Research Article

Exact Receptance Function of Cracked Beams and Its Application for Crack Detection

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The receptance function is very important which interrelates the harmonic excitation and the response of a structure in the frequency domain. This paper presents the exact receptance function of cracked beams. In this work, the “receptance curvature” is defined as the second derivative of the receptance. The influence of the crack on the receptance curvature is investigated. The results show that when there are cracks, the receptance curvature is influenced significantly at crack positions. This might be useful for the detection of cracks. In this paper, the derivation of exact receptance of the beam with general boundary conditions is presented, and the numerical simulations are provided.

1. Introduction

As is known the receptance method has been applied widely in mechanical system and structural dynamics. Yang [1] presented the exact receptances of nonproportionally damped dynamic systems. Based on a decomposition of the damping matrix, an iteration procedure is developed which does not require matrix inversion. Lin and Lim [2] developed a new and effective method to derive the structural design sensitivities which include the receptance sensitivity with respect to mass modification and stiffness modification from the limited vibration test data. Mottershead [3] investigated the measured zeros from frequency response functions and their application to model assessment and updating. Gurgoze [4] was concerned with receptance matrices of viscously damped systems subject to several constraint equations. The frequency response matrix of the constrained system was established in terms of the frequency response matrix of the unconstrained system and the coefficient vectors of the constraint equations. Karakas and Gurgoze [5] presented a formulation of the receptance matrix of nonproportionally damped dynamic systems. The

receptance matrix was obtained directly without using the iterations. Mottershead et al. [6] and Tehrani et al. [7] addressed the problem of pole-zero assignment using the receptance method in active vibration control. Bernal [8] presented a receptance-based formulation for modal scaling using mass perturbations. Bonello and Minh Hai [9] used the receptance harmonic balance technique for the computation of the vibration of a whole aeroengine model with nonlinear bearings. Albertelli et al. [10] proposed a method using the receptance coupling substructure method to improve chatter free cutting conditions prediction. Muscolino et al. [11] presented the explicit frequency response functions of discretized structures with uncertain parameters. Recently, Muscolino and Santoro [12] presented the explicit frequency response function of beams with cracks of uncertain depths in order to evaluate the main statistics as well as the upper and lower bounds of the response.

Early crack detection is extremely important in mechanical systems and engineering structures, and this issue has attracted many researchers in the last three decades. Since the cracks may influence significantly on the dynamic characteristics of structures such as natural frequencies and

mode shapes, these dynamic characteristics have been investigated and applied widely for crack detection of structures. Chondros and Dimarogonas [13] and Lee and Chung [14] studied the change in natural frequencies of beams caused by a crack. Hu and Liang [15] presented an integrated method using the knowledge of changes in natural frequencies for crack detection. Zheng and Kessissoglou [16] investigated the relationship between natural frequency of a cracked beam to the depth and location of the crack. The results in these papers showed that the natural frequency of the cracked beam decreases as the crack depth increases. Gudmundson [17] studied the influence of cracks on the natural frequencies of slender structures using a flexibility matrix approach. Ruotolo and Surace [18], Vakil-Baghmisheh et al. [19] applied genetic algorithms for crack detection of structures using natural frequencies. In other work, Thalopil and Maiti [20] investigated the change in natural frequencies caused by longitudinal cracks for crack detection. Gillich et al. [21] proposed new mathematical expressions applying the multimodal analysis for damage detection. The proposed method makes the frequency-based damage detection method robust and not influenced by temperature variations. Along with the change in natural frequencies, the change in mode shapes caused by cracks has been investigated in many previous works. Some authors presented methods to calculate and apply the mode shape of cracked structures for crack detection purpose. Very early works by Pandey et al. [22] and Abdel Wahab [23] used the change in curvature mode shapes for determining the location of cracks in beam structures. The author of this paper [24] applied 3D finite elements to investigate the change in mode shapes at the crack positions. However, the mode shape-based methods for damage detection have the disadvantage that the mode shapes cannot be measured directly but they have to be calculated from the frequency response functions (FRF). To overcome this, Sampaio et al. [25] presented a method using the frequency response function curvature for crack detection which is the extension of the method in [22]. However, the curvatures of mode shape and FRF in [22, 25] were calculated by approximation methods, and no mathematical formulas for the curvature mode shape and FRF curvature were presented. Recently, Caddemi and Calio [26] presented the exact closed-form solution for the mode shapes of the Euler–Bernoulli beam with multiple open cracks. This result can be applied to establish the exact formula of the receptance function to improve the accuracy of the methods using the receptance matrix.

In previous works, the receptance of cracked beams was derived approximately while the exact form of the receptance of cracked beams has not been presented. Mode shape and its curvature have been applied to investigate the influence of the crack on the dynamic response of structures. However, these works were based on approximate methods. The aim of this work is to establish the exact formulas of the receptance function and receptance curvature of a cracked beam using the exact mode shape. The exact receptance function provides convenient tool to calculate and study the distribution of response amplitude of the beam corresponding to harmonic force acting along the beam at any

forcing frequency. The influence of cracks on the receptance curvature is investigated for the crack detection purpose. Simulation results show that the receptance curvature has significant changes at crack positions. It is concluded that only the low forcing frequency is recommended for crack detection while the high forcing frequency is not recommended. The experiment has been carried out to justify the formulas obtained in this work.

2. Derivation of the Receptance Function of a Cracked Beam

2.1. Intact Beam. In this work, the undamped simply supported Euler–Bernoulli beam is considered. The forced vibration equation of the undamped beam can be written as follows:

$$EI \frac{\partial^4 v(\xi, t)}{\partial \xi^4} + L^4 m \frac{\partial^2 v(\xi, t)}{\partial t^2} = L^4 P(\xi, t), \quad (1)$$

where $\xi = x/L$ is the nondimensional coordinate.

The solution of equation (1) can be found in the form

$$v(\xi, t) = \sum_{n=1}^{\infty} \phi_n(\xi) Y_n(t), \quad (2)$$

where $\phi_n(x)$ is the n^{th} mode shape of the beam and $Y_n(t)$ is the time-dependent amplitude which is referred to as generalized coordinate. Substituting equation (2) into equation (1) yields

$$\sum_{n=1}^{\infty} EI \frac{d^4 \phi_n}{d\xi^4} Y_n + \sum_{n=1}^{\infty} L^4 m \phi_n \ddot{Y}_n = L^4 P. \quad (3)$$

Multiplying both sides of equation (3) with $\phi_k(x)$ and integrating give

$$m \sum_{n=1}^{\infty} \ddot{Y}_n L^4 \int_0^1 \phi_n \phi_k d\xi + EI \sum_{n=1}^{\infty} Y_n(t) \int_0^1 \phi_k \frac{d^4 \phi_n}{d\xi^4} d\xi = L^4 \int_0^1 \phi_k P d\xi. \quad (4)$$

The orthogonality conditions of the beam can be expressed as

$$m \int_0^1 \phi_k \phi_n d\xi = 0, \quad \text{if } k \neq n, \quad (5)$$

$$EI \int_0^1 \phi_n \frac{d^4 \phi_k}{d\xi^4} d\xi = 0, \quad \text{if } k \neq n.$$

Applying this orthogonality conditions yields

$$m \ddot{Y}_k \int_0^1 \phi_k^2 d\xi + \frac{EI Y_k}{L^4} \int_0^1 \phi_k \frac{d^4 \phi_k}{d\xi^4} d\xi = \int_0^1 \phi_k P d\xi. \quad (6)$$

If the force is harmonic $P = \hat{P} \sin \omega t$ exciting at the point $\xi = \hat{\xi}$, then $Y_k(t) = \bar{Y}_k \sin \omega t$ and from (6) we have

$$(\omega_k^2 - \omega^2) m \bar{Y}_k \int_0^1 \phi_k^2(\xi) d\xi = \phi_k(\hat{\xi}) \hat{P}. \quad (7)$$

Therefore, the receptance at ξ due to the force at $\hat{\xi}$ can be obtained as

$$\alpha_{\xi\xi}(\omega) = \sum_{k=1}^{\infty} \frac{1}{\omega_k^2 - \omega^2} \frac{\phi_k(\xi)\phi_k(\xi)}{m \int_0^1 \phi_k^2(\xi) d\xi}. \quad (8)$$

The receptance curvature matrix is defined as the second derivative of the receptance function with respect to ξ variable as follows:

$$\frac{\partial^2 \alpha_{\xi\xi}}{\partial \xi^2} = \sum_{n=1}^{\infty} \frac{1}{\omega_n^2 - \omega^2} \frac{\phi_n(\xi)}{m \int_0^1 \phi_n^2(\xi) d\xi} \frac{d^2 \phi_n(\xi)}{d\xi^2}. \quad (9)$$

As can be seen from equation (19), the receptance curvature is dependent on the curvature of the mode shape. It is well known that when there is a crack, the mode shape is changed at the crack position but it is small when the crack size is small. However, the receptance curvature may have significant change at the crack position in comparison with that of the receptance.

