Research Article

Nonlinear Vibrations of a Rotor-Active Magnetic Bearing System with 16-Pole Legs and Two Degrees of Freedom

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The asymptotic perturbation method is used to analyze the nonlinear vibrations and chaotic dynamics of a rotor-active magnetic bearing (AMB) system with 16-pole legs and the time-varying stiffness. Based on the expressions of the electromagnetic force resultants, the influences of some parameters, such as the cross-sectional area A_a of one electromagnet and the number N of windings in each electromagnet coil, on the electromagnetic force resultants are considered for the rotor-AMB system with 16-pole legs. Based on the Newton law, the governing equation of motion for the rotor-AMB system with 16-pole legs is obtained and expressed as a two-degree-of-freedom system with the parametric excitation and the quadratic and cubic nonlinearities. According to the asymptotic perturbation method, the four-dimensional averaged equation of the rotor-AMB system is derived under the case of 1:1 internal resonance and 1:2 subharmonic resonances. Then, the frequency-response curves are employed to study the steady-state solutions of the modal amplitudes. From the analysis of the frequency responses, both the hardening-type nonlinearity and the softening-type nonlinearity are observed in the rotor-AMB system. Based on the numerical solutions of the averaged equation, the changed procedure of the nonlinear dynamic behaviors of the rotor-AMB system with the control parameter is described by the bifurcation diagram. From the numerical simulations, the periodic, quasiperiodic, and chaotic motions are observed in the rotor-active magnetic bearing (AMB) system with 16-pole legs, the time-varying stiffness, and the quadratic and cubic nonlinearities.

1. Introduction

As a kind of new support device, active magnetic bearings (AMBs) can realize the active control and have many advantages, including little friction, absent lubrication, high rotatory speed, extreme working conditions, and better dynamic characteristics. Because the active magnetic bearings provide higher bearing surface speeds, the rotor can obtain greater dynamic stiffness and stability [1]. Therefore, many engineering fields are using active magnetic bearings, for example, machine tools and manufacture, mechanical engineering, and aeronautic and astronautic engineering. However, the current active magnetic bearings can only provide the small support force. Thus, the application ranges are restricted for the support of high rotatory speed for the rotor. We propose a rotor-AMB system with 16-pole legs and the time-varying stiffness [2] to replace a rotor-AMB

system with 8-pole legs. The rotor-AMB system with 16-pole legs and the time-varying stiffness has a better bearing capacity than one of the rotor-AMB systems with 8-pole legs.

Researches on the nonlinear vibrations under the case of the resonance for multi-degree-of-freedom nonlinear systems with the quadratic and cubic nonlinearities have been at the forefront and an important theoretical problem. In the last century, various asymptotic perturbation methods, such as the averaging method, the KBM method, the method of multiple scales, and the harmonic balance method [3–5], have been presented and widely used to construct the approximate solutions of the single- or multi-degree-of-freedom weakly nonlinear dynamical systems. In general situations, in order to obtain better qualitative and quantitative vibratory characteristics of the nonlinear dynamical systems including the quadratic and cubic nonlinearities simultaneously, the high-order averaging

method or high-order method of multiple scales must be utilized to obtain the averaged equations. The asymptotic perturbation method was firstly proposed by Maccari [6–9] to study the nonlinear vibrations of two-degree-of-freedom nonlinear dynamic systems with the quadratic and cubic nonlinearities. The asymptotic perturbation method is used to transform two-degree-of-freedom nonlinear dynamic systems into a four-dimensional nonlinear averaged equation governing the amplitudes and phases of the approximation solutions. In a certain sense, the asymptotic perturbation method can be regarded as an attempt to link the most useful characteristics of the harmonic balance method and the method of multiple scales. It is noticed that the asymptotic perturbation method can be extended to two-degree-of-freedom nonlinear dynamic systems under the case of 1 : 1 internal resonance because the approximate solutions of two modes have the same form. Therefore, the asymptotic perturbation method is a better technique to research the nonlinear vibrations of two-degree-of-freedom nonlinear dynamic systems with the quadratic and cubic nonlinearities.

The magnetic bearing becomes a hot topic of research area during these years. Many researchers draw their attentions to the relevant aspects, such as the dynamic behaviors, vibration, stability, unbalance responses, and parametric identification. Some excellent pertinent works have been done for concerning the magnetic bearing system. Ji et al. [10] utilized the normal form method to investigate the nonlinear vibrations in the horizontal and vertical directions of a rotor suspended by active magnetic bearings. Zhang and Zhan [11] studied the periodic and chaotic motions of a rotor-AMB system with 8-pole legs and the time-varying stiffness. Kumar et al. [12] investigated the effect of the vibration attenuation for a 12-pole radial AMB in the case of the rotor unbalance. Kluykens et al. [13, 14] proposed a model of the magnetic bearing to analyze its dynamic behaviors. The parameters of the model were identified by using the finite element method. Zhou et al. [15, 16] gave a method for identifying the closed-loop AMB stiffness and damping coefficients based on the rotor unbalance responses. Ebrahimi et al. [17] gave the chaotic vibration analysis of a coaxial rotor system in the active magnetic bearings and contact with the auxiliary bearings. Zhou et al. [18] gave a research on the steady-state characteristics of the centrifugal pump rotor system with the weak nonlinear stiffness.

The typical components of the rotor-AMB system are the position sensors, power amplifiers, magnetic bearings, a feedback controller, and a rotor. Since the most of the components in the AMB system are of the nonlinear characteristics, the dynamics of the rotor-AMB system are very complicated. The inherent nonlinear dynamic properties of the rotor-AMB system may cause the large amplitude oscillations of the rotor in some parametric regions. Thus, analyzing the nonlinear dynamics of the rotor-AMB system plays an important role in engineering fields. There were a number of literature studies to study the nonlinear vibrations of the magnetic bearing systems. Ji [19, 20] presented the theoretical and

experimental researches for the nonlinear dynamics of a Jeffcott rotor-magnetic bearing system with the time delays. In addition, Ji and Leung [21] also studied the nonlinear oscillations of a rotor-magnetic bearing system subjected to a superharmonic resonance. Eissa et al. [22, 23] analyzed the nonlinear dynamic behaviors of a rotor-AMB system with the time-varying stiffness under combined excitations. Furthermore, Amer and Hegazy [24] investigated the nonlinear dynamic behaviors of a rotor-AMB system subjected to a periodically time-varying stiffness under the simultaneous primary resonance. Kamel and Bauomy [25] employed the method of multiple scales to analyze the nonlinear vibrations of a rotor-AMB system with the multi-parametric excitations. Zhang et al. [26, 27] investigated the transient and steady nonlinear dynamic responses and the global bifurcations and chaos of a rotor-active magnetic bearing system with the time-varying stiffness. Yang et al. [28] investigated the nonlinear vibrations of the rotor-AMB system with 8-pole pairs and found three types of motions in two-degree-of-freedom nonlinear dynamic systems.

Some researchers contributed to the nonlinear dynamic phenomena of the rotor-AMB systems. Zhang et al. [29] discussed the Shilnikov-type multipulse jumping chaotic dynamics of a rotor-AMB system with the quadratic and cubic nonlinear terms. Li et al. [30, 31] found that at least 17, 19, 21, and 22 limit cycles exist, respectively, in a rotor-AMB system under different controlling conditions. Awrejcewicz and Dzyubak [32] investigated the chaos and saturation phenomena of the vibrations for a rotor supported by the magnetohydrodynamic bearings. Saeed et al. [33] applied the method of multiple scales to construct an analytical approximate solution of a rotor-AMB system subjected to the primary resonance and 1 : 1 internal resonance. Inayat-Hussain [34] discussed the geometric coupling effect on the bifurcations of a rotor response in the active magnetic bearings using numerical methods. Saeed and El-Ganaini [35] investigated the time-delayed control to suppress the nonlinear vibrations of a horizontally suspended Jeffcott rotor system. Ghazavi and Sun [36] studied the bifurcation onset delay in the magnetic bearing systems by the time-varying stiffness. Ebrahimi et al. [37] researched the effects of some design parameters on the bifurcation behaviors of a magnetically supported coaxial rotor in the auxiliary bearings.

As an electromechanical coupling dynamic system, the magnetic bearing system is meeting a variety of engineering problems and factors which may affect its working performance. In addition to the aforementioned literature studies, other projects have been done by some scholars. Messaoud et al. [38] proposed a theoretical model of two identical AMBs and studied the impacts of angular misalignment on the dynamic behavior of a misaligned rotor. Bouaziz et al. [39] used the numerical method to study the dynamic behaviors of a misaligned rotor supported by AMBs and the influence of angular misalignment on the transient responses in the system. Halminen et al. [40] utilized a numerical model to discuss

the necessity of the backup bearings when a failure occurs in the rotor-AMB system. Shelke [41] proposed a control design methodology for the radial magnetic bearing with four-pole pairs and indicated the optimal range of the air gap. Hutterer et al. [42] presented a new method for the magnetic bearing system to maintain better dynamic behaviors by using a self-sensing structure. Saeed and Kamel [43] gave an active magnetic bearing-based tuned controller to suppress lateral vibrations of a nonlinear Jeffcott rotor system. Wojna et al. [44] gave the numerical and experimental study of a double physical pendulum with the magnetic interaction. Sun et al. [45] investigated the nonlinear dynamic characteristics of the active magnetic bearing system based on the cell-mapping method with a case study.

As the aforementioned researches, there are many investigations focusing on the nonlinear dynamics of the rotor-AMB systems with 8-pole legs, but few researchers make contribution to a rotor-AMB system with 16-pole legs, the time-varying stiffness, and the quadratic and cubic nonlinearities. The number of poles in a rotor-AMB system plays an important role in the bearing force and heat dissipation for different parameters in the governing equations of motion. This paper investigates the nonlinear and chaotic dynamics of a rotor-AMB system with 16-pole legs, the time-varying stiffness, and the quadratic and cubic nonlinearities by using the asymptotic perturbation method [6–9]. Based on the expressions of the electromagnetic force resultants, the influences of some parameters, such as the cross-sectional area A_α of one electromagnet and the number N of windings in each electromagnet coil, on the electromagnetic force resultants are considered for the rotor-AMB system with 16-pole legs and the time-varying stiffness. The motion of the rotor-AMB system is modeled by two-degree-of-freedom nonlinear ordinary differential governing equations with the quadratic and cubic nonlinearities and the parametric excitation. According to the asymptotic perturbation method, the four-dimensional averaged equation of the two-degree-of-freedom nonlinear system is obtained near the 1:1 internal resonance and 1:2 subharmonic resonances. From the analysis of the frequency responses, both the hardening-type nonlinearity and the softening-type nonlinearity are observed in the rotor-AMB system with 16-pole legs and the time-varying stiffness. The numerical results of the averaged equation under a specific set of parameters demonstrate that the modulations of the amplitudes can be changed by the parameter b_{14} closely related to the eccentricity of the rotor in the AMB system with 16-pole legs. The periodic, quasiperiodic, and chaotic motions on a slow time scale are presented by phase portraits and waveforms. From the numerical simulation, it can be concluded that the parameter b_{14} related to the eccentricity of the rotor in the AMB system has significant influences on the vibration states of the rotor-AMB system with 16-pole legs, and the influences should be considered in the structural design and optimization.

