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Research Article

Parameters Approach Applied on Nonlinear Oscillators

Najeeb Alam Khan, Fatima Riaz, and Nadeem Alam Khan

Department of Mathematical Sciences, University of Karachi, Karachi 75270, Pakistan

Correspondence should be addressed to Najeeb Alam Khan; njbalam@yahoo.com

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We applied an approach to obtain the natural frequency of the generalized Duffing oscillator $\ddot{u}+u+\alpha_3u^3+\alpha_5u^5+\alpha_7u^7+\cdots+\alpha_nu^n=0$ and a nonlinear oscillator with a restoring force which is the function of a noninteger power exponent of deflection $\ddot{u}+\alpha u|u|^{n-1}=0$. This approach is based on involved parameters, initial conditions, and collocation points. For any arbitrary power of n, the approximate frequency analysis is carried out between the natural frequency and amplitude. The solution procedure is simple, and the results obtained are valid for the whole solution domain.

1. Introduction

Although a large amount of the efforts on dynamical systems are related to second-order differential equations, some dynamical systems can be described by nonlinear (second-order) differential equations. Attention in nonlinear oscillator equations involving the second temporal derivative of displacement has recently been focused on the existence of periodic solutions. The study of nonlinear periodic oscillator is of interest to many researchers and various methods of solution have been suggested. Several approaches have been proposed to deal with different kinds of oscillator equations, for example, [1–7]. He in [8] used Hamiltonian method to calculate the analytical approximate periodic solutions of nonlinear oscillator equations. The approximations to the periodic solution and the angular frequency obtained by He were not accurate enough.

Yildirim et al. [9] and Khan et al. [10], respectively, applied a higher order Hamiltonian formulation combined with parameters for nonlinear oscillators. Our concern in this work is the derivation of amplitude-frequency relationship for the nonlinear oscillator equations $\ddot{u}+u+\alpha_3u^3+\alpha_5u^5+\alpha_7u^7+\cdots+\alpha_nu^n=0$ and $\ddot{u}+\alpha u|u|^{n-1}=0$. The attention here has been restricted primarily to odd positive integer power for the first equation and rational powers greater than unity for the second equation. There are examples of systems, however, for which these exponents can be of noninteger order, for instance, the flexible elements of vibration isolators made of wire-mesh and felt materials, cable isolators, and radially loaded rubber cylinder.

In the present work, the mentioned parameters are the undetermined values in the assumed solution. In the parameters technique, the motion has been assumed as $u=\sum_{k=0}^{N}A_{2k+1}\cos(2k+1)\,\omega t$ where ω, A_k, and $k=0,1,2,\ldots$ are the angular frequency of motion and Fourier coefficients, respectively. The method in this approach to obtain the parameters is quite different from the method in He's Hamiltonian technique. Hence, the present technique is not similar to He's Hamiltonian technique. Finally, the paper provides some accurate results for the angular frequency ω of the motion.

2. Analysis of the Method

2.1. Generalized Duffing Oscillator. First, we consider a general form of nonlinear oscillator

$$\ddot{u}+u+\alpha_3u^3+\alpha_5u^5+\alpha_7u^7+\cdots+\alpha_nu^n=0 \tag{1}$$

with initial conditions

$$u(0)=A,\qquad \dot{u}(0)=0, \tag{2}$$

where A, α_{2k+1}, and $k = 1, 2, 3, \ldots$ are constants. Multiplying both sides of (1) by $2\dot{u}$ and integrating, with initial conditions, we get

$$\dot{u}^2 + u^2 + \alpha_3 \frac{u^4}{2} + \alpha_5 \frac{u^6}{3} + \cdots + 2\alpha_n \frac{u^{n+1}}{n+1} = A^2 + \alpha_3 \frac{A^4}{2} + \alpha_5 \frac{A^6}{3} + \cdots + 2\alpha_n \frac{A^{n+1}}{n+1}. \quad (3)$$

In this approach the solution of the problem is assumed to be

$$u = \sum_{k=0}^{N} A_{2k+1} \cos(2k+1)\omega t. \quad (4)$$

Differentiating (4) leads to the results

$$\dot{u} = -\omega \sum_{k=0}^{N} A_{2k+1} (2k+1) \sin(2k+1)\omega t \quad (5)$$

$$\ddot{u} = -\omega^2 \sum_{k=0}^{N} A_{2k+1} (2k+1)^2 \cos(2k+1)\omega t. \quad (6)$$

From the initial condition equations (2) and (4), we have

$$A = \sum_{i=0}^{N} A_{2k+1}. \quad (7)$$

Table 1: Collocation points for different ωt.

