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Research Article

On Weak-BCC-Algebras

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We describe weak-BCC-algebras (also called BZ-algebras) in which the condition $(x*y)*z=(x*z)*y$ is satisfied only in the case when elements x, y belong to the same branch. We also characterize ideals, nilradicals, and nilpotent elements of such algebras.

1. Introduction

BCK-algebras which are a generalization of the notion of algebra of sets with the set subtraction as the only fundamental nonnullary operation and on the other hand the notion of implication algebra (cf. [1]) were defined by Imai and Iséki in [2]. The class of all BCK-algebras does not form a variety. To prove this fact, Komori introduced in [3] the new class of algebras called BCC-algebras. In view of strong connections with a BIK^+-logic, BCC-algebras are also called BIK^+-algebras (cf. [4] or [5]). Nowadays, many mathematicians, especially from China, Japan, and Korea, have been studying various generalizations of BCC-algebras. All these algebras have one distinguished element and satisfy some common identities playing a crucial role in these algebras.

One of very important identities is the identity $(x*y)*z=(x*z)*y$. It holds in BCK-algebras and in some generalizations of BCK-algebras, but not in BCC-algebras. BCC-algebras satisfying this identity are BCK-algebras (cf. [6] or [7]). Therefore, it makes sense to consider such BCC-algebras and some of their generalizations for which this identity is satisfied only by elements belonging to some subsets. Such study has been initiated by Dudek in [8].

In this paper, we will study weak-BCC-algebras in which the condition $(x*y)*z=(x*z)*y$ is satisfied only in the case when elements x, y belong to the same branch. We describe some endomorphisms of such algebras, ideals, nilradicals, and nilpotent elements.

2. Basic Definitions and Facts

Definition 1. A weak-BCC-algebra is a system $(G;*,0)$ of type $(2,0)$ satisfying the following axioms:

(i) $((x*y)*(z*y))*(x*z)=0$,

(ii) $x*x=0$,

(iii) $x*0=x$,

(iv) $x*y=y*x=0\Rightarrow x=y$.

Weak-BCC-algebras are called *BZ-algebras* by many mathematicians, especially from China and Korea (cf. [9] or [10]), but we save the first name because it coincides with the general concept of names presented in the book [11] for algebras of logic.

A weak-BCC-algebra satisfying the identity

(v) $0*x=0$

is called a *BCC-algebra*. A BCC-algebra with the condition

(vi) $(x*(x*y))*y=0$

is called a *BCK-algebra*.

One can prove (see [6] or [7]) that a BCC-algebra is a BCK-algebra if and only if it satisfies the identity

(vii) $(x*y)*z=(x*z)*y$.

An algebra $(G;*,0)$ of type $(2,0)$ satisfying the axioms (i), (ii), (iii), (iv), and (vi) is called a *BCI-algebra*. A BCI-algebra

satisfies also (vii). A weak-BCC-algebra is a BCI-algebra if and only if it satisfies (vii).

Any weak-BCC-algebra can be considered as a partially ordered set. In any weak-BCC-algebra, we can define a natural partial order $\leqslant$ putting

$$x \leqslant y \Longleftrightarrow x * y = 0. \tag{1}$$

This means that a weak-BCC-algebra can be considered as a partially ordered set with some additional properties.

Proposition 2. *An algebra* $(G; *, 0)$ *of type* $(2, 0)$ *with a relation* $\leqslant$ *defined by* (1) *is a weak-BCC-algebra if and only if for all* $x, y, z \in G$ *the following conditions are satisfied:*

(i′) $(x * y) * (z * y) \leqslant x * z$,

(ii′) $x \leqslant x$,

(iii′) $x * 0 = x$,

(iv′) $x \leqslant y$ *and* $y \leqslant x$ *imply* $x = y$.

From (i′), it follows that in weak-BCC-algebras, implications

$$x \leqslant y \Longrightarrow x * z \leqslant y * z, \tag{2}$$

$$x \leqslant y \Longrightarrow z * y \leqslant z * x \tag{3}$$

are satisfied by all $x, y, z \in G$.

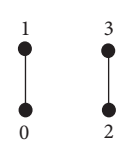
A weak-BCC-algebra which is neither BCC-algebra nor BCI-algebra is called *proper*. Proper weak-BCC-algebras have at least four elements (see [12]). But there are only two weak-BCC-algebras of order four which are not isomorphic:

$*$	0	1	2	3
0	0	0	2	2
1	1	0	2	2
2	2	2	0	0
3	3	3	1	0

(4)

$*$	0	1	2	3
0	0	0	2	2
1	1	0	3	3
2	2	2	0	0
3	3	3	1	0

(5)



(6)

They are proper, because in both cases $(3 * 2) * 1 \neq (3 * 1) * 2$.

Since two nonisomorphic weak-BCC-algebras may have the same partial order, they cannot be investigated as algebras with the operation induced by partial order. For example, weak-BCC-algebras defined by (4) and (5) have the same partial order but they are not isomorphic.

The methods of construction of weak-BCC-algebras proposed in [12] show that for every $n \geqslant 4$, there exist at least two proper weak-BCC-algebras of order n which are not isomorphic.

The set of all minimal (with respect to $\leqslant$) elements of G is denoted by $I(G)$. Elements belonging to $I(G)$ are called *initial*.

In the investigation of algebras G connected with various types of logics, an important role plays the so-called *Dudek's map* $\varphi : G \to G$ defined by $\varphi(x) = 0 * x$. The main properties of this map in the case of weak-BCC-algebras are collected in the following theorem proved in [13].

Theorem 3. *Let* G *be a weak-BCC-algebra. Then,*

(1) $\varphi^2(x) \leqslant x$,

(2) $x \leqslant y \Rightarrow \varphi(x) = \varphi(y)$,

(3) $\varphi^3(x) = \varphi(x)$,

(4) $\varphi^2(x * y) = \varphi^2(x) * \varphi^2(y)$,

(5) $\varphi^2(x * y) = \varphi(y * x)$,

(6) $\varphi(x) * (y * x) = \varphi(y)$

for all $x, y \in G$.

Theorem 4. $I(G) = \{a \in G : \varphi^2(a) = a\}$.

The proof of this theorem is given in [14]. Comparing this result with Theorem 3(4), we see that $I(G)$ is a subalgebra of G; that is, it is closed under the operation $*$. In some situations (see Theorem 21), $I(G)$ is a BCI-algebra.

Corollary 5. $I(G) = \varphi(G)$ *for any weak-BCC-algebra* G.

Proof. Indeed, if $x \in \varphi(G)$, then $x = \varphi(y)$ for some $y \in G$. Thus, by Theorem 3, $\varphi^2(x) = \varphi^3(y) = \varphi(y) = x$. Hence, $\varphi^2(x) = x$; that is, $x \in I(G)$. So, $\varphi(G) \subset I(G)$.

