Research Article
Energy Flow Analysis of High-Frequency Flexural Vibration of Wedge Beam Structures

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Wedge members with variable thickness are widely used in ship structures, aerospace structures, and building structures. Considering their application scenarios, their high-frequency vibration characteristics have important research value. Energy finite element analysis (EFEA) is a powerful tool to predict high-frequency vibration response. EFEA is essentially applying finite element analysis to solve the energy density governing equation. However, the equation for wedge beam is missing. Energy flow analysis aims to obtain energy density governing equation. The energy flow model of wedge beam with variable thickness is established in this paper. Firstly, according to the geometric acoustic approximation method, the general displacement solution of the bending deformation equation for a wedge beam is deduced. Furthermore, the wave dispersion relation of bending deformation of wedge beam is derived. Then, the relationship between the time-averaged vibration energy density and energy flow of the wedge beam is also derived. Also, the governing equation taking energy density as a variable of the wedge beam structure is derived through the energy balance relationship in the microelement body. The governing equation is numerically solved, and the energy density distribution on the wedge beam structure is calculated. Finally, the energy density distribution of uniform beam and wedge beam under the same excitation is analyzed and summarized; the influence of different excitation frequencies and power exponent of thickness change on the performance of wedge beam structure is also summarized. Geometrically, the wedge beam satisfies the one-dimensional acoustic black hole structure. The calculation results show that the energy density on the beam structure increases with the decrease of thickness and reaches its maximum near the tip. The wedge acoustic black hole beam structure has a good energy absorption effect with a frequency between 250 Hz and 2000 Hz. With the further increase of excitation frequency, the energy absorption effect of the wedge beam structure has dropped significantly.

1. Introduction
The variable-thickness structures have some advantages, such as effectively saving materials, improving material utilization, reducing the structure’s weight, and increasing the rigidity. They are widely used in various engineering structures, such as ship structures, aerospace structures, and building structures. For example, the wing of an airplane is a variable-thickness shell structure. Variable-thickness structures used in ships or aircraft structures tend to become thinner due to lightweight requirements, together with faster and faster flight speed, so this type of structure may induce high-frequency vibrations. The high-frequency vibrations may damage structures and the electronic components carried by structures, affecting the entire system’s smooth operation. The study of high-frequency vibration characteristics plays a vital role in the structure’s safety. Therefore, in the structural design stage, predicting the high-frequency vibration response of such structures has vital engineering significance for the safety of ships and aircraft.

The finite element analysis (FEA) is an effective numerical analysis method to deal with vibration analysis of complex structures [1]. Specially, in order to obtain more accurate vibration response, governing equation for beam
with variable thickness, with multiple jumped discontinuities, was derived [2]. FEA is adopted to solve the governing equation. However, with the increase of frequency, the modal deformation of the structure becomes more and more complex. Due to the limitation of mesh size, the FEA has an upper limit of frequency in the application of variable-thickness structures.

Energy finite element analysis (EFEA) is an auspicious method for predicting high-frequency vibration response. This method takes the energy density averaged in time and space as the variable of the governing differential equation and applies FEA to solve the governing differential equation. It can catch the spatial energy distribution in each subsystem and obtain the energy flow between subsystems. For predicting high-frequency vibration response, it makes up for the finite element method and other traditional vibration analysis methods that require a large amount of calculation and calculation time. Besides, it can better predict the internal energy of the substructure that cannot be accurately predicted in the statistical energy analysis. In short, the energy finite element method can clearly predict the mutual energy transfer between each unit, and it can also predict the power flow and vibration response distribution in the entire structure. The amount of information is more abundant, and the vibration analysis can be performed in higher frequency [3].

In 1989, Nefskse and Sung first proposed that the finite element method can be used to solve the energy flow equation and obtain the distribution characteristics of the energy density on the beam [4]. In the same year, Wholeser’s group studied the energy flow in vibrating rods and beams; they proposed a method similar to the heat transfer problem to deal with the heat flow to simulate the energy flow of the rod and beam system [5]. These works laid the foundation for the further development of the energy finite element method. Park and Hong combined the energy finite element method with Timoshenko’s theory to study beam vibration problems [6, 7]. Sun studied the application of the energy finite element method in ship structures, deduced the energy density governing equation related to bending vibration if the rod, beam, and thin plate structure were excited, and carried out the finite element solution [8]. Xie et al. deduced the energy density governing equation of flexural vibration of the cylindrical shell under symmetrical excitation and used the finite element method to calculate the energy density equation numerically [9]. In particular, the energy finite element method of beam structure research is constantly enriched. In 2010, Cai et al. established the energy density governing equations to study the lateral vibration of composite laminated beams under excitation [10].

