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Research Article

A Class of Shock Wave Solution to Singularly Perturbed Nonlinear Time-Delay Evolution Equations

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Nonlinear singularly perturbed problem for time-delay evolution equation with two parameters is studied. Using the variables of the multiple scales method, homogeneous equilibrium method, and approximation expansion method from the singularly perturbed theories, the structure of the solution to the time-delay problem with two small parameters is discussed. Under suitable conditions, first, the outer solution to the time-delay initial boundary value problem is given. Second, the multiple scales variables are introduced to obtain the shock wave solution and boundary layer corrective terms for the solution. Then, the stretched variable is applied to get the initial layer correction terms. Finally, using the singularly perturbed theory and the fixed point theorem from functional analysis, the uniform validity of asymptotic expansion solution to the problem is proved. In addition, the proposed method possesses the advantages of being very convenient to use.

1. Introduction

There are many important applications of nonlinear singular perturbation in applied mathematics, engineering, and physics [1, 2]. Recently, some scholars have done a great deal of work, for example, Nicaise and Pignotti [3] considered a stabilization problem for abstract second-order evolution equations with dynamic boundary feedback laws with a time delay and distributed structural damping. They proved an exponential stability result under a suitable condition between the internal damping and the boundary layers. The proof of the main result is based on an identity with multipliers that allows to obtain a uniform decay estimate for a suitable energy functional. Some concrete examples are detailed. Some counterexamples suggest that this condition is optimal.

Jeong et al. [4] considered a quasilinear wave equation $u_{tt} - \Delta u_t - div(|\nabla U|^{\alpha-2}\nabla u) - div(|\nabla u_t|^{\beta-2}\nabla u_t) + a|u_t|^{m-2} + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = b|u|^{p-2}u$, associated with initial and Dirichlet boundary conditions at one part and acoustic boundary conditions at another part, respectively. They proved under suitable conditions and for negative initial energy, a global nonexistence of solutions.

Feng [5] studied the following Cauchy problem with a time-delay term in the internal feedback:

$$\begin{aligned}
&u_{tt}(x,t) - \varphi(x)\left(\Delta u(x,t) - \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s\right) \\
&\quad + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \\
&u(x,0) = u_0(x), \\
&u_t(x,0) = u_1(x), \\
&x \in R^n, \\
&u_t(x,t-\tau) = f_0(x,t-\tau), \\
&x \in R^n, \\
&0 < t < \tau,
\end{aligned} \tag{1}$$

and in order to solve the problem in the noncompactness of some operators, they introduced some weighted spaces.

Under suitable assumptions on the relaxation function, they established a general decay result of solution for the initial-value problem by using the energy perturbation method and their result extends earlier results.

Nicaise and Valein [6] considered abstract second-order evolution equations with unbounded feedback with time delay. Existence results are obtained under some realistic assumptions. Sufficient and explicit conditions are derived that guarantee the exponential or polynomial stability. Some new examples that enter into our abstract framework are presented.

Weidenfeld and Frankel [7] focused on the early evolution of small (linear) perturbations following the sudden (step function) exposure of a liquid layer to a cold adjacent atmosphere. On a short time scale relative to that characterizing thermal relaxation across the liquid layer, the temperature distribution is nonlinear and highly transient. Thus, the conduction reference state may not be regarded quasisteady. They accordingly considered the initial-value problem and obtained a Volterra-type integral equation governing the evolution of surface-temperature perturbations.

Many approximate methods have been improved, such as Graef and Kong [8], Hovhannisyan and Vulanovic [9], Bonfoh et al. [10], Barbu and Cosma [11], Faye et al. [12], Samusenko [13], Mo [14], Das et al. [15, 16], and so on [17–26]. By using the singular perturbation theories, Feng et al. also studied a class of nonlinear singular perturbation problems [27–34].

For instance, Feng and Mo [32] in 2015 considered the nonlinear elliptic boundary value problem with two parameters:

$$\varepsilon^{2m}L^m u + \mu^{2k}L^k u = f(x,u), \quad x \in \Omega,$$
$$\frac{\partial^r u}{\partial n^r} = g_r(x), \quad x \in \partial\Omega, \tag{2}$$

where L is the uniform elliptic operator which can be expressed as follows:

$$L \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j},$$
$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda \sum_{i=1}^{n}\xi_i^2, \tag{3}$$
$$\xi_i \in R,$$
$$\lambda > 0,$$

and by introducing stretched variables, setting local coordinate systems, and using the differential inequalities, the authors proved the existence of the shock solution for boundary value problem and studied the asymptotic behavior of the solution.

