A closed form solution for nonlinear oscillators frequencies using amplitude-frequency formulation

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Abstract. Many nonlinear systems in industry including oscillators can be simulated as a mass-spring system. In reality, all kinds of oscillators are nonlinear due to the nonlinear nature of springs. Due to this nonlinearity, most of the studies on oscillation systems are numerically carried out while an analytical approach with a closed form expression for system response would be very useful in different applications. Some analytical techniques have been presented in the literature for the solution of strong nonlinear oscillators as well as approximate and numerical solutions. In this paper, Amplitude-Frequency Formulation (AFF) approach is applied to analyze some periodic problems arising in classical dynamics. Results are compared with another approximate analytical technique called Energy Balance Method developed by the authors (EBM) and also numerical solutions. Close agreement of the obtained results reveal the accuracy of the employed method for several practical problems in engineering.

Keywords: Amplitude Frequency Formulation (AFF), nonlinear oscillators, analytical investigations, Energy Balanced Method (EBM)

1. Introduction

Due to difficulties in solving nonlinear differential equations analytically the exploration of plausible solutions to nonlinear differential problems has always been an ongoing challenge for scientists and engineers [1–3]. For examples some problems involves oscillators [4–6], where we need to know the frequency of the oscillator, and the Improved Amplitude Frequency Formulation Method (IAFF) leads to obtain an analytical expression [7–9]. Since we are not provided with proper boundary conditions in the problem, it is not possible to use the Homotopy Perturbation Method, (HPM) [11–14] or the Variational Iteration Method (VIM) [15,16]. Zhang [17], presented an application of He's amplitude-frequency formulation in order to show the application of this method to nonlinear oscillations even to some equations with fractional term. Recently Geng and Cai [18] modified the method and made a discussion on the crucial point of the solutions to improve the amplitude-frequency formulation. Other applications of approximate methods to nonlinear oscillation problems can also be found within [19–22].

In real-world systems, the second law of thermodynamics dictates that there is some continual and inevitable conversion of energy into the thermal energy of the environment. Thus, oscillations tend to decay (become “damped”)

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with time unless there is some net source of energy into the system. The simplest description of this decay process can be illustrated by oscillation decay of the harmonic oscillator. In addition, an oscillating system may be subjected to some external force, as when an AC circuit is connected to an outside power source. In this case the oscillation is said to be driven. Some systems can be excited by energy transfer from the environment. This transfer typically occurs where systems are embedded in some fluid flows. For example, the phenomenon of flutter in aerodynamics occurs when an arbitrarily small displacement of an aircraft wing (from its equilibrium) results in an increase in the angle of attack of the wing on the air flow and a consequential increase in lift coefficient, leading to a still greater displacement. At sufficiently large displacements, the stiffness of the wing dominates to provide the restoring force that enables an oscillation. Pendulums are samples of oscillators; Pendulums in clocks are usually made of a weight or bob suspended by a rod of wood or metal [23,24]. In quality clocks the bob is made as heavy as the suspension can support and the movement can drive, since this improves the regulation of the clock. Instead of hanging from a pivot, clock pendulums are usually supported by a short straight spring of flexible metal ribbon. This avoids the friction and play caused by a pivot, and the slight bending force of the spring merely adds to the pendulum's restoring force.

The largest source of error in early pendulums was slight changes in length due to thermal expansion and contraction of the pendulum rod with changes in ambient temperature. This was discovered when people noticed that pendulum clocks ran slower in summer, by as much as a minute per week [25]. A pendulum with a steel rod will get about 6.3 parts per million (*ppm*) longer with each 1°F temperature increase, causing it to lose about 0.27 seconds/day, or 16 seconds/day for a 60°F (33°C) change. Another sample of oscillators is spring-mass system. The basic, open-loop accelerometer consists of a mass attached to a spring and constrained to move only in-line with the spring. Acceleration causes deflection of the mass and the offset distance is measured. The acceleration is derived from the values of deflection distance, mass, and the spring constant. The system must also be damped to avoid oscillation. A closed-loop accelerometer achieves higher performance by using a feedback loop to cancel the deflection, thus keeping the mass nearly stationary. Whenever the mass deflects, the feedback loop causes an electric coil to apply an equally negative force on the mass, canceling the motion. Acceleration is derived from the amount of negative force applied. Because the mass barely moves, the non-linearity of the spring and damping system are greatly reduced. In addition, this accelerometer provides for increased bandwidth past the natural frequency of the sensing element [26].

