Research Article

Temperature Effects on Dynamic Properties of Suspended Cables Subjected to Dual Harmonic Excitations

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The paper aims at studying the influences of temperature on the suspended cables' dynamical behaviors subjected to dual harmonic excitations in thermal environments. Significantly, the quadratic nonlinearity and the corresponding secondary resonances are considered. By introducing a tension variation factor, the nonlinear vibration equations of motion could be obtained based on the condensation model. By using Galerkin's procedure, the continuous model of the nonlinear system is reduced to a set of infinite models with quadratic and cubic nonlinearities. By using the multiple scales method, the resultant reduced model is solved and the stability analysis is also presented in two simultaneous resonance cases. Nonlinear dynamical behaviors with thermal effects are presented using bifurcation diagrams, time-history curves, phase portraits, frequency spectrums, and Poincaré sections. The numerical results show that thermal effects induce different scenarios. The sensitivities of linear (natural frequency) and nonlinear (quadratic and cubic) coefficients to temperature variations are different. The temperature may increase or decrease the response amplitudes depending on the excitation amplitude and the sag-to-span ratio. The inflection point is shifted and exhibited at a smaller or larger excitation amplitude in thermal environments. The resonant range between two Pitchfork bifurcations seems to be reduced when the temperature is decreasing. The response amplitude is very sensitive to temperature, and even an opposite spring behavior may be exhibited due to warming/cooling conditions. However, the periodic motions seem independent of temperature variations.

1. Introduction

With the increased significance of multiphysics phenomena occurring in many technological contexts, linear and nonlinear dynamic responses of materials and structures in thermal environments become a research topic of significant interest recently [1, 2]. Temperature becomes a significant factor that may change the dynamic characteristics and responses of some essential structure elements, for example, beam [3], shells [4], pipes [5], and cables [6].

In the past few years, structures with flexible cables have attracted significant attention. The potential application of the fundamental research in practical industries is exhibited, especially the shape-finding [7] and dynamic analysis [8] as well as sliding mode control [9] of pantograph-catenary in

high-speed railway engineering. These flexible systems especially possess low structural damping and small stiffness, and thus any disturbances may cause large-amplitude vibrations [10]. Therefore, a great deal of research has been conducted on the mathematical modeling and linear and nonlinear vibration analyses [11, 12].

Long-span cables experience a significant changing thermal environment during the construction and operation stages. The resultant temperature induces structural deformations that may affect the dynamical behaviors of the nonlinear system. Hence, it is necessary to investigate the linear and nonlinear vibration characteristics of cables after significant deformation induced by temperature effects [13–17]. To be more specific, the static analysis of a cable in thermal environments was presented in Irvine's monograph

[18], and Treysède [19] extended to a new thermal-stressed configuration. Then, a nonlinear dynamic model of the horizontal and inclined cables was established by Lepidi and Gattulli [20]. Furthermore, to explore the suspended cable's nonlinear vibration behaviors in thermal environments more fully and deeply, Zhao et al. presented some studies on the nonlinear primary [21] and secondary [22] resonant responses of suspended cables. Then, the temperature-dependent materials were illustrated in [23]. The influences of temperature on suspended cables' nonlinear vibration behaviors subjected to multifrequency excitations were also discussed by Zhao et al. [24].

On the other hand, considering different types of excitations, for example, direct/indirect excitation and auto-/parametric excitation, the loads applied on structures are very complicated and always contain multifrequency harmonic forces. The rich and complex dynamics of nonlinear oscillations attracted the attention of the research community, leading to exploring the possibility of understanding dynamical behaviors in nonlinear systems subjected to multifrequency excitations [25–39].

Specifically, Ding and Zu [25] presented a study in periodic and chaotic responses of an axially accelerating viscoelastic beam subject to two-frequency excitations. Breslavsky and Amabili [26] investigated the nonlinear vibration of a water-filled, thin circular cylindrical shell, simply supported at the edges, to multiple harmonic excitations. Bauomy and EL-Sayed [27] studied the vibration behaviors of a vertical shaking conveyor subjected to multiple harmonic excitations. A Duffing system with multifrequency slow parametric excitations was investigated to report two novel bursting patterns by Han et al. [28]. Based on recent experimental observations, an inclined marine riser may experience multifrequency vortex-induced vibrations, and Alfosail and Younis [29] investigated the nonlinear motions considering the internal resonances. Briend et al. [30] analyzed the dynamics of an on-board rotor mounted on hydrodynamic finite-length bearings, excited by multifrequency parametric excitations induced by support motions. Ghadiri et al. [31] presented a study on nonlinear vibrations of single-layer graphene sheets resting on a visco-Pasternak foundation subjected to multifrequency excitation and a thermomagnetic field. Kalita and Dwivedy [32] presented a study on the nonlinear dynamics of a pneumatic artificial muscle under simultaneous simple and principle parametric resonance conditions. The analytical solutions and vibration behaviors of a cable-stayed bridge and a cable-stayed cantilever beam subjected to two external harmonic excitations were obtained [33, 39]. Xu et al. [34] investigated the vibration behaviors of multiparametric excitations and multifrequency external excitations of the rolling mill under entry thickness fluctuation of the strip. Ren et al. [35] investigated the amplitude-information-frequency characteristics for an underwater active electrolocation composition system subjected to multifrequency excitations. Jabar et al. [36] analytically and experimentally investigated the nonlinear oscillations of a clamped-clamped microbeam under multifrequency excitations. Tzanov et al. [37] studied the nonlinear resonant

behaviors of a Si fractal NEMS resonator subjected to multifrequency excitations. Considering a piezoelectric nanoplate resting on a viscoelastic foundation, Ebrahimi and Hosseini [38] investigated its nonlinear vibration behaviors subjected to dual harmonic external excitation and thermoelectromechanical loads.

