

Research Article

Viscosity-Projection Method for a Family of General Equilibrium Problems and Asymptotically Strict Pseudocontractions in the Intermediate Sense

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In this paper, a Meir-Keeler contraction is introduced to propose a viscosity-projection approximation method for finding a common element of the set of solutions of a family of general equilibrium problems and the set of fixed points of asymptotically strict pseudocontractions in the intermediate sense. Strong convergence of the viscosity iterative sequences is obtained under some suitable conditions. Results presented in this paper extend and unify the previously known results announced by many other authors.

1. Introduction

Let *H* be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let *C* be a nonempty closed convex subset of *H*. Let $A : C \to H$ be a nonlinear mapping and F : $C \times C \to \mathbb{R}$ be a bifunction, where \mathbb{R} denotes the set of real numbers. We consider the following generalized equilibrium problem: Find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle \ge 0, \quad \forall y \in C.$$
(1)

We use EP(F, A) to denote the set of solution of problem (1). If $A \equiv 0$, the zero mapping, then the problem (1) reduces to the normal equilibrium problem: Find $x \in C$ such that

$$F(x, y) \ge 0, \quad \forall y \in C.$$
 (2)

We use EP(F) to denote the set of solution of problem (2). If $F \equiv 0$, then the problem (1) reduces to the classical variational inequality problem: Find $x \in C$ such that

$$\langle Ax, y - x \rangle \ge 0, \quad \forall y \in C.$$
 (3)

We use VI(C, A) to denote the set of solution of problem (3). The generalized equilibrium problem (1) is very general in the sense that it includes, as special cases, saddle point problems, variational inequalities, optimization problems, mini-max problems, the Nash equilibrium problem in noncooperative games, and others (see, e.g., [1–4]).

Recall that a nonlinear mapping $T: C \rightarrow C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

$$\tag{4}$$

T is said to be uniformly *L*-Lipschitz continuous if there exists a constant L > 0 such that

$$||T^{n}x - T^{n}y|| \le L ||x - y||, \quad n \ge 1, \ \forall x, y \in C.$$
 (5)

T is said to be asymptotically nonexpansive if there exists a sequence $k_n \in [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$||T^{n}x - T^{n}y|| \le k_{n} ||x - y||, \quad n \ge 1, \ \forall x, y \in C.$$
 (6)

T is said to be asymptotically nonexpansive in the intermediate sense [5] if it is continuous and the following inequality holds:

$$\limsup_{n \to \infty} \sup_{x, y \in C} \left(\left\| T^n x - T^n y \right\| - \left\| x - y \right\| \right) \le 0.$$
(7)

Putting $\xi_n = \max\{0, \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|)\}$, we see that $\xi_n \to 0$ as $n \to \infty$. Then scheme (7) is reduced to the following:

$$\left\|T^{n}x - T^{n}y\right\| \le \left\|x - y\right\| + \xi_{n}, \quad \forall x, y \in C.$$
(8)

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Kirk [5] as a generalization of the class of asymptotically nonexpansive mappings.

It is known that, if C is a nonempty bounded closed convex subset of a real Hilbert space H, then every asymptotically nonexpansive self-mapping in the intermediate sense has a fixed point (see, e.g., [6]).

Recall also that *T* is said to be a λ -strict pseudocontraction [7, 8] if there exists a coefficient $\lambda \in [0, 1)$ such that

$$\|Tx - Ty\|^{2} \le \|x - y\|^{2} + \lambda \|(I - T)x - (I - T)y\|^{2},$$

$$\forall x, y \in C.$$
(9)

T is said to be an asymptotically λ -strict pseudocontraction [9, 10] if there exists a sequence $k_n \in [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and a constant $\lambda \in [0, 1)$ such that

$$\|T^{n}x - T^{n}y\|^{2} \le k_{n}\|x - y\|^{2} + \lambda \|(x - T^{n}x) - (y - T^{n}y)\|^{2},$$

 $n \ge 1, \ \forall x, y \in C.$
(10)

T is said to be an asymptotically λ -strict pseudocontraction in the intermediate sense [11, 12] if there exists a sequence $k_n \in [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and a constant $\lambda \in [0, 1)$ such that

$$\limsup_{n \to \infty} \sup_{x, y \in C} \left(\|T^{n}x - T^{n}y\|^{2} - k_{n}\|x - y\|^{2} -\lambda\|(x - T^{n}x) - (y - T^{n}y)\|^{2} \right) \le 0,$$
(11)
$$\forall x, y \in C.$$

Putting $e_n = \max\{0, \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - k_n \|x - y\|^2 - \lambda \|(x - T^n x) - (y - T^n y)\|^2)\}$, we see that $e_n \to 0$ as $n \to \infty$. Then scheme (11) is reduced to the following:

$$\|T^{n}x - T^{n}y\|^{2} \le k_{n}\|x - y\|^{2} + \lambda\|(x - T^{n}x) - (y - T^{n}y)\|^{2} + e_{n},$$

$$n \ge 1, \ \forall x, y \in C.$$
(12)

We use Fix(T) to denote the set of fixed point of T, that is, $Fix(T) = \{x \in C : Tx = x\}$. The class of asymptotically strict pseudocontractions in the intermediate sense was introduced as a generalization of the asymptotically strict pseudocontractions and asymptotically nonexpansive in the intermediate sense. Clearly, a nonexpansive mapping is a 0strict pseudocontraction, and an asymptotically nonexpansive mapping is an asymptotically 0-strict pseudocontraction. (see, e.g., [7–12]).

Fixed point technique represent an important tool for finding the approximate solution of equilibrium problem and its variant forms, which have been studied extensively in recent years due to their applications in physics, economics, optimization, and pure and applied sciences. Some numerical methods have been proposed for finding a common element of the set of fixed point of various types of nonexpansive mappings and the set of solution of equilibrium problems with bifunctions satisfying certain conditions; see [8–20] and references therein.

In 2009, Qin et al. [10] introduced the following explicit iterative algorithm for finding a common fixed point of a finite family of asymptotically λ_i -strictly pseudocontractions T_i for each i = 1, 2, ..., N

$$x_{n} = \alpha_{n-1} x_{n-1} + (1 - \alpha_{n-1}) T_{i(n)}^{h(n)} x_{n-1}, \quad n \ge 1,$$
(13)

where $x_0 \in C$, $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in (0,1) and n = [h(n) - 1]N + i(n), i = i(n) = 1, 2, ..., N. They also obtain weak and strong convergence theorems based on the cyclic scheme above.

Recently, Sahu et al. [11] considered a new iterative scheme for asymptotically strictly pseudocontractive mappings in the intermediate sense. To be more precise, they proved the following theorem.

Theorem SXY. Let *C* be a nonempty closed and convex subset of a real Hilbert space *H* and *T* : *C* \rightarrow *C* be a uniformly continuous asymptotically λ -strictly pseudocontractive mapping in the intermediate sense with a sequence $\{k_n\}$ such that Fix(*T*) is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in [0, 1] such that $0 < \delta \le \alpha_n \le 1 - \lambda$ for all $n \in \mathbb{N}$. Let $\{x_n\} \subset C$ be a sequence generated by the following (CQ) algorithm:

$$u = x_{1} \in C,$$

$$y_{n} = (1 - \alpha_{n}) u_{n} + \alpha_{n} T^{n} x_{n},$$

$$C_{n} = \left\{ w \in C : \|y_{n} - w\|^{2} \le \|x_{n} - w\|^{2} + \theta_{n} \right\}, \quad (14)$$

$$Q_{n} = \left\{ w \in C : \langle x_{n} - w, u - x_{n} \rangle \ge 0 \right\},$$

$$x_{n+1} = P_{C_{n} \cap Q_{n}}(u),$$

where $\theta_n = (k_n - 1)\rho_n^2 + e_n$ and $\rho_n = \sup\{|| x_n - p ||: p \in Fix(T)\} < \infty$. Then, $\{x_n\}$ converges strongly to $P_{Fix(T)}(u)$, where $P_{Fix(T)}$ is metric projection from H onto Fix(T).

