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Research Article

Mathematical Formulation of Soft-Contact Problems for Various Rheological Models of Damper

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The paper deals with analysis of selected soft-contact problems in discrete mechanical systems. Elastic-dissipative rheological schemes representing dampers as well as the notion of unilateral constraints were used in order to model interaction between colliding bodies. The mathematical descriptions of soft-contact problems involving variational inequalities are presented. The main finding of the paper is a method of description of soft-contact phenomenon between rigid object and deformable rheological structure by the system of explicit nonlinear differential-algebraic equations easy for numerical implementation. The results of simulations, that is, time histories of displacements and contact forces as well as hysteretic loops, are presented.

1. Introduction

Impact is a short-lived phenomenon of energy exchange between colliding bodies. The velocities of colliding bodies change rapidly and the reactions are impulsive in nature which means that the interactions are short-lived and reach large values. Classical theory of impact mechanics applied for rigid bodies assumes that the collision phenomenon results with discontinuous change of velocities and the reaction are modeled as impulses. However, such idealization may not be valid in many mechanical problems especially when we need to evaluate a history of reactions within a short time of collision.

In the model of impact of two elastic spheres presented by Panovko and Stronge [1, 2] the local deformability of these spheres is assumed with use of nonlinear spring possessing the following property $f(x) := kx^{3/2}$, where k denotes a parameter depending on the spheres' radii and material. The Hertz model of impact can be used for modelling of elastic collisions.

The analysis of impact and contact problems in discrete mechanical systems has received a great deal of attention in the literature [3–5]. The mathematical description of such problems involves the notion of nonsmooth mechanics and needs a special numerical treatment [6–8].

In this paper we will apply rheological schemes representing deformable dampers and unilateral constraints for modeling of interaction between colliding bodies. The phenomenon of contact between rigid objects and deformable rheological structures is sometimes called "soft contact" [9]. The models we analyze in this paper allow evaluation of reactions during the time of collision.

The soft-contact models analyzed in our paper involve two cases of interaction between rigid bodies. The first case concerns the impact of two rigid bodies through a deformable built-in damper. In order to simplify mathematical description we assumed that one of these bodies possesses an infinite mass. The second case of interaction applying soft-contact models describes a rigid body interaction with discrete modelling of compliance for the contact region. In such models the compliance of the contact region is modelled with use of rheological schemes (see [2]).

We will analyze both linear viscoelastic schemes and nonlinear elastoplastic and viscoelastoplastic models of dampers.

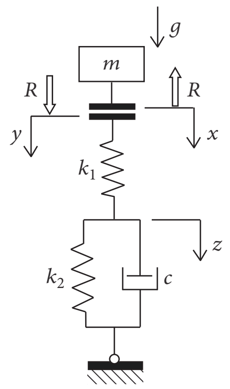

Figure 1: Visualization of viscoelastic system.

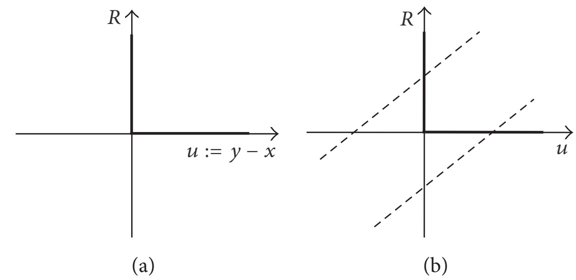

Figure 2: Graph of the mapping expressed via (1d) (a) and graphical solution leading to (4b) (b).

Using unilateral constraints one can model contact interactions between bodies. The mathematical descriptions of soft-contact problems will be presented. The notion of variational inequalities will be used in order to describe unilateral constraints and frictional properties of rheological models of dampers.

The main finding of the paper is a method we propose allowing description of the soft-contact problems with use of nonlinear explicit differential-algebraic equations. Numerical solution of such equations can be obtained using classical algorithms. We will show the results of numerical simulations in order to demonstrate the validity of the proposed formulation.

2. Soft-Contact Problem Formulation with Use of Viscoelastic Rheological Scheme

Let us consider the description of the soft-contact phenomenon in the system visualized in Figure 1. This system is composed of a rigid body (material point) possessing a mass m and of a mass-less linear viscoelastic rheological model of damper having stiffness parameters k_1 and k_2 and viscosity c. The unilateral constraints are visualized in Figure 1 by two horizontal bold lines representing mass-less bumpers. We also assume the gravity load acting where g denotes gravity constant.

The coordinates x, y, and z shown in Figure 1 determine the displacement of the body and deformation of the rheological structure. R denotes the force of mutual interaction between the body and the structure. The configuration shown in Figure 1 represents such a time instant when $x = 0$ and $y = 0$ and $z = 0$. In case of this configuration both bumpers are in contact but the springs' and dashpot's forces as well as the reaction R are equal to zero.

The description of the soft-contact problem for the analyzed system may be obtained by formulation of the equation of the body's motion and evolution of the structure's deformation.

