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*Research Article*

# An Iterative Algorithm for the Reflexive Solution of the General Coupled Matrix Equations

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The general coupled matrix equations (including the generalized coupled Sylvester matrix equations as special cases) have numerous applications in control and system theory. In this paper, an iterative algorithm is constructed to solve the general coupled matrix equations over reflexive matrix solution. When the general coupled matrix equations are consistent over reflexive matrices, the reflexive solution can be determined automatically by the iterative algorithm within finite iterative steps in the absence of round-off errors. The least Frobenius norm reflexive solution of the general coupled matrix equations can be derived when an appropriate initial matrix is chosen. Furthermore, the unique optimal approximation reflexive solution to a given matrix group in Frobenius norm can be derived by finding the least-norm reflexive solution of the corresponding general coupled matrix equations. A numerical example is given to illustrate the effectiveness of the proposed iterative algorithm.

## 1. Introduction

Let $P \in \mathscr{R}^{n\times n}$ be a generalized reflection matrix; that is, $P^T = P$ and $P^2 = I$. A matrix $A \in \mathscr{R}^{n\times n}$ is called reflexive with respect to the matrix $P$ if $PAP = A$. The set of all $n$-by-$n$ reflexive matrices with respect to the generalized reflection matrix $P$ is denoted by $\mathscr{R}_r^{n\times n}(P)$. Let $\mathscr{R}^{m\times n}$ denote the set of all $m \times n$ real matrices. We denote by the superscript $T$ the transpose of a matrix. In matrix space $\mathscr{R}^{m\times n}$, define inner product as; $\langle A, B\rangle = \operatorname{tr}(B^T A)$ for all $A, B \in \mathscr{R}^{m\times n}$; $\|A\|_F$ represents the Frobenius norm of $A$. $\mathscr{R}(A)$ represents the column space of $A$. $\operatorname{vec}(\cdot)$ represents the vector operator; that is, $\operatorname{vec}(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T \in \mathscr{R}^{mn}$ for the matrix $A = (a_1, a_2, \ldots, a_n) \in \mathscr{R}^{m\times n}$, $a_i \in R^m$, $i = 1, 2, \ldots, n$. $A \otimes B$ stands for the Kronecker product of matrices $A$ and $B$.

In this paper, we will consider the following two problems.

*Problem 1.* Let $P_j \in \mathscr{R}^{n_j\times n_j}$ be generalized reflection matrices. For given matrices $A_{ij} \in \mathscr{R}^{r_i\times n_j}$, $B_{ij} \in \mathscr{R}^{n_j\times s_i}$, and $M_i \in \mathscr{R}^{r_i\times s_i}$, find reflexive matrix solution group $(X_1, X_2, \ldots, X_q)$ with $X_j \in \mathscr{R}_r^{n_j\times n_j}(P_j)$ such that

$$\sum_{j=1}^{q} A_{ij} X_j B_{ij} = M_i, \quad i = 1, 2, \ldots, p. \tag{1}$$

*Problem 2.* When Problem 1 is consistent, let $S_E$ denote the set of the reflexive solution group of Problem 1; that is,

$$S_E = \left\{ \left(X_1, X_2, \ldots, X_q\right) \mid \sum_{j=1}^{q} A_{ij} X_j B_{ij} = M_i, \right. \\ \left. i = 1, 2, \ldots, p,\ X_j \in \mathscr{R}_r^{n_j\times n_j}\left(P_j\right) \right\}. \tag{2}$$

For a given reflexive matrix group

$$\left(X_1^0, X_2^0, \ldots, X_q^0\right) \in \mathscr{R}_r^{n_1\times n_1}\left(P_1\right) \\ \times \mathscr{R}_r^{n_2\times n_2}\left(P_2\right) \times \cdots \times \mathscr{R}_r^{n_q\times n_q}\left(P_q\right), \tag{3}$$

Find $(\widehat{X}_1, \widehat{X}_2, \dots, \widehat{X}_q) \in S_E$ such that

$$\sum_{j=1}^{q} \left\| \widehat{X}_j - X_j^0 \right\|^2 = \min_{(X_1, X_2, \dots, X_q) \in S_E} \left\{ \sum_{j=1}^{q} \left\| X_j - X_j^0 \right\|^2 \right\}. \tag{4}$$

The general coupled matrix equations (1) (including the generalized coupled Sylvester matrix equations as special cases) may arise in many areas of control and system theory.

Many theoretical and numerical results on (1) and some of its special cases have been obtained. Least-squares-based iterative algorithms are very important in system identification, parameter estimation, and signal processing, including the recursive least squares (RLS) and iterative least squares (ILS) methods for solving the solutions of some matrix equations, for example, the Lyapunov matrix equation, Sylvester matrix equations, and coupled matrix equations as well. For example, novel gradient-based iterative (GI) method [1–5] and least-squares-based iterative methods [3, 4, 6] with highly computational efficiencies for solving (coupled) matrix equations are presented and have good stability performances, based on the hierarchical identification principle, which regards the unknown matrix as the system parameter matrix to be identified. Ding and Chen [1] presented the gradient-based iterative algorithms by applying the gradient search principle and the hierarchical identification principle for (1) with $q = p$. Wu et al. [7, 8] gave the finite iterative solutions to coupled Sylvester-conjugate matrix equations. Wu et al. [9] gave the finite iterative solutions to a class of complex matrix equations with conjugate and transpose of the unknowns. Jonsson and Kågström [10, 11] proposed recursive block algorithms for solving the coupled Sylvester matrix equations and the generalized Sylvester and Lyapunov Matrix equations. By extending the idea of conjugate gradient method, Dehghan and Hajarian [12] constructed an iterative algorithm to solve (1) with $q = p$ over generalized bisymmetric matrices. Very recently, Huang et al. [13] presented a finite iterative algorithms for the one-sided and generalized coupled Sylvester matrix equations over generalized reflexive solutions. Yin et al. [14] presented a finite iterative algorithms for the two-sided and generalized coupled Sylvester matrix equations over reflexive solutions. For more results, we refer to [15–28]. However, to our knowledge, the reflexive solution to the general coupled matrix equations (1) and the optimal approximation reflexive solution have not been derived. In this paper, we will consider the reflexive solution of (1) and the optimal approximation reflexive solution.

This paper is organized as follows. In Section 2, we will solve Problem 1 by constructing an iterative algorithm. The convergence of the proposed algorithm is proved. For any arbitrary initial matrix group, we can obtain a reflexive solution group of Problem 1 within finite iteration steps in the absence of round-off errors. Furthermore, for a special initial matrix group, we can obtain the least Frobenius norm solution of Problem 1. Then in Section 3, we give the optimal approximate solution group of Problem 2 by finding the least Frobenius norm reflexive solution group of the corresponding general coupled matrix equations. In Section 4, a numerical example is given to illustrate the effectiveness of our method. At last, some conclusions are drawn in Section 5.

## 2. An Iterative Algorithm for Solving Problem 1

In this section, we will first introduce an iterative algorithm to solve Problem 1 then prove its convergence. We will also give the least-norm reflexive solution of Problem 1 when an appropriate initial iterative matrix group is chosen.

*Algorithm 3.*

*Step 1.* Input matrices $A_{ij} \in \mathscr{R}^{r_i \times n_j}$, $B_{ij} \in \mathscr{R}^{n_j \times s_i}$, $M_i \in \mathscr{R}^{r_i \times s_i}$, and generalized reflection matrices $P_j \in \mathscr{R}^{n_j \times n_j}$, $i = 1, \dots, p$, $j = 1, \dots, q$.

*Step 2.* Choose an arbitrary matrix group

$$\left(X_1(1), X_2(1), \dots, X_q(1)\right) \in \mathscr{R}_r^{n_1 \times n_1}(P_1) \times \mathscr{R}_r^{n_2 \times n_2}(P_2) \times \cdots \times \mathscr{R}_r^{n_q \times n_q}(P_q). \tag{5}$$

Compute

$$\begin{aligned} R(1) &= \operatorname{diag}\left( M_1 - \sum_{l=1}^{q} A_{1l} X_l(1) B_{1l}, M_2 - \sum_{l=1}^{q} A_{2l} X_l(1) B_{2l}, \dots, M_p - \sum_{l=1}^{q} A_{pl} X_l(1) B_{pl} \right), \\ S_j(1) &= \frac{1}{2}\left[ \sum_{i=1}^{p} A_{ij}^T \left( M_i - \sum_{l=1}^{q} A_{il} X_l(1) B_{il} \right) B_{ij}^T + \sum_{i=1}^{p} P_j A_{ij}^T \left( M_i - \sum_{l=1}^{q} A_{il} X_l(1) B_{il} \right) B_{ij}^T P_j \right], \\ & k := 1. \end{aligned} \tag{6}$$

*Step 3.* If $R(k) = 0$, then stop and $(X_1(k), X_2(k), \dots, X_q(k))$ is the solution group of (1); elseif $R(k) \neq 0$, but $S_j(k) = 0$, $j = 1, \dots, q$, then stop and (1) are not consistent over reflexive matrix group; else $k := k + 1$.

