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Effects of elastic foundation on the vibration of laminated non-homogeneous orthotropic circular cylindrical shells

A.H. Sofiyev[a,*], S.N. Keskin[a] and Ali H. Sofiyev[b]
[a]*Department of Civil Engineering of Suleyman Demirel University, Isparta, Turkey*
[b]*Technical Sciences Department of Kasakh Branch of Azerbaijan Teachers Institute, Kazakh, Azerbaijan*

Abstract. In this paper an analytical procedure is given to study the free vibration characteristics of laminated non-homogeneous orthotropic thin circular cylindrical shells resting on elastic foundation, accounting for Karman type geometric non-linearity. At first, the basic relations and modified Donnell type stability equations, considering finite deformations, have been obtained for laminated thin orthotropic circular cylindrical shells, the Young's moduli of which varies piecewise continuously in the thickness direction. Applying Galerkin method to the latter equations, a non-linear time dependent differential equation is obtained for the displacement amplitude. The frequency is obtained from this equation as a function of the shell displacement amplitude. Finally, the effect of elastic foundation, non-linearity, non-homogeneity, the number and ordering of layers on the frequency is found for different mode numbers. These results are given in the form of tables and figures. The present analysis is validated by comparing results with those in the literature.

Keywords: Elastic foundation, vibration, natural and nonlinear frequencies, cross-ply laminated cylindrical shells, orthotropic material

1. Introduction

Multi-layered composite shells composed of non-homogeneous materials with different elastic properties are being used extensively as structural elements in modern construction engineering, ship building, nuclear, space and aeronautical industries as well as the petroleum and petrochemical industries (pressure vessel, pipeline). These materials have properties that vary as a function of position in the body. Non-homogeneous materials can frequently be found in nature as well as in man-made structures. However, typically non-homogeneous materials seem to be those with elastic constants varying continuously in different spatial directions. Continuous non-homogeneity is a direct generalization of homogeneity in theory; besides, material non-homogeneity becomes essential and must sufficiently be considered in a number of practical situations. In all the referenced works, and in most of available solutions to elastic non-homogeneity, it is assumed that the material is isotropic or orthotropic, the Poisson's ratio is constant, and the Young's moduli is either an exponential or a power function of a spatial variable [5,8,12–14,28,32]. Cylindrical shells made of different materials that have continuous and thorough contact with an elastic medium, solid or liquid, either on an outer or inner surface is considered as cylindrical shells on an elastic foundation. Such components and structures are often subjected to dynamic loads. Flow-induced vibrations in heat exchangers and pipelines, wave loading on submarines, vibrations of fuel-filled drop tanks of fighter aircraft, underground and undersea pipelines, and tunnels and semicircular roofs of underground aircraft hangers subject to seismic forces,

*Corresponding author: Assoc. Prof. Dr. Abdullah Sofiyev, Department of Civil Engineering of Suleyman Demirel, University, Isparta, Turkey. Tel: +90 246 2111243; Fax: +90 246 2370859; E-mail: asofiyev@mmf.sdu.edu.tr.

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nuclear explosions, and other blasts are some of the numerous examples.

Significant contributions have been made in the field of non-linear vibrations of cylindrical shells in general. Some of these studies about this subject are given [9,15, 20,22,23,25,26,31,37]. Linear and non-linear free vibrations of laminated plates and shells have been studied only recently see [1–3,6,7,19,27,29,30,33,36,39]. However, vibrations of shells on elastic foundations have been studied only recently see [4,11,16–18,21,35, 38,40]. In most of these studies, the authors investigate the vibrations of an orthotropic cylindrical shell on an elastic foundation using membrane theory. It is known that, response of elastic media can be presented by using Winkler and Pasternak foundation models. In this study, response of elastic media is given by Winkler foundation model. It is known that Winkler represented an elastic foundation by a set of closely spaced, independent linear springs.

The effect of all three factors together that non-homogeneity, geometric non-linearity and elastic foundation to the vibration modes of laminated shells are not studied enough. In this study, vibration problem in large deformations of laminated non-homogeneous orthotropic cylindrical shells resting on an elastic foundation is taken up and the effect of all three factors together in question to the vibration frequency is researched.

2. Formulation of the problem

Consider a thin circular cylindrical shell as shown in Fig. 1, composed of N layers of equal thickness of non-homogeneous orthotropic composite material perfectly bonded together. The shell is on elastic foundation and of length L, total thickness 2h and radius R. In Fig. 1, the x and y axes are in the middle plane of the shell in the axial and tangential directions, respectively, and the z axis normal to them. The axes of orthotropy in all layers are parallel to x and y axes.

The equations of motion of circular cylindrical thin shells resting on an elastic foundation are as follows [10,38]:

$$\begin{aligned}
&N_{11,x} + N_{12,y} = \rho_1 h_1 u_{,tt} \\
&N_{21,x} + N_{22,y} = \rho_1 h_1 v_{,tt} \\
&M_{11,xx} + 2M_{12,xy} + M_{22,yy} + N_{22}/R \\
&+N_{11}w_{,xx} + 2N_{12}w_{,xy} + N_{22}w_{,yy} - k_0 w \\
&= \rho_1 h_1 w_{,tt}
\end{aligned} \tag{1}$$

where $h_1 = 2h$, a comma denotes partial differentiation with respect to the corresponding coordinates, N_{11}, N_{22} and N_{12} are, respectively, the axial and circumferential normal forces and the accompanying shear force; M_{11}, M_{22} and M_{12} are, respectively, the bending moments in axial and circumferential directions and the accompanying twisting moment, u, v and w are, respectively, the displacements on the reference surface in the directions of x, y and z axes, t is time coordinate, k_0 is foundation modulus and the following definitions apply:

$$\rho_1 = \sum_{k=1}^{N} \rho_0^{(k)}/N, \tag{2}$$

in which $\rho_0^{(k)}$ are the densities of the homogeneous materials, in the kth layer.

