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Research Article

Power Spectral Density Response of Bridge-Like Structures Loaded by Stochastic Moving Forces

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In this study, simple and manageable closed form expressions are obtained for the mean value, the spectral density function, and the standard deviation of the deflection induced by stochastic moving loads on bridge-like structures. As a basic case, a simply supported beam is considered, loaded by a sequence of concentrated forces moving in the same direction, with random instants of arrival, constant random crossing speeds, and constant random amplitudes. The loads are described by three stochastic processes, representing an idealization of vehicular traffic on a bridge in case of negligible inertial coupling effects between moving masses and structure. System's responses are analytically determined in terms of mean values and power spectral density functions, yielding standard deviations, with the possibility to easily extend the results to more refined models of single span bridge-like structures. Potential applications regard structural analysis, vibration control, and condition monitoring of traffic excited bridges.

1. Introduction

The problem of vibrations induced by moving loads has long been studied, being an issue of great importance in structural dynamics [1]. A vast literature exists devoted to the analysis of structures loaded by deterministic travelling forces. The most commonly adopted models consist of beams or plates, traversed by either concentrated [2, 3] or distributed moving loads [4, 5], leading to the definition of specific dynamic amplification factors and characteristic response spectra [6]. Main applications regard the dynamic responses of bridges to traffic, aimed at identifying vehicle-bridge dynamic interactions [7], at reducing structural vibrations by means of passive or active devices, as dynamic absorbers [8], and also at studying traffic induced ground vibrations, which may cause significant environmental problems [9].

Excitation due to stochastic moving loads was also investigated, however to a lesser extent. The response of a beam to a sequence of concentrated forces with random amplitudes and velocities was studied in [10], providing explicit expressions for the expected value and the variance of the deflection. Beam-like bridges excited by random traffic were considered in [11], mainly with condition monitoring purposes, analysing a simply supported beam loaded by sequences of concentrated moving loads with same magnitude, travelling at constant speed and separated by random time delays using computer generated random data. Transverse vibrations of elastic homogeneous isotropic beams with general boundary conditions due to a moving random force with a constant mean value, travelling with either constant, accelerating or decelerating speed, were studied in [12], presenting closed form expressions for the variance of the deflection. The dynamic response of an infinite beam and a plate resting on a Pasternak foundation to the passage of a train of random forces, according either to Poisson's distribution or Erlang process, all travelling at the same speed, was analysed in [13], deriving explicit formulas for the expected value, the variance, and the n-th cumulant of flexural displacements. A spectral analysis of a beam's vibration with uncertain parameters under a random train of moving forces which forms a filtered Poisson process was studied in [14], assuming uncertain natural frequencies which were modelled by fuzzy numbers, random variables, or fuzzy random variables.

In the present contribution, novel developments are presented regarding the description of structural responses to stochastic moving loads. Closed form expressions are obtained for the mean value, spectral density function, and standard deviation of the deflection induced by stochastic moving loads on bridge-like structures.

A simply supported beam is considered, loaded by a sequence of concentrated forces moving in the same direction, with random instants of arrival, constant random crossing speeds, and constant random amplitudes. The loads are described by three stochastic processes, which represent an idealization of traffic on a bridge, in the case in which the relative magnitude of moving masses introduces negligible coupling effects with the structure. These stochastic processes are described by means of a distribution of the number of incoming concentrated moving forces per unit time, a distribution of the associated crossing times, and a distribution of the associated amplitudes.

Based on modal analysis of the structure, the system's response is analytically determined in terms of mean values and power spectral density functions, yielding standard deviations, with the possibility to easily extend the results to more refined models of bridge-like structures than the simply supported beam herein considered. Possible applications (enhanced by the simple expressions which can be obtained for the responses, identifying and highlighting the role of stochastic parameters and main modal contributions) are in the fields of structural analysis (fatigue estimation), vibration control (optimization of damping properties), and condition monitoring (fault detection [15]) of structures excited by stochastic moving loads, like traffic-excited bridges.

2. Model of the Structure

A straight homogeneous beam is considered, simply supported at both ends as represented in Figure 1. The Euler–Bernoulli beam model is adopted, loaded by a sequence of concentrated forces moving in the same direction, with random instants of arrival, constant random crossing speeds, and constant random intensities.

The resulting equation of motion reads as follows:

$$M\frac{\partial^2 w(z,t)}{\partial t^2} + C\frac{\partial w(z,t)}{\partial t} + EJ\frac{\partial^4 w(z,t)}{\partial z^4} = \sum_{k=1}^{\mathrm{Num}(0,t)} A_k \delta\left[z - V_k\left(t-t_k\right)\right], \tag{1}$$

where $w(z,t)$ is the beam deflection, M is the constant mass per unit length, C is the constant viscous damping coefficient, E is Young's modulus, J is the cross-sectional moment of inertia, A_k is the random intensity of the k-th concentrated moving force, V_k is its constant travelling speed, t_k is its instant of arrival, $\delta(\cdot)$ is the Dirac distribution, and Num $(0, t)$ is the counter for the forces present on the beam at instant t. For the sake of convenience, the random variable $T_k = l/V_k$, i.e., the crossing time on the beam, will replace the speed V_k.

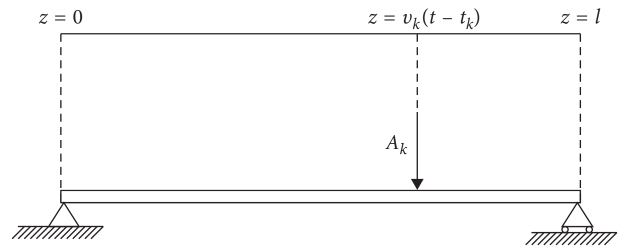


Figure 1: Schematic of the adopted model.