The equations describing the soft-contact problem are as follows:

$$m\ddot{x} = mg - R \tag{1a}$$

$$k_1 (y - z) = R \tag{1b}$$

$$c\dot{z} + k_2 z = R \tag{1c}$$

$$\begin{gathered} u := y - x, \quad u \geq 0, \\ R(\tilde{u} - u) \geq 0 \quad \forall \tilde{u} \geq 0, \end{gathered} \tag{1d}$$

where (1a) describes the motion of the material point m while (1b) and (1c) represent the evolution of the structure's deformation. The system of equations (1d) defines the relationships between a coordinate $u := y - x$ and the reaction R. It should be emphasized that despite the linearity of viscoelastic scheme to be analyzed, the system of equations (1) is nonlinear because of the form of constraints defined in (1d) involving variational inequality.

We will prove that (1d) can be visualized via a mapping shown in Figure 2(a). Let us divide the set $\{u \geq 0\}$ into two subsets $\{u = 0\} \cup \{u > 0\}$. If $u > 0$, then the inequality $R(\tilde{u} - u) \geq 0$ $\forall \tilde{u} \geq 0$ is satisfied only for $R = 0$. If $u = 0$, the inequality transforms to the following form $R\tilde{u} \geq 0$ $\forall \tilde{u} \geq 0$ being satisfied for $R \geq 0$. The result of the proof is shown in Figure 2(a).

Our objective is to transform (1a), (1b), (1c), and (1d) to the system of explicit nonlinear differential-algebraic equations. The expected system of equations should have the following form:

$$\ddot{x} = -\frac{R}{m} + g \tag{2a}$$

$$\dot{z} = \frac{1}{c} (R - k_2 z) \tag{2b}$$

$$y = f_y (x, z) \tag{2c}$$

$$R = f_R (x, z), \tag{2d}$$

where (2a) and (2b) can be easily obtained using (1a) and (1c), respectively.

The fundamental problem is to find the functions f_y and f_R describing time histories of displacement y and reaction R, respectively. Let us note that (1b) can be rewritten in the following equivalent form:

$$k_1 (y - x) + k_1 (x - z) = R. \tag{3}$$

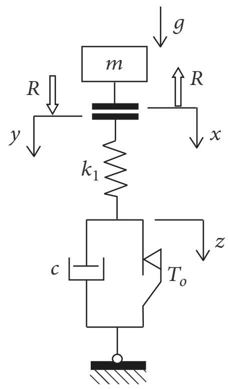

FIGURE 3: Visualization of elastic-viscoplastic system.

Equation (3) is a linear relation between R and $u := y - x$. Thus, using (3) and the mapping shown in Figure 2(a) leads to the solution as follows:

$$f_y(x,z) = \begin{cases} x & \text{if } x - z > 0 \\ z & \text{if } x - z \le 0 \end{cases} \tag{4a}$$

$$f_R(x,z) = \begin{cases} k_1(x - z) & \text{if } x - z > 0 \\ 0 & \text{if } x - z \le 0. \end{cases} \tag{4b}$$

Graphical solution leading to (4b) is visualized in Figure 2(b) where the dashed lines represent two possible locations of the function expressed by (3) for $x - z > 0$ and $x - z < 0$.

3. Elastic-Viscoplastic Rheological Scheme

This section is devoted to analysis of a soft-contact problem with use of elastic-viscoplastic rheological scheme shown in Figure 3. The scheme in Figure 3 was obtained using Figure 1 and replacing the spring k_2 by the slider modelling dry-friction or plasticity phenomena [10]. We assume that the slider force is denoted by T while the limit force is T_o.

The equations defining the soft-contact problem have the following form:

$$m\ddot{x} = mg - R \tag{5a}$$

$$k_1(y - z) = R \tag{5b}$$

$$c\dot{z} + T = R \tag{5c}$$

$$\begin{gathered} u := y - x, \quad u \ge 0, \\ R(\tilde{u} - u) \ge 0 \quad \forall \tilde{u} \ge 0 \end{gathered} \tag{5d}$$

$$\begin{gathered} T \in [-T_o, +T_o], \\ \dot{z}\left(T - \widetilde{T}\right) \ge 0 \quad \forall \widetilde{T} \in [-T_o, +T_o]. \end{gathered} \tag{5e}$$

The relationships expressed via (5e) describe constitutive characteristics of the slider. A graphical visualization of (5e) is shown in Figure 4 [8, 11].

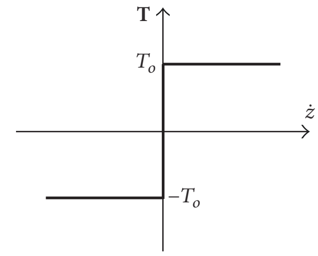

FIGURE 4: Constitutive relationship of frictional element (slider).

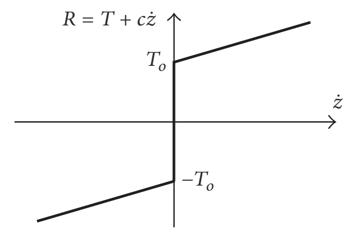

FIGURE 5: Constitutive relationship of viscoplastic system (slider and dashpot in parallel) (see Figure 3).

In order to prove the equivalence of (5d) and the mapping in Figure 4 let us divide the set $[-T_o, +T_o]$ into three subsets $\{-T_o\} \cup \{(-T_o, +T_o)\} \cup \{+T_o\}$.

