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Free and forced vibrations of an axially-loaded Timoshenko multi-span beam carrying a number of various concentrated elements

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Abstract. In the existing reports regarding free and forced vibrations of the beams, most of them studied a uniform beam carrying various concentrated elements using Bernoulli-Euler Beam Theory (BET) but without axial force. The purpose of this paper is to utilize the numerical assembly technique to determine the exact frequency-response amplitudes of the axially-loaded Timoshenko multi-span beam carrying a number of various concentrated elements (including point masses, rotary inertias, linear springs and rotational springs) and subjected to a harmonic concentrated force and the exact natural frequencies and mode shapes of the beam for the free vibration analysis. The model allows analyzing the influence of the shear and axial force and harmonic concentrated force effects and intermediate concentrated elements on the dynamic behavior of the beams by using Timoshenko Beam Theory (TBT). At first, the coefficient matrices for the intermediate concentrated elements, an intermediate pinned support, applied harmonic force, left-end support and right-end support of Timoshenko beam are derived. After the derivation of the coefficient matrices, the numerical assembly technique is used to establish the overall coefficient matrix for the whole vibrating system. Finally, solving the equations associated with the last overall coefficient matrix one determines the exact dynamic response amplitudes of the forced vibrating system corresponding to each specified exciting frequency of the harmonic force. Equating the determinant of the overall coefficient matrix to zero one determines the natural frequencies of the free vibrating system (the case of zero harmonic force) and substituting the corresponding values of integration constants into the related eigenfunctions one determines the associated mode shapes. The calculated vibration amplitudes of the forced vibrating systems and the natural frequencies of the free vibrating systems are given in tables for different values of the axial force. The dynamic response amplitudes and the mode shapes are presented in graphs. The effects of axial force and harmonic concentrated force on the vibration analysis of Timoshenko multi-span beam are also investigated.

Keywords: Axial force effect, dynamic response amplitudes, exact natural frequency, free and forced vibrations, numerical assembly technique.

1. Introduction

The free and forced vibration characteristics of a uniform beam carrying various concentrated elements (such as point masses, rotary inertias, linear springs, rotational springs, etc.) is an important problem in engineering. The situation of structural elements supporting motors or engines attached to them is usual in technological applications.

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The operation of the machine and its dynamic force may introduce severe dynamic stresses on the beam. Thus, a lot of studies have been published in this area.

The normal mode summation technique to determine the fundamental frequency of the cantilevered beams carrying masses and springs was used by Gürgöze [1,2]. Hamdan and Jubran investigated the free and forced vibrations of a restrained uniform beam carrying an intermediate lumped mass and a rotary inertia [3]. Gürgöze et al. solved the eigenfrequencies of a cantilevered beam with attached tip mass and a spring-mass system and studied the effect of an attached spring-mass system on the frequency spectrum of a cantilevered beam [4–6]. Moreover, they studied on two alternative formulations of the frequency equation of a Bernoulli-Euler beam to which several spring-mass systems being attached in-span and then solved for the eigenfrequencies. Liu et al. formulated the frequency equation for beams carrying intermediate concentrated masses by using the Laplace Transformation Technique [7]. Wu and Chou obtained the exact solution of the natural frequency values and mode shapes for a beam carrying any number of spring masses [8]. Gürgöze and Erol investigated the forced vibration responses of a cantilevered beam with single intermediate support [9,10]. Naguleswaran obtained the natural frequency values of the beams on up to five resilient supports including ends and carrying several particles by using Bernoulli-Euler Beam Theory and a fourth-order determinant equated to zero [11,12]. Abu-Hilal studied forced vibration of Bernoulli-Euler beams by using Green functions [13]. Lin and Chang studied the free vibration analysis of a multi-span Timoshenko beam with an arbitrary number of flexible constraints by considering the compatibility requirements on each constraint point and using a transfer matrix method [14]. Lin and Tsai determined the exact natural frequencies together with the associated mode shapes for Bernoulli-Euler multi-span beam carrying multiple point masses [15]. In the other study, Lin and Tsai investigated the free vibration characteristics of Bernoulli-Euler multiple-step beam carrying a number of intermediate lumped masses and rotary inertias [16]. The natural frequencies and mode shapes of Bernoulli-Euler multi-span beam carrying multiple spring-mass systems were determined by Lin and Tsai [17]. Wang et al. studied the natural frequencies and mode shapes of a uniform Timoshenko beam carrying multiple intermediate spring-mass systems with the effects of shear deformation and rotary inertia [18]. Yesilce et al. investigated the effects of attached spring-mass systems on the free vibration characteristics of the 1–4 span Timoshenko beams [19]. In the other study, Yesilce and Demirdag described the determination of the natural frequencies of vibration of Timoshenko multi-span beam carrying multiple spring-mass systems with axial force effect [20]. Lin investigated the free and forced vibration characteristics of Bernoulli-Euler multi-span beam carrying a number of various concentrated elements [21]. Yesilce investigated the effect of axial force on the free vibration of Reddy-Bickford multi-span beam carrying multiple spring-mass systems [22]. Lin investigated the free vibration characteristics of non-uniform Bernoulli-Euler beam carrying multiple elastic-supported rigid bars [23].

Unfortunately, a suitable example that studies the free and forced vibration analysis of axially-loaded Timoshenko multi-span beam carrying a number of various concentrated elements and subjected to a harmonic force has not been investigated by any of the studies in open literature so far. In the presented paper, we describe the determination of the exact natural frequencies and the exact frequency-response amplitudes of the uniform axially-loaded Timoshenko multi-span beam carrying a number of various concentrated elements. The dynamic response amplitudes of the forced vibrating systems and the natural frequencies of the free vibrating systems are calculated, the dynamic response amplitudes and the first three mode shapes are plotted and the effects of the axial force and the harmonic concentrated force and the influence of the shear are investigated by using the computer package, Matlab.

2. The mathematical model and formulation

An axially-loaded Timoshenko uniform beam supported by k pins by including those at the two ends of beam, carrying intermediate concentrated elements (n including point masses, rotary inertias, s linear springs and/or rotational springs) and subjected to j harmonic forces of the same frequency over bar is presented in (Fig. 1). From (Fig. 1), the total number of stations is $M' = k + n + s + f$.

The kinds of position coordinates which are used in this study are given below:

$x_{v'}$ are the position coordinates for the stations, $(1 \leqslant v' \leqslant M')$,
x_p^* are the position coordinates of the intermediate point masses with rotary inertias, $(1 \leqslant p \leqslant n)$,
$\bar{x}_r$ are the position coordinates of the pinned supports, $(1 \leqslant r \leqslant k)$,
x'_u are the position coordinates of the linear springs and/or rotational springs, $(1 \leqslant u \leqslant s)$,
$\hat{x}_f$ are the position coordinates of the applied harmonic forces, $(1 \leqslant f \leqslant j)$.

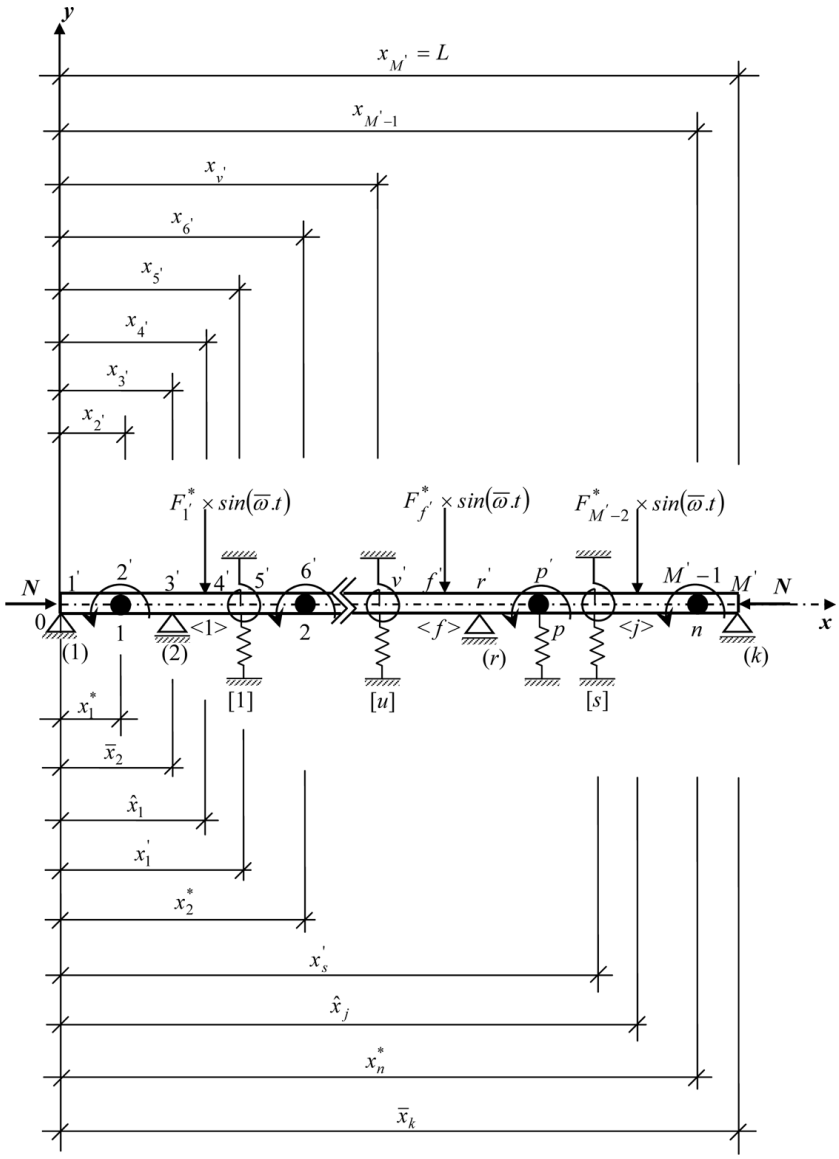


Fig. 1. The axially-loaded Timoshenko uniform beam supported by k pins and carrying various intermediate concentrated elements and subjected to j harmonic concentrated forces.

