Hindawi Publishing Corporation
The Scientific World Journal
Volume 2013, Article ID 798468, 7 pages
http://dx.doi.org/10.1155/2013/798468

Research Article

Stacked Central Configurations for the Spatial Nine-Body Problem

Su Xia and Deng Chunhua

Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai'an 223003, China

Correspondence should be addressed to Su Xia; dchsx2005@163.com

Received 25 April 2013; Accepted 3 June 2013

Academic Editors: Y.-S. Piao and M. Shibata

We show the existence of the twisted stacked central configurations for the 9-body problem. More precisely, the position vectors x_1, x_2, x_3, x_4, and x_5 are at the vertices of a square pyramid Σ; the position vectors x_6, x_7, x_8, and x_9 are at the vertices of a square Π.

1. Introduction and Main Results

The classical n-body problem [1, 2] concerns the motion of n mass points moving in space according to Newton's law:

$$m_i \ddot{x}_i = -\sum_{j=1, j\neq i}^{n} \frac{m_i m_j \left(x_i - x_j\right)}{r_{ij}^3}, \quad i = 1, 2, \ldots, n. \tag{1}$$

Here, $x_i \in \mathbb{R}^d$ is the position of mass $m_i > 0$, the gravitational constant is taken equal to 1, and $r_{ij} = |x_i - x_j|$ is the Euclidean distance between x_i and x_j.

The space of configuration is defined by

$$X = \left\{ \left(x_1, \ldots, x_n\right) \in \left(\mathbb{R}^d\right)^n : x_i \neq x_j \ \forall i \neq j \right\}, \tag{2}$$

while the center of mass is given by

$$c = \frac{m_1 x_1 + \cdots + m_n x_n}{M}, \tag{3}$$

where $M = m_1 + \cdots + m_n$ is the total mass.

A configuration $x = (x_1, \ldots, x_n) \in X$ is called a *central configuration* [2, 3] if there exists a constant λ, called the multiplier, such that

$$-\lambda \left(x_i - c\right) = \sum_{j=1, j\neq i}^{n} \frac{m_j \left(x_j - x_i\right)}{r_{ij}^3}, \quad i = 1, 2, \ldots, n. \tag{4}$$

It is easy to see that a central configuration remains a central configuration after a rotation in $\mathbb{R}^d$ and a scalar multiplication. More precisely, let $A \in SO(d)$ and $a > 0$, if $x = (x_1, \ldots, x_n)$ is a central configuration, so are $Ax = (Ax_1, \ldots, Ax_n)$ and $ax = (ax_1, \ldots, ax_n)$.

Two central configurations are said to be equivalent if one can be transformed to the other by a scalar multiplication and a rotation. In this paper, when we say a central configuration, we mean a class of central configurations as defined by the above equivalent relation.

Central configurations of the n-body problem are important because they allow the computation of homographic solutions; if the n bodies are heading for a simultaneous collision, then the bodies tend to a central configuration (see [3, 4]); there is a relation between central configurations and the bifurcations of the hypersurfaces of constant energy and angular momentum (see [5]).

In this paper, we are interested in spatial central configurations, that is, $d = 3$. In 2005, Hampton [6] provides a new family of planar central configurations for the 5-body problem with an interesting property: the central configuration has a subset of three bodies forming a central configuration of the 3-body problem. The authors [7] find new classes of central configurations of the 5-body problem which are the ones studied by Hampton [6] having three bodies in the vertices of an equilateral triangle, but the other two, instead of being located symmetrically with respect to a perpendicular bisector, are on the perpendicular bisector. The stacked central configurations studied by Hampton [6] were completed by Llibre et al. [8] (see also [9]).

Zhang and Zhou [10] showed the existence of double pyramidal central configurations of $N+2$-body problem. The authors [11–13] provided new examples of stacked central configurations for the spatial 7-body problem where four bodies are at the vertices of a regular tetrahedron and the other three bodies are located at the vertices of an equilateral triangle.

In this paper, we find new classes of stacked spatial central configurations for the 9-body problem which have five bodies at the vertices of a square pyramid, and the other four bodies are located at the vertices of a square. More precisely, the spatial central configurations considered here satisfy the following (see Figure 1): the position vectors x_1, x_2, x_3, x_4, and x_5 are at the vertices of a square pyramid Σ; the position vectors x_6, x_7, x_8, and x_9 are at the vertices of a square Π.