If $T \in (-T_o, +T_o)$ or $|T| < T_o$ then the inequality $\dot{z}(T - \widetilde{T}) \ge 0$ $\forall \widetilde{T} \in [-T_o, +T_o]$ is satisfied only if $\dot{z} = 0$. If $T \in \{+T_o\}$ then the inequality $\dot{z}(T_o - \widetilde{T}) \ge 0$ $\forall \widetilde{T} \in [-T_o, +T_o]$ is satisfied for $\dot{z} \ge 0$. Finally, if $T \in \{-T_o\}$ then the inequality $\dot{z}(-T_o - \widetilde{T}) \ge 0$ $\forall \widetilde{T} \in [-T_o, +T_o]$ is satisfied for $\dot{z} \le 0$.

We will demonstrate that the system of equations (5a), (5b), (5c), (5d), and (5e) can be replaced by the following differential-algebraic equations:

$$\ddot{x} = -\frac{R}{m} + g \tag{6a}$$

$$\dot{z} = f_z(y,z) \tag{6b}$$

$$y = f_y(x,z) \tag{6c}$$

$$R = f_R(x,z) \tag{6d}$$

$$T = f_T(y,z). \tag{6e}$$

Let us note that the functions f_y and f_R were formulated in previous section (see (4a) and (4b)). Our objective in this section is to find the mappings f_z and f_T describing time history of velocity $\dot{z}$ and friction force T, respectively. Substituting (5b) into (5c) gives

$$T + c\dot{z} = k_1(y - z). \tag{7}$$

Using Figure 4 the left-hand side of (7) can be visualized in a graph as it was shown in Figure 5. The multivalued function shown in Figure 5 is a constitutive relationship of the system of two elements joined in parallel, the slider and the dashpot (see Figure 3).

The unknown functions f_z and f_T can be determined using (7) and the graph shown in Figure 5. The results are as follows:

$$f_z(y,z) = \begin{cases} 0 & \text{if } k_1|y-z| \le T_o \\ \dfrac{k_1|y-z| - T_o}{c} \cdot \operatorname{sign}(y-z) & \text{if } k_1|y-z| > T_o \end{cases} \quad (8a)$$

$$f_T(y,z) = \begin{cases} k_1(y-z) & \text{if } k_1|y-z| \le T_o \\ T_o \cdot \operatorname{sign}(y-z) & \text{if } k_1|y-z| > T_o. \end{cases} \quad (8b)$$

4. Elastoplastic Rheological Scheme

Let us analyze nonlinear elastoplastic rheological scheme shown in Figure 4. The model contains two elastic elements (springs k_1 and k_2) and one friction element (slider). We will demonstrate that the solution of this soft-contact problem is more complicated that it was in case of the systems analyzed in previous sections.

The system of equations describing the problem is similar to (5a), (5b), (5c), (5d), and (5e). The only difference is that we should replace (5c) by the following relation:

$$k_2 z + T = R. \quad (9)$$

Moreover, we will formulate a differential-algebraic system of equations similar to (6a), (6b), (6c), (6d), and (6e). The fundamental issue is to find the values of mappings describing time history of velocity $\dot{z}$ and friction force T.

Let us note that using (9) and the graph shown in Figure 4 representing constitutive relations of frictional element (slider), it is possible to evaluate $\dot{z}$ and T if $|R - k_2 z| < T_o$. The solution is as follows:

$$\dot{z} = 0, \ T = R - k_2 z \ \text{if} \ |R - k_2 z| < T_o. \quad (10)$$

It should be emphasized that evaluation of $\dot{z}$ and T if $|R - k_2 z| = T_o$ is complicated and needs a special treatment. First of all one should establish additional relationships constituting so-called differential successions of relationships described in (5d) and (5e). The notion of differential successions was used by authors in various mechanical problems [12–14]. Using differential successions we will define additional relations being satisfied by the time derivatives of variables in (5d) and (5e).

We will try to establish relationships between the rates $\dot{u}$ and $\dot{R}$ based on (5d) and the graph shown in Figure 2. We will formulate such relations in case of three subsets being defined below:

$$\{u, R\} = A \cup B \cup C \quad (11a)$$

$$\begin{aligned} A &:= \{u > 0, \ R = 0\} \\ B &:= \{u = 0, \ R = 0\} \quad (11b) \\ C &:= \{u = 0, \ R > 0\}. \end{aligned}$$

Let us note that subsets A and C correspond to horizontal and vertical branches of the graph in Figure 2. The subset B corresponds to the origin of the graph in Figure 2.

It is easy to note that at points $(u, R) \in A$, the value $R = 0$ is constant; thus $\dot{R} = 0$. On the other hand at the same points $(u, R) \in A$, the value of $\dot{u}$ can be arbitrary which gives $\dot{u} \in (-\infty, +\infty)$. A similar analysis leads to the conclusion that if $(u, R) \in C$ then $\dot{u} = 0$ and $\dot{R} \in (-\infty, +\infty)$. At points $(u, R) \in B$ there exist two options. If u does not vary in time then $\dot{u} = 0$ and $\dot{R} \ge 0$. The second case we should consider concerns a situation when R does not vary over time then $\dot{R} = 0$ and $\dot{u} \ge 0$.

Summarizing the above analysis leads to the following relationships:

$$\{\dot{u}, \dot{R}\} \in \begin{cases} \{\dot{u} \in (-\infty, +\infty), \ \dot{R} = 0\} & \text{if } (u,R) \in A \\ \{\dot{u} \ge 0, \ \dot{R} = 0\} \cup \{\dot{u} = 0, \ \dot{R} \ge 0\} & \text{if } (u,R) \in B \\ \{\dot{u} = 0, \ \dot{R} \in (-\infty, +\infty)\} & \text{if } (u,R) \in C. \end{cases} \quad (12)$$

The relationships described in (12) constitute time differential successions of the relations in (5d). The graphs of (12) are shown in Figure 7.