In this paper it is tried present an analytical approach to analyze some practical nonlinear oscillators using amplitude frequency formulation. The results of AFF is compared with fourth-order Runge–Kutta as numerical solution (Appendix 2) and another approximate analytical technique namely EBM (Appendix 1). On the other hand the results are compared with some other references such as Ganji et al. [27] and Wu et al. [28].

2. Mathematical technique

He's amplitude-frequency formulation approach is one of the newest tools that is appropriate for solving some non-linear differential equations.

We can use the following equation for a generalized non-linear oscillator:

$$u'' + f(u) = 0, \quad u(0) = A, \quad u'(0) = 0 \quad (1)$$

We use two trial functions, substitute them into the main equation, then rearrange it and determine some parameters such as: R_1, R_2, ω .

Considering the trial functions as:

$$u_1(t) = A \cos \omega_1 t \quad (2)$$

$$u_2(t) = A \cos \omega_2 t \quad (3)$$

The residuals can be obtained as:

$$R_1(t) = -A\omega_1^2 \cos \omega_1 t + f(A \cos \omega_1 t) \quad (4)$$

$$R_2(t) = -A\omega_2^2 \cos \omega_2 t + f(A \cos \omega_2 t) \quad (5)$$

Fig. 1. Geometry of case 1.

The original frequency-amplitude formulation reads [29–32]

$$\omega^2 = \frac{\omega_1^2 R_2 - \omega_2^2 R_1}{R_2 - R_1} \quad (6)$$

Generally we can locate at

$$\cos \omega_1 t = \cos \omega_2 t = k \quad (7)$$

Its approximate solution reads

$$u(t) = A \cos \omega t \quad (8)$$

In view of the approximate solution, Eq. (8), we re-write Eq. (1) in the form

$$u'' + f(u) \pm \omega^2 u = 0 \rightarrow u'' + \omega^2 u = \omega^2 u - f(u) \quad (9)$$

If, by chance, Eq. (8) is the exact solution, then the right hand side of Eq. (9) is vanishing completely. Since our approach is only an approximation to the exact solution, we set

$$\int_0^{T/4} (\omega^2 u(t) - f(u(t))) \times \cos(\omega t) dt = 0 \quad (10)$$

Where $T = \frac{2\pi}{\omega}$ and by substituting (8) in (10), we can obtain k and then using Eq. (6) gives ω .

3. Applications of AFF

3.1. Case 1

In this problem we have a rigid frame (as shown in Fig. 1) which is forced to rotate at the fixed rate Ω (see Appendix 3).

While the frame rotates, the simple pendulum oscillates [33].

The governing equation can be simply derived as follows:

$$\frac{d^2 \theta}{dt^2} + (1 - \Lambda \cos \theta) \sin \theta = 0, \quad \theta(0) = A, \quad \frac{d\theta}{dt}(0) = 0 \quad (11)$$

Where

$$\Lambda = \frac{\Omega^2 r}{g}$$

The centrifugal acceleration from rotating reference frames can be used which is a readily-observable physical fact. The magnitude and direction of the centrifugal acceleration varies from place to place in the rotating frame. Therefore it is called the centrifugal field which is zero at the pivot. It should be noted that, by going farther from the pivot, the centrifugal acceleration increases. It is everywhere directed radially outwards from the pivot. Centrifugal force has relation with Coriolis force in rotating frameworks [34,35].

Pendulums require great mechanical stability: a length change of only 0.02%, 1/5 millimeter in a clock pendulum, will cause an error of a minute per week [36].

