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Research Article

The Study of Identification Method for Dynamic Behavior of High-Dimensional Nonlinear System

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The dynamic behavior of nonlinear systems can be concluded as chaos, periodicity, and the motion between chaos and periodicity; therefore, the key to study the nonlinear system is identifying dynamic behavior considering the different values of the system parameters. For the uncertainty of high-dimensional nonlinear dynamical systems, the methods for identifying the dynamics of nonlinear nonautonomous and autonomous systems are treated. In addition, the numerical methods are employed to determine the dynamic behavior and periodicity ratio of a typical hull system and Rössler dynamic system, respectively. The research findings will develop the evaluation method of dynamic characteristics for the high-dimensional nonlinear system.

1. Introduction

With the development of science and technology, the study to the dynamics of low-dimensional nonlinear dynamic systems can hardly satisfy the requirements of actual engineering [1, 2]. To explore the exact dynamics in nonlinear systems, it is necessary to study the identification method of dynamic behavior for high-dimensional nonlinear systems [3–5]. Nowadays, the principal method for identifying the dynamic behavior of the nonlinear system includes the Lyapunov index method, Poincaré mapping method, bifurcation theory, etc. The Lyapunov index method is an important method for identifying chaotic signals of nonlinear dynamical systems. The important characteristic of the nonlinear dynamical systems is that the final value of the system is sensitively depended on the initial value; therefore, the Lyapunov index method represents the average exponential rates of divergence or convergence of closed orbits of the vibrating object in phase space of a dynamic system. The Lyapunov exponent is an efficient tool for estimating whether a dynamic system is periodic or chaotic. However, the Lyapunov exponent is not suitable for determining the dynamic behavior that is neither periodic nor chaotic. The Poincaré mapping method can diagnose the dynamic characteristics of the nonlinear system based on the fixed points in the Poincaré section; however, it is unsuitable to determine the global dynamics and periodicity [6]. Although global bifurcation and local bifurcation theory can discern the dynamic characteristics of higher-dimensional dynamical systems, the prerequisite for this approach is BP normalization for the dynamic systems [7–10]. Theoretically speaking, we can calculate the canonical form of any order from a given dynamical system. Actually, it is very difficult to calculate the high order normal form for the higher-dimensional dynamical system because the process of the normalization computation is complex. For the limitations of identifying characteristics of the dynamic system based on the methods above, Dai and Singh proposed a periodicity ratio method to distinguish the dynamic characteristics of one-dimensional nonlinear dynamic system [11–14]. However, the identification method of the dynamic characteristics for high-dimensional nonlinear system is reported in the recent years. Mahmoud introduced a new theorem used to construct approximate analytical solutions

for n-dimensional strongly nonlinear dynamical system, and then passive control method is also used to control n-dimensional chaotic complex nonlinear systems [15–17].

In this paper, considering the principle of Poincaré mapping, the periodicity ratio methods for diagnosing the dynamic characteristics of the high-dimensional nonlinear systems are proposed. This method is employed to determine the dynamics and periodicity of the hull and Rössler system, respectively. The research findings will develop the evaluation method of dynamic characteristics for the high-dimensional nonlinear dynamic system.

2. The Periodicity Ratio of Nonautonomous Systems

Consider the following second order of the n-dimensional nonautonomous system,

$$\ddot{\mathbf{x}} = \mathbf{f}(\dot{\mathbf{x}}, \mathbf{x}, t), \quad \mathbf{x} \in \mathbf{R}^{\mathbf{n}}. \tag{1}$$

Suppose that the system is subjected to an external excitation with period time T, meanwhile, $\mathbf{x}$ is the periodic solution of the differential system above, i.e., $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$. Usually, $\mathbf{x}$ is the solution of the harmonic vibration related to multiple period of T satisfying the following relationship:

$$\mathbf{x}(t_0) = \mathbf{x}(t_0 + jT), \tag{2}$$

where t_0 is the reference time and j is the number of period points of the system in the Poincaré section. For a completely periodic nonautonomous dynamic system, no matter how long the vibration is sustained, only j finite points are appeared in Poincaré section $\mathbf{x} - \dot{\mathbf{x}}(x_r - \dot{x}_r, (r = 1, 2, \ldots, n))$. If the phase points are infinite in Poincaré section $\mathbf{x} - \dot{\mathbf{x}}$, the n-dimensional nonautonomous system is aperiodic.

According to equation (2), the overlapping points in n Poincaré sections can describe the periodicity of n-dimensional nonautonomous dynamic system. The number of overlapping points in the rth Poincaré section can be determined by

$$X_{r,ki} = x_r(t_0 + kT) - x_r(t_0 + iT), \tag{3a}$$

$$\dot{X}_{r,ki} = \dot{x}_r(t_0 + kT) - \dot{x}_r(t_0 + iT), \tag{3b}$$

where k and i are integers; $k \in [1, j]$, $i \in [1, m]$. Equation (3a) represents the displacement difference between phase point i and k in the Poincaré section when the phase trajectories pass through the rth section. Equation (3b) describes the velocity difference between phase point i and k in the Poincaré section when the phase trajectories pass through the rth section. m is the sum of the phase points in the Poincaré section, including the overlapping and nonoverlapping points. Therefore, the total number of phase points in the Poincaré section can be denoted as

$$S_{\mathrm{a}} = n \cdot m. \tag{4}$$

According to the above definition, the following conclusions are stressed:

(1) In the Poincaré section, the so-called overlapping phase points $i(x_i, \dot{x}_i)$ and $j(x_j, \dot{x}_j)$ represent $x_i = x_j$ and $\dot{x}_i = \dot{x}_j$.

