Research Article

Bifurcation Analysis for a Simple Dual-Rotor System with Nonlinear Intershaft Bearing Based on the Singularity Method

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This paper aims to classify bifurcation modes for two interrelated primary resonances of a simple dual-rotor system under double frequency excitations. The four degree-of-freedom (4DOF) dynamic equations of the system considering the nonlinearity of the intershaft bearing can be obtained by using the assumed mode method (AMM) and Lagrange's equation. A simplified method for dynamic equations is developed due to the symmetry of rotors, based on which the amplitude frequency equations for two interrelated primary resonances are obtained by using the multiple scales method. Furthermore, the validity of the simplified method for dynamic equations and the amplitude frequency equations solved by the multiple scales method are confirmed by numerical verification. Afterwards, the bifurcation analysis for two interrelated primary resonances is carried out according to the two-state-variable singularity method. There exist a total of three different types of bifurcation modes because of double frequency excitations of the dual-rotor system and the nonlinearity of the intershaft bearing. The second primary resonance is more prone to have nonlinear dynamic characteristics than the first primary resonance. This discovery indicates that two interrelated primary resonances of the dual-rotor system may have different bifurcation modes under the same dynamic parameters.

1. Introduction

With the development of aero-engines towards higher thrust-weight ratio and better aerodynamic stability [1], the dual-rotor structure has been widely used in the field of aero-engines. As a vital part between the lower pressure (LP) rotor and the higher pressure (HP) rotor in the dual-rotor system especially in aero-engines of fighters, the nonlinearity of the intershaft bearing [2] has achieved considerable attention [3]. There exist double frequency excitations, i.e., unbalanced excitations of LP and HP rotors, in the dual-rotor system due to the complexity of its structure. Moreover, under the nonlinear influence of the intershaft bearing, the dynamic behaviors of dual-rotor system will become much more complex and difficult to predict. Therefore, to clarify bifurcation modes of the dual-rotor system under different dynamic parameters is very contributive to avoid some unexpected nonlinear dynamic behaviors [4, 5], so as to improve the operation safety of aero-engines.

There indeed exists some literature about the dynamic behaviors of the dual-rotor system in recent decades. Hibner [6] proposed a unique method, namely, the transfer-matrix method, to estimate vibratory responses of a dual-rotor system with the nonlinear damping coefficient in an aero-engine. Gupta et al. [7] developed a dual-rotor test rig to simulate the two spool aero-engine dynamically and compared experimental results and theoretical results for unbalance responses of the dual-rotor system. Ferraris et al. [8] presented a simple but realistic nonsymmetric coaxial dual-rotor system and focused on the dynamic behaviors of the dual-rotor system under corotation and counterrotation. Different from Gupta et al. [7], a dual-rotor test rig composed of two coaxial shafts was developed by Guskov et al. [9], where two shafts are linked by an intershaft bearing and rotate individually. Zhang et al. [10, 11] successively put forward the whole-beat correlation method and the non-whole beat correlation method to identify unbalance responses of a dual-rotor system with a slight rotation speed difference. Yang et al. [12] established a dual-rotor system

with imbalance-rubbing coupled fault to study dynamic responses of aero-engines with fixed-point rubbing. Wang et al. [13, 14] also concentrated on the dynamic responses a dual-rotor structure of a whole aero-engine with blade-casing rubbing. Sun et al. [15, 16] investigated the nonlinear dynamical behaviors of a dual-rotor aero-engine with rub impact by means of an analytic method, namely, MHB-AFT (multiharmonic balance-alternating frequency/time domain). Nevertheless, the nonlinearity of the intershaft bearing is not considered in these dual-rotor models, and none of the above literature have classified bifurcation modes of the dual-rotor system.

The singularity theory is a mathematical method to study the morphology and classification of differentiable mapping near singular points [17]. Until 1985, Golubitsky and Schaeffer [18] elaborated the mathematical deductions and applications about the singularity theory in their books, based on which the singularity theory became a powerful mathematical tool to investigate bifurcation modes of the nonlinear dynamic system. Chen et al. [19, 20] applied the steady-state bifurcation theory of the dynamic system to investigate topological structures of bifurcation modes for a nonlinear dynamic system, and they proposed a new method, namely, C-L method, to study bifurcation modes for the nonlinear dynamic system of a single bifurcation parameter. Furthermore, Qin et al. [21, 22] developed the single state-variable singularity method into the two-state-variable singularity method through rigorous mathematical proof and gave mathematical formulas of the transition sets, i.e., the bifurcation set, the hysteresis set, and the double limit set. This research applied the two-state-variable singularity method to investigate bifurcation modes for the nonlinear dual-rotor system.

The purpose of this paper is to investigate bifurcation modes for two interrelated primary resonances of a dual-rotor system based on the two-state-variable singularity method, where the double frequency excitations of the dual-rotor system and the nonlinearity of the intershaft bearing are taken into consideration. A simplified method for dynamic equations is developed due to the symmetry of rotors, and the bifurcation parameters of the dual-rotor system are also reduced because of the amplitude frequency equations solved by the multiple scales method. Afterwards, the numerical integral is performed to confirm the validity of the simplified method and the amplitude frequency equations. Furthermore, the bifurcation analysis for two interrelated primary resonances of the dual-rotor system could be carried out based on the two-state-variable singularity method. It shows the second primary resonance is more prone to have nonlinear dynamic characteristics than the first primary resonance.

2. Dynamic Equations of the Dual-Rotor System

2.1. Dual-Rotor System. A simple two-disk dual-rotor system supported by four points with an intershaft bearing [15, 23] is depicted in Figure 1. The system is consisted of one LP rotor, one HP rotor, and an intershaft bearing. The LP rotor comprises a flexible shaft and a disk, while the HP rotor comprises a rigid rotor and a disk. The intershaft bearing, located between the LP rotor and HP rotor, is considered as a nonlinear elastic

FIGURE 1: A simple two-disk dual-rotor system supported by four points with an intershaft bearing.

spring, which is the nonlinear source of the dual-rotor system. Herein, the subscript 1 denotes the LP rotor and the subscript 2 denotes the HP rotor without special instructions; k , c , ω , and O are the stiffness coefficient, the damping coefficient of the shaft, the rotation speed of the rotor, and the geometric center of the disk; l_i ($i = 1 \sim 4$) and l are the lengths of shafts.

Figure 2 illustrates the lateral vibration of the dual-rotor system. The LP rotor is assumed as a flexible rotor while the HP rotor is assumed as a rigid rotor [24], since the LP shaft is much more slender than the HP shaft. Herein, x_1 and y_1 represent vertical and horizontal displacements of the LP disk; x_2 and y_2 represent vertical and horizontal displacements of the HP disk.

The LP shaft and HP shaft are considered as the Euler-Bernoulli beam, in which the deformations of shear and torsion are ignored. According to the assumed mode method [25], the LP shaft is a flexible shaft; thus, the modal shape function is assumed as a sine function, as

$$f_1(z_1) = \sin\left(\frac{\pi z_1}{l}\right), \quad (z_1 \in [0, l]). \quad (1)$$

At any portion z_1 , the vertical displacement u_1 and horizontal displacement v_1 of the LP shaft can be represented by vertical displacement x_1 and horizontal displacement y_1 of the LP disk due to the modal shape function equation (1), as

$$\begin{cases} u_1(z_1, t) = \frac{f_1(z_1)}{f_1(l_1)} x_1(t) = \frac{f_1(z_1)}{f_1(l_1)} x_1, \\ v_1(z_1, t) = \frac{f_1(z_1)}{f_1(l_1)} y_1(t) = \frac{f_1(z_1)}{f_1(l_1)} y_1. \end{cases} \quad (2)$$

The rotational angles of the LP shaft around x -axis θ_x and y -axis θ_y are both very small, and thus, they can also be represented by x_1 and y_1 , as

$$\begin{cases} \theta_x(z_1, t) \approx -\frac{\partial v_1(z_1, t)}{\partial z_1} = -\frac{df_1(z_1)}{f_1(l_1)dz_1} y_1(t) = -h_1(z_1) y_1, \\ \theta_y(z_1, t) \approx \frac{\partial u_1(z_1, t)}{\partial z_1} = \frac{df_1(z_1)}{f_1(l_1)dz_1} x_1(t) = h_1(z_1) x_1, \end{cases} \quad (3)$$

where $h_1(z_1) = (df_1(z_1)/f_1(l_1)dz_1)$.

