Collisions of constrained rigid body systems with friction

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A new approach is developed for the general collision problem of two rigid body systems with constraints (e.g., articulated systems, such as massy linkages) in which the relative tangential velocity at the point of contact and the associated friction force can change direction during the collision. This is beyond the framework of conventional methods, which can give significant and very obvious errors for this problem, and both extends and consolidates recent work. A new parameterization and theory characterize if, when and how the relative tangential velocity changes direction during contact. Elastic and dissipative phenomena and different values for static and kinetic friction coefficients are included. The method is based on the explicitly physical analysis of events at the point of contact. Using this method, Example 1 resolves (and corrects) a paradox (in the literature) of the collision of a double pendulum with the ground. The method fundamentally subsumes other recent models and the collision of rigid bodies; it yields the same results as conventional methods when they would apply (Example 2). The new method reformulates and extends recent approaches in a completely physical context.

Keywords: Collisions, impact, rigid body systems, constraints, motion reversal, friction

1. Introduction

In rigid body mechanics, a collision of two bodies (or systems) is assumed to be of very brief duration, with a single point of contact. The dominant forces during contact are induced by the collision, therefore impulsive in nature and internal to the combined system, and are so large that other applied and external forces (e.g., gravity) can be ignored during the collision. (Derivation of the equations of motion below validate this assumption.) At the point of contact, elastic, dissipative and frictional effects are significant and can be treated by approximations. During the collision, the system coordinates do not change, but the two colliding systems experience step changes in the velocities. Before and after contact, each system undergoes independent motion. (This includes the case where the bodies adhere at the point of contact.) Velocities and geometries prior to contact are specified; post-collision velocities are to be determined.

Most analyses of such collisions still follow Whittaker [15], e.g., Groesberg [2] and Kane and Levinson [4]. They focus on the pre- and post-collision velocities and generally eliminate, or ignore, what happens during the contact. With a few exceptions, these "conventional methods" are formulated in terms of two unconstrained rigid bodies and for systems with zero or unidirectional friction forces. However, these methods can and do give significant and very obvious errors when the relative tangential velocity at the point of contact (slip velocity) can change direction during a collision; this requires that the friction force also reverses, which is beyond the framework of conventional methods. This can happen when constrained and multi-body systems collide: an "interesting" example is presented in Kane and Levinson [4, problem 14.6] in which the end of a double pendulum hits a stationary surface. Their analysis shows two instances of increased kinetic energy during the collision for selected (and reasonable) values of the elastic and frictional dissipative parameters. (In their analysis the friction force resists motion in the reversed (rebound) direction, a reasonable observation. Normally, by the conventional analysis [2,4,15], friction resists the initial (approach) relative motion; this solution still shows one such instance of energy increase. We discovered this by trial-and-error. Results and more discussion are presented in Example 1.)

Kane previously wrote this up as "A Dynamics Puzzle" for the *Stanford Mechanics Alumni Newsletter* in 1984, which stimulated an analysis by Keller [5] in

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the context of two rigid bodies and a single coefficient of friction. To consider the reversal, he used the normal impulse as a new independent variable (instead of time) "to eliminate the normal force from the equations of motion". This is a highly mathematical analysis, somewhat non-physical, and restricted in the sense that some situations (and quantities) are difficult to evaluate. Only a single friction coefficient (presumably kinetic) is used, so either the static and kinetic coefficients must be the same, or the physical effect of different kinetic and static friction on the tangential impulse when the slip velocity becomes zero is lost. Not all possibilities of slip reversal are presented (see case (i) later). Kane's problem is not "corrected", nor could it be: Keller's method would give incorrect (or incomplete) results, since different values of the static and kinetic friction coefficients are not considered. Recently, Stronge [9] employed a similar parameterization and logic to address possible reversal of motion; he too addresses primarily rigid bodies and very simple constrained systems. His formulation generally follows Whittaker [15], and it would appear to be more complicated for complex systems than a Lagrange–Kane type approach developed here (cf. [2] or that of Rosen and Edelstein [6]. In our method, all of these limitations are overcome with fewer, and no new, assumptions.

In addition, Stronge [8–12] has introduced and vigorously promoted a new "internal dissipation" model for the collision process. However, it appears that the choice of model is somewhat arbitrary, based on the value of the coefficient of restitution, and that a case can be made for any of the collision models [1,7,12–14]; for convenience and from the various arguments, we employ the Poisson impulse hypothesis, as discussed in the next section.

This paper addresses the general collision problem of two rigid body systems with constraints (such as massy linkages), in an explicitly physical context, including different friction coefficients. The method inherently includes the collision of rigid bodies, eccentric collisions, etc., and will yield the same results as the conventional methods when they would apply. Indeed, the conventional as well as the methods in the recent literature clearly can be embedded in the new method.

In the following, first the motions and forces at the point of contact are characterized, then equations of motion are developed for a completely general problem. Next, a new parameterization and theory are developed for a finite, but brief, collision, in terms of normal and tangential relative velocity changes and the associated impulsive forces. This leads directly to the solution (algorithm) for the post-collision system motions. Two examples are presented, with commentary to show what happens during the collision. Example 1 is the Kane and Levinson problem; Example 2 is a standard collision with friction of two "large" spheres, from Groesberg [2]. Comparisons with the conventional methods are made.

2. Analysis

The basic assumptions were given in the opening paragraph of the Introduction. By hypothesis, the two systems collide at a single point of contact P, identified by P_1 in system 1 and P_2 in system 2, per Fig. 1(a). At P we can define a tangent plane and its normal, such that the relative velocity of the surfaces at P can be written in terms of tangential and normal components:

$$\mathbf{v}^{P_2} - \mathbf{v}^{P_1} = u_\mathrm{n}\mathbf{n} + u_\mathrm{t}\mathbf{t}. \tag{1}$$

The tangential component u_t is also called the slip velocity. Both u_t and the normal (approach) velocity u_n are variables during the collision. The unit vectors **t** and **n** are determined by $\mathbf{v}^{P_2} - \mathbf{v}^{P_1}$ at the onset of the collision. Note that the collision geometry can always be characterized in terms of two such intersecting planes.

