Hindawi
Shock and Vibration
Volume 2022, Article ID 1496358, 16 pages
https://doi.org/10.1155/2022/1496358

Research Article

Uncertainty Evaluation of Stochastic Structural Response with Correlated Random Variables

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Received 31 December 2021; Revised 17 February 2022; Accepted 2 April 2022; Published 6 June 2022

Academic Editor: Xiao Wang

It has been realized that the influence of system parameter uncertainties may be very significant, even dominant, in stochastic response evaluation. Nevertheless, in reality, this evaluation process may be difficult to conduct due to these parameter variables (viz. structural property parameters, such as stiffness, damping, and strength, and excitation characteristics parameters, such as frequency content and duration) that are usually correlated with each other. Therefore, this study devotes to develop a method for evaluating stochastic response uncertainty involving correlated system parameter variables. In this method, the evaluation expression for the mean and standard deviation of the maximum response including uncertainty parameter variables are provided first; subsequently, a third-moment pseudo-correlation normal transformation is able to be performed for converting the correlated and non-normal system parameter variables with unknown joint probability density function (PDF) or marginal PDF into the mutually independent standard normal ones; ultimately, a point estimate procedure (PEP) based on univariate dimension reduction integration can be carried out for evaluating the structural stochastic response including uncertainty system parameters. Several numerical examples with an engineering background involving correlated system parameter variables are analyzed and discussed under stochastic excitation, and their results are compared with those yielded by Monte Carlo simulation (MCS) so as to demonstrate the effectiveness of the approach proposed. It indicated that the method proposed, in this study, provides an effective path to deal with uncertainty evaluation of stochastic structural response involving correlated random variables.

1. Introduction

Stochastic response analysis is crucially significant for structural performance evaluation and probabilistic risk assessment [1–6]. Nevertheless, to get a structural realistic and accurate evaluation in practical engineering is always challenging due to the uncertainties that are generally considered primarily to be limited to stochastic excitation, in fact, the effect of uncertainty system parameters, which include structural property parameters (such as stiffness, mass, and intensity) and excitation characteristic parameters (such as spectrum and duration) on response evaluation that is equally important, even dominate [7–12]. Besides, it also makes evaluation difficult owing to these parameters that usually are mutually correlated in reality [13–16]. Thus, what the present study mainly attempts is to evaluate stochastic uncertainty response involving correlated random variables.

To evaluate structural stochastic uncertainty response, a lot of effort has been put into it by scholars and many celebrated works are able to be traced toward this topic. Generally, the uncertainty response evaluation can be classified into two categories: the time-domain analysis and the frequency-domain analysis [17–19]. In the frequency-domain analysis, the direct integration method and Monte Carlo simulation, two accurate evaluation methods, are adopted by some scholars in a linear single degree of freedom; however, when the dynamic system is complex or high safety, those methods are difficult to repeatedly apply

on account of that plenty of complicated analysis, such as eigenvalue analysis, computation of participation factor, and solution of linear equations, which are necessary, especially nonlinear system [20, 21]. Additionally, the response surface method is also capable to be performed to evaluate the uncertainty response for avoiding the difficulties in the computation of derivatives; however, some studies found that accurate results yielded by this approximation method are mainly depending on the location of the fitting points [10, 15, 22]. Other scholars [23, 24] found the first-order approximate of mean response for uncertain linear systems, which is obtained by applying Taylor series expansion, which is coincident with the given result; thus, this mean solution is treated to equivalent uncertain stochastic response. However, other research studies [3, 10, 25] are realizing the greater the coefficient of variance, the greater the error between the mean and truth value of uncertainty response; therefore, the variance of response is taken into account for improving the mean result. In the time-domain analysis, their analysis procedure and conclusion of applying these methods mentioned above are fundamentally similar to the frequency domain except that the first two moments of the response are obtained by statistic counting all the response results, which are obtained by multiple time-dependent inputs acting on the structure [26].

In summary, no matter what domain is used, the uncertainty response expressions are capable to be approximated with respect to the mean values of parameter variables by utilizing Taylor series expansion [1, 2]. Nevertheless, the Taylor series expansion method is requiring sensitivity analyses of response, which involves several complex calculation procedures, such as eigenvalue analysis, the computation of participation factor and spectral moment, and inverse distribution function; besides, the result is also not ideal when this method has been applied in the nonlinear system; what's more, the method only considers the influence of structural property parameters on the uncertain response under determinate multiple inputs; however, the effect of excitation characteristic parameters on the uncertainty response is equally significant [10, 25].

Specifically, Zhao et al. [10] propose a point estimation procedure (PEP), which is evaluating random response related to system uncertainty parameter by several repetitions of random response analysis under determination parameters, to directly evaluate the first two order moments of the structural maximum response distribution including uncertainty system parameters without any sensitivity analysis. The stochastic response of a 15-floor nonlinear structure, considering four structural property parameters and three characteristic parameters, in his paper, is steadily evaluated. What their result reveals is that PEP, a conceptually more straightforward and computationally effective method, can avoid the three main difficulties mentioned above form the series expansion method. However, in his paper, while applying the PEP, the system parameter variables are viewed as mutually independent ones. As a matter of fact, most of the system parameter variables, in actual engineering, are usually involving correlation [27, 28]. Therefore, prior to adopting the PEP, it is necessary to convert partially correlated system parameter variables into independent ones.

In response to this difficulty, in general, the Rosenblatt transformation is a classical method to implement the aforementioned correlated transformation process [29]; however, different results may be yielded if the integral order with respect to system parameters variables is exchanged; furthermore, the premise of this method applied is that the information for joint PDF of system parameter variables needs to remain completeness [30]. Thus, the Rosenblatt transformation, in a certain situation, is just an ideal method that is not easy to be implemented. On the other hand, if partial information about joint PDF of system parameter variables is capable to be acquired, the Nataf transformation, which just requires the marginal PDFs and correlation matrix of system parameter variables, is a useful alternative approach available to handle those variables related to correlation [16, 30–32]. Nevertheless, in engineering practice, both the methods above may not be achieved, if encountering the entire information for PDF of system parameter variables can scarcely be obtained except their statistical moments and correlation matrix. Under the circumstances, Lu et al. [33] recently provided a third-moment pseudo-correlation normal transformation, with the aid of the first three statistic moments (mean, standard deviation, and skewness) and correlation matrix of system parameter variables, for realizing the conversion process about correlated and non-normal system parameter variables with unknown joint PDF and marginal PDFs to mutual independent standard normal ones. However, at present, applying this effective transformation technique for evaluating stochastic uncertainty response involving correlated system parameter variables has not been investigated yet.

In this study, based on the studies above, the main objective is to extend the point estimation procedure based on a third-moment pseudo-correlation normal transformation for evaluating stochastic uncertainty response involving correlated system parameter variables. The remainder of this study is organized as follows: in Section 2, the statistical analytical expression of stochastic response evaluations with determination parameters is reviewed first; on this basis, stochastic response with independent system parameters can be evaluated utilizing the PEP. In Section 3, a third-moment pseudo-correlation normal transformation is introduced into PEP, for solving the correlated and non-normal system parameter variables to mutual independent and standard normal ones. In Section 4, the flowchart and main calculative procedure for evaluating stochastic response with uncertainty parameters are illustrated. It is then followed by Section 5, in which several examples, which involve the response uncertainty evaluation with correlated system parameter variables, are analyzed and discussed utilizing the presented approach. Eventually, the conclusions of this study are presented in Section 6.

2. Stochastic Response Evaluation with Uncertainty Parameters

To evaluate the structural uncertainty response, in general, is not easy on account of the structural stochastic response $R(t)$ cannot be explicitly descripted. From another perspective, its maximum response value, i.e., $R_{\max}$, is a random variable; then, the structural uncertainty response can be conveniently evaluated on the condition that the statistical characteristic values of $R_{\max}$, several concrete analytical expressions, are able to be known.