The Kirchhoff hypothesis on non-deformable normal element and Karman type geometric non-linearity are taken into account. In that case, in large deformation the stress-strain relations for a thin laminated layer, which has non-uniform Young's moduli with respect to the thickness coordinate, are given as follows

$$\begin{pmatrix} \sigma_{11}^{(k)} \\ \sigma_{22}^{(k)} \\ \sigma_{12}^{(k)} \end{pmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & 0 \\ Q_{12}^{(k)} & Q_{22}^{(k)} & 0 \\ 0 & 0 & Q_{66}^{(k)} \end{bmatrix} \begin{bmatrix} u_{,x} + 0.5(w_{,x})^2 - zw_{,xx} \\ v_{,y} - w/R + 0.5(w_{,y})^2 - zw_{,yy} \\ 0.5(u_{,y} + v_{,x}) + w_{,x}w_{,y} - zw_{,xy} \end{bmatrix} \tag{3}$$

where $\sigma_{11}^{(k)}, \sigma_{22}^{(k)}$ and $\sigma_{12}^{(k)}$ are the stresses in the layers. The quantities $Q_{ij}^{(k)}, i, j = 1, 2, 6$ for orthotropic lamina are

$$\begin{aligned}
&Q_{11}^{(k)} = \frac{E_{01}^{(k)}\overline{\varphi}^{(k)}(\overline{z})}{1 - \nu_{12}^{(k)}\nu_{21}^{(k)}},\ Q_{22}^{(k)} = \frac{E_{02}^{(k)}\overline{\varphi}^{(k)}(\overline{z})}{1 - \nu_{12}^{(k)}\nu_{21}^{(k)}}, \\
&Q_{12}^{(k)} = \nu_{21}^{(k)}Q_{11}^{(k)} = \nu_{12}^{(k)}Q_{22}^{(k)}, \\
&Q_{66}^{(k)} = G_0^{(k)}\overline{\varphi}^{(k)}(\overline{z}) \\
&\quad -h + (k-1)\delta \leqslant z \leqslant -h + k\delta, \overline{z} = z/h, \\
&\quad k = 1, 2, \ldots, N,\ \delta = 2hN^{-1}
\end{aligned} \tag{4}$$

wherein the superscript k denotes the kth layer. The quantity $E_{01}^{(k)}$ and $E_{02}^{(k)}$ are Young's modulus of the homogeneous material in the x and y directions for the layer k, $G_0^{(k)}$ are the shear modulus of the homogeneous material in the x-y plane of the layer k, $\nu_{12}^{(k)}$ and $\nu_{21}^{(k)}$ are the Poisson's ratio for contraction in the y and x directions due to tension in the x and y directions for the

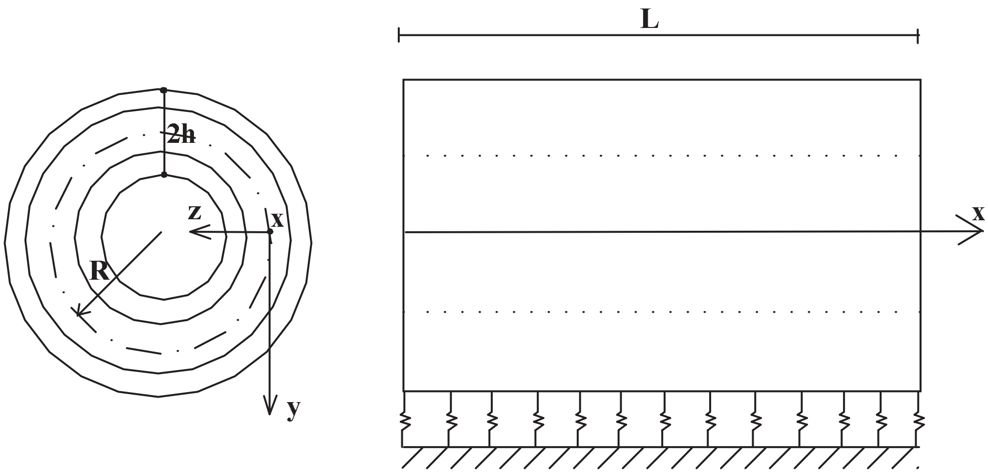

Fig. 1. Laminated cylindrical thin shell on elastic foundation.

layer k, respectively. There are apparently five material constants per layer; however, because of the reciprocal relations ($\nu_{12}^{(k)} E_{01}^{(k)} = \nu_{21}^{(k)} E_{02}^{(k)}$) there are actually only four independent constants. $\overline{\varphi}^{(k)}(\overline{z}) = 1 + \mu\varphi^{(k)}(\overline{z})$ is continuous functions expressing the variation of the Young's moduli for the layer k and $|\overline{\varphi}^{(k)}(\overline{z})| \leqslant 1$. In the above expression, μ is the variation coefficient of Young's moduli and $0 \leqslant \mu < 1$. δ is the equal thickness of the layers.

In the large deformation the strain compatibility equation on the reference surface is given as follows [10]:

$$e_{11,yy} + e_{22,xx} - 2e_{12,xy} = (w_{,xy})^2 - w_{,xx}w_{,yy} - w_{,xx}/R \quad (5)$$

where e_{11} and e_{22} are the normal strains in the curvilinear coordinate directions x and y, respectively, whereas e_{12} is the corresponding shear strain.

The force and moment resultants are defined by the following integrals [1,24,27,33,36,39]:

$$[(N_{11}, N_{22}, N_{12}), (M_{11}, M_{22}, M_{12})] = \sum_{k=1}^{N} \int_{-h+(k-1)\delta}^{-h+k\delta} (1, z)(\sigma_{11}^{(k)}, \sigma_{22}^{(k)}, \sigma_{12}^{(k)})dz \quad (6)$$

Let $\overline{\phi} = \phi/h_1$ be the stress function for the stress resultants defined by

$$[N_{11}, N_{22}, N_{12}] = [\phi_{,yy}, \phi_{,xx}, -\phi_{,xy}] \quad (7)$$

Considering relations (2–7) in Eq. (1) for compatibility and dynamic stability equations of laminated circular cylindrical shells resting on an elastic foundation, after some mathematical operations, one gets

$$L_1(w) + L_2(\phi) - \phi_{,xx}/R = L(\phi, w) \quad (8)$$

$$L_3(\phi) + L_4(w) + w_{,xx}/R = -0.5L(w, w) \quad (9)$$

where

$$\begin{aligned}
L_1(\bullet) &= C_3(\bullet)_{,xxxx} + (C_4 + 2C_{10} \\
&\quad + C_7)(\bullet)_{,xxyy} + C_8(\bullet)_{,yyyy} + k_0(\bullet) \\
&\quad + \rho_1 h_1(\bullet)_{,tt} \\
L_2(\bullet) &= -C_2(\bullet)_{,xxxx} - (C_1 - 2C_9 \\
&\quad + C_6)(\bullet)_{,xxyy} - C_5(\bullet)_{,yyyy} \\
L_3(\bullet) &= C_{16}(\bullet)_{,xxxx} + (C_{12} + 2C_{19} \\
&\quad + C_{15})(\bullet)_{,xxyy} + C_{11}(\bullet)_{,yyyy} \\
L_4(\bullet) &= -C_{17}(\bullet)_{,xxxx} - (C_{13} - 2C_{20} \\
&\quad + C_{18})(\bullet)_{,xxyy} - C_{14}(\bullet)_{,yyyy} \\
L(\bullet) &= (\bullet)_{,yy}(\bullet)_{,xx} + (\bullet)_{,xx}(\bullet)_{,yy} \\
&\quad - 2(\bullet)_{,xy}(\bullet)_{,xy}
\end{aligned} \quad (10)$$

