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A Modified Lindstedt–Poincaré Method for a Strongly Nonlinear System with Quadratic and Cubic Nonlinearities

A modified Lindstedt–Poincaré method is presented for extending the range of the validity of perturbation expansion to strongly nonlinear oscillations of a system with quadratic and cubic nonlinearities. Different parameter transformations are introduced to deal with equations with different nonlinear characteristics. All examples show that the efficiency and accuracy of the present method are very good. © *1996 John Wiley & Sons, Inc.*

INTRODUCTION

Recently the intense interest in the response of the strongly nonlinear system leads one to consider whether the classical perturbation methods can somehow be recast to be applicable to certain nonlinear systems. Some researchers utilize the simple equation

$$\ddot{x} + \omega_0^2 x + \varepsilon f(x) = p\cos(\omega t), \tag{1}$$

in which the overdots denote differentiation with respect to time t, ω_0 is the natural frequency, and $f(x)$ is a nonlinear function of x. Jones (1978), Burton and Hamdan (1983), and Burton (1984) developed some different perturbation procedures for the strongly nonlinear response of Eq. (1). However, $f(x)$ is restricted to be an odd function of x in their methods. Cheung et al. (1991) previously presented a modified Lindstedt–Poincaré (MLP) method to extend the validity of Lindstedt–Poincaré (LP) method for Eq. (1), in which $f(x)$ can be an even or odd nonlinear function. In the MLP method, a new parameter transformation $\alpha = \alpha(\varepsilon, \omega_0, \omega_i)$ is defined, where ε is a parameter of the original equation and ω_0, ω_i are two components of the expansion of the frequency ω, such that the value of α is always kept small regardless of the magnitude of the original parameter ε. Therefore, a strongly nonlinear system with a large value of $\varepsilon\omega_1$ is transformed into a small parameter system with respect to α. Consequently, the MLP method can extend the application range of the LP method. As a continuation of the previous work, in this article the MLP procedure is employed to study the system (1) in which $f(x)$ contains quadratic and cubic nonlinearities at the same time. Different parameter

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transformations $\alpha(\varepsilon, \omega_0, \omega_i)$ are employed to deal with equations with different nonlinear characteristics.

METHOD OF ANALYSIS

Consider the following simple nonlinear system with quadratic and cubic nonlinearities simultaneously.

$$\ddot{x} + \omega_0^2 x + \bar{k}_2 x^2 + \bar{k}_3 x^3 = \bar{p}\cos(\omega t), \tag{2}$$

where ω_0 is the linear frequency, $\bar{k}_2$ and $\bar{k}_3$ are positive quadratic and cubic nonlinear stiffness coefficients, and $\bar{p}$ and ω are the forced amplitude and forced frequency.

LP Method

For the sake of comparison, the classical LP method will be discussed briefly first. We need to order the nonlinearity and the excitation so that they appear at the same time in the perturbation scheme. We express $\bar{k}_2$ as εk_2, $\bar{k}_3$ as $\varepsilon^2 k_3$, $\bar{p}$ as $\varepsilon^2 p$, and introduce a new variable

$$\tau = \omega t, \tag{3}$$

so that the governing equation, Eq. (2), becomes

$$\omega^2 x'' + \omega_0^2 x + \varepsilon k_2 x^2 + \varepsilon^2 k_3 x^3 = \varepsilon^2 p\cos(\tau), \tag{4}$$

where primes denote differentiation with respect to τ and ε is a small parameter. According to the standard LP method, both ω and x are usually expanded in powers of ε,

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2^2 + \cdots, \tag{5}$$

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2^2 + \cdots. \tag{6}$$

Substituting Eqs. (5) and (6) into (4) and equating the coefficients of ε^0, ε, and ε^2 to zero, we obtain

$$x_0'' + x_0 = 0, \tag{7}$$

$$\omega_0^2 x_1'' + \omega_0^2 x_1 = -2\omega_0\omega_1 x_0'' - k_2 x_0^2, \tag{8}$$

$$\begin{aligned}\omega_0^2 x_2'' + \omega_0^2 x_2 = {} & -2\omega_0\omega_1 x_1'' - 2k_2 x_0 x_1 \\ & - (\omega_1^2 + 2\omega_0\omega_2)x_0'' \\ & - k_3 x_0^3 + p\cos(\tau).\end{aligned} \tag{9}$$