Let G_1 represent the forced response to the k-th moving force, defined for $t_k \le t \le t_k + T_k$, and let G_2 represent the free response, starting when the k-th moving force leaves the beam, defined for $t \ge t_k + T_k$. They satisfy the following differential equations:

$$\begin{cases} M\dfrac{\partial^2 G_1}{\partial t^2} + C\dfrac{\partial G_1}{\partial t} + EJ\dfrac{\partial^4 G_1}{\partial z^4} = A_k\delta\left[z - \dfrac{l}{T_k}\left(t-t_k\right)\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad t_k \le t \le t_k + T_k, \\ M\dfrac{\partial^2 G_2}{\partial t^2} + C\dfrac{\partial G_2}{\partial t} + EJ\dfrac{\partial^4 G_2}{\partial z^4} = 0, \qquad t \ge t_k + T_k. \end{cases} \tag{2}$$

Modal analysis yields the following:

$$\begin{cases} G_1\left(z, t-t_k, T_k\right) = \displaystyle\sum_{n=1}^{\infty} q_{1n}\left(t-t_k, T_k\right) W_n(z), \\ G_2\left(z, t-t_k-T_k, T_k\right) = \displaystyle\sum_{n=1}^{\infty} q_{2n}\left(t-t_k-T_k, T_k\right) W_n(z), \end{cases} \tag{3}$$

where W_n is the n-th mode shape and q_{1n}, q_{2n} are the modal co-ordinates for the n-th mode. In particular, assuming simply supported ends yields the values of modal parameters (M_n, C_n, and K_n), natural angular frequencies (ω_n), modal shapes (W_n), and forces (F_n) in the following form:

$$\begin{aligned} M_n &= \frac{1}{2}Ml, \\ C_n &= \frac{1}{2}Cl, \\ K_n &= \frac{(n\pi)^4}{2}\frac{EJ}{l^3}, \\ \omega_n &= \sqrt{\frac{K_n}{M_n}}, \\ W_n(z) &= \sin\left(\frac{n\pi}{l}z\right), \\ F_{nk}\left(\tau-t_k, T_k\right) &= A_k \sin\left[\frac{n\pi}{T_k}\left(\tau-t_k\right)\right]. \end{aligned} \tag{4}$$

Introducing the general expression for the unit impulse response function of an underdamped system,

$$h_n(\tau) = \frac{1}{M_n\Omega_n}\exp(-\zeta_n\omega_n\tau)\sin(\Omega_n\tau),$$
$$\Omega_n = \omega_n\sqrt{1-\zeta_n^2}, \tag{5}$$
$$\zeta_n = \frac{C_n}{2\sqrt{K_nM_n}},$$

then the modal co-ordinates in equation (3) can be found as convolution integrals:

$$q_{1nk}(t-t_k,T_k) = \int_{t_k}^{t} F_{nk}(\tau-t_k,T_k)h_n(t-\tau)d\tau, \quad t_k \le t \le t_k + T_k,$$
$$q_{2nk}(t-t_k-T_k,T_k) = \int_{t_k}^{t_k+T_k} F_{nk}(\tau-t_k,T_k)h_n(t-\tau)d\tau, \quad t \ge t_k + T_k, \tag{6}$$

while the modal transfer function (say H_n) takes the usual well known form, defined by natural frequencies and modal damping ratios. Equations (3), (5), and (6) are of general validity, while the expressions of the modal shapes (W_n), and consequently those of the related modal forces (F_n) in equation (4), depend on the adopted structural model.

It is worth pointing out that a single span structure is studied, in case of null or negligible influence of adjacent spans and of null or negligible compliance of the supports (i.e., in the realistic case of negligible contribution given to flexural behaviour by displacement of the revolute joints modelling the supports). Therefore, modal shapes can be expressed in any case on the basis of the sine function, and this allows the validity of the adopted model to be extended to a more general case. In this way, for example, different boundary conditions other than simply supported may be studied by introducing torsional springs in the supports (describing the range of all possible intermediate cases between simply supported and clamped ends). Or, in case of more complicated, realistic single span bridge-like structures, either a spectral approach or the results of a finite element analysis may be easily taken into account. In the former case, modal shapes may be expressed directly as linear combinations of sine functions due to Ritz–Galerkin expansions. In the latter case, since only the lowest modal shapes are of interest for the present study (as it will be clarified), once they have been computed they may be immediately expressed and approximated with the desired accuracy by Fourier analysis on the basis of the sine function. In principle, then, also the general case of multiple spans with intermediate supports may be studied by combining a finite element formulation with the proposed method, if the lowest modal shapes can be approximated with the necessary accuracy as linear combinations of sine functions by Fourier analysis.

3. Model of the Moving Loads

Three stochastic processes contribute to define the moving loads. These processes can be, respectively, described by means of a distribution of the number of incoming concentrated moving forces per unit time, a distribution of the associated crossing times on the structure, and a distribution of the associated amplitudes. In order to simplify the subsequent stages of analysis, these stochastic processes are assumed to be weak ergodic [10, 16].