*Step 4.* Compute

$$\begin{aligned} X_j(k) &= X_j(k-1) + \frac{\|R(k-1)\|_F^2}{\sum_{l=1}^{q} \|S_l(k-1)\|_F^2} \times S_j(k-1), \quad j = 1, \dots, q, \\ R(k) &= \operatorname{diag}\left( M_1 - \sum_{l=1}^{q} A_{1l} X_l(k) B_{1l}, M_2 - \sum_{l=1}^{q} A_{2l} X_l(k) B_{2l}, \dots, M_p - \sum_{l=1}^{q} A_{pl} X_l(k) B_{pl} \right) \end{aligned}$$

$$= R(k-1) - \frac{\|R(k-1)\|_F^2}{\sum_{l=1}^{q}\|S_l(k-1)\|_F^2} \cdot \operatorname{diag}\left(\sum_{l=1}^{q} A_{1l}S_l(k-1)B_{1l}, \sum_{l=1}^{q} A_{2l}S_l(k-1)B_{2l}, \ldots, \times \sum_{l=1}^{q} A_{pl}S_l(k-1)B_{pl}\right),$$

$$S_j(k) = \frac{1}{2}\left[\sum_{i=1}^{p} A_{ij}^T\left(M_i - \sum_{l=1}^{q} A_{il}X_l(k)B_{il}\right)B_{ij}^T + \sum_{i=1}^{p} P_jA_{ij}^T\left(M_i - \sum_{l=1}^{q} A_{il}X_l(k)B_{il}\right)B_{ij}^TP_j\right] + \frac{\|R(k)\|_F^2}{\|R(k-1)\|_F^2}S_j(k-1). \tag{7}$$

*Step 5*. Go to Step 3.

Obviously, it can be seen that $X_j(k), S_j(k) \in R_r^{n_j\times n_j}(P_j)$ for all $j = 1, \ldots, q$ and $k = 1, 2, \ldots$.

**Lemma 4.** *For the sequences $\{R(k)\}, \{S_j(k)\}$ $(j = 1, 2, \ldots, q)$ generated by Algorithm 3, and $m \geq 2$, we have*

$$\operatorname{tr}\left((R(s))^T R(t)\right) = 0, \qquad \sum_{j=1}^{q}\operatorname{tr}\left(\left(S_j(s)\right)^T S_j(t)\right) = 0, \qquad s, t = 1, 2, \ldots, m, s \neq t. \tag{8}$$

The proof of Lemma 4 is presented in the appendix.

**Lemma 5.** *Suppose that $(X_1^*, X_2^*, \ldots, X_q^*)$ is an arbitrary reflexive solution group of Problem 1; then for any initial reflexive matrix group $(X_1(1), X_2(1), \ldots, X_q(1))$, one has*

$$\sum_{j=1}^{q}\operatorname{tr}\left(\left(X_j^* - X_j(k)\right)^T S_j(k)\right) = \|R(k)\|_F^2, \quad k = 1, 2, \ldots, \tag{9}$$

*where the sequences $\{X_j(k)\}$, $\{S_j(k)\}$, and $\{R(k)\}$ are generated by Algorithm 3.*

The proof of Lemma 5 is presented in the appendix.

*Remark 6.* If there exists a positive number $k$ such that $S_j(k) = 0$, $j = 1, 2, \ldots, q$ but $R(k) \neq 0$, then, by Lemma 5, we get that (1) are not consistent over reflexive matrices.

**Theorem 7.** *Suppose that Problem 1 is consistent; then for an arbitrary initial matrix group $(X_1, X_2, \ldots, X_q)$ with $X_j \in \mathscr{R}_r^{n_j\times n_j}(P_j)$, a reflexive solution group of Problem 1 can be obtained with finite iteration steps in the absence of round-off errors.*

*Proof.* If $R(k) \neq 0$, $k = 1, 2, \ldots, m = \sum_{i=1}^{p} r_i s_i$, then by Lemma 5 and Remark 6 we have $S_j(k) \neq 0$ for all $j = 1, 2, \ldots, q$ and $k = 1, 2, \ldots, m$. Thus we can compute $R(m+1)$ and $(X_1(m+1), X_2(m+1), \ldots, X_q(m+1))$ by Algorithm 3.

By Lemma 4, we have

$$\operatorname{tr}\left((R(m+1))^T R(k)\right) = 0, \quad k = 1, 2, \ldots, m,$$
$$\operatorname{tr}\left((R(k))^T R(l)\right) = 0, \quad k, l = 1, 2, \ldots, m, \ k \neq l. \tag{10}$$

It can be seen that the set of $R(1), R(2), \ldots, R(m)$ is an orthogonal basis of the matrix subspace

$$S = \left\{L \mid L = \operatorname{diag}\left(L_1, L_2, \ldots, L_p\right), \ L_i \in \mathscr{R}^{r_i\times s_i}, \ i = 1, 2 \ldots, p\right\}, \tag{11}$$

which implies that $R(m+1) = 0$; that is, $(X_1(m+1), X_2(m+1), \ldots, X_q(m+1))$ with $X_j(m+1) \in \mathscr{R}_r^{n_j\times n_j}(P_j)$ is a reflexive solution group of Problem 1. This completes the proof. □

To show the least Frobenius norm reflexive solution of Problem 1, we first introduce the following result.

**Lemma 8** (see [20, Lemma 2.4]). *Suppose that the consistent system of linear equation $Ax = b$ has a solution $x^* \in R(A^T)$; then $x^*$ is a unique least Frobenius norm solution of the system of linear equation.*

By Lemma 8, the following result can be obtained.

**Theorem 9.** *Suppose that Problem 1 is consistent. If one chooses the initial iterative matrices $X_j(1) = \sum_{i=1}^{p} A_{ij}^T K_i B_{ij}^T + \sum_{i=1}^{p} P_j A_{ij}^T K_i B_{ij}^T P_j$, $j = 1, 2, \ldots, q$, where $K_i \in \mathscr{R}^{r_i\times s_i}$, $i = 1, 2, \ldots, p$ are arbitrary matrices, especially, $X_j(1) = 0 \in \mathscr{R}^{n_j\times n_j}(P_j)$, then the solution group $(X_1^*, X_2^*, \ldots, X_q^*)$ generated by Algorithm 3 is the unique least Frobenius norm reflexive solution group of Problem 1.*

*Proof.* We know that the solvability of (1) over reflexive matrices is equivalent to the following matrix equations:

$$\sum_{j=1}^{q} A_{ij}X_jB_{ij} = M_i \quad (i = 1, 2, \ldots, p),$$
$$\sum_{j=1}^{q} A_{ij}P_jX_jP_jB_{ij} = M_i \quad (i = 1, 2, \ldots, p). \tag{12}$$

Then the system of matrix equations (12) is equivalent to

$$\begin{pmatrix} B_{11}^T \otimes A_{11} & \cdots & B_{1q}^T \otimes A_{1q} \\ \vdots & \cdots & \vdots \\ B_{p1}^T \otimes A_{p1} & \cdots & B_{pq}^T \otimes A_{pq} \\ B_{11}^T P_1 \otimes A_{11} P_1 & \cdots & B_{1q}^T P_q \otimes A_{1q} P_q \\ \vdots & \cdots & \vdots \\ B_{p1}^T P_1 \otimes A_{p1} P_1 & \cdots & B_{pq}^T P_q \otimes A_{pq} P_q \end{pmatrix} \times \begin{pmatrix} \operatorname{vec}(X_1) \\ \vdots \\ \operatorname{vec}(X_q) \end{pmatrix} = \begin{pmatrix} \operatorname{vec}(M_1) \\ \vdots \\ \operatorname{vec}(M_p) \\ \operatorname{vec}(M_1) \\ \vdots \\ \operatorname{vec}(M_p) \end{pmatrix}. \tag{13}$$

Let $X_j(1) = \sum_{i=1}^{p} A_{ij}^T K_i B_{ij}^T + \sum_{i=1}^{p} P_j A_{ij}^T K_i B_{ij}^T P_j$, $j = 1, 2, \ldots, q$, where $K_i \in \mathscr{R}^{r_i \times s_i}$ are arbitrary matrices; then

