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Vibration analysis of rectangular plates with one or more guided edges via bicubic B-spline method

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Abstract. A simple and accurate method is proposed for the vibration analysis of rectangular plates with one or more guided edges, in which bicubic B-spline interpolation in combination with a new type of basis cubic B-spline functions is used to approximate the plate deflection. This type of basis cubic B-spline functions can satisfy simply supported, clamped, free, and guided edge conditions with easy numerical manipulation. The frequency characteristic equation is formulated based on classical thin plate theory by performing Hamilton's principle. The present solutions are verified with the analytical ones. Fast convergence, high accuracy and computational efficiency have been demonstrated from the comparisons. Frequency parameters for 13 cases of rectangular plates with at least one guided edge, which are possible by approximate or numerical methods only, are presented. These results are new in literature.

Keywords: Cubic B-spline function, bicubic B-spline interpolation, vibration analysis, rectangular plate, guided edge condition

Notation

$[A_x], [B_x], [C_x], [F_x]$ $[A_y], [B_y], [C_y], [F_y]$	spline matrices;
a, b	Plate lengths in x and y directions, respectively;
a_i	Undetermined spline nodal coefficient of beam;
$a_{i,j}$	Undetermined spline nodal coefficient of plate;
$[D]$	Rigidity matrix of plate;
d	Plate thickness;
E	Young's moduli of plate material;
$[H], [N]$	Spline interpolation matrices;
h	Spline section length of beam;
h_x, h_y	Spline section length in x and y directions, respectively;
$[K]$	Stiffness matrix;
l	Beam length;
$[M]$	Mass matrix;
N, M	Spline section in x and y directions, respectively;

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r	$=$ b/a, aspect ratio;
T	Kinetic energy;
U	Strain energy;
$w(x)$	Displacement of beam;
$w(x, y, t)$	Plate deflection;
$W(x, y)$	Amplitude of the plate deflection;
ω	Circular natural frequency;
$\{\delta\}$	Undetermined spline nodal coefficient vector;
$\{\chi\}$	Curvature vector;
$\prod$	Functional of plate;
μ	Poisson's ratios of plate material;
ρ	Mass per unit area of plate;
$\varphi_3(x)$	Standard cubic B-spline function;
$\phi_i(x), \psi_j(y)$	Local basis cubic B-spline functions in x and y directions, respectively;
$\otimes$	Kronecker product of two matrices or two vectors.

1. Introduction

Flexural vibration of rectangular plates under various combinations of the three classical edge conditions, i.e. simply supported, clamped and free edges, has been studied extensively. The three classical edge conditions are simply three of the four possible combinations of essential and natural conditions. The fourth mathematically possible boundary condition has zero rotation (essential condition) and zero effective shear force (natural condition), which has been referred to in the literature as the guided [2] or sliding edge [11]. Although the fourth edge condition is not as important as the three classical ones, it may be encountered in various disciplines of engineering. For example, in mechanical engineering, the bearings may have guided contact with the supported member. The boundary of a piston inside a circular cylinder with narrow clearance may appropriately be modeled as a guided edge.

There exist 21 possible combinations of the three classical edge conditions for rectangular plates, and accurate results for the free vibration of rectangular plates with all 21 combinations of edge conditions have been presented by Leissa [6]. The number of all possible combinations gives rise to 34 additional ones when at least one guided edge is involved. Of all these 34 cases, analytical solution is possible for 21 cases only, and the analytical solutions of the frequency parameters for the 21 cases have been presented by Bert and Malik [2]. The solutions of the remaining 13 cases are possible by approximate or numerical methods only, however, no investigation has been reported. The main purpose of this study is to provide accurate results of frequency parameters for the remaining 13 cases via bicubic B-spline method.

The concept of bicubic spline interpolation was first introduced by Boor [3]. The first application of bicubic B-splines in engineering analysis was reported by Antes [1] for bending analysis of plates. Shen, He and Le [9] presented so-called multivariable spline element method for vibration analysis of rectangular plates. However, their work was limited to plates with clamped or simply supported edges only due to the restrictions of the basis cubic B-spline functions adopted [5].

In this paper, firstly, a new type of basis cubic B-spline functions is constructed. This type of basis cubic B-spline functions can effectively satisfy simply supported, clamped, free and guided edge conditions with easy numerical manipulation. Any beam functions can be accurately approximated by the new type of basis B-spline functions. The plate deflection is chosen as field function and expressed as the product of basis cubic B-spline functions in both x and y directions. The frequency equation is formulated based on classical thin plate theory by performing Hamilton's principle. The validity of the present approach is well established by demonstrating the excellent agreement between the present results and those of analytical solutions by Bert and Malik [2]. As an application of the proposed method, frequency parameters of 13 cases of rectangular plates with at least one guided edge, which are possible by approximate or numerical methods only, are presented.

2. Functional of free vibrating plate

Consider a thin, isotropic rectangular plate of length a, width b, uniform thickness d, and mass per unit area ρ. For small amplitude free vibration, the plate deflection $w(x, y, t)$ may be written by

$$w(x, y, t) = W(x, y) \sin \omega t \tag{1}$$

where $W(x, y)$ is the amplitude and ω is the circular natural frequency of the plate. The curvature is given by

$$\{\chi\} = \begin{Bmatrix} -\dfrac{\partial^2 W}{\partial x^2} \\ -\dfrac{\partial^2 W}{\partial y^2} \\ -2\dfrac{\partial^2 W}{\partial x \partial y} \end{Bmatrix} \tag{2}$$

The strain energy U and the kinetic energy T are expressed, respectively, by

$$U = \frac{1}{2} \int_0^a \int_0^b \{\chi\}^T [D] \{\chi\} dydx \tag{3}$$

$$T = \frac{1}{2} \int_0^a \int_0^b \rho \left(\frac{\partial w}{\partial t} \right)^2 dydx \tag{4}$$

in which $[D]$ is flexural rigidity matrix, as follows:

$$[D] = \frac{Ed^3}{12(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & (1-\mu)/2 \end{bmatrix} \tag{5}$$

where E is Young's modulus, μ is Poisson's ratio.