In (Fig. 1), the symbols of $1', 2', \ldots, v', \ldots, M'-1, M'$ above the x-axis refer to the numbering of stations. The symbols of $1, 2, \ldots, p, \ldots, n$ below the x-axis refer to the numbering of the intermediate point masses with rotary inertias. The symbols of $(1), (2), \ldots, (r), \ldots, (k)$ below the x-axis refer to the numbering of the pinned supports. The symbols of $[1], [2], \ldots, [u], \ldots, [s]$ below the x-axis refer to the numbering of the linear springs and/or rotational springs. The symbols of $< 1 >, < 2 >, \ldots, < f >, \ldots, < j >$ below the x-axis refer to the numbering of the applied harmonic forces.

Using Hamilton's principle, the equations of motion for axially-loaded Timoshenko beam can be written as:

$$EI_x \cdot \frac{\partial^2 \theta(x,t)}{\partial x^2} + \frac{AG\cdot}{\bar{k}} \cdot \left(\frac{\partial y\,(x,t)}{\partial x} - \theta\,(x,t) \right) - m \cdot \frac{\partial^2 \theta\,(x,t)}{\partial t^2} = F_i\,(t) \cdot \delta\,(x - x_i) \tag{1a}$$

$$\frac{AG\cdot}{\bar{k}} \cdot \left(\frac{\partial^2 y\,(x,t)}{\partial x^2} - \frac{\partial \theta\,(x,t)}{\partial x} \right) - N \cdot \frac{\partial^2 y\,(x,t)}{\partial x^2} - \frac{m \cdot I_x}{A} \cdot \frac{\partial^2 y\,(x,t)}{\partial t^2} = 0 \quad (0 \leqslant x \leqslant L) \tag{1b}$$

where $y(x,t)$ represents transverse deflection of the beam; $\theta(x,t)$ is the rotation angle due to bending moment; m is mass per unit length of the beam; N is the axial compressive force; A is the cross-section area; I_x is moment of inertia; $\bar{k}$ is the shape factor due to cross-section geometry of the beam; E, G are Young's modulus and shear modulus of the beam, respectively; x is the beam position; t is time variable; $F_i(t)$ is a force (with its magnitude equal to external load per unit length) at time t; $\delta(x - x_i)$ is the Dirac delta with x_i denoting the coordinate at which the concentrated harmonic force $F_i(t)$ applied. The details for the application of Hamilton's principle and the derivation of the equations of motion are presented in Appendix-A at the end of the paper.

The parameters appearing in the foregoing expressions obey the following relationships:

$$\frac{\partial y(x,t)}{\partial x} = \theta(x,t) + \gamma(x,t) \tag{2a}$$

$$\bar{M}(x,t) = EI_x \cdot \frac{\partial \theta(x,t)}{\partial x} \tag{2b}$$

$$T(x,t) = \frac{AG}{\bar{k}} \cdot \gamma(x,t) = \frac{AG}{\bar{k}} \cdot \left(\frac{\partial y(x,t)}{\partial x} - \theta(x,t) \right) \tag{2c}$$

where $\bar{M}(x,t)$ and $T(x,t)$ are the bending moment function and shear force function, respectively, and $\gamma(x,t)$ is the associated shear deformation.

After some manipulations by using Eqs (1) and (2), one obtains the following uncoupled equations of motion for the axially-loaded Timoshenko beam as:

$$\begin{aligned}\left(1 - \tfrac{N\cdot\bar{k}}{AG}\right) \cdot EI_x \cdot \tfrac{\partial^4 y(x,t)}{\partial x^4} + N \cdot \tfrac{\partial^2 y(x,t)}{\partial x^2} + m \cdot \tfrac{\partial^2 y(x,t)}{\partial t^2} - \left(1 + \tfrac{E\cdot\bar{k}}{G} - \tfrac{N\cdot\bar{k}}{AG}\right) \cdot \tfrac{\partial^4 y(x,t)}{\partial x^2 \cdot \partial t^2} \\ + \tfrac{m^2 \cdot I_x \cdot \bar{k}}{A^2 \cdot G} \cdot \tfrac{\partial^4 y(x,t)}{\partial t^4} = F_i(t) \cdot \delta(x - x_i)\end{aligned} \tag{3a}$$

$$\begin{aligned}\left(1 - \tfrac{N\cdot\bar{k}}{AG}\right) \cdot EI_x \cdot \tfrac{\partial^4 \theta(x,t)}{\partial x^4} + N \cdot \tfrac{\partial^2 \theta(x,t)}{\partial x^2} + m \cdot \tfrac{\partial^2 \theta(x,t)}{\partial t^2} - \left(1 + \tfrac{E\cdot\bar{k}}{G} - \tfrac{N\cdot\bar{k}}{AG}\right) \cdot \tfrac{\partial^4 \theta(x,t)}{\partial x^2 \cdot \partial t^2} \\ + \tfrac{m^2 \cdot I_x \cdot \bar{k}}{A^2 \cdot G} \cdot \tfrac{\partial^4 \theta(x,t)}{\partial t^4} = 0\end{aligned} \tag{3b}$$

If the applied concentrated force takes the harmonic form as

$$F_i(t) = F^* \cdot \sin(\bar{\omega} \cdot t) \tag{4}$$

then, the displacement and slope functions of the beam can be written by using the method of separation of variables as:

$$y(x,t) = \phi(x) \cdot \sin(\bar{\omega} \cdot t) \tag{5a}$$

$$\theta(x,t) = \bar{\theta}(x) \cdot \sin(\bar{\omega} \cdot t) \tag{5b}$$

where $\phi(x)$ and $\bar{\theta}(x)$ are the amplitudes of the displacement function and slope function, respectively, and F^* is the amplitude of the applied concentrated harmonic force, $\bar{\omega}$ is the exciting frequency of the applied concentrated harmonic force.

The solution of Eq. (1) is obtained as:

$$y(z,t) = \phi(z) \cdot \sin(\bar{\omega} \cdot t) - \frac{F^* \cdot L^4}{\left(1 - N_r \cdot \pi^2 \cdot EI_x \cdot \bar{k}/AG\right) \cdot \alpha^4} \cdot \delta(z - z_i) \tag{6a}$$

$$\theta(z,t) = \bar{\theta}(z) \cdot \sin(\bar{\omega} \cdot t) \quad (0 \leqslant z \leqslant 1) \tag{6b}$$

in which

$$\phi(z) = [C_1 . \cosh(D_1 . z) + C_2 . \sinh(D_1 . z) + C_3 . \cos(D_2 . z) + C_4 . \sin(D_2 . z)];$$

$$\bar{\theta}(z) = [K_3 \cdot C_1 . \sinh(D_1 . z) + K_3 \cdot C_2 . \cosh(D_1 . z) + K_4 \cdot C_3 . \sin(D_2 . z) - K_4 \cdot C_4 . \cos(D_2 . z)];$$

$$D_1 = \sqrt{\frac{1}{2} \cdot \left(-\beta + \sqrt{\beta^2 + 4 \cdot \alpha^4}\right)}; \quad D_2 = \sqrt{\frac{1}{2} \cdot \left(\beta + \sqrt{\beta^2 + 4 \cdot \alpha^4}\right)};$$

$$\beta = \frac{\left[\frac{N_r \cdot \pi^2 \cdot EI_x}{L^2} + \left(1 + \frac{E\cdot\bar{k}}{AG} - \frac{N_r \cdot \pi^2 \cdot EI_x \cdot \bar{k}}{AG \cdot L^2}\right) \cdot \frac{m \cdot I_x}{A} \cdot \bar{\omega}^2\right] \cdot L^2}{\left(1 - \frac{N_r \cdot \pi^2 \cdot EI_x \cdot \bar{k}}{AG \cdot L^2}\right) \cdot EI_x};$$

$$\alpha^4 = \frac{\lambda^4 \cdot EI_x - \frac{m^2 \cdot I_x \cdot \bar{k} \cdot \bar{\omega}^4 \cdot L^4}{A^2 \cdot G}}{\left(1 - \frac{N_r \cdot \pi^2 \cdot EI_x \cdot \bar{k}}{AG \cdot L^2}\right) \cdot EI_x};$$

$$N_r = \frac{N \cdot L^2}{\pi^2 \cdot EI_x} \text{ (nondimensionalized multiplication factor for the axial compressive force);}$$

$$\lambda = \sqrt[4]{\frac{m.\bar{\omega}^2.L^4}{EI_x}} \text{ (frequency parameter)}$$

$$K_3 = \frac{AG \cdot D_1}{\bar{k} \cdot \left(-EI_x \cdot D_1^2 - \frac{m \cdot I_x}{A} \cdot \bar{\omega}^2 + \frac{AG}{k}\right)}; \quad K_4 = \frac{-AG \cdot D_2}{\bar{k} \cdot \left(EI_x \cdot D_2^2 - \frac{m \cdot I_x}{A} \cdot \bar{\omega}^2 + \frac{AG}{k}\right)};$$

$z = \frac{x}{L}; C_1, \ldots, C_4 =$ constants of integration; $L =$ total length of the beam.