2. Equations of Motion and Energy

In this paper, the rotor-AMB system is a uniform, symmetric rigid rotor suspended by two radial AMBs at its both ends. It is assumed that each AMB has a stator of 16-pole legs and identical structure. A cross-sectional diagram of the rotor-AMB system is shown in Figure 1. For simplicity, the magnetic flux leakage, the eddy current loss, the fringe magnetic flux, the saturation and hysteresis of the magnetic core material, and the coupling effects between the electromagnets are neglected.

According to the electromagnetic theory, the electromagnetic force produced by the i th opposed pair of electromagnet coils can be expressed as [46]

$$F_i(I_i, w_i) = \frac{\mu_0 A_\alpha N^2}{4} \left[\frac{(I_0 + I_i)^2}{(C_0 + w_i)^2} - \frac{(I_0 - I_i)^2}{(C_0 - w_i)^2} \right] \cos \theta, \quad (1)$$

$$I_1 = i_1, I_i = i_0 + i_i, i = 2, \dots, 8,$$

where μ_0 is the permeability of a vacuum, A_α is the cross-sectional area of one electromagnet, N is the number of windings in each electromagnet coil, θ is the corresponding half-angle of the radial electromagnetic circuit, C_0 is the air gap between the stator and the shaft, I_0 represents the bias current which plays a role of producing electromagnetic field, I_i is the control current in the i direction, i_0 is the static component of the control current which produces a force to balance the weight of the rotor, i_i denotes the feedback component of the control current, and w_i represents the displacement of the rotor in the i direction.

Considering the proportional-derivative (PD) controller in the rotor-AMB system, we let

$$i_i = K(w) = k_1 w + k_2 \dot{w}, \quad i = 1, 2, \dots, 8, \quad (2)$$

where $\dot{w}$ is the velocity in the w direction, k_1 is the proportional gain, and k_2 denotes the derivative gain.

In order to increase the stability and controllability, we take into account the influence of the time-varying stiffness on the nonlinear dynamics of the rotor-AMB system. The PD controller can be physically referred to as a spring-damper element [46]. For studying the nonlinear dynamic behaviors of the rotor-AMB system in detail, it is assumed that the proportional gain in the PD controller is a time-varying periodic coefficient:

$$k_1 = k_0 + k \cos \omega t, \quad (3)$$

where ω is the frequency of varying proportional gain.

The eccentricity of the rotor from the geometrical center of the magnetic bearings is represented by the Cartesian coordinates x and y . As shown in Figure 1, it is known that $2\alpha = \pi/8$, and the radial displacements of the rotor and the control current are, respectively, expressed as

FIGURE 1: Cross-sectional diagram of the AMB with 16-pole legs.

$$\begin{aligned}
 w_1 &= x, \\
 I_1 &= i_1, \\
 w_2 &= x \cos\left(\frac{\pi}{8}\right) + y \sin\left(\frac{\pi}{8}\right), \\
 I_2 &= i_0 + i_2, \\
 w_3 &= x \cos\left(\frac{\pi}{4}\right) + y \sin\left(\frac{\pi}{4}\right), \\
 I_3 &= i_0 + i_3, \\
 w_4 &= x \cos\left(\frac{3\pi}{8}\right) + y \sin\left(\frac{3\pi}{8}\right), \\
 I_4 &= i_0 + i_4, \\
 w_5 &= y, \\
 I_5 &= i_0 + i_5, \\
 w_6 &= x \cos\left(\frac{5\pi}{8}\right) + y \sin\left(\frac{5\pi}{8}\right), \\
 I_6 &= i_0 + i_6, \\
 w_7 &= x \cos\left(\frac{3\pi}{4}\right) + y \sin\left(\frac{3\pi}{4}\right), \\
 I_7 &= i_0 + i_7, \\
 w_8 &= x \cos\left(\frac{7\pi}{8}\right) + y \sin\left(\frac{7\pi}{8}\right), \\
 I_8 &= i_0 + i_8.
 \end{aligned} \tag{4}$$

From Figure 1, it is seen that the electromagnetic force resultants in the horizontal and the vertical directions can be written as

$$\begin{aligned}
 F_x &= F_1 + F_2 \cos \frac{\pi}{8} + F_3 \cos \frac{\pi}{4} + F_4 \cos \frac{3\pi}{8} + F_6 \cos \frac{5\pi}{8} \\
 &\quad + F_7 \cos \frac{6\pi}{8} + F_8 \cos \frac{7\pi}{8} \\
 &= a \left\{ \frac{(1-c_1)^2}{(1-b_1)^2} - \frac{(1+c_1)^2}{(1+b_1)^2} + \cos \frac{\pi}{8} \left[\frac{(1-c_2)^2}{(1-b_2)^2} - \frac{(1+c_2)^2}{(1+b_2)^2} \right. \right. \\
 &\quad \left. \left. - \frac{(1-c_3)^2}{(1-b_3)^2} + \frac{(1+c_3)^2}{(1+b_3)^2} \right] \right. \\
 &\quad + \frac{\sqrt{2}}{2} \left[\frac{(1-c_4)^2}{(1-b_4)^2} - \frac{(1+c_4)^2}{(1+b_4)^2} - \frac{(1-c_5)^2}{(1-b_5)^2} + \frac{(1+c_5)^2}{(1+b_5)^2} \right. \\
 &\quad \left. \left. + \cos \frac{3\pi}{8} \left[\frac{(1-c_6)^2}{(1-b_6)^2} - \frac{(1+c_6)^2}{(1+b_6)^2} - \frac{(1-c_7)^2}{(1-b_7)^2} + \frac{(1+c_7)^2}{(1+b_7)^2} \right] \right\}, \tag{5a}
 \end{aligned}$$

$$\begin{aligned}
 F_y &= F_5 + F_2 \sin \frac{\pi}{8} + F_3 \sin \frac{\pi}{4} + F_4 \sin \frac{3\pi}{8} + F_6 \sin \frac{5\pi}{8} \\
 &\quad + F_7 \sin \frac{6\pi}{8} + F_8 \sin \frac{7\pi}{8} \\
 &= a \left\{ \frac{(1-c_8)^2}{(1-b_1)^2} - \frac{(1+c_8)^2}{(1+b_1)^2} + \sin \frac{\pi}{8} \left[\frac{(1-c_2)^2}{(1-b_2)^2} - \frac{(1+c_2)^2}{(1+b_2)^2} \right. \right. \\
 &\quad \left. \left. - \frac{(1-c_3)^2}{(1-b_3)^2} - \frac{(1+c_3)^2}{(1+b_3)^2} \right] \right. \\
 &\quad + \frac{\sqrt{2}}{2} \left[\frac{(1-c_4)^2}{(1-b_4)^2} - \frac{(1+c_4)^2}{(1+b_4)^2} + \frac{(1-c_5)^2}{(1-b_5)^2} - \frac{(1+c_5)^2}{(1+b_5)^2} \right. \\
 &\quad \left. \left. + \sin \frac{3\pi}{8} \left[\frac{(1-c_6)^2}{(1-b_6)^2} - \frac{(1+c_6)^2}{(1+b_6)^2} + \frac{(1-c_7)^2}{(1-b_7)^2} - \frac{(1+c_7)^2}{(1+b_7)^2} \right] \right\}, \tag{5b}
 \end{aligned}$$

where

$$\begin{aligned}
 b_1 &= \frac{x}{C_0}, \\
 b_2 &= \frac{1}{C_0} \left(x \cos \frac{\pi}{8} + y \sin \frac{\pi}{8} \right), \\
 b_3 &= \frac{1}{C_0} \left(y \sin \frac{\pi}{8} - x \cos \frac{\pi}{8} \right), \\
 b_4 &= \frac{\sqrt{2}(x+y)}{2C_0},
 \end{aligned}$$

$$\begin{aligned}
b_5 &= \frac{\sqrt{2}(y-x)}{2C_0}, \\
b_6 &= \frac{1}{C_0} \left(x \cos \frac{3\pi}{8} + y \sin \frac{3\pi}{8} \right), \\
b_7 &= \frac{1}{C_0} \left(y \sin \frac{3\pi}{8} - x \cos \frac{3\pi}{8} \right), \\
c_1 &= k_1 \frac{x}{I_0} + k_2 \frac{\dot{x}}{I_0}, \\
c_2 &= \frac{i_0}{I_0} + \frac{1}{I_0} \left[k_1 \left(x \cos \frac{\pi}{8} + y \sin \frac{\pi}{8} \right) + k_2 \left(\dot{x} \cos \frac{\pi}{8} + \dot{y} \sin \frac{\pi}{8} \right) \right], \\
c_3 &= \frac{i_0}{I_0} + \frac{1}{I_0} \left[k_1 \left(y \sin \frac{\pi}{8} - x \cos \frac{\pi}{8} \right) + k_2 \left(\dot{y} \sin \frac{\pi}{8} - \dot{x} \cos \frac{\pi}{8} \right) \right], \\
c_4 &= \frac{i_0}{I_0} + \frac{\sqrt{2}(k_1(x+y) + k_2(\dot{x} + \dot{y}))}{2I_0}, \\
c_5 &= \frac{i_0}{I_0} + \frac{\sqrt{2}(k_1(y-x) + k_2(\dot{y} - \dot{x}))}{2I_0}, \\
c_6 &= \frac{i_0}{I_0} + \frac{1}{I_0} \left[k_1 \left(x \cos \frac{3\pi}{8} + y \sin \frac{3\pi}{8} \right) \right. \\
&\quad \left. + k_2 \left(\dot{x} \cos \frac{3\pi}{8} + \dot{y} \sin \frac{3\pi}{8} \right) \right], \\
c_7 &= \frac{i_0}{I_0} + \frac{1}{I_0} \left[k_1 \left(y \sin \frac{3\pi}{8} - x \cos \frac{3\pi}{8} \right) \right. \\
&\quad \left. + k_2 \left(\dot{y} \sin \frac{3\pi}{8} - \dot{x} \cos \frac{3\pi}{8} \right) \right], \\
c_8 &= \frac{i_0}{I_0} + k_1 \frac{y}{I_0} + k_2 \frac{\dot{y}}{I_0}, \\
a &= \frac{\mu_0 A_\alpha N^2 I_0^2}{4C_0^2} \cos \theta,
\end{aligned} \tag{6}$$

where $\dot{x}$ and $\dot{y}$, respectively, represent the velocities of the rotor in the x and y directions and a is the coefficient of the electromagnetic force.