As regards the stochastic process of the arrival times, let $dL(t)$ be the number of incoming concentrated moving forces in the period $(t, t+dt)$, let the probability that a single force enters the system be considered proportional to dt, and let the probability that two or more forces enter in the same time period be regarded as negligible. Designating the cumulative probability as P, the probability (say, Pr) that a concentrated moving force enters in the infinitesimal period $(t,\ t+dt)$ is

$$\Pr|_{dL(t)=1} = dP_L(t) = \lambda(t)dt = \lambda dt, \tag{7}$$

where λ represents the (constant) expected number of entering vehicles per unit time, as well as the probability density of the stochastic process. Indicating with $E[\cdot]$ the expected value, then according to the Poissonian distribution the mean value and the second cumulant of the number of moving loads entering in the time interval $(t,\ t+dt)$ are

$$\mathrm{E}[dL(t)] = \lambda dt,$$
$$\mathrm{E}\left[dL^2(t)\right] = \lambda_2 dt = \left(\lambda+\lambda^2\right)dt. \tag{8}$$

Conversely, with $t_1 \ne t_2$:

$$E[dL(t_1)dL(t_2)] = \varphi_L(t_1,t_2)dt_1dt_2,$$
$$\psi_L(t_1,t_2) = \varphi_L(t_1,t_2) - \lambda^2, \tag{9}$$

where φ is the second-order probability density and ψ is the correlation function. Since the instants of entry are supposed to be uncorrelated, it is $\psi_L = 0\ \forall\ t_1,\ t_2$.

As regards the crossing times over the structure, they are considered independent from the instants in which the associated concentrated moving forces start acting on the structure itself. Let $dDt_k(T)$ be the stochastic process defined by $dDt_k(T)=0$ for $T<T_k$ and $dDt_k(T)=1$ for $T \ge T_k$. The probability that the concentrated force arriving at instant t_k will travel over the structure in a crossing time greater than or equal to T_k is expressed by

$$\Pr|_{D_{t_k}(T)=1} = dP_D(T) = \xi(T)dT, \tag{10}$$

where ξ represents the probability density of the stochastic process. For further developments, it is convenient to express the density ξ (which is nonnegative over a range $[T_{\min}, T_{\max}]$, and zero outside of this range) as a function of the two sets of N discrete values $\{T_1, T_2, \dots, T_N\}$ and $\{p_1, p_2, \dots, p_N\}$, rather than as a function of the continuous variable T, i.e,

$$\xi(T) \cong \xi^{(N)}(T) = \sum_{k=1}^{N} p_k\delta(T-T_k), \tag{11}$$

where p_k represents the probability that the crossing time T will be equal to T_k and $\delta(\cdot)$ is the Dirac distribution, while $T_1 \equiv T_{\min}$ and $T_N \equiv T_{\max}$ represent the minimum and maximum crossing times, respectively. The link between $\xi^{(N)}(T)$ and the function $\xi(T)$ to be approximated can be explicitly stated by defining p_k as

$$p_k = \int_{T_{k-1/2}}^{T_{k+1/2}} \xi(T)dt,$$
$$T_{k-1/2} = \frac{T_k + T_{k-1}}{2}, \tag{12}$$
$$T_{k+1/2} = \frac{T_{k+1} + T_k}{2}.$$

The mean value and the second cumulant of the stochastic process $dDt_k(T)$ are

$$E\left[dD_{t_k}(T)\right] = \xi(T)dT,$$
$$E\left[dD_{t_k}^2(T)\right] = T\xi(T)dT, \tag{13}$$
$$E\left[dD_{t_k}(T_1)dD_{t_k}(T_2)\right] = 0, \quad T_1 \neq T_2.$$

Conversely, for $t_k \neq t_l$, it is

$$E\left[dD_{t_k}(T_1)dD_{t_l}(T_2)\right] = \varphi_D(T_1, T_2, t_k, t_l)dT_1 dT_2,$$
$$\psi_D(T_1, T_2, t_k, t_l) = \varphi_D(T_1, T_2, t_k, t_l) - \xi(T_1)\xi(T_2). \tag{14}$$

Since the crossing times over the structure are assumed to be uncorrelated, it is $\psi_D = 0 \;\forall\; T_1,\ T_2$.

As regards the stochastic process for the amplitudes of the concentrated moving forces, which are considered to be independent from both the instants of entry and the crossing times over the structure, the only assumptions regard the expected value and the second cumulant, which are assumed to be constant over time, is $E[A_k] = E[A]\ \forall k$; $E[A_k^2] = E[A^2]\ \forall k$.

4. Mean Value of the Response

The n-th modal force acting on the system at any instant $t > T_N$ can be expressed by adding the contributions of each concentrated moving force by means of stochastic Stieltjes integrals [10]. Equation (4) yields

$$F_n(t) = \int_{t-T_1}^{t} \int_{T_1}^{T_N} A(\tau)\sin\left[\frac{n\pi}{T}(t-\tau)\right] dD_\tau(T)dL(\tau) + \int_{t-T_N}^{t-T_1} \int_{t-\tau}^{T_N} A(\tau)\sin\left[\frac{n\pi}{T}(t-\tau)\right] dD_\tau(T)dL(\tau), \tag{15}$$

where the first double integral sums the contributions of all the concentrated moving forces which enter between instants $t-T_1$ and t (in fact, as T_1 is the minimum crossing time, any concentrated moving force arriving after instant $t-T_1$, at time t will still be acting on the structure). The second double integral sums only the contributions of those forces arriving between instants $t-T_N$ and $t-T_1$, which have not yet left the structure at time t. The mean value of the n-th modal input is

$$E\left[Q_n(t)\right] = \frac{[1-(-1)^n]}{n\pi} \frac{E[A]E[T]\lambda}{K_n},$$
$$Q_n(t) = \frac{F_n(t)}{K_n}, \tag{16}$$
$$E[T] = \sum_{k=1}^{N} p_k T_k.$$

