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Research Article

An Improved Principle of Rapid Oscillation Suppression of a Pendulum by a Controllable Moving Mass: Theory and Simulation

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An improved principle is proposed in this paper to achieve a more efficient amplitude suppression of an oscillating pendulum by a controllable moving mass. By establishing the governing equation of the pendulum with a moving mass, several suppression rules are presented to stipulate the motion parameters of the moving mass. The damping ratio is introduced to quantify the suppression effects generated by the mass intermittent motion. With the verification of the accuracy of the present model, some simulation has been taken to get the maximum damping ratio and solve the problem of the nonsynchronous motion. The simulation results show that the new principle is a more powerful control method for the oscillation problem of a pendulum. It is suggested that the new method should carry out the relevant active control experiment in the future research.

1. Introduction

The pendulum is a classical model in mechanics. There are many pendulum-like structures that can oscillate with a fixed point in the field of construction and aviation, such as the oscillation of the crane payload and the space tether-net system. However, the oscillation of these structures often has a negative influence on them. The unexpected oscillation can reduce the working efficiency and may cause serious accidents. Therefore, the study of the rapid oscillation suppression of the pendulum is of great significance in terms of improving working efficiency, reducing workload, and protecting safety.

In recent decades, some scholars have carried out the theoretical analysis research on the dynamic analysis and amplitude control of the pendulum with a moving mass. Aslanov [1] obtained the analytical solution of the averaged equation for the amplitude of the pendulum with a moving mass based on the averaging method and proved the asymptotic stability of the equilibrium position of the pendulum with a relative motion of the moving mass. Szyszkowski and Stilling [2] proposed a continuous motion of the moving mass to attenuate the oscillating pendulum. The attenuation effects of the designed motion of the moving mass were analyzed by the damping ratio. The results showed that the pendulum could be attenuated by a continuous mass motion. Szyszkowski and Sharbati [3] employed the finite element method to analyze the effects of the moving mass on the oscillation of the pendulum. Sharbati and Szyszkowski [4] proved that the FEM approach was capable of analyzing the vibration of the system for any predefined pattern of the mass motion. Yoshida et al. [5] designed a control law of the moving mass by the Lyapunov theorem to stabilize the pendulum in the vertical position. The motion of the moving mass could be controlled according to the pendulum amplitude and its velocity. Sogo et al. [6] proposed a control law to control the oscillating pendulum based on the energy method by considering the external disturbance of the system. The control experiment was used to verify the effectiveness of the proposed control law. Xin and Liu [7] designed a trajectory tracking control strategy to control the pendulum to rotate in a desired trajectory by sliding a point mass along the pendulum. The effectiveness of the control law was also discussed when the system had linear damping. Gutiérrez-Frias et al. [8] proposed a proportional derivative controller plus gravity compensation based on the Lyapunov method together with the LaSalle's theorem to suppress the oscillation of the pendulum by controlling the moving mass.

The related study results of the variable length pendulum can also provide a useful research basis for the analysis of this paper. Therefore, some researches on the variable length pendulum are presented as an extension of the investigation. The variable length pendulum can be regarded as a parametric vibration model. Many studies on the variable length pendulum focus on the periodic solutions. Pinsky and Zevin [9] considered a pendulum with a periodically varying length. The existence and stability of periodic solutions were proved when the maximum amplitude was less than π. Zevin and Filonenko [10] studied the stability and instability conditions of the periodic solution of a variable length pendulum. The parametric and self-excited oscillator model was discussed by the obtained conditions. Belyakov et al. [11] analyzed the regular motion and the chaos phenomenon of a pendulum with a periodically varying length by the averaging method and Lyapunov's theorem. Yang et al. [12] studied the approximate solution of the variable length pendulum by the homotopy analysis method and proved that the homotopy method could supply a more accurate result than the averaging method for predicting the behavior of the pendulum in the long term. Markeev and Krasil'nikov [13, 14] investigated the stability of the periodic solution of some motion states of the variable length pendulum such as the uniform rotation of the pendulum and the regular oscillation of the pivot. Kumar and Parul [15] studied the periodic solutions of the pendulum with a small periodic change in length by the multiple scale method, and the accuracy of the conclusions was proved by numerical simulation.

According to the research on the variable length pendulum mentioned above, the pendulum behavior depends on the change of the pendulum length. Based on this characteristic, some experts oscillate the pendulum to a desired motion by appropriately changing the pendulum length. Savi et al. [16] established an experimental device of the variable length pendulum and studied the chaos phenomenon of the pendulum which was generated by controlling the length. It was proved that the chaos in mechanical systems can be controlled. Reguera et al. [17] presented a control law for the variable length pendulum to keep the pendulum in a stable rotation when the pivot was in sinusoidal and stochastic excitations.

Similarly, if the desired motion is set at the vertical position of the pendulum, the oscillation suppression of the pendulum can be achieved. Some experts have employed the method of changing length to stabilize the pendulum oscillation. Yoshida et al. [18] proposed a control law based on the energy method to suppress the oscillation of the pendulum by simultaneously controlling the motion of the pivot and the pendulum length. Stilling and Szyszkowski [19] explained the effects of the Coriolis force and the work-energy transformation of the system when the pendulum length was changed. A continuous and sinusoidal change of the pendulum length was used to attenuate the pendulum oscillation.

As can be seen from the previous references, the characteristics of the pendulum with a moving mass or a variable length have been applied to control the pendulum amplitude. However, few studies focus on the investigation of the most efficient suppression of the pendulum. In consideration of some practical applications, the pendulum length may not be able to change. Thus, a controllable moving mass is employed to suppress the oscillating pendulum in this paper. Comparing with the analysis in previous studies, the present work aims to study how to use an intermittent motion of the moving mass to achieve the oscillation suppression of a pendulum more rapidly. The analysis of the damping ratio and the continuous behavior of the pendulum verify the high suppression efficiency of the intermittent mass motion.

This paper is organized as follows: Section 2 presents the governing equation of the pendulum with a moving mass and a new mass motion based on suppression rules. The damping ratio is also introduced in this section. Section 3 verifies the accuracy of the present model in this paper. Section 4 shows some simulation results of the damping ratio and the pendulum behavior. Section 5 makes some conclusions.

2. Governing Equation and Improved Suppression Principle

2.1. Governing Equation. The pendulum with a moving mass is shown in Figure 1. The system consists of a homogeneous frictionless pendulum with the pivot O and a moving mass that can slide along the pendulum. It is assumed that the motion of the moving mass is driven by a motor at the pivot O. The moving mass is connected with the motor by a rope. The tension F of the rope acting on the mass can be measured by the tension pickup. The speed of the motor can be controlled to change the tension F, and then, the motion of the mass $l(t)$ can also be controlled. Therefore, F can be considered as an external controlling force acting on the mass.

As shown in Figure 1(a), L and M are the length and the mass of the pendulum, the centroid of the pendulum is in the median position; l_0 and m are the initial position and the mass of the moving mass which can slide between two predefined positions $l_{\min}$ and $l_{\max}$. In Figure 1(b), θ is the amplitude between the pendulum and the vertical position; the symbol N represents the normal component of the force acting on the mass; $ml\dot{\theta}^2$ and $F_c = 2m\dot{l}\dot{\theta}$ represent the inertial centrifugal force and the Coriolis inertia force, respectively. The combined radial action of the external controlling force F, $ml\dot{\theta}^2$, and $mg\cos\theta$ generates the acceleration $\ddot{l}$ of the moving mass and thus controls the mass motion $l(t)$.