In 2011, Hu and Cai [12] modified schemes (13) and (14) to the case of asymptotically strictly pseudocontractive mappings in the intermediate sense concerning the equilibrium problem and proposed the following modified hybrid method:

$$x_{0} \in C, \quad u_{0} \in C,$$

$$F(u_{n}, y) + \langle Ax_{n}, y - u_{n} \rangle$$

$$+ \frac{1}{r_{n}} \langle y - u_{n}, u_{n} - x_{n} \rangle \ge 0, \quad \forall y \in C,$$

$$z_{n} = (1 - \beta_{n}) u_{n} + \beta_{n} T_{i(n)}^{h(n)} u_{n},$$

$$y_{n} = (1 - \alpha_{n}) u_{n} + \alpha_{n} z_{n},$$

$$C_{n} = \left\{ w \in C : \left\| y_{n} - w \right\|^{2} \leq \left\| x_{n} - w \right\|^{2} + \theta_{n} \right\},$$

$$Q_{n} = \left\{ w \in C : \left\langle x_{n} - w, x_{0} - x_{n} \right\rangle \geq 0 \right\},$$

$$x_{n+1} = P_{C_{n} \cap Q_{n}} \left(x_{0} \right),$$
(15)

where $\theta_n = (k_{h(n)}-1)\rho_n^2 + e_{h(n)} \to 0 \text{ as } n \to \infty \text{ and } \rho_n = \sup\{\|x_n - p\|: p \in \Omega\} < \infty$. Moreover, they obtained convergence theorems under some suitable conditions.

On the other hand, Moudafi [13] introduced the following viscosity approximation method for fixed point problem of nonexpansive mapping

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n, \tag{16}$$

where f is a contractive mapping. He proved that the viscosity iterative sequence $\{x_n\}$ convergence strongly to a fixed point of T, which is the unique solution of the variational inequality:

$$\langle (I-f) x, z-x \rangle \ge 0, \quad \forall z \in \operatorname{Fix}(T).$$
 (17)

Furthermore, S. Takahashi and W. Takahashi [14] and Inchan [15] modified the viscosity approximation methods for finding a common element of the set of fixed point problems and equilibrium problems.

In 2012, Kimura and Nakajo [16] introduced a Meir-Keeler contraction and proposed a modified viscosity approximations by the shrinking projection method in Hilbert spaces, the socalled viscosity-projection method. To be more precise, they proved the following theorem.

Theorem KN. Let *C* be a nonempty closed convex subset of *H*, and let $\{T_n\}$ be a sequence of mappings of *C* into itself with $\Omega = \bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n) \neq \emptyset$ which satisfies the following condition: there exists $\{a_n\} \subset \mathbb{R}$ with $\liminf_{n \to \infty} a_n > -1$ such that $\|T_n x - z\|^2 \le \|x - z\|^2 - a_n \|x - T_n x\|^2$ for every $n \in \mathbb{N}, x \in C$, and $z \in \Omega$. Let *f* be a Meir-Keeler contraction of *C* into itself, and let $\{x_n\}$ be a sequence generated by

$$x_{1} = x \in C, \qquad C_{1} = C,$$

$$y_{n} = T_{n}x_{n},$$

$$C_{n+1} = \left\{ w \in C_{n} : \|y_{n} - w\|^{2} \le \|x_{n} - w\|^{2} - a_{n}\|x_{n} - y_{n}\|^{2} \right\},$$

$$x_{n+1} = P_{C_{n+1}}f(x_{n}),$$
(18)

for each $n \in \mathbb{N}$. For every sequence $\{z_n\} \in C$ and $z_n \to z \in C$ and $T_n z_n \to z$ imply that $z \in \Omega$. Then, $\{x_n\}$ converges strongly to $q \in \Omega$, which satisfies $q = P_\Omega f(q)$.

In this paper, inspired and motivated by research going on in this area, we introduce a new viscosity-projection method for a family of general equilibrium problems and asymptotically strict pseudocontractions in the intermediate sense, which is defined in the following way:

$$x_{1} \in C, \quad C_{1} = C,$$

$$u_{n} = F_{r_{M,n}}^{A_{M}} F_{r_{M-1,n}}^{A_{M-1}} \cdots F_{r_{2,n}}^{A_{2}} F_{r_{1,n}}^{A_{1}} x_{n},$$

$$z_{n} = (1 - \beta_{n}) u_{n} + \beta_{n} T_{i(n)}^{h(n)} u_{n},$$

$$y_{n} = (1 - \alpha_{n}) u_{n} + \alpha_{n} z_{n},$$

$$C_{n+1} = \left\{ w \in C_{n} : \|y_{n} - w\|^{2} \le \|x_{n} - w\|^{2} + \theta_{n} \right\},$$

$$x_{n+1} = P_{C_{n+1}} f(x_{n}),$$
(19)

where $\theta_n = (k_{h(n)} - 1)\rho_n^2 + e_{h(n)} \rightarrow 0$ as $n \rightarrow \infty$ and $\rho_n = \sup\{||x_n - p||: p \in \Omega\} < \infty$.

Our purpose is not only to extend the viscosity-projection method with a Meir-Keeler contraction to the case of a family of general equilibrium problems and asymptotically strict pseudocontractions in the intermediate sense, but also to obtain a strong convergence theorem by using the proposed schemes under some appropriate conditions. Results presented in this paper extend and unify the corresponding ones of [10–13, 16].

2. Preliminaries

Let *C* be a nonempty closed convex subset of a real Hilbert space *H* with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We use notation \rightarrow for weak convergence and \rightarrow for strong convergence of a sequence. For every point $x \in H$, there exists a unique nearest point in *C*, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C.$$

$$(20)$$

 P_C is called the metric projection of H onto C defined by $P_C(x) = \arg \min_{y \in C} || x - y ||$. It is well known that P_C is nonexpansive mapping, and $u = P_C x$ is equivalent to (see, e.g., [21]) the following:

$$\langle x - u, u - y \rangle \ge 0, \quad \forall y \in C.$$
 (21)

Recall that a mapping $A: C \to H$ is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \ge 0, \quad \forall x, y \in C.$$
 (22)

A is said to be *r*-strongly monotone if there exists a constant r > 0 such that

$$\langle Ax - Ay, x - y \rangle \ge r ||x - y||^2, \quad \forall x, y \in C.$$
 (23)

A is said to be α -inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \ge \alpha ||Ax - Ay||^2, \quad \forall x, y \in C.$$
 (24)

It is easy to see that if *A* is an α -inverse strongly monotone mapping from *C* into *H*, then *A* is $1/\alpha$ -Lipschitz continuous.

To study the generalized equilibrium problem (1), we may assume that the bifunction $F : C \times C \rightarrow \mathbb{R}$ satisfies the following conditions:

- (A1) F(x, x) = 0 for all $x \in C$;
- (A2) *F* is monotone, that is, $F(x, y) + F(y, x) \le 0$ for all $x, y \in C$;
- (A3) for each $x, y, z \in C$, $\lim_{t \to 0} F(tz + (1 t)x, y) \le F(x, y)$;
- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 1 (see [1, 3]). Let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Then, for any r > 0 and $x \in H$, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \ge 0, \quad \forall y \in C.$$
 (25)

Further, if $F_r x = \{z \in C : F(z, y) + (1/r)\langle y - z, z - x \rangle \ge 0, \forall y \in C\}$, then the following hold:

(1) F_r is single-valued;

(2) F_r is firmly nonexpansive, that is, $||F_r x - F_r y||^2 \le \langle F_r x - F_r y, x - y \rangle$ for all $x, y \in H$;

(3) $Fix(F_r) = EP(F);$

(4) EP(F) is closed and convex.