in which the expressions $C_j (j = 1, 2, \ldots, 20)$ are:

$$C_1 = a_{111}C_{11} + a_{121}C_{15},$$

$$C_2 = a_{111}C_{12} + a_{121}C_{16},$$

$$C_3 = a_{111}C_{13} + a_{121}C_{17} + a_{112},$$

$$C_4 = a_{111}C_{15} + a_{121}C_{18} + a_{122},$$

$$C_5 = a_{211}C_{11} + a_{221}C_{15},$$

$$C_6 = a_{211}C_{12} + a_{221}C_{16},$$

$$C_7 = a_{211}C_{13} + a_{221}C_{17} + a_{212},$$

$$C_8 = a_{211}C_{14} + a_{221}C_{18} + a_{222},$$

$$\begin{aligned}
&C_9 = a_{661}C_{19},\\
&C_{10} = a_{661}C_{20} + a_{662},\\
&C_{11} = a_{220}D,\ C_{12} = -a_{120}D,\\
&C_{13} = (a_{120}a_{211} - a_{111}a_{220})D,\\
&C_{14} = (a_{120}a_{221} - a_{121}a_{220})D,\\
&C_{15} = -a_{210}D,\\
&C_{16} = a_{110}D,\\
&C_{17} = (a_{210}a_{111} - a_{211}a_{110})D,\\
&C_{18} = (a_{210}a_{121} - a_{221}a_{110})D,\\
&C_{19} = 1/a_{660}, C_{20} = -a_{661}/a_{660},\\
&D = 1/(a_{110}a_{220} - a_{210}a_{120})
\end{aligned} \tag{11}$$

Finally, the expressions for the factors $a_{ij\gamma},\ i,j = 1,2,6$ and $\gamma = 0,1,2$ are:

$$\begin{aligned}
a_{11\gamma} &= h^{\gamma+1}\sum_{k=1}^{N}\frac{E_{01}^{(k)}\Psi^{(k)}}{1-v_{12}^{(k)}v_{21}^{(k)}},\\
a_{12\gamma} &= h^{\gamma+1}\sum_{k=1}^{N}\frac{v_{21}^{(k)}E_{01}^{(k)}\Psi^{(k)}}{1-v_{12}^{(k)}v_{21}^{(k)}},\\
a_{22\gamma} &= h^{\gamma+1}\sum_{k=1}^{N}\frac{E_{02}^{(k)}\Psi^{(k)}}{1-v_{12}^{(k)}v_{21}^{(k)}},\\
a_{21\gamma} &= h^{\gamma+1}\sum_{k=1}^{N}\frac{v_{12}^{(K)}E_{02}^{(k)}\Psi^{(k)}}{1-v_{12}^{(k)}v_{21}^{(k)}},\\
a_{66\gamma} &= h^{\gamma+1}\sum_{k=1}^{N}G_0^{(k)}\Psi^{(k)},\\
\Psi^{(k)} &= \int_{-1+2(k-1)/N}^{-1+2k/N}\overline{z}^{\gamma}\overline{\varphi}^{(k)}(\overline{z})d\overline{z}
\end{aligned} \tag{12}$$

3. Analytic solution of the problem

Assuming that the cylindrical shell is simply supports at both ends, the solution of equation set (8–9) is sought in the following form [10]:

$$w = q(t)\sin\alpha x\sin\beta y \tag{13}$$

where $\alpha = m\pi/L, \beta = n/R, m$ is the half wave length in the direction of the x axis, n is the wave number in the direction of the y axis and $q(t)$ is the time dependent amplitude. Substituting expressions (13) in the Eq. (9) and eliminating

$$\begin{aligned}
\phi &= A_1\cos 2\alpha x + A_2\cos 2\beta y\\
&\quad + A_3\sin\alpha x\sin\beta y
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
&A_1 = B_1q^2(t), A_2 = B_2q^2(t),\ A_3 = B_3q(t),\\
&B_1 = \frac{\beta^2}{32C_{16}\alpha^2},\ B_2\frac{\alpha^2}{32C_{11}\beta^2}\\
&B_3 = [C_{17}\alpha^4(C_{13} - 2C_{20} + C_{18})\alpha^2\beta^2\\
&+C_{14}\beta^4 + \alpha^2/R]/[C_{16}\alpha^4 +\\
&(C_{12} + 2C_{19} + C_{15})\alpha^2\beta^2 + C_{11}\beta^4]
\end{aligned} \tag{15}$$

Substituting expressions (13) and (14) in Eq. (8) and applying Galerkin method in the ranges $0 \leqslant x \leqslant L$ and $0 \leqslant y \leqslant 2\pi R$, the following nonlinear time differential equation obtained as:

$$q_{1,\tau\tau} + \lambda_1 q_1(\tau) + \lambda_2 q_1^2(\tau) + \lambda_3 q_1^3(\tau) = 0 \tag{16}$$

where $q_1 = q/h_1, \tau = t\omega$, ω is frequency parameter, τ is dimensionless parameter and the following definitions apply:

$$\begin{aligned}
&\lambda_1 = \frac{\Lambda_1 + \overline{k}_0}{\omega^2}, \quad \overline{k}_0 = \frac{k_0}{\rho_1 h_1},\\
&\lambda_2 = \frac{8\beta^2\alpha^2 B_2}{\rho_1\omega^2}, \quad \lambda_3 = \frac{8\alpha^2\beta^2 B_1 h_1}{\rho_1\omega^2}
\end{aligned} \tag{17a}$$

$$\begin{aligned}
\Lambda_1 = \frac{1}{\rho_1 h_1}\{&[C_3\alpha^4 + (C_4 + 2C_{10} + C_7)\alpha^2\beta^2\\
&+C_8\beta^4] + B_3[\alpha^2/R - C_2\alpha^4 - (C_1\\
&-2C_9 + C_6)\alpha^2\beta^2 - C_5\beta^4]\}
\end{aligned} \tag{17b}$$

in which

$$k_0 = a_{110}k_1/R^2 \tag{18}$$

where k_1 is a non-dimensional foundation modulus.

The expression (18) for the single layer shell made of homogeneous isotropic and orthotropic material are in the following form [16,18]:

$$k_{01} = E_0k_1h_1/[R^2(1-\nu^2)] \tag{19}$$

$$k_{02} = E_{01}k_1h_1/[R^2(1-\nu_{12}\nu_{21})] \tag{20}$$

where E_0, ν are the Young's modulus and Poisson's ratio of the homogeneous isotropic material and E_{01}, ν_{12}, ν_{21} are the Young's modulus and Poisson's ratios of the homogeneous orthotropic material, respectively.

