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Research Article

Some Endpoint Results for β-Generalized Weak Contractive Multifunctions

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We introduce β-generalized weak contractive multifunctions and give some results about endpoints of the multifunctions. Also, we give some results about role of a point in the existence of endpoints.

1. Introduction

Let (X, d) be a metric space, $CB(X)$ the collection of all nonempty bounded and closed subsets of X, and H the Hausdorff metric with respect to d; that is, $H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$ for all $A, B \in CB(X)$, where $d(x, B) = \inf_{y\in B} d(x, y)$. Let $T : X \to 2^X$ be a multifunction. An element $x \in X$ is said to be a fixed point of T whenever $x \in Tx$. Also, an element $x \in X$ is said to be an endpoint of T whenever $Tx = \{x\}$ [1]. We say that T has the approximate endpoint property whenever $\inf_{x\in X} \sup_{y\in Tx} d(x, y) = 0$ [1]. Let $f : X \to X$ be a mapping. We say that f has the approximate endpoint property whenever $\inf_{x\in X} d(x, fx) = 0$ [1]. Also, the function $g : \mathbb{R} \to \mathbb{R}$ is called upper semicontinuous whenever $\limsup_{n\to\infty} g(\lambda_n) \le g(\lambda)$ for all sequences $\{\lambda_n\}_{n\ge1}$ with $\lambda_n \to \lambda$ [2]. In 2010, Amini-Harandi defined the concept of approximate endpoint property for multifunctions and proved the following result (see [1]).

Theorem 1. *Let $\psi : [0, \infty) \to [0, \infty)$ be an upper semicontinuous function such that $\psi(t) < t$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$ for all $t > 0$, (X, d) a complete metric space, and $T : X \to CB(X)$ a multifunction satisfing $H(Tx, Ty) \le \psi(d(x, y))$ for all $x, y \in X$. Then T has a unique endpoint if and only if T has the approximate endpoint property.*

Then Moradi and Khojasteh introduced the concept of generalized weak contractive multifunctions and improved Theorem 1 by providing the following result [3].

Theorem 2. *Let $\psi : [0, \infty) \to [0, \infty)$ be an upper semicontinuous function such that $\psi(t) < t$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$ for all $t > 0$, (X, d) a complete metric space, and $T : X \to CB(X)$ a generalized weak contractive multifunction; that is, T satisfies $H(Tx, Ty) \le \psi(N(x, y))$ for all $x, y \in X$, where $N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), (d(x, Ty) + d(y, Tx))/2\}$. Then T has a unique endpoint if and only if T has the approximate endpoint property.*

In this paper, we introduce β-generalized weak contractive multifunctions, and by adding some conditions to assumptions of the results, we give some results about endpoints of β-generalized weak contractive multifunctions. In 2012, the technique of α-ψ-contractive mappings was introduced by Samet et al. [4]. Later, some authors used it for some subjects in fixed point theory (see for example [5–8]) or generalized it by using the method of β-ψ-contractive multifunctions (see e.g., [9–12]).

Let (X, d) be a metric space and $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping. A multifunction $T : X \to 2^X$ is called β-generalized weak contraction whenever there

exists a nondecreasing, upper, semicontinuous function $\psi : [0,+\infty) \to [0,+\infty)$ such that $\psi(t) < t$ for all $t > 0$ and

$$\beta(Tx,Ty)\,H(Tx,Ty) \le \psi(N(x,y)) \tag{1}$$

for all $x, y \in X$. We say that T is β-admissible whenever $\beta(A,B) \ge 1$ implies that $\beta(Tx,Ty) \ge 1$ for all $x \in A$, and $y \in B$, where A and B are subsets of X. We say that T has the property (R) whenever for each convergent sequence $\{x_n\}$ in X with $x_n \to x$ and $\beta(Tx_{n-1},Tx_n) \ge 1$ for all $n \ge 1$, we have $\beta(Tx_n,Tx) \ge 1$. One can find idea of the property (R) for mappings in [13]. We say that T has the property (K) whenever for each sequence $\{x_n\}$ in X with $\beta(Tx_{n-1},Tx_n) \ge 1$ for all $n \ge 1$, there exists a natural number k such that $\beta(Tx_m,Tx_n) \ge 1$ for all $m > n \ge k$. Finally, we say that T has the property (H) whenever for each $\varepsilon > 0$, there exists $z \in X$ such that $\sup_{a\in Tz} d(z,a) < \varepsilon$ implies that for every $x \in X$ there exists $y \in Tx$ such that $H(Tx,Ty) = \sup_{b\in Ty} d(y,b)$. A multifunction $T : X \to 2^X$ is called lower semicontinuous at $x_0 \in X$ whenever for each sequence $\{x_n\}$ in X with $x_n \to x_0$ and every $y \in Tx_0$, there exists a sequence $\{y_n\}$ in X with $y_n \in Tx_n$ for all $n \ge 1$ such that $y_n \to y$ [14].

2. Main Results

Now, we are ready to state and prove our main results.