The kinetic energy K and the potential energy P of the system are

$$
\begin{aligned}
K &= \frac{1}{6}ML^2\dot{\theta}^2 + \frac{1}{2}m\left(\dot{l}^2 + l^2\dot{\theta}^2\right),\\
P &= -\frac{1}{2}MgL\cos\theta - mgl\cos\theta.
\end{aligned}
\tag{1}
$$

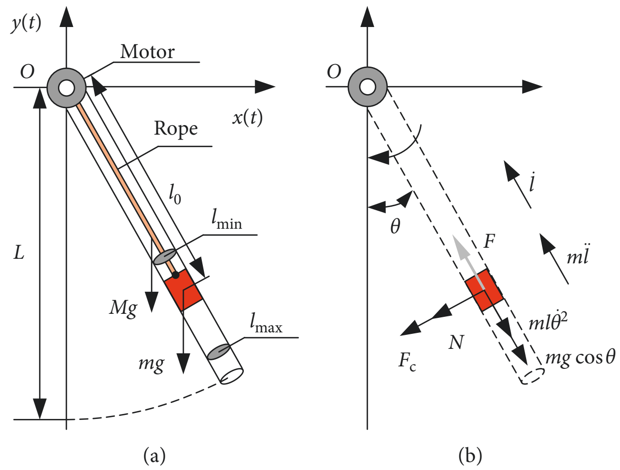


Figure 1: The pendulum with a moving mass. (a) Coordinate system; (b) forces involved.

Using the Euler–Lagrange equation, the governing equation of the pendulum with a moving mass can be obtained as follows:

$$\left(\frac{ML^2}{3} + ml^2\right)\ddot{\theta} + 2ml\dot{l}\dot{\theta} + g\left(M\frac{L}{2} + ml\right)\sin\,\theta = 0. \quad (2)$$

In equation (2), symbol $\dot{l}$ represents the motion velocity of the moving mass. The external controlling force F always acts on the radial direction; therefore, F will not be involved in the rotation equation of the pendulum equation (2). The first and third terms represent the inertia and stiffness forces, respectively. It is worth noting that the second term of equation (2) can be regarded as the work done by the Coriolis force, which can generate the amplification or attenuation effects on the amplitude. As can be seen from equation (2), the amplitude θ is determined by the motion of the mass $l(t)$. Therefore, if the parameter $l(t)$ is designedly controlled, the oscillation of the pendulum will be suppressed purposefully and gradually from an initial amplitude.

2.2. Suppression Rules. It is confirmed in Reference [19] that the mass upward process increases the amplitude of the pendulum, and the mass downward process decreases the amplitude. What is more, the pendulum oscillation will be attenuated if the mass moves upwards when θ is close to maximum and moves downwards when θ is close to minimum. Figure 2 shows this basic motion of the moving mass that can attenuate the pendulum oscillation. This section will present some new rules that can suppress the pendulum oscillation more rapidly.

To simplify the analysis in this paper, we stipulate that the motion distance of the mass upward or downward process is the same as $\Delta L = l_{max} - l_{min}$, and the corresponding elapsed time is the same as ΔT. In order to achieve better suppression effects generated by the motion of the moving mass, several new suppression rules are followed.

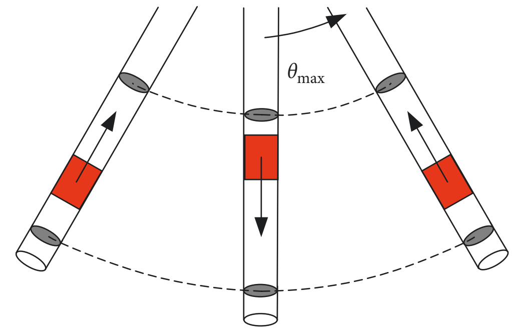


Figure 2: The basic motion of the moving mass that can attenuate the pendulum oscillation.

2.2.1. Rule 1. The elapsed time of the mass upward or downward process ΔT is required to be as small as possible to ensure that the mass motion process is close to the maximum or minimum of the amplitude to generate the best effects on oscillation suppression. This conclusion will be verified in Section 2.4.

2.2.2. Rule 2. The mass initial position l_0 is required to be as close to l_{min} as possible. The proof of this rule is as follows.

As mentioned above, the mass upward process from the initial position l_0 to l_{min} ($l_{min} < l_0 < l_{max}$) will increase the pendulum amplitude θ. According to equation (3), the total energy E of the system will also increase, where k in equation (3) represents the stiffness coefficient of the system. It indicates that, although the initial energy of the system is lower when the mass initial position is at l_0, the total energy of the system increased by the mass upward process is always higher than the initial energy difference between the mass initial position at l_0 and l_{min}. Therefore, Rule 2 is proposed to stipulate the mass initial position:

$$E = \frac{1}{2}k\theta_{max}^2. \quad (3)$$

2.2.3. Rule 3. The proportion of the mass decelerated process to the mass upward and downward process is required to be as large as possible. The proof of Rule 3 is as follows.

The mass will be accelerated from $\dot{l} = 0$ to a certain velocity and then be decelerated to $\dot{l} = 0$ in the actual upward and downward process. The radial forces acting on the mass in the mass upward and downward process are shown in Figure 3.

In Figure 3(a), the trajectory from l_1 to l_2 shows the upward accelerated process of the moving mass, and the symbols F_1 and $\ddot{l}_1$ are the external controlling force and the acceleration of the moving mass in this process, respectively. Here, we define Δl_1 and Δt_1 as the motion distance and the elapsed time of the mass upward accelerated process. The trajectory from l_2 to l_3 represents the upward decelerated process, and the parameters F_2 and $\ddot{l}_2$ are the external controlling force and the deceleration of the moving mass.

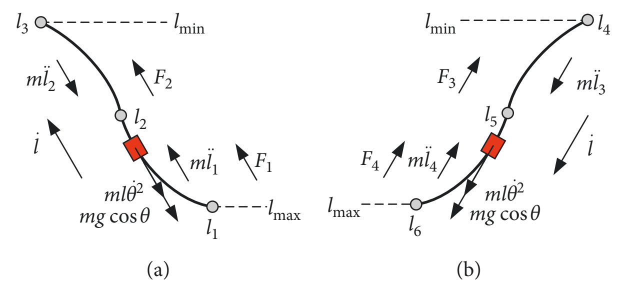


Figure 3: Radial forces acting on the mass. (a) The mass upward process; (b) the mass downward process.

The symbols Δl_2 and Δt_2 are defined as the upward decelerated motion distance and the corresponding time. Thus, we can obtain $\Delta L = \Delta l_1 + \Delta l_2$ and $\Delta T = \Delta t_1 + \Delta t_2$, where ΔL and ΔT are the total motion distance and the total elapsed time of the mass upward process.

