2}{2}\ddot{T}_n + \cdots. \tag{17}$$

Limiting ourselves to the second-order expansion, we get the expression for the temperature at the $(n + 1)$st instant of time in the form

$$T_{n+1} = T_n + \Delta t_n \dot{T}_n + \frac{\Delta t_n^2}{2}\ddot{T}_n. \tag{18}$$

The temperature expressed by (18) is assumed to be exact. So the error for the nth step is described as follows

$$\varepsilon = T_{n+1} - T^*_{n+1} = \frac{\Delta t_n^2}{2}\ddot{T}_n \cong \frac{\Delta t_n^2}{2}\frac{\dot{T}_n - \dot{T}_{n-1}}{\Delta t_{n-1}}. \tag{19}$$

After approximating the temperature by the single-step method (16), we get for each time step:

$$\varepsilon_n = \frac{\Delta t_n^2}{2}\frac{\alpha_{n-1} - \alpha_{n-2}}{\Delta t_{n-1}}, \tag{20}$$

and replacing the nth with the next time step in Eq. (20), we can assume the error at the $n + 1$ time step as follows:

$$\varepsilon_{n+1} = \frac{\Delta t_{n+1}^2}{2}\frac{\alpha_n - \alpha_{n-1}}{\Delta t_n}. \tag{21}$$

The relations (16–21) describe the changes of temperature at any point of the analyzed domain. Assuming that this temperature is calculated at m points of the domain, the error can be calculated on the basis of Eq. (21) and can be expressed as follows:

$$\varepsilon = \frac{\Delta t_{n+1}^2}{2}\frac{\|[\alpha_n] - [\alpha_{n-1}]\|}{\Delta t_n} \tag{22}$$

and

$$\Delta t_{n+1} = \sqrt{\frac{2\Delta t_n \varepsilon}{\|[\alpha_n] - [\alpha_{n-1}]\|}}. \tag{23}$$

As proved in our previous article (Korytkowski and Wincenciak (1993), this method of calculating the time step for the heat equation provides the best results. The time step Δt_{n+1} described by (23) uses the time step values calculated earlier and assumes a constant error ε at each time step.

If we replace the second temporal derivative of temperature with the forward final difference, which is possible as $[\alpha_n]$ is known after Eq. (14) has been solved, and if we take into account the whole analyzed domain, we can express the coefficient ε_A in the following equation:

$$\varepsilon_A = \frac{\Delta t_n}{2}\max_i |\alpha_{n_i} - \alpha_{n-1_i}|, \tag{24}$$

where the index i is valid for all the vector elements $[\alpha_n]$ and $[\alpha_{n-1}]$. The value of the coefficient ε_A defines the maximal error that is made while determining the temperature in the $(n + 1)$st instant of time.

A very small error ε results in a very short time step and consequently in a longer calculation time. But in this way the nonlinear calculation described by Eqs. (13) and (14) can be avoided. Then the material parameters can be assumed constant for a given time step and calculated on the basis of the $\mathbf{T}_n$ value.

It is possible to assume a higher value of the error ε, but then a nonlinear calculation will be necessary at some time steps. They will occur

just at those time steps where the inequality $\varepsilon_A > \varepsilon$ is satisfied.

When considering the nonlinear iteration, the temperature chosen for the calculation of the material parameters at the next iteration is:

$$T_{\text{av}}^k = \frac{T_{n+1}^k + T_n}{2}, \tag{25}$$

where T_{n+1}^k denotes the kth iteration in the $(n + 1)$st instant of time for the temperature, or

$$T_{\text{av}}^k = T_n + \frac{1}{2}\alpha_n^k \Delta t_n. \tag{26}$$

The procedure agrees with the assumed temperature approximation for the nth time step. Nonlinear calculations are repeated using the Newton–Raphson method with the Kirchhoff replacement until the inequalities given below hold simultaneously:

$$\frac{\|[\alpha_n^k] - [\alpha_n^{k-1}]\|}{\|[\alpha_n^{k-1}]\|} \leq \eta_t \tag{27}$$

$$\frac{\|\mathbf{R}_{n+1}^k - \mathbf{R}_{n+1}^{k-1}\|}{\|\mathbf{R}_{n+1}^{k-1}\|} \leq \eta_r, \tag{28}$$

where η_t and η_r are the assumed values of the maximal error for the variable α and the residuum of Eq. (14), respectively.