(2) According to the overlapping property of the phase points, the phase points of the nonlinear periodic system in the rth Poincaré section should satisfy the following conditions:

$$X_{r,ki} = 0, \tag{5a}$$

$$\dot{X}_{r,ki} = 0. \tag{5b}$$

(3) If the phase point in arbitrary Poincaré sections cannot satisfy equation (5), the phase points is nonoverlapping.

Applying equations (5a) and (5b), the total number $\zeta(k)$ of the kth overlapping phase piont in the rth Poincaré section can be expressed as follows:

$$\zeta_r(k) = \left\{ \sum_{i=k}^{n} Q\left(X_{r,ki}\right) Q\left(\dot{X}_{rki}\right) \right\} P\left(\sum_{i=k}^{n} \left[Q\left(X_{r,ki}\right) Q\left(\dot{X}_{r,ki}\right) \right] - 1 \right), \tag{6}$$

where $\zeta_r(k)$ is applied to calculate the number of all phase points overlapping to the kth phase point. Q and P are step functions as follows:

$$Q(y) = \begin{cases} 1, & \text{if } y = 0, \\ 0, & \text{if } y \neq 0, \end{cases} \qquad p(z) = \begin{cases} 1, & \text{if } z = 0, \\ 0, & \text{if } z \neq 0. \end{cases} \tag{7}$$

Considering equation (6), after the total number k of the overlapping phase points is determined, the number of the jth visible point corresponding to overlapping point can be calculated. Assign N_r as overlapping points in the rth Poincaré section; thus N_r can be expressed by

$$N = N_r(1) + \sum_{k=2}^{n} \zeta_r(k) P\left(\prod_{l=1}^{k-1} \left(X_{r,kl} + \dot{X}_{r,kl} \right) \right), \tag{8}$$

where $\prod$ is the symbol for multiplication and $P(\cdot)$ is the step function as defined previously. This equation ensures that the duplicate includes in the calculations for N_r or missing in any overlapping point is prevented. If the response of a dynamic system is completely periodic, all the points in the Poincaré map must be overlapping, and the corresponding N_r can be simply expressed in the following form:

$$N_r = \sum_{k=1}^{j} \zeta_r(k). \tag{9}$$

For periodicity of the nonlinear dynamics system, N_r represents the overlapping phases points in the rth Poincaré section. Therefore, the total number of overlapping points and phase points in n-dimensional space can be independently denoted by

$$S = \sum_{r=1}^{n} N_r, \tag{10a}$$

$$S_a = \sum_{r=1}^{n} S_r, \tag{10b}$$

where S_r is the number of phase points in the rth Poincaré section. Therefore, the periodicity ratio of the nonlinear nonautonomous system with n-dimensional space can be denoted by

$$\gamma = \lim_{S_a \longrightarrow \infty} \frac{S}{S_a}. \tag{11}$$

It can be known that the number of the overlapping points is less than or equal to the all phase points, i.e., $0 \le S \le S_a$, in this case, $0 \le \gamma \le 1$. If the dynamic responses of the nonlinear system are periodic, all the phase points in the Poincaré section must be overlapping, and periodicity ratio γ is equal to 1. If the dynamic responses of the nonlinear system are chaotic, all the phase points in the Poincaré section must be nonoverlapping, and periodicity ratio γ is equal to zero. Through the definition for the periodicity ratio γ of the nonlinear dynamic system, it is easy to find that periodicity ratio γ can describe the periodicity of the nonlinear dynamic system.

3. The Periodicity Ratio of Autonomous Systems

Consider the following second order of the n-dimensional autonomous system:

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad \mathbf{x} \in \mathbf{R}^{\mathbf{n}}. \tag{12}$$

As the system is an autonomous system without external excitation, the phase points in the Poincaré section cannot be determined by equation (2). For the n-dimensional nonlinear autonomous system, no matter how long the vibration of the system is sustained, only j finite phase points are appeared in Poincaré section $x_{r-1} - x_r\,(r = 2, 3, \ldots, n)$. Therefore, the periodicity of the nonlinear system can be described by $n-1$ Poincaré sections. If there are infinite phase points in the $n-1$ Poincaré sections, the nonlinear dynamic system is nonperiodic.

If the vibration behavior of the autonomous system is periodic, period T_1 of x_1 can be estimated with the maximum method. As numerical solutions of x_1 are a series of points related to time, i.e., $x_{1(\tau)} = x_1(t_\tau), \tau = 1, 2, 3, \ldots, \infty$. Thus, the search procedure can be employed to determine the maximum value of x_1. If $x_{1(\tau)}$ satisfy the following condition:

$$x_{1(\tau-1)} > x_{1(\tau)}, x_{1(\tau)} > x_{1(\tau+1)}, \ldots, x_{1(\tau+1)} > \frac{1}{2}\left(x_{1(\tau)} + x_{1(\tau+2)}\right), \tag{13}$$

$x_{1(\tau)}$ is the maximum point of x_1, and the point of x_2 corresponding to $x_{1(\tau)}$ in this time is $x_{2(\tau)}$. Thus, the phase point of the first Poincaré section is expressed by $(x_{1(\tau)}, x_{2(\tau)})$. The amount of the overlapping phase points in the first Poincaré section is written by

$$N_1(k) = \zeta(1) + \sum_{k=2}^{n} \zeta_1(k) P\left(\prod_{l=1}^{k-1} x_{1(\tau),kl}\right), \tag{14}$$

where $x_{1(\tau),kl} = |x_{1(\tau),k} - x_{1(\tau),l}|$. Therefore, the period ratio can be determined through the phase point in the first Poincaré section:

$$\gamma_1 = \lim_{S_a \longrightarrow \infty} \frac{N_1}{S_1}. \tag{15}$$

If $\gamma_1 = 1$, the period T of the system can be confirmed as the following:

(1) Assume that the number of the visible phase points in the first Poincaré section is j. Choosing the kth visible point of j, the number of the overlapping points is q, and so the span of average time between two arbitrary adjacent points can be expressed as

$$\eta = \frac{t_{k,q} - t_{k,1}}{q-1}, \tag{16}$$

where $t_{k,q}$ represents the time of the qth overlapping points in the kth visible points and $t_{k,1}$ represents the time of the qth overlapping points in the first visible points. If all the overlapping points are periodic, parameter η is equivalent to the vibration period of the dynamic system.