The work done by the external controlling force F determines the input energy and the output energy of the system. Assume that $\ddot{l}_1$ and $\ddot{l}_2$ are constant. The external work done by F_1 and F_2 can be expressed as

$$\begin{aligned} W_1 &= F_1 \Delta l_1, \\ W_2 &= F_2 \Delta l_2, \end{aligned} \tag{4}$$

where $F_1 = mg\cos\theta + ml\dot{\theta}^2 + m\ddot{l}_1$, $F_2 = mg\cos\theta + ml\dot{\theta}^2 - m\ddot{l}_2$, and $F_1 > F_2$.

The total external work done in the mass upward process can be expressed as

$$W_{up} = W_1 + W_2 = F_1 \Delta L - (F_1 - F_2)\Delta l_2. \tag{5}$$

From equation (5), it is clear that the mass upward process increases the system energy because of $W_{up} > 0$, and W_{up} decreases with the increase of Δl_2. According to the mass upward process, the relationship between the motion distance and the corresponding elapsed time can be obtained as follows:

$$\frac{\Delta l_1}{\Delta t_1} = \frac{\Delta l_2}{\Delta t_2}. \tag{6}$$

It can be inferred from equations (5) and (6) that W_{up} decreases with the increase of the proportion of Δl_2 to ΔL (or the proportion of Δt_2 to ΔT). Therefore, the total energy increased by the mass upward process will decrease with the increase of the proportion of the mass upward decelerated process to the total upward process.

In Figure 3(b), the trajectory from l_4 to l_5 represents the mass downward accelerated process, and the symbols F_3 and $\ddot{l}_3$ are the external controlling force and the mass acceleration. Define Δl_3 and Δt_3 as the motion distance and the elapsed time of the mass downward accelerated process. The trajectory of the mass downward decelerated process is from l_5 to l_6. The parameters F_4 and $\ddot{l}_4$ are the external controlling force and the deceleration in this process. Define Δl_4 and Δt_4 as the downward decelerated distance and the corresponding elapsed time. It can also be obtained that $\Delta L = \Delta l_3 + \Delta l_4$ and $\Delta T = \Delta t_3 + \Delta t_4$.

Similarly, the total external work done in the mass downward process can be expressed as follows:

$$\begin{aligned} W_{down} &= F_3(-\Delta l_3) + F_4(-\Delta l_4) \\ &= -[F_3 \Delta L + (F_4 - F_3)\Delta l_4], \end{aligned} \tag{7}$$

where $F_3 = mg\cos\theta + ml\dot{\theta}^2 - m\ddot{l}_3$, $F_4 = mg\cos\theta + ml\dot{\theta}^2 + m\ddot{l}_4$, and $F_4 > F_3$.

In equation (7), W_{down} is always negative and decreases with the increase of the proportion of Δl_4 to ΔL (or the proportion of Δt_4 to ΔT). Thus, the mass downward process will consume more energy of the system with the increase of the proportion of the downward decelerated process to the total mass downward process.

2.3. Improved Suppression Principle. It can be inferred that the mass motion should be intermittent to meet the suppression rules of Section 2.2. Comparing with the previous study, we combine the suppression rules to improve the principle of using the moving mass to suppress the oscillation, and a new intermittent motion of the moving mass is proposed which can be expressed as follows:

$$l(t) = \begin{cases} l_0\left[1 - \varepsilon \sin\left(\omega_m\left(t - \dfrac{n\pi}{\omega_0}\right)\right)\right], \\ \quad \dfrac{n\pi}{\omega_0} \le t < \dfrac{n\pi}{\omega_0} + \Delta t, \\ l_0(1-\varepsilon), \\ \quad \dfrac{n\pi}{\omega_0} + \Delta t \le t < \dfrac{(2n+1)\pi}{2\omega_0} - \Delta t, \\ l_0\left[1 - \varepsilon \sin\left(\omega_m\left(t - \dfrac{(2n+1)\pi}{2\omega_0} - 2\Delta t\right)\right)\right], \\ \quad \dfrac{(2n+1)\pi}{2\omega_0} - \Delta t \le t < \dfrac{(2n+1)\pi}{2\omega_0} + \Delta t, \\ l_0(1+\varepsilon), \\ \quad \dfrac{(2k+1)\pi}{2\omega_0} + \Delta t \le t < \dfrac{(n+1)\pi}{\omega_0} - \Delta t, \\ l_0\left[1 - \varepsilon \sin\left(\omega_m\left(t - \dfrac{(n+1)\pi}{\omega_0}\right)\right)\right], \\ \quad \dfrac{(n+1)\pi}{\omega_0} - \Delta t \le t < \dfrac{(n+1)\pi}{\omega_0}, \end{cases} \tag{8}$$

where $n = 0, 1, 2, 3, \ldots$, the symbol $l_0 = (l_{max} + l_{min})/2$ represents the mass initial position, the symbol ε is a nondimensional motion distance parameter, and $\Delta L = 2l_0\varepsilon$. The elapsed time of the accelerated and decelerated process of the mass upward and downward motion is the same as $\Delta t = \Delta T/2$, and $0 < \Delta t < \pi/4\omega_0$ is stipulated to meet the requirement of the intermittent mass motion. ω_0 is the initial motion frequency of the oscillating pendulum, and $\omega_m = \pi/2\Delta t$. In equation (8), the first and fifth equations represent the mass upward process, the third equation represents the mass downward process, and the second and fourth equations represent that the position of the mass remains stationary at $l_0(1-\varepsilon)$ and $l_0(1+\varepsilon)$, respectively.

In order to facilitate the analysis, Figure 4 intuitively shows the position and time parameters of the first two cycles of the mass motion represented by equation (8) (when $n = 0$ and 1 in equation (8)). Figure 4 indicates that the pendulum oscillates for one cycle corresponding to two cycles of the mass motion. The mass motions in the following analysis are equation (8) and the adjustment of equation (8).

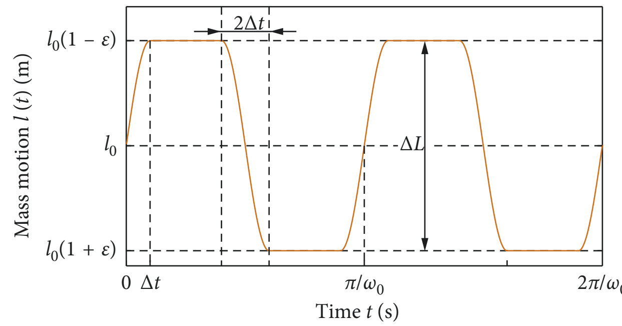


Figure 4: The first two cycles of the mass motion represented by equation (8).

2.4. Damping Ratio of the Predetermined Mass Motion. The effects of the moving mass can be regarded as the effects of a variable system damping. The damping ratio is an important parameter for quantifying the amplitude suppression. The suppression effects increase with the damping ratio. There are two means of expressions of the damping ratio: the exact damping ratio and the equivalent damping ratio. The exact damping ratio can be expressed as

$$\zeta_e = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}, \quad (9)$$

where $\delta = \ln(A_0/A_1)$, A_0 and A_1 represent the initial pendulum amplitude and the amplitude at the end of a cycle, respectively. ζ_e can be obtained from the numerical simulation of the pendulum behavior.