In 2013, Zheng, based on the Euler–Bernoulli beam considering the damping coefficient of the structure, established the wave equation of a simple trigraminal beam structure coupled at any angle and obtained the dissipated power coefficient, reflection coefficient, and propagation coefficient in the near field at coupling point according to the relationship between the vibration wave amplitude and the vibration power [11]. In 2018, Wang et al. used the energy finite element method to study the flexural vibration characteristics of functionally graded beams and coupled beams and deduced the energy density governing equations of functionally graded material beams and the coupled beam to get the energy density and energy flow in the beam [12]. In 2021, Liu et al. analyzed the vibration response of a beam with a temperature gradient under high-frequency excitation and established an energy flow model for the thermal gradient beam [13].

In the above-mentioned beam structure, the beam structure is considered to be a uniform thickness beam structure. However, the energy distribution in variable-thickness beams is different from uniform-thickness beams; the derivation of the energy density governing equation for variable-thickness beam structures is necessary. In this paper, we focus on the wedge beam, a kind of variable-thickness beam.

This article assumes that the wedge beam structure is excited by the vertical incident bending wave, and the general solution of the motion equation for variable-thickness beam is derived by the method of geometric acoustic approximation. According to the general solution obtained, the energy density governing equation of the wedge beam structure is obtained by catching the relationship between the energy density and energy flow value averaged in time and space. The governing equation is solved numerically, and the energy density distribution on the wedge beam under point excitation is obtained. At the same time, the energy density distribution results of uniform beams are solved. Through comparison and analysis, the energy density distribution law of wedge beams and uniform beams is summarized. Also, the influence of different excitation frequencies and power exponent of thickness change on the performance of wedge beams is summarized. Compared with the existing theories, the established energy flow model can get a more accurate energy distribution of wedge beams under high-frequency excitation.

Section 2 gives the derivation process of the energy density governing equation of the wedge beam. Section 3 numerically solves the energy density governing equation of the wedge beam, verifies the model’s effectiveness, and explores the influence of other factors on the vibration characteristics of the beam structure. Section 4 summarizes the full text.

2. Derivation of the Energy Density Governing Equation

In this paper, the derivation of the energy density governing equation is based on the following assumptions:

1. All the studies in this paper assume that the propagation direction of the bending wave is perpendicular to the edge of the wedge beam, and the oblique incidence of the bending wave is not considered.

2. Assume that the damping coefficient of the structure is far less than $1(\eta \ll 1)$. 

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2. Assume that the damping coefficient of the structure is far less than $1(\eta \ll 1)$.
Bending wave in structure forms the reverberant plane wave field; in the derivation process, the far-field solution in the displacement solution is ignored.

This research model is based on Euler and Bernoulli’s beam theory.

The wedge beam structure studied in this paper is shown in Figure 1. The lower surface is a plane and the height of the upper surface changes in the form of a power function. The thickness variation law satisfies $h(x) = \varepsilon x^m$, where $m \geq 2$. Its structure is just in line with the one-dimensional acoustic black hole structure. For an ideal one-dimensional acoustic black hole structure, the phase and group velocities of flexural waves in a beam decrease with the decreasing thickness. Thus, if the thickness varies smoothly and reaches a zero value, the corresponding wave velocities also reduce to zero. It follows that the wave can never reach the tip of the wedge; hence, it cannot be reflected. This concept of a retarding structure was originally proposed by Krylov [14]. For its thickness variation rule $h(x) = \varepsilon x^m$, $m$ is a real constant defining the power-law profile, $x$ is the axial coordinate along with the wedge, and $\varepsilon$ is a constant.