The functional of a thin plate for free vibration is:

$$\Pi = \frac{2\pi}{\omega} \int_0^a \int_0^b (\{\chi\}^T [D] \{\chi\} - \rho \omega^2 W^2) dydx \tag{6}$$

3. Bicubic B-spline interpolation

The mathematical expression of standard cubic B-spline function can be expressed as

$$\varphi_3(x) = \begin{cases} \dfrac{1}{6}(x+2)^3 & x \in [-2, -1] \\ \dfrac{1}{6}(x+2)^3 - \dfrac{2}{3}(x+1)^3 & x \in [-1, 0] \\ \dfrac{1}{6}(2-x)^3 - \dfrac{2}{3}(1-x)^3 & x \in [0, 1] \\ \dfrac{1}{6}(2-x)^3 & x \in [1, 2] \\ 0 & |x| > 2 \end{cases} \tag{7}$$

The graphical representation of standard cubic B-spline function is shown in Fig. 1. It's seen from Fig. 1 that standard cubic B-spline function is a piecewise cubic polynomial. Cubic B-spline function has mathematical properties which are advantageous in numerical analysis. The mathematical properties of cubic B-spline functions are well documented by Boor [4] and Schumaker [8].

Consider a uniform beam of length l. The beam is divided into N equivalent spline sections by means of spline nodes. The beam deflection can be taken as the summation of $N + 3$ terms and each term can be represented by a

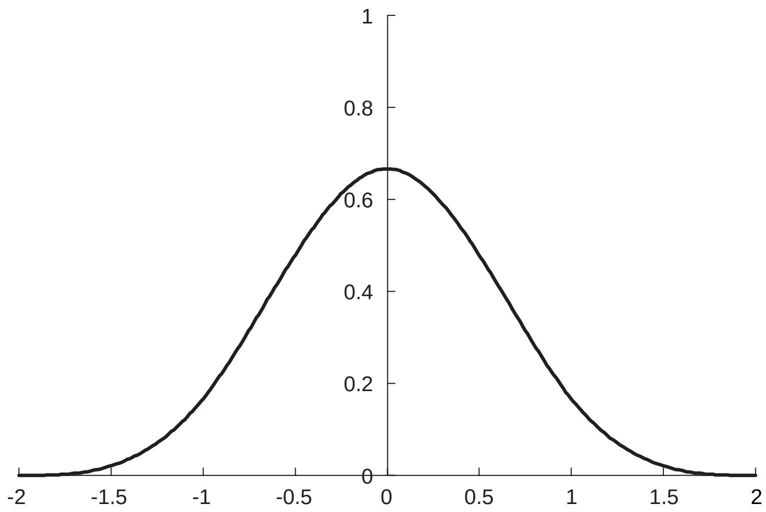

Fig. 1. Standard cubic B-spline function.

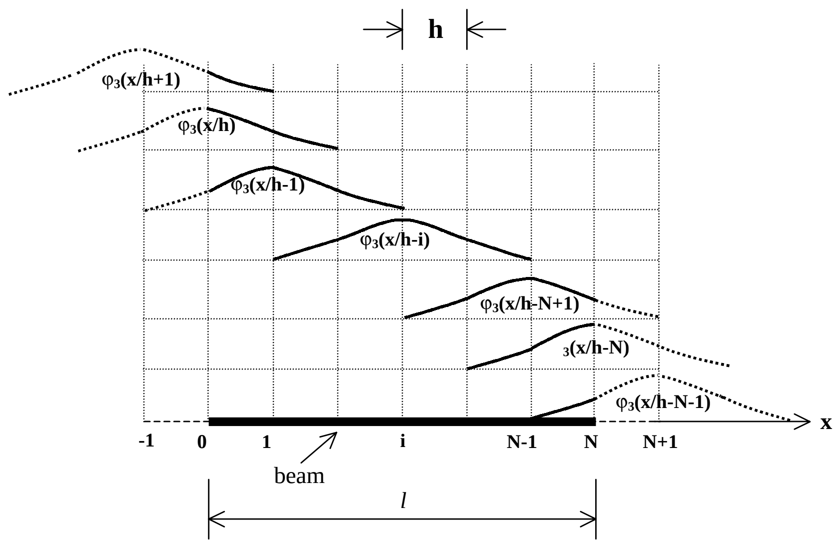

Fig. 2. Beam function interpolation by ordinary cubic B-spline functions.

local dimensionless cubic B-spline function, as shown in Fig. 2. It's observed from Fig. 2 that, at any spline node, the displacement expression has three non-zero terms only, e.g. only $\varphi_3(\frac{x}{h}-(i-1))$, $\varphi_3(\frac{x}{h}-i)$, and $\varphi_3(\frac{x}{h}-(i+1))$ have non-zero contributions to the displacement at the ith spline node. However, it's difficult to account for different boundary conditions using ordinary cubic B-spline interpolation. The three local B-spline functions at the boundary, therefore, must be constructed to accommodate different edge conditions. Several kinds of basis cubic B-spline functions have been proposed by Qin [7] and one of which has been commonly used by other researchers [9,10, 12]. This commonly used kind of basis cubic B-spline functions can satisfy simply supported and clamped edges conveniently, however, free and guided edges may not be satisfied [5].