The concept of the equivalent damping ratio is introduced in Reference [2], and it can be calculated by equation (10). The equivalent damping ratio ζ can be applied to predict the pendulum behavior for any predetermined mass motion $l(t)$ when other system parameters are determined:

$$\zeta = \frac{\int_0^T 2\left(ml\dot{l}/(ML^2/3+ml^2)\right)\left(\dot{\theta}^2 + (g/2l)\left((ML(l-L/3)+ml^2)/(ML^2/3+ml^2)\right)(1-\cos\theta)\right)dt}{2\pi\omega_0^2\theta_0^2}, \quad (10)$$

where T represents the pendulum oscillation period.

If $\zeta \ll 1$ and the initial amplitude $\theta_0 \ll 1$, the motion frequency of the pendulum can be approximately replaced by its initial frequency (i.e., $\omega = \omega_0$). Thus, the pendulum amplitude θ can be written as

$$\theta(t) \approx \theta_0 \cos(\omega_0 t). \quad (11)$$

One cycle of the mass motion can be divided into two parts: the upward process (the first and fifth equations in equation (8)) and the downward process (the third equation in equation (8)). The equivalent damping ratio of these two processes can be obtained by substituting equations (8) and (11) into equation (10). The equivalent damping ratio of the mass upward process ζ_{up} and the mass downward process ζ_{down} is obtained as follows:

$$\zeta_{up} = \frac{\varepsilon m l_0\left[3\pi^2\cos(2\omega_0\Delta t) - 5\pi^2 + 80\omega_0^2\Delta t^2\right]\left[ML(l_0+L/3)+3ml_0^2\right]}{12\pi(\pi^2-16\omega_0^2\Delta t^2)(ML/2+ml_0)\left(ML^2/3+ml_0^2\right)}, \quad (12)$$

$$\zeta_{down} = \frac{\varepsilon m l_0\left[3\pi^2\cos(2\omega_0\Delta t) + 5\pi^2 - 80\omega_0^2\Delta t^2\right]\left[ML(l_0+L/3)+3ml_0^2\right]}{12\pi(\pi^2-16\omega_0^2\Delta t^2)(ML/2+ml_0)\left(ML^2/3+ml_0^2\right)}. \quad (13)$$

It can be inferred from equations (12) and (13) that, if ε and l_0 are constant, ζ_{up} and ζ_{down} are only determined by Δt. Figure 5 shows the value of ζ_{up} and ζ_{down} with the variety of Δt.

In Figure 5, ζ_{up} is always negative and ζ_{down} is positive. It indicates that the mass upward process generates amplification effect and the mass downward process generates suppression effect. Besides, the amplification effect decreases

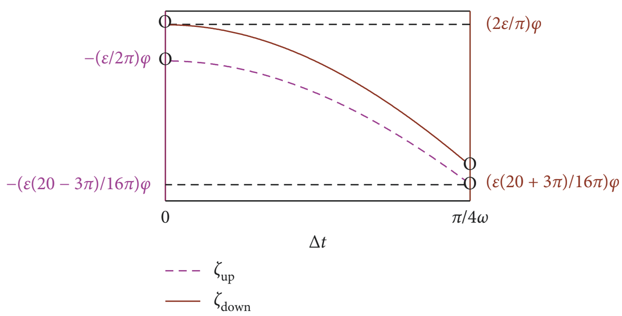


FIGURE 5: The value of ζ_{up} and ζ_{down} with the variety of Δt. Note: $\varphi = (ml_0(ML(l_0+L/3)+3ml_0^2))/(3(ML/2+ml_0)(ML^2/3+ml_0^2)) > 0$ is a constant.

and the suppression effect increases both with the decrease of Δt. Thus, better suppression of the pendulum oscillation can be obtained with the decrease of Δt. Moreover, the conclusion of Rule 1 can also be verified.

The equivalent damping ratio for one cycle of the pendulum behavior can be obtained by doubling equations (12) and (13) as follows:

$$\zeta = 2\left(\zeta_{up}+\zeta_{down}\right) = \frac{\varepsilon\pi m l_0 \cos(2\omega_0\Delta t)\left[ML\left(l_0+L/3\right)+3ml_0^2\right]}{\left(\pi^2-16\omega_0^2\Delta t^2\right)\left(ML/2+ml_0\right)\left(ML^2/3+ml_0^2\right)}. \tag{14}$$

If the parameters in equation (14) are determined, the attenuation of the pendulum amplitude per cycle can be directly calculated.

3. Model Verification

The continuous mass motion proposed in Reference [2] is expressed as

$$l(t) = l_0\left[1-\varepsilon\sin(2\omega_0 t)\right]. \tag{15}$$

The mass motion represented by equation (8) is an improved motion of equation (15). Note that when $\Delta t = \pi/4\omega_0$, equation (8) is identical to equation (15). Thus, the accuracy of the present model in this paper can be verified by the comparison between the pendulum behavior provided by Reference [2] and that for the mass motion equation (8) when $\Delta t = \pi/4\omega_0$.

In this paper, the time laws of the pendulum behaviors are obtained for the mass motion given by equation (8) by integrating numerically equation (2). The fourth-order Runge–Kutta procedure with fixed time step (0.001 s) available from the MATLAB program has been used for the numerical integration. Taking the same system parameters as Reference [2], $\theta_0 = 0.1$ rad, $L = 1.4$ m, $M = 2.7484$ kg, $m = 1$ kg, $l_0 = 1.1$ m, and $\Delta L = 0.44$ m. These system parameters together with $\Delta t = \pi/4\omega_0$ are substituted into equations (8) and (2). Figure 6 shows the first cycle of the pendulum behavior provided by Reference [2] and that calculated by the model in this paper.

In Figure 6, the pendulum behaviors are almost coincided. It can be seen from the partial enlargement of

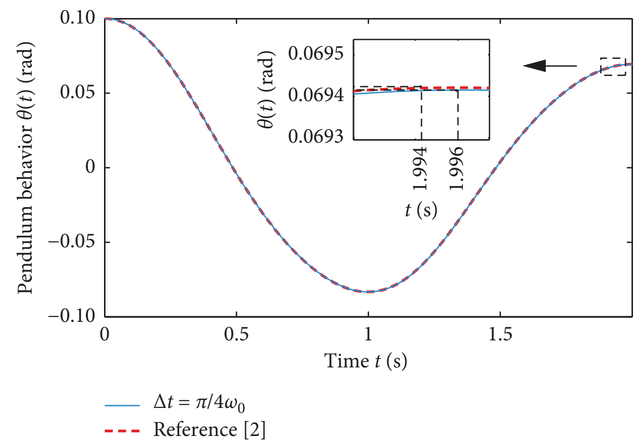


FIGURE 6: The comparison of the pendulum behavior (the first cycle).

Figure 6 that the errors of the amplitude and period of the pendulum behavior in Reference [2] and that of this paper are extremely small. Therefore, the present model of this paper can be considered accurate.