2.1. Collision system and forces

The collision-induced forces can now be expressed in terms of normal and tangential forces, i.e., $F\mathbf{n} + \tau\mathbf{t}$, per Fig. 1(b). F and τ are internal to the collision system and comprise equal and opposing forces: F resists the compression at the point of contact, and τ acts on each body to oppose the tangential relative velocity at the contact point. By hypothesis, these forces are very large and not calculable without subsidiary assumptions. However, over the duration of the collision, $\Delta t = t_2 - t_1$, their impulses, defined by the integrals

$$\mathcal{F} = \lim_{\Delta t\to 0}\int_{t_1}^{t_2}(F\mathbf{n} + \tau\mathbf{t})\,\mathrm{d}t \,\underline{\triangle}\, \widehat{F}\mathbf{m} + \widehat{\tau}\mathbf{t}, \tag{2}$$

where

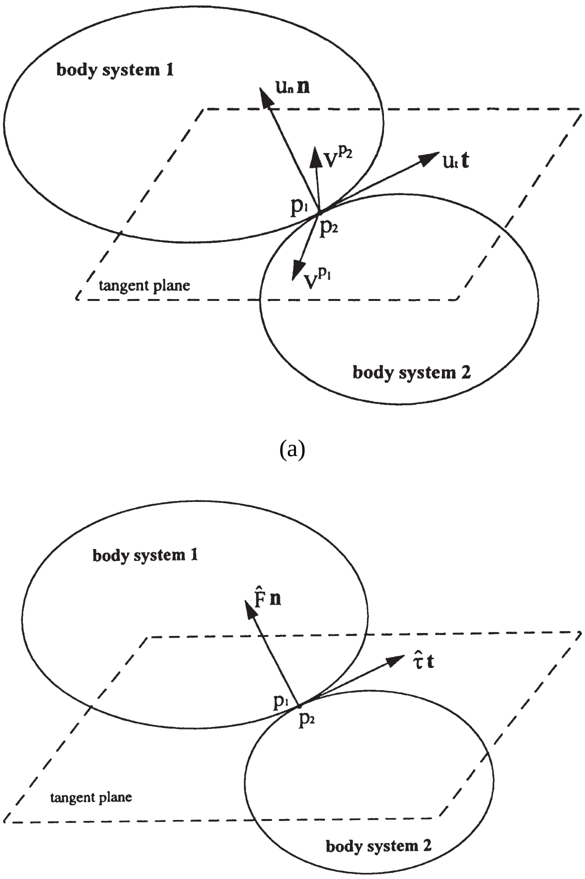

Fig. 1. (a) Collision system geometry: relative velocities. (b) Impulsive forces.

$$
\begin{aligned}
\widehat{F} &= \lim_{\Delta t \to 0} \int_{t_1}^{t_2} F \, \mathrm{d}t, \\
\widehat{\tau} &= \lim_{\Delta t \to 0} \int_{t_1}^{t_2} \tau \, \mathrm{d}t,
\end{aligned}
\tag{3}
$$

are assumed to be well-defined and finite. In the analysis, it is assumed that Δt is very small, but nonzero.

In the normal direction, the collision is assumed to be unimodally elastic, so the general phenomenon can be modeled as consisting of a compression phase followed by a restitution (or rebound) phase, subscripted c and r, respectively, per Fig. 2. Without loss of generality, let the collision start at $t_1 = 0$. The compression phase terminates when the normal force is a maximum; at this "time" (t_c) the relative velocity u_n becomes zero (then reverses). At the end of the collision (at t_s) the total normal impulse during contact $\widehat{F}_\mathrm{s}$ can be expressed as the sum of the normal impulses during the compression and restitution phases, $\widehat{F}_\mathrm{c} + \widehat{F}_\mathrm{r}$, where the latter is defined over the period $\Delta t_\mathrm{r} = t_\mathrm{s} - t_\mathrm{c}$. The elastic and dissipative character of a collision is conventionally expressed in terms of these phases by a coefficient of restitution e:

$$
e = \frac{\lim_{\Delta t_\mathrm{r} \to 0} \int_{t_\mathrm{c}}^{t_\mathrm{s}} F \, \mathrm{d}t}{\lim_{\Delta t_\mathrm{c} \to 0} \int_0^{t_\mathrm{c}} F \, \mathrm{d}t} = \frac{\widehat{F}_\mathrm{r}}{\widehat{F}_\mathrm{c}},
\tag{4}
$$

where $0 \leqslant e \leqslant 1$. Therefore, the total normal impulse over the duration of the collision is

$$
\widehat{F}_\mathrm{s} = \widehat{F}_\mathrm{c} + \widehat{F}_\mathrm{r} = (1+e)\widehat{F}_\mathrm{c}.
\tag{5}
$$

This is the Poisson, or impulse, hypothesis (cf. [8]). There are other models. From the literature, this is probably superior to Newtonian (relative velocity) model [1,7,12–14]. Stronge in [8,12] makes a strong case for an "internal dissipation" hypothesis. However, the coefficient of restitution is an approximation, so any of these assumptions can, in principle, be accommodated. From the discussion of Wang and Mason [13,14] and Stoianovici and Hurmuzlu [7], this model has certain advantages. For our purposes, the Poisson model is both consistent with the dynamic analysis (below) and has a solid physical basis.

There seems to be reasonable agreement on the assumption that resistance to motion in the tangential direction can be modeled by Coulomb friction. In this analysis, this is characterized by a kinetic coefficient of friction μ when slipping and a static coefficient of friction μ_s during sticking or rolling; in general, $\mu \leqslant \mu_\mathrm{s}$. During slipping, the tangential force $\tau = \mu F$ and opposes the relative tangential velocity of each body. If, to sustain equilibrium, it is found that $\tau < \mu F$, sticking occurs, whence u_t in Eq. (1) is zero. Subsequently, if a value of τ to maintain sticking attains $\mu_\mathrm{s} F$, slipping resumes with $\tau = \mu F$ in opposition to the relative tangential motion, as before. Since the forces and impulses are proportional, their impulses obey the same rules. Therefore, over any interval in which slipping occurs,

$$
\widehat{\tau} = \mu \widehat{F}.
\tag{6}
$$

Comment. Absolute precision about the sense of τ and $\widehat{\tau}$ throughout the collision is essential (per erroneous and "anomalous" results in the literature). τ and $\widehat{\tau}$ always oppose incipient as well as actual tangential relative motion. For example, Eq. (1) is the velocity of

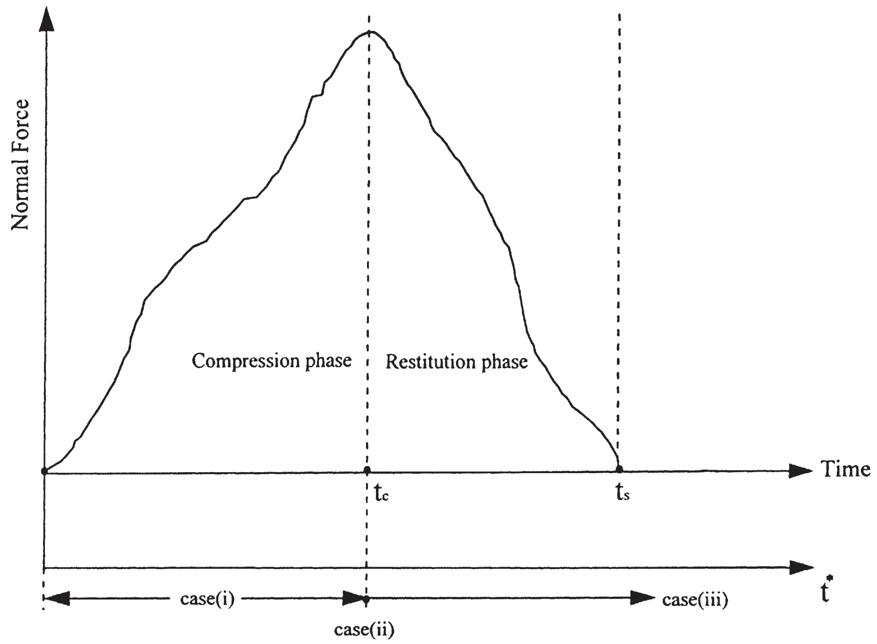

Fig. 2. Collision normal force vs time and slip time parameter.