Theorem 3. *Let (X,d) be a complete metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping, and $T : X \to CB(X)$ a β-admissible, β-generalized weak contractive multifunction which has the properties (R), (K), and (H). Suppose that there exist a subset A of X and $x_0 \in A$ such that $\beta(A,Tx_0) \ge 1$. Then T has an endpoint if and only if T has the approximate endpoint property.*

Proof. It is clear that if T has an endpoint, then T has the approximate endpoint property. Conversely, suppose that T has the approximate endpoint property. Choose $A \subset X$ and $x_0 \in A$ such that $\beta(A,Tx_0) \ge 1$. Since T has the approximate endpoint property, for each $\varepsilon > 0$, there exists $z \in X$ such that $\sup_{a\in Tz} d(z,a) < \varepsilon$. Now by using the condition (H), choose $x_1 \in Tx_0$ such that $H(Tx_0,Tx_1) = \sup_{a\in Tx_1} d(x_1,a)$. Also, choose $x_2 \in Tx_1$ such that $H(Tx_1,Tx_2) = \sup_{a\in Tx_2} d(x_2,a)$, and by continuing this process, we find a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$ and

$$H(Tx_{n-1},Tx_n) = \sup_{a\in Tx_n} d(x_n,a), \tag{2}$$

for all $n \ge 1$. Since $\beta(A,Tx_0) \ge 1$ and T is β-admissible, $\beta(Tx_0,Tx_1) \ge 1$. By using induction, it is easy to see that $\beta(Tx_{n-1},Tx_n) \ge 1$ for all $n \ge 1$. Thus, we obtain

$$\begin{aligned} d(x_n,x_{n+1}) &\le \sup_{a\in Tx_n} d(x_n,a) = H(Tx_{n-1},Tx_n) \\ &\le \beta(Tx_{n-1},Tx_n)\,H(Tx_{n-1},Tx_n) \\ &\le \psi(N(x_{n-1},x_n)) \end{aligned} \tag{3}$$

for all $n \ge 1$. If $N(x_{n-1},x_n) = d(x_{n-1},x_n)$, then

$$d(x_n,x_{n+1}) \le \psi(d(x_{n-1},x_n)). \tag{4}$$

If $N(x_{n-1},x_n) = d(x_{n-1},Tx_{n-1})$, then

$$d(x_n,x_{n+1}) \le \psi(d(x_{n-1},Tx_{n-1})) \le \psi(d(x_{n-1},x_n)). \tag{5}$$

If $N(x_{n-1},x_n) = d(x_n,Tx_n)$, then

$$d(x_n,x_{n+1}) \le \psi(d(x_n,Tx_n)) \le \psi(d(x_n,x_{n+1})), \tag{6}$$

and so $d(x_n,x_{n+1}) = 0$. Thus, $d(x_n,x_{n+1}) \le \psi(d(x_{n-1},x_n))$. If

$$\begin{aligned} N(x_{n-1},x_n) &= \frac{d(x_n,Tx_{n-1}) + d(x_{n-1},Tx_n)}{2} \\ &= \frac{d(x_{n-1},Tx_n)}{2}\,\frac{d(x_{n-1},Tx_n)}{2} \\ &\le \frac{d(x_{n-1},x_{n+1})}{2} \\ &\le \frac{d(x_{n-1},x_n) + d(x_n,x_{n+1})}{2} \\ &\le \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}, \end{aligned} \tag{7}$$

then $d(x_n,x_{n+1}) \le \psi(d(x_{n-1},x_n))$ (other case implies that $d(x_n,x_{n+1}) = 0$). Thus,

$$d(x_n,x_{n+1}) \le \psi(d(x_{n-1},x_n)) \tag{8}$$

for all $n \ge 1$. We claim that $\psi(0) = 0$. If $\psi(0) > 0$, then $\psi^2(0) \ge \psi(0) > 0$ because ψ is nondecreasing. On the other hand, since $\psi(t) < t$ for all $t > 0$, we have $\psi^2(0) < \psi(0)$ which is a contradiction. Hence, $\psi(0) = 0$. Let $d_n = d(x_n,x_{n+1})$ for all n. If there exists a natural number n_0 such that $d_{n_0} = 0$, then it is easy to see that $d_n = 0$ for all $n \ge n_0$, and so $\lim_{n\to\infty} d_n = 0$. Now suppose that $d_n \ne 0$ for all n. In this case, we have $d_n \le \psi(d_{n-1}) < d_{n-1}$ for all n. Hence, $\{d_n\}$ is a decreasing sequence, and so there exists $d \ge 0$ such that $\lim_{n\to\infty} d_n = d$. If $d > 0$, then $d_n > 0$ for all n, and so $d_n \le \psi(d_{n-1}) < d_{n-1}$ for all n. Since ψ is upper and semicontinuous, we obtain $d = \lim_{n\to\infty} d_n \le \lim_{n\to\infty} \psi(d_{n-1}) \le \psi(\lim_{n\to\infty} d_{n-1}) = \psi(d) < d$ which is a contradiction. Thus, $\lim_{n\to\infty} d_n = 0$. Now, we prove that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and natural numbers m_k, n_k such that $m_k > n_k \ge k$ and $d(x_{m_k},x_{n_k}) \ge \varepsilon$ for all $k \ge 1$. Also, we choose m_k as small as possible such that

$$d(x_{m_k-1},x_{n_k}) < \varepsilon. \tag{9}$$

Thus, $\varepsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \leq d_{m_k-1} + \varepsilon$ for all k. Hence, $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$. Since T has the property (K), we obtain

