Research Article

Dynamic Analysis of Rectangular Plate Stiffened by Any Number of Beams with Different Lengths and Orientations

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Received 4 June 2019; Revised 19 August 2019; Accepted 17 October 2019; Published 7 November 2019

Academic Editor: Angelo Marcelo Tusset

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The present work is concerned with dynamic characteristics of beam-stiffened rectangular plate by an improved Fourier series method (IFSM), including mobility characteristics, structural intensity, and transient response. The artificial coupling spring technology is introduced to establish the clamped or elastic connections at the interface between the plate and beams. According to IFSM, the displacement field of the plate and the stiffening beams are expressed as a combination of the Fourier cosine series and its auxiliary functions. Then, the Rayleigh–Ritz method is applied to solve the unknown Fourier coefficients, which determines the dynamic characteristics of the coupled structure. The Newmark method is adopted to obtain the transient response of the coupled structure, where the Rayleigh damping is taken into consideration. The rapid convergence of the current method is shown, and good agreement between the predicted results and FEM results is also revealed. On this basis, the effects of the factors related to the stiffening beam (including the length, orientations, and arrangement spacing of beams) and elastic parameters, as well as damping coefficients on the dynamic characteristics of the stiffened plate are investigated.

1. Introduction

The stiffened plate component can be regarded as a coupling structure that is composed of a plate and several beams. In practical engineering, the connection between the plate and beam does not only involve the classical coupling, but the elastic coupling is also frequently encountered. In addition, due to the complexity of the actual working conditions, the stiffened plates will be often subjected to the elastic boundary restrictions. A good understanding of the dynamics of the stiffened plate with general boundary restrains can provide direct benefits to the structural design of complex systems. However, there are only a few literatures on the dynamic analysis of the rectangular plate with some beams of arbitrary lengths and orientations. The current work is to present the dynamic analysis of the beam-stiffened rectangular plate with general boundary restrains.

In the past decades, many scholars have made a lot of efforts on free vibration characteristics of the stiffened plate. Mukherjee and Mukhopadhyay [1] presented the free vibration

of the isoparametric stiffened plate element, which could accommodate irregular boundaries. Holopainen [2] proposed a new finite element model to consider free vibration of eccentrically stiffened plate. In this research, a plate with any number of arbitrarily oriented stiffeners was presented. Based on the Rayleigh–Ritz method, Liew et al. [3] and Lee and Ng [4] both investigated the free vibration of the stiffened plate with arbitrarily oriented stiffeners. The difference was that Liew studied the stiffened plate with various shapes, while Lee only examined the stiffened rectangular plate. A hierarchical finite element in conjunction with local trigonometric interpolation functions was employed by Barrette et al. [5] to investigate the vibration performances of the stiffened plate. Employing the differential quadrature technique, free vibration of plate with eccentric stiffeners was performed by Zeng and Bert [6]. And their theoretical solutions were in good agreement with the experimental results. Using the finite element method, Srivastava et al. [7] explored vibration of the stiffened plate subjected to partial edge loading. However, only the simply supported and clamped cases were considered in their research.

On the basis of the first-order shear deformable theory, Peng et al. [8] used mesh-free Galerkin method to carry out the free vibration and stability analyses of the stiffened plate. On the framework of the assumed-modes method, low-frequency free vibration analysis of the thin rectangular plate with a few light stiffeners was made by Dozio and Ricciardi [9]. Based on the improved Fourier series method, Xu et al. [10] examined the free vibration of the beam-stiffened rectangular plate, where beams of arbitrary lengths and placement angles were taken into consideration. Three different modelling approaches were reported by Bhar et al. [11] to carry out the component-wise vibration analysis of the stiffened plate, and comparative studies between different models were also given. The static, free vibration and buckling analysis of the stiffened plate were carried out by Nguyen-Thoi et al. [12] using CS-FEM-DSG3. In Ref. [13], the CS-FEM-DSG3 method was also extended to analyze the static and free vibration of stiffened folded plates, where three-node triangular element was adopted. Four hierarchical models were developed by Bhaskar and Pydah [14] to obtain the natural frequencies of the isotropic and orthotropic stiffened plate with simply supported boundary condition. On the basis of the elastic approach, Pydah and Bhaskar [15] established an analytical model to investigate the statics and dynamics of the stiffened plate with simply supported case. Based on the assumed mode method, Cho et al. [16] investigated natural frequencies and modes of various opening stiffened panels under different boundary conditions, where the effect of the parameters of the rib on the frequency was checked. Then, he and other co-authors [17, 18] carried out the free vibration analysis of the stiffened panel with lumped mass and stiffness attachments, including rectangular plates without opening and with various opening shapes. Utilizing the NURBS based on isogeometric analysis approach, Qin et al. [19] examined the frequency characteristics of curved stiffened plate under in-plane loading. Considering a beam-stiffened plate, Yin et al. [20] examined its vibration transmission by applying the dynamic stiffness method, where the transmission mode was presented. Meanwhile, Damnjanović et al. [21] adopted the same approach to perform the free vibration analysis of the stiffened composite plate based on the high-order shear deformation theory, and the results were in good agreement with the available data and FEM results.

Through the abovementioned review of literatures, the predecessors have conducted in-depth research on the free vibration of the stiffened plate. Nevertheless, many scholars are not satisfied with research in the field of free vibration. Harik and Salamoun [22] applied an analytical strip method to examine bending deflection of the stiffened plate, where the type of load and the number of ribs were used to determine the number of strips. Transient response of the simply supported panel reinforced by stiffeners was predicted by Louca et al. [23] using the finite element technique, where the effect of boundary condition on transient response was considered. Sheikh and Mukhopadhyay [24] applied the spline finite strip method to analyze the linear and nonlinear transient vibration of the stiffened plate with arbitrary shapes, in which stiffeners had arbitrary orientation and eccentricity. Based on finite element analysis, Srivastava et al. [25] presented the dynamic instability of the stiffened plate,

where both in-plane partial and concentrated loadings at the edge were examined. Xu et al. [26] developed the structural intensity technique to analyze energy flow for the stiffened plate applied in the field of marine structures. Later, the same authors [27] investigated the structural intensity of the stiffened plate by means of the finite element method. In this research, they draw a conclusion that the energy flow of the whole structure was closely related to the natural frequency. Employing the harmonic compound strip method, Borković et al. [28] obtained the transient response of the beam-stiffened plate, where Bernoulli–Euler beam and Kirchhoff–Love thin plate were both applied. Combining the Mindlin plate theory and Timoshenko beam theory, Cho et al. [29] extended the assumed mode method to predict forced vibration of the stiffened plate and rectangular plate with various boundary restrains. Tian and Jie [30] carried out the research on the dynamic response of the finite ribbed plate under point force moment excitations, and he concluded that the input mobilities were affected by the distance between the excitation point and the beam. Then, he [31] performed the vibration response analysis and experiment of the stiffened plates subjected to the clamped support at edge. In 2018, he [32] introduced a double cosine integral transform technique to study the free vibration and forced response of the stiffened plate with free boundary restrains. Meanwhile, the energy flow of stiffened panels was reported by Cho et al. [33], where the structural intensity technique was applied. In order to efficiently solve the dynamic responses of the stiffened plate in wide frequency domain, Jia et al. [34] proposed a Composite B-spline Wavelet Elements Method (CBWEM) and verified the reliability of the method by comparison with experimental results. Applying the Mindlin plate and Timoshenko beam theory, Liu et al. [35] analyzed the dynamic power transmission characteristics of a finite stiffened plate with various classical cases, where the effects number and geometric dimensions on the stop band in the low-frequency domain were shown.

