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Research Article

An Exact Series Solution for the Vibration of Mindlin Rectangular Plates with Elastically Restrained Edges

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An analysis method is proposed for the vibration analysis of the Mindlin rectangular plates with general elastically restrained edges, in which the vibration displacements and the cross-sectional rotations of the mid-plane are expressed as the linear combination of a double Fourier cosine series and four one-dimensional Fourier series. The use of these supplementary functions is to solve the possible discontinuities with first derivatives at each edge. So this method can be applied to get the exact solution for vibration of plates with general elastic boundary conditions. The matrix eigenvalue equation which is equivalent to governing differential equations of the plate can be derived through using the boundary conditions and the governing equations based on Mindlin plate theory. The natural frequencies can be got through solving the matrix equation. Finally the numerical results are presented to validate the accuracy of the method.

1. Introduction

Rectangular plates are important structural elements and the analysis of the vibration is very important for the design of plate-type structures in aerospace, electronic, mechanical, marine, nuclear, and structural engineering. Thus, many researchers have done much work and got a lot of results. However, this research is based on the classical Kirchhoff hypothesis. This theory neglects the effect of shear deformation and rotary inertia which result in the over-estimation of vibration frequencies. This deviation will increase with increasing plate thickness. To improve the results, the Mindlin first-order plate can be employed. So the vibration of Mindlin rectangular plates has begun to gain attention.

The effective methods used to analyze the vibration of Mindlin rectangular plates include Rayleigh-Ritz method [1–4] and some numerical methods, such as differential quadrature method (DQM) [5, 6] and discrete singular convolution method (DSC) [7, 8]. Several analysis methods were also proposed by some researchers. Hashemi et al. derived the exact close form characteristic equations and their associated eigenfunctions for the thick rectangular plates with two opposite sides simply supported [9, 10]. Gorman used the superposition method to obtain a solution for the Mindlin plates [11, 12].

Most existing studies are limited to the classical homogeneous boundary conditions. Recently, Li proposed a Fourier series method for the vibration analysis of arbitrarily supported beam [13]. The flexural displacement of the beam is sought as the linear combination of a Fourier series and an auxiliary polynomial function. Subsequently, this method is extended to the flexural and in-plane vibration of rectangular plates under general boundary conditions. The flexural and in-plane displacement of the plate is sought as the linear combination of a double Fourier cosine series and auxiliary series functions [14–16]. It has been shown that this solution method works very well for various edge supports.

In this paper, an improved Fourier series method is employed to analyze the free vibration of Mindlin rectangular plates with general elastic boundary supports, in which the vibration displacements and the cross-sectional rotations of the mid-plane are expressed as the linear combination of a double Fourier cosine series and four one-dimensional Fourier series. The possible discontinuities problems, which maybe encountered in the displacement and rotations partial differentials along the edges, can be solved by these

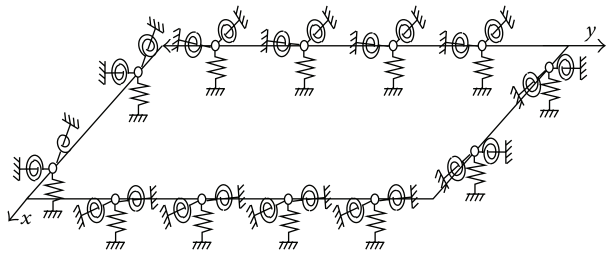

Figure 1: A Mindlin plate with general elastic boundary support.

supplementary functions. Then, an exact solution for Mindlin rectangular plates with arbitrary elastically restrained edges can be obtained. Finally several numerical examples and the comparisons with those results reported in the literature are presented to validate the accuracy of the present approach.

2. Mathematical Modeling and Solution Methodology

Consider a rectangular Mindlin plate elastically restrained along all edges, as shown in Figure 1. The boundary conditions are physically realized in terms of three kinds of restraining springs (translational, rotational, and torsional springs) attached to each edge. Different boundary conditions can be directly obtained by changing the stiffness of springs. The governing differential equations for free vibration of a Mindlin plate are given by

$$\begin{gathered}
kGh\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial \psi_x}{\partial x}+\frac{\partial \psi_y}{\partial y}\right)-\rho h\frac{\partial^2 w}{\partial t^2}=0,\\
D\left(\frac{\partial^2 \psi_x}{\partial x^2}+\frac{1-\mu}{2}\frac{\partial^2 \psi_x}{\partial y^2}+\frac{1+\mu}{2}\frac{\partial^2 \psi_y}{\partial x\partial y}\right)\\
-kGh\left(\frac{\partial w}{\partial x}+\psi_x\right)-\rho h\frac{\partial^2 \psi_x}{\partial t^2}=0,\\
D\left(\frac{\partial^2 \psi_y}{\partial y^2}+\frac{1-\mu}{2}\frac{\partial^2 \psi_y}{\partial x^2}+\frac{1+\mu}{2}\frac{\partial^2 \psi_x}{\partial x\partial y}\right)\\
-kGh\left(\frac{\partial w}{\partial y}+\psi_y\right)-\rho h\frac{\partial^2 \psi_y}{\partial t^2}=0,
\end{gathered}\tag{1}$$

where w is the transverse displacement, ψ_x and ψ_y are the slope due to bending alone in the respective planes, k is the shear correction factor to account for the fact, $G = E/2(1+\mu)$ is the shear modulus, μ is the Poisson's ratio, $D = Eh^3/(12(1-\mu^2))$ is the flexural rigidity, ρ is the mass density, and h is the thickness of the plate.

In terms of transverse displacements and slope, the bending and twisting moments and the transverse shearing forces in plates can be expressed as

$$\begin{gathered}
M_x = D\left(\frac{\partial \psi_x}{\partial x}+\mu\frac{\partial \psi_y}{\partial y}\right),\\
M_y = D\left(\frac{\partial \psi_y}{\partial y}+\mu\frac{\partial \psi_x}{\partial x}\right),\\
M_{xy} = \frac{1-\mu}{2}D\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right),\\
Q_y = kGh\left(\frac{\partial w}{\partial y}+\psi_y\right),\\
Q_x = kGh\left(\frac{\partial w}{\partial x}+\psi_x\right).
\end{gathered}\tag{2}$$

There are three forces along every edge and they are the bending moment, the twisting moment, and the shearing forces. Three kinds of restraining springs (rotational, torsional, and translational springs) along every edge can be corresponding to these three forces. The boundary conditions for an elastically restrained rectangular plate are as follows:

$$\begin{gathered}
k_{x0}w = -Q_x,\\
K_{x0}\psi_x = -M_x,\\
K_{yx0}\psi_y = -M_{xy},\\
\text{at } x = 0;\\
k_{xa}w = Q_x,\\
K_{xa}\psi_x = M_x,\\
K_{yxa}\psi_y = M_{xy},\\
\text{at } x = a;\\
k_{y0}w = -Q_y,\\
K_{y0}\psi_y = -M_y,\\
K_{xy0}\psi_x = -M_{xy},\\
\text{at } y = 0;\\
k_{yb}w = Q_y,\\
K_{yb}\psi_y = M_y,\\
K_{xyb}\psi_x = M_{xy},\\
\text{at } y = b,
\end{gathered}\tag{3}$$

where k_{x0} and k_{xa} (k_{y0} and k_{yb}) are the translation spring constants, K_{x0} and K_{xa} (K_{y0} and K_{yb}) are the rotational spring constants, and K_{yx0} and K_{yxa} (K_{xy0} and K_{xyb}) are the torsional spring constants at $x = 0$ and $x = a$ ($y = 0$ and $y = b$), respectively. All classical homogeneous boundary conditions can be easily derived by simply setting each of the spring constants to be infinite or zero.