Furthermore, vibration absorber and control considering the multifrequency vibrations are also investigated. Pai et al. [40] presented a study on a new acoustic metamaterial beam consisting of a uniform isotropic beam and small two-mass spring-mass-damper subsystems and acting as a multifrequency absorber. As to a nonlinear system consisting of a cantilever beam with a tip mass, Rezaei et al. [41] analyze the efficient energy harvesting from nonlinear oscillations of PZT beam excited by two simultaneous hard base accelerations. Zheng et al. [42] designed a multifrequency adaptive disturbance cancellation controller for a 6DOF microvibration excitation system. An active control strategy is applied by Sayed et al. [43] in the nonlinear vibrations of a buckled beam. Many simultaneous resonant cases have been investigated, and the worst case is the simultaneous primary and principal parametric in the existence of internal resonance cases.

Whether in practical applications or in laboratory experiments, the suspended cable is always subjected to multifrequency excitations and thermal loads. Hence, understanding the cables' dynamical behaviors considering multifrequency excitations and thermal effects is essential, since the temperature can affect the linear and nonlinear vibration characteristics. It is usually conducted to provide crucial information for damage evaluation and condition assessment. The suspended cable is a classical flexible nonlinear system with quadratic and cubic nonlinearities simultaneously. The initial curvature induces the quadratic nonlinearity, and a new thermal-stressed configuration will be generated in thermal environments. As to a string with a large initial tension force, only the pure cubic nonlinearity is considered. With the decrease in cable tension force, the effect of mixed quadratic and cubic nonlinearities should be considered due to the static configuration's change. In some circumstances, the quadratic nonlinearity may even overcome the cubic one and dominates the vibration behaviors. Besides, it is found out that the sensitivity of quadratic and cubic nonlinearity coefficients to temperature changes is different, and it depends on the cable tension force or the sag-to-span ratio.

In the present work, we still focus our attention on the nonlinear forced vibration response of suspended cables in thermal environments. In particular, we concentrate on the quadratic nonlinearity due to the curvature of the nonlinear system. To the best of the authors' knowledge, considering the primary, the second-order superharmonic, and the one-half-order subharmonic resonances simultaneously, no previous study regarding the nonlinear dynamic behaviors of suspended cables with thermal effects has been reported before. The introduction of the research background and the literature review are illustrated in Section 1. The mathematical model of suspended cable excited by dual harmonic forces in thermal environments is described in Section 2. The

perturbation procedures, as well as the stability analysis, are presented in Section 3. Section 4 gives some numerical examples and discussions about the temperature effects on vibration behaviors. A conclusion closes the whole work in Section 5.

2. Mathematical Modeling

A Cartesian coordinate system $O - xy$ is shown here. The nonlinear flexible system under consideration shown in Figure 1 contains a uniform horizontal cable of length L , initial sag b , thermal-stressed sag $b_{\Delta T}$, Young's modulus E , cross-sectional area A , and mass per length m . The static configuration is the original one without thermal effects. Besides, an extra new static configuration, a thermal-stressed one, is generated due to the cables' expanding or contracting in thermal environments. Therefore, all the tension force, sag, and cable length are closely related to temperature variations. Under the external excitations, the displacements along two directions are expressed as $u(x, t)$ and $v(x, t)$, respectively. Dual distributed harmonic excitation forces along the cable length are applied along the suspended cable.

Generally, a suspended cable in thermal environments is affected by temperature changes in many resources, for example, tension forces (sag-to-span ratios), material properties, and boundary conditions. Here, material properties are assumed to be independent of temperature variations, and the cross-sectional area remains plane and uniform in thermal environments. Hence, by using the direct force method, a cubic equation including the cable tension force is obtained [18, 19].

$$\Delta h^3 + \left(2 + \theta_{\Delta T} + \frac{\lambda^2}{24}\right)\Delta h^2 + \left(1 + 2\theta_{\Delta T} + \frac{\lambda^2}{12}\right)\Delta h + \theta_{\Delta T} = 0, \quad (1)$$

where

$$\begin{aligned} \Delta h &= \frac{h}{H}, \\ \theta_{\Delta T} &= \frac{\alpha \Delta T L_t}{H L_e} E A, \\ \lambda^2 &= \left(\frac{m g L}{H}\right)^2 \frac{E A}{H}, \\ L_t &= L \left(1 + \frac{m^2 g^2 L^2}{8 H^2}\right), \\ L_e &= L \left(1 + \frac{m^2 g^2 L^2}{12 H^2}\right), \end{aligned} \quad (2)$$

where h is an increment/reduction of the tension force induced by temperature changes; H is the initial horizontal tension force; α is the thermal expansion coefficient; m is the mass per unit length; A , E , and λ^2 are the cross-sectional area, Young's modulus, and Irvine parameter, respectively.