2.1. Stochastic Response Evaluation under Deterministic Parameters. The PDF of maximum peak value for zero-mean normal stationary random process $R(t)$, in time period T, does not exceed a certain threshold that is considered to approximately obey the Poisson distribution [34]. According to Davenport's equation, its mean value and standard deviation can be obtained as follows:

$$\mu_m = \left(\sqrt{2\mathrm{In}\nu T} + \frac{0.5772}{\sqrt{2\mathrm{In}\nu T}}\right)\sigma_R. \tag{1}$$

$$\sigma_m = \frac{\pi}{\sqrt{12\mathrm{In}(\nu T)}}\sigma_R, \tag{2}$$

where μ_m and σ_m denote the mean value and standard deviation of the maximum response, respectively; T denotes the duration of stochastic excitation; ν denotes the mean cross ratio; and it can be calculated by the following equation:

$$\nu = \frac{1}{2\pi}\frac{\sigma_{\dot{R}}}{\sigma_R}. \tag{3}$$

In equations (1)–(3), σ_R denotes the standard deviation of the stochastic response, which can be acquired from the state-of-the-art techniques, e.g., those based on random process and vibration theory, and $\sigma_{\dot{R}}$ is corresponding derivative. For a linear single or multiple degrees of freedom system, σ_R is capable to be obtained based on the mode decomposition method, in which its main calculation principles are as follows [1, 2]:

$$\left[S_{yy}(\omega)\right] = [H(\omega)]^*\left[S_{xx}(\omega)\right][H(\omega)]^T. \tag{4}$$

$$\left[S_{yx}(\omega)\right] = [H(\omega)]^*\left[S_{xx}(\omega)\right]. \tag{5}$$

$$\left[S_{xy}(\omega)\right] = \left[S_{xx}(\omega)\right][H(\omega)]^T. \tag{6}$$

Equations (4)-(6) represent when the linear system is subjected to the stationary random excitation $\{x(t)\}$, whose power spectrum matrix is $[S_{xx}(\omega)]$; then, the power spectrum matrix of system response $\{y(t)\}$ can be expressed as $[S_{yy}(\omega)]$, and its mutual power spectrum matrix is able to be expressed as $[S_{yx}(\omega)]$ and $[S_{xy}(\omega)]$, respectively. $[H(\omega)]$ denotes the response matrix in the frequency domain. In addition, for a nonlinear single or multiple degrees of freedom system, the equivalent linearization method is recommended to calculate response standard deviation σ_R [3–5].

2.2. Stochastic Response Evaluation under System Parameter Uncertainties. In the structural stochastic response evaluation described in the previous section, these system parameters, including structural property parameters and excitation characteristics parameters, are premised to be given. While these system parameters are considered to treat as a group of random variables $\mathbf{X}$, then, the maximum uncertainty response of the structure, $R_{\max}$, can be described as a function of $\mathbf{X}$:

$$R_{\max} = R_m(\mathbf{X}). \tag{7}$$

Generally, the maximum response variable $R_{\max}$ cannot be explicitly descripted by random vibration theory, except it is the mean μ_m and standard deviation σ_m. Therefore, this study's primary aim is to evaluate structural uncertainty response by adopting the statistics of maximum response variable $R_{\max}$, i.e., equations (1)-(2).

While the uncertainty system parameters are described by random variables $\mathbf{X}$, the expression of mean μ_m and standard deviation σ_m become a function about $\mathbf{X}$, i.e., $\mu_m(\mathbf{X})$ and $\sigma_m(\mathbf{X})$, respectively. For a group of determination values of $\mathbf{X} = \mathbf{x}$, the conditional mean and standard deviation of maximum response variables can be evaluated by directly substituting equations (1)-(2), i.e., $\mu_m(\mathbf{X} = \mathbf{x})$ and $\sigma_m(\mathbf{X} = \mathbf{x})$. Subsequently, when system parameters $\mathbf{X}$ are considered as uncertainty random variables, the overall uncertainty response, i.e, mean μ_{M} and standard deviation σ_{M}, can be evaluated by integrating over the whole area of $\mathbf{X}$. The entire evaluation procedure mentioned above is shown in the following equations.

In the first step, the conditional mean of maximum response $\mu_m(\mathbf{X})$ and its standard deviation $\sigma_m(\mathbf{X})$ can be evaluated as follows [35]:

$$\mu_m(\mathbf{X}) = \int R_m(\mathbf{X}) f(R_m|\mathbf{X})\mathrm{d}R_m. \tag{8}$$

$$\sigma_m^2(\mathbf{X}) = \int \left[R_m(\mathbf{X}) - \mu_m(\mathbf{X})\right]^2 f(R_m|\mathbf{X})\mathrm{d}R_m. \tag{9}$$

Subsequently, integrating over the whole area of system parameter variables $\mathbf{X}$, the overall mean of maximum response μ_M and its standard deviation σ_M can be obtained [10]:

$$\mu_M = \int \mu_m(\mathbf{X}) f(\mathbf{X})\mathrm{d}\mathbf{X} = E\left[\mu_m(\mathbf{X})\right]. \tag{10}$$

$$\begin{aligned}\sigma_M^2 &= \int \left[R_m(\mathbf{X}) - \mu_M(\mathbf{X})\right]^2 f(R_m|\mathbf{X}) f(\mathbf{X})\mathrm{d}R_m\mathrm{d}\mathbf{X} \\ &= E\left[\mu_m^2(\mathbf{X})\right] + E\left[\sigma_m^2(\mathbf{X})\right] - \mu_M^2.\end{aligned} \tag{11}$$

It is worth noting that the result of $E[\mu_m^2(\mathbf{X})]$ is not equal to μ_M^2 owing to different integral variables; similarly, the result of $E[\sigma_m^2(\mathbf{X})]$ is not equal to σ_M^2; in addition, the conditional probability density function $f(R_m|\mathbf{X})$ in equations (8)-(11) is no need to be evaluated due to it is just facilitating description about the whole assessment process.

As long as the results of $E[\mu_m(\mathbf{X})]$, $E[\mu_m^2(\mathbf{X})]$ and $E[\sigma_m^2(\mathbf{X})]$ can be obtained, the structural uncertainty

response, in terms of equations (8)-(11), is able to be stably evaluated. Based on this view, in order to facilitate description and comprehension of PEP to be applied for uncertainty response evaluation in subsequent studies, supposing R is just a symbol, and R represents any one of μ_m, ${\mu_m}^2$, and ${\sigma_m}^2$. Then, while $E[\mu_m(\mathbf{X})]$, $E[\mu_m^2(\mathbf{X})]$ and $E[\sigma_m^2(\mathbf{X})]$ are capable to be written in an uniform form as shown in equation (12).

$$E[R(\mathbf{X})] = \int R(\mathbf{X}) f(\mathbf{X}) \mathrm{d}\mathbf{X}. \tag{12}$$

in which $R(\mathbf{X})$ denotes conditional maximum response statistics, i.e., any one of $\mu_m(\mathbf{X})$, ${\mu_m}^2(\mathbf{X})$, and ${\sigma_m}^2(\mathbf{X})$; $\mathbf{X}$ denotes a set of arbitrary system parameter variables.

2.3. Point Estimation Procedure Based on Univariate Dimension Reduction Integration. In this section, the PEP can be conducted for evaluating stochastic uncertainty response. By observing equation (12), it can be found that the essence of assessing the uncertain stochastic response is to evaluate the expectation of $R(\mathbf{X})$, in which $\mathbf{X}$ represents a set of arbitrary system parameter vectors.

In general, evaluating this expectation process based on PEP is capable to be divided into two steps [36, 37]: To begin with, in order to avoid a large number of calculations, the univariate dimension reduction integration is first performed for approximating conditional maximum response statistics $R(\mathbf{X})$ as a series of sum functions, in which function $R(\mathbf{X})$ just includes one random variable X_i as follows:

$$R(\mathbf{X}) \cong \sum_{i=1}^{n} R_i - (n-1) R_{\boldsymbol{\mu}}, \tag{13}$$

where

$$R_i = R(X_i) = R\left[T^{-1}(U_i)\right]. \tag{14}$$

$$R_{\boldsymbol{\mu}} = R(\boldsymbol{\mu}). \tag{15}$$

in which $\mathbf{X}_i = (\mu_1^X, \ldots, \mu_{i-1}^X, X_i, \mu_{i+1}^X, \ldots, \mu_n^X)^T$; μ_i^X denotes the mean of the ith term of $R(\mathbf{X})$ in the original space. Note that standard vertical character $\mathbf{X}_i$ represents a vector, in which this vector is consisted of an independent random variable, i.e., italic character X_i and the mean of all the remaining arbitrary variables; similarly, $\mathbf{U}_i = (\mu_1^U, \ldots, \mu_{i-1}^U, U_i, \mu_{i+1}^U, \ldots, \mu_n^U)^T$; μ_i^U denotes the mean of the ith term of $R(\mathbf{U})$ in the standard normal space; and standard vertical character $\mathbf{U}_i$ represents a vector, in which this vector is consisted of an independent standard normal random variable, i.e., italic character U_i and the mean of all the remaining standard normal variables. T^{-1} denotes the inverse of the standard normal space; $\boldsymbol{\mu}$ denotes the vector whose system parameter variables take their corresponding mean value.