4. Numerical Simulation and Discussion

4.1. Calculation of the Maximum Damping Ratio. If $\Delta t \longrightarrow 0$ in equation (14), the maximum equivalent damping ratio for one cycle can be theoretically obtained as

$$\zeta_{max} = \frac{\varepsilon m l_0\left[ML\left(l_0+L/3\right)+3ml_0^2\right]}{\pi\left(ML/2+ml_0\right)\left(ML^2/3+ml_0^2\right)}. \tag{16}$$

It is clear that ζ_{max} increases with ε in equation (16), and there is no need to discuss the effects of ε on it. But it is found that the mass of the moving mass m and the position of the mass motion range will also affect the maximum equivalent damping ratio. Rewrite equation (14) into a dimensionless form as follows:

$$\zeta = \frac{(C/l_0)\pi\cos(2\omega_0\Delta t)}{\pi^2-16\omega_0^2\Delta t^2}\,\frac{\eta\left(\mu^2+\mu/3+3\eta\mu^3\right)}{((1/2)+\eta\mu)((1/3)+\eta\mu^2)}, \tag{17}$$

where the symbol $C = \varepsilon l_0 = \Delta L/2$ represents half of the mass motion distance, the parameter $\eta = m/M$ is a nondimensional mass, and $\mu = l_0/L$ represents the position of the mass motion range. For example, if $\mu = 0.5$, the mass motion range is from $0.5L - C$ to $0.5L + C$.

Assume that $\Delta t = 0.1$ s, $C = 0.2$ m, and the pendulum length $L = 1.5$ m, then the value of μ is approximately from 0.13 to 0.87. The mass m does not exceed M, i.e., $\eta \leq 1$. The effects of μ and η on the equivalent damping ratio ζ and the maximum damping ratio ζ_{max} are shown in Figure 7.

As can be seen from Figure 7, when μ is constant, ζ will always increase with η. However, when η is constant, the value of μ corresponding to ζ_{max} (represented by the dots) is not always the largest or the smallest, but a certain value. And this value increases with the decrease of η until the maximum value of the span of μ. For example, when $\eta = 0.8$, the value of μ corresponding to ζ_{max} is 0.49, and μ (ζ_{max}) = 0.68 when $\eta = 0.5$. Therefore, when the mass of the

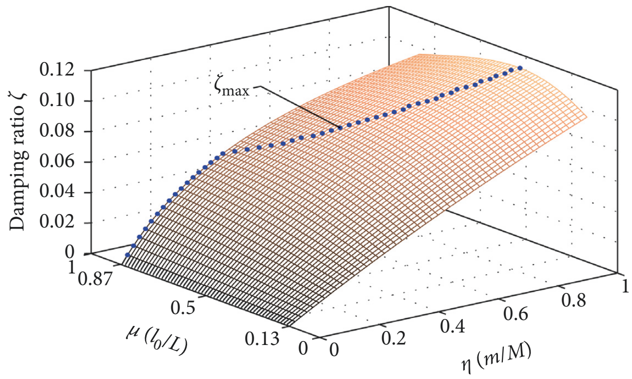


Figure 7: The effects of η and μ on the equivalent damping ratio ζ and the maximum damping ratio $\zeta_{\max}$.

pendulum M and the moving mass m are determined, it is necessary to select an appropriate position of the mass motion range to generate the best suppression effects.

4.2. Nonsynchronous Motion Problem in Simulation. It is well known that the damping creates the phase shift in a damped system. The phase shift caused by the mass motion generates the nonsynchronous motion between the moving mass and pendulum and thus affects the pendulum amplitude.

As an example, the system parameters are set as $L = 1.5$ m, $M = 1.6$ kg, $m = 0.8$ kg, and then, $\eta = 0.5$. According to Figure 7, let $l_0 = 1$ m in order to maximize the suppression effects. The pendulum oscillates for one cycle with the mass motion $l(t)$ given by equation (8) when $\Delta t = 0.1$ s, and the effects of the motion distance ΔL on the equivalent damping ratio ζ in equation (14) and the exact damping ratio ζ_e in equation (9) are shown in Figure 8(a). And the effects of the initial amplitude θ_0 on the exact damping ratio ζ_e are shown in Figure 8(b). The x-coordinate of Figure 8(a) has been done nondimensional treatment, where $\Delta L/L$ represents the nondimensional motion distance of the moving mass.

In Figure 8(a), ζ is always larger than ζ_e. When $\Delta L/L < 4/15$, the error between ζ and ζ_e is less than 1.02%. However, when $\Delta L/L = 10/15$, the error is up to 8.2%. It indicates that the increase of ΔL aggravates the phase shift and has a negative effect on the exact damping ratio. What is more, the initial amplitude θ_0 should have no effect on the damping ratio in equation (14), but Figure 8(b) shows that ζ_e continues to decrease when $\theta_0 > 0.5$. This phenomenon is extremely disadvantageous to the oscillation suppression.

The continuous pendulum behavior θ is shown in Figure 9(a) with its corresponding mass motion l when $\theta_0 = 0.1$ rad and $\Delta L/L = 4/15$. The coordinate axes of Figure 9 are also nondimensional parameters, where T represents the period of the pendulum oscillation and θ/θ_0 and l/l_0 represent the nondimensional behavior of the pendulum and the nondimensional motion of the moving mass, respectively. Figure 9(b) is a partial enlargement of the square frame in Figure 9(a), which shows that the nonsynchronous motion increases with time t. And Figure 9(a) shows the beating phenomenon which has been explained in Reference [2].

The phenomena in Figures 8 and 9 indicate that the increase of the mass motion distance ΔL and the initial amplitude θ_0 of the pendulum will aggravate the nonsynchronous motion and have negative effects on the exact damping ratio. Although the mass motion given by equation (8) can attenuate the oscillation for a short time, it cannot generate the best suppression effects and cannot suppress the pendulum asymptotically and rapidly. Thus, the mass motion given by equation (8) has to be adjusted to synchronize the motion of the mass with the pendulum to offset the effects of the phase shift.

4.3. Numerical Simulation of Synchronous Motion. In Section 4.3, a new method is proposed to synchronize the motion of the moving mass with the pendulum. First, we define the concept of *Synchronous Motion* of this paper: when the pendulum amplitude θ is minimum, the mass moves downwards to l_0 exactly; when the pendulum amplitude θ is maximum, the mass moves upwards to l_0 exactly. It is only needed to adjust the start moment of the mass upward or downward process slightly to synchronize the mass motion with the pendulum (i.e., it is only needed to adjust the span of the time t in equation (8) without changing the motion process of the moving mass represented by the first, third, and fifth equations in equation (8)). The definite means are as follows.

As shown in Figure 10, if the moving mass is fixed at $l_{\min}$ after the initial upward process from l_0 at $t = t_0$, the moment t_1 when the amplitude θ first time equal to 0 can be obtained (assistant motion). We define $t_1 - \Delta t$ as the start moment of the first mass downward process (synchronous motion). After the first mass downward process, the mass is fixed at $l_{\max}$ from $t = t_1 + \Delta t$, and the moment t_2 can be obtained when the pendulum amplitude θ is maximum (assistant motion). And $t_2 - \Delta t$ can be defined as the next start moment of the mass upward process (synchronous motion). In this method, a series of the start moments of the mass motion process t_0, $t_1 - \Delta t$, $t_2 - \Delta t$, … can be obtained. These start moments can be calculated by programming arithmetic when the system parameters are determined. The synchronous motion of the moving mass with the pendulum can be achieved if the mass motion follows the start moments obtained above. Then, the synchronous damping ratio and behavior of the pendulum can be obtained by replacing equation (8) with the synchronous mass motion.