2.2. Cracked Beam. The orientation of cracks changes with location and loading conditions and is not always vertical or normal to the beam central line. The orientation of crack influences the change in stiffness at the crack location and the position of the crack as presented in [27]. In this study, for simplicity reason, we consider only cracks which are open and perpendicular to the beam surface. In order to derive the exact receptance function of the cracked beam, the exact closed form of the mode shape of the simply supported Euler–Bernoulli beam with n cracks is adopted from [26] as follows:

$$\begin{aligned} \phi_k(\xi) = C_1 \left\{ \sum_{i=1}^n \frac{\lambda_i \mu_i}{2\alpha_k} \left[\sin \alpha_k \tilde{\xi}_i + \sinh \alpha_k \tilde{\xi}_i \right] U_i + \sin \alpha_k \xi \right\} \\ + \left\{ \sum_{i=1}^n \frac{\lambda_i \zeta_i}{2\alpha_k} \left[\sin \alpha_k \tilde{\xi}_i + \sinh \alpha_k \tilde{\xi}_i \right] U_i + \sinh \alpha_k \xi \right\}, \end{aligned} \quad (10)$$

where

$$C_1 = - \frac{\sum_{i=1}^n \lambda_i \zeta_i / 2\alpha_k \left[\sin \alpha_k \tilde{\xi}_i^1 + \sinh \alpha_k \tilde{\xi}_i^1 \right] + \sinh \alpha_k}{\sum_{i=1}^n \lambda_i \mu_i / 2\alpha_k \left[\sin \alpha_k \tilde{\xi}_i^1 + \sinh \alpha_k \tilde{\xi}_i^1 \right] + \sin \alpha_k}, \quad (11)$$

α_k is the dimensionless frequency parameter $\alpha_k^4 = \omega_k^2 m L^4 / EI$; ξ_i is the position of the i th crack, $0 < \xi_1 < \xi_2 < \dots < \xi_n < 1$.

For simplicity reason, some notations are presented as follows:

$$\begin{aligned} \tilde{\xi}_i &= \xi - \xi_i, \\ \tilde{\xi}_j &= \xi - \xi_j, \\ \tilde{\xi}_{ij} &= \xi_i - \xi_j, \\ \tilde{\xi}_{ji} &= \xi_j - \xi_i, \\ \tilde{\xi}_i^1 &= 1 - \xi_i, \\ \tilde{\xi}_j^1 &= 1 - \xi_j, \\ S_i &= \sin \alpha_k \tilde{\xi}_i + \sinh \alpha_k \tilde{\xi}_i, \\ S_j &= \sin \alpha_k \tilde{\xi}_j + \sinh \alpha_k \tilde{\xi}_j, \\ U_i &= U(\xi - \xi_i), \\ U_j &= U(\xi - \xi_j), \\ U_{ij} &= U(\xi_i - \xi_j), \\ U_{ji} &= U(\xi_j - \xi_i), \end{aligned} \quad (12)$$

where U is Heaviside function.

The parameters in (10) are presented as follows:

$$\begin{aligned} \mu_j &= \sum_{i=1}^{j-1} \frac{\alpha_k \lambda_i \mu_i}{2} \left[-\sin \alpha_k \tilde{\xi}_{ji} + \sinh \alpha_k \tilde{\xi}_{ji} \right] U_{ji} - \alpha_k^2 \sin \alpha_k \xi_j, \\ \zeta_j &= \sum_{i=1}^{j-1} \frac{\alpha_k \lambda_i \zeta_i}{2} \left[-\sin \alpha_k \tilde{\xi}_{ji} + \sinh \alpha_k \tilde{\xi}_{ji} \right] U_{ji} + \alpha_k^2 \sinh \alpha_k \xi_j, \\ \lambda_i &= \frac{h}{5.346 L g(s)}, \\ g(s) &= 1.86s_i^2 - 3.95s_i^3 + 16.375s_i^4 - 37.226s_i^5 + 76.81s_i^6 \\ &\quad - 126.9s_i^7 + 172s_i^8 - 143.97s_i^9 + 66.56s_i^{10}, \end{aligned} \quad (13)$$

where λ_i is the severity parameter of the crack and $s_i = d_i/h$ is defined as the ratio between the crack depth d_i and the beam height h .

From equations (10) and (11), the square of the mode shape can be calculated as

$$\begin{aligned} \phi_k^2(\xi) &= \sum_{i=1}^n \sum_{j=1}^n \frac{\lambda_i \lambda_j A_1}{4\alpha_k^2} S_i S_j U_i U_j \\ &\quad + \sum_{i=1}^n \left[\frac{\lambda_i A_2}{\alpha_k} \sin \alpha_k \xi S_i + \frac{\lambda_i A_3}{\alpha_k} \sinh \alpha_k \xi S_i \right] U_i \\ &\quad + C_1^2 \sin^2 \alpha_k \xi + \sinh^2 \alpha_k \xi + 2C_1 \sin \alpha_k \xi \sinh \alpha_k \xi, \end{aligned} \quad (14)$$

where

$$\begin{aligned} A_1 &= C_1^2 \mu_i \mu_j + \zeta_i \zeta_j + 2C_1 \mu_i \zeta_j; \\ A_2 &= C_1^2 \mu_i + C_1 \zeta_i; \\ A_3 &= \zeta_i + C_1 \mu_i. \end{aligned} \quad (15)$$

To integrate the terms including Heaviside function in the square of the mode shape, following properties of Heaviside function should be noted:

$$U_i U_j = \begin{cases} U_i, & i \geq j, \\ U_j, & i < j, \end{cases} \quad (16)$$

$$\int_0^1 f(\xi) U_i d\xi = \int_{\xi_{0i}}^1 f(\xi) d\xi = F(1) - F(\xi_i),$$

where F is the antiderivative function of f .

From equation (16), we have

$$\int_0^1 f(\xi) U_i U_j d\xi = \begin{cases} \int_0^1 f(\xi) U_i d\xi = F(1) - F(\xi_i), & i \geq j, \\ \int_0^1 f(\xi) U_j d\xi = F(1) - F(\xi_j), & i < j. \end{cases} \quad (17)$$

Equation (17) cannot be used directly to derive the explicit formula of $\int_0^1 \phi_k^2(\xi) d\xi$. However, we can express equation (17) in the form

$$\int_0^1 f(\xi) U_i U_j d\xi = F(1) - F(\xi_i) U_{ij} - F(\xi_j) U_{ji} + F(\xi_i) \delta_{ij}, \quad (18)$$

where δ_{ij} is the Kronecker delta.