Based on the expressions of the electromagnetic force resultants, we compared the rotor-AMB system with 8-pole legs and the rotor-AMB system with 16-pole legs given in reference [2]. The results from the comparisons demonstrated that the rotor-AMB system with 16-pole legs has a better bearing capacity than the rotor-AMB system with 8-pole legs. Figures 2 and 3 show the influences of some parameters such as the cross-sectional area of one electromagnet A_α and the number of windings in each electromagnet coil N in equations (5a) and (5b) on the electromagnetic force resultants in the rotor-AMB system with 16-pole legs. In Figures 2 and 3, different colors of the curved surfaces represent the results of electromagnetic force resultants under different values of N or A_α and with other main parameters chosen as

FIGURE 2: Electromagnetic force resultants of the rotor-AMB system with 16-pole legs obtained under different values of N : (a) electromagnetic force resultants F_x in the x direction; (b) electromagnetic force resultants F_y in the y direction.

$$\theta = \frac{\pi}{16},$$

$$\mu_0 = 4\pi \times 10^{-7},$$

$$I_0 = 1,$$

$$i_0 = 5,$$

$$k_1 = 1,$$

$$k_2 = 2.$$

(7)

It is found from Figure 2 that the electromagnetic force resultant in the horizontal direction is increased with the growth of the number of windings in each electromagnet coil in some location. The amplitude of the electromagnetic force resultant in the vertical direction for $N = 850$ is bigger than the result of $N = 950$ in the location when $y > 0.2$. Therefore, the electromagnetic force resultant in the vertical direction is not always increased with the growth of the number of windings in each electromagnet coil. Figure 3 shows that the electromagnetic force resultant in the horizontal direction is decreased with the growth of the cross-sectional area of one electromagnet. The electromagnetic force resultant in the vertical direction of the rotor-AMB system with 16-pole legs

FIGURE 3: Electromagnetic force resultants of the rotor-AMB system with 16-pole legs obtained under different values of A_α : (a) electromagnetic force resultants F_x in the x direction; (b) electromagnetic force resultants F_y in the y direction.

is also decreased when $y < 0.09$ and $y > 0.12$. From the aforementioned two figures, we can conclude that the number of windings in each electromagnet coil and the cross-sectional area of one electromagnet can be the structural design parameters to adjust the bearing capacity of the rotor-AMB system.

Using Taylor's series, the electromagnetic force resultants are expanded in the power series up to the third-order terms as

$$F_x = -\bar{\alpha}_1 x - \bar{f}_{11} x \cos \omega t + \bar{\alpha}_2 x^3 + \bar{\alpha}_3 x y^2 + \bar{\alpha}_4 x y + \bar{f}_{12} x y \cos \omega t + (\bar{\mu}_1 + 3\bar{\mu}_2 x^2 + 3\bar{\mu}_3 x \dot{x} + \bar{\mu}_2 y^2 + 2\bar{\mu}_3 y \dot{y}) \dot{x} + (2\bar{\mu}_2 x y + \bar{\mu}_3 x \dot{y}) \dot{y} + \bar{\mu}_4 x \dot{y} + \bar{\mu}_4 \dot{x} y, \quad (8a)$$

$$F_y = -\bar{\beta}_1 y + \bar{\beta}_2 y^3 + \bar{\beta}_3 x^2 y - \bar{\beta}_4 + \bar{\beta}_5 x^2 + \bar{f}_{22} x^2 \cos \omega t + \bar{\beta}_6 y^2 + (\bar{f}_{23} y^2 - \bar{f}_{21} y) \cos \omega t + (\bar{\mu}_1 + 3\bar{\mu}_2 y^2 + 3\bar{\mu}_3 y \dot{y} + \bar{\mu}_2 x^2 + 2\bar{\mu}_3 x \dot{x}) \dot{y} + (2\bar{\mu}_2 x y + \bar{\mu}_3 x \dot{y}) \dot{x} + \bar{\mu}_4 x \dot{x} + \bar{\mu}_5 y \dot{y}, \quad (8b)$$

where

$$\begin{aligned} \bar{\alpha}_1 &= \frac{4a}{C_0 I_0} (4C_0 k_0 - 4I_0 - 3\gamma^2 I_0), \\ \bar{\alpha}_2 &= \frac{4a}{C_0^3 I_0^2} (6I_0^2 - 9C_0 I_0 k_0 + 3C_0^2 k_0^2 + 4\gamma^2 I_0^2), \\ \bar{\alpha}_3 &= \bar{\beta}_2 = \bar{\beta}_3 = \frac{12a}{C_0^3 I_0^2} [C_0^2 k_0^2 - 3C_0 I_0 k_0 + 2(1 + \gamma^2) I_0^2], \\ \bar{f}_{11} &= \frac{16ak}{I_0}, \\ \bar{f}_{21} &= \frac{16ak}{I_0}, \\ \bar{\alpha}_4 &= \frac{4\sqrt{2}\gamma a}{C_0^2 I_0} (2C_0 k_0 - 3I_0) \left(1 + \cos \frac{\pi}{8} + \cos \frac{3\pi}{8}\right), \\ \bar{f}_{12} &= \frac{8\sqrt{2}\gamma ak}{C_0 I_0} \left(1 + \cos \frac{\pi}{8} + \cos \frac{3\pi}{8}\right), \\ \bar{\mu}_1 &= \frac{16ak_2}{I_0}, \\ \bar{\mu}_2 &= \frac{4ak_2}{C_0^2 I_0^2} (2C_0 k_0 - 3I_0), \\ \bar{\mu}_4 &= \frac{4\sqrt{2}\gamma ak_2}{C_0 I_0} \left(1 + \cos \frac{\pi}{8} + \cos \frac{3\pi}{8}\right), \\ \bar{\mu}_5 &= \frac{4\gamma k_2}{C_0 I_0} \left[2 + \sqrt{2} + \sin \frac{\pi}{8} (2 - \sqrt{2}) + \sin \frac{3\pi}{8} (2 + \sqrt{2})\right], \\ \bar{\mu}_3 &= \frac{4ak_2^2}{C_0 I_0^2}, \\ \bar{\beta}_1 &= \frac{16a}{C_0 I_0} [C_0 k_0 - (1 + \gamma^2) I_0], \\ \bar{\beta}_4 &= 4a\gamma \left(1 + \sqrt{2} + 2 \sin \frac{\pi}{8} + 2 \sin \frac{3\pi}{8}\right), \\ \bar{\beta}_5 &= \frac{2\sqrt{2}a\gamma}{C_0^2 I_0} (2C_0 k_0 - 3I_0) \left(1 + \cos \frac{\pi}{8} + \cos \frac{3\pi}{8}\right), \\ \bar{\beta}_6 &= \frac{\bar{\mu}_5}{2C_0 k_2} (2C_0 k_0 - 3I_0), \\ \gamma &= \frac{i_0}{I_0}, \\ \bar{f}_{22} &= \frac{4\sqrt{2}a\gamma k}{C_0 I_0} \left(1 + \cos \frac{\pi}{8} + \cos \frac{3\pi}{8}\right), \\ \bar{f}_{23} &= \frac{\bar{\mu}_5 k}{k_2}. \end{aligned} \quad (9)$$

Considering the weight of the rotor, the differential governing equations of motion for the rotor-AMB system with 16-pole legs are given as

$$m\ddot{x} = F_x - c\dot{x} + me\Omega^2 \cos \Omega t, \quad (10a)$$

$$m\ddot{y} = F_y - c\dot{y} + me\Omega^2 \sin \Omega t + mg, \quad (10b)$$

where m , e , c , and Ω , respectively, are the mass, the eccentricity of the rotor, the damping coefficient, and the angular velocity of the rotor and F_x and F_y , respectively, are the electromagnetic force resultants in the x and y directions.

When the condition $y = y_0 = 0$ is satisfied, the motion of the rotor performs a steady-state rotation. The electromagnetic force F_{y_0} in the vertical direction is equal to the weight of the rotor. Therefore, we have

$$mg = \frac{i_0 \mu_0 A_\alpha I_0 N^2}{C_0^2} \left(1 + \sqrt{2} + 2 \sin \frac{\pi}{8} + 2 \cos \frac{\pi}{8} \right) \cos \theta = 4 \left(1 + \sqrt{2} + 2 \sin \frac{\pi}{8} + 2 \cos \frac{\pi}{8} \right) a \gamma. \quad (11)$$

Introduce the following nondimensional variables and parameters:

$$\begin{aligned} x^* &= \frac{x}{C_0}, \\ y^* &= \frac{y}{C_0}, \\ t^* &= \left(\frac{a}{mC_0} \right)^{1/2} t, \\ \omega^* &= \left(\frac{mC_0}{a} \right)^{1/2} \omega, \\ \Omega^* &= \left(\frac{mC_0}{a} \right)^{1/2} \Omega. \end{aligned} \quad (12)$$

Substituting equations (8a), (8b), (11), and (12) into equations (10a) and (10b) and dropping the asterisks in the following analysis to simplify the notation, we obtain the dimensionless differential governing equations of motion for the rotor-AMB system with 16-pole legs and the time-varying stiffness as follows:

$$\begin{aligned} \ddot{x} + \omega_1^2 x + \mu \dot{x} - \alpha_2 x^3 - \alpha_3 x y^2 - \alpha_4 x y - (3\mu_2 x^2 + 3\mu_3 x \dot{x} \\ + \mu_2 y^2 + 2\mu_3 y \dot{y} + \mu_4 y) \dot{x} - (2\mu_2 x y + \mu_3 x \dot{y} + \mu_4 x) \dot{y} \\ + x \cos \omega t (2f_{11} - f_{12} y) = F \cos \Omega t, \end{aligned} \quad (13a)$$

$$\begin{aligned} \ddot{y} + \omega_2^2 y + \mu \dot{y} - \beta_2 y^3 - \beta_3 x^2 y - \beta_5 x^2 - \beta_6 y^2 - (f_{22} x^2 \\ + f_{23} y^2 - 2f_{21} y) \cos \omega t - (2\mu_2 x y + \mu_3 x \dot{y} + \mu_4 x) \dot{x} \\ - (3\mu_2 y^2 + 3\mu_3 y \dot{y} + \mu_2 x^2 + 2\mu_3 x \dot{x} + \mu_5 y) \dot{y} = F \sin \Omega t, \end{aligned} \quad (13b)$$

where

$$\begin{aligned} \mu &= \left(\frac{C_0}{ma} \right)^{1/2} \bar{\mu}, \\ \mu_2 &= \left(\frac{C_0}{ma} \right)^{1/2} C_0^2 \bar{\mu}_2, \\ \mu_3 &= \frac{C_0^2 \bar{\mu}_3}{m}, \\ f_{22} &= \frac{C_0^2 \bar{f}_{22}}{a}, \\ \mu_4 &= \left(\frac{C_0}{ma} \right)^{1/2} C_0 \bar{\mu}_4, \\ \mu_5 &= \left(\frac{C_0}{ma} \right)^{1/2} C_0 \bar{\mu}_5, \\ \omega_1^2 &= \frac{C_0 \bar{\alpha}_1}{a}, \\ \alpha_2 &= \frac{C_0^3 \bar{\alpha}_2}{a}, \\ \alpha_3 &= \frac{C_0^3 \bar{\alpha}_3}{a}, \\ \alpha_4 &= \frac{C_0^2 \bar{\alpha}_4}{a}, \\ f_{11} &= \frac{C_0 \bar{f}_{11}}{2a}, \\ f_{12} &= \frac{C_0^2 \bar{f}_{12}}{a}, \\ F &= \frac{e\Omega^2}{C_0}, \\ \omega_2^2 &= \frac{C_0 \bar{\beta}_1}{a}, \\ \beta_2 &= \frac{C_0^3 \bar{\beta}_2}{a}, \\ \beta_3 &= \frac{C_0^3 \bar{\beta}_3}{a}, \\ \beta_5 &= \frac{C_0^2 \bar{\beta}_5}{a}, \\ \beta_6 &= \frac{C_0^2 \bar{\beta}_6}{a}, \\ f_{21} &= \frac{C_0 \bar{f}_{21}}{2a}, \\ f_{23} &= \frac{C_0^2 \bar{f}_{23}}{a}. \end{aligned} \quad (14)$$