Without loss of generality, we can assume that

$$\begin{aligned} x_1 &= (1,0,0), & x_2 &= (0,1,0), & x_3 &= (-1,0,0), \\ x_4 &= (0,-1,0), & x_5 &= (0,0,h), & x_6 &= (x,0,y), \\ x_7 &= (0,x,y), & x_8 &= (-x,0,y), & x_9 &= (0,-x,y), \end{aligned} \tag{5}$$

where $x>0$, $y\in\mathbb{R}$, and $y\neq 0$; the positive constant h satisfies the equation

$$\frac{2}{r_{15}^3} = \frac{1}{r_{12}^3} + \frac{1}{r_{13}^3}, \tag{6}$$

(see [10] and the references therein); that is, $h = 1.26276522$.

The main results of this paper are the following.

Theorem 1. *Consider the spatial configurations according to Figure 1, in order that the nine mass points are in a central configuration, the following statements are necessary:*

(1) *the masses m_1, m_2, m_3, and m_4 must be equal;*

(2) *the masses m_6, m_7, m_8, and m_9 must be equal.*

Theorem 2. *There exist points $(x_0, y_0) \in T^{-1}(0) \cap D$ (see Figure 2) such that the nine bodies take the coordinates*

$$\begin{aligned} x_1 &= (1,0,0), & x_2 &= (0,1,0), \\ x_3 &= (-1,0,0), & x_4 &= (0,-1,0), \\ x_5 &= (0,0,h), & x_6 &= (x_0,0,y_0), \\ x_7 &= (0,x_0,y_0), & x_8 &= (-x_0,0,y_0), \\ & & x_9 &= (0,-x_0,y_0). \end{aligned} \tag{7}$$

Then, there are positive solutions of m_1, m_5, m_6 such that these bodies form a spatial central configuration according to Figure 1.

The proofs of the theorems are given in the next sections.

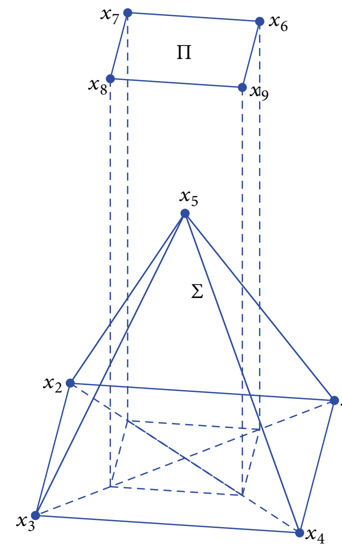

Figure 1: The configuration for the 9-body problem.

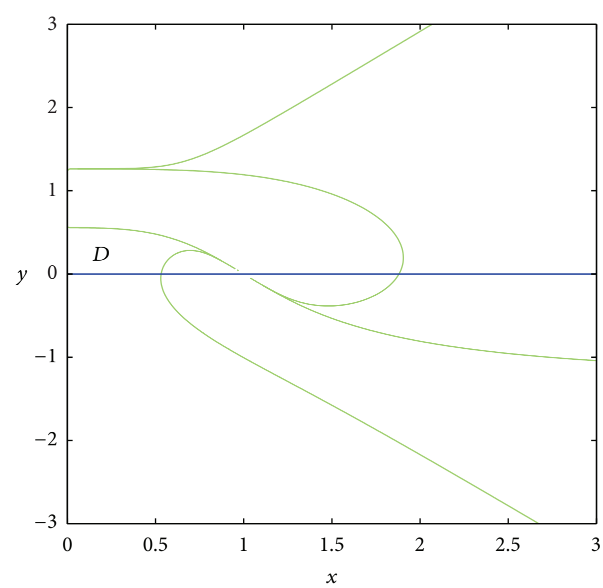

Figure 2: The region D.