In the framework of the Euler–Bernoulli assumptions and in the harmonic regime, when the wedge beam structure is excited by a vertical incident bending wave, its motion equation is as follows [15]:

$$\frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 w}{\partial x^2} \right] + \rho h b \frac{\partial^2 w}{\partial t^2} = F\delta(x - x_0), \tag{1}$$

where $w$ is the transverse displacement of the structure; $EI(x)$ is the bending stiffness; $E$ is Young’s modulus; and $I(x) = bh(x)^3/12$ is the moment of inertia of the section. In order to consider the effect of structural damping at high frequencies, the complex elastic modulus $E_c = E(1 + in)$ is introduced, where $n$ is the structural damping coefficient; $\rho$ is the structural density; $h(x)$ is the structural thickness; $b$ is the width of the beam cross section; $t$ is the time variable; $F$ is the amplitude of the point force exerted on the beam; and $\delta$ is the $\delta$ function.

The corresponding WKB transverse displacement solution is [16]

$$w = e^{-i \sum_{n=0}^{\infty} S_n \varepsilon^n}, \quad \varepsilon \longrightarrow 0, \tag{2}$$

where $S_n$ characterizes the temporal and spatial oscillation characteristics. $\varepsilon$ is the perturbation factor defined as $\varepsilon = \omega^{-1/2}$. The requirement that the perturbation factor of the solution tends to zero means that the WKB method can provide good results at higher frequencies. Taking the first-order WKB approximation (geometric acoustic approximation) method, the transverse displacement fluctuation solution of the wedge beam structure corresponding to position $x$ with time and spatial harmonics can be obtained as

$$w(x) = \sum A(x)e^{i\varphi(x)}e^{i\omega t}, \tag{3}$$

where $A(x)$ and $\varphi(x)$ are the amplitude and cumulative phase at the coordinate $x$, respectively, and $\omega$ is the circular frequency, which characterizes the time oscillation characteristics. $A(x)$ and $\varphi(x)$ are as follows:

$$A(x) = A\left( \frac{h_0}{h(x)} \right)^{\frac{3}{4}},$$

$$\varphi(x) = \int_0^x k(x)dx,$$

where $A$ is an arbitrary complex constant, $h_0$ is the maximum thickness of the wedge beam, and $k(x)$ is the wavenumber related to coordinate $x$. Incorporating equation (3) into the homogeneous equation corresponding to the motion equation (1), the wave dispersion relationship between the wavenumber and the circular frequency can be established.

Because equation (3) is considered to be the result of the first-order WKB approximation, directly introducing it into equation (1) will cause the error of the solution to be further increased. Here, the transverse displacement trial solution (2), which is directly expressed in the form of infinite exponential power series, is brought into the homogeneous equation corresponding to equation (1). A series of differential equations can be obtained, and the differential equation corresponding to the first-order WKB approximation is

$$\varepsilon^0: S_0 = \frac{\rho bh}{EI} = \frac{12}{c_p^4 \left[ h(x) \right]^2}, \tag{5}$$

where $c_p$ is the phase velocity of the longitudinal wave in the beam, $c_p = (EI\rho)^{1/2}$, and solving the equation shows that there are four algebraic roots to $S_0$.

$$S_0' = j \int_0^x k(x)dx.$$ \tag{6}

where the factor $j = \pm 1, \pm i$. From the first-order WKB solution form, we can get

$$\varepsilon^{-1} S_0 = i \int_0^x k(x)dx.$$ \tag{7}

Solving equation (7) can find the four roots of wave number $k(x)$: $k_{x,t} = \pm ik_r, k_{x,i} = \pm ik_i$, where

$$k_r = \frac{12}{c_p^2} \omega^{1/2} h(x)^{1/2}; \tag{8}$$

After substituting $k_{x,t}(x)$ into the general solution of transverse displacement, the far-field wave oscillating in.
space with wavenumber as the frequency will be obtained, and \( k_{c,d}(x) \) corresponds to the exponentially attenuated near-field wave. When the vibration of the variable-thickness structure system reaches a steady state, the near-field solution will decay to a small amount, and the far-field solution that only considers energy density and energy flow generally does not consider the influence of near-field waves in the energy flow [17]. Therefore, equation (3) can be expressed as

\[
\omega(x,t) = (A(x)e^{i\varphi(x)}+B(x)e^{-i\varphi(x)})e^{i\omega t}.
\] (9)