Feng et al. [33] in 2017 considered the singularly perturbed boundary value problem for a class of nonlinear integral-differential elliptic equation:

$$\varepsilon^{2m}L^m u - Tu = f(x,u), \quad x \in \Omega,$$
$$\frac{\partial^l u}{\partial n^l} = g_l(x), \quad l = 0, 1, \ldots, m-1,\ x \in \partial\Omega, \tag{4}$$

where L denotes the uniform elliptic type:

$$L = \sum_{i,j=1}^{n} \alpha_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} \beta_i(x)\frac{\partial}{\partial x_i}, \quad x \in R,$$
$$Tu = \int_{\Omega} K(x)u(x), \quad x \in \Omega, \tag{5}$$

and by using the multiple scales variable, the method of component expansion, and the singular perturbation theory, we proved the existence of solution to the problem and the uniformly valid asymptotic estimation.

Feng et al. [34] in 2018 considered a class of nonlinear differential-integral singular perturbation problem for the disturbed evolution equations. Using the singular perturbation method, the structure of solution to problem is discussed in the cases of two small parameters under suitable conditions.

The same authors considered the singular perturbation differential-integral initial boundary value problem of the form

$$\mu\frac{\partial^2 y}{\partial t^2} - \varepsilon L y + T y = f(t,x,\varepsilon y), \quad (t,x) \in (0,T_0)\times\Omega,$$
$$y = g(t,x), \quad x \in \partial\Omega,$$
$$y|_{t=0} = h_1(x),$$
$$\varepsilon\frac{\partial y}{\partial t}\bigg|_{t=0} = h_2(x), \tag{6}$$

where

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} b_i(x)\frac{\partial}{\partial x_i},$$
$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda_i \sum_{i=1}^{n}\xi_i^2, \quad \xi_i \in R,\ \lambda_i > 0,$$
$$T y \equiv \phi(x) + \int_{\partial\Omega} K(x)y(x)\mathrm{d}x, \quad K(x) \geq 0. \tag{7}$$

First, the outer solution to the boundary value problem is given. Second, by constructing the nonsingular coordinate system near the boundary, the variables of multiple scales are introduced to obtain the boundary layer corrective term for the solution. Then, the stretched variable is applied to get the initial layer correction term. Finally, using the fixed point theorem, the uniformly valid asymptotic expansion of the solution to problem is proved. The proposed method possesses the advantages of being convenient to use.

By introducing stretched variables, setting local coordinate systems, and using the differential inequalities, we proved the existence of the shock solution for boundary value problem and studied the asymptotic behavior of the solution.

In this paper, using the variables of the multiple scales method from the singularly perturbed theory, after simplifying the method, we consider a class of shock wave solution to the nonlinear singularly perturbed time-delay evolution equations initial boundary value problem with two parameters as follows, and the structure of the solution to the problem is discussed. In addition, the proposed method possesses the advantages of being very convenient to use.

$$\varepsilon^2\frac{\partial^2 w(t,x)}{\partial t^2} - \mu^2 Lw(t,x) + cw(t-\kappa,x) = F(t,x,w(t,x)), \quad (t,x) \in (-\kappa, T_0] \times \Omega, \tag{8}$$

$$w = g(t,x), \quad x \in \partial\Omega, \tag{9}$$

$$w|_{-k\le t\le 0} = h_1(x), \quad \varepsilon\frac{\partial w}{\partial t}\bigg|_{-k\le t\le 0} = h_2(x), \quad x \in \Omega_\mu, \tag{10}$$

where L signifies a uniformly elliptic operator:

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}, \quad \sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \ge \lambda_i \sum_{i=1}^{n}\zeta_i^2, \quad \forall \zeta_i \in R,\ \lambda > 0. \tag{11}$$

ε and μ are small parameters and $k = b\varepsilon$ is a time-delay parameter, $b > 0$ and c are constants, $x \equiv (x_1, x_2, \ldots, x_n) \in \Omega$, Ω is a bounded convex region in R^n, $\partial\Omega$ denotes boundary of Ω for class $C^{1+\alpha}$, where $\alpha \in (0,1)$ is Hölder exponent, T_0 is a large enough positive constant, and F is a disturbed term.