3.1. The new type of basis cubic B-spline functions

In the present paper, a new type of basis cubic B-spline functions is constructed, as follows

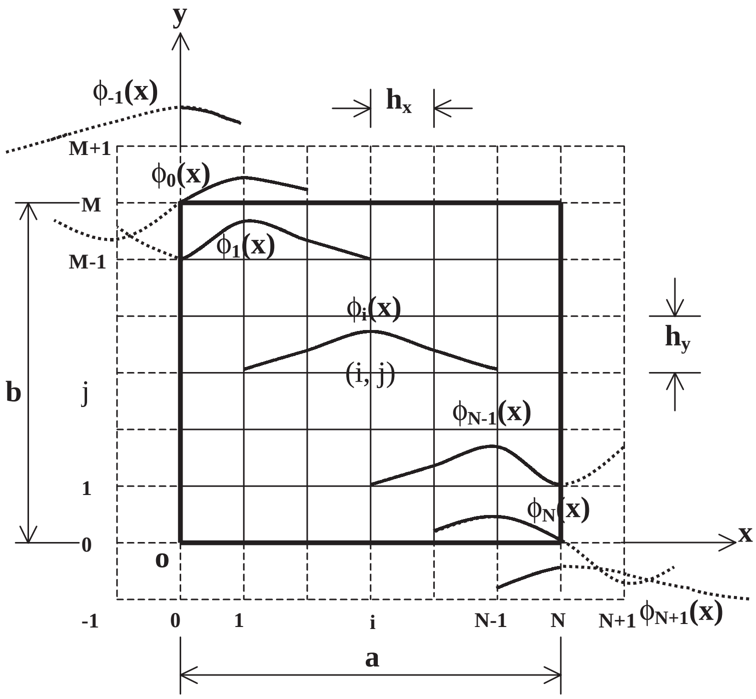

Fig. 3. Plate deflection interpolation by basis cubic B-spline functions (x-direction).

$$\begin{cases} \phi_{-1}(x) = \varphi_3(x/h_x + 1) + \varphi_3(x/h_x) + \varphi_3(x/h_x - 1) \\ \phi_0(x) = -\varphi_3(x/h_x + 1) + \varphi_3(x/h_x - 1) \\ \phi_1(x) = \dfrac{1}{3}\varphi_3(x/h_x + 1) - \dfrac{1}{6}\varphi_3(x/h_x) + \dfrac{1}{3}\varphi_3(x/h_x - 1) \\ \phi_2(x) = \varphi_3(x/h_x - 2) \\ \cdots \quad \cdots \quad \cdots \\ \phi_{N-2}(x) = \varphi_3(x/h_x - N + 2) \\ \phi_{N-1}(x) = \dfrac{1}{3}\varphi_3(x/h_x - N + 1) - \dfrac{1}{6}\varphi_3(x/h_x - N) + \dfrac{1}{3}\varphi_3(x/h_x - N - 1) \\ \phi_N(x) = \varphi_3(x/h_x - N + 1) - \varphi_3(x/h_x - N - 1) \\ \phi_{N+1}(x) = \varphi_3(x/h_x - N + 1) + \varphi_3(x/h_x - N) + \varphi_3(x/h_x - N - 1) \end{cases} \tag{8}$$

The three boundary cubic B-spline functions at both ends are linearly combined to satisfy different edge conditions, as sketched in Fig. 3. By virtue of the new type of basis cubic B-spline functions in Eq. (8), the beam deflection can be expressed as

$$w(x) = \sum_{i=-1}^{N+1} \phi_i(x) a_i \tag{9}$$

where $a_i (i = -1, 0, 1, \ldots,$ N $-$ 1, N, N $+$ 1) is undetermined coefficient at the ith spline node.

It's well known that in energy approaches admissible functions have to satisfy geometrical boundary conditions but need not satisfy natural boundary conditions. However, from a practical consideration of the rate of convergence, it is desirable to satisfy natural boundary conditions if possible [13]. The new type of basis cubic B-spline functions has desirable characteristics for numerical analysis since it can satisfy all geometrical boundary conditions and part of natural boundary conditions of the four kinds of edge conditions. For example, at the end $x = 0$, Eq. (9) yields

$$w(0) = a_{-1}\phi_{-1}(0) + a_0\phi_0(0) + a_1\phi_1(0) \tag{10}$$

Table 1
Convergence study of bicubic B-spline solutions of frequency parameters for SSSG square plates

Num. of sections	Frequency parameters 1st.	2nd.	3rd.	4th.	5th.	6th.
6×6	12.33745	32.08115	41.98588	61.71575	71.71681	91.92226
8×8	12.33714	32.07767	41.95718	61.69363	71.59787	91.45260
10×10	12.33706	32.07679	41.95024	61.68836	71.57095	91.35205
12×12	12.33703	32.07649	41.94789	61.68658	71.56216	91.32023
14×14	12.33701	32.07636	41.94692	61.68585	71.55858	91.30756
16×16	12.33701	32.07630	41.94646	61.68551	71.55691	91.30168
18×18	12.33701	32.07627	41.94621	61.68532	71.55604	91.29865
20×20	12.33701	32.07625	41.94608	61.68522	71.55554	91.29696
Exact*	12.33701	32.07622	41.94582	61.68503	71.55464	91.29385

*Bert and Malik [2].

Table 2
Comparison of frequency parameters for SCSG plates

Mode sequence	$r = 0.4$ Present	$r = 0.4$ Exact*	$r = 1.0$ Present	$r = 1.0$ Exact*	$r = 2.5$ Present	$r = 2.5$ Exact*
1	10.34459	10.34454	13.68579	13.68577	41.18469	41.18466
2	14.04479	14.04455	38.69439	38.69393	65.27708	65.27632
3	21.15134	21.15106	42.58779	42.58662	111.25348	111.24204
4	31.48613	31.47916	66.30066	66.29910	178.53572	178.45914
5	39.91245	39.91111	83.49563	83.48830	197.06605	197.06296
6	43.36284	43.36022	91.71810	91.70417	222.30722	222.30413
7	45.00561	44.97255	111.01629	111.00916	266.23690	266.16636
8	50.20771	50.20251	114.37324	114.35987	266.50405	266.22905
9	60.37257	60.36157	147.92791	147.87515	330.11816	330.07159

*Bert and Malik [2].

And at the end $x = 0$, Eq. (8) gives

$$\phi_{-1}(0) = 1, \phi_i(0) = 0 \; (i = 0, 1, 2, \cdots, N+1) \tag{11}$$

$$\phi'_0(0) = 1, \phi'_i(0) = 0 \; (i = -1, 1, 2, \cdots, N+1) \tag{12}$$

$$\phi''{}_1(0) = 1, \phi''{}_i(0) = 0 \; (i = -1, 0, 2, \cdots, N+1) \tag{13}$$

It's observed from Eqs (10–13) that only one term has contribution to the deflection, slope and second derivative of the deflection at the boundary. Therefore, except the prescribed values for third derivatives of the deflection at the boundary, the new type of basis cubic B-spline functions can satisfy geometrical boundary conditions and prescribed second derivative of deflection for the four kinds of edge conditions.