Through the review of the abovementioned literature, it can be found that the existing researches are limited to the dynamic of beam-stiffened plate with classical boundary conditions (like clamped, simply supported, and free). Only the free vibration analysis related to elastic boundary conditions and nonfixed connections between a plate and beams is carried out in Xu's et al. [10] research. However, the investigations on the mobility characteristic, structural intensity, and transient response analysis of the stiffened plate subjected to the elastic boundary are still blank. In an attempt to fill this gap, the current work presents the dynamic analysis of the rectangular beam-reinforced plate with general boundary restrains, including classical and the elastic restrains as well as elastic coupling of beams and plate. IFSM is extended to construct displacement field of the beam-stiffened plate, which has proven to be a very effective method from existing research [10, 36–38]. The artificial spring technology is introduced to obtain general boundary conditions and various coupling connections between a plate and beams. Based on Xu's et al. [10] research, the energy expressions of the coupling system is obtained, which can be solved by the Rayleigh–Ritz method. Transient

response of the stiffened plate with classical or elastic boundary restrains is also calculated by the Newmark method. The consistency of the current data and FEM results is revealed. Also some novel numerical results with respect to the structural intensity, structural mobility, and transient response of the stiffened plate are also reported.

2. Theoretical Formulations

2.1. Descriptions of the Model. The geometry of a typical beam-stiffened plate and the coordinate system of the coupling structure are depicted in Figure 1. For the sake of illustration, only a thin plate reinforced by a beam with any lengths and angles is shown in Figure 1(a), where a , b , and h denote the length, width, and thickness of the plate, respectively. In addition, the primary coordinate system (o - x - y - z) is placed on the middle surface of the plate, where u , v , and w represent the midsurface displacements of the plate in the primary coordinate system, respectively. The width and height of the cross section for a stiffening beam are denoted by b_b and h_b , respectively, which is in the local coordinate system (o' - x' - y' - z'). Four types of boundary springs are used on each edge of the plate to obtain arbitrary boundary conditions, in which three types (k_w , k_u , and k_v) of linear springs and one type of rotational spring (K_w) are included. For a thin plate, the clamped boundary condition will be obtained when the spring stiffness value of each side of the plate is infinity. Conversely, the free boundary is easily achieved when the stiffness value is 0. In current work, the stiffening beam and plate are described in separate systems. As shown in Figure 1(a), six sets of coupling springs are used to achieve the connection between the beam and plate, including three linear springs (k_{pb1} , k_{pb2} , and k_{pb3}) and three types of rotational springs (K_{pb1} , K_{pb2} , and K_{pb3}). By setting stiffness values of coupling springs, different coupling relationships between the beams and plate can be achieved, including various elastic coupling and clamped coupling. It should be noted that the coupling between two intersecting beams is ignored in this paper. From Figure 1(b), the orientation between the local coordinate system (o' - x' - y' - z') attached to each beam and the primary coordinate system (o - x - y - z) is denoted by the symbol of φ when the length of the beam is expressed by L . And the coordinates of the starting end of the beam in the primary coordinate system are represented by (x_0, y_0) .

2.2. Energy Functionals. In this study, both the in-plane displacement components (u , v) and the out-of-plane displacement component (w) of the plate are considered. Therefore, the total strain potential energy (V_p) of the plate is composed of the out-of-plane strain potential energy (V_p^{out}) and the in-plane strain potential energy (V_p^{in}). Specifically, the expression of the total potential energy is as follows:

$$V_p = V_p^{\text{out}} + V_p^{\text{in}}, \quad (1)$$

in which

$$V_p^{\text{out}} = \frac{D_p}{2} \iint_s \left(\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 2\mu_p \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2(1 - \mu_p) \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) dS, \quad (2)$$

$$V_p^{\text{in}} = \frac{G_p}{2} \iint_s \left\{ \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 - 2(1 - \mu_p) \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{1 - \mu_p}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right\} dS. \quad (3)$$

Similarly, the kinetic energy (T_p) expression of the plate can be written as

$$T_p = \frac{\rho_p h}{2} \iint_s \left[\left(\frac{\partial w}{\partial t} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 \right] dS \quad (4)$$

$$= \frac{\rho_p h \omega^2}{2} \iint_s [w^2 + u^2 + v^2] dS.$$

In equations (2)~(4), $D_p = E_p h^3 / (12(1 - \mu_p))$ is the bending stiffness of the plate and E_p , G_p , μ_p , ρ_p , and ω represent Young's modulus, extensional rigidity, Poisson's ratio, mass density, and circular frequency, respectively [10].

Since the boundary conditions of the plate are realized by the artificial virtual spring technology, the elastic potential energy (V_{sp}) stored in the boundary springs here should be taken into the potential energy of plate. V_{sp} can be expressed as

$$V_{sp} = \frac{1}{2} \int_0^b \left[k_{wx0} w^2 + K_{wx0} \left(\frac{\partial w}{\partial x} \right)^2 \right]_{x=0} dy + \frac{1}{2} \int_0^b \left[k_{wxa} w^2 + K_{wxa} \left(\frac{\partial w}{\partial x} \right)^2 \right]_{x=a} dy$$

$$+ \frac{1}{2} \int_0^a \left[k_{wy0} w^2 + K_{wy0} \left(\frac{\partial w}{\partial y} \right)^2 \right]_{y=0} dx + \frac{1}{2} \int_0^a \left[k_{wya} w^2 + K_{wya} \left(\frac{\partial w}{\partial y} \right)^2 \right]_{y=b} dx$$

$$+ \frac{1}{2} \int_0^b [k_{ux0} u^2 + k_{vx0} v^2]_{x=0} dy + \frac{1}{2} \int_0^b [k_{uxa} u^2 + k_{vxa} v^2]_{x=a} dy$$

$$+ \frac{1}{2} \int_0^a [k_{vy0} v^2 + k_{uy0} u^2]_{y=0} dx + \frac{1}{2} \int_0^a [k_{vya} v^2 + k_{uya} u^2]_{y=b} dx. \quad (5)$$

FIGURE 1: Schematic diagram of the coupling systems. (a) Geometry of rectangular plate reinforced by arbitrarily orientated beams. (b) Schematic of an arbitrarily placed beam.

In this paper, four degrees of freedom for a single beam are considered, which are two bending displacement components ($w_y b$ and $w_z b$) in the y' and z' directions and the axial displacement component (u_b) and torsional displacement component (θ_b) in the x' direction. Therefore, the strain potential energy (V_{bi}) of i th beam can be written as

$$V_{bi} = \frac{1}{2} \int_0^{L_i} \left[D_{bi}^z \left(\frac{\partial^2 w_{bi}^z}{\partial x'^2} \right)^2 + D_{bi}^y \left(\frac{\partial^2 w_{bi}^y}{\partial x'^2} \right)^2 + E_{bi} A_{bi} \left(\frac{\partial u_{bi}}{\partial x'} \right)^2 + G_{bi} J_{bi} \left(\frac{\partial \theta_{bi}}{\partial x'} \right)^2 \right] dx' \quad (6)$$

The kinetic energy of i th beam can be expressed as

$$T_{bi} = \frac{\rho_{bi}}{2} \int_0^{L_i} \left[A_{bi} \left(\frac{\partial w_{bi}^z}{\partial t} \right)^2 + A_{bi} \left(\frac{\partial w_{bi}^y}{\partial t} \right)^2 + A_{bi} \left(\frac{\partial u_{bi}}{\partial t} \right)^2 + J_{bi} \left(\frac{\partial \theta_{bi}}{\partial t} \right)^2 \right] dx' \quad (7)$$

$$= \frac{\rho_{bi} \omega^2}{2} \int_0^{L_i} \left[A_{bi} (w_{bi}^z)^2 + A_{bi} (w_{bi}^y)^2 + A_{bi} (u_{bi})^2 + J_{bi} \theta_{bi}^2 \right] dx'$$

In equations (6)~(7), D_{bi}^z and D_{bi}^y are, respectively, the bending rigidity in the x' - z' plane and x' - y' plane when torsional is denoted by the symbol of J_{bi} . In addition, Young's modulus, shear modulus, mass density, and cross-sectional area of i th beam are taken place of E_{bi} , G_{bi} , ρ_{bi} , and A_{bi} respectively.