In the second place, the mean of conditional maximum response statistics $R(\mathbf{X})$, i.e., overall maximum response statistics $E[R(\mathbf{X})]$, can be evaluated by using direct PEP with regard to R_i in equation (14). It can be formulated as follows:

$$E[R(\mathbf{X})] \cong \sum_{i=1}^{n} \mu_{R_i} - (n-1) R(\boldsymbol{\mu}), \tag{16}$$

in which μ_{R_i} represents a function of $R(\mathbf{X})$ with only one system parameter variable X_i, in which ith term of $R(\mathbf{X})$ is independent parameter variable X_i, and the remaining terms of $R(\mathbf{X})$ take the corresponding mean of parameter variables. $R(\boldsymbol{\mu})$ represents all system parameter variables of $R(\mathbf{X})$ that take their corresponding mean values; then, μ_{R_i} in equation (16) can be steadily directly evaluated by using PEP as follows:

$$\mu_{R_i} = E\left[R(X_i)\right] = E\left[R\left[T^{-1}(U_i)\right]\right] = \sum_{k=1}^{m} P_k R\left[T^{-1}\left(u_{i,k}\right)\right], \tag{17}$$

where $u_{i,k}$ denotes the estimation value for adopting k-point estimation at the ith coordinate position in the standard normal space, and P_k is their corresponding weight. They can be readily obtained by using the following equation (38):

$$u_{i,k} = \sqrt{2} x_k, \qquad P_k = \frac{w_k}{\sqrt{\pi}}, \tag{18}$$

in which x_k and w_k are the abscissas and weights for Hermite integration with the weight function $\exp(-x^2)$ that can be found in Abramowitz and Stegum. In particular, five-point and seven-point estimates in standard normal space are commonly used in practice; thus, their corresponding estimating point and weight are given in Tables 1 and 2 respectively.

3. The Third-Moment Pseudo-Correlation Normal Transformation for Correlated and Non-Normal Variables into Independent Normal Ones

In Section 2.3, while PEP is conducted, the system parameter variables X require to be treated as a set of mutually independent variables. However, in reality, most of these are regarded as correlated variables with unknown joint and marginal PDFs [13–16]. Thus, the third-moment pseudo-correlated normal transformation is introduced to resolve the difficulty procedure of transforming correlated variables into the mutually independent ones [33]. This transformation procedure mainly includes the following two steps.

3.1. The Third-Moment Transformation for Correlated Non-Normal System Parameter Variables into Correlated Standard Normal Ones. Without loss of generality, an arbitrary system parameter variable X_i is able to be standardized into X_{is}, whose mean and variance are 0 and 1, respectively. Meanwhile, according to third-moment transformation technology, X_{is} can be approximated by a second-order polynomial normal function. The whole process above can be expressed in the following equation [39, 40]:

$$X_{is} = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} = S_u(U_i) = a_i + b_i U_i + c_i U_i^2, \tag{19}$$

where μ_{Xi} and s_{Xi} represent the mean and standard deviation of a set of system parameter variables **X,** respectively. $S_u(U_i)$ is a second-order polynomial about U_i; a_i, b_i, and c_i represent polynomial relevant coefficients of $S_u(U_i)$, and these polynomial coefficients, in which their expression include the skewness α_{3X} of X_i, can be formulated by equations (23)-(26) in detail.

For the sake of facilitating derivation and comprehension hereinafter, the equation (19) is rewritten so as to reflect the relationship between X_i and U_i as follows:

$$X_i = \mu_{X_i} + \sigma_{X_i}\left(a_i + b_i U_i + c_i U_i^2\right). \tag{20}$$

Observing the equation (20), the independent non-normal random variables X_i with unknown joint PDF and marginal PDFs, on the basis of the third-moment transformation, have been converted into the standard normal ones U_i.

3.2. The Cholesky Decomposition for Correlated Standard Normal System Parameter Variable into Independent Ones. According to third-moment transformation technology, i.e., equation (19), in view of the correlation effect between system parameter variables X_i and X_j, in which their correlative coefficient is ρ_{ij}, then, their corresponding standardized variables X_{is} and X_{js}, respectively, can be expressed as follows:

$$X_{is} = a_i + b_i Z_i + c_i Z_i^2 = (a_i, b_i, c_i) \cdot \left(1, Z_i, Z_i^2\right)^T. \tag{21}$$

$$X_{js} = a_j + b_j Z_j + c_j Z_j^2 = \left(1, Z_j, Z_j^2\right) \cdot \left(a_j, b_j, c_j\right)^T. \tag{22}$$

in which the superscript T denotes the vector transpose; Z_i and Z_j, as those described hereinabove, are two correlated standard normal variables, and Z_i and Z_j, corresponding coefficients of a_i, b_i, and c_i in equation (21) and a_j, b_j, and c_j in (22) are capable to be determined as follows:

$$c_i = -a_i = \mathrm{Sgn}\left(\alpha_{3X_i}\right)\sqrt{2}\cos\left[\frac{\mathrm{Sgn}\left(\alpha_{3X_i}\right)\theta_i - \pi}{3}\right]. \tag{23}$$

$$b_i = \sqrt{1 - 2c_i^2}; \theta_i = \arctan\left(-\frac{\sqrt{8 - \alpha_{3X_i}^2}}{\alpha_{3X_i}}\right). \tag{24}$$

$$c_j = -a_j = \mathrm{Sgn}\left(\alpha_{3X_j}\right)\sqrt{2}\cos\left[\frac{\mathrm{Sgn}\left(\alpha_{3X_j}\right)\theta_j - \pi}{3}\right]. \tag{25}$$

$$b_j = \sqrt{1 - 2c_j^2}; \theta_j = \arctan\left(-\frac{\sqrt{8 - \alpha_{3X_j}^2}}{\alpha_{3X_j}}\right), \tag{26}$$

in which $\alpha_{3\mathbf{X}_i}$ and $\alpha_{3\mathbf{X}_j}$ denote skewness of X_i (or X_{is}) and X_j (or X_{js}), respectively, and by observing the equations (24) and (26), the range of $\alpha_{3\mathbf{X}}$ should be restricted in the efficient interval, i.e., $-2\sqrt{2} \le \alpha_{3\mathbf{X}} \le 2\sqrt{2}$. Fortunately, this bound is not restricted to general engineering applications.

Table 1: Five-point estimate corresponding the estimating point and weight.

	Estimation point	Weights
Five-point estimate	$u_0 = 0$	$p_0 = 0.53333$
	$u_{1+} = -u_{1-} = 1.35563$	$p_{1+} = p_{1-} = 0.22208$
	$u_{2+} = -u_{2-} = 2.85697$	$p_{2+} = p_{2-} = 1.12574 \times 10^{-2}$

Table 2: Seven-point estimate corresponding the estimating point and weight.