When these conditions hold, the nonlinear iteration ends. $\mathbf{R}_{n+1}^k$ is a vector of residua of Eq. (14) for the kth nonlinear iteration and the $(n + 1)$st instant of time.

If we use the Kirchhoff replacement

$$\theta = \int_{T_B}^{T} \lambda(T')\, dT' \tag{29}$$

then the partial differential equation, Eq. (2), describing the temperature distribution is expressed as:

$$\nabla^2\theta - \frac{\rho(\theta)C(\theta)}{\lambda(\theta)}\frac{\partial\theta}{\partial t} = -\gamma(\theta)(\text{grad }\varphi)^2. \tag{30}$$

This replacement results in the fact that the nonlinearity of the **A** matrix (7) is caused only by the influence of the boundary condition (3). Therefore, we can assume that for the variable θ the Jacobian matrix for the residuum of Eq. (14) in the kth iteration and at the nth time step will take the following form:

$$\mathbf{J}^k = \mathbf{D}([\theta_{\text{av}}^{k-1}]) + \delta\Delta t_n \mathbf{A}([\theta_{\text{av}}^{k-1}]). \tag{31}$$

In this equation the Jacobian matrix is symmetrical and can be determined in the same way as coefficient matrices in a linear problem.

In a nonlinear process, we look for the solution in the form:

$$[\alpha_n^k] = [\alpha_n^{k-1}] - \omega^k[\Delta\alpha_n^k] \tag{32}$$

where the coefficient ω^k is calculated from the condition

$$\omega^k = \min\left(\nu \frac{\|[\alpha_n^{k-1}]\|}{\|[\Delta\alpha_n^k]\|}, 1\right) \tag{33}$$

and $\nu \in (0.1; 0.2)$.

The correction $[\Delta\alpha_n^k]$ is derived from the matrix equation

$$\mathbf{J}^k[\Delta\alpha_n^k] = \mathbf{R}^{k-1}. \tag{34}$$

Introducing the variable θ has considerably shortened the calculation time in comparison with the calculation time directly for the temperature.

If the error ε_A is still bigger than the previously assumed error ε then we have to halve the time step Δt_n and repeat the calculation. The whole operation should be repeated until the condition $\varepsilon_A \leq \varepsilon$ is satisfied.

NUMERICAL EXPERIMENTS

The process of producing wolfram in a high temperature furnace was simulated numerically. The current passing through the molded bar of wolfram is the source of Joule's heat. As a result of heating, metallic wolfram is produced. The process would fail if the metal is overheated (temperature above 3550K) or if the gradients of temperature are too high, which causes a splitting of the heavy bar. In the article written by Korytkowski and Starzyński (1991), calculations of heating at continuous feed (current with the root-mean-square, rms, value 4250 A) were presented. In the present example, continuous feed has been replaced by an impulse one with a stronger current, in our case 6000 A. If the maxi-

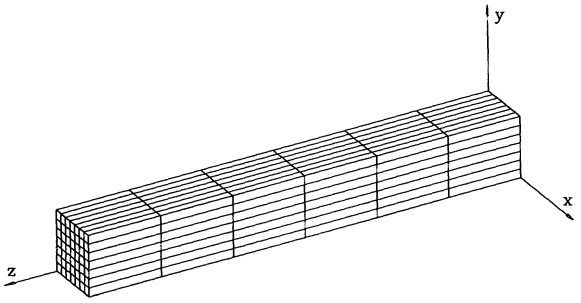

FIGURE 1 Discretization of the model.

mum temperature approaches the melting point, the feed is disconnected. Repeated connection takes place when the minimum temperature of the entire heavy bar attains a value of about 1070K. The starting point is the thermal steady state with continuous feed. The values of coefficients $\delta = 0.75$, $\Delta t_0 = 3$ s were assumed in the calculations.

If we consider that the electric and thermal fields are symmetrical only a quarter of the heavy bar has been examined. We assume that the phenomena on the boundary of the heavy bar are neglected and that the distribution of cooling along the side walls is identical and described by Eq. (3). The model has been discretized by 294 finite cubic 8-node elements (7 layers along the axes $0x$ and $0y$ and 6 layers along the axis $0z$; see Fig. 1).

