

Research Article

Differential Equations of Motion for Naturally Curved and Twisted Composite Space Beams

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The differential equations of motion for naturally curved and twisted elastic space beams made of anisotropic materials with noncircular cross sections, being a coupled system consisting of 14 second-order partial differential equations with variable coefficients, are derived theoretically. The warping deformation of beam's cross section, as a new design factor, is incorporated into the differential equations in addition to the anisotropy of material, the curvatures of the rod axis, the initial twist of the cross section, the rotary inertia, and the shear and axial deformations. Numerical examples show that the effect of warping deformation on the natural frequencies of the beam is significant under certain geometric and boundary conditions. This study focuses on improving and consummating the traditional theories to build a general curve beam theory, thereby providing new scientific research reference and design principle for curve beam designers.

1. Introduction

Composite space beams with noncircular cross sections have extensive applications in civil, mechanical, aircraft, and aerospace constructions given their outstanding engineering properties, such as specific strength, specific stiffness, weight, and fatigue life over metallic counterparts. The static and dynamic characteristics of composite beams have attracted considerable attention in recent decades.

Elastic couplings occur among all the forms of deformation given the anisotropy of the material. Moreover, in-plane and out-of-plane displacements may be coupled. Therefore, the differential equations of motion, obtained by the 3D elastic theory under isothermal conditions, become more complex by allowing initial curvature and torsion. Euler-Bernoulli and Timoshenko beam theories are unable to accurately describe the behavior of composite beams.

Many researchers have conducted profound and careful research on nonclassical influences, such as transverse shear deformations, torsion-related warping, and several elastic couplings that can develop into composite beams. The natural frequencies and modes of composite beams under various boundary conditions are obtained by using the first-order

shear deformation theory [1]. Based on the third-order shear theory, the forced vibration response of composite beams is investigated [2]. Li et al. [3] discussed the influence of different support conditions on the natural frequency of composite beams using many numerical examples. In this study, the effect of shear deformation is noted. Ramalingeswara and Ganesan [4] and Zapfe and Lesieutre [5] delivered the natural frequencies of laminated beams by using higher order shear theory and iterative method.

Most of abovementioned studies [1–5] focus on straight beams. The research on the dynamic behavior is far less in composite curved beams than in straight beams considering the curvature and torsion. Yildirim [6] studied the free vibration of symmetrically orthogonal laminated curved beams, in which rotary inertia and axial and shear deformation effects are considered. Tseng et al. [7] studied the free vibration of laminated curved beams with variable curvature based on Timoshenko theory. However, a transverse shear strain is constant along the thickness direction, and the shear shape factor was used in this research to adjust the lateral shear stiffness. The accuracy of the results significantly depends on a selected shear shape factor. Matsunaga [8, 9] used 1D high-order theory to solve the natural frequencies and

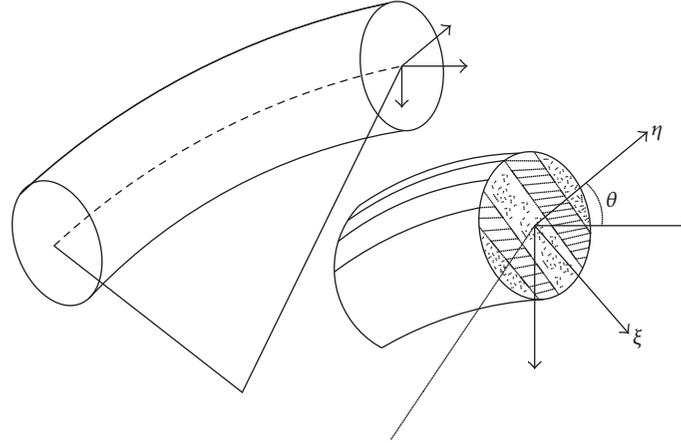


FIGURE 1: Geometry of a naturally curved and twisted anisotropic beam.

buckling loads of composite curved beams subjected to axial concentrated force considering the shear deformation and moment of inertia. Baba and Thoppul [10] explored the flexural strength and vibration characteristics of sandwich laminated curved beams determined the effects of laminated debonding and curvature on their natural frequencies. Yildirim [11] derived the differential equations of motion of spatial curved bars of anisotropic materials, but the examples in this research remain straight beams.

The abovementioned studies [1–11] included one or several items of the anisotropy of the rod material, the curvature and torsion of the rod axis, the effects of rotary inertia, and the shear and axial deformations. The effect of the warping deformation of the cross section on natural frequencies is disregarded. In recent years, many researchers have attempted to establish generalized curved beam theory with warping effect. Kim et al. [12] developed a finite element-based beam analysis for anisotropic beams with arbitrary shaped cross sections through asymptotic expansion method. Displacements at each order are split into fundamental and warping solutions in their research. The influence of the warping effect is analyzed through examples, thereby demonstrating the importance of including the warping degrees of freedom in the dynamic analysis [13]. Yu et al. [14] formulated a generalized finite element-based cross-sectional analysis for inhomogeneous, initial curved and twisted, anisotropic beams, in which the transverse shear deformation is classified as part of the warping field. Ghafari and Rezaeepazhand [15] discussed the free vibration of rotating composite beams with an arbitrary cross section on the basis of dimensional reduction method. These authors decomposed a 3D elasticity problem of the composite beam into a 2D cross-sectional and a 1D beam analysis. Recently, Sapountzakis and Tsipitsis [16–19] used the Analog Equation Method and the Isogeometric Tool to solve the static and dynamic analysis of the curved beams with arbitrary cross section taking into account nonuniform warping and shear deformation effects. Nonuniform warping distributions are taken into account by employing four independent warping parameters

multiplying a shear warping function in each direction and two torsional warping functions in these papers.

As an extension of [11, 20–22], the essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

- (i) The improved beam theory is capable of spatial curved beams with arbitrary cross section, variable curvature, and composite material considering warping effect.
- (ii) The warping function of unidirectional composite beam with rectangular cross section can be obtained using the separation of variables method discussed in [23].
- (iii) Numerical examples are presented for two 3D curved beams with constant curvature and torsion. The results show that the effect of the warping deformation on natural frequencies of the beam is significant under certain geometric and boundary conditions.
- (iv) The influence of the warping deformation on the natural frequency is enhanced with the increase in the anisotropy of materials.