Since the stiffening beam and plate is connected by a coupling spring, the coupling potential ($V_{pb Ci}$) of the coupling spring needs to be considered in the coupling system. According to references [10, 39], its specific expression can be written as

$$V_{Ci}^{pb} = \frac{1}{2} \int_0^{L_i} \left[k_{pb1} (w - w_{bi}^z)^2 + k_{pb2} (v \cos \varphi - u \sin \varphi - w_{bi}^y)^2 + k_{pb3} (v \sin \varphi + u \cos \varphi - u_{bi})^2 \right] dx' + \frac{1}{2} \int_0^{L_i} \left[K_{pb1} \left(w_x^{pb1} - \frac{\partial w_{bi}^z}{\partial x'} \right)^2 + K_{pb2} \left(\frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) - \frac{\partial w_{bi}^y}{\partial x'} \right)^2 + K_{pb3} (w_y^{pb1} - \theta_{bi})^2 \right] dx' \quad (8)$$

It should be noted that six degrees of freedom between the beam and the plate are considered here. As shown in equation (8), three sets of linear springs (k_{pb1} , k_{pb2} , and k_{pb3}) are used for linear displacement constraints and three sets of rotational springs (K_{pb1} , K_{pb2} , and K_{pb3}) are used for angular displacement constraints. Besides, w_{bi}^z and w_{bi}^y are shown as follows

$$w_x^{pb1} = \frac{\partial w}{\partial x} \cos \varphi + \frac{\partial w}{\partial y} \sin \varphi, \quad (9)$$

$$w_y^{pb1} = -\frac{\partial w}{\partial x} \sin \varphi + \frac{\partial w}{\partial y} \cos \varphi.$$

2.3. Displacement Function. In the present research, the admissible displacement functions of a plate and a beam are established by IFSM, which has been successfully applied to solve the vibration of beams [40, 41], plates [42, 43], and shells [44]. The improved Fourier series consists of main functions and auxiliary functions. For the auxiliary functions, theoretically, they may have infinite forms. Considering programming and computational efficiency, a set of sine functions is used as auxiliary functions. Hence, the displacement functions of the plate can be written as follows:

$$\begin{aligned}
u(x, y, t) &= \mathbf{U} \mathbf{A}_u e^{j\omega t}, & \text{in which} \\
v(x, y, t) &= \mathbf{V} \mathbf{B}_v e^{j\omega t}, & (10) \\
w(x, y, t) &= \mathbf{W} \mathbf{C}_w e^{j\omega t},
\end{aligned}$$

$$\mathbf{U} = \mathbf{V} = \begin{bmatrix} \cos(\lambda_0 x) \cos(\lambda_0 y), \dots, \cos(\lambda_0 x) \cos(\lambda_n y), \dots, \cos(\lambda_0 x) \cos(\lambda_N y), \dots, \\ \cos(\lambda_m x) \cos(\lambda_n y), \dots, \cos(\lambda_m x) \cos(\lambda_N y), \dots, \cos(\lambda_M x) \cos(\lambda_N y) \\ \cos(\lambda_0 x) \sin(\lambda_1 y), \dots, \cos(\lambda_m x) \sin(\lambda_1 y), \dots, \cos(\lambda_M x) \sin(\lambda_1 y), \dots, \\ \cos(\lambda_0 x) \sin(\lambda_2 y), \dots, \cos(\lambda_m x) \sin(\lambda_2 y), \dots, \cos(\lambda_M x) \sin(\lambda_2 y), \\ \sin(\lambda_1 x) \cos(\lambda_0 y), \dots, \sin(\lambda_1 x) \cos(\lambda_n y), \dots, \sin(\lambda_1 x) \cos(\lambda_N y), \dots, \\ \sin(\lambda_2 x) \cos(\lambda_0 y), \dots, \sin(\lambda_2 x) \cos(\lambda_n y), \dots, \sin(\lambda_2 x) \cos(\lambda_N y) \end{bmatrix}, \quad (11a)$$

$$\mathbf{W} = \begin{bmatrix} \cos(\lambda_0 x) \cos(\lambda_0 y), \dots, \cos(\lambda_0 x) \cos(\lambda_n y), \dots, \cos(\lambda_0 x) \cos(\lambda_N y), \dots, \\ \cos(\lambda_m x) \cos(\lambda_n y), \dots, \cos(\lambda_m x) \cos(\lambda_N y), \dots, \cos(\lambda_M x) \cos(\lambda_N y) \\ \cos(\lambda_0 x) \sin(\lambda_1 y), \dots, \cos(\lambda_m x) \sin(\lambda_1 y), \dots, \cos(\lambda_M x) \sin(\lambda_1 y), \dots, \\ \cos(\lambda_0 x) \sin(\lambda_n y), \dots, \cos(\lambda_m x) \sin(\lambda_n y), \dots, \cos(\lambda_M x) \sin(\lambda_n y), \dots, \\ \cos(\lambda_0 x) \sin(\lambda_4 y), \dots, \cos(\lambda_m x) \sin(\lambda_4 y), \dots, \cos(\lambda_M x) \sin(\lambda_4 y), \\ \sin(\lambda_1 x) \cos(\lambda_0 y), \dots, \sin(\lambda_1 x) \cos(\lambda_n y), \dots, \sin(\lambda_1 x) \cos(\lambda_N y), \dots, \\ \sin(\lambda_m x) \cos(\lambda_0 y), \dots, \sin(\lambda_m x) \cos(\lambda_n y), \dots, \sin(\lambda_m x) \cos(\lambda_N y), \dots, \\ \sin(\lambda_4 x) \cos(\lambda_0 y), \dots, \sin(\lambda_4 x) \cos(\lambda_n y), \dots, \sin(\lambda_4 x) \cos(\lambda_N y) \end{bmatrix}, \quad (11b)$$

$$\mathbf{A}_u = \begin{bmatrix} A_{00}^u, \dots, A_{0n}^u, \dots, A_{0N}^u, \dots, A_{mn}^u, \dots, A_{mN}^u, \dots, A_{MN}^u, \\ a_{01}, \dots, a_{m1}, \dots, a_{M1}, a_{02}, \dots, a_{m2}, \dots, a_{M2}, \\ b_{10}, \dots, b_{1n}, \dots, b_{1N}, b_{20}, \dots, b_{2n}, \dots, b_{2N} \end{bmatrix}^T, \quad (12a)$$

$$\mathbf{B}_v = \begin{bmatrix} B_{00}^v, \dots, B_{0n}^v, \dots, B_{0N}^v, \dots, B_{mn}^v, \dots, B_{mN}^v, \dots, B_{MN}^v, \\ c_{01}, \dots, c_{m1}, \dots, c_{M1}, c_{02}, \dots, c_{m2}, \dots, c_{M2}, \\ d_{10}, \dots, d_{1n}, \dots, d_{1N}, d_{20}, \dots, d_{2n}, \dots, d_{2N} \end{bmatrix}^T, \quad (12b)$$

$$\mathbf{C}_w = \begin{bmatrix} C_{00}^w, \dots, C_{0n}^w, \dots, C_{0N}^w, \dots, C_{mn}^w, \dots, C_{mN}^w, \dots, C_{MN}^w, \\ e_{01}, \dots, e_{m1}, \dots, e_{M1}, e_{0n}, \dots, e_{mn}, \dots, e_{Mn}, \dots, e_{M4}, \\ f_{10}, \dots, f_{1n}, \dots, f_{1N}, \dots, f_{m0}, \dots, f_{mn}, \dots, f_{mN}, \dots, f_{4N} \end{bmatrix}^T, \quad (12c)$$

where $\lambda_m = m\pi/a$, $\lambda_n = n\pi/b$, and M, N represent the number of truncations. Fourier coefficients of displacement functions for the plate are expressed by $A_{u\ mn}$, a_{mb} , b_{ln} , $B_{v\ mn}$, c_{mb} , d_{ln} , $C_{w\ mn}$, e_{mp} , and f_{pn} ($l=1, 2$; $p=1, 2, 3, 4$), respectively.