FIGURE 2: Lateral vibration of the dual-rotor system.

The LP disk is fixed with the LP shaft at $z_1 = l_1$; thus, displacements and rotational angles of LP disk are just equal to displacements and rotational angles of LP shaft at $z_1 = l_1$.

The kinetic energy of the LP rotor contains two parts, one is the kinetic energy of the LP disk T_{d1} and the other is the kinetic energy of the LP shaft T_{s1} , as

$$\begin{aligned} T_{d1} &= \frac{1}{2} m_1 [\dot{u}_1^2(l_1) + \dot{v}_1^2(l_1)] + \frac{1}{2} J_{d1} [\dot{\theta}_x^2(l_1) + \dot{\theta}_y^2(l_1)] + \frac{1}{2} J_{p1} [\omega_1^2 - 2\omega_1 \theta_x(l_1) \dot{\theta}_y(l_1)] \\ &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} J_{d1} h_1^2(l_1) (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} J_{p1} (\omega_1^2 + 2\omega_1 h_1^2(l_1) \dot{x}_1 y_1), \end{aligned} \quad (4a)$$

$$\begin{aligned} T_{s1} &= \frac{\rho_1 \pi r_1^2}{2} \int_0^l (\dot{u}_1^2 + \dot{v}_1^2) dz_1 + \frac{\rho_1 I_1}{2} \int_0^l (\dot{\theta}_x^2 + \dot{\theta}_y^2) dz_1 + \rho_1 I_1 \int_0^l (\omega_1^2 - 2\omega_1 \theta_x \dot{\theta}_y) dz_1 \\ &= \frac{\rho_1 \pi r_1^2}{2 f_1^2(l_1)} (\dot{x}_1^2 + \dot{y}_1^2) \int_0^l f_1^2 dz_1 + \frac{\rho_1 I_1}{2} (\dot{x}_1^2 + \dot{y}_1^2) \int_0^l h_1^2 dz_1 + \rho_1 I_1 \left(\omega_1^2 l + 2\omega_1 \dot{x}_1 y_1 \int_0^l h_1^2 dz_1 \right), \end{aligned} \quad (4b)$$

where m_1 is the mass of the LP rotor; J_{p1} and J_{d1} are the LP disk's polar and diameter moment of inertia; r_1 and ρ_1 are the radius and the density of the LP shaft (solid shaft); and the LP shaft's area moment of inertia is $I_1 = (\pi r_1^4/4)$.

The HP shaft is a stubby shaft, which can be considered as a rigid rotor; thus, the modal shape function is

$$f_2(z_2) = z_2 - l_2, \quad z_2 \in (l_2, l_4). \quad (5)$$

At any portion z_2 , the vertical displacement u_2 and the horizontal displacement v_2 of the HP shaft can be represented by vertical displacement x_2 and horizontal displacement y_2 of the HP disk due to the modal shape function equation (5), as

$$\begin{cases} u_2(z_2, t) = \frac{f_2(z_2)}{f_2(l_3)} x_2(t) = \frac{f_2(z_2)}{f_2(l_3)} x_2, \\ v_2(z_2, t) = \frac{f_2(z_2)}{f_2(l_3)} y_2(t) = \frac{f_2(z_2)}{f_2(l_3)} y_2. \end{cases} \quad (6)$$

The rotational angles of HP shaft around x -axis φ_x and y -axis φ_y are both very small; thus, they can also be represented by x_2 and y_2 , as

$$\begin{cases} \varphi_x(z_2, t) \approx -\frac{\partial v_2(z_2, t)}{\partial z_2} = -\frac{df_2(z_2)}{f_2(l_3)dz_2} y_2(t) = -h_2(z_2) y_2, \\ \varphi_y(z_2, t) \approx \frac{\partial u_2(z_2, t)}{\partial z_2} = \frac{df_2(z_2)}{f_2(l_3)dz_2} x_2(t) = h_2(z_2) x_2, \end{cases} \quad (7)$$

where $h_2(z_2) = (df_2(z_2)/f_2(l_3)dz_2) = (1/(l_3 - l_2))$.

The HP disk is fixed with the HP shaft at $z_2 = l_3$; thus, displacements and rotational angles of HP disk are just equal to displacements and rotational angles of HP shaft at $z_2 = l_3$. The kinetic energy of the HP rotor contains two parts, one is the kinetic energy of the HP disk T_{d2} and the other is the kinetic energy of the HP shaft T_{s2} , as

$$\begin{aligned} T_{d2} &= \frac{1}{2} m_2 [\dot{u}_2^2(l_3) + \dot{v}_2^2(l_3)] + \frac{1}{2} J_{d2} (\dot{\varphi}_x^2(l_3) + \dot{\varphi}_y^2(l_3)) + \frac{1}{2} J_{p2} (\omega_2^2 - 2\omega_2 \varphi_x(l_3) \dot{\varphi}_y(l_3)) \\ &= \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} J_{d2} h_2^2(l_3) (\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} J_{p2} (\omega_2^2 + 2\omega_2 h_2^2(l_3) \dot{x}_2 y_2), \end{aligned} \quad (8a)$$

$$\begin{aligned} T_{s2} &= \frac{\rho_2 \pi (r_3^2 - r_2^2)}{2} \int_{l_2}^{l_4} (\dot{u}_2^2 + \dot{v}_2^2) dz_2 + \frac{\rho_2 I_2}{2} \int_{l_2}^{l_4} (\dot{\varphi}_x^2 + \dot{\varphi}_y^2) dz_2 + \rho_2 I_2 \int_{l_2}^{l_4} (\omega_2^2 - 2\omega_2 \varphi_x \dot{\varphi}_y) dz_2 \\ &= \frac{\rho_2 \pi (r_3^2 - r_2^2)}{2 f_2^2(l_3)} (\dot{x}_2^2 + \dot{y}_2^2) \int_{l_2}^{l_4} f_2^2 dz_2 + \frac{\rho_2 I_2}{2} (\dot{x}_2^2 + \dot{y}_2^2) \int_{l_2}^{l_4} h_2^2 dz_2 + \rho_2 I_2 \left[\omega_2^2 (l_4 - l_2) + 2\omega_2 \dot{x}_2 y_2 \int_{l_2}^{l_4} h_2^2 dz_2 \right], \end{aligned} \quad (8b)$$

where m_2 is the mass of the HP rotor; J_{p2} and J_{d2} are the HP disk's polar and diameter moment of inertia; and r_2 , r_3 , and ρ_2 are the inner and the outer radiuses, and the density of the HP shaft (hollow shaft); the HP shaft's area moment of inertia is $I_2 = (\pi(r_3^4 - r_2^4)/4)$.

Thus, the kinetic energy of the system can be obtained as

$$T = T_{d1} + T_{s1} + T_{d2} + T_{s2}. \quad (9)$$

The potential energy of the system is

$$V = \frac{1}{2}k_1(x_1^2 + y_1^2) + \frac{1}{2}k_2(x_2^2 + y_2^2), \quad (10)$$

where k_1 and k_2 are the stiffness coefficients of LP shaft and HP shaft.

Rayleigh's dissipation energy of the system is

$$D = \frac{1}{2}c_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}c_2(\dot{x}_2^2 + \dot{y}_2^2), \quad (11)$$

where c_1 and c_2 are the damping coefficients of LP shaft and HP shaft.