P_2 relative to P_1; in Fig. 1(b), the tangential impulse $\widehat{\tau}$ acts on P_1 in the positive **t** direction according to sign(u_t), and oppositely on P_2. That is, for this example $\widehat{\tau}_1 = \widehat{\tau}$, and $\widehat{\tau}_2 = -\widehat{\tau}_1$.

2.2. Equations of motion

The equations of motion during the collision can be set up by any method, e.g., Newton, Lagrange, Kane, etc. The Lagrange formulation (to which Kane's [4] is essentially equivalent) is used here; for multibody systems, it is completely general and avoids consideration of constraint forces. (The implementation of Newton's method is given in a later, so-named, section.)

Body system 1 is defined by m generalized coordinates and speeds, $\{q_i\}$ and $\{\dot{q}_i\}$, $i = 1, m$, and kinetic energy T_1; similarly, system 2 is defined by n-m generalized coordinates and speeds $\{q_i\}$ and $\{\dot{q}_i\}$, $i = m+1, n$, and T_2. For each system k, Lagrange's equations of the first kind are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_k}{\partial \dot{q}_i}\right) - \frac{\partial T_k}{\partial q_i} = Q_i, \quad i(k) = 1, n. \tag{7}$$

The notation $i(k) = 1, n$ means $k = 1$ for $i = 1, m$ $k = 2$ for $i = m+1, n$: each system is identified with (and by) its own generalized coordinates. Eq. (7) is n equations in the n generalized coordinates. The totality of the generalized coordinates and speeds are denoted by $\mathbf{q} = \{q_i\}$ and $\dot{\mathbf{q}} = \{\dot{q}_i\}$, $i = 1, n$. As the generalized forces Q_i are associated with the respective generalized coordinates, they do not require the explicit body designation k. Therefore, each body system is considered individually and related to the other system only through the collision forces' contributions to the generalized forces during the contact. That is, the Q_i are

$$Q_i = Q_i^k + a_i(\mathbf{q})F_k + b_i(\mathbf{q})\tau_k, \quad i(k) = 1, n;\ k = 1, 2. \tag{8}$$

The Q_i^k are the usual applied and field generalized forces associated with system k. F_k and τ_k are the respective reactions of the collision forces on each body; the $a_i(\mathbf{q})$ and $b_i(\mathbf{q})$ arise from forming their contributions to the generalized forces.

To get the equations of motion during the collision, each of Eqs (7) is multiplied by dt and integrated over $\Delta t = t_2 - t_1$

$$\int_{t_1}^{t_2} \mathrm{d}\left(\frac{\partial T_k}{\partial \dot{q}_i}\right) = \int_{t_1}^{t_2} \left(\frac{\partial T_k}{\partial q_i} + Q_i\right) \mathrm{d}t, \quad i(k) = 1, n. \tag{9}$$

Using Eq. (8) and $\Delta t \to 0$ (Δt "small"),

$$\frac{\partial T_k}{\partial \dot{q}_i}(t_2) - \frac{\partial T_k}{\partial \dot{q}_i}(t_1) = \lim_{\Delta t \to 0} \int_{t_1}^{t_2} \left(\frac{\partial T_k}{\partial q_i} + Q_i^k + a_i(\mathbf{q}) F_k + b_i(\mathbf{q}) \tau_k \right) \mathrm{d}t, \quad i(k) = 1, n. \tag{10}$$

On the right side of Eq. (10) the Q_i^k and $\partial T_k / \partial q_i$ are finite, so as $\Delta t \to 0$ their integrals become arbitrarily small (negligible) relative to impulses of the collision forces defined in Eqs (3). The $a_i(\mathbf{q})$ and $b_i(\mathbf{q})$ are unaffected by the integration since the system coordinates do not change during the collision. Hence, the right side of Eqs (10) is dominated by the collision forces, and Eq. (9) is

$$\mathcal{L}(t_2)\colon\ \frac{\partial T_k}{\partial \dot{q}_i}(t_2) - \frac{\partial T_k}{\partial \dot{q}_i}(t_1) = \widehat{Q}_i(\widehat{F}, \widehat{\tau}), \quad i(k) = 1, n, \tag{11}$$

where $\widehat{Q}_i(\widehat{F}, \widehat{\tau}) = a_i(\mathbf{q}) \widehat{F}_k + b_i(\mathbf{q}) \widehat{\tau}_k$, $i(k) = 1, n$. (The subscript k on the collision forces and impulses is superfluous; the individual reactions are equal and opposite and can be expressed in terms of $\widehat{F}$ and $\widehat{\tau}$ in the sense of **n** and **t**.)

Eq. (11) is the fundamental dynamical equation of the system during the collision; it provides n equations which are linear in the $n + 2$ unknowns $\dot{\mathbf{q}}(t_2)$ and collision impulses $\widehat{F}$ and $\widehat{\tau}$, since the kinetic energy T_k is quadratic in $\dot{\mathbf{q}}$. Eq. (11) will appear in many guises in the following. To distinguish these, and as a mnemonic, they are designated by $\mathcal{L}(t)$, as indicated. The argument indicates a specific time t after t_1.

Finally, when the relative velocity in Eq. (1) is expressed in terms of $(\mathbf{q}, \dot{\mathbf{q}})$, the components $u_\mathrm{n} = u_\mathrm{n}(\mathbf{q}, \dot{\mathbf{q}})$ and $u_\mathrm{t} = u_\mathrm{t}(\mathbf{q}, \dot{\mathbf{q}})$ are also linear in $\dot{\mathbf{q}}$.

Rosen and Edelstein [6] present a different approach Lagrangian development, using Lagrange multipliers instead of force models. For the case of collisions, it would appear that the present method is somewhat simpler, particularly when analysing events during the collision (below).

3. A new parameterization of a collision

The collision starts at $t_1 = 0$. Unless otherwise stated, the generalized impulses $\widehat{Q}_i(\widehat{F}, \widehat{\tau})$ are for a specific interval $(0, t)$; the arguments $(\widehat{F}, \widehat{\tau})$ will carry subscripts identified with that interval. In general, at any time, there are $n+2$ unknowns: $\dot{\mathbf{q}}(t)$, $\widehat{F}$ and $\widehat{\tau}$. Eqs (11) provide n relations, so two more must be developed.