3.1.1. Solution of case 1 using frequency formulation

The above mentioned approach is applied herein to the non-linear differential equation introduced. The equation takes the form

$$\frac{d^2}{dt^2}u(t) + (1 - \Lambda \cos u(t)) \sin u(t) = 0 \quad (12)$$

Most useful trial-functions are [37]

$$u_1(t) = A \cos t \quad (13)$$

$$u_2(t) = A \cos 2t \quad (14)$$

Substituting u_1 , u_2 in Eq. (12) separately results in R_1 , R_2 :

$$\begin{aligned} R_1(t) &= \frac{d^2}{dt^2}(A \cos t) + (1 - \Lambda \cos(A \cos t)) \sin(A \cos t) = \\ &= -A \cos t + (1 - \Lambda \cos(A \cos t)) \sin(A \cos t) \end{aligned} \quad (15)$$

And

$$\begin{aligned} R_2(t) &= \frac{d^2}{dt^2}(A \cos 2t) + (1 - \Lambda \cos(A \cos 2t)) \sin(A \cos 2t) = \\ &= -4A \cos 2t + (1 - \Lambda \cos(A \cos 2t)) \sin(A \cos 2t) \end{aligned} \quad (16)$$

So

$$\omega^2 = \frac{-4A \cos 2t + (1 - \Lambda \cos(A \cos 2t)) \sin(A \cos 2t) + 4A \cos t - 4(1 - \Lambda \cos(A \cos t)) \sin(A \cos t)}{-4A \cos 2t + (1 - \Lambda \cos(A \cos 2t)) \sin(A \cos 2t) + A \cos t - (1 - \Lambda \cos(A \cos t)) \sin(A \cos t)} \quad (17)$$

Now we form this equation and put $\cos t = \cos 2t = k$, so:

$$R_1 = -Ak + (1 - \Lambda \cos(Ak)) \sin(Ak) \quad (18)$$

$$R_2 = -4Ak + (1 - \Lambda \cos(Ak)) \sin(Ak) \quad (19)$$

The period may be obtained as:

$$\omega = \sqrt{-\frac{(-1 + \Lambda \cos(Ak)) \sin(Ak)}{Ak}} \quad (20)$$

Write the following integral as Eq. (10):

$$\begin{aligned} &\int_0^{\frac{\pi}{4}} \left[\left(\left(-\frac{(-1 + \Lambda \cos(Ak)) \sin(Ak) \times \cos\left(\sqrt{-\frac{(-1 + \Lambda \cos(Ak)) \sin(Ak)}{Ak}} \times t\right)}{k} \right) \right. \right. \\ &\quad \left. \left. - \left(1 - \Lambda \cos \left(A \cos \left(\sqrt{-\frac{(-1 + \Lambda \cos(Ak)) \sin(Ak)}{Ak}} \times t \right) \right) \right) \right) \times \right. \\ &\quad \left. \sin \left(A \cos \left(\sqrt{-\frac{(-1 + \Lambda \cos(Ak)) \sin(Ak)}{Ak}} \times t \right) \right) \right) \times \cos \sqrt{-\frac{(-1 + \Lambda \cos(Ak)) \sin(Ak)}{Ak}} t \right] dt \end{aligned} \quad (21)$$

Fig. 2. Comparison of periods between three methods (case 1).

Where $g = 9.81 \frac{m}{s^2}$, $r = 1m$, $\Omega = 3 \frac{rad}{s}$, $\Lambda = 0.9174$, $A = \frac{\pi}{18}$

If we equate Eq. (21) to zero, we can find k as, $k = -0.8657421586$

We may then substitute k in Eq. (20) to find ω :

$$\omega_{AFF} = 0.7599599730 \sqrt{\frac{2 \cdot \sin(0.8657421586 \cdot A) - \Lambda \cdot \sin(1.731484317 \cdot A)}{A}} \quad (22)$$

Another analytical technique for this kind of problems used previously in the technical literature is the Energy Balance Method (EBM). This method is described briefly in the Appendix 1.

Using EBM we can find the parameter above as:

$$\omega_{EBM} = \frac{2}{A} \sqrt{\cos\left(\frac{\sqrt{2}}{2}A\right) - \cos A + \frac{\Lambda}{2} \left(\cos^2 A - \cos^2\left(\frac{\sqrt{2}}{2}A\right)\right)} \quad (23)$$

The comparison of results obtained by AFF, EBM, analytical solution and also numeric is illustrated in Figs 2 and 3. The analytical solution can be obtained using the energy integral as follows. From the Eq. (11) it can be written:

$$\frac{1}{2}\dot{\theta}^2 + \Pi = 0, \Pi = -\frac{1}{2}(\cos A - \cos \theta)(\Lambda \cos A + \Lambda \cos \theta - 2) \quad (24)$$

where the oscillation period is:

$$T = 4 \int_0^A \frac{d\theta}{\sqrt{-2\Pi}} \quad (25)$$

and the oscillation frequency can be easily obtained by:

$$\omega = \frac{2\pi}{T} \quad (26)$$

As it was noticed before, we can't consider every quantity for A . Because for some amounts of A , the value of ω^2 becomes negative. By continuing this procedure, the amplitudes of ω decreases and tends to zero.