2. Geometrical Characteristics of a Space Anisotropic Beam

Considering a naturally curved and twisted anisotropic beam in space in Figure 1, \mathbf{t} , \mathbf{n} , \mathbf{b} indicate the tangential, normal, and binormal unit vectors of the beam axis, respectively. A second rectangular Cartesian frame (s, ξ, η) is introduced to consider the initial twist of the cross section. The angle between the ξ - and η -axes is $\theta(s)$. For a smooth curve, the Frenet–Serret formulae are

$$\begin{aligned} \mathbf{t}' &= k_1 \mathbf{n}, \\ \mathbf{n}' &= -k_1 \mathbf{t} + k_2 \mathbf{b}, \\ \mathbf{b}' &= -k_2 \mathbf{n}, \end{aligned} \quad (1)$$

where $(\prime) = d/ds$ and s, k_1, k_2 are the arc coordinate, the curvature, and torsion of the curve, respectively. In order to take into account the initial twist of the cross section, a new Cartesian reference frame (x_1, ξ, η) is introduced such that the x_1 -axis is in the direction of \mathbf{t} . If the unit vectors of $O\xi$ and $O\eta$ are represented by \mathbf{i}_ξ and \mathbf{i}_η , then

$$\mathbf{i}_\xi = \mathbf{n} \cos \theta + \mathbf{b} \sin \theta \quad (2)$$

$$\mathbf{i}_\eta = -\mathbf{n} \sin \theta + \mathbf{b} \cos \theta.$$

With simultaneous equations (1) and (2), we have

$$\begin{Bmatrix} \mathbf{t}' \\ \mathbf{i}'_\xi \\ \mathbf{i}'_\eta \end{Bmatrix} = \begin{bmatrix} 0 & k_\eta & -k_\xi \\ -k_\eta & 0 & k_s \\ k_\xi & -k_s & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{t} \\ \mathbf{i}_\xi \\ \mathbf{i}_\eta \end{Bmatrix}, \quad (3)$$

where $k_\xi = k_1 \sin \theta$, $k_\eta = k_1 \cos \theta$, $k_s = k_2 + \theta'$.

3. Constitutive Equations

The following assumptions are used in the present study: the beam material is linearly elastic, homogeneous, and anisotropic. The pretwisting of the cross section is neglected ($k_\xi = 0$). The generalized Hooke's law for linearly elastic materials is

$$\sigma_i = \mathbf{C}_{ij} \mathbf{e}_j \quad (4)$$

or

$$\mathbf{e}_i = \mathbf{S}_{ij} \sigma_j \quad (i, j = 1, 2, \dots, 6), \quad (5)$$

where \mathbf{C}_{ij} , \mathbf{S}_{ij} are the stiffness and flexibility matrices, correspondingly. Let

$$\begin{aligned} \sigma_1 &= \sigma_s, \\ \sigma_2 &= \sigma_\xi, \\ \sigma_3 &= \sigma_\eta, \\ \sigma_4 &= \tau_{\xi\eta}, \\ \sigma_5 &= \tau_{s\eta}, \\ \sigma_6 &= \tau_{s\xi}, \\ e_1 &= e_{ss}, \\ e_2 &= e_{\xi\xi}, \\ e_3 &= e_{\eta\eta}, \\ e_4 &= 2e_{\xi\eta}, \\ e_5 &= 2e_{s\eta}, \\ e_6 &= 2e_{s\xi}. \end{aligned} \quad (6)$$

The following assumptions on stresses are valid in the classical rod theory.

$$\sigma_2 = \sigma_3 = \sigma_4 = 0. \quad (7)$$

σ_β ($\beta = 1, 5, 6$) can be represented as e_β ($\beta = 1, 5, 6$) by solving (7) and (5) simultaneously. Then, the results are substituted to the residual expressions of (5), and the strain components e_m ($m = 2, 3, 4$) are related to e_β ($\beta = 1, 5, 6$) by

$$e_m = S_{mn} \alpha_{n\beta} e_\beta \quad (m = 2, 3, 4; n, \beta = 1, 5, 6). \quad (8)$$

In (8),

$$\begin{aligned} \alpha_{11} &= \frac{S_{55}S_{66} - S_{56}^2}{\Delta}, \\ \alpha_{55} &= \frac{S_{11}S_{66} - S_{16}^2}{\Delta}, \\ \alpha_{66} &= \frac{S_{11}S_{55} - S_{15}^2}{\Delta}, \\ \alpha_{15} &= \alpha_{51} = \frac{S_{16}S_{56} - S_{15}S_{66}}{\Delta}, \\ \alpha_{16} &= \alpha_{61} = \frac{S_{15}S_{56} - S_{16}S_{55}}{\Delta}, \\ \alpha_{56} &= \alpha_{65} = \frac{S_{15}S_{16} - S_{11}S_{56}}{\Delta}, \\ \Delta &= \det \begin{bmatrix} S_{11} & S_{15} & S_{16} \\ S_{15} & S_{55} & S_{56} \\ S_{16} & S_{56} & S_{66} \end{bmatrix}. \end{aligned} \quad (9)$$

So, σ_β ($\beta = 1, 5, 6$) is only related to e_β ($\beta = 1, 5, 6$) but unrelated to e_m ($m = 2, 3, 4$) when (8) is substituted to (4). Then, (4) is reduced to

$$\sigma_k = (C_{k\beta} + C_{km} S_{mn} \alpha_{n\beta}) e_\beta \triangleq Q_{k\beta} e_\beta; \quad (k, \beta, n = 1, 5, 6; m = 2, 3, 4). \quad (10)$$

The displacement components of an arbitrary point of the beam are given as [20–22, 24]

$$\begin{aligned} \mathbf{u}(s, \xi, \eta, t) &= W(s, \xi, \eta, t) \mathbf{t} + U(s, \xi, \eta, t) \mathbf{i}_\xi \\ &\quad + V(s, \xi, \eta, t) \mathbf{i}_\eta. \end{aligned} \quad (11)$$

$\mathbf{u}(s, \xi, \eta, t)$ is not only a function of coordinates s, ξ, η , but also a function of time under dynamic loads, in which

$$\begin{aligned} U(s, \xi, \eta, t) &= u_\xi(s, t) - \eta \varphi_s(s, t), \\ V(s, \xi, \eta, t) &= u_\eta(s, t) + \xi \varphi_s(s, t) \\ W(s, \xi, \eta, t) &= u_s(s, t) + \eta \varphi_\xi(s, t) - \xi \varphi_\eta(s, t) \\ &\quad + \alpha(s, t) \varphi(\xi, \eta), \end{aligned} \quad (12)$$

where $u_\xi(s, t)$, $u_\eta(s, t)$, $u_s(s, t)$ are the three translations, $\varphi_\xi(s, t)$, $\varphi_\eta(s, t)$, $\varphi_s(s, t)$ are the three rotations, $\alpha(s, t)$ is a generalized warping coordinate which is used to describe the decay of deformation along beam length, and $\varphi(\xi, \eta)$ is the

warping function of Saint-Venant's torsion. Let $e_{ss}, e_{s\xi}, e_{s\eta}$ be the axial and shear strain components, respectively,