(2) To improve the accuracy and reliability for identifying the periodicity of the dynamic system, identification parameter ρ is defined as

$$\rho^2 = \frac{\sum_1^{q-1}\left(t_{k,i+1} - t_{k,i} - \eta\right)}{q-1}, \quad i = 1, 2, 3, \ldots, q. \tag{17}$$

From the equation above, it follows that when the identifying parameter $\rho = 0$, the vibration behavior of the dynamic system is periodic. In this case, the period T_1 of the system can be expressed by

$$T_1 = t_{k,i+1} - t_{k,i}. \tag{18}$$

Therefore, the number of overlapping points in the rth Poincaré section can be determined by

$$X_{r,ki} = x_r(t_0 + kT_1) - x_r(t_0 + iT_1), \tag{19a}$$

$$X_{r+1,ki} = x_{r+1}(t_0 + kT_1) - x_{r+1}(t_0 + iT_1), \tag{19b}$$

where k and i are integers, $k \in [1, j]$, $i \in [1, m]$. Equation (19) represents the displacement difference between phase points i and k. m is the number of the phase points in an arbitrary Poincaré sections. In this case, the total number of the phase points for the n-dimensional system can be denoted by

$$S_a = (n-1) \cdot m. \tag{20}$$

According to the definition above, the following conclusions can be stressed:

(1) The so-called overlapping phase points represent $x_i = x_j$ and $x_{i+1} = x_{j+1}$.
(2) According to the characteristics of the overlapping points, the phase points of the periodic system satisfy the following conditions:

$$X_{r,ki} = 0, \tag{21a}$$

$$X_{r+1,ki} = 0. \tag{21b}$$

(3) If equation (19) is not satisfied, these phase points cannot be overlapping.

Employing equations (21a) and (21b), the amount $\zeta(k)$ of the phase points overlapping with the kth phase point can be determined with

$$\zeta_r(k) = \left\{ \sum_{i=k}^{n} Q\left(X_{r,ki}\right) Q\left(X_{r+1,ki}\right) \right\} P \cdot \left(\sum_{i=k}^{n} \left[Q\left(X_{r,ki}\right) Q\left(X_{r+1,ki}\right) \right] - 1 \right). \tag{22}$$

According to equation. (22), the overlapping points in the rth Poincaré sections can be defined by

$$N_r = N_r(1) + \sum_{k=2}^{n} \zeta_r(k) P\left(\prod_{l=1}^{k-1} \left(X_{r,kl} + X_{r+1,kl} \right) \right), \tag{23}$$

where $\prod$ is multiplication and $P(\cdot)$ is step function. If the responses of the nonlinear dynamic system are periodic, N_r can be simply expressed by

$$N_r = \sum_{k=1}^{j} \zeta_r(k). \tag{24}$$

Therefore, the amount of the overlapping point and the all points in the Poincaré section can be represented by

$$S = \sum_{r=1}^{n-1} N_r, \tag{25a}$$

$$S_a = \sum_{r=1}^{n-1} S_r, \tag{25b}$$

where S_r represents the amount of the phase points in the rth Poincaré section. Therefore, the periodicity ratio of the n-dimensional nonlinear autonomous system is written by

$$\gamma = \lim_{S_a \longrightarrow \infty} \frac{S}{S_a}. \tag{26}$$

4. Numerical Analysis

4.1. Periodicity of the Nonautonomous System. In the second section, the computation method of periodicity ratio for the nonlinear autonomous system is described. Here, the 4th order Runge–Kutta method is employed to determine the periodicity ratio of the hull system, as shown in equation (27). This model is applied to describe the nonlinear coupling characteristics of the pitching and rolling of the hull [1].

$$\begin{cases} \ddot{x}_1 + 2\mu_1 \dot{x}_1 + \omega_1^2 x_1 = \alpha_1 x_1 x_2 + F(\cos \omega t), \\ \ddot{x}_2 + 2\mu_2 \dot{x}_2 + \omega_2^2 x_2 = \alpha_2 x_1^2 + F(\cos \omega t). \end{cases} \tag{27}$$

Consider F and ω as control parameters, and the initial value of the system is assumed to be $x_1(0) = 0.1$, $\dot{x}_1(0) = 0.2$, $x_2(0) = 0.3$, and $\dot{x}_2(0) = 0.4$. The other parameters of the system are defined by $\mu_1 = 0.1$, $\mu_2 = 0.1$, $\alpha_1 = 0.5$, $\omega_1 = 5.5$, and $\omega_2 = 5.5$. Figure 1 shows the periodicity ratio when the parameters of the system satisfy $\alpha_2 = 1.0$, $\alpha_2 = 1.5$, $\alpha_2 = 2.0$, and $\alpha_2 = 2.5$. The red region in this figure represents that the dynamic characteristics of the system are periodic, i.e., $\gamma = 1$; the blue region denotes that the dynamic characteristics of the system are chaotic, i.e., $\gamma = 0$; and the other color region signifies that the dynamic characteristics of the system are neither periodic nor chaotic, i.e., $0 < \gamma < 1$. The figure reveals the dynamic behavior of the system with the change of system parameters. As shown in this figure, when ω is located in region of $(0, 2]$, the vibration behavior of the system is transferred from chaos to periodicity with the increase of the external excitations; when ω is located in region of $(5.2, 5.8]$, the probability of nonperiodic vibration of the system is increased with the increase of coupling coefficient α_2; when ω is located in the other region, the vibration behavior of the system is periodic. It can be seen that the system is super near resonance, near resonance, sharp resonance, and far resonance when $\omega \in (0, 2]$, $\omega \in (2, 5.2]$, $\omega \in (5.2, 5.8]$, and $\omega \in (5.8, 10]$, respectively. It can be concluded that the dynamic characteristics are chaotic when the system is sharp resonance or super near resonance.