A similar analysis can be used in order to determine relationships between variables $\dot{z}$ and $\dot{R}$ based on (5e) and the graph shown in Figure 4. We assume the following decomposition of (5e) into five subsets:

$$\{\dot{z}, T\} = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \quad (13a)$$

$$\begin{aligned} S_1 &:= \{\dot{z} = 0, \ |T| < T_o\} \\ S_2 &:= \{\dot{z} = 0, \ T = T_o\} \\ S_3 &:= \{\dot{z} > 0, \ T = T_o\} \quad (13b) \\ S_4 &:= \{\dot{z} = 0, \ T = -T_o\} \\ S_5 &:= \{\dot{z} < 0, \ T = -T_o\}. \end{aligned}$$

If $\{\dot{z}, T\} \in S_1$ then the value of $\dot{z} = 0$ does not change in time but the velocity of the slider force T is arbitrary; thus $\dot{T} \in (-\infty, +\infty)$. If $\{\dot{z}, T\} \in S_2$ then one should consider two cases $\dot{z} = 0$ and $\dot{T} \le 0$ or $\dot{z} \ge 0$ and $\dot{T} = 0$. If $\{\dot{z}, T\} \in S_3$ then $\dot{z} \ge 0$ and $\dot{T} = 0$. Similarly, we can define the relationships $\{\dot{z}, \dot{T}\}$ in case of subsets S_4 and S_5.

Summarizing all cases gives the following formula:

$$\{\dot{z}, \dot{T}\} = \begin{cases} \{\dot{z} = 0, \ \dot{T} \in (-\infty, +\infty)\} & \text{if } (\dot{z}, T) \in S_1 \\ \{\dot{z} = 0, \ \dot{T} \le 0\} \cup \{\dot{z} \ge 0, \ \dot{T} = 0\} & \text{if } (\dot{z}, T) \in S_2 \\ \{\dot{z} \ge 0, \ \dot{T} = 0\} & \text{if } (\dot{z}, T) \in S_3 \\ \{\dot{z} = 0, \ \dot{T} \ge 0\} \cup \{\dot{z} \le 0, \ \dot{T} = 0\} & \text{if } (\dot{z}, T) \in S_4 \\ \{\dot{z} \le 0, \ \dot{T} = 0\} & \text{if } (\dot{z}, T) \in S_5. \end{cases} \quad (14)$$

Let us move back to the problem of evaluation of $\dot{z}$ and T if $|R - k_2 z| = T_o$. We will begin with the case $R - k_2 z = T_o$ which is equivalent to the condition $T = T_o$. We will use the relationships given in (12) along with the following equation resulting from (9):

$$k_2 \dot{z} + \dot{T} = \dot{R}. \tag{15}$$

Additionally, let us note that using (13b) gives $\dot{z} \geq 0$ for $T = T_o$ (subsets S_2 and S_3). Moreover, for subsets S_2 and S_3 the differential successions expressed in (14) can be visualized in one graph shown in Figure 8.

If $(u, R) \in A$ (see (11b)) then using (12) gives $\dot{R} = 0$. Applying (15) along with the graph shown in Figure 8 gives $\dot{z} = 0$.

If $(u, R) \in C$ (see (11b)) then using (12) gives $\dot{u} = 0 \Rightarrow \dot{x} = \dot{y}$. Applying (5b) leads to the following equation:

$$\dot{R} = -k_1 (\dot{z} - \dot{x}). \tag{16}$$

Substituting (16) into (15) gives the following relationship:

$$\dot{T} = -(k_1 + k_2)\dot{z} + k_1 \dot{x}. \tag{17}$$

Using (17) and the graph shown in Figure 8 leads to the solution

$$\dot{z} = \begin{cases} 0 & \text{if } \dot{x} \leq 0 \\ \dfrac{k_1}{k_1 + k_2}\dot{x} & \text{if } \dot{x} > 0. \end{cases} \tag{18}$$

It can be proved that the solution expressed via (18) is valid also if $(u, R) \in B$.

Using a similar procedure it is possible to evaluate $\dot{z}$ if $R - k_2 z = -T_o$ which is equivalent to the condition $T = -T_o$.

Finally, the total set of differential-algebraic equations defining the problem of a material point m impacting on elastoplastic rheological structure is as follows:

$$\ddot{x} = -\frac{R}{m} + g \tag{19a}$$

$$\dot{z} = f_z(\dot{x}, u, z, R) \tag{19b}$$

$$y = f_y(x, z) \tag{19c}$$

$$R = f_R(x, z) \tag{19d}$$

$$T = f_T(R, z), \tag{19e}$$

Table 1: Parameters of mechanical systems shown in Figures 1, 3, and 6.

g (m/sec^2)	m (kg)	k_1 (N/m)	k_2 (N/m)	c (N·sec/m)	T_o (N)
9.81	100	3924	1962	251	2000