$$\begin{aligned} d\left(x_{m_k}, x_{n_k}\right) &\leq d\left(x_{m_k}, x_{m_k+1}\right) + d\left(x_{m_k+1}, x_{n_k+1}\right) \\ &\quad + d\left(x_{n_k+1}, x_{n_k}\right) \\ &\leq d_{m_k} + H\left(Tx_{m_k}, Tx_{n_k}\right) + d_{n_k} \\ &\leq d_{m_k} + \beta\left(Tx_{m_k}, Tx_{n_k}\right) \\ &\quad \times H\left(Tx_{m_k}, Tx_{n_k}\right) + d_{n_k} \\ &\leq d_{m_k} + \psi\left(N\left(x_{m_k}, x_{n_k}\right)\right) + d_{n_k} \end{aligned} \tag{$*$}$$

for all k. Since $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$, $\lim_{k\to\infty} N(x_{m_k}, x_{n_k}) = \varepsilon$. In fact,

$$\begin{aligned} &d\left(x_{m_k}, x_{n_k}\right) \\ &\quad \leq N\left(x_{m_k}, x_{n_k}\right) \\ &\quad = \max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{m_k}, Tx_{m_k}\right), d\left(x_{n_k}, Tx_{n_k}\right),\right. \\ &\qquad \left.\frac{d\left(x_{m_k}, Tx_{n_k}\right) + d\left(x_{n_k}, Tx_{m_k}\right)}{2}\right\} \\ &\quad \leq \max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{m_k}, x_{m_{k+1}}\right), d\left(x_{n_k}, x_{n_{k+1}}\right),\right. \\ &\qquad \left.\frac{d\left(x_{m_k}, x_{n_{k+1}}\right) + d\left(x_{n_k}, x_{m_{k+1}}\right)}{2}\right\} \\ &\quad \leq \max\left\{d\left(x_{m_k}, x_{n_k}\right), d\left(x_{m_k}, x_{m_{k+1}}\right), d\left(x_{n_k}, x_{n_{k+1}}\right),\right. \\ &\qquad \left(d\left(x_{m_k}, x_{n_k}\right) + d\left(x_{n_k}, x_{n_{k+1}}\right)\right. \\ &\qquad\quad \left.+ d\left(x_{n_k}, x_{m_k}\right) + d\left(x_{m_k}, x_{m_{k+1}}\right)\right) \\ &\qquad \left.\times (2)^{-1}\right\}, \end{aligned} \tag{10}$$

and so $\varepsilon = \lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) \leq \lim_{k\to\infty} N(x_{m_k}, x_{n_k}) \leq \varepsilon$. Since ψ is upper semicontinuous, by using $(*)$ we obtain

$$\begin{aligned} \varepsilon &= \lim_{k\to\infty} d\left(x_{m_k}, x_{n_k}\right) \\ &\leq \lim_{k\to\infty} \psi\left(N\left(x_{m_k}, x_{n_k}\right)\right) \leq \psi(\varepsilon) < \varepsilon, \end{aligned} \tag{11}$$

which is a contradiction, and so $\{x_n\}$ is a Cauchy sequence. Choose $x^* \in X$ such that $x_n \to x^*$. Now, note that

$$\begin{aligned} H\left(\{x_n\}, Tx_n\right) &= \max\left\{d\left(x_n, Tx_n\right), \sup_{y\in Tx_n} d\left(x_n, y\right)\right\} \\ &= H\left(Tx_{n-1}, Tx_n\right) \end{aligned} \tag{12}$$

for all n, and so

$$\begin{aligned} H\left(\{x_n\}, Tx_n\right) &= H\left(Tx_{n-1}, Tx_n\right) \\ &\leq \beta\left(Tx_{n-1}, Tx_n\right) H\left(Tx_{n-1}, Tx_n\right) \\ &\leq \psi\left(N\left(x_{n-1}, x_n\right)\right) \\ &\leq \psi\left(d\left(x_{n-1}, x_n\right)\right) \leq d\left(x_{n-1}, x_n\right) \end{aligned} \tag{13}$$

for all n, and so $\lim_{n\to\infty} H(\{x_n\}, Tx_n) = 0$. Since T has the property (R), we obtain

$$\begin{aligned} H\left(\{x^*\}, Tx^*\right) &\leq d\left(x^*, x_n\right) \\ &\quad + H\left(\{x_n\}, Tx_n\right) + H\left(Tx_n, Tx^*\right) \\ &\leq d\left(x^*, x_n\right) + H\left(\{x_n\}, Tx_n\right) \\ &\quad + \beta\left(Tx_n, Tx^*\right) H\left(Tx_n, Tx^*\right) \\ &\leq d\left(x^*, x_n\right) + H\left(\{x_n\}, Tx_n\right) \\ &\quad + \psi\left(N\left(x_n, x^*\right)\right) \end{aligned} \tag{14}$$

for all n. If $N(x_n, x^*) = d(x^*, Tx^*)$, then we have

$$\begin{aligned} &H\left(\{x^*\}, Tx^*\right) \\ &\quad \leq d\left(x^*, x_n\right) + H\left(\{x_n\}, Tx_n\right) + \psi\left(H\left(\{x^*\}, Tx^*\right)\right) \end{aligned} \tag{15}$$

for all n. This implies that $H(\{x^*\}, Tx^*) \leq \psi(H(\{x^*\}, Tx^*))$, and so

$$H\left(\{x^*\}, Tx^*\right) = 0. \tag{16}$$

If $N(x_n, x^*) = d(x_n, x^*)$ or $N(x_n, x^*) \leq d(x_n, x_{n+1})$, then it is easy to see that $H(\{x^*\}, Tx^*) = 0$. Thus, x^* is an endpoint of T. □

Next example shows that a β-generalized weak contractive multifunction is not necessarily a generalized weak contractive multifunction.