$$\begin{aligned} g e_{ss} &= \varepsilon_s + \eta \omega_\xi - \xi \omega_\eta + g \varphi \alpha' \\ &+ k_s \left[\left(\frac{\partial \varphi}{\partial \xi} \right) \eta - \left(\frac{\partial \varphi}{\partial \eta} \right) \xi \right] \alpha \\ 2g e_{s\xi} &= G_\xi \varepsilon_\xi - \varphi_\eta - \eta \omega_s + \left[g \left(\frac{\partial \varphi}{\partial \xi} \right) + k_\eta \varphi \right] \alpha \\ 2g e_{s\eta} &= G_\eta \varepsilon_\eta + \varphi_\xi + \xi \omega_s + \left[g \left(\frac{\partial \varphi}{\partial \eta} \right) + k_\xi \varphi \right] \alpha, \end{aligned} \quad (13)$$

in which G_ξ, G_η are shear correction factors, and $g = 1 - \xi k_\eta + \eta k_\xi$. Based on the assumption that the initial curvature of the beam is gentle, then $g \approx 1$. Let

$$\begin{aligned} \varepsilon_s &= u'_s - k_\eta u_\xi + k_\xi u_\eta, \\ \varepsilon_\xi &= u'_\xi - k_s u_\eta + k_\eta u_s, \\ \varepsilon_\eta &= u'_\eta - k_\xi u_s + k_s u_\xi, \\ \omega_s &= \varphi'_s - k_\eta \varphi_\xi + k_\xi \varphi_\eta, \\ \omega_\xi &= \varphi'_\xi - k_s \varphi_\eta + k_\eta \varphi_s, \\ \omega_\eta &= \varphi'_\eta - k_\xi \varphi_s + k_s \varphi_\xi. \end{aligned} \quad (14)$$

Let, for convenience, the resultant forces and moments on the cross section be expressed as $Q_1 = Q_s, Q_5 = Q_\eta, Q_6 = Q_\xi, M_1 = M_s, M_5 = M_\eta, M_6 = M_\xi$ and shown as

$$\begin{aligned} Q_i &= \iint_A \sigma_i dA; \\ M_i &= \varepsilon_{ijk} \iint_A x_j \sigma_k dA; \end{aligned} \quad (15)$$

$(i, j, k = 1, 5, 6),$

where ε_{ijk} is the permutation tensor and $x_1 = s; x_5 = \eta; x_6 = \xi$.

The generalized warping moment is

$$T = \iint_A \varphi \sigma_1 dA. \quad (16)$$

The constitutive equations expressed by six displacement functions $u_i(s), \varphi_i(s)$ ($i = s, \xi, \eta$) and the generalized warping coordinate $\alpha(s)$ can be obtained by substituting (10), (13), and (14) into the above formulations (15)-(16).

4. Equations of Motion

The external force and moments per unit length of the beam axis are expressed as $\mathbf{q}^{\text{ex}}(s, t) = \{q_s^{\text{ex}}(s, t), q_\xi^{\text{ex}}(s, t), q_\eta^{\text{ex}}(s, t)\}$, $\mathbf{m}^{\text{ex}}(s, t) = \{m_s^{\text{ex}}(s, t), m_\xi^{\text{ex}}(s, t), m_\eta^{\text{ex}}(s, t)\}$. Also let $\mathbf{q}^{\text{in}}(s, t) = \{q_s^{\text{in}}(s, t), q_\xi^{\text{in}}(s, t), q_\eta^{\text{in}}(s, t)\}$, $\mathbf{m}^{\text{in}}(s, t) = \{m_s^{\text{in}}(s, t), m_\xi^{\text{in}}(s, t), m_\eta^{\text{in}}(s, t)\}$ be the inertial force and the inertial moment per unit length of the rod axis. Owing to the displacements

and the rotations that are assumed to be infinitesimal, the equations of motion can be written as [11, 24]

$$\begin{aligned} \frac{\partial \mathbf{Q}(s, t)}{\partial s} + \mathbf{q}^{\text{ex}}(s, t) &= \mathbf{q}^{\text{in}}(s, t), \\ \frac{\partial \mathbf{M}(s, t)}{\partial s} + \mathbf{t} \times \mathbf{Q}(s, t) + \mathbf{m}^{\text{ex}}(s, t) &= \mathbf{m}^{\text{in}}(s, t), \end{aligned} \quad (17)$$

in which

$$\begin{aligned} q_s^{\text{in}}(s, t) &= \iint_A \rho \frac{\partial^2 u_s(s, t)}{\partial t^2} d\xi d\eta; \\ q_\xi^{\text{in}}(s, t) &= \iint_A \rho \frac{\partial^2 u_\xi(s, t)}{\partial t^2} d\xi d\eta; \\ q_\eta^{\text{in}}(s, t) &= \iint_A \rho \frac{\partial^2 u_\eta(s, t)}{\partial t^2} d\xi d\eta; \\ m_s^{\text{in}}(s, t) &= \iint_A \rho \xi \frac{\partial^2 u_\eta(s, t)}{\partial t^2} d\xi d\eta \\ &- \iint_A \rho \eta \frac{\partial^2 u_\xi(s, t)}{\partial t^2} d\xi d\eta; \\ m_\xi^{\text{in}}(s, t) &= \iint_A \rho \eta \frac{\partial^2 u_s(s, t)}{\partial t^2} d\xi d\eta; \\ m_\eta^{\text{in}}(s, t) &= - \iint_A \rho \xi \frac{\partial^2 u_s(s, t)}{\partial t^2} d\xi d\eta. \end{aligned} \quad (18)$$

The equation of motion in a scalar form can be obtained by substituting (12)–(14) into (17). The equations of $\partial u_s(s, t)/\partial s, \partial u_\xi(s, t)/\partial s, \partial u_\eta(s, t)/\partial s, \partial \varphi_s(s, t)/\partial s, \partial \varphi_\xi(s, t)/\partial s, \partial \varphi_\eta(s, t)/\partial s$ can be obtained by solving the above-mentioned equations simultaneously. The obtained results are substituted into the derivatives of the generalized warping moment to determine the final differential equations of motion of naturally curved and twisted elastic space beams of anisotropic materials.

$$\frac{\partial T(s, t)}{\partial s} = \left(\iint \sigma_s \varphi d\xi d\eta \right)'. \quad (19)$$

5. Applications

5.1. Differential Equations of Motion for the Examples. A simple example is developed here for illustration. Two cylindrical helical springs with rectangular cross section can be regarded as 3D curved beams with constant curvature and torsion (Figure 2). The warping function of unidirectional composite beam with rectangular cross section (Figure 3) can be obtained using the separation of variables method [23]:

$$\begin{aligned} \varphi(\xi, \eta) &= \xi \eta - \sum_{n=1,3,5}^{\infty} \left(\frac{c_n e^{\alpha \gamma_n b}}{\alpha \gamma_n (1 + e^{2\alpha \gamma_n b})} e^{\alpha \gamma_n \eta} \right. \\ &\left. - \frac{c_n e^{\alpha \gamma_n b}}{\alpha \gamma_n (1 + e^{2\alpha \gamma_n b})} e^{-\alpha \gamma_n \eta} \right) \sin(\gamma_n \xi), \end{aligned} \quad (20)$$

where $\alpha = \sqrt{G_{x\xi}/G_{x\eta}}$, $\gamma_n = n\pi/2a$, $c_n = 16a^2(-1)^{(n-1)/2}/n^2\pi^2$, ($n = 1, 3, 5, \dots$).