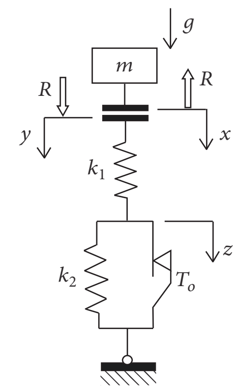

Figure 6: Visualization of elastoplastic system.

where functions f_y and f_R are given in (4a) and (4b), respectively, while functions f_z and f_T have the following definitions:

$$f_z(\dot{x}, u, z, R) = \begin{cases} 0 & \text{if } |R - k_2 z| < T_o \\ 0 & \text{if } |R - k_2 z| = T_o \wedge \{u > 0,\ R = 0\} \\ \dfrac{k_1}{k_1 + k_2}[\dot{x}]^+ & \text{if } R - k_2 z = T_o \wedge \{u = 0,\ R \geq 0\} \\ \dfrac{k_1}{k_1 + k_2}[\dot{x}]^- & \text{if } R - k_2 z = -T_o \wedge \{u = 0,\ R \geq 0\} \end{cases} \tag{20a}$$

$$f_T(R, z) = \begin{cases} R - k_2 z & \text{if } |R - k_2 z| < T_o \\ T_o \cdot \operatorname{sign}(R - k_2 z) & \text{if } |R - k_2 z| = T_o \end{cases} \tag{20b}$$

$$[\xi]^+ := \begin{cases} 0 & \text{if } \xi \leq 0 \\ \xi & \text{if } \xi > 0 \end{cases} \tag{20c}$$

$$[\xi]^- := \begin{cases} 0 & \text{if } \xi \geq 0 \\ \xi & \text{if } \xi < 0. \end{cases} \tag{20d}$$

5. Numerical Simulations

Computer simulations were carried out for the systems analysed in previous sections. The parameters of these systems are presented in Table 1 (compare Figures 1, 3, and 6). We assume that a material point of mass m falling freely in the earth's gravitational field impacts a rheological structure. The initial position of the material point equals $x_o = -3$ m while its initial velocity is $v_o = 5$ m/sec. The solutions of nonlinear differential equations defining the soft-contact problems were obtained applying the classical 4th-order fixed-time step Runge-Kutta algorithm.

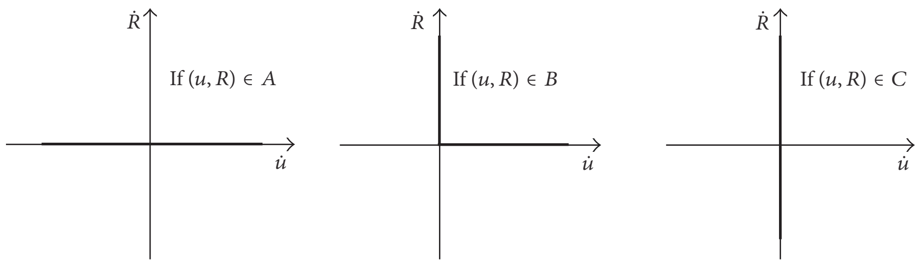

Figure 7: Graphs of mappings expressed via (12).

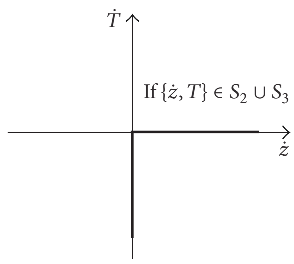

Figure 8: Graphs of mappings expressed via (14) for subsets S_2 and S_3.

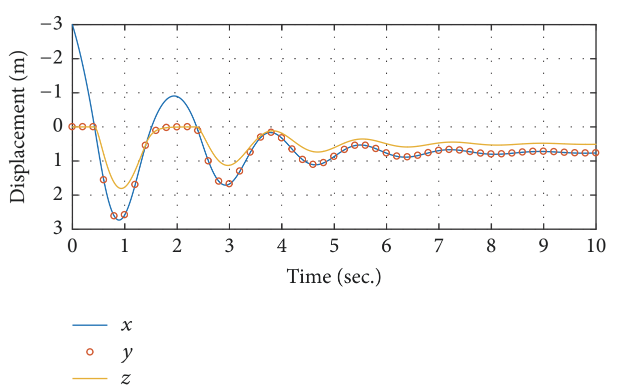

Figure 9: Time history of displacements in viscoelastic system shown in Figure 1.

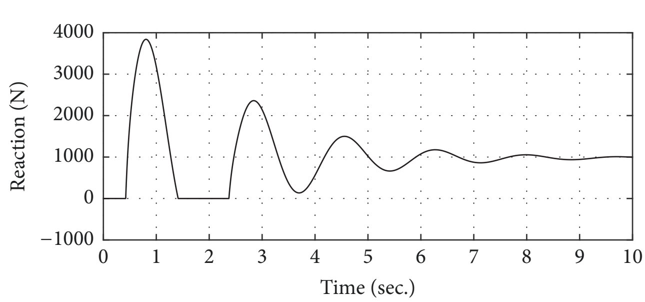

Figure 10: Time history of reaction R in viscoelastic system shown in Figure 1.