We have the hypotheses that

$[H_1]\sigma = (\mu/\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

$[H_2]a_{ij}$ and b_i with regard to x are Hölder continuous functions and g and h_i are sufficiently smooth functions in correspondence ranges.

$[H_3]F$ is a sufficiently smooth function in correspondence ranges except $x_0 \in \Omega$, $F_w(t,x,w) \le -\bar{c} < 0\ (x \ne x_0)$, where $\bar{c} > 0$ is a constant.

$[H_4]$ For $cw - F(t,x,w) = 0$, there exists a solution $W_{00}(t,x)$ and $\lim_{x\longrightarrow x_0} W_{00}(t,x) \ne W_{00}(t,x_0)$.

The rest of this paper is organized as follows. In Section 2, we construct the outer solution to the initial boundary value problem (8)–(10). In Sections 3 and 4, we set up a local coordinate system, and then we construct the spike layer corrective term and boundary layer corrective term. In Section 5, we obtain the formal asymptotic expansion solution for the nonlinear singular perturbation time-delay evolution equation initial boundary value problem (8)–(10) with two parameters. At last, in Section 6, we prove that the formal asymptotic expansion solution is uniformly valid.

2. Outer Solution

Now, we construct the outer solution to the problem (8)–(10). First, we develop $w(t-k,x,y)$ in small parameter $k = b\varepsilon$:

$$w(t-k,x) = w(t,x) + \sum_{l=1}^{\infty}\frac{(-b)^l}{l!}\frac{\partial^l w}{\partial k^l}\bigg|_{k=0}\varepsilon^l. \tag{12}$$

The degradation of problem (8)–(10) is

$$cw = F(t,x,w), \quad (x,y) \in \Omega. \tag{13}$$

From the hypotheses, there is a solution $W_{00}(t,x)\ (x \ne x_0)$ to equation (13).

We set $W(t,x)$ as the outer solution to problem (8)–(10), and we have

$$W(t,x) = \sum_{i.j=1}^{\infty} W_{ij}(t,x)\varepsilon^i\mu^j. \tag{14}$$

Substitute equation (14) into equation (8), develop the nonlinear term F in ε and μ, and equate coefficients of the powers $\varepsilon^i\mu^j\ (i,j = 0,1,\ldots, i+j \ne 0)$, respectively.

$$cW_{ij} = F_w(t,x,W_{00})W_{ij} - c\left(\sum_{l=1}^{i}\frac{(-b\varepsilon)^l}{l!}\frac{\partial^l W_{(i-l)j}}{\partial k^l}\bigg|_{k=0}\right) + F_{ij}, \quad (x,y) \in \Omega, \tag{15}$$

where

$$F_{ij} = \frac{1}{i!j!}\frac{\partial^{i+j}F}{\partial\varepsilon^i\partial\mu^j}\bigg|_{\varepsilon=\mu=0}, \quad i,j = 0,1,2,\ldots, i+j \ne 0. \tag{16}$$

From equation (15), we can obtain $W_{ij}(t,x), (i,j = 0,1,\ldots, i+j \ne 0)$. Substituting $W_{00}(t,x)$ and $W_{ij}(t,x), (i,j = 0,1,\ldots, i+j \ne 0)$ into equation (14), we have the outer solution $W(t,x)$ to the original problem. But it does not continue at (t,x_0) and may not satisfy the boundary and initial conditions (9) and (10), so we need to construct the spike layer, boundary layer, and initial layer corrective functions.

3. Spike Layer Corrective Term

Set up a local coordinate system (ρ,ϕ) near $x_0 \in \Omega$. Define the coordinate of every point Q in the neighborhood of x_0 in the following way: the coordinate $\rho(\le \rho_0)$ is the distance from the point Q to x_0, where ρ_0 is small enough. $\phi = (\phi_1, \phi_2, \ldots, \phi_{n-1})$ is a nonsingular coordinate.