2. Proof of Theorem 1

For the spatial central configurations, instead of working with (4), we consider the Dziobek-Laura-Andoyer equations (see [9, 11–13] and the references therein):

$$f_{ijk} = \sum_{l=1, l\neq i,j,k}^{n} m_l \left(d_{il} - d_{jl}\right) \Delta_{ijkl} = 0 \tag{8}$$

for $1 \le i < j \le n$, $k = 1, \ldots, n$, $k \neq i, j$. Here, $d_{ij} = 1/r_{ij}^3$ and $\Delta_{ijkl} = (x_i - x_j) \wedge (x_i - x_k) \cdot (x_i - x_l)$. Thus, Δ_{ijkl} gives six times the signed volume of the tetrahedron formed by the bodies with positions x_i, x_j, x_k, and x_l; (8) is a system of $n(n-1)(n-2)/2$ equations.

For the 9-body problem, (8) is a system of 252 equations. According to Figure 1, our class of configurations with nine bodies must satisfy

$$\begin{aligned} r_{12} &= r_{23} = r_{34} = r_{14} = \sqrt{2}, \qquad r_{13} = r_{24} = 2, \\ r_{67} &= r_{78} = r_{89} = r_{69} = \sqrt{2}x, \qquad r_{68} = r_{79} = 2x, \\ r_{16} &= r_{27} = r_{38} = r_{49} = \sqrt{(x-1)^2 + y^2}, \\ r_{17} &= r_{19} = r_{26} = r_{28} = r_{37} = r_{39} = r_{46} \\ &= r_{48} = \sqrt{x^2 + 1 + y^2}, \\ r_{18} &= r_{29} = r_{36} = r_{47} = \sqrt{(x+1)^2 + y^2}, \\ r_{15} &= r_{25} = r_{35} = r_{45} = \sqrt{1 + h^2}, \\ r_{56} &= r_{57} = r_{58} = r_{59} = \sqrt{x^2 + (y-h)^2}. \end{aligned} \tag{9}$$

Due to assumption (5) and the definition of Δ_{ijkl}, we have several symmetries in the signed volumes.

By using the symmetries and the properties of Δ_{ijkl}, we obtain the following results.

Lemma 3. *In order to have a spatial central configuration according to Figure 1, a necessary condition is that the masses m_1, m_2, m_3, and m_4 must be equal.*

Proof. It is sufficient to consider the equations $f_{687} = 0$ and $f_{796} = 0$:

$$\begin{aligned} f_{687} &= (m_1 - m_3)(d_{16} - d_{18})\Delta_{6871} = 0, \\ f_{796} &= (m_2 - m_4)(d_{16} - d_{18})\Delta_{7962} = 0. \end{aligned} \tag{10}$$

For our class of central configurations, we have $d_{16} - d_{18} \neq 0$, $\Delta_{6871} \neq 0$, and $\Delta_{7962} \neq 0$. So the above equations hold if and only if $m_1 = m_3$, $m_2 = m_4$. Consider the expression of $f_{678} = 0$:

$$\begin{aligned} f_{678} = {} & (m_1 - m_2)(d_{16} - d_{17})\Delta_{6781} \\ & + (m_3 - m_4)(d_{18} - d_{17})\Delta_{6783} = 0. \end{aligned} \tag{11}$$

Substituting $m_1 = m_3$, $m_2 = m_4$ into the above equation, we have

$$f_{678} = (m_1 - m_2)(d_{16} + d_{18} - 2d_{17})\Delta_{6781} = 0. \tag{12}$$

For our class of central configurations, we have $d_{16} + d_{18} - 2d_{17} \neq 0$, since the function $g(x) = x^{-3/2}$ is convex for all $x > 0$, and $\Delta_{6781} \neq 0$. So the above equation holds if and only if $m_1 = m_2$. So statement 1 of Theorem 1 is proved. □

Lemma 4. *If the configuration, according to Figure 1, is a central configuration, a necessary condition is that the masses m_6, m_7, m_8, and m_9 must be equal.*

Proof. It is sufficient to consider the equations $f_{132} = 0$ and $f_{241} = 0$:

$$\begin{aligned} f_{132} &= (m_6 - m_8)(d_{16} - d_{18})\Delta_{1326} = 0, \\ f_{241} &= (m_7 - m_9)(d_{16} - d_{18})\Delta_{2417} = 0. \end{aligned} \tag{13}$$