Applying equations (14)–(18), the integral of the square of the mode shape can be derived. It is interesting that the term $F(\xi_i) \delta_{ij}$ will vanish, and the following formula is obtained:

$$\begin{aligned} \int_0^1 \phi_k^2(\xi) d\xi &= \sum_{i=1}^n \sum_{j=1}^n \frac{\lambda_i \lambda_j A_1}{8\alpha_k^2} \times \left\{ \cos \alpha_k \tilde{\xi}_{ij} - \frac{\sin(\tilde{\xi}_i^1 + \tilde{\xi}_j^1)}{2\alpha_k} - \cosh \alpha_k \tilde{\xi}_{ij} + \frac{\sin \alpha_k \tilde{\xi}_i^1 \cosh \alpha_k \tilde{\xi}_i^1}{\alpha_k} \right. \\ &\quad - \frac{\sinh \alpha_k \tilde{\xi}_j^1 \cos \alpha_k \tilde{\xi}_i^1}{\alpha_k} + \frac{\sin \alpha_k \tilde{\xi}_j^1 \cosh \alpha_k \tilde{\xi}_i^1}{\alpha_k} - \frac{\sinh \alpha_k \tilde{\xi}_i^1 \cos \alpha_k \tilde{\xi}_j^1}{\alpha_k} + \frac{\sinh \alpha_k \tilde{\xi}_i^1 \cosh \alpha_k \tilde{\xi}_j^1}{\alpha_k} \\ &\quad - \left[\xi_i \cos \alpha_k \tilde{\xi}_{ij} + \frac{\sin \alpha_k \tilde{\xi}_{ij}}{2\alpha_k} - \xi_i \cosh \alpha_k \tilde{\xi}_{ij} - \frac{\sinh \alpha_k \tilde{\xi}_{ij}}{\alpha_k} \right] U_{ij} \\ &\quad - \left[\xi_j \cos \alpha_k \tilde{\xi}_{ji} + \frac{\sin \alpha_k \tilde{\xi}_{ji}}{2\alpha_k} - \xi_j \cosh \alpha_k \tilde{\xi}_{ji} \right] U_{ji} \} \\ &\quad + \sum_{i=1}^n \frac{\lambda_i A_2}{2\alpha_k} \left[\tilde{\xi}_i^1 \cos \alpha_k \xi_i - \frac{\sin \alpha_k (2 - \xi_i)}{2\alpha_k} - \frac{\sin \alpha_k \xi_i}{2\alpha_k} + \frac{\sin \alpha_k \cosh \alpha_k \tilde{\xi}_i^1}{\alpha_k} - \frac{\sinh \alpha_k \tilde{\xi}_i^1 \cos \alpha_k}{\alpha_k} \right] \\ &\quad + \sum_{i=1}^n \frac{\lambda_i A_3}{2\alpha_k} \left[-\tilde{\xi}_i^1 \cosh \alpha_k \xi_i + \frac{\sinh \alpha_k (2 - \xi_i)}{2\alpha_k} + \frac{\sinh \alpha_k \xi_i}{2\alpha_k} + \frac{\sin \alpha_k \tilde{\xi}_i^1 \cosh \alpha_k}{\alpha_k} - \frac{\sinh \alpha_k \cos \alpha_k \tilde{\xi}_i^1}{\alpha_k} \right] \\ &\quad + \frac{(C_1^2 - 1)}{2} - \frac{C_1^2 \sin 2\alpha_k}{4\alpha_k} + \frac{\sinh 2\alpha_k}{4\alpha_k} + \frac{C_1 \sin \alpha_k \cosh \alpha_k}{\alpha_k} - \frac{C_1 \sinh \alpha_k \cos \alpha_k}{\alpha_k}. \end{aligned} \quad (19)$$

Substituting equations (10) and (19) into equation (8), the exact receptance of the cracked beam will be obtained.

In order to derive the exact formulas of curvature receptance, the second derivative of the mode shape needs to be calculated. It is noted that

$$(SU)'' = S''U + SU'' + 2S'U'. \quad (20)$$

Applying the following properties of Heaviside function and Dirac delta function [27]

$$\begin{aligned} U'(\xi) &= \delta(\xi), \\ f(\xi) \delta'(\xi) &= -f'(\xi) \delta(\xi), \end{aligned} \quad (21)$$

yields

$$(SU)'' = S''U - S'\delta + 2S'\delta = S''U + S'\delta. \quad (22)$$

From equations (9) and (22), the second derivative of the mode shape can be derived as follows:

$$\begin{aligned} \phi_k''(\xi) = C_1 \left\{ \sum_{i=1}^n \frac{\lambda_i \mu_i}{2} \left[\alpha_k (\sinh \alpha_k \tilde{\xi}_i - \sin \alpha_k \tilde{\xi}_i) U_i + (\cos \alpha_k \tilde{\xi}_i \right. \right. \\ \left. \left. + \cosh \alpha_k \tilde{\xi}_i) \delta(\tilde{\xi}_i) \right] - \alpha_k^2 \sin \alpha_k \xi \right\} \left\{ \sum_{i=1}^n \frac{\lambda_i \nu_i}{2} \left[\alpha_k \right. \right. \\ \left. \left. \cdot (\sinh \alpha_k \tilde{\xi}_i - \sin \alpha_k \tilde{\xi}_i) U_i + (\cos \alpha_k \tilde{\xi}_i + \cosh \alpha_k \tilde{\xi}_i) \right. \right. \\ \left. \left. \cdot \delta(\tilde{\xi}_i) \right] + \alpha_k^2 \sinh \alpha_k \xi \right\}. \quad (23) \end{aligned}$$

Substituting equations (19) and (23) into (9), the exact receptance curvature will be determined.

The exact formulas of receptance and its curvature of cracked beams with other general boundary conditions can be obtained by the same procedure and can be seen in Appendix.

3. Numerical Simulation and Discussions

Numerical simulations of a simply supported beam with two cracks are presented in this section. Parameters of the beam are mass density $\rho = 7800 \text{ kg/m}^3$; modulus of elasticity $E = 2.0 \times 10^{11} \text{ N/m}^2$; $L = 1 \text{ m}$; and $b = 0.02 \text{ m}$; $h = 0.01 \text{ m}$. The first three natural frequencies of the intact beam are calculated and listed in Table 1.

Figure 1 presents the 3D graphs of normalized receptance matrices at the first two natural frequencies of the intact beam. As can be seen from Figure 1(a), when the forcing frequency is equal to the first natural frequency, the receptance is zero at the ends of the beam and then increases and reaches the maximum when the response point moves from the ends to the middle of beam. When the forcing frequency is equal to the second natural frequency, the receptance matrix is minimum at the middle of the beam and becomes maximum at positions $L/4$ and $3L/4$ as depicted in Figure 1(b). When the forcing frequency is equal to the third natural frequency, there are three peaks at $L/6$, $3L/6$, and $5L/6$ as illustrated in Figure 1(c). The distribution of response amplitude of the beam can be investigated clearer when the force position is fixed. Figure 2 illustrates the 2D graphs of receptance when the force position is at $0.58L$. As can be seen from this figure, the receptance curves are smooth except the minimum position. The positions of maxima and minima of these receptance graphs coincide with the positions of nodes of the corresponding mode shapes. These results imply that when the beam is excited at natural frequencies, the distribution of response amplitude along the beam can be predicted easily by using the receptance matrices as well as using the mode shapes.

However, when the forcing frequency is in between any two natural frequencies, the distribution of response

TABLE 1: Three natural frequencies of the intact beam.

Mode	Natural frequency (Rad/s)
1	146.4
2	585.7
3	1317.8

amplitude is more complicated. For example, as presented in Figure 3(a), when the forcing frequency is 400 Rad/s, between the first and the second frequencies, the receptance matrix has two main peaks in the middle area of the beam and two small peaks close to the ends. The shape of this receptance matrix suggests that it is a hybrid shape between the shapes of receptance of mode 1 and mode 2 where the two main peaks correspond to mode 1 and two other peaks correspond to mode 2. This means that in this case, mode 1 and mode 2 are both excited, but mode 1 is dominant. When the forcing frequency is 950 Rad/s, between the second and the third natural frequencies, the receptance matrix has six main peaks and two other small peaks as can be observed from Figure 3(b). In this case, modes 2 and 3 are both excited, but mode 3 is dominant. These results show that the distribution of response amplitude is complicated at arbitrary frequency and cannot be predicted using mode shapes, but it can be predicted easily by using the 3D graph of the receptance matrix.