According to the Mindlin plate theory, the transverse displacement of the plate median surface and the rotations of the cross-section, respectively, along the x direction and the y direction are utilized. In this study, these quantities are expressed in form of improved Fourier series expansions [16]:

$$\begin{aligned}
&w(x,y)\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\cos(\lambda_m x)\cos(\lambda_n y)\\
&\quad+\sum_{l=1}^{2}\left(\xi_{lb}(y)\sum_{m=0}^{\infty}d_{lm}^{1}\cos(\lambda_m x)+\xi_{la}(x)\sum_{n=0}^{\infty}f_{ln}^{1}\cos(\lambda_n y)\right),
\end{aligned}$$

$$\begin{aligned}
\psi_x(x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}\cos(\lambda_m x)\cos(\lambda_n y) \\
&\quad + \sum_{l=1}^{2}\left(\xi_{lb}(y)\sum_{m=0}^{\infty} d_{lm}^2\cos(\lambda_m x) + \xi_{la}(x)\sum_{n=0}^{\infty} f_{ln}^2\cos(\lambda_n y)\right), \\
\psi_y(x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\cos(\lambda_m x)\cos(\lambda_n y) \\
&\quad + \sum_{l=1}^{2}\left(\xi_{lb}(y)\sum_{m=0}^{\infty} d_{lm}^3\cos(\lambda_m x) + \xi_{la}(x)\sum_{n=0}^{\infty} f_{ln}^3\cos(\lambda_n y)\right),
\end{aligned} \tag{4}$$

where A_{mn}, d_{lm}^1, f_{ln}^1, B_{mn}, d_{lm}^2, f_{ln}^2, C_{mn}, d_{lm}^3, and f_{ln}^3 are the expansion coefficients, $\lambda_m = m\pi/a$, $\lambda_n = n\pi/b$, a and b are the length and width, respectively, and

$$\begin{aligned}
\xi_{1a}(x) &= \frac{a}{2\pi}\sin\frac{\pi x}{2a} + \frac{a}{2\pi}\sin\frac{3\pi x}{2a}, \\
\xi_{2a}(x) &= -\frac{a}{2\pi}\cos\frac{\pi x}{2a} + \frac{a}{2\pi}\cos\frac{3\pi x}{2a}, \\
\xi_{1b}(y) &= \frac{b}{2\pi}\sin\frac{\pi y}{2b} + \frac{b}{2\pi}\sin\frac{3\pi y}{2b}, \\
\xi_{2b}(y) &= -\frac{b}{2\pi}\cos\frac{\pi y}{2b} + \frac{b}{2\pi}\cos\frac{3\pi y}{2b}.
\end{aligned} \tag{5}$$

Theoretically, there is an infinite number of these supplementary functions. However, one needs to ensure that the selected functions will not nullify any of the boundary conditions. It is easy to verify that $\xi 1a(0) = \xi 1a(a) = \xi 1a'(a) = 0$, $\xi 1a'(0) = 1$, $\xi 2a(0) = \xi 2a(a) = \xi 2a'(0) = 0$, $\xi 2a'(a) = 1$, similar conditions exist for the supplementary function in y-direction. Though these conditions are not necessary, they can simplify the subsequent mathematical expressions and the corresponding solution procedures.

One will notice from (4) that beside the standard double Fourier series, four single Fourier series are also included. The potential discontinuity associated with the x-derivative and y-derivative of the original function along the four edges can be transferred onto these auxiliary series functions. Then, the Fourier series would be smooth enough in the whole solving domain. Therefore, not only is this Fourier series representation of solution applicable to any boundary conditions but also the convergence of the series expansion can be improved.

Substituting (4) into the boundary conditions, for example, $k_{x0}w = -Q_x$, at $x = 0$; one can have

$$\begin{aligned}
&k_{x0}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}\cos\lambda_n y + \xi_{1b}(y)\sum_{m=0}^{\infty} d_{1m}^1 + \xi_{2b}(y)\sum_{m=0}^{\infty} d_{2m}^1\right) \\
&= -kGh\left(\sum_{n=0}^{\infty} f_{1n}^1\cos\lambda_n y + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}\cos\lambda_n y + \xi_{1b}(y)\sum_{m=0}^{\infty} d_{1m}^2 + \xi_{2b}(y)\sum_{m=0}^{\infty} d_{2m}^2\right).
\end{aligned} \tag{6}$$

In order to derive the constraint equations for the unknown coefficients, all the sine terms and the auxiliary series functions will be expanded into Fourier cosine series. The related formulas are provided in Appendix A. Then, by equating the coefficients for the like terms on both sides, one can obtain the following equations:

$$\begin{aligned}
&k_{x0}\left(\sum_{m=0}^{\infty} A_{mn} + \beta_{1n}\sum_{m=0}^{\infty} d_{1m}^1 + \beta_{2n}\sum_{m=0}^{\infty} d_{2m}^1\right) \\
&= -kGh\left(f_{1n}^1 + \sum_{m=0}^{\infty} B_{mn} + \beta_{1n}\sum_{m=0}^{\infty} d_{1m}^2 + \beta_{2n}\sum_{m=0}^{\infty} d_{2m}^2\right).
\end{aligned} \tag{7}$$

Similarly, the substitution of (4) into the remaining boundary conditions will lead to eleven equations that can be obtained from (3):

$$\begin{aligned}
&K_{x0}\left(\sum_{m=0}^{\infty} B_{mn} + \beta_{1n}\sum_{m=0}^{\infty} d_{1m}^2 + \beta_{2n}\sum_{m=0}^{\infty} d_{2m}^2\right) \\
&= -D\left(f_{1n}^2 + \mu\left(\sum_{m=0}^{\infty}\sum_{q=0}^{\infty}(-\lambda_q)C_{mq}\tau_q + \eta_{1n}\sum_{m=0}^{\infty} d_{1m}^3 + \eta_{2n}\sum_{m=0}^{\infty} d_{2m}^3\right)\right), \\
&K_{yx0}\left(\sum_{m=0}^{\infty} C_{mn} + \beta_{1n}\sum_{m=0}^{\infty} d_{1m}^3 + \beta_{2n}\sum_{m=0}^{\infty} d_{2m}^3\right) \\
&= -\frac{1-\mu}{2}D\left(f_{1n}^3 + \sum_{m=0}^{\infty}\sum_{q=0}^{\infty}(-\lambda_q)B_{mq}\tau_q + \eta_{1n}\sum_{m=0}^{\infty} d_{1m}^2 + \eta_{2n}\sum_{m=0}^{\infty} d_{2m}^2\right), \\
&k_{xa}\left(\sum_{m=0}^{\infty}(-1)^m A_{mn} + \sum_{l=1}^{2}\left(\beta_{ln}\sum_{m=0}^{\infty}(-1)^m d_{lm}^1\right)\right) \\
&= kGh\left(f_{2n}^1 + \sum_{m=0}^{\infty}(-1)^m B_{mn} + \sum_{l=1}^{2}\left(\beta_{ln}\sum_{m=0}^{\infty}(-1)^m d_{lm}^2\right)\right),
\end{aligned}$$