For our class of central configurations, we have $d_{16} - d_{18} \neq 0$, $\Delta_{1326} \neq 0$, and $\Delta_{2417} \neq 0$. So the above equations hold if and only if $m_6 = m_8$, $m_7 = m_9$. Consider the expression of $f_{123} = 0$:

$$\begin{aligned} f_{123} = {} & (m_6 - m_7)(d_{16} - d_{17})\Delta_{1236} \\ & + (m_8 - m_9)(d_{18} - d_{17})\Delta_{1238} = 0. \end{aligned} \tag{14}$$

Substituting $m_6 = m_8$, $m_7 = m_9$ into the above equation, we have

$$f_{123} = (m_6 - m_7)(d_{16} + d_{18} - 2d_{17})\Delta_{1236} = 0. \tag{15}$$

For our class of central configurations, we have $d_{16} + d_{18} - 2d_{17} \neq 0$, and $\Delta_{1236} \neq 0$. So the above equation holds if and only if $m_6 = m_7$. Hence, statement 2 of Theorem 1 is proved.

The proof Theorem 1 is completed. □

We restrict the set of admissible masses to $m_1 = m_2 = m_3 = m_4 = \alpha$ and $m_6 = m_7 = m_8 = m_9 = \beta$. Substituting $m_1 = m_2 = m_3 = m_4 = \alpha$ and $m_6 = m_7 = m_8 = m_9 = \beta$ into (8), they reduce to the following 4 equations:

$$\begin{aligned} f_{152} = {} & \beta((d_{16} + d_{17} - 2d_{56})\Delta_{1526} \\ & + (d_{17} + d_{18} - 2d_{56})\Delta_{1528}) = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} f_{162} = {} & \alpha(d_{12} + d_{13} - d_{17} - d_{18})\Delta_{1623} \\ & + m_5(d_{15} - d_{56})\Delta_{1625} \\ & + \beta(d_{17} + d_{18} - d_{67} - d_{68})\Delta_{1628} = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} f_{175} = {} & \alpha((d_{12} - d_{16})\Delta_{1752} + (d_{13} - d_{17})\Delta_{1753} \\ & + (d_{12} - d_{18})\Delta_{1754}) \\ & + \beta((d_{16} - d_{67})\Delta_{1756} + (d_{18} - d_{67})\Delta_{1758} \\ & + (d_{17} - d_{68})\Delta_{1759}) = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} f_{562} = {} & \alpha((d_{15} - d_{16})\Delta_{5621} + (d_{15} - d_{18})\Delta_{5623} \\ & + (d_{15} - d_{17})\Delta_{5624}) \\ & + \beta((d_{56} - d_{67})\Delta_{5627} + (d_{56} - d_{68})\Delta_{5628} \\ & + (d_{56} - d_{67})\Delta_{5629}) = 0. \end{aligned} \tag{19}$$

If we write $f_{152} = \beta T = \beta((d_{16} + d_{17} - 2d_{56})\Delta_{1526} + (d_{17} + d_{18} - 2d_{56})\Delta_{1528}) = 0$, it follows that $T = 0$ in order to have central configurations. So in the following, we restrict our central configurations to the set $T^{-1}(0)$.

Lemma 5. *According to one's assumptions and the set $T^{-1}(0)$, (8) is satisfied if (17) and (18) are satisfied.*

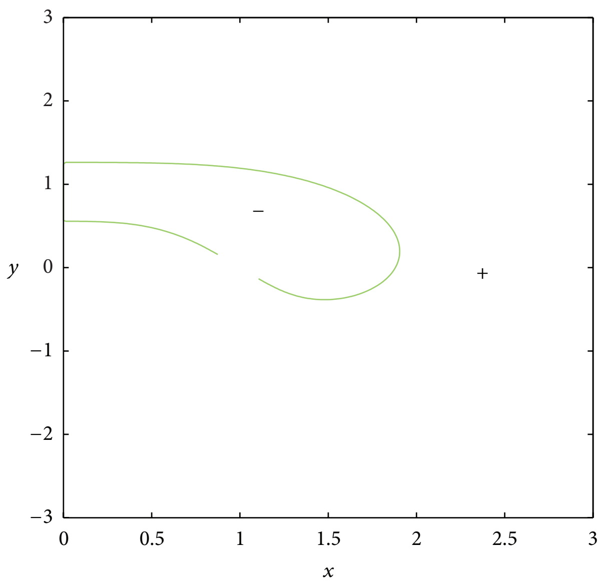

FIGURE 3: The curve $a_{21} = 0$.