The bending moment and shear force functions of the beam with respect to z are given below:

$$\bar{M}(z,t) = \frac{EI_x}{L} \cdot \frac{d\theta(z)}{dz} \cdot \sin(\bar{\omega} \cdot t) \tag{7a}$$

$$T(z,t) = \frac{AG}{\bar{k}} \cdot \left(\frac{1}{L} \cdot \frac{d\phi(z)}{dz} - \theta(z)\right) \cdot \sin(\bar{\omega} \cdot t) \tag{7b}$$

The solutions of the free vibrating system can be written as:

$$y(z,t) = [C_1.\cosh(D_1.z) + C_2.\sinh(D_1.z) + C_3.\cos(D_2.z) + C_4.\sin(D_2.z)] \cdot \sin(\omega \cdot t) \tag{8a}$$

$$\bar{\theta}(z,t) = [K_3 \cdot C_1.\sinh(D_1.z) + K_3 \cdot C_2.\cosh(D_1.z) + K_4 \cdot C_3.\sin(D_2.z) - K_4 \cdot C_4.\cos(D_2.z)] \cdot \sin(\omega \cdot t) \tag{8b}$$

where ω is the natural circular frequency of the free vibrating system.

For the free vibrating systems, the frequency values of the applied concentrated harmonic forces are changed by the natural frequency values of the free vibrating systems in all equations.

3. Determination of natural frequencies, mode shapes and frequecny-response curves of the beam

The state is written in terms of the values of the displacement, slope, bending moment and shear force functions at the locations of z and t for Timoshenko beam, as:

$$\{S(z,t)\}^T = \left\{\phi(z) \quad \bar{\theta}(z) \quad \bar{M}(z) \quad T(z)\right\} \cdot \sin(\bar{\omega}.t) \tag{9}$$

where $\{S(z,t)\}$ shows the state vector.

If the left-end support of the beam is pinned, the boundary conditions for the left-end support are written as:

$$\phi_{1'}(z=0) = 0 \tag{10a}$$

$$\bar{M}_{1'}(z=0) = 0 \tag{10b}$$

From Eqs (6a) and (7a), the boundary conditions for the left-end pinned support can be written in matrix equation form as:

$$[B_{1'}] \cdot \{C_{1'}\} = \{0\} \tag{11a}$$

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ K_1 & 0 & -K_2 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \cdot \begin{Bmatrix} C_{1',1} \\ C_{1',2} \\ C_{1',3} \\ C_{1',4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{11b}$$

where $K_1 = \frac{EI_x \cdot K_3 \cdot D_1}{L}$; $K_2 = -\frac{EI_x \cdot K_4 \cdot D_2}{L}$

If the left-end support of the beam is clamped, the boundary conditions are written as:

$$\phi_{1'}(z=0) = 0 \tag{12a}$$

$$\bar{\theta}_{1'}(z=0) = 0 \tag{12b}$$

From Eqs (6a) and (6b), the boundary conditions for the left-end clamped support can be written in matrix equation form as:

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & K_3 & 0 & -K_4 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \cdot \begin{Bmatrix} C_{1',1} \\ C_{1',2} \\ C_{1',3} \\ C_{1',4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{13}$$

The boundary conditions for the pth intermediate concentrated element including point mass, rotary inertia, linear springs and rotational spring are written by using continuity of deformations, slopes and equilibrium of bending moments and shear forces, as (the station numbering corresponding to the pth intermediate concentrated element is represented by p'):

$$\phi_{p'}^{L}(z_{p'}) = \phi_{p'}^{R}(z_{p'}) \tag{14a}$$

$$\bar{\theta}_{p'}^{L}(z_{p'}) = \bar{\theta}_{p'}^{R}(z_{p'}) \tag{14b}$$

$$\bar{M}_{p'}^{L}(z_{p'}) + \left(J_p - I_p \cdot \bar{\omega}^2\right) \cdot \theta_{p'}^{L}(z_{p'}) = \bar{M}_{p'}^{R}(z_{p'}) \tag{14c}$$

$$T_{p'}^{L}(z_{p'}) - \left(R_p - m_p \cdot \bar{\omega}^2\right) \cdot \phi_{p'}^{L}(z_{p'}) = T_{p'}^{R}(z_{p'}) \tag{14d}$$

where m_p is the magnitude of the lumped mass; I_p is the rotary inertia; R_p is the linear spring constant; J_p is the rotational spring constant; L and R refer to the left side and right side of the pth intermediate concentrated element, respectively.

In Appendix-B, the boundary conditions for the pth intermediate concentrated element are presented in matrix equation form.

The boundary conditions for the rth intermediate pinned support are written by using continuity of deformations, slopes and equilibrium of bending moments, as (the station numbering corresponding to the rth intermediate pinned support is represented by r'):

$$\phi_{r'}^{L}(z_{r'}) = \phi_{r'}^{R}(z_{r'}) = 0 \tag{15a}$$

$$\bar{\theta}_{r'}^{L}(z_{r'}) = \bar{\theta}_{r'}^{R}(z_{r'}) \tag{15b}$$

$$\bar{M}_{r'}^{L}(z_{r'}) = \bar{M}_{r'}^{R}(z_{r'}) \tag{15c}$$

In Appendix-B, the boundary conditions for the rth intermediate pinned support are presented in matrix equation.

The boundary conditions for the fth intermediate harmonic concentrated force are written by using continuity of deformations, slopes and equilibrium of bending moments and shear forces, as (the station numbering corresponding to the fth intermediate harmonic concentrated force is represented by f'):

$$\phi_{f'}^{L}(z_{f'}) = \phi_{f'}^{R}(z_{f'}) \tag{16a}$$

$$\bar{\theta}_{f'}^{L}(z_{f'}) = \bar{\theta}_{f'}^{R}(z_{f'}) \tag{16b}$$

$$\bar{M}_{f'}^{L}(z_{f'}) = \bar{M}_{f'}^{R}(z_{f'}) \tag{16c}$$

$$T_{f'}^{L}(z_{f'}) + F_{f'}^{*} = T_{f'}^{R}(z_{f'}) \tag{16d}$$

In Appendix-B, the boundary conditions for the fth intermediate harmonic concentrated force are presented in matrix equation.

If the right-end support of the beam is pinned, the boundary conditions for the right-end support are written as:

$$\phi_{M'}\left(z=1\right)=0 \tag{17a}$$

$$\bar{M}_{M'}\left(z=1\right)=0 \tag{17b}$$

From Eqs (6a) and (7a), the boundary conditions for the right-end pinned support can be written in matrix equation form as:

$$\left[B_{M'}\right]\cdot\left\{C_{M'}\right\}=\left\{0\right\} \tag{18a}$$

$$\begin{array}{c} \begin{array}{cccc} 4\mathrm{M}_{\mathrm{i}}'+1 & 4\mathrm{M}_{\mathrm{i}}'+2 & 4\mathrm{M}_{\mathrm{i}}'+3 & 4\mathrm{M}_{\mathrm{i}}'+4 \end{array} \\ \begin{bmatrix} \cosh\left(D_1\right) & \sinh\left(D_1\right) & \cos\left(D_2\right) & \sin\left(D_2\right) \\ K_1\cdot\cosh\left(D_1\right) & K_1\cdot\sinh\left(D_1\right) & -K_2\cdot\cos\left(D_2\right) & -K_2\cdot\sin\left(D_2\right) \end{bmatrix} \end{array} \begin{array}{c} q-1 \\ q \end{array}\cdot\begin{Bmatrix} C_{M',1} \\ C_{M',2} \\ C_{M',3} \\ C_{M',4} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{18b}$$

If the right-end support of the beam is free, the boundary conditions are written as:

$$\bar{M}_{M'}\left(z=1\right)=0 \tag{19a}$$

$$T_{M'}\left(z=1\right)=0 \tag{19b}$$

From Eqs (7a) and (7b), the boundary conditions for the right-end support can be written in matrix equation form as:

$$\begin{array}{c} \begin{array}{cccc} 4\mathrm{M}_{\mathrm{i}}'+1 & 4\mathrm{M}_{\mathrm{i}}'+2 & 4\mathrm{M}_{\mathrm{i}}'+3 & 4\mathrm{M}_{\mathrm{i}}'+4 \end{array} \\ \begin{bmatrix} K_1\cdot\cosh\left(D_1\right) & K_1\cdot\sinh\left(D_1\right) & -K_2\cdot\cos\left(D_2\right) & -K_2\cdot\sin\left(D_2\right) \\ K_5\cdot\sinh\left(D_1\right) & K_5\cdot\cosh\left(D_1\right) & K_6\cdot\sin\left(D_2\right) & -K_6\cdot\cos\left(D_2\right) \end{bmatrix} \end{array} \begin{array}{c} q-1 \\ q \end{array}\cdot\begin{Bmatrix} C_{M',1} \\ C_{M',2} \\ C_{M',3} \\ C_{M',4} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{20}$$

where $K_5=\frac{AG}{k}\cdot\left(\frac{D_1}{L}-K_3\right)$; $K_6=\frac{AG}{k}\cdot\left(-\frac{D_2}{L}-K_4\right)$

In Eqs (18) and (20), M_i' is the total number of intermediate stations and is given by:

$$M_i'=M'-2 \tag{21a}$$

with

$$M'=k+n+s+f \tag{21b}$$

In Eq. (21b), M' is the total number of stations.

In Eqs (18b) and (20), q denotes the total number of equations for integration constants given by

$$q=2+4\cdot\left(M'-2\right)+2 \tag{22}$$

From Eq. (22), it can be seen that; the left-end support of the beam has two equations, each intermediate station of the beam has four equations and the right-end support of the beam has two equations.