At the end of the compression phase, at t_c, the normal relative velocity u_n becomes zero,

$$u_\mathrm{n}(t_\mathrm{c}) = u_\mathrm{n}\big(\mathbf{q}, \dot{\mathbf{q}}(t_\mathrm{c})\big) = 0. \tag{12}$$

With a normal impulse $\widehat{F}_\mathrm{c}$ and tangential impulse $\widehat{\tau}_\mathrm{c}$, Eqs (11) give

$$\mathcal{L}(t_\mathrm{c})\colon\ \frac{\partial T_k}{\partial \dot{q}_i}(t_\mathrm{c}) - \frac{\partial T_k}{\partial \dot{q}_i}(0) = \widehat{Q}_i(\widehat{F}_\mathrm{c}, \widehat{\tau}_\mathrm{c}), \quad i(k) = 1, n. \tag{13}$$

The collision ends at t_s, with $\widehat{F}_\mathrm{s}$ and $\widehat{\tau}_\mathrm{s}$; Eqs (11) are

$$\mathcal{L}(t_\mathrm{c})\colon\ \frac{\partial T_k}{\partial \dot{q}_i}(t_\mathrm{s}) - \frac{\partial T_k}{\partial \dot{q}_i}(0) = \widehat{Q}_i(\widehat{F}_\mathrm{s}, \widehat{\tau}_\mathrm{s}), \quad i(k) = 1, n. \tag{14}$$

Eqs (14) give $\dot{\mathbf{q}}(t_\mathrm{s})$, the solution to the problem.

Now, consider a "time" t^* at which the tangential relative velocity u_t becomes zero. Since the collision equations are linear, this can happen at most once during a collision. The tangential component then may pass through zero (change sign), reverse (same sign), or stay zero (stick). t^* depicts "when" things happen during the collision. Clearly $t^* \geqslant 0$; t^* can be infinity, e.g., collision of two rigid bodies with zero friction. As it turns out, t^* is never actually calculated (nor is t_c). In a sense, t^* is a contrivance to advance the solution.

Associated with t^*, i.e., the period $(0, t^*)$, are normal and tangential impulses $\widehat{F}^*$ and $\widehat{\tau}^*$. Prior to the vanishing of the relative tangential velocity, slipping occurs, so that from $t = 0$ to t^*, from Eq. (6), the impulses are related by

$$\widehat{\tau}^* = \mu \widehat{F}^*. \tag{15}$$

At t^*,

$$\mathcal{L}(t^*)\colon\ \frac{\partial T_k}{\partial \dot{q}_i}(t^*) - \frac{\partial T_k}{\partial \dot{q}_i}(0) = \widehat{Q}_i(\widehat{F}^*, \widehat{\tau}^*), \quad i(k) = 1, n, \tag{16}$$

$$u_\mathrm{t}^* \mathrel{\underline{\triangle}} u_\mathrm{t}(t^*) = u_\mathrm{t}\big(\mathbf{q}, \dot{\mathbf{q}}(t^*)\big) = 0. \tag{17}$$

Eqs (15)–(17) are $n+2$ equations (which are linear) in the $n + 2$ unknowns at t^*: $\dot{\mathbf{q}}^*(= \dot{\mathbf{q}}(t^*))$, $\widehat{F}^*$ and $\widehat{\tau}^*$.

We now relate t^* to t_c, i.e., the end of slipping to the end of the compression phase. From the $\dot{\mathbf{q}}^*$ obtained from Eqs (15)–(17), we can calculate the relative normal velocity, $u_n^* = u_n(\mathbf{q}, \dot{\mathbf{q}}(t^*))$. From the initial conditions, $u_n(0) = u_n(\mathbf{q}, \dot{\mathbf{q}}(0))$ is known. There are three cases to address:

(i) $t^* < t_c$. This occurs if $\mathrm{sign}(u_n^*) = \mathrm{sign}(u_n(0))$: the relative tangential velocity vanishes before the compression phase is over.
(ii) $t^* = t_c$. Then $u_n^* = 0$: the relative tangential and normal velocities vanish simultaneously.
(iii) $t^* > t_c$. Now $\mathrm{sign}(u_n^*) = -\mathrm{sign}(u_n(0))$: slipping occurs into, and possibly through, the restitution phase.

These "times" when the normal and tangential relative velocities vanish (or pass through zero) are shown in Fig. 2. The subsequent motion (slipping or sticking) and complete solutions depend upon the magnitudes of the forces (impulses), as follows.

4. Theory and solutions

To start the solution, we have the parameters e, μ and μ_s, the pre- and post-collision geometry $\mathbf{q}$ (they are the same), and the precollision values $\dot{\mathbf{q}}(0)$, hence, $u_n(0)$ and $u_t(0)$. We also have values at t^* (defined by $u_t^* = 0$) from Eqs (15)–(17): $\dot{\mathbf{q}}^*$, u_n^*, and $\widehat{F}^*$ and $\widehat{\tau}^*$ for the interval $(0, t^*)$ during which sliding occurs. Therefore, we know the case: (i), (ii), or (iii) from above.

The solution (theory) to get the n postcollision variables $\dot{\mathbf{q}}_s = \dot{\mathbf{q}}(t_s)$ for each case generally has three parts: (a) consider sticking at t^*; (b) check if sticking is sustained; if not, (c) determine the character of the subsequent sliding motion.

Case (i). If $\mathrm{sign}(u_n^*) = \mathrm{sign}(u_n(0))$, $t^* < t_c$: the tangential relative velocity becomes zero before the end of the compression phase.

(a) If sticking occurs at t^*, u_t will be zero thereafter. Subsequently, at the end of the compression phase, both the tangential and normal relative velocities are zero; the tentative solution is given by $u_n(t_c) = 0$ and $\mathcal{L}(t_c)$, per Eqs (12) and (13), plus

$$u_t(t_c) = u_t\big(\mathbf{q}, \dot{\mathbf{q}}(t_c)\big) = 0. \tag{18}$$

These are $n+2$ equations in the $n+2$ unknowns $\dot{\mathbf{q}}(t_c)$, $\widehat{F}_c$ and $\widehat{\tau}_c$.