Let us begin with the results obtained in case of viscoelastic system shown in Figure 1. Time history graphs of displacements x, y, and z are presented in Figure 9, while Figure 10 visualizes the reaction. These figures represent the solutions of (2a), (2b), (2c), and (2d) with initial conditions $x(0) = x_o$, $\dot{x}(0) = v_o$, and $z(0) = 0$. Let us note that for time sections [0, 0.4 sec] and [1.4 sec, 2.4 sec] the reaction equals zero, $R = 0$ and $x(t) < y(t)$, which means that the bumpers are not in contact. In other time instants, the reaction is bigger than zero, $R > 0$ and $x(t) = y(t)$. Thus, the bumpers are in contact.

We can also note analyzing Figures 9 and 10 that the variables x, y, z, and R tend to the following values:

$$\begin{aligned} x_\infty &= y_\infty = \frac{k_1 + k_2}{k_1 k_2} mg = 0.75\,\text{m}, \\ z_\infty &= \frac{mg}{k_2} = 0.5\,\text{m}, \\ R &= mg = 981\,\text{N}. \end{aligned} \tag{21}$$

Graphs shown in Figure 11 represent the solutions (hysteretic loops) on planes (R, y) and (R, z). Analysing both loops one can see limit points $(R, y) = (981\,\text{N}, 0.75\,\text{m})$ and $(R, z) = (981\,\text{N}, 0.5\,\text{m})$.

In case of elastic-viscoplastic model shown in Figure 3 the solutions of (6a), (6b), (6c), (6d), and (6e) are presented in Figures 12, 13, and 14. Let us note that for $t > 1$ sec the variable z does not change over time but the structure vibrates. During these vibrations there are short time instants when $R = 0$ (see Figure 13). During such time instants the bumpers are not in contact.

The first time period of loading for $t < 1$ sec is represented in hysteretic loops shown in Figure 14 by a thin line. The thick line in Figure 14 represents the second period of vibrations for $t > 1$ sec.

The solutions of (19a), (19b), (19c), (19d), and (19e) describing elastoplastic system shown in Figure 6 are presented in Figure 15 (displacements), Figure 16 (reaction), and Figure 17 (hysteretic loops). Let us note that previously analyzed graphs shown in Figure 12 are similar to that of Figure 15. After a short transient time period, the coordinate z is constant while y and z change over time. For such time periods when $x < y$ (see Figure 15) and $R = 0$ (see Figure 16) the bumpers are not in contact.

Analyzing hysteretic loops shown in Figure 17 leads to the conclusion that the process of energy dissipation is formed in the first period of impact when z changes over time. This time period is represented in Figure 17 by broken line OABCD. The next period of vibrations when the structure deforms elastically is represented by section DE in Figure 17.

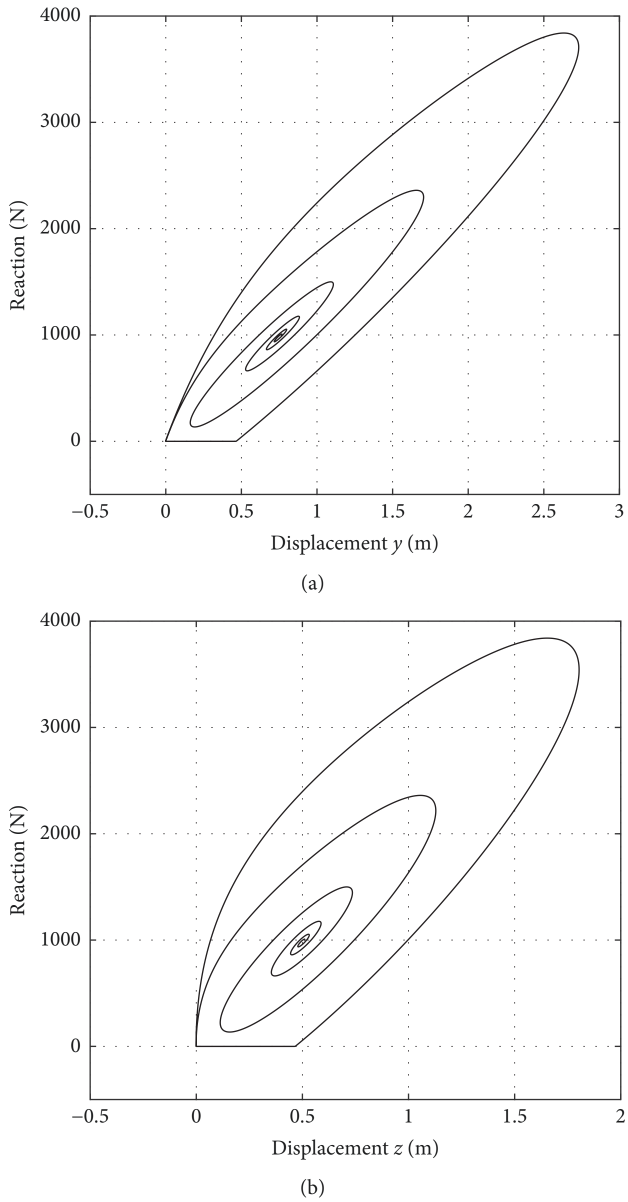

Figure 11: Hysteretic loops in viscoelastic system: relationships (R, y) (a) and (R, z) (b).