Similarly, the displacement field functions of the i th beam can be obtained by IFSM as follows:

$$\begin{aligned}
w_{bi}^z(x') &= \Psi_{bi}^z \mathbf{D}_{mi} e^{j\omega t}, \\
w_{bi}^y(x') &= \Psi_{bi}^y \mathbf{E}_{mi} e^{j\omega t}, \\
u_{bi}(x') &= \Psi_{bi}^u \mathbf{F}_{mi} e^{j\omega t}, \\
\theta_{bi}(x') &= \Psi_{bi}^\theta \mathbf{G}_{mi} e^{j\omega t},
\end{aligned} \quad (13)$$

where

$$\Psi_{bi}^z = \Psi_{bi}^y = \begin{bmatrix} \sin(\lambda_{-4}x'), \dots, \sin(\lambda_{m1}x'), \dots, \sin(\lambda_{-1}x') \\ \cos(\lambda_0x'), \dots, \cos(\lambda_{m1}x'), \dots, \cos(\lambda_{M1}x') \end{bmatrix}, \quad (14a)$$

$$\Psi_{bi}^u = \Psi_{bi}^\theta = [\sin(\lambda_{-2}x'), \sin(\lambda_{-1}x'), \cos(\lambda_0x'), \dots, \cos(\lambda_{m1}x'), \dots, \cos(\lambda_{M1}x')], \quad (14b)$$

$$\mathbf{D}_{mi} = [D_{-4} \dots D_{-1} D_0 \dots D_{M1}]^T, \quad (15a)$$

$$\mathbf{E}_{mi} = [E_{-4} \dots E_{-1} E_0 \dots E_{M1}]^T, \quad (15b)$$

$$\mathbf{F}_{mi} = [F_{-2} F_{-1} F_0 \dots F_{M1}]^T, \quad (15c)$$

$$\mathbf{G}_{mi} = [G_{-2} G_{-1} G_0 \dots G_{M1}]^T. \quad (15d)$$

In the above equations (14a)~(15d), $\lambda_{m1} = m_1\pi/L_i$ and $M1$ indicates the truncated number. D_{mi} , E_{mi} , F_{mi} , and G_{mi} represent the unknown coefficient of i th beam's displacement functions.

2.4. Steady-State Vibration Solution. For the calculation of the steady-state response for the beam-stiffened plate under external excitation, it is necessary to consider the work done by the external force. In terms of the coupling structure shown in Figure 1(a), the Lagrange function (L) can be expressed as follows:

$$L = V_p + V_{sp} + \sum_{i=0}^N (V_{bi} + V_{Ci}^{pb}) - \left(T_p + \sum_{i=0}^N T_{bi} \right) - W_{exc}, \quad (16)$$

in which the number of stiffening beams is denoted by N , $i=0$ means unreinforced bare plate, and W_{exc} represents the work done by the external excitation force acting on the plate:

$$W_{exc} = \iint_S (f_u u + f_v v + f_w w) dS. \quad (17)$$

In the above equations (14a) and (14b), the external load distribution function on the plate is denoted by the symbols of f_u , f_v , and f_w . In the current research, f_w is taken into consideration, which expresses the lateral load associated with out-of-plane displacement (w) of the plate. In particular, for external point excitation, f_w is represented by the magnitude (F) of the point excitation force and the 2D delta function (δ):

$$f_w = F\delta(x - x_{exc})\delta(y - y_{exc}). \quad (18)$$

Substituting equations (1)~(8), (10), (13), and (17) into equation (16), linear equations in the matrix form are available by employing the Rayleigh-Ritz method:

$$(\mathbf{K} - \omega^2 \mathbf{M}) \vec{\mathbf{H}} = \vec{\mathbf{F}}, \quad (19)$$

where the vector $\vec{\mathbf{H}} = [\mathbf{A}_u^T \ \mathbf{B}_v^T \ \mathbf{C}_w^T \ \dots \ \mathbf{D}_{mi}^T \ \mathbf{E}_{mi}^T \ \mathbf{F}_{mi}^T \ \mathbf{G}_{mi}^T \ \dots]^T$ is composed of the unknown Fourier coefficient,

$\vec{\mathbf{F}} = [\mathbf{0} \ \mathbf{0} \ f_w \mathbf{C}_w^T \ \dots \ \mathbf{0} \ \mathbf{0} \ \mathbf{0} \ \dots]^T$ denotes the external force vector, and the mass matrix and stiffness matrix of the coupled systems are displayed by $\mathbf{M}$ and $\mathbf{K}$, where their elements are only related to the material properties, geometric dimensions, and boundary constraints of the beam-stiffened plate structure. And their expressions are presented as follows:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{uu} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{vv} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{ww} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{bi} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots \end{bmatrix}, \quad (20)$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_u & \mathbf{K}_{uv} & \mathbf{0} & \dots & \mathbf{K}_{bi}^u & \dots \\ \mathbf{K}_{uv}^T & \mathbf{K}_v & \mathbf{0} & \dots & \mathbf{K}_{bi}^v & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_w & \dots & \mathbf{K}_{bi}^w & \dots \\ \vdots & \vdots & \vdots & \ddots & \mathbf{0} & \dots \\ [\mathbf{K}_{bi}^u]^T & [\mathbf{K}_{bi}^v]^T & [\mathbf{K}_{bi}^w]^T & \mathbf{0} & \mathbf{K}_{bi} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \mathbf{0} & \ddots \end{bmatrix}. \quad (21)$$

For the external excitation force with any frequency, the unknown series expansion coefficient related to displacement functions of the coupled structure can be directly calculated from equation (19). That is,

$$\vec{\mathbf{H}} = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \vec{\mathbf{F}}. \quad (22)$$

Substituting equation (22) into equation (10), the displacement response of plate reinforced by beams under some specific excitation will be obtained. And vibration speed can also be obtained by $v = j\omega w$. Particularly, the complex Young's modulus ($\bar{E}$) is introduced in the current numerical solution process. $\bar{E} = E_p(1 + j\eta)$ is composed of Young's modulus (E_p) and structural loss factor (η), where $j = \sqrt{-1}$ is the imaginary unit.

The structure mobility can quantitatively describe the law of power flow in the structure. Therefore, understanding the admittance characteristics of the stiffened plate is of great significance for structural vibration control. The structure mobility can be calculated by the following equation:

$$Y_{nk} = \frac{V_k}{F_n}, \quad (23)$$

in which Y_{nk} represents the transfer mobility from point n to point k when V_k is utilized to express the velocity at point k . And the magnitude of the excitation force on point n is presented by F_n . The mobilities of the drive point n can be achieved by setting $n = k$.

The vibration energy of the structure can be described by the structural intensity vector, which helps the designers understand the energy distribution of the structure. The definition of structural intensity can be found in references [26, 37, 45]. In this paper, the structural intensity vector $I(x, y)$ of any point in the stiffened plate can be obtained by vector superposition of the component $I_x(x, y)$ along the x -axis and component $I_y(x, y)$ along the y -axis. Their relationship is seen in the following equation:

$$|I(x, y)| = \sqrt{|I_x(x, y)|^2 + |I_y(x, y)|^2}. \quad (24)$$

In equation (24), the structural intensity at any point includes both bending vibration components and in-plane vibration components, as shown in equations (25a) and (25b):

$$I_x(x, y) = I_w^x(x, y) + I_{uv}^x(x, y), \quad (25a)$$

$$I_y(x, y) = I_w^y(x, y) + I_{uv}^y(x, y). \quad (25b)$$

In the above equations (25a) and (25b), the structural intensity components associated with structural bending vibrations in both directions are

$$I_w^x(x, y) = -\frac{1}{2} \text{Re} \left\{ Q_x \left(\frac{\partial w}{\partial t} \right)^* - M_x \left(\frac{\partial^2 w}{\partial x \partial t} \right)^* - M_{xy} \left(\frac{\partial^2 w}{\partial y \partial t} \right)^* \right\}, \quad (26a)$$