The aforementioned equations, which include the quadratic and cubic nonlinear terms and the parametric excitations, describe the nonlinear vibrations of the rotor-AMB system with 16-pole legs and the time-varying stiffness. In the following analysis, we will use the asymptotic perturbation method to obtain the averaged equations of the rotor-AMB system with 16-pole legs and the time-varying stiffness. Numerical simulations are utilized to study the modulation of the amplitudes and the periodic and chaotic motions for the rotor-AMB system with 16-pole legs and the time-varying stiffness.

3. Asymptotic Perturbation Method

In order to utilize the asymptotic perturbation method [6–9] analyzing the nonlinear dynamic responses of equations (13a) and (13b), the scale transformations are introduced as

$$\begin{aligned}\mu &\longrightarrow \varepsilon^2 \mu, \\ f_{11} &\longrightarrow \varepsilon^2 f_{11}, \\ f_{21} &\longrightarrow \varepsilon^2 f_{21}, \\ F &\longrightarrow \varepsilon^2 F,\end{aligned}\quad (15)$$

where ε is a small perturbation parameter.

Based on equations (9) and (14), the relationship of the internal resonances for the rotor-AMB system can be obtained under different values of A_α or i_0 and with other main parameters chosen as $\mu_0 = 4\pi \times 10^{-7}$, $N = 10000$, $I_0 = 1$, and $k_0 = 1$, as shown in Figure 4. It is found from Figure 4 that there exists 1 : 1 internal resonance in the rotor-AMB system with 16-pole legs and the time-varying stiffness. Considering the case of the 1 : 1 internal resonance and 1 : 2 subharmonic resonances, there are the following resonant relations:

$$\begin{aligned}\omega_1 &= \frac{\Omega}{2} + \varepsilon^2 \sigma_1, \\ \omega_2 &= \frac{\Omega}{2} + \varepsilon^2 \sigma_2, \\ \omega &= \Omega,\end{aligned}\quad (16)$$

where σ_1 and σ_2 are two different detuning parameters.

Substituting equations (15) and (16) into equations (13a) and (13b), the two-degree-of-freedom dimensionless nonlinear system under combined parametric and forcing excitations is obtained as follows:

$$\begin{aligned}\ddot{x} &+ (\varepsilon^2 \mu - 3\mu_2 x^2 - 3\mu_3 x \dot{x} - \mu_2 y^2 - 2\mu_3 y \dot{y}) \dot{x} \\ &- (2\mu_2 x y + \mu_3 x \dot{y}) \dot{y} - \mu_4 x \dot{y} - \mu_4 \dot{x} y - \alpha_3 x y^2 \\ &+ \left(\frac{\Omega}{2} + \varepsilon^2 \sigma_1 \right)^2 x + 2\varepsilon^2 f_{11} x \cos \omega t - \alpha_2 x^3 - \alpha_4 x y \\ &- f_{12} x y \cos \omega t = \varepsilon^2 F \cos \Omega t,\end{aligned}\quad (17a)$$

$$\begin{aligned}\ddot{y} &+ (\varepsilon^2 \mu - 3\mu_2 y \dot{y} - \mu_2 x^2 - 2\mu_3 x \dot{x}) \dot{y} - (2\mu_2 x y + \mu_3 \dot{x} y) \dot{x} \\ &- \mu_4 x \dot{x} - \mu_5 y \dot{y} + \left(\frac{\Omega}{2} + \varepsilon^2 \sigma_2 \right)^2 y + 2\varepsilon^2 f_{21} y \cos \omega t - \beta_2 y^3 \\ &- \beta_3 x^2 y - \beta_5 x^2 - \beta_6 y^2 - (f_{22} x^2 + f_{23} y^2) \cos \omega t \\ &= \varepsilon^2 F \sin \Omega t.\end{aligned}\quad (17b)$$

The aforementioned equation includes the parametric and forcing excitations of the rotor-AMB system with 16-pole legs and the time-varying stiffness. We will use the asymptotic perturbation method [6–9] to obtain the averaged equation.

Now we introduce the temporal rescaling

$$\tau = \varepsilon^q t, \quad (18)$$

where q is a rational positive number which will be fixed afterwards.

The value of q fixes the magnitude order of the temporal asymptotic limit in such a way that the nonlinear effects become consistent and not negligible. If $t \rightarrow \infty$, we set $\varepsilon \rightarrow 0$, and the value of τ remains finite.

Taking $\varepsilon = 0$ in equations (17a) and (17b) and neglecting all nonlinear terms, it is found that the resulting linear equation has a simple harmonic solution:

$$x(t) = A_1 e^{-i(\Omega/2)t} + \text{cc}, \quad (19a)$$

$$y(t) = A_2 e^{-i(\Omega/2)t} + \text{cc}, \quad (19b)$$

where A_1 and A_2 are two constants depending on initial conditions and cc represents the parts of the complex conjugate.

It is considered that the nonlinear effects will induce the modulation of the amplitudes A_1 and A_2 and the appearance of the higher harmonics. The slow modulation that resulted from the nonlinear terms can be determined by means of the rescaled time. Assume that solutions $x(t)$ and $y(t)$ of equations (17a) and (17b) can be expressed as

$$x(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{r_n} \psi_n(\tau, \varepsilon) e^{-in(\Omega/2)t}, \quad (20a)$$

$$y(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{r_n} \varphi_n(\tau, \varepsilon) e^{-in(\Omega/2)t}, \quad (20b)$$

where $r_n = |n|$ for $n \neq 0$ and $r_0 = r$ is a positive number which will be fixed later on.

Considering the real part of solutions $x(t)$ and $y(t)$ yields

$$\psi_n(\tau, \varepsilon) = \psi_n^*(\tau, \varepsilon), \quad (21a)$$

$$\varphi_n(\tau, \varepsilon) = \varphi_n^*(\tau, \varepsilon), \quad (21b)$$

where the asterisk denotes the complex conjugate.

It is found that the solutions have been considered the combination of different harmonics with the coefficients depending on both τ and ε . Suppose that functions $\psi_n(\tau, \varepsilon)$ and $\varphi_n(\tau, \varepsilon)$ can be expressed as

$$\psi_n(\tau, \varepsilon) = \sum_{i=0}^{+\infty} \varepsilon^i \psi_n^{(i)}(\tau), \quad (22a)$$

$$\varphi_n(\tau, \varepsilon) = \sum_{i=0}^{+\infty} \varepsilon^i \varphi_n^{(i)}(\tau). \quad (22b)$$

FIGURE 4: Possible relationship of internal resonances: (a) frequencies obtained under different values of A_α ; (b) frequencies obtained under different values of i_0 .

It is also assumed that the limits of the functions $\psi_n(\tau, \varepsilon)$ and $\varphi_n(\tau, \varepsilon)$ exist and are finite when $\varepsilon \rightarrow 0$. For simplicity of analysis, we use abbreviations $\psi_n^{(0)} = \psi_n$ and $\varphi_n^{(0)} = \varphi_n$ for $n \neq 1$, and $\psi_1^{(0)} = \psi$ and $\varphi_1^{(0)} = \varphi$ for $n = 1$. It is noticed that the introduction of equation (18) implies that

$$\frac{d}{dt}(\psi_n e^{-in(\Omega/2)t}) = \left(-in\frac{\Omega}{2}\psi_n + \varepsilon^q \frac{d\psi_n}{d\tau}\right) e^{-in(\Omega/2)t}, \quad (23a)$$

$$\frac{d}{dt}(\varphi_n e^{-in(\Omega/2)t}) = \left(-in\frac{\Omega}{2}\varphi_n + \varepsilon^q \frac{d\varphi_n}{d\tau}\right) e^{-in(\Omega/2)t}. \quad (23b)$$

In order to determine the coefficients $\psi_n(\tau, \varepsilon)$ and $\varphi_n(\tau, \varepsilon)$, we substitute equations (20a) and (20b) into equations (17a) and (17b) and obtain equations for each harmonic with the order n and for a fixed approximation on the perturbation parameter ε .