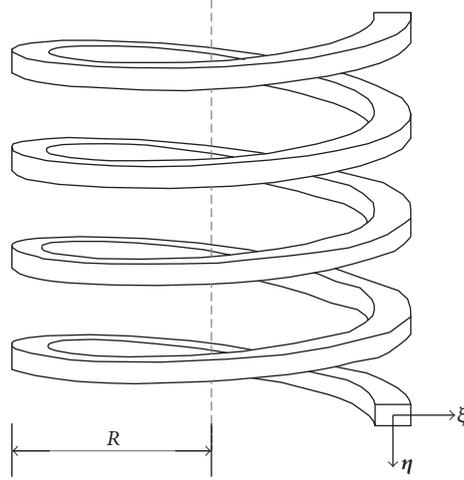


FIGURE 2: Cylindrical helical spring.

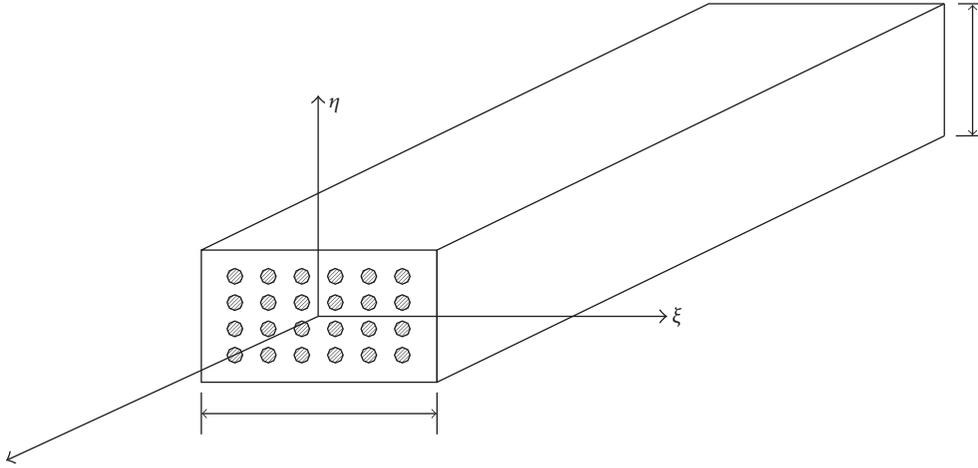


FIGURE 3: Unidirectional composite beam.

The differential equations of motion for cylindrical helical springs with rectangular cross section made of unidirectional composite materials are described as follows:

$$\begin{aligned} \frac{\partial u_s(s, t)}{\partial s} = & k_\eta u_\xi(s, t) + \left[\frac{k_s Q_{11} D_1 (D_{10} D_3 I_2 + D_2 D_9 I_3 - D_{12} I_3 I_2)}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3)} - \frac{k_s Q_{11} D_4}{A_{11}} \right] \alpha(s, t) + \left[\frac{1}{A_{11}} \right. \\ & \left. - \frac{Q_{11} D_1^2 I_2 I_3}{A_{11} (A_{11} D_3^2 I_2 + A_{11} D_2^2 I_3 - A_{11} D_{11} I_2 I_3 + Q_{11} D_1^2 I_2 I_3)} \right] Q_s(s, t) \\ & - \left(\frac{D_1 D_2 I_3}{A_{11} D_3^2 I_2 + A_{11} D_2^2 I_3 - A_{11} D_{11} I_2 I_3 + Q_{11} D_1^2 I_2 I_3} \right) M_\xi(s, t) + \left(\frac{D_1 D_3 I_2}{A_{11} D_3^2 I_2 + A_{11} D_2^2 I_3 - A_{11} D_{11} I_2 I_3 + Q_{11} D_1^2 I_2 I_3} \right) \\ & \cdot M_\eta(s, t) + \left(\frac{D_1 I_2 I_3}{A_{11} D_3^2 I_2 + A_{11} D_2^2 I_3 - A_{11} D_{11} I_2 I_3 + Q_{11} D_1^2 I_2 I_3} \right) T(s, t), \end{aligned}$$

$$\frac{\partial u_\xi(s, t)}{\partial s} = -k_\eta u_s(s, t) + k_s u_\eta(s, t) + \varphi_\eta(s, t) - \frac{D_5 Q_{66} + k_\eta D_1 Q_{66}}{A_{66} G_\xi} \alpha(s, t) + \frac{1}{A_{66} G_\xi} Q_\xi(s, t),$$

$$\frac{\partial u_\eta(s, t)}{\partial s} = -k_s u_\xi(s, t) - \varphi_\xi(s, t) - \frac{D_6 Q_{55}}{A_{55} G_\eta} \alpha(s, t) + \frac{1}{A_{55} G_\eta} Q_\eta(s, t),$$

$$\frac{\partial \varphi_s(s, t)}{\partial s} = k_\eta \varphi_\xi(s, t) + \frac{-D_7 Q_{55} + D_8 Q_{66} + k_\eta D_2 Q_{66}}{I_3 Q_{55} + I_2 Q_{66}} \alpha(s, t) + \frac{1}{I_3 Q_{55} + I_2 Q_{66}} M_s(s, t),$$

$$\begin{aligned} \frac{\partial \varphi_\xi(s, t)}{\partial s} &= -k_\eta \varphi_s + k_s \varphi_\eta - \left(\frac{D_1 D_2 I_3}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} \right) Q_s(s, t) \\ &+ \frac{k_s [A_{11} (D_{10} D_3 D_2 - D_3^2 D_9 - D_2 D_{12} I_3 + D_{11} D_9 I_3) + Q_{11} I_3 D_1 (D_2 D_4 - D_1 D_9)]}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} \alpha(s, t) + \left[\frac{1}{I_2 Q_{11}} \right. \\ &- \left. \frac{A_{11} D_2^2 I_3}{Q_{11} I_2 [A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3]} \right] M_\xi(s, t) + \frac{A_{11} D_2 D_3}{Q_{11} [A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3]} \\ &\cdot M_\eta(s, t) + \frac{A_{11} D_2 I_3}{Q_{11} [A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3]} T(s, t), \end{aligned}$$