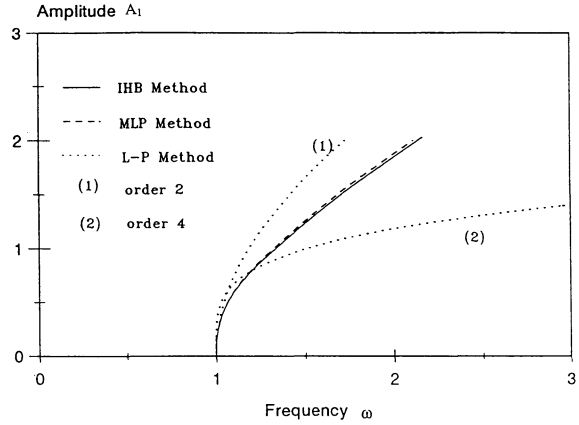

FIGURE 1 Free vibration response curve ω–A_1 for $\varepsilon = 1$, $\omega_0 = 1$, $k_2 = 0.8$, $k_3 = 1.2$, and $p = 0$.

x_0 and ω_1, x_1 and ω_2, and x_2 can be solved sequentially. Finally the solution is expressed as

$$\begin{aligned}x = {} & a\cos(\tau) - \frac{1}{2}\varepsilon a^2 k_2 \omega_0^{-2}\left[1 - \frac{1}{3}\cos(2\tau)\right] \\ & + \frac{1}{96}\varepsilon^2 a^3 \omega_0^{-4}(2k_2^2 + 3k_3\omega_0^2)\cos(3\tau) \\ & + 0(\varepsilon^3),\end{aligned} \tag{10}$$

$$\omega = \omega_0\left[1 + \frac{9k_3\omega_0^2 - 10k_2^2}{24\omega_0^2}\varepsilon^2 a^2\right] - \frac{p}{2\omega_0 a} + (\varepsilon^3). \tag{11}$$

Nayfeh and Mook (1979) proved that if the value of εa is small, this solution is in full agreement with the harmonic balance method solution. If the value of εa is large, this solution has some obvious errors. For example, one only needs to examine the response curves shown in Figs. 1–6

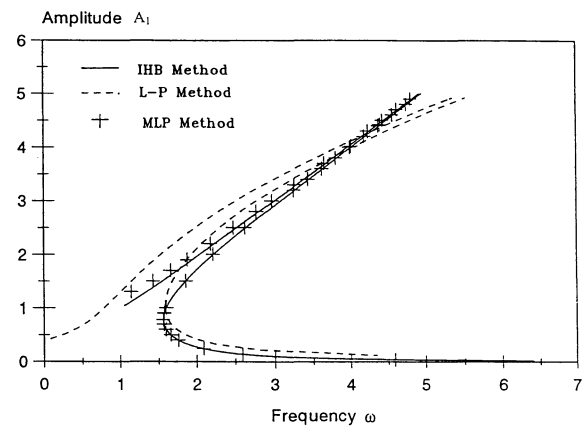

FIGURE 2 Forced vibration response curve ω–A_1 for $\varepsilon = 1$, $\omega_0 = 1$, $k_2 = 0.8$, $k_3 = 1.2$, and $p = 0.8$.