As shown in Figure 1(d), the system parameters are located in the blue region when $\omega = 0.2$, $\alpha_2 = 2.5$, and $F = 0.9$, and the vibration behavior is chaotic because of $\gamma = 0$; the system parameters are located in the red region when $\omega = 8$, $\alpha_2 = 2.5$, and $F = 20$, and the vibration behavior is chaotic because of $\gamma = 1$. The vibration characteristics of the hull is shown in Figure 2 when $\omega = 0.2$, $\alpha_2 = 2.5$, and $F = 0.9$. Figures 2(a) and 2(c) follow that the trajectory of the system is periodic in x_1 direction; and the trajectory is chaotic in x_2 direction as shown in Figures 2(b) and 2(d). Therefore, the vibration characteristics of the system are chaotic in the condition of $\omega = 0.2$, $\alpha_2 = 2.5$, and $F = 0.9$. As a result, the vibration characteristics of the system are chaotic in the condition of $\omega = 0.2$, $\alpha_2 = 2.5$, and $F = 0.9$. Figure 3 shows the vibration characteristics of the hull system considering $\omega = 8$, $\alpha_2 = 2.5$, and $F = 20$. It can be seen that the trajectory of the system is periodic in x_1 and x_2 directions. And so the vibration characteristics of the system are periodic in the condition of $\omega = 8$, $\alpha_2 = 2.5$, and $F = 0.9$. According to the analysis above, if the vibration characteristics in the all dimensionality are periodic, the system is periodic or deterministic. Conversely, it is an uncertain system.

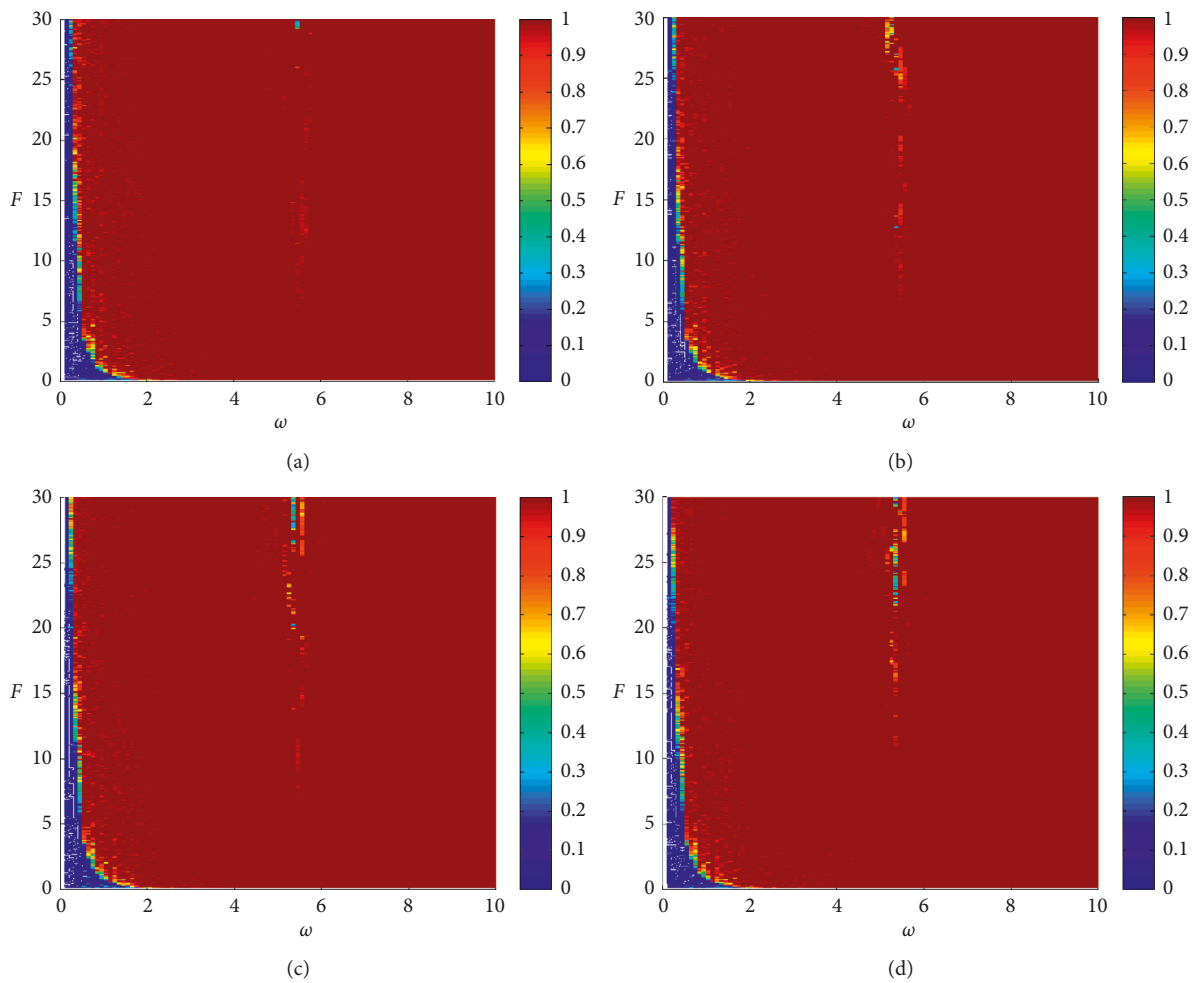

Figure 1: Periodicity of hull dynamic system. (a) $\alpha_2 = 1.0$. (b) $\alpha_2 = 1.5$. (c) $\alpha_2 = 2.0$. (d) $\alpha_2 = 2.5$.