(b) We now calculate the impulsive forces for which sticking continues after t^* through the end of the collision (at t_s). With sticking at t_s, then the tangential velocity at t_s is zero:

$$u_t(t_s) = u_t\big(\mathbf{q}, \dot{\mathbf{q}}(t_s)\big) = 0. \tag{19}$$

Another relation is from $\mathcal{L}(t_s) - \mathcal{L}(t^*)$, per Eqs (14) and (16):

$$\mathcal{L}(t_s) - \mathcal{L}(t^*)\colon\ \frac{\partial T_k}{\partial \dot{q}_i}(t_s) - \frac{\partial T_k}{\partial \dot{q}_i}(t^*) = \widehat{Q}_i\big(\widehat{F}_{res}, \widehat{T}_{res}\big), \quad i(k) = 1, n, \tag{20}$$

where

$$\widehat{F}_{res} = \widehat{F}_s - \widehat{F}^* = (1+e)\widehat{F}_c - \widehat{F}^*, \tag{21}$$

$$\widehat{\tau}_{res} = \widehat{\tau}_s - \widehat{\tau}^* \tag{22}$$

are the impulses in (t^*, t_s). With $\widehat{F}_c$ from i(a) above and $\widehat{F}^*$ from Eqs (15)–(17), Eqs (19)–(22) can be solved in terms of $\widehat{\tau}_{res}$.

If $|\widehat{\tau}_{res}/\widehat{F}_{res}| < \mu_s$, sticking continues after t^*, and Eqs (19)–(22) give the solution. Otherwise:

(c) If $|\widehat{\tau}_{res}/\widehat{F}_{res}| < mu_s$, slipping occurs after t^*, but in the opposite direction. The solution is recomputed as follows.

Since $t_c > t^*$, the $\widehat{F}_c$ and $\widehat{\tau}_c$ are (re)computed for slipping; n relations are obtained from the difference of Eqs (13) and (16):

$$\mathcal{L}(t_c) - \mathcal{L}(t^*)\colon\ \frac{\partial T_k}{\partial \dot{q}_i}(t_c) - \frac{\partial T_k}{\partial \dot{q}_i}(t^*) = \widehat{Q}_i\big(\widehat{F}_c - \widehat{F}^*, \widehat{\tau}_c - \widehat{\tau}^*\big), \quad i(k) = 1, n, \tag{23}$$

where

$$-\mu\big(\widehat{F}_c - \widehat{F}^*\big) = \widehat{\tau}_c - \widehat{\tau}^*. \tag{24}$$

The one more needed relation is from Eq. (12), viz., $u_n(t_c) = 0$. These equations are solved for $\widehat{F}_c$. The postcollision solution is now easily expressed in terms of the restitution phase values:

$$\mathcal{L}(t_s) - \mathcal{L}(t_c)\colon\ \frac{\partial T_k}{\partial \dot{q}_i}(t_s) - \frac{\partial T_k}{\partial \dot{q}_i}(t_c) = \widehat{Q}_i\big(\widehat{F}_r, \widehat{\tau}_r\big), \quad i(k) = 1, n, \tag{25}$$

where

$$\widehat{F}_r = e\widehat{F}_c \qquad \widehat{\tau}_r = -\mu\widehat{F}_r. \tag{26}$$

Case (ii). If $u_{\mathrm{n}}^* = u_{\mathrm{n}}(t^*) = 0$, $t^* = t_{\mathrm{c}}$. The relative tangential and normal velocities vanish simultaneously. For this case, the $*$ and c values are the same.

(a) If sticking occurs at $t^* = t_{\mathrm{c}}$, the tangential relative velocity is zero thereafter; hence, the postcollision tangential velocity $u_{\mathrm{t}}(t_{\mathrm{s}}) = 0$. Now, $\widehat{F}_{\mathrm{c}} = \widehat{F}^*$ from Eqs (15)–(17).

(b) The impulsive forces necessary to sustain sticking in the restitution phase are given by Eqs (19) and (25), viz.

$$u_{\mathrm{t}}(t^*) = u_{\mathrm{t}}(t_{\mathrm{s}}) = u_{\mathrm{t}}\big(\mathbf{q}, \dot{\mathbf{q}}(t_{\mathrm{s}})\big) = 0, \tag{27}$$

$$\begin{aligned}\mathcal{L}(t_{\mathrm{s}}) - \mathcal{L}(t_{\mathrm{c}})\colon\ & \frac{\partial T_k}{\partial \dot{q}_i}(t_{\mathrm{s}}) - \frac{\partial T_k}{\partial \dot{q}_i}(t_{\mathrm{c}}) \\ &= \widehat{Q}_i\big(\widehat{F}_{\mathrm{r}}, \widehat{\tau}_{\mathrm{r}}\big), \quad i(k) = 1, n,\end{aligned} \tag{28}$$

where $\mathcal{L}(t_{\mathrm{c}}) = \mathcal{L}(t^*)$; since $\widehat{F}_{\mathrm{c}} = \widehat{F}^*$,

$$\widehat{F}_{\mathrm{r}} = e\widehat{F}_{\mathrm{c}}. \tag{29}$$

Eqs (27)–(29) are $n + 2$ equations in $\dot{\mathbf{q}}(t_{\mathrm{s}})$, $\widehat{F}_{\mathrm{r}}$ and $\widehat{\tau}_{\mathrm{r}}$;

If $|\widehat{\tau}_{\mathrm{r}}/e\widehat{F}_{\mathrm{c}}| < \mu_{\mathrm{s}}$, sticking is maintained, and Eqs (27)–(29) give the solution. Otherwise:

(c) If$|\widehat{\tau}_{\mathrm{r}}/e\widehat{F}_{\mathrm{c}}| > \mu_{\mathrm{s}}$, slipping occurs in the opposite direction after $t^*(= t_{\mathrm{c}})$.

The solution is given by recomputing Eqs (28) and (29) with

$$\widehat{\tau}_{\mathrm{r}} = -\widehat{F}_{\mathrm{r}} = -\mu e\widehat{F}_{\mathrm{c}}. \tag{30}$$

Case (iii). If $\mathrm{sign}(u_{\mathrm{n}}^*) = -\mathrm{sign}(u_{\mathrm{n}}(0))$, $t_{\mathrm{c}} < t^*$. The tangential relative velocity vanishes after the normal relative velocity: slipping occurs through the compression phase and into (and possibly through) the restitution phase. Therefore, through t_{c},

$$\widehat{\tau}_{\mathrm{c}} = \mu\widehat{F}_{\mathrm{c}} \tag{31}$$

Eqs (31), (12) and (13) ($u_{\mathrm{n}}(t_{\mathrm{c}}) = 0$ and $\mathcal{L}(t_{\mathrm{c}})$) yield $\widehat{F}_{\mathrm{c}}$, as well as $\dot{\mathbf{q}}(t_{\mathrm{c}})$ and $\widehat{\tau}_{\mathrm{c}}$. Therefore, for the entire collision, the total normal impulse is $(1 + e)\widehat{F}_{\mathrm{c}} = \widehat{F}_{\mathrm{s}}$.