	Estimation point	Weights
Seven-point estimate	$u_0 = 0$	$p_0 = 0.45714$
	$u_{1+} = -u_{1-} = 1.15441$	$p_{1+} = p_{1-} = 0.24012$
	$u_{2+} = -u_{2-} = 2.36676$	$p_{2+} = p_{2-} = 0.03076$
	$u_{3+} = -u_{3-} = 3.75044$	$p_{3+} = p_{3-} = 5.48269 \times 10^{-4}$

Let ρ_{0ij} be the correlation coefficient between Z_i and Z_j, after to relevantly convert and derivate, ρ_{0ij} and ρ_{ij} are able to be determined as [33].

$$\rho_{0ij} = \frac{-\sqrt{\left(1 - 2c_i^2\right)\left(1 - 2c_j^2\right)} + \sqrt{\left(1 - 2c_i^2\right)\left(1 - 2c_j^2\right) + 8c_i c_j \rho_{ij}}}{4c_i c_j}. \tag{27}$$

When obtained the equation (27), two correlated variables with known statistical moments and correlation coefficient can be converted into two correlated standard normal variables. Meanwhile, this procedure can be easily extended to n variables with known statistical moments and correlation matrix, and their equivalent correlation matrix of standard normal variables, C_z, can be summarized as follows:

$$\mathbf{C_Z} = \begin{bmatrix} 1 & \rho_{012} & \cdots & \rho_{012} \\ \rho_{012} & 1 & \cdots & \rho_{02n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{012} & \rho_{0n2} & \cdots & 1 \end{bmatrix}, \tag{28}$$

in which ρ_{0ij} represents the correlative coefficient of Z_i and Z_j, $i, j = 1, 2, \ldots, n$. By utilizing Cholesky decomposition, C_z can be rewritten as follows [41]:

$$\mathbf{C_Z} = \mathbf{L}_0 \mathbf{L}_0^T, \tag{29}$$

where $\mathbf{L}_0$ denotes the lower triangular matrix acquired from Cholesky decomposition, and $\mathbf{L}_0^{\mathbf{T}}$ is its transpose matrix. Then, the correlated standard normal random vector Z can be transformed into the independent standard normal vector $U = (U_1, U_2, \ldots, U_n)$ adopting the lower triangular matrix $\mathbf{L}_0$ as follows:

$$\mathbf{U} = \mathbf{L}_0^{-1}\mathbf{Z}, \tag{30}$$

where $\mathbf{L}_0^{-1}$ is the corresponding inverse matrix of $\mathbf{L}_0$, which can be expressed as follows:

$$\mathbf{L}_0 = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}. \tag{31}$$

After relevant transforming the equation (30), Z_i can be formulated by substituting the equation (31) into equation (30) as follows:

$$Z_i = \sum_{k=1}^{i} l_{ik} U_k, \quad (i = 1, 2, \ldots, n). \tag{32}$$

Combining the equation (20), then, the correlated and non-normal system parameter variables X_i can be expressed by the independent standard normal ones U_i as follows:

$$X_i = \mu_{X_i} + \sigma_{X_i}\left[a_i + b_i \sum_{k=1}^{i} l_{ik} U_k + c_j \left(\sum_{k=1}^{i} l_{ik} U_k\right)^2\right], \quad (i = 1, 2, \ldots, n). \tag{33}$$

Substituting this equation above into the expression of conditional maximum response statistics $R(\mathbf{X})$, the expression comes from equation (14) in Section 2.3. Then, evaluating uncertainty stochastic response is able to be expressed as follows:

$$E[R(\mathbf{X})] = E\left[R\left(X_1, X_2, \ldots, X_n\right)\right] = E\left[R\left(U_1, U_2, \ldots, U_n\right)\right] = E[R(\mathbf{U})]. \tag{34}$$

in which $X = (X_1, X_2, \ldots, X_n)^T$ is a system parameter vector involving the correlated and non-normal coefficient matrix C_X; $E[R(U)]$ denotes the overall maximum response statistics, which includes independent standard normal system parameter variables, i.e., $U = (U_1, U_2, \ldots, U_n)$.

4. Point Estimation Procedure for Uncertainty Response Evaluation Involving Correlated System Parameter Variables Based on the Third-Moment Pseudo-Correlation Normal Transformation

Combining the PEP with the third-moment pseudo-correlation transformation, as those are described hereinabove, uncertainty evaluation of stochastic structural response involving correlated system parameter variables is capable to be readily conducted. The entire procedure is illustrated in Figure 1, and its corresponding explanatory items are able to be described as follows:

(1) The analytical expression is acquired for uncertainty evaluation of mean and standard deviation of maximum stochastic response, i.e., equations (1) and (2), involving correlated system parameter variables X with unknown joint PDF and marginal PDFs.

(2) The expressions of $E[\mu_m(X)]$, $E[\mu_{\mathbf{m}}^2(X)]$ and $E[\sigma_{\mathbf{m}}^2(X)]$ are unified, which can be derived by utilizing equations (8) and (9) into an integration form for easy describing conditional maximum response statistics $R(X)$ with regard to arbitrary system parameter variables, i.e., equation (12).

(3) The expression of $E[R(X)]$, in which X is a set of arbitrary system parameter vectors, is converted to the expression of $E[R(U)]$, in which U is a group of mutually independent ones, based on the third-moment pseudo-correlation normal transformation.

(4) Conditional maximum response statistics $R(U)$ are approximated as a series of sum functions, in which these functions just include one standard normal random variable U_i, based on the univariate dimension reduction integration.

(5) The ultimate overall maximum response statistics $E[R(X)]$ are obtained, i.e., the mean of conditional statistics of maximum response $R(X)$, by directly utilizing PEP.

5. Numerical Examples and Investigations

To verify the simplicity, efficiency, and accuracy of the proposed method for uncertainty evaluation of stochastic response involving correlated system parameter variables, two numerical cases are presented. In example 1, a nonlinear single degree of freedom system, with known marginal PDF and correlation matrix of system parameter variables, subject to Gaussian white noise excitation is investigated, and their evaluation process of uncertainty response involving two correlated parameters by the presented method is illustrated step by step; in example 2, the third-moment pseudo-correlation normal transformation for a two degree of freedom linear simply isolated bridge, just known the first three moments of system parameter variables instead of joint PDF and marginal PDF, taking into account uncertainties of three correlated structural property parameters and two correlated excitation characteristic parameters, subjected to dynamic excitation, is first applied, and then, the effects of correlation and uncertainties are evaluated and discussed in

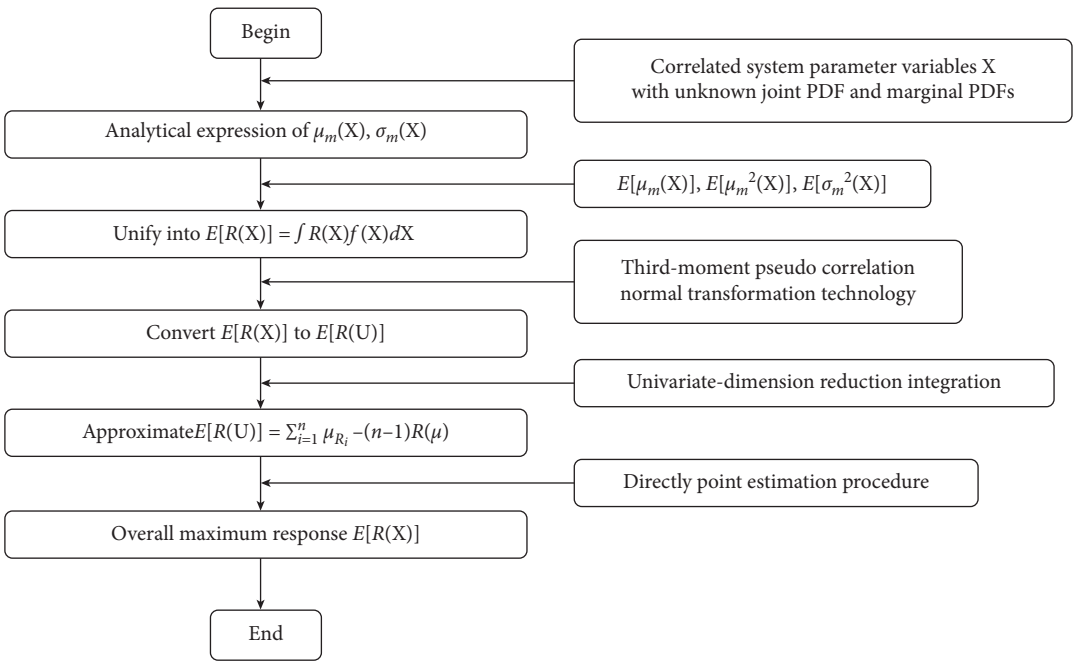

Figure 1: Flowchart for uncertainty evaluation of stochastic response with correlated system parameter variables.

detail. In these examples, a comparative analysis and discussion are, respectively, conducted between given parameter variables and uncertainty ones, and correlated parameter variables and independent ones. Meanwhile, MCS, a supplement approach, is used to investigate the accuracy of the presented method.

5.1. Example 1: A Nonlinear SDOF Subject to Gaussian White Noise Excitation. Considering an example of a nonlinear SDOF system under dynamic excitation, the excitation is modeled as Gaussian white noise, in which its intensity S_0, the duration T, the natural frequency f, and damping ratio ξ are uncertain. Their probability models are known as marginal PDFs as shown in Table 3. Assuming that the correlation coefficient between the intensity S_0 and the duration T is 0.3, the remaining variables are mutually independent.