For $n = 0$, we obtain

$$\varepsilon^r \frac{\Omega^2}{4} \psi_0 - \varepsilon^2 \alpha_4 (\psi \varphi^* + \psi^* \varphi) - \frac{1}{2} \varepsilon^2 f_{12} (\psi \varphi + \psi^* \varphi^*) = 0, \quad (24a)$$

$$\begin{aligned} \varepsilon^r \frac{\Omega^2}{4} \varphi_0 - 2\varepsilon^2 \beta_5 |\psi|^2 - 2\varepsilon^2 \beta_6 |\varphi|^2 - \frac{1}{2} \varepsilon^2 f_{22} (\psi^2 + \psi^{*2}) \\ - \frac{1}{2} \varepsilon^2 f_{23} (\varphi^2 + \varphi^{*2}) = 0. \end{aligned} \quad (24b)$$

The correct balance of the terms leads to $r = 2$, and the following relations are obtained:

$$\psi_0 = \frac{[4\alpha_4 (\psi \varphi^* + \psi^* \varphi) + 2f_{12} (\psi \varphi + \psi^* \varphi^*)]}{\Omega^2}, \quad (25a)$$

$$\varphi_0 = \frac{[8\beta_5 |\psi|^2 + 8\beta_6 |\varphi|^2 + 2f_{22} (\psi^2 + \psi^{*2}) + 2f_{23} (\varphi^2 + \varphi^{*2})]}{\Omega^2}. \quad (25b)$$

For $n = 2$, taking into account equations (23a) and (23b) yields the correspondent relations

$$\psi_2 = -\frac{[(4\alpha_4 - 4i\Omega\mu_4)\psi\varphi + 2f_{12}(\psi\varphi^* + \psi^*\varphi) + 2F]}{3\Omega^2}, \quad (26a)$$

$$\varphi_2 = -\frac{[(4\beta_5 - 2i\Omega\mu_4)\psi^2 + (4\beta_6 - 2i\Omega\mu_5)\varphi^2 + 4f_{22}|\psi|^2 + 4f_{23}|\varphi|^2 + 2iF]}{3\Omega^2}. \quad (26b)$$

The complex conjugates of ψ_2 and φ_2 are given as follows:

$$\psi_2^* = -\frac{[(4\alpha_4 + 4i\Omega\mu_4)\psi^*\varphi^* + 2f_{12}(\psi\varphi^* + \psi^*\varphi) + 2F]}{3\Omega^2}, \quad (27a)$$

$$\varphi_2^* = -\frac{[(4\beta_5 + 2i\Omega\mu_4)\psi^{*2} + (4\beta_6 + 2i\Omega\mu_5)\varphi^{*2} + 4f_{22}|\psi|^2 + 4f_{23}|\varphi|^2 - 2iF]}{3\Omega^2}. \quad (27b)$$

For $n = 1$, based on equations (17a) and (17b), we have

$$\begin{aligned} & -\varepsilon^{q+1}i\Omega\dot{\psi} - \frac{1}{2}\varepsilon^3i\Omega\mu\psi + \varepsilon^3i\Omega\mu_2\left(\frac{3}{2}|\psi|^2\psi + \frac{1}{2}\varphi^2\psi^* + |\varphi|^2\psi\right) + \varepsilon^3\sigma_1\Omega\psi + \varepsilon^3f_{11}\psi^* \\ & - \varepsilon^3\Omega^2\mu_3\left(\frac{3}{4}|\psi|^2\psi + \frac{1}{4}\varphi^2\psi^* + \frac{1}{2}|\varphi|^2\psi\right) + \frac{1}{2}\varepsilon^{r+1}i\Omega\mu_4(\psi_0\varphi + \varphi_0\psi) - 3\varepsilon^3\alpha_2|\psi|^2\psi \\ & + \frac{1}{2}\varepsilon^3i\Omega\mu_4(\psi_2\varphi^* + \psi^*\varphi_2) - \varepsilon^3\alpha_3(2|\varphi|^2\psi + \varphi^2\psi^*) - \varepsilon^{r+1}\alpha_4(\psi_0\varphi + \psi\varphi_0) - \varepsilon^3\alpha_4\psi^*\varphi_2 \\ & - \varepsilon^3\alpha_4\psi_2\varphi^* - \frac{1}{2}\varepsilon^{r+1}f_{12}(\psi_0\varphi^* + \psi^*\varphi_0) - \frac{1}{2}\varepsilon^3f_{12}(\psi\varphi_2^* + \varphi\psi_2 + \varphi\psi_2^*) = 0, \end{aligned} \quad (28a)$$

$$\begin{aligned} & -\varepsilon^{q+1}i\Omega\dot{\varphi} - \frac{1}{2}\varepsilon^3i\Omega\mu\varphi + \varepsilon^3i\Omega\mu_2\left(\frac{3}{2}|\varphi|^2\varphi + \frac{1}{2}\psi^2\varphi^* + |\psi|^2\varphi\right) - \varepsilon^3\Omega^2\mu_3\left(\frac{3}{4}|\varphi|^2\varphi + \frac{1}{2}|\psi|^2\varphi\right) \\ & - \varepsilon^3\Omega^2\mu_3\frac{1}{4}\psi^2\varphi^* + \frac{1}{2}\varepsilon^{r+1}i\Omega\mu_4\psi_0\psi + \frac{1}{2}\varepsilon^3i\Omega\mu_4\psi_2\psi^* + \frac{1}{2}\varepsilon^{r+1}i\Omega\mu_5\varphi_0\varphi + \frac{1}{2}\varepsilon^3i\Omega\mu_5\varphi_2\varphi^* \\ & + \varepsilon^3\sigma_2\Omega\varphi + \varepsilon^3f_{21}\varphi^* - 3\varepsilon^3\beta_2|\varphi|^2\varphi - \varepsilon^3\beta_3(2|\psi|^2\varphi + \psi^2\varphi^*) - 2\beta_5(\varepsilon^{r+1}\psi_0\psi + \varepsilon^3\psi^*\psi_2) \\ & - 2\beta_6(\varepsilon^{r+1}\varphi_0\varphi + \varepsilon^3\varphi^*\varphi_2) - f_{22}(\varepsilon^{r+1}\psi_0\psi^* + \varepsilon^3\psi_2\psi + \varepsilon^3\psi_2^*\psi) \\ & - f_{23}(\varepsilon^{r+1}\varphi_0\varphi^* + \varepsilon^3\varphi_2\varphi + \varepsilon^3\varphi_2^*\varphi) = 0. \end{aligned} \quad (28b)$$

It is found that, in the case $q = 2$, the first-order terms of equations (28a) and (28b) have the same magnitude order as all other nonlinear terms.

Based on equations (25a) and (25b), and (27a) and (27b), the differential equation of the complex amplitudes ψ and φ can be derived as

$$\begin{aligned} \frac{d\psi}{d\tau} &= (g_1 - i\sigma_1)\psi + ih_1\varphi + (g_2 + ig_3)\psi^* + (h_2 + ig_2)\varphi^* + ie_1\varphi^{*2}\psi^* + (g_4 + ie_2)\varphi^2\psi^* \\ &+ (g_5 + ie_3)|\psi|^2\psi + (g_6 + ie_4)|\psi|^2\psi^* + (g_7 + ie_5)\psi^3 + ie_6\psi^{*3} + (g_8 + ie_7)|\varphi|^2\psi \\ &+ (g_9 + ih_3)\psi\varphi^{*2} + (g_{10} + ih_4)\psi\varphi^2 + (g_{11} + ih_5)|\varphi|^2\psi^*, \end{aligned} \quad (29a)$$

$$\begin{aligned} \frac{d\varphi}{d\tau} &= (g_1 - i\sigma_2)\varphi + (k_1 + ij_1)\varphi^* + iz_1\psi + (h_2 + iz_2)\psi^* + ij_2\psi^{*2}\varphi^* + (k_2 + iz_3)|\psi|^2\varphi^* \\ &+ (k_3 + iz_4)|\psi|^2\varphi + (k_4 + iz_5)\varphi\psi^{*2} + (k_5 + iz_6)\psi^2\varphi^* + ij_3\varphi^{*3} + (k_6 + iz_7)\psi^2\varphi \\ &+ (k_7 + iz_8)|\varphi|^2\varphi^* + (k_8 + iz_9)\varphi^3 + (k_9 + iz_{10})|\varphi|^2\varphi, \end{aligned} \quad (29b)$$

where

$$\begin{aligned}
g_1 &= \frac{1}{2}\mu, \\
g_2 &= \frac{2\alpha_4 F}{3\Omega^3}, \\
g_3 &= -\frac{f_{11}}{\Omega} - \frac{\mu_4 F}{3\Omega^2}, \\
g_4 &= \frac{\mu_2}{2} + \frac{2\mu_4\alpha_4}{\Omega^2} - \frac{2\mu_4\beta_6 + 2\mu_5\alpha_4}{3\Omega^2}, \\
g_5 &= \frac{3\mu_2}{2} + \frac{10\mu_4\beta_5 - \mu_4\alpha_4}{3\Omega^2}, \\
g_6 &= \frac{\mu_4(f_{12} + f_{22})}{3\Omega^2}, \\
g_7 &= \frac{\mu_4(f_{12} + 3f_{22})}{3\Omega^2}, \\
g_8 &= \mu_2 + \frac{4\mu_4\beta_6}{\Omega^2}, \\
g_9 &= \frac{\mu_4 f_{23}}{\Omega^2} - \frac{f_{12}(\mu_5 - \mu_4)}{3\Omega^2}, \\
g_{10} &= \frac{\mu_4 f_{23}}{\Omega^2} + \frac{f_{12}(\mu_4 - \mu_5)}{3\Omega^2}, \\
g_{11} &= \frac{\mu_4(4f_{12} - 2f_{23})}{3\Omega^2}, \\
h_1 &= -\frac{2f_{12}F}{3\Omega^3}, \\
h_2 &= -\frac{\mu_4 F}{3\Omega^2}, \\
h_3 &= \frac{2\alpha_4 f_{23}}{\Omega^3} + \frac{f_{12}(4\alpha_4 - 2\beta_6)}{3\Omega^3}, \\
e_1 &= \frac{f_{12}^2 + f_{12}f_{23}}{\Omega^3}, \\
h_4 &= \frac{4\beta_6 f_{12}}{\Omega^3} + \frac{4\alpha_4(2f_{12} - f_{23})}{3\Omega^3}, \\
e_4 &= \frac{10f_{12}\beta_5 + 2\alpha_4 f_{22}}{3\Omega^3}, \\
e_5 &= \frac{2\alpha_4 f_{22}}{\Omega^3} - \frac{2f_{12}\beta_5}{3\Omega^3}, \\
e_2 &= \frac{\Omega\mu_3}{4} + \frac{\mu_4\mu_5 + 3\alpha_3}{3\Omega} + \frac{f_{12}f_{23}}{\Omega^3} - \frac{4\alpha_4\beta_6 + 2f_{12}^2}{3\Omega^3}, \\
e_6 &= \frac{f_{12}f_{22}}{\Omega^3}, \\
j_1 &= -\frac{f_{21}}{\Omega} - \frac{\mu_5 F}{3\Omega^2}, \\
e_3 &= \frac{3\Omega\mu_3}{4} + \frac{3\alpha_2}{\Omega} + \frac{\mu_4^2}{3\Omega} + \frac{20\alpha_4\beta_5 - f_{12}f_{22}}{3\Omega^3}, \\
j_2 &= \frac{2f_{22}(f_{12} + f_{23})}{\Omega^3}, \\
k_1 &= \frac{4\beta_6 F}{3\Omega^3}, \\
e_7 &= \frac{\Omega\mu_3}{2} + \frac{2\alpha_3}{\Omega} + \frac{2\mu_4^2}{3\Omega} + \frac{24\alpha_4\beta_6 - 4f_{12}f_{22}}{3\Omega^3} + \frac{f_{12}^2 + 8\alpha_4^2}{3\Omega^3}, \\
k_2 &= \frac{2\mu_4 f_{12}}{3\Omega^2} + \frac{f_{22}(4\mu_4 - 2\mu_5)}{3\Omega^2}, \\
k_3 &= \mu_2 + \frac{4\mu_5\beta_5}{\Omega^2} + \frac{4\mu_4\alpha_4 - 8\mu_4\beta_5}{3\Omega^2}, \\
k_4 &= \frac{\mu_5 f_{22}}{\Omega^2} + \frac{\mu_4(2f_{23} - f_{12})}{3\Omega^2}, \\
k_7 &= \frac{\mu_5 f_{23}}{\Omega^2}, \\
k_5 &= \frac{\mu_2}{2} + \frac{2\mu_4\alpha_4}{\Omega^2} - \frac{2\mu_5\beta_5 + 4\mu_4\beta_6}{3\Omega^2}, \\
k_6 &= -\frac{4\mu_4 f_{22}}{3} + \frac{\mu_4 f_{12} + \mu_5 f_{22}}{\Omega^2} - \frac{2f_{23}\mu_4}{3\Omega^2}, \\
k_8 &= \frac{\mu_5 f_{23}}{3\Omega^2}, \\
k_9 &= \frac{3\mu_2}{2} + \frac{2\beta_5\beta_6}{\Omega^2}, \\
z_1 &= -\frac{4f_{22}F}{3\Omega^3}, \\
z_2 &= \frac{4\beta_5 F}{3\Omega^3}, \\
j_3 &= \frac{2f_{23}^2}{\Omega^3}, \\
z_3 &= \frac{8f_{22}(\alpha_4 - \beta_6)}{3\Omega^3} + \frac{8\beta_5(f_{12} + 3f_{23})}{3\Omega^3}, \\
z_4 &= \frac{\Omega\mu_3}{2} + \frac{2\beta_3}{\Omega} + \frac{2\mu_4^2}{3\Omega} + \frac{2f_{22}f_{12} - 8f_{23}f_{22}}{3\Omega^3} \\
&\quad + \frac{16\beta_5\beta_6}{\Omega^3} + \frac{16\beta_5\alpha_4}{3\Omega^3}, \\
z_5 &= \frac{4f_{22}(\alpha_4 + \beta_6)}{\Omega^3} - \frac{4\beta_5(f_{12} + f_{23})}{3\Omega^3}, \\
z_8 &= \frac{8f_{23}\beta_6}{\Omega^3}, \\
z_6 &= \frac{\Omega\mu_3}{4} + \frac{\beta_3}{\Omega} + \frac{\mu_4\mu_5}{3\Omega} + \frac{2f_{22}f_{23} - 4f_{12}f_{22}}{\Omega^3} + \frac{8\beta_5(\alpha_4 - \beta_6)}{\Omega^3}, \\
z_9 &= \frac{8f_{23}\beta_6}{3\Omega^3}, \\
z_7 &= \frac{f_{22}(12\beta_6 - 4\alpha_4)}{3\Omega^3} + \frac{\beta_5(12f_{12} - 4f_{23})}{3\Omega^3}, \\
z_{10} &= \frac{3\Omega\mu_3}{4} + \frac{3\beta_2}{\Omega} + \frac{\mu_5^2}{3\Omega} + \frac{40\beta_6^2 - 2f_{23}^2}{3\Omega^3}.
\end{aligned} \tag{30}$$