First, we can check whether t^* occurs during the collision, before t_{s}. From Eqs (15)–(17), we have $\widehat{F}^*$. Therefore, if $\widehat{F}^* > (1 + e)\widehat{F}_{\mathrm{c}}$, the relative tangential velocity never becomes zero during the collision; slipping at P occurs throughout. The postcollision velocities are given by Eqs (14), viz.

$$\begin{aligned}\mathcal{L}(t_{\mathrm{s}})\colon\ & \frac{\partial T_k}{\partial \dot{q}_i}(t_{\mathrm{s}}) - \frac{\partial T_k}{\partial \dot{q}_i}(0) = \widehat{Q}_i\big(\widehat{F}_{\mathrm{s}}, \widehat{\tau}_{\mathrm{s}}\big), \\ & i(k) = 1, n,\end{aligned} \tag{32}$$

where, with $\widehat{F}_{\mathrm{c}}$ from above,

$$\begin{aligned}&\widehat{F}_{\mathrm{s}} = (1 + e)\widehat{F}_{\mathrm{c}}, \\ &\widehat{\tau}_{\mathrm{s}} = \mu\widehat{F}_{\mathrm{s}}.\end{aligned} \tag{33}$$

These values of $\widehat{F}^*$ and $\widehat{\tau}^*$ corresponding to t^* do not (cannot) really occur – they are values which would make $u_{\mathrm{t}} = 0$ with a friction force throughout $(0, t^*)$. Another way of seeing this is that for this case, $\widehat{F}^* = \widehat{\tau}^*/\mu > \widehat{F}_{\mathrm{s}}$, so slipping occurs up to and at t_{s}. Otherwise, for the general procedure: if $\widehat{F}^* < (1 + e)\widehat{F}_{\mathrm{c}}$, the relative tangential velocity will become zero sometime during the restitution phase. Then, (a,b) To check if sticking occurs at t^*, we solve as in Case (i), part (b), viz., Eqs (19)–(22), to see if the necessary forces (impulses) are developed. Recall that $\widehat{F}_{\mathrm{res}}$ and $\widehat{\tau}_{\mathrm{res}}$ determine whether subsequent motions are sticking or slipping.

If $\widehat{\tau}_{\mathrm{res}}/\widehat{F}_{\mathrm{res}} < \mu_{\mathrm{s}}$, sticking holds, and Eqs (19)–(22) give the solution. Otherwise:

(c) If $\widehat{\tau}_{\mathrm{res}}/\widehat{F}_{\mathrm{res}} > \mu_{\mathrm{s}}$, slipping occurs in the opposite direction after t^*; the direction of the friction force reverses to oppose the incipient relative tangential motion.

The solution is obtained from Eq. (20), viz.,

$$\begin{aligned}\mathcal{L}(t_{\mathrm{s}}) - \mathcal{L}(t^*)\colon\ & \frac{\partial T_k}{\partial \dot{q}_i}(t_{\mathrm{s}}) - \frac{\partial T_k}{\partial \dot{q}_i}(t^*) \\ &= \widehat{Q}_i\big(\widehat{F}_{\mathrm{res}}, \widehat{\tau}_{\mathrm{res}}\big), \quad i(k) = 1, n,\end{aligned} \tag{34}$$

where

$$\widehat{F}_{\mathrm{res}} = \widehat{F}_{\mathrm{s}} - \widehat{F}^* = (1 + e)\widehat{F}_{\mathrm{c}} - \widehat{F}^*, \tag{35}$$

$$\widehat{\tau}_{\mathrm{res}} = -\mu\widehat{F}_{\mathrm{res}}. \tag{36}$$

The total impulses are:

$$\widehat{F}_{\mathrm{s}} = (1 + e)\widehat{F}_{\mathrm{c}}, \tag{37}$$

$$\begin{aligned}\widehat{\tau}_{\mathrm{s}} &= \mu\widehat{F}^* - \mu\big[(1 + e)\widehat{F}_{\mathrm{c}} - \widehat{F}^*\big] \\ &= 2\mu\widehat{F}^* - \mu(1 + e)\widehat{F}_{\mathrm{c}}.\end{aligned} \tag{38}$$

Special cases

The foregoing three cases address all contingencies. For example, we did not explicitly address whether t^* occurs before or after t_{s}. This was "automatically" taken care of in Case (iii) – the first calculation – by checking the value of $\widehat{F}^*$ against the total normal impulse.

If the collision is frictionless, $\widehat{\tau} = 0$ at all times. This can be implemented explicitly in the various equations, thereby reducing the number of equations to be solved by one. In the above framework, setting $\mu = \mu_s = 0$ achieves the same result. For the frictionless collision of simple rigid bodies, t^* is infinite; the initial calculation in Case (iii) covers this. However, for a constrained system, a finite t^* can be calculated, in terms of finite u_n^* and u_t^*, which may occur during the collision. So, even though sticking never occurs, unimpeded sliding may not be a correct solution. Example 1 shows this.

5. Newtonian formulation – modifications

For the collision of individual rigid bodies, the Newtonian formulation is straightforward and can be easier. For two bodies of masses m_k and central moments of inertia $\bar{\mathbf{I}}_k$, $k = 1, 2$, subject to the impulsive force $\mathcal{F}$ defined in Eqs (2)–(4), the relevant equations are, for each body: linear momentum of the center of mass

$$m_k\big(\mathbf{v}^k(t_2) - \mathbf{v}^k(t_1)\big) = \mathcal{F}_k, \quad k = 1, 2, \tag{39}$$

angular momentum with respect to the center of mass:

$$\begin{aligned} \mathbf{H}_k(t_2) - \mathbf{H}_k(t_1) &= \bar{\mathbf{I}}_k\big(\boldsymbol{\omega}^k(t_2) - \boldsymbol{\omega}^k(t_1)\big) \\ &= \mathbf{r}^{P_k} \times \mathcal{F}_k, \quad k = 1, 2, \end{aligned} \tag{40}$$

where $\mathcal{F}_2 = -\mathcal{F}_1$, and $\mathcal{F}_1 = \mathcal{F}$ of Eq. (2). $\mathbf{v}^k(t)$ is the velocity of the center of mass of body k at time t. $\boldsymbol{\omega}^k(t)$ is the angular velocity of body k at time t. $\mathbf{r}^{P_k}$ is the vector from the center of mass of body k to the contact point P.

Clearly, we can define $\mathbf{v}^k(t)$ and $\boldsymbol{\omega}^k(t)$ in terms of six generalized coordinates for each body k. Together, Eqs (39) and (40) replace $\mathcal{L}(t)$ of Eqs (11) in whatever form that Eqs (11) are applied. Everything else in the previous analysis applies directly.

6. Examples

These are intended to demonstrate the algorithm and to correct solutions in the literature. Accordingly, the examples are presented in some detail and with commentary. The "conventional method" referred to is the method presented by the respective authors, as noted. These methods are applicable to collisions of individual rigid bodies, but they can be incorrect for more complicated systems. The examples clearly show this.

