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Analytical stationary acoustic wave in a liquid over which a moving pressure runs

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Abstract. This paper presents an analytical study of the stationary response of a liquid loaded on its free surface by an ideal pressure step moving in a constant direction at a constant velocity. The acoustic pressure in the liquid is found, in four different examples, by means of the Fourier Transform. Two loading regimes are considered; subsonic and supersonic. Two configurations of liquid domains are also studied, the first one is a half infinite space while the second one is bounded by a rigid bottom at a finite depth.

For the two supersonic cases, a simple reasoning based on the existence of a front of discontinuity in the liquid and on the property of reflection of waves confirms the result of the mathematical investigations.

The results obtained for the steady state case are of intererest, even when the loading is not exactly stationary, such as the presure produced by an explosion occurring in the vicinity of the surface of a liquid. Two numerically resolved examples are presented, which confirm this assumption.

Keywords: Wave equation, sliding pressure, transform methods, analytical solutions

1. Introduction

The objective of this work is to determine, under rational hypothesis, the analytical formulas for the pressure induced in a liquid by a loading pressure wave travelling over its surface at a high uniform speed and amplitude.

Such loading pressure may be typically produced by a detonation. In this case, the loading velocity can reach very high values, perhaps higher than the velocity of the acoustic waves in the liquid.

Detonations and deflagrations are two possible modes of explosion. The detonations create shock waves travelling at supersonic speeds and represent high exposure to pressure as well as a very short raising time (possibly 10^{9} Pa.s^{-1}). In spite of its violence, the detonation process is essentially deterministic. That is not the case with a deflagration process, which propagates at subsonic speed and is essentially stochastic.

The complete problem of the interaction of shock waves with a fluid interface involves the shock dynamics in the two media in contact with each other. Grove [1] has presented a mathematical analysis based on the front-tracking method. Into this method, he incorporated wave interactions and bifurcation of the front topology, which dominate the early period of the shock/fluid surface interaction. The phenomena encountered over longer times (interface instability, chaotic mixing) could then be computed to give an enhanced resolution.

Experimental results of the effect of a moving shock wave over the surface of a liquid are presented by Borisov et al. [2]. They consider the detonation of a gaseous explosive mixture confined in a tube partially filled with liquid. In these experiments, the velocity and amplitude of the shock are constant (before reflection at the ends),

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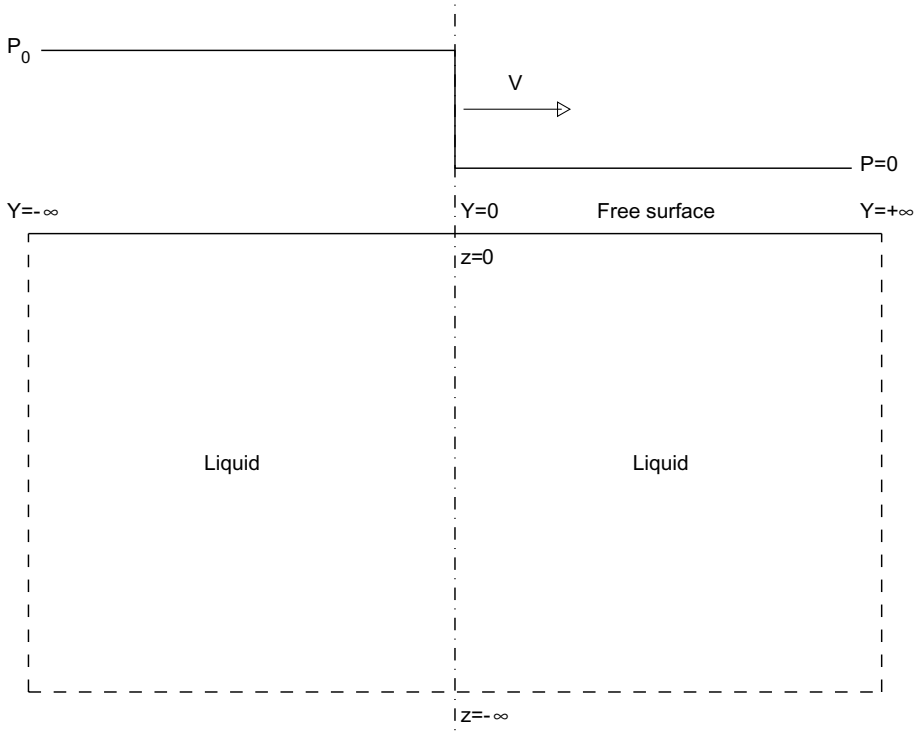

Fig. 1. The moving pressure step running on the liquid.

determined by the experimental conditions. The case of subsonic schocks (where the shock velocity is lower than the speed of sound in the liquid) and supersonic shocks are treated by the authors. In both cases, the different stages of the phenomena are identified. For example, in the supersonic case, during the first stage, a typical wave pattern takes place in the fluid, which is exactly that calculated analytically in the present paper. A more recent experimental study was presented by Dengel et al. [3] concerning the detonation and deflagration of gas over a liquid surface. High speed images (up to 200 000 frames per second) of the detonation front are presented, together with pressure measurements,. The images clearly show that the shape of the fluid surface remains almost unchanged when traversed by the detonation shock front.

An analytical work has been carried out by Kistler [4]. He considers the effect of a spherical shock wave hitting the surface of an incompressible fluid with the assumption of linear irrotational fluid dynamics. The gravitational effect is taken into account. The assumption of incompressibility is not adapted when the load front velocity is of the same order of magnitude as the speed of sound in the liquid (or is even greater). Despite its limitation, the work of Kistler is interesting, since it shows the application of Fourier transforms and the importance of the loading pressure function P, that is to say, the reflected pressure over the surface.

In effect, the reflection of spherical shocks over flat surfaces has been widely studied. From previous works providing the pressure-time history on the surfaces of reflection (in particular Brossard et al. [5]), it is well known that the time of expansion of the shock waves is very short as compared to the time needed by the liquid to move significantly; as a consequence, the response of the liquid can be considered as acoustic and linear, at least for a short duration (corresponding to the first stages described in [2]).

Every explosion has its own characteristics and requires a specific study; nevertheless, a general problem can be set which is that of a pressure front moving quickly over the surface of the liquid (Fig. 1). This realistic phenomenon is vastly different from that of a liquid loaded instantaneously by a pressure which is applied everywhere on its whole surface.

If the explosion is a detonation occurring in the vicinity of a plane surface, the moving pressure [5] may be approached by a pressure step of quasi constant amplitude, running for significant distances at a quasi constant velocity. This velocity can reach up to several thousand meters per second as shown by the experiments.

Therefore, the problem is idealised by the assumption of stationarity. It is merely necessary for the load to be represented by an ideal Heaviside step running at constant velocity for a very long time on the surface of a very large extent of liquid. The pressure front may be assumed to be plane and progressing in a constant perpendicular direction (Fig. 1).

This article presents the exact analytical solution of this idealised problem. This theoretical solution is obtained by Fourier transforms, either in the subsonic or in the supersonic case.

2. The problem for an infinite depth of liquid

Figure 1 presents a liquid loaded on its free surface by a plane pressure front of constant amplitude P_0 moving at a constant velocity V in direction x. The fundamental equations are recalled below:

$$\ddot{\varphi} = C^2 \Delta\varphi \tag{1}$$

where C is the velocity of acoustic waves in the liquid, ρ its mass density, and φ is the velocity potential function defined by:

$$\mathbf{v} = -\,\mathbf{grad}\,\varphi \tag{2}$$

and

$$p = \rho\,\dot{\varphi}, \tag{3}$$

with the notations $\dot{()} \equiv \partial\,/\,\partial t$ and Δ the Laplacian operator.

With the stationary assumption, all the functions describing the state of the liquid are frozen in coordinates moving at the constant velocity V in the direction x. In the moving frame, all the functions depend only on the local coordinate $Y = x - Vt$, so that the partial derivative operators can be replaced: $\partial/\partial x$ by d/dY and $\partial/\partial t$ by $-V\,\mathrm{d}/\mathrm{d}Y$.