4.2. Periodicity of the Autonomous System. Consider the famous Rössler system, shown in the following equation:

$$\begin{cases} \dot{x}_1 = -x_2 - x_3, \\ \dot{x}_2 = x_1 + ax_2, \\ \dot{x}_3 = b + (x_1 - c)x_3. \end{cases} \tag{28}$$

Assuming b and c to be control parameters, the periodicity of the Rössler system can be determined with the method in Section 4. Considering the initial value of the system as $x_1(0) = 0.1$, $x_2(0) = 0.1$, and $x_3(0) = 0.3$, Figure 4 shows the periodicity of the Rössler system with different values of a. The red region in this figure represents that the dynamic characteristics of the system is periodic, i.e., $\gamma = 1$; the blue region denotes that the dynamic characteristics of the system are chaotic, i.e., $\gamma = 0$; the other color region signifies that the dynamic characteristics of the system are neither periodic nor chaotic, i.e., $0 < \gamma < 1$. The figure can intuitively determine the process of the dynamic behavior changed with parameters of the system. Because of the nonlinear characteristics of Rössler system, the dynamic characteristics of the system are sensitive to the system parameters.

As shown in Figure 4(d), the parameters are located in the red region when $a = 0.20$, $b = 0.20$, and $c = 3.50$, and in this case, the vibration characteristics of the system are periodic; when $a = 0.20$, $b = 0.20$, and $c = 5.50$, the parameters are located in the blue region, and in this case, the vibration characteristics of the system are chaotic. Figure 5 shows the periodic vibration of the Rössler system. It can be found that the phase trajectory is periodic when $a = 0.20$, $b = 0.20$, and $c = 3.50$, as shown in Figure 5(e), and the number of the visible points in the Poincaré sections of x_1-x_2 and x_2-x_3 is two, respectively. However, the phase trajectory is chaotic when $a = 0.20$, $b = 0.20$, and $c = 5.50$; therefore, the number of the visible points in the sections of x_1-x_2 and x_2-x_3 is infinite, as shown in Figure 6.

5. Conclusions

Through the discussions of theoretical research and numerical analysis, it can be found that periodicity ratio is an effective tool to identify the dynamic behavior of high-dimensional nonlinear systems. The conclusions of the study on the periodicity ratio are as follows:

(1) If the dynamic responses of the nonlinear system are periodic, the phase points in the Poincaré sections are overlapping. In this case, the value of the periodicity ratio is equal to 1.