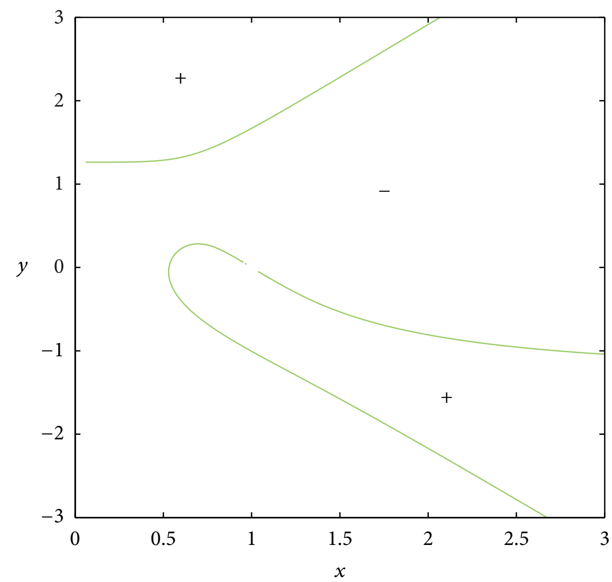

FIGURE 4: The curve $a_{23} = 0$.

Proof. Under the assumptions (5), we have

$$T = (d_{16} + 2d_{17} + d_{18} - 4d_{56})(y - h) + hx(d_{16} - d_{18}) = 0; \quad (20)$$

that is,

$$4(y - h) d_{56} = (y - h)(d_{16} + 2d_{17} + d_{18}) + hx(d_{16} - d_{18}). \quad (21)$$

Substituting (21) into (19), we obtain the equation $f_{175} = 0$.

Hence in the set $T^{-1}(0)$, $f_{175} = 0$ implies $f_{562} = 0$. This completes the proof. □

From Lemma 5, in order to study central configurations according to Figure 1 in the set $T^{-1}(0)$, it is sufficient to study the following 2 equations:

$$f_{162} = 0, \qquad f_{175} = 0. \quad (22)$$

Denote by $A = (a_{ij})$ the matrix of the coefficients of the homogeneous linear system in the variables α, m_4, β defined by (22). Thus,

$$\begin{aligned} a_{11} &= (d_{12} + d_{13} - d_{17} - d_{18})\Delta_{1623} \\ &= -2y\left(\frac{1}{2\sqrt{2}} + \frac{1}{8} - \frac{1}{(x^2+1+y^2)^{3/2}} - \frac{1}{((x+1)^2+y^2)^{3/2}}\right), \end{aligned}$$

$$\begin{aligned} a_{12} &= (d_{15} - d_{56})\Delta_{1625} \\ &= (-y - hx + h) \times \left(\frac{1}{(1+h^2)^{3/2}} - \frac{1}{(x^2+(y-h)^2)^{3/2}}\right), \end{aligned}$$

$$\begin{aligned} a_{13} &= (d_{17} + d_{18} - d_{67} - d_{68})\Delta_{1628} \\ &= -2xy\left(\frac{1}{(x^2+1+y^2)^{3/2}} + \frac{1}{((x+1)^2+y^2)^{3/2}} - \frac{1}{8x^3} - \frac{1}{2\sqrt{2}x^3}\right), \end{aligned}$$

$$\begin{aligned} a_{21} &= (d_{12} - d_{16})\Delta_{1752} + (d_{13} - d_{17})\Delta_{1753} + (d_{12} - d_{18})\Delta_{1754} \\ &= (y - h)\left(\frac{1}{((x+1)^2+y^2)^{3/2}} - \frac{1}{((x-1)^2+y^2)^{3/2}}\right) \\ &\quad + hx\left(\frac{1}{4} + \frac{1}{\sqrt{2}} - \frac{1}{((x-1)^2+y^2)^{3/2}} - \frac{1}{((x+1)^2+y^2)^{3/2}} - \frac{2}{(x^2+1+y^2)^{3/2}}\right), \end{aligned}$$