Taking these changes into account, Eq. (1) becomes:

$$V^2\,\partial^2_{,YY}\,\varphi(Y,z) = C^2\,[\partial^2_{,YY}\,\varphi(Y,z) + \partial^2_{,zz}\,\varphi(Y,z)] \tag{4}$$

and Eq. (3):

$$p = -\rho V\,\partial_{,Y}\,\varphi(Y,z) \tag{5}$$

By defining:

$$\Omega^2 = \left|C^2 - V^2\right|/C^2 \tag{6}$$

it becomes

$$\partial^2_{,zz}\varphi + \Omega^2\,\partial^2\varphi_{,YY} = 0 \text{ for } V < C \tag{7}$$

and

$$\partial^2_{,zz}\varphi - \Omega^2\,\partial^2\varphi_{,YY} = 0 \text{ for } V > C \tag{8}$$

<u>Note:</u>

As there is no damping Eqs (7, 8) do not take into account the direction of the movement (forward or backward). These stationary equations are identical, whether V is positive or negative. However, the solution in the supersonic case must verify the condition that no perturbations can propagate ahead of the load front. In the subsonic case, this condition no longer exists: the conditions of pressure at the infinite boundaries ($Y = \pm\infty$) are not altered by the direction of movement. Therefore, the solution procedure must distinguish two cases: the subsonic and the supersonic.

The moving pressure can be written $p(Y,0) = P(Y) = P_0 H(-Y)$ where $H(.)$ is the Heaviside step function. To introduce the symmetry facility, as has already been done in a previous work [6], this can be split into symmetrical and anti-symmetrical parts:

$$P_0\, H(-Y) = \frac{P_0}{2} + P_0 \left(H(-Y) - \frac{1}{2} \right) = P_s(Y) \,+\, P_a(Y) \tag{9}$$

The research of the solution can be made first for the anti-symmetrical part $P_a(Y)$. Finally, to obtain the real pressure, the constant symmetrical pressure $p_s(Y,z) = P_0/2$ will be added to the pressure result. $\varphi_s(Y,z)$ and $\varphi_a(Y,z)$ are the potentials which respectively correspond to $p_s(Y,z)$ and $p_a(Y,z)$.

To find the solution, the Fourier transform is used, according to the following notations and definitions:

$$\overline{f}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(Y)\, e^{i\xi Y} dY \tag{10}$$

$$f(Y) = \int_{-\infty}^{+\infty} \bar{f}(\xi)\, e^{-i\xi Y} d\xi \tag{11}$$

2.1. The subsonic case

After applying the Fourier transform to Eq. (7) of elliptic type, we obtain:

$$-\,\Omega^2 \xi^2\, \bar{\varphi}_a + \partial_{,zz} \bar{\varphi}_a = 0 \tag{12}$$

z being non positive (the liquid is in the half plane $z < 0$), the general solution is

$$\bar{\varphi}_a(\xi, z) = C(\xi) e^{|\xi|\Omega z} \tag{13}$$

(further explanations can be found in Haberman [7, p. 475]).

According to Eq. (5), the pressure transform is deduced:

$$\bar{p}_a(\xi, z) = i\xi\, \rho V C(\xi) e^{|\xi|\Omega\, z} \tag{14}$$

The forcing pressure transform being

$$\bar{P}_a(\xi) \;= \frac{P_0}{2\pi} \frac{1}{i\xi} = \bar{p}_a(\xi, 0), \tag{15}$$

$C(\xi)$ can be identified:

$$C(\xi) \;= -\frac{P_0}{2\pi} \frac{1}{\rho V} \frac{1}{\xi^2} \tag{16}$$

Finally, the potential transform is known everywhere:

$$\bar{\varphi}_a(\xi, z) = -\frac{P_0}{2\pi} \frac{1}{\rho V} \frac{1}{\xi^2} e^{|\xi|\Omega z} \tag{17}$$

and by inversion,

$$\varphi_a(Y, z) = -\frac{P_0}{2\pi} \frac{1}{\rho V} \int_{-\infty}^{+\infty} \frac{1}{\xi^2} e^{|\xi|\Omega z} e^{-i\xi Y} d\xi \tag{18}$$

Using the symmetry, this becomes:

$$\varphi_a(Y, z) = -\frac{P_0}{\pi} \frac{1}{\rho V} \int_0^{+\infty} \frac{1}{\xi^2} e^{\xi\Omega z} \cos(\xi Y) d\xi \tag{19}$$

According to Eq. (5), the pressure in the fluid is obtained:

$$p_a(Y, z) = -\frac{P_0}{\pi} \int_0^{+\infty} \frac{1}{\xi} e^{\xi\Omega z} \sin(\xi Y) d\xi \tag{20}$$

After evaluation of the integral, Eq. (20), and adding the constant symmetrical pressure, the final value is obtained:

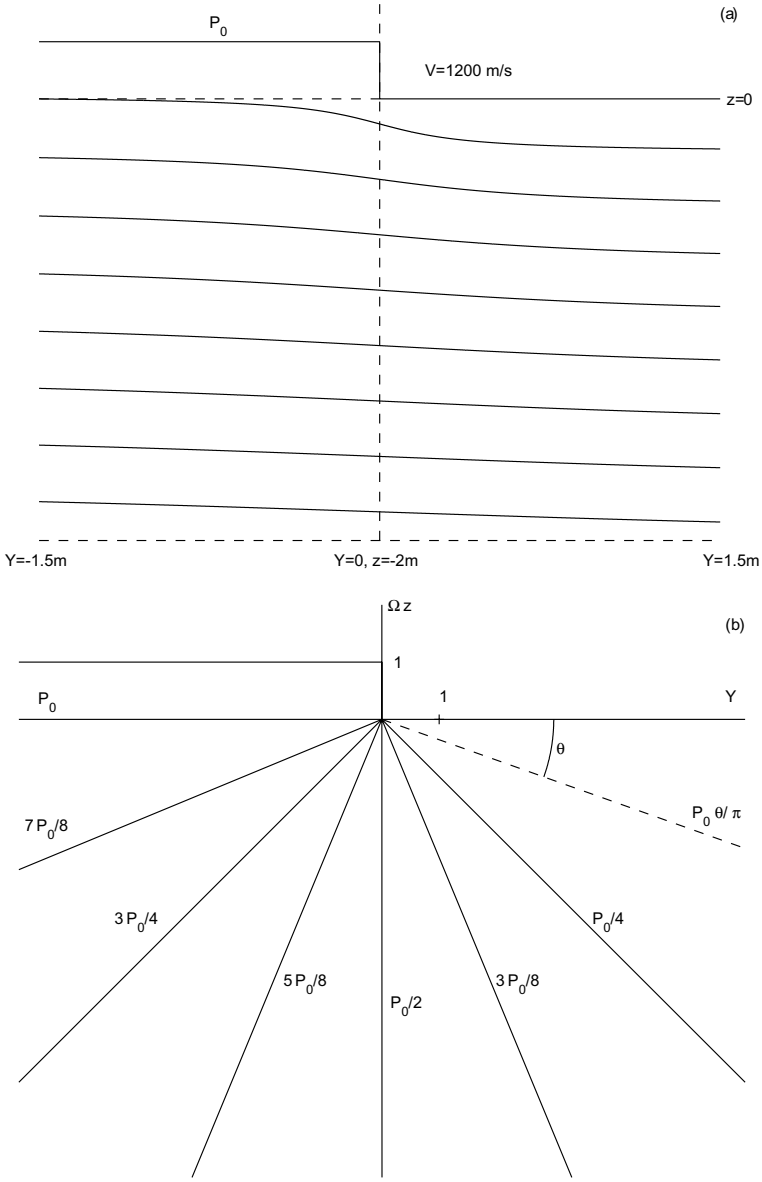

Fig. 2. Acoustic pressure in a infinite half space of liquid loaded by a subsonic pressure step (a), and equal-pressure curves (b).

$$p(Y,z) = P_0 \left(\frac{1}{2} + \frac{1}{\pi} \arctan \left(\frac{Y}{\Omega\, z} \right) \right) \text{ (z is negative)} \tag{21}$$

The corresponding pressure curves are presented in Fig. 2(a).