Just like the n-th modal force, also the response $w(z,t)$ at an arbitrary instant $t > T_N$ can be expressed by means of stochastic Stieltjes integrals. Referring to G_1 and G_2 (equation (3)) which give the response for a single concentrated moving force yields

$$\begin{aligned} w(z,t) = &\int_{t-T_1}^{t} \int_{T_1}^{T_N} A(\tau)G_1(z, t-\tau, T)dD_\tau(T)dL(\tau) \\ &+ \int_{t-T_N}^{t-T_1} \int_{t-\tau}^{T_N} A(\tau)G_1(z, t-\tau, T)dD_\tau(T)dL(\tau) \\ &+ \int_{t-T_N}^{t-T_1} \int_{T_1}^{t-\tau} A(\tau)G_2(z, t-\tau-T, T)dD_\tau(T)dL(\tau) \\ &+ \int_{0}^{t-T_N} \int_{T_1}^{T_N} A(\tau)G_2(z, t-\tau-T, T)dD_\tau(T)dL(\tau), \end{aligned} \tag{17}$$

where the first couple of double integrals represents the forced vibrations caused by the load on the structure at instant t, while the second couple represents the free vibrations induced by the forces which have already left the structure. Its mean value is

$$\mathrm{E}[w(z,t)] = \sum_{n=1}^{\infty} \left\{ \frac{[1-(-1)^n]}{n\pi} \frac{\mathrm{E}[A]\mathrm{E}[T]\lambda}{K_n} \sin\left(\frac{n\pi}{l}z\right) \right\}, \tag{18}$$

which amplitude is modulated in z by modal shapes (linear combinations of sine functions, as already discussed regarding the model of the structure).

5. Power Spectral Density of the Moving Loads

By definition, assuming τ as an independent time variable, the autocorrelation function R_{Q_n} for the n-th modal input Q_n is

$$R_{Q_n}(\tau) = \lim_{\Lambda \longrightarrow \infty} \frac{1}{\Lambda} \int_{-\Lambda/2}^{\Lambda/2} Q_n(t)Q_n(t+\tau)dt. \tag{19}$$

Due to the assumptions $\psi_L = \psi_D = 0$, after some passages, it can be expressed as

$$\begin{cases} R_{Q_n}(\tau) = R_{Q_{n,r}}(\tau), & \text{for } -T_r \le \tau < -T_{r-1} \text{ and } T_{r-1} < \tau \le T_r, \\ & r = 1, 2, \ldots, L, T_0 = 0, \\ R_{Q_n}(\tau) = 0, & \text{for } \tau < -T_L \text{ and } \tau > T_L, \end{cases}$$

$$R_{Q_n,r}(\tau) = \frac{\mathrm{E}[A^2]\lambda_2}{2K_n^2} \sum_{k=r}^{N} p_k T_k \Big[(T_k - \tau)\cos(\beta_{nk}\tau) + \beta_{nk}^{-1}\sin(\beta_{nk}\tau) \Big], \quad \beta_{nk} = \frac{n\pi}{T_k}. \tag{20}$$

Note that although Q_n has a nonzero mean value (with respect to time), the mean value of the associated autocorrelation function is zero so that this function coincides with the autocovariance of the n-th modal input. Similarly, the cross-correlation function between the n-th and m-th modal inputs $R_{Q_{nm}}$ can be written as follows:

$$\begin{cases} R_{Q_{nm}}(\tau) = R_{Q_{nm},r}(\tau), & \text{for } T_{r-1} < \tau \le T_r, r = 1, 2, \ldots, N, \\ & T_0 = 0, \\ R_{Q_{nm}}(\tau) = 0, & \text{for } \quad \tau > T_N, \\ R_{Q_{nm}}(-\tau) = R_{Q_{mn}}(\tau), & \end{cases}$$

$$R_{Q_{nm},r}(\tau) = \frac{E[A^2]\lambda_2}{K_n K_m} \frac{nm}{(n^2 - m^2)} \sum_{k=r}^{N} p_k T_k \Big[\beta_{mk}^{-1}\sin(\beta_{mk}\tau) - \beta_{nk}^{-1}\sin(\beta_{mk}\tau) \Big]. \tag{21}$$

Finally, the variance $\sigma^2[Q_n]$ of the n-th modal input Q_n can be readily found from the autocorrelation function R_{Q_n}:

$$\sigma^2[Q_n] = R_{Q_n}(0) - E^2[Q_n] = \frac{1}{2K_n^2}\left\{ E[A^2]E[T^2]\lambda_2 - \frac{4[1-(-1)^n]}{(n\pi)^2} E^2[A]E^2[T]\lambda^2 \right\},$$

$$E[T^2] = \sum_{k=1}^{N} p_k T_k^2. \tag{22}$$

Equation (17) are of general validity, while the expressions descending from the n-th modal input Q_n, such as its expected value, its autocorrelation function, its variance, and the expected value of the response, they all depend on the adopted structural model. In the case of single span bridgelike structures, however, these functions may be represented in a more general form as linear combinations of the expressions given in equations (16), (18), (20), and (22) due to expansions on the basis of the sine function (as already discussed regarding the model of the structure).