By making the following transformation in Eq. (16),

$$q_1(\tau) = (A_p/h_1)f(\tau) \tag{21}$$

below equation is obtained:

$$f_{,\tau\tau}(\tau) + \lambda_1 f(\tau) + \lambda_2 (A_p/h_1) f^2(\tau) + \lambda_3 (A_p/h_1)^2 f^3(\tau) = 0 \tag{22}$$

An approximating function will be chosen as a first approximation as [10]:

$$f(\tau) = \cos\tau \tag{23}$$

satisfying the initial conditions

$$f(0) = 1,\ f_{,\tau}(0) = 0 \tag{24}$$

Substituting expression (23) in Eq.(22) and solving the resulting equation, satisfying the orthogonality condition

$$\int_0^{\pi/2} [f_{,\tau\tau}(\tau) + \lambda_1 f(\tau) + \lambda_2 (A_p/h_1) f^2(\tau) + \lambda_3 (A_p/h_1)^2 f^3(\tau)] \cos\tau d\tau = 0 \tag{25}$$

the amplitude-frequency relation for the finite deformations of laminated non-homogeneous orthotropic cylindrical thin shells resting on elastic foundation is obtained in the following form:

$$\frac{\omega_{NL}}{\omega_L} = \left[1 + \frac{8}{3\pi}\frac{A_p}{h_1}\frac{\lambda_2}{\lambda_1} + \frac{3}{4}\left(\frac{A_p}{h_1}\right)^2 \frac{\lambda_3}{\lambda_1}\right]^{1/2} \tag{26}$$

where

$$\omega_L = (\Lambda_1 + \overline{k}_0)^{1/2} \tag{27}$$

ω_L is the linear frequency, ω_{NL} is the nonlinear frequency and ω_{NL}/ω_L relative frequency of vibrating shell.

The dimensionless frequency parameter defined in the following form:

$$\Delta = \omega_L [\rho_1 R^2 h / a_{110}]^{0.5} \tag{28}$$

The expression (28) for a single layer shell made of homogeneous orthotropic and isotropic material are in the following form [18,31]:

$$\Delta_1 = \omega_L [\rho_0 R^2 (1 - \nu_{12}\nu_{21})/E_{01}]^{0.5} \tag{29}$$

$$\Delta_2 = \omega_L [\rho_0 R^2 (1 - \nu^2)/E_0]^{0.5} \tag{30}$$

where ρ_0 density of the homogeneous material in a single layer shell.

The solution of Eq. (19) with initial conditions $f(0) = 1, f_{,\tau}(0) = 0$ have been given in [9] and the ratio of nonlinear and linear frequencies takes the form

$$\frac{\omega_{1NL}}{\omega_{1L}} = \left[1 + \left(\frac{A_p}{h_1}\right)^2 \left(\frac{3\lambda_3}{4\lambda_1} - \frac{5}{6}\left(\frac{\lambda_2}{\lambda_1}\right)^2\right)\right]^{1/2} \tag{31}$$

When $\lambda_2 = \lambda_3 = 0$, $k_1 = 0$ expression (26) yields the amplitude-frequency relation for the geometric linear free vibration analysis of a laminated non-homogeneous orthotropic cylindrical thin shell as a special case. When $\mu = 0$, $N = 1$ expression (26) yields the amplitude-frequency relation for a single layer non-homogeneous orthotropic cylindrical thin shell resting on an elastic foundation, as another special case.

Table 1
Comparing the results obtained in [16] (the lowest eigenfrequency Δ_2) with the dimensionless frequency parameter when the effect of elastic foundation is taken into consideration ($m_1 = m\pi R/L$, $n = 2$, $k_1 = 0.5$, $h_1/R = 0.002$, $\nu = 0.3$)

	$\Delta_2 = \omega_L[\rho_0 R^2(1-\nu^2)/E_0]^{0.5}$			
m_1	1	2	3	4
Paliwal and Pandey [16]	0.670	0.780	0.910	1.010
Present study	0.725	0.799	0.884	0.955

Table 2
Comparison of frequency parameters Δ_2 for an isotropic cylindrical shell ($m = 1$, $R/h_1 = 500$, $L/R = 6$, $\nu = 0.3$)

	$\Delta_2 = \omega_L[\rho_0 R^2(1-\nu^2)/E_0]^{0.5}$		
n	Present study	Naeem and Sharma [31], (Number of polynamials N_1)	
		$N_1 = 2$	$N_1 = 8$
2	0.05696	0.05976	0.054323
3	0.027715	0.029967	0.027074
4	0.01829	0.019339	0.017776
5	0.01776	0.017804	0.017088
6	0.020746	0.021587	0.021303
7	0.028878	0.028213	0.028089
8	0.03727	0.036524	0.036469
9	0.04698	0.046195	0.046174
10	0.057896	0.057105	0.057088

4. Results and discussions

To validate the analysis, for simply supported one layered orthotropic cylindrical shells, the values of relative frequency are compared with the analytical results obtained in [22] and the results obtained in [3] by using finite elements method, see in Fig. 2. For one layered isotropic cylindrical shells, a) by taking the effect of foundation into consideration, the values of dimensionless frequency parameter are compared with the analytical results obtained in Ref. [16], see Table 1, b) the values of dimensionless frequency pa-

Table 3
Comparison of experimental and theoretical natural frequencies (Hz) of an isotropic cylindrical shell ($h_1 = 2.29 \times 10^{-4}$ (m), $R = 0.377$ (m), $L = 0.234$ (m), $E_0 = 2 \times 10^5$ (MPa), $\rho_0 = 7.8 \times 10^3$(kg/m^3), $\nu = 0.3$)

(m,n)	Lakis et al. [3]	Exper. Study Lindholm et al. [37]	Present study
(1,5)	942	995	1012
(1,6)	1353	1430	1429
(1,7)	1853	1938	1935
(2,3)	2067	2070	2000
(2,4)	1368	1430	1369
(2,5)	1248	1313	1290
(2,6)	1489	1570	1551
(2,7)	1927	2050	1998

rameter are compared with the analytical results obtained in Ref. [31], see Table 2, c) the values of natural frequency are compared with the experimental results obtained in references [23,37,39], analytical results obtained in Ref. [31] and the results obtained in [3] by using finite elements method, see Tables 3–4 and Fig. 3. The comparisons show that the present results are in accommodation with the results in literature.

Figure 2 shows, the influence of geometrical nonlinear effects on the free vibrations of a simply supported orthotropic cylindrical shell, along with corresponding results given in references [3] and [22]. The given results in [22] were obtained based on Donnell's simplified non-linear method where only lateral displacement was considered. The finite element method based on an energy formulation is used in [3]. The comparisons were carried out for non-dimensional foundation modulus $k_1 = 0$ and for the following material properties, shell parameters and mode numbers:

$$E_{01} = 2 \times 10^5 \text{ (MPa)},\ E_{02} = 0.05 \times E_{01},$$
$$\nu_{12} = 0.2,\ \nu_{21} = 0.05 \times \nu_{12},$$
$$\rho_0 = 7.8 \times 10^3 \text{ (kg/m}^3),\ R = 0.254 \text{ (m)},$$
$$L = 0.40 \text{ (m)},\ h_1 = 0.00254 \text{ (m)},\ m = 1,$$
$$n = 4.$$

The values obtained in this study for the free vibration frequencies are greater than the values obtained in references [3] and [22].