5.1.1. Response Evaluation under Determination Parameters. When system uncertainty just is considered to be limited to stochastic excitation, to evaluate the uncertainty response is needed to substitute the means of these parameter variables into equations (1)-(2), respectively; then, the response evaluation results under given parameters can be obtained as equations (35)-(36). Noting ν, in equations (1)-(2), represents the average number of times that threshold limit is exceeded per unit time, i.e., ν is equivalent to natural frequency f:

$$\mu_m = \left(\sqrt{2\mathrm{In}(fT)} + \frac{0.5772}{\sqrt{2\mathrm{In}(fT)}}\right)\sigma_R = 2.683\sigma_R. \quad (35)$$

$$\sigma_m = \frac{\pi}{\sqrt{12\mathrm{In}(fT)}}\sigma_R = 0.524\sigma_R, \quad (36)$$

where σ_R denotes the system displacement response standard deviation. While ignoring the higher-order terms of nonlinearity coefficient ε, its analytical expression can be approximately obtained based on the perturbation method, a nonlinear analysis method of random vibration theory, as follows:

$$\sigma_R \approx \frac{\pi S_0}{2(2\pi f)^3\xi}\sqrt{1 - 3\varepsilon\left(\frac{\pi S_0}{2(2\pi f)^3\xi}\right)^2}. \quad (37)$$

Supposing nonlinearity coefficient ε is taken as 0.01; equation (37) is substituted into equations (35) and (36), respectively; then, the mean and standard deviation of maximum response, under the given parameter, are obtained as follows: $\mu_M{}^{\mathrm{det}} = 4.602 \times 10^{-2}$ and $\sigma_M{}^{\mathrm{det}} = 8.986 \times 10^{-3}$.

5.1.2. Response Evaluation with Uncertainty Independent Parameters. When uncertainties in system parameters S_0, f, ξ, and T are considered, on condition that they will be treated as mutually independent random variables, then, the overall

Table 3: Marginal PDFs and correlation matrix of system parameter variables.

Parameter	Distribution	Mean	COV	Skewness	Correlation matrix
T	Lognormal	10 s	0.3	0.93	$\begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}$
S_0	Type II extreme value	$0.65\,dm^2/s^3$	0.6	1.14	
ξ	Lognormal	0.03	0.3	0.93	
f	Normal	2 Hz	0.2	0	

mean of the maximum response analytical expression, μ_M, can be expressed as follows:

$$\mu_M = \int \mu_m(S_0, f, \xi, T) f(S_0, f, \xi, T) \mathrm{d}S_0 \mathrm{d}f \mathrm{d}\xi \mathrm{d}T. \tag{38}$$

in which

$$\mu_m(S_0, f, \xi, T) = \left(\frac{0.088 S_0 \sqrt{\mathrm{In} T}}{\pi^2 f^{5/2} \xi} + \frac{0.025 S_0}{\pi^2 f^3 \xi \sqrt{\mathrm{In} f T}} \right) \sqrt{1 - \frac{0.03 S_0^2}{256 \pi^4 f^6 \xi^2}}. \tag{39}$$

Univariate dimension reduction integration can be directly used for equation (39); then, the conditional mean of maximum response $\mu_m(\mathbf{X})$ can be expressed as follows:

$$\mu_m(\mathbf{X}) \cong \sum_{i=1}^{4} \mu_{mi} - (4-1)\mu_{m\boldsymbol{\mu}}, \tag{40}$$

where

$$\begin{aligned}
\mu_{m1} &= \mu_m(S_0) = 0.071 S_0 \sqrt{1 - 2.0 \times 10^{-5} S^2}, \\
\mu_{m2} &= \mu_m(f) = 0.053 f^{-3} \sqrt{\left(1 - 8.355 \times 10^{-5} f^{-6}\right)} \left[\frac{0.406}{\sqrt{\mathrm{In}(10 f)}} + \sqrt{2} \sqrt{\mathrm{In}(10 f)} \right], \\
\mu_{m3} &= \mu_m(\xi) = 5.311 \times 10^{-4} \xi^{-2} \sqrt{\xi^2 - 1.175 \times 10^{-9}}, \\
\mu_{m4} &= \mu_m(T) = \frac{9.156 \times 10^{-3} + 9.329 \times 10^{-3} \mathrm{In}(T)}{\sqrt{\mathrm{In}(2T)}}, \\
\mu_{m\boldsymbol{\mu}} &= \mu_m(\boldsymbol{\mu}) = 1.770 \times 10^{-2}.
\end{aligned} \tag{41}$$

In this example, a five-point estimation in standard normal space is selected as given in Table 1. For mutually independent system parameters, corresponding coordinates of estimation point $\mu\mu_{mi}$ in original space can be readily calculated by the Rosenblatt transformation. The result is shown in Table 4.

Substituting the estimation point coordinates in original space from Table 4 and its weight (see Table 1) into the (17), the evaluated result of univariate conditional mean response $\mu_{mi}(\mathbf{X})$ and its result from the corresponding five estimation points can be obtained in Table 5.

Eventually, combining Table 5 and equation (16), the overall mean of maximum response is able to be evaluated as follows:

$$\mu_M^{\mathrm{Ind}} = \mathrm{E}[\mu_m(\mathbf{X})] \cong \sum_{i=1}^{4} \mu_{mi} - (4-1)\mu_m(\boldsymbol{\mu}) = 6.463 \times 10^{-2}. \tag{42}$$

According to equations (9) and (11), the analytical overall variance expression of the maximum response, $\sigma_M{}^2$, in this example, can be expressed as follows:

Table 4: Five estimation point coordinates in original space for example 1.

Parameter variables	The original coordinates corresponding to five estimated points				
	X_{2-}	X_{1-}	X_0	X_{1+}	X_{2+}
T	4.140	6.434	9.578	1.426×10^{1}	2.216×10^{1}
S_0	2.804×10^{-1}	3.741×10^{-1}	5.553×10^{-1}	1.039	3.349
ξ	1.242×10^{-2}	1.930×10^{-2}	2.873×10^{-2}	4.278×10^{-2}	6.665×10^{-2}
f	8.572×10^{-1}	1.458	2.000	2.542	3.142

Table 5: Estimation results for $\mu_{mi}(\mathbf{X})$ in original space.

Parameter variables	μ_{mi}	The estimation results corresponding to five estimated points				
		X_{2+}	X_{1+}	X_0	X_{1+}	X_{2+}
T	4.571×10^{-2}	4.001×10^{-2}	4.315×10^{-2}	4.575×10^{-2}	4.822×10^{-2}	5.082×10^{-2}
S_0	4.601×10^{-2}	1.985×10^{-2}	2.649×10^{-2}	3.918×10^{-2}	7.358×10^{-2}	2.371×10^{-1}
ξ	5.017×10^{-2}	1.111×10^{-1}	7.153×10^{-2}	4.799×10^{-2}	3.228×10^{-2}	2.077×10^{-2}
f	6.081×10^{-2}	5.118×10^{-1}	1.136×10^{-1}	4.602×10^{-2}	2.313×10^{-2}	1.257×10^{-2}

$$\begin{aligned}
\sigma_M^2 &= T_1 + T_2 - T_3,\\
T_1 &= E\left[\mu_m^2(\mathbf{X})\right] = \int \mu_m^2(S_0, f, \xi, T) f(S_0, f, \xi, T)\mathrm{d}S_0\mathrm{d}f\mathrm{d}\xi\mathrm{d}T,\\
T_2 &= E\left[\sigma_m^2(\mathbf{X})\right] = \int \sigma_m^2(S_0, f, \xi, T) f(S_0, f, \xi, T)\mathrm{d}S_0\mathrm{d}f\mathrm{d}\xi\mathrm{d}T,\\
T_3 &= \mu_M^2.
\end{aligned} \tag{43}$$

in which

$$\begin{aligned}
\mu_m^2(S_0, f, \xi, T) &= \left[\left(\frac{0.088S_0\sqrt{\mathrm{In}T}}{\pi^2 f^{5/2}\xi} + \frac{0.025S_0}{\pi^2 f^3\xi\sqrt{\mathrm{In}fT}}\right)\sqrt{1 - \frac{0.03S_0^2}{256\pi^4 f^6\xi^2}}\right]^2,\\
\sigma_m^2(S_0, f, \xi, T) &= \left\{\frac{S_0}{32\sqrt{3}\,\pi f^3\xi\sqrt{\mathrm{In}(fT)}}\sqrt{1 - \frac{0.03S_0^4}{256\pi^4 f^6\xi^2}}\right\}^2.
\end{aligned} \tag{44}$$

The entire evaluation process is similar as the overall mean of maximum response μ_M above is analyzed; thus, the evaluation results of $\mu_{mi}^2(\mathbf{X})$ and $\sigma_{mi}^2(\mathbf{X})$, hereafter, are directly provided in Table 6.