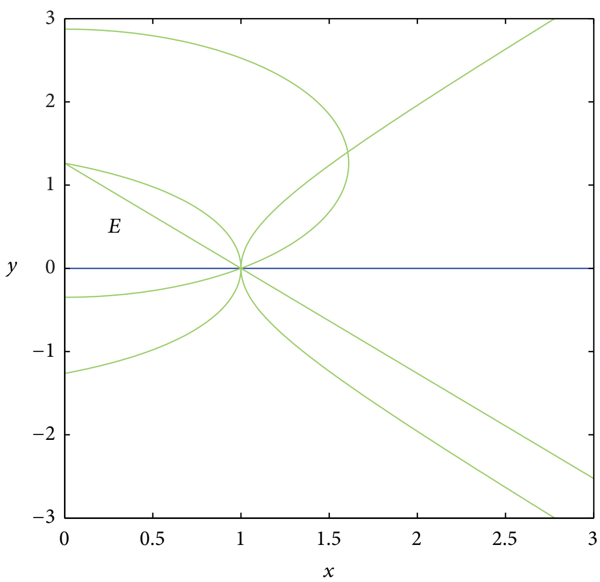

Figure 5: The region E.

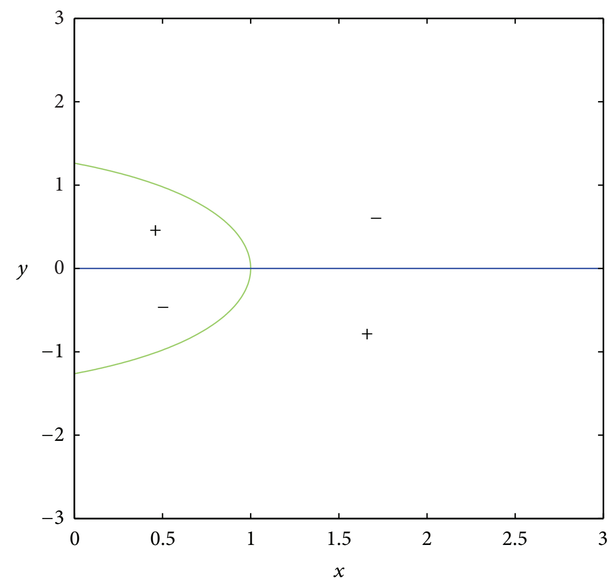

Figure 6: The curve $a_{11} = 0$.

$$
\begin{aligned}
a_{22} &= 0,\\
a_{23} &= (d_{16} - d_{67})\,\Delta_{1756} + (d_{18} - d_{67})\,\Delta_{1758}\\
&\quad + (d_{17} - d_{68})\,\Delta_{1759}\\
&= hx^2\left(\frac{1}{\left((x+1)^2 + y^2\right)^{3/2}} - \frac{1}{\left((x+1)^2 + y^2\right)^{3/2}}\right)\\
&\quad - x\,(y - h)\\
&\quad \times\left(\frac{1}{\left((x+1)^2 + y^2\right)^{3/2}} + \frac{1}{\left((x+1)^2 + y^2\right)^{3/2}}\right.\\
&\qquad \left. + \frac{2}{\left(x^2 + 1 + y^2\right)^{3/2}} - \frac{1}{4x^3} - \frac{1}{\sqrt{2}x^3}\right).
\end{aligned}
\tag{23}
$$

Let $x = \begin{pmatrix} \alpha \\ m_4 \\ \beta \end{pmatrix}$. Then in order to get the spatial central configuration as Figure 1, we need to find a positive solution α, m_4, β of the following system:

$$Ax = 0, \tag{24}$$

where $T = 0$.

3. The Existence of Spatial Central Configurations

In order to prove the existence of positive solutions of (24) in the set $T^{-1}(0)$, it is sufficient to prove that the entries in each row of A change the signs. So if the entries of some row of A have the same signs, there are no admissible masses such that the bodies are in a central configuration according to Figure 1.