Lemma 2 (see [8]). In a Hilbert space H, there hold the following identities:

Lemma 3 (see [8]). Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x, y, z \in H$ and given also a real number $a \in \mathbb{R}$, the set

$$\left\{ v \in C : \left\| y - v \right\|^2 \le \left\| x - v \right\|^2 + \langle z, v \rangle + a \right\}$$
(26)

is closed and convex.

Lemma 4 (see [11]). Let C be a nonempty closed convex subset of a real Hilbert space H and T : $C \rightarrow C$ be a uniformly L-Lipschitz continuous and asymptotically λ -strict pseudocontraction in the intermediate sense. Then Fix(T) is closed and convex.

Lemma 5 (see [11]). Let C be a nonempty closed convex subset of a real Hilbert space H and T : $C \rightarrow C$ be a uniformly L-Lipschitz continuous and asymptotically λ -strict pseudocontraction in the intermediate sense. Then I - T is demiclosed at zero, that is, if the sequence $\{x_n\} \in C$ such that $x_n \rightarrow x$ and $x_n - Tx_n \rightarrow 0$ as $n \rightarrow \infty$, then $x \in Fix(T)$.

Lemma 6 (see [11]). Let *C* be a nonempty closed convex subset of a real Hilbert space *H* and $T : C \rightarrow C$ be an asymptotically

 λ -strict pseudocontraction in the intermediate sense with $\gamma_n = k_n - 1$. Then

$$\begin{aligned} \left\| T^{n} x - T^{n} y \right\| \\ &\leq \frac{1}{1 - \lambda} \left\{ \lambda \left\| x - y \right\| \right. \\ &+ \sqrt{\left[1 + (1 - \lambda) \gamma_{n} \right] \left\| x - y \right\|^{2} + (1 - \lambda) e_{n}} \right\}, \\ &n \geq 1, \ \forall x, y \in C. \end{aligned}$$

$$(27)$$

Recall also that a mapping f of a complete metric space (X, d) into itself is called a contraction with coefficient $r \in (0, 1)$ if $|| f(x) - f(y) || \le r || x - y ||$, for all $x, y \in X$. It is known that f has a unique fixed point (see, e.g., [22]). On the other hand, Meir and Keeler [23] defined the following mapping, called the Meir-Keeler contraction. A mapping $f : X \to X$ is called a Meir-Keeler contraction if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \epsilon + \delta$ implies that

$$d(f(x), f(y)) < \epsilon, \quad \forall x, y \in X.$$
(28)

We know that Meir-Keeler contraction is a generalization of contraction, and the following result, which extends the Banach contraction principle, is proved in [23].

Lemma 7 (see [23]). *A Meir-Keeler contraction defined on a complete metric space has a unique fixed point.*

Lemma 8 (see [24]). Let f be a Meir-Keeler contraction on a convex subset C of a Banach space E. Then, for every $\epsilon > 0$, there exists $r \in (0, 1)$ such that $|| x - y || \ge \epsilon$ implies that

$$\left\|f\left(x\right) - f\left(y\right)\right\| \le r \left\|x - y\right\|, \quad \forall x, y \in C.$$
(29)

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. We define a subset s-Li_n C_n of H as follows: $x \in s$ -Li_n C_n if and only if there exists $\{x_n\} \subset H$ such that $x_n \to x$ and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, a subset w-Ls_n C_n of H is defined by $y \in w$ -Ls_n C_n if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H$ such that $y_i \to y$ and $y_i \in C_n$ for all $i \in \mathbb{N}$. If $C_0 \subset H$ satisfies

$$C_0 = s - Li_n C_n = w - Ls_n C_n, \tag{30}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [25], and we write $C_0 = M-\lim_n C_n$. One of the simplest examples of Mosco convergence is a decreasing sequence $\{C_n\}$ with respect to inclusion. The Mosco limit of such a sequence is $\bigcap_{n=1}^{\infty} C_n$. For more details, see [26].

Lemma 9 (see [27]). Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. If $C_0 = M-\lim_n C_n$ exists and is nonempty, then, for each $x \in H$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$.

For the rest of this paper, let $F_m : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and $A_m : C \rightarrow H$ be an α_m -inverse strongly monotone mapping, for some m = 1, 2, ..., M.

For each $r_m > 0$ and $x \in H$, define a mapping $F_{r_m}^{A_m} : H \to C$ as follows:

$$F_{r_m}^{A_m}(x) = \left\{ z \in C : F_m(z, y) + \langle A_m x, y - z \rangle + \frac{1}{r_m} \langle y - z, z - x \rangle \ge 0, \ \forall y \in C \right\}.$$
(31)

It follows from Lemma 1 that $F_{r_m}^{A_m} = F_{r_m}(I - r_m A_m)$ for each m = 1, 2, ..., M.

Let $T_i : C \to C$ be a uniformly L_i -Lipschitz continuous and asymptotically λ_i -strict pseudocontractive mapping in the intermediate sense with the sequences $\{k_{n,i}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$ and $\{e_{n,i}\} \subset [0, \infty)$ such that $\lim_{n\to\infty} e_{n,i} = 0$, for some i = 1, 2, ..., N, that is,

$$\|T_{i}^{n}x - T_{i}^{n}y\|^{2} \leq k_{n,i}\|x - y\|^{2} + \lambda_{i}\|(x - T_{i}^{n}x) - (y - T_{i}^{n}y)\|^{2} + e_{n,i},$$

$$\forall x, y \in C.$$
(32)

Remark that $L = \max\{L_i : 1 \le i \le N\}$, $\lambda = \max\{\lambda_i : 1 \le i \le N\}$, $k_n = \max\{k_{n,i} : 1 \le i \le N, n \in \mathbb{N}\}$ and $e_n = \max\{e_{n,i} : 1 \le i \le N, n \in \mathbb{N}\}$.

3. Main Results

Theorem 10. Let *C* be a nonempty closed convex subset of Hilbert space *H*. Let $F_m : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)–(A4), and let $A_m : C \rightarrow H$ be an α_m inverse strongly monotone mapping, for each m = 1, 2, ..., M. Let $T_i : C \rightarrow C$ be a uniformly L_i -Lipschitz continuous and asymptotically λ_i -strict pseudocontractive mapping in the intermediate sense with the sequences $\{k_{n,i}\}$ and $\{e_{n,i}\}$ for each i = 1, 2, ..., N. If f is a Meir-Keeler contraction of C into itself and $\Omega = (\bigcap_{i=1}^N \operatorname{Fix}(T_i)) \cap (\bigcap_{m=1}^M EP(F_m, A_m))$ is nonempty and bounded. Assume that $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0, 1] such that $0 < a \le \alpha_n \le 1, 0 < b \le \beta_n \le 1 - \lambda$ and $\{r_{m,n}\} \in (0, \infty)$ such that $r_{m,n} \in [c, d] \subset (0, 2\alpha_m)$, for each m = 1, 2, ..., Mand $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ generated by (19) converges strongly to $q = P_\Omega f(q)$.

Proof. We split the proof into six steps.