Figure 11 shows the effects of the motion distance ΔL and the initial amplitude θ_0 on the synchronous exact damping ratio with the same parameters as Figure 8. ζ_e in Figure 11(a) is always larger than that in Figure 8(a) with the same ΔL, and it is closer to the equivalent damping ratio ζ. The error between ζ and ζ_e is only 4.1% when $\Delta L/L = 10/15$. It indicates that the simulation results can be better replaced by equation (14) when the mass motion is synchronized with the pendulum. What is more, different from Figure 8(b), the exact damping ratio ζ_e in Figure 11(b) increases slightly with

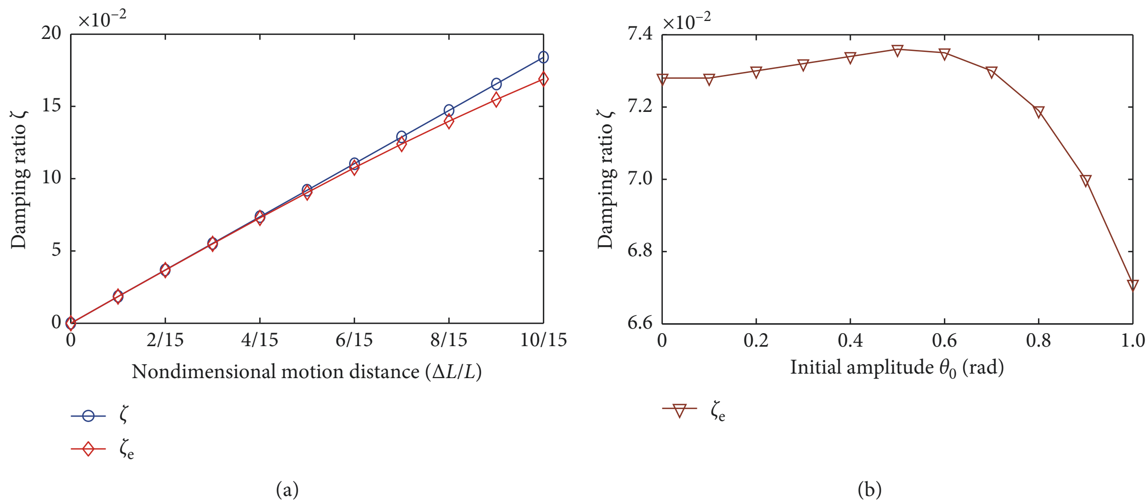


Figure 8: Nonsynchronous motion. (a) $\theta_0 = 0.1$ rad, the effects of the motion distance ΔL on the equivalent damping ratio ζ and the exact damping ratio ζ_e; (b) $\Delta L/L = 4/15$, the effect of the initial amplitude θ_0 on the exact damping ratio ζ_e.

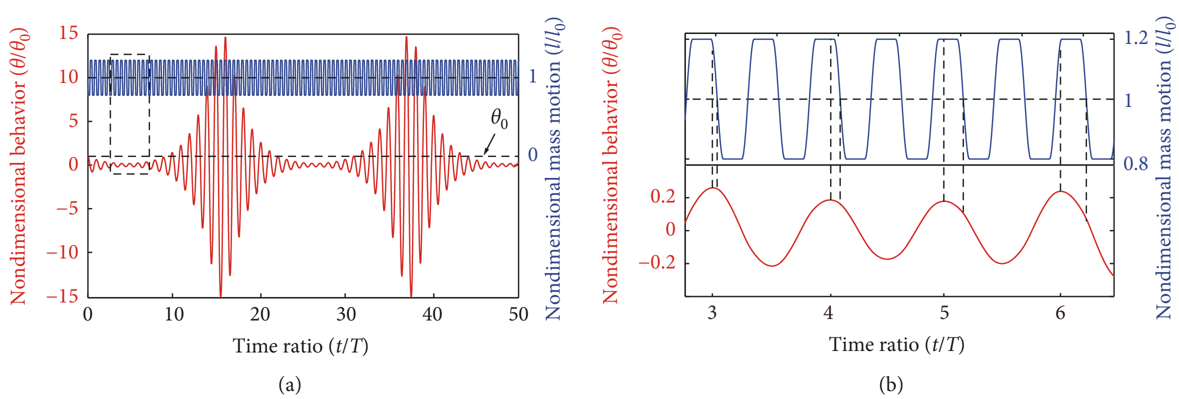


Figure 9: Nonsynchronous motion. (a) The continuous pendulum behavior θ and the mass motion l; (b) the partial enlargement of (a).

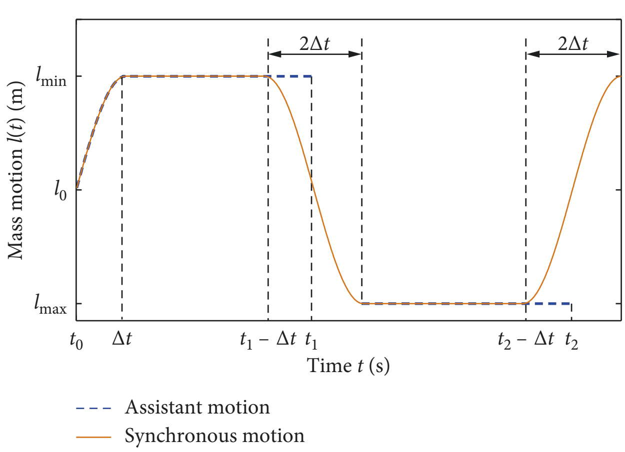


Figure 10: Diagram of synchronous motion method.

the increase of the initial amplitude θ_0 without negative effects.

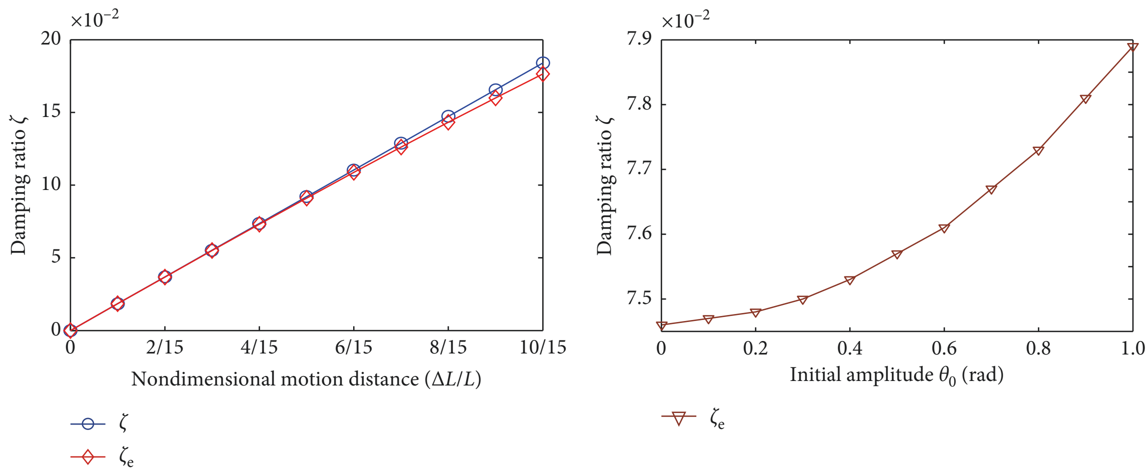
Figure 12(a) shows about 10 cycles of the synchronous behavior of the pendulum, which is a continuous and rapid suppression behavior without beating phenomenon. In Figure 12(b), there is almost no phase shift between the motion of the mass and the pendulum. All results of Section 4.3 indicate that synchronous motion will generate better suppression effects.