Let

$$\psi = a_1(\tau)e^{i\gamma_1(\tau)}, \tag{31a}$$

$$\varphi = a_2(\tau)e^{i\gamma_2(\tau)}, \tag{31b}$$

where the variables a_n and γ_n ($n = 1, 2$) are the real functions with respect to τ .

Substituting equations (31a) and (31b) into equations (29a) and (29b) and separating the real and imaginary parts, we have the following equations:

$$\begin{aligned} a_1' = & [g_4 \cos(2\gamma_1 - 2\gamma_2) + e_2 \sin(2\gamma_1 - 2\gamma_2) + e_1 \sin(2\gamma_1 + 2\gamma_2)]a_1a_2^2 + h_1a_2 \sin(\gamma_1 - \gamma_2) \\ & + (g_6a_1^2 + g_{11}a_2^2 + g_2)a_1 \cos(2\gamma_1) + (e_4a_1^2 + h_5a_2^2 + g_3)a_1 \sin(2\gamma_1) + g_7a_1^2 \cos(\gamma_1) \\ & + (h_3 - h_4)a_1a_2^2 \sin(2\gamma_2) + h_2a_2 \cos(\gamma_1 + \gamma_2) + g_2a_2 \sin(\gamma_1 + \gamma_2) + e_6a_1^3 \sin(4\gamma_1) \\ & + (g_9 + g_{10})a_1a_2^2 \cos(2\gamma_2) + g_5a_1^3 - e_5a_1^2 \sin(\gamma_1) + g_8a_1a_2^2 + g_1a_1, \end{aligned} \quad (32a)$$

$$\begin{aligned} a_1\gamma_1' = & [e_2 \cos(2\gamma_1 - 2\gamma_2) + e_1 \cos(2\gamma_1 + 2\gamma_2) - g_4 \sin(2\gamma_1 - 2\gamma_2)]a_1a_2^2 + h_1a_2 \cos(\gamma_1 - \gamma_2) \\ & + (e_4a_1^2 + h_5a_2^2 + g_3)a_1 \cos(2\gamma_1) - (g_6a_1^2 + g_{11}a_2^2 + g_2)a_1 \sin(2\gamma_1) + g_7a_1^2 \sin(\gamma_1) \\ & + (h_3 + h_4)a_1a_2^2 \cos(2\gamma_2) - h_2a_2 \sin(\gamma_1 + \gamma_2) + g_2a_2 \cos(\gamma_1 + \gamma_2) + e_6a_1^3 \cos(4\gamma_1) \\ & - (g_9 - g_{10})a_1a_2^2 \sin(2\gamma_2) + e_3a_1^3 + e_5a_1^2 \cos(\gamma_1) + e_7a_1a_2^2 - \sigma_1a_1, \end{aligned} \quad (32b)$$

$$\begin{aligned} a_2' = & [k_5 \cos(2\gamma_1 - 2\gamma_2) - z_6 \sin(2\gamma_1 - 2\gamma_2) + j_2 \sin(2\gamma_1 + 2\gamma_2)]a_2a_1^2 - z_1a_1 \sin(\gamma_1 - \gamma_2) \\ & + [(k_7 + k_8)a_2^2 + k_2a_1^2 + k_1]a_2 \cos(2\gamma_2) + [(z_8 - z_9)a_2^2 + z_3a_1^2 + j_1]a_2 \sin(2\gamma_2) \\ & + [(k_4 + k_6)\cos(2\gamma_1) + (z_5 - z_7)\sin(2\gamma_1)]a_2a_1^2 + h_2a_1 \cos(\gamma_1 + \gamma_2) + z_2a_1 \sin(\gamma_1 + \gamma_2) \\ & + j_3a_2^3 \sin(4\gamma_2) + k_9a_2^3 + k_3a_2a_1^2 + g_1a_2, \end{aligned} \quad (32c)$$

$$\begin{aligned} a_2\gamma_2' = & [k_5 \sin(2\gamma_1 - 2\gamma_2) + z_6 \cos(2\gamma_1 - 2\gamma_2) + j_2 \cos(2\gamma_1 + 2\gamma_2)]a_2a_1^2 + z_1a_1 \cos(\gamma_1 - \gamma_2) \\ & + [(z_8 + z_9)a_2^2 + z_3a_1^2 + j_1]a_2 \cos(2\gamma_2) - [(k_7 - k_8)a_2^2 + k_2a_1^2 + k_1]a_2 \sin(2\gamma_2) \\ & + [(z_5 + z_7)\cos(2\gamma_1) + (k_6 - k_4)\sin(2\gamma_1)]a_2a_1^2 + z_2a_1 \cos(\gamma_1 + \gamma_2) - h_2a_1 \sin(\gamma_1 + \gamma_2) \\ & + j_3a_2^3 \cos(4\gamma_2) + z_{10}a_2^3 + z_4a_2a_1^2 - \sigma_2a_2. \end{aligned} \quad (32d)$$

In order to transform equations (29a) and (29b) into the Cartesian form, let

$$\psi = x_1 + ix_2, \quad (33a)$$

$$\psi^* = x_1 - ix_2,$$

$$\varphi = x_3 + ix_4, \quad (33b)$$

$$\varphi^* = x_3 - ix_4,$$

where the variables x_n ($n = 1, 2, 3, 4$) are the real functions with respect to T_1 .

Substituting equations (33a) and (33b) into equations (29a) and (29b), the averaged equations of the rotor-AMB system with 16-pole legs in the Cartesian form are obtained as follows:

$$\begin{aligned} \frac{dx_1}{d\tau} = & b_{11}x_1 + (b_{12} + \sigma_1)x_2 + b_{13}x_3 + b_{14}x_4 + d_{11}x_1(x_1^2 + x_2^2) + d_{12}x_2(x_1^2 + x_2^2) \\ & + d_{13}x_1(x_3^2 + x_4^2) + d_{14}x_2(x_3^2 + x_4^2) + d_{15}x_1(x_3^2 - x_4^2) + d_{16}x_2(x_3^2 - x_4^2) \\ & + c_{11}x_1x_3x_4 + c_{12}x_2x_3x_4 + c_{13}x_1(x_1^2 - 3x_2^2) + c_{14}x_2(x_2^2 - 3x_1^2), \end{aligned} \quad (34a)$$

$$\begin{aligned} \frac{dx_2}{d\tau} = & (b_{12} - \sigma_1)x_1 + b_{21}x_2 + b_{22}x_3 - b_{13}x_4 + d_{21}x_1(x_1^2 + x_2^2) + d_{22}x_2(x_1^2 + x_2^2) \\ & + d_{23}x_1(x_3^2 + x_4^2) + d_{24}x_2(x_3^2 + x_4^2) + d_{25}x_1(x_3^2 - x_4^2) + d_{26}x_2(x_3^2 - x_4^2) \\ & - c_{11}x_2x_3x_4 + c_{21}x_1x_3x_4 + c_{22}x_1(x_1^2 - 3x_2^2) - c_{13}x_2(x_2^2 - 3x_1^2), \end{aligned} \quad (34b)$$

$$\begin{aligned} \frac{dx_3}{d\tau} = & b_{13}x_1 + b_{31}x_2 + b_{32}x_3 + (b_{33} + \sigma_2)x_4 + d_{31}x_3(x_1^2 + x_2^2) + d_{32}x_4(x_1^2 + x_2^2) \\ & + d_{33}x_3(x_3^2 + x_4^2) + d_{34}x_4(x_3^2 + x_4^2) + d_{35}x_3(x_1^2 - x_2^2) + d_{36}x_4(x_1^2 - x_2^2) + c_{31}x_1x_2x_4 \\ & + c_{32}x_1x_2x_3 + c_{33}x_3(x_3^2 - 3x_4^2) + c_{34}x_4(x_4^2 - 3x_3^2), \end{aligned} \quad (34c)$$