Combining Table 6 and equation (16), the overall mean square of maximum response is able to be evaluated as follows:

$$E\left[\mu_m^2(\mathbf{X})\right] \cong \sum_{i=1}^{4}\mu_{mi}^2 - (4-1)\,\mu_m^2(\boldsymbol{\mu}) = 8.360\times10^{-3}. \tag{45}$$

Combining Table 7 and equation (16), the overall standard deviation square of maximum response is able to be evaluated as follows:

$$E\left[\sigma_m^2(\mathbf{X})\right] \cong \sum_{i=1}^{4}\sigma_{mi}^2 - (4-1)\,\sigma_m^2(\boldsymbol{\mu}) = 4.373\times10^{-4}. \tag{46}$$

Then, substituting the result of $E[\mu_{\mathbf{m}}^2(X)]$ and $E[\sigma_{\mathbf{m}}^2(X)]$ into equation (11), the overall standard deviation square of maximum response can be evaluated as follows:

$$\sigma_M^{\mathrm{Ind}} = \sqrt{E\left[\mu_m^2(\mathbf{X})\right] + E\left[\sigma_m^2(\mathbf{X})\right] - \left(\mu_M^{\mathrm{ind}}\right)^2} = 6.797\times10^{-2}. \tag{47}$$

To investigate the evaluation process of response uncertainty in detail, the uncertainties of four cases, i.e., given parameters, structural property parameters (ξ, f), excitation characteristics parameters (T, S_0), and all parameters (T, S_0, ξ, f), are considered, respectively. Meanwhile, the accuracy of their results is able to be confirmed by utilizing the MCS with a size of 106. The results of these cases are listed in Table 8.

From Table 8, some conclusions, in this example, can be revealed: (I) the parameter uncertainties have a great in-

Table 6: Estimation results for $\mu_{mi}^2(\mathbf{X})$ in original space.

Parameter variables	μ_{mi}^2	The estimation results corresponding to five estimated points				
		X_{2+}	X_{1+}	X_0	X_{1+}	X_{2+}
T	2.093×10^{-3}	1.606×10^{-3}	1.861×10^{-3}	2.093×10^{-3}	2.325×10^{-3}	2.582×10^{-3}
$S0$	2.815×10^{-3}	3.941×10^{-4}	7.018×10^{-4}	1.535×10^{-3}	5.414×10^{-3}	5.625×10^{-2}
Ξ	2.743×10^{-3}	1.236×10^{-2}	5.118×10^{-3}	2.309×10^{-3}	1.041×10^{-3}	4.315×10^{-4}
F	7.063×10^{-3}	2.620×10^{-1}	1.290×10^{-2}	2.118×10^{-3}	5.353×10^{-4}	1.582×10^{-4}

Table 7: Estimation results for $\sigma_{mi}^2(\mathbf{X})$ in original space.

Parameter variables	σ_{mi}^2	The estimation results corresponding to five estimated points				
		X_{2+}	X_{1+}	X_0	X_{1+}	X_{2+}
T	8.277×10^{-5}	1.144×10^{-4}	9.470×10^{-5}	8.193×10^{-5}	7.220×10^{-5}	6.381×10^{-5}
S_0	1.073×10^{-4}	1.502×10^{-5}	2.676×10^{-5}	5.853×10^{-5}	2.064×10^{-4}	2.141×10^{-3}
ξ	1.046×10^{-4}	4.710×10^{-4}	1.951×10^{-4}	8.802×10^{-5}	3.971×10^{-5}	1.644×10^{-5}
f	3.851×10^{-4}	1.813×10^{-2}	6.021×10^{-4}	8.076×10^{-5}	1.773×10^{-5}	4.661×10^{-6}

Table 8: Comparison between the MCS and the presented method under different parameter cases.

Considered cases	PEP method		MCS		Method error (%)	
	μ_M^{Ind}	σ_M^{Ind}	μ_M^{MCS}	σ_M^{MCS}	$\mathrm{E}\mu_M$	$\mathrm{E}\sigma_M$
Given parameters	4.602×10^{-2}	8.986×10^{-3}	—	—	—	—
Structure parameters	2.498×10^{-2}	7.473×10^{-3}	2.630×10^{-2}	7.120×10^{-3}	5.016	4.957
Excitation parameters	4.570×10^{-2}	8.001×10^{-4}	4.792×10^{-2}	7.777×10^{-4}	4.632	2.880
All parameters	6.463×10^{-2}	6.797×10^{-2}	6.221×10^{-2}	7.151×10^{-2}	3.890	4.950

fluence on response evaluation relative to determination parameters, and the effect of structural parameters on the overall mean evaluation of maximum response is more obvious than that of excitation parameters. (II) All uncertainty parameters are simultaneously considered, and these uncertainties, relative to other cases, have the greatest effect on the response evaluation. (III) The results from PEP are basically in agreement with those acquired using MCS, in which the maximum error between them is just 5.016%.

5.1.3. Response Evaluation Involving Uncertainty Correlated Parameters. To investigate the evaluation process of response uncertainty involving correlated parameter variables, the example, hereinabove, considering all uncertainty parameters (T, S_0, ξ, and f), will be further analyzed.

While considering the correlation between the duration T and the intensity S_0, the third-moment pseudo-correlation transformation is applied first, so as to convert their correlated variables into independent standard normal variables. The correlated matrix including all parameters can be written as follows:

$$\rho = \begin{pmatrix} 1 & 0.6 & 0 & 0 \\ 0.6 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{48}$$

According to equations (23)-(30), the polynomial coefficients of each parameter can be calculated, and the result is provided in Table 9.

Combining equations (27)-(29) and Cholesky decomposition, the lower triangular matrix $\mathbf{L}_0$ can be obtained as follows:

$$\mathbf{L}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.615 & 0.788 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{49}$$

Substituting the lower triangular matrix $\mathbf{L}_0$ into equation (33), these correlation variables, utilizing the five-point estimation, were transferred as mutually independent estimation point coordinates X_i in the original space. These transformation results are shown in Table 10.

Substituting the estimation point coordinates in original space from Table 10 and their weight (see Table 1) into equation (17), the evaluated result of univariate conditional mean, conditional mean square, and conditional standard variance square of response, i.e., $\mu_{mi}(\mathbf{X})$, $\mu_{mi}^2(\mathbf{X})$, and $\sigma_{mi}^2(\mathbf{X})$, and their function value corresponding five estimation point in original space can be obtained in Tables 11-13.

Having taken into account the effect of correlation of duration T and spectral S_0 of the excitation, the overall mean and standard deviation of maximum response, combing Tables 11–13 and equation (16), are able to be evaluated as follows:

Table 9: Polynomial coefficients of each parameter.

Variables	a_i	b_i	c_i
T	−0.156	0.975	0.156
S_0	−0.675	0.296	0.675
ξ	−0.156	0.975	0.156
f	0	1	0

Table 10: Five-point estimated coordinates in original space after correlation transformation.

Parameter variables	The original space coordinates corresponding to five estimated points				
	X_{2+}	X_{1+}	X_0	X_{1+}	X_{2+}
T	1.429	5.933	10.000	14.070	18.571
S_0	-9.13×10^{-1}	-9.208×10^{-2}	6.500×10^{-1}	1.392	2.213
ξ	4.287×10^{-3}	1.180×10^{-2}	3.000×10^{-2}	4.220×10^{-2}	5.571×10^{-2}
f	8.572×10^{-1}	1.458	2.000	2.542	3.143

Table 11: Estimation results for $\mu_{mi}(\mathbf{X})$ in original space after correlation transformation.

