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Research Article

On Satnoianu-Wu's Inequality

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By applying techniques in the theory of convex functions and Schur-geometrically convex functions, the author investigates a conjecture of Satnoianu on an algebraic inequality and generalizes some known results in recent years.

1. Introduction

For $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $x_i \in \mathbb{R}_+ = (0, \infty)$ with $i = 1, 2, \ldots, n$, let

$$A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad G_n(x) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \qquad H_n(x) = \frac{n}{\sum_{i=1}^{n}(1/x_i)}. \tag{1}$$

These quantities are, respectively, called the arithmetic, geometric, and harmonic means of a positive sequence $x = (x_1, x_2, \ldots, x_n)$. For more information on the theory of means, see the monograph [1] or the papers [2–5] and plenty of references therein.

For convenience, in what follows, we use the notation $x^q = (x_1^q, x_2^q, \ldots, x_n^q)$ for

$$0 < m \le x_1^q, x_2^q, \ldots, x_n^q \le M, \tag{2}$$

and $q \in \mathbb{R} = (-\infty, \infty)$.

In [6], Satnoianu posed the following conjecture.

Conjecture 1. *For $n \ge 2$, $x_i > 0$, and $\lambda \ge n^{n-1} - 1$, it is valid that*

$$\sum_{i=1}^{n}\left(\frac{x_i^{n-1}}{x_i^{n-1} + \lambda\prod_{k=1,k\ne i}^{n} x_k}\right)^{1/(n-1)} \ge \frac{n}{(1+\lambda)^{1/(n-1)}}. \tag{3}$$

This conjecture has been solved and researched in [7–10]. Among them, Wu obtained in [10] the following result.

Theorem 2 (see [10, Theorem 1]). *Let n be a positive integer and let α, β, and x_i for $i = 1, 2, \ldots, n$ be positive numbers. If $p \ne 0$ and*

$$\beta \ge \left(n^{\max\{p,1\}} - 1\right)\alpha, \tag{4}$$

then

$$\sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta G(x^q)}\right]^{1/p} \ge \frac{n}{(\alpha+\beta)^{1/p}}. \tag{5}$$

It is clear that inequality (5) generalizes (3). Therefore, we would like to call inequality (5) Satnoianu-Wu's inequality.

Now we naturally pose the following questions:

(1) Can one improve the condition (4)?

(2) Does the reversed inequality of (5) exist?

(3) Are there other types of inequalities of Satnoianu-Wu type?

The goal of this paper is to answer these questions.

2. Definitions and Lemmas

We need the following definitions and lemmas.

Definition 3 (see [11, page 8]). Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. We say that x is majorized by y (in symbols $x \prec y$) if

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, \qquad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i, \tag{6}$$

for $k = 1, 2, \ldots, n-1$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of x and y in a descending order.

Definition 4 (see [12]). Let $\Omega \subseteq \mathbb{R}_+^n$.

(1) The set Ω is said to be geometrically convex if $(x_1^\lambda y_1^{1-\lambda}, \ldots, x_n^\lambda y_n^{1-\lambda}) \in \Omega$ for every $x, y \in \Omega$ and $\lambda \in [0, 1]$.

(2) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur-geometrically convex on Ω if $\ln x = (\ln x_1, \ldots, \ln x_n) \prec \ln y = (\ln y_1, \ldots, \ln y_n)$ implies $\varphi(x) \leq \varphi(y)$ for every $x, y \in \Omega$.

(3) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur-geometrically concave on Ω if $\ln x = (\ln x_1, \ldots, \ln x_n) \prec \ln y = (\ln y_1, \ldots, \ln y_n)$ implies $\varphi(x) \geq \varphi(y)$ for every $x, y \in \Omega$.