On the surface, the fluid pressure must match the external pressure:

$$p(Y,0) = P_0 \left(\frac{1}{2} - \frac{1}{2} Sgn(Y) \right) = P_0\, H(-Y) \tag{22}$$

For a given value of Ω, the pressure in the liquid only depends on the ratio Y/z. In normalised coordinates (Y, Ωz), the equal-pressure curves are inclined lines as can be seen in Fig. 2(b). According to Eq. (21), the pressure is proportional to the inclination angle of the line. Therefore, this normalised representation of the pressure is valid for any subsonic value of the pressure front velocity.

2.2. The supersonic case

After applying Fourier transform to Eq. (8), we obtain:

$$\partial^2_{,zz}\bar{\varphi}_a + \Omega^2\xi^2\bar{\varphi}_a = 0 \tag{23}$$

which leads to the general solution:

$$\bar{\varphi}_a(\xi, z) = a(\xi)\ \cos(\Omega\,\xi\,z) + b(\xi)\ \sin(\Omega\,\xi\,z) \tag{24}$$

Respecting the forcing pressure on the free surface,

$$\bar{P}_a(\xi)\ = \frac{P_0}{2\pi}\frac{1}{i\xi} = \bar{p}_a(\xi, 0) = i\xi\,\rho V\,\bar{\varphi}_a(\xi, 0) \tag{25}$$

the following can be deduced:

$$a(\xi) = -\frac{P_0}{2\pi}\frac{1}{\rho V}\frac{1}{\xi^2} \tag{26}$$

From Eq. (23) written in the form

$$(\partial_{,z} - \Omega\,\partial_{,Y}\ )\ (\partial_{,z} + \Omega\partial_{,Y}\ )\bar{\varphi}_a = 0 \tag{27}$$

this becomes the general expression

$$\varphi_a(Y, z) = f(Y - \Omega\,z) + g(Y + \Omega\,z) \tag{28}$$

For a forward-moving pressure front, the stationary response cannot contain upwardly propagating waves, so, only the terms of type $f(Y - \Omega\,z)$ are to be retained. The following relation between derivatives is deduced:

$$\partial\varphi_{a,z} = -\Omega\,\partial\varphi_{a,Y} \tag{29}$$

Thus, the previous remark concerning the direction of the movement is taken into account at this point in the development.

After applying this relation to Eq. (24), we obtain:

$$b(\xi)\ = i\,a(\xi) \tag{30}$$

so that the final expression is:

$$\bar{\varphi}_a(\xi, z)\ = -\frac{P_0}{2\pi}\frac{1}{\rho V}\frac{1}{\xi^2}e^{i\xi\Omega z} \tag{31}$$

The inverse transform leads to:

$$\varphi_a(Y, z)\ = -\frac{P_0}{2\pi}\frac{1}{\rho V}\int_{-\infty}^{+\infty}\frac{1}{\xi^2}e^{-i\xi(Y-\Omega z)}d\xi \tag{32}$$

which can be expressed, using the symmetry, as the Fourier cosine transform:

$$\varphi_a(Y, z)\ = -\frac{P_0}{\pi}\frac{1}{\rho V}\int_{0}^{+\infty}\frac{1}{\xi^2}\cos(\xi(Y - \Omega\,z))d\xi \tag{33}$$

The pressure takes the form, by differentiation, of the Fourier sine transform

$$p_a(Y, z)\ = -\frac{P_0}{\pi}\int_{0}^{+\infty}\frac{1}{\xi}\sin(\xi(Y - \Omega\,z))d\xi \tag{34}$$

the result of which is simply:

$$p_a(Y, z)\ = -\frac{P_0}{2}\,Sgn(Y - \Omega\,z) \tag{35}$$

The final pressure, with the addition of the constant symmetrical term, becomes:

$$p(Y, z)\ = \frac{P_0}{2}(1 - Sgn(Y - \Omega\,z)) \tag{36}$$

This result agrees well with the supersonic character of the loading, which generates a front of discontinuity in the liquid. It would be even possible to omit the demonstration and to present Eq. (36) as the only physically realizable steady-state pressure in the supersonic regime.

This is illustrated in Fig. 3 for a unit pressure step moving at the supersonic velocity $V = 1700$ m/s ($C = 1500$ m/s).

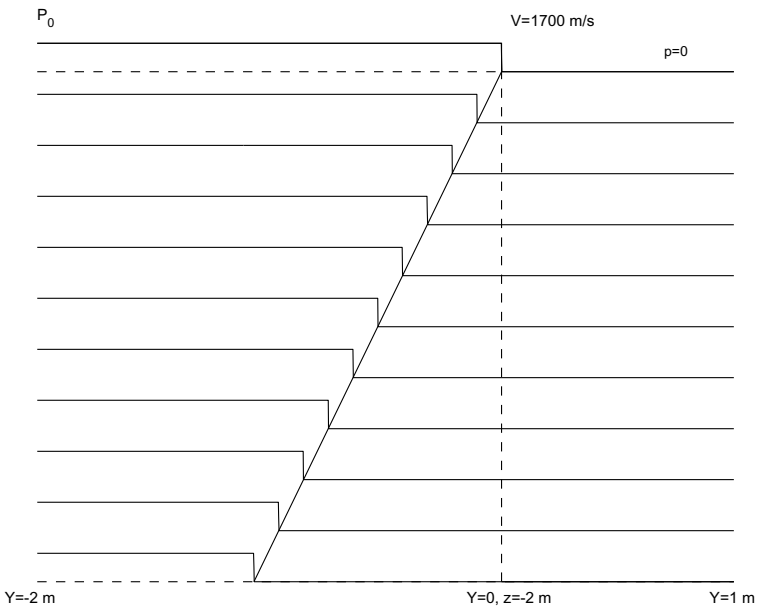

Fig. 3. Acoustic pressure in a half space of liquid loaded by a supersonic pressure step.

3. Taking into account a finite depth of liquid

The liquid is now assumed to be limited by a rigid wall, parallel to the surface, at the depth $z = -h$. The same loading cases – subsonic and supersonic – are considered. The same break down of the forcing pressure into symmetrical and anti-symmetrical parts is repeated.

The only difference with the previous study is that the new boundary condition is taken into account, which is to say the zero normal velocity at the bottom of the liquid.

3.1. The subsonic case

Contrary to the previous study, the Fourier cosine transform is preferred and applied, according to the following notations and definitions:

$$C\,[f(Y)] = \overline{f}(\omega) = \frac{2}{\pi}\int_0^{+\infty} f(Y)\,\cos(\omega Y)dY \tag{37}$$

$$f(Y) = \int_0^{+\infty} \bar{f}(\omega)\,\cos(\omega Y)d\omega \tag{38}$$

The rule of second derivative transform is recalled below:

$$C[\,\partial^2_{,YY}\,f] = -\frac{2}{\pi}(\partial_{,Y}\,f)_0 - \omega^2\overline{f}(\omega) \tag{39}$$

and a useful relation with the Fourier sine transform,

$$S[\,\partial_{,Y}\,f] = -\omega\,C[f(Y)] \tag{40}$$

By transformation of Eq. (7), we obtain:

$$\partial^2_{,zz}\,\bar{\varphi}_a + \Omega^2\left(-\frac{2}{\pi}\,\partial_{,Y}\,\varphi_{a(0,z)} - \omega^2\bar{\varphi}_a\right) = 0 \tag{41}$$

φ_a being an even function of Y, $\partial_{,Y}\,\varphi_a$ is an odd one and Eq. (41) is reduced to:

$$\partial^2_{,zz}\,\bar{\varphi}_a - \Omega^2\omega^2\bar{\varphi}_a = 0 \tag{42}$$