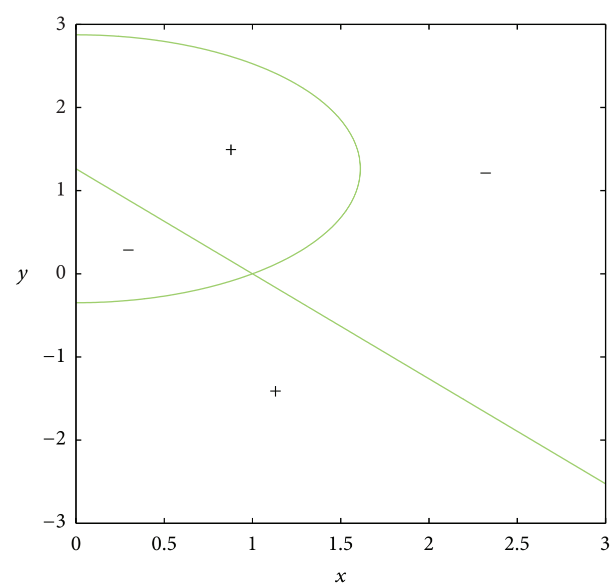

Figure 7: The curve $a_{12} = 0$.

Proof of Theorem 2. Since the rank of matrix A is two in the set $T^{-1}(0)$, there are nontrivial solutions of (24) in the set $T^{-1}(0)$.

Now we prove the existence of spatial central configurations according to Figure 1 for some points in the set D (see Figure 2). In order to prove the existence of positive solutions of (24) in the set $T^{-1}(0)$, the entries a_{21}, a_{23} of the second line in the matrix A should have opposite signs. Thus, we consider the following set D, where D is surrounded by curves $x = 0$, $y = 0$, $a_{21} = 0$, and $a_{23} = 0$.

In the set D, the entries of matrix A have the following signs: $a_{21} > 0$, $a_{23} < 0$ (see Figures 3 and 4); $a_{11} > 0$, $a_{12} < 0$, $a_{13} > 0$ because the set D is included in the set E, where E is

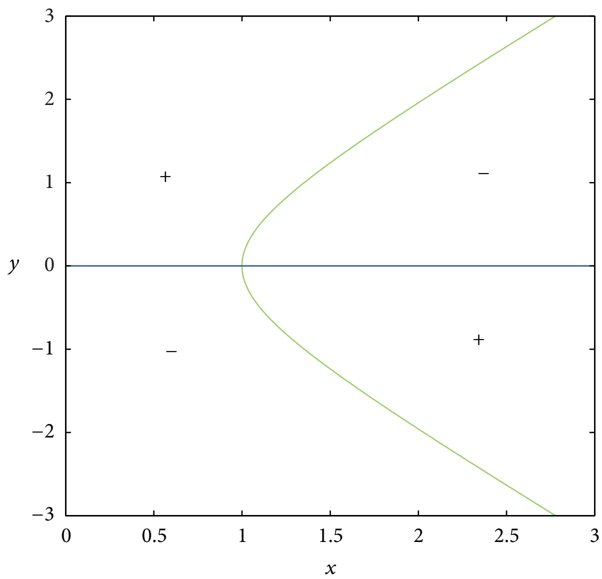

FIGURE 8: The curve $a_{13} = 0$.

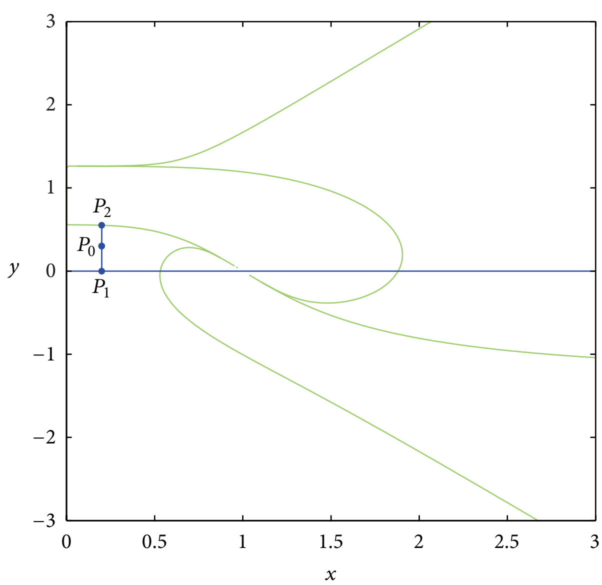

FIGURE 9: The existence of central configurations for the 9-body problem.