Since the complex stiffness is used to simulate the hysteretic damping effect of the structure, where \( c_p = c_p(1 + i\eta)^{1/4} \), \( k_r \) is also a complex number, that is, \( k_r = k_r(1 + i\eta)^{1/4} \). Since the damping coefficient \( \eta \ll 1 \) in most cases, \( k_r \) can be simplified as

\[
k_r = \frac{12^{1/4}}{c_p^{1/2}h(x)^{1/2}} \left( 1 - i\frac{\eta}{4} \right),
\] (10)

where

\[
k_1 = \frac{12^{1/4}}{c_p^{1/2}h(x)^{1/2}},
\] (11)

\[
k_2 = -\frac{1}{4}k_1.
\]

The phase velocity can be calculated as

\[
C_{ph} = \frac{\omega}{Re(k)^{1/2}} = \frac{\omega}{12^{1/4}c_p^{1/2}h(x)^{1/2}},
\] (12)

\[
= \frac{\omega^{1/2}}{12^{1/4}}\frac{c_p}{h(x)^{1/2}}.
\]

Simultaneously, equation (5) can calculate the phase velocity as

\[
C_{ph} = \left( \frac{2\omega^2}{12\rho} \right)^{1/4} h(x)^{1/2}.
\] (13)

Therefore, the group wave velocity is

\[
C_{gr}(x) = 2\left( \frac{2\omega^2}{12\rho} \right)^{1/4} h(x)^{1/2}.
\] (14)

For the entire wedge beam structure, the power of the transverse vibration is transmitted through the shear force and the bending moment. The power corresponding to the shear force and the bending moment is, respectively [18],

\[
q_s = \int EI \left( \frac{3}{2} \frac{\partial w(x,t)}{\partial x} \right) \left( \frac{\partial w(x,t)}{\partial t} \right) dx,
\] (15)

\[
q_m = \int EI \left( \frac{3}{2} \frac{\partial^2 w(x,t)}{\partial x^2} \right) \left( \frac{\partial^2 w(x,t)}{\partial t^2} \right) dx.
\]

The kinetic energy density and potential energy density are, respectively,

\[
T = \frac{1}{2} \rho bh^2 \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx,
\]

\[
V = \frac{1}{2} EI \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx.
\]

Because this article studies the steady-state response, it is concerned with the relationship between the energy flow and the mean value of the energy density. Therefore, the time-averaged power flow and energy density in the differential unit can be obtained as [19]

\[
\langle q_s \rangle = Re \left( \frac{EI\omega}{2} \left( \frac{\partial w(x,t)}{\partial x} \right) \left( \frac{\partial w(x,t)}{\partial x} \right)^* \right),
\]

\[
\langle q_m \rangle = Re \left( \frac{EI\omega}{2} \left( \frac{\partial^2 w(x,t)}{\partial x^2} \right) \left( \frac{\partial w(x)}{\partial x} \right)^* \right).
\]

\[
\langle T \rangle = Re \left( \frac{1}{4} \rho bh^2 \left( \frac{\partial w}{\partial x} \right)^* \right),
\]

\[
\langle V \rangle = Re \left( \frac{1}{4} EI \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right).
\]

where \( \ast \) means to take the conjugate. Bring equation (9) into equations (17) and (18). As calculated in the differential unit, the thickness changes little, and \( m = \) small; the derivative of \( A(x), B(x) \) can be ignored. Knowing that \( k_1 \gg k_2 \), it ignores all the higher power of \( k_3 \) (2nd and 3rd). After reorganizing, the time-averaged total power flow in the microelement can be obtained as

\[
\langle q \rangle = \langle q_s \rangle + \langle q_m \rangle = EI\omega k_1^2 \left( -\frac{1}{2} \left| A(x) \right|^2 e^{-2\int k_2(x)dx} + \left| B(x) \right|^2 e^{2\int k_2(x)dx} \right).
\] (21)
In the same way, by substituting formula (9) into equations (19) and (20), the time-averaged total energy density in the microelement can be obtained as

\[
\langle e \rangle = \langle T \rangle + \langle V \rangle = \frac{1}{2} \rho bh \omega^2 \left( |A(x)|^2 e^{-2\gamma} \int k_z(x) dx + |B(x)|^2 e^{2\gamma} \int k_z(x) dx \right).
\]  