In Table 1, the results obtained in this study for dimensionless frequency parameter are compared with the theoretical results obtained in [16] for the non-dimensional foundation modulus of $k_1 = 0.5$. It is observed that there is an agreement with the results obtained in this study and the results in [16].

In Table 2, the values of dimensionless frequency parameter for one layered isotropic cylindrical shell for which the effect of foundation is not taken into consideration are compared with the values of dimensionless frequency parameter obtained analytically in Ref. [31]. It is observed that the results are in a well accommodation. Besides, in both of two studies, the minimum values of dimensionless frequency parameter versus the circumferential wave number $n = 5$ and these values are exactly the same.

In Fig. 3, the values of natural frequencies for one layered isotropic cylindrical shell for which the effect of foundation is not taken into consideration are compared with experimental values in [23] and with analytical values obtained in [31]. There is an agreement with the results obtained in this study, analytical results obtained in [31] and experimental results obtained in Ref. [23]. Besides, although in the studies which are compared, the minimum values of natural frequencies covers the value of circumferential wave number when $n = 7$, in the present study $n = 6$. But in all three studies, the values of natural frequencies are approximately the same. When the results obtained in this study are compared with the theoretical results obtained in [31] and the experimental results obtained in [23], the maximum difference is 5.5% and it is covering the value of circumferential wave number $n = 4$.

The values of the non-dimensional fundamental frequencies obtained from the present study are shown in Fig. 4 along with corresponding values given in references [29,30], for a four layer cross-ply cylindrical shell to demonstrate the accuracy and range of applicability of the present study. All layers, for Fig. 4, are assumed to have the same geometric and material parameters and the individual layer is assumed to be orthotropic with the following material properties:

$$E_{01} = 25 \times E_{02},\ G_0 = 0.5 \times E_{02},$$
$$\nu_{12} = 0.25,\ \rho_0 = 1,\ m = 1,\ n = 4$$

The results obtained in this study are in a well accommodation with the theoretical results obtained in [29]. However, the results obtained in [30] by using finite element are a little smaller. It is because of that, also the effect of transverse shear deformation is taken into consideration in [30]. In this study, medium length shells are used, so the values of L/R are taken into consideration in comparison which are showed in figure.

In Table 3, the values of natural frequency for one layered isotropic cylindrical shell for which the effect of foundation is not taken into consideration are compared with experimental values obtained in Ref. [37] and with the values obtained in [3] by using finite elements method. The comparison shows that the re-

Table 4
Comparison of experimental frequency spectra of a monocoque cylindrical shell ($R/h_1 = 400$, $L/R = 0.465$, $\rho_0 = 8.538 \times 10^3$ (kg/m^3), $E_0 = 2.06843 \times 10^5$ (MPa), $\nu = 0.315$)

	ω_L(cps)									
(m, n)	(1,3)	(1,4)	(1,8)	(1,11)	(2,5)	(2,12)	(3,9)	(3,11)	(4,9)	(4,11)
Experimental study Weingarten [39]	1648	1266	590	765	2168	1067	1797	1560	2557	2320
Present study	1626	1056	586	944	2103	1252	1775	1576	2528	2143

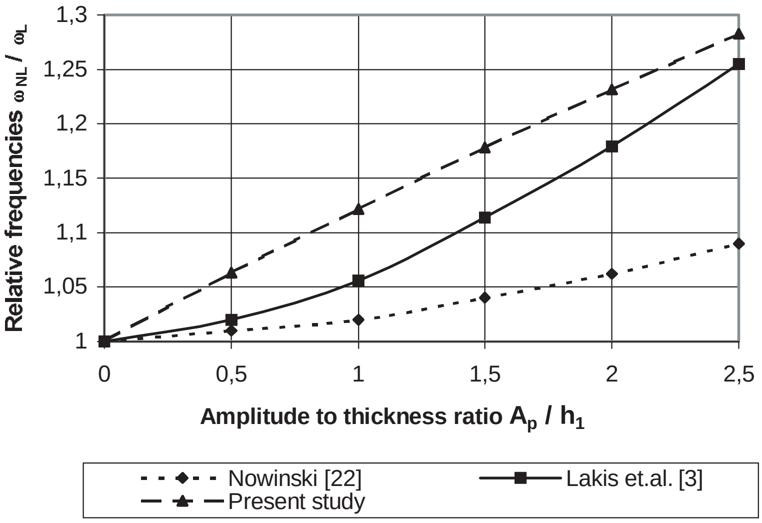

Fig. 2. Relative frequencies versus relative amplitude A_p/h_1 for nonlinear vibration of an orthotropic cylindrical shell.

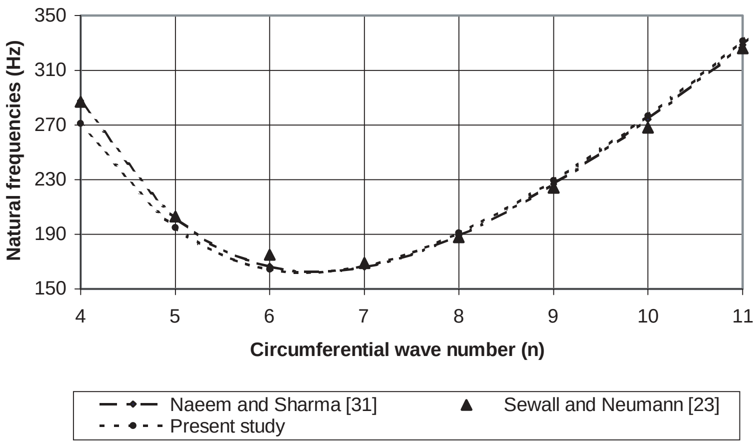

Fig. 3. Comparison of experimental and theoretical natural frequencies (Hz) of an isotropic cylindrical shell ($h_1 = 0.000648$ (m), $R = 0.2423$ (m), $L = 0.6096$ (m), $m = 1$, $k_1 = 0$, $\rho_0 = 2.7145 \times 10^3$ (kg/m^3), $E_0 = 68.95 \times 10^3$ (MPa), $\nu = 0.315$).

sults obtained in this study and the experimental values obtained in [37] are more approximate than the results which are obtained in [3] by using finite elements method.

In Table 4, the values of frequency spectra for one layered isotropic cylindrical shell for which the effect of foundation is not taken into consideration are compared with the values of frequency spectra obtained in [39] experimentally. It is observed that the results are in accommodation.