$$\begin{aligned} &\begin{pmatrix} \operatorname{vec}(X_1(1)) \\ \vdots \\ \operatorname{vec}(X_q(1)) \end{pmatrix} \\ &= \begin{pmatrix} \operatorname{vec}\left(\sum_{i=1}^{p} A_{i1}^T K_i B_{i1}^T + \sum_{i=1}^{p} P_1 A_{i1}^T K_i B_{i1}^T P_1\right) \\ \vdots \\ \operatorname{vec}\left(\sum_{i=1}^{p} A_{iq}^T K_i B_{iq}^T + \sum_{i=1}^{p} P_q A_{iq}^T K_i B_{iq}^T P_q\right) \end{pmatrix} \\ &= \begin{pmatrix} B_{11} \otimes A_{11}^T & \cdots & B_{p1} \otimes A_{p1}^T & P_1 B_{11} \otimes P_1 A_{11}^T & \cdots & P_1 B_{p1} \otimes P_1 A_{p1}^T \\ \vdots & \cdots & \vdots & \cdots & \cdots & \vdots \\ B_{1q} \otimes A_{1q}^T & \cdots & B_{pq} \otimes A_{pq}^T & P_q B_{1q} \otimes P_q A_{1q}^T & \cdots & P_q B_{pq} \otimes P_q A_{pq}^T \end{pmatrix} \begin{pmatrix} \operatorname{vec}(K_1) \\ \vdots \\ \operatorname{vec}(K_p) \\ \operatorname{vec}(K_1) \\ \vdots \\ \operatorname{vec}(K_p) \end{pmatrix} \\ &= \begin{pmatrix} B_{11}^T \otimes A_{11} & \cdots & B_{1q}^T \otimes A_{1q} \\ \vdots & \vdots & \vdots \\ B_{p1}^T \otimes A_{p1} & \cdots & B_{pq}^T \otimes A_{pq} \\ B_{11}^T P_1 \otimes A_{11} P_1 & \cdots & B_{1q}^T P_q \otimes A_{1q} P_q \\ \vdots & \vdots & \vdots \\ B_{p1}^T P_1 \otimes A_{p1} P_1 & \cdots & B_{pq}^T P_q \otimes A_{pq} P_q \end{pmatrix}^T \begin{pmatrix} \operatorname{vec}(K_1) \\ \vdots \\ \operatorname{vec}(K_p) \\ \operatorname{vec}(K_1) \\ \vdots \\ \operatorname{vec}(K_p) \end{pmatrix} \\ &\in R\left(\begin{pmatrix} B_{11}^T \otimes A_{11} & \cdots & B_{1q}^T \otimes A_{1q} \\ \vdots & \vdots & \vdots \\ B_{p1}^T \otimes A_{p1} & \cdots & B_{pq}^T \otimes A_{pq} \\ B_{11}^T P_1 \otimes A_{11} P_1 & \cdots & B_{1q}^T P_q \otimes A_{1q} P_q \\ \vdots & \vdots & \vdots \\ B_{p1}^T P_1 \otimes A_{p1} P_1 & \cdots & B_{pq}^T P_q \otimes A_{pq} P_q \end{pmatrix}^T\right). \end{aligned} \tag{14}$$

Furthermore, we can see that all reflexive matrix solution groups $(X_1(k), X_2(k), \ldots, X_q(k))$ generated by Algorithm 3 satisfy

$$\begin{pmatrix} \operatorname{vec}\left(X_1(1)\right) \\ \vdots \\ \operatorname{vec}\left(X_q(1)\right) \end{pmatrix} \in R\left(\begin{pmatrix} B_{11}^T \otimes A_{11} & \cdots & B_{1q}^T \otimes A_{1q} \\ \vdots & \vdots & \vdots \\ B_{p1}^T \otimes A_{p1} & \cdots & B_{pq}^T \otimes A_{pq} \\ B_{11}^T P_1 \otimes A_{11} P_1 & \cdots & B_{1q}^T P_q \otimes A_{1q} P_q \\ \vdots & \vdots & \vdots \\ B_{p1}^T P_1 \otimes A_{p1} P_1 & \cdots & B_{pq}^T P_q \otimes A_{pq} P_q \end{pmatrix}^T\right); \tag{15}$$

by Lemma 8 we know that $(X_1^*, X_2^*, \ldots, X_q^*)$ is the least Frobenius norm reflexive solution group of the system of linear equation (13). Since vector operator is isomorphic, $(X_1^*, X_2^*, \ldots, X_q^*)$ is the unique least Frobenius norm reflexive solution group of the system of matrix equations (12). Thus $(X_1^*, X_2^*, \ldots, X_q^*)$ is the unique least Frobenius norm reflexive solution group of Problem 1. This completes the proof. □

## 3. The Solution of Problem 2

In this section, we will show that the reflexive solution group of Problem 2 to a given reflexive matrix group can be derived by finding the least Frobenius norm reflexive solution group of the corresponding general coupled matrix equations.

When Problem 1 is consistent, the set of the reflexive solution groups of Problem 1 denoted by $S_E$ is not empty. For a given matrix pair $(X_1^0, X_2^0, \ldots, X_q^0)$ with $X_j^0 \in \mathscr{R}_r^{n_j \times n_j}(P_j)$, $j = 1, 2, \ldots, q$, we have

$$\begin{aligned} \sum_{j=1}^{q} A_{ij} X_j B_{ij} = M_i \Longleftrightarrow \sum_{j=1}^{q} A_{ij} \left(X_j - X_j^0\right) B_{ij} \\ = M_i - \sum_{j=1}^{q} A_{ij} X_j^0 B_{ij}, \\ i = 1, 2, \ldots, p. \end{aligned} \tag{16}$$

Set $\widetilde{X}_j = X_j - X_j^0$ and $\widetilde{M}_i = M_i - \sum_{j=1}^{q} A_{ij} X_j^0 B_{ij}$; then solving Problem 2 is equivalent to finding the least Frobenius norm reflexive solution group $(\widetilde{X}_1^*, \widetilde{X}_2^*, \ldots, \widetilde{X}_q^*)$ of the corresponding general coupled matrix equations

$$\sum_{j=1}^{q} A_{ij} \widetilde{X}_j B_{ij} = \widetilde{M}_i, \quad i = 1, 2, \ldots, p. \tag{17}$$

By using Algorithm 3, let initial iteration matrices

$$\begin{aligned} \widetilde{X}_j(1) = \sum_{i=1}^{p} A_{ij}^T K_i B_{ij}^T \\ + \sum_{i=1}^{p} P_j A_{ij}^T K_i B_{ij}^T P_j, \quad j = 1, 2, \ldots, q, \end{aligned} \tag{18}$$

where $K_i \in \mathscr{R}^{r_i \times s_i}$, $i = 1, 2, \ldots, p$ are arbitrary matrices, especially, $\widetilde{X}_j(1) = 0 \in \mathscr{R}^{n_j \times n_j}(P_j)$, $j = 1, 2, \ldots, q$; then we can get the the least Frobenius norm reflexive solution group $(\widetilde{X}_1^*, \widetilde{X}_2^*, \ldots, \widetilde{X}_q^*)$ of (17). Thus the reflexive solution group of Problem 2 can be represented as

$$\left(\widehat{X}_1, \widehat{X}_2, \ldots, \widehat{X}_q\right) = \left(\widetilde{X}_1^* + X_1^0, \widetilde{X}_2^* + X_2^0, \ldots, \widetilde{X}_q^* + X_q^0\right). \tag{19}$$

## 4. A Numerical Example

In this section, we will show a numerical example to illustrate our results. All the tests are performed by MATLAB 7.8.

*Example 10.* Consider the reflexive solution of the general coupled matrix equations

$$A_{11} X_1 B_{11} + A_{12} X_2 B_{12} = M_1, A_{21} X_1 B_{21} + A_{22} X_2 B_{22} = M_2, \tag{20}$$

where

$$A_{11} = \begin{pmatrix} 1 & 3 & -5 & 7 & -9 \\ 2 & 0 & 4 & 6 & -1 \\ 0 & -2 & 9 & 6 & -8 \\ 3 & 6 & 2 & 2 & -3 \\ -5 & 5 & -22 & -1 & -11 \\ 8 & 4 & -6 & -9 & -9 \end{pmatrix}, \tag{21}$$

$$B_{11} = \begin{pmatrix} 3 & 5 & 6 & 7 \\ 4 & 8 & -5 & 4 \\ -1 & 5 & -2 & 3 \\ 3 & 9 & 2 & -6 \\ -2 & 7 & -8 & 1 \end{pmatrix}, \tag{22}$$

$$A_{12} = \begin{pmatrix} 6 & -5 & 7 & -9 \\ 2 & 4 & 6 & -11 \\ 9 & -12 & 3 & -8 \\ 13 & 6 & 4 & -15 \\ -5 & 15 & -13 & -11 \\ 2 & 9 & -6 & -9 \end{pmatrix},$$

$$B_{12} = \begin{pmatrix} 5 & 1 & 9 & -6 \\ -4 & 5 & -2 & 3 \\ 3 & -12 & 0 & 8 \\ -5 & 8 & -2 & 9 \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} 14 & 5 & -1 & 7 & 1 \\ -2 & 3 & -2 & 5 & 4 \\ 13 & 4 & 2 & -3 & 6 \\ -8 & 1 & -5 & 4 & 8 \end{pmatrix},$$

$$B_{21} = \begin{pmatrix} 6 & 5 & 2 & 3 & 7 \\ 1 & 3 & -5 & 8 & 2 \\ -11 & 5 & -6 & 2 & 5 \\ 13 & 2 & 7 & -9 & 7 \\ -9 & 6 & -5 & 12 & 1 \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} 1 & 2 & -5 & 8 \\ -5 & 5 & -7 & 3 \\ 2 & 4 & 9 & -6 \\ -3 & 7 & -12 & 11 \end{pmatrix},$$

$$B_{22} = \begin{pmatrix} 7 & -1 & 5 & -2 & 3 \\ 6 & 3 & 9 & 2 & -6 \\ 5 & -2 & 7 & -8 & 1 \\ 1 & 4 & -3 & -2 & 6 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} -406 & 123 & 16 & -74 \\ 79 & 290 & 408 & -71 \\ -891 & 597 & -664 & 720 \\ 6 & 205 & 147 & 349 \\ 651 & -2638 & 625 & -131 \\ 652 & -1923 & 634 & -106 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 2814 & -239 & 1455 & -1634 & 845 \\ 439 & 287 & 480 & -550 & 890 \\ 2500 & -126 & 1199 & -720 & 376 \\ -1000 & 630 & -266 & -24 & 1042 \end{pmatrix}. \tag{23}$$

Let

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{24}$$

be the generalized reflection matrices.