4.4. Discussion on Suppression Efficiency. The high suppression efficiency of the improved principle proposed in Section 2.3 can be verified by comparing the synchronous pendulum behavior for two different elapsed time Δt of this paper with the pendulum behavior in the literature.

The mass motion proposed in Reference [2] is equation (15), and the corresponding equivalent damping ratio ζ_{eqv} for one cycle of the pendulum oscillation is introduced as

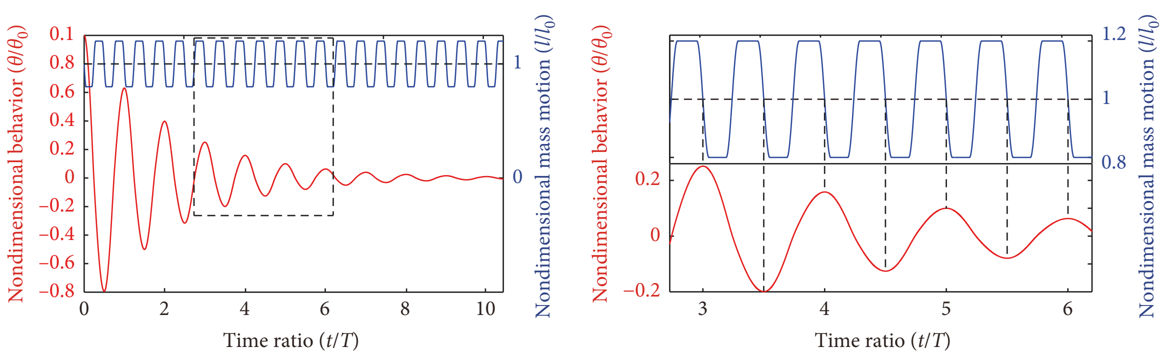
$$\zeta_{\mathrm{eqv}} = \frac{\varepsilon m l_0 \left[ML\left(l_0 + L/3\right) + 3ml_0^2\right]}{4\left(ML/2 + ml_0\right)\left(ML^2/3 + ml_0^2\right)}. \tag{18}$$

The mass motion frequency $2\omega_0$ in equation (15) is invariable, which can also cause the nonsynchronous motion of the mass with the pendulum. Thus, equation (15) is also needed to be adjusted synchronously. The synchronous method proposed in Reference [2] is replacing ω_0 by the



(a) (b)

Figure 11: Synchronous motion. (a) $\theta_0 = 0.1$ rad, the effects of the motion distance ΔL on the equivalent damping ratio ζ and the exact damping ratio ζ_e; (b) $\Delta L/L = 4/15$, the effect of the initial amplitude θ_0 on the exact damping ratio ζ_e.



(a) (b)

Figure 12: Synchronous motion. (a) The continuous pendulum behavior θ and the mass motion l; (b) the partial enlargement of (a).

pervious cycle frequency $\omega_1 = 2\pi/T_1$ in every current cycle to obtain a continuous attenuation behavior of the pendulum.

The synchronous pendulum behavior of Reference [2] can be obtained by substituting the synchronized equation (15) into equation (2). In the cases of this paper, we take the same parameters as Reference [2] mentioned in Section 3. And $\Delta t = 0.05$ s and 0.15 s are set as the elapsed time of the upward and downward process of the mass synchronous motion, where the motion processes of the mass are also represented by the first, third, and fifth equations of equation (8). In the above 3 cases, the synchronous pendulum behaviors for 15 cycles are shown in Figure 13, and the corresponding amplitudes are shown in Table 1.

Take 1% of the initial amplitude (0.001 rad) as the suppression standard. It can be clearly seen from the partial enlargement of Figure 13 that it only needs 10 cycles to meet the standard when $\Delta t = 0.05$ s and 11 cycles when $\Delta t = 0.15$ s. However, the synchronous pendulum behavior of Reference [2] needs 13 cycles.

It can be calculated from Table 1 that the exact damping ratio ζ_e for each cycle in Reference [2] is 0.058. When $\Delta t = 0.05$ s, ζ_e is approximately 0.0732, which is 26.21% higher than that in Reference [2]. Using equations (16) and (18) yields $\zeta_{max} = (4/\pi)\zeta_{eqv}$. It indicates that the maximum damping ratio of the system for the intermittent mass motion is theoretically 27% higher than the equivalent damping ratio in Reference [2]. Therefore, when $\Delta t = 0.05$ s, the suppression effect is already close to the best.

In order to verify the high suppression efficiency of the intermittent mass motion proposed in this paper, Δt is set to be very small (0.05 s and 0.15 s), which may not be able to achieve in practice. However, there will usually be enough time to generate effective effects on oscillation suppression in engineering applications.

5. Conclusions

In this paper, an improved principle is proposed to address the rapid suppression problem of an oscillating pendulum by controlling a moving mass intermittently. Some conclusions are as follows:

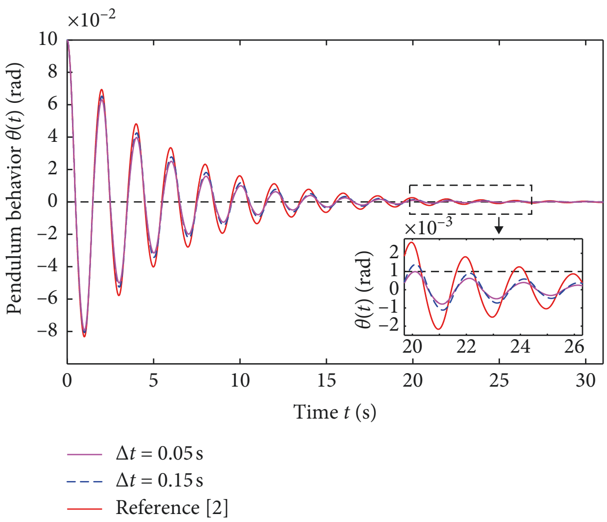


FIGURE 13: The comparison of the synchronous pendulum behaviors.

TABLE 1: Amplitudes of the pendulum for 15 cycles.

i	θ_i		
	$\Delta t = 0.05$ s	$\Delta t = 0.15$ s	Reference [2]
0	0.100000	0.100000	0.10000
1	0.063050	0.065210	0.06943
2	0.039750	0.042510	0.04821
3	0.025070	0.027710	0.03347
4	0.015810	0.018070	0.02324
5	0.009968	0.011780	0.01614
6	0.006286	0.007679	0.01121
7	0.003964	0.005007	0.00778
8	0.002500	0.003264	0.00541
9	0.001576	0.002128	0.00375
10	0.000994	0.001387	0.00261
...	...	...	...
15	0.000099	0.000163	0.00041

(1) In order to get better oscillation suppression effects, the motion of the moving mass should be intermittent. The elapsed time of the mass upward and downward process ΔT is required to be as small as possible when the corresponding motion distance ΔL is unchanged. The initial position of the moving mass l_0 is required to be as close to $l_{\min}$ as possible if conditions permit. The proportion of the mass decelerated process to the mass upward and downward process is required to be large. And it is necessary to select an appropriate position of the mass motion range to generate the best suppression effects.