$$\begin{aligned} \frac{\partial \varphi_\eta(s, t)}{\partial s} &= -k_s \varphi_\xi(s, t) + \left(\frac{D_1 D_3 I_2}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} \right) Q_s(s, t) \\ &+ \frac{k_s [A_{11} (D_{10} D_2^2 - D_3 D_2 D_9 + D_3 D_{12} I_2 - D_{10} D_{11} I_2) + Q_{11} I_2 D_1 (D_1 D_{10} - D_3 D_4)]}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} \alpha(s, t) \\ &+ \frac{A_{11} D_2 D_3}{Q_{11} [A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3]} M_\xi(s, t) + \left[\frac{1}{I_3 Q_{11}} \right. \\ &- \left. \frac{A_{11} D_3^2 I_2}{Q_{11} I_3 [A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3]} \right] M_\eta(s, t) - \frac{A_{11} D_3 I_2}{Q_{11} [A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3]} \\ &\cdot T(s, t), \end{aligned}$$

$$\begin{aligned} \frac{\partial \alpha(s, t)}{\partial s} &= -\frac{k_s [A_{11} (D_{10} D_3 I_2 + I_3 D_2 D_9 - I_3 D_{12} I_2) + Q_{11} D_1 D_4 I_2 I_3]}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} \alpha(s, t) \\ &+ \frac{D_1 I_3 I_2}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} Q_s(s, t) + \frac{A_{11} D_2 I_3}{Q_{11} [A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3]} M_\xi(s, t) \\ &- \frac{A_{11} D_3 I_2}{Q_{11} [A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3]} M_\eta(s, t) - \frac{A_{11} I_3 I_2}{Q_{11} [A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3]} \\ &\cdot T(s, t), \end{aligned}$$

$$\frac{\partial Q_s(s, t)}{\partial s} = k_\eta Q_\xi(s, t) + \rho A \frac{\partial^2 u_s(s, t)}{\partial t^2} + \rho D_1 \frac{\partial^2 \alpha(s, t)}{\partial t^2},$$

$$\frac{\partial Q_\xi(s, t)}{\partial s} = -k_\eta Q_s(s, t) + k_s Q_\eta(s, t) + \rho A \frac{\partial^2 u_\xi(s, t)}{\partial t^2},$$

$$\frac{\partial Q_\eta(s, t)}{\partial s} = -k_s Q_\xi(s, t) + \rho A \frac{\partial^2 u_\eta(s, t)}{\partial t^2},$$

$$\frac{\partial M_s(s, t)}{\partial s} = k_\eta M_\xi(s, t) + \rho I_1 \frac{\partial^2 \varphi_s(s, t)}{\partial t^2},$$

$$\begin{aligned}
\frac{\partial M_\xi(s, t)}{\partial s} &= Q_\eta(s, t) - k_\eta M_s(s, t) + k_s M_\eta(s, t) + \rho I_2 \frac{\partial^2 \varphi_\xi(s, t)}{\partial t^2} + \rho D_2 \frac{\partial^2 \alpha(s, t)}{\partial t^2}, \\
\frac{\partial M_\eta(s, t)}{\partial s} &= -Q_\xi - k_s M_\xi + \rho I_3 \frac{\partial^2 \varphi_\eta(s, t)}{\partial t^2} - \rho D_3 \frac{\partial^2 \alpha(s, t)}{\partial t^2}, \\
\frac{\partial T(s, t)}{\partial s} &= \rho D_{11} \ddot{\alpha} + \rho D_1 \ddot{u}_s + \rho D_2 \ddot{\varphi}_\xi - \rho D_3 \ddot{\varphi}_\eta + \left\{ D_{16} Q_{55} + D_{14} Q_{66} + (2D_{15} + D_{11} k_\eta) k_\eta Q_{66} + D_{13} k_s^2 Q_{11} \right. \\
&\quad - \frac{(D_7 Q_{55} - (D_8 + D_2 k_\eta) Q_{66})^2}{I_3 Q_{55} + I_2 Q_{66}} + \frac{D_3^2 D_{10}^2 I_2 k_s^2 Q_{11} A_{11}}{I_3 (A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3)} \\
&\quad - \frac{2k_s^2 Q_{11} A_{11} (D_{12} D_{10} D_3 I_2 + D_9 D_{12} D_2 I_3)}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} + \frac{2D_9 D_{10} D_3 D_4 k_s^2 Q_{11} A_{11}}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} \\
&\quad + \frac{D_{12}^2 I_2 I_3 k_s^2 Q_{11} A_{11}}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} + \frac{2k_s^2 Q_{11}^2 D_1 D_4 (D_{10} D_3 I_2 + D_2 D_9 I_3)}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} \\
&\quad + \frac{D_2^2 D_9 I_3 k_s^2 Q_{11} A_{11}}{I_2 (A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3)} - \frac{k_s^2 Q_{11} D_9^2}{I_2} - \frac{k_s^2 Q_{11} D_{10}^2}{I_3} - \frac{k_s^2 Q_{11} D_4^2}{A_{11}} \\
&\quad + \frac{D_1^2 D_4^2 I_2 I_3 k_s^2 Q_{11}^3}{A_{11} (A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3)} - \frac{D_6^2 Q_{55}^2}{A_{55}} - \frac{(D_5 + D_1 k_\eta)^2 Q_{66}^2}{A_{66}} \left. \right\} \alpha(s, t) \\
&\quad + \frac{k_s Q_{11} [D_4 (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) - D_1 (D_{10} D_3 I_2 + D_2 D_9 I_3 - D_{12} I_3 I_2)]}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} Q_s(s, t) + \frac{(D_5 + D_1 k_\eta) Q_{66}}{A_{66}} Q_\xi(s, t) \\
&\quad + \frac{D_6 Q_{55}}{A_{55}} Q_\eta(s, t) + \frac{D_7 Q_{55} - (D_8 + D_2 k_\eta) Q_{66}}{I_3 Q_{55} + I_2 Q_{66}} M_s(s, t) \\
&\quad - \frac{k_s A_{11} (D_{10} D_2 D_3 - D_3^2 D_9 - D_{12} D_2 I_3 + D_{11} D_9 I_3) + k_s Q_{11} D_1 I_3 (D_2 D_4 - D_1 D_9)}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} M_\xi(s, t) \\
&\quad - \frac{k_s A_{11} (D_9 D_2 D_3 + D_2^2 D_{10} + D_{12} D_3 I_2 - D_{11} D_{10} I_2) + k_s Q_{11} D_1 I_2 (D_1 D_{10} - D_3 D_4)}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} M_\eta(s, t) \\
&\quad + \frac{k_s A_{11} (D_9 D_2 I_3 + D_{10} D_3 I_2 - D_{12} I_2 I_3) + k_s Q_{11} D_4 I_2 I_3}{A_{11} (D_3^2 I_2 + D_2^2 I_3 - D_{11} I_2 I_3) + Q_{11} D_1^2 I_2 I_3} T(s, t),
\end{aligned} \tag{21}$$