(22)

Since it is calculated in the differential unit, the first derivative of \( S(x) \) can be ignored in the same way. Compare equation (21) and (22) to find

\[
\langle q \rangle = \frac{EI \omega}{\rho bh \omega^2} k_z \frac{d\langle e \rangle}{dx}.
\]  

(23)

Substituting equations (11) and (12) into equation (23), we can get

\[
\langle q \rangle = -\frac{C_{gr}(x)^2}{\eta \omega} \frac{d\langle e \rangle}{dx}.
\]  

(24)

In energy flow analysis, the energy density governing equation can be derived from the energy flow balance in the control volume of the differential unit body. As shown in Figure 2, for any microsegment \( dx \) of the thermal gradient beam, there is a balance between energy input, energy dissipation, and energy conduction in the steady state [20].

\[
\pi_{in} = \pi_{diss} + \nabla \langle q \rangle.
\]  

(25)

For the hysteresis damping model adopted, due to the fact that vibration period \( T = 2\pi/\omega \), research by Cremer et al.

\[
\int_{x_i}^{x_f} \nabla \cdot \left( \frac{d^2 \langle e \rangle}{dx^2} + \frac{m}{x} \frac{d\langle e \rangle}{dx} \right) dx \right) + \left( \int_{x_i}^{x_f} \frac{\eta \omega}{C \cdot h(x)} \langle e \rangle dx \right) = \int_{x_i}^{x_f} \nabla \cdot \left( \frac{\pi_{in}}{C \cdot h(x)} \right) dx.
\]  

(29)

The energy density is interpolated as follows:

\[
e = \sum_{j=1}^{n} e \phi_j.
\]  

(30)

Substituting equation (30) into equation (29), we can obtain the Galerkin weighted residual equation:

\[
\sum_{j=1}^{n} \left( \int_{x_i}^{x_f} \frac{d \phi_i}{dx} \frac{d \phi_j}{dx} + \frac{m}{x} \frac{d \phi_i}{dx} \frac{d \phi_j}{dx} + \frac{\eta \omega}{C \cdot h(x)} \phi_i \phi_j \right) dx \right) e_j - \phi_i q_i \bigg|_{x_i}^{x_f} - \int_{x_i}^{x_f} \frac{\pi_{in}}{C \cdot h(x)} dx = 0.
\]  

(31)

Write equation (31) as a matrix as follows:

\[
[K^{(e)}] \{e^{(e)}\} = \{F^{(e)}\} + \{Q^{(e)}\},
\]  

(32)

\[
K^{(e)}_{ij} = \int_{x_i}^{x_f} \left( \frac{d \phi_i}{dx} \frac{d \phi_j}{dx} + \frac{m}{x} \frac{d \phi_i}{dx} \frac{d \phi_j}{dx} + \frac{\eta \omega}{C \cdot h(x)} \phi_i \phi_j \right) dx,
\]  

(33)

Equation (27) is the energy density governing equation of the wedge beam. In order to obtain the numerical solution to the energy flow analysis governing differential equation (27), the matrix form of the differential equation should be obtained via the variation method. Construct appropriate shape functions \( v = \varphi_i \), for equation (27), and the weak integral form of a differential equation can be derived by using the Galerkin weighted residual method as follows:

\[
\langle q \rangle = -\frac{C_{gr}(x)^2}{\eta \omega} \frac{d\langle e \rangle}{dx}.
\]  

(26)

Substituting equations (24) and (26) into equation (25), the energy density governing equation of the wedge beam structure can be obtained as

\[
\frac{d^2 \langle e \rangle}{dx^2} + \frac{m}{x} \frac{d\langle e \rangle}{dx} + \frac{\eta \omega}{C \cdot h(x)} \langle e \rangle = \frac{\pi_{in}}{C \cdot h(x)}.
\]  

(27)

where

\[
C = \frac{12^{1/2} E^{1/2}}{3 \eta p^{1/2}}.
\]  

(28)