We will find the reflexive solution of the the general coupled matrix equations (20) by using Algorithm 3. It can be verified that the matrix equations (20) are consistent over reflexive matrices and the solution is

$$X_1^* = \begin{pmatrix} 3 & 0 & -6 & 3 & -4 \\ 4 & 3 & -6 & 4 & -2 \\ 0 & 2 & 4 & 0 & -2 \\ 3 & -4 & 6 & 3 & 0 \\ 4 & -2 & 6 & 4 & 3 \end{pmatrix},$$
$$X_2^* = \begin{pmatrix} -5 & 2 & -1 & 1 \\ 2 & -1 & 2 & -3 \\ -1 & -1 & -5 & -2 \\ -2 & -3 & -2 & -1 \end{pmatrix}. \tag{25}$$

Because of the influence of the error of calculation, the residual $R(k)$ is usually unequal to zero in the process of the iteration, where $k = 1, 2, \ldots$. For any chosen positive number $\varepsilon$, however small enough, for example, $\varepsilon = 1.0000e - 010$, whenever $\|R(k)\| < \varepsilon$, stop the iteration; $(X_1(k), X_2(k))$ is regarded to be the reflexive solution of the matrix equations (20). Choose an initially iterative matrix group $(X_1(1), X_2(1))$, such as

$$X_1(1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_2(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \tag{26}$$

by Algorithm 3, we have

$$\begin{aligned} X_1^* &= X_1(31) \\ &= \begin{pmatrix} 3.0000 & -0.0000 & -6.0000 & 3.0000 & -4.0000 \\ 4.0000 & 3.0000 & -6.0000 & 4.0000 & -2.0000 \\ 0.0000 & 2.0000 & 4.0000 & -0.0000 & -2.0000 \\ 3.0000 & -4.0000 & 6.0000 & 3.0000 & -0.0000 \\ 4.0000 & -2.0000 & 6.0000 & 4.0000 & 3.0000 \end{pmatrix}, \\ X_2^* &= X_2(31) = \begin{pmatrix} -5.0000 & 2.0000 & -1.0000 & 1.0000 \\ 2.0000 & -1.0000 & 2.0000 & -3.0000 \\ -1.0000 & -1.0000 & -5.0000 & -2.0000 \\ -2.0000 & -3.0000 & -2.0000 & -1.0000 \end{pmatrix}, \\ &\|R(31)\| = 3.1869e - 011 < \varepsilon. \end{aligned} \tag{27}$$

So we obtain the reflexive solution of the matrix equations (20). The relative error of the solution and the residual are shown in Figure 1, where the relative error $REk = (\|X_1(k) - X_1^*\| + \|X_2(k) - X_2^*\|)/(\|X_1^*\| + \|X_2^*\|)$ and the residual $Rk = \|R(k)\|$.

Let $S_E$ denote the set of all reflexive solution group of the matrix equations (20). For two given reflexive matrices,

$$X_1^0 = \begin{pmatrix} 2 & 3 & -5 & 3 & 3 \\ -1 & 3 & 3 & -5 & 2 \\ 5 & -2 & 2 & -5 & 2 \\ 3 & 3 & 5 & 2 & 3 \\ -5 & 2 & -3 & -1 & 3 \end{pmatrix},$$
$$X_2^0 = \begin{pmatrix} -3 & -3 & 4 & 2 \\ 0 & 1 & 1 & 2 \\ 4 & -2 & -3 & 3 \\ -1 & 2 & 0 & 1 \end{pmatrix}, \tag{28}$$

we will find $(\widehat{X}_1, \widehat{X}_2) \in S_E$, such that

$$\begin{aligned} &\left\|\widehat{X}_1 - X_1^0\right\| + \left\|\widehat{X}_2 - X_2^0\right\| \\ &\quad = \min_{(X_1, X_2) \in S_E} \left\|X_1 - X_1^0\right\| + \left\|X_2 - X_1^0\right\|; \end{aligned} \tag{29}$$

that is, find the optimal approximate reflexive solution group to the given matrix group $(X_1^0, X_2^0)$ in $S_E$ in Frobenius norm.

Let $\widetilde{X}_1 = X_1 - X_1^0$, $\widetilde{X}_2 = X_2 - X_2^0$, $\widetilde{M}_1 = M_1 - A_{11}X_1^0B_{11} - A_{12}X_2^0B_{12}$, $\widetilde{M}_2 = M_2 - A_{21}X_1^0B_{21} - A_{22}X_2^0B_{22}$, by the method mentioned in Section 3, we can obtain the least-norm reflexive solution group $(\widetilde{X}_1^*, \widetilde{X}_2^*)$ of the matrix equations $A_{11}\widetilde{X}_1B_{11} + A_{12}\widetilde{X}_2B_{12} = \widetilde{M}_1$ and $A_{21}\widetilde{X}_1B_{21} + A_{22}\widetilde{X}_2B_{22} = \widetilde{M}_2$ by choosing the initially iterative matrices $\widetilde{X}_1(1) = 0$ and $\widetilde{X}_2(1) = 0$; then by Algorithm 3 we have that

$$\widetilde{X}_1^* = \widetilde{X}_1(29) = \begin{pmatrix} 1.0000 & -3.0000 & -1.0000 & -0.0000 & -7.0000 \\ 5.0000 & 0.0000 & -9.0000 & 9.0000 & -4.0000 \\ -5.0000 & 4.0000 & 2.0000 & 5.0000 & -4.0000 \\ -0.0000 & -7.0000 & 1.0000 & 1.0000 & -3.0000 \\ 9.0000 & -4.0000 & 9.0000 & 5.0000 & -0.0000 \end{pmatrix},$$

$$\widetilde{X}_2^* = \widetilde{X}_2(29) = \begin{pmatrix} -2.0000 & 5.0000 & -5.0000 & -1.0000 \\ 2.0000 & -2.0000 & 1.0000 & -5.0000 \\ -5.0000 & 1.0000 & -2.0000 & -5.0000 \\ -1.0000 & -5.0000 & -2.0000 & -2.0000 \end{pmatrix},$$

$$\|R(30)\| = 3.6134e-011 < \varepsilon = 1.0000e-010, \tag{30}$$

and the optimal approximate reflexive solution to the matrix group $(X_1^0, X_2^0)$ in Frobenius norm are

$$\widehat{X}_1 = \widetilde{X}_1^* + X_1^0 = \begin{pmatrix} 3.0000 & 0.0000 & -6.0000 & 3.0000 & -4.0000 \\ 4.0000 & 3.0000 & -6.0000 & 4.0000 & -2.0000 \\ 0.0000 & 2.0000 & 4.0000 & -0.0000 & -2.0000 \\ 3.0000 & -4.0000 & 6.0000 & 3.0000 & 0.0000 \\ 4.0000 & -2.0000 & 6.0000 & 4.0000 & 3.0000 \end{pmatrix},$$

$$\widehat{X}_2 = \widetilde{X}_2^* + X_2^0 = \begin{pmatrix} -5.0000 & 2.0000 & -1.0000 & 1.0000 \\ 2.0000 & -1.0000 & 2.0000 & -3.0000 \\ -1.0000 & -1.0000 & -5.0000 & -2.0000 \\ -2.0000 & -3.0000 & -2.0000 & -1.0000 \end{pmatrix}. \tag{31}$$

The relative error and the residual of the solution are shown in Figure 2, where the relative error $REk = (\|\widetilde{X}_1(k) + X_1^0 - X_1^*\| + \|\widetilde{X}_2(k) + X_2^0 - X_2^*\|)/(\|X_1^*\| + \|X_2^*\|)$ and the residual $Rk = \|R(k)\|$.

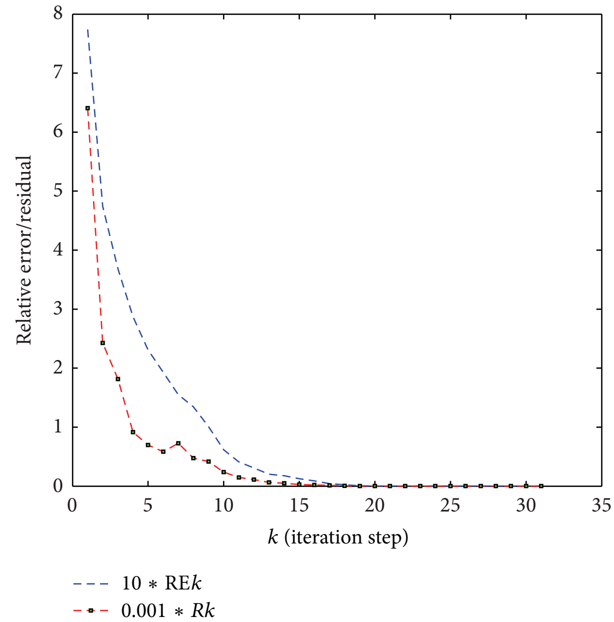


Figure 1: The relative error of the solutions and the residual for Example 10 with $X_1(1) = 0$ and $X_2(1) = 0$.