In this paper, the coefficient matrices for left-end support, each intermediate concentrated element, each intermediate pinned support, each applied harmonic force and right-end support of a Timoshenko beam are derived, respectively. In the next step, the numerical assembly technique is used to establish the overall coefficient matrix for the whole vibrating system as is given in Eq. (23).

$$\left[B\right]\cdot\left\{C\right\}=\left\{D\right\} \tag{23}$$

For the case of the free vibrations (in such a case, one must set $\bar{\omega}=\omega$), the applied harmonic force is zero and Eq. (23) reduces to

$$\left[B\right]\cdot\left\{C\right\}=\left\{0\right\} \tag{24}$$

For non-trivial solution of the free vibrating system, equating the determinant of the last overall coefficient matrix to zero one determines the natural frequencies of the free vibrating system as is given in Eq. (25) and substituting of the last integration constants into the related eigenfunctions one determines the associated mode shapes.

$$\left|B\right|=0 \tag{25}$$

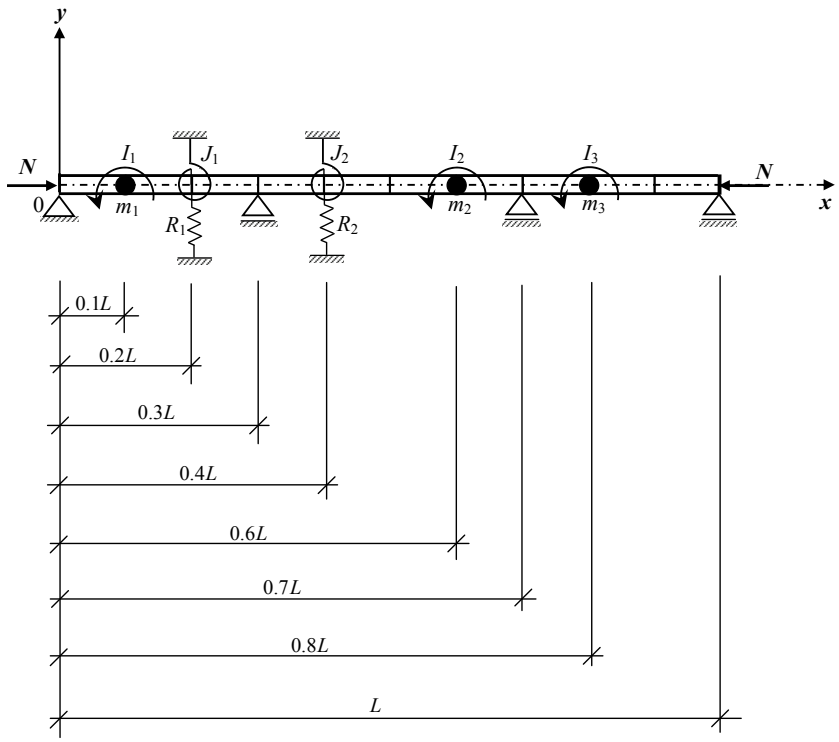


Fig. 2. A pinned-pinned Timoshenko beam carrying three intermediate point masses, three rotary inertias, two linear springs, two rotational springs and with two intermediate pinned supports.

4. Numerical analysis and discussions

In this study, three numerical examples are considered. For the example of the free vibrating system, the first five frequency parameters, λ_i $(i = 1, \ldots, 5)$ are calculated by using a computer program prepared by author. In this program, the secant method is used in which determinant values are evaluated for a range (ω_i) values. The (ω_i) value causing a sign change between the successive determinant values is a root of frequency equation and means a frequency for the system. The iterative computations are determined when the value of the determinant changed sign due to a change of 10^{-6} in the value of ω_i.

For the examples of the forced vibrating systems, the dimensionless vibration amplitudes are calculated at different locations of the axially-loaded Timoshenko beam carrying various intermediate concentrated elements and subjected to harmonic concentrated forces.

All numerical results of this paper are obtained based on a uniform, circular Timoshenko beam with the following data as:

Diameter $d = 0.05$ m; $EI_x = 6.34761 \times 10^4$ Nm^2; $m = 15.3875$ kg/m; $L = 1.0$ m; for the shear effect, $\bar{k} = 4/3$ and $AG_=$ 1.562489231 $\times$ 10^8 N; for the axial force effect, $N_r = 0$, 0.25, 0.50 and 0.75.

4.1. Free vibration analysis of the axially-loaded Timoshenko beam carrying three point masses, three rotary inertias, two linear springs, two rotational springs and with two intermediate pinned supports

In the first numerical example (see Figs 2 and 3), the uniform pinned-pinned and clamped-free Timoshenko beams carrying three point masses with rotary inertias at three locations, two linear springs and two rotational springs at the other two locations and having two intermediate pinned supports are considered. In this numerical example, the magnitudes and locations of the intermediate point masses are taken as: $m_1 = (0.10 \cdot m \cdot L)$, $m_2 = (0.60 \cdot m \cdot L)$ and $m_3 = (0.80 \cdot m \cdot L)$ located at $z_1^* = 0.10$, $z_2^* = 0.60$ and $z_3^* = 0.80$, respectively. The given data for the three

Table 1
The first five frequency parameters of the uniform Timoshenko beam with two intermediate pinned supports and carrying three point masses, three rotary inertias, two linear springs, two rotational springs for different values of N_r

Boundary conditions	λ_i	(BET) $N_r = 0.00$ Ref. [21]	(TBT) $N_r = 0.00$	$N_r = 0.25$	$N_r = 0.50$	$N_r = 0.75$
Pinned-pinned	λ_1	6.613083	6.517031	6.477434	6.436984	6.395638
	λ_2	8.214078	8.018377	7.991200	7.963509	7.935285
	λ_3	9.235993	8.957345	8.929905	8.902323	8.874599
	λ_4	11.506641	11.062908	11.054649	11.046312	11.037896
	λ_5	13.353669	12.164230	12.154437	12.144547	12.134560
Clamped-free	λ_1	4.089879	4.058147	4.057505	4.056363	4.054690
	λ_2	7.633916	7.491407	7.467522	7.443442	7.419168
	λ_3	10.002664	9.599952	9.590746	9.581441	9.572034
	λ_4	10.948062	10.518501	10.498724	10.478822	10.458793
	λ_5	11.986887	11.680263	11.678383	11.676463	11.674500

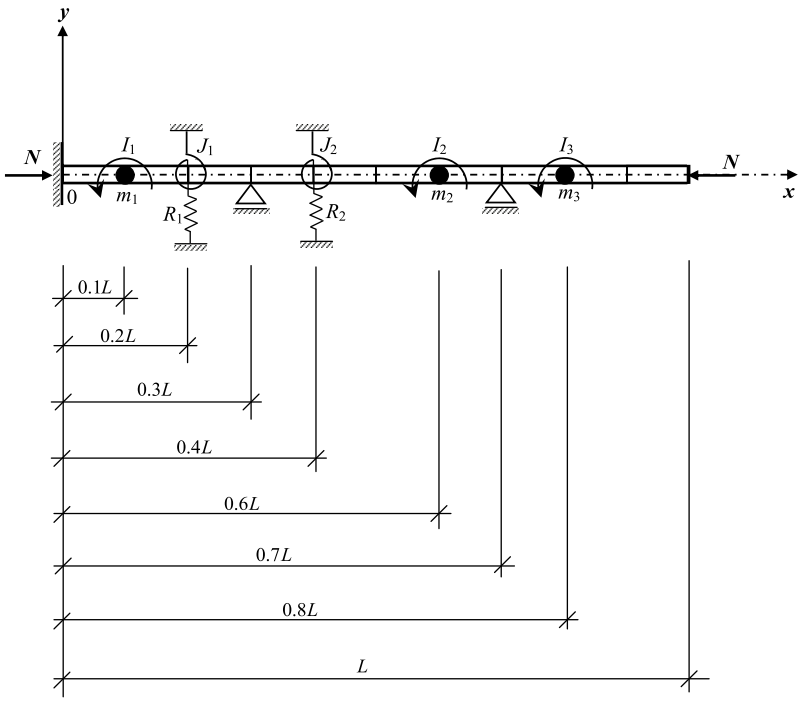


Fig. 3. A cantilevered Timoshenko beam carrying three intermediate point masses, three rotary inertias, two linear springs, two rotational springs and with two intermediate pinned supports.

rotary inertias are: $I_1 = 0.001 \cdot \left(m \cdot L^3\right)$, $I_2 = 0.002 \cdot \left(m \cdot L^3\right)$ and $I_3 = 0.003 \cdot \left(m \cdot L^3\right)$; those for the two linear springs and two rotational springs are: $R_1 = 10 \cdot \left(EI_x/L^3\right)$, $R_2 = 20 \cdot \left(EI_x/L^3\right)$ and $J_1 = 3 \cdot (EI_x/L)$, $J_2 = 4 \cdot (EI_x/L)$ located at $z_1' = 0.20$ and $z_2' = 0.40$, respectively; and two intermediate pinned supports are located at $\bar{z}_1 = 0.3$ and $\bar{z}_2 = 0.7$. The frequency parameters obtained for the first five modes are presented in (Table 1) being compared with the frequency parameters obtained for $N_r = 0$, 0.25, 0.50, and 0.75 and for $N_r = 0.75$, the first three mode shapes of the axially-loaded Timoshenko beam with pinned-pinned and clamped-free boundary conditions are presented in (Figs 4 and 5), respectively.

From (Table 1) one can sees that increasing N_r causes a decrease in the first five mode frequency parameters for two different boundary conditions, as expected. The frequency parameters obtained for the Timoshenko beam without the axial force effect in this study are a little less than the values obtained for the Bernoulli-Euler beam in the reference [21], as expected, since the shear deformation is considered in Timoshenko Beam Theory.