$$K_{xa}\left(\sum_{m=0}^{\infty}(-1)^m B_{mn}+\sum_{l=1}^{2}\left(\beta_{ln}\sum_{m=0}^{\infty}(-1)^m d_{lm}^2\right)\right)$$
$$=-D\left(f_{2n}^2+\mu\left(\sum_{m=0}^{\infty}\sum_{q=0}^{\infty}\left(-\lambda_q\right)(-1)^m C_{mq}\tau_q\right.\right.$$
$$\left.\left.+\sum_{l=1}^{2}\left(\eta_{ln}\sum_{m=0}^{\infty}(-1)^m d_{lm}^3\right)\right)\right),$$
$$K_{yxa}\left(\sum_{m=0}^{\infty}(-1)^m C_{mn}+\sum_{l=1}^{2}\left(\beta_{ln}\sum_{m=0}^{\infty}(-1)^m d_{lm}^3\right)\right)$$
$$=\frac{1-\mu}{2}D\left(f_{2n}^3+\sum_{m=0}^{\infty}\sum_{q=0}^{\infty}\left(-\lambda_q\right)(-1)^m B_{mq}\tau_q\right.$$
$$\left.+\sum_{l=1}^{2}\left(\eta_{ln}\sum_{m=0}^{\infty}(-1)^m d_{lm}^2\right)\right),$$
$$k_{y0}\left(\sum_{n=0}^{\infty}A_{mn}+\alpha_{1m}\sum_{n=0}^{\infty}f_{1n}^1+\alpha_{2m}\sum_{n=0}^{\infty}f_{2n}^1\right)$$
$$=-kGh\left(d_{1m}^1+\sum_{n=0}^{\infty}C_{mn}+\alpha_{1m}\sum_{n=0}^{\infty}f_{1n}^3+\alpha_{2m}\sum_{n=0}^{\infty}f_{2n}^3\right),$$
$$K_{y0}\left(\sum_{n=0}^{\infty}C_{mn}+\alpha_{1m}\sum_{n=0}^{\infty}f_{1n}^3+\alpha_{2m}\sum_{n=0}^{\infty}f_{2n}^3\right)$$
$$=-D\left(d_{1m}^3\right.$$
$$+\mu\left(\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\left(-\lambda_p\right)B_{mp}\varepsilon_p\right.$$
$$\left.\left.+\gamma_{1m}\sum_{n=0}^{\infty}f_{1n}^2+\gamma_{2m}\sum_{n=0}^{\infty}f_{2n}^2\right)\right),$$
$$K_{xy0}\left(\sum_{n=0}^{\infty}B_{mn}+\alpha_{1m}\sum_{n=0}^{\infty}f_{1n}^2+\alpha_{2m}\sum_{n=0}^{\infty}f_{2n}^2\right)$$
$$=-\frac{1-\mu}{2}D\left(d_{1m}^2+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\left(-\lambda_p\right)C_{pn}\varepsilon_p\right.$$
$$\left.+\gamma_{1m}\sum_{n=0}^{\infty}f_{1n}^3+\gamma_{2m}\sum_{n=0}^{\infty}f_{2n}^3\right),$$
$$k_{yb}\left(\sum_{n=0}^{\infty}(-1)^n A_{mn}+\sum_{l=1}^{2}\left(\alpha_{lm}\sum_{n=0}^{\infty}(-1)^n f_{ln}^1\right)\right)$$
$$=kGh\left(d_{2m}^1+\sum_{n=0}^{\infty}(-1)^n C_{mn}+\sum_{l=1}^{2}\left(\alpha_{lm}\sum_{n=0}^{\infty}(-1)^n f_{ln}^3\right)\right),$$
$$K_{yb}\left(\sum_{n=0}^{\infty}(-1)^n C_{mn}+\sum_{l=1}^{2}\left(\alpha_{lm}\sum_{n=0}^{\infty}(-1)^n f_{ln}^3\right)\right)$$
$$=D\left(d_{2m}^3+\mu\left(\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\left(-\lambda_p\right)(-1)^n B_{mp}\varepsilon_p\right.\right.$$
$$\left.\left.+\sum_{l=1}^{2}\left(\gamma_{lm}\sum_{n=0}^{\infty}(-1)^n f_{ln}^2\right)\right)\right),$$
$$K_{xyb}\left(\sum_{n=0}^{\infty}(-1)^n B_{mn}+\sum_{l=1}^{2}\left(\alpha_{lm}\sum_{n=0}^{\infty}(-1)^n f_{ln}^2\right)\right)$$
$$=\frac{1-\mu}{2}D\left(d_{2m}^2+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\left(-\lambda_p\right)(-1)^n C_{pn}\varepsilon_p\right.$$
$$\left.+\sum_{l=1}^{2}\left(\gamma_{lm}\sum_{n=0}^{\infty}(-1)^n f_{ln}^3\right)\right). \tag{8}$$

When all the series expansions are truncated to $m = M$ and $n = N$ in numerical calculations, the twelve equations can be rewritten in a matrix form as

$$\mathbf{HB} = \mathbf{QA}, \tag{9}$$

where

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{1,1} & \mathbf{H}_{1,2} & \cdots & \mathbf{H}_{1,12} \\ \mathbf{H}_{2,1} & \mathbf{H}_{2,2} & \cdots & \mathbf{H}_{2,12} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_{12,1} & \mathbf{H}_{12,2} & \cdots & \mathbf{H}_{12,12} \end{bmatrix},$$
$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{1,1} & \mathbf{Q}_{1,2} & \mathbf{Q}_{1,3} \\ \mathbf{Q}_{2,1} & \mathbf{Q}_{2,2} & \mathbf{Q}_{2,3} \\ \vdots & \vdots & \vdots \\ \mathbf{Q}_{12,1} & \mathbf{Q}_{12,2} & \mathbf{Q}_{12,3} \end{bmatrix},$$
$$\mathbf{A} = \{A_{00}, A_{01}, \ldots, A_{MN}, B_{00}, B_{01}, \ldots, B_{MN}, C_{00}, C_{01}, \ldots, C_{MN}\}^T,$$
$$\mathbf{B} = \{d_{10}^1, d_{11}^1, \ldots, d_{2M}^1, f_{10}^1, f_{11}^1, \ldots, f_{2N}^1, d_{10}^2, d_{11}^2, \ldots, d_{2M}^2, f_{10}^2, f_{11}^2, \ldots, f_{2N}^2, d_{10}^3, d_{11}^3, \ldots, d_{2M}^3, f_{10}^3, f_{11}^3, \ldots, f_{2N}^3\}^T. \tag{10}$$

The elements of the matrices $\mathbf{H}$ and $\mathbf{Q}$ are defined in Appendix B.