Conversely, for $x \in I(G)$, we have $x = \varphi^2(x) = \varphi(\varphi(x)) = \varphi(y)$, where $y = \varphi(x) \in G$. Thus, $I(G) \subset \varphi(G)$, which completes the proof. □

This means that an element $a \in G$ is an initial element of a weak-BCC-algebra G if and only if it is mentioned in the first row (i.e., in the row corresponding to 0) of the multiplication table of G.

Let G be a weak-BCC-algebra. For each $a \in I(G)$, the set

$$B(a) = \{x \in G : a \leqslant x\} \tag{7}$$

is called *a branch of* G *initiated by* a. A branch containing only one element is called *trivial*. The branch $B(0)$ is the greatest BCC-algebra contained in a weak-BCC-algebra G ([8]).

According to [1, 15], we say that a subset A of a BCK-algebra G is an ideal of G if (1) $0 \in A$, (2) $y \in A$ and $x * y \in A$ imply $x \in A$. If A is an ideal, then the relation θ defined by

$$x\theta y \Longleftrightarrow x * y,\ y * x \in A \tag{8}$$

is a congruence on a BCK-algebra G. Unfortunately, it is not true for weak-BCC-algebras (cf. [16]). In connection with this fact, Dudek and Zhang introduced in [16] the new concept of ideals. These new ideals are called *BCC-ideals*.

Definition 6. A nonempty subset A of a weak-BCC-algebra G is called a *BCC-ideal* if

(1) $0 \in A$,

(2) $y \in A$ and $(x * y) * z \in A$ imply $x * z \in A$.

By putting $z = 0$, we can see that a BCC-ideal is a BCK-ideal. In a BCK-algebra, any ideal is a BCC-ideal, but in BCC-algebras, there are BCC-ideals which are not ideals in the above sense (cf. [16]). It is not difficult to see that $B(0)$ is a BCC-ideal of each weak-BCC-algebra.

The equivalence classes of a congruence θ defined by (8), where $A = B(0)$, coincide with branches of G; that is, $B(a) = C_a$ for any $a \in I(G)$ (cf. [14]). So,

$$\begin{aligned} B(a) * B(b) &= \{x * y : x \in B(a), y \in B(b)\} \\ &= B(a * b). \end{aligned} \tag{9}$$

In the following part of this paper, we will need those two propositions proved in [14].

Proposition 7. *Elements $x, y \in G$ are in the same branch if and only if $x * y \in B(0)$.*

Proposition 8. *If $x, y \in B(a)$, then also $x * (x * y)$ and $y * (y * x)$ are in $B(a)$.*

One of the important classes of weak-BCC-algebras is the class of the so-called *group-like weak-BCC-algebras* called also *antigrouped BZ-algebras* [9], that is, weak-BCC-algebras containing only trivial branches. A special case of such algebras is group-like BCI-algebras described in [17].

From the results proved in [17] (see also [9]), it follows that such weak-BCC-algebras are strongly connected with groups.

Theorem 9. *An algebra $(G; *, 0)$ is a group-like weak-BCC-algebra if and only if $(G; \cdot, ^{-1}, 0)$, where $x \cdot y = x * (0 * y)$, is a group. Moreover, in this case, $x * y = x \cdot y^{-1}$.*

Corollary 10. *A group $(G; \cdot, ^{-1}, 0)$ is abelian if and only if the corresponding weak-BCC-algebra G is a BCI-algebra.*

Corollary 11. *$I(G)$ is a maximal group-like subalgebra of each weak-BCC-algebra G.*

3. Solid Weak-BCC-Algebras

As it is well known in the investigations of BCI-algebras, the identity (vii) plays a very important role. It is used in the proofs of almost all theorems, but as Dudek noted in his paper [8], many of these theorems can be proved without this identity. Just assume that this identity is fulfilled only by elements belonging to the same branch. In this way, we obtain a new class of weak-BCC-algebras which are called *solid*.

Definition 12. A weak-BCC-algebra G is called *solid*, if the equation

(vii) $(x * y) * z = (x * z) * y$

is satisfied by all x, y belonging to the same branch and arbitrary $z \in G$.

Any BCI-algebra and any BCK-algebra are solid weak-BCC-algebras. A solid weak-BCC-algebra containing only one branch is a BCK-algebra. To see examples of solid weak-BCC-algebras which are not BCI-algebras, one can find them in [8].

Theorem 13. *Dudek's map φ is an endomorphism of each solid weak-BCC-algebra.*

Proof. Indeed,

$$\begin{aligned} \varphi(x) * \varphi(y) &= (0 * x) * (0 * y) \\ &= (((x * y) * (x * y)) * x) * (0 * y) \\ &= (((x * y) * x) * (x * y)) * (0 * y) \\ &= (((x * x) * y) * (x * y)) * (0 * y) \\ &= ((0 * y) * (x * y)) * (0 * y) \\ &= ((0 * y) * (0 * y)) * (x * y) \\ &= 0 * (x * y) = \varphi(x * y) \end{aligned} \tag{10}$$

for all $x, y \in G$. □

Corollary 14. *$I(G)$ is a maximal group-like BCI-subalgebra of each solid weak BCC-algebra.*

Proof. Comparing Corollaries 5 and 11, we see that $I(G)$ is a maximal group-like subalgebra of each weak BCC-algebra G. Thus, by Theorem 9, there exists a group $(I(G); \cdot, ^{-1}, 0)$ such that $a * b = a \cdot b^{-1}$ for $a, b \in I(G)$. Since G is solid, φ is its endomorphism. Hence, $(0 * a) * (0 * b) = 0 * (a * b)$ for $a, b \in I(G)$; that is, $a^{-1} \cdot b = (a \cdot b^{-1})^{-1} = b \cdot a^{-1}$ in the corresponding group. The last is possible only in an abelian group, but in this case, $(a * b) * c = (a * c) * b$, which means that $I(G)$ is a BCI-algebra. □

Definition 15. For $x, y \in G$ and nonnegative integers n, we define

$$xy^0 = x, \qquad x * y^{n+1} = (x * y^n) * y. \tag{11}$$

Theorem 16. *In solid weak-BCC-algebras, the following identity*

$$(0 * x^k) * (0 * y^k) = 0 * (x * y)^k \tag{12}$$

is satisfied for each nonnegative integer k.

Proof. Let $x \in B(a)$. Then, by Theorem 3, $a \leqslant x$ implies $0 * x = 0 * a$. Suppose that $0 * x^k = 0 * a^k$ for some nonnegative integer k. Then, also $(0 * a^k) * x \leqslant (0 * a^k) * a$, by (3). Consequently,

$$\begin{aligned} 0 * x^{k+1} &= (0 * x^k) * x \\ &= (0 * a^k) * x \leqslant (0 * a^k) * a = 0 * a^{k+1}, \end{aligned} \tag{13}$$

which means that $0 * x^{k+1} = 0 * a^{k+1}$ because $0 * a^{k+1} \in I(G)$. So, $0*a^k = 0*x^k$ is valid for all $x \in B(a)$ and each nonnegative integer k.