3.1. Influence of the Crack on the Receptance. When there are cracks, the mode shapes will have changes at the crack positions leading to the changes in the receptance at the crack positions. These changes are dependent on the crack depths and the length-to-height ratio of the beam. Simulation results show that the changes in the receptance are small when the crack depth is small, and they only become significant when the crack depth is large. In our simulation, when the length-to-height ratio is fixed, the changes in the receptance are very difficult to be detected visually when the crack depth is smaller than 40% of the beam height. Figure 4 depicts the distortions of the receptance matrix at the first natural frequency when the crack depth is up to 50% of the beam height. As can be seen from this figure, the receptance matrix is distorted at the crack positions. The distorted positions coincide with the crack positions are at $0.4L$ and $0.76L$ as can be observed from Figure 4(b).

3.2. Influence of the Crack on the Receptance Curvature. In this section, in order to investigate the influence of the crack on the receptance curvature matrices, two cracks with the same depths are made at arbitrary positions of $0.4L$ and $0.76L$ from the left end of the beam. Three crack depth levels ranging from 10% to 30% have been applied. The receptance and receptance curvature matrices are calculated at 100 points spaced equally on the beam while the force moves along these points.

In order to amplify influence of the crack, power 4 of the receptance curvature is presented.

Figures 5–7 present the graphs of normalized receptance curvature matrices of the cracked beam with different levels

FIGURE 1: Receptance matrices of the intact beam: (a) $\omega = \omega_1$, (b) $\omega = \omega_2$, and (c) $\omega = \omega_3$.

FIGURE 2: Continued.

FIGURE 2: Receptance curves of the intact beam, $\hat{x} = 0.58(L)$: (a) $\omega = \omega_1$, (b) $\omega = \omega_2$, and (c) $\omega = \omega_3$.

FIGURE 3: Receptance matrices of the intact beam: (a) $\omega = 400$ Rad/s and (b) $\omega = 950$ Rad/s.

FIGURE 4: Receptance of the cracked beam. (a) Receptance matrix. (b) Receptance curve, $\hat{x} = 0.58(L)$.

FIGURE 5: Receptance curvature matrices of the cracked beam, crack depth = 10%: (a) $\omega = \omega_1$ and (b) $\omega = \omega_2$.

FIGURE 6: Receptance curvature matrices of the cracked beam, crack depth = 20%, $\hat{x} = 0.58(L)$: (a) $\omega = \omega_1$ and (b) $\omega = \omega_2$.

FIGURE 7: Receptance curvature matrices of the cracked beam, crack depth = 30%: (a) $\omega = \omega_1$ and (b) $\omega = \omega_2$.

FIGURE 8: Receptance curvatures of the cracked beam, crack depth = 10%, $\hat{x} = 0.58(L)$: (a) $\omega = \omega_1$ and (b) $\omega = \omega_2$.

of the crack depth when the forcing frequencies are at the first three natural frequencies, respectively. As can be seen from these figures, there are sharp peaks in the receptance curvature matrices at crack positions. In order to determine exactly the sharp peak positions, the receptance curvatures along the beam with a fixed force at the position of $0.58L$ are presented as shown in Figures 8–10. As can be observed from these figures, the positions of the sharp peaks are clearly detected at $0.4L$ and $0.76L$ which are the same with the crack positions. It is noted from Figures 5–10 that when the crack position is closer to the maxima of the receptance curvature matrices, the heights of sharp peaks are greater. When the crack position is far from the maxima of the receptance curvature, the heights of sharp peaks in receptance curvature are smaller. As can be observed from Figures 8–10, when the crack depth increases from 10% to 30%, the height of sharp peaks in receptance curvature increases and becomes clearer.

3.3. Influence of Forcing Frequency. When forcing frequency is equal to 146 Rad/s which is close to the first natural frequency, there are clear peaks at the crack position of $0.4L$ and $0.76L$ as presented in Figure 11(b). However, in higher-order modes, if the crack position coincides with a minimum of the receptance, the vibration at this position may be zero, so no vibration at this point can be measured; thus, the influence of the crack cannot be measured as well. If the crack position is close to the minimum of the receptance curvature, the influence of the crack will be small. Figure 11(a) presents the overview of the receptance curvature matrix obtained when the forcing frequency is 950 Rad/s. As can be observed from this figure, the peaks caused by the crack are disappeared at some minima of the receptance curvature. The receptance extracted when the force is fixed at $0.8L$ with the frequency of force 950 Rad/s is presented by the dotted line in Figure 11(b). The crack at $0.4L$ coincides with the minimum of the receptance curvature, so no peak at this position is presented. Meanwhile,

the crack at $0.76L$ causes a small peak at this position in the receptance curvature since this crack is close to another minimum of receptance curvature. Thus, the receptance curvature corresponding to the low forcing frequency is recommended for crack detection while the receptance curvatures corresponding to the high frequency are not recommended. This conclusion has not been reported in previous works yet.

3.4. Influence of the Boundary Conditions. The proposed method can be applied for different boundary conditions since this method is just used to detect the disruption in the receptance curvature. However, the influence of the cracks on the receptance curvature will be different with different boundary conditions. Figure 12 presents the receptance curvatures obtained at the first natural frequency of the beam with clamped-free and clamped-clamped conditions. As can be seen from this figure, two peaks at crack positions are presented in both cases. For the cantilever beam, the peak at $0.76L$ which is close to the free end of the beam is smaller than the peak at $0.4L$ which is close to the fixed end. For the clamped-clamped beam, the peak at $0.76L$ which is close to the minimum of the receptance curvature is smaller than the peak far from the minimum as mentioned above.

4. Experiment Validations

The experimental setup is depicted in Figure 13. The simply supported steel beam with the same parameters presented in Section 3 has been tested. Two saw-cut cracks perpendicular to the beam surface with the same depths ranging from 10% to 30% were made at $0.4L$ and $0.76L$. The widths of the cracks are 1.2 mm, and these cracks work as open cracks. The forcing frequencies are 147.65 Rad/s and 146 Rad/s which are the first natural frequencies measured from the intact beam and the cracked beam, respectively. The position of the exciter is selected at $0.58L$ which is close to the middle of the beam to excite the first vibration mode. This position is also

FIGURE 9: Receptance curvatures of the cracked beam, crack depth = 20%, $\hat{x} = 0.58(L)$: (a) $\omega = \omega_1$ and (b) $\omega = \omega_2$.

FIGURE 10: Receptance curvatures of the cracked beam, crack depth = 30%: (a) $\omega = \omega_1$ and (b) $\omega = \omega_2$.

FIGURE 11: Receptance curvature at different forcing frequencies, crack depth = 30%: solid line $\omega = \omega_1$ and dotted line $\omega = 950$ Rad/s: (a) receptance curvature matrix and (b) receptance curvature when $\hat{x} = 0.8(L)$.

FIGURE 12: Receptance curvature of the beam with different boundary conditions, crack depth = 30%, $\hat{x} = 0.58(L)$: (a) cantilever and (b) clamped-clamped.

FIGURE 13: Experimental setup.

the midpoint of two cracks to limit the influence of exciter on these cracks. The beam is excited by the vibration exciter Bruel & Kjaer 4808, and the response is measured by the polytec laser vibrometer PVD-100. The receptance is measured at 99 points spaced equally along the beam when the forcing frequency is set to the first natural frequency of the cracked beam. The receptance curvature is then calculated as the second derivative of the receptance.

It is noted that the excitation amplitude is important to obtain high-quality data and should be chosen suitably. If the excitation amplitude is too small, the vibration signal will be polluted by the measurement noise. If the amplitude is too large, the vibration signal will be nonlinear, so the proposed method cannot be applied since the theory in this paper is only applied for linear problems.

In order to reduce the presence of measurement noise and amplify the peaks at crack positions, the fourth power of the receptance curvature is presented instead of the receptance curvature. The reason is that by presenting the fourth power of the receptance curvature, the low values including noise in the receptance curvature will approach

zero earlier than the large values at crack positions. Figure 14 depicts the measured receptance and the fourth power of receptance curvature graphs measured at the first natural frequency with the crack depths ranging from 10% to 30%. As can be seen from this figure, the simulations and the experimental results are in very good agreement. The influence of the cracks on the receptance is small and cannot be detected visually. However, when the crack depth is as small as 10%, two peaks in the receptance curvature are presented at crack positions although it is polluted due to the measurement noise. When the crack depth is greater than 10%, the significant peaks are presented clearly at the positions of the cracks. These experimental results prove the correctness of the mathematical formulas and the numerical simulations presented in this work.