Step 1. We prove that $P_{\Omega}f$ exists a unique fixed point. To do this, we first show that $(I - r_{m,n}A_m)$ is nonexpansive for each m = 1, 2, ..., M. Indeed,

$$\begin{aligned} \left\| \left(I - r_{m,n} A_m \right) x - \left(I - r_{m,n} A_m \right) y \right\|^2 \\ &= \left\| \left(x - y \right) - r_{m,n} \left(A_m x - A_m y \right) \right\|^2 \\ &= \left\| x - y \right\|^2 - 2r_{m,n} \left\langle A_m x - A_m y, x - y \right\rangle \\ &+ r_{m,n}^2 \left\| A_m x - A_m y \right\|^2 \\ &\leq \left\| x - y \right\|^2 - r_{m,n} \left(2\alpha_m - r_{m,n} \right) \left\| A_m x - A_m y \right\|^2 \\ &\leq \left\| x - y \right\|^2. \end{aligned}$$
(33)

It follows that $(I - r_{m,n}A_m)$ is nonexpansive. By Lemma I, we know that $\bigcap_{m=1}^{M} \operatorname{EP}(F_m, A_m)$ is closed and convex. We also know from Lemma 4 that $\bigcap_{i=1}^{N} \operatorname{Fix}(T_i)$ is closed, and convex. Hence, $\Omega = (\bigcap_{i=1}^{N} \operatorname{Fix}(T_i)) \cap (\bigcap_{m=1}^{M} \operatorname{EP}(F_m, A_m))$ is a nonempty, closed and convex subset of *C*. Consequently, P_{Ω} is well-defined. Since P_{Ω} is nonexpansive, the composed mapping $P_{\Omega}f$ of *C* into itself is a Meir-Keeler contraction on *C*; see [24, Proposition 3]. By Lemma 7, there exists a unique fixed point $z \in \Omega$ of $P_{\Omega}f$.

Step 2. We show that C_n is closed convex subset of C for each $n \ge 1$. By the assumption of C_{n+1} , it is easy to see that C_n is closed for each $n \ge 1$. We only show that C_n is convex for each $n \ge 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \ge 1$. For any $w \in C_k$, we see that

$$\|y_k - w\|^2 \le \|x_k - w\|^2 + \theta_k, \quad k \ge 1,$$
 (34)

is equivalent to

$$2\langle x_{k} - y_{k}, w \rangle \leq ||x_{k}||^{2} - ||y_{k}||^{2} + \theta_{k}.$$
 (35)

Taking w_1 and w_2 in C_{k+1} and putting $\overline{w} = tw_1 + (1-t)w_2$, it follows that $w_1, w_2 \in C_k$, and so

$$2\langle x_{k} - y_{k}, w_{1} \rangle \leq ||x_{k}||^{2} - ||y_{k}||^{2} + \theta_{k},$$
(36)

$$2\langle x_{k} - y_{k}, w_{2} \rangle \leq ||x_{k}||^{2} - ||y_{k}||^{2} + \theta_{k}.$$
 (37)

Combing (36) and (37), we obtain that

$$2\langle x_{k} - y_{k}, \overline{w} \rangle \leq ||x_{k}||^{2} - ||y_{k}||^{2} + \theta_{k}.$$
 (38)

That is,

$$\left\|y_{k} - \overline{w}\right\|^{2} \le \left\|x_{k} - \overline{w}\right\|^{2} + \theta_{k}.$$
(39)

In view of the convexity of C_k , we see that $\overline{w} \in C_k$. This implies that $\overline{w} \in C_{k+1}$. Therefore, C_{k+1} is convex. Consequently, C_n is closed and convex for each $n \ge 1$.

Step 3. We show that $\Omega \subset C_n$ for each $n \ge 1$. Put $\Theta_n^m = F_{r_{m,n}}^{A_m}$ $F_{r_{m-1,n}}^{A_{m-1}} \cdots F_{r_{2,n}}^{A_2} F_{r_{1,n}}^{A_1}$ for every $m = 1, 2, \ldots, M$ and $\Theta_n^0 = I$ for all $n \in \mathbb{N}$. Note that $F_{r_{m,n}}^{A_m} = F_{r_{m,n}}(I - r_{m,n}A_m)$ is nonexpansive. Therefore, $u_n = \Theta_n^M x_n$. It is obvious that $\Omega \subset C_1 = C$. Suppose that $\Omega \subset C_k$ for some $k \ge 1$. Taking $p \in \Omega$, it follows from Lemma 1 that

$$\begin{aligned} \|u_{n} - p\| &= \left\| \Theta_{n}^{M} x_{n} - \Theta_{n}^{M} p \right\| \\ &= \left\| F_{r_{M,n}} \left(I - r_{M,n} A_{M} \right) \Theta_{n}^{M-1} x_{n} \right. \\ &\left. - F_{r_{M,n}} \left(I - r_{M,n} A_{M} \right) \Theta_{n}^{M-1} x_{n} \right. \\ &\left. - \left(I - r_{M,n} A_{M} \right) \Theta_{n}^{M-1} x_{n} \right. \\ &\left. - \left(I - r_{M,n} A_{M} \right) \Theta_{n}^{M-1} p \right\| \\ &\leq \left\| \Theta_{n}^{M-1} x_{n} - \Theta_{n}^{M-1} p \right\| \\ &\vdots \\ &\leq \left\| \Theta_{n}^{0} x_{n} - \Theta_{n}^{0} p \right\| \\ &= \left\| x_{n} - p \right\| . \end{aligned}$$
(40)

From (19), we observe that

$$\begin{aligned} \left\| z_{n} - p \right\|^{2} &= \left\| (1 - \beta_{n}) u_{n} + \beta_{n} T_{i(n)}^{h(n)} u_{n} - p \right\|^{2} \\ &= (1 - \beta_{n}) \left\| u_{n} - p \right\|^{2} + \beta_{n} \left\| T_{i(n)}^{h(n)} u_{n} - p \right\|^{2} \\ &- \beta_{n} (1 - \beta_{n}) \left\| u_{n} - T_{i(n)}^{h(n)} u_{n} \right\|^{2} \\ &\leq (1 - \beta_{n}) \left\| u_{n} - p \right\|^{2} \\ &+ \beta_{n} \left[k_{h(n)} \left\| u_{n} - p \right\|^{2} + \lambda \left\| u_{n} - T_{i(n)}^{h(n)} u_{n} \right\|^{2} + e_{h(n)} \right] \\ &- \beta_{n} (1 - \beta_{n}) \left\| u_{n} - T_{i(n)}^{h(n)} u_{n} \right\|^{2} \\ &\leq k_{h(n)} \left\| u_{n} - p \right\|^{2} - \beta_{n} (1 - \beta_{n} - \lambda) \\ &\times \left\| u_{n} - T_{i(n)}^{h(n)} u_{n} \right\|^{2} + \beta_{n} e_{h(n)} \\ &\leq k_{h(n)} \left\| u_{n} - p \right\|^{2} + \beta_{n} e_{h(n)}. \end{aligned}$$
(41)

By virtue of convexity of $\|\cdot\|^2$, combining (40) and (41), we have

$$\|y_{n} - p\|^{2} = \|(1 - \alpha_{n})u_{n} + \alpha_{n}z_{n} - p\|^{2}$$

$$\leq (1 - \alpha_{n})\|u_{n} - p\|^{2} + \alpha_{n}\|z_{n} - p\|^{2}$$

$$\leq (1 - \alpha_{n})\|u_{n} - p\|^{2}$$

$$+ \alpha_{n}[k_{h(n)}\|u_{n} - p\|^{2} + \beta_{n}e_{h(n)}]$$

$$\leq \|u_{n} - p\|^{2} + (k_{h(n)} - 1)\|u_{n} - p\|^{2} + e_{h(n)}$$

$$= \|u_{n} - p\|^{2} + \theta_{n}$$

$$\leq \|x_{n} - p\|^{2} + \theta_{n},$$
(43)

where $\theta_n = (k_{h(n)} - 1)\rho_n^2 + e_{h(n)} \rightarrow 0$ as $n \rightarrow \infty$ and $\rho_n = \sup\{|| x_n - p ||: p \in \Omega\} < \infty$. Setting n = k in (40)–(43), it follows that $p \in C_{k+1}$. Therefore, $\Omega \in C_n$ for each $n \ge 1$.