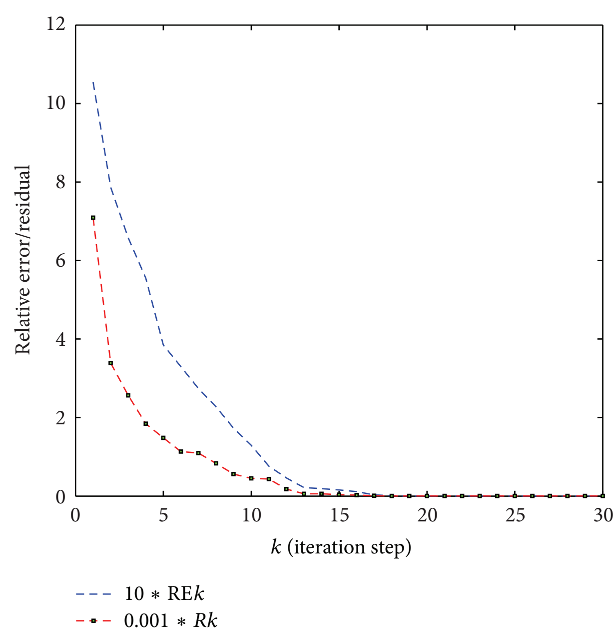


Figure 2: The relative error of the solutions and the residual for Example 10 with $X_1^0$ and $X_2^0$.

## 5. Conclusions

In this paper, an iterative algorithm is presented to solve the general coupled matrix equations $\sum_{j=1}^{q} A_{ij}X_jB_{ij} = M_i$ $(i = 1, 2, \ldots, p)$ over reflexive matrices. When the general coupled matrix equations are consistent over reflexive matrices, for any initially reflexive matrix group, the reflexive solution group can be obtained by the iterative algorithm within finite iterative steps in the absence of round-off errors. When a special kind of initial iteration matrix group is given, the unique least-norm reflexive solution of the general coupled matrix equations can be derived. Furthermore, the optimal approximate reflexive solution of the general coupled matrix equations to a given reflexive matrix group can be derived by finding the least-norm reflexive solution of new corresponding general coupled matrix equations. Finally, a numerical example is given in Section 4 to illustrate that our iterative algorithm is quite effective.

## Appendices

### A. The Proof of Lemma 4

Since $\mathrm{tr}((R(s))^T R(t)) = \mathrm{tr}((R(t))^T R(s))$ and $\mathrm{tr}((S_j(s))^T S_j(t)) = \mathrm{tr}((S_j(t))^T S_j(s))$ for all $s, t = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, q$, we only need to prove that

$$\mathrm{tr}\left((R(s))^T R(t)\right) = 0, \quad \sum_{j=1}^{q} \mathrm{tr}\left(\left(S_j(s)\right)^T S_j(t)\right) = 0, \quad 1 \le t < s \le m. \tag{A.1}$$

We prove the conclusion by induction, and two steps are required.

*Step 1.* we will show that

$$\mathrm{tr}\left((R(k+1))^T R(k)\right) = 0, \quad \sum_{j=1}^{q} \mathrm{tr}\left(\left(S_j(k+1)\right)^T S_j(k)\right) = 0, \quad k = 1, 2, \ldots, m-1. \tag{A.2}$$

To prove this conclusion, we also use induction.

For $k = 1$, by Algorithm 3, we have that

$$\begin{aligned}
&\mathrm{tr}\left((R(2))^T R(1)\right)\\
&= \mathrm{tr}\Bigg(\Bigg[R(1) - \frac{\|R(1)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(1)\right\|_F^2} \times \mathrm{diag}\Bigg(\sum_{j=1}^{q} A_{1j} S_j(1) B_{1j}, \sum_{j=1}^{q} A_{2j} S_j(1) B_{2j}, \ldots, \sum_{j=1}^{q} A_{pj} S_j(1) B_{pj}\Bigg)\Bigg]^T R(1)\Bigg)\\
&= \|R(1)\|_F^2 - \frac{\|R(1)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(1)\right\|_F^2} \times \mathrm{tr}\Bigg(\Bigg[\mathrm{diag}\Bigg(\sum_{j=1}^{q} A_{1j} S_j(1) B_{1j}, \sum_{j=1}^{q} A_{2j} S_j(1) B_{2j}, \ldots, \sum_{j=1}^{q} A_{pj} S_j(1) B_{pj}\Bigg)\Bigg]^T\\
&\qquad \cdot \mathrm{diag}\Bigg(M_1 - \sum_{l=1}^{q} A_{1l} X_l(1) B_{1l}, M_2 - \sum_{l=1}^{q} A_{2l} X_l(1) B_{2l}, \ldots, M_p - \sum_{l=1}^{q} A_{pl} X_l(1) B_{pl}\Bigg)\Bigg)\\
&= \|R(1)\|_F^2 - \frac{\|R(1)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(1)\right\|_F^2} \times \mathrm{tr}\Bigg(\mathrm{diag}\Bigg(\Bigg(\sum_{j=1}^{q} A_{1j} S_j(1) B_{1j}\Bigg)^T \times \Bigg(M_1 - \sum_{l=1}^{q} A_{1l} X_l(1) B_{1l}\Bigg),\\
&\qquad \Bigg(\sum_{j=1}^{q} A_{2j} S_j(1) B_{2j}\Bigg)^T \times \Bigg(M_2 - \sum_{l=1}^{q} A_{2l} X_l(1) B_{2l}\Bigg), \ldots, \times \Bigg(\sum_{j=1}^{q} A_{pj} S_j(1) B_{pj}\Bigg)^T \times \Bigg(M_p - \sum_{l=1}^{q} A_{pl} X_l(1) B_{pl}\Bigg)\Bigg)\Bigg)\\
&= \|R(1)\|_F^2 - \frac{\|R(1)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(1)\right\|_F^2} \times \mathrm{tr}\Bigg(\sum_{i=1}^{p}\Bigg(\Bigg(\sum_{j=1}^{q} B_{ij}^T \left(S_j(1)\right)^T A_{ij}^T\Bigg) \times \Bigg(M_i - \sum_{l=1}^{q} A_{il} X_l(1) B_{il}\Bigg)\Bigg)\Bigg)\\
&= \|R(1)\|_F^2 - \frac{\|R(1)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(1)\right\|_F^2} \times \mathrm{tr}\Bigg(\sum_{j=1}^{q}\left(S_j(1)\right)^T \Bigg[\sum_{i=1}^{p} A_{ij}^T \Bigg(M_i - \sum_{l=1}^{q} A_{il} X_l(1) B_{il}\Bigg) B_{ij}^T\Bigg]\Bigg)\\
&= \|R(1)\|_F^2 - \frac{\|R(1)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(1)\right\|_F^2} \times \mathrm{tr}\Bigg(\sum_{j=1}^{q}\left(S_j(1)\right)^T \times \Bigg[\frac{\sum_{i=1}^{p} A_{ij}^T \left(M_i - \sum_{l=1}^{q} A_{il} X_l(1) B_{il}\right) B_{ij}^T}{2}\\
&\qquad + \frac{\sum_{i=1}^{p} P_j A_{ij}^T \left(M_i - \sum_{l=1}^{q} A_{il} X_l(1) B_{il}\right) B_{ij}^T P_j}{2}\Bigg]\Bigg)
\end{aligned}$$

$$= \|R(1)\|_F^2 - \frac{\|R(1)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(1)\right\|_F^2} \times \operatorname{tr}\left(\sum_{j=1}^{q}\left(S_j(1)\right)^T S_j(1)\right) = 0,$$