In the calculations presented in Table (5–10), that are done for the formulas (26)–(31) by considering the cases of cross-ply laminated orthotropic cylindrical shells up to five layers, variation function of Young's

Table 5
Variation of the natural and relative frequencies with respect to number and ordering of layers for non-dimensional foundation modulus $k_1 = 0.01$ and variation coefficient of Young's moduli $\mu = 0$ and $\mu = 0.9$ ($m = 1$, $n = 4$, $h_1 = 5.08 \times 10^{-4}$ (m), $A_p/h_1 = 3$)

		$k_1 = 0, \mu = 0$			$k_1 = 0.01, \mu = 0$		
N	Stacking of layers	$\overline{\omega}_L$(Hz)	ω_{NL}/ω_L	ω_{1NL}/ω_{1L}	$\overline{\omega}_L$(Hz)	ω_{NL}/ω_L	ω_{1NL}/ω_{1L}
1	(0°)	790	1.681	1.164	1846	1.155	1.032
1	(90°)	618	2.892	2.858	1779	1.374	1.365
2	(0°/90°) (90°/0°)	796	2.033	1.829	1848	1.257	1.198
3	(0°/90°/0°)	778	1.953	1.661	1841	1.226	1.146
3	(90°/0°/90°)	803	2.124	1.989	1851	1.288	1.247
4	(0°/90°/0°/90°)	796	2.033	1.829	1848	1.257	1.198
4	(90°/0°/90°/0°)	796	2.033	1.829	1848	1.257	1.198
4	(0°/90°/90°/0°)	766	2.092	1.878	1836	1.260	1.200
4	(90°/0°/0°/90°)	824	1.980	1.785	1861	1.254	1.195
5	(0°/90°/0°/..)	785	1.987	1.732	1844	1.238	1.167
5	(90°/0°/90°..)	803	2.082	1.922	1851	1.276	1.227
N		$k_1 = 0, \mu = 0.9$			$k_1 = 0.01, \mu = 0.9$		
1	(0°)	1077	1.681	1.165	1986	1.240	1.051
1	(90°)	840	2.897	2.863	1868	1.580	1.567
2	(0°/90°) (90°/0°)	1083	2.034	1.830	1989	1.389	1.302
3	(0°/90°/0°)	1058	1.958	1.668	1976	1.347	1.230
3	(90°/0°/90°)	1094	2.120	1.984	1995	1.432	1.372
4	(0°/90°/0°/90°)	1083	2.034	1.823	1989	1.389	1.302
4	(90°/0°/90°/0°)	1083	2.034	1.823	1989	1.389	1.302
4	(0°/90°/90°/0°)	1043	2.098	1.886	1968	1.398	1.311
4	(90°/0°/0°/90°)	1122	1.977	1.614	2011	1.381	1.294
5	(0°/90°/0°/..)	1069	1.988	1.735	1981	1.364	1.259
5	(90°/0°/90°..)	1093	2.083	1.922	1994	1.415	1.345

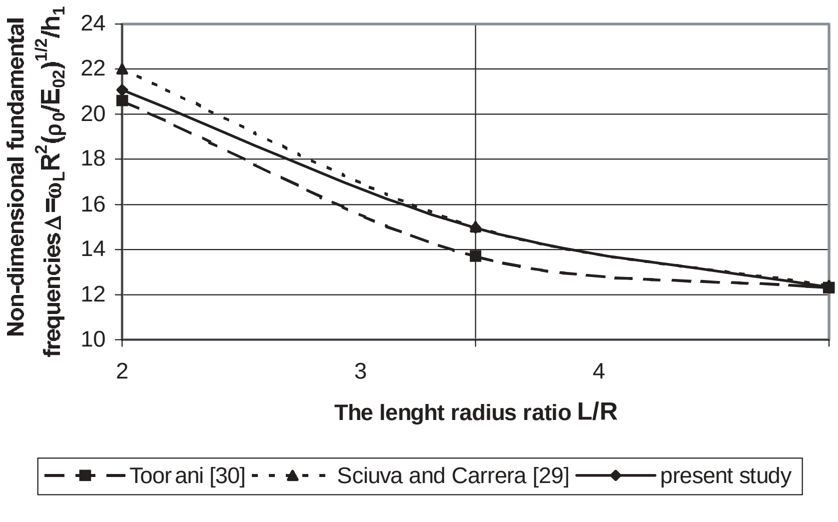

Fig. 4. Non-dimensional fundamental frequencies of simply supported cylindrical shell with symmetric cross-ply (0°/90°/90°/0°).

moduli in layers is taken into consideration in the form $\varphi^{(k)}(\overline{z}) = \exp(-0.1|\overline{z}|)$ and natural frequency is taken into consideration in the form $\overline{\omega}_L = \omega_L/(2\pi)$ (Hz). Furthermore, the values of variation coefficient of Young's moduli μ and non-dimensional foundation modulus k_1 are indicated in Tables (5–10). In all layers of cross-ply cylindrical shell, the material properties are same and the material properties in [2], shell parameters in Ref. [39] and non-dimensional foundation modulus in Ref. [16–18,35,38] are given as:

$$E_{01} = 1.3237 \times 10^5 \text{ (MPa)},$$

$$E_{02} = 1.0755 \times 10^4 \text{ (MPa)},$$

$$G_0 = 5.6537 \times 10^3 \text{ (MPa)},\ \nu_{12} = 0.24,$$

$$\nu_{21} = 0.0195,\ \rho_0 = 1.308 \times 10^3 \text{ (kg/m}^3\text{)},$$

Table 6
Variation of the relative frequency with respect to amplitude of the frequency, number and ordering of layers for $k_1 = 0.005$ and $\mu = 0$ ($h_1 = 2.54 \times 10^{-4}$ (m), $m = 1$, $n = 4$)

A_p/h_1	ω_{NL}/ω_L							
	$k_1 = 0, \mu = 0$				$k_1 = 0.005, \mu = 0$			
	(0°)	2 Layers	(0°/90°/0°)	(90°/0°/90°)	(0°)	2 Layers	(0°/90°/0°)	(90°/0°/90°)
0	1	1	1	1	1	1	1	1
0.5	1.062	1.044	1.050	1.039	1.0193	1.0131	1.0152	1.0131
1.0	1.122	1.103	1.109	1.098	1.0390	1.0311	1.0338	1.0286
1.5	1.182	1.174	1.176	1.176	1.0590	1.0538	1.0555	1.0524
2.0	1.241	1.255	1.250	1.268	1.0793	1.0808	1.0803	1.0940
2.5	1.299	1.345	1.328	1.371	1.1000	1.1120	1.1080	1.1170
3.0	1.356	1.441	1.412	1.484	1.1210	1.1468	1.1382	1.1568