$$\begin{aligned} \frac{dx_4}{d\tau} = & b_{41}x_1 - b_{13}x_2 + (b_{33} - \sigma_2)x_3 + b_{42}x_4 + d_{41}x_4(x_1^2 + x_2^2) + d_{42}x_3(x_1^2 + x_2^2) \\ & + d_{43}x_3(x_3^2 + x_4^2) + d_{44}x_4(x_3^2 + x_4^2) + d_{45}x_3(x_1^2 - x_2^2) + d_{46}x_4(x_1^2 - x_2^2) + c_{41}x_1x_2x_3 \\ & + c_{42}x_1x_2x_4 + c_{43}x_3(x_3^2 - 3x_4^2) - c_{33}x_4(x_4^2 - 3x_3^2), \end{aligned} \quad (34d)$$

where

$$\begin{aligned} b_{11} &= g_1 + g_2, \\ b_{12} &= -g_3, \\ b_{13} &= h_2, \\ b_{14} &= g_2 - h_1, \\ d_{11} &= g_5 + g_6, \\ d_{12} &= e_4 - e_3, \\ d_{13} &= g_8 + g_{11}, \\ d_{14} &= h_4 - e_7, \\ d_{15} &= g_4 + g_9 + g_{10}, \\ d_{16} &= e_1 + e_2 - 2h_3, \\ c_{11} &= 2(e_1 - e_2), \\ c_{12} &= 2(g_4 + g_9 - g_{10}), \\ c_{13} &= g_7, \\ c_{14} &= e_5 - e_6, \\ b_{21} &= g_1 - g_2, \\ b_{22} &= g_2 + h_1, \\ d_{21} &= e_4 + e_3, \\ d_{22} &= g_5 - g_6, \\ d_{23} &= h_4 + e_7, \\ d_{24} &= g_8 - g_{11}, \\ d_{25} &= e_1 + e_2 + 2h_3, \\ d_{26} &= g_9 + g_9 - g_4, \\ c_{21} &= 2(g_4 - g_9 + g_{10}), \\ c_{22} &= e_5 + e_6, \\ b_{31} &= z_2 - z_1, \\ b_{32} &= k_1 + g_1, \\ b_{33} &= j_1, \\ d_{31} &= k_2 + k_3, \\ d_{32} &= z_3 - z_4, \\ d_{33} &= k_7 + k_9, \\ d_{34} &= z_8 - z_{10}, \end{aligned}$$

$$\begin{aligned} d_{35} &= k_4 + k_5 + k_6, \\ d_{36} &= z_6 - z_5 - z_7 + j_2, \\ c_{32} &= 2(z_5 - z_6 - z_7 + j_2), \\ c_{33} &= k_8, \\ c_{31} &= 2(k_4 + k_5 - k_6), \\ c_{34} &= z_9 - j_3, \\ b_{41} &= z_1 + z_2, \\ b_{42} &= g_1 - k_1, \\ d_{41} &= k_3 - k_2, \\ d_{42} &= z_3 + z_4, \\ d_{43} &= z_8 + z_{10}, \\ d_{44} &= k_9 - k_7, \\ d_{45} &= z_5 + z_6 + z_7 + j_2, \\ d_{46} &= k_4 - k_5 + k_6, \\ c_{41} &= 2(k_5 + k_6 - k_4), \\ c_{42} &= 2(z_5 + z_6 - z_7 - j_2), \\ c_{43} &= z_9 + j_3. \end{aligned} \quad (35)$$

4. Analysis of Frequency Responses

In this section, we investigate the frequency responses of the rotor-AMB system with 16-pole legs and the time-varying stiffness. Setting the left parts of equations (32a) and (32b) equal to zero for the steady-state responses of ($a'_1 = \gamma'_1 = a'_2 = \gamma'_2 = 0$) of the rotor-AMB system with 16-pole legs, we obtain the frequency-response functions and the influences of the nonlinearity terms in the function on the rotor-AMB system with 16-pole legs and the time-varying stiffness, which can be described by the frequency-response curves.

Figures 5–8 show the mode amplitudes a_1 and a_2 as the change of σ_2 for different parameter values in the frequency-response functions. In these figures, the blue and red lines, respectively, are the stable and unstable solutions for a_1 , while black and magenta lines, respectively, are the stable and unstable solutions for a_2 . The parameters in Figure 5 are chosen as follows:

$$\begin{aligned}
g_1 &= -3, \\
g_2 &= 1.3, \\
g_3 &= -3, \\
g_4 &= 0.12, \\
g_5 &= -0.05, \\
g_6 &= 0.02, \\
g_7 &= -0.13, \\
g_8 &= -0.35, \\
g_9 &= -1.13, \\
g_{10} &= -0.05, \\
g_{11} &= 0.05, \\
h_1 &= -1.3, \\
h_2 &= -2.8, \\
h_3 &= 0.05, \\
h_4 &= -0.02, \\
h_5 &= 1.03, \\
e_1 &= -1.1, \\
e_2 &= 0.5, \\
e_3 &= 0.03, \\
e_4 &= 0.03, \\
e_5 &= 0.8, \\
e_6 &= 0.01, \\
e_7 &= -1.53, \\
j_1 &= -1.8, \\
j_2 &= -0.15, \\
j_3 &= 0.5, \\
k_1 &= 2, \\
k_2 &= -0.16, \\
k_3 &= -0.38, \\
k_4 &= 0.32, \\
k_5 &= -0.06, \\
k_6 &= 0.12, \\
k_7 &= 0.03, \\
k_8 &= -0.012, \\
k_9 &= -2.03, \\
z_1 &= -0.01, \\
z_2 &= -0.03, \\
z_3 &= 0.1, \\
z_4 &= 3.3, \\
z_5 &= 0.02, \\
z_6 &= 0.02, \\
z_7 &= 0.13, \\
z_8 &= 0.15, \\
z_9 &= 0.1, \\
z_{10} &= -0.5, \\
\sigma_1 &= 0.
\end{aligned}
\tag{36}$$

It is observed from Figure 5 that the solutions of a_1 and a_2 are multivalued when the detuning σ_2 is changed. The hardening-type nonlinearity exists in the rotor-AMB system with 16-pole legs and the time-varying stiffness when the excitation frequency changes. It is also indicated that there are only the stable steady-state solutions in the range of -100 and -20 near σ_2 . From Figure 5, we observe that as σ_2 increases, the amplitudes of the stable steady-state solutions increase for the rotor-AMB system with 16-pole legs and the time-varying stiffness.

Figure 6 shows the modal amplitudes as the change of σ_2 for $g_1 = -1$, $g_9 = -0.13$, $h_4 = 0.02$, $h_5 = 0.03$, $z_4 = 0.12$, $z_5 = -0.02$, $z_6 = 0.2$, and $z_7 = 0.03$ and keeping other parameters

FIGURE 5: Frequency-response curves obtained when $g_1 = -3$, $g_2 = 1.3$, and $\sigma_1 = 0$.

FIGURE 6: Frequency-response curves obtained when $g_1 = -1$, $g_9 = -0.13$, and $\sigma_1 = 0$.

FIGURE 7: Frequency-response curves obtained when $g_2 = -0.3$, $j_1 = -0.8$, and $\sigma_1 = 3$.

the same as in Figure 4. The big difference between Figures 6 and 5 is that there exists the softening-type nonlinearity in Figure 6 for the rotor-AMB system with 16-pole legs and the time-varying stiffness when σ_2 changes in the range of ± 100 .

FIGURE 8: Frequency-response curves obtained when $g_1 = -0.1$, $j_1 = 0.08$, and $\sigma_1 = 0$.

Based on Figure 6, we find that as σ_2 increases, the amplitudes of the stable steady-state solutions decrease for the rotor-AMB system with 16-pole legs and the time-varying stiffness.

Continuously changing the parameters to $g_2 = -0.3$, $g_3 = 3$, $g_4 = 0.02$, $g_{10} = 0.05$, $h_1 = 1.3$, $e_1 = -0.1$, $e_2 = 0.1$, $e_5 = 0.36$, $j_1 = -0.8$, $k_4 = 0.02$, $z_2 = 0.03$, and $\sigma_1 = 3$ and keeping other parameters the same as in Figure 6, the frequency-response curves are depicted, as shown in Figure 7. It is seen from Figure 7 that both hardening-type nonlinearity and softening-type nonlinearity exist in the rotor-AMB system with 16-pole legs and the time-varying stiffness when the frequency changes. Moreover, the multivalued solutions and jumping phenomena are also observed under these parameter conditions. The amplitudes of the horizontal mode are greater than those of the vertical mode. As the excitation frequency increases, the amplitudes of the frequency-response curves first decrease and then increase for the rotor-AMB system with 16-pole legs and the time-varying stiffness.

The parameters in Figure 8 are chosen as $g_1 = -0.1$, $j_1 = 0.08$, and $\sigma_1 = 0$, and other parameters are the same as in Figure 7. It is found from Figure 8 that the unstable solutions only exist in the range of ± 5 near σ_2 , and in the rest of the regions, all solutions are the stable steady-state solutions. Based on Figure 8, as the excitation frequency increases, the amplitudes of the stable steady-state solutions first decrease and then increase for the rotor-AMB system with 16-pole legs and the time-varying stiffness.