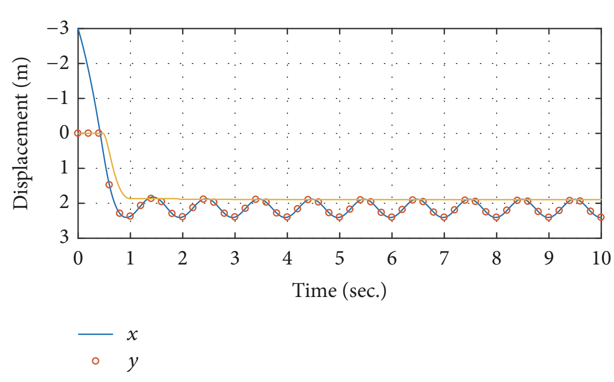

Figure 12: Time history of displacements in elastic-viscoplastic system shown in Figure 3.

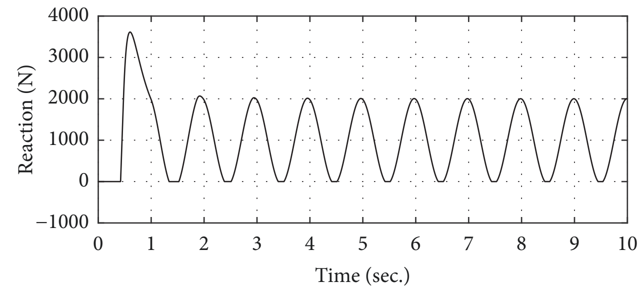

Figure 13: Time history of reaction R in elastic-viscoplastic system shown in Figure 3.

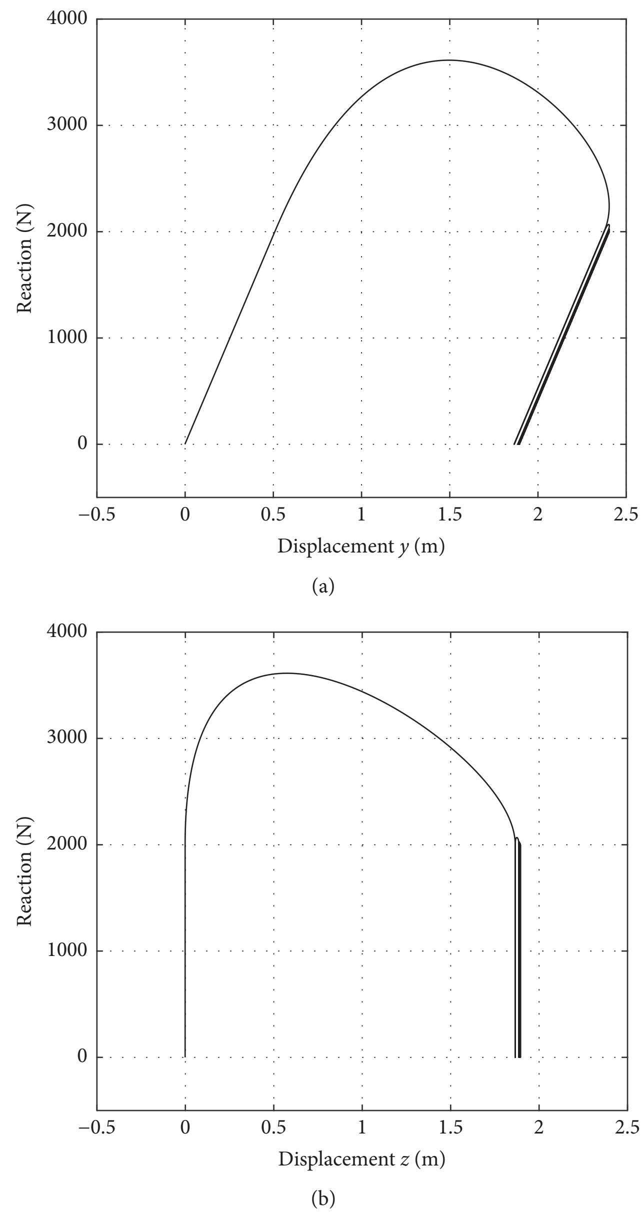

Figure 14: Hysteretic loops in elastic-viscoplastic system: relationships (R, y) (a) and (R, z) (b).

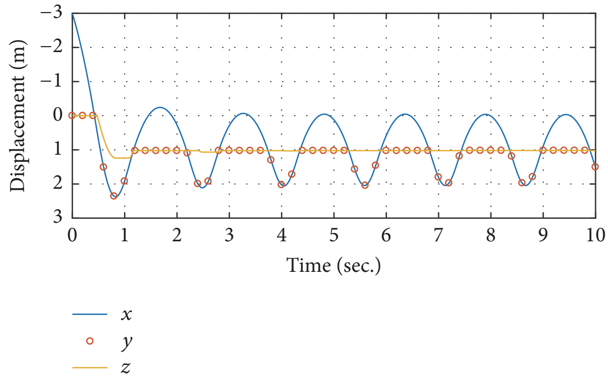

FIGURE 15: Time history of displacements in elastoplastic system shown in Figure 6.

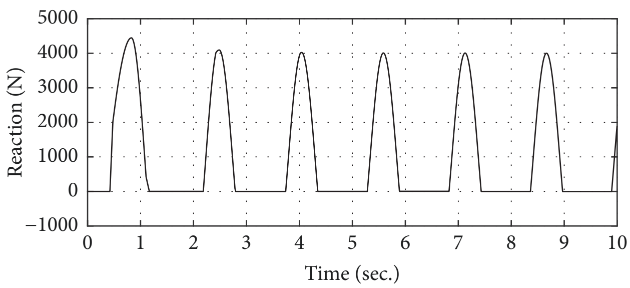

FIGURE 16: Time history of reaction R in elastoplastic system shown in Figure 6.