Example 4. Let $X = \mathbb{R}$. Define $T : X \to CB(X)$ by $Tx = [x, x+2]$ for all $x \in X$. Suppose that $\psi : [0, +\infty) \to [0, +\infty)$ is an arbitrary upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. If $x = 0$ and $y = 2$, then $H(Tx, Ty) = H([0, 2], [2, 4]) = 2$ and $N(x, y) = 2$. Hence,

$$H\left(Tx, Ty\right) = 2 \gneqq \psi(2) = \psi\left(N\left(x, y\right)\right). \tag{17}$$

Thus, T is not a generalized weak contractive multifunction. Now, suppose that $\psi(t) = t/2$ for all $t \geq 0$ and define $\beta : 2^X \times 2^X \to [0, \infty)$ by $\beta(A, B) = 1/2$ for all subsets A and B of X. Then, we have

$$\begin{aligned} \beta\left(Tx, Ty\right) H\left(Tx, Ty\right) &= \frac{1}{2} d\left(x, y\right) \\ &= \psi\left(d\left(x, y\right)\right) = \psi\left(N\left(x, y\right)\right) \end{aligned} \tag{18}$$

for all $x, y \in \mathbb{R}$. Thus, T is a β-generalized weak contractive multifunction.

Next example shows that there are multifunctions which satisfy the conditions of Theorem 3, while they are not generalized weak contractive multifunctions.

Example 5. Let $X = [0, 9/2]$ and let $d(x, y) = |x - y|$. Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \left\{\frac{x}{2}\right\} & 0 \le x \le 1 \\ \left\{4x - \frac{3}{2}\right\} & 1 < x \le \frac{3}{2} \\ \{0\} & \frac{3}{2} < x \le \frac{9}{2}. \end{cases} \tag{19}$$

If $x = 1$ and $y = 3/2$, then

$$\begin{aligned} H(Tx, Ty) &= H\left(\left\{\frac{1}{2}\right\}, \left\{\frac{9}{2}\right\}\right) = 4 > 3 \\ &= N(x, y) > \psi(N(x, y)), \end{aligned} \tag{20}$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is an arbitrary upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. Thus, T is not a generalized weak contractive multifunction. Now, we show that T satisfies all conditions of Theorem 3. For this aim, define $\psi(t) = t/2$ and $\beta(A, B) = 1$ whenever A and B are subsets of $[0, 1]$ and $\beta(A, B) = 0$ otherwise. First suppose that $x \notin [0, 1]$ or that $y \notin [0, 1]$. If $x, y \in (3/2, 9/2]$, then $Tx, Ty \subset [0, 1]$ and $\beta(Tx, Ty) = 1$. But, $H(Tx, Ty) = 0$, and so $\beta(Tx, Ty)H(Tx, Ty) \le \psi(N(x, y))$. If $x \in (1, 3/2]$ or $y \in (1, 3/2]$, then $Tx \not\subseteq [0, 1]$, or $Ty \not\subseteq [0, 1]$ and so $\beta(Tx, Ty) = 0$. Hence, $\beta(Tx, Ty)H(Tx, Ty) \le \psi(N(x, y))$. Now, suppose that $x, y \in [0, 1]$. In this case, we have $\beta(Tx, Ty) \ge 1$, $H(Tx, Ty) = H(\{x/2\}, \{y/2\}) = (1/2)d(x, y)$, and $N(x, y) = \max\{d(x, y), x/2, y/2, (d(x, y/2) + d(y, x/2))/2\}$. Thus, $d(x, y) \le N(x, y)$, and so

$$\begin{aligned} \beta(Tx, Ty)\, H(Tx, Ty) &= \frac{1}{2} d(x, y) \\ &\le \psi(d(x, y)) \le \psi(N(x, y)). \end{aligned} \tag{21}$$