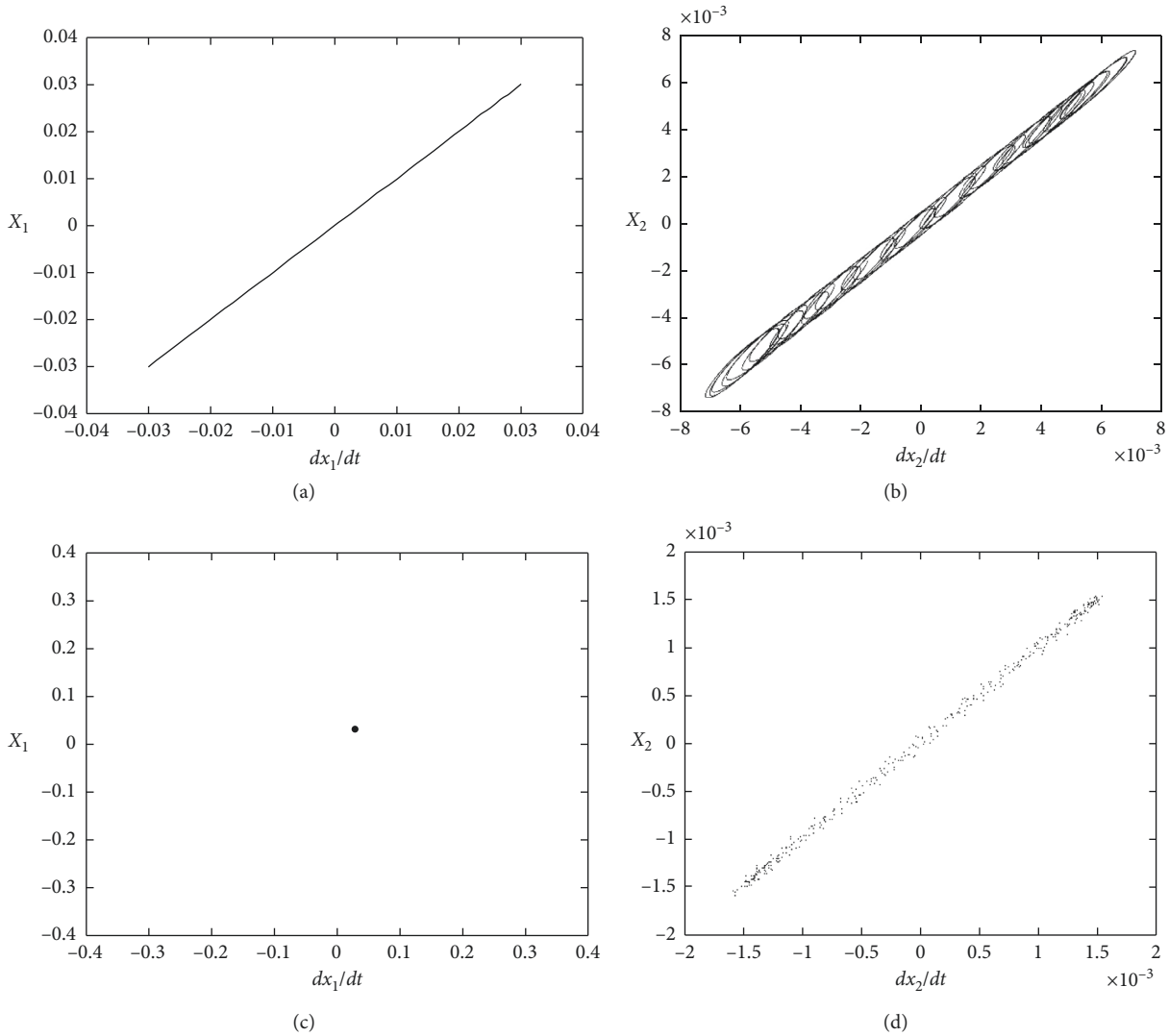

Figure 2: Vibration characteristics of hull dynamic system for $\omega = 0.2$, $\alpha_2 = 2.5$, and $F = 0.9$. (a) Phase diagram in x_1 direction. (b) Phase diagram in x_2 direction. (c) Poincaré map in x_1 direction. (d) Poincaré map in x_2 direction.

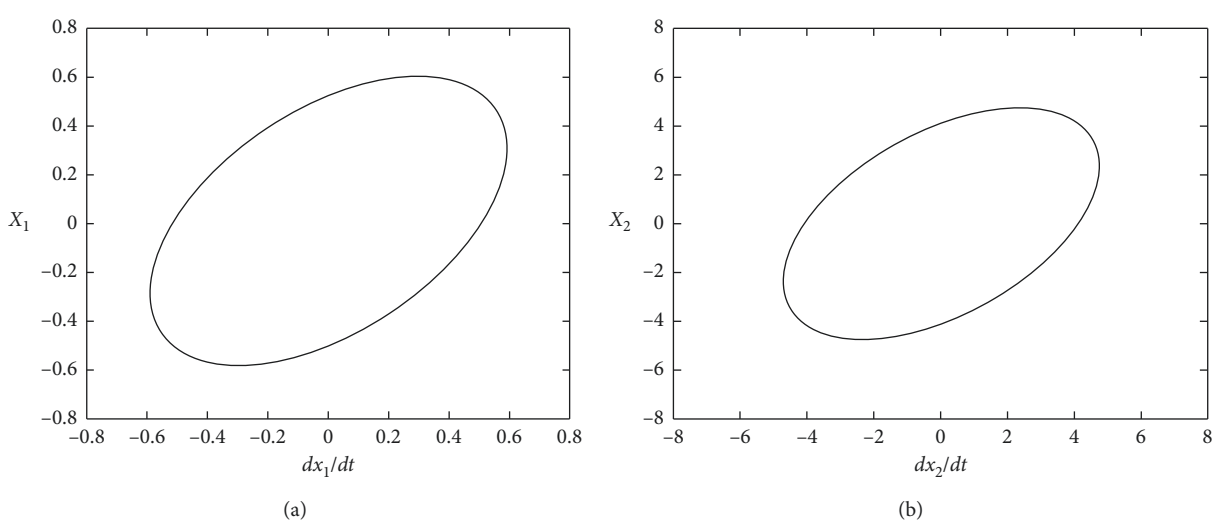

Figure 3: Continued.

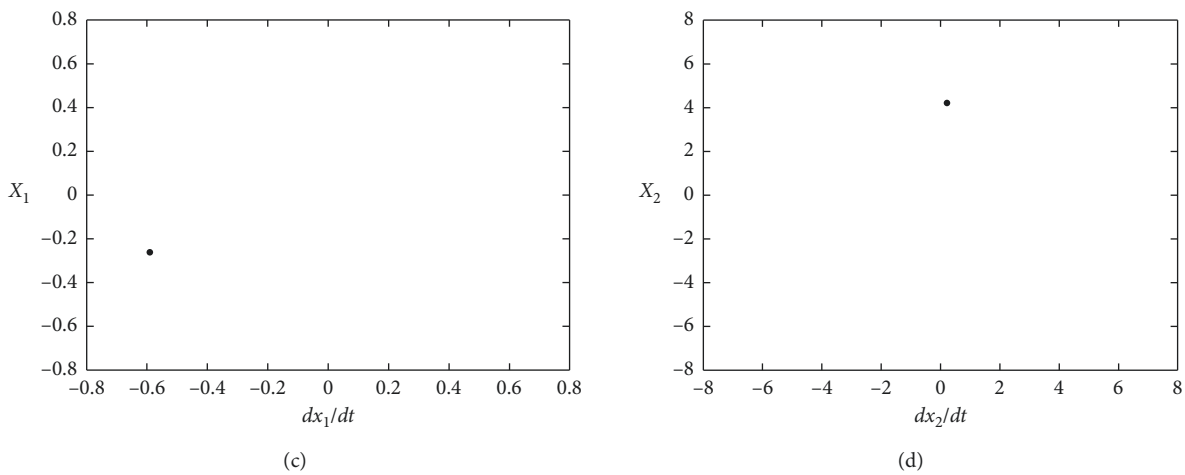

Figure 3: Vibration characteristics of hull dynamic system for $\omega = 8$, $\alpha_2 = 2.5$, and $F = 20$. (a) Phase diagram in x_1 direction. (b) Phase diagram in x_2 direction. (c) Poincaré map in x_1 direction. (d) Poincaré map in x_2 direction.