Step 4. Next, we prove that $\lim_{n\to\infty} x_n = q$, where $q = P_{\bigcap_{n=1}^{\infty} C_n} f(q)$. Note that C_n is a closed convex subset of H and $\emptyset \neq \Omega \subset C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$. Thus, $\{x_n\}$ is well-defined. Since the composed mapping $P_{\bigcap_{n=1}^{\infty} C_n} f$ is a Meir-Keeler contraction on C, there exists a unique fixed point $q = P_{\bigcap_{n=1}^{\infty} C_n} f(q) \in \bigcap_{n=1}^{\infty} C_n$ by Lemma 7. Let $w_n = P_{C_n} f(q)$ for each $n \in \mathbb{N}$. We get $\bigcap_{n=1}^{\infty} C_n = M$ -lim_n C_n , since $\Omega \subset C_{n+1} \subset C_n$ for every $n \in \mathbb{N}$. Thus, from Lemma 9, we get

$$w_n \longrightarrow P_{\bigcap_{n=1}^{\infty} C_n} f(q) = q. \tag{44}$$

We prove that $x_n \to q$ in the following. If it were so, it would hold that $\limsup_{n\to\infty} ||x_n - q|| > 0$. Let $\epsilon > 0$, such that $\limsup_{n\to\infty} ||x_n - q|| > \epsilon$. By the definition of Meir-Keeler contraction, there exists $\delta > 0$ with $\limsup_{n\to\infty} ||x_n - q|| > \epsilon + \delta$ such that $||x - y|| < \epsilon + \delta$ implies that

$$\left\| f\left(x\right) - f\left(y\right) \right\| < \epsilon, \quad \forall x, y \in C.$$
(45)

From Lemma 8, there exists $r \in (0, 1)$ such that $||x-y|| \ge \epsilon + \delta$ implies that

$$||f(x) - f(y)|| < r ||x - y||, \quad \forall x, y \in C.$$
 (46)

By (44), there exists $n_0 \in \mathbb{N}$ such that $|| w_n - q || < \delta$ for each $n \ge n_0$. As in the proof of [24, Theorem 8], we consider the following two cases:

- (i) There exists $n_1 \ge n_0$ such that $|| x_{n_1} q || < \epsilon + \delta$.
- (ii) $||x_n q|| \ge \epsilon + \delta$ for every $n \ge n_0$.

In case (i), it holds that

$$\|x_{n_1+1} - w_{n_1+1}\| \le \|f(x_{n_1}) - f(q)\| < \epsilon.$$
 (47)

Thus we get

$$\|x_{n_{1}+1} - q\| \le \|x_{n_{1}+1} - w_{n_{1}+1}\| + \|w_{n_{1}+1} - q\| < \epsilon + \delta,$$
(48)

which means that

$$\limsup_{n \to \infty} \|x_n - q\| \le \epsilon + \delta < \limsup_{n \to \infty} \|x_n - q\|.$$
⁽⁴⁹⁾

This is a contradiction. In case (ii), we have

$$||f(x_n) - f(q)|| < r ||x_n - q||, \quad n \ge n_0.$$
 (50)

Thus we get

$$\begin{aligned} \left\| x_{n_{1}+1} - w_{n_{1}+1} \right\| &\leq \left\| f\left(x_{n} \right) - f\left(q \right) \right\| \\ &< r \left\| x_{n} - q \right\| \leq r \left(\left\| x_{n} - w_{n} \right\| + \left\| w_{n} - q \right\| \right), \end{aligned}$$
(51)

It follows from (44) that

$$\begin{split} \limsup_{n \to \infty} \|x_n - w_n\| &= \limsup_{n \to \infty} \|x_{n_1+1} - w_{n_1+1}\| \\ &\leq r \limsup_{n \to \infty} \|x_n - w_n\| \\ &< \limsup_{n \to \infty} \|x_n - w_n\| \,. \end{split}$$
(52)

This is a contradiction again. Therefore, we obtain that $\lim_{n\to\infty} x_n = q$. Moreover, since $x_{n+1} = P_{C_{n+1}} f(x_n)$, we have $\langle f(x_n) - x_{n+1}, x_{n+1} - y \rangle \ge 0$ for each $y \in C_{n+1}$. By $\Omega \subset C_{n+1}$, we have

$$\langle f(q) - q, q - y \rangle \ge 0, \quad \forall y \in \Omega,$$
 (53)

which is equivalent to $q = P_{\Omega}f(q)$ from the property of metric projection.

Step 5. Now, we prove that $\lim_{n \to \infty} || x_n - u_n || = 0$. From $x_n \to q$ as $n \to \infty$, we have

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.$$
 (54)

Since $x_{n+1} = P_{C_{n+1}} f(x_n) \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n.$$
 (55)

It follows from (55) and Lemma 2 that

$$\|y_{n} - x_{n}\|^{2} = \|y_{n} - x_{n+1} + x_{n+1} - x_{n}\|^{2}$$

$$= \|y_{n} - x_{n+1}\|^{2} + 2 \langle y_{n} - x_{n+1}, x_{n+1} - x_{n} \rangle$$

$$+ \|x_{n+1} - x_{n}\|^{2}$$

$$\leq 2 \|x_{n} - x_{n+1}\|^{2} + 2 \|y_{n} - x_{n+1}\|$$

$$\times \|x_{n+1} - x_{n}\| + \theta_{n}.$$
(56)

Since $\theta_n = (k_{h(n)} - 1)\rho_n^2 + e_{h(n)} \to 0$ as $n \to \infty$ and (54), we obtain

$$\lim_{n \to \infty} \left\| x_n - y_n \right\| = 0.$$
⁽⁵⁷⁾

For each $p \in \Omega$, m = 1, 2, ..., M, it follows from (33) and (40) that

$$\begin{split} \left\| \Theta_{n}^{m} x_{n} - p \right\|^{2} \\ &= \left\| F_{r_{m,n}} \left(I - r_{m,n} A_{m} \right) \Theta_{n}^{m-1} x_{n} - F_{r_{m,n}} \left(I - r_{m,n} A_{m} \right) p \right\|^{2} \\ &\leq \left\| \left(I - r_{m,n} A_{m} \right) \Theta_{n}^{m-1} x_{n} - \left(I - r_{m,n} A_{m} \right) p \right\|^{2} \\ &\leq \left\| \Theta_{n}^{m-1} x_{n} - p \right\|^{2} - r_{m,n} \left(2\alpha_{m} - r_{m,n} \right) \\ &\times \left\| A_{m} \Theta_{n}^{m-1} x_{n} - A_{m} p \right\|^{2} \\ &\leq \left\| x_{n} - p \right\|^{2} - r_{m,n} \left(2\alpha_{m} - r_{m,n} \right) \\ &\times \left\| A_{m} \Theta_{n}^{m-1} x_{n} - A_{m} p \right\|^{2} . \end{split}$$
(58)

By (40), (42), and (58), we have

$$\|y_{n} - p\|^{2} \leq \|u_{n} - p\|^{2} + \theta_{n}$$

$$= \|\Theta_{n}^{M}x_{n} - \Theta_{n}^{M}p\| + \theta_{n}$$

$$\leq \|\Theta_{n}^{m}x_{n} - \Theta_{n}^{m}p\| + \theta_{n}$$

$$\leq \|x_{n} - p\|^{2} - r_{m,n}(2\alpha_{m} - r_{m,n})$$

$$\times \|A_{m}\Theta_{n}^{m-1}x_{n} - A_{m}p\|^{2} + \theta_{n},$$
(59)

which implies that

$$r_{m,n} (2\alpha_m - r_{m,n}) \|A_m \Theta_n^{m-1} x_n - A_m p\|^2$$

$$\leq \|x_n - p\|^2 - \|y_n - p\|^2 + \theta_n$$
(60)

$$\leq \|x_n - y_n\| (\|x_n - p\| + \|y_n - p\|) + \theta_n.$$

Since $r_{m,n} \in [c,d] \subset (0, 2\alpha_m)$ for each m = 1, 2, ..., M and $\theta_n \to 0$ as $n \to \infty$. From (57), we have

$$\lim_{n \to \infty} \left\| A_m \Theta_n^{m-1} x_n - A_m p \right\| = 0, \quad m = 1, 2, \dots, M.$$
(61)