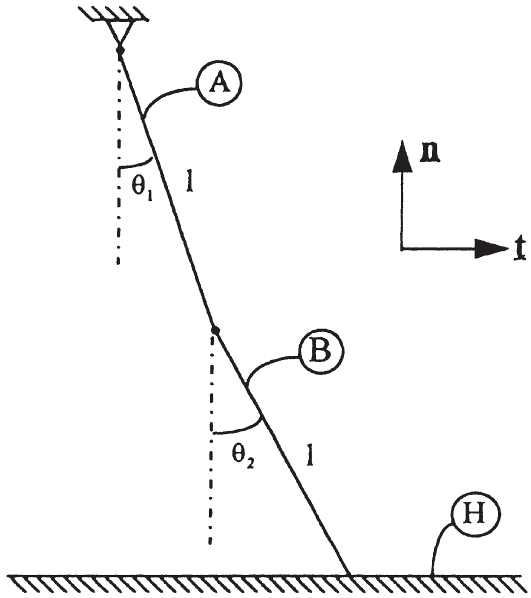

Fig. 3. Example 1.

6.1. *Example 1*

This is Problem 14.6 posed by Kane and Levinson [4], whose "anomalous" results stimulated this analysis. In Fig. 3, a double pendulum is formed by the uniform rods *A* and *B*, each of length 2 m and mass 3 kg. The free end of B strikes the horizontal surface *H* with $q_1 \triangleq \theta_1 = 20^\circ$, $q_2 \triangleq \theta_2 = 30^\circ$, and initial conditions $\dot{q}_1(0) = -0.1$ rad/s, $\dot{q}_2(0) = -0.2$ rad/s.

The collision equations of motion, Eqs (11), are:

$$\begin{aligned} &C_1\big(\dot{q}_1(t_2) - \dot{q}_1(t_1)\big) + D_1\big(\dot{q}_2(t_2) - \dot{q}_2(t_1)\big) \\ &\quad = a_1\widehat{F} + b_1\widehat{\tau}, \\ &C_2\big(\dot{q}_1(t_2) - \dot{q}_1(t_1)\big) + D_2\big(\dot{q}_2(t_2) - \dot{q}_2(t_1)\big) \\ &\quad = a_2\widehat{F} + b_2\widehat{\tau}, \end{aligned}$$

where

$$\begin{aligned} &C_1 = 4D_2, \quad D_1 = C_2 = \tfrac{1}{2}\mathrm{m}\ell^2\cos(q_2 - q_1), \\ &D_2 = \tfrac{1}{3}\mathrm{m}\ell^2, \quad a_i = \ell\sin(q_i),\ b_i = \ell\cos(q_i), \\ &\quad i = 1, 2. \end{aligned}$$

The results of Kane and Levinson [4] are shown in Table 1. Kinetic energy increases during the collision in two cases, clearly an impossibility with nonzero friction and $e < 1$. Actually, solution of the problem using a "conventional" implementation of Kane and Levin-

Table 1
Example 1: results given by Kane and Levinson [4]

e	μ_s	μ	Slipping	Energy (N m)
0.5	0.25	0.2	Yes	0.03 Decrease
0.5	0.5	0.4	No	0.16 Increase
0.3	0.5	0.4	No	0.12 Decrease
0.7	0.51	0.5	Yes	0.49 Increase

Table 2
Example 1: results using "conventional method" (see text)

e	μ_s	μ	Slipping	Energy (N m)
0.5	0.25	0.2	Yes	0.1034 Decrease
0.5	0.5	0.4	No	0.16 Increase
0.3	0.5	0.4	No	0.12 Decrease
0.7	0.51	0.5	Yes	0.1011 Decrease

son's [4] formulation gives a slightly different set of results, per Table 2. The difference is in the assumed direction of the friction force. In the conventional method (which were our initial results), the friction force resists the initial tangential motion. After some experimentation, we were able to replicate Kane and Levinson's results by letting $\widehat{\tau} = -\mu\widehat{F}$, i.e., by reversing the sign of the friction force of the conventional analysis. This resists motion in the restitution phase, but has the a priori assumption that the pendulum tangential velocity reverses direction. However, this gives the wrong direction for the friction force for the initial relative motion, so this assumption is always wrong, as are all the results in Table 1. Note that the conventional method can be correct if the slip velocity does not change direction. If the slip velocity reverses direction, clearly both approaches are wrong.

In this example, the conventional method results might be considered "better" since "only" one increase in energy appears, although this is still not a satisfactory state of affairs.

As will now be shown, all of these parameter values indicate a reversal of the slip velocity and the friction force. Due to the constraints the tangential velocity changes direction, or tries to (incipience), so that the friction force (impulse) also must change direction during the collision. For a constrained system like this, an "overall" assumption does not consider this. Indeed, all conventional methods fail to capture this property of friction impulse accurately [2,4,15]. This also affects whether sticking or slipping occurs, so those cases should also be noted, in Table 2.

The problem is worked in detail to demonstrate the new method. The results are shown in Table 3: energy always decreases (as it should) and sticking never occurs. It turns out that the value of static coefficient of friction for which sticking will occur can be calculated explicitly (shortly).

Table 3
Example 1: results using proposed method

e	μ_s	μ	Slipping/ u_t reverses	Energy (N m)	Case
0.5	0.25	0.2	Yes/Yes	0.11 Decrease	(iii)
0.5	0.5	0.4	Yes/Yes	0.12 Decrease	(iii)
0.3	0.5	0.4	Yes/Yes	0.13 Decrease	(iii)
0.7	0.51	0.5	Yes/Yes	0.12 Decrease	(iii)

From Eqs (15)–(17), case (iii) obtains for all values of the specified parameters, per Kane and Levinson [4]. Suppose sticking occurs after the tangential relative velocity becomes zero. Then, per Case iii(a,b), solve Eqs (19)–(22), which from the above are

$$C_1\big(\dot{q}_1(t_s) - \dot{q}_1^*\big) + D_1\big(\dot{q}_2(t_s) - \dot{q}_2^*\big) = a_1\widehat{F}_{res} + b_1\widehat{\tau}_{res},$$
$$C_2\big(\dot{q}_1(t_s) - \dot{q}_1^*\big) + D_2\big(\dot{q}_2(t_s) - \dot{q}_2^*\big) = a_2\widehat{F}_{res} + b_2\widehat{\tau}_{res}$$

and

$$u_t^* = 0, \qquad u_t(t_s) = 0.$$

These can be solved for $\widehat{F}_{res}$ and $\widehat{\tau}_{res}$ and

$$\left|\frac{\widehat{\tau}_{res}}{\widehat{F}_{res}}\right| = \left|\frac{\gamma\ell\sin(\theta_2) - \ell\sin(\theta_1)}{\gamma\ell\cos(\theta_2) - \ell\cos(\theta_1)}\right|,$$

where

$$\gamma = \frac{C_1\cos(\theta_2) - D_1\cos(\theta_1)}{C_2\cos(\theta_2) - D_2\cos(\theta_1)}.$$

Substitution of the values of the parameters and initial conditions gives $|\widehat{\tau}_{res}/\widehat{F}_{res}| = 0.6234$. Therefore, if $\mu_s > 0.6234$, the surfaces stick thereafter; if $\mu_s < 0.6234$, slipping will resume. This is determined solely by the system structure. (This may not be obvious for the double pendulum. However, for the simpler case of a single pendulum, there is an obvious structure-defined fixed number for μ_s, i.e., tan $\theta(0)$, where $\theta(0)$ is the angle from the vertical at the onset of contact.)