In the neighborhood of x_0: $(0 \le \rho \le \rho_0) \subset \Omega$, we have

$$L = \tilde{a}_{nn}\frac{\partial^2}{\partial\rho^2} + \sum_{i=1}^{n-1}\tilde{a}_{ni}\frac{\partial^2}{\partial\rho\,\partial\phi_i} + \sum_{i,j=1}^{n-1}\tilde{a}_{ij}\frac{\partial^2}{\partial\phi_i\partial\phi_j} + \tilde{b}_n\frac{\partial}{\partial\rho} + \sum_{i=1}^{n-1}\tilde{b}_i\frac{\partial}{\partial\phi_i}, \tag{17}$$

where

$$\tilde{a}_{nn} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j},$$
$$\tilde{a}_{ni} = 2\sum_{j,k=1}^{n} a_{jk}\frac{\partial\rho}{\partial x_j}\frac{\partial\phi_i}{\partial x_k},$$
$$\tilde{a}_{ij} = \sum_{k,l=1}^{n} a_{kl}\frac{\partial\phi_i}{\partial x_k}\frac{\partial\phi_j}{\partial x_l}, \tag{18}$$
$$\tilde{b}_n = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2\rho}{\partial x_i\partial x_j},$$
$$\tilde{b}_i = \sum_{j,k=1}^{n} a_{jk}\frac{\partial^2}{\partial x_j}\frac{\partial\phi_i}{\partial x_k}.$$

We introduce the variables of multiple scales [1, 2] on $(0 \le \rho \le \rho_0) \subset \Omega$:

$$\tilde{\sigma} = \frac{h(\rho,\phi)}{\mu}, \quad \tilde{\rho} = \rho,\ \tilde{\phi} = \phi, \tag{19}$$

where $h(\rho,\phi)$ is a function to be determined. For convenience, we still substitute ρ, ϕ for $\tilde{\rho}, \tilde{\phi}$ below, respectively. From equation (17), we have

$$L = \frac{1}{\mu^2}K_0 + \frac{1}{\mu}K_1 + K_2, \tag{20}$$

while $K_0 = \tilde{a}_{nn}h_\rho^2(\partial^2/\partial\tilde{\sigma}^2)$ and K_1, K_2 are determined operators and their constructions are omitted.

Let $h_\rho = (1/\sqrt{\tilde{a}_{nn}})$, and the solution w to the original problem (8)–(10) is

$$w = W(t,x) + W_1(t,\rho,\phi), \tag{21}$$

where W_1 is the spike layer corrective term, and

$$W_1 \sim \sum_{i,j=0}^{\infty} w_{1ij}(t,\rho,\phi)\varepsilon^i\mu^j. \tag{22}$$

Substituting equations (17)–(22) into equation (8), expanding nonlinear terms in ε and μ, and equating the coefficients of $\varepsilon^i\mu^j\,(i,j = 0,1,\ldots)$, respectively, we obtain

$$K_{10}w_{100} = 0, \quad (0 \le \rho \le \rho_0), \tag{23}$$

$$w_{100}\big|_{\rho=0} = W_{00}(t,x_0), \tag{24}$$

$$K_{10}w_{1ij} = G_{ij}, \quad (0 \le \rho \le \rho_0),\ i = 0,1,\ldots,\ i+j \ne 0, \tag{25}$$

$$w_{1ij}\big|_{\rho=0} = W_{ij}(t,x_0), \quad i = 0,1,\ldots,\ i+j \ne 0, \tag{26}$$

where G_{ij}, $(i = 0,1,\ldots,i+j \ne 0)$ are determined functions and their constructions are omitted.

From problems (23)-(24), we can have w_{100}, From w_{100} and equations (25)-(26), we can obtain solutions $w_{1ij}\,(i,j = 0,1,\ldots,\ i+j \ne 0)$ successively.

From the hypotheses, it is easy to see that $w_{1ij}\,(i,j = 0,1,\ldots)$ possesses spike layer behavior:

$$w_{1ij} = O\left(\exp\left(-\delta_{ij}\frac{\rho}{\mu}\right)\right), \quad i,j = 0,1,\ldots, \tag{27}$$

where $\delta_{ij} > 0\,(i,j = 0,1,2,\ldots)$ are constants.

Let $\overline{w}_{1ij} = \psi(\rho)w_{1ij}$, where $\psi(\rho)$ is a sufficiently smooth function in $0 \le \rho \le \rho_0$ and satisfies

$$\psi(\rho) = \begin{cases} 1, & 0 \le \rho \le \frac{1}{3}\rho_0, \\ 0, & \rho \ge \frac{2}{3}\rho_0. \end{cases} \tag{28}$$

For convenience, we still substitute w_{1ij} for $\overline{w}_{1ij}$ below. Then from equation (22), we have the spike layer corrective term W_1 near $(0 \le \rho \le \rho_0) \subset \Omega$.