Numerical examples analyzed in this section illustrate the solutions of the soft-contact problems defined via differential-algebraic equations derived in Sections 2, 3, and 4. Based on these equations it is possible to carry out an optimization procedure in order to find an optimal set of parameters for rheological structures modelling energy dissipators.

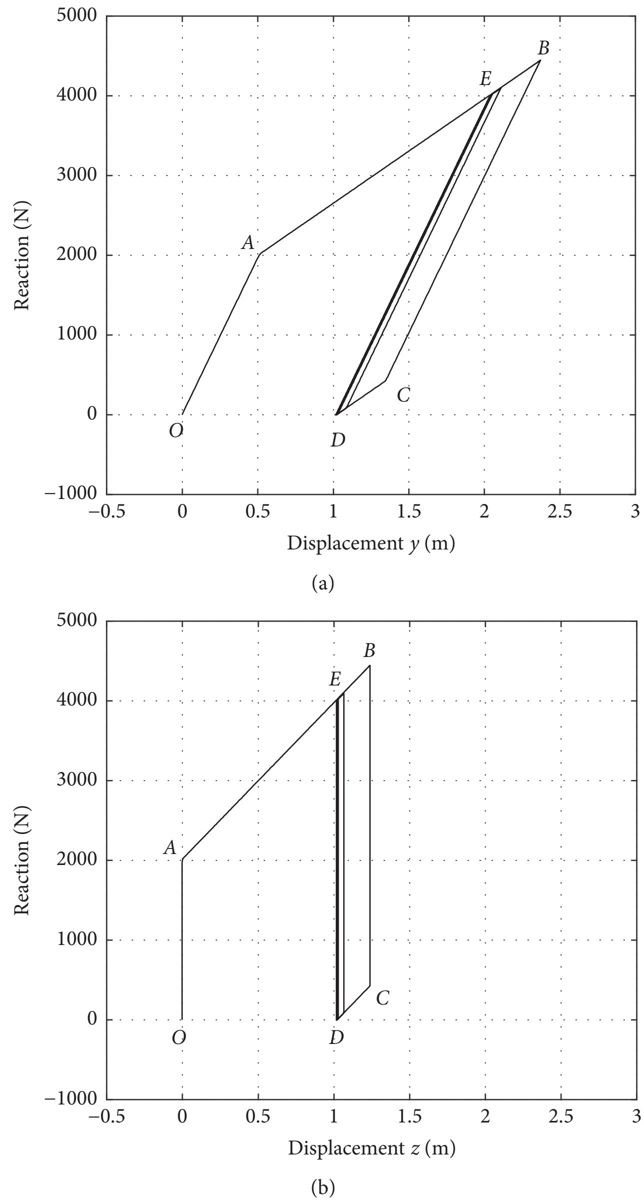

FIGURE 17: Hysteretic loops in elastoplastic system: relationships (R, y) (a) and (R, z) (b).

6. Final Remarks

Discrete rheological schemes analyzed in this paper are used to model energy-absorbing devices in machinery, vehicles, and buildings. The scheme visualized in Figure 6 was considered in [12] as a model of rail bumper while a viscoplastic scheme, similar to that shown in Figure 3, was analyzed in [15] as a model of magnetorheological fluid damper for seismic retrofit of buildings. Finding the response of rheological models exhibiting elasticity, viscosity, and plasticity phenomena against impact loading seems to be very important in order to investigate dissipative capabilities of these systems.

The fundamental characteristics of each damper model are its hysteretic loop (see Figures 11(a), 14(a), and 17(a)). For example the hysteretic loop visualized in Figure 11(a) is typical for viscoelastic rheological models. The loops shown in Figures 14(a) and 17(a) relate to rheological schemes modeling permanent deformations. Such phenomena are observed in elastic-friction dampers. On the other hand, rheological schemes with permanent deformations can be also used for the modelling of the contact region compliance between elastoplastic deformable bodies.

Thanks to the method proposed in this paper the three analyzed soft-contact problems were formulated with use of nonlinear explicit differential-algebraic equations. As it was demonstrated numerical solution of such equations can be obtained using classical Runge-Kutta algorithm. Thus, there was no need to use more advanced solvers. Moreover, we did not apply any regularization procedure as it is sometimes suggested in the literature where multivalued mappings describing friction and plasticity phenomena (see Figure 4) are arbitrarily replaced by regular functions (see [16] and cited literature).

In case of viscoelastic and elastic-viscoplastic systems analyzed in Sections 2 and 3, respectively, explicit forms of equations defining the problems were obtained directly. In case of elastoplastic system presented in Section 4 explicit form of equations was obtained after formulation of additional relationships being satisfied by the time derivatives of variables defining the problem.

The rheological schemes analyzed in this paper were composed of finite number of elastic, viscous, and plastic elements. Because of strongly nonlinear nature of the analyzed problems, the method proposed herein cannot be simply generalized to any number of elements. Each scheme should be treated individually as it was shown analyzing three separate soft-contact problems in this paper.

It is worth mentioning that the method presented in this paper can also be applied in case of soft-contact viscoelastic problems described by fractional derivatives (see [17]).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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