$N = 1$	$\pi/2$	—	—	—
$N = 2$	$\pi/4$	$\pi/2$	—	—

Substituting (4) and (5) into (3) at different values of ωt is mentioned in Table 1.

For $N = 1$ the following equation is obtained:

$$\omega^2 \left(\sum_{k=0}^{N} A_{2k+1} (2k+1) \right)^2 = A^2 + \alpha_3 \frac{A^4}{2} + \alpha_5 \frac{A^6}{3} + \cdots + 2\alpha_n \frac{A^{n+1}}{n+1}. \quad (8)$$

For $N = 2$, corresponding number of equations will be obtained, respectively. Considering the acceleration at $\omega t = 0$, from (1), (2), and (6), we get the following equation:

$$\omega^2 \sum_{k=0}^{N} A_{2k+1} (2k+1)^2 = \left(A + \alpha_3 A^3 + \alpha_5 A^5 + \alpha_7 A^7 + \cdots + \alpha_n A^n \right). \quad (9)$$

The numerical solution of these algebraic equations can be obtained by symbolic software or an iterative scheme. From these system of equations, the approximate frequency-amplitude relationship of a nonlinear oscillator up to higher order will be attained. From (3), the period of the motion is given by

$$T = \int_{-A}^{A} \frac{2du}{\sqrt{A^2 + \alpha_3 (A^4/2) + \alpha_5 (A^6/3) + \cdots + 2\alpha_n (A^{n+1}/(n+1)) - (u^2 + \alpha_3 (u^4/2) + \alpha_5 (u^6/3) + \cdots + 2\alpha_n (u^{n+1}/(n+1)))}}. \quad (10)$$

The angular frequency of the motion can be expressed by the relation

$$\omega = \frac{2\pi}{T}. \quad (11)$$

2.2. Noninteger Order Force-Deflection Oscillator. Now, consider a nonlinear oscillator of the form

$$\ddot{u} + \alpha u |u|^{n-1} = 0, \quad n > 1 \quad (12)$$

with initial conditions

$$u(0) = A, \qquad \dot{u}(0) = 0. \quad (13)$$

Multiply both sides of (12) by $2\dot{u}$ and integrate, with initial conditions

$$\dot{u}^2 + 2\alpha \frac{u^2 |u|^{n-1}}{n+1} = 2\alpha \frac{A^2 |A|^{n-1}}{n+1}. \quad (14)$$

Proceeding in the same way as in the previous section, the obtained algebraic equations can be solved by symbolic software. From these system of equations, one can obtain approximate frequency-amplitude relationship of a nonlinear oscillator up to higher order. From (14), the period of the motion is obtained as

$$T = \int_{-A}^{A} \frac{2du}{\sqrt{2\alpha \left(A^2 |A|^{n-1} / (n+1) \right) - 2\alpha \left(u^2 |u|^{n-1} / (n+1) \right)}}. \quad (15)$$

The angular frequency of the motion can be expressed by the relation

$$\omega = 2\pi \left(\int_{-A}^{A} 2du \times \left(2\alpha \frac{A^2 |A|^{n-1}}{n+1} - 2\alpha \frac{u^2 |u|^{n-1}}{n+1} \right)^{-1/2} \right)^{-1}. \quad (16)$$

It should be pointed out that the square root under the integral sign in (16) should be positive.

3. Numerical Experiments

3.1. For Duffing Oscillator $n = 3$. Consider the following nonlinear Duffing oscillator:

$$\ddot{u} + u + \alpha_3 u^3 = 0 \tag{17}$$

with initial conditions $u(0) = A$ and $\dot{u}(0) = 0$. Assume that the solution can be expressed as

$$u = A_1 \cos \omega t + A_3 \cos 3\omega t. \tag{18}$$

According to the initial conditions,

$$A_1 + A_3 = A. \tag{19}$$

Substituting (18) and its derivative into (3) at $\omega t = \pi/2$, the following equation is obtained:

$$\omega^2 (A_1 - 3A_3)^2 = A^2 + \alpha_3 \frac{A^4}{2}. \tag{20}$$

From (17) the acceleration at $\omega t = 0$ has been obtained as follows:

$$\omega^2 (A_1 + 9A_3) = A + \alpha_3 A^3. \tag{21}$$

Using *MATHEMATICA* the simplified values will be in the following form:

$$\begin{gathered} A_1 = \frac{5A}{8} - \frac{A}{8(1+\alpha_3 A^2)} + \frac{A\sqrt{16 + 22\alpha_3 A^2 + 7{\alpha_3}^2 A^4}}{8(1+\alpha_3 A^2)}, \\ A_3 = \frac{3A}{8} + \frac{A}{8(1+\alpha_3 A^2)} - \frac{A\sqrt{16 + 22\alpha_3 A^2 + 7{\alpha_3}^2 A^4}}{8(1+\alpha_3 A^2)}, \\ \omega(A) = \frac{1}{3}\left(\sqrt{1 + \frac{\alpha_3}{2}A^2} + \sqrt{4 + \frac{7}{2}\alpha_3 A^2}\right). \end{gathered} \tag{22}$$

The frequency-amplitude relationship of nonlinear oscillator for $\alpha_3 = 1$, $A = 1$, has been obtained as

$$\omega_{2(\text{app})} = 1.32111921. \tag{23}$$

From (3), the frequency of the motion for $A = 1$ is

$$\omega_{(\text{Exact})} = \frac{\sqrt{(3/2)}\pi}{2\text{Elliptic}\,K[-1/3]} \approx 1.31777606, \tag{24}$$

where Elliptic $K[m]$ is the complete elliptic integral of the first kind. The three-parameter approach provides good approximations to the exact frequency and the relative error lower than 0.2536%.

Since the accuracy of the obtained results in three-parameter technique is not so high, the four-parameter technique has been introduced as follows:

$$u = A_1 \cos \omega t + A_3 \cos 3\omega t + A_5 \cos 5\omega t, \tag{25}$$

where A_1, A_3, A_5, and ω are four undetermined parameters. Four equations can be formulated for the solution of four parameters. According to the initial conditions,

$$A = A_1 + A_3 + A_5. \tag{26}$$

Substituting (25) and its derivative into (3) at the times $\omega t = \pi/2$ and $\omega t = \pi/4$ as mentioned in Table 1, the following equations have been obtained:

$$\begin{gathered} \frac{\omega^2}{2}(A_1 + 3A_3 - 5A_5)^2 + u_1^2 + \alpha_3 \frac{u_1^4}{2} = A^2 + \alpha_3 \frac{A^4}{2} \\ \omega^2 (A_1 - 3A_3 + 5A_5)^2 = A^2 + \alpha_3 \frac{A^4}{2}, \end{gathered} \tag{27}$$

where $u_1 = (1/\sqrt{2})(A_1 - A_3 - A_5)$.

The acceleration at $\omega t = 0$, from (17), resulted with the following equation:

$$\omega^2 (A_1 + 9A_3 + 25A_5)^2 = A + \alpha_3 A^3. \tag{28}$$

It is difficult to obtain the analytical expression for the unknown values; we derived the numerical results for the unknown parameters ω, A_1, A_3, and A_5 by using (26)–(28). After some mathematical simplification by taking $A = 1, \alpha_3 = 1$, the following values have been obtained:

$$\begin{gathered} A_1 = 0.981716, \qquad A_3 = 0.0179609, \\ A_5 = 0.000322698, \qquad \omega = 1.31777939 \end{gathered} \tag{29}$$

which is very close to the exact solution and highly accurate in comparison with [11, 12, 17, 18].

3.2. For Duffing Oscillator $n = 5$. The following nonlinear Duffing oscillator is considered:

$$\ddot{u} + u + \alpha_3 u^3 + \alpha_5 u^5 = 0 \tag{30}$$

with initial conditions $u(0) = A$ and $\dot{u}(0) = 0$. Let the solution be of the form

$$u = A_1 \cos \omega t + A_3 \cos 3\omega t. \tag{31}$$

According to the initial conditions

$$A_1 + A_3 = A. \tag{32}$$

Substituting (31) and its derivatives into (3) at $\omega t = \pi/2$, the following equation is obtained:

$$\omega^2 (A_1 - 3A_3)^2 = A^2 + \alpha_3 \frac{A^4}{2} + \alpha_5 \frac{A^6}{3}. \tag{33}$$