By substituting (4) into the governing differential equation (1), as mentioned earlier, all the sine terms and the auxiliary series functions are expanded into Fourier cosine series.

Then, by equating the coefficients for the like terms on both sides, one can obtain the following equations:

$$
\begin{aligned}
&A_{mn}\left(-\lambda_m^2\right) \\
&+\sum_{l=1}^{2}\left(d_{lm}^1\left(-\lambda_m^2\right)\beta_{ln}+f_{ln}^1\phi_{lm}\right)+A_{mn}\left(-\lambda_n^2\right) \\
&+\sum_{l=1}^{2}\left(d_{lm}^1\varphi_{ln}+\left(-\lambda_n^2\right)f_{ln}^1\alpha_{lm}\right)+\sum_{p=0}^{\infty}B_{mn}\left(-\lambda_m\right)\varepsilon_{mp} \\
&+\sum_{l=1}^{2}\left(\sum_{p=0}^{\infty}d_{lm}^2\left(-\lambda_m\right)\varepsilon_{mp}\beta_{ln}+f_{ln}^2\gamma_{lm}\right) \\
&+\sum_{q=0}^{\infty}C_{mn}\left(-\lambda_n\right)\tau_{nq}+\sum_{l=1}^{2}\left(d_{lm}^3\eta_{ln}+\sum_{q=0}^{\infty}f_{ln}^3\left(-\lambda_n\right)\tau_{nq}\alpha_{lm}\right) \\
&+\frac{\rho\omega^2}{kG}\left(A_{mn}+\beta_{1n}d_{1m}^1+f_{1n}^1\alpha_{1m}+\beta_{2n}d_{2m}^1+f_{2n}^1\alpha_{2m}\right)=0, \\
&D\left(B_{mn}\left(-\lambda_m^2\right)+\sum_{l=1}^{2}\left(\beta_{ln}d_{lm}^2\left(-\lambda_m^2\right)+\phi_{lm}f_{ln}^2\right)\right. \\
&\quad+\frac{1-\mu}{2}\left(B_{mn}\left(-\lambda_n^2\right)\right. \\
&\quad\quad\left.+\sum_{l=1}^{2}\left(\varphi_{ln}d_{lm}^2+\alpha_{lm}f_{ln}^2\left(-\lambda_n^2\right)\right)\right) \\
&\quad+\frac{1+\mu}{2}\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}C_{mn}\lambda_m\lambda_n\varepsilon_{mp}\tau_{nq}\right. \\
&\quad\quad+\sum_{l=1}^{2}\left(\sum_{p=0}^{\infty}d_{lm}^3\eta_{ln}\left(-\lambda_m\right)\varepsilon_{mp}\right. \\
&\quad\quad\quad\left.\left.\left.+\sum_{q=0}^{\infty}f_{ln}^3\gamma_{lm}\left(-\lambda_n\right)\tau_{nq}\right)\right)\right) \\
&-kGh\left(\sum_{p=0}^{\infty}A_{mn}\left(-\lambda_m\right)\varepsilon_{mp}\right. \\
&\quad+\sum_{l=1}^{2}\left(\sum_{p=0}^{\infty}\beta_{ln}d_{lm}^1\left(-\lambda_m\right)\varepsilon_{mp}+\gamma_{ln}f_{ln}^1\right) \\
&\quad\left.+B_{mn}+\sum_{l=1}^{2}\left(\beta_{ln}d_{lm}^2+\alpha_{lm}f_{ln}^2\right)\right) \\
&+\rho h\omega^2\left(B_{mn}+\sum_{l=1}^{2}\left(\beta_{ln}d_{lm}^2+\alpha_{lm}f_{ln}^2\right)\right)=0, \\
&D\left(C_{mn}\left(-\lambda_n^2\right)+\sum_{l=1}^{2}\left(\varphi_{ln}d_{lm}^3+\alpha_{lm}f_{ln}^3\left(-\lambda_n^2\right)\right)\right. \\
&\quad+\frac{1-\mu}{2}\left(C_{mn}\left(-\lambda_m^2\right)\right. \\
&\quad\quad\left.+\sum_{l=1}^{2}\left(\beta_{ln}d_{lm}^3\left(-\lambda_m^2\right)+\phi_{lm}f_{ln}^3\right)\right) \\
&\quad+\frac{1+\mu}{2}\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}B_{mn}\lambda_m\lambda_n\varepsilon_{mp}\tau_{nq}\right. \\
&\quad\quad+\sum_{l=1}^{2}\left(\sum_{p=0}^{\infty}d_{lm}^2\eta_{ln}\left(-\lambda_m\right)\varepsilon_{mp}\right. \\
&\quad\quad\quad\left.\left.\left.+\sum_{q=0}^{\infty}f_{ln}^2\gamma_{lm}\left(-\lambda_n\right)\tau_{nq}\right)\right)\right) \\
&-kGh\left(\sum_{q=0}^{\infty}A_{mn}\left(-\lambda_n\right)\tau_{nq}\right. \\
&\quad+\sum_{l=1}^{2}\left(d_{lm}^1\eta_{ln}+\sum_{q=0}^{\infty}f_{ln}^1\alpha_{lm}\left(-\lambda_n\right)\tau_{nq}\right) \\
&\quad\left.+C_{mn}+\sum_{l=1}^{2}\left(d_{lm}^3\beta_{ln}+f_{ln}^3\alpha_{lm}\right)\right) \\
&+\rho h\omega^2\left(C_{mn}+\sum_{l=1}^{2}\left(d_{lm}^3\beta_{ln}+f_{ln}^3\alpha_{lm}\right)\right)=0.
\end{aligned}
\tag{11}
$$

Writing in matrix form, we have

$$\mathbf{CA}+\mathbf{DB}+\frac{\rho h\omega^2}{kG}\left(\mathbf{EA}+\mathbf{FB}\right)=\mathbf{0}. \tag{12}$$

Substituting (9), the final system equations can be obtained as

$$\left(\mathbf{K}+\frac{\rho h\omega^2}{kG}\mathbf{M}\right)\mathbf{A}=\mathbf{0}, \tag{13}$$

where

$$
\mathbf{C}=\begin{bmatrix}\mathbf{C}_{1,1} & \mathbf{C}_{1,2} & \mathbf{C}_{1,3}\\ \mathbf{C}_{2,1} & \mathbf{C}_{2,2} & \mathbf{C}_{2,3}\\ \mathbf{C}_{3,1} & \mathbf{C}_{3,2} & \mathbf{C}_{3,3}\end{bmatrix},
$$

$$
\mathbf{D}=\begin{bmatrix}\mathbf{D}_{1,1} & \mathbf{D}_{1,2} & \cdots & \mathbf{D}_{1,12}\\ \mathbf{D}_{2,1} & \mathbf{D}_{2,2} & \cdots & \mathbf{D}_{2,12}\\ \mathbf{D}_{3,1} & \mathbf{D}_{3,2} & \cdots & \mathbf{D}_{3,12}\end{bmatrix},
$$

$$
\mathbf{E}=\begin{bmatrix}\mathbf{E}_{1,1} & \mathbf{E}_{1,2} & \mathbf{E}_{1,3}\\ \mathbf{E}_{2,1} & \mathbf{E}_{2,2} & \mathbf{E}_{2,3}\\ \mathbf{E}_{3,1} & \mathbf{E}_{3,2} & \mathbf{E}_{3,3}\end{bmatrix},
$$

Table 1: The first seven frequency parameters $\Omega = (\omega b^2/\pi^2)(\rho h/D)^{1/2}$ for F-F-F-F Mindlin rectangular plates.