In [4], all four values of $\mu_s < 0.6234$, so sticking does not occur in any case, per Table 3. Also, in all cases u_t reverses direction. Therefore, the conventional method will not give correct results. The modification of Kane and Levinson [4] to address the reversed motion (Table 1) has a certain logic, but it never gives correct results. Neither will Keller's [5] method, since

Table 4

Example 1: two special cases (proposed method)

e	μ_s	μ	$\widehat{F}^*$	$(1+e)\widehat{F}_c$	u_t reverses	Energy (N m)	Case
0.5	0	0	1.0537	1.1964	Yes	0.08 Decrease	(iii)
0.2	0.5	0.4	0.6419	0.5971	No	0.13 Decrease	(iii)

there is no accommodation of different values of static and kinetic friction coefficients.

A few more cases are of interest: zero friction, and no change in the direction of the relative tangential velocity. These are shown in Table 4. Both agree with the conventional method – the former since the friction force is zero, and therefore not an issue, and the latter because the tangential relative velocity and friction force do not reverse direction.

Finally, a case where sticking would prevail was studied, with $\mu_s = 0.63$ (just slightly over the theoretical limit of 0.6234), $\mu = 0.5$, $e = 0.7$. Sticking does occur, as predicted, with a kinetic energy decrease of 0.12. The conventional method will give a 0.59 increase.

6.2. Example 2

This example considers the collision of two identical spheres, per Groesberg [2, p. 176]. In the text, the equations indicated that the balls are 4 units in diameter (not radius as stated); in the following we have changed the length units from inches to cm. The geometry and initial conditions are shown in Fig. 4: ball 2 is initially at rest and ball 1 has $\mathbf{v}_1(0) = 30\mathbf{a}_2$ cm/s and $\omega_1 = -11\mathbf{k}$ rad/s; $e = 0.80$ and $\mu = 0.30$. (μ_s is not specified, but will be equal to μ if the issue arises).

With this change, Eqs (4.5–17) and (4.5–22) in [2] are correct Newtonian collision equations by the conventional method. However, with these equations, there is a numerical mistake in the calculations. The correct results at t_s are:

$$\mathbf{v}_1(t_s) = 1.5\mathbf{i} + 28.555\mathbf{j},$$

$$\mathbf{v}_2(t_s) = 13.5\mathbf{i} - 2.5742\mathbf{j},$$

$$\omega_1(t_s) = -9.391\mathbf{k},$$

$$\omega_2(t_s) = 1.6089\mathbf{k},$$

$$\widehat{F} = 13.5,$$

$$\widehat{\tau} = 2.5742.$$

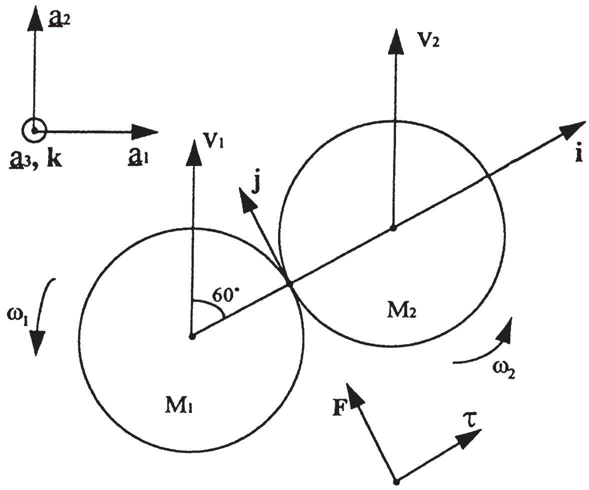

Fig. 4. Example 2.

Solution by the proposed method (modified Newtonian formulation)

When the tangential relative velocity vanishes (t^*)

$$\mathbf{v}_1^* = 6.4194\mathbf{i} + 28.555\mathbf{j},$$

$$\mathbf{v}_2^* = 8.5806\mathbf{i} - 2.5742\mathbf{j},$$

$$\boldsymbol{\omega}_1^* = -9.391\mathbf{k},$$

$$\boldsymbol{\omega}_2^* = 1.6089\mathbf{k},$$

$$\widehat{F}^* = 8.5806.$$

The normal relative velocity is

$$u_n = 6.4194 - 8.5806 = -2.1612$$

and changes sign, which is Case (iii).

At the end of the compression phase,

$$\mathbf{v}_1(t_c) = 7.5\mathbf{i} + 28.2308\mathbf{j},$$

$$\mathbf{v}_2(t_c) = 7.5\mathbf{i} - 2.25\mathbf{j},$$

$$\boldsymbol{\omega}_1(t_c) = -9.5938\mathbf{k},$$

$$\boldsymbol{\omega}_2(t_c) = 1.4062\mathbf{k},$$

$$\widehat{F}_c = 7.5.$$

Since $(1+e)\widehat{F}_c > \widehat{F}^*$, the tangential relative velocity becomes zero in the restitution phase and remains zero, i.e., sticking. At the point when the tangential relative velocity becomes zero, all the angular velocities and tangential velocities are the same as obtained from the conventional method when sticking is assumed, because afterwards, there is no force in the tangential direction. Indeed, from Eq. (35)

$$\widehat{F}_{\text{res}} = \widehat{F}_{\text{s}} - \widehat{F}^{*} = (1+e)\widehat{F}_{\text{c}} - \widehat{F}^{*} = 4.9194$$

will drive the normal velocities to

$$v_{1i} = 6.4194 - 4.9194 = 1.5,$$
$$v_{2i} = 8.5806 - 4.9194 = 13.5.$$

These are the same as given by the conventional method. The same results are obtained since there is no reversal in the direction of the tangential relative velocity during the collision.

7. Discussion

When there is no tangential relative velocity direction change, or zero friction, the new theory (method) is comparable to the conventional methods, as demonstrated in the two examples, although it gives considerably greater detail about the process of the collision. When there is, or can be, a change in the slip velocity direction, the new method gives both more reasonable results and a proper "snapshot" of the collision process.

The algorithm follows exactly the physics of the conditions during contact (within the confines of the simplifying assumptions of restitution and friction). It is not a "thought process" – as with many algorithms – but strictly an analysis of possible motions due to the forces which are "automatically" solved by the physical system.

8. Conclusions

A new analysis and algorithm for the solution to the collision of constrained rigid body systems has been presented. It is based on the physical principles applicable at the point of contact and completely captures the characteristics of a collision. The new algorithm successfully solves this problem and other more traditional problems, as well. It matches conventional methods in the literature when the conventional methods apply.

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