The acceleration will be procured from (30) at $\omega t = 0$ as follows:

$$\omega^2 (A_1 + 9A_3) = A + \alpha_3 A^3 + \alpha_5 A^5. \tag{34}$$

After some mathematical simplification using *MATHEMATICA*, the following values were achieved:

$$A_1 = \frac{5A}{8} + \frac{-3A + \alpha_5 A^5}{24\left(1 + \alpha_3 A^2 + \alpha_5 A^4\right)} + \frac{A\sqrt{144 + 198\alpha_3 A^2 + 63\alpha_3^2 A^4 + 168\alpha_5 A^4 + 102\alpha_3\alpha_5 A^6 + 40\alpha_5^2 A^8}}{24\left(1 + \alpha_3 A^2 + \alpha_5 A^4\right)},$$

$$A_3 = \frac{3A}{8} + \frac{3A - \alpha_5 A^5}{24\left(1 + \alpha_3 A^2 + \alpha_5 A^4\right)} - \frac{A\sqrt{144 + 198\alpha_3 A^2 + 63\alpha_3^2 A^4 + 168\alpha_5 A^4 + 102\alpha_3\alpha_5 A^6 + 40\alpha_5^2 A^8}}{24\left(1 + \alpha_3 A^2 + \alpha_5 A^4\right)}, \tag{35}$$

$$\omega = \sqrt{\frac{5}{9} + \frac{4A^2}{9}\alpha_3 + \frac{11A^4}{27}\alpha_5 + \frac{1}{27}\sqrt{144 + 198\alpha_3 A^2 + 63\alpha_3^2 A^4 + 168\alpha_5 A^4 + 102\alpha_3\alpha_5 A^6 + 40\alpha_5^2 A^8}}.$$

From (3), the exact period of the motion for $A = 1$ has been obtained as

$$\omega_{(\text{Exact})} = 1.52358602. \tag{36}$$

The three-parameter approach provides good approximations to the exact frequency and the relative error lower than 1.6332%.

Since the accuracy of the obtained results in three-parameter technique is not so high, the four-parameter technique has been introduced as follows:

$$u = A_1 \cos\omega t + A_3 \cos 3\omega t + A_5 \cos 5\omega t, \tag{37}$$

where A_1, A_3, A_5, and ω are four undetermined parameters. Four equations can be formulated for the solution of four parameters. According to the initial conditions,

$$A = A_1 + A_3 + A_5. \tag{38}$$

Substituting (37) and its derivative into (3) at the times $\omega t = \pi/2$ and $\omega t = \pi/4$ as mentioned in Table 1, the following equations have been obtained:

$$\omega^2\left(A_1 + 3A_3 - 5A_5\right)^2 + u_1^2 + \alpha_3\frac{u_1^4}{2} + \alpha_5\frac{u_1^6}{3} = A^2 + \alpha_3\frac{A^4}{2} + \alpha_5\frac{A^6}{3}$$
$$\omega^2\left(A_1 - 3A_3 + 5A_5\right)^2 = A^2 + \alpha_3\frac{A^4}{2} + \alpha_5\frac{A^6}{3}, \tag{39}$$

where $u_1 = (1/\sqrt{2})(A_1 - A_3 - A_5)$.

By the acceleration at $\omega t = 0$, from (30), the following equation will get

$$\omega^2\left(A_1 + 9A_3 + 25A_5\right)^2 = A + \alpha_3 A^3 + \alpha_5 A^5. \tag{40}$$

Similarly, from (38)–(40), four unknowns ω, A_1, A_3, and A_5 can be solved numerically by using MATHEMATICA taking $A = 1$ to get

$$A_1 = 0.968372, \qquad A_3 = 0.0296105,$$
$$A_5 = 0.00201743, \qquad \omega = 1.52376242 \tag{41}$$

which is very close to the exact solutions. The four-parameter approach provides good approximations to the exact frequency and the relative error lower than 0.0115%.