where

$$A_{11} = \iint_A Q_{11} d\xi d\eta,$$

$$A_{55} = \iint_A Q_{55} d\xi d\eta,$$

$$A_{66} = \iint_A Q_{66} d\xi d\eta,$$

$$D_1 = \iint_A \varphi d\xi d\eta,$$

$$D_2 = \iint_A \varphi \eta d\xi d\eta,$$

$$D_3 = \iint_A \varphi \xi d\xi d\eta,$$

$$D_4 = \iint_A \left[\left(\frac{\partial \varphi}{\partial \xi} \right) \eta - \left(\frac{\partial \varphi}{\partial \eta} \right) \xi \right] d\xi d\eta,$$

$$D_5 = \iint_A \frac{\partial \varphi}{\partial \xi} d\xi d\eta$$

$$D_6 = \iint_A \frac{\partial \varphi}{\partial \eta} d\xi d\eta,$$

$$D_7 = \iint_A \xi \frac{\partial \varphi}{\partial \eta} d\xi d\eta,$$

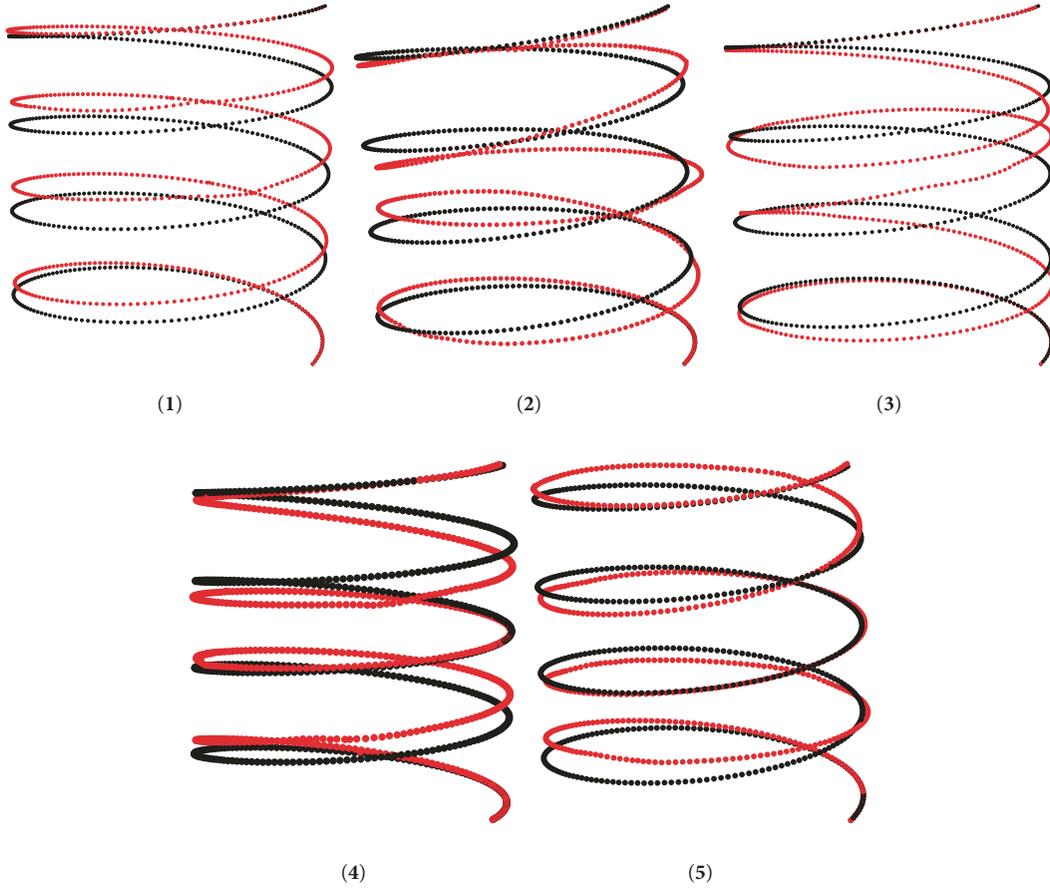


FIGURE 4: First five mode shapes of Beam 1: (1) first mode (523.25 Hz); (2) second mode (897.71 Hz); (3) third mode (914.36 Hz); (4) fourth mode (973.88 Hz); (5) fifth mode (1118.41 Hz).

$$\begin{aligned}
 D_8 &= \iint_A \eta \frac{\partial \varphi}{\partial \xi} d\xi d\eta, \\
 D_{12} &= \iint_A \varphi \left[\left(\frac{\partial \varphi}{\partial \xi} \right) \eta - \left(\frac{\partial \varphi}{\partial \eta} \right) \xi \right] d\xi d\eta \\
 D_9 &= \iint_A \eta \left[\left(\frac{\partial \varphi}{\partial \xi} \right) \eta - \left(\frac{\partial \varphi}{\partial \eta} \right) \xi \right] d\xi d\eta, \\
 D_{10} &= \iint_A \xi \left[\left(\frac{\partial \varphi}{\partial \xi} \right) \eta - \left(\frac{\partial \varphi}{\partial \eta} \right) \xi \right] d\xi d\eta, \\
 D_{11} &= \iint_A \varphi^2 d\xi d\eta \\
 D_{13} &= \iint_A \left[\left(\frac{\partial \varphi}{\partial \xi} \right) \eta - \left(\frac{\partial \varphi}{\partial \eta} \right) \xi \right]^2 d\xi d\eta, \\
 D_{14} &= \iint_A \left(\frac{\partial \varphi}{\partial \xi} \right)^2 d\xi d\eta, \\
 D_{15} &= \iint_A \varphi \left(\frac{\partial \varphi}{\partial \xi} \right) d\xi d\eta \\
 D_{16} &= \iint_A \left(\frac{\partial \varphi}{\partial \eta} \right)^2 d\xi d\eta.
 \end{aligned} \tag{22}$$

5.2. *Effects of Material on the Natural Frequencies with Warping.* The geometric parameters and finite element meshes of the two beams are exactly the same: the cylinder diameter $R = 5$ mm, the helix pitch angle $\bar{\alpha} = 5^\circ$, the number of active turns $n = 4$, and the rectangular cross section of $2a = 1$ mm in the ξ -direction and $2b = 0.4$ mm in the η -direction. The only difference between them is the material properties. Beam 1 is made of isotropic materials (the density is $\rho = 7900$ kg/m³, the modulus of elasticity is $E = 206$ GPa, and Poisson's ratio is $\mu = 0.3$), while Beam 2 is made of unidirectional composite materials (the modulus of elasticity in the three main directions is $E_1 = 181$ GPa, $E_2 = E_3 = 10.3$ GPa, the modulus of shearing of the three planes is $G_{23} = 3.433$ GPa, $G_{12} = G_{13} = 7.17$ GPa, the main Poisson's ratio is $\mu_{23} = 0.5$, $\mu_{12} = \mu_{13} = 0.28$, and the density is $\rho = 1600$ kg/m³). We partitioned Beam 1 (Beam 2) into 720 Solid45 (Solid46) elements using ANSYS for analysis. The total number of nodal points is 1452. The first five natural frequencies of Beam 1 and Beam 2 are listed in Tables 1 and 2. The first five normal mode shapes of beams and the generalized warping coordinate are demonstrated in Figures 4–7, respectively.