5. Conclusions

In this paper, exact formula of the simply supported cracked beam is presented. The graphs of receptance matrices are used to present the overall picture of the

FIGURE 14: Receptance and its curvature: (a)-(b): crack depth = 10%; (c)-(d): crack depth = 20%; (e)-(f): crack depth = 30%. (a) Receptance. (b) Fourth power of receptance curvature. (c) Receptance. (d) Fourth power of receptance curvature. (e) Receptance. (f) Fourth power of receptance curvature.

vibration amplitude distribution of the beam. When the forcing frequency is equal to the natural frequency, the vibration amplitude distribution is similar to the vibration mode shape. However, the vibration amplitude distribution is complicated when the forcing frequency is arbitrary.

Receptance curvature-based techniques have been applied recently for crack detection. However, no exact mathematical expression has been established so far. In this study, the exact formula of the receptance curvature of the cracked beam is established and applied to investigate the crack influence on the dynamic characteristics of the beam.

When there are cracks, the receptance curvature is changed significantly at the crack positions, and it can be used for crack detection. Only receptance curvatures corresponding to low forcing frequencies can be applied efficiently for crack detection while the receptance curvatures corresponding to the high frequencies are not recommended. This result has not been reported yet since no detailed investigation of receptance curvature graphs at different forcing frequencies has been carried out.

The experimental results and numerical simulations of receptance and its curvature are in very good agreement. This justifies that the formulas of the receptance and

receptance curvature derived in this paper are reliable. Again, the experimental results confirm that the receptance curvature can be applied for crack detection. In this experiment, the crack can be detected when the depth is as small as 10% of the beam height with measurement noise.

Appendix

Cantilever Beam

$$C_1 = -C_3 = 1;$$

$$C_2 = -C_4 = -\frac{(1/2\alpha_k)\sum_{i=1}^n \lambda_i (\mu_i - \zeta_i) [-\sin \alpha_k (1 - \xi_{0i}) + \sinh \alpha_k (1 - \xi_{0i})] - (\sin \alpha_k + \sinh \alpha_k)}{(1/2\alpha_k)\sum_{i=1}^n \lambda_i (\nu_i - \eta_i) [-\sin \alpha_k (1 - \xi_{0i}) + \sinh \alpha_k (1 - \xi_{0i})] - (\cos \alpha_k + \cosh \alpha_k)}, \quad (\text{A.1})$$

$$\begin{aligned} \phi_k(\xi) = & \left\{ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i \mu_i [\sin \alpha_k (\xi - \xi_{0i}) + \sinh \alpha_k (\xi - \xi_{0i})] U(\xi - \xi_{0i}) + \sin \alpha_k \xi \right\} \\ & + C_2 \left\{ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i \nu_i [\sin \alpha_k (\xi - \xi_{0i}) + \sinh \alpha_k (\xi - \xi_{0i})] U(\xi - \xi_{0i}) + \cos \alpha_k \xi \right\} \\ & - \left\{ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i \zeta_i [\sin \alpha_k (\xi - \xi_{0i}) + \sinh \alpha_k (\xi - \xi_{0i})] U(\xi - \xi_{0i}) + \sinh \alpha_k \xi \right\} \\ & - C_2 \left\{ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i \eta_i [\sin \alpha_k (\xi - \xi_{0i}) + \sinh \alpha_k (\xi - \xi_{0i})] U(\xi - \xi_{0i}) + \cosh \alpha_k \xi \right\}, \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \phi_k''(\xi) = & \left\{ \frac{1}{2} \sum_{i=1}^n \lambda_i \mu_i [\alpha_k (\sinh \alpha_k (\xi - \xi_{0i}) - \sin \alpha_k (\xi - \xi_{0i})) U(\xi - \xi_{0i}) + (\cos \alpha_k (\xi - \xi_{0i}) \right. \\ & \left. + \cosh \alpha_k (\xi - \xi_{0i})) \delta(\xi - \xi_{0i})] - \alpha_k^2 \sin \alpha_k \xi \right\} \\ & + C_2 \left\{ \frac{1}{2} \sum_{i=1}^n \lambda_i \nu_i [\alpha_k (\sinh \alpha_k (\xi - \xi_{0i}) - \sin \alpha_k (\xi - \xi_{0i})) U(\xi - \xi_{0i}) + (\cos \alpha_k (\xi - \xi_{0i}) \right. \\ & \left. + \cosh \alpha_k (\xi - \xi_{0i})) \delta(\xi - \xi_{0i})] - \alpha_k^2 \cos \alpha_k \xi \right\} \\ & - \left\{ \frac{1}{2} \sum_{i=1}^n \lambda_i \zeta_i [\alpha_k (\sinh \alpha_k (\xi - \xi_{0i}) - \sin \alpha_k (\xi - \xi_{0i})) U(\xi - \xi_{0i}) + (\cos \alpha_k (\xi - \xi_{0i}) \right. \\ & \left. + \cosh \alpha_k (\xi - \xi_{0i})) \delta(\xi - \xi_{0i})] \right\} + \alpha_k^2 \sinh \alpha_k \xi \\ & - C_2 \left\{ \frac{1}{2} \sum_{i=1}^n \lambda_i \eta_i [\alpha_k (\sinh \alpha_k (\xi - \xi_{0i}) - \sin \alpha_k (\xi - \xi_{0i})) U(\xi - \xi_{0i}) + (\cos \alpha_k (\xi - \xi_{0i}) \right. \\ & \left. + \cosh \alpha_k (\xi - \xi_{0i})) \delta(\xi - \xi_{0i})] + \alpha_k^2 \cosh \alpha_k \xi \right\}, \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} A_1 &= C_1^2 \mu_i \mu_j + \nu_i \nu_j + C_1^2 \zeta_i \zeta_j + \eta_i \eta_j + 2C_1 \mu_i \nu_j - 2C_1^2 \mu_i \zeta_j - 2C_1 \mu_i \eta_j - 2C_1 \nu_i \zeta_j - 2\nu_i \eta_j + 2C_1 \zeta_i \eta_j, \\ A_2 &= C_1^2 \mu_i + C_1 \nu_i - C_1^2 \zeta_i - C_1 \eta_i, \\ A_3 &= \nu_i + C_1 \mu_i - C_1 \zeta_i - \eta_i, \\ A_4 &= C_1^2 \zeta_i - C_1^2 \mu_i - C_1 \nu_i + C_1 \eta_i, \\ A_5 &= \eta_i - C_1 \mu_i - \nu_i + C_1 \zeta_i, \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned}
\int_0^1 \phi_k^2(\xi) d\xi &= \frac{1}{8\alpha_k^2} \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j A_1 \\
&\times \left\{ \cos \alpha_k (\xi_{0i} - \xi_{0j}) - \cosh \alpha_k (\xi_{0i} - \xi_{0j}) - \frac{1}{2\alpha_k} \sin \alpha_k (2 - \xi_{0i} - \xi_{0j}) + \frac{1}{\alpha_k} \sin \alpha_k (1 - \xi_{0i}) \cosh \alpha_k (1 - \xi_{0j}) \right. \\
&- \frac{1}{\alpha_k} \cos \alpha_k (1 - \xi_{0i}) \sinh \alpha_k (1 - \xi_{0j}) + \frac{1}{\alpha_k} \sin \alpha_k (1 - \xi_{0j}) \cosh \alpha_k (1 - \xi_{0i}) \\
&- \frac{1}{\alpha_k} \cos \alpha_k (1 - \xi_{0j}) \sinh \alpha_k (1 - \xi_{0i}) + \frac{1}{\alpha_k} \sinh \alpha_k (1 - \xi_{0i}) \cosh \alpha_k (1 - \xi_{0j}) \\
&- \left[\xi_{0i} \cos \alpha_k (\xi_{0i} - \xi_{0j}) - \xi_{0i} \cosh \alpha_k (\xi_{0i} - \xi_{0j}) + \frac{1}{2\alpha_k} \sin \alpha_k (\xi_{0i} - \xi_{0j}) - \frac{1}{\alpha_k} \sinh \alpha_k (\xi_{0i} - \xi_{0j}) \right] U(\xi_{0i} - \xi_{0j}) \\
&- \left[\xi_{0j} \cos \alpha_k (\xi_{0j} - \xi_{0i}) - \xi_{0j} \cosh \alpha_k (\xi_{0j} - \xi_{0i}) + \frac{1}{2\alpha_k} \sin \alpha_k (\xi_{0j} - \xi_{0i}) \right] U(\xi_{0j} - \xi_{0i}) \Big\} \\
&+ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i A_2 \left[(1 - \xi_{0i}) \cos \alpha_k \xi_{0i} - \frac{1}{2\alpha_k} \sin \alpha_k \xi_{0i} - \frac{1}{2\alpha_k} \sin \alpha_k (2 - \xi_{0i}) \right. \\
&+ \left. \frac{1}{\alpha_k} \sin \alpha_k \cosh \alpha_k (1 - \xi_{0i}) - \frac{1}{\alpha_k} \cos \alpha_k \sinh \alpha_k (1 - \xi_{0i}) \right] \\
&+ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i A_3 \left[(\xi_{0i} - 1) \sin \alpha_k \xi_{0i} - \frac{1}{2\alpha_k} \cos \alpha_k \xi_{0i} - \frac{1}{2\alpha_k} \cos \alpha_k (2 - \xi_{0i}) \right. \\
&+ \left. \frac{1}{\alpha_k} \sin \alpha_k \sinh \alpha_k (1 - \xi_{0i}) + \frac{1}{\alpha_k} \cos \alpha_k \cosh \alpha_k (1 - \xi_{0i}) \right] \\
&+ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i A_4 \left[(\xi_{0i} - 1) \cosh \alpha_k \xi_{0i} + \frac{1}{2\alpha_k} \sinh \alpha_k \xi_{0i} + \frac{1}{2\alpha_k} \sinh \alpha_k (2 - \xi_{0i}) \right. \\
&+ \left. \frac{1}{\alpha_k} \sin \alpha_k (1 - \xi_{0i}) \cosh \alpha_k - \frac{1}{\alpha_k} \cos \alpha_k (1 - \xi_{0i}) \sinh \alpha_k \right] \\
&+ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i A_5 \left[(\xi_{0i} - 1) \sinh \alpha_k \xi_{0i} + \frac{1}{2\alpha_k} \cosh \alpha_k \xi_{0i} + \frac{1}{2\alpha_k} \cosh \alpha_k (2 - \xi_{0i}) \right. \\
&+ \left. \frac{1}{\alpha_k} \sin \alpha_k (1 - \xi_{0i}) \sinh \alpha_k - \frac{1}{\alpha_k} \cos \alpha_k (1 - \xi_{0i}) \cosh \alpha_k \right] \\
&+ \frac{C_1}{\alpha_k} \sin^2 \alpha_k - \frac{C_1}{\alpha_k} \sinh^2 \alpha_k - \frac{C_1^2 + 1}{\alpha_k} \sin \alpha_k \cosh \alpha_k \\
&+ \frac{C_1^2 - 1}{\alpha_k} \cos \alpha_k \sinh \alpha_k - \frac{2C_1}{\alpha_k} \sin \alpha_k \sinh \alpha_k + \frac{1 - C_1^2}{4\alpha_k} \sin 2\alpha_k + \frac{C_1^2 + 1}{4\alpha_k} \sinh 2\alpha_k + 1.
\end{aligned}
\tag{A.5}$$