$$I_w^y(x, y) = -\frac{1}{2} \text{Re} \left\{ Q_y \left(\frac{\partial w}{\partial t} \right)^* - M_y \left(\frac{\partial^2 w}{\partial y \partial t} \right)^* - M_{yx} \left(\frac{\partial^2 w}{\partial x \partial t} \right)^* \right\}, \quad (26b)$$

in which the symbol of $(*)$ represents the conjugate of a complex number. And the transverse shear, bending moment, and torque in the two directions are, respectively,

$$Q_x = -D_p \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right),$$

$$M_x = -D_p \left(\frac{\partial^2 w}{\partial x^2} + \mu_p \frac{\partial^2 w}{\partial y^2} \right), \quad (27a)$$

$$M_{xy} = -(1 - \mu_p) D_p \frac{\partial^2 w}{\partial x \partial y},$$

$$Q_y = -D_p \left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y} \right),$$

$$M_y = -D_p \left(\frac{\partial^2 w}{\partial y^2} + \mu_p \frac{\partial^2 w}{\partial x^2} \right), \quad (27b)$$

$$M_{yx} = -(1 - \mu_p) D_p \frac{\partial^2 w}{\partial x \partial y}.$$

The structural intensity associated with the in-plane vibration components in equations (25a) and (25b) is presented as follows:

$$I_{uv}^x(x, y) = -\frac{1}{2} \text{Re} \left\{ \sigma_x \left(\frac{\partial u}{\partial t} \right)^* + \tau_{xy} \left(\frac{\partial v}{\partial t} \right)^* \right\} = -\frac{1}{2} \text{Re} \left\{ \frac{E_p h}{1 - \mu_p^2} \left(\frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial y} \right) \left(\frac{\partial u}{\partial t} \right)^* + \frac{E_p h}{2(1 + \mu_p)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \left(\frac{\partial v}{\partial t} \right)^* \right\}, \quad (28a)$$

$$I_{uv}^y(x, y) = -\frac{1}{2} \text{Re} \left\{ \sigma_y \left(\frac{\partial v}{\partial t} \right)^* + \tau_{yx} \left(\frac{\partial u}{\partial t} \right)^* \right\} = -\frac{1}{2} \text{Re} \left\{ \frac{E_p h}{1 - \mu_p^2} \left(\mu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \left(\frac{\partial v}{\partial t} \right)^* + \frac{E_p h}{2(1 + \mu_p)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \left(\frac{\partial u}{\partial t} \right)^* \right\}. \quad (28b)$$

Substituting equations (10), (22), and (26a)~(28b) into (25a) and (25b), the structural intensity at any point of the stiffened plate can be obtained.

2.5. Transient Vibration Solution. The mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ of the coupled structure have been presented in equations (20)~(21) in Section 2.4. Hence, the dynamic equation of the structural system in the time domain can be described as [46]

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}_p, \quad (29)$$

where $\mathbf{F}_p$ is the external excitation vector, which is the function of time, and x denotes the displacement at time t .

Generally, damping constants have frequency-varying characteristics. Therefore, it is difficult to accurately define the damping matrix in numerical simulation. For the sake of simplicity, only Rayleigh damping of the coupling structure is considered in this study, namely, $\mathbf{C} = \alpha_0 \mathbf{M} + \beta_0 \mathbf{K}$. And α_0

and β_0 are the Rayleigh damping coefficients independent of frequency.

In order to solve the transient response of the system, the Newmark method is adopted, which is an extensive implicit algorithm. Its integration method is as follows:

$$\dot{x}_{n+1} = \dot{x}_n + (1 - \gamma)\Delta t \ddot{x}_n + \gamma\Delta t \ddot{x}_{n+1}, \quad (30)$$

$$x_{n+1} = x_n + \Delta t \dot{x}_n + \left(\frac{1}{2} - \beta\right)\Delta t^2 \ddot{x}_n + \beta\Delta t^2 \ddot{x}_{n+1}. \quad (31)$$

In equations (30)~(31), when the velocity function and the displacement function are expanded by Taylor series, the expansion is retained to the second derivative and equations (32a)~(33) can be calculated as follows:

$$\ddot{x}_{n+1} = \frac{1}{\beta\Delta t^2} (x_{n+1} - x_n) - \frac{1}{\beta\Delta t} \dot{x}_n - \left(\frac{1}{2\beta} - 1\right) \ddot{x}_n, \quad (32a)$$

$$\dot{x}_{n+1} = \frac{\gamma}{\beta\Delta t} (x_{n+1} - x_n) + \left(1 - \frac{\gamma}{\beta}\right) \dot{x}_n + \left(1 - \frac{\gamma}{2\beta}\right) \Delta t^2 \ddot{x}_n. \quad (32b)$$

Combining equations (29), (32a), and (32b), the following matrix equation (33) can be obtained:

$$\bar{\mathbf{K}}\mathbf{x}_{n+1} = \mathbf{F}_p^{n+1}, \quad (33)$$

in which

$$\bar{\mathbf{K}} = \mathbf{K} + \frac{1}{\beta\Delta t^2} \mathbf{M}, \quad (34)$$

$$\mathbf{F}_p^{n+1} = \mathbf{f}_{n+1} + \mathbf{M} \left[\frac{1}{\beta\Delta t^2} \mathbf{x}_n + \frac{1}{\beta\Delta t} \dot{\mathbf{x}}_n + \left(\frac{1}{2\beta} - 1\right) \ddot{\mathbf{x}}_n \right].$$

The accuracy and stability of the Newmark method depends on β and γ . In this study, $\beta = 0.25$ and $\gamma = 0.5$ are used, which is known as the average acceleration Newmark method. In addition, the force in the above equation (29) can be either point force or pressure.

3. Results and Discussions

In this section, a series of numerical results for dynamic behaviors of the stiffened plate with general boundary restraints are carried out based on the theoretical model established in Section 2. Further analysis will be given in the following sections. Particularly, unless otherwise stated, the beam and the plate are assumed to have the same material properties, i.e., $E_p = E_{bi} = 2.07$ GPa, $\rho_p = \rho_{bi} = 7800$ kg/m³, $\mu_p = \mu_{bi} = 0.3$, and $\eta = 0.01$. It should also be noted that for the convenience of research, two reinforced plates of different geometric parameters are included in the current research, which are rectangular plate and square plate, respectively. For a rectangular plate, its geometric parameters are $b = 1$ m, $a/b = 2$, and $h = 0.01$ m, where the measured points can be expressed as follows according to the o - x - y coordinate in Figure 1: point A: (1 m, 0.5 m) and point B: (0.5 m, 0.5 m). For the square plate, except for $a/b = 1$, the other parameters are the same as the rectangular plate, of

which the measured points are point C: (0.5 m, 0.5 m), point D: (0.25 m, 0.25 m), and point E: (0.4 m, 0.4 m). In the following analysis, the geometric and material parameters of the two plates are not repeatedly described. It is worth emphasizing that the response at the above-measured points is the lateral vibration response of the stiffened plate.

In addition, for the sake of simplicity, the symbols of C , S , F , and E are utilized to represent the clamped, simply supported, free, and elastic boundary conditions, respectively, all of which can be achieved by changing the spring stiffness. The spring stiffness values for different boundary conditions used in this paper are presented as follows:

At $x = \text{constant}$ or $y = \text{constant}$

C : $k_u = 10^{12}$ N/m, $k_v = 10^{12}$ N/m, $k_w = 10^{12}$ N/m, and $K_w = 10^{12}$ N·m/rad.

S : $k_u = 10^{12}$ N/m, $k_v = 10^{12}$ N/m, $k_w = 10^{12}$ N/m, and $K_w = 0$ N·m/rad.

F : $k_u = 0$ N/m, $k_v = 0$ N/m, $k_w = 0$ N/m, and $K_w = 0$ N·m/rad.