Therefore, T is a β-generalized weak contractive multifunction. Now, we show that T is β-admissible. If $\beta(A, B) \ge 1$, then $A, B \subset [0, 1]$, and so $Tx = \{x/2\} \in [0, 1]$ and $Ty = \{y/2\} \in [0, 1]$ for all $x \in A$ and $y \in B$. Thus, $\beta(Tx, Ty) \ge 1$ for all $x \in A$ and $y \in B$. Now, suppose $A = [0, 1/2]$ and $x_0 = 1/4$. Then, $Tx_0 = \{1/8\} \in [0, 1]$ and $[0, 1/2] \subset [0, 1]$. Hence, $\beta(A, Tx_0) \ge 1$. Now, we show that T satisfies the condition (H). First note that, for each $\varepsilon > 0$, there exists $z \in X$ such that $\sup_{a \in Tz} d(z, a) < \varepsilon$. Now, we show that for each $x \in X$ there exists $y \in Tx$ such that $H(Tx, Ty) = \sup_{b \in Ty} d(y, b)$. If $0 \le x \le 1$, then $Tx = \{x/2\}$, $T(x/2) = \{x/4\}$, and

$$\begin{aligned} H\left(Tx, T\left(\frac{x}{2}\right)\right) &= H\left(\left\{\frac{x}{2}\right\}, \left\{\frac{x}{4}\right\}\right) \\ &= \frac{x}{4} = \sup_{b \in T(x/2)} d\left(\frac{x}{2}, b\right). \end{aligned} \tag{22}$$

Since for $1 < x \le 3/2$ we have $5/2 < 4x - (3/2) \le 9/2$, $T(4x - (3/2)) = \{0\}$. Thus,

$$\begin{aligned} H\left(Tx, T\left(4x - \frac{3}{2}\right)\right) &= H\left(\left\{4x - \frac{3}{2}\right\}, \{0\}\right) \\ &= 4x - \frac{3}{2} = \sup_{b \in T(4x - 3/2)} d\left(4x - \frac{3}{2}, b\right). \end{aligned} \tag{23}$$

If $3/2 < x \le 9/2$, then $Tx = \{0\}$ and $T(0) = \{0\}$. Hence,

$$\begin{aligned} H(Tx, T(0)) &= H(\{0\}, \{0\}) \\ &= 0 = \sup_{b \in T(0)} d(0, b). \end{aligned} \tag{24}$$

It is easy to check that T satisfies the conditions (R) and (K). Note that, 0 is the endpoint of T.

Now, we add an assumption to obtain uniqueness of endpoint. In this way, we introduce a new notion. Let X be a set and $\beta : 2^X \times 2^X \to [0, \infty)$ a map. We say that the set X has the property (G_β) whenever $\beta(A, B) \ge 1$ for all subsets A and B of X with $A \not\subseteq B$ or $B \not\subseteq A$.

Corollary 6. *Let (X, d) be a complete metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping, and $T : X \to CB(X)$ a β-admissible, β-generalized weak contractive multifunction which has the properties (R), (K), and (H). Suppose that there exist a subset A of X and $x_0 \in A$ such that $\beta(A, Tx_0) \ge 1$. If T has the approximate endpoint property and X has the property (G_β), then T has a unique endpoint.*

Proof. By using Theorem 3, T has a endpoint. If T has two distinct endpoints x^* and y^*, then $\beta(Tx^*, Ty^*) = \beta(\{x^*\}, \{y^*\}) \ge 1$ because X has the property (G_β). Hence,

$$\begin{aligned} d(x^*, y^*) &\le H(Tx^*, Ty^*) \\ &\le \beta(Tx^*, Ty^*)\, H(Tx^*, Ty^*) \\ &\le \psi(N(x^*, y^*)) < N(x*, y*) \\ &= d(x^*, y^*), \end{aligned} \tag{25}$$

which is a contradiction. Thus, T has a unique endpoint. □

In Example 5, T has a unique endpoint, while X does has not the property (G_β). Also, T has the property (R), while T is not lower semicontinuous. To see this, consider the sequence $\{x_n\}$ defined by

$$x_n = \begin{cases} 1 - \frac{1}{n} & n = 2k \\ 1 + \frac{1}{n} & n = 2k - 1 \end{cases} \tag{26}$$

for $k \ge 1$ and put $y = 1/2$ and $x_0 = 1$. Then $x_n \to 1$ and $y \in Tx_0 = \{1/2\}$. Let $\{y_n\}$ be an arbitrary sequence in X such that $y_n \in Tx_n$ for all $n \ge 1$. Then, $y_{2k-1} \in Tx_{2k-1}$ and $y_{2k} \in Tx_{2k}$ for all k. But, $y_{2k-1} = 4x_{2k-1} - (3/2)$ for sufficiently large k and $y_{2k} = x_{2k}/2$ for all k since $y_{2k-1} \to 5/2$, $y_n \nrightarrow 1/2$. This implies that T is not lower semicontinuous.

Corollary 7. *Let (X,d) be a complete metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping, and $T : X \to CB(X)$ a β-admissible multifunction which has the properties (R), (K), and (H). Suppose that X has the property (G_β), and there exist a subset A of X, $x_0 \in A$ and $k \in [0,1)$ such that $\beta(A,Tx_0) \geq 1$ and $\beta(Tx,Ty)H(Tx,Ty) \leq kN(x,y)$ for all $x,y \in X$. Then T has a unique endpoint if and only if T has the approximate endpoint property.*

Proof. It is sufficient that we define $\psi(t) = kt$ for all $t \geq 0$. Then, Theorem 3 and Corollary 6 guarantee the result. □

It has been proved that lower semicontinuity of the multifunction T and the property (R) are independent conditions [9]. We can replace lower semicontinuity of the multifunction instead of the property (R) to obtain the next result. Its proof is similar to the proof of Theorem 3.