Similarly $0 * y^k = 0 * b^k$ and $0 * (x * y)^k = 0 * (a * b)^k$ for $y \in B(b)$ and nonnegative integer k. Thus, a weak-BCC-algebra G satisfies the identity (12) if and only if

$$\left(0 * a^k\right) * \left(0 * b^k\right) = 0 * (a * b)^k \tag{14}$$

holds for $a, b \in I(G)$. But in view of Corollary 11 and Theorem 9 in the group $(I(G); \cdot, ^{-1}, 0)$, the last equation can be written in the following form:

$$a^{-k} \cdot b^k = \left(a \cdot b^{-1}\right)^{-k}. \tag{15}$$

Since a weak-BCC-algebra G is solid, by Corollary 14, $I(G)$ is a BCI-algebra. So, the group $(I(G); \cdot, ^{-1}, 0)$ is abelian. Thus, the above equation is valid for all $a, b \in I(G)$. Hence, (12) is valid for all $x, y \in G$ and all nonnegative integers k. □

Corollary 17. *The map $\varphi_k(x) = 0 * x^k$ is an endomorphism of each solid weak-BCC-algebra.*

Definition 18. A weak-BCC-algebra for which φ_k is an endomorphism is called *k-strong*. In the case $k = 1$, we say that it is strong.

A solid weak-BCC-algebra is strong for every k. The converse statement is not true.

Example 19. The weak-BCC-algebra defined by (4) is not solid because $(3 * 2) * 1 \neq (3 * 1) * 2$, but it is strong for every k. Indeed, in this weak-BCC-algebra, we have $0 * x = 0$ for $x \in B(0)$, $0 * x = 2$ for $x \in B(2)$, and $0 * x^2 = 0$ for all $x \in G$. So, it is 1-strong and 2-strong. Since in this algebra $0 * x^k = 0$ for even k, and $0 * x^k = 0 * x$ for odd k, it is strong for every k.

Example 20. Direct computations show that the group-like weak-BCC-algebra induced by the symmetric group S_3 (Theorem 9) is k-strong for $k = 5$ and $k = 6$ but not for $k = 1, 2, 3, 4, 7, 8$.

Theorem 21. *A weak-BCC-algebra G is strong if and only if $I(G)$ is a BCI-algebra, that is, if and only if $(I(G); \cdot, ^{-1}, 0)$ is an abelian group.*

Proof. Indeed, if G is strong, then $(0 * a) * (0 * b) = 0 * (a * b)$ holds for all $a, b \in I(G)$. Thus, in the group $(I(G); \cdot, ^{-1}, 0)$, we have $a^{-1} \cdot b = (a \cdot b^{-1})^{-1} = b \cdot a^{-1}$, which means that the group $(I(G); \cdot, ^{-1}, 0)$ is abelian. Hence,

$$\begin{aligned}(a * b) * c &= a \cdot b^{-1} \cdot c^{-1} \\ &= a \cdot c^{-1} \cdot b^{-1} = (a * c) * b\end{aligned} \tag{16}$$

for all $a, b, c \in I(G)$. So, $(I(G); *, 0)$ is a BCI-algebra.

On the other hand, according to Theorem 3, for any $x \in B(a)$, $y \in B(b)$, we have $0 * x = 0 * a$ and $0 * y = 0 * b$. So, if $I(G)$ is a BCI-algebra, then for any $a, b, c \in I(G)$, we have $(a * b) * c = (a * c) * b$. Consequently,

$$\begin{aligned}(0 * x) * (0 * y) &= (0 * a) * (0 * b) \\ &= (((a * b) * (a * b)) * a) * (0 * b) \\ &= (((a * b) * a) * (a * b)) * (0 * b) \\ &= (((a * a) * b) * (a * b)) * (0 * b) \\ &= ((0 * b) * (a * b)) * (0 * b) \\ &= ((0 * b) * (0 * b)) * (a * b) \\ &= 0 * (a * b) = 0 * (x * y),\end{aligned} \tag{17}$$

because $x * y \in B(a * b)$. This completes the proof. □

Corollary 22. *A strong weak-BCC-algebra is k-strong for every k.*

Proof. In a strong weak-BCC-algebra G, the group $(I(G); \cdot, ^{-1}, 0)$ is abelian and $0 * z^k = 0 * c^k$ for every $z \in B(c)$. Thus,

$$\begin{aligned}\left(0 * x^k\right) * \left(0 * y^k\right) &= \left(0 * a^k\right) * \left(0 * b^k\right) = a^{-k} \cdot b^k \\ &= \left(a \cdot b^{-1}\right)^{-k} = 0 * (a * b)^k \\ &= 0 * (x * y)^k\end{aligned} \tag{18}$$

for all $x \in B(a)$ and $y \in B(b)$. □

Example 20 shows that the converse statement is not true; that is, there are weak-BCC-algebras which are strong for some k but not for $k = 1$.

Corollary 23. *A weak-BCC-algebra in which $I(G)$ is a BCI-algebra is strong for every k.*

Corollary 24. *In any strong weak-BCC-algebra, we have*

$$0 * \left(0 * x^k\right) = 0 * (0 * x)^k \tag{19}$$

for every $x \in G$ and every natural k.

4. Ideals of Weak-BCC-Algebras

To avoid repetitions, all results formulated in this section will be proved for BCC-ideals. Proofs for ideals are almost identical to proofs for BCC-ideals.

Theorem 25. *Let G be a weak-BCC-algebra. Then, $A \subset I(G)$ is an ideal (BCC-ideal) of $I(G)$ if and only if the set theoretic union of branches $B(a)$, $a \in A$, is an ideal (BCC-ideal) of G.*

Proof. Let $S(A)$ denote the set theoretic union of some branches initiated by elements belonging to $A \subset I(G)$; that is,

$$S(A) = \bigcup_{a \in A} B(a) = \{x \in G : x \in B(a), a \in A\}. \tag{20}$$

By Corollary 11, $I(G)$ is a weak-BCC-algebra contained in G.

If A is a BCC-ideal of $I(G)$, then obviously $0 \in A$. Consequently, $0 \in S(A)$ because $0 \in B(0) \subset S(A)$. Now let $y \in S(A)$ and $(x * y) * z \in S(A)$ for some $x, z \in G$. Then, $x \in B(a)$, $y \in B(b)$, $z \in B(c)$, and $(x * y) * z \in B(d)$ for some $a, c \in I(G)$ and $b, d \in A$. Thus, $(x*y)*z \in (B(a)*B(b))*B(c) = B((a*b)*c)$, which means that $B((a * b) * c) = B(d)$ since two branches are equal or disjoint. Hence, $(a * b) * c = d \in A$, so $a * c \in A$. Therefore, $x * z \in B(a) * B(c) = B(a * c) \subset S(A)$. This shows that $S(A)$ is a BCC-ideal of G.