5. Numerical Simulation of Dynamic Responses

In this section, the averaged equations (34a) and (34b) are used to perform numerical simulations of investigating the bifurcation diagrams and the periodic and chaotic motions of the rotor-AMB system with 16-pole legs and the time-varying stiffness. The parameter b_{14} closely related to the eccentricity is chosen as a controlling parameter to investigate the nonlinear dynamics of the rotor-AMB system with 16-pole legs. The bifurcation diagrams, waveforms, and two-dimensional and three-dimensional phase portraits are presented when the parameter b_{14} varies from 10 to 50 and other parameters and the initial conditions are chosen as follows:

$$\begin{aligned}
 \sigma_1 &= -1.25, \\
 \sigma_2 &= 0.57, \\
 b_{11} &= -17.3, \\
 b_{12} &= -4.25, \\
 b_{13} &= -12.5, \\
 d_{11} &= 3.76, \\
 d_{12} &= -19.76, \\
 d_{13} &= 8.72, \\
 d_{14} &= 7.29, \\
 d_{15} &= -0.7, \\
 d_{16} &= 0.25, \\
 c_{11} &= -10.57, \\
 c_{12} &= 5.17, \\
 c_{13} &= 7, \\
 c_{14} &= -9.1, \\
 b_{21} &= 1.13, \\
 b_{22} &= -38, \\
 d_{21} &= 5.23, \\
 d_{22} &= -2.23, \\
 d_{23} &= 2.9, \\
 d_{24} &= 1.72, \\
 d_{25} &= -10.15, \\
 d_{26} &= 1.16, \\
 c_{21} &= 1.67, \\
 c_{22} &= -1.5, \\
 b_{31} &= 50.3, \\
 b_{32} &= -4.3, \\
 b_{33} &= -75.6, \\
 d_{31} &= 13.93, \\
 d_{32} &= -2.4, \\
 d_{33} &= -2.7, \\
 d_{34} &= -2, \\
 d_{35} &= 2.89, \\
 d_{36} &= -1.2, \\
 c_{31} &= -2.1, \\
 c_{32} &= 3.69, \\
 c_{33} &= -4.05, \\
 c_{34} &= 10.57,
 \end{aligned}$$

$$\begin{aligned}
b_{41} &= -40, \\
b_{42} &= -24.15, \\
d_{41} &= -15.06, \\
d_{42} &= 50, \\
d_{43} &= 4.9, \\
d_{44} &= -1.54, \\
d_{45} &= -2, \\
d_{46} &= 23.44, \\
c_{41} &= 2.8, \\
c_{42} &= -2.25, \\
c_{43} &= 2.7, \\
x_{10} &= 0.34, \\
x_{20} &= 0.15, \\
x_{30} &= -0.35, \\
x_{40} &= -0.179.
\end{aligned} \tag{37}$$

Figure 9 shows the bifurcation diagram in which the horizontal axis represents the change of the parameter b_{14} and the vertical axis represents the displacement in the x direction or y direction of the rotor-AMB system with 16-pole legs and the time-varying stiffness. From the bifurcation diagram, it is demonstrated that the rotor-AMB system begins to change from the chaotic motion to the periodic motion, then to the chaotic motion, and finally to the periodic motion when the value of the parameter b_{14} changes from 10 to 50. There are many periodic windows among the regions of the chaotic motions, which mean the frequencies of the responses are related to the value of the parameter b_{14} . It is helpful for diagnosing the working condition of the rotor-AMB system with 16-pole legs and the time-varying stiffness. It is also found that the amplitude changes are different in the x direction and y direction when the parameter b_{14} increases. This means that choosing the proper values of some parameters could avoid the large amplitudes of the nonlinear vibrations in a specific direction.

The changing procedure of the nonlinear dynamic responses is investigated for the rotor-AMB system with 16-pole legs and the time-varying stiffness, as shown in Figures 10–19. Figure 10 illustrates that the chaotic motion of the rotor-AMB system occurs when the parameter b_{14} reaches 14 and other parameters are the same as those given in Figure 9. Figures 9(a)–9(b), respectively, represent the phase portraits on the planes (x_1, x_2) and (x_3, x_4) , the waveforms on the planes (t, x_1) and (t, x_3) , and the three-dimensional phase portraits. It is observed from

Figures 10(c) and 10(d) that the largest amplitudes of x_1 and x_3 are nearly equal to 1.2–1.4 and 2.2–2.4.

The chaotic motion of the rotor-AMB system with 16-pole legs and the time-varying stiffness also occurs when the parameter $b_{14} = 18.3$, as shown in Figure 11. It is seen from Figures 11(c) and 11(d) that the largest amplitudes of x_1 and x_3 are nearly equal to 1.5–1.6 and 2.2–2.3. Figure 12 demonstrates the existence of the periodic-4 motion for the horizontal and the vertical modes of the rotor-AMB system with 16-pole legs and the time-varying stiffness when the parameter $b_{14} = 20$. It is found from Figures 12(c) and 12(d) that the largest amplitudes of x_1 and x_3 are nearly equal to 1.3–1.4 and 2.2–2.3.

Continuously increasing the parameter b_{14} to 23, we find that chaotic motion again exists in the rotor-AMB system with 16-pole legs and the time-varying stiffness, as shown in Figure 13. It is known from Figures 13(c) and 13(d) that the largest amplitudes of x_1 and x_3 are nearly equal to 1.5–1.6 and 2.1–2.2. Figure 14 indicates that the quasiperiodic response of the rotor-AMB system with 16-pole legs and the time-varying stiffness occurs when the parameter $b_{14} = 25$. From Figures 14(c) and 14(d), we see that the largest amplitudes of x_1 and x_3 are nearly equal to 1.5–1.6 and 2.1–2.2. Figure 15 illustrates that the periodic-7 motion of the rotor-AMB system occurs when the parameter b_{14} changes to 31. From Figures 15(c) and 15(d), we observe that the largest amplitudes of x_1 and x_3 are nearly equal to 1.5–1.6 and 1.9–2.0. These phenomena given in Figures 10–15 are in good agreement with that given in Figure 9 for the rotor-AMB system with 16-pole legs and the time-varying stiffness.

As the parameter b_{14} increases to 38, the quasiperiodic response of the rotor-AMB system reoccurs, as shown in Figure 16. It is observed from Figures 16(c) and 16(d) that the asymmetry periodic vibration around the center occurs for the rotor-AMB system with 16-pole legs and the time-varying stiffness. The positive largest amplitudes of x_1 and x_3 are nearly equal to 1.5 and 1.3–1.4. The negative largest amplitudes of x_1 and x_3 are nearly equal to 1.2 and 1.8–1.9. In Figure 17, the periodic-2 motion occurs in the rotor-AMB system with 16-pole legs and the time-varying stiffness when the parameter $b_{14} = 40$. Based on Figures 17(c) and 17(d), we also find that the asymmetry periodic vibration around the center occurs for the rotor-AMB system with 16-pole legs and the time-varying stiffness.

It is found from Figure 18 that the periodic-3 motion occurs when the parameter b_{14} changes to 43. As the parameter b_{14} reaches 50, the rotor-AMB system performs a reverse double bifurcation to a periodic-1 motion, which can be seen in Figure 19. From Figures 18(c), 18(d), 19(c), and 19(d), it is seen that the asymmetry periodic vibration around the center occurs for the rotor-AMB system with 16-pole legs and the time-varying stiffness. Based on the bifurcation diagrams given in Figures 9–19, it is concluded that the periodic, quasiperiodic, and chaotic motions occur alternately in the rotor-AMB system with 16-pole legs and the time-varying stiffness.

FIGURE 9: Bifurcation diagrams of the (a) horizontal mode and (b) vertical mode via the change of the parameter b_{14} .

FIGURE 10: Continued.

FIGURE 10: Chaotic motion exists when $b_{14} = 14$: (a) phase portrait on the plane (x_1, x_2) ; (b) phase portrait on the plane (x_3, x_4) ; (c) waveform on the plane (t, x_1) ; (d) waveform on the plane (t, x_3) ; (e) phase portrait in the three-dimensional space (x_1, x_2, x_3) ; (f) phase portrait in the three-dimensional space (x_1, x_3, x_4) .

FIGURE 11: Continued.

FIGURE 11: Chaotic motion occurs when $b_{14} = 18.3$.

FIGURE 12: Continued.

FIGURE 12: Periodic-4 motion occurs when $b_{14} = 20$.

FIGURE 13: Continued.

FIGURE 13: Chaotic motion exists when $b_{14} = 23$.

FIGURE 14: Continued.

FIGURE 14: Quasiperiodic motion occurs when $b_{14} = 25$.

FIGURE 15: Continued.

FIGURE 15: Periodic-7 motion occurs when $b_{14} = 31$.

FIGURE 16: Continued.

FIGURE 16: Quasiperiodic motion occurs when $b_{14} = 38$.

FIGURE 17: Continued.

FIGURE 17: Periodic-2 motion occurs when $b_{14} = 40$.

FIGURE 18: Continued.

FIGURE 18: Periodic-3 motion occurs when $b_{14} = 43$.

FIGURE 19: Continued.

FIGURE 19: Periodic-1 motion occurs when $b_{14} = 50$.

6. Conclusions

In this paper, we use the asymptotic perturbation method to analyze the frequency responses, the bifurcation diagrams, and the periodic and chaotic motions of a rotor-AMB system with 16-pole legs, the time-varying stiffness, and the quadratic and cubic nonlinearities. The magnetic force resultants are obtained using the electromagnetic theory, which include the quadratic and cubic nonlinearities. According to the classical Newton law, the ordinary differential governing equation of motion is derived for the rotor-AMB system with 16-pole legs and the time-varying stiffness. It can be observed that the rotor-AMB system is modeled as a two-degree-of-freedom nonlinear dynamic system. The four-dimensional averaged equation of the two-degree-of-freedom system is obtained through the asymptotic perturbation method under the case of 1:1 internal resonance and 1:2 subharmonic resonances. The stability of the steady-state solutions of the modal amplitudes for the rotor-AMB system with 16-pole legs is investigated under different parameters by using the frequency-response curves. The influences of the external excitation on nonlinear dynamic behaviors of the rotor-AMB system with 16-pole legs are numerically studied. It is found that the occurrence of the periodic, quasiperiodic, and chaotic motions depends on the magnitude of the external excitation in the rotor-AMB system with 16-pole legs and the time-varying stiffness under certain parameter conditions.

From the frequency-response analysis, it is found that the hardening-type nonlinearity and softening-type nonlinearity are observed in the rotor-AMB system with 16-pole legs and the quadratic and cubic nonlinearities. The jumping and multivalued phenomena of the amplitudes for the responses under different parameter conditions are investigated. It is demonstrated that the detuning parameters for the frequencies of the excitations have significant influences on the amplitudes and can control the vibration state of the rotor-AMB system with 16-pole legs, the time-varying stiffness, and the quadratic and cubic nonlinearities.

It is demonstrated from the numerical simulation results that the periodic, quasiperiodic, and chaotic motions for the

rotor-AMB system with 16-pole legs, the time-varying stiffness, and the quadratic and cubic nonlinearities can be controlled by adjusting the parameter b_{14} under the certain parametric conditions. The different nonlinear dynamic responses of the rotor-AMB system occurred under different values of the parameter b_{14} , and the specific vibration state can be diagnosed by monitoring the frequency change. Since the parameter b_{14} investigated here represents the external excitation, the required nonlinear dynamic response can be obtained by changing the magnitude of the external excitation. Moreover, the parameter b_{14} is closely related to the eccentricity of the rotor in the rotor-AMB system. Thus, the unbalance eccentricity can be considered an important factor which can induce or avoid the large amplitudes of the nonlinear vibrations for the rotor-AMB system in a specific direction. The results obtained above indicate that the parameter b_{14} related to the unbalance eccentricity and the external excitation has significant influence on the nonlinear dynamic responses of the rotor-AMB system with 16-pole legs, the time-varying stiffness, and the quadratic and cubic nonlinearities. The influences of this parameter must be considered in the structural dynamic design and optimization of the rotor-AMB system.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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