Lemma 5 (see [12]). *Let $\Omega \subseteq \mathbb{R}_+^n$ be a symmetric and geometrically convex set with inner points and $\varphi : \Omega \to \mathbb{R}_+$ a symmetric and differentiable function in Ω°. Then φ is a Schur-geometrically convex (or Schur-geometrically concave, resp.) function on Ω if and only if*

$$(\ln x_1 - \ln x_2)\left[x_1 \frac{\partial \varphi(x)}{\partial x_1} - x_2 \frac{\partial \varphi(x)}{\partial x_2}\right] \geq 0 \quad (\text{or} \leq 0, \text{resp.}), \quad x \in \Omega^\circ. \tag{7}$$

Lemma 6 (see [1, page 4, Bernoulli's inequality]). *The inequality*

$$(1+t)^r \geq 1 + rt \tag{8}$$

holds for $r \geq 1$ and $t \geq -1$ or for $r \leq 0$ and $t > -1$. If $0 < r < 1$ and $t \geq -1$, inequality (8) is reversed.

3. Main Results

Now we start off to demonstrate our main results.

Theorem 7. *Let m, M be defined as in (2) and $n \geq 2$, let $\alpha, \beta, x_i \in \mathbb{R}_+$ for $i = 1, 2, \ldots, n$, and let $p, q \in \mathbb{R}$ with $p \neq 0$. If $(1-p)m\beta \geq 2pM\alpha$, then*

$$\sum_{i=1}^n \left[\frac{x_i^q}{\alpha x_i^q + \beta A_n(x^q)}\right]^{1/p} \geq \frac{n}{(\alpha+\beta)^{1/p}}. \tag{9}$$

If $(1-p)M\beta \leq 2pm\alpha$, inequality (9) is reversed.

Proof. For $M' > m' > 0$ and $r \in \mathbb{R}$ with $r \neq 0$, since

$$\left[\left(\frac{t}{1+t}\right)^r\right]'' = \frac{r(r-1-2t)t^{r-1}}{(1+t)^{r+2}}, \quad t \in [m', M'], \tag{10}$$

then

(1) for $r < 0$, or for $r > 1$ and $r-1-2M' > 0$, the function $(t/(1+t))^r$ is convex on $[m', M']$;

(2) for $0 < r \leq 1$, or for $r > 1$ and $r-1-2m' < 0$, the function $(t/(1+t))^r$ is concave on $[m', M']$.

By Jensen's inequality, when $r < 0$, or when $r > 1$ and $r-1-2M' > 0$, we have

$$\sum_{i=1}^n \left(\frac{u_i}{1+u_i}\right)^r \geq n\left[\frac{A_n(u)}{1+A_n(u)}\right]^r, \tag{11}$$

where $u = (u_1, u_2, \ldots, u_n) \in [m', M']^n$. When $0 < r \leq 1$, or when $r > 1$ and $r-1-2m' < 0$, inequality (11) is reversed.

Let $u_i = \alpha x_i^q / \beta A_n(x^q)$ for $i = 1, 2, \ldots, n$, $r = 1/p$, $m' = \alpha m/\beta M$, and $M' = \alpha M/\beta m$. Then it is easy to show that $A_n(u) = \alpha/\beta$. Making use of (11), we obtain inequality (9). The proof of Theorem 7 is complete. □

Corollary 8. *Under the conditions of Theorem 7 and when $M = (n-1)m$, if $(1-p)\beta > 2p(n-1)\alpha$, then*

$$\sum_{i=1}^n \left[\frac{x_i^q}{\alpha x_i^q + \beta A_n(x^q)}\right]^{1/p} \geq \frac{n}{(\alpha+\beta)^{1/p}}; \tag{12}$$

if $(1-p)(n-1)\beta < 2p\alpha$, inequality (12) is reversed.

Corollary 9. *Let $n \geq 2$, $\alpha, \beta, x_i \in \mathbb{R}_+$ for $i = 1, 2, \ldots, n$, and $p, q \in \mathbb{R}$. If $p < 0$, then*

$$\sum_{i=1}^n \left[\frac{x_i^q}{\alpha x_i^q + \beta A_n(x^q)}\right]^{1/p} \geq \frac{n}{(\alpha+\beta)^{1/p}}; \tag{13}$$

if $p \geq 1$, inequality (13) is reversed.