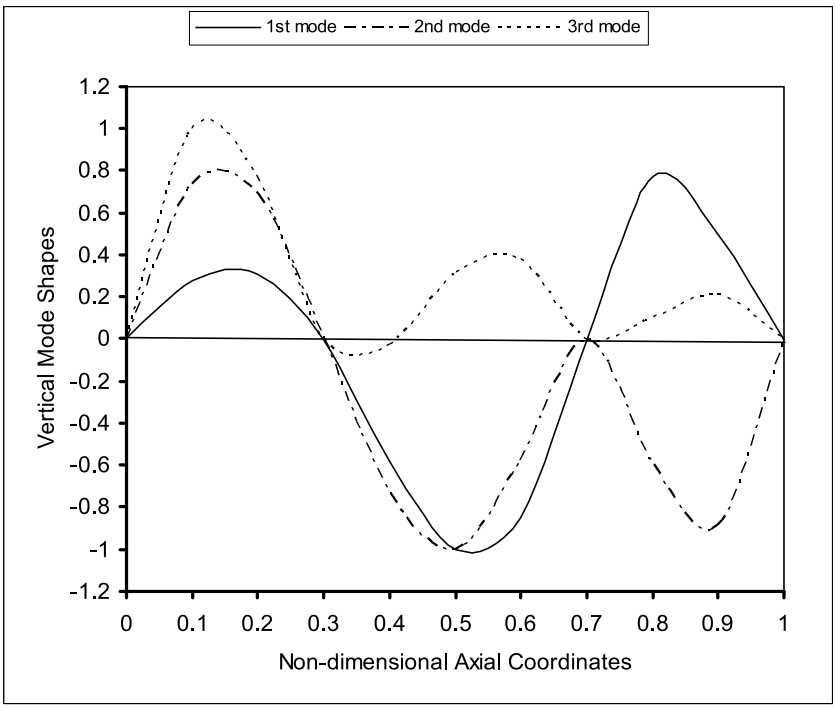


Fig. 4. The first three mode shapes of the pinned-pinned Timoshenko beam carrying three intermediate point masses, three rotary inertias, two linear springs, two rotational springs and with two intermediate pinned supports, $N_r = 0.75$.

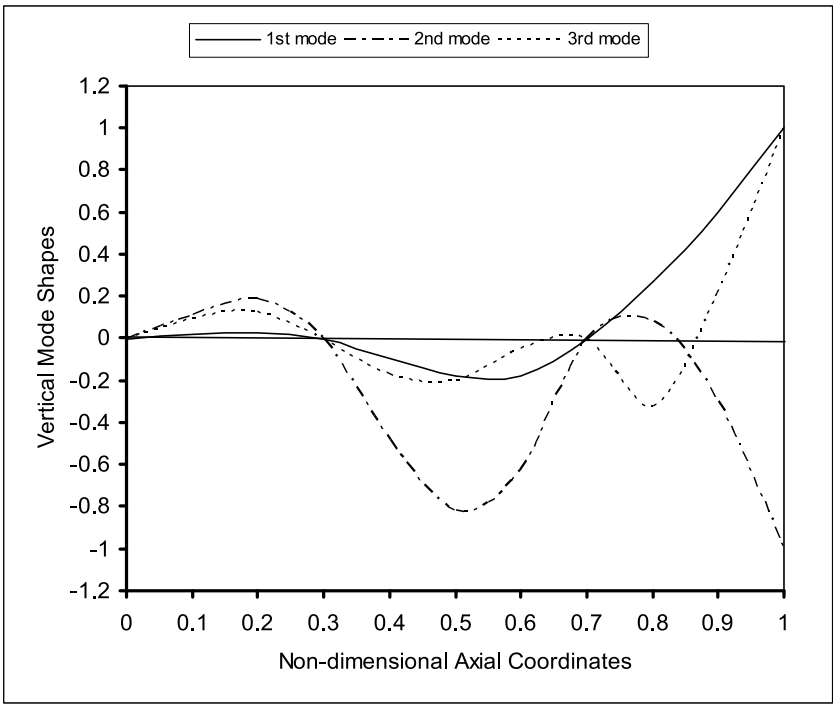


Fig. 5. The first three mode shapes of the cantilevered Timoshenko beam carrying three intermediate point masses, three rotary inertias, two linear springs, two rotational springs and with two intermediate pinned supports, $N_r = 0.75$.

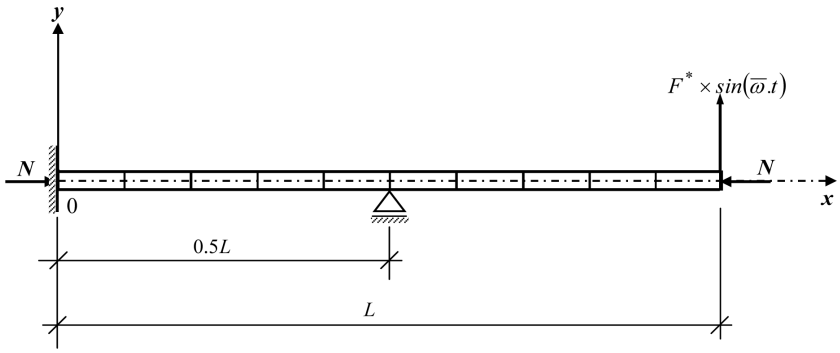


Fig. 6. A cantilevered Timoshenko beam with single intermediate pinned support and subjected to a harmonic force.

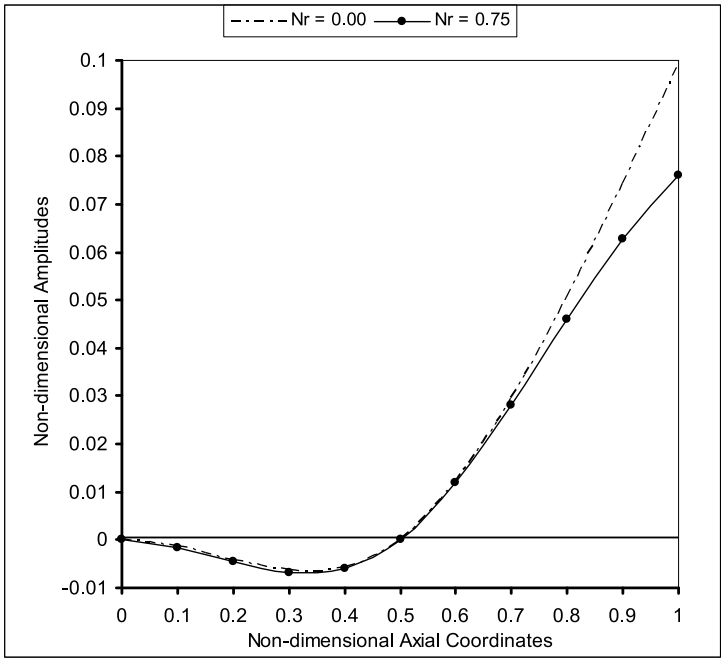


Fig. 7. The dimensionless vibration amplitudes of the cantilevered Timoshenko beam with single intermediate pinned support and subjected to a harmonic force for different values of N_r.

4.2. Forced vibration responses of the cantilevered Timoshenko beam with an intermediate pinned support and subjected to a harmonic concentrated force

In the second numerical example (see Fig. 6), the axially-loaded cantilevered Timoshenko beam with one intermediate pinned support and subjected to a harmonic concentrated force (located at $\hat{z}_1 = 1.0$) is considered. In this numerical example, the location of intermediate pinned support is at $\bar{z}_1 = 0.5$. The frequency parameter and magnitude of the applied force are $\lambda = \sqrt{5}$ and $F^* = 1.0$ N, respectively. The dimensionless vibration amplitudes $\left[\bar{y}\left(z\right) = y\left(z\right) \cdot EI_x / \left(F^* \cdot L^3\right)\right]$ obtained for the different locations of Timoshenko beam are presented in (Table 2) being compared with the dimensionless vibration amplitudes obtained for $N_r = 0, 0.25, 0.50$, and 0.75 and for $N_r =$ 0 and 0.75, the dimensionless vibration amplitudes of the cantilevered Timoshenko beam with single intermediate pinned support and subjected to a harmonic force are presented in (Fig. 7).

From (Table 2) one can sees that, as the axial compressive force acting to the beam is increased, the dimensionless vibration amplitudes of the first span of the beam are increased, but, the dimensionless vibration amplitudes of the second span are decreased, as expected. The dimensionless vibration amplitudes obtained for the Timoshenko beam

Table 2
The dimensionless vibration amplitudes at different locations of the cantilevered Timoshenko beam subjected to a harmonic concentrated force and with an intermediate pinned support for different values of N_r

Dimensionless coordinates $z = x/L$	Dimensionless vibration amplitudes, $\bar{y}(z)$ (BET) $N_r = 0.00$ Ref. [21]	(TBT) $N_r = 0.00$	$N_r = 0.25$	$N_r = 0.50$	$N_r = 0.75$
0.00	0.000000	0.000000	0.000000	0.000000	0.000000
0.10	−0.001380	−0.001468	−0.001497	−0.001529	−0.001564
0.20	−0.004136	−0.004266	−0.004353	−0.004450	−0.004557
0.30	−0.006197	−0.006324	−0.006441	−0.006571	−0.006715
0.40	−0.005501	−0.005583	−0.005661	−0.005748	−0.005847
0.50	0.000000	0.000000	0.000000	0.000000	0.000000
0.60	0.011747	0.012056	0.012024	0.012007	0.012000
0.70	0.028814	0.029455	0.029008	0.028600	0.028230
0.80	0.049712	0.050696	0.049062	0.047516	0.046054
0.90	0.073026	0.074353	0.070292	0.066452	0.062814
1.00	0.097467	0.099130	0.090900	0.083219	0.076042