E^2 : $k_u = 10^{12}$ N/m, $k_v = 10^{12}$ N/m, $k_w = 10^{12}$ N/m, and $K_w = 10^2$ N·m/rad.

E^5 : $k_u = 10^{12}$ N/m, $k_v = 10^{12}$ N/m, $k_w = 10^{12}$ N/m, and $K_w = 10^5$ N·m/rad.

E^7 : $k_u = 10^{12}$ N/m, $k_v = 10^{12}$ N/m, $k_w = 10^{12}$ N/m, and $K_w = 10^7$ N·m/rad.

E^{55} : $k_u = 10^5$ N/m, $k_v = 10^5$ N/m, $k_w = 10^5$ N/m, and $K_w = 10^5$ N·m/rad.

E^{77} : $k_u = 10^7$ N/m, $k_v = 10^7$ N/m, $k_w = 10^7$ N/m, and $K_w = 10^7$ N·m/rad.

Another special note is the type and magnitude of the external load used in this paper. The excitation forces applied in the current research are point excitation and uniform pressure in the normal direction of the stiffened plate, which are denoted by the symbol of q_w and p_w , respectively. And the location of the concentrated point force is at point Q in Figure 2(a), when locally acting area of uniform pressure is shown by Ω in Figure 2(b). What needs special mention is $q_w = 1$ and $p_w = 1$ in the following calculation.

3.1. Steady-State Vibration Analysis

3.1.1. Convergence and Correctness Analysis. In this section, the stiffened rectangular plate with CCCC case is first selected to check the convergence of the current method. The cross-section parameters of the beam are $b_b/h = 1$ and $h_b/h = 1$. Figure 3 shows convergence of velocity response for the rectangular plate with one central x -wise beam and one central y -wise beam under $q_w = 1$, when the similar curves for the same coupling structure under $p_w = 1$ are presented in Figure 4. It can be observed from Figures 3~4 that when the truncated numbers M and N are equal to 12 or 14, the response curves of the coupling structure under the action of point force or surface force are basically coincident and match well with the results obtained by FEM software Ansys15.0. The above analysis fully proves that the current method has good convergence. In order to balance both

FIGURE 2: External load types for the stiffened plates: (a) point force, (b) surface force, and (c) rectangular pulse.

FIGURE 3: Convergence of velocity response for rectangular plate with one central x-wise beam and one central y-wise beam under unit force ($q_w = 1$). (a) Point A (1 m, 0.5 m). (b) Point B (0.5 m, 0.5 m).

calculation efficiency and calculation accuracy, the displacement series expressions of beams and plates are truncated by $M = N = M1 = 12$ in the following calculations. Besides, it can be seen by comparing Figures 3 and 4 that the trend of the response of drive point A or transfer point B does not change under different forms of force, and only the magnitude of the response is different.

Next, a square plate with one beam located at its diagonal line is utilized to implement a comparative study between the current method and FEM result (Ansys15.0). It is clarified that the configurations of the reinforcing component are $L_1 = \sqrt{2} m$, $b_b/h = 1$, $h_b/h = 1$, $x_0 = y_0 = 0$, and $\varphi = 45^\circ$, and two points (point C and point D) on the square plate supported by CCCC are selected as analysis points. The comparison examples are depicted in Figures 5~6 which present the comparison of velocity response for the stiffened square plate under $q_w = 1$ and under $p_w = 1$, respectively. It is easy to be observed from the two figures that the current calculation method and the finite element result are in good agreement, which proves that the construction of the steady-state response dynamic analysis

model by IFSM is correct. Simultaneously, compared with the finite element method, element mesh is not required in the current calculation model, which greatly saves the memory resources and improves the solving efficiency.

3.1.2. Structural Mobility Analysis. In this part, the effect of the stiffening beam with different lengths on mobility characteristics of the reinforced plate subjected to SSSS will be investigated at first, where $q_w = 1$ is adopted. As shown in Figure 7, four different lengths of beams are selected for the analysis of the mobility characteristics of the coupled structure, including $L_1 = 0 m$, 0.2 m, 0.5 m, and 1 m. Particularly, except for the length of the stiffening beam and the cross-section parameter $h_b/h = 1.5$, the remaining geometric configurations are consistent with Figure 5 in this example. From Figure 7, the following points can be found: Firstly, regardless of the drive point or the transfer point, the number of peak points of mobility curves for the plate with different lengths of stiffening beam compared with the bare

FIGURE 4: Convergence of velocity response for rectangular plate with one central x -wise beam and one central y -wise beam under local uniform pressure ($p_w = 1$). (a) Point A (1 m, 0.5 m). (b) Point B (0.5 m, 0.5 m).

FIGURE 5: Comparison of velocity response for square plate with one beam located at its diagonal line under unit force ($q_w = 1$). (a) Point C (0.5 m, 0.5 m). (b) Point D (0.25 m, 0.25 m).

plate will increase, which may be because the beam changes the local stiffness and mass distribution of the plate; Secondly, when the reinforcing beam passes through the transfer point D or the drive point C corresponding to $L_1 = 0.5$ m or $L_1 = 1$ m, the peak of their mobility curves will shift to the high frequency and magnitude of some peaks will appear to decrease in the range of 200–800 Hz, especially $L_1 = 1$ m, which shows that the rib increases the ratio of stiffness to mass of the plate in high frequency. The above phenomena also reveals that in structural vibration control,

reinforcement helps to reduce the response peak of the plate in the midhigh-frequency range and changes the structural resonance frequency, but has a little effect on the response in low-frequency domain.

Figure 8 studies the mobility characteristics of the diagonally stiffened square plate subjected to local uniform pressure with various elastic boundary conditions (i.e., E^2SE^2S , $E^2E^5E^2E^5$, $E^5E^5E^5E^5$, and CE^5CE^5). The points C and D are selected as the observed points, and a local uniform pressure p_w acts on the coupling system containing two

FIGURE 6: Comparison of velocity response for square plate with one beam located at its diagonal line under local uniform pressure ($p_w = 1$). (a) Point C (0.5 m, 0.5 m). (b) Point D (0.25 m, 0.25 m).

FIGURE 7: Mobility characteristics for square plate with one beam of different lengths located at its diagonal line under unit force ($q_w = 1$). (a) Point C (0.5 m, 0.5 m). (b) Point D (0.25 m, 0.25 m).

reinforcing beams. For convenience of explanation, (x_0, y_0, φ) in Figure 1(b) is used to indicate the arrangement of the ribs. Hence, the arrangement of the beams used in Figure 8 are expressed as $(0 \text{ m}, 0 \text{ m } 45^\circ)$ and $(1 \text{ m}, 0 \text{ m } 135^\circ)$. The geometric configurations of the stiffening beams are $L_1 = L_2 = \sqrt{2}$, $b_{b1}/h = 1$, $b_{b2}/h = 1$, $h_{b1}/h = 1$, and $h_{b2}/h = 1$. It can be seen from Figure 8 that as the boundary restrains vary from E^2SE^2S to CE^5CE^5 , the resonant peak of the structural mobility response moves to the high-frequency direction and the peak value of the response curves will show a

downward trend, especially under CE^5CE^5 . Therefore, it can be known that the frequency position corresponding to the vibration formant can be adjusted by modifying the boundary constraint spring stiffness to achieve structural vibration control. Another interesting finding is that the number of formants of the mobility responses increases when the stiffened plate is subjected to $E^2E^5E^2E^5$ and CE^5CE^5 boundary conditions.