3.3. Force-Deflection Oscillator (for Noninteger n). Consider the following nonlinear *non-integer* force-deflection oscillator:

$$\ddot{u} + u\,|u|^{n-1} = 0, \quad n > 1 \tag{42}$$

with initial conditions

$$u(0) = A, \qquad \dot{u}(0) = 0. \tag{43}$$

The differential (42) with (43) has an exact analytical solution in the form of the Ateb cam function [13, 14]:

$$u = \text{Acam}\left(t\sqrt{\frac{|A|^{n-1}(n+1)}{2}}\right) \tag{44}$$

which is the inverse incomplete Euler Beta function. The exact period of the oscillation is

$$T = \frac{4}{\sqrt{2|A|^{n-1}(n+1)}} B\left(\frac{1}{n+1}, \frac{1}{2}\right), \tag{45}$$

where $B(m, n)$ is the complete Beta function. Consider that the solution can be written as

$$u = A_1 \cos\omega t + A_3 \cos 3\omega t. \tag{46}$$

Table 2: The frequencies ω_{LP} and ω_{MLP} given by [15], $\omega^{*}_{Present}$ and $\omega^{**}_{Present}$ given by three-parameter and four-parameter approaches, ω_{Exact} given by (16) as a function of various values of the parameters n and $A = 1$.

n	ω_{LP}	ω_{MLP}	$\omega^{*}_{Present}$	$\omega^{**}_{Present}$	ω_{Exact}
1	1	1	1	1	1
4/3	0.97013	0.95452	0.83543	0.96326	0.96916
3/2	0.95671	0.93298	0.82546	0.94793	0.95469
5/3	0.94418	0.91212	0.81649	0.93417	0.94081
2	0.92130	0.87214	0.80103	0.91045	0.91468