3.2. Case 2

Consider the following nonlinear oscillator [33].

Fig. 3. Comparison of AFF and EBM solution with numerical results for $\theta(t)$ (case 1).

Fig. 4. Geometry of case 2.

This oscillator is very applicable in automobile design where a horizontal motion is converted into a vertical one or vice versa.

The equation of motion and appropriate initial conditions for this case can be gained as,

$$(1 + Ru(t)^2) \left(\frac{d^2}{dt^2} u(t) \right) + Ru(t) \left(\frac{d}{dt} u(t) \right)^2 + \omega_0^2 u(t) + \frac{1}{2} \frac{Rgu(t)^3}{l} = 0 \quad (27)$$

$$u(0) = A, \frac{du}{dt}(0) = 0$$

Where

$$\omega_0^2 = \frac{k}{m_1} + \frac{Rg}{l}, \quad R = \frac{m_2}{m_1} \quad (28)$$

The specific dynamics of spring-mass system are described mathematically by the simple harmonic oscillator and the regular periodic motion is known as simple harmonic motion. In the spring-mass system, oscillations occur because, at the static equilibrium displacement, the mass has kinetic energy which is converted into potential energy stored in the spring at the extremes of its path. The spring-mass system illustrates some common features of oscillation,

namely the existence of equilibrium and the presence of a restoring force getting stronger when the system deviates from equilibrium.

3.2.1. Solution of case 2 using AFF

According to He's amplitude frequency formulation we choose two trial functions: $u_1(t) = A \cos(t)$, $u_2(t) = A \cos(2t)$ substituting u_1, u_2 into Eq. (27) we obtain, respectively, the following residuals:

$$R_1 = (1 + RA^2 \cos^2 t) \left(\frac{d^2}{dt^2} (A \cos t) \right) + RA \cos t \left(\frac{d}{dt} (A \cos t)^2 \right) + \omega_0 A \cos t + \frac{RgA^3 \cos^3 t}{l} - (1 + RA^2 \cos^2 t) A \cos t + RA^3 \cos t \sin^2 t + \omega_0 A \cos t \quad (29)$$

And

$$R_2 = (1 + RA^2 \cos^2 2t) \left(\frac{d^2}{dt^2} (A \cos 2t) \right) + RA \cos t \left(\frac{d}{dt} (A \cos 2t)^2 \right) + \omega_0 A \cos 2t + \frac{RgA^3 \cos^3 2t}{l} - (1 + RA^2 \cos^2 2t) A \cos 2t + RA^3 \cos 2t \sin^2 2t + \omega_0 A \cos 2t \quad (30)$$

The angular rate is:

$$\begin{aligned} \omega^2 = & (-4(1 + RA^2 \cos(2t)^2) A \cos(2t) + 4RA^3 \cos(2t) \sin(2t)^2 \\ & + \omega_0^2 A \cos(2t) + \frac{1}{2} \frac{RgA^3 \cos(2t)^3}{l} + 4(1 + RA^2 \cos(t)^2) A \cos(t) - 4RA^3 \cos(t) \\ & \sin(t)^2 - 4\omega_0^2 A \cos(t) - \frac{2RgA^3 \cos(t)^3}{l}) / (-4(1 + RA^2 \cos(2t)^2) A \cos(2t) \\ & + 4RA^3 \cos(2t) \sin(2t)^2 + \omega_0^2 A \cos(2t) + \frac{1}{2} \frac{RgA^3 \cos(2t)^3}{l} \\ & + (1 + RA^2 \cos(t)^2) A \cos(t) - RA^3 \cos(t) \sin(t)^2 - \omega_0^2 A \cos(t) - \frac{1}{2} \frac{RgA^3 \cos(t)^3}{l}) \end{aligned} \quad (31)$$