The generalized force virtual work of the system is

$$\begin{aligned} \delta W = & m_1\omega_1^2 e_1 \cos(\omega_1 t) \delta x_1 + m_1\omega_1^2 e_1 \sin(\omega_1 t) \delta y_1 \\ & + m_2\omega_2^2 e_2 \cos(\omega_2 t) \delta x_2 + m_2\omega_2^2 e_2 \sin(\omega_2 t) \delta y_2 \\ & - F_x(F_1 \delta x_1 - F_2 \delta x_2) - F_y(F_1 \delta y_1 - F_2 \delta y_2), \end{aligned} \quad (12)$$

where $F_1 = (f_1(l_4)/f_1(l_1))$ and $F_2 = (f_2(l_4)/f_2(l_3))$; F_x and F_y represent vertical and horizontal nonlinear restoring forces of the intershaft bearing; e_1 and e_2 represent unbalances of LP and HP rotors; and $\delta \bullet (x_1, y_1, x_2, y_2)$ represents

the corresponding virtual displacement in the corresponding generalized coordinate.

2.2. Nonlinear Restoring Forces of the Intershaft Bearing.

The inner race and the LP rotor are interference fit at $z_1 = l_4$; thus, vertical displacement x_i and horizontal displacement y_i of the inner race are just equal to vertical displacement and horizontal displacement of the LP shaft at $z_1 = l_4$; substituting $z_1 = l_4$ into equation (2), we obtain

$$\begin{cases} x_i = u_1(l_4, t) = \frac{f_1(l_4)}{f_1(l_1)}x_1 = F_1x_1, \\ y_i = v_1(l_4, t) = \frac{f_1(l_4)}{f_1(l_1)}y_1 = F_1y_1. \end{cases} \quad (13)$$

The outer race and the HP rotor are interference fit at $z_2 = l_4$; thus, vertical displacement x_o and horizontal displacement y_o of the outer race are just equal to vertical displacement and horizontal displacement of the HP shaft at $z_2 = l_4$; substituting $z_2 = l_4$ into equation (6), we obtain

$$\begin{cases} x_o = u_2(l_4, t) = \frac{f_2(l_4)}{f_2(l_1)}x_2 = F_2x_2, \\ y_o = v_2(l_4, t) = \frac{f_2(l_4)}{f_2(l_1)}y_2 = F_2y_2. \end{cases} \quad (14)$$

The intershaft bearing can be seen as a nonlinear elastic spring [26, 27], which is consisted of a linear stiffness k_s and a cubic nonlinear stiffness k_n . Therefore, vertical and horizontal nonlinear restoring forces of the intershaft bearing are

$$\begin{aligned} F_x = & k_s(x_i - x_o) + k_n(x_i - x_o)[(x_i - x_o)^2 - 3(y_i - y_o)^2] \\ = & k_s(F_1x_1 - F_2x_2) + k_n(F_1x_1 - F_2x_2)[(F_1x_1 - F_2x_2)^2 - 3(F_1y_1 - F_2y_2)^2], \end{aligned} \quad (15a)$$

$$\begin{aligned} F_y = & k_s(y_i - y_o) + k_n(y_i - y_o)[(y_i - y_o)^2 - 3(x_i - x_o)^2] \\ = & k_s(F_1y_1 - F_2y_2) + k_n(F_1y_1 - F_2y_2)[(F_1y_1 - F_2y_2)^2 - 3(F_1x_1 - F_2x_2)^2], \end{aligned} \quad (15b)$$

where k_s denotes the linear stiffness of the intershaft bearing while k_n denotes the cubic nonlinear stiffness of the intershaft bearing.

Comparing equations (15a) and (15b) with conventional cubic nonlinear elastic spring, the symmetry of the intershaft bearing in vertical and horizontal directions is taken into consideration for the nonlinear restoring forces of the intershaft bearing.

2.3. Dynamic Equations of the Dual-Rotor System. The 4DOF dynamic equations of the dual-rotor system can be obtained with the aid of Lagrange's equation of the second kind, as

$$M_1\ddot{x}_1 + c_1\dot{x}_1 + \omega_1 g_1 \dot{y}_1 + k_1x_1 = m_1\omega_1^2 e_1 \cos \omega_1 t - F_1F_x, \quad (16a)$$

$$M_1\ddot{y}_1 + c_1\dot{y}_1 - \omega_1 g_1 \dot{x}_1 + k_1y_1 = m_1\omega_1^2 e_1 \sin \omega_1 t - F_1F_y, \quad (16b)$$

$$M_2\ddot{x}_2 + c_2\dot{x}_2 + \omega_2 g_2 \dot{y}_2 + k_2x_2 = m_2\omega_2^2 e_2 \cos \omega_2 t + F_2F_x, \quad (16c)$$

$$M_2\ddot{y}_2 + c_2\dot{y}_2 - \omega_2 g_2 \dot{x}_2 + k_2y_2 = m_2\omega_2^2 e_2 \sin \omega_2 t + F_2F_y, \quad (16d)$$

where $M_1 = m_1 + H_1^2 J_{d1} + (\rho_1 \pi r_1^2 / f_1^2(l_1)) \int_0^l f_1^2 dz_1 + \rho_1 I_1 \int_0^l h_1^2 dz_1$, $M_2 = m_2 + H_2^2 J_{d2} + (\rho_2 \pi (r_3^2 - r_2^2) / f_2^2(l_3)) \int_{l_2}^{l_3} f_2^2 dz_2 + \rho_2 I_2 \int_{l_2}^{l_3} h_2^2 dz_2$, $g_1 = H_1^2 J_{p1} + 2\rho_1 I_1 \int_0^l h_1^2 dz_1$, and $g_2 = H_2^2 J_{p2} + 2\rho_2 I_2 \int_{l_2}^{l_3} h_2^2 dz_2$. Moreover, the gyroscopic effect is reflected in the parameters M_1, M_2, g_1 , and g_2 .

The dimensionless dynamic equations of the dual-rotor system are

$$X_1'' + C_1 X_1' + G_1 Y_1' + \frac{k_1}{M_1 \omega_1^2} X_1 = E_1 \cos \tau - \frac{F_1}{M_1 \omega_1^2} F_X, \quad (17a)$$

$$Y_1'' + C_1 Y_1' - G_1 X_1' + \frac{k_1}{M_1 \omega_1^2} Y_1 = E_1 \sin \tau - \frac{F_1}{M_1 \omega_1^2} F_Y, \quad (17b)$$

$$X_2'' + C_2 X_2' + G_2 Y_2' + \frac{k_2}{M_2 \omega_1^2} X_2 = \lambda^2 E_2 \cos(\lambda \tau) + \frac{F_2}{M_2 \omega_1^2} F_X, \quad (17c)$$

$$Y_2'' + C_2 Y_2' - G_2 X_2' + \frac{k_2}{M_2 \omega_1^2} Y_2 = \lambda^2 E_2 \sin(\lambda \tau) + \frac{F_2}{M_2 \omega_1^2} F_Y, \quad (17d)$$

where $\tau = \omega_1 t$, $\lambda = (\omega_2 / \omega_1)$ (rotation speed ratio), $E_1 = (m_1 / M_1)$, $E_2 = (m_2 / M_2) \cdot (e_2 / e_1)$, $X_1 = (x_1 / e_1)$, $Y_1 = (y_1 / e_1)$, $X_2 = (x_2 / e_1)$, $Y_2 = (y_2 / e_1)$, $C_1 = (c_1 / M_1 \omega_1)$, $C_2 = (c_2 / M_2 \omega_1)$, $G_1 = (g_1 / M_1)$, and $G_2 = (\lambda g_2 / M_2)$ are the dimensionless parameters.