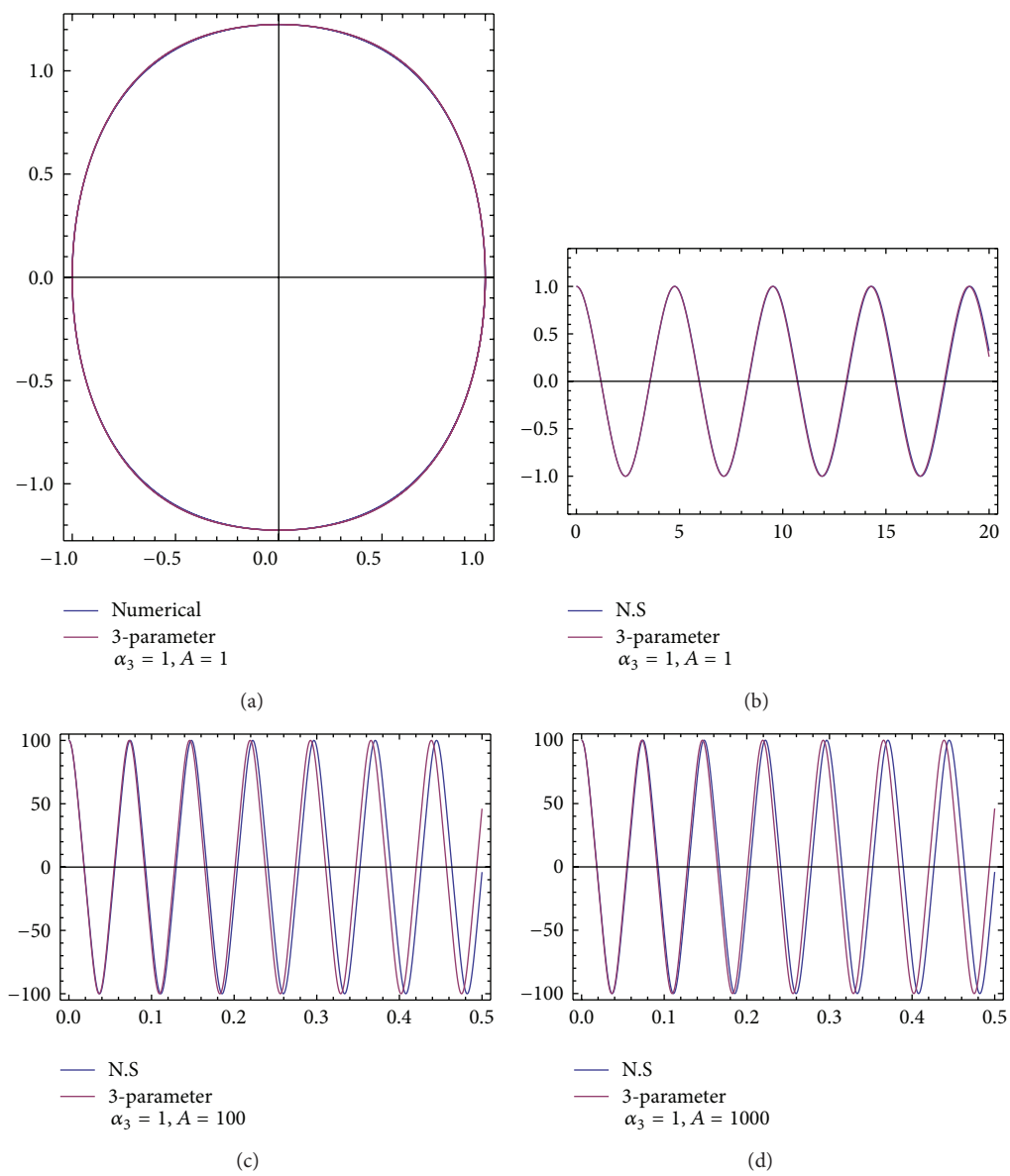

Figure 1: (a) Comparison for the u versus $\dot{u}$ trajectory for the case of $A = 1$, (b) t versus u (displacement $A = 1$), (c) t versus u (displacement $A = 100$), (d) t versus u (displacement $A = 1000$).

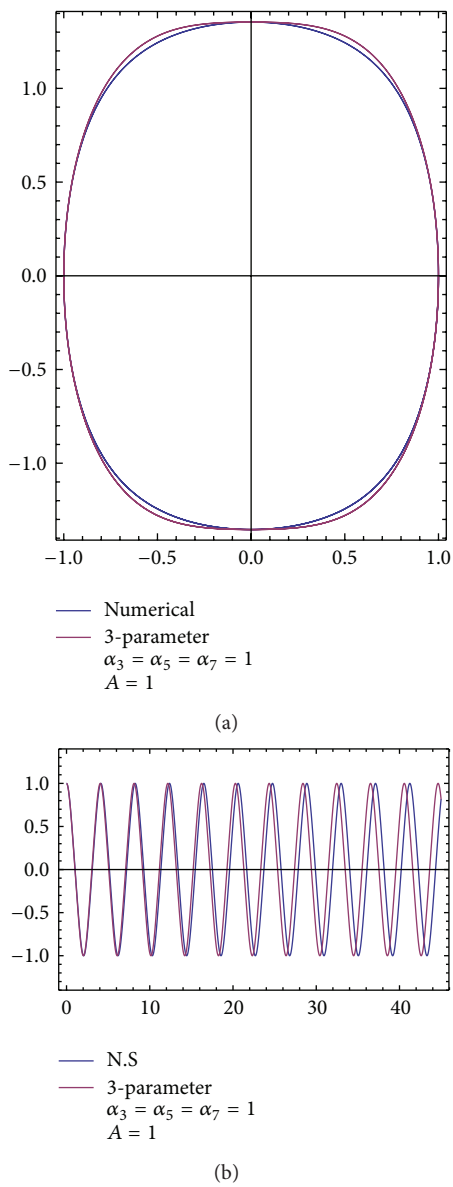

FIGURE 2: (a) Comparison for the u versus $\dot{u}$ trajectory for the case of $A = 1$ and (b) t versus u (displacement $A = 1$).

According to the initial conditions,

$$A_1 + A_3 = A. \tag{47}$$

Substituting (46) into (14) at $\omega t = \pi/2$, the following equation has been obtained:

$$\omega^2 (A_1 - 3A_3)^2 = \frac{A^2 |A|^{n-1}}{n+1}. \tag{48}$$

By the acceleration at $\omega t = 0$, from (46) and (42), the following equation will be achieved:

$$\omega^2 (A_1 + 9A_3)^2 = A^n. \tag{49}$$

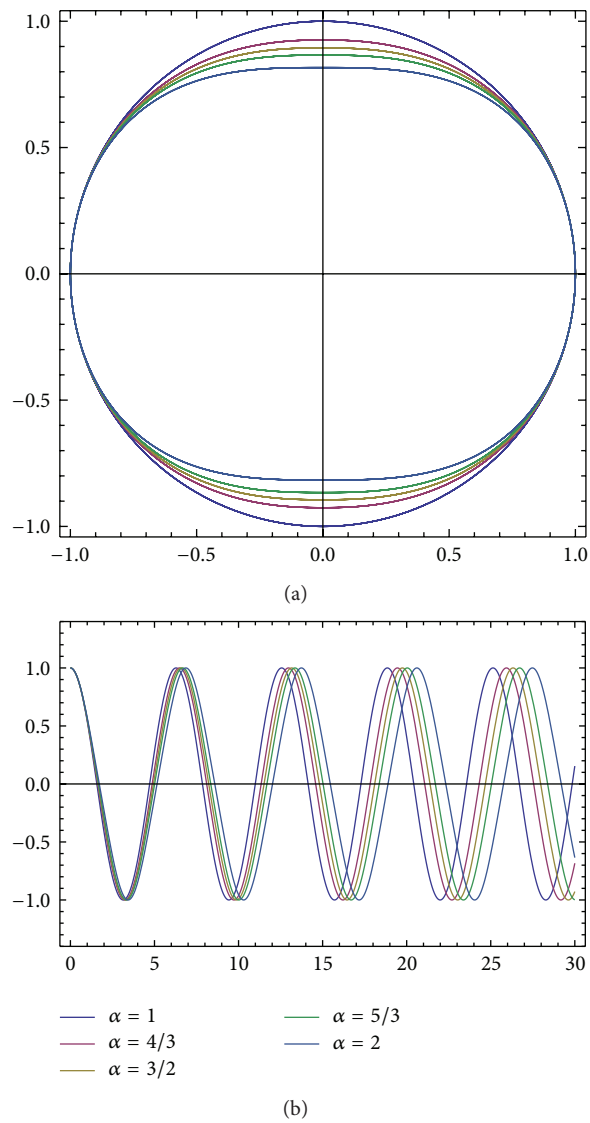

FIGURE 3: (a) Comparison for the u versus $\dot{u}$ trajectory for the case of $A = 1$ and (b) t versus u (displacement $A = 1$).

An analytical frequency-amplitude relation has been heeded from (47)–(49), by *MATHEMATICA 8* built-in utilities, as

$$\begin{aligned} A_1 &= \frac{3A}{8} - \frac{A}{n+1} - \frac{(7+3n)A}{8(n+1)} \\ &\quad - \frac{2\sqrt{2}A|A|^{1-n}\sqrt{(5+3n)|A|^{2n+2}}}{(n+1)|A|^2}, \\ A_3 &= \frac{5A}{8} + \frac{A}{n+1} \\ &\quad - \frac{(7+3n)A}{8(n+1)} - \frac{A|A|^{1-n}\sqrt{(5+3n)|A|^{2n+2}}}{2\sqrt{2}(n+1)|A|^2}, \\ \omega &= \frac{1}{3}\sqrt{\frac{|A|^{n-1}}{(n+1)}}\left(1 + \sqrt{4+3n}\right). \end{aligned} \tag{50}$$