Now we form this equation and put $\cos t = \cos 2t = k$, so:

$$\omega = \frac{\sqrt{2} \sqrt{\frac{2\omega_0^2 l + RgA^2 k^2}{l(1+2RA^2 k^2 - RA^2)}}}{2} \quad (32)$$

We choose $u(t) = A \cos(\omega t)$ and put it into the Eqs (10) and (27), then we determine the following integral:

$$\begin{aligned} & \int_0^{T/4} \left[\left(\left(1 + RA^2 \cos \left(\frac{\sqrt{2} \sqrt{\frac{2\omega_0^2 l + RgA^2 k^2}{l(1+2RA^2 k^2 - RA^2)}} \times t \right) \right)^2 \right) \left(\frac{d^2}{dt^2} \left(A \cos \left(\frac{\sqrt{2} \sqrt{\frac{2\omega_0^2 l + RgA^2 k^2}{l(1+2RA^2 k^2 - RA^2)}} \times t \right) \right) \right) \right) \\ & + RA \cos \left(\frac{\sqrt{2} \sqrt{\frac{2\omega_0^2 l + RgA^2 k^2}{l(1+2RA^2 k^2 - RA^2)}} \times t \right) \left(\frac{d}{dt} \left(A \cos \left(\left(\frac{1}{2} \left(\frac{\sqrt{2} \sqrt{\frac{2\omega_0^2 l + RgA^2 k^2}{l(1+2RA^2 k^2 - RA^2)}} \times t \right) \right) \right) \right) \right) \right) \\ & + \omega_0^2 A \cos \left(\frac{\sqrt{2} \sqrt{\frac{2\omega_0^2 l + RgA^2 k^2}{l(1+2RA^2 k^2 - RA^2)}} \times t \right) + \frac{1}{2} \frac{RgA^3 \cos \left(\frac{\sqrt{2} \sqrt{\frac{2\omega_0^2 l + RgA^2 k^2}{l(1+2RA^2 k^2 - RA^2)}} \times t \right)^3}{l} \times \cos(\omega t) \right] dt \end{aligned} \quad (33)$$

Where $g = 9.81 \frac{m}{s^2}$, $k = 200 \frac{N}{m}$, $m_1 = 10kg$, $m_2 = 1kg$, $l = 0.5m$, $A = 0.5 \tan \frac{\pi}{18}$

We can find k if we put Eq. (33) equal zero. So we have:

Fig. 5. Comparison of periods between two methods (case 2).

Fig. 6. Comparison of AFF and EBM solution with numerical solution for $u(t)$ (case 2).

$$k^2 = \frac{3}{4} \quad (34)$$

Now, the exact expression for ω can be found by substituting k into Eq. (32). Subsequently we have:

$$\omega_{AFF} = \frac{1}{2} \times \sqrt{\frac{8\omega_0^2 l + 3A^2 Rg}{l(2 + RA^2)}} \quad (35)$$

Another analytical technique for this kind of problems used previously in the technical literature is the Energy Balance Method (EBM). Using EBM we can find the parameter above as (please see the Appendix 1):

$$\omega_{EBM} = \sqrt{\frac{\frac{1}{4}\omega_0^2 A^2 + \frac{3Rg}{32l} A^4}{\frac{1}{4}A^2 + \frac{1}{8}A^4}} \quad (36)$$

To evaluate the ability of the method and show how the value of mass ratio affects the frequency, the variations of the frequency versus R is demonstrated in Fig. 5 and comparison of the results obtained by AFF, EBM and numerical solution is illustrated in Fig. 6.

The results obtained here show the high-efficiency and accuracy of the AFF although this method is very simple. Utilizing this method can be very useful in different problems arising in engineering.

4. Conclusions

Amplitude Frequency Formulation (AFF) was used to solve analytically some practical nonlinear equations of oscillators. Two examples were chosen and solved using AFF. The influence of constant parameters on system response and frequency was studied. It was shown that AFF results are in very good agreement with those obtained by EBM and fourth-order Runge–Kutta. It can be concluded that AFF is very accurate technique which is valid for a wide range of constant parameters and oscillation domain. The same approach can be used and extended to solve analytically many nonlinear problems arising in nonlinear dynamics, vibration and also fluid dynamics.