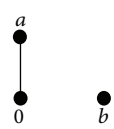
Conversely, let $S(A)$ be a BCC-ideal of G. If $a, (b * a) * c \in A$ for some $a \in A$ and $b, c \in I(G)$, then $a \in B(a) \subset S(A)$, $(b * a) * c \in B((b * a) * c) \subset S(A)$. Hence, $b * c \in S(A)$. Since $b * c \in I(G)$ and $S(A) \cap I(G) = A$, the above implies $b * c \in A$. Thus, A is a BCC-ideal of $I(G)$. □

$I(G)$ is a subalgebra of each weak-BCC-algebra G, but it is not an ideal, in general.

Example 26. It is easy to check that in the weak-BCC-algebra G defined by

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

(21)



(22)

$I(G) = \{0, b\}$ is not an ideal because $a * b = b \in I(G)$, but $a \notin I(G)$.

The above example suggests the following.

Theorem 27. *If $I(G)$ is a proper ideal or a proper BCC-ideal of a weak-BCC-algebra G, then G has at least two nontrivial branches.*

Proof. Since $\{0\} \neq I(G) \neq G$, at least one branch of G is not trivial. Suppose that only $B(b)$ has more than one element. Then, for any $0 \neq a \in I(G)$ and $x \in B(b)$, $x \neq b$, we have $x * a \in B(b) * B(a) = B(b * a)$. But, by Corollary 11, $I(G)$ is a maximal group-like subalgebra contained in G. Thus, $b * a \in I(G)$ and $b * a \neq b$, because in the case $b * a = b$ in the corresponding group $(G; \cdot, ^{-1}, 0)$, we obtain $b = b \cdot a$ which is impossible for $a \neq 0$. Therefore, $B(b*a) \neq B(b)$ and $B(b*a)$ has only one element. So, $x*a = b*a$. Hence, $x*a \in I(G)$, which according to the assumption on $I(G)$ implies $x \in I(G)$. The obtained contradiction shows that $I(G)$ cannot be an ideal of G. Consequently, it cannot be a BCC-ideal, too. □

Definition 28. A nonempty subset A of a weak-BCC-algebra G is called an *(m, n)-fold p-ideal* of G if it contains 0 and

$$(x * z^m) * (y * z^n),\ y \in A \Longrightarrow x \in A. \tag{23}$$

An (n, n)-fold p-ideal is called an *n-fold p-ideal*. Since $(0, 0)$-fold p-ideals coincide with BCK-ideals, we will consider (m, n)-fold p-ideals only for $m \geqslant 1$ and $n \geqslant 1$. Moreover, it will be assumed that $m \neq n + 1$ because for $m = n + 1$ we have $(x * x^{n+1}) * (0 * x^n) = (0 * x^n) * (0 * x^n) = 0 \in A$, which implies $x \in A$. So, $A = G$ for every $(n + 1, n)$-fold p-ideal A of G. Note, that the concept of $(1, 1)$-fold p-ideals coincides with the concept of p-ideals studied in BCI-algebras (see e.g., [18] or [19]).

Example 29. It is easy to see that in the weak-BCC-algebra defined by (4), the set $A = \{0, 1\}$ is an n-fold p-ideal for every $n \geqslant 1$. It is not an (m, n)-fold p-ideal, where m is odd and n is even because in this case $(2 * 2^m) * (0 * 2^n) \in A$ and $0 \in A$, but $2 \notin A$.

Putting $z = 0$ in (23), we see that each (m, n)-fold p-ideal of a weak-BCC-algebra is an ideal. The converse statement is not true since, as it follows from Theorem 30 proved below, each (m, n)-fold ideal contains the branch $B(0)$ which for BCC-ideals is not true.

Theorem 30. *Any (m, n)-fold p-ideal contains $B(0)$.*

Proof. Let A be an (m, n)-fold p-ideal of a weak-BCC-algebra G. Since for every $x \in B(0)$ from $0 \leqslant x$ it follows that $0 * x = 0$, we have

$$(x * x^m) * (0 * x^n) = \left(0 * x^{m-1}\right) * (0 * x^n) = 0 \in A, \tag{24}$$

which, according to (23), gives $x \in A$. Thus, $B(0) \subseteq A$. □

Corollary 31. *An (m, n)-fold p-ideal A together with an element $x \in A$ contains whole branch containing this element.*

Proof. Let $x \in A$ and y be an arbitrary element from the branch $B(a)$ containing x. Then, according to Proposition 7, we have $y * x \in B(0) \subset A$. Since A is also an ideal, the last implies $y \in A$. Thus, $B(a) \subset A$. □

Corollary 32. *For any n-fold p-ideal A from $x \leqslant y$ and $x \in A$, it follows that $y \in A$.*

Theorem 33. *A nonempty subset A of a solid weak-BCC-algebra G is its (m, n)-fold p-ideal if and only if*

(a) *$I(A)$ is an (m, n)-fold p-ideal of $I(G)$,*

(b) $A = \bigcup\{B(a) : a \in I(A)\}$.

Proof. Let A be an (m, n)-fold p-ideal of G. Then, clearly $I(A) = A \cap I(G) \neq \emptyset$ is an (m, n)-fold p-ideal of $I(G)$. By Corollary 31, A is the set theoretic union of all branches $B(a)$ such that $a \in I(A)$. So, any (m, n)-fold p-ideal A satisfies the above two conditions.

Suppose now that a nonempty subset A of G satisfies these two conditions. Let $x, y, z \in G$. If $x \in B(a)$, $y \in B(b)$, $z \in B(c)$, and $y, (x * z^m) * (y * z^n) \in A$, then $(x * z^m) * (y * z^n) \in B((a * c^m) * (b * c^n))$, which, by (b), implies $b, (a * c^m) * (b * c^n) \in I(A)$. This, by (a), gives $a \in I(A)$. So, $B(a) \subset A$. Hence, $x \in A$. □

Note that in some situations, the converse of Theorem 30 is true.

Theorem 34. *An ideal A of a weak-BCC-algebra G is its n-fold p-ideal if and only if $B(0) \subset A$.*

Proof. By Theorem 30, any n-fold p-ideal contains $B(0)$. On the other hand, if A is an ideal of G and $B(0) \subset A$, then from $y \in A$ and $(x * z^n) * (y * z^n) \in A$, by (i′), it follows that

$$\begin{aligned}(x * z^n) * (y * z^n) &\leqslant \left(x * z^{n-1}\right) * \left(y * z^{n-1}\right) \\ &\leqslant \left(x * z^{n-2}\right) * \left(y * z^{n-2}\right) \leqslant \cdots \leqslant x * y,\end{aligned} \tag{25}$$

so $(x * z^n) * (y * z^n)$ and $x * y$, as comparable elements, are in the same branch. Hence, $(x * y) * ((x * z^n) * (y * z^n)) \in B(0) \subset A$, by Proposition 7. Since $(x * z^n) * (y * z^n) \in A$ and A is a BCC-ideal (or a BCK-ideal), $(x * y) * ((x * z^n) * (y * z^n)) \in A$ implies $x * y \in A$. Consequently, $x \in A$. So, A is an n-fold p-ideal. □

Corollary 35. *Any ideal containing an n-fold p-ideal is also an n-fold p-ideal.*

Proof. Suppose that an ideal B contains some n-fold p-ideal A. Then, $B(0) \subset A \subset B$, which completes the proof. □

Corollary 36. *An ideal A of a weak-BCC-algebra G is its n-fold p-ideal if and only if the implication*

$$(x * z^n) * (y * z^n) \in A \Longrightarrow x * y \in A \tag{26}$$

is valid for all $x, y, z \in G$.