Its general solution is:

$$\bar{\varphi}_a(\omega, z) = A(\omega)\,\cosh(\omega\,\Omega z) + B(\omega)\,\sinh(\omega\,\Omega z) \tag{43}$$

which is more convenient in this equivalent form:

$$\bar{\varphi}_a(\omega, z) = a(\omega)\,\cosh(\omega\,\Omega\,z) + b(\omega)\,\cosh\left[\omega\,\Omega\,(z+h)\right] \tag{44}$$

The boundary condition at the bottom implies that:

$$\left(\partial_{,z}\,\bar{\varphi}_a\right)_{z=-h} = 0 \tag{45}$$

This results in:

$$a(\omega) = 0 \tag{46}$$

and:

$$\bar{\varphi}_a(\omega, z) = b(\omega)\,\cosh(\omega\Omega(z+h)) \tag{47}$$

On the free surface, this becomes:

$$-\,\omega\,C[\,\varphi_a(Y,0)] = -\omega\,b(\omega)\,\cosh(\omega\Omega h) \tag{48}$$

Using Eq. (39), we obtain:

$$-\,\omega\,b(\omega)\,\cosh(\omega\Omega h) = S\,[\partial_{,Y}\,\varphi_a] = S\left[\frac{P_0}{\rho V}\frac{1}{2}Sgn(Y)\right] = \frac{P_0}{\rho V}\frac{1}{\pi}\frac{1}{\omega} \tag{49}$$

which enables us to identify:

$$b(\omega) = -\frac{P_0}{\pi}\frac{1}{\rho V}\frac{1}{\omega^2}\frac{1}{\cosh(\omega\Omega h)} \tag{50}$$

to obtain:

$$\bar{\varphi}_a(\omega, z) = -\frac{P_0}{\pi}\frac{1}{\rho V}\frac{1}{\omega^2}\frac{\cosh(\omega\Omega(z+h))}{\cosh(\omega\Omega h)} \tag{51}$$

and finally, the potential in the whole field:

$$\varphi_a(Y, z) = -\frac{P_0}{\pi}\frac{1}{\rho V}\int_0^\infty \frac{1}{\omega^2}\frac{\cosh(\omega\Omega(z+h))}{\cosh(\omega\Omega h)}\cos(\omega Y)d\omega \tag{52}$$

By differentiation, the pressure is deduced and, adding the constant symmetrical term, this becomes:

$$p(Y, z) = P_0\left(\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\frac{\cosh(\omega\Omega(z+h))}{\cosh(\omega\Omega h)}\sin(\omega Y)d\omega\right) \tag{53}$$

On the fluid surface, the expression of the forcing pressure can easily be verified:

$$p(Y, 0) = P_0\left(\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\sin(\omega Y)d\omega\right) = \frac{P_0}{2}(1 - Sgn(Y)) = P_0\,H(-Y) \tag{54}$$

Returning to the whole field of liquid, the pressure can be explicitly expressed. By differentiation of $p(Y, z)$, Eq. (53), a more common Fourier cosine transform is obtained:

$$\frac{\partial p(Y, z)}{\partial Y} = -\frac{P_0}{\pi}\int_0^\infty \frac{\cosh(\omega\Omega(z+h))}{\cosh(\omega\Omega h)}\cos(\omega Y)d\omega \tag{55}$$

The inverse transform is usually found in table pairs [8]:

$$\frac{\partial p(Y, z)}{\partial Y} = -\frac{P_0}{2\Omega h}\frac{\cos\left(\frac{\pi}{2}\frac{(z+h)}{h}\right)\cosh\left(\frac{\pi Y}{2\Omega h}\right)}{\cos\left(\pi\frac{(z+h)}{h}\right) + \cosh\left(\frac{\pi Y}{\Omega h}\right)} \tag{56}$$

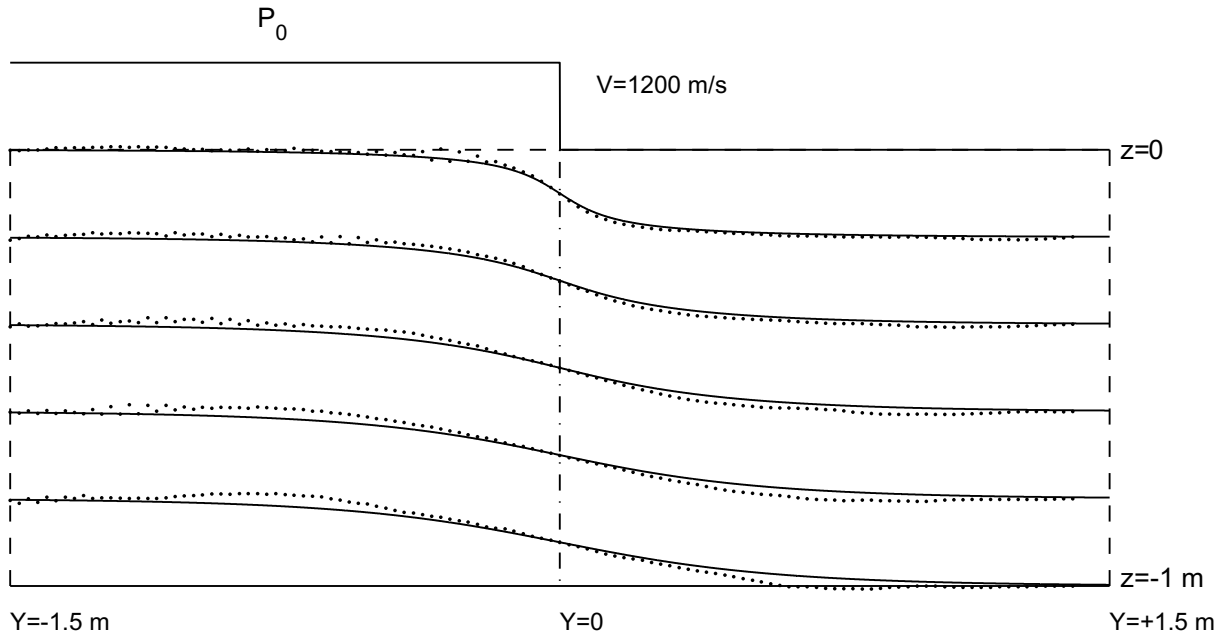

Fig. 4. Acoustic pressure in a strip of liquid loaded by a subsonic stationary pressure step, plotted at a 3 meter width interval around the load front. Solid lines: exact stationary pressure. Dotted-lines: transient pressure (numerical) after a travel distance of 100 meters of the load front.

This can be transformed to integrate:

$$p(Y,z) = -\frac{P_0}{2\Omega h}\int \frac{\cos\left(\frac{\pi}{2h}(z+h)\right)\cosh\left(\frac{\pi Y}{2\Omega h}\right)}{\cos^2(\frac{\pi}{2h}(z+h)) + \sinh^2\left(\frac{\pi Y}{2\Omega h}\right)} dY + Const \tag{57}$$

Reintroducing the right constant of the integration taken in Eq (53), the pressure is finally obtained:

$$p(Y,z) = P_0\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{\sinh\left(\frac{\pi Y}{2\Omega h}\right)}{\cos\left(\frac{\pi}{2h}(z+h)\right)}\right) \tag{58}$$

On the bottom, the pressure takes the form:

$$p(Y,-h) = P_0\left(\frac{1}{2} - \frac{1}{\pi}\arctan\left(\sinh(\frac{\pi Y}{2\Omega h})\right)\right) \tag{59}$$

It can also be expressed in non dimensional coordinates.

Defining: $\eta = Y/h$ and $\chi = z/h$, Eq. (58) takes the form:

$$p(Y,z) = P_0\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{\sinh(\frac{\pi}{2\Omega}\eta)}{\cos(\frac{\pi}{2}(1+\chi))}\right) \tag{60}$$

Figure 4 presents, in solid lines, the result of pressure obtained by Eq. (60). Depending on the depth in the liquid, the passage can be seen from pressure P_0 on the rear part, to pressure zero, on the front part. The wall boundary condition slightly changes the distribution of pressure if compared to the case without a bottom such as described in Fig. 2. The shallower the liquid, the greater the change.