For example, for clamped end of the beam at $x = 0$, the boundary conditions are $w(0) = 0, w'(0) = 0$ and $w''(0) \neq 0$. Those undetermined coefficients corresponding to $\phi_{-1}(0)$ and $\phi_0(0)$, i.e. a_{-1} and a_0, thus should be equal to zero. For simply supported end of the beam at $x = 0$, the boundary condition is $w(0) = 0, w'(0) \neq 0, w''(0) = 0$, thus a_{-1} and a_1 should be equal to zero. For free end of the beam at $x = 0$, the boundary condition is $w(0) \neq 0, w'(0) \neq 0, w''(0) = 0$, thus a_1 should vanish. Likewise, for guided end of the beam at $x = 0$, the boundary condition is $w(0) \neq 0, w'(0) = 0, w''(0) \neq 0$, thus a_0 should vanish. The case for the end at $a = l$ is similar. Hence, simply supported, clamped, free and guided edges can be implemented with ease by using the new type of basis cubic B-spline functions. It should be noted that the numerical manipulation to eliminate the corresponding coefficients can be easily carried out in computer programme.

3.2. *Bicubic B-spline interpolation*

To express the plate deflection by bicubic B-spline approximation, the plate is divided into N and M equivalent spline sections in x and y directions, respectively, as shown in Fig. 3. Thus

Table 3
Comparison of frequency parameters for SGGF plates

Mode	$r = 0.4$		$r = 1.0$		$r = 2.5$	
sequence	Present	Exact *	Present	Exact*	Present	Exact*
1	2.44325	2.44001	2.41070	2.40785	2.37357	2.37104
2	3.77012	3.76564	9.18348	9.18141	21.76868	21.74211
3	7.79626	7.79427	22.02673	21.99667	39.03164	39.03119
4	14.82359	14.82200	30.55345	30.51000	60.93475	60.85150
5	22.14358	22.10998	33.42682	33.42615	65.94330	65.92023
6	23.60167	23.55383	56.21210	56.18961	110.63301	110.53142
7	25.01826	25.00849	61.40349	61.31043	119.88639	119.68813
8	28.09377	28.05593	70.35809	70.11123	172.64695	172.39071
9	35.52962	35.50419	77.61083	77.60132	192.17749	192.17367

*Bert and Malik [2].

$$0 = x_0 < x_1 < \cdots < x_i < \cdots < x_N = a, x_i = x_0 + ih_x, h_x = a/N$$

$$0 = y_0 < y_1 < \cdots < y_i < \cdots < y_M = b, y_i = y_0 + ih_y, h_y = b/M$$

where N and M are integers and $N, M > 4$, h_x and h_y are spline section lengths in x and y directions, respectively.

Based on the technique of separation of variables, the amplitude of the plate deflection may be represented in the form

$$W(x, y) = \sum_{i=-1}^{N+1} \sum_{j=-1}^{M+1} \psi_j(y)\phi_i(x)a_{i,j} \tag{14}$$

where $a_{i,j}$ is undetermined coefficient at the spline node (i, j), $\phi_i(x)$ and $\psi_j(y)$ are the ith and jth local basis cubic B-spline functions in x and y directions, respectively, as in Eq. (8). In matrix form, Eq. (14) can be written as

$$W(x, y) = [\psi(y)] \otimes [\phi(x)] \{\delta\} \tag{15}$$

where $[\phi(x)]$ and $[\psi(y)]$ are basis cubic B-spline functions vectors in x and y directions, respectively, as follows

$$[\phi(x)] = [\phi_{-1}(x)\phi_0(x)\phi_1(x)\phi_2(x) \cdots \phi_{N-1}(x)\phi_N(x)\phi_{N+1}(x)] \tag{16}$$

$$[\psi(y)] = [\psi_{-1}(y)\psi_0(y)\psi_1(y)\psi_2(y) \cdots \psi_{M-1}(y)\psi_M(y)\psi_{M+1}(y)] \tag{17}$$

and

$$\{\delta\} = \left\{\{a\}_{-1}^T \ \{a\}_0^T \ \{a\}_1^T \ \cdots \{a\}_M^T \ \{a\}_{M+1}^T\right\}^T \tag{18}$$

is undetermined spline nodal coefficient vector, in which

$$\{a\}_j = \{a_{-1,j} \ a_{0,j} \ a_{1,j} \ \cdots a_{N,j} \ a_{N+1,j}\}^T, j = -1, 0, 1, \cdots, M, M+1$$

and the symbol, $\otimes$, means the Kronecker product of two matrices or two vectors.

4. Frequency characteristic equation

The amplitude of the plate deflection in Eq. (15) can be further written as

$$W(x, y) = [\psi(y)] \otimes [\phi(x)] \{\delta\} = [N] \{\delta\} \tag{19}$$

The curvature vector in Eq. (2) can, therefore, be expressed as

$$\{\chi\} = \begin{Bmatrix} -\dfrac{\partial^2 W}{\partial x^2} \\ -\dfrac{\partial^2 W}{\partial y^2} \\ -2\dfrac{\partial^2 W}{\partial x \partial y} \end{Bmatrix} = [H]\{\delta\} \tag{20}$$

Table 4
Comparison of frequency parameters for GGGG plates

Mode	$r = 0.4$		$r = 1.0$		$r = 2.5$	
sequence	Present	Exact*	Present	Exact*	Present	Exact*
1	1.57914	1.57914	9.86962	9.86960	9.86962	9.869620
2	6.31673	6.31655	19.73923	19.73921	39.47958	39.47842
3	9.86962	9.86960	39.47958	39.47842	61.68514	61.68503
4	11.44876	11.44874	49.34896	49.34802	71.55473	71.55464
5	14.21448	14.21223	78.95801	78.95684	88.84053	88.82644
6	16.18624	16.18615	88.84053	88.82644	101.16397	101.16345
7	24.08319	24.08184	98.70876	98.69605	150.51997	150.51148
8	25.27991	25.26619	128.31508	128.30486	157.99945	157.91368
9	35.14584	35.13579	157.99945	157.91368	219.66152	219.59871

*Bert and Malik [2].