Equation (27) is the energy density governing equation of the wedge beam. In order to obtain the numerical solution to the energy flow analysis governing differential equation (27), the matrix form of the differential equation should be obtained via the variation method. Construct appropriate shape functions \( v = \varphi_i \), for equation (27), and the weak integral form of a differential equation can be derived by using the Galerkin weighted residual method as follows:

\[
\sum_{j=1}^{n} \left( \int_{x_i}^{x_f} \frac{d \phi_i}{dx} \frac{d \phi_j}{dx} + \frac{m}{x} \frac{d \phi_i}{dx} \frac{d \phi_j}{dx} + \frac{\eta \omega}{C \cdot h(x)} \phi_i \phi_j \right) dx \right) e_j - \phi_i q_i \bigg|_{x_i}^{x_f} - \int_{x_i}^{x_f} \frac{\pi_{in}}{C \cdot h(x)} dx = 0.
\]  

(31)

Substituting equation (30) into equation (29), we can obtain the Galerkin weighted residual equation:

\[
K^{(e)}_{ij} = \int_{x_i}^{x_f} \left( \frac{d \phi_i}{dx} \frac{d \phi_j}{dx} + \frac{m}{x} \frac{d \phi_i}{dx} \frac{d \phi_j}{dx} + \frac{\eta \omega}{C \cdot h(x)} \phi_i \phi_j \right) dx = \int_{x_i}^{x_f} \varphi_i \pi_{in} dx,
\]  

(34)

\[
Q^{(e)} = \varphi_i q_i \bigg|_{x_f} - \varphi_i q_i \bigg|_{x_i},
\]  

(35)


\[ e^{(c)} = \{ e_1^{(c)}, e_2^{(c)}, \ldots, e_n^{(c)} \}^T, \quad i, j = 1, 2, \ldots, n \]  

where \( K_{ij}^{(c)} \) is the equivalent coefficient matrix of stiffness and mass related to material properties, \( F^{(c)} \) is the input term of energy, and \( Q^{(c)} \) is the power flow at the intersection nodes between elements. The energy density at any node of the structure can be obtained by introducing the known quantities and boundary conditions into the obtained energy finite element matrix equation.

### 3. Numerical Examples and Discussion

#### 3.1. Verification of Model Validity

In the previous section, the energy density governing equation of the wedge beam structure is derived and obtained. However, for complex structures, it is usually challenging to obtain an analytical solution and can only be solved numerically. For the calculation of the energy density of complex structures, matrix equation (32) needs to be solved to obtain the energy density at each node. However, for some structures with simple shapes, a numerical solution of the governing equation can be directly used to obtain the energy density distribution. In the following section, we will use the fourth-fifth-order Runge–Kutta algorithm to solve the energy density governing equation. It uses the fourth-order method to provide candidate solutions and the fifth-order method to control the error. It is a numerical solution method for ordinary differential equations with adaptive step size (variable step size), and its overall truncation error is \( \Delta x^5 \). The calculation process is as follows [21]:

\[
\begin{align*}
  k_1 &= hf(x_n, y_n), \\
  k_2 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}), \\
  k_3 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}), \\
  k_4 &= hf(x_n + h, y_n + k_3), \\
  y_{n+1} &= y_n + \frac{k_1 + k_2 + k_3 + k_4}{6} + O(h^5).
\end{align*}
\]

The energy density governing equation solved in this paper is a second-order variable coefficient non-homogeneous ordinary differential equation, and its coefficient terms have no special relationship between them. After setting the following model geometric parameters, material parameters, and initial conditions, we solve the numerical solution of the differential equation.

For an ideal one-dimensional acoustic black hole structure, if the thickness tends to zero, no reflection will occur so that the incident wave can be completely absorbed. However, for the actual structure, it is not possible to fabricate a wedge having a residual thickness equal to zero, as shown in Figure 3. Even realizing finite but very thin wedges poses significant complexities due to the deformation produced by the mechanical and thermal loads associated with the manufacturing case. An easy way to circumvent this difficulty is to modify the profile slightly by setting \( h(x) = \varepsilon(x + x_0)^m; \) therefore, in this example, the thickness of the wedge beam is set as \( h(x) = h_0(1 - (x/x_0))^2 \) [22].