The pressure plotted in dotted lines in Fig. 4 will be explained in §4.

3.2. The supersonic case.

As was verified earlier, the supersonic case is somewhat different from the subsonic one, because there is a front of discontinuity in the liquid beyond which no perturbation can propagate. This property is always true, even if the liquid is bounded by a bottom.

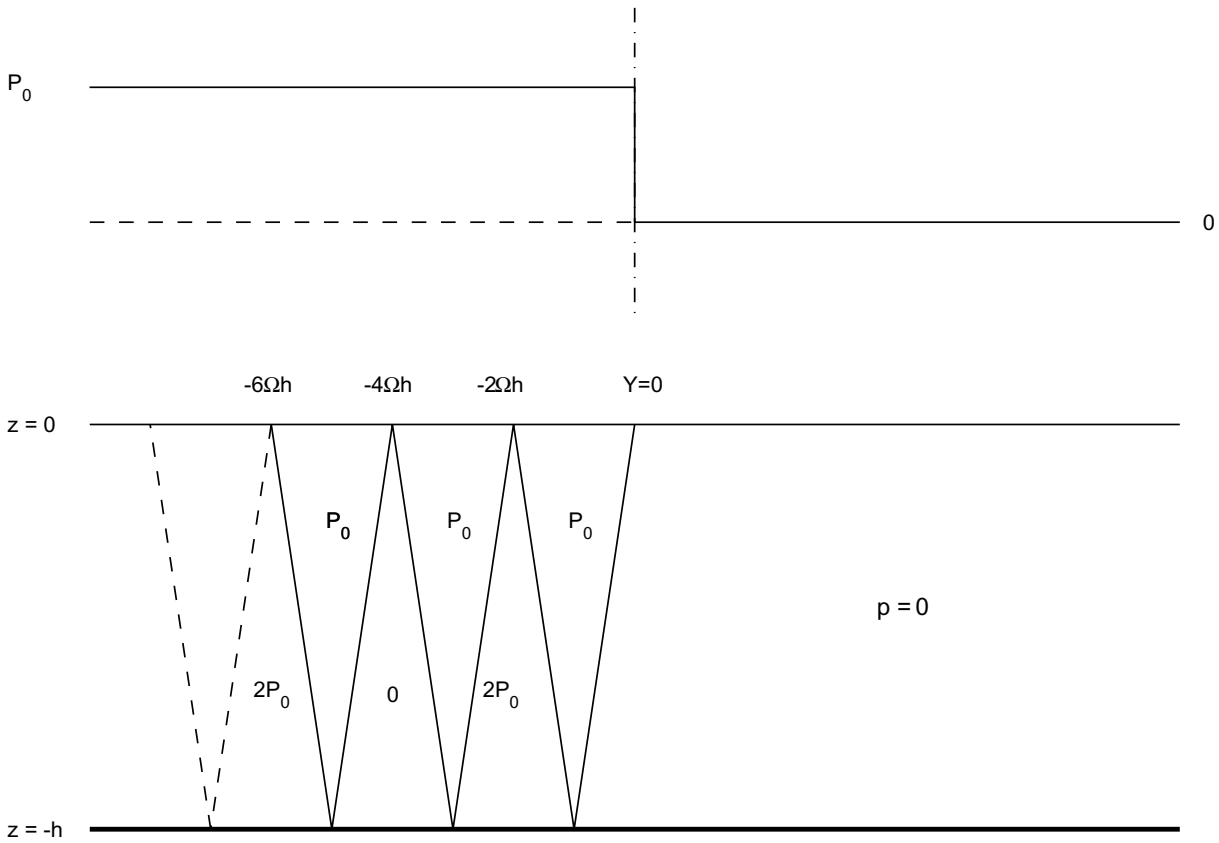

Fig. 5. Acoustic pressure in a strip of liquid loaded by a supersonic stationary pressure step.

3.2.1. A straightforward result by direct reasoning based on the physics of wave reflection

The presence of a rigid bottom compels any pressure wave hitting it to be reflected upwards. A compressive wave is then reflected into a compressive one. When this reflected compressive wave reaches the free surface, it is reflected backwards into an extensive wave.

So, following this explanation on Fig. 5, one may easily construct the pressure pattern in the strip of liquid.

Behind the first pressure front, the pressure takes the form of the discontinuity:

$$P(Y,z) = P_0\, H(-Y + \Omega\, z) \tag{61}$$

After the first reflection on the bottom, a reflected pressure is added. This results in:

$$P(Y,z) = P_0\, H(-Y + \Omega\, z) + P_0\, H(-Y - \Omega\, z - 2\Omega\, h) \tag{62}$$

After the first reflection on the free surface, a reflected pressure is subtracted. This results in:

$$P(Y,z) = P_0\, H(-Y + \Omega\, z) + P_0\, H(-Y - \Omega\, z - 2\Omega\, h) - P_0\, H(-Y + \Omega\, z - 2\Omega\, h) \tag{63}$$

and so on, for an infinite number of reflections, so that the final pressure takes the form:

$$P(Y,z) = P_0\, H(-Y + \Omega\, z) + P_0 \sum_{n=1}^{\infty} (H(-Y - \Omega\, z - 2n\Omega\, h) - H(-Y + \Omega\, z - 2n\Omega\, h)) \tag{64}$$

The final description of the pressure agrees with the forcing pressure front of constant value P_0. The pressure resulting on the bottom is periodic, arranged like a crenel ranging from zero to twice P_0, and of average value P_0. The period is proportional to the height h of the liquid.

This direct reasoning based on the physical combination of reflected waves could be considered as a means to check the analytical solution, which will now be found by means of integral transforms.

3.2.2. An analytical response in three steps.

The first step gives the anti-symmetrical part of the response, as previously defined. The general form for the potential given in Eq. (24) is still valid, but another equivalent expression is more convenient to to take into account the boundary condition on the bottom:

$$\bar{\varphi}_a(\xi, z) = a(\xi)\ \cos(\xi\Omega(z+h)) + b(\xi)\ \sin(\xi\Omega(z+h)) \tag{65}$$

The boundary condition at the bottom is a wall condition (the normal velocity component is zero):

$$(\partial_{,z}\,\bar{\varphi}_a)_{z=-h} = 0 \tag{66}$$

This results in:

$$b(\xi) = 0 \tag{67}$$

Respecting the forcing pressure on the surface, we obtain:

$$\bar{p}_a(\xi, 0)\ = \frac{P_0}{2\pi}\frac{1}{i\xi} = -i\xi\rho V a(\xi)\ \cos(\xi\Omega h) \tag{68}$$

which leads to:

$$a(\xi)\ = -\frac{P_0}{2\pi}\frac{1}{\rho V}\frac{1}{\xi^2}\frac{1}{\cos(\xi\Omega h)} \tag{69}$$

Then the pressure transform is deduced:

$$\bar{p}_a(\xi, z)\ = \frac{P_0}{2\pi}\frac{1}{i\xi}\frac{\cos(\xi\Omega(h+z))}{\cos(\xi\Omega h)} \tag{70}$$

After applying the inversion process described in Appendix A, the anti-symmetrical part of the pressure is found. It is visible in Fig. 6.

The second step is very simple. The constant pressure $P_0/2$ merely needs to be added to the previous one in the whole field of liquid. The resulting pressure is presented in Fig. 7.

It is clear that the pressure obtained is not yet conform to reality because no perturbation can be present ahead of the supersonic front.

It is also clear that nothing, up to now, has taken into account the positive or negative direction of the movement of the load.

The solution of the system is not complete. Mathematically it is possible to complete the solution by adding a free wave to it. A free wave is a non trivial solution of the system corresponding to a homogeneous load, that is to say no load. Such solutions may exist and are analogous to free vibrations for finite dynamic systems. Their amplitudes are arbitrary. They merely have to be adjusted so that the final solution verifies the boundary conditions. A previous work concerning plates coupled with liquid also used free waves [9].