Parameter variables	μ_{mi}	The estimation results corresponding to five estimated points				
		X_{2+}	X_{1+}	X_0	X_{1+}	X_{2+}
T	4.561×10^{-2}	3.169×10^{-2}	4.260×10^{-2}	4.602×10^{-2}	4.814×10^{-2}	4.979×10^{-2}
S_0	4.602×10^{-2}	-6.464×10^{-2}	-6.519×10^{-3}	4.602×10^{-2}	9.856×10^{-2}	1.566×10^{-1}
ξ	6.170×10^{-2}	3.220×10^{-1}	1.170×10^{-1}	4.602×10^{-2}	3.272×10^{-2}	2.578×10^{-2}
f	6.079×10^{-2}	5.118×10^{-1}	1.135×10^{-1}	4.602×10^{-2}	2.314×10^{-2}	1.257×10^{-2}

$$\begin{aligned}\mu_M^{\mathrm{Cor}} &= \mathrm{E}E\left[\mu_m(\mathbf{X})\right] \cong \sum_{i=1}^{4}\mu_{mi} - (4-1)\mu_m(\boldsymbol{\mu}) = 7.606\times10^{-2},\\ \sigma_M^{\mathrm{Cor}} &= \sqrt{E\left[\mu_m^2(\mathbf{X})\right] + E\left[\sigma_m^2(\mathbf{X})\right] - \left(\mu_M^{\mathrm{cor}}\right)^2} = 0.869.\end{aligned} \tag{50}$$

In order to further investigate the uncertainty effect on stochastic response involving correlated parameter variables in detail, four cases involving correlated parameter variables, i.e., given parameters, structure parameters (ξ and f), excitation parameters (T and S_0), and all parameters (T, S_0, ξ, and f), are considered, respectively. Meanwhile, the accuracy of their results is able to be confirmed by utilizing the MCS with a size of 10^6. The results of these cases are listed in Table 14.

From Table 14, some conclusions, in this example, can be revealed: (I) the correlation in this example has a great influence on response evaluation relative to the independent situation: if ignored the effect of correlation, the evaluation result in overall mean of maximum response will be underestimated, especially for structural property parameters and all parameters. When considering the excitation characteristic parameter, the overall standard deviation of the maximum result will be overestimated; when considering all parameters, the opposite conclusion will be obtained. (II)While the correlation is taken into account, the effect of excitation characteristic parameters on the overall mean of maximum response and the overall standard deviation is more obvious than in other cases. (III) The calculation results including correlation are basically in agreement with those obtained using MCS, in which the maximum error between them is just 4.124%.

5.2. Example 2: 2-DOF Linear Simply Isolated Bridge Subjected to Dynamic Excitation. Figure 2 shows a 2-DOF linear isolated bridge in which one of the masses is subjected to dynamic vertical excitation $P(t)$.

Assuming their structure-property parameters, that is, mass m, elastic modulus E, and initial moment I, are the same, the natural frequency ω_i of the simply supported isolated bridge, based on linear vibration theory, can be provided as follows:

$$\begin{aligned}\omega_1 &= 5.69\sqrt{\frac{EI}{mL^3}},\\ \omega_2 &= 22\sqrt{\frac{EI}{mL^3}},\end{aligned} \tag{51}$$

where E, I, and L are the elastic modulus, initial moment, and span of the bridge, respectively.

Table 12: Estimation results for $\mu_{mi}^2(\mathbf{X})$ in original space after correlation transformation.

Parameter variables	μ_{mi}^2	The estimation results corresponding to five estimated points				
		X_{2+}	X_{1+}	X_0	X_{1+}	X_{2+}
T	2.086×10^{-3}	1.004×10^{-3}	1.815×10^{-3}	2.118×10^{-3}	2.317×10^{-3}	2.479×10^{-3}
S_0	3.620×10^{-3}	4.179×10^{-3}	4.251×10^{-5}	2.118×10^{-3}	9.714×10^{-3}	2.455×10^{-2}
ξ	5.582×10^{-3}	1.037×10^{-1}	1.369×10^{-2}	2.118×10^{-3}	1.071×10^{-3}	6.143×10^{-4}
f	7.062×10^{-3}	2.620×10^{-1}	1.289×10^{-2}	2.118×10^{-3}	5.356×10^{-4}	1.582×10^{-4}

Table 13: Estimation results for $\sigma_{mi}^2(\mathbf{X})$ in original space after correlation transformation.

Parameter variables	σ_{mi}^2	The estimation results corresponding to five estimated points				
		X_{2+}	X_{1+}	X_0	X_{1+}	X_{2+}
T	8.523×10^{-5}	2.303×10^{-4}	9.779×10^{-5}	8.076×10^{-5}	7.249×10^{-5}	6.693×10^{-5}
S_0	1.380×10^{-4}	1.593×10^{-4}	1.621×10^{-6}	8.075×10^{-5}	3.704×10^{-4}	9.360×10^{-4}
ξ	2.128×10^{-4}	3.953×10^{-3}	5.220×10^{-4}	8.076×10^{-5}	4.081×10^{-5}	2.342×10^{-5}
f	3.850×10^{-4}	1.181×10^{-2}	6.015×10^{-4}	8.076×10^{-5}	1.773×10^{-5}	4.659×10^{-6}

Table 14: Result comparison between independent variables and correlation variables.

Considered cases	μ_M^{Ind}	σ_M^{Ind}	μ_M^{Cov}	σ_M^{Cov}	Error with MCS (%)	
					$E\mu_M^{\text{Cov}}$	$E\sigma_M^{\text{Cov}}$
Given parameters	4.602×10^{-2}	8.986×10^{-3}	—	—		
Structural parameters	2.498×10^{-2}	7.473×10^{-3}	7.647×10^{-2}	7.202×10^{-2}	3.730	3.311
Excitation parameters	4.570×10^{-2}	8.001×10^{-4}	4.561×10^{-2}	4.061×10^{-2}	4.124	2.895
All parameters	6.463×10^{-2}	6.797×10^{-2}	7.606×10^{-2}	8.238×10^{-2}	2.578	3.193

In the frequency domain, the standard deviation of the displacement responses, based on the random vibration theory, is obtained as follows:

$$\begin{aligned}\sigma_{y1}^2 &= \frac{\pi S_0}{4m^2}\left\{\frac{1}{2\xi_1\omega_1^3}+\frac{1}{2\xi_2\omega_2^3}+\frac{8\left(\xi_1\omega_1+\xi_2\omega_2\right)}{\left(\omega_1^2-\omega_2^2\right)^2+4\left[\xi_1\xi_2\omega_1\omega_2\left(\omega_1^2+\omega_2^2\right)+\left(\xi_1^2+\xi_2^2\right)\omega_1^2\omega_2^2\right]}\right\},\\ \sigma_{y2}^2 &= \frac{\pi S_0}{4m^2}\left\{\frac{1}{2\xi_1\omega_1^3}+\frac{1}{2\xi_2\omega_2^3}-\frac{8\left(\xi_1\omega_1+\xi_2\omega_2\right)}{\left(\omega_1^2-\omega_2^2\right)^2+4\left[\xi_1\xi_2\omega_1\omega_2\left(\omega_1^2+\omega_2^2\right)+\left(\xi_1^2+\xi_2^2\right)\omega_1^2\omega_2^2\right]}\right\},\end{aligned} \tag{52}$$

where σ_{yi} is the standard deviation of the displacement responses of each mass, S_0 is the spectral intensity of the excitation, which is modeled as Gaussian white noise, and T is its corresponding duration $\nu_i \approx 2\pi/\omega_i$ is approximately the mean cross ratio corresponding to ith mass, and both masses have the same damping ratio ξ. The joint and marginal PDFs of these uncertainty system parameters are unknown, except for the first three moments and the correlation matrix, as listed in Table 15.

To investigate the evaluation influence of both masses on response uncertainty involving correlated parameter variables, the mean and standard deviation of the maximum response evaluation under the given parameters are contrasted with the correlation parameters. Meanwhile, for the uncertainty response evaluation, the following three cases were analyzed:

Case 1. Considering only uncertainties contained in excitation parameters, S_0 and T.

Case 2. Considering only uncertainties contained in the structural parameters L, I, E, m, and ξ.

Case 3. Considering uncertainties of all uncertainty parameters, S_0, T, L, I, E, m, and ξ.

Partial calculation results with regard to the correlation transformation process (third-moment pseudo-correlation normal transformation technology) are given, such as polynomial coefficients of each parameter (see Table 16), the lower triangular matrix L_0, and five estimation point coordinates, in the original space (see Table 17).