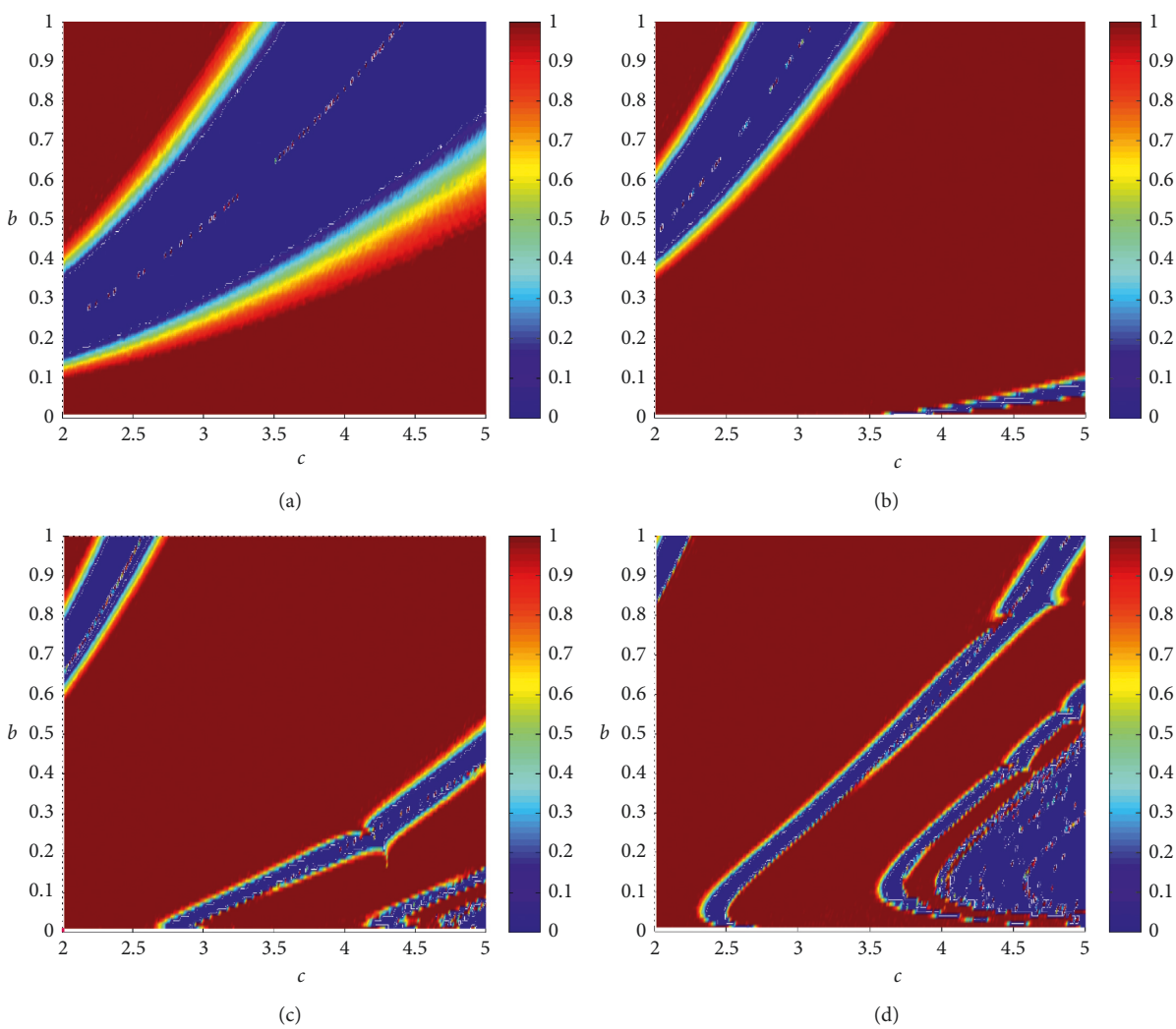

Figure 4: Periodicity of Rössler dynamic system. (a) $a = 0.05$. (b) $a = 0.10$. (c) $a = 0.15$. (d) $a = 0.20$.