Table 5
Bicubic B-spline solutions of frequency parameters for CCCG plates

Mode	Aspect ratio r = a/b				
sequence	2/5	2/3	1	3/2	5/2
1	22.63272	23.19655	24.57795	28.70100	46.70085
2	24.86870	30.74236	44.77075	67.54214	82.05066
3	29.97003	47.96869	63.98760	80.19072	139.49409
4	38.50371	62.62624	83.27823	115.82573	199.23378
5	50.64564	70.44395	87.26174	126.64734	217.67627
6	61.99886	75.06327	123.28343	173.24566	230.55915
7	64.60543	86.90657	123.69770	179.13252	283.30350
8	66.37193	111.59241	142.40142	205.67903	316.34459
9	69.97816	112.67860	150.60841	212.52122	357.55418

Table 6
Bicubic B-spline solutions of frequency parameters for CCSG plates

Mode	Aspect ratio r = a/b				
sequence	2/5	2/3	1	3/2	5/2
1	15.77075	16.53117	18.34855	23.45847	43.44845
2	18.69489	25.78157	41.25078	56.70824	72.92613
3	24.86585	44.61565	52.63283	77.98472	124.64896
4	34.44482	51.06220	74.08709	108.68130	197.38366
5	47.41460	59.93062	85.15311	110.58450	198.09013
6	50.33792	72.68709	106.86083	160.52812	226.12866
7	53.32496	77.97819	116.82442	177.82731	274.27814
8	59.39298	105.28626	127.68015	184.49038	290.73972
9	63.75092	105.37240	149.14356	207.77138	343.21661

Thus we have

$$[N] = [\psi(y)] \otimes [\phi(x)] \tag{21}$$

$$[H] = \left\{ \begin{array}{l} -[\psi(y)] \otimes [\phi''(x)] \\ -[\psi''(y)] \otimes [\phi(x)] \\ -2[\psi'(y)] \otimes [\phi'(x)] \end{array} \right\} \tag{22}$$

Upon substitution of Eqs (19) and (20) into Eq. (6), the following expression is obtained

$$\Pi = \frac{2\pi}{\omega} \{\delta\}^T \left([K] - \omega^2 [M]\right) \{\delta\} \tag{23}$$

where $[K]$ and $[M]$ are the stiffness matrix and the mass matrix of the plate, respectively, as follows

$$[K] = \int_0^a \int_0^b [H]^T [D][H] dy dx \tag{24}$$

Table 7
Bicubic B-spline solutions of frequency parameters for CCGF plates

Mode	Aspect ratio $r =$ a/b				
sequence	2/5	2/3	1	3/2	5/2
1	5.82781	6.39955	7.78816	11.52535	24.93200
2	8.25672	13.98630	25.85303	35.53868	47.46232
3	13.87844	30.89554	32.29520	53.12424	90.53861
4	22.92983	31.03902	51.23587	78.60378	141.03696
5	30.46421	39.20468	64.92146	79.56457	153.69023
6	33.27805	55.74725	76.53554	123.99143	166.47386
7	35.33239	56.95635	89.29651	141.87340	213.48687
8	39.00428	75.36847	95.95572	143.37222	236.93056
9	47.78803	81.08415	123.94247	165.66525	279.53734

Table 8
Bicubic B-spline solutions of frequency parameters for CCGG plates

Mode	Aspect ratio $r =$ a/b				
sequence	2/5	2/3	1	3/2	5/2
1	5.91096	6.75124	8.99631	15.19030	36.94347
2	8.85439	16.63070	32.89572	37.41907	55.33993
3	15.37489	31.32443	33.05174	70.47996	96.09306
4	25.35216	36.05379	55.00942	81.12103	158.45099
5	30.59011	40.30721	77.23363	90.69122	191.18820
6	33.54399	58.66116	77.29907	131.98762	209.64994
7	38.64062	64.45888	98.19975	145.03248	241.50389
8	39.62308	75.74074	98.48365	170.41666	247.64425
9	48.98082	84.55412	140.54532	190.24676	306.13010

$$[M] = \int_0^a \int_0^b \rho [N]^T [N] dy dx \tag{25}$$

By Hamilton's principle, the functional Π in Eq. (23) can be minimized with respect to the undetermined coefficient vector $\{\delta\}$. The minimization procedure results in the generalized eigenvalue equation as follows

$$([K] - \omega^2 [M]) \{\delta\} = 0 \tag{26}$$

With the definition of the following spline matrices

$$[A_x] = \int_0^a [\phi''(x)]^T [\phi''(x] dx \tag{27}$$

$$[B_x] = \int_0^a [\phi(x)]^T [\phi''(x] dx \tag{28}$$

$$[C_x] = \int_0^a [\phi'(x)]^T [\phi'(x] dx \tag{29}$$

$$[F_x] = \int_0^a [\phi(x)]^T [\phi(x] dx \tag{30}$$

Equations (24) and (25) can be written as

$$\begin{aligned} [K] = \frac{Ed^3}{12(1-\mu^2)} \big(& [F_y] \otimes [A_x] + [A_y] \otimes [F_x] + \mu [B_y]^T \otimes [B_x] + \mu [B_y] \otimes [B_x]^T \\ & + 2(1-\mu)[C_y] \otimes [C_x]) \end{aligned} \tag{31}$$

$$[M] = \rho [F_y] \otimes [F_x] \tag{32}$$

Table 9
Bicubic B-spline solutions of frequency parameters for CGCF plates

Mode sequence	Aspect ratio r = a/b 2/5	2/3	1	3/2	5/2
1	22.34398	22.31794	22.28514	22.23789	22.15524
2	23.16445	24.62271	27.53553	34.19715	55.85023
3	25.98568	33.21762	48.55258	61.36610	61.14986
4	31.42507	50.46883	61.47959	76.53181	101.62957
5	40.05130	61.55633	68.31690	84.88648	120.12019
6	52.13970	64.61253	90.30401	120.46857	165.04976
7	61.61644	74.50666	90.89954	128.07138	198.93900
8	62.71886	77.23182	120.64817	137.17803	204.37870
9	66.23537	92.06798	128.15189	182.36948	246.49511