$\Delta T = T - T_0$, where T_0 is the initial temperature and ΔT denotes the temperature variation before loading.

Hence, the variation of the tension force and sag could be described by the following nondimensional parameter [20]:

$$\chi_{\Delta T}^2 = 1 + \Delta h = \frac{h + H}{H} = \frac{H_{\Delta T}}{H} = \frac{b}{b_{\Delta T}}, \quad (3)$$

where $b_{\Delta T}(b)$ and $H_{\Delta T}(H)$ are the cable sag and tension force with (without) thermal effects, respectively.

Neglecting temperature effects and adopting a kinematic condensation procedure, the partial differential equations of the suspended cable's planar motions could be obtained [11]. Moreover, an analytical model, including the horizontal suspended and inclined cables, incorporating the temperature effects, has been built by Lepidi and Gattulli [20]. Based on the obtained condensed mathematical model, the following continuous expressions for the in-plane displacements are obtained when considering thermal effects.

$$\begin{aligned} m v_{tt} + c v_t - \chi_{\Delta T}^2 H v_{xx} - \frac{E A}{L} \left(\frac{y_{xx}}{\chi_{\Delta T}^2} + v_{xx} \right) \int_0^L \left(\frac{y_x v_x}{\chi_{\Delta T}^2} + \frac{1}{2} v_x^2 \right) dx \\ = \sum_{m=1}^2 K_m \cos(\Omega_m t + \theta_m), \end{aligned} \quad (4)$$

where the subscript index denotes the partial differential to x and t , respectively. c is the viscous damping coefficient; $yA(x)$ is the initial cable profile. Two external distributed loads are denoted by $K_m \cos(\Omega_m t + \theta_m)$, ($m = 1, 2$), where K_m , Ω_m , and θ_m are excitation amplitudes, frequencies, and phases, respectively. For the sake of simplicity, the influences of temperature on the material properties E and cross-sectional area A are neglected.

The following dimensionless parameters and variables are adopted:

$$\begin{aligned} v^*(x^*, y^*, f^*) &= \frac{v(x, y, b)}{L}, \\ t^* &= \sqrt{\frac{g}{8b}} t, \\ c^* &= \sqrt{\frac{8b}{g}} \frac{c}{m}, \\ \Theta &= \frac{E A}{H}, \\ K_m^* &= \frac{K_m L}{H}, \\ \Omega_m^* &= \sqrt{\frac{8b}{g}} \Omega_m. \end{aligned} \quad (5)$$

Here, considering the temperature effect, substituting equations (5) into (4) and neglecting asterisk notations, one obtains [20]

FIGURE 1: A schematic figure of a suspended cable in thermal environments.

$$v_{tt} + cv_t - \chi_{\Delta T}^2 v_{xx} - \Theta \left(\frac{y_{xx}}{\chi_{\Delta T}^2} + v_{xx} \right) \int_0^L \left(\frac{y_x v_x}{\chi_{\Delta T}^2} + \frac{1}{2} v_x^2 \right) dx$$

$$= \sum_{m=1}^2 K_m \cos(\Omega_m t + \theta_m). \quad (6)$$

The in-plane displacement is decomposed by a sum of the generalized coordinates $q_n(t)$ and mode shapes $\varphi_n(x)$.

$$v(x, t) = \sum_{n=1}^{\infty} \varphi_n(x) q_n(t). \quad (7)$$

Substituting equation (7) into (6), multiplying the results with $\varphi_n(x)$, and integrating the outcomes from 0 to 1, the following discretized equation is obtained:

$$q_{n,tt} + \omega_n^2 q_n + 2\mu_n q_{n,t} + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \Lambda_{jkn} q_j q_k + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \Gamma_{jkl n} q_j q_k q_l$$

$$= 2 \sum_{m=1}^2 K_{mn} \cos(\Omega_m t + \theta_m), \quad (8)$$

in which coefficients depending on the cable properties with thermal effects are given in Appendix A. An eigenanalysis on the linear parts of equation (8) is performed in Appendix B.

3. Perturbation Analysis

The presence of both quadratic and cubic nonlinear terms in equation (8) gives rise to a rich and complex set of responses. A variety of phenomena are associated with the occurrence of the resonance conditions for the system, namely, the primary one, and the order one-half/one-third subharmonic resonances, and the two-order/three-order superharmonic resonances.

Here, considering the quadratic nonlinearity induced by initial curvature, two simultaneous resonances are investigated: the primary and the two-order superharmonic resonances: $2\Omega_1 \approx \omega_1$ and $\Omega_2 \approx \omega_1$ (Case A), and the two-order superharmonic and the order one-half subharmonic resonances: $2\Omega_1 \approx \omega_1$ and $\Omega_2 \approx 2\omega_1$ (Case B). As to a weakly flexible nonlinear system, the method of multiple scales is used to obtain the steady-state solutions of multitype

simultaneous resonance responses. Following a similar procedure in [44], the perturbation analysis is presented as follows.

Firstly, the quick and slow times are introduced as follows:

$$T_0 = t,$$

$$T_1 = \varepsilon t, \quad (9)$$

$$T_2 = \varepsilon^2 t,$$

and the solution of the nonlinear system is in the form of expansion

$$q_n(t; \varepsilon) = q_{n0}(T_0, T_1, T_2) + \varepsilon q_{n1}(T_0, T_1, T_2) + \varepsilon^2 q_{n2}(T_0, T_1, T_2), \dots, \quad (10)$$

where ε is a small nondimensional parameter.