The dimensionless nonlinear restoring force of the intershaft bearing is

$$\begin{aligned} \begin{bmatrix} F_X \\ F_Y \end{bmatrix} &= \begin{bmatrix} \frac{F_x}{e_1} & \frac{F_y}{e_1} \end{bmatrix}^T = \begin{bmatrix} k_s (X_i - X_o) + e_1^2 k_n (X_i - X_o) [(X_i - X_o)^2 - 3(Y_i - Y_o)^2] \\ k_s (Y_i - Y_o) + e_1^2 k_n (Y_i - Y_o) [(Y_i - Y_o)^2 - 3(X_i - X_o)^2] \end{bmatrix} \\ &= \begin{bmatrix} k_s (F_1 X_1 - F_2 X_2) + e_1^2 k_n (F_1 X_1 - F_2 X_2) [(F_1 X_1 - F_2 X_2)^2 - 3(F_1 Y_1 - F_2 Y_2)^2] \\ k_s (F_1 Y_1 - F_2 Y_2) + e_1^2 k_n (F_1 Y_1 - F_2 Y_2) [(F_1 Y_1 - F_2 Y_2)^2 - 3(F_1 X_1 - F_2 X_2)^2] \end{bmatrix}. \end{aligned} \quad (18)$$

The parameters of the dual-rotor system [23] are shown as follows:

$$\begin{aligned} m_1 &= 97.37 \text{ kg}, \\ J_{p1} &= 3.6907 \text{ kg} \cdot \text{m}^2, \\ J_{d1} &= 1.8454 \text{ kg} \cdot \text{m}^2, \\ m_2 &= 108.3 \text{ kg}, \\ J_{p2} &= 4.0114 \text{ kg} \cdot \text{m}^2, \\ J_{d2} &= 2.0060 \text{ kg} \cdot \text{m}^2, \\ k_1 &= k_2 = 8 \times 10^7 \text{ N/m}, \\ c_1 &= c_2 = 600 \text{ N} \cdot \text{s/m}, \\ \rho_1 &= \rho_2 = 7800 \text{ kg/m}^3, \\ e_1 &= 1 \mu\text{m}, \\ e_2 &= 3 \mu\text{m}, \\ l &= 1.5913 \text{ m}, \\ l_1 &= 0.2250 \text{ m}, \\ l_2 &= 0.3555 \text{ m}, \\ l_3 &= 0.8675 \text{ m}, \\ l_4 &= 1.4918 \text{ m}, \\ r_1 &= 20 \text{ mm}, \\ r_2 &= 30 \text{ mm}, \\ r_3 &= 35 \text{ mm}, \\ \lambda &= 1.2, \\ k_s &= 3 \times 10^8 \text{ N/m}, \\ k_n &= 1 \times 10^{23} \text{ N/m}^3. \end{aligned} \quad (19)$$

3. Multiple Scales Solution

3.1. Dynamic Equations Transferring. Considering the symmetry of LP and HP rotors in vertical and horizontal directions, the 4DOF equations (17a)–(17d) can be simplified into a 2DOF equation with the help of a complex-number coordinate [28]. Let $r_1 = X_1 + iY_1$, $r_2 = X_2 + iY_2$, and $F_b = F_X + iF_Y$ ($i^2 = -1$); equations (17a)–(17d) are transformed into

$$r_1'' + (C_1 - iG_1)r_1' + \frac{k_1}{M_1 \omega_1^2} r_1 = E_1 e^{i\tau} - \frac{F_1}{M_1 \omega_1^2} F_b, \quad (20a)$$

$$r_2'' + (C_2 - iG_2)r_2' + \frac{k_2}{M_2 \omega_1^2} r_2 = \lambda^2 E_2 e^{i\lambda\tau} + \frac{F_2}{M_2 \omega_1^2} F_b. \quad (20b)$$

The nonlinear force of the intershaft bearing in the complex-number coordinate (deduction process in Appendix) is

$$\begin{aligned} F_b &= F_X + iF_Y = k_s (r_i - r_o) + e_1^2 k_n (\bar{r}_i - \bar{r}_o)^3 \\ &= k_s (F_1 r_1 - F_2 r_2) + e_1^2 k_n (F_1 \bar{r}_1 - F_2 \bar{r}_2)^3. \end{aligned} \quad (21)$$

Substitute the nonlinear force equation (21) into equations (20a) and (20b) and transpose terms as

$$r_1'' + \frac{k_1 + F_1^2 k_s}{M_1 \omega_1^2} r_1 - \frac{F_1 F_2 k_s}{M_1 \omega_1^2} r_2 = E_1 e^{i\tau} - (C_1 - iG_1) r_1' - \frac{F_1 e_1^2 k_n}{M_1 \omega_1^2} (F_1 \bar{r}_1 - F_2 \bar{r}_2)^3, \quad (22a)$$

$$r_2'' - \frac{F_1 F_2 k_s}{M_2 \omega_1^2} r_1 + \frac{k_2 + F_2^2 k_s}{M_2 \omega_1^2} r_2 = \lambda^2 E_2 e^{i\lambda\tau} - (C_2 - iG_2) r_2' + \frac{F_2 e_2^2 k_n}{M_2 \omega_1^2} (F_1 \bar{r}_1 - F_2 \bar{r}_2)^3. \quad (22b)$$

Equations (22a) and (22b) in a matrix form is

$$\mathbf{r}'' + \mathbf{K}\mathbf{r} = \mathbf{Q}, \quad (23)$$

where $\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$, $\mathbf{K} =$

$$(1/\omega_1^2) \begin{bmatrix} (k_1 + F_1^2 k_s)/M_1 & -(F_1 F_2 k_s)/M_1 \\ -(F_1 F_2 k_s)/M_2 & (k_2 + F_2^2 k_s)/M_2 \end{bmatrix}, \quad \text{and}$$

$$\mathbf{Q} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} E_1 e^{i\tau} - (C_1 - iG_1) r_1' - (F_1 e_1^2 k_n / M_1 \omega_1^2) \\ (F_1 \bar{r}_1 - F_2 \bar{r}_2)^3 \lambda^2 E_2 e^{i\lambda\tau} - (C_2 - iG_2) r_2' + (F_2 e_2^2 k_n / M_2 \omega_1^2) \end{bmatrix}.$$

$$(F_1 \bar{r}_1 - F_2 \bar{r}_2)^3 \lambda^2 E_2 e^{i\lambda\tau} - (C_2 - iG_2) r_2' + (F_2 e_2^2 k_n / M_2 \omega_1^2)$$

$$(F_1 \bar{r}_1 - F_2 \bar{r}_2)^3].$$

Eigenvectors and eigenvalues of the matrix $\mathbf{K}$ are

$$(\Phi, \Omega) = \text{eig}(\mathbf{K}), \quad (24)$$

where $\text{eig}(\cdot)$ is a function to compute eigenvectors and eigenvalues of the matrix.

Let eigenvalues be $\Omega = (1/\omega_1^2) \begin{bmatrix} \Omega_1^2 & 0 \\ 0 & \Omega_2^2 \end{bmatrix}$, where Ω_1 and

Ω_2 are the first- and second-order critical speed of the sys-

tem. And let eigenvectors be $\Phi = \begin{bmatrix} p_1 & p_2 \\ 1 & 1 \end{bmatrix}$; the inverse

matrix of eigenvectors is $\Phi^{-1} = \begin{bmatrix} q_1 & 1 - q_2 \\ -q_1 & q_2 \end{bmatrix}$, where $q_1 =$

$$(1/(p_1 - p_2)) \text{ and } q_2 = (p_1/(p_1 - p_2)).$$

Let both sides of equation (23) be premultiplied by Φ^{-1} and then expanded into a 2DoF equation as

$$R_1'' + \frac{\Omega_1^2}{\omega_1^2} R_1 = q_1 Q_1 + (1 - q_2) Q_2, \quad (25a)$$

$$R_2'' + \frac{\Omega_2^2}{\omega_1^2} R_2 = -q_1 Q_1 + q_2 Q_2, \quad (25b)$$

where $R_1 = q_1 r_1 + (1 - q_2) r_2$ and $R_2 = -q_1 r_1 + q_2 r_2$.