On the other hand, it follows from the nonexpansive $I - r_{m,n}A_m$ and Lemma 1 that

$$\begin{split} \left\| \Theta_{n}^{m} x_{n} - p \right\|^{2} \\ &= \left\| F_{r_{m,n}} \left(I - r_{m,n} A_{m} \right) \Theta_{n}^{m-1} x_{n} - F_{r_{m,n}} \left(I - r_{m,n} A_{m} \right) \Theta_{n}^{m-1} p \right\|^{2} \\ &\leq \left\langle \left(I - r_{m,n} A_{m} \right) \Theta_{n}^{m-1} x_{n} - \left(I - r_{m,n} A_{m} \right) p, \Theta_{n}^{m} x_{n} - p \right\rangle \\ &= \frac{1}{2} \left[\left\| \left(I - r_{m,n} A_{m} \right) \Theta_{n}^{m-1} x_{n} - \left(I - r_{m,n} A_{m} \right) p \right\|^{2} \\ &+ \left\| \Theta_{n}^{m} x_{n} - p \right\|^{2} \\ &- \left\| \left(I - r_{m,n} A_{m} \right) \Theta_{n}^{m-1} x_{n} - \left(I - r_{m,n} A_{m} \right) p - \left(\Theta_{n}^{m} x_{n} - p \right) \right\|^{2} \right] \\ &\leq \frac{1}{2} \left[\left\| \Theta_{n}^{m-1} x_{n} - p \right\|^{2} + \left\| \Theta_{n}^{m} x_{n} - p \right\|^{2} \\ &- \left\| \Theta_{n}^{m-1} x_{n} - \Theta_{n}^{m} x_{n} - r_{m,n} \left(A_{m} \Theta_{n}^{m-1} x_{n} - A_{m} p \right) \right\|^{2} \right], \end{split}$$

$$\tag{62}$$

which implies that

$$\begin{split} \left\| \Theta_{n}^{m} x_{n} - p \right\|^{2} \\ \leq \left\| \Theta_{n}^{m-1} x_{n} - p \right\|^{2} \\ - \left\| \Theta_{n}^{m-1} x_{n} - \Theta_{n}^{m} x_{n} - r_{m,n} \left(A_{m} \Theta_{n}^{m-1} x_{n} - A_{m} p \right) \right\|^{2} \\ = \left\| \Theta_{n}^{m-1} x_{n} - p \right\|^{2} - \left\| \Theta_{n}^{m-1} x_{n} - \Theta_{n}^{m} x_{n} \right\|^{2} \\ - r_{m,n}^{2} \left\| A_{m} \Theta_{n}^{m-1} x_{n} - A_{m} p \right\|^{2} \\ + 2r_{m,n} \left\langle \Theta_{n}^{m-1} x_{n} - \Theta_{n}^{m} x_{n}, A_{m} \Theta_{n}^{m-1} x_{n} - A_{m} p \right\rangle \\ \leq \left\| x_{n} - p \right\|^{2} - \left\| \Theta_{n}^{m-1} x_{n} - \Theta_{n}^{m} x_{n} \right\|^{2} \\ + 2r_{m,n} \left\| \Theta_{n}^{m-1} x_{n} - \Theta_{n}^{m} x_{n} \right\|^{2} \\ + 2r_{m,n} \left\| \Theta_{n}^{m-1} x_{n} - \Theta_{n}^{m} x_{n} \right\| \\ \cdot \left\| A_{m} \Theta_{n}^{m-1} x_{n} - A_{m} p \right\|. \end{split}$$

$$(63)$$

From (40), (42), and (63), we obtain

$$\|y_{n} - p\|^{2} \leq \|u_{n} - p\|^{2} + \theta_{n}$$

$$= \|\Theta_{n}^{M}x_{n} - p\| + \theta_{n}$$

$$\leq \|\Theta_{n}^{m}x_{n} - p\| + \theta_{n}$$

$$\leq \|x_{n} - p\|^{2} - \|\Theta_{n}^{m-1}x_{n} - \Theta_{n}^{m}x_{n}\|^{2}$$

$$+ 2r_{m,n} \|\Theta_{n}^{m-1}x_{n} - \Theta_{n}^{m}x_{n}\|$$

$$\cdot \|A_{m}\Theta_{n}^{m-1}x_{n} - A_{m}p\| + \theta_{n},$$
(64)

which implies that

$$\begin{split} \left\| \Theta_{n}^{m-1} x_{n} - \Theta_{n}^{m} x_{n} \right\|^{2} \\ &\leq \left\| x_{n} - p \right\|^{2} - \left\| y_{n} - p \right\|^{2} + 2r_{m,n} \left\| \Theta_{n}^{m-1} x_{n} - \Theta_{n}^{m} x_{n} \right\| \\ &\times \left\| A_{m} \Theta_{n}^{m-1} x_{n} - A_{m} p \right\| + \theta_{n} \\ &\leq \left\| x_{n} - y_{n} \right\| \left(\left\| x_{n} - p \right\| + \left\| y_{n} - p \right\| \right) \\ &+ 2r_{m,n} \left\| \Theta_{n}^{m-1} x_{n} - \Theta_{n}^{m} x_{n} \right\| \left\| A_{m} \Theta_{n}^{m-1} x_{n} - A_{m} p \right\| + \theta_{n}. \end{split}$$
(65)

It follows from (57) and (65) that

$$\lim_{n \to \infty} \left\| \Theta_n^{m-1} x_n - \Theta_n^m x_n \right\| = 0, \quad m = 1, 2, \dots, M.$$
 (66)

Note that

$$\|x_{n} - u_{n}\| \leq \|\Theta_{n}^{0}x_{n} - \Theta_{n}^{1}x_{n}\| + \|\Theta_{n}^{1}x_{n} - \Theta_{n}^{2}x_{n}\| + \dots + \|\Theta_{n}^{M-1}x_{n} - \Theta_{n}^{M}x_{n}\|.$$
(67)

Therefore,

$$\lim_{n \to \infty} \|x_n - u_n\| = 0.$$
 (68)

Step 6. Finally, we prove that $q \in (\bigcap_{i=1}^{N} \operatorname{Fix}(T_i)) \cap (\bigcap_{m=1}^{M} \operatorname{EP}(F_m, A_m))$. To do this, we first show that $q \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)$. Note that

$$\|u_{n+1} - u_n\| \le \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - u_n\|.$$
(69)

From (54) and (68), we get

$$\lim_{n \to \infty} \|u_{n+1} - u_n\| = 0.$$
 (70)

It follows that

$$\lim_{n \to \infty} \|u_{n+i} - u_n\| = 0, \quad i = 1, 2, \dots, N.$$
(71)

For any positive integer $n \ge N$, note that n = [h(n)-1]N+i(n), where i = i(n) = 1, 2, ..., N. By (19) and the conditions $0 < a \le \alpha_n \le 1$ and $0 < b \le \beta_n \le 1 - \lambda$, we have

$$\|u_{n} - T_{i(n)}^{h(n)}u_{n}\| = \frac{1}{\beta_{n}} \|z_{n} - u_{n}\|$$

$$= \frac{1}{\alpha_{n}\beta_{n}} \|y_{n} - u_{n}\|$$

$$\leq \frac{1}{ab} (\|y_{n} - x_{n}\| + \|x_{n} - u_{n}\|).$$
(72)

From (57) and (68), we get

$$\lim_{n \to \infty} \left\| u_n - T_{i(n)}^{h(n)} u_n \right\| = 0.$$
(73)