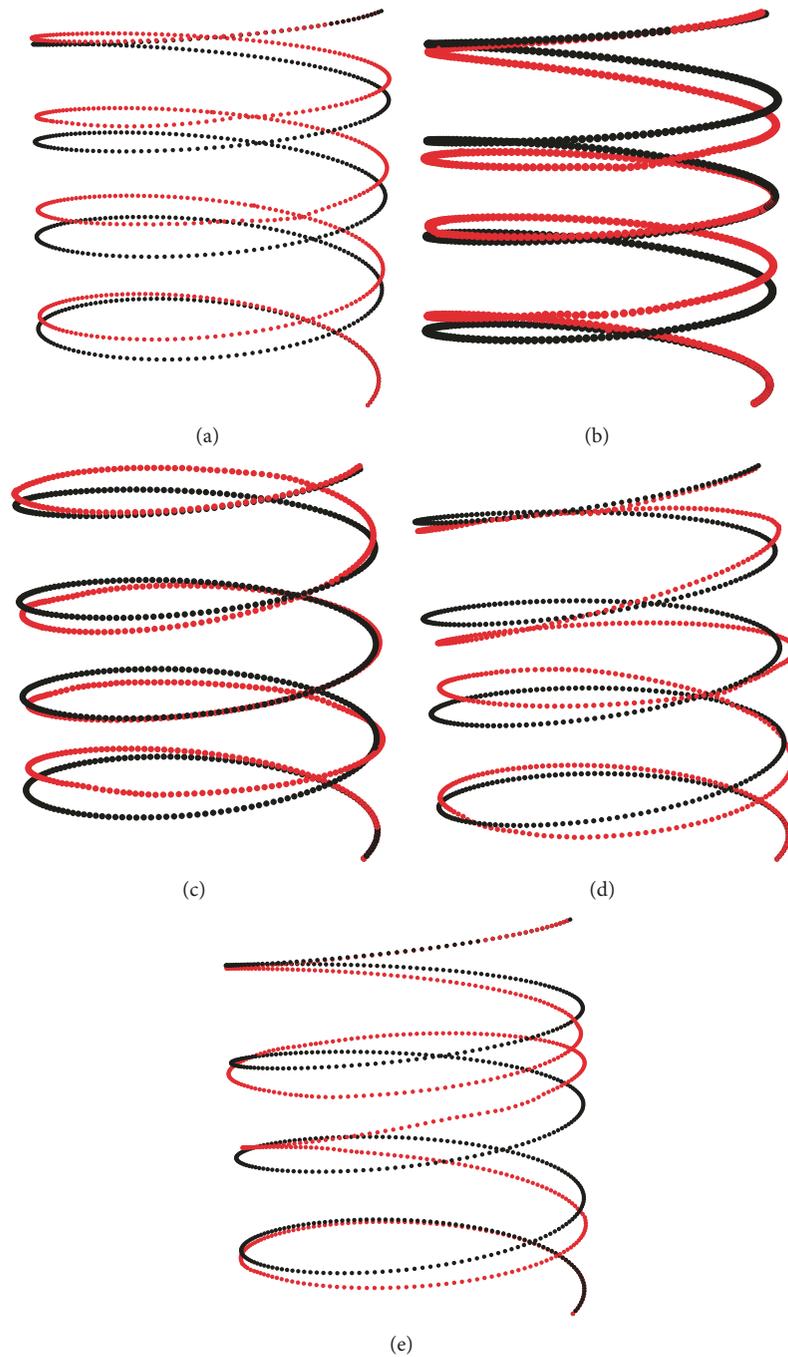


FIGURE 5: First five mode shapes of Beam 2 (made of unidirectional composite material): (a) first mode (358.21 Hz); (b) second mode (687.63 Hz); (c) third mode (805.74 Hz); (d) fourth mode (903.08 Hz); (e) fifth mode (958.56 Hz).

6. Conclusion

The differential equations of motion for elastic space beams with noncircular cross sections and anisotropic materials have been derived systematically based on naturally curved and twisted beam theory. In the equations, the warping deformation of the cross section, the initial twist of the cross section, the shear and axial

deformations, and the rotary inertia are studied theoretically.

The results show that the effect of warping deformations on the natural frequencies of the beam is very significant under certain geometric and boundary conditions. The errors can be excessively large when the warping effect is disregarded. The influence of warping on the natural frequency is enhanced with the increase in the anisotropy of materials.

TABLE 1: The first five frequencies of Beam 1 (made of isotropic material).

Mode number (Hz)	FEM	Present study (PS)			
		Warping ignored	Errors (%)	Warping included	Errors (%)
1	521.34	745.84	43.06	523.25	0.37
2	899.54	1009.30	12.20	897.71	0.20
3	915.97	1044.17	13.99	914.36	0.17
4	970.21	1122.22	15.67	973.88	0.38
5	1129.5	1338.65	18.52	1118.41	0.98

TABLE 2: The first five frequencies of Beam 2 (made of unidirectional composite material).

Mode number (Hz)	FEM	Present study (PS)			
		Warping ignored	Errors (%)	Warping included	Errors (%)
1	351.54	512.80	45.87	358.21	1.89
2	675.19	981.32	45.34	687.63	1.84
3	787.86	1117.87	41.89	805.74	2.27
4	880.95	1200.09	36.23	903.08	2.51
5	940.96	1357.22	44.24	958.56	1.87