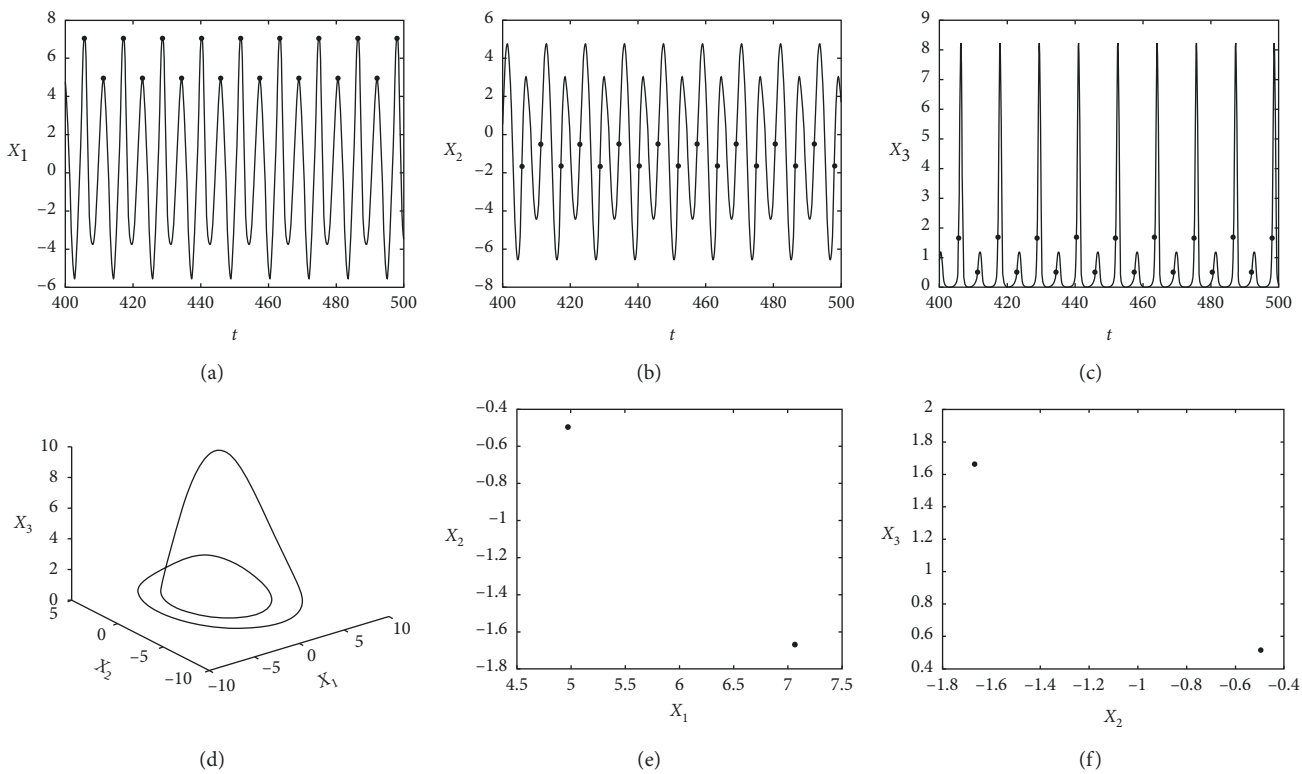

Figure 5: Vibration characteristics of Rössler dynamic system for $a = 0.20$, $b = 0.20$, and $c = 3.50$.

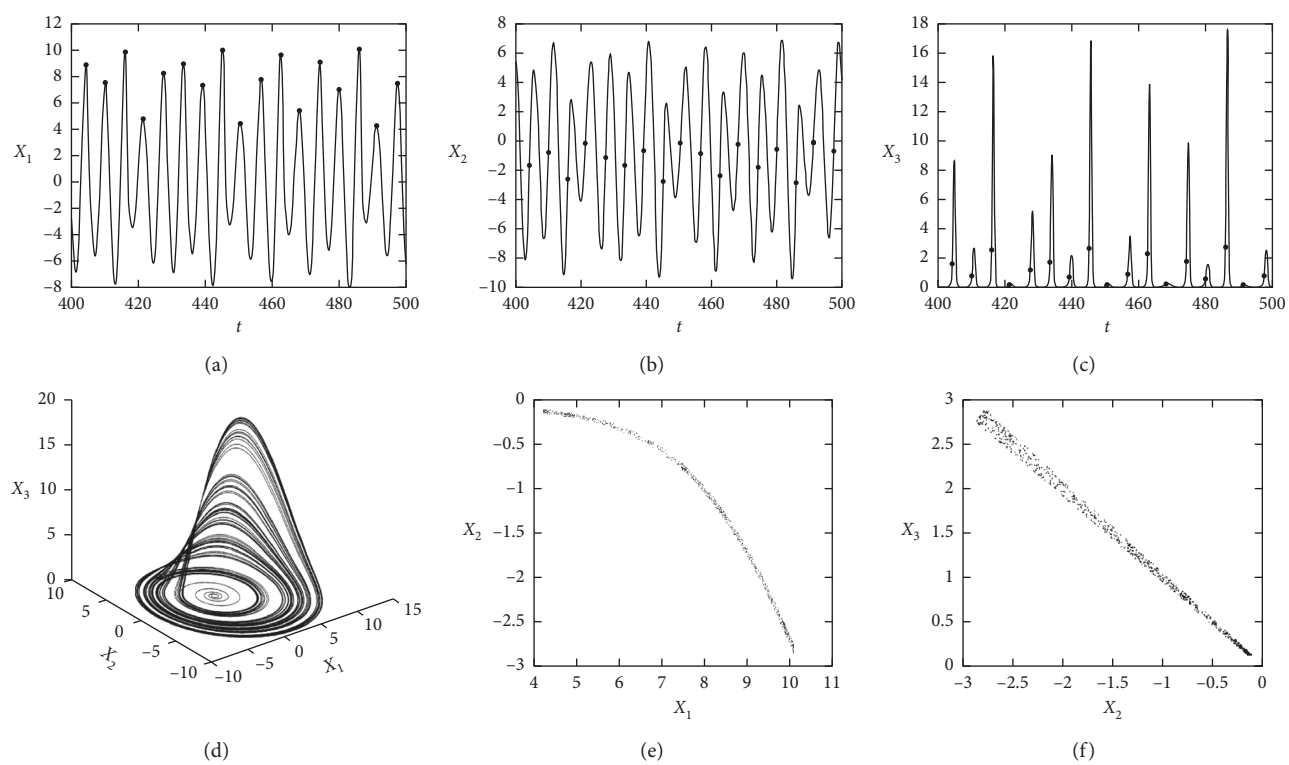

Figure 6: Vibration characteristics of Rössler dynamic system for $a = 0.20$, $b = 0.20$, and $c = 5.50$.

(2) If the dynamic responses of the nonlinear system are chaotic, the phase points in the Poincaré sections are nonoverlapping. In this case, the value of the periodicity ratio is equal to 0.

(3) For a nonlinear dynamic system, there may exist an infinite number of nonperiodic solutions, which are neither periodic nor chaotic. For these nonperiodic solutions, the corresponding periodicity ratio values are in the range of $0 < \gamma < 1$. The larger value of the periodicity ratio represents the dynamic characteristics close to periodic motion, and the smaller value of the periodicity ratio represents the dynamic characteristics close to chaotic motion.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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