Theorem 8. *Let (X,d) be a complete metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping, and $T : X \to CB(X)$ a lower semicontinuous, β-admissible, β-generalized weak contractive multifunction which has the properties (K) and (H). Suppose that there exist a subset A of X and $x_0 \in A$ such that $\beta(A,Tx_0) \geq 1$. Then T has the approximate endpoint property if and only if T has an endpoint.*

Corollary 9. *Let (X,d) be a complete metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping, and $T : X \to CB(X)$ a lower semicontinuous, β-admissible, β-generalized weak contractive multifunction which has the properties (K) and (H). Suppose that there exist a subset A of X and $x_0 \in A$ such that $\beta(A,Tx_0) \geq 1$. If T has the approximate endpoint property and X has the property (G_β), then T has a unique endpoint.*

Corollary 10. *Let (X,d) be a complete metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping, and $T : X \to CB(X)$ a β-admissible multifunction which has the properties (R), (K), and (H). Suppose that X has the property (G_β), and there exist a subset A of X, $x_0 \in A$ and $k \in [0,1)$ such that $\beta(A,Tx_0) \geq 1$ and $\beta(Tx,Ty)H(Tx,Ty) \leq kN(x,y)$ for all $x,y \in X$. If T has the approximate endpoint property, then* $\mathrm{Fix}(T) = \mathrm{End}(T) = \{x\}$.

Proof. If we put $\psi(t) = kt$, then, by using Theorem 2.10 in [9], T has a fixed point. Since T has the approximate endpoint property, by using Corollary 7, T has a unique endpoint such x. Let $y \in \mathrm{Fix}(T)$. If $Tx = Ty$, then $y = x$. If $Tx \neq Ty$, then $\beta(Tx,Ty) \geq 1$ because X has the property (G_β). Also, we have

$$\begin{aligned} d(x,y) &\leq H(\{x\},Ty) = H(Tx,Ty) \\ &\leq \beta(Tx,Ty)\,H(Tx,Ty) \\ &\leq kN(x,y). \end{aligned} \tag{27}$$

But, $N(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), (d(x,Ty) + d(y,Tx))/2\} = d(x,y)$. Thus, $d(x,y) = 0$, and so $\mathrm{Fix}(T) = \mathrm{End}(T) = \{x\}$. □

Next corollary shows us the role of a point in the existence of endpoints.

Corollary 11. *Let (X,d) be a complete metric space, $x^* \in X$ a fixed element, and $T : X \to CB(X)$ a multifunction such that T has the property (H) and $x^* \in Tx \cap Ty$ for all subsets A and B of X with $x^* \in A \cap B$ and all $x \in A$ and $y \in B$. Assume that $H(Tx,Ty) \leq \psi(N(x,y))$ for all $x,y \in X$ with $x^* \in Tx \cap Ty$, where $\psi : [0,+\infty) \to [0,+\infty)$ is a nondecreasing upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. Suppose that there exist a subset A_0 of X and $x_0 \in A_0$ such that $x^* \in A_0 \cap Tx_0$. Assume that for each convergent sequence $\{x_n\}$ in X with $x_n \to x$ and $x^* \in Tx_{n-1} \cap Tx_n$, for all $n \geq 1$, one has $x^* \in Tx_n \cap Tx$. Also, for each sequence $\{x_n\}$ in X with $x^* \in Tx_{n-1} \cap Tx_n$ for all $n \geq 1$, there exists a natural number k such that $x^* \in Tx_m \cap Tx_n$ for all $m > n \geq k$. Then T has an endpoint if and only if T has the approximate endpoint property.*

Proof. It is sufficient we define $\beta : 2^X \times 2^X \to [0,\infty)$ by $\beta(A,B) = 1$ whenever $x^* \in A \cap B$ and $\beta(A,B) = 0$ otherwise, and then we use Theorem 3. □

Corollary 12. *Let (X,d) be a complete metric space, $x^* \in X$ a fixed element and $T : X \to CB(X)$ a lower semicontinuous multifunction such that T has the property (H) and $x^* \in Tx \cap Ty$ for all subsets A and B of X with $x^* \in A \cap B$ and all $x \in A$ and $y \in B$. Assume that*

$$H(Tx,Ty) \leq \psi(N(x,y)) \tag{28}$$

for all $x,y \in X$ with $x^* \in Tx \cap Ty$, where $\psi : [0,+\infty) \to [0,+\infty)$ is a nondecreasing upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. Suppose that there exist a subset A_0 of X and $x_0 \in A_0$ such that $x^* \in A_0 \cap Tx_0$. Assume that for each convergent sequence $\{x_n\}$ in X with $x_n \to x$ and $x^* \in Tx_{n-1} \cap Tx_n$ for all $n \geq 1$, we have $x^* \in Tx_n \cap Tx$. Then T has an endpoint if and only if T has the approximate endpoint property.

Proof. It is sufficient to define $\beta : 2^X \times 2^X \to [0,\infty)$ by $\beta(A,B) = 1$ whenever $x^* \in A \cap B$ and $\beta(A,B) = 0$ otherwise, and then we use Theorem 8. □

Let $(X,d,\leq)$ be an ordered metric space. Define the order $\preceq$ on arbitrary subsets A and B of X by $A \preceq B$ if and only if for each $a \in A$ there exists $b \in B$ such that $a \leq b$. It is easy to check that $(CB(X),\preceq)$ is a partially ordered set.