$$\begin{aligned}
&\sum_{j=1}^{q}\operatorname{tr}\left(\left(S_j(2)\right)^T S_j(1)\right) \\
&= \sum_{j=1}^{q}\operatorname{tr}\left(\left[\frac{1}{2}\left(\sum_{i=1}^{p}A_{ij}^T\left(M_i-\sum_{l=1}^{q}A_{il}X_l(2)B_{il}\right)B_{ij}^T + \sum_{i=1}^{p}P_jA_{ij}^T\left(M_i-\sum_{l=1}^{q}A_{il}X_l(2)B_{il}\right) \times B_{ij}^TQ_j\right) + \frac{\|R(2)\|_F^2}{\|R(1)\|_F^2}S_j(1)\right]^T S_j(1)\right) \\
&= \sum_{j=1}^{q}\operatorname{tr}\left(\left(\sum_{i=1}^{p}A_{ij}^T\left(M_i-\sum_{l=1}^{q}A_{il}X_l(2)B_{il}\right)B_{ij}^T + \frac{\|R(2)\|_F^2}{\|R(1)\|_F^2}S_j(1)\right)^T S_j(1)\right) \\
&= \sum_{j=1}^{q}\operatorname{tr}\left(\left(S_j(1)\right)^T\sum_{i=1}^{p}A_{ij}^T\left(M_i-\sum_{l=1}^{q}A_{il}X_l(2)B_{il}\right)B_{ij}^T\right) + \frac{\|R(2)\|_F^2}{\|R(1)\|_F^2}\sum_{j=1}^{q}\operatorname{tr}\left(\left(S_j(1)\right)^T S_j(1)\right) \\
&= \sum_{i=1}^{p}\operatorname{tr}\left(\sum_{j=1}^{q}\left(M_i-\sum_{l=1}^{q}A_{il}X_l(2)B_{il}\right)^T A_{ij}S_j(1)B_{ij}\right) + \frac{\|R(2)\|_F^2}{\|R(1)\|_F^2}\sum_{j=1}^{q}\left\|S_j(1)\right\|_F^2 \\
&= \operatorname{tr}\left(\operatorname{diag}\left(\left(M_1-\sum_{l=1}^{q}A_{1l}X_l(2)B_{1l}\right)^T, \left(M_2-\sum_{l=1}^{q}A_{2l}X_l(2)B_{2l}\right)^T, \dots, \left(M_p-\sum_{l=1}^{q}A_{pl}X_l(2)B_{pl}\right)^T\right)\right. \\
&\quad \left. \times \operatorname{diag}\left(\sum_{j=1}^{q}A_{1j}S_j(1)B_{1j}, \sum_{j=1}^{q}A_{2j}S_j(1)B_{2j}, \dots, \sum_{j=1}^{q}A_{pj}S_j(1)B_{pj}\right)\right) + \frac{\|R(2)\|_F^2}{\|R(1)\|_F^2}\sum_{j=1}^{q}\left\|S_j(1)\right\|_F^2 \\
&= \frac{\sum_{j=1}^{q}\left\|S_j(1)\right\|_F^2}{\|R(1)\|_F^2}\operatorname{tr}\left(\left(R(2)\right)^T\left(R(1)-R(2)\right)\right) + \frac{\|R(2)\|_F^2}{\|R(1)\|_F^2}\sum_{j=1}^{q}\left\|S_j(1)\right\|_F^2 \\
&= \frac{\sum_{j=1}^{q}\left\|S_j(1)\right\|_F^2}{\|R(1)\|_F^2}\operatorname{tr}\left(\left(R(2)\right)^T R(1)\right) = 0.
\end{aligned} \tag{A.3}$$

Assume that (A.2) holds for $k = m - 1$; that is,

$$\begin{gathered}
\operatorname{tr}\left(\left(R(m)\right)^T R(m-1)\right) = 0, \\
\sum_{j=1}^{q}\operatorname{tr}\left(\left(S_j(m)\right)^T S_j(m-1)\right) = 0.
\end{gathered} \tag{A.4}$$

When $k = m$, we have that

$$\begin{aligned}
&\operatorname{tr}\left(\left(R(m+1)\right)^T R(m)\right) \\
&= \operatorname{tr}\left(\left[R(m) - \frac{\|R(m)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(m)\right\|_F^2} \times \operatorname{diag}\left(\sum_{j=1}^{q}A_{1j}S_j(m)B_{1j}, \sum_{j=1}^{q}A_{2j}S_j(m)B_{2j}, \dots, \sum_{j=1}^{q}A_{pj}S_j(m)B_{pj}\right)\right]^T R(m)\right) \\
&= \|R(m)\|_F^2 - \frac{\|R(m)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(m)\right\|_F^2} \times \operatorname{tr}\left(\left[\operatorname{diag}\left(\sum_{j=1}^{q}A_{1j}S_j(m)B_{1j}, \sum_{j=1}^{q}A_{2j}S_j(m)B_{2j}, \dots, \sum_{j=1}^{q}A_{pj}S_j(m)B_{pj}\right)\right]^T\right. \\
&\quad \left. \cdot \operatorname{diag}\left(M_1-\sum_{l=1}^{q}A_{1l}X_l(m)B_{1l}, M_2-\sum_{l=1}^{q}A_{2l}X_l(m)B_{2l}, \dots, M_p-\sum_{l=1}^{q}A_{pl}X_l(m)B_{pl}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \|R(m)\|_F^2 - \frac{\|R(m)\|_F^2}{\sum_{j=1}^{q} \left\|S_j(m)\right\|_F^2} \\
&\quad \times \operatorname{tr}\left( \operatorname{diag}\left( \left( \sum_{j=1}^{q} A_{1j} S_j(m) B_{1j} \right)^T \right.\right. \\
&\qquad\qquad \times \left( M_1 - \sum_{l=1}^{q} A_{1l} X_l(m) B_{1l} \right), \\
&\qquad\qquad \left( \sum_{j=1}^{q} A_{2j} S_j(m) B_{2j} \right)^T \\
&\qquad\qquad \times \left( M_2 - \sum_{l=1}^{q} A_{2l} X_l(m) B_{2l} \right), \ldots, \\
&\qquad\qquad \left( \sum_{j=1}^{q} A_{pj} S_j(m) B_{pj} \right)^T \\
&\qquad\qquad \left.\left. \times \left( M_p - \sum_{l=1}^{q} A_{pl} X_l(m) B_{pl} \right) \right) \right) \\
&= \|R(m)\|_F^2 - \frac{\|R(m)\|_F^2}{\sum_{j=1}^{q} \left\|S_j(m)\right\|_F^2} \\
&\quad \times \operatorname{tr}\left( \sum_{i=1}^{p} \left( \left( \sum_{j=1}^{q} B_{ij}^T \left(S_j(m)\right)^T A_{ij}^T \right) \right.\right. \\
&\qquad\qquad \left.\left. \times \left( M_i - \sum_{l=1}^{q} A_{il} X_l(m) B_{il} \right) \right) \right) \\
&= \|R(m)\|_F^2 - \frac{\|R(m)\|_F^2}{\sum_{j=1}^{q} \left\|S_j(m)\right\|_F^2} \\
&\quad \times \operatorname{tr}\left( \sum_{j=1}^{q} \left(S_j(m)\right)^T \right. \\
&\qquad\qquad \left. \times \left[ \sum_{i=1}^{p} A_{ij}^T \left( M_i - \sum_{l=1}^{q} A_{il} X_l(m) B_{il} \right) B_{ij}^T \right] \right) \\
&= \|R(m)\|_F^2 - \frac{\|R(m)\|_F^2}{\sum_{j=1}^{q} \left\|S_j(m)\right\|_F^2} \\
&\quad \times \operatorname{tr}\left( \sum_{j=1}^{q} \left(S_j(m)\right)^T \right. \\
&\qquad \times \left[ \frac{\sum_{i=1}^{p} A_{ij}^T \left(M_i - \sum_{l=1}^{q} A_{il} X_l(m) B_{il}\right) B_{ij}^T}{2} \right. \\
&\qquad \left.\left. + \frac{\sum_{i=1}^{p} P_j A_{ij}^T \left(M_i - \sum_{l=1}^{q} A_{il} X_l(m) B_{il}\right) B_{ij}^T P_j}{2} \right] \right) \\
&= \|R(m)\|_F^2 - \frac{\|R(m)\|_F^2}{\sum_{j=1}^{q} \left\|S_j(m)\right\|_F^2} \\
&\quad \times \operatorname{tr}\left( \sum_{j=1}^{q} \left(S_j(m)\right)^T \left( S_j(m) - \frac{\|R(m)\|_F^2}{\|R(m-1)\|_F^2} S_j(m-1) \right) \right) = 0,
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
&\sum_{j=1}^{q} \operatorname{tr}\left( \left(S_j(m+1)\right)^T S_j(m) \right) \\
&= \sum_{j=1}^{q} \operatorname{tr}\left( \left[ \frac{1}{2} \left( \sum_{i=1}^{p} A_{ij}^T \left( M_i - \sum_{l=1}^{q} A_{il} X_l(m+1) B_{il} \right) B_{ij}^T \right.\right.\right. \\
&\qquad\qquad + \sum_{i=1}^{p} P_j A_{ij}^T \left( M_i - \sum_{l=1}^{q} A_{il} X_l(m+1) B_{il} \right) \\
&\qquad\qquad \left. \times B_{ij}^T P_j \right) \\
&\qquad\qquad \left.\left. + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} S_j(m) \right]^T S_j(m) \right) \\
&= \sum_{j=1}^{q} \operatorname{tr}\left( \left( \sum_{i=1}^{p} A_{ij}^T \left( M_i - \sum_{l=1}^{q} A_{il} X_l(m+1) B_{il} \right) B_{ij}^T \right.\right. \\
&\qquad\qquad \left.\left. + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} S_j(m) \right)^T S_j(m) \right) \\
&= \sum_{j=1}^{q} \operatorname{tr}\left( \left(S_j(m)\right)^T \sum_{i=1}^{p} A_{ij}^T \left( M_i - \sum_{l=1}^{q} A_{il} X_l(m+1) B_{il} \right) B_{ij}^T \right) \\
&\quad + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} \sum_{j=1}^{q} \operatorname{tr}\left( \left(S_j(m)\right)^T S_j(m) \right) \\
&= \sum_{i=1}^{p} \operatorname{tr}\left( \sum_{j=1}^{q} \left( M_i - \sum_{l=1}^{q} A_{il} X_l(m+1) B_{il} \right)^T A_{ij} S_j(m) B_{ij} \right) \\
&\quad + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} \sum_{j=1}^{q} \left\|S_j(m)\right\|_F^2 \\
&= \operatorname{tr}\left( \operatorname{diag}\left( \left( M_1 - \sum_{l=1}^{q} A_{1l} X_l(m+1) B_{1l} \right)^T, \right.\right. \\
&\qquad\qquad \left( M_2 - \sum_{l=1}^{q} A_{2l} X_l(m+1) B_{2l} \right)^T, \ldots, \\
&\qquad\qquad \left.\left( M_p - \sum_{l=1}^{q} A_{pl} X_l(m+1) B_{pl} \right)^T \right) \\
&\qquad \times \operatorname{diag}\left( \sum_{j=1}^{q} A_{1j} S_j(m) B_{1j}, \sum_{j=1}^{q} A_{2j} S_j(m) B_{2j}, \ldots, \right.
\end{aligned}
$$

$$\sum_{j=1}^{q} A_{pj} S_j(m) B_{pj} \Bigg)\Bigg)$$

$$+ \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} \sum_{j=1}^{q} \left\|S_j(m)\right\|_F^2$$

$$= \frac{\sum_{j=1}^{q} \left\|S_j(m)\right\|_F^2}{\|R(m)\|_F^2}$$

$$\times \operatorname{tr}\left((R(m+1))^T (R(m) - R(m+1))\right)$$

$$+ \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} \sum_{j=1}^{q} \left\|S_j(m)\right\|_F^2$$

$$= \frac{\sum_{j=1}^{q} \left\|S_j(m)\right\|_F^2}{\|R(m)\|_F^2} \operatorname{tr}\left((R(m+1))^T R(m)\right) = 0. \tag{A.6}$$