Table 7
Variation of the natural frequency with respect to half axial wave number m, number and ordering of layers for $k_1 = 0.01$, $\mu = 0$ and $\mu = 0.9$ ($n = 4$, $h_1 = 2.54 \times 10^{-4}$ (m), $A_p/h_1 = 3$)

m	$\overline{\omega}_L$(Hz)							
	$k_1 = 0, \mu = 0$				$k_1 = 0.01, \mu = 0$			
	(0°)	2 Layers	(0°/90°/0°)	(90°/0°/90°)	(0°)	2 Layers	(0°/90°/0°)	(90°/0°/90°)
1	783	759	768	738	1843	1833	1837	1824
2	1575	1634	1637	1619	2295	2335	2337	2325
3	2228	2465	2454	2461	2784	2977	2967	2974
4	2750	3257	3224	3266	3217	3660	3630	3668
m	$k_1 = 0, \mu = 0.9$				$k_1 = 0.01, \mu = 0.9$			
1	1067	1033	1046	1006	1980	1962	1969	1948
2	2146	2226	2230	2207	2718	2782	2785	2767
3	3036	3359	3344	3354	3464	3750	3737	3746
4	3747	4438	4394	4450	4101	4741	4700	4752

Table 8
Variation of the natural frequency with respect to the number of circumferential waves and number and ordering of layers for $k_1 = 0.005$, $\mu = 0$ and $\mu = 0.9$ ($m = 1, h_1 = 2.54 \times 10^{-4}$ (m), $A_p/h_1 = 3$)

n	$\overline{\omega}_L$(Hz)								
	$k_1 = 0, \mu = 0$					$k_1 = 0.005, \mu = 0$			
	(0o)	2 Layers	(0°/90°/0°)	(90°/0°/90°)		(0°)	2 Layers	(0°/90°/0°)	(90°/0°/90°)
4	783	759	768	737	1416	1403	1408	1391	
5	612	602	592	602		1329	1324	1320	1324
6	501	536	483	578		1282	1296	1275	1314
7	431	546	421	643		1256	1300	1253	1344
8	396	616	399	771		1245	1331	1246	1410
9	391	728	412	943		1258	1387	1250	1510
10	412	872	452	1147		1250	1467	1263	1646
11	454	1041	513	1379		1264	1573	1287	1815
n	$k_1 = 0, \mu = 0.9$					$k_1 = 0.005, \mu = 0.9$			
4	1067	1033	1046	1006		1590	1568	1576	1551
5	833	819	806	820		1444	1436	1429	1437
6	682	729	657	786		1363	1387	1350	1418
7	587	741	573	873		1318	1393	1311	1468
8	539	835	543	1046		1297	1445	1299	1577
9	531	987	560	1278		1294	1538	1306	1739
10	559	1182	614	1553		1306	1670	1330	1951
11	615	1410	698	1868		1331	1839	1371	2209

$L/R = 0.465,\ R = 0.1016$ (m),

$h_1 = 2.54 \times 10^{-4}$ (m) or

$h_1 = 5.08 \times 10^{-4}$ (m)

In Table 5, by taking the number and ordering of layers, elastic foundation and the effect of non-homogeneity into consideration, the values of natural frequency and ω_{NL}/ω_L relative frequency are pre-

Table 9
Variation of the relative frequency with respect to the number of circumferential waves and number and ordering of layers for $k_1 = 0.005$ and $\mu = 0$ ($m = 1, h_1 = 2.54 \times 10^{-4}$ (m), $A_p/h_1 = 3$)

$\mu = 0$	ω_{NL}/ω_L							
n	$k_1 = 0$				$k_1 = 0.005$			
	(0°)	2 Layers	(0°/90°/0°)	(90°/0°/90°)	(0°)	2 Layers	(0°/90°/0°)	(90°/0°/90°)
4	1.356	1.442	1.412	1.484	1.121	1.147	1.138	1.157
5	1.609	2.047	1.941	2.169	1.156	1.288	1.247	1.328
6	1.988	2.899	2.855	2.959	1.205	1.506	1.423	1.581
7	2.503	3.676	4.107	3.497	1.273	1.791	1.671	1.890
8	3.096	4.162	5.449	3.755	1.368	2.120	1.987	2.219
9	3.649	4.404	6.563	3.863	1.490	2.465	2.360	2.535
10	4.062	4.513	7.308	3.907	1.638	2.801	2.776	2.816
11	4.321	4.463	7.743	3.925	1.810	3.110	3.222	3.053
12	4.465	4.585	7.981	3.933	2.000	3.381	3.682	3.244

Table 10
Variation of the natural and relative frequencies with respect to non-dimensional foundation modulus k_1, number and ordering of layers for $\mu = 0$ ($m = 1, n = 7, h_1 = 2.54 \times 10^{-4}$ (m), $A_p/h_1 = 3$)

$\mu = 0$	$\overline{\omega}_L$(Hz)				ω_{NL}/ω_L			
k_1	(0°)	2 Layers	(0°/90°/0°)	(90°/0°/90°)	(0°)	2 Layers	(0°/90°/0°)	(90°/0°/90°)
0	431	546	421	643	2.5032	3.6764	4.1074	3.497
0.005	1256	1300	1253	1344	1.2733	1.7907	1.6712	1.890
0.010	1723	1755	1721	1788	1.1533	1.4866	1.3965	1.5660
0.015	2088	2115	2086	2142	1.1067	1.3541	1.2831	1.4184

sented. When the effect of elastic foundation isn't taken into consideration, it is observed that the effect of the variation of number and ordering of layers on the natural frequency ω_{NL}/ω_L relative frequency values are very important. Besides, although the effect of non-homogeneity on natural frequency values is very considerable, the effect of non-homogeneity on ω_{NL}/ω_L relative frequency values is very little. When the effect of elastic foundation is taken into consideration, values of the natural frequency increase, however values of the relative frequency decrease and approximates to one. Consequently, the effect of non-linearity on the values of vibration frequencies decreases. Besides this, the effect of non-homogeneity and variation of layers number on the values of vibration frequency is very small. Furthermore, when the calculations done according to the formulas (26) and (31) are compared, it is observed that the values of ω_{NL}/ω_L are greater than the values of ω_{1NL}/ω_{1L}. The formulas used for ω_{1NL}/ω_{1L} relative frequency values is obtained from [9].

In Table 6, by taking various number and ordering of layers and elastic foundation into consideration, the relative frequency values are presented dependent on the variation of A_p/h_1. When the effect of elastic foundation isn't taken into consideration, if A_p/h_1 increases, the effect of geometrical non-linearity on frequency values increases. The effect of the number and ordering of layers variation on frequency values is considerable. When the effect of elastic foundation is taken into consideration, there is no effect of variation of number and ordering of layers on the frequency values.