There are some assumptions as follows:

- (1) The damping of the system is weak damping, and the nonlinearity of the system is weak nonlinearity
- (2) $\Omega_2 \gg \Omega_1$ ($\Omega_2 = \eta \Omega_1$, $\eta \approx 5.10$), i.e., second-order critical speed is far greater than first-order critical speed. This paper focuses on primary resonances, i.e., first-order critical speed

The dual-rotor system is excited by double frequency excitations, i.e., the unbalances of LP rotor and HP rotor. Therefore, first-order critical speed contains two interrelated primary resonances, one of which is excited by the unbalance of the HP rotor and the other is excited by the unbalance of the LP rotor [23]. When the dual-rotor system rotates at a constant rotation speed ratio $\lambda = (\omega_2/\omega_1)$, in general, $\lambda > 1$ because the HP rotor rotates faster than the LP rotor. Thus, with the increase of the rotation speed, the first primary resonance is excited by the unbalance of the HP rotor at $\omega_1 \approx (\Omega_1/\lambda)$ ($\omega_2 \approx \Omega_1$) while the second primary resonance is excited by the unbalance of the LP rotor at $\omega_1 \approx \Omega_1$.

3.2. First Primary Resonance at $\omega_1 \approx (\Omega_1/\lambda)$ ($\omega_2 \approx \Omega_1$). Time scales $T_1 = \tau$ and $T_2 = \varepsilon\tau$, and the tuning parameter σ_1 satisfies

$$\frac{\Omega_1^2}{\omega_1^2} = \lambda^2 - \varepsilon\sigma_1. \quad (26)$$

Assume that solutions of equations (25a) and (25b) are

$$R_1 = R_{11} + \varepsilon R_{12}, \quad (27a)$$

$$R_2 = R_{21} + \varepsilon R_{22}. \quad (27b)$$

Substitute equations (26)–(27b) into equations (25a) and (25b) and equate coefficients of ε^0 and ε^1 to zero, as ε^0 is

$$D_1^2 R_{11} + \lambda^2 R_{11} = q_1 E_1 e^{iT_1}, \quad (28a)$$

$$D_1^2 R_{21} + (\lambda\eta)^2 R_{21} = -q_1 E_1 e^{iT_1}. \quad (28b)$$

Also, ε^1 is

$$\begin{aligned} D_1^2 R_{12} + \lambda^2 R_{12} = & -2D_1 D_2 R_{11} + \sigma_1 R_{11} + (1 - q_2) \lambda^2 E_2 e^{i\lambda T_1} \\ & - q_1 (C_1 - iG_1) (p_1 D_1 R_{11} + p_2 D_1 R_{21}) \\ & - (1 - q_2) (C_2 - iG_2) (D_1 R_{11} + D_1 R_{21}) \\ & + \gamma_1 [(p_1 F_1 - F_2) \bar{R}_{11} + (p_2 F_1 - F_2) \bar{R}_{21}]^3, \end{aligned} \quad (29a)$$

$$\begin{aligned} D_1^2 R_{22} + (\lambda\eta)^2 R_{22} = & -2D_1 D_2 R_{21} + \sigma_1 \eta^2 R_{21} - q_2 \lambda^2 E_2 e^{i\lambda T_1} \\ & + q_1 (C_1 - iG_1) (p_1 D_1 R_{11} + p_2 D_1 R_{21}) \\ & - q_2 (C_2 - iG_2) (D_1 R_{11} + D_1 R_{21}) \\ & + \gamma_2 [(p_1 F_1 - F_2) \bar{R}_{11} + (p_2 F_1 - F_2) \bar{R}_{21}]^3, \end{aligned} \quad (29b)$$

where $D_1 = (\partial(\cdot)/\partial T_1)$, $D_2 = (\partial(\cdot)/\partial T_2)$, $\gamma_1 = [(1 - q_2) F_2 / M_2 \omega_1^2] - (q_1 F_1 / M_1 \omega_1^2) e_1^2 k_n$, $\gamma_2 = [(q_2 F_2 / M_2 \omega_1^2) + (q_1 F_1 / M_1 \omega_1^2)] e_1^2 k_n$, and $\eta = (\Omega_2/\Omega_1)$.

Assume that general solutions of equations (28a) and (28b) are

$$R_{11} = A_1 e^{i(\lambda T_1 + \varphi_1)} + cc + a_1 e^{iT_1}, \quad (30a)$$

$$R_{21} = A_2 e^{i(\lambda\eta T_1 + \varphi_2)} + cc + a_2 e^{iT_1}, \quad (30b)$$

where $a_1 = (q_1 E_1 / (\lambda^2 - 1))$ and $a_2 = -(q_1 E_1 / ((\lambda\eta)^2 - 1))$.

Substitute equations (30a) and (30b) into equations (29a) and (29b), and then, eliminate secular terms and separate real and imaginary parts, as

$$D_2 A_1 = -\frac{1}{2\lambda} [\lambda \alpha_4 A_1 + \lambda^2 (1 - q_2) E_2 \sin \varphi_1], \quad (31a)$$

$$A_1 D_2 \varphi_1 = -\frac{1}{2\lambda} [\alpha_1 A_1^3 + \alpha_2 A_1 A_2^2 + (\lambda \alpha_3 + \sigma_1) A_1 + \lambda^2 (1 - q_2) E_2 \cos \varphi_1], \quad (31b)$$

$$D_2 A_2 = -\frac{1}{2} \beta_4 A_2, \quad (31c)$$

$$A_2 D_2 \varphi_2 = -\frac{1}{2\lambda \eta} A_2 [\beta_1 A_2^2 + \beta_2 A_1^2 + (\lambda \beta_3 + \eta \sigma_1) \eta], \quad (31d)$$

where $\alpha_1 = 3\gamma_1 (p_1 F_1 - F_2)^3$, $\alpha_2 = 6\gamma_1 (p_1 F_1 - F_2) (p_2 F_1 - F_2)^2$, $\alpha_3 = -q_1 p_1 G_1 - (1 - q_2) G_2$, $\alpha_4 = q_1 p_1 C_1 + (1 - q_2) C_2$, $\beta_1 = 3\gamma_2 (p_2 F_1 - F_2)^3$, $\beta_2 = 6\gamma_2 (p_1 F_1 - F_2)^2 (p_2 F_1 - F_2)$, $\beta_3 = q_1 p_2 G_1 - q_2 G_2$, and $\beta_4 = -q_1 p_2 C_1 + q_2 C_2$.

Equate right sides of equations (31a)–(31d) to zero; we obtain $A_2 = 0$ due to equation (31c). Since A_2 is the amplitude of the harmonic term with frequency Ω_2 at $\omega_1 \approx (\Omega_1/\lambda)$, $A_2 = 0$ is correct when $\Omega_2 \gg \Omega_1$ (assumption 2). Therefore, equations (31a)–(31d) are simplified as

$$D_2 A_1 = -\frac{1}{2\lambda} [\lambda \alpha_4 A_1 + \lambda^2 (1 - q_2) E_2 \sin \varphi_1], \quad (32a)$$

$$A_1 D_2 \varphi_1 = -\frac{1}{2\lambda} [\alpha_1 A_1^3 + (\lambda \alpha_3 + \sigma_1) A_1 + \lambda^2 (1 - q_2) E_2 \cos \varphi_1]. \quad (32b)$$

Finally, eliminate φ_1 in equations (32a) and (32b); the amplitude frequency equation for the first primary resonance at $\omega_1 \approx (\Omega_1/\lambda)$ is

$$\alpha_1^2 A^3 + 2\alpha_1 (\lambda \alpha_3 + \sigma_1) A^2 + [(\lambda \alpha_3 + \sigma_1)^2 + \lambda^2 \alpha_4^2] A - \lambda^4 (1 - q_2)^2 E_2^2 = 0, \quad (33)$$

where $A = A_1^2$.