Two detuning parameters σ_1 and σ_2 are introduced for dual harmonic excitations to describe the nearness between them.

Case A:

$$2\Omega_1 = \omega_1 + \varepsilon\sigma_1, \quad (11)$$

$$\Omega_2 = \omega_1 + \varepsilon\sigma_2.$$

Case B:

$$2\Omega_1 = \omega_1 + \varepsilon\sigma_1, \quad (12)$$

$$\Omega_2 = 2\omega_1 + \varepsilon\sigma_2.$$

The damping, dual excitations, and quadratic and cubic nonlinearities are rescaled as follows.

Case A:

$$\mu_n \Rightarrow \varepsilon\mu_n,$$

$$\Lambda_{jkn} \Rightarrow \varepsilon\Lambda_{jkn},$$

$$\Gamma_{jkl n} \Rightarrow \varepsilon^2\Gamma_{jkl n}, \quad (13)$$

$$K_{1n} \Rightarrow \varepsilon^0 K_{1n},$$

$$K_{2n} \Rightarrow \varepsilon^1 K_{2n}.$$

Case B:

$$\begin{aligned}
 \mu_n &\Rightarrow \varepsilon^2 \mu_n, \\
 \Lambda_{jkn} &\Rightarrow \varepsilon \Lambda_{jkn}, \\
 \Gamma_{jkl n} &\Rightarrow \varepsilon^2 \Gamma_{jkl n}, \\
 K_{1n} &\Rightarrow \varepsilon^1 K_{1n}, \\
 K_{2n} &\Rightarrow \varepsilon^1 K_{2n}.
 \end{aligned} \tag{14}$$

Substituting equations (9)–(10) and (13)–(14) into (8), equating coefficients of the same powers of ε , one obtains the following.

Case A:

$$\begin{aligned}
 D_0^2 q_{n0} + \omega_n^2 q_{n0} &= 2K_{1n} \cos(\Omega_1 T_0 + \theta_1), \\
 D_0^2 q_{n1} + \omega_n^2 q_{n1} &= -2D_0 D_1 q_{n0} - 2\mu_n D_0 q_{n0} - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \Lambda_{jkn} q_{j0} q_{k0} + 2K_{2n} \cos(\Omega_2 T_0 + \theta_2), \\
 D_0^2 q_{n2} + \omega_n^2 q_{n2} &= -2D_0 D_1 q_{n1} - (2D_0 D_2 + D_1^2) q_{n0} - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (\Lambda_{jkn} + \Lambda_{k jn}) q_{j1} q_{k0} - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \Gamma_{jkl n} q_{j0} q_{k0} q_{l0}.
 \end{aligned} \tag{15}$$

Case B:

$$\begin{aligned}
 D_0^2 q_{n0} + \omega_n^2 q_{n0} &= 0, \\
 D_0^2 q_{n1} + \omega_n^2 q_{n1} &= -2D_0 D_1 q_{n0} - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \Lambda_{jkn} q_{j0} q_{k0} + 2 \sum_{m=1}^2 K_{mn} \cos(\Omega_m t + \theta_m), \\
 D_0^2 q_{n2} + \omega_n^2 q_{n2} &= -2D_0 D_1 q_{n1} - (2D_0 D_2 + D_1^2 + 2\mu_n D_0) q_{n0} - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (\Lambda_{jkn} + \Lambda_{k jn}) q_{j1} q_{k0} - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \Gamma_{jkl n} q_{j0} q_{k0} q_{l0}.
 \end{aligned} \tag{16}$$

The generating solution of the first equation in equations (15) and (16) is obtained, respectively.

Case A:

$$q_{n0} = A_n e^{i\omega_n T_0} + E_{1n} e^{i\Omega_1 T_0} + \text{c.c.}, \tag{17}$$

where $E_{1n} = (e^{i\theta_1} K_{1n} / (\omega_n^2 - \Omega_1^2))$.

Case B:

$$q_{n0} = A_n e^{i\omega_n T_0} + \text{c.c.}, \tag{18}$$

where c.c. denotes the complex conjugate of the preceding terms.

Substituting equations (17) and (18) into the second equation in equations (15) and (16) and eliminating the secular terms from q_{n1} , the solution for q_{n1} is obtained. Then, secular terms from q_{n2} in the third equation in equations (15) and (16) could be eliminated by following the same procedure. Finally, the solvability condition is expressed as follows.

Case A:

$$\begin{aligned}
 -2i\omega_n (D_1 A_n + \mu_n A_n) - 2\delta_{n1} \omega_1 \bar{F}_1 e^{i(\sigma_1 T_1 + 2\theta_1)} \\
 + \delta_{n1} K_{21} e^{i(\sigma_2 T_1 + \theta_2)} = 0.
 \end{aligned} \tag{19}$$

Case B:

$$i(D_2 A_n + \mu_n A_n) - 4A_n \sum_{j=1}^{\infty} \alpha_{jn} A_j \bar{A}_j + \delta_{n1} I_{21} \bar{A}_1 e^{i(\sigma_2 T_2 + \theta_2)} = 0, \tag{20}$$

where δ_{n1} is the Kronecker delta, and coefficients $\bar{F}_1$, α_{jn} , and I_{21} are expressed in Appendix C.