$M = N$	1	2	3	4	5	6	7
5	1.2978	1.9202	2.3644	3.2556	3.2556	5.6148	5.6148
7	1.2924	1.9196	2.3637	3.2438	3.2438	5.6105	5.6105
9	1.2904	1.9195	2.3635	3.2388	3.2388	5.6092	5.6092
10	1.2903	1.9194	2.3634	3.2371	3.2371	5.6089	5.6089
11	1.2895	1.9194	2.3634	3.2366	3.2366	5.6087	5.6087
12	1.2895	1.9194	2.3633	3.2358	3.2358	5.6085	5.6085
13	1.2892	1.9194	2.3633	3.2355	3.2355	5.6085	5.6085
14	1.2892	1.9194	2.3633	3.2355	3.2355	5.6084	5.6084

Table 2: The first seven frequency parameters $\Omega = (\omega a^2)(\rho h/D)^{1/2}$ for S-F-S-F Mindlin rectangular plates.

a/b	h/a	Method	1	2	3	4	5	6	7
0.4	0.1	Present	9.5829	10.7868	14.5793	20.7593	29.3643	36.7449	37.8085
		[9]	9.5814	10.7809	14.5672	20.7245	29.3338	36.736	37.7879
	0.2	Present	9.1316	10.1983	13.5314	18.7785	25.8073	31.7024	32.4788
		[9]	9.1313	10.1968	13.5287	18.7728	25.8016	31.7000	32.4730
1	0.1	Present	9.4461	15.4109	33.9251	36.4266	42.8986	62.3508	66.3889
		[9]	9.4458	15.4054	33.9160	36.4246	42.8870	62.3304	66.3720
	0.2	Present	8.9997	14.1349	29.2566	31.434	36.1663	49.8976	52.8026
		[9]	8.9997	14.1341	29.2558	31.4338	36.1646	49.8953	52.8012
2.5	0.1	Present	9.3065	29.6407	35.8486	64.0871	75.9262	105.417	125.3433
		[9]	9.3065	29.6389	35.8486	64.0850	75.9234	105.4121	125.3334
	0.2	Present	8.8835	24.2058	30.9976	49.7612	59.4412	77.0664	90.361
		[9]	8.8835	24.2054	30.9976	49.7609	59.4404	77.0654	90.3658

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{1,1} & \mathbf{F}_{1,2} & \cdots & \mathbf{F}_{1,12} \\ \mathbf{F}_{2,1} & \mathbf{F}_{2,2} & \cdots & \mathbf{F}_{2,12} \\ \mathbf{F}_{3,1} & \mathbf{F}_{3,2} & \cdots & \mathbf{F}_{3,12} \end{bmatrix}, \quad (14)$$

where $\mathbf{K} = \mathbf{C} + \mathbf{D}\mathbf{H}^{-1}\mathbf{Q}$ and $\mathbf{M} = \mathbf{E} + \mathbf{F}\mathbf{H}^{-1}\mathbf{Q}$. The element of the matrices **C**, **D**, **E**, and **F** are defined in Appendix B. The natural frequencies and eigenvectors can be obtained through solving (13). Then, the physical mode shapes also can be got using (4) and (9).

3. Result and Discussion

Several examples involving various boundary conditions will be discussed in this section. To avoid any comparison of the roundoff results, which might be unrealistic, the nondimensional frequency is used. For the analysis, Poisson's ratio $\mu = 0.3$ and shear correction factor $k = 5/6$ are used.

First, convergence studies are carried out. Table 1 gives the frequencies calculated by using different number of terms in the series expansion for F-F-F-F Mindlin rectangular plates (for $a/b = 1$ and $h/a = 0.1$). It shows that the results are very accurate when M and N are small numbers. When M and N are larger than 10, results are almost invariant. So this method has a very good convergence characteristic.

In order to evaluate the accuracy of the present method, the comparisons with those results reported in the literature are carried out. First of all, considering a plate with two opposite simple edges and the other two edges free (S-F-S-F). A simple edge can be obtained by setting the translational and torsional spring constants to be infinite (here the infinite is represented by a very large number, $D \times 10^7$, and D is the flexural rigidity) and the rotational spring constants to be zero. A free edge can be obtained when all the spring constants are zero. In Table 2, the first seven nondimensional frequency parameters, $\Omega = (\omega a^2)(\rho h/D)^{1/2}$, are given with different aspect ratios and thickness ratios. At the same time, a comparison of the exact solution in [9] is also presented. The results show that the calculated frequencies are excellent. As mentioned earlier, the series expansion will have to be truncated in numerical calculations. In this example and all the subsequent calculations, the setting $M = N = 12$ is used. Specially, in order to compare with the results in [9], shear correction factor $k = 0.86667$ is used.

Three more classical cases (C-F-F-F, C-F-S-F, and C-S-S-F) were considered, and the corresponding frequency parameters are listed in Tables 3, 4, and 5. A clamped edge can be viewed as all the spring constants set to be infinite. The frequency parameters solved by Ritz method and DSC method also are given as a comparison. A good agreement is also observed among these solutions.

This method not only can solve the Mindlin rectangular plates with classical boundary conditions but also can solve plates with the elastical supports. Xiang et al. [3] and Gorman [11] and Zhou [4] also researched Mindlin plates with elastically restrained edges, but they both used two kinds of springs (rotational and translational springs) to realize the elastical

Table 3: The first seven frequency parameters $\Omega = (\omega b^2/\pi^2)(\rho h/D)^{1/2}$ for C-F-F-F Mindlin rectangular plates.

a/b	h/b	Method	1	2	3	4	5	6	7
0.4	0.1	Present	2.1006	2.7297	4.3933	7.2648	10.4212	11.1567	11.4449
		[1]	2.1006	2.7294	4.3921	7.2622	10.4206	11.1554	11.4433
	0.2	Present	1.8579	2.2911	3.4911	5.5123	6.9021	7.3873	8.077
		[1]	1.8579	2.2910	3.4909	5.5119	6.9020	7.3871	8.0767
1	0.1	Present	0.3476	0.8171	2.0357	2.584	2.8633	4.8186	5.4836
		[1]	0.3476	0.8168	2.0356	2.5836	2.8620	4.8162	5.4834
	0.2	Present	0.3384	0.7446	1.7806	2.2766	2.4207	3.8853	4.3156
		[1]	0.3384	0.7445	1.7806	2.2765	2.4205	3.8851	4.3168
2.5	0.1	Present	0.0554	0.2798	0.3432	0.8865	0.9517	1.6251	1.8326
		[1]	0.0554	0.2795	0.3431	0.8854	0.9516	1.6228	1.8320
	0.2	Present	0.0550	0.2626	0.3337	0.8196	0.8958	1.4639	1.6566
		[1]	0.0550	0.2625	0.3337	0.8192	0.8958	1.4632	1.6564