(2) Because of the phase shift caused by the motion of the moving mass, there will be nonsynchronous motion between the moving mass and the pendulum. The negative effects of nonsynchronous motion increase with the increase of the mass motion distance and the initial amplitude of the pendulum. And it also causes the beating phenomenon of the continuous pendulum behavior so that the pendulum oscillation cannot be rapidly suppressed. Therefore, nonsynchronous motion has to be avoided.

(3) The start moments of the mass upward and downward process have to be adjusted slightly each time to synchronize the moving mass with the pendulum. When the synchronous motion is achieved, the exact damping ratio can be replaced by the equivalent damping ratio more accurately. What is more, the oscillation of the pendulum can also be rapidly suppressed. Therefore, the intermittent mass motion can generate better oscillation suppression effects after the synchronous adjustment.

Nomenclature

C:	Half of the mass motion distance
E:	Total energy of the system
F:	External controlling force
F_c:	The Coriolis inertia force
K:	Kinetic energy of the system
L:	Pendulum length
M:	The mass of the pendulum
N:	The normal component force acting on the mass
O:	The pivot of the pendulum
P:	Potential energy of the system
T:	The period of the pendulum oscillation
W:	The work done by F
ΔL:	Motion distance of the mass upward and downward process
ΔT:	Elapsed time of the mass upward and downward motion
k:	System stiffness
$l(t)$:	The motion of the moving mass
l_0:	Initial position of the moving mass
$\dot{l}$ and $\ddot{l}$:	The velocity and the acceleration of the moving mass
$l_{\max}$ and $l_{\min}$:	Maximum and minimum displacement of the moving mass
l_1, l_2, and l_3:	The trajectory of the mass upward process
l_4, l_5, and l_6:	The trajectory of the mass downward process
m:	The mass of the moving mass
Δl:	Motion distance of the accelerated or decelerated process of the mass upward or downward motion
Δt:	Elapsed time of the accelerated or decelerated process of the mass upward or downward motion
θ:	Pendulum amplitude
θ_0:	Pendulum initial amplitude
$\theta_{\max}$:	Maximum amplitude of the pendulum
$\dot{\theta}$ and $\ddot{\theta}$:	Angular velocity and angular acceleration of the moving mass
ω_0:	Initial motion frequency of the pendulum
ω_m:	Motion frequency of the moving mass
ζ:	The equivalent damping ratio
ζ_e:	The exact damping ratio

ζ_{max}: The maximum equivalent damping ratio
ζ_{eqv}: The equivalent damping ratio of Reference [2]
ζ_{up} and ζ_{down}: The equivalent damping ratio of the mass upward and downward process
ε: Nondimensional motion distance of the moving mass
μ: Nondimensional mass of the system
η: Position of the mass motion range.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] V. S. Aslanov, "Stability of a pendulum with a moving mass: the averaging method," *Journal of Sound and Vibration*, vol. 445, pp. 261–269, 2019.
[2] W. Szyszkowski and D. S. D. Stilling, "On damping properties of a frictionless physical pendulum with a moving mass," *International Journal of Non-Linear Mechanics*, vol. 40, no. 5, pp. 669–681, 2005.
[3] W. Szyszkowski and E. Sharbati, "On the FEM modeling of mechanical systems controlled by relative motion of a member: a pendulum-mass interaction test case," *Finite Elements in Analysis and Design*, vol. 45, no. 10, pp. 730–742, 2009.
[4] E. Sharbati and W. Szyszkowski, "A new FEM approach for analysis of beams with relative movements of masses," *Finite Elements in Analysis and Design*, vol. 47, no. 9, pp. 1047–1057, 2011.
[5] K. Yoshida, I. Kawanishi, and H. Kawabe, "Stabilizing control for a single pendulum by moving the center of gravity: theory and experiment," in *Proceedings of American Control Conference*, vol. 5, pp. 3405–3410, Albuquerque, NM, USA, June 1997.
[6] H. Sogo, A. Matsumoto, and T. Yamamoto, "Self-tuning swing control of a variable-length pendulum," *IFAC Proceedings Volumes*, vol. 34, no. 14, pp. 377–382, 2001.
[7] X. Xin and Y. Liu, "Trajectory tracking control of variable length pendulum by partial energy shaping," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 5, pp. 1544–1556, 2014.
[8] O. O. Gutiérrez-Frias, J. C. Martinez Garcia, and R. Garrido, "PD control for vibration attenuation in a physical pendulum with moving mass," *Mathematical Problems in Engineering*, vol. 2009, Article ID 179724, 11 pages, 2009.
[9] M. A. Pinsky and A. A. Zevin, "Oscillations of a pendulum with a periodically varying length and a model of swing," *International Journal of Non-Linear Mechanics*, vol. 34, no. 1, pp. 105–109, 1999.
[10] A. A. Zevin and L. A. Filonenko, "A qualitative investigation of the oscillations of a pendulum with a periodically varying length and a mathematical model of a swing," *Journal of Applied Mathematics and Mechanics*, vol. 71, no. 6, pp. 892–904, 2007.
[11] A. O. Belyakov, A. P. Seyranian, and A. Luongo, "Dynamics of the pendulum with periodically varying length," *Physica D: Nonlinear Phenomena*, vol. 238, no. 16, pp. 1589–1597, 2009.
[12] T. Yang, B. Fang, S. Li, and W. Huang, "Explicit analytical solution of a pendulum with periodically varying length," *European Journal of Physics*, vol. 31, no. 5, pp. 1089–1096, 2010.
[13] A. P. Markeev, "Uniform rotations of a variable-length pendulum," *Doklady Physics*, vol. 56, no. 4, pp. 240–243, 2011.
[14] P. S. Krasil'nikov, "The non-linear oscillations of a pendulum of variable length on a vibrating base," *Journal of Applied Mathematics and Mechanics*, vol. 76, no. 1, pp. 25–35, 2012.
[15] M. Kumar and Parul, "Mathematical modelling and numerical simulation of child swing motion," *National Academy Science Letters*, vol. 36, no. 2, pp. 225–232, 2012.
[16] M. A. Savi, F. H. I. Pereira-Pinto, and A. M. Ferreira, "Chaos control in mechanical systems," *Shock and Vibration*, vol. 13, no. 4-5, pp. 301–314, 2006.
[17] F. Reguera, F. E. Dotti, and S. P. Machado, "Rotation control of a parametrically excited pendulum by adjusting its length," *Mechanics Research Communications*, vol. 72, pp. 74–80, 2016.
[18] K. Yoshida, S. Okanouchi, and H. Kawabe, "Vibration suppression control for a variable length pendulum with a pivot movable in a restricted range," in *Proceedings of the 2006 SICE-ICASE International Joint Conference*, vol. 18, pp. 4538–4544, Busan, Korea, October 2006.
[19] D. S. D. Stilling and W. Szyszkowski, "Controlling angular oscillations through mass reconfiguration: a variable length pendulum case," *International Journal of Non-Linear Mechanics*, vol. 37, no. 1, pp. 89–99, 2002.


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