Similarly, the complex amplitude A_n can be written in the polar form including the amplitude a_n and phase β_n , so equations (19) to (20) will be separated into the real and imaginary parts when $n = 1$ (a_n decay for $n \geq 2$).

Case A:

$$\begin{aligned}\omega_1 a_1' + \omega_1 \mu_1 a_1 + 2\omega_1 \bar{F}_1 \sin(\sigma_1 T_1 + 2\theta_1 - \beta_1) - K_{21} \sin(\sigma_2 T_1 + \theta_2 - \beta_1) &= 0, \\ \omega_1 a_1 \beta_1' - 2\omega_1 \bar{F}_1 \cos(\sigma_1 T_1 + 2\theta_1 - \beta_1) + K_{21} \cos(\sigma_2 T_1 + \theta_2 - \beta_1) &= 0.\end{aligned}\quad (21)$$

Case B:

$$\begin{aligned}a_1' + \mu_1 a_1 + I_{21} a_1 \sin(\sigma_2 T_2 + \theta_2 - 2\beta_1) &= 0, \\ a_1(\beta_1' + \alpha_{11} a_1^2) + I_{21} a_1 \cos(\sigma_2 T_2 + \theta_2 - 2\beta_1) &= 0.\end{aligned}\quad (22)$$

In Case A, it leads to the requirement that $\sigma_2 = \sigma_1$ and $\Omega_2 = 2\Omega_1$ as to the steady-state response, and one defines $\sigma = \sigma_1 = \sigma_2$ and $\gamma = \sigma T_1 - \beta_1$. Accordingly, the governing equations are obtained.

Case A:

$$\begin{aligned}\mu_1 a_1 + 2\omega_1 \bar{F}_1 \sin(\gamma + 2\theta_1) - K_{21} \sin(\gamma + \theta_2) &= 0, \\ \omega_1 a_1 \sigma - 2\omega_1 \bar{F}_1 \cos(\gamma + 2\theta_1) + K_{21} \cos(\gamma + \theta_2) &= 0.\end{aligned}\quad (23)$$

Similarly, in Case B, one defines $\sigma = (\sigma_2/2)$ and $\gamma = \sigma T_2 - \beta_1$ and obtains the following.

Case B:

$$\begin{aligned}\mu_1 a_1 + I_{21} a_1 \sin(2\gamma + \theta_2) &= 0, \\ a_1(\sigma + \alpha_{11} a_1^2) + I_{21} a_1 \cos(2\gamma + \theta_2) &= 0.\end{aligned}\quad (24)$$

Hence, the relationship between the detuning parameter σ and the response amplitude a_1 could be expressed as follows.

Case A:

$$a_1 = \pm \frac{1}{\omega_1} \sqrt{\sigma^2 + \mu_1^2} (2\omega_1 \bar{F}_1 - K_{21}). \quad (25)$$

Substituting $\bar{F}_1$ into equation (25), one obtains

$$a_1 = \pm \frac{16\Lambda_{111}K_{11}^2 - 9\omega_1^4 K_{21}}{9\omega_1^5 \sqrt{\sigma^2 + \mu_1^2}}, \quad (26)$$

where $\theta_2 = 2\theta_1$.

Case B:

$$a_1^2 = \frac{-\sigma \pm \sqrt{I_{21}^2 - \mu_1^2}}{\alpha_{11}}. \quad (27)$$

Introducing two small disturbances ($a_1 = a_{10} + a_{11}$ and $\gamma = \gamma_0 + \gamma_1$), substituting them into (21) and (22), expanding for small a_{11} and γ_1 , noticing that a_{10} and γ_0 satisfy equations (23) and (24), and finally keeping linear terms in a_{11} and γ_1 , one obtains the following.

Case A:

$$\begin{aligned}a_{11}' &= -\mu_1 a_{11} + \left[-2\bar{F}_1 \cos(\gamma_0 + 2\theta_1) + \frac{K_{21}}{\omega_1} \cos(\gamma_0 + \theta_2) \right] \gamma_1, \\ \gamma_1' &= \left[\frac{2\bar{F}_1}{a_{10}} \sin(\gamma_0 + 2\theta_1) - \frac{K_{21}}{a_{10}\omega_1} \sin(\gamma_0 + \theta_2) \right] \gamma_1.\end{aligned}\quad (28)$$

Case B:

$$\begin{aligned}a_{11}' &= [-\mu_1 - I_{21} \sin(2\gamma_0 + \theta_2)] a_{11} \\ &\quad + [-2I_{21} a_{10} \cos(2\gamma_0 + \theta_2)] \gamma_1, \\ \gamma_1' &= 2\alpha_{11} a_{10} a_{11} - 2I_{21} \sin(2\gamma_0 + \theta_2) \gamma_1,\end{aligned}\quad (29)$$

where the stability of the nontrivial solutions is determined by investigating the eigenvalues of the coefficients matrix.