Proof. Let A be an n-fold p-ideal of G. Since $(x * z^n) * (y * z^n) \leqslant x * y$, from $(x * z^n) * (y * z^n) \in A$ and by Corollary 32, we obtain $x * y \in A$. So, any n-fold p-ideal satisfies this implication.

The converse statement is obvious. □

Theorem 37. *An n-fold p-ideal is a k-fold p-ideal for any $k \leqslant n$.*

Proof. Similarly, as in the previous proof, we have

$$\begin{aligned}(x * z^n) * (y * z^n) &\leqslant \left(x * z^{n-1}\right) * \left(y * z^{n-1}\right) \\ &\leqslant \cdots \leqslant \left(x * z^k\right) * \left(y * z^k\right)\end{aligned} \tag{27}$$

for every $1 \leqslant k \leqslant n$. Thus, $(x * z^n) * (y * z^n)$ and $(x * z^k) * (y * z^k)$ are in the same branch. Hence, if A is an n-fold p-ideal and $(x * z^k) * (y * z^k) \in A$, then, by Corollary 31, also $(x * z^n) * (y * z^n) \in A$. This, together with $y \in A$, implies $x \in A$. Therefore, A is a k-fold ideal. □

Theorem 38. *$B(0)$ is the smallest n-fold p-ideal of each weak-BCC-algebra.*

Proof. Obviously, $0 \in B(0)$. If $y \in B(0)$, then $0 \leqslant y$, $0 * z^n \leqslant y * z^n$ and

$$\begin{aligned}(x * z^n) * (y * z^n) &\leqslant (x * z^n) * (0 * z^n) \\ &\leqslant \left(x * z^{n-1}\right) * \left(0 * z^{n-1}\right) \\ &\leqslant \cdots \leqslant x * 0 = x.\end{aligned} \tag{28}$$

Thus, $(x * z^n) * (y * z^n) \leqslant x$. Since $(x * z^n) * (y * z^n) \in B(0)$ means $0 \leqslant (x * z^n) * (y * z^n)$, from the above, we obtain $0 \leqslant x$. So, $x \in B(0)$. Hence, $B(0)$ is an n-fold p-ideal. By Theorem 30, it is the smallest n-fold p-ideal of each weak-BCC-algebra. □

Theorem 39. *Let G be a weak-BCC-algebra. If $I(G)$ has k elements and k divides $|m - n|$, then $B(0)$ is an (m, n)-fold p-ideal of G.*

Proof. By Corollary 11, $I(G)$ is a group-like subalgebra of G. Hence, if $I(G)$ has k elements, then in the group $(I(G); \cdot, ^{-1}, 0)$ connected with $I(G)$ (Theorem 9), we have $b^{ks} = 0$ for every $b \in I(G)$ and any integer s.

At first, we consider the case $m \geqslant n$. If $(x * z^m) * (y * z^n) \in B(0)$ for some $x \in B(a)$, $y \in B(0)$, $z \in B(c)$, then, by (i′), we have $(x * z^m) * (y * z^n) \leqslant (x * z^{m-n}) * y$. Hence, $(x * z^{m-n}) * y$ and $(x * z^m) * (y * z^n)$, as comparable elements, are in the same branch. Consequently, $((x * z^{m-n}) * y) * ((x * z^m) * (y * z^n)) \in B(0)$ (Proposition 7). Since, $B(0)$ is an ideal in each weak-BCC-algebra, from the last, we obtain $(x * z^{m-n}) * y \in B(0)$, and consequently, $x * z^{m-n} \in B(0)$. But, $x * z^{m-n} \in B(a * c^{m-n})$, so $B(0) = B(a * c^{m-n})$; that is, $0 = a * c^{m-n}$. This in the group $(I(G); \cdot, ^{-1}, 0)$ connected with $I(G)$ gives $0 = a \cdot c^{n-m} = a$. So, $x \in B(0)$.

Now let $m < n$. Then $(x * z^m) * (y * z^n) \leqslant x * (y * z^{n-m})$. This, similarly as in the previous case, for $(x * z^m) * (y * z^n) \in B(0)$ gives $(x * (y * z^{n-m})) * ((x * z^m) * (y * z^n)) \in B(0)$. Consequently, $x * (y * z^{n-m}) \in B(0) \cap B(a * (0 * c^{n-m}))$. So, $0 = a * (0 * c^{n-m})$. This in the group $(I(G); \cdot, ^{-1}, 0)$ implies $0 = a \cdot c^{n-m} = a$. Hence, $x \in B(0)$.

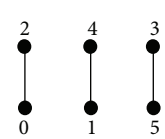
The proof is complete. □

The assumption on the number of elements of the set $I(G)$ is essential; if k is not a divisor of $|m - n|$, then $B(0)$ may not be an (m, n)-fold p-ideal.

Example 40. The solid weak-BCC-algebra G defined by

$*$	0	1	2	3	4	5
0	0	5	0	1	5	1
1	1	0	1	5	0	5
2	2	5	0	1	5	1
3	3	4	5	0	4	2
4	4	2	1	5	0	5
5	5	1	5	0	1	0

(29)



(30)

is proper, because $(3*1)*4 \neq (3*4)*1$. The set $I(G)$ has three elements. The set $B(0) = \{0, 2\}$ is an n-fold p-ideal for every natural n but it is not a $(3, 2)$-fold ideal because $(1*1^3)*(0*1^2) \in B(0)$ and $1 \notin B(0)$.

In the case when $B(0)$ has only one element, the equivalence relation induced by $B(0)$ has one-element equivalence classes. Since these equivalence classes are branches, a weak-BCC-algebra with this property is group-like. Direct computations show that in this case, $B(0)$ is an n-fold p-ideal for every natural n.

This observation together with the just proved results suggests simple characterization of group-like weak-BCC-algebras.

Theorem 41. *A weak-BCC-algebra G is group-like if and only if for some $n \geqslant 1$ and all $x, z \in G$*

$$(x*z^n)*(0*z^n) = 0 \Longrightarrow x = 0. \tag{31}$$

Proof. Let G be a weak-group-like BCC-algebra. Then, $G = I(G)$, which means that G has a discrete order; that is, $x \leqslant y$ implies $x = y$. Since for $x, y, z \in G$ we have $(x*z^n)*(y*z^n) \leqslant x*y$, a group-like weak-BCC-algebra satisfies the identity $(x*z^n)*(y*z^n) = x*y$. In particular, for $y = 0$, we have $(x*z^n)*(0*z^n) = x*0 = x$. So, $(x*z^n)*(0*z^n) = 0$ implies $x = 0$.