Hence, (A.2) holds for $k = m$. Therefore, (A.2) holds by the principle of induction.

*Step 2.* We show that

$$\operatorname{tr}\left((R(k+1))^T R(t)\right) = 0,$$
$$\sum_{j=1}^{q} \operatorname{tr}\left(\left(S_j(k+1)\right)^T S_j(t)\right) = 0, \tag{A.7}$$
$$t = 1, 2, \ldots, k, \quad \forall k \geq 1.$$

When $k = 1$, (A.7) holds.

Assume that

$$\operatorname{tr}\left((R(k))^T R(t)\right) = 0, \qquad \sum_{j=1}^{q} \operatorname{tr}\left(\left(S_j(k)\right)^T S_j(t)\right) = 0,$$
$$t = 1, 2, \ldots, k-1, \quad \forall k \geq 2; \tag{A.8}$$

then we show that

$$\operatorname{tr}\left((R(k+1))^T R(t)\right) = 0,$$
$$\sum_{j=1}^{q} \operatorname{tr}\left(\left(S_j(k+1)\right)^T S_j(t)\right) = 0, \tag{A.9}$$
$$t = 1, 2, \ldots, k.$$

In fact, we have that

$$\operatorname{tr}\left((R(k+1))^T R(t)\right)$$

$$= \operatorname{tr}\Bigg(\Bigg[R(k) - \frac{\|R(k)\|_F^2}{\sum_{j=1}^{q} \left\|S_j(k)\right\|_F^2}$$

$$\times \operatorname{diag}\left(\sum_{j=1}^{q} A_{1j} S_j(k) B_{1j}, \sum_{j=1}^{q} A_{2j} S_j(k) B_{2j}, \ldots, \right.$$

$$\left.\sum_{j=1}^{q} A_{pj} S_j(k) B_{pj}\right)\Bigg]^T R(t)\Bigg)$$

$$= \operatorname{tr}\left((R(k))^T R(t)\right) - \frac{\|R(k)\|_F^2}{\sum_{j=1}^{q} \left\|S_j(k)\right\|_F^2}$$

$$\times \operatorname{tr}\Bigg(\Bigg[\operatorname{diag}\Bigg(\sum_{j=1}^{q} A_{1j} S_j(k) B_{1j},$$

$$\sum_{j=1}^{q} A_{2j} S_j(k) B_{2j}, \ldots, \sum_{j=1}^{q} A_{pj} S_j(k) B_{pj}\Bigg)\Bigg]^T$$

$$\cdot \operatorname{diag}\Bigg(M_1 - \sum_{l=1}^{q} A_{1l} X_l(t) B_{1l},$$

$$M_2 - \sum_{l=1}^{q} A_{2l} X_l(t) B_{2l}, \ldots,$$

$$M_p - \sum_{l=1}^{q} A_{pl} X_l(t) B_{pl}\Bigg)\Bigg)$$

$$= -\frac{\|R(k)\|_F^2}{\sum_{j=1}^{q} \left\|S_j(k)\right\|_F^2}$$

$$\times \operatorname{tr}\Bigg(\operatorname{diag}\Bigg(\left(\sum_{j=1}^{q} A_{1j} S_j(k) B_{1j}\right)^T$$

$$\times \left(M_1 - \sum_{l=1}^{q} A_{1l} X_l(t) B_{1l}\right),$$

$$\left(\sum_{j=1}^{q} A_{2j} S_j(k) B_{2j}\right)^T$$

$$\times \left(M_2 - \sum_{l=1}^{q} A_{2l} X_l(t) B_{2l}\right), \ldots,$$

$$\left(\sum_{j=1}^{q} A_{pj} S_j(k) B_{pj}\right)^T$$

$$\times \left(M_p - \sum_{l=1}^{q} A_{pl} X_l(t) B_{pl}\right)\Bigg)\Bigg)$$

$$= -\frac{\|R(k)\|_F^2}{\sum_{j=1}^{q} \left\|S_j(k)\right\|_F^2}$$

$$\times \operatorname{tr}\left(\sum_{i=1}^{p}\left(\left(\sum_{j=1}^{q}B_{ij}^{T}\left(S_j(k)\right)^{T}A_{ij}^{T}\right)\right.\right.$$
$$\left.\left.\times\left(M_i-\sum_{l=1}^{q}A_{il}X_l(t)B_{il}\right)\right)\right)$$
$$=-\frac{\|R(k)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(k)\right\|_F^2}$$
$$\times\operatorname{tr}\left(\sum_{j=1}^{q}\left(S_j(k)\right)^{T}\left[\sum_{i=1}^{p}A_{ij}^{T}\left(M_i-\sum_{l=1}^{q}A_{il}X_l(t)B_{il}\right)B_{ij}^{T}\right]\right)$$
$$=-\frac{\|R(k)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(k)\right\|_F^2}$$
$$\times\operatorname{tr}\left(\sum_{j=1}^{q}\left(S_j(k)\right)^{T}\right.$$
$$\times\left[\frac{\sum_{i=1}^{p}A_{ij}^{T}\left(M_i-\sum_{l=1}^{q}A_{il}X_l(t)B_{il}\right)B_{ij}^{T}}{2}\right.$$
$$\left.\left.+\frac{\sum_{i=1}^{p}P_jA_{ij}^{T}\left(M_i-\sum_{l=1}^{q}A_{il}X_l(t)B_{il}\right)B_{ij}^{T}P_j}{2}\right]\right)$$
$$=-\frac{\|R(k)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(k)\right\|_F^2}$$
$$\times\operatorname{tr}\left(\sum_{j=1}^{q}\left(S_j(k)\right)^{T}\left(S_j(t)-\frac{\|R(t)\|_F^2}{\|R(t-1)\|_F^2}S_j(t-1)\right)\right)$$
$$=-\frac{\|R(k)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(k)\right\|_F^2}\sum_{j=1}^{q}\operatorname{tr}\left(\left(S_j(k)\right)^{T}S_j(t)\right)$$
$$+\frac{\|R(k)\|_F^2}{\sum_{j=1}^{q}\left\|S_j(k)\right\|_F^2}\frac{\|R(t)\|_F^2}{\|R(t-1)\|_F^2}$$
$$\times\sum_{j=1}^{q}\operatorname{tr}\left(\left(S_j(k)\right)^{T}S_j(t-1)\right)=0. \tag{A.10}$$

From the above results, we have $\operatorname{tr}(R(k+1)^{T}R(t+1))=0$, $t=1,2,\ldots,k-1$, and