Theorem 13. *Let $(X,d,\leq)$ be a complete ordered metric space and T a closed and bounded valued multifunction on X such that T has the property (H) and $Tx \preceq Ty$ for all subsets A and B of X with $A \preceq B$ and all $x \in A$ and $y \in B$. Assume that $H(Tx,Ty) \leq \psi(N(x,y))$ for all $x,y \in X$ with $Tx \preceq Ty$, where $\psi : [0,+\infty) \to [0,+\infty)$ is a nondecreasing upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. Suppose that there exist a subset A_0 of X and $x_0 \in A_0$ such that $A_0 \preceq Tx_0$. Assume that for each convergent sequence $\{x_n\}$ in X with $x_n \to x$ and $Tx_{n-1} \preceq Tx_n$, for all $n \geq 1$, one has $Tx_n \preceq Tx$. Also, for each sequence $\{x_n\}$ in X with $Tx_{n-1} \preceq Tx_n$ for all $n \geq 1$, there exists a natural number k such that $Tx_m \preceq Tx_n$ for all $m > n \geq k$. Then T has an endpoint if and only if T has the approximate endpoint property.*

Proof. Define $\beta(A,B)=1$ whenever $A \preceq B$ and $\beta(A,B)=0$ otherwise, and then we use Theorem 3. □

Corollary 14. *Let $(X,d,\leq)$ be a complete ordered metric space and T a closed and bounded valued multifunction on X such that T has the property (H) and $Tx \preceq Ty$ for all subsets A and B of X with $A \preceq B$, all $x \in A$, and $y \in B$. Assume that $H(Tx,Ty) \leq \psi(N(x,y))$ for all $x,y \in X$ with $Tx \preceq Ty$, where $\psi : [0,+\infty) \to [0,+\infty)$ is a nondecreasing upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. Suppose that there exist a subset A_0 of X and $x_0 \in A_0$ such that $A_0 \preceq Tx_0$. Assume that for each convergent sequence $\{x_n\}$ in X with $x_n \to x$ and $Tx_{n-1} \preceq Tx_n$, for all $n \geq 1$, one has $Tx_n \preceq Tx$. Also, for each sequence $\{x_n\}$ in X with $Tx_{n-1} \preceq Tx_n$ for all $n \geq 1$, there exists a natural number k such that $Tx_m \preceq Tx_n$ for all $m > n \geq k$. If T has the approximate endpoint property and $A \preceq B$ for all subsets A and B of X with $A \nsubseteq B$ or $B \nsubseteq A$, then T has a unique endpoint.*

Proof. Define $\beta(A,B)=1$ whenever $A \preceq B$ and $\beta(A,B)=0$ otherwise, and then we use Corollary 6. □

Let (X,d) be a metric space and $T : X \to 2^X$ a multifunction. We say that T is an HS-multifunction whenever for each $x \in X$ there exists $y \in Tx$ such that $H(Tx,Ty) = \sup_{b\in Ty} d(y,b)$. It is obvious that each HS-multifunction is an multifunction which has the property (H). Thus, one can conclude similar results to above ones for HS-multifunctions. Here, we provide some ones. Although by considering HS-multifunction we restrict ourselves, we obtain strange results with respect to above ones. One can prove the following by reading exactly the proofs of similar above results.

Theorem 15. *Let (X,d) be a complete metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping, and $T : X \to CB(X)$ a β-admissible, β-generalized weak contractive HS-multifunction which has the properties (R) and (K). Suppose that there exist a subset A of X and $x_0 \in A$ such that $\beta(A,Tx_0) \geq 1$. Then T has an endpoint, and so T has the approximate endpoint property.*

Theorem 16. *Let (X,d) be a complete metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping, and $T : X \to CB(X)$ a lower semicontinuous, β-admissible, and β-generalized weak contractive HS-multifunction which has the property (K). Suppose that there exist a subset A of X and $x_0 \in A$ such that $\beta(A,Tx_0) \geq 1$. Then T has an endpoint, and so T has the approximate endpoint property.*

The next result is a consequence of Theorem 15.

Corollary 17. *Let (X,d) be a complete metric space, $x^* \in X$ a fixed element, and $T : X \to CB(X)$ an HS-multifunction such that $x^* \in Tx \cap Ty$ for all subsets A and B of X with $x^* \in A \cap B$, all $x \in A$, and $y \in B$. Assume that $H(Tx,Ty) \leq \psi(N(x,y))$ for all $x,y \in X$ with $x^* \in Tx \cap Ty$, where $\psi : [0,+\infty) \to [0,+\infty)$ is a nondecreasing upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. Suppose that there exist a subset A_0 of X and $x_0 \in A_0$ such that $x^* \in A_0 \cap Tx_0$. Assume that for each convergent sequence $\{x_n\}$ in X with $x_n \to x$ and $x^* \in Tx_{n-1} \cap Tx_n$ for all $n \geq 1$ one has $x^* \in Tx_n \cap Tx$. Also, for each sequence $\{x_n\}$ in X with $x^* \in Tx_{n-1} \cap Tx_n$ for all $n \geq 1$, there exists a natural number k such that $x^* \in Tx_m \cap Tx_n$ for all $m > n \geq k$. Then T has an endpoint, and so T has the approximate endpoint property.*

The next result is a consequence of Theorem 16.