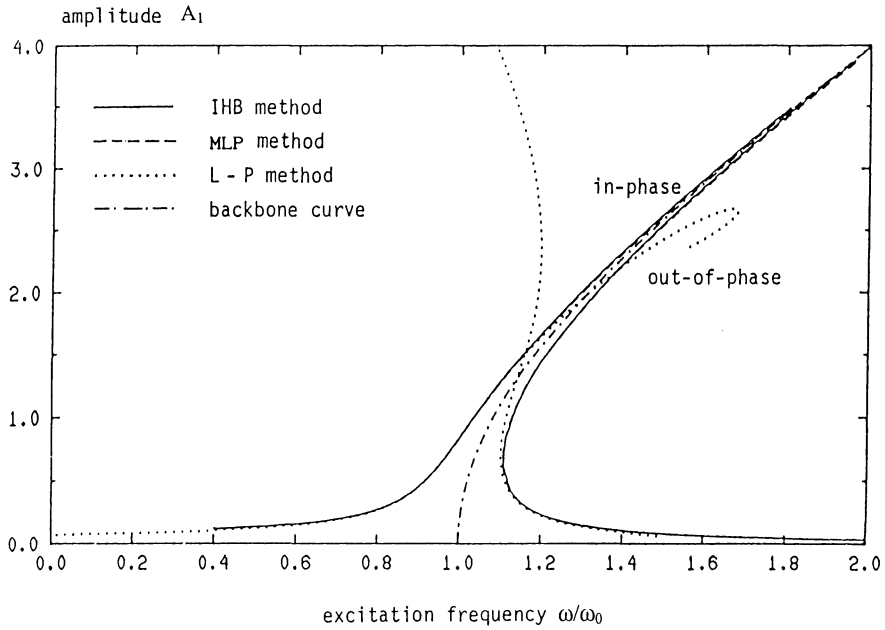

FIGURE 3 Forced vibration response curve ω/ω_0–A_1 for $\varepsilon = 1$, $\omega_0 = 9.97$, $k_2 = 14.6$, $k_3 = 24.4$, and $p = 10$.

to find that the range of validity of the LP method in fact is very narrow. This means that the LP method is only suitable for the weakly nonlinear vibrations. Therefore, it is necessary and important to extend the range of the validity of the LP method to strongly nonlinear vibrations.

MLP Method

Cheung et al. (1991) presented an MLP method to extend the validity of the LP method. To apply this MLP method to Eq. (2), three procedure steps differing from the standard LP method are

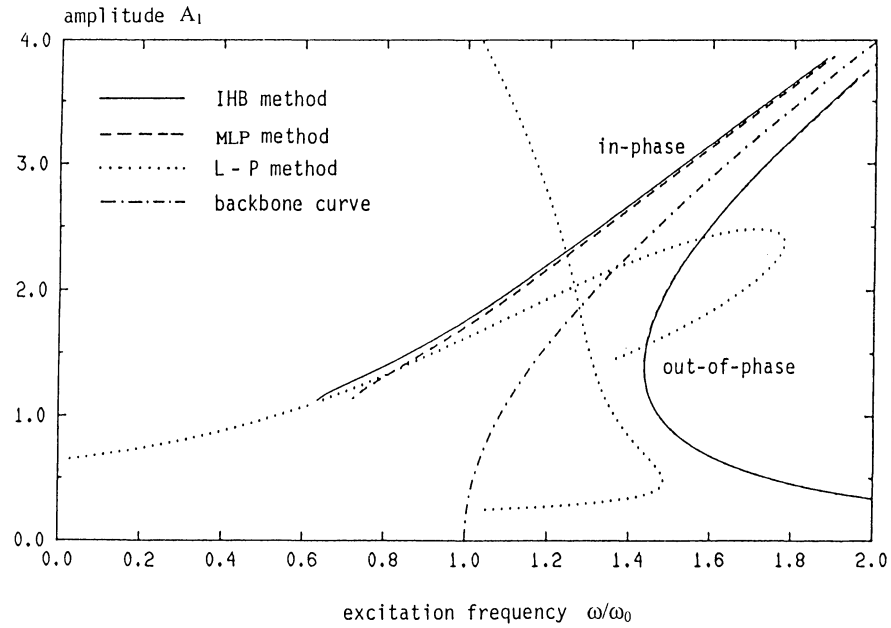

FIGURE 4 Forced vibration response curve ω/ω_0–A_1 for $\varepsilon = 1$, $\omega_0 = 9.97$, $k_2 = 14.6$, $k_3 = 24.4$, and $p = 100$.

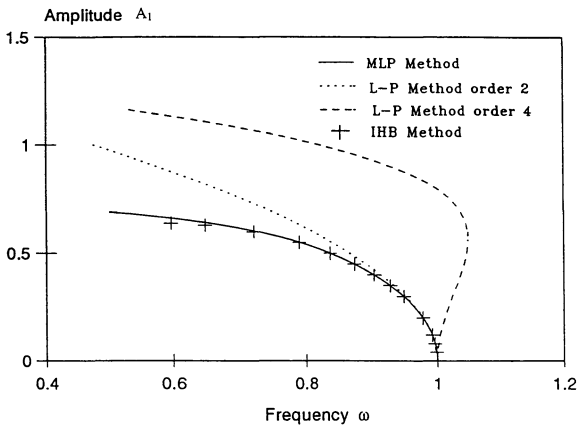

FIGURE 5 Free vibration response curve ω–A_1 for $\varepsilon = 1$, $\omega_0 = 1$, $k_2 = 1.2$, $k_3 = 0.2$, and $p = 0$.