4. Boundary Layer Corrective Term

Now, we set up a local coordinate system $(\overline{\rho},\overline{\phi})$ in the neighborhood near $\partial\Omega: 0 \le \overline{\rho} \le \overline{\rho}_0$ as Ref. [14], where $\overline{\phi} = (\overline{\phi}_1,\overline{\phi}_2,\ldots,\overline{\phi}_{n-1})$. In the neighborhood of $\partial\Omega:\ 0 \le \overline{\rho} \le \overline{\rho}_0$,

$$L = \overline{a}_{nn}\frac{\partial^2}{\partial\overline{\rho}^2} + \sum_{i=1}^{n-1}\overline{a}_{ni}\frac{\partial^2}{\partial\overline{\rho}\partial\overline{\phi}_i} + \sum_{i,j=1}^{n-1}\overline{a}_{ij}\frac{\partial^2}{\partial\overline{\phi}_i\partial\overline{\phi}_j} + \overline{b}_n\frac{\partial}{\partial\overline{\rho}} + \sum_{i=1}^{n-1}\overline{b}_i\frac{\partial}{\partial\overline{\phi}_i}, \tag{29}$$

where

$$\overline{a}_{nn} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial\overline{\rho}}{\partial x_i}\frac{\partial\overline{\rho}}{\partial x_j},$$
$$\overline{a}_{ni} = 2\sum_{j,k=1}^{n} a_{jk}\frac{\partial\overline{\rho}}{\partial x_j}\frac{\partial\overline{\phi}_i}{\partial x_k},$$
$$\overline{a}_{ij} = \sum_{k,l=1}^{n} a_{kl}\frac{\partial\overline{\phi}_i}{\partial x_k}\frac{\partial\overline{\phi}_j}{\partial x_l}, \tag{30}$$
$$\overline{b}_n = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2\overline{\rho}}{\partial x_i\partial x_j},$$
$$\overline{b}_i = \sum_{j,k=1}^{n} a_{jk}\frac{\partial^2\overline{\phi}_i}{\partial x_j\partial x_k}.$$

We introduce the variables of multiple scales, see the Ref [1, 2], on $(0 \le \overline{\rho} \le \overline{\rho}_0) \subset \Omega$:

$$\overline{\sigma} = \frac{\overline{h}(\overline{\rho},\overline{\phi})}{\mu}, \quad \tilde{\rho} = \overline{\rho},\ \tilde{\phi} = \overline{\phi}, \tag{31}$$

where $\overline{h}(\overline{\rho},\overline{\phi})$ is a function to be determined. For convenience, we still substitute $\overline{\rho},\overline{\phi}$ for $\widetilde{\rho},\widetilde{\phi}$ below, respectively. From (29), we have

$$L=\frac{1}{\mu^2}\overline{K}_0+\frac{1}{\mu}\overline{K}_1+\overline{K}_2, \tag{32}$$

where $\overline{K}_0=\overline{a}_{nn}\overline{h}_{\overline{\rho}}^2(\partial^2/\partial\overline{\sigma}^2)$ and $\overline{K}_1$, $\overline{K}_2$ are determined operators and their constructions are omitted too.

We set the solution w to original boundary value problem (8)–(10), where

$$w=W+U, \tag{33}$$

where U is a boundary layer corrective function.

Set $\overline{h}_\rho(\overline{\rho},\overline{\phi})=(1/\sqrt{\overline{a}_{nn}})$. Thus, we have $\overline{K}_0=(\partial^2/\partial\overline{\sigma}^2)$. And let

$$U\sim\sum_{i,j=0}^{\infty}u_{ij}(t,\overline{\rho},\overline{\phi})\varepsilon^i\mu^j. \tag{34}$$

Substituting equation (34) into equations (8) and (9) and expanding nonlinear terms in ε, μ, we equate the coefficients of the same powers for $\varepsilon^i\mu^j$ $(i,j=0,1,\ldots)$. And we obtain

$$\overline{K}_0u_{00}=0,\quad (0\le\overline{\rho}\le\overline{\rho}_0), \tag{35}$$

$$u_{00}\big|_{\overline{\rho}=0}=g(t,x)-W_{00}(t,x), \tag{36}$$

$$\overline{K}_0u_{ij}=\overline{G}_{ij},\quad (0\le\overline{\rho}\le\overline{\rho}_0),\ i,j=0,1,\ldots,\ i+j\neq0, \tag{37}$$

$$u_{ij}\big|_{\overline{\rho}=0}=-W_{ij}(t,x),\quad i,j=0,1,\ldots,\ i+j\neq0, \tag{38}$$

where $\overline{G}_{ij}\,(i,j=0,1,\ldots,k-1,\ i+j\neq0)$ are determined functions successively, and their constructions are omitted too.