In order to investigate the effect of coupling spring stiffnesses on the structural mobility characteristics, a

FIGURE 8: Mobility characteristics of diagonally stiffened square plate subjected to local uniform pressure with various elastic boundary conditions ($p_w = 1$). (a) Point C (0.5 m, 0.5 m). (b) Point D (0.25 m, 0.25 m).

stiffened rectangular plate with $E^2E^2E^2E^2$ case is presented in Figure 9. The analytical model used in Figure 9 is a rectangular plate with one central x -wise beam and three y -wise evenly distributed beams. The geometric parameters of the reinforced beam used are $L_1 = 2$ m in the x direction, $L_2 = L_3 = L_4 = 1$ m in the y direction, $b_{b1}/h = 1$, $b_{b2} = b_{b3} = b_{b4} = b_{b1}$, $h_{b1}/h = 1.5$, and $h_{b2} = h_{b3} = h_{b4} = h_{b1}$. And point A and point B are selected as the drive point and transfer point, respectively. In addition, it should be noted that the coupling springs (including k_{pb1} , k_{pb2} , k_{pb3} , K_{pb1} , K_{pb2} , and K_{pb3}) shown in Section 2 are replaced by the symbol of k_b . The curves in Figure 9 show that with the increase of the coupled spring stiffness, the resonant peaks of the structural velocity mobility responses move faster and faster in the high-frequency direction. And the resonant peaks of the structural mobility move faster in the range of 150 Hz to 500 Hz than the range of 0 to 100 Hz, which indicates that the structural mobility responses in the high frequency are more sensitive to the coupled springs. In addition, we can also see that an increase in coupling spring stiffness will result in an increase in the peak values of the mobility response in the range of 0 to 150 Hz, which indicates that the reinforcement of the beam and the plate will have a negative effect on the low-frequency vibration reduction.

In the analytical model constructed in this paper, structural damping is introduced by means of complex Young's modulus. Damping effects on the mobility characteristics of a rectangular plate stiffened by three y -wise evenly distributed beams are illustrated in Figure 10. Except for removing a beam in the x direction, the remaining geometric parameters of the current model with $E^5E^5E^5E^5$ boundary restrains and measured points follow the same values of the cases studied in Figure 9. It can be seen from

Figure 10 that due to the existence of structural damping, the peak values of the vibration mobility responses are reduced. It is found after further observation that the magnitude of resonance peaks for mobility responses shows a decreasing trend as the structural damping increases, and the peak reduction in the range of 150~500 Hz is more obvious than that in the range of 0~150 Hz. Besides, the structural damping of the stiffened plate can only effectively reduce the mobility response amplitude of the resonance peak, while the effect of structural damping on the mobility response in the nonresonant region is not obvious.

3.1.3. Structural Intensity Analysis. In the previous section, the mobility characteristics of the stiffened plates are clearly described. Undoubtedly, these numerical results provide reference data for further research. However, for the specific structural design, it is extremely necessary to understand the distribution and transmission of vibration energy in the structural system. To this end, the following analysis of structural intensity will be performed to show the flow strength and direction of the vibrational energy in the structure.

Firstly, structural intensity of a rectangular plate attached by beams with different configurations under two-point excitation is carried out, where four symmetrical boundary conditions are employed, namely, $CSCS$, $SFSF$, $E^{77}E^{77}E^{77}E^{77}$, and $E^{55}E^{55}E^{55}E^{55}$. The geometric parameters of the beams (see Figure 11) are $L_1 = L_2 = L_3 = 1$, $h_{b1}/h = 2$, $h_{b2} = h_{b3} = h_{b1}$, and $b_{b1} = b_{b2} = b_{b3} = h$. In this example, regardless of the arrangement spacing of the beams, the orientation φ of all beams and their coordinates y_0 in the y direction (see Figure 1(b)) are the same, $\varphi = 90^\circ$ and $y_0 = 0$. Therefore, the coordinates x_0 in the x direction will be used

FIGURE 9: Mobility characteristics of beam-reinforced rectangular plate with different coupling spring stiffnesses ($p_w = 1$). (a) Point A (1 m, 0.5 m). (b) Point B (0.5 m, 0.5 m).

FIGURE 10: Damping effects on the mobility characteristics of a rectangular plates stiffened by three y -wise evenly distributed beams ($p_w = 1$). (a) Point A (1 m, 0.5 m). (b) Point B (0.5 m, 0.5 m).

to show the arrangements of the three beams at the spacing of each beam. For example, (0, 0.3, 1) means that the arrangements of the three beams attached to the rectangular plate are (0 m, 0 m 90°), (0.3 m, 0 m, 90°), and (1 m, 0 m, 90°). According to the above description, the arrangements of the three-pitch reinforcing beams are (0.6 m, 0.8 m, 1 m), (0.6 m, 0.9 m, 1.2 m), and (0.6 m, 1 m, 1.4 m), which are represented by Type I, Type II, and Type III, respectively. In addition, the excitation frequency of the two-point forces is 72.54 Hz (denoting 2nd frequency of the rectangular plate with CSCS

case), the amplitude of which are 1 N. For symmetrical two-point forces (see Figure 11), the locations of two-points force are point G: (0.5 m, 0.5 m) and point H: (1.5 m, 0.5 m), respectively. The results in Figure 11 indicate that as the stiffness of the boundary spring weakens, the presence of the reinforcement will intensify the effect of the vibrational energy gathering toward the edges of the plate and the reflection and backflow of vibrational energy occur at the edges of the plate. Moreover, regardless of the boundary conditions, the increased spacing of the ribs will have a

FIGURE 11: Continued.

FIGURE 11: Structural intensity of a rectangular plate attached by beams with different configurations under two-point excitation: (a) CSCS; (b) SFSE; (c) $E^{77}E^{77}E^{77}E^{77}$; (d) $E^{55}E^{55}E^{55}E^{55}$.

positive increasing effect on the structural intensity of the plate between the reinforcements, especially for elastic boundary cases ($E^5E^5E^5E^5$ and $E^7E^7E^7E^7$). The above phenomenon may be because reinforcements hinder energy flow on the plate, resulting in energy accumulation in some areas. By further observing, it is found that in the case of symmetrical boundary cases and symmetrical point forces, there is always at least a section where the vibrational energy cancels each other out, which is called the power-insulation section. Furthermore, the number of such power-insulation

section can be increased by setting an appropriate reinforcement spacing. Also, it can be intuitively found that the vibration energy is transmitted from the excitation to the periphery of the plate, but the force source is not always output source of the vibration energy (see Figure 11(d)).

Next, structural intensity of an SSSS square plate stiffened by one beam with various orientations is shown in Figure 12, where two-point forces at point S: (0.75 m, 0.25 m) and point T: (0.25 m, 0.75 m) are adopted. The geometric parameters of the attaching beam are $L_1 = 1$, $b_{b1}/h = 1$, and

FIGURE 12: Structural intensity of an SSSS square plate stiffened by one beam with various orientations ($q_w=1$): (a) 2nd frequency (122.44 Hz); (b) 4th frequency (195.90 Hz).

$h_{b1}/h=2$, and four arrangements of the beams are taken into consideration, including $(0, 0, 0^\circ)$, $(0, 0, 20^\circ)$, $(0, 0, 30^\circ)$, and $(0, 0, 45^\circ)$, according to the beam's arrangement method adopted in Figure 8. Moreover, the excitation frequency is 122.44 Hz (see Figure 12(A~D)) and 195.90 Hz (see Figure 12(E~H)), respectively, which represent the 2nd frequency and 4th frequency for the square plate. From Figure 12, it is no difficult to find that as the azimuth angle of the reinforcement varies from 0° to 45° , the energy flow near the excitation source S first decreases and then increases and the structural intensity reaches a minimum when the reinforcement passes through the excitation source (that is, $\varphi = 20^\circ$). Besides, there are significant differences in the influence of the azimuth variation of the reinforcement on the structural density distribution of the plate at different excitation frequencies, which reflects that the effect of the reinforcement on the structural density of the plate depends dramatically on the excitation frequency.