One of the main assumptions of EFEA is the reverberant plane wave field in structures. For special structures, in order to satisfy these assumptions, exciting frequency should be greater than critical frequency. According to Reference [23], when \( k\eta L > 2 \), it can be considered as a high-frequency problem.

The model’s geometric and material parameters are shown in Table 1.

The initial conditions are as follows: \( \langle e \rangle x_n = 0.000001; \partial \langle e \rangle / \partial x | x_n = 0 \); the incident wave excitation frequency is 2000 Hz, the excitation point is located at \( x_n, x_n \) is the ideal truncation length, and \( x_n = 0.09 \) is the actual truncation length. In the table, \( E \) represents the elastic modulus, \( \rho \) represents the density, and \( \eta \) represents the structural damping coefficient. When the input power is \( 1 \times 10^{-5} \) W, the energy density distribution of each point on the beam is calculated.

In order to perform a simple verification of the results obtained in this paper, according to the energy density governing equation of the uniform beam [5] (equation (38)), we calculated the energy density distribution results on the uniform beam with the same material geometric parameters [24].

\[
\frac{C^2}{\eta \omega^2} \langle e \rangle + \eta \omega e = \pi_n.
\]

The numerical solution of the two calculation results is shown in Figure 4.

From Figure 4, it can be seen that under the same conditions, on the wedge beam structure, the energy presents a concentration effect in \( x = x_n \), and the energy density in the beam increases with the increase of the distance from the excitation position. For a uniform beam, as the propagation distance of the bending wave increases, the energy density in the beam gradually attenuates. Because the wedge beam studied in this paper is precisely in line with the one-dimensional acoustic black hole structure, the acoustic black hole (ABH) effect reduces the flexural wave velocity and achieves energy accumulation and absorption at the end through power exponential law tailoring thickness or material parameter gradient changes [25], which is consistent with the calculation results and proves the correctness of the energy density governing equation of the wedge beam model established in this paper.
3.2. The Influence of Excitation Frequency on Vibration Response. In order to explore whether different excitation frequencies will affect the vibration characteristics of the wedge beam structure, in this section, we discuss the influence of the excitation frequency on the vibration response of the wedge beam. In the calculation example in this section, except for the change of the excitation frequency of the wedge beam, the other parameters are the same as those in the previous section. Compare the energy density distribution on the beam under different excitation frequencies.

Figure 5 shows the energy density distribution of the wedge beam structure under 1000 Hz excitation. From the trend of the figure, it can be seen that the energy density on the wedge beam structure increases as the thickness of the wedge beam structure becomes thinner, and the energy density gets maximum value if $x$ reaches near the tip. This result is consistent with the conclusion of the one-dimensional acoustic black hole beam structure.

It can be seen from Figure 6 that the frequency at which the energy absorption effect of the variable-thickness acoustic black hole beam structure performs well is between 250 Hz and 2000 Hz. With the further increase of the excitation frequency, the energy absorption effect of the beam structure has been significantly reduced. The reason is that the tip volume of the wedge beam structure is relatively small, and the volume of energy is mainly determined by the volume at high frequencies. Figure 7 compares the energy density distribution of uniform beams under four excitation frequencies. From the figure, it can be seen that as the excitation frequency increases, due to the decrease in wavelength, the input power will experience more wavelength cycle attenuation in the space when it flows to the boundary. Therefore, the attenuation trend of energy density in space is more evident with the increase of excitation frequency in the figure.

Figure 6 shows the change of energy density on the wedge beam when bending waves with different incident frequencies are incident vertically because the wedge beam studied in this paper is just in line with the one-dimensional acoustic black hole structure in geometry. The acoustic black hole effect means that the phase velocity of the bending wave is gradually reduced to zero by cutting the thickness of the beam by a power law so that no reflection occurs at the tip. The acoustic black hole effect has a significant absorption effect on vibration energy. For the wedge beam, with the change of thickness, the energy presents an aggregation effect, and the energy density in the beam increases with the increase of the distance from the excitation position.

Table 1: Geometric parameters and material parameters.