Appendix 1. Description of energy balance method

In the present paper, we consider a general nonlinear oscillator in the form:

$$u'' + f(u(t)) = 0 \quad (\text{Ap.1.1})$$

in which u and t are generalized dimensionless displacement and time variables, respectively. Its variational principle can be easily obtained:

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + F(u) \right) dt \quad (\text{Ap.1.2})$$

Where $T = \frac{2\pi}{\omega}$ is period of the nonlinear oscillator, $F(u) = \int f(u)du$. Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + F(u) + F(A) \quad (\text{Ap.1.3})$$

$$R(t) = \frac{1}{2}u'^2 + F(u) - F(A) = 0 \quad (\text{Ap.1.4})$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation, A . So let us consider such initial conditions:

$$u(0) = 0, u'(0) = 0 \quad (\text{Ap.1.5})$$

We use the following trial function to determine the angular frequency ω :

$$u(t) = A \cos(\omega t) \quad (\text{Ap.1.6})$$

Substituting (1.6) into u term of (1.4), yield:

$$R(t) = \frac{1}{2}\omega^2 A^2 \sin^2 \omega t + F(A \cos \omega t) - F(A) = 0 \quad (\text{Ap.1.7})$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω . Since Eq. (1.5) is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega t = \frac{\pi}{4}$ gives:

$$\omega = \sqrt{\frac{2(F(A) - F(A \cos \omega t))}{A^2 \sin^2 \omega t}} \quad (\text{Ap.1.8})$$

$$T = \frac{2\pi}{\sqrt{\frac{2(F(A) - F(A \cos \omega t))}{A^2 \sin^2 \omega t}}} \quad (\text{Ap.1.9})$$

Appendix 2. Description of RK4

The Runge-Kutta method is used to solve differential equation systems [38]. Second order differential equations can be usually changed into first order equations and then are solved through Runge-Kutta method:

Consider an initial value problem as follows:

$$y' = f(t, y), \quad y(t_0) = y_0. \quad (\text{Ap.2.1})$$

Then RK4 method is given for this problem as below:

$$y_{n+1} = y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4), \quad (\text{Ap.2.2})$$

$$t_{n+1} = t_n + h \quad (\text{Ap.2.3})$$

Where y_{n+1} is the RK4 approximation of $y(t_{n+1})$ and

$$k_1 = f(t_n, y_n), \quad (\text{Ap.2.4})$$

$$k_2 = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \quad (\text{Ap.2.5})$$

Appendix 3

Motion of a rigid rod rocking back is presented as follows [28]:

$$\ddot{u} + \frac{3}{4}u^2\ddot{u} + \frac{3}{4}u\dot{u}^2 + \frac{3gu \cos u}{l} = 0 \quad (\text{Ap.3.1})$$

Assuming $\cos u = 1 - \frac{u^2}{2} + \frac{u^4}{24}$, (Ap.3.1) is rewritten as follows:

$$\ddot{u} + \frac{3}{4}u^2\ddot{u} + \frac{3}{4}u\dot{u}^2 + \frac{g}{8l}u(u^4 - 12u^2 + 24) = 0, \quad u(0) = A, \quad \dot{u}(0) = 0 \quad (\text{Ap.3.2})$$

The exact period ω_e by imposing the initial conditions is [28]

$$T_{ex} = 4 \left(\frac{1}{3g} \right)^{1/2} \int_0^{\pi/2} \left(\frac{(4 + 3A^2 \sin^2 \varphi) A^2 \cos^2 \varphi}{8(A \sin A + \cos A - A \sin \varphi \sin(A \sin \varphi) - \cos(A \sin \varphi))} \right)^{1/2} d\varphi \quad (\text{Ap.3.3})$$

After approximation, AFF gives [23]

$$\omega = \frac{\sqrt{2/(3A^2 + 8)(5A^4 - 72A^2 + 192)g}}{4/(3A^2 + 8)} \quad (\text{Ap.3.4})$$

$$T = \frac{8\pi/(3A^2 + 8)}{\sqrt{2/(3A^2 + 8)(5A^4 - 72A^2 + 192)g}} \quad (\text{Ap.3.5})$$

The comparisons between results obtained by Amplitude-Frequency Formulation [27] and exact solution [28] are presented in literature [27] and it is found that, Amplitude-Frequency Formulation is very powerful to solve highly nonlinear oscillators.

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