The third step concerns the search for free waves. After a laborious process developed in Appendix B, a free wave was found. Its representation is shown in Fig. 8. Its amplitude has been chosen so that the boundary conditions are respected.

If the results of the three steps are added, the resulting pressure obtained corresponds exactly to the pressure previously obtained quickly, Eq. (64), and plotted in Fig. 5.

4. Two examples of non exactly stationary loading

If all the conditions of a stationary moving load are not respected, the response cannot be exactly stationary. That is the case, for example, when the pressure step has moved only for a finite time or, equivalently, over a finite distance. Nevertheless, if the traveled distance is long enough, and if all other conditions are stationary, the response would converge towards the stationary one. Two examples are chosen here to illustrate this result.

The first one considers a subsonic loading ($V = 1200$ m/s, $C = 1500$ m/s). The loading pressure starts at a definite time and the observations are made when the pressure front has traveled over a distance of 100 meters. The liquid

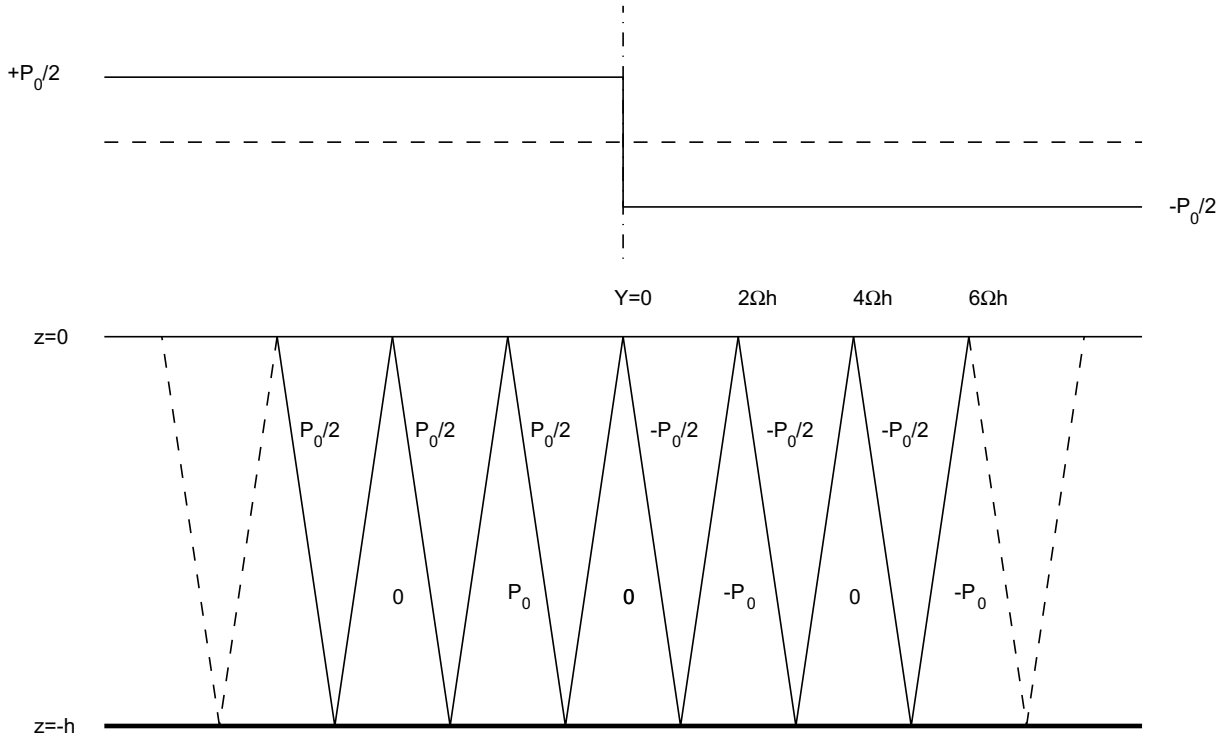

Fig. 6. Anti-symmetrical pressure distribution in the liquid.

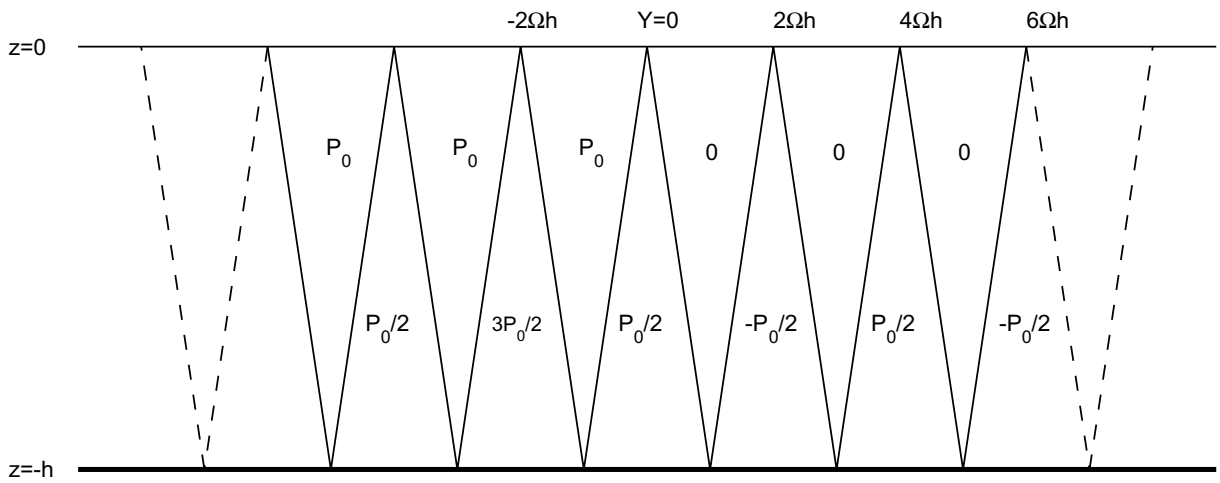

Fig. 7. Distribution of pressure accounting for anti-symmetrical and constant contributions.

is limited by a bottom at a depth of 1 meter. The pressure is computed by a finite-difference code which solves the propagation equations resulting from Eq. (1). The pressure is observed over an interval of 3 meters long, centered around the pressure front. The conditions of that transient loading are the same as those of the stationary loading corresponding to Fig. 2, except that the pressure has started at a finite distance. In Fig. 4, the transient pressure curves -calculated numerically- are plotted in dotted lines, and the stationary pressure curves, in solid lines. Effectively, the quasi stationary solution is closer and closer to the exactly stationary one when the running time increases.

The second example, Fig. 9, corresponds to the stationary supersonic case described in Fig. 5. All the conditions

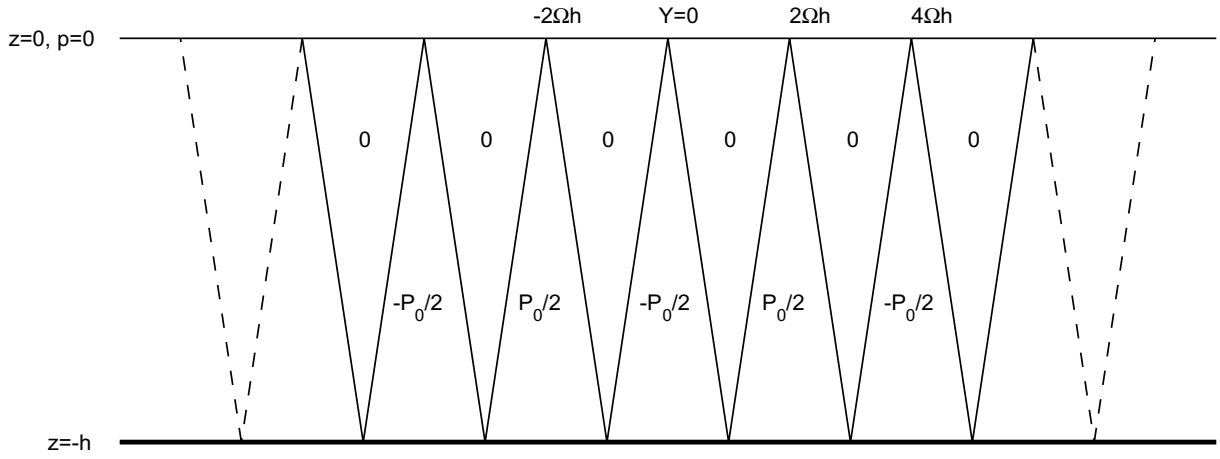

Fig. 8. Free pressure wave in the liquid.