Table 4: The first seven frequency parameters $\Omega = (\omega b^2/\pi^2)(\rho h/D)^{1/2}$ for C-F-S-F Mindlin rectangular plates.

a/b	h/b	Method	1	2	3	4	5	6	7
0.4	0.1	Present	7.8209	8.2262	9.6485	12.1337	15.6899	20.1409	20.358
		[8]	7.8285	8.2339	9.6566	12.1373	15.6917	20.1394	20.3607
	0.2	Present	5.5099	5.7895	6.7405	8.3478	10.4967	12.6125	12.8205
		[8]	5.5128	5.7926	6.7436	8.3495	10.4978	12.6141	12.8219
1	0.1	Present	1.4735	1.9497	3.6462	4.502	5.0407	6.7828	6.8161
		[7]	1.4738	1.9536	3.6499	4.5031	5.0483	6.7909	—
	0.2	Present	1.3254	1.7019	3.0527	3.6262	4.0033	5.2067	5.3485
		[7]	1.3255	1.7030	3.0534	3.6265	4.0054	5.2084	—
2.5	0.1	Present	0.2413	0.5789	0.776	1.2825	1.5996	2.1811	2.4288
		[7]	0.2413	0.5817	0.7761	1.2859	1.5995	2.1852	—
	0.2	Present	0.2362	0.5402	0.7365	1.1704	1.4595	1.9285	2.1794
		[7]	0.2362	0.5409	0.7366	1.1714	1.4594	1.9300	—

Table 5: The first seven frequency parameters $\Omega = (\omega b^2/\pi^2)(\rho h/D)^{1/2}$ for C-S-S-F Mindlin rectangular plates.

a/b	h/b	Method	1	2	3	4	5	6	7
0.4	0.1	Present	7.9414	8.9713	11.137	14.4651	18.7651	20.4622	21.36
		[8]	7.9491	8.9792	11.1430	14.4679	18.7554	20.4646	21.3629
	0.2	Present	5.5943	6.3055	7.752	9.8284	12.295	12.6792	13.2365
		[8]	5.5972	6.3085	7.7544	9.8298	12.2958	12.6807	13.2381
1	0.1	Present	1.6196	2.9171	4.6617	5.7683	5.9723	8.5764	8.8548
		[8]	1.6281	2.9271	4.6650	5.7741	5.9769	8.5795	9.8555
	0.2	Present	1.4451	2.4997	3.7407	4.6356	4.677	6.4325	6.4907
		[8]	1.4476	2.5027	3.7419	4.6375	4.6787	6.4338	6.4910
2.5	0.1	Present	0.3658	0.9415	1.746	1.7792	2.3445	2.8701	3.217
		[8]	0.3832	0.9487	1.7596	1.7828	2.3543	2.8721	3.2233
	0.2	Present	0.3526	0.8834	1.6006	1.6092	2.0839	2.4914	2.7667
		[8]	0.3574	0.8854	1.6043	1.6102	2.0867	2.4919	2.7685

supports along every edge. As mentioned earlier, three kinds of springs along every edge are needed to truly realize the general elastic supports, including all classical homogeneous boundary conditions. Table 6 gives the first seven frequency parameters for Mindlin rectangular plates (for $a/b = 1$ and $h/a = 0.2$) with two opposite free edges, the other two edges only elastically restrained against translation, and Table 7 gives the first seven frequency parameters for the plates with two opposite free edges, the other two edges symmetrically elastically restrained. Table 8 lists the frequency parameters when the translational, rotational, and torsional spring constants are all varying at $x = a$. For simplicity, all the restraining springs are assumed to have the same stiffness in the three examples. Table 8 shows that the results are invariant when K is larger than 10^7, so it is appropriate to set infinite to be $D \times 10^7$ in this paper.

Table 6: The first seven frequency parameters $\Omega = (\omega b^2/\pi^2)(\rho h/D)^{1/2}$ for Mindlin rectangular plates with two opposite free edges, the other two edges only elastically restrained against translation $k_{x0} = k_{xa} = K \times D$.

K	Method	1	2	3	4	5	6	7
5	Present	0.3051	0.3071	0.5397	1.2922	1.8248	2.2009	2.8356
	[4]	0.3052	0.3075	0.5396	1.3117	1.8447	2.2089	2.8913
50	Present	0.6926	0.8072	1.5785	1.9220	2.1374	2.7775	3.3443
	[4]	0.6941	0.8225	1.5797	1.9268	2.1634	2.7919	3.3823
500	Present	0.8782	1.2203	2.6177	2.8662	3.1461	4.2179	5.0805
	[4]	0.8831	1.3081	2.7583	2.8767	3.2089	4.3584	5.0980
5000	Present	0.905	1.306	2.7851	3.1319	3.4845	4.7069	5.1891
	[4]	0.9105	1.4254	2.9541	3.1485	3.6157	4.9856	5.3230

Table 7: The first seven frequency parameters $\Omega = (\omega b^2/\pi^2)(\rho h/D)^{1/2}$ for Mindlin rectangular plates with two opposite free edges, the other two edges symmetrically elastically restrained $k_{x0} = k_{xa} = K_1 \times D$ and $K_{x0} = K_{xa} = K_2 \times D$.

K_1	K_2	Method	1	2	3	4	5	6	7
5	10	Present	0.3105	0.3148	0.9155	1.4411	2.006	2.8874	2.9385
		[4]	0.3104	0.3148	0.9208	1.4505	2.0122	2.8989	2.9659
50	50	Present	0.8936	0.901	1.6281	1.9545	2.1865	3.2206	3.412
		[4]	0.8944	0.9032	1.6305	1.9599	2.1912	3.2374	3.4255
5000	5000	Present	1.7537	1.958	3.0634	3.9445	4.1976	5.1453	5.2879
		[4]	1.7607	1.9901	3.1318	3.9578	4.2553	5.2883	5.3548

Table 8: The first seven frequency parameters $\Omega = (\omega b^2/\pi^2)(\rho h/D)^{1/2}$ for C-C-F-C Mindlin rectangular plates with translational, rotational, and torsional restraints at $x = a$, $k_{xa} = K_{xa} = K_{yxa} = K \times D$.

K	1	2	3	4	5	6	7
0	2.2388	3.6105	5.4944	6.6135	6.8022	9.4188	9.7667
0.1	2.2476	3.6233	5.5034	6.6282	6.811	9.4295	9.775
1	2.3132	3.7213	5.5721	6.7447	6.8826	9.5158	9.8391
10^2	2.878	4.562	6.0184	7.641	7.6707	10.2284	10.3535
10^4	3.2854	6.2286	6.2735	8.7475	10.2438	10.4305	12.3813
10^6	3.2954	6.2857	6.2861	8.8102	10.3788	10.4778	12.5528
10^7	3.2955	6.2863	6.2863	8.8109	10.3797	10.4786	12.5545
10^9	3.2955	6.2863	6.2863	8.8109	10.3797	10.4786	12.5546
∞	3.2955	6.2863	6.2863	8.8109	10.3797	10.4786	12.5546

4. Conclusions

An improved Fourier series method is proposed to analyze the free vibration of Mindlin rectangular plates with general elastic boundary supports. The general boundary conditions are physically realized with the uniform distribution of springs on each boundary edge, and different boundary conditions can be directly obtained by changing the stiffness of the springs. The vibration displacements and the cross-sectional rotations of the mid-plane are sought as the linear combination of a double Fourier cosine series and four single auxiliary series functions, respectively. The use of these supplementary functions is to solve the discontinuity problems which were encountered in the displacement and rotations partial differentials along the edges. The unknown expansion coefficients can be solved through using the boundary conditions and the governing equations. In this method, analytical solution is derived for the vibrations of Mindlin rectangular plates with general elastic boundary support. Finally, the numerical results and the comparisons with those reported in the literature are presented to validate the accuracy of the method.