From equations (35) and (36), we can have solution u_{00}. And from equations (37) and (38), we can obtain u_{ij}, $(i,j=0,1,2,\ldots,i+j\neq0)$ successively. Substituting u_{ij}, $(i,j=0,1,2,\ldots)$ into equation (34), we obtain U.

From the hypotheses, it is easy to see that u_{ij}, $(i,j=0,1,\ldots)$ possesses boundary layer behavior:

$$u_{ij}=O\left(\exp\left(-\overline{\delta}_{ij}\frac{\overline{\rho}}{\mu}\right)\right),\quad i,j=0,1,\ldots, \tag{39}$$

where $\overline{\delta}_{ij}>0$, $(i=1,2,\ldots)$ are constants.

Let $\overline{u}_{ij}=\overline{\psi}(\overline{\rho})u_{ij}$, where $\overline{\psi}(\overline{\rho})$ is a sufficiently smooth function in $0\le\overline{\rho}\le\overline{\rho}_0$ and satisfies

$$\overline{\psi}(\overline{\rho})=\begin{cases}1, & 0\le\overline{\rho}\le\frac{1}{3}\overline{\rho}_0,\\ 0, & \overline{\rho}\ge\frac{2}{3}\overline{\rho}_0.\end{cases} \tag{40}$$

For convenience, we still substitute u_{ij} for $\overline{u}_{ij}$ below. Then, from equation (34), we have the boundary corrective term U near $0\le\overline{\rho}\le\overline{\rho}_0$.

5. Initial Layer Corrective Term

The solution w to the original problem (8)–(10) is

$$w=\cdot W+W_1+U+V, \tag{41}$$

where V is an initial layer corrective term. Substituting (41) into equations (8)–(10), we have

$$\begin{aligned}&\varepsilon^2V_{tt}(t,x)-\mu^2LV(t,x)+cV(t-k,x)\\&\quad=F(t,x,W(t,x)+W_1(t,x)+U(t,x)+V(t,x))\\&\qquad-F(t,x,W(t,x)+W_1(t,x)+U(t,x))\\&\qquad-\varepsilon^2(W(t,x)+W_1(t,x)+U(t,x))\\&\qquad-\mu^2L(W(t,x)+W_1(t,x)+U(t,x))\\&\qquad+c(W(t-k,x)+W_1(t-k,x)+U(t-k,x)),\end{aligned} \tag{42}$$

$$V=g(t,x)-W(t,x)-W_1(t,x)-U(t,x),\quad x\in\partial\Omega, \tag{43}$$

$$V|_{-k\le t\le0}=h_1(x)-W|_{-k\le t\le0}-W_1|_{-k\le t\le0}-U|_{-k\le t\le0},\quad x\in\Omega, \tag{44}$$

$$\varepsilon\frac{\partial V}{\partial t}\bigg|_{-k\le t\le0}=h_2(x)-\varepsilon\frac{\partial}{\partial t}(W+W_1+U)\bigg|_{-k\le t\le0},\quad x\in\Omega. \tag{45}$$

We introduce a stretched variable, see [1, 2]: $\tau=(t/\varepsilon)$ and let

$$V\sim\sum_{i,j=0}^{\infty}v_{ij}(\tau,x)\varepsilon^i\mu^j. \tag{46}$$

Substituting equations (14), (22), (34), and (46) into equations (42)–(45), expanding nonlinear terms in ε and μ, and equating the coefficients of like powers of $\varepsilon^i\mu^j$, respectively, for $i,j=0,1,\ldots$, we have

$$\begin{aligned}(v_{00})_{\tau\tau}+cv_{00}&=F(0,x,W_{00}+W_{100}+u_{00}+v_{00})\\&\quad-F(0,x,W_{00}+W_{100}+u_{00}),\end{aligned} \tag{47}$$

$$v_{00}\big|_{x\in\partial\Omega}=0, \tag{48}$$

$$v_{00}\big|_{-b\le\tau\le0}=h_1(x)-(W_{00}(0,x)+W_{100}(0,x)+u_{00}(0,x)),\quad x\in\Omega, \tag{49}$$

$$\frac{\partial v_{00}}{\partial\tau}\bigg|_{-b\le\tau\le0}=h_2(x)-\overline{H}_{00}|_{-b\le\tau\le0}, \tag{50}$$