included. The first one is to expand ω^2, instead of ω, in a power series of ε,

$$\omega^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots. \tag{12}$$

The point to be made is that Eq. (4) is a second-order differential equation. The frequency perturbation therefore appears naturally as a perturbation of ω^2 rather than ω. The second one is to define a new parameter

$$\alpha = \alpha(\varepsilon, \omega_0, \omega_i). \tag{13}$$

It is required that $0 < \alpha < 1$ for any value of ε and ω_i. So α can be selected as a new small perturbation parameter. Then ε can be written in terms of α from Eq. (13),

$$\varepsilon = \varepsilon(\alpha, \omega_0, \omega_i). \tag{14}$$

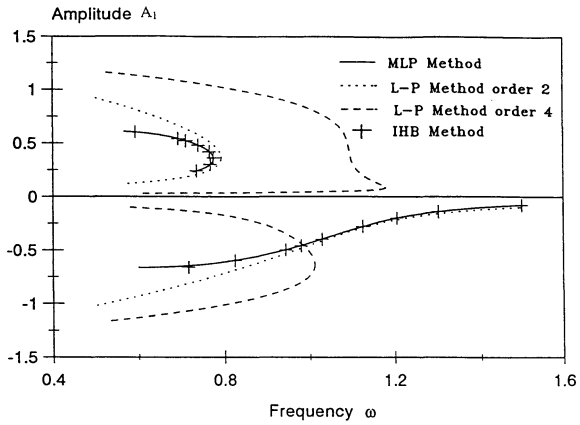

FIGURE 6 Forced vibration response curve ω–A_1 for $\varepsilon = 1$, $\omega_0 = 1$, $k_2 = 1.2$, $k_3 = 0.2$, and $p = 0.1$.

Hence ω^2 is rewritten in terms of α from Eq. (12),

$$\omega = \sum_{n=0} \delta_n \alpha^n. \tag{15}$$

The third one is to expand x into a power series of α,

$$x = x_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2^2 + \cdots. \tag{16}$$

Substituting Eqs. (14)–(16) into (4) and equating like powers of α, we obtain a set of perturbation equations as Eqs. (7)–(9). After that, the usual steps in the LP method may be applied to solve these perturbation equations.

According to Eq. (11), if $9k_3\omega_0^2 > 10k_2^2$, i.e., $9\bar{k}_3\omega_0^2 > 10\bar{k}_2^2$, the nonlinearity has a hardening effect that tends to bend the frequency–amplitude response curve to higher frequency. Conversely, if $9\bar{k}_3\omega_0^2 < 10\bar{k}_2^2$, the nonlinearity has a softening effect and the frequency–amplitude response curve will bend to the side of lower frequency. We will discuss these two cases separately and use different parameter transformations to deal with them. In another situation, when $9\bar{k}_3\omega_0^2 = 10\bar{k}_2^2$, the nonlinearity has no effect on the response to this order because the effect of the quadratic and the cubic nonlinearities cancel each other.

Hardening Spring Characteristic Equation

If $9\bar{k}_3\omega_0^2 > 10\bar{k}_2^2$, the cubic nonlinearity will play an important role in the nonlinear response. To use the perturbation method, we express $\bar{k}_2$ as εk_2, $\bar{k}_3$ as εk_3, and $\bar{p}$ as εp in this case so that the governing equation, Eq. (2), becomes

$$\omega^2 x'' + \omega_0^2 x + \varepsilon k_2 x^2 + \varepsilon k_3 x^3 = \varepsilon p \cos(\tau). \tag{17}$$

Next we introduce a parameter transformation,

$$\alpha = \frac{\varepsilon\omega_1}{\omega_0^2 + \varepsilon\omega_1} \quad \omega_1 > 0. \tag{18}$$

It can be seen from Eq. (18) that the value of α is always kept small regardless of the magnitude of $\varepsilon\omega_1$. When $\varepsilon\omega_1 \to 0$ then $\alpha \to 0$, while $\alpha \to 1$ as $\varepsilon\omega_1 \to \infty$, i.e., $0 < \alpha < 1$. Therefore, α can be selected as a new small parameter. Then ε can be written in terms of α from Eq. (18),

$$\varepsilon = \frac{\omega_0^2\alpha}{\omega_1(1-\alpha)} \tag{19}$$

and

$$\omega_0^2 + \varepsilon\omega_1 = \frac{\omega_0^2}{(1-\alpha)}. \tag{20}$$

ω^2 can also now be expressed in terms of α from Eqs. (12), (19), and (20),

$$\omega^2 = \frac{\omega_0^2}{(1-\alpha)}(1 + \delta_2\alpha^2 + \delta_3\alpha^3 + \cdots). \tag{21}$$