By the fact that h(n) = h(n - N) + 1 and i(n) = i(n - N), we observe that

$$\begin{aligned} \left\| u_{n} - T_{i(n)}^{h(n)-1} u_{n} \right\| &\leq \left\| u_{n} - u_{n-N} \right\| \\ &+ \left\| u_{n-N} - T_{i(n-N)}^{h(n-N)} u_{n-N} \right\| \\ &+ \left\| T_{i(n-N)}^{h(n-N)} u_{n-N} - T_{i(n)}^{h(n)-1} u_{n} \right\|. \end{aligned}$$
(74)

Applying (71), (73), and Lemma 6, we obtain

$$\lim_{n \to \infty} \left\| u_n - T_{i(n)}^{h(n)-1} u_n \right\| = 0.$$
 (75)

By the uniformly *L*-Lipschitzian of T_i , we have

$$\begin{aligned} \|u_n - T_n u_n\| &\leq \|u_n - T_{i(n)}^{h(n)} u_n\| + \|T_{i(n)}^{h(n)} u_n - T_{i(n)} u_n\| \\ &\leq \|u_n - T_{i(n)}^{h(n)} u_n\| + L \|u_n - T_{i(n)}^{h(n)-1} u_n\|. \end{aligned}$$
(76)

It follows from (73) and (75) that

$$\lim_{n \to \infty} \|u_n - T_n u_n\| = 0.$$
 (77)

Since

$$\begin{aligned} \|u_{n} - T_{n+i}u_{n}\| &\leq \|u_{n} - u_{n+i}\| + \|u_{n+i} - T_{n+i}u_{n+i}\| \\ &+ \|T_{n+i}u_{n+i} - T_{n+i}u_{n}\| \\ &\leq (1+L) \|u_{n} - u_{n+i}\| + \|u_{n+i} - T_{n+i}u_{n+i}\|. \end{aligned}$$
(78)

Combining (71) and (77), we obtain

$$\lim_{n \to \infty} \|u_n - T_i u_n\| = 0, \quad i = 1, 2, \dots, N.$$
(79)

Moreover, for each i = 1, 2, ..., N, we have

$$\|x_n - T_i x_n\| \le \|x_n - u_n\| + \|u_n - T_i u_n\| + \|T_i u_n - T_i x_n\|.$$
(80)

This implies that

$$\lim_{n \to \infty} \|x_n - T_i x_n\| = 0, \quad i = 1, 2, \dots, N.$$
(81)

Note that $x_n \to q$ as $n \to \infty$. It follows from (81) and Lemma 5 that $q \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)$. Next, we show that $q \in \bigcap_{m=1}^{M} \operatorname{EP}(F_m, A_m)$. From Lemma 1

Next, we show that $q \in \bigcap_{m=1}^{M} EP(F_m, A_m)$. From Lemma 1 and since $\Theta_n^m x_n = F_{r_{m,n}}(I - r_{m,n}A_m)\Theta_n^{m-1}x_n, m = 1, 2, ..., M$, we have

$$F_{m}\left(\Theta_{n}^{m}x_{n}, y\right) + \left\langle A_{m}\Theta_{n}^{m-1}x_{n}, y - \Theta_{n}^{m}x_{n} \right\rangle$$
$$+ \frac{1}{r_{m,n}}\left\langle y - \Theta_{n}^{m}x_{n}, \Theta_{n}^{m}x_{n} - \Theta_{n}^{m-1}x_{n} \right\rangle \ge 0, \quad (82)$$
$$\forall y \in C.$$

By (A2), we have

$$\left\langle A_{m}\Theta_{n}^{m-1}x_{n}, y-\Theta_{n}^{m}x_{n}\right\rangle$$

+ $\frac{1}{r_{m,n}}\left\langle y-\Theta_{n}^{m}x_{n},\Theta_{n}^{m}x_{n}-\Theta_{n}^{m-1}x_{n}\right\rangle$ (83)
 $\geq F_{m}\left(y,\Theta_{n}^{m}x_{n}\right), \quad \forall y \in C.$

Let $z_t = ty + (1 - t)q$ for all $t \in (0, 1]$ and $y \in C$. This implies that $z_t \in C$. Then, we have

$$\begin{split} \langle A_m z_t, z_t - \Theta_n^m x_n \rangle \\ &\geq \langle A_m z_t, z_t - \Theta_n^m x_n \rangle - \langle A_m \Theta_n^{m-1} x_n, z_t - \Theta_n^m x_n \rangle \\ &- \langle z_t - \Theta_n^m x_n, \frac{\Theta_n^m x_n - \Theta_n^{m-1} x_n}{r_{m,n}} \rangle + F_m \left(z_t, \Theta_n^m x_n \right) \\ &\geq \langle A_m z_t - A_m \Theta_n^m x_n, z_t - \Theta_n^m x_n \rangle \\ &+ \langle A_m \Theta_n^m x_n - A_m \Theta_n^{m-1} x_n, z_t - \Theta_n^m x_n \rangle \\ &- \langle z_t - \Theta_n^m x_n, \frac{\Theta_n^m x_n - \Theta_n^{m-1} x_n}{r_{m,n}} \rangle + F_m \left(z_t, \Theta_n^m x_n \right). \end{split}$$

$$(84)$$

From (66), we have $||A_m \Theta_n^m x_n - A_m \Theta_n^{m-1} x_n|| \to 0$ as $n \to \infty$. Moreover, by (A4) and the monotonicity of A_m , we obtain

$$\langle A_m z_t, z_t - q \rangle \ge F_m(z_t, q), \quad m = 1, 2, \dots, M.$$
 (85)

Using (A1), (A4), and (85), we obtain

$$0 = F_m(z_t, z_t) \le tF_m(z_t, y) + (1 - t) F_m(z_t, q)$$

$$\le tF_m(z_t, y) + (1 - t) \langle A_m z_t, z_t - q \rangle$$
(86)
$$\le tF_m(z_t, y) + (1 - t) t \langle A_m z_t, y - q \rangle.$$

and hence

$$F_m(z_t, y) + (1-t) \left\langle A_m z_t, y - q \right\rangle \ge 0.$$
(87)

Let $t \rightarrow 0$, from (A3) and (87), we have

$$F_m(q, y) + \langle A_m q, y - q \rangle \ge 0, \quad \forall y \in C, \ m = 1, 2, \dots, M.$$
(88)

This implies that $q \in EP(F_m, A_m)$, m = 1, 2, ..., M. Therefore, $q \in \bigcap_{m=1}^{M} EP(F_m, A_m)$. Consequently, we obtain that $q \in \Omega$. This completes the proof.

We also obtain the following results by using the viscosityhybrid projection methods, which extend and improve the hybrid method (CQ) proposed by Sahu et al. [11] and Hu and Cai [12].