The numerical simulation has been done for a number of assumed error values ε. In all cases the coefficient δ [Eq. (11)] and time step Δt_n [Eq. (11)] have assigned values, $\delta = 0.75$ and $\Delta t_0 = 3$ s, respectively. All calculations were done on an IBM 486 personal computer with 33-MHz clock.

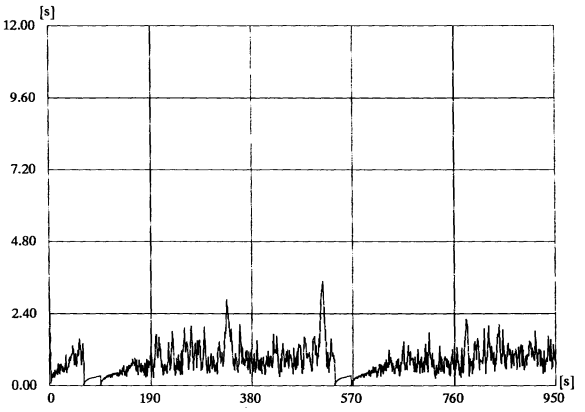

FIGURE 3 Change of time step during calculation process at error $\varepsilon = 15$ (W/m).

The diagram of Fig. 2 presents the change of maximum temperature with an error of $\varepsilon = 15$ (W/m) for the variable θ and Fig. 3 displays the change of time step during the calculation process. The simulation process uses 32,400 of CPU time.

Figure 4 presents the change of maximum temperature with an error of $\varepsilon = 60$ (W/m) and the diagram of Fig. 5 shows the change of time step during the calculations. The procedure used 14,600 of CPU time.

Figure 6 presents the change of maximum temperature with an error of $\varepsilon = 120$ (W/m) and the diagram of Fig. 7 displays the change of time step during calculations. The calculations took 21,300 of CPU time.

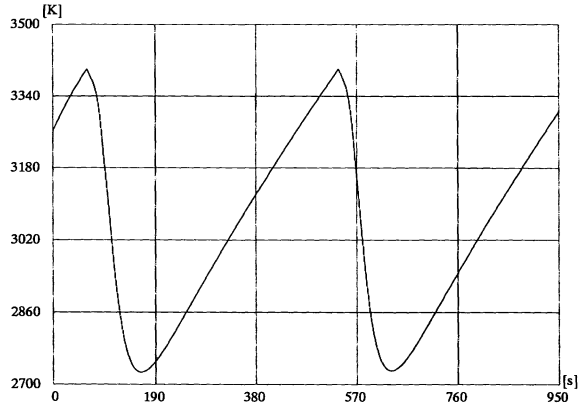

FIGURE 2 Change of maximum temperature at error $\varepsilon = 15$ (W/m).

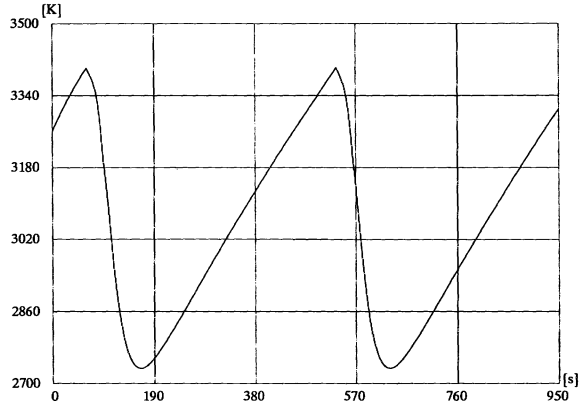

FIGURE 4 Change of maximum temperature at error $\varepsilon = 60$ (W/m).

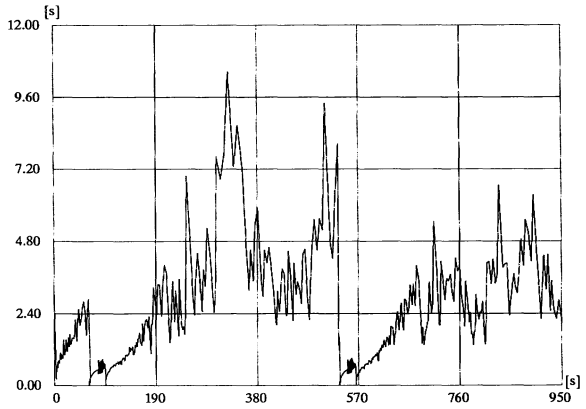

FIGURE 5 Change of time step during calculation process at error ε = 60 (W/m).