Clamped-Clamped Beam

$$\begin{aligned}
 C_1 &= -\frac{(1/2\alpha_k) \sum_{i=1}^n \lambda_i (\nu_i - \eta_i) [\sin \alpha_k (1 - \xi_{oi}) + \sinh \alpha_k (1 - \xi_{oi})] + \cos \alpha_k - \cosh \alpha_k}{(1/2\alpha_k) \sum_{i=1}^n \lambda_i (\mu_i - \zeta_i) [\sin \alpha_k (1 - \xi_{oi}) + \sinh \alpha_k (1 - \xi_{oi})] + \sin \alpha_k - \sinh \alpha_k}, \\
 C_2 &= -C_4 = 1; \\
 C_3 &= -C_1,
 \end{aligned} \tag{A.6}$$

$$\begin{aligned}
 \phi_k(\xi) &= C_1 \left\{ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i \mu_i [\sin \alpha_k (\xi - \xi_{oi}) + \sinh \alpha_k (\xi - \xi_{oi})] U(\xi - \xi_{oi}) + \sin \alpha_k \xi \right\} \\
 &\quad + \left\{ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i \nu_i [\sin \alpha_k (\xi - \xi_{oi}) + \sinh \alpha_k (\xi - \xi_{oi})] U(\xi - \xi_{oi}) + \cos \alpha_k \xi \right\} \\
 &\quad - C_1 \left\{ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i \zeta_i [\sin \alpha_k (\xi - \xi_{oi}) + \sinh \alpha_k (\xi - \xi_{oi})] U(\xi - \xi_{oi}) + \sinh \alpha_k \xi \right\} \\
 &\quad - \left\{ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i \eta_i [\sin \alpha_k (\xi - \xi_{oi}) + \sinh \alpha_k (\xi - \xi_{oi})] U(\xi - \xi_{oi}) + \cosh \alpha_k \xi \right\},
 \end{aligned} \tag{A.7}$$

$$\begin{aligned}
 \phi_k''(\xi) &= C_1 \left\{ \frac{1}{2} \sum_{i=1}^n \lambda_i \mu_i [\alpha_k (\sinh \alpha_k (\xi - \xi_{oi}) - \sin \alpha_k (\xi - \xi_{oi})) U(\xi - \xi_{oi}) + (\cos \alpha_k (\xi - \xi_{oi}) \right. \\
 &\quad \left. + \cosh \alpha_k (\xi - \xi_{oi})) \delta(\xi - \xi_{oi})] - \alpha_k^2 \sin \alpha_k \xi \right\} \\
 &\quad + \left\{ \frac{1}{2} \sum_{i=1}^n \lambda_i \nu_i [\alpha_k (\sinh \alpha_k (\xi - \xi_{oi}) - \sin \alpha_k (\xi - \xi_{oi})) U(\xi - \xi_{oi}) + (\cos \alpha_k (\xi - \xi_{oi}) \right. \\
 &\quad \left. + \cosh \alpha_k (\xi - \xi_{oi})) \delta(\xi - \xi_{oi}) - \alpha_k^2 \cos \alpha_k \xi] \right\} \\
 &\quad - C_1 \left\{ \frac{1}{2} \sum_{i=1}^n \lambda_i \zeta_i [\alpha_k (\sinh \alpha_k (\xi - \xi_{oi}) - \sin \alpha_k (\xi - \xi_{oi})) U(\xi - \xi_{oi}) + (\cos \alpha_k (\xi - \xi_{oi}) \right. \\
 &\quad \left. + \cosh \alpha_k (\xi - \xi_{oi})) \delta(\xi - \xi_{oi})] + \alpha_k^2 \sinh \alpha_k \xi \right\} \\
 &\quad - \left\{ \frac{1}{2} \sum_{i=1}^n \lambda_i \eta_i [\alpha_k (\sinh \alpha_k (\xi - \xi_{oi}) - \sin \alpha_k (\xi - \xi_{oi})) U(\xi - \xi_{oi}) + (\cos \alpha_k (\xi - \xi_{oi}) \right. \\
 &\quad \left. + \cosh \alpha_k (\xi - \xi_{oi})) \delta(\xi - \xi_{oi})] + \alpha_k^2 \cosh \alpha_k \xi \right\},
 \end{aligned} \tag{A.8}$$

$$\begin{aligned}
 A_1 &= C_1^2 \mu_i \mu_j + \nu_i \nu_j + C_1^2 \zeta_i \zeta_j + \eta_i \eta_j + 2C_1 \mu_i \nu_j - 2C_1^2 \mu_i \zeta_j - 2C_1 \mu_i \eta_j - 2C_1 \nu_i \zeta_j - 2\nu_i \eta_j + 2C_1 \zeta_i \eta_j, \\
 A_2 &= C_1^2 \mu_i + C_1 \nu_i - C_1^2 \zeta_i - C_1 \eta_i, \\
 A_3 &= C_1 \mu_i + \nu_i - C_1 \zeta_i - \eta_i, \\
 A_4 &= -C_1^2 \mu_i - C_1 \nu_i + C_1^2 \zeta_i + C_1 \eta_i, \\
 A_5 &= -C_1 \mu_i - \nu_i + C_1 \zeta_i + \eta_i,
 \end{aligned}$$