Proof. This follows from Theorem 7 by considering that if $p < 0$, we have $(1-p)m\beta > 2pM\alpha$ and that if $p \geq 1$, we have $(1-p)M\beta < 2pm\alpha$. □

Theorem 10. *Let $n \geq 2$, $\alpha, \beta, x_i \in \mathbb{R}_+$ for $i = 1, 2, \ldots, n$, $p, q \in \mathbb{R}$ with $p \neq 0$, and m, M defined by (2). If $m\beta \geq pM\alpha$, then*

$$\sum_{i=1}^n \left[\frac{x_i^q}{\alpha x_i^q + \beta G_n(x^q)}\right]^{1/p} \geq \frac{n}{(\alpha+\beta)^{1/p}}; \tag{14}$$

if $M\beta \leq mp\alpha$, inequality (14) is reversed.

Proof. For $M' > m' > 0$, let

$$f(u) = \sum_{i=1}^n (1+u_i)^r, \quad u \in [m', M']^n. \tag{15}$$

Then

$$\frac{\partial f(u)}{\partial u_i} = r(1+u_i)^{r-1}, \qquad \frac{\partial}{\partial u_i}\left[u_i \frac{\partial f(u)}{\partial u_i}\right] = r(1+u_i)^{r-2}(1+ru_i), \tag{16}$$

for $1 \leq i \leq n$. When $r > 0$, or when $r < 0$ and $1 + rm' < 0$, we have

$$(\ln u_1 - \ln u_2)\left[u_1 \frac{\partial f(u)}{\partial u_1} - u_2 \frac{\partial f(u)}{\partial u_2}\right] \geq 0, \quad u \in [m', M']^n; \tag{17}$$

when $r < 0$ and $1 + rM' > 0$, inequality (17) is reversed.

Using Lemma 5, we have the following conclusions:

(1) if $r > 0$, or if $r < 0$ and $1 + rm' < 0$, the function $f(u)$ is Schur-geometrically convex on $[m', M']^n$;

(2) if $r < 0$ and $1 + rM' > 0$, the function $f(u)$ is Schur-geometrically concave on $[m', M']^n$.

By the fact that

$$\begin{aligned}&\left(\ln G_n(u), \ln G_n(u), \ldots, \ln G_n(u)\right)\\ &\quad \prec \left(\ln u_1, \ln u_2, \ldots, \ln u_n\right), \quad u \in \mathbb{R}_+^n,\end{aligned} \tag{18}$$

and by Definition 4, if $r > 0$, or if $r < 0$ and $1 + rm' < 0$, we have

$$\sum_{i=1}^{n}(1+u_i)^r \geq n(1+G_n(u))^r, \quad u \in \left[m', M'\right]^n; \tag{19}$$

if $r < 0$ and $1 + rM' > 0$, inequality (19) is reversed.

Letting $u_i = \beta G_n(x^q)/\alpha x_i^q$ for $i = 1, 2, \ldots, n$ leads to $G_n(u) = \beta/\alpha$. Putting $r = -1/p$, $m' = \beta m/\alpha M$ and $M' = \beta M/\alpha m$ in inequality (19) results in inequality (14). The proof of Theorem 10 is complete. □

Corollary 11. *Under the conditions of Theorem 10 and when $M = (n-1)m$, if $\beta \geq p(n-1)\alpha$, then*

$$\sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta A_n(x^q)}\right]^{1/p} \geq \frac{n}{(\alpha+\beta)^{1/p}}; \tag{20}$$

if $(n-1)\beta \leq p\alpha$, inequality (20) is reversed.

Remark 12. When $n \geq 2$ and $p > 1$, from Lemma 6 and (4), it follows that

$$\beta \geq \left(n^{\max\{p,1\}} - 1\right)\alpha > p(n-1)\alpha. \tag{21}$$

Theorem 13. *Let $n \geq 2$, $\alpha, \beta, x_i \in \mathbb{R}_+$ for $i = 1, 2, \ldots, n$, and $p, q \in \mathbb{R}$ with $p \neq 0$. If $p \geq -1$, then*

$$\sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta H_n(x^q)}\right]^{1/p} \geq \frac{n}{(\alpha+\beta)^{1/p}}; \tag{22}$$

if $p \leq -1$, inequality (22) is reversed.