$$(v_{ij})_{\tau\tau}+cv_{ij}=\overline{G}_{ij},\quad i,j=0,1,\ldots,\ i+j\neq0, \tag{51}$$

$$v_{ij}\big|_{x\in\partial\Omega} = -\left(W_{ij} + W_{1ij} + u_{ij}\right)\big|_{x\in\partial\Omega}, \quad i, j = 0, 1, \ldots, i + j \neq 0, \tag{52}$$

$$v_{ij}\big|_{-b\le\tau\le 0} = \overline{H}_{ij}\big|_{-b\le\tau\le 0}, \quad x \in \Omega,\ i, j = 0, 1, \ldots, i + j \neq 0, \tag{53}$$

$$\frac{\partial v_{ij}}{\partial \tau}\bigg|_{-b\le\tau\le 0} = \overline{H}_{ij}\big|_{-b\le\tau\le 0}, \quad x \in \Omega,\ i, j = 0, 1, \ldots, i + j \neq 0, \tag{54}$$

where $\overline{G}_{ij}, \overline{H}_{ij}\ (i = 0, 1, \ldots, i + j \neq 0)$ are determined functions, and their constructions are omitted too.

From the problem equations (47)–(50), we can have v_{00}. From v_{00} and equations (51)–(54), we can obtain solutions v_{ij}, $(i, j = 0, 1, \ldots, i + j \neq 0)$ successively.

From the hypotheses, it is easy to see that v_{ij}, $(i, j = 0, 1, \ldots)$ possesses boundary layer behavior:

$$v_{ij} = O\left(\exp\left(-\widetilde{\delta}_{ij}\frac{t}{\varepsilon}\right)\right), \quad i, j = 0, 1, \ldots, \tag{55}$$

where $\widetilde{\delta}_{ij} > 0$, $(i, j = 1, 2, \ldots)$ are constants.

Then, from equation (46), we have the initial corrective term V.

From equation (41), we obtain the formal asymptotic expansion of solution w to the nonlinear singular perturbation time-delay evolution equation initial boundary value problem (8)–(10) with two parameters:

$$w \sim W_{00} + \sum_{i.j=0, i+j\neq 0}^{\infty} W_{ij}\varepsilon^i\mu^j + \sum_{i.j=0}^{\infty}\left(w_{1ij} + u_{ij} + v_{ij}\right)\varepsilon^i\mu^j, \quad 0 < \varepsilon, \mu \ll 1, \tag{56}$$

6. Main Result

Now, we prove that this expansion (56) is uniformly valid in Ω and we have the following theorem.

Theorem 1. *Under the hypotheses $[H_1] - [H_4]$, there exists a solution $w(t, x)$ to the nonlinear singular perturbation time-delay evolution equation initial boundary value problem (8)–(10) with two parameters which holds the uniformly valid asymptotic expansion (56) for ε and μ in $(t, x) \in [-k, T_0] \times \overline{\Omega}$.*

Proof. First, we get the remainder term $R(t, x)$ of the initial boundary value problem with two parameters (8)–(10). Let

$$w(t, x) = \vec{w}(t, x) + R(t, x), \tag{57}$$

where

$$\overline{w}(t, x) = W_{00} + \sum_{i,j=0, i+j\neq 0}^{m} W_{ij}\varepsilon^i\mu^j + \sum_{i.j=0}^{m}\left(w_{1ij} + u_{ij} + v_{ij}\right)\varepsilon^i\mu^j. \tag{58}$$

Using equations (14), (27), (39), (55), and (57), we obtain

$$\begin{aligned} H[R] &\equiv \varepsilon^2\frac{\partial^2 R(t, x)}{\partial t^2} - \mu LR(t, x) + cR(t - \tau, x) \\ &\quad - F(t, x, \overline{u} + R(t, x)) - F(t, x, \overline{u}(t, x)) \\ &= O\left(\lambda^{m+1}\right), \quad x \in \Omega,\ \lambda = \max(\varepsilon, \mu), \\ R &= O\left(\lambda^{m+1}\right), \quad x \in \partial\Omega,\ \lambda = \max(\varepsilon, \mu), \\ R|_{-k\le t\le 0} &= O\left(\lambda^{m+1}\right), \quad x \in \Omega,\ \lambda = \max(\varepsilon, \mu), \\ \varepsilon\frac{\partial R}{\partial t}\bigg|_{-k\le t\le 0} &= O\left(\lambda^{m+1}\right), \quad x \in \Omega,\ \lambda = \max(\varepsilon, \mu). \end{aligned} \tag{59}$$