Conversely, if the above implication is valid for all $x, z \in G$, then

$$0 = (x*z^n)*(0*z^n) \leqslant x*0 = x \tag{32}$$

gives $0 \leqslant x$. This, according to the assumption, implies $x = 0$. Hence, $B(0) = \{0\}$, which means that G is group-like. □

Remember that an ideal A of a weak-BCC-algebra is called *closed* if $0*x \in A$ for every $x \in A$, that is, if $\varphi(A) \subset A$.

Theorem 42. *For an (m, n)-fold p-ideal A of a solid weak-BCC-algebra G, the following statements are equivalent:*

(1) *A is a closed (m, n)-fold p-ideal of G,*

(2) *$I(A)$ is a closed (m, n)-fold p-ideal of $I(G)$,*

(3) *$I(A)$ is a subalgebra of $I(G)$,*

(4) *A is a subalgebra of G.*

Proof. The implication (1) ⇒ (2) follows from Theorem 33.

(2) ⇒ (3) Observe first that $I(A)$ is a closed BCK-ideal of $I(G)$ and $a*b = c \in I(G)$ for any $a, b \in I(A)$. Since $I(G)$ is a group-like subalgebra of G (Corollary 11), in the group $(I(G); \cdot, ^{-1}, 0)$, we have $c = a \cdot b^{-1}$ (Theorem 9), which means that $c \cdot b = a \in I(A)$. Thus,

$$\begin{aligned} c \cdot (0 \cdot b) &= c*(0 \cdot b)^{-1} = a \\ &= c*\left(b^{-1}\right)^{-1} = c*b \in I(A). \end{aligned} \tag{33}$$

Hence, $c*(0*b) \in I(A)$. But $0*b \in I(A)$ and $I(A)$ is a BCK-ideal of $I(G)$; therefore $c \in I(A)$. Consequently, $a*b \in I(A)$ for every $a, b \in I(A)$. So, $I(A)$ is a subalgebra of $I(G)$.

(3) ⇒ (4) $I(A) \subset A$, so $0 \in A$. Let $x \in B(a)$, $y \in B(b)$. If $x, y \in A$, then $a, b \in I(A)$, and by the assumption $a*b \in I(A)$. From this, we obtain $x*y \in B(a)*B(b) = B(a*b)$, which together with Theorem 33 proves $x*y \in A$. Hence, A is a subalgebra of G.

The implication (4) ⇒ (1) is obvious. □

5. Nilpotent Weak-BCC-Algebras

A special role in weak-BCC-algebras play elements having a finite "order," that is, elements for which there exists some natural k such that $0*x^k = 0$. We characterize sets of such elements and prove that the properties of such elements can be described by the properties of initial elements of branches containing these elements.

Definition 43. An element x of a weak-BCC-algebra G is called *nilpotent*, if there exists some positive integer k such that $0*x^k = 0$. The smallest k with this property is called the *nilpotency index* of x and is denoted by $n(x)$. A weak-BCC-algebra in which all elements are nilpotent is called *nilpotent*.

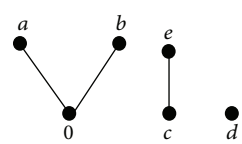
By $N_k(G)$, we denote the set of all nilpotent elements $x \in G$ such that $n(x) = k$. $N(G)$ denotes the set of all nilpotent elements of G. It is clear that $N_1(G) = B(0)$.

Example 44. In the weak-BCC-algebras defined by (4) and (5), we have $n(0) = n(1) = 1$, $n(2) = n(3) = 2$.

Example 45. In the weak-BCC-algebra defined by

$*$	0	a	b	c	d	e
0	0	0	0	d	c	d
a	a	0	a	d	c	d
b	b	b	0	d	c	d
c	c	c	c	0	d	0
d	d	d	c	0	c	
e	e	c	e	a	d	0

(34)



(35)

there are no elements with $n(x) = 2$, but there are three elements with $n(x) = 3$ and three with $n(x) = 1$.

Proposition 46. *Elements belonging to the same branch have the same nilpotency index.*

Proof. Let $x \in B(a)$. Then $a \leqslant x$, which, by Theorem 3, implies $0*a = 0*x$. This together with $a \leqslant x$ gives $0*x^2 \leqslant (0*a)*x \leqslant 0*a^2$. Hence, $0*x^2 \leqslant 0*a^2$. In the same manner from $0*x^k \leqslant 0*a^k$, it follows that $0*x^{k+1} \leqslant 0*a^{k+1}$, which by induction proves $0*x^m \leqslant 0*a^m$ for every $x \in B(a)$ and any natural m. Thus, $0*a^m = 0$ implies $0*x^m = 0$. On the other hand, from $0*x^m = 0$, we obtain $0 \leqslant 0*a^m$. This implies $0 = 0*a^m$ since $0, 0*a^m \in I(G)$ and elements of $I(G)$ are incomparable. Therefore, $0*x^m = 0$ if and only if $0*a^m = 0$. So, $n(x) = n(a)$ for every $x \in B(a)$. □

Corollary 47. *A weak-BCC-algebra G is nilpotent if and only if its subalgebra $I(G)$ is nilpotent.*

Corollary 48. $x \in B(a) \cap N_k(G) \Rightarrow B(a) \subset N_k(G)$.

The above results show that the study of nilpotency of a given weak-BCC-algebras can be reduced to the study of nilpotency of its initial elements.

Proposition 49. *Let G be a weak-BCC-algebra. If $I(G)$ is a BCI-algebra, then $N_k(G)$ is a subalgebra and a BCC-ideal of G for every k.*

Proof. Obviously, $0 \in N_k(G)$ for every k. Let $x, y \in N_k(G)$. Then $0 * x^k = 0 * y^k = 0$ and $0 * x^k = 0 * a^k = 0$, $0 * y^k = 0 * b^k = 0$ for some $a, b \in I(G)$. Since $I(G)$ is a BCI-algebra, by Theorem 16, we have $0 = (0 * a^k) * (0 * b^k) = 0 * (a * b)^k$. Hence, $a * b \in N_k(G)$. Consequently, $x * y \in B(a) * B(b) = B(a * b) \subset N_k(G)$. So, $N_k(G)$ is a subalgebra of G.