Moreover, we assume that x can be represented by an expansion having the form,

$$x = \sum_{n=0}^{\infty} \alpha^n x_n. \tag{22}$$

Substituting Eqs. (20)–(22) into (17) and equating the coefficients of α^0, α, α^2, and α^3 to zero, we obtain the following perturbation equations.

$$x_0'' + x_0 = 0 \tag{23}$$

$$x_1'' + x_1 = x_0 - \frac{1}{\omega_1}(k_2x_0^2 + k_3x_0^3 - p\cos\tau) \tag{24}$$

$$x_2'' + x_2 = x_1 - \delta_2x_0'' - \frac{1}{\omega_1}(2k_2x_0x_1 + 3k_3x_0^2x_1) \tag{25}$$

$$\begin{aligned} x_3'' + x_3 = {} & x_2 - \delta_3x_0'' - \delta_2x_1'' \\ & - \frac{1}{\omega_1}k_2(x_1^2 + 2x_0x_2) \\ & - \frac{3}{\omega_1}k_3(x_0^2x_2 + x_0x_1^2). \end{aligned} \tag{26}$$

ω_1, δ_2, δ_3, and x_i $(i = 0, 1, 2, 3)$ can be determined successively just like those in the LP method. Finally the solution of Eq. (17) to $0(\alpha^4)$ is expressed as

$$x = \sum_{i=0}^{7} A_i\cos(i\tau) \tag{27}$$

$$\omega_1 = \frac{3}{4}k_3A_1^2 - \frac{p}{A_1} \tag{28}$$

$$\delta_2 = \frac{1}{384\omega_1^2}(9k_3^2A_1^4 - 320k_2^2A_1^2) \tag{29}$$

$$\begin{aligned} \delta_3 = {} & \frac{-1}{\omega_1^2}\left(\frac{19}{18}k_2^2A_1^2 + \frac{3}{1024}k_3^2A_1^4\right) \\ & + \frac{1}{\omega_1^3}\left(\frac{53}{32}k_2^2k_3A_1^4 + \frac{3}{1024}k_3^2A_1^6\right), \end{aligned} \tag{30}$$

where A_i is not presented here in detail due to space limitation. From the above analysis one can find that Eqs. (18), (21), and (22) are the three main procedural steps of the MLP method that differ from the standard LP method. The key step is how to find the parameter transformation. Comparing Eq. (5) with Eq. (21), we can also find that the essential idea of the LP method is to expand the frequency ω in the power series near the linear frequency ω_0 in terms of ε. However, the essential idea of the MLP method is to expand ω^2 in the power series near $\omega_0^2 + \varepsilon\omega_1$ in terms of α. Therefore, the MLP method will yield better results than those of the LP method.

Softening Spring Characteristic Equation

If $9\bar{k}_3\omega_0^2 < 10\bar{k}_2^2$, the quadratic nonlinearity will play an important role in nonlinear response. The governing equation in this case is adopted as Eq. (4). If the expression of ω^2 is assumed in the form of Eq. (12), it will produce the result of $\omega_1 = 0$ and α will be zero according to Eq. (19). Therefore ω^2 should be assumed as

$$\omega^2 = \omega_0^2 + \varepsilon^2\omega_2 + \varepsilon^3\omega_3 + \cdots. \tag{31}$$

Note that ω_2 will be negative so that α should be defined as

$$\alpha^2 = \frac{-\varepsilon^2\omega_2}{\omega_0^2 + \varepsilon^2\omega_2} \quad \text{for} \quad -\frac{\omega_0^2}{2\varepsilon^2} < \omega_2 < 0, \tag{32}$$

then

$$\varepsilon^2 = \frac{\omega_0^2\alpha^2}{-\omega_2(1+\alpha^2)}, \tag{33}$$

$$\omega^2 = \frac{\omega_0^2}{(1+\alpha^2)}(1 + \delta_2\alpha^2 + \delta_3\alpha^3 + \cdots). \tag{34}$$