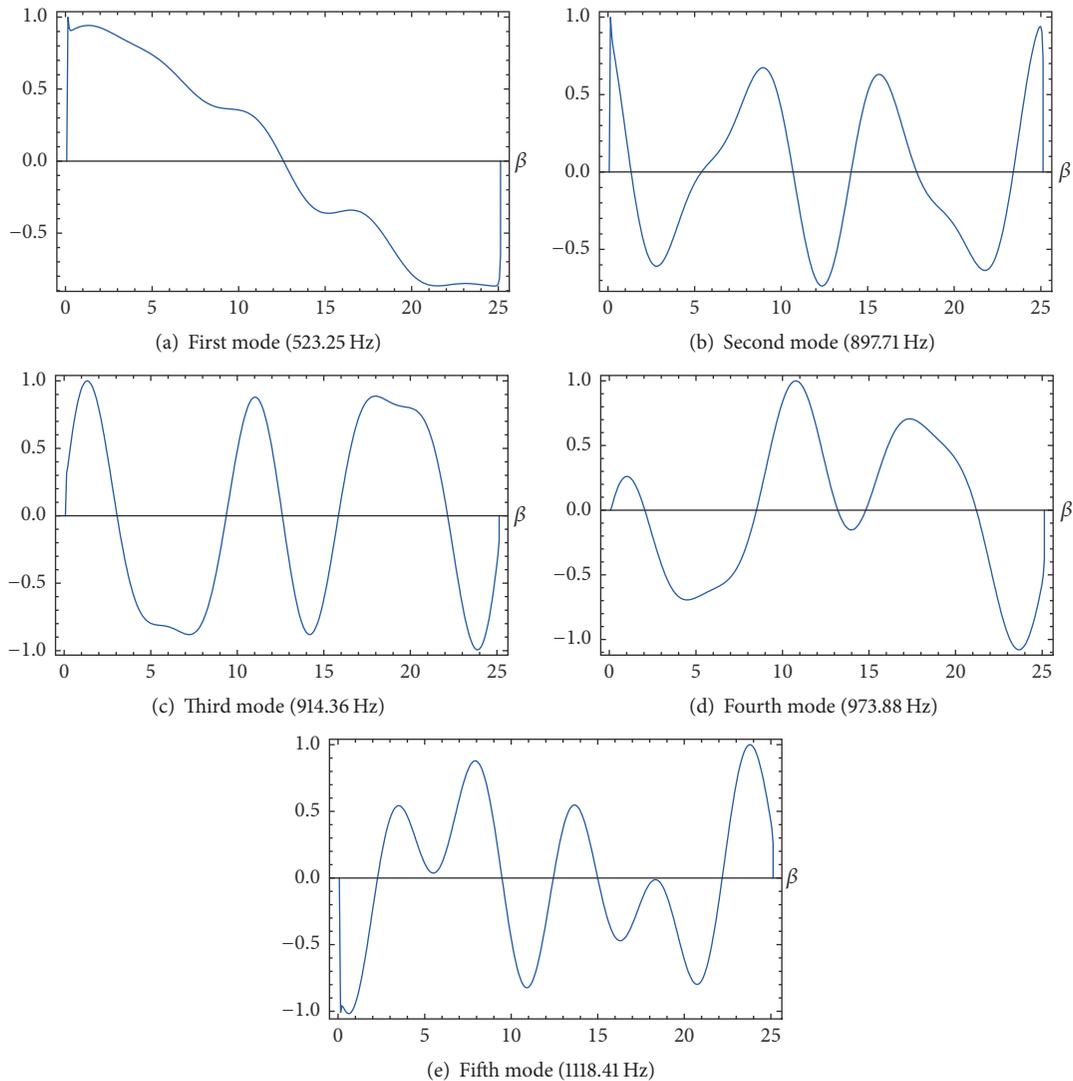


FIGURE 6: First five normal mode shapes of the generalized warping coordinate of Beam 1.

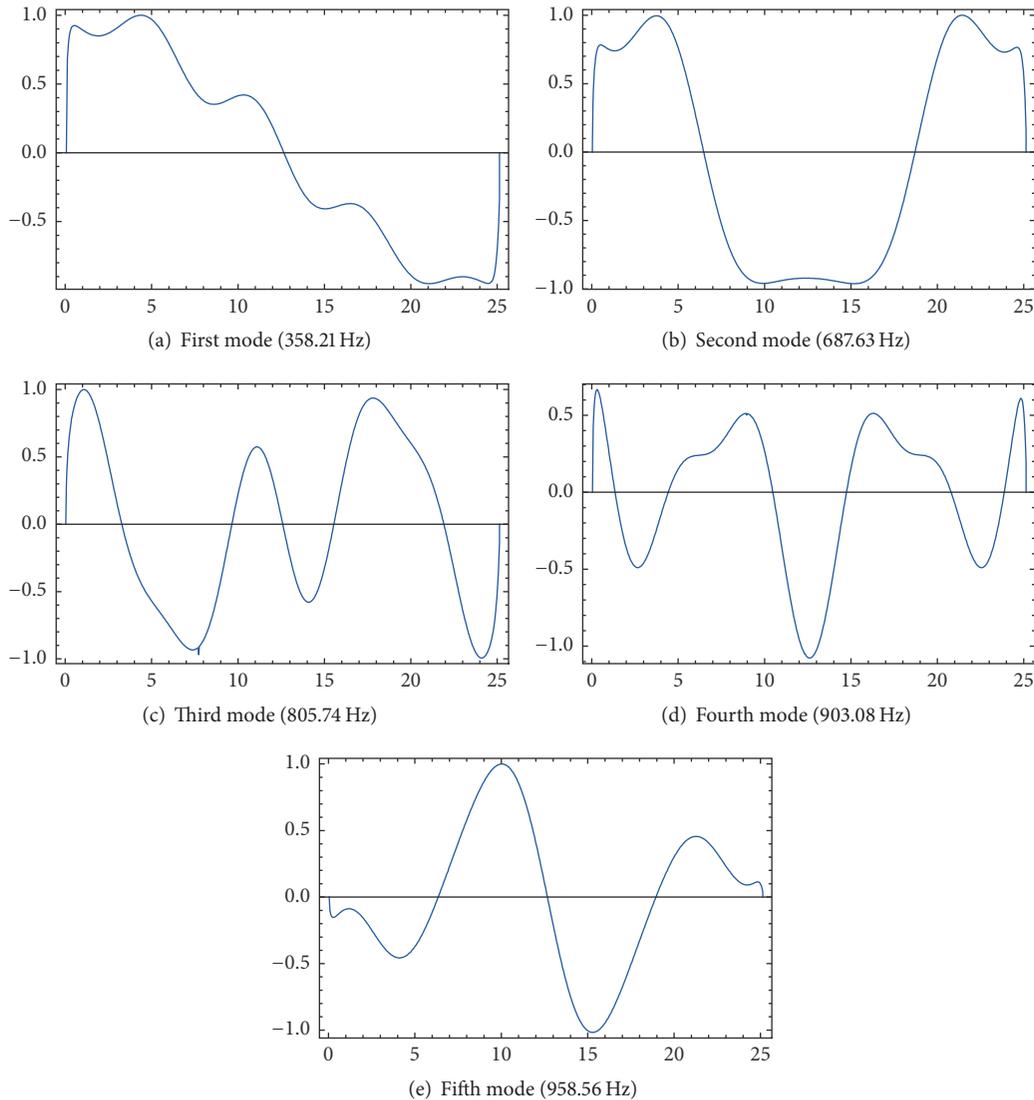


FIGURE 7: First five normal mode shapes of the generalized warping coordinate of Beam 2.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

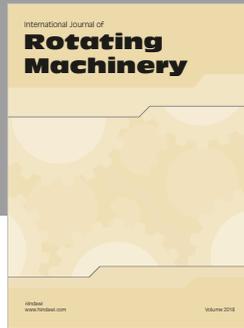
Acknowledgments

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