Proof. Since $(1+t)^r$ is a convex (or concave, resp.) function on $\mathbb{R}_+$ for $r \geq 1$ or $r < 0$ (or for $0 < r \leq 1$, resp.), by Jensen's inequality, if $r \geq 1$ or $r < 0$, we have

$$\sum_{i=1}^{n}(1+u_i)^r \geq n(1+A_n(u))^r, \quad u \in \mathbb{R}_+^n; \tag{23}$$

if $0 < r \leq 1$, inequality (23) is reversed.

Letting $u_i = \beta H_n(x^q)/\alpha x_i^q$ and $r = -1/p$ shows $A_n(u) = \beta/\alpha$. Further from (23), we obtain inequality (22). The proof of Theorem 13 is complete. □

Remark 14. It is clear that inequalities (9) and (22) both generalize inequality (3).

Corollary 15. *Let $n \geq 2$, $\alpha, \beta, x_i \in \mathbb{R}_+$ for $i = 1, 2, \ldots, n$, $p, q \in \mathbb{R}$ with $p \neq 0$, and m, M defined as in (2).*

(1) *When $-1 \leq p < 0$, one has*

$$\begin{aligned}\sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta A_n(x^q)}\right]^{1/p} &\geq \sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta G_n(x^q)}\right]^{1/p}\\ &\geq \sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta H_n(x^q)}\right]^{1/p}\\ &\geq \frac{n}{(\alpha+\beta)^{1/p}}.\end{aligned} \tag{24}$$

(2) *When $0 < p < 1$ and $(1-p)m\beta > 2pM\alpha$, one has*

$$\begin{aligned}\sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta H_n(x^q)}\right]^{1/p} &\geq \sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta G_n(x^q)}\right]^{1/p}\\ &\geq \sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta A_n(x^q)}\right]^{1/p}\\ &\geq \frac{n}{(\alpha+\beta)^{1/p}}.\end{aligned} \tag{25}$$

(3) *When $p > 0$ and $m\beta > pM\alpha$, one has*

$$\begin{aligned}\sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta H_n(x^q)}\right]^{1/p} &\geq \sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta G_n(x^q)}\right]^{1/p}\\ &\geq \frac{n}{(\alpha+\beta)^{1/p}}.\end{aligned} \tag{26}$$

(4) *When $p > 0$ and $M\beta < mp\alpha$, one has*

$$\begin{aligned}\sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta A_n(x^q)}\right]^{1/p} &\leq \sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta G_n(x^q)}\right]^{1/p}\\ &\leq \frac{n}{(\alpha+\beta)^{1/p}}.\end{aligned} \tag{27}$$

Proof. This follows from utilizing the well-known harmonic-geometric-arithmetic mean inequality

$$H_n(x) \leq G_n(x) \leq A_n(x) \tag{28}$$

and Theorems 7 to 13. □

Corollary 16. *Under the conditions of Corollary 15 and when* $M = (n-1)m$,

(1) *if* $0 < p < 1$ *and* $(1-p)\beta > 2p(n-1)\alpha$, *one has*

$$\sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta H_n(x^q)}\right]^{1/p} \geq \sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta G_n(x^q)}\right]^{1/p} \geq \sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta A_n(x^q)}\right]^{1/p} \geq \frac{n}{(\alpha+\beta)^{1/p}}; \tag{29}$$

(2) *if* $p > 0$ *and* $\beta > p(n-1)\alpha$, *one has*

$$\sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta H_n(x^q)}\right]^{1/p} \geq \sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta G_n(x^q)}\right]^{1/p} \geq \frac{n}{(\alpha+\beta)^{1/p}}; \tag{30}$$

(3) *if* $p > 0$ *and* $(n-1)\beta < p\alpha$, *one has*

$$\sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta A_n(x^q)}\right]^{1/p} \leq \sum_{i=1}^{n}\left[\frac{x_i^q}{\alpha x_i^q + \beta G_n(x^q)}\right]^{1/p} \leq \frac{n}{(\alpha+\beta)^{1/p}}. \tag{31}$$

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