6. Power Spectral Density and Standard Deviation of the Response

Since the power spectral density (PSD) function S_{Q_n} of the n-th modal input Q_n, amplified by the constant factor 2π is the Fourier transform of the autocorrelation (even) function R_{Q_n}, then the Wiener–Khintchine formulas [16] after some passages yield

$$S_{Q_n}(\omega) = \frac{1}{\pi}\int_0^\infty R_{Q_n}(\tau)\cos(\omega\tau)d\tau = \frac{\mathrm{E}[A^2]\lambda_2}{K_n^2} \sum_{k=1}^{N} p_k T_k \frac{1-(-1)^n\cos(\omega T_k)}{\pi\beta_{nk}^2\left(1-\beta_{nk}^{-2}\omega^2\right)^2}. \tag{23}$$

A similar procedure is adopted to determine the cross PSD function between the n-th and the m-th modal input $S_{Q_{nm}}$ associated with the cross-correlation (odd) function $R_{Q_{nm}}$:

$$S_{Q_{nm}}(\omega) = -\frac{i}{\pi}\int_0^\infty R_{Q_{nm}}(\tau)\sin(\omega\tau)d\tau = i\frac{E[A^2]\lambda_2}{K_n K_m}\frac{2nm}{(n^2-m^2)}\sum_{k=1}^{N}\frac{p_k T_k}{\pi}\left[\frac{(-1)^m\sin(\omega T_k)}{\beta_{mk}^2\left(1-\beta_{mk}^{-2}\omega^2\right)} - \frac{(-1)^n\sin(\omega T_k)}{\beta_{nk}^2\left(1-\beta_{nk}^{-2}\omega^2\right)}\right]. \tag{24}$$

The power spectral density S_w of the response $w(z,t)$ can then be computed from functions W_n (modal shape), H_n (modal transfer function, dimensionless receptance), and S_{Q_n}:

$$S_w(\omega,z) = \sum_{n=1}^{\infty}\left[|H_n(\omega)|^2 S_{Q_n}(\omega)W_n^2(z)\right] = \mathrm{E}[A^2]\lambda_2\sum_{n=1}^{\infty}\sum_{k=1}^{N}\left\{\frac{|H_n(\omega)|^2}{K_n^2} \cdot p_k T_k \frac{1-(-1)^n\cos(\omega T_k)}{\pi\beta_{nk}^2\left(1-\beta_{nk}^{-2}\omega^2\right)^2}\sin^2\left(\frac{n\pi}{l}z\right)\right\}. \tag{25}$$

To highlight the parameters defining the random processes, equation (25) is rewritten as

$$S_w(\omega,z) = E[A^2]\lambda_2\sum_{n=1}^{\infty}\sum_{k=1}^{N} p_k s_{nk}(\omega,z), \quad \omega \in (-\infty,+\infty), z \in [0,l],$$

$$s_{nk}(\omega,z) = \frac{|H_n(\omega)|^2}{K_n^2}\frac{\pi n^2 T_k{}^3[1-(-1)^n\cos(\omega T_k)]}{\left[(n\pi)^2-(\omega T_k)^2\right]^2} \cdot \sin^2\left(\frac{n\pi}{l}z\right). \tag{26}$$

The latter expression represents a sequence of functions that uniformly converge to zero as n tends to infinity (recall, for instance, the explicit expression of K_n given in equation (4)). Consequently, the series of functions which defines S_w is also uniformly convergent, which means that S_w is continuous over the entire domain. Note also that this series of functions is composed by positive terms, as expected for a PSD function.

Let now the number of steps N of the generic partition of $[T_1, T_N]$ tends to infinity. As N turns out to be a variable, it is assumed that now $T_k = T_k^{(N)}$, $p_k = p_k^{(N)}$, and $S_w = S_w^{(N)}$ are all functions of N, with $[T_1^{(N)}, T_N^{(N)}] \equiv [T_{\min}, T_{\max}]$. It can be proved that if the amplitude of each step of this partition tends to zero as N tends to infinity, then S_w can be expressed in the following integral form:

$$\begin{aligned}
&\lim_{N\longrightarrow\infty}\left\{\max_{k=1,\dots,N}\left[T_{k+1/2}^{(N)} - T_{k-1/2}^{(N)}\right]\right\} = 0 \\
&\Longrightarrow \lim_{N\longrightarrow\infty} S_w^{(N)}(\omega, z) \\
&= \mathrm{E}\left[A^2\right]\lambda_2 \sum_{n=1}^{\infty}\int_{T_{\min}}^{T_{\max}} s_n(T;\omega,z)\xi(T)dT,
\end{aligned} \tag{27}$$

where s_n is obtained from s_{nk} simply by introducing the continuous variable T in place of the discrete variable $T_k^{(N)}$ and ξ represents the probability density function, defined on the range $[T_{\min}, T_{\max}]$, to which the discretized approximation corresponds. The error $\varepsilon^{(N)}$ introduced with this approximation can be computed as

$$\begin{aligned}
\varepsilon^{(N)} &= \left| E\left[A^2\right]\lambda_2 \sum_{n=1}^{\infty}\int_{T_{\min}}^{T_{\max}} s_n(T;\omega,z)\xi(T)dT - S_w^{(N)}(\omega,z)\right| \\
&\le E\left[A^2\right]\lambda_2 \max_{[T_{\min},T_{\max}]}\left|\sum_{n=1}^{\infty}\frac{\partial s_n}{\partial T}\right| \cdot \max_{k=1,\dots,N}\left[T_{k+1/2}^{(N)} - T_{k-1/2}^{(N)}\right].
\end{aligned} \tag{28}$$

If an approximated function ξ for discrete values $T_k^{(N)}$ is adopted, and the error $\varepsilon^{(N)}$ must be contained within a certain limit, the partition of $[T_{\min}, T_{\max}]$ can be consequently optimized.