3.3. Second Primary Resonance at $\omega_1 \approx \Omega_1$. Time scales $T_1 = \tau$ and $T_2 = \varepsilon \tau$, and the tuning parameter σ_2 satisfies

$$\frac{\Omega_1^2}{\omega_1^2} = 1 - \varepsilon \sigma_2. \quad (34)$$

Assume that solutions of equations (25a) and (25b) are

$$R_1 = R_{11} + \varepsilon R_{12}, \quad (35a)$$

$$R_2 = R_{21} + \varepsilon R_{22}. \quad (35b)$$

Substitute equations (34)–(35b) into equations (25a) and (25b) and equate coefficients of ε^0 and ε^1 to zero, as ε^0 is

$$D_1^2 R_{11} + R_{11} = (1 - q_2) \lambda^2 E_2 e^{i\lambda T_1}, \quad (36a)$$

$$D_1^2 R_{21} + \eta^2 R_{21} = q_2 \lambda^2 E_2 e^{i\lambda T_1}. \quad (36b)$$

Also ε^1 is

$$\begin{aligned} D_1^2 R_{12} + R_{12} = & -2D_1 D_2 R_{11} + \sigma_2 R_{11} + q_1 E_1 e^{iT_1} \\ & - q_1 (C_1 - iG_1) (p_1 D_1 R_{11} + p_2 D_1 R_{21}) \\ & - (1 - q_2) (C_2 - iG_2) (D_1 R_{11} + D_1 R_{21}) \\ & + \gamma_1 [(p_1 F_1 - F_2) \bar{R}_{11} + (p_2 F_1 - F_2) \bar{R}_{21}]^3, \end{aligned} \quad (37a)$$

$$\begin{aligned} D_1^2 R_{22} + \eta^2 R_{22} = & -2D_1 D_2 R_{21} + \sigma_2 \eta^2 R_{21} - q_1 E_1 e^{iT_1} \\ & + q_1 (C_1 - iG_1) (p_1 D_1 R_{11} + p_2 D_1 R_{21}) \\ & - q_2 (C_2 - iG_2) (D_1 R_{11} + D_1 R_{21}) \\ & + \gamma_2 [(p_1 F_1 - F_2) \bar{R}_{11} + (p_2 F_1 - F_2) \bar{R}_{21}]^3. \end{aligned} \quad (37b)$$

Assume that general solutions of equations (36a) and (36b) are

$$R_{11} = B_1 e^{i(T_1 + \varphi_1)} + cc + b_1 e^{i\lambda T_1}, \quad (38a)$$

$$R_{21} = B_2 e^{i(\eta T_1 + \varphi_2)} + cc + b_2 e^{i\lambda T_1}, \quad (38b)$$

where $b_1 = ((1 - q_2) \lambda^2 E_2 / (1 - \lambda^2))$ and $b_2 = (q_2 \lambda^2 E_2 / (\eta^2 - \lambda^2))$.

Substitute equations (38a) and (38b) into equations (37a) and (37b), and then, eliminate secular terms and separate real and imaginary parts, as

$$D_2 B_1 = -\frac{1}{2} (\alpha_4 B_1 + q_1 E_1 \sin \varphi_2), \quad (39a)$$

$$B_1 D_2 \varphi_1 = -\frac{1}{2} [\alpha_1 B_1^3 + \alpha_2 B_1 B_2^2 + (\alpha_3 + \sigma_2) B_1 + q_1 E_1 \cos \varphi_2], \quad (39b)$$

$$D_2 B_2 = -\frac{1}{2} \beta_4 B_2, \quad (39c)$$

$$B_2 D_2 \varphi_2 = -\frac{1}{2\eta} B_2 [\beta_1 B_2^2 + \beta_2 B_1^2 + (\beta_3 + \eta \sigma_2) \eta]. \quad (39d)$$

Equate right sides of equations (39a)–(39d) to zero; we obtain $B_2 = 0$ due to equation (39c). Since B_2 is the amplitude of the harmonic term with frequency Ω_2 at $\omega_1 \approx \Omega_1$, $B_2 = 0$ is correct when $\Omega_2 \gg \Omega_1$ (assumption 2). Therefore, equations (39a)–(39d) are simplified as

$$D_2 B_1 = -\frac{1}{2} (\alpha_4 B_1 + q_1 E_1 \sin \varphi_2), \quad (40a)$$

$$B_1 D_2 \varphi_2 = -\frac{1}{2} [\alpha_1 B_1^3 + (\alpha_3 + \sigma_2) B_1 + q_1 E_1 \cos \varphi_2]. \quad (40b)$$

Finally eliminating φ_2 in equations (40a) and (40b), the amplitude frequency equation for the second primary resonance at $\omega_1 \approx \Omega_1$ is

$$\alpha_1^2 B^3 + 2\alpha_1 (\alpha_3 + \sigma_2) B^2 + [(\alpha_3 + \sigma_2)^2 + \alpha_4^2] B - E_1^2 q_1^2 = 0, \quad (41)$$

where $B = B_1^2$.

3.4. Stability Analysis. The Floquet theory [29] is applied to confirm the stability of results solved by the multiple scales method, and the Floquet index can be figured out according to Hsu's method [30, 31].

Let $\mathbf{X} = [R_1, R_2, R'_1, R'_2]^T = [X_1, X_2, X_3, X_4]^T$; transform equations (25a) and (25b) into $\mathbf{X}'(\tau) = \mathbf{F}(\mathbf{X}(\tau), \tau)$, and $\mathbf{X}^*$ denotes equilibriums of equations (25a) and (25b). Take a perturbation $\Delta\mathbf{X}$ to $\mathbf{X}^*$, as

$$\Delta\mathbf{X}' \approx \frac{\partial \mathbf{F}(\tau, \mathbf{X}^*)}{\partial \mathbf{X}^*} \Delta\mathbf{X} = \mathbf{A}(\tau, \mathbf{X}^*(\tau)) \Delta\mathbf{X}. \quad (42)$$

The stability of $\mathbf{X}^*$ can be confirmed according to the Jacobi matrix $\mathbf{A}(\tau, \mathbf{X}^*(\tau))$.

The approximate monodromy matrix can be obtained according to Hsu's method, as

$$\mathbf{M} = \Psi(T) = \prod_{n=1}^{N_n} e^{\mathbf{A}_n \Delta T} = \prod_{n=1}^{N_n} \left(\mathbf{I} + \sum_{j=1}^{N_j} \frac{(\mathbf{A}_n \Delta T)^j}{j!} \right), \quad (43)$$

where $\mathbf{I}$ represents the identity matrix, T denotes the time period of the solution, which is divided into N_n subintervals, ΔT is the length of one subinterval, $\mathbf{A}_n$ is a constant matrix which represents the time-varying matrix $\mathbf{A}(\tau, \mathbf{X}^*(\tau))$, and N_j represents the number of terms in the approximation of the constant matrix $\mathbf{A}_n$ exponential. The Floquet index of matrix $\mathbf{M}$ is available by calculating eigenvalues, and the stability of $\mathbf{X}^*$ can be confirmed.

3.5. Numerical Verification. In order to verify the validity of the simplified method for dynamic equations and the amplitude frequency equations solved by the multiple scales method, the multiple scales solutions of the simplified 2DOF equations (22a) and (22b) are compared with Runge-Kutta solutions of original 4DOF equations (17a) and (17b) around the first-order critical speed as shown in Figure 3. The ode45 function in MATLAB (Runge-Kutta method) is applied to solve original 4DOF equations (17a) and (17b). The initial states of motion for the first rotation speed $\omega_1 = 600$ rad/s are set as $[X_1 \ X'_1 \ Y_1 \ Y'_1 \ X_2 \ X'_2 \ Y_2 \ Y'_2]^T = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$, but for other rotation speeds, the initial states of motion are set as the final states of motion for the previous rotation speed. Integration duration should be long enough to ensure steady-state responses; for equations (17a) and (17b), integration duration are set as $\tau = [0, 2000\pi]$. The root mean square (RMS) [32] is applied to represent the amplitude of the vibration solved by the Runge-Kutta method. The amplitudes of LP and HP rotor

are $r_j = \sqrt{(\sum_{i=1}^N (X_j(i) - \bar{X}_j)^2 + (Y_j(i) - \bar{Y}_j)^2)/N}$, $j = 1, 2$, where j is the number of points in one period and $\bar{X}_j, \bar{Y}_j$ are the average values of corresponding response.