(A.9)

$$\begin{aligned}
\int_0^1 \phi_k^2(\xi) d\xi &= \frac{1}{8\alpha_k^2} \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j A_1 \\
&\times \left\{ \cos \alpha_k (\xi_{0i} - \xi_{0j}) - \frac{1}{2\alpha_k} \sin \alpha_k (2 - \xi_{0i} - \xi_{0j}) - \cosh \alpha_k (\xi_{0i} - \xi_{0j}) + \frac{1}{\alpha_k} \sinh \alpha_k (1 - \xi_{0i}) \cosh \alpha_k (1 - \xi_{0j}) \right. \\
&- \frac{1}{\alpha_k} \cos \alpha_k (1 - \xi_{0i}) \sinh \alpha_k (1 - \xi_{0j}) + \frac{1}{\alpha_k} \sin \alpha_k (1 - \xi_{0i}) \cosh \alpha_k (1 - \xi_{0j}) \\
&- \frac{1}{\alpha_k} \cos \alpha_k (1 - \xi_{0j}) \sinh \alpha_k (1 - \xi_{0i}) + \frac{1}{\alpha_k} \sin \alpha_k (1 - \xi_{0j}) \cosh \alpha_k (1 - \xi_{0i}) \\
&- \left[\xi_{0i} \cos \alpha_k (\xi_{0i} - \xi_{0j}) + \frac{1}{2\alpha_k} \sin \alpha_k (\xi_{0i} - \xi_{0j}) - \xi_{0i} \cosh \alpha_k (\xi_{0i} - \xi_{0j}) - \frac{1}{\alpha_k} \sinh \alpha_k (\xi_{0i} - \xi_{0j}) \right] U(\xi_{0i} - \xi_{0j}) \\
&- \left. \left[\xi_{0j} \cos \alpha_k (\xi_{0j} - \xi_{0i}) + \frac{1}{2\alpha_k} \sin \alpha_k (\xi_{0j} - \xi_{0i}) - \xi_{0j} \cosh \alpha_k (\xi_{0j} - \xi_{0i}) \right] U(\xi_{0j} - \xi_{0i}) \right\}, \\
&+ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i A_2 \left[(1 - \xi_{0i}) \cos \alpha_k \xi_{0i} - \frac{1}{2\alpha_k} \sin \alpha_k \xi_{0i} - \frac{1}{2\alpha_k} \sin \alpha_k (2 - \xi_{0i}) + \frac{1}{\alpha_k} \sin \alpha_k \cosh \alpha_k (1 - \xi_{0i}) \right. \\
&- \left. \frac{1}{\alpha_k} \cos \alpha_k \sinh \alpha_k (1 - \xi_{0i}) \right] \\
&+ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i A_3 \left[(\xi_{0i} - 1) \sin \alpha_k \xi_{0i} - \frac{1}{2\alpha_k} \cos \alpha_k \xi_{0i} - \frac{1}{2\alpha_k} \cos \alpha_k (2 - \xi_{0i}) + \frac{1}{\alpha_k} \sin \alpha_k \sinh \alpha_k (1 - \xi_{0i}) \right. \\
&+ \left. \frac{1}{\alpha_k} \cos \alpha_k \cosh \alpha_k (1 - \xi_{0i}) \right] \\
&+ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i A_4 \left[(\xi_{0i} - 1) \cosh \alpha_k \xi_{0i} + \frac{1}{2\alpha_k} \sinh \alpha_k \xi_{0i} + \frac{1}{2\alpha_k} \sinh \alpha_k (2 - \xi_{0i}) + \frac{1}{\alpha_k} \sin \alpha_k (1 - \xi_{0i}) \cosh \alpha_k \right. \\
&- \left. \frac{1}{\alpha_k} \cos \alpha_k (1 - \xi_{0i}) \sinh \alpha_k \right] \\
&+ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i A_5 \left[(\xi_{0i} - 1) \sinh \alpha_k \xi_{0i} + \frac{1}{2\alpha_k} \cosh \alpha_k \xi_{0i} + \frac{1}{2\alpha_k} \cosh \alpha_k (2 - \xi_{0i}) + \frac{1}{\alpha_k} \sin \alpha_k (1 - \xi_{0i}) \sinh \alpha_k \right. \\
&- \left. \frac{1}{\alpha_k} \cos \alpha_k (1 - \xi_{0i}) \cosh \alpha_k \right] \\
&+ \left[1 + \frac{1 - C_1^2}{4\alpha_k} \sin 2\alpha_k + \frac{C_1^2 + 1}{4\alpha_k} \sinh 2\alpha_k + \frac{C_1}{\alpha_k} \sin^2 \alpha_k + \frac{C_1}{\alpha_k} \sinh^2 \alpha_k - \frac{C_1^2 + 1}{\alpha_k} \sin \alpha_k \cosh \alpha_k + \frac{C_1^2 - 1}{\alpha_k} \cos \alpha_k \sinh \alpha_k \right. \\
&- \left. \frac{2C_1}{\alpha_k} \sin \alpha_k \sinh \alpha_k \right].
\end{aligned} \tag{A.10}$$

Free-Free Beam

$$C_1 = C_3 = 1;$$

$$C_2 = C_4 = -\frac{(1/2\alpha_k) \sum_{i=1}^n \lambda_i (\mu_i + \zeta_i) [-\sin \alpha_k (1 - \xi_{0i}) + \sinh \alpha_k (1 - \xi_{0i})] - (\sin \alpha_k - \sinh \alpha_k)}{(1/2\alpha_k) \sum_{i=1}^n \lambda_i (\nu_i + \eta_i) [-\sin \alpha_k (1 - \xi_{0i}) + \sinh \alpha_k (1 - \xi_{0i})] - (\cos \alpha_k - \cosh \alpha_k)}, \quad (\text{A.11})$$

$$\begin{aligned} \phi_k(\xi) = & \left\{ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i \mu_i [\sin \alpha_k (\xi - \xi_{0i}) + \sinh \alpha_k (\xi - \xi_{0i})] U(\xi - \xi_{0i}) + \sin \alpha_k \xi \right\} \\ & + C_2 \left\{ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i \nu_i [\sin \alpha_k (\xi - \xi_{0i}) + \sinh \alpha_k (\xi - \xi_{0i})] U(\xi - \xi_{0i}) + \cos \alpha_k \xi \right\} \\ & + \left\{ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i \zeta_i [\sin \alpha_k (\xi - \xi_{0i}) + \sinh \alpha_k (\xi - \xi_{0i})] U(\xi - \xi_{0i}) + \sinh \alpha_k \xi \right\} \\ & + C_2 \left\{ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i \eta_i [\sin \alpha_k (\xi - \xi_{0i}) + \sinh \alpha_k (\xi - \xi_{0i})] U(\xi - \xi_{0i}) + \cosh \alpha_k \xi \right\}, \end{aligned} \quad (\text{A.12})$$