If the sequences s_{nk}, s_n decrease with n^{-8} (as for the adopted Euler-Bernoulli beam model); the series can be truncated to the first term, and hence

$$\begin{aligned}
S_w(\omega, z) &= E\left[A^2\right]\lambda_2 \int_{T_{\min}}^{T_{\max}} s_1(T;\omega,z)\ \xi(T)dT \\
&\cong E\left[A^2\right]\lambda_2 \sum_{k=1}^{N} p_k s_{1k}(\omega,z),
\end{aligned} \tag{29}$$

where s_1 represents the first term of the sequence s_n, which means that the contribution of the first mode is largely dominant. Note that the PSD function S_w of the response as given in equation (25) may be expressed in more general form also in this case by introducing Ritz–Galerkin expansions on the basis of the sine function.

Finally, the variance of the response $\sigma^2[w]$ (and then its standard deviation) can be found from the mean value and the PSD function S_w:

$$\begin{aligned}
\sigma^2[w] &= \int_{-\infty}^{+\infty} S_w(\omega,z)d\omega - E^2[w], \\
\int_{-\infty}^{+\infty} S_w(\omega,z)d\omega &= \int_{-\infty}^{+\infty}\sum_{n=1}^{\infty}\left[\left|H_n(\omega)\right|^2 S_{Q_n}(\omega) W_n^2(z)\right]d\omega,
\end{aligned} \tag{30}$$

where due to uniform convergence of the series, the integral can be numerically computed, eventually taking into account the approximation given in equations (29).

The amplitudes of the PSD function of the response and of its variance are both modulated in z by squared modal shapes.

7. Results and Discussion

As a case study, a beam-like concrete bridge is considered, with a single span of length $l = 45$ m and first natural frequency $\omega_1 = 1.36$ Hz, as in [11], assuming a damping ratio $\zeta_1 = 0.02$ for the first mode. A normal distribution is adopted for approximating the crossing speed distribution, defined by a mean value $E[V] = 25$ m/s and by a standard deviation $\sigma[V] = 4$ m/s, yielding crossing times T with $E[T] = 1.849$ s and $E[T^2] = 3.526$ s. The process is discretized with $N = 9$ steps according to equations (11) and (12), as reported in Table 1.

In the present formulation, any other possible discrete representation for the distribution of crossing speeds or crossing times could be considered, as, for instance, results directly obtained from empirical observation.

Within the limits imposed by the adopted assumptions, the maximum mean value of the deflection w along z (as in equation (18)) can be expressed in a dimensionless form as

$$\max_z\left\{\frac{K_1 E[w]}{E[A]E[T]\lambda}\right\} = \sum_{n=1}^{\infty}\left\{\frac{\left[1-(-1)^n\right]}{n\pi}\frac{K_1}{K_n}\right\} \cong \frac{2}{\pi}, \tag{31}$$

where the term $E[A]E[T]\lambda$ at the left-hand side simply represents the mean total force due to the mean number $(E[T]\lambda)$ of concentrated forces simultaneously loading the structure. Clearly, the contribution of the first mode is largely dominant due to modal stiffness parameters (as, for instance, those given in equation (4)), leading to the approximation of the right-hand side.

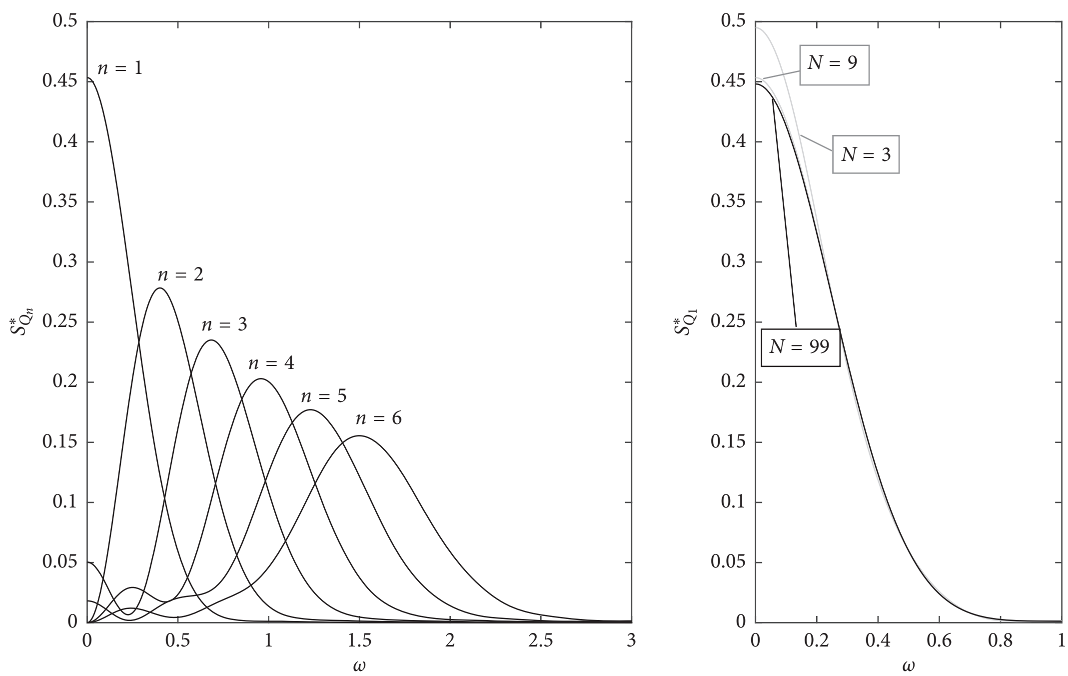
The PSD function S_{Q_n} of the n-th modal input (derived in equation (23)) is the key factor for characterizing both the PSD and the standard deviation of the output w. It can be rewritten in the following dimensional scaled form:

$$S_{Q_n}^{*} = \frac{K_n^2 S_{Q_n}(\omega)}{E[A^2]\ \lambda_2} = \sum_{k=1}^{N} p_k \frac{\pi n^2 T_k{}^3\ \left[1-(-1)^n\cos\left(\omega T_k\right)\right]}{\left[(n\pi)^2 - \left(\omega T_k\right)^2\right]^2}, \tag{32}$$

since the modal stiffness (K_n) and the second-order moments ($E[A^2]$, λ_2) simply play the role of scaling factors. This function modulates the modal transfer function H_n in the frequency domain, with windowing and filtering effects as shown in Figures 2–4. The shapes related to the first 6 modal terms are displayed in Figure 2(a), with parameters as in