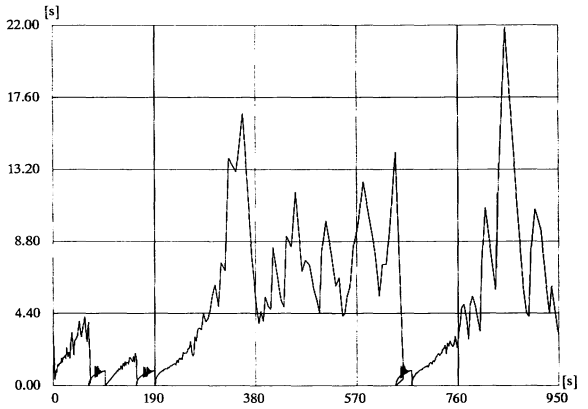

FIGURE 7 Change of time step during calculation process at error ε = 120 (W/m).

CONCLUSIONS

Comparison of the presented results of simulation shows that assuming a higher error value shortens the calculation time but the nature of the temperature changes remains the same (compare Figs. 2 and 4). However, a considerable increase in the error results in temperature changes of an unpredictable character during the simulation (compare Figs. 2 and 6).

When calculating the transient state with great temperature variations, the following definition of the relative error ε can be used

$$\varepsilon^* = \frac{\varepsilon}{\|T_n\|}. \tag{35}$$

The time variation of the temperature in the calculations presented was so small that use of formula (35) was unnecessary.

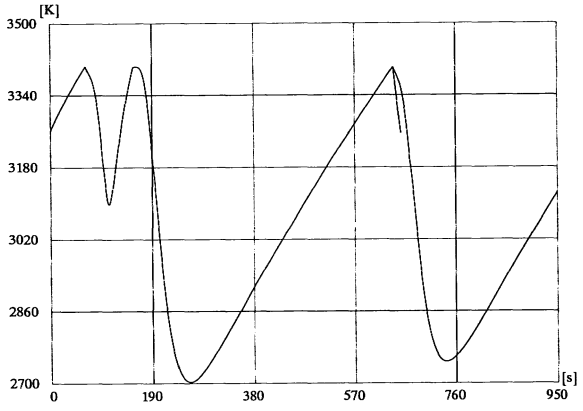

FIGURE 6 Change of maximum temperature at error ε = 120 (W/m).

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MATHEMATICAL SYMBOLS

$\gamma(T)$	conductance
$\lambda(T)$	heat conductivity
$\rho(T)$	mass density
$C(T)$	specific heat
$\alpha_z(T)$	equivalent coefficient that takes into consideration heat convection and heat radiation

$\alpha_n^{(p)}$ average value of the pth temporal derivative of temperature in nth time interval

$[\alpha_n]$ vector of coefficients α_n in points of discretization

$[\Delta\alpha_n^k]$ correction of vector $[\alpha_n]$ in the kth iteration of calculations

$\|\cdot\|$ the Euclidean metric

T_0 the temperature of conductor's surrounding

T_{av}^k average value of temperature for kth iteration and in nth time step

$\alpha(T)$ coefficient that takes into consideration heat transfer by convection

$\varepsilon(T)$ coefficient that takes into consideration heat transfer by radiation

ε error of calculations

$\dot{\mathbf{T}}$ vector of derivatives of temperature in discretization nodes of the domain

$\mathbf{J}$ the Jacobian matrix

$\mathbf{R}$ vector of residua

$V^{(e)}$ volume of the eth element

$S^{(e)}$ surface of the eth element that builds a part of the examined conductor flank

N_i shape functions inside each discretization element

δ constant $\in \langle 0.5; 1\rangle$

ν constant $\in (0.1; 0.2)$

ω coefficient of underrelaxation

θ auxiliary variable for Kirchhoff replacement

Δt_n the length of the nth time step

η_t value of maximum error for temperature

η_r value of maximum error for residuum