Now let $x \in B(a)$, $y \in B(b)$, $z \in B(c)$. If $y, (x * y) * z \in N_k(G)$, then also $b, (a * b) * c \in N_k(G)$. Thus, $0 * b^k = 0$ and

$$\begin{aligned} 0 * (a * c)^k &= \left(0 * a^k\right) * \left(0 * c^k\right) \\ &= \left(\left(0 * a^k\right) * \left(0 * b^k\right)\right) * \left(0 * c^k\right) \\ &= 0 * ((a * b) * c)^k = 0, \end{aligned} \tag{36}$$

which implies $a * c \in N_k(G)$. This together with Corollary 48 implies $x * z \in B(a * c) \subset N_k(G)$. Therefore, $N_k(G)$ is a BCC-ideal of G. Clearly, it is a BCK-ideal, too. □

Corollary 50. *$N_k(G)$ is a subalgebra of each solid weak-BCC-algebra.*

Proposition 51. *$N(G)$ is a subalgebra of each weak-BCC-algebra G in which $I(G)$ is a BCI-algebra.*

Proof. Since $N(G) = \bigcup_{k \in N} N_k(G)$ and $0 \in N_k(G)$ for every k, the set $N(G)$ is nonempty. Let $x \in B(a)$, $y \in B(b)$. If $x, y \in N(G)$ and $n(x) = m$, $n(y) = n$, then $0 * x^m = 0 * y^n = 0$. From this, by Proposition 46, we obtain $0 * a^m = 0 * b^n = 0$, which in the group $(I(G); \cdot, ^{-1}, 0)$ can be written in the form $a^{-m} = b^{-n} = 0$. But $I(G)$ is a BCI-algebra; hence, $(I(G); \cdot, ^{-1}, 0)$ is an abelian group. Thus,

$$\begin{aligned} 0 = \left(a^{-m}\right)^n \cdot \left(b^n\right)^m &= a^{-mn} \cdot b^{mn} \\ &= \left(a \cdot b^{-1}\right)^{-mn} = 0 * (a \cdot b)^{mn}, \end{aligned} \tag{37}$$

by Theorem 9. Hence, $a * b \in N(G)$. This implies $x * y \in B(a * b) \subset N(G)$. Therefore, $N(G)$ is a subalgebra of G. □

Corollary 52. *$N(G)$ is a subalgebra of each solid weak-BCC-algebra.*

Corollary 53. *Any solid weak-BCC-algebra G with finite $I(G)$ is nilpotent.*

Proof. Indeed, $I(G)$ is a maximal group-like BCI-algebra contained in any solid weak-BCC-algebra. Hence, the group $(I(G); \cdot, ^{-1}, 0)$ is abelian. If it is finite, then each of its element has finite order k. Thus, $0 * a^k = 0 \cdot a^{-k} = 0$ for every $a \in I(G)$. Consequently, $B(a) \subset N_k(G) \subset N(G)$ for every $a \in I(G)$. Therefore, $G = N(G)$. □

Corollary 54. *A solid weak-BCC-algebra G is nilpotent if and only if each element of the group $(I(G); \cdot, ^{-1}, 0)$ has finite order.*

Corollary 55. *In a solid weak-BCC-algebra G, the nilpotency index of each $x \in N(G)$ is a divisor of $Card(I(G))$.*

6. k-Nilradicals of Solid Weak-BCC-Algebras

The theory of radicals in BCI-algebras was considered by many mathematicians from China (cf. [18]). Obtained results show that this theory is almost parallel to the theory of radicals in rings. But results proved for radicals in BCI-algebras cannot be transferred to weak-BCC-algebras.

In this section, we characterize one analog of nilradicals in weak-BCC-algebras. Further, this characterization will be used to describe some ideals of solid weak-BCC-algebras.

We begin with the following definition.

Definition 56. Let A be a subset of solid weak-BCC-algebra G. For any positive integer k by a *k-nilradical of A*, denoted by $[A; k]$, we mean the set of all elements of G such that $0 * x^k \in A$; that is,

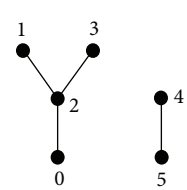
$$[A; k] = \left\{x \in G : 0 * x^k \in A\right\}. \tag{38}$$

Example 57. In the weak-BCC-algebra G defined in Example 44 for $A = \{0, a\}$ and any natural k, we have $[A; 3k + 1] = [A; 3k + 2] = B(0)$, $[A; 3k] = G$. But for $B = \{a, e\}$, we get $[B; 3k + 1] = \{d\}$, $[B; 3k + 2] = B(c)$. The set $[B; 3k]$ is empty.

Example 58. The solid weak-BCC-algebra G defined by

$*$	0	1	2	3	4	5
0	0	0	0	0	5	5
1	1	0	2	2	4	4
2	2	0	0	0	5	5
3	3	2	2	0	4	4
4	4	5	5	5	0	2
5	5	5	5	5	0	0

(39)



(40)

is proper, because $(3 * 4) * 5 \neq (3 * 5) * 4$. In this algebra, each k-nilradical of $A = \{0, 5\}$ is equal to G; each k-nilradical of $B = \{1, 4\}$ is empty.

The first question is when for a given nonempty set A its k-nilradical is also nonempty? The answer is given in the following proposition.

Proposition 59. *A k-nilradical $[A;k]$ of a nonempty subset A of a weak-BCC-algebra G is nonempty if and only if A contains at least one element $a \in I(G)$.*

Proof. From the proof of Theorem 16, it follows that $0 * x^k = 0 * a^k$ for every $x \in B(a)$ and any positive k. So, $x \in [A;k]$ if and only if $0 * a^k \in A$. The last means that $0 * a^k \in A \cap I(G)$ because $I(G)$ is a subalgebra of G. □

Corollary 60. $[I(G);k] = G$ *for every* k.

Proof. Indeed, $0 * x^k = 0 * a^k \in I(G)$ for every $x \in G$. Thus, $x \in [I(G);k]$. □

Corollary 61. *If $I(G)$ has n elements, then $[A;n] = G$ for any subset A of G containing 0, and $[A;n] = \emptyset$ if $0 \notin A$.*

Proof. Similarly, as in previous proofs, we have $0 * x^k = 0 * a^k$ for every $x \in B(a)$ and any k. Since $0 * a^k \in I(G)$ and $I(G)$ is a group-like subalgebra of G, $0 * x^k = a^{-k}$ in the group $(I(G);\cdot,^{-1},0)$ (Theorem 9). If $I(G)$ has n elements, then obviously $0 * x^n = a^{-n} = 0 \in A$. Hence, $x \in [A;n]$. This completes the proof. □

Corollary 62. *Let $x \in B(a)$. Then $x \in [A;k]$ if and only if $B(a) \subset [A;k]$.*

Proof. Since $0 * x^k = 0 * a^k$, we have $x \in [A;k] \Leftrightarrow a \in [A;k]$. □

Corollary 63. $[A;k] = \bigcup\{B(a) : 0 * a^k \in A\}$.