Theorem 11. Let *C* be a nonempty closed convex subset of Hilbert space *H*. Let $F_m : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4), and let $A_m : C \rightarrow H$ be an α_m inverse strongly monotone mapping, for each m = 1, 2, ..., M. Let $T_i : C \rightarrow C$ be a uniformly L_i -Lipschitz continuous and asymptotically λ_i -strict pseudocontractive mapping in the intermediate sense with the sequences $\{k_{n,i}\}$ and $\{e_{n,i}\}$ for each i = 1, 2, ..., N. If f is a Meir-Keeler contraction of C into itself and $\Omega = (\bigcap_{i=1}^N \operatorname{Fix}(T_i)) \cap (\bigcap_{m=1}^M \operatorname{EP}(F_m, A_m))$ is nonempty and bounded. Let $\{x_n\}$ be a sequence defined by

$$x_{1} \in C, \quad Q_{1} = C,$$

$$u_{n} = F_{r_{M,n}}^{A_{M}} F_{r_{M-1,n}}^{A_{M-1}} \cdots F_{r_{2,n}}^{A_{2}} F_{r_{1,n}}^{A_{1}} x_{n},$$

$$z_{n} = (1 - \beta_{n}) u_{n} + \beta_{n} T_{i(n)}^{h(n)} u_{n},$$

$$y_{n} = (1 - \alpha_{n}) u_{n} + \alpha_{n} z_{n},$$

$$C_{n} = \left\{ w \in C : \|y_{n} - w\|^{2} \le \|x_{n} - w\|^{2} + \theta_{n} \right\},$$

$$Q_{n} = \left\{ u \in Q_{n-1} : \left\langle f(x_{n-1}) - x_{n}, x_{n} - u \right\rangle \ge 0 \right\},$$

$$x_{n+1} = P_{C_{n} \cap Q_{n}} f(x_{n}),$$
(89)

where $\theta_n = (k_{h(n)} - 1)\rho_n^2 + e_{h(n)} \rightarrow 0$ as $n \rightarrow \infty$ and $\rho_n = \sup\{\|x_n - p\|: p \in \Omega\} < \infty$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1] such that $0 < a \le \alpha_n \le 1, 0 < b \le \beta_n \le 1 - \lambda$ and $\{r_{m,n}\} \subset (0, \infty)$ such that $r_{m,n} \in [c, d] \subset (0, 2\alpha_m)$, for each $m = 1, 2, \ldots, M$. Then the sequence $\{x_n\}$ generated by (89) converges strongly to $q = P_\Omega f(q)$.

Proof. We have that C_n and Q_n are closed convex subsets of H and $\Omega \subset C_n$ for every $n \in \mathbb{N}$. We only prove that $\Omega \subset Q_n$ for every $n \in \mathbb{N}$ and that a sequence $\{x_n\}$ is well-defined. We have $x_1 \in C$ and $\Omega \subset Q_1 = C$. Assume that $x_k \in C$ and $\Omega \subset Q_k$ for some $k \in \mathbb{N}$. Since $\Omega \subset C_k \cap Q_k$, there exists a unique element $x_{k+1} = P_{C_k \cap Q_k} f(x_k)$, and hence

$$\left\langle f\left(x_{k}\right) - x_{k+1}, x_{k+1} - y\right\rangle \ge 0, \quad \forall y \in C_{k} \cap Q_{k}.$$

$$(90)$$

This implies that

$$\langle f(x_k) - x_{k+1}, x_{k+1} - y \rangle \ge 0, \quad \forall y \in \Omega.$$
 (91)

That is, $\Omega \subset Q_{k+1}$. Therefore, we prove that $\Omega \subset Q_n$.

On the other hand, $P_{\bigcap_{n=1}^{\infty}Q_n}f$ is a Meir-Keeler contraction on *C*, there exists a unique element $q = P_{\bigcap_{n=1}^{\infty}Q_n}f(q) \in \bigcap_{n=1}^{\infty}Q_n$ by Lemma 7. Let $z_n = P_{Q_n}f(q)$ for each $n \in \mathbb{N}$. Since $\Omega \subset Q_{n+1} \subset Q_n$, it follows from Lemma 9 that $z_n \rightarrow q = P_{\bigcap_{n=1}^{\infty}Q_n}f(q)$. We also have $x_n = P_{Q_n}f(x_{n-1})$ by the definition of Q_n . Therefore, as in the proof of Theorem 10, we get $x_n \rightarrow q$, and the desired conclusion follows immediately from Theorem 10. This completes the proof.

If M = N = 1, we obtain the following corollary for a general equilibrium problem and asymptotically strict pseudocontraction in the intermediate sense as a special cases.

Theorem 12. Let *C* be a nonempty closed convex subset of Hilbert space *H*. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and $A : C \to H$ be an α -inverse strongly monotone mapping. Let $T : C \to C$ be a uniformly *L*-Lipschitz continuous and asymptotically λ -strict pseudocontractive mapping in the intermediate sense with the sequences $\{k_n\}$ and $\{e_n\}$. If *f* is a Meir-Keeler contraction of *C* into itself and $\Omega = \text{Fix}(T) \cap$ EP(*F*, *A*) is nonempty and bounded. Let $\{x_n\}$ be a sequence defined by

$$x_{1} \in C, \quad C_{1} = C,$$

$$F(u_{n}, y) + \langle Ax_{n}, y - u_{n} \rangle$$

$$+ \frac{1}{r_{n}} \langle y - u_{n}, u_{n} - x_{n} \rangle \ge 0, \quad \forall y \in C,$$

$$z_{n} = (1 - \beta_{n}) u_{n} + \beta_{n} T^{n} u_{n},$$

$$y_{n} = (1 - \alpha_{n}) u_{n} + \alpha_{n} z_{n},$$

$$C_{n+1} = \left\{ w \in C_{n} : \|y_{n} - w\|^{2} \le \|x_{n} - w\|^{2} + \theta_{n} \right\},$$

$$x_{n+1} = P_{C_{n+1}} f(x_{n}),$$
(92)

where $\theta_n = (k_n - 1)\rho_n^2 + e_n \rightarrow 0$ as $n \rightarrow \infty$ and $\rho_n = \sup\{\|x_n - p\| : p \in \Omega\} < \infty$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1] such that $0 < a \le \alpha_n \le 1, 0 < b \le \beta_n \le 1 - \lambda$ and $\{r_n\} \subset (0, \infty)$ such that $r_n \in [c, d] \subset (0, 2\alpha)$. Then the sequence $\{x_n\}$ generated by (92) converges strongly to $q = P_\Omega f(q)$.

If M = N = 1 and F = 0, the general equilibrium problem (1) reduces into the classical variational inequality problem (3), we obtain the following corollary as a special case of Theorems 10 and 12.

Theorem 13. Let C be a nonempty closed convex subset of Hilbert space H. Let $A : C \to H$ be an α -inverse strongly monotone mapping. Let $T : C \to C$ be a uniformly L-Lipschitz continuous and asymptotically λ -strict pseudocontractive mapping in the intermediate sense with the sequences $\{k_n\}$ and $\{e_n\}$. If f is a Meir-Keeler contraction of C into itself and $\Omega =$ Fix $(T) \cap VI(C, A)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence defined by

$$x_{1} \in C, \quad C_{1} = C,$$

$$y_{n} = (1 - \alpha_{n}) x_{n} + \alpha_{n} T^{n} P_{C} (x_{n} - r_{n} A x_{n}),$$

$$C_{n+1} = \left\{ w \in C_{n} : \|y_{n} - w\|^{2} \le \|x_{n} - w\|^{2} + \theta_{n} \right\},$$

$$x_{n+1} = P_{C_{n+1}} f(x_{n}),$$
(93)

where $\theta_n = (k_n - 1)\rho_n^2 + e_n \rightarrow 0$ as $n \rightarrow \infty$ and $\rho_n = \sup\{\|x_n - p\| : p \in \Omega\} < \infty$. Assume that $\{\alpha_n\}$ is a sequence in [0, 1] such that $0 < b \le \alpha_n \le 1 - \lambda$ and $\{r_n\} \subset (0, \infty)$ such that $r_n \in [c, d] \subset (0, 2\alpha)$. Then the sequence $\{x_n\}$ generated by (93) converges strongly to $q = P_\Omega f(q)$.

Proof. If F = 0, the general equilibrium problem (1) reduces into the classical variational inequality problem (3), and

$$\langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in C,$$
(94)

which is equivalent to

$$\langle y - u_n, u_n - (x_n - r_n A x_n) \rangle \ge 0, \quad \forall y \in C.$$
 (95)

Therefore, we have $u_n = P_C(x_n - r_nAx_n)$. The desired conclusion follows immediately from Theorem 10 (Set $\alpha_n = 1$, $\beta_n = \alpha_n$). This completes the proof.

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