We can have the linearized differential operator $\overline{L}$ as follows:

$$\overline{L}[p] = \varepsilon^2\frac{\partial^2 p(t, x)}{\partial t^2} - \mu^2 L[p(t, x)] + cp(t - \tau, x), \tag{60}$$

and therefore,

$$\begin{aligned} \Psi[p] &\equiv H[p(t, x)] - \overline{L}[p(t, x)] \\ &\quad - F(t, x, \overline{u}(t, x)) - F(t, x, (\overline{u}(t, x) + p(t, x))) \\ &\quad + F_{\overline{u}}(t, x, (\overline{u}(t, x) + p(t, x)))p(t, x). \end{aligned} \tag{61}$$

For fixed ε, μ, the normed linear space N is chosen as

$$N = \left\{p \mid p \in C^2((-k, T_0] \times \Omega), p|_{\partial\Omega} = g, p|_{-k\le t\le 0} = h_1, p_t|_{-k\le t\le 0} = h_2\right\}, \tag{62}$$

with norm

$$\|p\| = \max_{t\in(-k,T_0], x\in\Omega} |p|, \tag{63}$$

and the Banach space B as

$$B = \left\{q \mid q \in C((-k, T_0] \times \Omega)\right\}, \tag{64}$$

with norm

$$\|q\| = \max_{t\in(-k,T_0], x\in\Omega} |q|. \tag{65}$$

From the hypotheses, we may show that the condition

$$\left\|L^{-1}[g]\right\| \le l^{-1}\|g\|, \quad \forall g \in B, \tag{66}$$

by the fixed point theorem [1, 2] is fulfilled where l^{-1} is independent of ε and μ, i.e., L^{-1} is continuous. The Lipschitz condition of the fixed point theorem becomes

$$\begin{aligned} &\left\|\Psi[p_2] - \Psi[p_1]\right\| \\ &< C_1 \max_{t\in(-k,T_0], x\in\Omega}\left\{\left(|p_1| + |p_2|\right)|p_2 - p_1|\right\} \\ &\quad + C_2 \max_{t\in(-k,T_0], x\in\Omega}\left\{|p_1|^2 \cdot |p_2 - p_1|\right\} \\ &< C_3 r\|p_2 - p_1\|, \end{aligned} \tag{67}$$

where C_1, C_2, and C_3 are constants; this inequality is valid for all p_1, p_2 in a ball $K_N(r)$ with $\|r\| \leq 1$. Finally, we obtain the result that the remainder term exists; moreover,

$$\max_{t\in(-k,T_0],x\in\Omega} |R(t,x)| = O\left(\lambda^{m+1}\right), \quad \lambda = \max(\varepsilon, \mu). \tag{68}$$

From equation (56), we have

$$\begin{aligned} w(t,x) \equiv{}& W_{00} + \sum_{i,j=0,i+j\neq 0}^{m} W_{ij}\varepsilon^i\mu^j \\ &+ \sum_{i,j=0}^{m}\left(w_{1ij} + u_{ij} + v_{ij}\right)\varepsilon^i\mu^j \\ &+ O\left(\lambda^{m+1}\right), \quad \lambda = \max(\varepsilon,\mu). \end{aligned} \tag{69}$$

The proof of the theorem is completed. □

7. Conclusions

Nonlinear singularly perturbed problem for time-delay evolution equation with two parameters is an attractive investigated subject. In this paper, we use the variables of the multiple scales method, homogeneous equilibrium method, and approximation expansion method from the singularly perturbed theories, and then we discuss the structure of the solution to the time-delay problem with two small parameters; after setting up a local coordinate system, we obtained the formal asymptotic expansion solution for the nonlinear singular perturbation time-delay evolution equation initial boundary value problem with two parameters; using the singularly perturbed theory and the fixed point theorem from functional analysis, the uniform validity of asymptotic expansion solution to the problem is proved. In addition, the proposed method possesses the advantages of being very convenient to use.

Data Availability

The authors declare that the data included in the article are available, shareable, and referenced. Readers can check each article at http://apps.webofknowledge.com/.

Conflicts of Interest

The authors declare that they have no known conflicts of financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Authors' Contributions

Yihu Feng and Lei Hou wrote the original draft.

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