Appendices

A. Supplementary Series

We have

$$\xi_{1a}(x) = \frac{a}{2\pi}\sin\frac{\pi x}{2a} + \frac{a}{2\pi}\sin\frac{3\pi x}{2a} = \sum_{m=0}^{\infty}\alpha_{1m}\cos\lambda_m x,$$

$$\alpha_{1m} = \begin{cases} \dfrac{4a}{3\pi^2}, & m = 0, \\ \dfrac{2a}{(1-4m^2)\pi^2} + \dfrac{6a}{(9-4m^2)\pi^2}, & m \neq 0, \end{cases}$$

$$\xi_{2a}(x) = -\frac{a}{2\pi}\cos\frac{\pi x}{2a} + \frac{a}{2\pi}\cos\frac{3\pi x}{2a} = \sum_{m=0}^{\infty}\alpha_{2m}\cos\lambda_m x,$$

$$\alpha_{2m} = \begin{cases} -\dfrac{2a}{3\pi^2}, & m = 0, \\ -\dfrac{2a(-1)^m}{(1-4m^2)\pi^2} + \dfrac{6a(-1)^{m+1}}{(9-4m^2)\pi^2}, & m \neq 0, \end{cases}$$

$$\xi_{1b}(y) = \frac{b}{2\pi}\sin\frac{\pi y}{2b} + \frac{b}{2\pi}\sin\frac{3\pi y}{2b} = \sum_{n=0}^{\infty}\beta_{1n}\cos\lambda_n y,$$

$$\beta_{1n} = \begin{cases} \dfrac{4b}{3\pi^2}, & n = 0, \\ \dfrac{2b}{(1-4n^2)\pi^2} + \dfrac{6b}{(9-4n^2)\pi^2}, & n \neq 0, \end{cases}$$

$$\xi_{2b}(y) = -\frac{b}{2\pi}\cos\frac{\pi y}{2b} + \frac{b}{2\pi}\cos\frac{3\pi y}{2b} = \sum_{n=0}^{\infty}\beta_{2n}\cos\lambda_n y,$$

$$\beta_{2n} = \begin{cases} -\dfrac{2b}{3\pi^2}, & n = 0, \\ -\dfrac{2b(-1)^n}{(1-4n^2)\pi^2} + \dfrac{6b(-1)^{n+1}}{(9-4n^2)\pi^2}, & n \neq 0, \end{cases}$$

$$\xi'_{1a}(x) = \frac{1}{4}\cos\frac{\pi x}{2a} + \frac{3}{4}\cos\frac{3\pi x}{2a} = \sum_{m=0}^{\infty}\gamma_{1m}\cos\lambda_m x,$$

$$\gamma_{1m} = \begin{cases} \dfrac{1}{\pi}, & m = 0, \\ \dfrac{(-1)^m}{(1-4m^2)\pi} + \dfrac{9(-1)^{m+1}}{(9-4m^2)\pi}, & m \neq 0, \end{cases}$$

$$\xi'_{2a}(x) = \frac{1}{4}\sin\frac{\pi x}{2a} - \frac{3}{4}\sin\frac{3\pi x}{2a} = \sum_{m=0}^{\infty}\gamma_{2m}\cos\lambda_m x,$$

$$\gamma_{2m} = \begin{cases} 0, & m = 0, \\ \dfrac{1}{(1-4m^2)\pi} - \dfrac{9}{(9-4m^2)\pi}, & m \neq 0, \end{cases}$$

$$\xi'_{1b}(y) = \frac{1}{4}\cos\frac{\pi y}{2b} + \frac{3}{4}\cos\frac{3\pi y}{2b} = \sum_{n=0}^{\infty}\eta_{1n}\cos\lambda_n y,$$

$$\eta_{1n} = \begin{cases} \dfrac{1}{\pi}, & n = 0, \\ \dfrac{(-1)^n}{(1-4n^2)\pi} + \dfrac{9(-1)^{n+1}}{(9-4n^2)\pi}, & n \neq 0, \end{cases}$$

$$\xi'_{2b}(y) = \frac{1}{4}\sin\frac{\pi y}{2b} - \frac{3}{4}\sin\frac{3\pi y}{2b} = \sum_{n=0}^{\infty}\eta_{2n}\cos\lambda_n y,$$

$$\eta_{2n} = \begin{cases} 0, & n = 0, \\ \dfrac{1}{(1-4n^2)\pi} - \dfrac{9}{(9-4n^2)\pi}, & n \neq 0, \end{cases}$$

$$\xi''_{1a}(x) = -\frac{\pi}{8a}\sin\frac{\pi x}{2a} - \frac{9\pi}{8a}\sin\frac{3\pi x}{2a} = \sum_{m=0}^{\infty}\phi_{1m}\cos\lambda_m x,$$

$$\phi_{1m} = \begin{cases} -\dfrac{1}{a}, & m = 0, \\ -\dfrac{1}{2(1-4m^2)a} - \dfrac{27}{2(9-4m^2)a}, & m \neq 0, \end{cases}$$

$$\xi''_{2a}(x) = \frac{\pi}{8a}\cos\frac{\pi x}{2a} - \frac{9\pi}{8a}\cos\frac{3\pi x}{2a} = \sum_{m=0}^{\infty}\phi_{2m}\cos\lambda_m x,$$

$$\phi_{2m} = \begin{cases} -\dfrac{1}{2a}, & m = 0, \\ \dfrac{(-1)^m}{2(1-4m^2)a} - \dfrac{27(-1)^{m+1}}{2(9-4m^2)a}, & m \neq 0, \end{cases}$$

$$\xi''_{1b}(y) = -\frac{\pi}{8b}\sin\frac{\pi y}{2b} - \frac{9\pi}{8b}\sin\frac{3\pi y}{2b} = \sum_{n=0}^{\infty}\varphi_{1n}\cos\lambda_n y,$$

$$\varphi_{1n} = \begin{cases} -\dfrac{1}{b}, & n = 0, \\ -\dfrac{1}{2(1-4n^2)b} - \dfrac{27}{2(9-4n^2)b}, & n \neq 0, \end{cases}$$

$$\xi''_{2b}(y) = \frac{\pi}{8b}\cos\frac{\pi y}{2b} - \frac{9\pi}{8b}\cos\frac{3\pi y}{2b} = \sum_{n=0}^{\infty}\varphi_{2n}\cos\lambda_n y,$$

$$\varphi_{2n} = \begin{cases} -\dfrac{1}{2b}, & n = 0, \\ \dfrac{(-1)^n}{2(1-4n^2)b} - \dfrac{27(-1)^{n+1}}{2(9-4n^2)b}, & n \neq 0, \end{cases}$$