Table 10
Bicubic B-spline solutions of frequency parameters for CGSF plates

Mode sequence	Aspect ratio r = a/b 2/5	2/3	1	3/2	5/2
1	15.38026	15.34938	15.31182	15.25870	15.16871
2	16.47848	18.36153	21.92477	29.54055	49.40109
3	20.06276	28.49989	45.06977	49.63005	52.58593
4	26.48754	47.12309	49.75065	66.56775	93.34186
5	36.05374	49.83304	57.50045	82.53528	103.40695
6	48.90650	53.32429	82.09721	103.77033	151.55991
7	49.89882	64.37345	88.09754	121.01804	177.24671
8	51.16345	74.79419	103.95881	121.97817	203.01177
9	55.16032	83.37999	112.18399	177.74621	227.91396

Table 11
Bicubic B-spline solutions of frequency parameters for CSGF plates

Mode sequence	Aspect ratio r = a/b 2/5	2/3	1	3/2	5/2
1	5.75830	6.05969	6.61272	7.70546	10.39906
2	7.65028	11.69383	19.95748	33.68319	39.29193
3	12.42087	26.15263	31.74735	38.96296	84.96153
4	20.57671	30.86898	47.07998	67.27730	100.38533
5	30.43234	37.80435	53.63521	78.32008	129.82887
6	32.10642	49.83552	76.18947	114.79794	149.52489
7	32.95762	52.39028	80.10300	115.91148	181.24692
8	38.15401	75.27012	92.85304	142.32923	233.63307
9	46.23710	75.40192	107.56692	142.49455	251.08769

where $[A_y], [B_y], [C_y]$, $[F_y]$ are the spline matrices similar to $[A_x], [B_x], [C_x]$ and $[F_x]$, but in the y direction. The dimension of the spline matrices $[A_x], [B_x], [C_x]$ and $[F_x]$ is $(N+3)\times(N+3)$ and the dimension of spline matrices $[A_y], [B_y], [C_y]$ and $[F_y]$ is $(M+3)\times(M+3)$. The spline matrices are given in Appendix.

Edge conditions can be imposed by eliminating rows and columns of the spline matrices associated with zero spline nodal coefficients at the boundaries of the plate. For example, for a clamped edge at $x = 0$, as stated earlier in last section, the spline nodal coefficients $a_{-1,j}$ and $a_{0,j}$ ($j = -1, 0, 1, \cdots$, M,M $+ 1$) should be equal to zero. Accordingly, the first two rows and columns of the spline matrices $[A_x], [B_x], [C_x]$ and $[F_x]$ should be eliminated.

Similarly, for a simply supported edge at $x = 0$, the first and third rows and columns of the spline matrices should be eliminated; and for a free edge at $x = 0$, the third row and column of the spline matrices should be eliminated. Likewise, for a guided edge at $x = 0$, the second row and column of the spline matrices should be eliminated. The case for the end at $x = a$, $y = 0, y = b$ is similar. Upon application of edge conditions, the free vibration problem of the rectangular plate is reduced to generalized eigenvalue problem in Eq. (26).

Table 12
Bicubic B-spline solutions of frequency parameters for CGGF plates

Mode	Aspect ratio $r =$ a/b				
sequence	2/5	2/3	1	3/2	5/2
1	5.57607	5.56300	5.54792	5.52898	5.50107
2	6.41026	7.91727	10.90588	17.72148	29.81596
3	9.62376	17.75265	30.06285	29.97633	39.83580
4	16.07221	30.12290	34.22535	45.84235	70.73926
5	25.92440	33.43747	37.38449	71.77492	73.93128
6	30.17148	36.85899	61.21624	74.22668	120.24959
7	31.37563	44.08551	74.38073	91.89574	137.83999
8	35.20760	62.80136	78.09442	99.22643	187.40569
9	39.11793	65.07106	82.41132	138.27153	192.48870

Table 13
Bicubic B-spline solutions of frequency parameters for CFGF plates

Mode	Aspect ratio $r =$ a/b				
sequence	2/5	2/3	1	3/2	5/2
1	5.56171	5.54016	5.51922	5.49518	5.46053
2	6.26823	7.32162	9.00242	11.87375	18.03545
3	8.96880	15.13892	27.36397	29.77589	29.55495
4	14.57315	30.03512	29.92623	42.76309	58.71603
5	23.51690	31.75483	36.29256	55.13088	73.36098
6	30.12570	33.01511	57.04120	73.86955	111.28386
7	31.22434	42.73659	65.79679	87.00912	136.99304
8	34.87247	57.58303	74.15486	89.40988	144.42500
9	35.82138	59.73655	81.41000	137.78089	178.21389

Table 14
Bicubic B-spline solutions of frequency parameters for SGFF plates

Mode	aspect ratio $r =$ a/b				
sequence	2/5	2/3	1	3/2	5/2
1	2.833583	5.153985	8.710524	14.44969	14.92445
2	7.197001	15.03607	15.29102	16.77704	38.26339
3	14.13896	16.25319	16.39972	37.02406	49.48403
4	15.43609	20.89188	32.93346	49.50993	64.47378
5	17.51225	34.45655	49.63856	69.95977	102.9047
6	23.17969	35.70995	53.91768	73.52203	106.1977
7	24.56755	49.77296	60.83301	93.63335	162.7142
8	31.83681	54.58633	76.89855	103.8243	176.6324
9	37.78603	55.81457	91.10221	126.9152	191.6661

5. Numerical results

Based on the bicubic B-spline method presented in the foregoing sections, a general unified computational program is developed for vibration analysis of rectangular plates with one or more guided edges. For the purpose of description, a notation will be adopted as follows. Consider the rectangular plate shown in Fig. 3, the symbolism FSGC, for example, will identify a rectangular plate with edges $x = 0, y = 0, x = a, y = b$ having free(F), simply supported(S), guided(G), and clamped(C) edge conditions, respectively.