surrounded by curves $x = 0$, $y = 0$, and $y = h(1 - x)$ (see Figures 5, 6, 7, and 8). In short, the signs of the entries of the matrix A restricted to the set D are the following:

$$A = \begin{pmatrix} + & - & + \\ + & 0 & - \end{pmatrix}. \qquad (25)$$

In the rest of the proof, we show that the set $T^{-1}(0)$ has intersection with the set D. We consider the subset of D:

$$L = \{(x, y) : x = x_1, 0 < y < y_1\}, \qquad (26)$$

where $x_1 \in (0, 1)$. Obviously L is a segment with endpoints

$$P_1 = (x_1, 0), \qquad P_2 = (x_1, y_1), \qquad (27)$$

(see Figure 9), and the point (x_1, y_1) satisfies the equation $a_{21} = 0$. Evaluating the function T at these points, we have

$$T(P_1) < 0, \qquad T(P_2) > 0. \qquad (28)$$

Thus, there exists a point $P_0 = (x_0, y_0) \in L$, such that $T(P_0) = 0$. So at the point P_0 we have nontrivial positive solutions of (24), since the signs of the entries of the matrix A at this point are the following:

$$A(P_0) = \begin{pmatrix} + & - & + \\ + & 0 & - \end{pmatrix}. \qquad (29)$$

Thus, the proof of Theorem 2 is completed. □

Acknowledgments

The authors are supported by the Natural Science Foundation of China (NFSC11201168) and the Scientific Research Foundation of Huaiyin Institute of Technology (HGA1102).

References

[1] V. Arnold, V. Kozlov, and A. Neishtadt, *Dynamical Systems, Mathematical Aspects of Classical and Celestial Machanics*, Springer, Berlin, Germany, 1988.

[2] A. Wintner, *Analytical Foundations of Celestial Mechanics*, Princeton University Press, Princeton, NJ, USA, 1941.

[3] R. Moeckel, "On central configurations," *Mathematische Zeitschrift*, vol. 205, no. 4, pp. 499–517, 1990.

[4] D. Sarri, "On the role and the properties of n-body central configurations," *Celestial Mechanics*, vol. 21, no. 1, pp. 9–20, 1980.

[5] S. Smale, "Topology and mechanics II: the planar n-body problem," *Inventiones Mathematicae*, vol. 11, no. 1, pp. 45–64, 1970.

[6] M. Hampton, "Stacked central configurations: new examples in the planar five-body problem," *Nonlinearity*, vol. 18, no. 5, pp. 2299–2304, 2005.

[7] J. Llibre and L. F. Mello, "New central configurations for the planar 5-body problem," *Celestial Mechanics and Dynamical Astronomy*, vol. 100, no. 2, pp. 141–149, 2008.

[8] J. Llibre, L. F. Mello, and E. Perez-Chavela, "New stacked central configurations for the planar 5-body problem," *Celestial Mechanics and Dynamical Astronomy*, vol. 110, no. 1, pp. 43–52, 2011.

[9] L. F. Mello, F. E. Chaves, A. C. Fernandes, and B. A. Garcia, "Stacked central configurations for the spatial six-body problem," *Journal of Geometry and Physics*, vol. 59, no. 9, pp. 1216–1226, 2009.

[10] S. Q. Zhang and Q. Zhou, "Double pyramidal central configurations," *Physics Letters A*, vol. 281, no. 4, pp. 240–248, 2001.

[11] M. Hampton and M. Santoprete, "Seven-body central configurations: a family of central configurations in the spatial seven-body problem," *Celestial Mechanics and Dynamical Astronomy*, vol. 99, no. 4, pp. 293–305, 2007.

[12] L. F. Mello and A. C. Fernandes, "Stacked central configurations for the spatial seven-body problem," *Qualitative Theory of Dynamical Systems*, vol. 12, no. 1, pp. 101–114, 2013.

[13] X. Su and T. Q. An, "Twisted stacked central configurations for the spatial seven-body problem," *Journal of Geometry and Physics*, vol. 70, pp. 164–171, 2013.