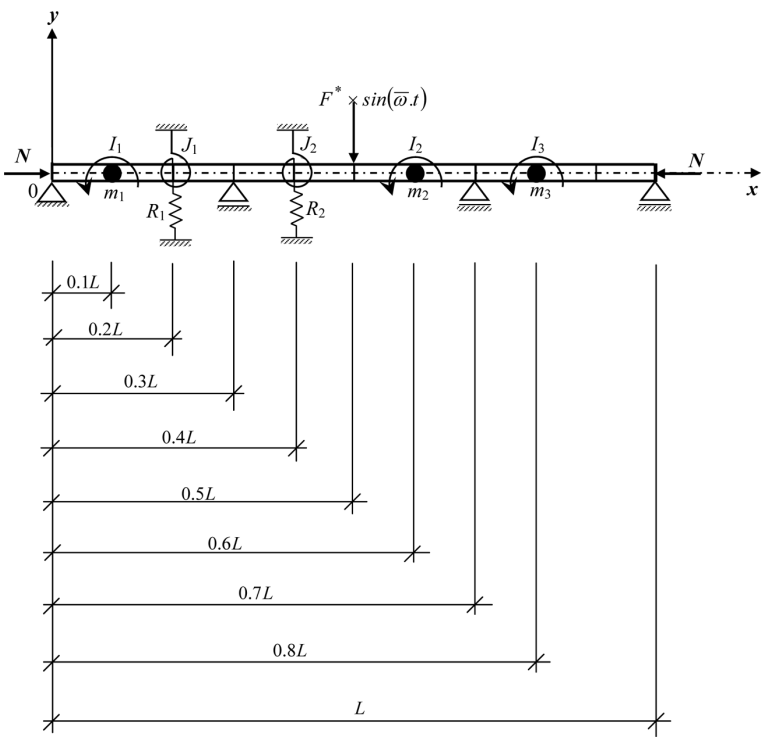


Fig. 8. The pinned-pinned Timoshenko beam shown in (Fig. 2) subjected to a harmonic force.

without the axial force effect in this study are a little more than the values obtained for the Bernoulli-Euler beam in the reference [21].

4.3. Forced vibration responses of the axially-loaded Timoshenko beam carrying three point masses, three rotary inertias, two linear springs, two rotational springs with two intermediate pinned supports and subjected to a harmonic concentrated force

In the third numerical example, the uniform Timoshenko beams shown in (Figs 2 and 3) subjected to a harmonic concentrated force located at various positions along the beam length are considered (see Figs 8 and 9). For a

Table 3
The dimensionless vibration amplitudes at different locations of the uniform pinned-pinned Timoshenko beam with two intermediate pinned supports, subjected to a harmonic concentrated force and carrying three point masses, three rotary inertias, two linear springs, two rotational springs for different values of N_r and $F^* = -1.0$ N

Dimensionless coordinates $z = x/L$	Dimensionless vibration amplitudes, $\bar{y}(z) \times 10^{-3}$			
	$N_r = 0.00$	$N_r = 0.25$	$N_r = 0.50$	$N_r = 0.75$
0.00	0.000000	0.000000	0.000000	0.000000
0.10	0.134273	0.139518	0.145091	0.151020
0.20	0.169573	0.175575	0.181936	0.188686
0.30	0.000000	0.000000	0.000000	0.000000
0.40	−0.409739	−0.419338	−0.429441	−0.440089
0.50	−0.666766	−0.681219	−0.696413	−0.712410
0.60	−0.414009	−0.423772	−0.434049	−0.444885
0.70	0.000000	0.000000	0.000000	0.000000
0.80	0.172071	0.178272	0.184849	0.191836
0.90	0.137850	0.143352	0.149204	0.155438
1.00	0.000000	0.000000	0.000000	0.000000

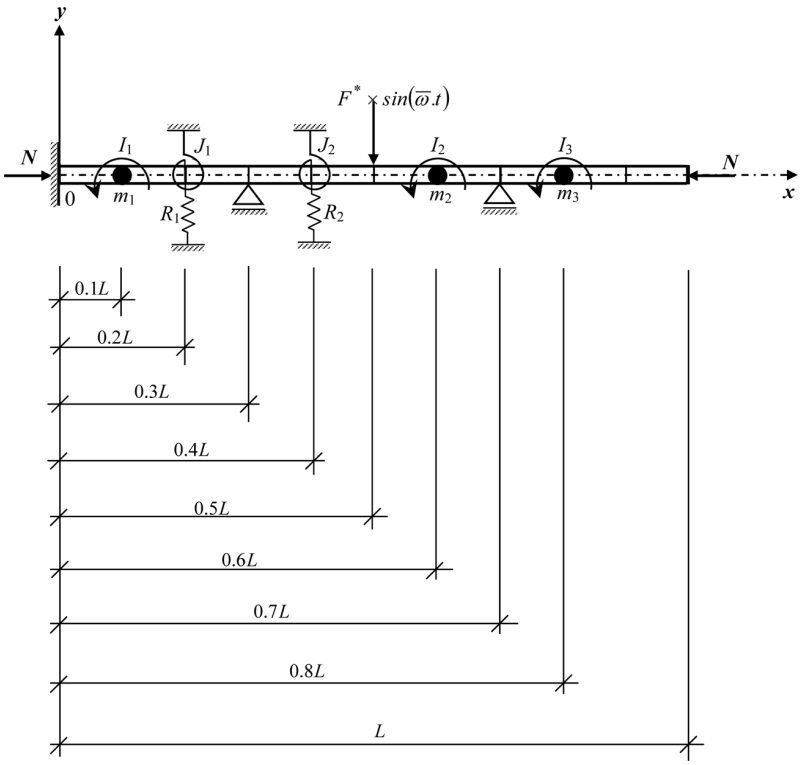


Fig. 9. The cantilevered Timoshenko beam shown in (Fig. 3) subjected to a harmonic force.

harmonic force with frequency parameter $\lambda = \sqrt{5}$ and force amplitude $F^* = -1.0$ N or $F^* = -4.0$ N located at $\hat{z}_1 = 0.5$. The dimensionless vibration amplitudes $\left[\bar{y}(z) = y(z) \cdot EI_x / \left(F^* \cdot L^3\right)\right]$ obtained for the different locations of pinned-pinned Timoshenko beam are presented in (Table 3) for $F^* = -1.0$ N, in (Table 4) for $F^* = -4.0$ N being compared with the dimensionless vibration amplitudes obtained for $N_r = 0, 0.25, 0.50$, and 0.75. The dimensionless vibration amplitudes of cantilevered Timoshenko beam are presented in (Table 5) for $F^* = -1.0$ N, in (Table 6) for $F^* = -4.0$ N being compared with the dimensionless vibration amplitudes obtained for $N_r = 0$, 0.25, 0.50, and 0.75. The dimensionless vibration amplitudes of the pinned-pinned Timoshenko beam are presented in (Fig. 10a) for different values of N_r and $F^* = -1.0$ N, in (Fig. 10b) for different values of F^* and $N_r = 0.75$. Similarly, the

Table 4
The dimensionless vibration amplitudes at different locations of the uniform pinned-pinned Timoshenko beam with two intermediate pinned supports, subjected to a harmonic concentrated force and carrying three point masses, three rotary inertias, two linear springs, two rotational springs for different values of N_r and $F^* = -4.0$ N

Dimensionless coordinates $z = x/L$	Dimensionless vibration amplitudes, $\bar{y}(z) \times 10^{-3}$			
	$N_r = 0.00$	$N_r = 0.25$	$N_r = 0.50$	$N_r = 0.75$
0.00	0.000000	0.000000	0.000000	0.000000
0.10	0.537093	0.558073	0.580364	0.604081
0.20	0.678293	0.702301	0.727743	0.754745
0.30	0.000000	0.000000	0.000000	0.000000
0.40	−1.638958	−1.677352	−1.717762	−1.760356
0.50	−2.667065	−2.724877	−2.785654	−2.849641
0.60	−1.656036	−1.695088	−1.736198	−1.779541
0.70	0.000000	0.000000	0.000000	0.000000
0.80	0.688284	0.713087	0.739396	0.767345
0.90	0.551401	0.573410	0.596818	0.621752
1.00	0.000000	0.000000	0.000000	0.000000

Table 5
The dimensionless vibration amplitudes at different locations of the uniform cantilevered Timoshenko beam with two intermediate pinned supports, subjected to a harmonic concentrated force and carrying three point masses, three rotary inertias, two linear springs, two rotational springs for different values of N_r and $F^* = -1.0$ N

Dimensionless coordinates $z = x/L$	Dimensionless vibration amplitudes, $\bar{y}(z) \times 10^{-3}$			
	$N_r = 0.00$	$N_r = 0.25$	$N_r = 0.50$	$N_r = 0.75$
0.00	0.000000	0.000000	0.000000	0.000000
0.10	0.079386	0.082091	0.084951	0.087978
0.20	0.151405	0.156423	0.161725	0.167335
0.30	0.000000	0.000000	0.000000	0.000000
0.40	−0.492855	−0.505438	−0.518696	−0.532685
0.50	−0.853524	−0.874725	−0.897057	−0.920616
0.60	−0.615733	−0.632234	−0.649628	−0.667991
0.70	0.000000	0.000000	0.000000	0.000000
0.80	0.687902	0.704425	0.721850	0.740253
0.90	1.388322	1.404107	1.420919	1.438844
1.00	2.091165	2.071629	2.052563	2.033985

dimensionless vibration amplitudes of the cantilevered Timoshenko beam are presented in (Fig. 11a) for different values of N_r and $F^* = -1.0$ N, in (Fig. 11b) for different values of F^* and $N_r = 0.75$.

It can be seen from (Tables 3 and 4) that, as the axial compressive force acting to the beam is increased, all dimensionless vibration amplitudes of the pinned-pinned Timoshenko beam are increased. From (Tables 5 and 6) one can see that, as the axial force is increased, the dimensionless vibration amplitudes of the cantilevered Timoshenko beam (apart from free end of the beam) are increased. But, the dimensionless vibration amplitudes of the free end of the cantilevered Timoshenko beam are decreased, as expected. It can be seen from the tables that, as the harmonic concentrated force applied to the beam is increased for N_r is being constant, all dimensionless vibration amplitudes of Timoshenko beams are increased for two boundary conditions.