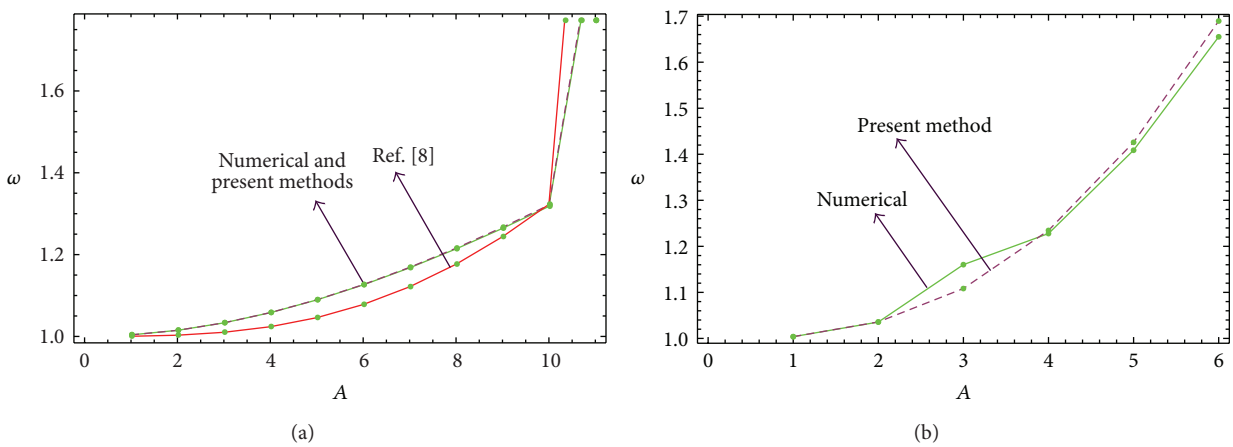

Figure 4: Comparison of the present method with numerical values and [18] for ω versus A of Duffing Oscillator (a) with $n = 3$, $\alpha_3 = 1$ (b) with $n = 3$, $\alpha_3 = 1$, $\alpha_5 = 1$.

The four-parameter approach provides good approximations to the exact frequency. The computed results and its comparison with exact frequency, Lindstedt-Poincaré (LP) method "ω_{LP}", and modified Lindstedt-Poincaré (MLP) method "ω_{MLP}" of second order have been tabulated in Table 2.

4. Conclusions

The parameter method [16] gives the approximate solution for the generalized Duffing and non-integer order oscillator equations. Accuracy and validity of the obtained results have been examined by comparing it with the exact ones in time histories and table. Figures 1 and 2 are depicted for cubic and quanta oscillator equations and Figure 3 for non-integer oscillator equation. The frequency of vibration depends on the initial amplitude, the coefficient of nonlinearity, and the value of the fractional power. The main results of the paper obtained by this method can be summarized as follows.

(i) It has been observed that if numbers of parameters are increased, then the method will give the better results.

(ii) The nonlinear oscillator equation has converted to nonlinear algebraic equation by this approach which can be solved by numerical methods that leads to a better result.

(iii) It has also been observed that presented results for oscillator equations (as in numerical experiments Sections 3.1 and 3.2) are accurate in comparison with [11, 12, 17, 18].

(iv) The parameters method may also provide for large values of amplitude of the motion. The method is also valid for any arbitrary values of α_3, α_5, α, and n. It will provide liberty to solve any kind of nonlinear oscillator problem which is not suitable for established methods.

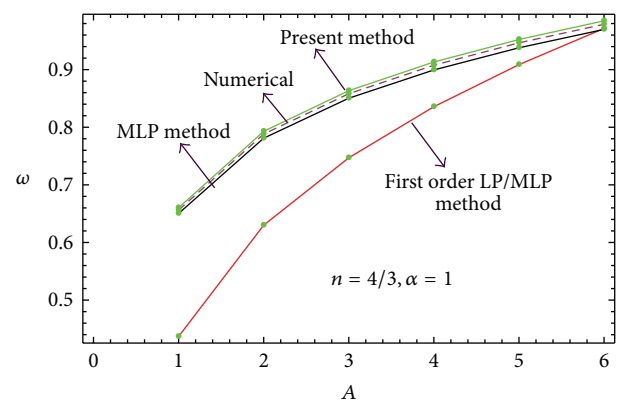

Figure 5: Comparison of various methods for ω versus A of force-deflection oscillator.

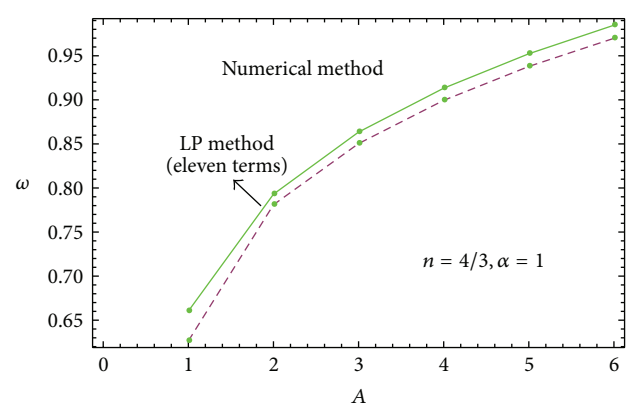

Figure 6: Comparison of LP method and numerical values for ω versus A of force-deflection oscillator.

(v) Figures 4–6 have been plotted for comparison of the numerical results with presented and existing methods in the literature. The LP and MLP methods give the same results for motion in the first approximation.

(vi) The higher order MLP gives minor corrections in LP method. It has been observed that four-parameter approximations give the better results as shown in Figure 5.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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