In Figure 3, it can be seen that the amplitude frequency curve of the LP rotor is basically the same as that of the HP rotor in appearance, but the amplitude of the LP rotor is much greater than that of the HP rotor, which is because the LP rotor is much more slender than the HP rotor. There are two primary resonances A and B caused by double frequency excitations, which are controlled by amplitude frequency equations (33) and (41). Furthermore, the multiple scales solutions almost coincide with the Runge-Kutta solutions, which verify the validity of the simplified method for dynamic equations proposed in this paper and provide a theoretical fundament for the multiple scales method or other analytical methods to solve the nonlinear dynamic equations of the dual-rotor system. The Runge-Kutta solutions are slightly larger than the multiple scales solutions, because dynamic responses solved by the multiple scales method only contain one frequency excitation which plays the most important role while others are ignored. Moreover, the unstable solutions can be attained by this method, while these cannot be attained by the Runge-Kutta method. It can save a lot of calculation time and cost. The results solved by this method can well reveal the inherent mechanism of some unexpected nonlinear phenomena, such as the vibration jump and the resonance hysteresis. All in all, the validity of the simplified method for dynamic equations and the amplitude frequency equations solved by the multiple scales method has been confirmed, based on which the bifurcation analysis can be carried out afterwards.

4. Bifurcation Analysis

The bifurcation characteristics for two interrelated primary resonances caused by double frequency excitations of the dual-rotor system are investigated in this section. Nevertheless, the local bifurcation characteristics for two primary resonances mostly depend on dynamic parameters of the dual-rotor and intershaft bearing systems, in which the linear stiffness and nonlinear stiffness of the intershaft bearing play the most important roles. Therefore, this section focuses on the effect of the intershaft bearing's linear stiffness and nonlinear stiffness on the bifurcation characteristics for two interrelated primary resonances. Considering the complexity of the amplitude frequency equations, a two-state-variable singularity method [21, 22, 33] is applied to describe bifurcation modes for two primary resonances of the dual-rotor system in an engineering way. Amplitude frequency equations (33) and (41) are directly taken as the engineering unfolding, instead of the universal unfolding. The engineering unfolding may not be logical in mathematics, but it is of very important significance for engineering.

Substitute $\sigma_1 = \lambda^2 - \sigma$ into equation (33) and $\sigma_2 = 1 - \sigma$ into equation (41), and take $\sigma = (\Omega_1^2/\omega_1^2)$ as the bifurcation parameter and α_1 and α_4 as the unfolding parameters; the left side of equations (33) and (41) could be viewed as the engineering unfolding for two primary resonances of the dual-rotor system, as

FIGURE 3: Amplitude frequency curves solved by the multiple scales method and the Runge-Kutta method. (a) For the LP rotor. (b) For the HP rotor. Solid lines represent stable solutions, and dotted lines represent unstable solutions.

$$\begin{aligned} F_1 = & \alpha_1^2 A^3 + 2\alpha_1(\lambda\alpha_3 + \lambda^2 - \sigma)A^2 \\ & + \left[(\lambda\alpha_3 + \lambda^2 - \sigma)^2 + \lambda^2\alpha_4^2 \right] A - \lambda^4(1 - q_2)^2 E_2^2, \end{aligned} \quad (44a)$$

$$\begin{aligned} F_2 = & \alpha_1^2 B^3 + 2\alpha_1(\alpha_3 + 1 - \sigma)B^2 \\ & + \left[(\alpha_3 + 1 - \sigma)^2 + \alpha_4^2 \right] B - E_1^2 q_1^2. \end{aligned} \quad (44b)$$

The derivatives of F_1 and F_2 can be expressed as

$$F_{1A} = 3\alpha_1^2 A^2 + 4\alpha_1(\lambda\alpha_3 + \lambda^2 - \sigma)A + (\lambda\alpha_3 + \lambda^2 - \sigma)^2 + \lambda^2\alpha_4^2, \quad (45a)$$

$$F_{1AA} = 6\alpha_1^2 A + 4\alpha_1(\lambda\alpha_3 + \lambda^2 - \sigma), \quad (45b)$$

$$F_{1\sigma} = -2\alpha_1 A^2 - 2(\lambda\alpha_3 + \lambda^2 - \sigma)A, \quad (45c)$$

$$F_{2B} = 3\alpha_1^2 B^2 + 4\alpha_1(\alpha_3 + 1 - \sigma)B + (\alpha_3 + 1 - \sigma)^2 + \alpha_4^2, \quad (45d)$$

$$F_{2BB} = 6\alpha_1^2 B + 4\alpha_1(\alpha_3 + 1 - \sigma), \quad (45e)$$

$$F_{2\sigma} = -2\alpha_1 B^2 - 2(\alpha_3 + 1 - \sigma)B, \quad (45f)$$

$$F_{1B} = F_{2A} = F_{1AB} = F_{2AB} = 0. \quad (45g)$$

Transition sets are consisted of three parts, which are bifurcation set, hysteresis set, and double limit set. The bifurcation set indicates that unfolding equations (44a) and (44b) contain a singular point. The hysteresis set indicates that the unfolding makes at least quadratic contact with the vertical plane of a constant bifurcation parameter. The double limit set indicates that the unfolding exists for two different limit points for the same bifurcation parameter [21]. According to the two-variable singularity method, transition sets for the bifurcation equations can be expressed as follows:

Bifurcation set is as follows:

$$\text{Bi} = \left\{ (\alpha_1, \alpha_4) \in R^2 \mid \begin{aligned} & \exists (A, B, \sigma) \text{ s.t. } F_1 = 0, F_2 = 0, \\ & F_{1A}F_{2B} - F_{1B}F_{2A} = 0, \\ & F_{1A}F_{2\sigma} - F_{1\sigma}F_{2A} = 0 \end{aligned} \right\}. \quad (46a)$$

Hysteresis set is as follows:

$$\text{Hy} = \left\{ (\alpha_1, \alpha_4) \in R^2 \mid \begin{aligned} & \exists (A, B, \sigma) \text{ s.t. } F_1 = 0, F_2 = 0, F_{1A}F_{2B} - F_{1B}F_{2A} = 0, \\ & F_{1A}A' - F_{1B}B' = 0, F_{1A}f_2 - F_{2A}f_1 = 0, W' \neq 0, W' = (A', B'), \\ & f_1 = F_{1AA}A'^2 + 2F_{1AB}A'B' + F_{1BB}B'^2, \\ & f_2 = F_{2AA}A'^2 + 2F_{2AB}A'B' + F_{2BB}B'^2 \end{aligned} \right\}. \quad (46b)$$

Double limit set is as follows:

FIGURE 4: Transition sets in $k_n - k_s$ parameters plane. FIGURE 5: Amplitude frequency curves for the bifurcation mode of the region I in Figure 4 ($k_n = 1 \times 10^{23} \text{ N/m}^3$, $k_s = 12 \times 10^8 \text{ N/m}$). (a) For the LP rotor. (b) For the HP rotor. Solid lines represent stable solutions, and dotted lines represent unstable solutions.

$$DL = \left\{ (\alpha_1, \alpha_4) \in R^2 \mid \begin{array}{l} \exists (W_1, W_2, \sigma) \text{ s.t. } \mathbf{F}_1 = 0, \mathbf{F}_2 = 0, \\ \mathbf{F}_{1A}\mathbf{F}_{2B} - \mathbf{F}_{1B}\mathbf{F}_{2A} = 0, \\ W_1 \neq W_2, W = (A, B) \end{array} \right\}. \quad (46c)$$

According to above calculation procedure equations (44a)–(46c), the transition sets for two primary resonances of the dual-rotor system can be attained. The transition sets contain two hysteresis set curves H_1 and H_2 , as shown in the $k_n - k_s$ parameters plane in Figure 4 with k_s from 0 to 15 (10^8 N/m) and k_n from 0 to 150 (10^{23} N/m^3). There are three regions divided by H_1 and H_2 in the $k_n - k_s$ parameters plane, which indicates that two primary resonances of the dual-rotor system have three different bifurcation modes. Three different bifurcation modes for three regions I, II, and III can be obtained, as shown in Figures 5–7. Moreover, two critical bifurcation modes for two hysteresis set curves H_1 and H_2 can also be obtained, as shown in Figures 8 and 9.