5. Conclusion

In this study, frequency values, frequency parameters and the mode shapes for free vibration of the axially-loaded Timoshenko multi-span beam carrying a number of various concentrated elements (including point masses, rotary inertias, linear springs and rotational springs) for different values of axial compressive force. In the first numerical example, the frequency parameters are determined for Timoshenko beams with and without the axial force effect and are presented in the tables. The frequency parameters obtained for the Timoshenko beam without the axial force

Table 6
The dimensionless vibration amplitudes at different locations of the uniform cantilevered Timoshenko beam with two intermediate pinned supports, subjected to a harmonic concentrated force and carrying three point masses, three rotary inertias, two linear springs, two rotational springs for different values of N_r and $F^* = -4.0$ N

Dimensionless coordinates $z = x/L$	Dimensionless vibration amplitudes, $\bar{y}(z) \times 10^{-3}$			
	$N_r = 0.00$	$N_r = 0.25$	$N_r = 0.50$	$N_r = 0.75$
0.00	0.000000	0.000000	0.000000	0.000000
0.10	0.317545	0.328364	0.339802	0.351913
0.20	0.605620	0.625691	0.646898	0.669339
0.30	0.000000	0.000000	0.000000	0.000000
0.40	−1.971420	−2.021752	−2.074784	−2.130741
0.50	−3.414098	−3.498898	−3.588229	−3.682466
0.60	−2.462934	−2.528935	−2.598513	−2.671966
0.70	0.000000	0.000000	0.000000	0.000000
0.80	2.751607	2.817698	2.887400	2.961011
0.90	5.553290	5.616427	5.683678	5.755377
1.00	8.364662	8.286516	8.210254	8.135938

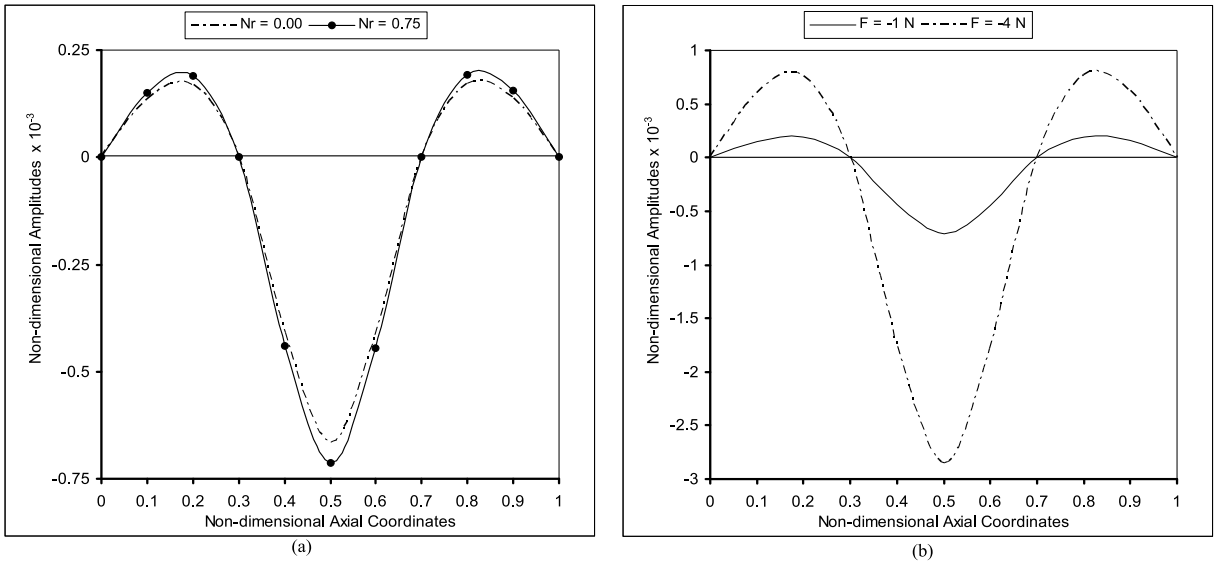
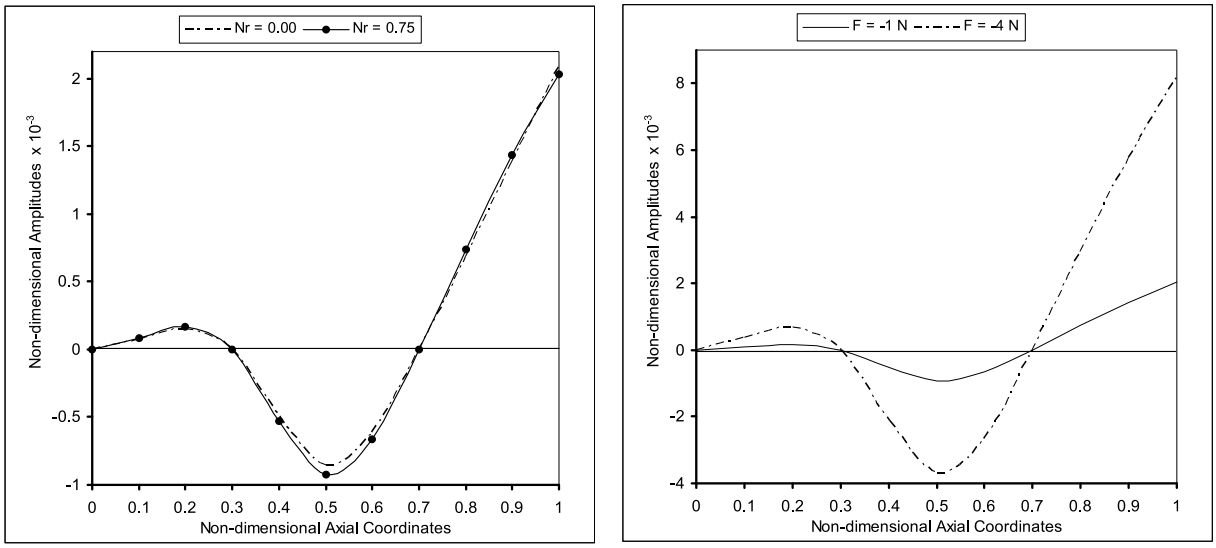


Fig. 10. The dimensionless vibration amplitudes of the pinned-pinned Timoshenko beam carrying five concentrated elements with two intermediate pinned supports and subjected to a harmonic force. a. For different values of N_r and $F^* = -1.0$ N; b. For different values of F^* and $N_r = 0.75$.

effect in this study are a little less than the values obtained for the Bernoulli-Euler beam in the reference [21]. The increase in the value of axial force also causes a decrease in the frequency parameters.

In addition, the exact frequency-response amplitudes of the axially-loaded Timoshenko multi-span beam carrying a number of various concentrated elements and subjected to a harmonic concentrated force with a specified exciting frequency are determined in this study for different values of N_r and F^*. As the axial compressive force acting to Timoshenko beam is increased, the dimensionless vibration amplitudes (apart from free end of the cantilevered beam) are increased. This result indicates that, the increasing for the axial compressive force leads to augmentation in the frequency-response amplitudes of the axially-loaded Timoshenko multi-span beam carrying a number of various concentrated elements and subjected to a harmonic concentrated force. But, in the frequency-response amplitudes of the free end of the cantilevered Timoshenko beam, the increasing for the axial compressive force leads to reduction.

As the harmonic concentrated force applied to the beam is increased for N_r is being constant, all frequency-response amplitudes of axially-loaded Timoshenko multi-span beam carrying a number of various concentrated elements and subjected to a harmonic concentrated force are increased for two boundary conditions.



(a) (b)

Fig. 11. The dimensionless vibration amplitudes of the cantilevered Timoshenko beam carrying five concentrated elements with two intermediate pinned supports and subjected to a harmonic force. a. For different values of N_r and $F^* = -1.0$ N; b. For different values of F^* and $N_r = 0.75$.

Appendix A

The details for the application of Hamilton's principle and the derivation of the equations of motion are presented below.

The virtual kinetic energy δV and the virtual potential energy $\delta\Pi$ can be written for an axially-loaded Timoshenko beam as:

$$\delta V = \int_0^L \left[m \cdot \frac{\partial y(x,t)}{\partial t} \cdot \frac{\partial \delta y(x,t)}{\partial t} + \frac{m \cdot I_x}{A} \cdot \frac{\partial \theta(x,t)}{\partial t} \cdot \frac{\partial \delta\theta(x,t)}{\partial t} \right] \cdot dx \tag{A1}$$

$$\delta \prod = \int_0^L \left[EI_x \cdot \frac{\partial \theta(x,t)}{\partial x} \cdot \frac{\partial \delta\theta(x,t)}{\partial x} + \frac{AG}{\bar{k}} \cdot \left(\frac{\partial y(x,t)}{\partial x} - \theta(x,t) \right) \cdot \left(\frac{\partial \delta y(x,t)}{\partial x} - \delta\theta(x,t) \right) \right.$$
$$\left. -N \frac{\partial y(x,t)}{\partial x} \cdot \frac{\partial \delta y(x,t)}{\partial x} \right] \cdot dx \tag{A2}$$

The virtual work δW due to the external dynamic load $F_i(t) \cdot \delta(x - x_i)$ is given by

$$\delta W = \int_0^L \left[F_i(t) \cdot \delta(x - x_i) \right] \cdot \delta y(x,t) \cdot dx \tag{A3}$$

The equations of motion for an axially-loaded Timoshenko beam are derived by applying Hamilton's principle, which is given by

$$\delta \int_{t_1}^{t_2} \int_0^L L_g \cdot dx \cdot dt = 0 \tag{A4}$$

where

$$L_g = V - \prod + W \tag{A5}$$

is termed as the Lagrangian density function.