Corollary 18. *Let $(X,d,\leq)$ be a complete ordered metric space and T a closed and bounded valued lower semicontinuous HS-multifunction on X such that $Tx \preceq Ty$ for all subsets A and B of X with $A \preceq B$, all $x \in A$, and $y \in B$. Assume that $H(Tx,Ty) \leq \psi(N(x,y))$ for all $x,y \in X$ with $Tx \preceq Ty$, where $\psi : [0,+\infty) \to [0,+\infty)$ is a nondecreasing upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. Suppose that there exist a subset A_0 of X and $x_0 \in A_0$ such that $A_0 \preceq Tx_0$. Assume that for each sequence $\{x_n\}$ in X with $Tx_{n-1} \preceq Tx_n$ for all $n \geq 1$, there exists a natural number k such that $Tx_m \preceq Tx_n$ for all $m > n \geq k$. Then T has an endpoint, and so T has the approximate endpoint property.*

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References

[1] A. Amini-Harandi, "Endpoints of set-valued contractions in metric spaces," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 72, no. 1, pp. 132–134, 2010.

[2] R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*, Springer, New York, NY, USA, 2009.

[3] S. Moradi and F. Khojasteh, "Endpoints of multi-valued generalized weak contraction mappings," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 74, no. 6, pp. 2170–2174, 2011.

[4] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for α-ψ-contractive type mappings," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[5] D. Baleanu, H. Mohammadi, and Sh. Rezapour, "Some existence results on nonlinear fractional differential equations," *Philosophical Transactions of the Royal Society*, vol. 371, Article ID 20120144, 2013.

[6] M. Jleli and B. Samet, "Best proximity points for α-ψ-proximal contractive type mappings and applications," *Bulletin des Sciences Mathématiques*, 2013.

[7] M. A. Miandaragh, M. Postolache, and Sh. Rezapour, "Some approximate fixed point results for generalized-contractive mappings ," To appear in. *Scientific Bulletin-University Politehnica of Bucharest A*.

[8] Sh. Rezapour and J. H. Asl, "A simple method for obtaining coupled fixed points of α-ψ-contractive type mappings," *International Journal of Analysis*, Article ID 438029, 7 pages, 2013.

[9] H. Alikhani, V. Rakočević, Sh. Rezapour, and N. Shahzad, "Fixed points of proximinal valued β-ψ-contractive multifunctions," To appear in. *Journal of Nonlinear and Convex Analysis*.

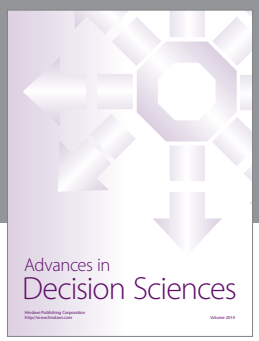
[10] H. Alikhani, Sh. Rezapour, and N. Shahzad, "Fixed points of a new type contractive mappings and multifunctions," To appear in. *Filomat*.

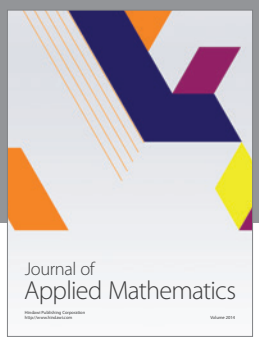
[11] S. M. A. Aleomraninejad, Sh. Rezapour, and N. Shahzad, "On fixed points of α-ψ-contractive multifunctions," *Fixed Point Theory and Applications*, vol. 2012, article 212, 8 pages, 2012.

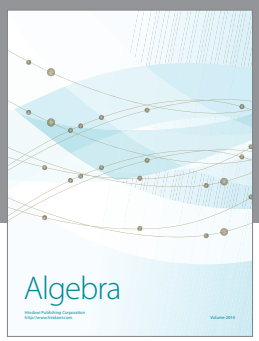
[12] B. Mohammadi, Sh. Rezapour, and N. Shahzad, "Some results on fixed points of α-ψ-ciric generalized multifunctions," *Fixed Point Theory and Applications*, vol. 2013, article 24, 2013.

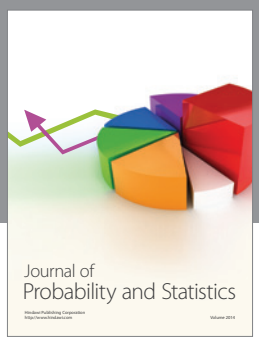
[13] S. M. A. Aleomraninejad, Sh. Rezapour, and N. Shahzad, "Some fixed point results on a metric space with a graph," *Topology and Its Applications*, vol. 159, no. 3, pp. 659–663, 2012.

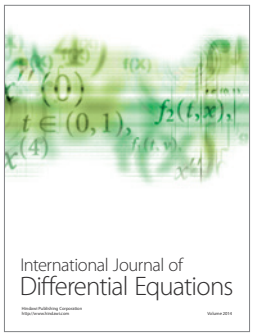
[14] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization: Theory and Examples*, Springer, New York, NY, USA, 2000.











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