Instead of Eqs. (23)–(26), the perturbation equations in this case are

$$x_0'' + x_0 = 0 \tag{35}$$

$$x_1'' + x_1 = -\frac{1}{\omega_0\sqrt{-\omega_2}}k_2x_0^2 \tag{36}$$

$$\begin{aligned} x_2'' + x_2 = {} & -x_0 - \frac{2}{\omega_0\sqrt{-\omega_2}}k_2x_0x_1 \\ & + \frac{1}{\omega_2}k_3x_0^3 - \frac{p}{\omega_2}\cos\tau \end{aligned} \tag{37}$$

$$x_3'' + x_3 = -\delta_3 x_0'' - x_1 - \frac{1}{\omega_0\sqrt{-\omega_2}} k_2\left(x_1^2 + 2x_0x_2 + \frac{1}{2}x_0^2\right) + \frac{3}{\omega_2}k_3x_0^2x_1 \tag{38}$$

$$x_4'' + x_4 = -\delta_3 x_1'' - \delta_4 x_0'' - x_2 - \frac{1}{\omega_0\sqrt{-\omega_2}} k_2(2x_0x_3 + 2x_1x_2 + x_0x_1) + \frac{3}{\omega_2}k_3(x_0^2x_2 + x_0x_1^2). \tag{39}$$

The solution to $O(\alpha^5)$ is given by following:

$$x = \sum_{i=0}^{5} A_i \cos(i\tau) \tag{40}$$

$$\omega_2 = \left(\frac{3}{4}k_3 - \frac{5}{6}k_2^2\omega_0^{-2}\right)A_1^2 - \frac{p}{A_1} \tag{41}$$

$$\delta_3 = 0 \tag{42}$$

$$\delta_4 = -\frac{2}{9\omega_0^2\omega_2}k_2^2A_1^2 - \frac{55}{96\omega_0^4\omega_2^2}k_2^4A_1^4 + \frac{53}{32\omega_0^2\omega_2^2}k_2^2k_3A_1^4 + \frac{3}{128\omega_2^2}k_3^2A_1^4. \tag{43}$$

A_i is not presented here in detail due to space limitation.

It is worth pointing out that the restriction of $-\omega_0^2/2\varepsilon^2 < \omega_2 < 0$ in transformation (32) is to ensure $0 < \alpha^2 < 1$, so that α can be employed as a small parameter for the later perturbation procedure.

EXAMPLES AND DISCUSSION

To illustrate the effectiveness and accuracy of the MLP method mentioned above, some examples are presented and discussed in this section. Three methods, the LP method, the MLP method, and the incremental harmonic balance (IHB) method derived by Cheung and colleagues (1990) are used to calculate the solution for comparing the accuracy of these methods. The IHB method has been shown to be an efficient and reliable numerical method for treating the vibration of strongly nonlinear systems. Therefore, it can be used to check the accuracy of various perturbation methods.

EXAMPLE 1. Consider the free and forced vibration of Eq. (2) for $\omega_0 = 1$, $\bar{k}_2 = 0.8$, $\bar{k}_3 = 1.2$, $\bar{p} = 0$, and $\bar{p} = 0.8$, respectively. Because $\bar{k}_2$, $\bar{k}_3$, and ω_0 satisfy the relation $9\bar{k}_3\omega_0^2 - 10\bar{k}_2^2 > 0$, Eq. (2) has hardening spring characteristics so that the transformation $T1$ (18) is employed. The free vibration response curves ω–A_1 obtained by using various methods are shown in Fig. 1 while the forced response curves ω–A_1 are shown in Fig. 2.

EXAMPLE 2. Consider the forced vibration of Eq. (2) for $\omega_0 = 9.97$, $\bar{k}_2 = 14.6$, $\bar{k}_3 = 24.4$, $\bar{p} = 10$, and $\bar{p} = 100$, respectively. This is a practical example of the nonlinear vibration of an elastic shallow arch subjected to a very strong force. Obviously, Eq. (2) in this case also has hardening spring characteristics and therefore the $T1$ transformation is adopted. The fundamental resonance for $\bar{p} = 10$ is shown in Fig. 3 and the resonance for $\bar{p} = 100$ is shown in Fig. 4.