By taking the variation of the Lagrangian density function; integrating Eq. (A4) by parts, and then collecting all the terms of the integrand with respect to $\delta y(x,t)$ and $\delta\theta\,(x,t)$, one can derive the following equations of motion as the coefficients of $\delta y(x,t)$ and $\delta\theta\,(x,t)$:

$$EI_x \cdot \frac{\partial^2\theta(x,t)}{\partial x^2} + \frac{AG\cdot}{\bar{k}} \cdot \left(\frac{\partial y\,(x,t)}{\partial x} - \theta\,(x,t)\right) - m \cdot \frac{\partial^2\theta\,(x,t)}{\partial t^2} = F_i\,(t) \cdot \delta\,(x - x_i) \tag{A6}$$

$$\frac{AG\cdot}{\bar{k}} \cdot \left(\frac{\partial^2 y\,(x,t)}{\partial x^2} - \frac{\partial\theta\,(x,t)}{\partial x}\right) - N \cdot \frac{\partial^2 y\,(x,t)}{\partial x^2} - \frac{m \cdot I_x}{A} \cdot \frac{\partial^2 y\,(x,t)}{\partial t^2} = 0 \quad (0 \leqslant x \leqslant L) \tag{A7}$$

Appendix B

From Eqs (6)and (7), the boundary conditions for the pth intermediate concentrated element can be written in matrix equation form as:

$$\left[B_{p'}\right] \cdot \{C_{p'}\} = \{0\} \tag{B1}$$

where

$$\left\{C'_p\right\}^T = \left\{C_{p'-1,1}\; C_{p'-1,2}\; C_{p'-1,3}\; C_{p'-1,4}\; C_{p',1}\; C_{p',2}\; C_{p',3}\; C_{p',4}\right\} \tag{B2}$$

$$\left[B'_p\right] = \begin{array}{c} \begin{array}{cccccccc} 4p'-3 & 4p'-2 & 4p'-1 & 4p' & 4p'+1 & 4p'+2 & 4p'+3 & 4p'+4 \end{array} \\ \begin{bmatrix} ch_1 & sh_1 & cs_2 & sn_2 & -ch_1 & -sh_1 & -cs_2 & -sn_2 \\ K_3 \cdot sh_1 & K_3 \cdot ch_1 & K_4 \cdot sn_2 & -K_4 \cdot cs_2 & -K_3 \cdot sh_1 & -K_3 \cdot ch_1 & -K_4 \cdot sn_2 & K_4 \cdot cs_2 \\ K_7 & K_8 & K_9 & K_{10} & -K_1 \cdot ch_1 & -K_1 \cdot sh_1 & K_2 \cdot cs_2 & K_2 \cdot sn_2 \\ K_{11} & K_{12} & K_{13} & K_{14} & -K_5 \cdot sh_1 & -K_5 \cdot ch_1 & -K_6 \cdot sn_2 & K_6 \cdot cs_2 \end{bmatrix} \end{array} \begin{array}{c} \\ 4p'-1 \\ 4p' \\ 4p'+1 \\ 4p'+2 \end{array} \tag{B3}$$

$$ch_1 = \cosh\left(D_1 \cdot z_{p'}\right); \quad ch_2 = \cosh\left(D_2 \cdot z_{p'}\right); \quad sh_1 = \sinh\left(D_1 \cdot z_{p'}\right); \quad sh_2 = \sinh\left(D_2 \cdot z_{p'}\right);$$

$$cs_1 = \cos\left(D_1 \cdot z_{p'}\right); \quad cs_2 = \cos\left(D_2 \cdot z_{p'}\right); \quad sn_1 = \sin\left(D_1 \cdot z_{p'}\right); \quad sn_2 = \sin\left(D_2 \cdot z_{p'}\right);$$

$$K_7 = K_1 \cdot ch_1 - K_3 \cdot \left(I_p \cdot \bar{\omega}^2 - J_p\right) \cdot sh_1; \quad K_8 = K_1 \cdot sh_1 - K_3 \cdot \left(I_p \cdot \bar{\omega}^2 - J_p\right) \cdot ch_1;$$

$$K_9 = -K_2 \cdot cs_2 - K_4 \cdot \left(I_p \cdot \bar{\omega}^2 - J_p\right) \cdot sn_2; \quad K_{10} = -K_2 \cdot sn_2 + K_4 \cdot \left(I_p \cdot \bar{\omega}^2 - J_p\right) \cdot cs_2;$$

$$K_{11} = K_5 \cdot sh_1 - \left(R_p - m_p \cdot \bar{\omega}^2\right) \cdot ch_1; \quad K_{12} = K_5 \cdot ch_1 - \left(R_p - m_p \cdot \bar{\omega}^2\right) \cdot sh_1;$$

$$K_{13} = K_6 \cdot sn_2 - \left(R_p - m_p \cdot \bar{\omega}^2\right) \cdot cs_2; \quad K_{14} = -K_6 \cdot cs_2 - \left(R_p - m_p \cdot \bar{\omega}^2\right) \cdot sn_2$$

From Eqs (6) and (7), the boundary conditions for the rth intermediate pinned support can be written in matrix equation form as:

$$\left[B_{r'}\right] \cdot \{C_{r'}\} = \{0\} \tag{B4}$$

where

$$\left\{C'_r\right\}^T = \left\{C_{r'-1,1}\; C_{r'-1,2}\; C_{r'-1,3}\; C_{r'-1,4}\; C_{r',1}\; C_{r',2}\; C_{r',3}\; C_{r',4}\right\} \tag{B5}$$

$$\left[B'_r\right] = \begin{array}{c} \begin{array}{cccccccc} 4r'-3 & 4r'-2 & 4r'-1 & 4r' & 4r'+1 & 4r'+2 & 4r'+3 & 4r'+4 \end{array} \\ \begin{bmatrix} chr_1 & shr_1 & csr_2 & snr_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & chr_1 & shr_1 & csr_2 & snr_2 \\ K_3 \cdot shr_1 & K_3 \cdot chr_1 & K_4 \cdot snr_2 & -K_4 \cdot csr_2 & -K_3 \cdot shr_1 & -K_3 \cdot chr_1 & -K_4 \cdot snr_2 & K_4 \cdot csr_2 \\ K_1 \cdot chr_1 & K_1 \cdot shr_1 & -K_2 \cdot csr_2 & -K_2 \cdot snr_2 & -K_1 \cdot chr_1 & -K_1 \cdot shr_1 & K_2 \cdot csr_2 & K_2 \cdot snr_2 \end{bmatrix} \end{array} \begin{array}{c} \\ 4r'-1 \\ 4r' \\ 4r'+1 \\ 4r'+2 \end{array} \tag{B6}$$

$$chr_1 = \cosh\left(D_1 \cdot z_{r'}\right); \quad chr_2 = \cosh\left(D_2 \cdot z_{r'}\right); \quad shr_1 = \sinh\left(D_1 \cdot z_{r'}\right); \quad shr_2 = \sinh\left(D_2 \cdot z_{r'}\right);$$

$$csr_1 = \cos\left(D_1 \cdot z_{r'}\right); \quad csr_2 = \cos\left(D_2 \cdot z_{r'}\right); \quad snr_1 = \sin\left(D_1 \cdot z_{r'}\right); \quad snr_2 = \sin\left(D_2 \cdot z_{r'}\right)$$

From Eqs (6) and (7), the boundary conditions for the fth intermediate harmonic concentrated force can be written in matrix equation form as:

$$\left[B_{f'}\right]\cdot\{C_{f'}\}=\{D_{f'}\} \tag{B7}$$

where

$$\{C_{f'}\}^T=\left\{C_{f'-1,1}\;C_{f'-1,2}\;C_{f'-1,3}\;C_{f'-1,4}\;C_{f',1}\;C_{f',2}\;C_{f',3}\;C_{f',4}\right\} \tag{B8}$$

$$\{D_{f'}\}^T=\left\{0\;0\;0\;-F^*\right\} \tag{B9}$$

$$[B_{f'}]=\begin{array}{c} \begin{array}{cccccccc} 4f'-3 & 4f'-2 & 4f'-1 & 4f' & 4f'+1 & 4f'+2 & 4f'+3 & 4f'+4 \end{array} \\ \begin{bmatrix} chf_1 & shf_1 & csf_2 & snf_2 & -chf_1 & -shf_1 & -csf_2 & -snf_2 \\ K_3\cdot shf_1 & K_3\cdot chf_1 & K_4\cdot snf_2 & -K_4\cdot csf_2 & -K_3\cdot shf_1 & -K_3\cdot chf_1 & -K_4\cdot snf_2 & K_4\cdot csf_2 \\ K_1\cdot chf_1 & K_1\cdot shf_1 & -K_2\cdot csf_2 & -K_2\cdot snf_2 & -K_1\cdot chf_1 & -K_1\cdot shf_1 & K_2\cdot csf_2 & K_2\cdot snf_2 \\ K_5\cdot shf_1 & K_5\cdot chf_1 & K_6\cdot snf_2 & -K_6\cdot csf_2 & -K_5\cdot shf_1 & -K_5\cdot chf_1 & -K_6\cdot snf_2 & K_6\cdot csf_2 \end{bmatrix} \end{array} \begin{array}{c} 4f'-1 \\ 4f' \\ 4f'+1 \\ 4f'+2 \end{array} \tag{B10}$$

$$chf_1=\cosh\left(D_1\cdot z_{f'}\right);\quad chf_2=\cosh\left(D_2\cdot z_{f'}\right);\quad shf_1=\sinh\left(D_1\cdot z_{f'}\right);\quad shf_2=\sinh\left(D_2\cdot z_{f'}\right);$$

$$csf_1=\cos\left(D_1\cdot z_{f'}\right);\quad csf_2=\cos\left(D_2\cdot z_{f'}\right);\quad snf_1=\sin\left(D_1\cdot z_{f'}\right);\quad snf_2=\sin\left(D_2\cdot z_{f'}\right)$$

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