Figure 5 shows the bifurcation mode of region I in Figure 4, in which the linear stiffness k_s is rather large and the nonlinear stiffness k_n is rather small. It can be found that two

primary resonances A and B have a linear characteristics. The dynamic characteristics of the dual-rotor system are almost the same with the linear system in this case. Figure 6 shows the bifurcation mode of region II in Figure 4, in which the linear stiffness is slightly smaller or the nonlinear stiffness is slightly larger. It can be found that the second primary resonance B shows vibration jump and resonance hysteresis while the first primary resonance A does not, which means B shows the hardening characteristics while A does not. This phenomenon indicates that B is more prone to show nonlinear phenomena than A. Figure 7 shows the bifurcation mode of region III in Figure 4, in which the linear stiffness is rather small and the nonlinear stiffness is rather large. It can be found that both primary resonances A and B show vibration jump and resonance hysteresis, which indicates that both A and B show the hardening characteristics. The dynamic characteristics of the dual-rotor system have essential differences with the linear system in this case.

Furthermore, two critical bifurcation modes for two hysteresis set curves H_1 and H_2 are investigated. Figure 8 shows the bifurcation mode of hysteresis set curve H_1 , in

FIGURE 6: Amplitude frequency curves for the bifurcation mode of the region II in Figure 4 ($k_n = 30 \times 10^{23} \text{N/m}^3$, $k_s = 9 \times 10^8 \text{N/m}$). (a) For the LP rotor. (b) For the HP rotor. Solid lines represent stable solutions, and dotted lines represent unstable solutions.

FIGURE 7: Amplitude frequency curves for the bifurcation mode of the region III in Figure 4 ($k_n = 80 \times 10^{23} \text{N/m}^3$, $k_s = 8 \times 10^8 \text{N/m}$). (a) For the LP rotor. (b) For the HP rotor. Solid lines represent stable solutions, and dotted lines represent unstable solutions.

FIGURE 8: Amplitude frequency curves for the bifurcation mode of the hysteresis set curve H_1 in Figure 4 ($k_n = 20 \times 10^{23} \text{N/m}^3$, $k_s = 10 \times 10^8 \text{N/m}$). (a) For the LP rotor. (b) For the HP rotor. Solid lines represent stable solutions, and dotted lines represent unstable solutions.

FIGURE 9: Amplitude frequency curves for the bifurcation mode of the hysteresis set curve H_2 in Figure 4 ($k_n = 45 \times 10^{23} \text{ N/m}^3$, $k_s = 10 \times 10^8 \text{ N/m}$). (a) For the LP rotor. (b) For the HP rotor. Solid lines represent stable solutions, and dotted lines represent unstable solutions.

which the second primary resonance B just begins to show vibration jump in this case, while the first primary resonance A has linear characteristics. Figure 9 shows the bifurcation mode of hysteresis set curve H_2 , in which the first primary resonance A just begins to show vibration jump in this case, while the second primary resonance B has nonlinear characteristics.

5. Conclusions

In this paper, a simplified method for dynamic equations under double frequency excitations of a simple dual-rotor system considering the nonlinearity of the intershaft bearing has been proposed, based on which bifurcation modes for two interrelated primary resonances caused by double frequency excitations have been investigated theoretically. The multiple scales method has been applied to calculate amplitude frequency equations for two interrelated primary resonances, which have been taken as the engineering unfolding directly to investigate bifurcation modes of the dual-rotor system based on the two-state-variable singularity method. The results show that the linear stiffness and the nonlinear stiffness of the intershaft bearing have a significant influence on bifurcation modes of the system. The transition set, including two hysteresis set curves, divides the parameters plane $k_n - k_s$ into three regions, indicating that there exists three different types of bifurcation modes for two interrelated primary resonances, which are, respectively,

for both primary resonances having linear characteristics and for second primary resonance having hardening characteristics while first primary resonance having linear characteristics and for both primary resonances having hardening characteristics. The future work will concentrate on the experimental study of the dual-rotor system.

Appendix

The nonlinear force of the intershaft bearing in the complex-number coordinate is

$$\begin{aligned}
 F_b &= \frac{F_x}{e_1} + i \frac{F_y}{e_1} \\
 &= k_s (X_i - X_o) + e_1^2 k_n (X_i - X_o) [(X_i - X_o)^2 \\
 &\quad - 3(Y_i - Y_o)^2] + i \{k_s (Y_i - Y_o) + e_1^2 k_n (Y_i - Y_o) \\
 &\quad \cdot [(Y_i - Y_o)^2 - 3(X_i - X_o)^2]\} \\
 &= k_s (r_i - r_o) + e_1^2 k_n \{ (X_i - X_o) [(X_i - X_o)^2 \\
 &\quad - 3(Y_i - Y_o)^2] + i(Y_i - Y_o) [(Y_i - Y_o)^2 - 3(X_i - X_o)^2] \}, \quad (\text{A.1})
 \end{aligned}$$

where the brace $\{ (X_i - X_o) [(X_i - X_o)^2 - 3(Y_i - Y_o)^2] + i(Y_i - Y_o) [(Y_i - Y_o)^2 - 3(X_i - X_o)^2] \}$ is formulated as

$$\begin{aligned}
\{\} &= (X_i^3 - X_o^3 - 3X_i^2X_o + 3X_iX_o^2) - 3(X_iY_i^2 - X_oY_o^2 - X_oY_i^2 - 2X_iY_iY_o + X_iY_o^2 + 2X_oY_iY_o) + i(Y_i^3 - Y_o^3 - 3Y_i^2Y_o + 3Y_iY_o^2) \\
&\quad - 3i(X_i^2Y_i - X_o^2Y_o - X_i^2Y_o - 2X_iX_oY_i + X_o^2Y_i + 2X_iX_oY_o) \\
&= (X_i^3 + iY_i^3 - 3X_iY_i^2 - 3iX_i^2Y) - (X_o^3 + iY_o^3 - 3X_oY_o^2 - 3iX_o^2Y_o) - 3(X_i^2X_o + iY_i^2Y_o - X_oY_i^2 - iX_i^2Y_o - 2X_iY_iY_o - 2iX_iX_oY_i) \\
&\quad + 3(X_iX_o^2 + iY_iY_o^2 - X_iY_o^2 - iX_o^2Y_i - 2X_oY_iY_o - 2iX_iX_oY_o) \\
&= (X_i - iY_i)^3 - (X_o - iY_o)^3 - 3(X_i - iY_i)^2(X_o - iY_o) + 3(X_i - iY_i)(X_o - iY_o)^2 \\
&= \bar{r}_i^3 - \bar{r}_o^3 - 3\bar{r}_i^2\bar{r}_o + 3\bar{r}_i\bar{r}_o^2 \\
&= (\bar{r}_i - \bar{r}_o)^3.
\end{aligned}$$

(A.2)

Thus, the nonlinear force of the intershaft bearing in the complex-number coordinate is

$$\begin{aligned}
F_b &= F_X + iF_Y = k_s(r_i - r_o) + e_1^2 k_n(\bar{r}_i - \bar{r}_o)^3 \\
&= k_s(F_1 r_1 - F_2 r_2) + e_1^2 k_n(F_1 \bar{r}_1 - F_2 \bar{r}_2)^3.
\end{aligned}
\tag{A.3}$$

Data Availability

The dynamic simulation data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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