4. Numerical Examples and Discussions

The following physical parameters are adopted in the numerical examples, as shown in Table 1. Based on the parabolic assumption of the suspended cables' configuration, four sag-to-span ratios (e.g., 0.0025, 0.006, 0.01, and 0.015) are adopted to exhibit the influence of temperature on the vibration behaviors. It was shown that the nonlinear dynamic behaviors of the suspended cable are not sensitive to the temperature-dependent materials between $[-40^\circ\text{C}, 40^\circ\text{C}]$, such as Young's modulus [23]. Here, material properties are assumed to be independent of temperature variations, and the cross-sectional area remains plane and uniform in thermal environments.

As illustrated in the previous sections, the infinite-dimensional model is considered. Nevertheless, in the following numerical examples, just a single-degree-of-freedom model is adopted for simplicity. Table 2 gives the tension variation factor, natural frequency, and quadratic and cubic nonlinear coefficients in the warming ($\Delta T = 40^\circ\text{C}$) and cooling ($\Delta T = -40^\circ\text{C}$) conditions. The temperature effects on linear (natural frequency) and nonlinear (quadratic and cubic) coefficients are closely related to the sag-to-span ratios. To be more specific, both positive and negative relationships between temperature changes and natural frequencies are found. If the sag-to-span ratio is tiny, for example, 0.0025 and 0.006, the temperature effects on the cubic coefficients seem to be neglected. As an increase in the sag-to-span ratio, both quadratic and cubic nonlinearity coefficients are increased (decreased) in the warming (cooling) condition. Due to the variations of these parameters and coefficients, the influences of temperature on vibration behaviors are discussed.

The response amplitudes a_1 as functions of two excitation amplitudes K_{11} and K_{21} for this nonlinear system at a given detuning parameter σ are presented in Figures 2 and 3 by solving equation (26), respectively. First of all, it is assumed that the excitation amplitude K_{21} is a constant

TABLE 1: Notations and values of physical parameters.

Physical meaning (notation)	Value (unit)
Density (ρ)	7800 (kg/m ³)
Cable span (L)	200 (m)
Young's modulus (E)	2.0×10^{11} (Pa)
Area of cross section (A)	7.069×10^{-2} (m ²)
Thermal expansion coefficient (α)	1.2×10^{-5} (/°C)
Temperature variations (ΔT)	-40, 0, 40 (°C)
Damping ratio (μ)	0.005 (-)
Sag-to-span ratio (f)	0.0025, 0.006, 0.01, 0.015 (-)

TABLE 2: Values of corresponding parameters for the first symmetric mode.

f	ΔT (°C)	$\chi_{\Delta T}^2$	ω_n^2	Λ_{nnn}	Γ_{nnnn}
0.0025	-40	1.1345	11.2803	71.86	13771.7
	0	1.0000	9.9769	81.52	13771.8
	+40	0.8661	8.6912	94.12	13771.8
0.006	-40	1.2783	13.3356	314.66	30607.1
	0	1.0000	11.0444	401.86	30618.3
	+40	0.7464	9.4694	537.45	30690.5
0.01	-40	1.3713	16.4205	812.71	51085.7
	0	1.0000	15.2646	1112.82	51571.5
	+40	0.7352	17.0703	1538.31	55321.5
0.015	-40	1.3407	22.9867	1826.90	77269.8
	0	1.0000	27.0093	2539.48	87587.5
	+40	0.7930	33.6275	3980.64	158065.0

FIGURE 2: Temperature effects on relationships between excitation amplitude K_{11} and response amplitude a_1 when $K_{21} = 0.0001$ in Case (A). (a) $f = 0.0025$, (b) $f = 0.006$, (c) $f = 0.010$, and (d) $f = 0.015$.

FIGURE 3: Temperature effects on relationships between excitation amplitude K_{21} and response amplitude a_1 when $K_{11} = 0.005$ in Case (A). (a) $f = 0.0025$, (b) $f = 0.006$, (c) $f = 0.010$, and (d) $f = 0.015$.

FIGURE 4: Continued.

FIGURE 4: Temperature effects on relationships between detuning parameter σ and response amplitude a when $K_{21} = K_{11} = 0.015$ in Case (B). (a) $f = 0.0025$, (b) $f = 0.006$, (c) $f = 0.010$, and (d) $f = 0.015$.

FIGURE 5: Temperature effects on time history curves, phase portraits, frequency spectrum, and Poincaré sections when $f = 0.01$, $\sigma = 0.05$, and $K_{21} = K_{11} = 0.015$ in Case (B). (a) 40°C , (b) 0°C , and (c) -40°C .

(0.0001), and the detuning parameter σ is zero. As shown in Figure 2, with an increase in K_{11} from zero, the response amplitude a_1 decreases to zero at the beginning and then increases after the inflection point.

To be more specific, Figure 2(a) shows that the response amplitude a_1 increases (decreases) in the warming (cooling) condition when $K_{11} < 0.01$. On the contrary, provided that

$0.01 < K_{11} < 0.02$, the response amplitude a_1 is reduced in the warming condition. The inflection point will be exhibited earlier in the warming condition when the excitation amplitude increases. Equation (26) shows that the inflection point is exhibited when $K_{11} = \sqrt{(9\omega_1^4 K_{21}/16\Lambda_{111})}$. Here, as to the cable with $f = 0.0025$, the natural frequency increases

when $\Delta T = -40^\circ\text{C}$ and decreases when $\Delta T = 40^\circ\text{C}$. The quadratic coefficient will increase significantly when the temperature is increased. Then, if K_{11} keeps increasing, the amplitude of responses a_1 increases accordingly in the warming condition.