<table>
<thead>
<tr>
<th>Geometric parameters</th>
<th>Value (m)</th>
<th>Material parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
<td>0.1</td>
<td>$E$</td>
<td>210 GPa</td>
</tr>
<tr>
<td>$x_a$</td>
<td>0.01</td>
<td>$\rho$</td>
<td>7800 kg/m³</td>
</tr>
<tr>
<td>$h_0$</td>
<td>0.008</td>
<td>$\eta$</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Figure 3: Schematic diagram of the model.
Figure 7 shows the change of energy density on a beam structure with uniform thickness when bending waves with different incident frequencies are incident perpendicularly. For a beam with uniform thickness, there is no energy absorption effect, and the energy density in the beam gradually decreases with the increase of the propagation distance of bending waves in the beam. Therefore, changes in curvature will occur in Figures 6 and 7.

3.3. The Influence of Power Exponent Describing Thickness Variation. Power exponent describing thickness variation of the wedge beam structure is another essential factor that affects the vibration characteristics of the beam structure. This section will discuss the influence of the power exponent on the vibration characteristics of the wedge beam structure. In the calculation examples in this section, except for the change of the power exponent of the wedge beam, the other parameters remain unchanged. Compare the energy density of beams with different thickness varying power exponents.

It can be seen from Figure 8 that with the increase of the power exponent, the tip energy density of the wedge beam structure increases, which also indicates the enhancement of its energy absorption effect. When \( m \geq 2 \), the energy absorption effect of the structure is relatively significant, and the energy absorption effect is proportional to the size of the power exponent. When \( m < 2 \), the energy accumulation effect of this structure is relatively weak and is not considered.

3.4. The Strong and Weak Sides of the Proposed Model. The model in this paper can model the energy density distribution on a wedge beam. In addition, this model belongs to the energy finite element method, so this model has the advantages of the energy finite element method:

1. The established energy flow model can accurately predict the vibration energy distribution of wedge beams under high-frequency excitation with a small amount of calculation.
2. The model can predict the mutual energy transfer between each unit and predict the distribution of power flow and vibration response in the whole structure, which is richer in information.
3. The requirements for the grid number of the structure are not high, so the solution speed is fast, and it is convenient for engineering applications.

However, due to some assumptions used in the derivation of the energy density governing equation, the application of the model will be limited:

1. It is suitable for wedge beams with a small thickness change rate.
2. It is suitable for high frequency.
3. It is suitable for structures with a damping coefficient far less than 1.
4. Conclusion

In this paper, the energy density governing equation of the wedge beam is deduced by the method of geometric acoustic approximation. The equivalent wave group velocity in the wedge beam is derived, and the energy density governing equation is numerically solved. Compared with the numerical solution of the energy density governing equation of the uniform beam, it is found that the energy absorption effect of the wedge beam structure is significant when it is close to the tip of the wedge beam structure, which is consistent with the conclusion of the one-dimensional acoustic black hole, thus verifying the deduction of the correctness of the energy density governing equation. A simple analysis about the influence of the excitation frequency and the power exponent describing thickness variation on the structural vibration characteristics is performed.

Conclusions on the derived governing equations are as follows:

(1) Since the analysis, frequency is assumed to be a high frequency in the derivation process, and \( \eta \ll 1 \); the near-field solution term in the displacement solution is ignored. The derivative of \( A(x) \) and \( B(x) \) is ignored in the calculation, so the derived energy density governing equation is suitable for high-frequency, variable-thickness beam structures with little thickness change and small \( m \). However, when used in low-frequency bands and other shapes and variable-thickness beam structures, the calculation accuracy will be affected.

(2) Compared with the energy density governing equation of the uniform thickness beam, the energy density governing equation of the wedge beam structure has an additional first-order derivative term of the energy density to \( x \), which is the influence term brought by the thickness change. Besides, it is also mainly different from the uniform beam energy density governing equation term.

With the help of the equation, the accuracy of predicting vibration response of wedge beam by EFEA can be improved greatly. However, the energy density governing equation derived in this paper is only for bending vibration of special wedge beam. There is still much work to be done if the energy finite element method is to be used in more general variable-thickness structures. Nevertheless, this paper still promotes the application of the energy finite element method in variable-thickness structures.

Data Availability

The Matlab program data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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References