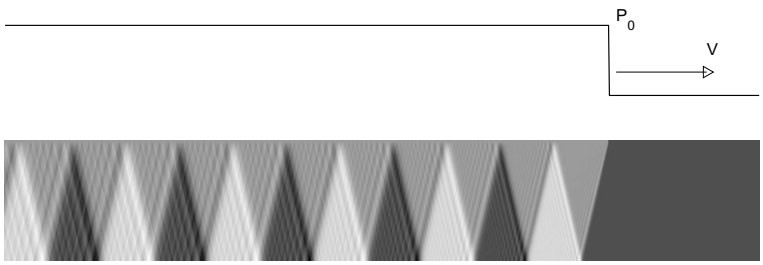

Fig. 9. Acoustic pressure in a strip of liquid loaded by a supersonic quasi-stationary pressure step (height: 0.05 m, length: 1 m, V = 2000 m/s).

are the same, except for the fact that the loading does not come from an infinitely far origin, but from a finite distance of 5 meters. The depth of the liquid is 5 cm. The velocity of the supersonic loading is $V = 2000$ m/s. The numerical calculations, solving the propagation equation of Eq. (1), highlight a pressure chart exactly similar to the theoretical one visible in Fig. 5. We find again the triangles of height h and of base $2\Omega h$. The clear triangles correspond to a pressure close to zero, the grey ones to a pressure close to P_0 and the dark ones to a pressure close to $2P_0$.

5. Conclusion

The physical problem considered in this paper concerns the movement of a fluid when a pressure is suddenly applied locally on its surface, and then moves on the surface at a constant speed and amplitude. When the velocity of the moving pressure is likely to be in the magnitude of the speed of sound, C, or even greater, as observed when detonations occur near the surface of a fluid, it is justified to seek the response of the fluid in the linear acoustic domain.

It is not common to seek the steady-state solution to the wave equation, with a forcing pressure spreading on one boundary of the fluid. Physically, there would be a transient solution to be found. In fact, as is clear in the responses of extended structures to moving loads, a steady-state solution can always be considered as the limit of the transient one, as the time and the load position tend towards large values, say infinite. In practice, the higher the load velocity, the shorter the time and the extent in space required to observe a steady-state solution, growing from the load front. For the high velocities of the sliding pressure encountered in schock wave dynamics, it is interesting to search the steady-state solution for the acoustic pressure in the liquid.

The steady-state analysis necessarily involves the characteristic velocities of the fluid or of the structures under consideration. In the fluid, the sound velocity C appears as the characteristic one, defining the subsonic and the

supersonic regimes. When $V \to C$ the limit of the solution can be easily found from the pressure equation, or by solving for the potential when $\Omega^2 = 0$, which implies that: $\varphi = Const$ on both sides of a line $X = ct$; for the pressure: $p = p_0$ if $X < ct$, $p = 0$ if $X > ct$.

The analysis presented in this paper, conducted analytically, either for a deep liquid or for a liquid limited by a horizontal rigid wall, has permitted us to obtain the closed-form expressions of the pressure, by means of the transforms method, as it was previously succesfully performed for a plate coupled with a fluid, and loaded by a moving pressure [9]. The results remain fairly close to the real conditions of loading, even if the steady-state conditions are not exactly fulfilled. For this reason, they can also be used to estimate the acoustic pressure induced by real and fast loading such as explosions, and especially detonations, near the liquid surface.

Appendix A

The aim of this appendix is to find the original pressure $p(Y, z)$ corresponding to the Fourier transform defined by Eq. (51) and recalled here,

$$\bar{p}_a(\xi, z) = \frac{P_0}{2\pi}\frac{1}{i\xi}\frac{\cos(\xi\Omega(h+z))}{\cos(\xi\Omega h)} \tag{71}$$

A simpler form is obtained after derivation,

$$\overline{\left(\frac{\partial p_a}{\partial Y}\right)} = -\frac{P_0}{2\pi}\int_{-\infty}^{\infty}\frac{\cos(\xi\Omega(z+h))}{\cos(\xi\Omega h)}e^{-i\xi Y}d\xi \tag{72}$$

Let $f(Y, z)$ be the original transform of

$$F(\xi, z) = \frac{\cos(\xi\Omega(h+z))}{\cos(\xi\Omega h)} \tag{73}$$

$F(\xi, z)$ being real and even vs. ξ, $f(Y, z)$ is also a real and even function of Y.

Using the following relation between Laplace and Fourier transforms,

$$F[f(x);\, \xi] = L[f(x);\, -i\,\xi] + L[f(x);\, i\,\xi] \tag{74}$$

and taking into account the parity of the functions, this becomes:

$$F(\xi, z) = \frac{1}{2}\left(\frac{\cosh(-i\xi\Omega(h+z))}{\cosh(-i\xi\Omega h)} + \frac{\cosh(i\xi\Omega(h+z))}{\cosh(i\xi\Omega h)}\right) \tag{75}$$

Let $f^+(Y, z)$ be the part of f restricted to Y>0; then one can identify the Laplace Transform of f^+:

$$L[f^+(Y, z); s] = \frac{1}{2}\frac{\cosh(s\Omega(h+z))}{\cosh(s\Omega h)} \tag{76}$$

$$f^+(Y, z) = L^{-1}\left(\frac{1}{2}\frac{\cosh(s\Omega(h+z))}{\cosh(s\Omega h)}\right) = \frac{1}{2}\frac{1}{2i\pi}\oint\frac{\cosh(s\Omega(h+z))}{\cosh(s\Omega h)}\,e^{-sY}ds \tag{77}$$

Applying the Residues Theorem,

$$f^+(Y, z) = \frac{1}{2}\sum_n res(s_n) = \sum_n \frac{p(s_n)}{q'(s_n)}e^{s_n Y} \tag{78}$$

with:

$$p(s_n) = \cos(is_n\Omega(z+h)), \quad q(s_n) = \cos(is_n\Omega(h)) \tag{79}$$

For:

$$s_n = i\frac{\pi}{2}\frac{1}{\Omega h}(1+2n), \quad n = -\infty, \ldots \; +\infty \tag{80}$$

$$q'(s_n) = (-1)^n \, i\Omega h \tag{81}$$

then,

$$f^+(Y,z) = \frac{1}{2}\frac{1}{i\Omega h} \sum_{n=-\infty}^{+\infty} (-1)^n \cos(\frac{\pi}{2}\frac{z+h}{h}(1+2n)) \, e^{i\frac{\pi}{2}\frac{1}{\Omega h}(1+2n)Y} \tag{82}$$

$$f^+(Y,z) = \frac{1}{2}\frac{1}{i\Omega h} \sum_{n=-\infty}^{+\infty} (-1)^n \, \frac{e^{i\frac{\pi}{2}\frac{1}{\Omega h}(Y+\Omega(z+h))(1+2n)} + e^{i\frac{\pi}{2}\frac{1}{\Omega h}(Y-\Omega(z+h))(1+2n)}}{2} \tag{83}$$