As shown in Figure 2(b), there are some similarities of the temperature effects on the vibration behaviors between $f = 0.0025$ and 0.006 . Because the increase of quadratic coefficient is much more significant than the increase of natural frequency, it is noted that all the inflection points are exhibited in a smaller value of K_{11} . However, if f is chosen as 0.01 , some differences are exhibited. The influences of temperature on the vibration characteristics seem more obvious in the cooling condition than in the warming one. The primary reason comes from the natural frequency with thermal effects, because it increases in both warming and cooling conditions. Furthermore, in the case of $f = 0.015$, both the natural frequency and the quadratic coefficient increase (decrease) with the increase (decrease) of temperature. Therefore, no matter before or after the inflection point, the response amplitudes a_1 will be reduced in the warming condition. Here, it should be pointed out that the temperature effects on the vibration behaviors seem reduced significantly when f is chosen as 0.015 .

Moreover, suppose that the first excitation amplitude K_{11} is a constant. In that case, the temperature effects on the relationships between the excitation amplitude K_{21} and the response one a_1 are described in Figure 3 for a given detuning parameter. As the previous case, the response amplitude a_1 decreases initially, followed by an increase with K_{21} . Similarly, there is an inflection point in the response amplitudes, and it is exhibited when $K_{21} = (16\Lambda_{111}K_{11}^2/9\omega_1^4)$. On the contrary, the point seems to be delayed in the warming condition when K_{21} increases from zero.

As to the case of $f = 0.0025$, 0.006 , and 0.01 , the response amplitude a_1 increases in the warming condition before the inflection point. Nevertheless, after the inflection point, a_1 decreases in the warming condition. It is also observed that the discrepancies between different conditions are reduced when K_{21} keeps increasing in the latter. As to the case of $f = 0.015$, no significant change on the inflection point could be observed. The response amplitude a_1 would be reduced in the warming condition no matter before or after the inflection point.

Furthermore, the temperature effects on relationships between detuning parameter σ and response amplitude a are described in Figure 4 as to Case B. Similarly, four cases are discussed, and two excitation amplitudes are constant: $K_{21} = K_{11} = 0.015$. Solid lines denote the stable steady-state solutions, and dashed lines denote the unstable ones. Excited by certain harmonic excitations, the suspended cable exhibits softening or hardening behaviors depending on the contribution of quadratic and cubic nonlinearities and the corresponding configurations.

In the case of $f = 0.0025$ in Figure 4(a), all branches are bent to right, and the nonlinear system presents a hardening-type nonlinearity. Provided that the excitation frequency continues to decrease from -0.1 , the trivial fixed point loses its stability at the Pitchfork bifurcation. In the

cooling condition, the bifurcation point seems to be delayed, and the resonant range between two Pitchfork bifurcations is reduced significantly ($\Delta T = -40^\circ\text{C}$). With an increase in the detuning parameter σ , the influences of temperature on the stable solutions seem reduced more and more.

Regarding the dynamical behaviors when $f = 0.006$ in Figure 4(b), the resonant curves show that the nonlinear system exhibits a hardening-type behavior when no temperature is considered. The lower the temperature drops (e.g., $\Delta T = -40^\circ\text{C}$), the more the curves are bending to the right, and the response amplitudes are reduced. However, if the temperature increases to 40°C , a softening-type characteristic is exhibited due to the temperature effects on the quadratic nonlinearity coefficient.

However, when f is chosen as 0.01 in Figure 4(c), at a specific low temperature (e.g., $\Delta T = -40^\circ\text{C}$), the nonlinear system exhibits a softening-type behavior. In contrast, for sufficiently high temperature (e.g., $\Delta T = 40^\circ\text{C}$), the nonlinear behavior is changed to a softening type. As shown in Figure 4(d) ($f = 0.015$), all branches are bent to the left, and they all present a softening-type nonlinearity in different thermal environments. It is observed that the nonlinear system predicts a weaker softening behavior in the cooling condition.

Moreover, details of the nonlinear system responses are described in Figure 5 through some time traces, phase-plane portraits, fast Fourier transforms (FFTs), and Poincaré sections in the case of $f = 0.01$, $\sigma = 0.05$, and $K_{21} = K_{11} = 0.015$. On the one hand, temperature increasing causes a smaller peak of the response amplitude. On the other hand, as shown in the phase portraits, two circles could be observed in different conditions. To verify the frequency components of the time traces, the fast Fourier transform (FFT) is adopted here. A FFT analysis suggests that the motion mainly comprises two frequencies. At the same time, the Periodic-2 motions are also confirmed, because two isolated dots in the Poincaré sections could be observed. Nevertheless, the peak value of the response amplitude is sensitive to temperature variations, and the periodic motions seem to be independent of thermal effects.