Proposition 64. *Let G be a solid weak-BCC-algebra. Then for every positive integer k and any subalgebra A of G a k-nilradical $[A;k]$ is a subalgebra of G such that $A \subseteq [A;k]$.*

Proof. Let $x, y \in [A;k]$. Then $0 * x^k, 0 * y^k \in A$ and $0 * (x * y)^k = (0 * x^k) * (0 * y^k) \in A$, by Theorem 16. Hence, $x * y \in [A;k]$. Clearly $A \subseteq [A;k]$. □

Proposition 65. *In a solid weak-BCC-algebra, a k-nilradical of an ideal is also an ideal.*

Proof. Let A be a BCC-ideal of G. If $y \in [A;k]$ and $(x * y) * z \in [A;k]$, then $0 * y^k \in A$ and $A \ni 0 * ((x * y) * z)^k = ((0 * x^k) * (0 * y^k)) * (0 * z^k)$, by Theorem 16. Hence, $A \ni (0 * x^k) * (0 * z^k) = 0 * (x * z)^k$. Thus, $x * z \in [A;k]$. □

Note that the last two propositions are not true for weak-BCC-algebras which are not solid.

Example 66. The weak-BCC-algebra G induced by the symmetric group S_3 is not solid because S_3 is not an abelian group (Corollary 14). Routine calculations show that $A = \{0, 3\}$ is a subalgebra and a BCC-ideal of this weak-BCC-algebra, but $[A;3] = \{0, 1, 2, 3\}$ is neither ideal nor subalgebra.

Theorem 67. *In a solid weak-BCC-algebra, a k-nilradical of an (m,n)-fold p-ideal is also an (m,n)-fold p-ideal.*

Proof. By Proposition 65, a k-nilradical of an (m,n)-fold p-ideal A of G is an ideal of G. If $y, (x * z^m) * (y * z^n) \in [A;k]$, then $0 * y^k, 0 * ((x * z^m) * (y * z^n)^k) \in A$. Hence, applying Theorem 16, we obtain

$$\left(\left(0 * x^k\right) * \left(\left(0 * z^k\right)^m\right)\right) * \left(\left(0 * y^k\right) * \left(\left(0 * z^k\right)^n\right)\right) = 0 * \left(\left(x * z^m\right) * \left(y * z^n\right)^k\right) \in A. \tag{41}$$

Thus, $0 * x^k \in A$. So, $x \in [A;k]$. □

Note that in general, a k-nilradical $[A;k]$ of an ideal A does not save all properties of an ideal A. For example, if an ideal A is a horizontal ideal, that is, $x \in A \cap B(0) \Leftrightarrow x = 0$, then a k-nilradical $[A;k]$ may not be a horizontal ideal. Such situation takes place in a weak-BCC-algebra defined by (34). In this algebra, we have $0 * x^3 = 0$ for all elements. Hence, $x \in [A;3] \cap B(0)$ means that $0 * x^3 \in A$ and $x \in B(0)$ which is also true for $x \neq 0$.

Nevertheless, properties of many main types of ideals are saved by their k-nilradicals. Below, we present the list of the main types of ideals considered in BCI-algebras and weak-BCC-algebras.

Definition 68. An ideal A of a weak-BCC-algebra G is called

(i) *antigrouped*, if

$$\varphi^2(x) \in A \Longrightarrow x \in A, \tag{42}$$

(ii) *associative*, if

$$(x * y) * z,\ y * z \in A \Longrightarrow x \in A, \tag{43}$$

(iii) *quasiassociative* if

$$x * (y * z),\ y \in A \Longrightarrow x * z \in A, \tag{44}$$

(iv) *closed*, if

$$x \in A \Longrightarrow 0 * x \in A, \tag{45}$$

(v) *commutative*, if

$$x * y \in A \Longrightarrow x * (y * (y * x)) \in A, \tag{46}$$

(vi) *subcommutative*, if

$$y * (y * (x * (x * y))) =\in A \Longrightarrow x * (x * y) \in A, \tag{47}$$

(vii) *implicative* if

$$(x * y) * z,\ y * z \in A \Longrightarrow x * z \in A, \tag{48}$$

(viii) *subimplicative* if

$$(x * (x * y)) * (y * x) \in A \Longrightarrow y * (y * x) \in A, \tag{49}$$

(ix) *weakly implicative* if

$$(x * (y * x)) * (0 * (y * x)) \in A \Longrightarrow x \in A, \tag{50}$$

(x) *obstinate*, if

$$x, y \notin A \Longrightarrow x * y,\ y * x \in A, \tag{51}$$

(xi) *regular*, if

$$x * y,\ x \in A \Longrightarrow y \in A, \tag{52}$$

(xii) *strong*, if

$$x \in A,\ y \in X - A \Longrightarrow x * y \in X - A, \tag{53}$$

for all $x, y, z \in G$.

Definition 69. We say that an ideal A of a weak-BCC-algebra G has the property $\mathscr{P}$ if it is one of the above types, that is, if it satisfies one of implications mentioned in the above definition.

Theorem 70. *If an ideal A of a solid weak-BCC-algebra G has the property $\mathscr{P}$, then its k-nilradical $[A;k]$ also has this property.*

Proof. (1) A is antigrouped. Let $\varphi^2(x) \in [A;k]$. Then $0 * (\varphi^2(x))^k \in A$. Since, by Theorem 3, φ^2 is an endomorphism of each weak-BCC-algebra, we have

$$\begin{aligned} \varphi^2\left(0 * x^k\right) &= \varphi^2(0) * \left(\varphi^2(x)\right)^k \\ &= 0 * \left(\varphi^2(x)\right)^k \in A. \end{aligned} \tag{54}$$

Thus, $\varphi^2(0 * x^k) \in A$, which according to the definition implies $0 * x^k \in A$. Hence, $x \in [A;k]$.

(2) A is associative. If $(x * y) * z,\ y * z \in [A;k]$, then $0 * ((x * y) * z)^k \in A$ and $0 * (y * z)^k \in A$ which, in view of Theorem 16, means that $((0 * x^k) * (0 * y^k)) * (0 * z^k) \in A$ and $(0 * y^k) * (0 * z^k) \in A$. Since an ideal A is associative, this implies $0 * x^k \in A$; that is, $x \in [A;k]$.

(3) A is quasiassociative. Similarly as in the previous case $x * (y * z),\ y \in [A;k]$ means that $0 * (x * (y * z))^k \in A$ and $0 * y^k \in A$. Hence, $(0 * x^k) * ((0 * y^k) * (0 * z^k)) \in A$. This implies $0 * (x * z)^k = (0 * x^k) * (0 * z^k) \in A$. Consequently, $x * z \in [A;k]$.

(4) A is closed. Let $x \in [A;k]$. Then, $0 * x^k \in A$. Thus,

$$0 * (0 * x)^k = 0 * \left(0 * x^k\right) \in A. \tag{55}$$

So, $0 * x \in [A;k]$.

(5) A is commutative. Let $x * y \in [A;k]$. Then, $0 * (x * y)^k \in A$. From this, we obtain $(0 * x^k) * (0 * y^k) \in A$, which gives $0 * (x * (y * (y * x)))^k = (0 * x^k) * ((0 * y^k) * ((0 * y^k) * (0 * x^k))) \in A$. Hence, $x * (y * (y * x)) \in [A;k]$.

For other types of ideals, the proof is very similar. □

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