$$\sin\lambda_m x = \sum_{p=0}^{\infty}\varepsilon_{mp}\cos\lambda_p x$$

$$p = 0, \quad \varepsilon_{mp} = 0,$$

$$p \neq 0, \quad \varepsilon_{mp} = \begin{cases} m = 0, & \dfrac{1-(-1)^p}{p\pi}, \\ m \neq 0 & \begin{cases} m = p, & 0, \\ m \neq p, & \dfrac{2p\left((-1)^{m+p}-1\right)}{(m^2-p^2)\pi}, \end{cases} \end{cases}$$

$$\sin\lambda_n y = \sum_{q=0}^{\infty}\varepsilon_{nq}\cos\lambda_q y,$$

$$q = 0, \quad \varepsilon_{nq} = 0,$$

$$q \neq 0, \quad \varepsilon_{nq} = \begin{cases} n = 0, & \dfrac{1-(-1)^q}{q\pi} \\ n \neq 0, & \begin{cases} n = q, & 0, \\ n \neq q, & \dfrac{2q\left((-1)^{n+q}-1\right)}{(n^2-q^2)\pi}. \end{cases} \end{cases} \tag{A.1}$$

B. Definitions of Matrices

We have

$$\{\mathbf{C}_{1,1}\}_{s,t} = -\left(\lambda_m^2 + \lambda_n^2\right)\delta_{st},$$

$$\{\mathbf{C}_{1,2}\}_{s,t} = -\lambda_m^2\delta_{st}\sum_{p=0}^{\infty}\varepsilon_{mp},$$

$$\{\mathbf{C}_{1,3}\}_{s,t} = -\lambda_n^2\delta_{st}\sum_{q=0}^{\infty}\varepsilon_{nq},$$

$$\{\mathbf{D}_{1,1}\}_{s,m'+1} = \left(-\lambda_m^2\beta_{1n} + \varphi_{1n}\right)\delta_{mm'},$$

$$\{\mathbf{D}_{1,2}\}_{s,m'+1} = \left(-\lambda_m^2\beta_{2n} + \varphi_{2n}\right)\delta_{mm'},$$

$$\{\mathbf{D}_{1,3}\}_{s,n'+1} = \left(-\lambda_n^2\alpha_{1m} + \phi_{1m}\right)\delta_{nn'},$$

$$\{\mathbf{D}_{1,4}\}_{s,n'+1} = \left(-\lambda_n^2\alpha_{2m} + \phi_{2m}\right)\delta_{nn'},$$

$$\{\mathbf{D}_{1,5}\}_{s,m'+1} = -\lambda_m\beta_{1n}\delta_{mm'}\sum_{p=0}^{\infty}\varepsilon_{mp},$$

$$\{\mathbf{D}_{1,6}\}_{s,m'+1} = -\lambda_m\beta_{2n}\delta_{mm'}\sum_{p=0}^{\infty}\varepsilon_{mp},$$

$$\{\mathbf{D}_{1,7}\}_{s,n'+1} = \gamma_{1m}\delta_{nn'}, \qquad \{\mathbf{D}_{1,8}\}_{s,n'+1} = \gamma_{2m}\delta_{nn'},$$

$$\{\mathbf{D}_{1,9}\}_{s,m'+1} = \eta_{1n}\delta_{mm'}, \qquad \{\mathbf{D}_{1,10}\}_{s,m'+1} = \eta_{2n}\delta_{mm'},$$

$$\{\mathbf{D}_{1,11}\}_{s,n'+1} = -\lambda_n\alpha_{1m}\delta_{nn'}\sum_{q=0}^{\infty}\tau_{nq},$$

$$\{\mathbf{D}_{1,12}\}_{s,n'+1} = -\lambda_n\alpha_{2m}\delta_{nn'}\sum_{q=0}^{\infty}\tau_{nq},$$

$$\{\mathbf{E}_{1,1}\}_{s,t} = \delta_{st}, \qquad \{\mathbf{E}_{1,2}\}_{s,t} = 0,$$

$$\{\mathbf{E}_{1,3}\}_{s,t} = 0,$$

$$\{\mathbf{F}_{1,1}\}_{s,m'+1} = \beta_{1n}\delta_{mm'},$$

$$\{\mathbf{F}_{1,1}\}_{s,m'+1} = \beta_{2n}\delta_{mm'},$$

$$\{\mathbf{F}_{1,3}\}_{s,n'+1} = \alpha_{1m}\delta_{nn'},$$

$$\{\mathbf{F}_{1,4}\}_{s,n'+1} = \alpha_{2m}\delta_{nn'},$$

$$\{\mathbf{F}_{1,5}\}_{s,m'+1} = 0, \qquad \{\mathbf{F}_{1,6}\}_{s,m'+1} = 0,$$

$$\{\mathbf{F}_{1,7}\}_{s,n'+1} = 0, \qquad \{\mathbf{F}_{1,8}\}_{s,n'+1} = 0,$$

$$\{\mathbf{F}_{1,9}\}_{s,m'+1} = 0, \qquad \{\mathbf{F}_{1,10}\}_{s,m'+1} = 0,$$

$$\{\mathbf{F}_{1,11}\}_{s,n'+1} = 0, \qquad \{\mathbf{F}_{1,12}\}_{s,n'+1} = 0,$$

$$\{\mathbf{H}_{1,1}\}_{n+1,m+1} = -k_{x0}\beta_{1n},$$

$$\{\mathbf{H}_{1,2}\}_{n+1,m+1} = -k_{x0}\beta_{2n},$$

$$\{\mathbf{H}_{1,3}\}_{n+1,n'+1} = -kch\delta_{nn'},$$

$$\{\mathbf{H}_{1,4}\}_{n+1,n'+1} = 0,$$

$$\{\mathbf{H}_{1,5}\}_{n+1,m+1} = -kch\beta_{1n},$$

$$\{\mathbf{H}_{1,6}\}_{n+1,m+1} = -kch\beta_{2n},$$

$$\{\mathbf{H}_{1,7}\}_{n+1,n'+1} = 0, \qquad \{\mathbf{H}_{1,8}\}_{n+1,n'+1} = 0,$$

$$\{\mathbf{H}_{1,9}\}_{n+1,m+1} = 0, \qquad \{\mathbf{H}_{1,10}\}_{n+1,m+1} = 0,$$

$$\{\mathbf{H}_{1,11}\}_{n+1,n'+1} = 0, \qquad \{\mathbf{H}_{1,12}\}_{n+1,n'+1} = 0,$$

$$\{\mathbf{Q}_{1,1}\}_{s,t} = k_{x0}\delta_{mm'}, \qquad \{\mathbf{Q}_{1,2}\}_{s,t} = kGh\delta_{mm'},$$

$$\{\mathbf{Q}_{1,3}\}_{s,t} = 0, \tag{B.1}$$

where $m' = 0, 1, \ldots, M$, $m = 0, 1, \ldots, M$, $n' = 0, 1, \ldots, N$, $n = 0, 1, \ldots, N$, $s = m(N+1)+n+1$, and $t = m'(N+1)+n'+1$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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