5. Conclusions

The sensitivities of linear (natural frequency) and nonlinear (quadratic and cubic) coefficients to temperature variations are different, and they are closely related to the suspended cable's sag-to-span ratios. In general, as to a string with a very large initial tension force and a suspended cable with a very large sag, the temperature's influence on their vibration behaviors seems to be negligible. The temperature may increase or decrease the response amplitudes, depending on the excitation amplitude and the initial tension force. Considering two excitation amplitudes, if one keeps constant, the response amplitude decreases to zero firstly and then increases after the inflection point when another excitation amplitude increases continuously. The inflection point is shifted due to thermal effects and exhibited at a smaller or larger excitation amplitude in thermal environments. Temperature variations may affect the hardening or

softening spring characteristics. Even an opposite behavior may be exhibited in warming/cooling conditions due to the temperature effects on the cable tension force. Temperature variations significantly affect the response amplitudes and affect the stable region of the nonlinear system. The resonant range between two Pitchfork bifurcations seems reduced when the temperature is decreasing. It is found out that the response amplitude is very sensitive to thermal effects, but

the periodic motions seem to be independent of temperature in these cases.

Appendix

A. The Coefficients Depending on the Cable Properties with Thermal Effect in Equation (8)

$$\begin{aligned}
 \mu_n &= \frac{1}{2}c, \\
 K_{nn} &= \frac{K_m}{2} \int_0^1 \varphi_n dx, \\
 \Lambda_{jklm} &= -\frac{\Theta}{2} \int_0^1 \left[\varphi_{k,xx} \int_0^1 \varphi_{j,x} \varphi_{l,x} dx \right] \varphi_n dx, \\
 \omega_n^2 &= -\chi_{\Delta T}^2 \int_0^1 \varphi_{n,xx} \varphi_n dx - \frac{\Theta}{\chi_{\Delta T}^4} \int_0^1 y_{xx} \left[\int_0^1 y_x \varphi_{n,x} dx \right] \varphi_n dx, \\
 \Gamma_{jkn} &= -\frac{\Theta}{\chi_{\Delta T}^2} \int_0^1 \left[\varphi_{k,xx} \int_0^1 y_x \varphi_{j,x} dx \right] \varphi_n dx - \frac{\Theta}{2\chi_{\Delta T}^2} \int_0^1 y_{xx} \int_0^1 \varphi_{k,x} \varphi_{j,x} dx \varphi_n dx.
 \end{aligned} \tag{A.1}$$

B. An Eigenvalue Analysis on the Linear Parts of Equation (8)

By solving the transcendental equations, one obtains the in-plane symmetric mode frequencies

$$\tan\left(\frac{\omega_n}{2\chi_{\Delta T}}\right) = \frac{\omega_n}{2\chi_{\Delta T}} - \frac{4}{\lambda_{\Delta T}^2} \left(\frac{\omega_n}{2\chi_{\Delta T}}\right)^3, \quad (n = 1, 3, 5, \dots), \tag{B.1}$$

where

$$\lambda_{\Delta T}^2 = \frac{\lambda^2}{\chi_{\Delta T}^6}, \tag{B.2}$$

$$\text{and } \lambda^2 = \left(\frac{mgL}{H}\right)^2 \frac{EA}{H},$$

and the in-plane symmetric mode shapes are expressed as

$$\begin{aligned}
 \varphi_n(x) &= \xi_n \left[1 - \cos\left(\frac{\omega_n}{\chi_{\Delta T}} x\right) \right. \\
 &\quad \left. - \tan\left(\frac{\omega_n}{2\chi_{\Delta T}}\right) \sin\left(\frac{\omega_n}{\chi_{\Delta T}} x\right) \right], \quad (n = 1, 3, 5, \dots),
 \end{aligned} \tag{B.3}$$

where ξ_n would be ascertained through $\int_0^1 \varphi_n^2 dx = 1$.

In-plane antisymmetric mode shapes and frequencies are expressed as follows, respectively:

$$\begin{aligned}
 \omega_n &= \chi_{\Delta T} n\pi, \\
 \varphi_n(x) &= \sqrt{2} \sin(n\pi x), \quad n = 2, 4, 6, \dots,
 \end{aligned} \tag{B.4}$$

where all these mode shapes are independent of temperature effects.

C. The Coefficients in Equations (19)-(20)

$$\begin{aligned}
 \bar{F}_m &= \frac{1}{2\omega_m} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \Lambda_{jkm} \bar{E}_{1j} \bar{E}_{1k}, \\
 \bar{E}_{mn} &= \frac{K_{mn}}{\omega_n^2 - \Omega_m^2}, \\
 I_{ml} &= \frac{1}{2\omega_l} \sum_{j=1}^{\infty} \bar{\Lambda}_{jll} \bar{E}_{mj}, \\
 \alpha_{jn} &= \frac{1}{8(1 + \delta_{jn})\omega_n} \left\{ -\bar{\Gamma}_{jjnn} + \sum_{k=1}^{\infty} [\bar{\Lambda}_{knn} \bar{\Lambda}_{jjk} S_{knn} + \bar{\Lambda}_{jkn} \bar{\Lambda}_{jnk} (R_{kjn} + S_{kjn})] \right\}, \\
 \bar{\Gamma}_{jklm} &= \Gamma_{jklm} + \Gamma_{kjlm} + \Gamma_{jlkm} + \Gamma_{kljm} + \Gamma_{ljkm} + \Gamma_{lkjm}, \quad \bar{\Lambda}_{jkn} = \Lambda_{jkn} + \Lambda_{kjn}, \\
 R_{njk} &= \frac{1}{\omega_n^2 - (\omega_j + \omega_k)^2}, \\
 S_{njk} &= \frac{1}{\omega_n^2 - (\omega_j - \omega_k)^2}.
 \end{aligned} \tag{C.1}$$

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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