3.2. Transient Response Analysis. This section is concerned with the transient vibration analysis of the reinforced plate under point force and local uniform pressure. The convergence and accuracy of transient responses for the plate with beams attached are first validated. For a validation case, transient responses of a coupling structure identical to the reinforced rectangular plate used in Figures 3~4 are examined in Figures 13~14 and the external excitation loads in time domain are the point excitation force q_w and local uniform pressure p_w , respectively, the time of which is

$t_0=0.2$ s shown in Figure 2(c). It is well revealed from Figures 13~14 that the results calculated by the current method converge very rapidly as M , N , and $M1$ are increased. Convergence for transient responses of the stiffened plate is achieved by $M=N=M1=12$. In addition, it can be seen through further observation that the present results are highly consistent with FEM results, which verifies the correctness of the transient response model of the stiffened plates established in this paper. In order to further verify the correctness of the damping treatment method in this paper, a comparison between the current method and FEM results is implemented in Figure 15, where only the Rayleigh damping is added and the other parameters are consistent with these from Figure 13. And the selected Rayleigh damping coefficients are set as $\alpha_0 = 0.999$ and $\beta_0 = 0.001$. As can be seen from Figure 15, a good agreement between the current results and those obtained by FEM is achieved, which indicates that the presented damping treatment of stiffened plates is reasonable.

Next, transient responses of an SSSS square plate stiffened by a beam with various orientations are presented in Figure 16. For the present analysis, the time increment is taken to be 0.33 ms and normal point load q_w located at point C is applied. The geometric parameters and arrangement of the beam are the same as those in Figure 12, but the arrangement of the beam with $(0, 0, 20^\circ)$ is replaced by $(0, 0, 15^\circ)$ here. The curves in Figure 16 reveal that regardless of the measured point A or B, when the orientation of the reinforcing beam changes from 0° to 45° , the oscillation periods of the time response curve are shortened and

FIGURE 13: Convergence of transient responses for rectangular plate with one central x -wise beam and one central y -wise beam under unit force ($q_w = 1$). (a) Point A (1 m, 0.5 m). (b) Point B (0.5 m, 0.5 m).

FIGURE 14: Convergence of transient responses for rectangular plate with one central x -wise beam and one central y -wise beam under local uniform pressure ($p_w = 1$). (a) Point A (1 m, 0.5 m). (b) Point B (0.5 m, 0.5 m).

the amplitude of the displacement responses shows a decreasing trend. This is undoubtedly expected because the closer the stiffening beam is to the measured points, the greater the bending stiffness at the measured points is.

As mentioned in Section 2.5, the Rayleigh damping is selected to implement the transient vibration analysis of the beam-stiffened plate and it can be obtained by setting the

values of α_0 and β_0 . Therefore, the influence of the Rayleigh damping coefficients on transient performances of the stiffened plate will receive attention in the following example. As shown in Figure 17, a SCSC square plate reinforced by two identical beams is considered to perform current analysis, of which the configurations are (0 m, 0 m, 45°) and (0 m, 0 m, 135°), respectively. And the geometric

FIGURE 15: Comparison of transient responses for rectangular plate with one central x -wise beam and one central y -wise beam under local uniform pressure ($p_w = -1$). (a) Point A (1 m, 0.5 m). (b) Point B (0.5 m, 0.5 m).

FIGURE 16: Transient response of an SSSS square plate stiffened by one beam with various orientations ($q_w = 1$). (a) Point C (0.5 m, 0.5 m). (b) Point E (0.4 m, 0.4 m).

parameters of the used beams are set as $L_1 = L_2 = \sqrt{2}$, $h_{b1}/h = 1$, $h_{b2} = h_{b1}$, and $b_{b1} = b_{b2} = h$. In this investigation, the same form and incentive position of normal load as Figure 17 is utilized, where the observed point is point C. From Figure 17, it can be intuitively seen that due to the existence of damping, the vibration energy of the coupling structure will be continuously consumed and the transient vibration of the coupled structure exhibits a completely different behavior than the undamped transient vibration. Secondly, the change of the mass damping coefficient α_0 has a little effect on the transient vibration of the stiffened plate (see

Figure 17(a)). However, a small change in the stiffness damping coefficient β_0 will cause a significant change in the transient response (see Figure 17(b)). Specifically, when the parameter β_0 is smaller than a certain value denoted by the symbol β_0^1 , the transient amplitude of the coupled structure will gradually decrease and finally stabilize. When β_0 exceeds the value β_0^1 , the vibration response of the reinforced structure will gradually increase and finally stabilize at a certain value. Also, as the value of β_0 increases, the time for the transient response to reach the steady state will be extended.

FIGURE 17: Effect of Rayleigh damping coefficient on transient response of stiffened square plate with SCSC case ($q_w = 1$). (a) Mass damping coefficient α_0 . (b) Stiffness damping coefficient β_0 .

In the last transient study, transient responses of a rectangular plate with three x -wise and one y -wise evenly distributed beams are examined in Figure 18, in which normal local uniform pressure ($p_w = 1$) is considered. Except for the difference of the arrangement of the beams, the parameters of the beams employed in Figure 3 are used here and the measured point is point B. What needs to be explained here is that when the same type of springs at edges of the stiffened plate is studied, the stiffness values of the remaining types of springs are set to infinity replaced by 10^{12} ; for instance, while the elastic parameters k_w varies from 0 to 10^{12} , the stiffness values of k_w , k_v , and K_w are designated as 10^{12} . Figure 18 shows that as the stiffness value of transverse springs k_w varies from 10^5 to 10^{11} , the amplitude and phase of the lateral displacement response will change significantly. Specifically, as the stiffness value increases, the oscillation period of the response curve will be shortened and the vibration amplitude will decrease. When the stiffness value of spring k_w is greater than 10^{11} , the transient displacement response remains unchanged and is close to the response under CCCC case. For a rotating spring K_w , the lateral displacement response will only change significantly if its stiffness value varies from 10^3 to 10^8 , but its effect is not as obvious as that of the transverse spring k_w . In addition, it can be seen from Figures 18(a) and 18(b) that the lateral displacement response is not affected by the variation of the in-plane linear spring k_u and k_v , which is consistent with the current classical thin plate theory. In addition, the results in Figure 18 indicate that the definition of the boundary conditions at the beginning of this section is reasonable.

4. Conclusions

An analysis model is established for investigating the linear dynamic analysis of rectangular plates reinforced by beams of any lengths and orientations. Employing IFSM, displacement functions of the plate and reinforcements are obtained, respectively. Based on Rayleigh–Ritz method, dynamic response of the coupling structure is obtained, in which various boundary restrains and beam–plate coupling relationships are achieved using the artificial spring technology. Several numerical analyses verify the effectiveness of the current method, and a series of novel numerical results are also given. Besides, it should also be noted that even though only the linear dynamic performances of the beam–stiffened plate are reported here, the presented methods can also be extended to perform other dynamics analysis related to the stiffened plate, such as the vibration control of the stiffened plate in electrothermal-magnetic field and vibration analysis of reinforced coupling plates.

In addition, there are some conclusions that are stated as follows: (1) The mobility characteristics of the beam–stiffened plate are not only related to the length of the beams, but also to coupling spring stiffness and structural damping, especially in the high-frequency range, which provides a design basis for structural vibration control. (2) The arrangement spacing and orientation of stiffening beams have a significant effect on the structural intensity distribution of the plate. Moreover, the structural intensity distribution of the stiffened plates is sensitive to the frequency of the excitation force and boundary conditions, which give a reference for discriminating the position of the load and the transmission path of the vibration energy. (3) Reducing the distance between the measured point and the reinforcing

FIGURE 18: Effect of the boundary spring stiffness on transient response of stiffened plate under local uniform pressure ($p_w = 1$). (a) Linear spring: k_u . (b) Linear spring: k_v . (c) Linear spring: k_w . (d) Rotational spring: K_w .

beams or increasing the spring stiffness (k_w and K_w) will have a negative effect on the amplitude and oscillation period of the transverse transient response of the measured point. However, the in-plane springs (k_u and k_v) have no effect on the lateral transient vibration of the stiffened plate. (4) When the Rayleigh damping is considered in the transient analysis of the stiffened plate, the transient response is heavily affected by the stiffness damping coefficient rather than the mass damping coefficient.

Data Availability

All the underlying data related to this article are available upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The authors gratefully acknowledge the financial support from the National Natural Science Foundation of China (grant no. 51905511).

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