Table 1: Discretization of crossing speeds distribution, with $N = 9$.

k	V_k (m/s)	V_k (km/h)	T_k (m/s)	p_k
1	11.7	42	3.857	0.001680
2	15.0	54	3.000	0.016841
3	18.3	66	2.454	0.087038
4	21.7	78	2.077	0.232809
5	25.0	90	1.800	0.323075
6	28.3	102	1.588	0.232809
7	31.7	114	1.421	0.087038
8	35.0	126	1.286	0.016841
9	38.3	138	1.174	0.001680



(a) (b)

Figure 2: Scaled modal input PSD $S^*_{Q_n}$ [s^3] (equation (32)) as a function of ω (Hz) for modes $n = 1$–6 (a) and different discretization values N (b).

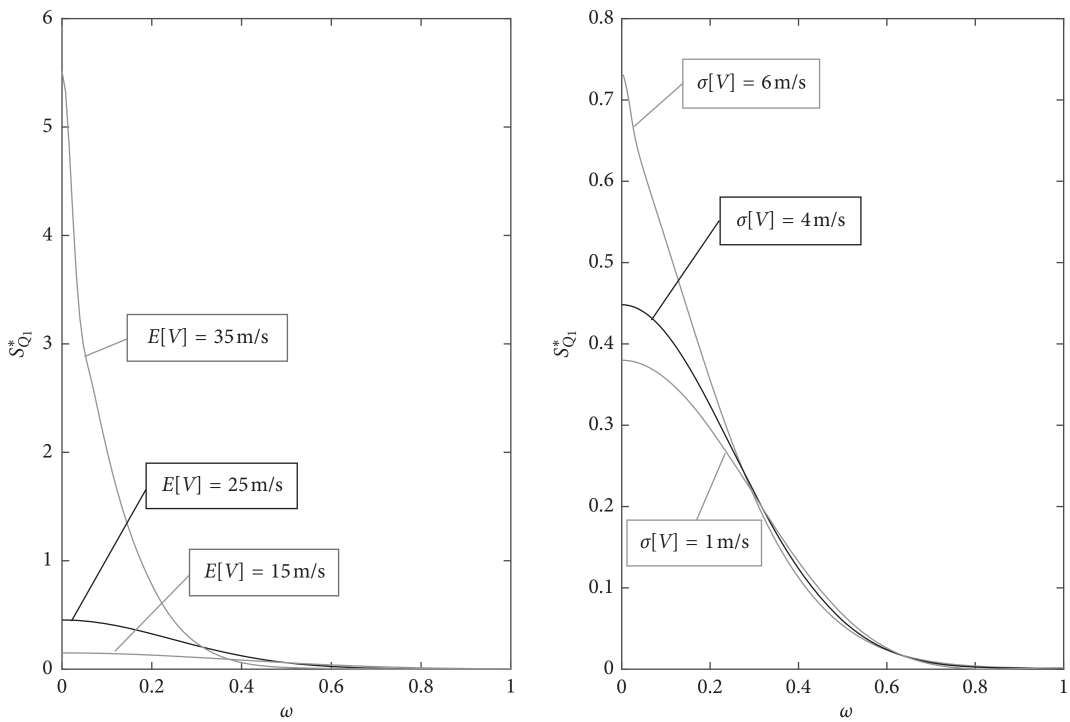
Table 1, while Figure 2(b) evidences the approximation errors induced by adopting different discretization values N of the continuous probability density distribution. Figure 3 highlights the effects on the first modal component (the most important one) of varying mean value (Figure 3(a)) and standard deviation (Figure 3(b)) of the crossing speeds normal distribution (black curves with parameters as in Table 1). Windowing is not significantly altered in its frequency extension, especially by the standard deviation, but strongly influenced in its intensity, which is reduced by either lowering the mean value or contracting the standard deviation.

Figure 4 shows the modulating effects on the square amplitude of the first mode transfer function H_1. The plot is obtained by superimposing 9 curves, combining 3 different values of the first natural frequency ω_1 (1, 1.36, and 2 Hz) with 3 different values of the first modal damping ratio ζ_1 (0.02, 0.03, and 0.05; black curves with intermediate value $\zeta_1 = 0.03$). Even relatively small variations of modal natural frequency and damping ratio can result in large power spectral density modifications due to filtering effects.

In the PSD function of the deflection S_w given in equation (26), the first modal component is largely dominant since the contribution of all other modes tends to become almost totally negligible due to squared modal stiffness parameters, leading to the approximation given in equation (29). Consequently, the standard deviation of the response $\sigma[w]$ given in equation (30) can be characterized by a dimensional function defined as

$$f\left[T^2\right] = \int_{-\infty}^{+\infty} \left|H_1(\omega)\right|^2 S^*_{Q_1}(\omega, T)\, d\omega. \qquad (33)$$

The other stochastic parameters play the role of scaling factors. Figure 5 displays f as a function of either the first natural frequency (Figure 5(a)) or the first modal damping



(a) (b)

Figure 3: Scaled modal input PSD $S^*_{Q_1}$ [s^3] (equation (32)) as a function of ω (Hz) for different mean values (a) and standard deviations (b) of crossing speeds V.