The pressure, restricted to $Y < 0$, is obtained by integration:

$$p_a^+(Y,z) \; = -P_0 \int f^+(Y,z)dY + cst \tag{84}$$

$$p_a^+(Y,z) \; = cst + \frac{P_0}{2\pi} \sum_{n=0}^{n=\infty} \frac{(-1)^n}{2n+1} \begin{pmatrix} e^{i\frac{\pi}{2}\frac{1}{\Omega h}(Y+\Omega(z+h))(1+2n)} + e^{-i\frac{\pi}{2}\frac{1}{\Omega h}(Y+\Omega(z+h))(1+2n)} \\ +e^{i\frac{\pi}{2}\frac{1}{\Omega h}(Y-\Omega(z+h))(1+2n)} + e^{-i\frac{\pi}{2}\frac{1}{\Omega h}(Y-\Omega(z+h))(1+2n)} \end{pmatrix} \tag{85}$$

Recognising the series development, the following simplification is possible,

$$p_a^+(Y,z) \; = -\frac{P_0}{2} + \frac{P_0}{2\pi} \begin{pmatrix} \arctan \, e^{i\frac{\pi}{2}\frac{1}{\Omega h}(Y+\Omega(z+h))} + \arctan \, e^{-i\frac{\pi}{2}\frac{1}{\Omega h}(Y+\Omega(z+h))} \\ + \arctan \, e^{i\frac{\pi}{2}\frac{1}{\Omega h}(Y-\Omega(z+h))} + \arctan \, e^{-i\frac{\pi}{2}\frac{1}{\Omega h}(Y-\Omega(z+h))} \end{pmatrix} \tag{86}$$

The constant has been chosen equal to $-P_0/2$ to match the value of the forcing pressure for $Y > 0$.

Taking into account the relations:

$$\arctan \, e^{it} + \arctan \, e^{-it} = \frac{\pi}{2} \qquad \text{for } -\frac{\pi}{2} + 2k\pi < t \leqslant \frac{\pi}{2} + 2k\pi \tag{87}$$

$$\arctan \, e^{it} + \arctan \, e^{-it} = -\frac{\pi}{2} \qquad \text{for } \frac{\pi}{2} + 2k\pi < t \leqslant \frac{3\pi}{2} + 2k\pi \tag{88}$$

this finally becomes, restricted to $Y > 0$:

$$p_a^+(Y,z) \; = \frac{P_0}{2}\left(\frac{c_1+c_2}{2} - 1\right) \tag{89}$$

with the conditions:

$$c_1 = 1 \text{ if } 0 < \; Y - \Omega z \leqslant 2\Omega h (\text{mod } 4\Omega h) c_1 = -1 \text{ if not} \tag{90}$$

$$c_2 = 1 \text{ if } 2\Omega h < \; Y + \Omega z \leqslant 4\Omega h (\text{mod } 4\Omega h) c_2 = -1 \text{ if not} \tag{91}$$

The pressure in the whole field of liquid is obtained by anti-symmetry. Its representation is visible in Fig. 6.

Appendix B

The aim of this appendix is to find free waves, possible solutions of the equation in absence of any loading.

The valid equation is recalled in terms of Fourier transform, the indicia f being the symbol of the free solution:

$$\Omega^2\xi^2 \, \bar{\varphi}_f + \partial_{,zz}\bar{\varphi}_f = 0 \tag{92}$$

The general form of its solution can be written:

$$\bar{\varphi}_f(\xi,z) = a(\xi)\sin(\xi\Omega z) + b(\xi) \, \cos(\xi\Omega z) \tag{93}$$

The pressure is deduced by derivation,

$$\bar{p}_f(\xi,z) = i\xi \, \rho V \, (a(\xi)\sin(\xi\Omega z) + b(\xi) \, \cos(\xi\Omega z)) \tag{94}$$

On the free surface, for $z = 0$, the forcing pressure is null, then:

$$b(\xi) = 0. \tag{95}$$

On the bottom, for $z = -h$, the normal velocity is zero, then:

$$\left(\frac{\partial \bar{\varphi}_f}{\partial z}\right)_{z=-h} = \xi\Omega\, a(\xi)\cos(\xi\Omega h) = 0 \tag{96}$$

The challenge is to fulfil this boundary condition with $a(\xi)$ different from the null function.

Equation (96) is fulfilled for any value of $a(\xi)$ if $\cos(\xi\Omega h)$ is zero, that is to say for the discrete values:

$$\xi_n = \frac{\pi}{2}\frac{1}{\Omega h}(2n+1) \quad n = -\infty, \ldots \; +\infty \tag{97}$$

Among the different possibilities, a distribution is chosen which leads to a pressure function which is real, even and containing only first kind discontinuities, as the result will prove.

Assuming:

$$\rho V\, \xi^2\, a(\xi) = i \sum_{n=-\infty}^{n=+\infty} \delta(\xi - \xi_n) \tag{98}$$

This becomes, for the derivative of the pressure

$$\overline{\left(\frac{\partial p_f}{\partial Y}\right)}(\xi, z) = -\rho V\, \xi^2\, a(\xi)\sin\xi\Omega z = -i\ \sin\xi\Omega z \sum_{n=-\infty}^{n=+\infty} \delta(\xi - \xi_n) \tag{99}$$

The inverse Fourier transform is given by the integral:

$$\frac{\partial p_f}{\partial Y}(Y, z) = i \int_{-\infty}^{+\infty} -\sin\xi\Omega z \sum_{n=-\infty}^{n=+\infty} \delta(\xi - \xi_n)\, e^{-i\xi Y} d\xi \tag{100}$$

Taking into account the general result:

$$\int_{-\infty}^{+\infty} f(x)\,\delta(a - x)dx = f(a) \tag{101}$$

this becomes

$$\frac{\partial p_f}{\partial Y}(Y, z) = i \sum_{n=-\infty}^{n=+\infty} \sin(\xi_n \Omega z)\, e^{-i\xi_n Y} \tag{102}$$

$$\frac{\partial p_f}{\partial Y}(Y, z) = \sum_{n=-\infty}^{n=+\infty} \frac{e^{-\frac{i\pi}{2\Omega h}(Y-\Omega z)(1+2n)} - e^{-\frac{i\pi}{2\Omega h}(Y+\Omega z)(1+2n)}}{2} \tag{103}$$

$$\frac{\partial p_f}{\partial Y}(Y, z) = \frac{1}{2}\left(e^{-\frac{i\pi}{2\Omega h}(Y-\Omega z)} \sum_{n=-\infty}^{n=+\infty} e^{-\frac{2i\pi}{2\Omega h}(Y-\Omega z)n} - e^{-\frac{i\pi}{2\Omega h}(Y+\Omega z)} \sum_{n=-\infty}^{n=+\infty} e^{-\frac{2i\pi}{2\Omega h}(Y+\Omega z)n}\right) \tag{104}$$

We recognise the Fourier series development of the periodic function

$$\frac{\partial p_f}{\partial Y}(Y, z) = \frac{1}{2}\left(e^{-\frac{i\pi}{2\Omega h}(Y-\Omega z)} \sum_{n=-\infty}^{n=+\infty} \delta(Y - \Omega z - 2n\Omega h) - e^{-\frac{i\pi}{2\Omega h}(Y+\Omega z)} \sum_{n=-\infty}^{n=+\infty} \delta(Y + \Omega z - 2n\Omega h)\right) \tag{105}$$

Retaining only the discrete values corresponding to the different Dirac functions abscises,

$$\frac{\partial p_f}{\partial Y}(Y,z) = \frac{1}{2}\left(\sum_{n=-\infty}^{n=+\infty}(-1)^n\delta(Y-\Omega z-2n\Omega h) - \sum_{n=-\infty}^{n=+\infty}(-1)^n\delta(Y+\Omega z-2n\Omega h)\right) \tag{106}$$

These can now be integrated to finally obtain the free pressure wave

$$p_f(Y,z) = \frac{1}{2}\sum_{n=-\infty}^{n=+\infty}(-1)^n[H(Y-\Omega z-2n\Omega h) - H(Y+\Omega z-2n\Omega h)] \tag{107}$$

The representation of this distribution of pressure is shown in Fig. 8.

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