5.1. Convergence study

Before making detailed computations it is considered imperative to decide the suitable number of spline sections used and investigate the accuracy of the solutions. The rectangular SSSG plate is used for convergence study. The exact solutions of the dimensionless natural frequency $\Omega = \omega a^2\sqrt{\rho/D}$. are given by Bert and Malik [2]. These results are used to check the accuracy of the results obtained from the present method and determine the suitable number of spline sections.

Table 15
Bicubic B-spline solutions of frequency parameters for CGFF plates

Mode	Aspect ratio $r = $ a/b				
sequence	2/5	2/3	1	3/2	5/2
1	3.50985	3.50385	3.49611	3.48488	3.46558
2	4.83473	6.84778	10.19646	16.97070	21.48501
3	8.61754	16.95434	21.86353	22.05229	39.27830
4	15.24552	21.98069	31.48405	42.08476	60.95759
5	21.95659	26.73682	34.08862	61.11062	69.00911
6	23.65443	36.08935	58.15338	71.06789	114.27546
7	25.30255	39.78884	61.49358	82.81370	120.13440
8	29.00165	59.63750	71.25972	97.34350	175.95907
9	36.85770	61.43003	77.55365	120.40683	190.98657

Table 16
Bicubic B-spline solutions of frequency parameters for GGFF plates

Mode	Aspect ratio $r = $ a/b				
sequence	2/5	2/3	1	3/2	5/2
1	0.85908	2.38190	4.90505	5.35928	5.36923
2	4.66279	5.55157	6.07341	12.49104	29.14244
3	5.61770	10.99370	15.93748	24.73582	35.11061
4	7.87228	13.35196	29.32022	30.04191	49.20175
5	11.80246	23.47643	30.64494	52.82196	73.76539
6	13.34751	30.02651	40.44186	67.55964	83.42192
7	21.13354	32.91008	42.17589	74.04768	132.08461
8	22.03992	35.47088	70.36587	79.80949	137.74953
9	30.09736	43.39231	73.55692	97.63271	188.10852

Table 17
Bicubic B-spline solutions of frequency parameters for GFFF plates

Mode	Aspect ratio $r = $ a/b				
sequence	2/5	2/3	1	3/2	5/2
1	3.40206	5.34374	5.37730	5.37070	5.35875
2	5.60836	9.91575	14.63677	20.81380	29.24565
3	7.61444	10.61721	22.02883	29.37640	33.31023
4	9.87651	21.63366	29.75201	50.06930	72.76770
5	12.89197	27.08493	36.08762	50.25363	72.96237
6	19.17776	30.00296	40.16609	64.59443	124.18050
7	20.14829	34.78236	61.35288	74.25586	133.93105
8	29.46422	39.62941	66.93716	95.03981	141.18779
9	30.49951	48.55998	74.01804	101.69672	155.95143

The rate of convergence of the frequencies corresponding to the first six modes is presented in Table 1. The number of spline sections in both x and y directions is varied from 6 $\times$ 6 to 20 $\times$ 20. It can be observed from Table 1 that the rate of convergence for the fundamental frequency is very fast. Full convergence is achieved with $N \times M =$ 14 $\times$ 14. Based on the convergence study, 14 $\times$ 14 spline sections are used hereafter for calculating frequencies.

5.2. Numerical examples

Bert and Malik [2] presented analytical solutions of frequency parameters for 21 cases of rectangular plates with one or more guided edges. The 21 cases have been sorted into three groups, i.e. plates with two opposite edges simply supported, plates with one edge simply supported and opposite edge guided, and plates with two opposite edges guided. The first nine frequency parameters of three cases from the three groups, i.e. SCSG, SGGF, and GGGG, are computed by the present method and compared with the analytical solutions in Tables 2–4, respectively. The agreement between the present solutions and analytical ones is impressive.

5.3. Bicubic B-spline solutions for plates with guided edges

As has been stated in the introduction, for rectangular plates with at least one guided edge there exist 34 combinations of edge conditions, among which 21 cases have analytical solutions and have been given by Bert and Malik [2]. However, no study has been reported for the remaining 13 cases that have approximate or numerical solutions only. Nevertheless, the numerical accuracy and versatility of the present bicubic B-spline approach have been well validated from the above comparisons. Bicubic B-spline solutions for the first nine frequency parameters of the 13 cases of rectangular plate over a range of aspect ratios are given in Tables 5–17 for future comparison. The plate geometrical description is the same as shown in Fig. 3 and the aspect ratio $r = a/b$. The frequency parameter is $\omega a^2\sqrt{\rho/D}$.

6. Conclusions

This work concerns the free vibration of thin isotropic rectangular plates with at least one guided edge. A versatile type of basis cubic B-spline functions has been first developed to satisfy simply supported, clamped, free, and guided edge conditions with easy numerical manipulation. Incorporating the new type of basis cubic B-spline functions, bicubic B-spline interpolation is used to represent the plate deflection. The frequency characteristic equation is derived on classical thin plate theory by performing Hamilton's principle.

There exist 34 combinations of edge conditions for rectangular plates, with at least one guided edge, among which 21 cases have analytical solutions and have been given by Bert and Malik [2]. The present solutions are compared with the analytical ones by Bert and Malik [2]. Rapid convergence, high accuracy and computational efficiency of the present method have been demonstrated from the excellent agreement of the comparisons.

By using the present bicubic B-spline approach, extensive computations were performed for natural frequencies of 13 cases of rectangular plates having one or more guided edges with various aspect ratios. These numerical results provided in this paper are believed to fill the gap in literature on rectangular plates with guided edge(s).