In Table 7, taking various number and ordering of layers, elastic foundation and non-homogeneity into consideration, the values of natural frequency for the half axial wave number are presented. When the effect of elastic foundation isn't taken into consideration ($k_1 = 0$) and the half axial wave number increases, the values of natural frequency increases importantly and the effect of non-homogeneity, variation of number and ordering of layers is very considerable. When the effect of elastic foundation is taken into consideration ($k_1 = 0.01$), the effect of the changes in the Young's moduli in layers on natural frequency values decrease. There is a very little effect of the variation number and ordering of layers.

By comparing Tables 5 and 7, it can be observed that, the ratio of thickness to radius (h_1/R) variation effect on natural frequency values is considerable.

In Table 8, by taking various number and ordering of layers, elastic foundation and non-homogeneity into consideration, natural frequency values for different circumferential wave number values are presented. When the effect of elastic foundation isn't taken into consideration, the effect of number and ordering of layers on natural frequency values in the great values of wave number is very large. For example, for the wave numbers (1,11), in one layered (0°) ordering and three layered (90°/0°/90°) ordering shells, difference between the values of natural frequencies go up to 3 times of the first value. When the effect of elastic foundation is taken into consideration, the factors in question are not effective and the values of natural frequency are approximate with the values of one layered shell. Besides, the minimum values of natural frequency are obtained in different wave circumferential numbers by the effect of in question factors. For example, the minimum value of natural frequency is obtained for $k_1 = 0$, $\mu = 0$ in the circumferential wave number $n = 8$ for (0°/90°/0°) ordering shells and for $k_1 = 0$, $\mu = 0$ in the circumferential wave number $n = 6$ for (90°/0°) and (90°/0°/90°) ordering shells.

In Table 9, taking various number and ordering of layers and elastic foundation into consideration, ω_{NL}/ω_L relative frequency values are presented for different circumferential wave number values. If the effect of elastic foundation is not taken into consideration, when circumferential wave number values increase, relative frequency values increase and for the great values of wave number, the effect of number and ordering of layers is important. For example, for (1,12) wave numbers, in three layered (0°/90°/0°) and (90°/0°/90°) ordering shells, the difference between the relative frequency values is 51%. When the effect of elastic foundation is taken into consideration, in small values of circumferential wave number, the factors in question are not effective and the relative frequency values are approximate with the values of one layered shells.

In Table 10, natural frequency and ω_{NL}/ω_L relative frequency values for various non-dimensional foundation modulus, various number and ordering of layers are presented. When the foundation modulus increase, the effect of non-linearity on natural frequency and relative frequency values decreases. But in larger values of foundation modulus, this effect may not taken into consideration.

5. Conclusion

The present study considers the vibration problem of laminated non-homogeneous orthotropic cylindrical thin shells resting on elastic foundation, accounting for Karman type geometric non-linearity. Obtaining the fundamental relations employing finite deformation analysis, the equations of compatibility and dynamic stability are derived and solved simultaneously to establish an analytical relation between amplitude and frequency. The following conclusions have been drawn from the numerical analysis carried out using the general formulas obtained from the analytical study:

a) When the effect of elastic foundation is not taken into consideration, the effects of geometrical non-linearity, non-homogeneity, number and ordering of layers are very important on frequency values.
b) When the value of foundation modulus increases, the effects of the factors in question on the frequency values decrease and these factors do not have effects on frequency values in larger values of foundation modulus.
c) When the ratio of shell thickness to radius (h_1/R) increases, the values of natural frequency increase.

A validation of the analysis has been carried out by comparing results with those in the literature and has found to be accurate.

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Nomenclature

$A_p, A_p/h_1$	Amplitude and dimensionless or relative amplitude of motion, respectively
$C_j, a_{ij\gamma}$	The constants depend on properties of material and shell and included by Eq. (9)
$G_0^{(k)}$	Shear moduli of the homogeneous orthotropic materials in the kth layer
G_0	Shear modulus of the homogeneous material in a single layer shell
$E_{01}^{(k)}, E_{02}^{(k)}$	Young's moduli of the homogeneous orthotropic materials in the kth layer
E_{01}, E_{02}	Young's moduli of the homogeneous orthotropic material in a single layer
E_0	Young's modulus of the homogeneous isotropic material in a single layer
e_{11}, e_{22}, e_{12}	Axial, circumferential and shear strains on the reference surface of the shell, respectively
$f(\tau)$	Time dependent amplitude
$h_1 = 2h$	Thickness of the cylindrical shell
k_0, k_1	Foundation modulus and non-dimensional foundation modulus, respectively
k	Denotes the kth layer
L	Length of the cylindrical shell
m	Half axial wave number
$m_1 = m\pi R/L$	Axial wave parameter
M_{11}, M_{22}, M_{12}	Axial, circumferential and twisting moments, respectively
N_{11}, N_{22}, N_{12}	Axial, circumferential and shear forces, respectively
N, N_i	Number of layers and number of polynomials, respectively
n	Circumferential wave number
$Oxyz$	Coordinate system with the origin on the reference surface of the shell

$Q_{ij}^{(k)(i,j=1,2,6)}$	Coefficients defined by Eq (4)
$q(\tau)$	Time dependent amplitude
R	Mean radius of the cylindrical shell
t	Time coordinate
u, v, w	Axial, circumferential and radial displacements, respectively
x, y, z	Rectangular coordinates (axial, circumferential and radial, respectively)
$\alpha = m\pi/L$	Parameter
$\beta = n/R$	Parameter
ϕ	Stress function
$\varphi^{(k)}(\overline{z})$	Variation function of Young's moduli in the kth layer
$\nu_{12}^{(k)}, \nu_{21}^{(k)}$	Poisson's ratios of the orthotropic materials in the kth layer
ν_{12}, ν_{21}	Poisson's ratios of the orthotropic materials in a single layer shell
ν	Poisson's ratio of the isotropic material in a single layer shell
$\lambda_1, \lambda_2, \lambda_3$	Coefficients defined by Eq. (20a)
μ	Variation coefficient of the Young's moduli
ρ_0	Density of the homogeneous material in a single layer shell
$\rho_0^{(k)}$	Densities of the homogeneous orthotropic materials in the kth layer
$\sigma_{ij}^{(k)}(i, j = 1, 2)$	Stress components in the kth layer
τ	Dimensionless time parameter
ω	Frequency parameter
$\omega_L, \overline{\omega}_L,$	Linear frequency and natural frequency (Hz), respectively
ω_{NL}	Non-linear frequency of free vibrations
ω_{NL}/ω_L	Relative frequency defined by Eq (26)
ω_{1NL}/ω_{1L}	Relative frequency defined by Eq (31)
$\Delta, \Delta_1, \Delta_2$	Dimensionless frequency parameters defined by Eqs (28–30)