$$\begin{aligned} \phi_k''(\xi) = & \left\{ \frac{1}{2} \sum_{i=1}^n \lambda_i \mu_i [\alpha_k (\sinh \alpha_k (\xi - \xi_{0i}) - \sin \alpha_k (\xi - \xi_{0i})) U(\xi - \xi_{0i}) + (\cos \alpha_k (\xi - \xi_{0i}) \right. \\ & \left. + \cosh \alpha_k (\xi - \xi_{0i})) \delta(\xi - \xi_{0i})] - \alpha_k^2 \sin \alpha_k \xi \right\} \\ & + C_2 \left\{ \frac{1}{2} \sum_{i=1}^n \lambda_i \nu_i [\alpha_k (\sinh \alpha_k (\xi - \xi_{0i}) - \sin \alpha_k (\xi - \xi_{0i})) U(\xi - \xi_{0i}) + (\cos \alpha_k (\xi - \xi_{0i}) \right. \\ & \left. + \cosh \alpha_k (\xi - \xi_{0i})) \delta(\xi - \xi_{0i})] - \alpha_k^2 \cos \alpha_k \xi \right\} \\ & + \left\{ \frac{1}{2} \sum_{i=1}^n \lambda_i \zeta_i [\alpha_k (\sinh \alpha_k (\xi - \xi_{0i}) - \sin \alpha_k (\xi - \xi_{0i})) U(\xi - \xi_{0i}) + (\cos \alpha_k (\xi - \xi_{0i}) \right. \\ & \left. + \cosh \alpha_k (\xi - \xi_{0i})) \delta(\xi - \xi_{0i})] + \alpha_k^2 \sinh \alpha_k \xi \right\} \\ & + C_2 \left\{ \frac{1}{2} \sum_{i=1}^n \lambda_i \eta_i [\alpha_k (\sinh \alpha_k (\xi - \xi_{0i}) - \sin \alpha_k (\xi - \xi_{0i})) U(\xi - \xi_{0i}) + (\cos \alpha_k (\xi - \xi_{0i}) \right. \\ & \left. + \cosh \alpha_k (\xi - \xi_{0i})) \delta(\xi - \xi_{0i})] + \alpha_k^2 \cosh \alpha_k \xi \right\}, \end{aligned} \quad (\text{A.13})$$

$$A_1 = \mu_i \mu_j + C_2^2 \nu_i \nu_j + \zeta_i \zeta_j + C_2^2 \eta_i \eta_j + 2C_2 \mu_i \nu_j + 2\mu_i \zeta_j + 2C_2 \mu_i \eta_j + 2C_2 \nu_i \zeta_j + 2C_2^2 \nu_i \eta_j + 2C_2 \zeta_i \eta_j,$$

$$A_2 = \mu_i + C_2 \nu_i + \zeta_i + C_2 \eta_i,$$

$$A_3 = C_2 \mu_i + C_2^2 \nu_i + C_2 \zeta_i + C_2^2 \eta_i,$$

$$A_4 = \mu_i + C_2 \nu_i + \zeta_i + C_2 \eta_i,$$

$$A_5 = C_2 \mu_i + C_2^2 \nu_i + C_2 \zeta_i + C_2^2 \eta_i,$$

(A.14)

$$\begin{aligned}
\int_0^1 \phi_k^2(\xi) d\xi &= \frac{1}{8\alpha_k^2} \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j A_1 \\
&\times \left\{ \cos \alpha_k (\xi_{0i} - \xi_{0j}) - \cosh \alpha_k (\xi_{0i} - \xi_{0j}) - \frac{1}{2\alpha_k} \sin \alpha_k (2 - \xi_{0i} - \xi_{0j}) + \frac{1}{\alpha_k} \sin \alpha_k (1 - \xi_{0i}) \cosh \alpha_k (1 - \xi_{0j}) \right. \\
&+ \frac{1}{\alpha_k} \sin \alpha_k (1 - \xi_{0j}) \cosh \alpha_k (1 - \xi_{0i}) - \frac{1}{\alpha_k} \cos \alpha_k (1 - \xi_{0i}) \sinh \alpha_k (1 - \xi_{0j}) \\
&- \frac{1}{\alpha_k} \cos \alpha_k (1 - \xi_{0j}) \sinh \alpha_k (1 - \xi_{0i}) + \frac{1}{\alpha_k} \cosh \alpha_k (1 - \xi_{0j}) \sinh \alpha_k (1 - \xi_{0i}) \\
&- \left[\xi_{0i} \cos \alpha_k (\xi_{0i} - \xi_{0j}) - \xi_{0i} \cosh \alpha_k (\xi_{0i} - \xi_{0j}) + \frac{1}{2\alpha_k} \sin \alpha_k (\xi_{0i} - \xi_{0j}) - \frac{1}{\alpha_k} \sinh \alpha_k (\xi_{0i} - \xi_{0j}) \right] U(\xi_{0i} - \xi_{0j}) \\
&- \left[\xi_{0j} \cos \alpha_k (\xi_{0j} - \xi_{0i}) - \xi_{0j} \cosh \alpha_k (\xi_{0j} - \xi_{0i}) + \frac{1}{2\alpha_k} \sin \alpha_k (\xi_{0j} - \xi_{0i}) \right] U(\xi_{0j} - \xi_{0i}) \} \\
&+ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i A_2 \left[(1 - \xi_{0i}) \cos \alpha_k \xi_{0i} - \frac{1}{2\alpha_k} \sin \alpha_k \xi_{0i} - \frac{1}{2\alpha_k} \sin \alpha_k (2 - \xi_{0i}) + \frac{1}{\alpha_k} \sin \alpha_k \cosh \alpha_k (1 - \xi_{0i}) \right. \\
&- \left. \frac{1}{\alpha_k} \cos \alpha_k \sinh \alpha_k (1 - \xi_{0i}) \right] \\
&+ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i A_3 \left[(\xi_{0i} - 1) \sin \alpha_k \xi_{0i} - \frac{1}{2\alpha_k} \cos \alpha_k \xi_{0i} - \frac{1}{2\alpha_k} \cos \alpha_k (2 - \xi_{0i}) + \frac{1}{\alpha_k} \sin \alpha_k \sinh \alpha_k (1 - \xi_{0i}) \right. \\
&+ \left. \frac{1}{\alpha_k} \cos \alpha_k \cosh \alpha_k (1 - \xi_{0i}) \right] \\
&+ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i A_4 \left[(\xi_{0i} - 1) \cosh \alpha_k \xi_{0i} + \frac{1}{2\alpha_k} \sinh \alpha_k \xi_{0i} + \frac{1}{2\alpha_k} \sinh \alpha_k (2 - \xi_{0i}) + \frac{1}{\alpha_k} \sin \alpha_k (1 - \xi_{0i}) \cosh \alpha_k \right. \\
&- \left. \frac{1}{\alpha_k} \cos \alpha_k (1 - \xi_{0i}) \sinh \alpha_k \right] \\
&+ \frac{1}{2\alpha_k} \sum_{i=1}^n \lambda_i A_5 \left[(\xi_{0i} - 1) \sinh \alpha_k \xi_{0i} + \frac{1}{2\alpha_k} \cosh \alpha_k \xi_{0i} + \frac{1}{2\alpha_k} \cosh \alpha_k (2 - \xi_{0i}) + \frac{1}{\alpha_k} \sin \alpha_k (1 - \xi_{0i}) \sinh \alpha_k \right. \\
&- \left. \frac{1}{\alpha_k} \cos \alpha_k (1 - \xi_{0i}) \cosh \alpha_k \right] \\
&+ \left[C_2^2 + \frac{C_2^2 - 1}{4\alpha_k} \sin 2\alpha_k + \frac{C_2^2 + 1}{4\alpha_k} \sinh 2\alpha_k + \frac{C_2}{\alpha_k} \sin^2 \alpha_k + \frac{C_2}{\alpha_k} \sinh^2 \alpha_k + \frac{C_2^2 + 1}{\alpha_k} \sin \alpha_k \cosh \alpha_k \right. \\
&+ \left. \frac{C_2^2 - 1}{\alpha_k} \cos \alpha_k \sinh \alpha_k + \frac{2C_2}{\alpha_k} \sin \alpha_k \sinh \alpha_k \right].
\end{aligned} \tag{A.15}$$

Data Availability

All data included in this study are available upon request by contacting the corresponding author.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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