Appendix

The elements of the spline matrices in Eqs (27)–(30), i.e. $[A_x], [B_x], [C_x]$ and $[F_x]$, are as following. The dimension of the spline matrices is $(N+3) \times (N+3)$. The spline matrices $[A_y], [B_y], [C_y]$ and $[F_y]$ are in similar form as $[A_x], [B_x], [C_x]$ and $[F_x]$, and the dimension is $(M+3) \times (M+3)$.

$$[A_x] = \frac{1}{36h_x^3}\begin{bmatrix} 36 & 54 & 15 & -48 & 6 & 6 & 0 & 0 \\ & 96 & 27 & -60 & 0 & 6 & 0 & 0 \\ & & 18 & -16 & -1 & 2 & 0 & 0 \\ & & & 96 & 54 & 0 & 6 & 0 \\ Sym. & & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}_{(N+3)\times(N+3)}$$

$$[B_x] = \frac{1}{720h_x}\begin{bmatrix} -396 & -954 & -57 & 240 & 150 & 6 & 0 & 0 \\ -234 & -624 & -165 & 84 & 144 & 6 & 0 & 0 \\ & & -54 & 8 & 47 & 2 & 0 & 0 \\ & & & -480 & 90 & 144 & 6 & 0 \\ Sym. & & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}_{(N+3)\times(N+3)}$$

$$[C_x] = \frac{1}{720h_x}\begin{bmatrix} 396 & 234 & 57 & -240 & -150 & -6 & 0 & 0 \\ & 624 & 165 & -84 & -144 & -6 & 0 & 0 \\ & & 54 & -8 & -47 & -2 & 0 & 0 \\ & & & 480 & -90 & -144 & -6 & 0 \\ Sym. & & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}_{(N+3)\times(N+3)}$$

$$[F_x] = \frac{h_x}{30240}\begin{bmatrix} 36756 & 19854 & 5055 & 7872 & 726 & 6 & 0 & 0 \\ & 13776 & 3819 & 7140 & 720 & 6 & 0 & 0 \\ & & 1098 & 2264 & 239 & 2 & 0 & 0 \\ & & & 14496 & 7146 & 720 & 6 & 0 \\ Sym. & & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}_{(N+3)\times(N+3)}$$

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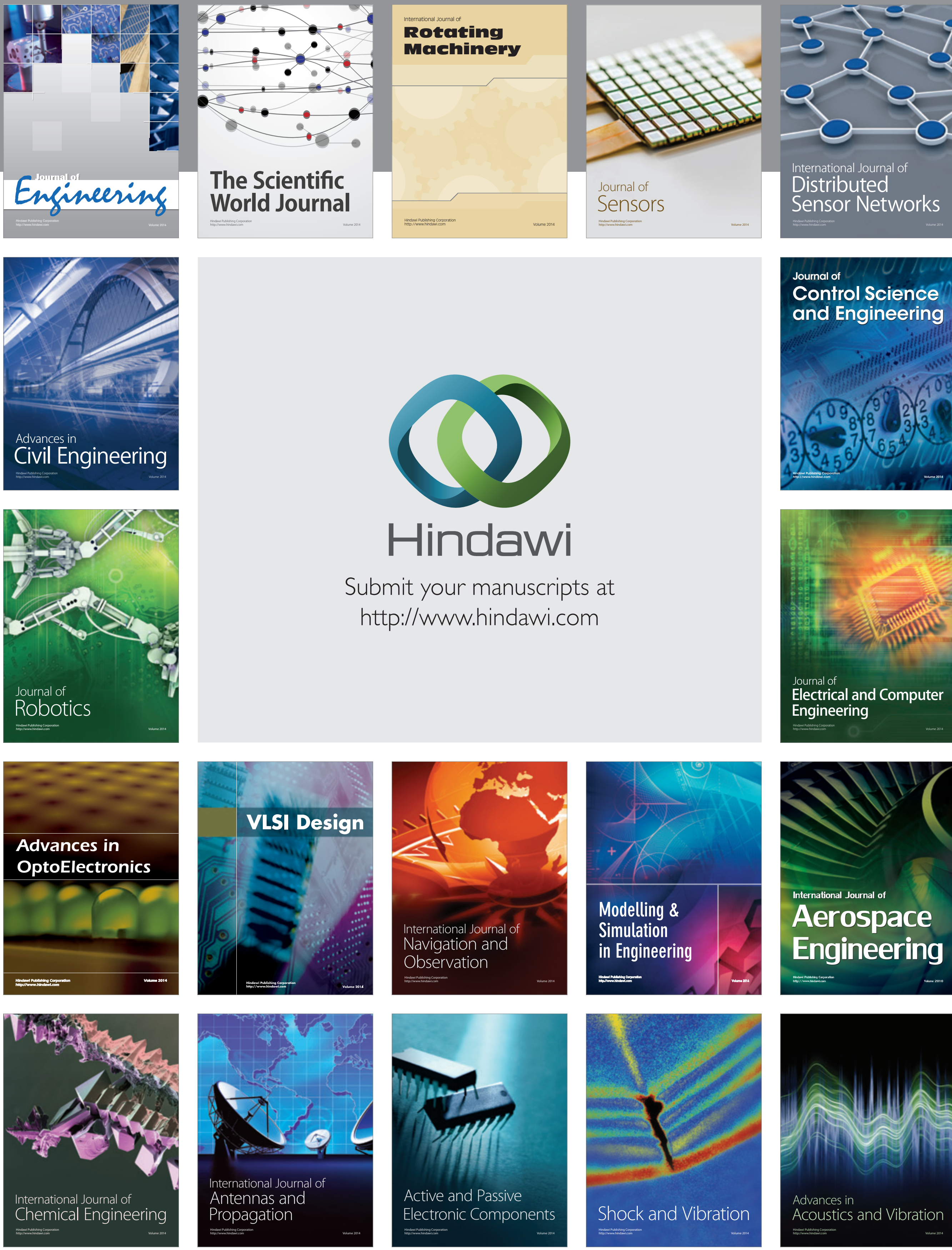
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