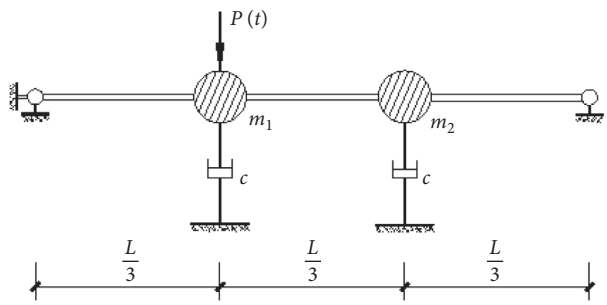

Figure 2: The calculation diagrams of simply supported isolated bridges.

Table 15: Statistic moments and correlation matrix of uncertain system parameters.

Parameter	Mean	COV	Skewness	Correlation matrix
T	15 s	0.4	0.431	$\begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$
S_0	$780\,\mathrm{cm}^2/\mathrm{s}^3$	0.6	1.608	
ξ_1	0.043	0.3	0.431	
ξ_2	0.069	0.3	0.431	
m	4.616×10^5 kg	0.1	0.301	
L	1.500×10^4 mm	0.1	0.301	$\begin{pmatrix} 1.0 & 0.3 & 0.2 \\ 0.3 & 1.0 & 0.1 \\ 0.2 & 0.1 & 1.0 \end{pmatrix}$
I	$7.087\times10^{11}\,\mathrm{mm}^4$	0.1	0.301	
E	3.150×10^4 MPa	0.1	0	

Table 16: Polynomial coefficients of each parameter.

Variables	a_i	b_i	c_i
T	−0.072	0.995	0.072
S_0	−0.283	0.916	0.283
ξ_1	−0.072	0.995	0.072
ξ_2	−0.072	0.995	0.072
E	−0.0503	0.997	0.050
m	−0.0503	0.997	0.050
L	−0.0503	0.997	0.050
I	0	1	0

Table 17: Five estimation point coordinates in original space after correlation transformation.

Parameter variables	The original space coordinates corresponding to five estimated points				
	X_{2+}	X_{1+}	X_0	X_{1+}	X_{2+}
T	32.140	6.866	15.000	23.134	32.142
S_0	2.625×10^5	-9.557×10^3	7.800×10^4	1.656×10^5	2.625×10^5
ξ_1	7.985×10^{-2}	2.551×10^{-2}	4.310×10^{-2}	6.112×10^{-2}	7.985×10^{-2}
ξ_2	12.811	4.093×10^{-2}	6.930×10^{-2}	9.710×10^{-2}	1.281×10^{-1}
E	5.934×10^5	3.990×10^5	4.616×10^5	5.242×10^5	5.935×10^5
m	1.928×10^4	1.296×10^4	1.500×10^4	1.703×10^4	1.929×10^4
L	9.263×10^{11}	5.882×10^{11}	7.087×10^{11}	8.292×10^{11}	9.627×10^{11}
I	4.249×10^4	2.629×10^4	3.150×10^4	3.671×10^4	4.249×10^4

Table 18: Response evaluation result comparison between presented method and MCS for Mass.1.

Considered cases	Presented method		MCS		Method error (%)	
	μ_M^{Cov}	σ_M^{Cov}	μ_M^{MCS}	σ_M^{MCS}	$E_{\mu M}$	$E_{\sigma M}$
Given parameters	5.141×10^{-3}	4.256×10^{-3}	—	—	—	—
Structure parameters	3.952×10^{-3}	1.466×10^{-2}	3.767×10^{-3}	1.538×10^{-2}	4.911	4.681
Excitation parameters	3.657×10^{-3}	5.061×10^{-3}	3.493×10^{-3}	5.327×10^{-3}	4.500	4.993
All parameters	2.466×10^{-3}	1.452×10^{-2}	2.553×10^{-3}	1.412×10^{-2}	3.416	2.832

Table 19: Response evaluation result comparison between presented method and MCS for Mass.2.

Considered cases	Presented method		MCS		Method error (%)	
	μ_M^{Cov}	σ_M^{Cov}	μ_M^{MCS}	σ_M^{MCS}	$E_{\mu M}$	$E_{\sigma M}$
Given parameters	7.251×10^{-3}	2.186×10^{-3}	—	—	—	—
Structure parameters	1.808×10^{-3}	1.835×10^{-2}	1.886×10^{-3}	1.905×10^{-2}	4.136	3.675
Excitation parameters	6.403×10^{-3}	4.512×10^{-3}	6.379×10^{-3}	$4.370\times10^{-}$	3.762	3.249
All parameters	9.595×10^{-4}	1.850×10^{-2}	9.255×10^{-4}	1.768×10^{-2}	3.673	4.638

Combiningequations (27)–(29) and the Cholesky decomposition, thelower triangular matrix L_0 can be obtained as follows:

$$\mathbf{L}_0 = \begin{pmatrix} 1 & & & & & & \\ 0.536 & 0.839 & & & & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0.301 & 0.953 \\ 0 & 0 & 0 & 0 & 0 & 0.201 & 0.042 & 0.979 \end{pmatrix}. \quad (53)$$

Having been converted by the third-moment pseudo-normal transformation technology, the five estimation point coordinates in the original space are listed in Table 17.

Both masses' calculation results of the response evaluation, including correlated parameter variables, are provided inTables 18 and 19. Meanwhile, the accuracy of their results can be confirmed by utilizing an MCS with a size of 10^6, which are listed in Tables 18 and 19 as well.

After observing and contrasting the results in Table 18, some conclusions on Mass 1 can be drawn as follows: (I) To consider the uncertain and correlated system parameter variables, relative to the determination parameter situation, the evaluation results of overall mean and standard deviation of the maximum response have a great influence; for the overall mean of maximum response, the evaluation results will be overestimated; for the overall standard deviation of maximum response, the evaluation result will be overestimated under considering structure-property and all parameters, while the evaluation results, under excitation-characteristics parameters considered, will be underestimated. (II) For the overall mean of maximum response, the effect of the excitation-characteristic parameters was more remarkable than structure-property parameters; for the overall standard deviation of maximum response, the opposite conclusion is obtained. (III) Their calculation results are in agreement with those obtained using MCS, in which the maximum error between them is merely 4.993%. Similarly, the conclusions of Mass.1 are suitable for Mass.2 inTable 19.

6. Conclusions

This study focused on evaluating uncertainty response with correlated random variables, the primary conclusion, on the basis of two examples, which were able to be drawn:

(1) Without being any sensitivity analysis with regard to the maximum response, the first two statistical moments of the maximum response, including correlated system parameter variables, were able to be fast evaluated.

(2) With just needing the first three moments and correlation matrix of system parameter variables instead of knowing their joint PDF or marginal PDF, the mean value and standard deviation of the maximum response, including correlated system parameter variables, were capable to be steadily evaluated.

(3) While taking into account the correlated system parameter variables, the overall mean and standard deviation's evaluation results of maximum response, relative to the determination parameter situation, have a great influence: ignoring the effect of these parameter variables will underestimate or overestimate the actual response evaluation.

(4) When considering the effect of correlation among system parameter variables, one can find the effect of statistic maximum response evaluation with respect to structural property parameters is relatively obvious than excitation characteristic parameters in example 1, whereas the opposite conclusion can be obtained in example 2.

(5) Compared with the ones using MCS, the accuracy of response evaluation by the presented method was trustworthy.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Qiang Fu and Jiang jun Liu conceived, designed, and performed the study. Xiao Li and Xueji Cai collected statistical sample dates of engineering examples used in the paper. Zilong Meng and Jiarui Shi wrote and revised the paper together. The authors have read and approved the final published manuscript.

Acknowledgments

The research article was supported by the National Natural Science Foundations of China, whose grant numbers are U1134209 and U1434204, respectively, and the Natural Science Foundation Project of Hunan Province in 2022, which was named as "Research on the Influencing Factors of Dynamic Stiffness of Cutting Machine-Foundation-Soil System." It was also supported by the Scientific Research Project of Hengyang Science and Technology Bureau, grant no. 2020jh012854, the Scientific Research Project of Hengyang Science and Technology Bureau, grant no. 2019jhzx0639, and the Education and Science Research Program for Fujian Middle Aged and Young Teachers, grant no. JT180504. These supports are gratefully acknowledged.

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