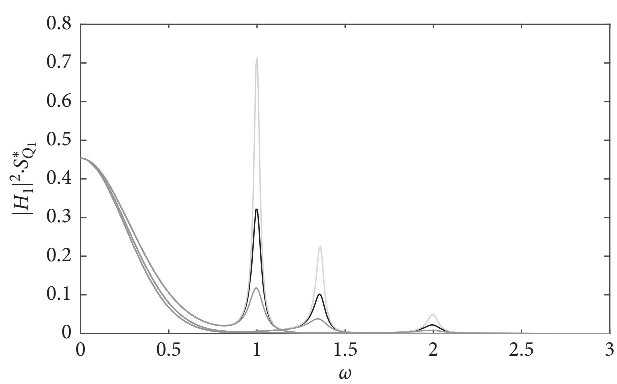


Figure 4: Scaled modal component of the deflection PSD $|H_1|^2 \cdot S^*_{Q_1}$ [s^3] as a function of ω (Hz) for 3 different values of natural frequency ($\omega_1 = 1$, 1.36, and 2 Hz) and 3 different values of damping ratio ($\zeta_1 = 0.02$, 0.03, and 0.05).

ratio (Figure 5(b)), providing a measure of the filtering effects due to the input PSD function S_{Q1} already observed in Figure 4.

In defining the shape of the deflection PSD (S_w), therefore, apart from the input stochastic parameters $E[A^2]$ and λ_2 (second cumulants) which within the adopted assumptions simply play the role of scaling factors, two aspects are of primary importance: first, the difference of phase between modal resonance peak and limited window frequency domain of modal input and, second, the amplitudes of both resonance peak (modal damping ratio) and modal input window (mean value and standard deviation of crossing speeds or crossing times).

8. Conclusions

A statistical approach has been adopted to analyse the input characteristics and the related responses of a simply supported beam loaded by a sequence of concentrated forces moving in the same direction, with random instants of arrival, constant random crossing speeds (with the possibility of adopting any possible continuous or discrete distribution as results directly obtained from empirical observation), and constant random amplitudes, the whole representing an idealization of vehicular traffic on a bridge.

The main contribution of this study is the analytical derivation of simple and manageable closed form

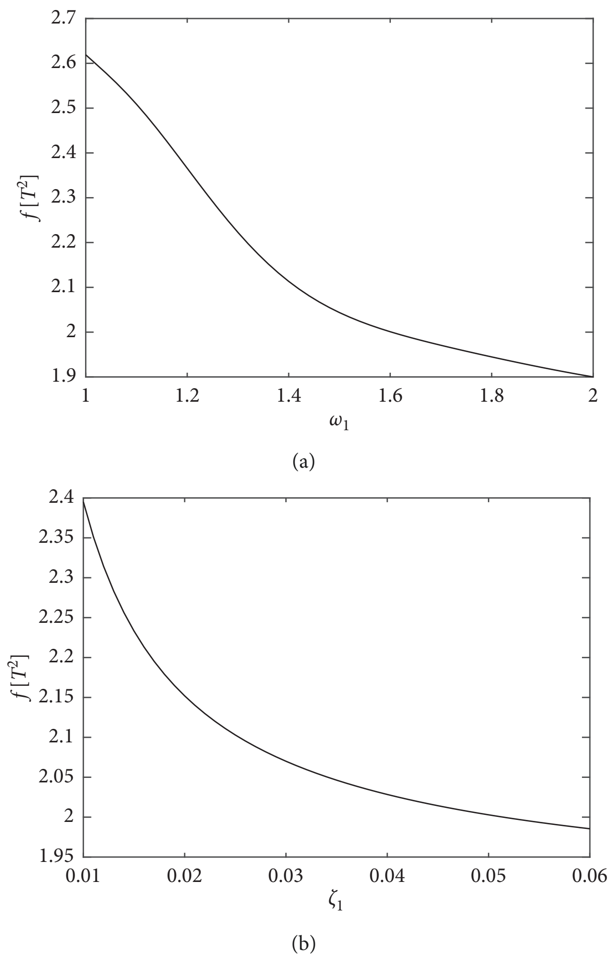


Figure 5: $f\,[T^2]$ [s^2] (equation (33)) as a function of natural frequency ω_1 (Hz) and modal damping ratio ζ_1.

expressions for mean value, spectral density function, and standard deviation of the deflection induced by stochastic moving loads on bridge-like structures. As a result, the most influential factors in shaping the response power spectral density functions have been evidenced: the difference of phase between the first mode resonance peak and the limited frequency domain of the window due to the first modal input and the amplitudes of both the first mode resonance peak (modal damping ratio) and the first mode input window (mean value and standard deviation of crossing speeds or crossing times).

With very few *a priori* assumptions on the adopted stochastic processes and the possibility to easily extend the results to more refined models of bridge-like structures (finite element models) than the simply supported beam herein considered, possible applications (enhanced by the simple expressions which can be obtained for the responses, identifying and highlighting the role of stochastic parameters and main modal contributions) are in the fields of structural analysis (fatigue estimation), vibration control (optimization of damping properties), and condition monitoring (fault detection) of structures excited by stochastic moving loads, like traffic excited bridges. In particular, the theory of models may be applied for linking dynamic measurements performed over real structures (power spectral density and other statistical response functions) and those performed with scaled models, in order to obtain appropriate traffic simulations.

Data Availability

All data included in the submitted manuscript are available in a public domain.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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