EXAMPLE 3. Consider the free and forced vibration of Eq. (2) for $\omega_0 = 1$, $\bar{k}_2 = 1.2$, $\bar{k}_3 = 0.8$, $\bar{p} = 0$, and $\bar{p} = 0.1$, respectively. Because $\bar{k}_2$, $\bar{k}_3$, and ω_0 in this case satisfy the relation $9\bar{k}_3\omega_0^2 - 10\bar{k}_2^2 < 0$, then Eq. (2) has softening spring characteristics. Therefore the transformation $T2$ (32) is employed. The response curves ω–A_1 for free vibration are shown in Fig. 5 and the forced response curves ω–A_1 are shown in Fig. 6.

It can be seen from Figs. 1 to 6 that the efficiency and accuracy of the MLP method solution are very good. The solutions of the MLP method are all in good agreement with those of the IHB method in the whole domain, but the solutions of the LP method are very far away from the correct solutions except for a very narrow range. Obviously, the errors of the LP method are unacceptable in most of the domain.

It is a pleasant surprise to find from Fig. 4 that the response curves of the MLP method under very strong excitation are still in good agreement with those of the IHB method. However, the errors of both the in-phase and the out of phase resonance of the LP method, either for the small value or the large value of $A1$, are not acceptable at all.

Figures 1, 5, and 6 also reveal one interesting phenomenon, i.e., the higher order approximate solution of the LP method in fact cannot improve the error of the solution. The LP method requires that the ε must be a small parameter and $\varepsilon\omega_1 < \omega_0$. Under these conditions the lower order solution is good enough to obtain the correct solu-

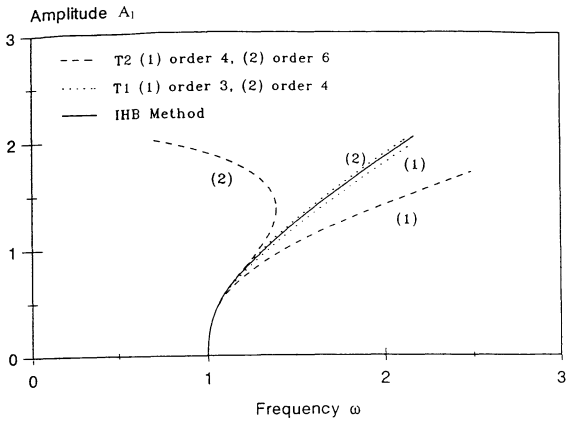

FIGURE 7 Comparison between transformations $T1$ and $T2$ for $\varepsilon = 1$, $\omega_0 = 1$, $k_2 = 0.8$, $k_3 = 1.2$, and $p = 0$.

tion. If $\varepsilon\omega_1 > \omega_0$, then the lower order solution fails and the higher order solution also loses its effectiveness, and at times may even yield a worse solution. Therefore, any effort to try to obtain a more accurate solution by carrying out higher order perturbation is useless in the range of large amplitudes.

It is worth pointing out that transformation $T1$ is suitable for the hardening characteristics equation. If we employ it to solve the softening characteristic equation, the result is not so good. Conversely, the transformation $T2$ is suitable for the softening characteristic equation but not for the hardening one. To illustrate this, we use $T2$ to solve the free vibration of the hardening characteristic equation, which is discussed in example 1. The response curves ω–A_1 obtained by $T1$ and $T2$ are shown in Fig. 7. Obviously, the results of $T1$ are close to that of the IHB method while the results of $T2$ are far away from the correct solution in the range of large amplitudes. Furthermore, the higher order approximation solution of $T1$ is more accurate than that of the lower order. However, the higher order approximation solution of $T2$ is worse than that of the lower order. Similarly, if $T1$ is used to calculate the response of a softening characteristic equation, such as example 3, then it will fail just as for the case of using $T2$ to calculate the hardening spring equation.

CONCLUDING REMARKS

1. The MLP method is an effective method for extending the range of validity of the conventional perturbation method to strongly nonlinear oscillation. The procedure is just as simple as the LP method and is easy to use.
2. It should be identified first whether the system with quadratic and cubic nonlinearities simultaneously is in fact hardening or softening before using the MLP method. Different parameter transformations are adopted for different nonlinear characteristics.
3. The range of validity of the classical LP method is in fact very narrow for strongly nonlinear vibration. Furthermore, the higher order approximate solutions cannot improve the error of solution in the range of large amplitudes.

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