$$\sum_{j=1}^{q}\operatorname{tr}\left(\left(S_j(k+1)\right)^{T}S_j(t)\right)$$
$$=\sum_{j=1}^{q}\operatorname{tr}\left(\left[\frac{1}{2}\left(\sum_{i=1}^{p}A_{ij}^{T}\left(M_i-\sum_{l=1}^{q}A_{il}X_l(k+1)B_{il}\right)B_{ij}^{T}\right.\right.\right.$$
$$\left.+\sum_{i=1}^{p}P_jA_{ij}^{T}\left(M_i-\sum_{l=1}^{q}A_{il}X_l(k+1)B_{il}\right)B_{ij}^{T}P_j\right)$$
$$\left.\left.+\frac{\|R(k+1)\|_F^2}{\|R(k)\|_F^2}S_j(k)\right]^{T}S_j(t)\right)$$
$$=\sum_{j=1}^{q}\operatorname{tr}\left(\left(\left(\sum_{i=1}^{p}A_{ij}^{T}\left(M_i-\sum_{l=1}^{q}A_{il}X_l(k+1)B_{il}\right)B_{ij}^{T}\right.\right.\right.$$
$$\left.\left.+\frac{\|R(k+1)\|_F^2}{\|R(k)\|_F^2}S_j(k)\right)^{T}S_j(t)\right)$$
$$=\sum_{j=1}^{q}\operatorname{tr}\left(\left(S_j(t)\right)^{T}\sum_{i=1}^{p}A_{ij}^{T}\left(M_i-\sum_{l=1}^{q}A_{il}X_l(k+1)B_{il}\right)B_{ij}^{T}\right)$$
$$+\frac{\|R(k+1)\|_F^2}{\|R(k)\|_F^2}\sum_{j=1}^{q}\operatorname{tr}\left(\left(S_j(k)\right)^{T}S_j(t)\right)$$
$$=\sum_{i=1}^{p}\operatorname{tr}\left(\sum_{j=1}^{q}\left(M_i-\sum_{l=1}^{q}A_{il}X_l(k+1)B_{il}\right)^{T}A_{ij}S_j(t)B_{ij}\right)$$
$$=\operatorname{tr}\left(\operatorname{diag}\left(\left(M_1-\sum_{l=1}^{q}A_{1l}X_l(k+1)B_{1l}\right)^{T},\right.\right.$$
$$\left(M_2-\sum_{l=1}^{q}A_{2l}X_l(k+1)B_{2l}\right)^{T},\ldots,$$
$$\left.\left(M_p-\sum_{l=1}^{q}A_{pl}X_l(k+1)B_{pl}\right)^{T}\right)$$
$$\times\operatorname{diag}\left(\sum_{j=1}^{q}A_{1j}S_j(t)B_{1j},\right.$$
$$\left.\left.\sum_{j=1}^{q}A_{2j}S_j(t)B_{2j},\ldots,\sum_{j=1}^{q}A_{pj}S_j(t)B_{pj}\right)\right)$$
$$=\frac{\sum_{j=1}^{q}\left\|S_j(t)\right\|_F^2}{\|R(t)\|_F^2}$$
$$\times\operatorname{tr}\left(\left(R(k+1)\right)^{T}\left(R(t)-R(t+1)\right)\right)$$
$$=\frac{\sum_{j=1}^{q}\left\|S_j(t)\right\|_F^2}{\|R(t)\|_F^2}$$
$$\times\operatorname{tr}\left(\left(R(k+1)\right)^{T}R(t)\right)=0. \tag{A.11}$$

By the principle of induction, (A.7) holds.

Note that (A.1) is implied in Steps 1 and 2 by the principle of induction. This completes the proof.

## B. The Proof of Lemma 5

We prove the conclusion by induction for the positive integer $k$.

For $k = 1$, we have that

$$\begin{aligned}
&\sum_{j=1}^{q} \operatorname{tr}\left(\left(X_j^* - X_j(1)\right)^T S_j(1)\right) \\
&= \sum_{j=1}^{q} \operatorname{tr}\Bigg(\left(X_j^* - X_j(1)\right)^T \\
&\qquad \times \Bigg[\frac{1}{2}\Bigg(\sum_{i=1}^{p} A_{ij}^T\left(M_i - \sum_{l=1}^{q} A_{il} X_l(1) B_{il}\right) B_{ij}^T \\
&\qquad + \sum_{i=1}^{p} P_j A_{ij}^T\left(M_i - \sum_{l=1}^{q} A_{il} X_l(1) B_{il}\right) B_{ij}^T P_j\Bigg)\Bigg]\Bigg) \\
&= \sum_{j=1}^{q} \operatorname{tr}\Bigg(\left(X_j^* - X_j(1)\right)^T \\
&\qquad \times \left[\sum_{i=1}^{p} A_{ij}^T\left(M_i - \sum_{l=1}^{q} A_{il} X_l(1) B_{il}\right) B_{ij}^T\right]\Bigg) \\
&= \sum_{i=1}^{p} \operatorname{tr}\Bigg(\left(M_i - \sum_{l=1}^{q} A_{il} X_l(1) B_{il}\right)^T \\
&\qquad \times \sum_{j=1}^{q} A_{ij}\left(X_j^* - X_j(1)\right) B_{ij}\Bigg) \\
&= \operatorname{tr}\Bigg(\operatorname{diag}\Bigg(\left(M_1 - \sum_{l=1}^{q} A_{1l} X_l(1) B_{1l}\right)^T, \\
&\qquad \left(M_2 - \sum_{l=1}^{q} A_{2l} X_l(1) B_{2l}\right)^T, \dots, \\
&\qquad \left(M_p - \sum_{l=1}^{q} A_{pl} X_l(1) B_{pl}\right)^T\Bigg) \\
&\qquad \times \operatorname{diag}\Bigg(\sum_{j=1}^{q} A_{1j}\left(X_j^* - X_j(1)\right) B_{1j}, \\
&\qquad \sum_{j=1}^{q} A_{2j}\left(X_j^* - X_j(1)\right) B_{2j}, \dots, \\
&\qquad \sum_{j=1}^{q} A_{pj}\left(X_j^* - X_j(1)\right) B_{pj}\Bigg)\Bigg) \\
&= \operatorname{tr}\Bigg(\operatorname{diag}\Bigg(\left(M_1 - \sum_{l=1}^{q} A_{1l} X_l(1) B_{1l}\right)^T, \\
&\qquad \left(M_2 - \sum_{l=1}^{q} A_{2l} X_l(1) B_{2l}\right)^T, \dots, \\
&\qquad \left(M_p - \sum_{l=1}^{q} A_{pl} X_l(1) B_{pl}\right)^T\Bigg) \\
&\qquad \times \operatorname{diag}\Bigg(M_1 - \sum_{j=1}^{q} A_{1j} X_j(1) B_{1j}, \\
&\qquad M_2 - \sum_{j=1}^{q} A_{2j} X_j(1) B_{2j}, \dots, \\
&\qquad M_p - \sum_{j=1}^{q} A_{pj} X_j(1) B_{pj}\Bigg)\Bigg) \\
&= \|R(1)\|^2.
\end{aligned} \tag{B.1}$$

Assume that (9) holds for $k = m$. When $k = m + 1$, by Algorithm 3, we have that

$$\begin{aligned}
&\sum_{j=1}^{q} \operatorname{tr}\left(\left(X_j^* - X_j(m+1)\right)^T S_j(m+1)\right) \\
&= \sum_{j=1}^{q} \operatorname{tr}\Bigg(\left(X_j^* - X_j(m+1)\right)^T \\
&\qquad \times \Bigg[\frac{1}{2}\Bigg(\sum_{i=1}^{p} A_{ij}^T\left(M_i - \sum_{l=1}^{q} A_{il} X_l(m+1) B_{il}\right) B_{ij}^T \\
&\qquad + \sum_{i=1}^{p} P_j A_{ij}^T\left(M_i - \sum_{l=1}^{q} A_{il} X_l(m+1) B_{il}\right) B_{ij}^T P_j\Bigg) \\
&\qquad + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} S_j(m)\Bigg]\Bigg) \\
&= \sum_{j=1}^{q} \operatorname{tr}\Bigg(\left(X_j^* - X_j(m+1)\right)^T \\
&\qquad \times \left[\sum_{i=1}^{p} A_{ij}^T\left(M_i - \sum_{l=1}^{q} A_{il} X_l(m+1) B_{il}\right) B_{ij}^T\right]\Bigg) \\
&\quad + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} \\
&\quad \times \sum_{j=1}^{q} \operatorname{tr}\left(\left(X_j^* - X_j(m+1)\right)^T S_j(m)\right) \\
&= \sum_{i=1}^{p} \operatorname{tr}\Bigg(\left(M_i - \sum_{l=1}^{q} A_{il} X_l(m+1) B_{il}\right)^T
\end{aligned}$$

$$\times \sum_{j=1}^{q} A_{ij} \left( X_j^* - X_j (m+1) \right) B_{ij} \Bigg)$$

$$+ \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} \sum_{j=1}^{q} \operatorname{tr}\left( \left( X_j^* - X_j(m) \right)^T S_j(m) \right)$$

$$- \frac{\|R(m+1)\|_F^2}{\sum_{j=1}^{q} \left\| S_j(m) \right\|_F^2} \sum_{j=1}^{q} \operatorname{tr}\left( \left( S_j(m) \right)^T S_j(m) \right)$$

$$= \operatorname{tr}\Bigg( \operatorname{diag}\Bigg( \left( M_1 - \sum_{l=1}^{q} A_{1l} X_l (m+1) B_{1l} \right)^T, \left( M_2 - \sum_{l=1}^{q} A_{2l} X_l (m+1) B_{2l} \right)^T, \ldots, \left( M_p - \sum_{l=1}^{q} A_{pl} X_l (m+1) B_{pl} \right)^T \Bigg)$$

$$\times \operatorname{diag}\Bigg( \sum_{j=1}^{q} A_{1j} \left( X_j^* - X_j (m+1) \right) B_{1j}, \sum_{j=1}^{q} A_{2j} \left( X_j^* - X_j (m+1) \right) B_{2j}, \ldots, \sum_{j=1}^{q} A_{pj} \left( X_j^* - X_j (m+1) \right) B_{pj} \Bigg) \Bigg)$$

$$+ \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} \|R(m)\|_F^2$$

$$- \frac{\|R(m+1)\|_F^2}{\sum_{j=1}^{q} \left\| S_j(m) \right\|_F^2} \sum_{j=1}^{q} \left\| S_j(m) \right\|_F^2$$

$$= \operatorname{tr}\Bigg( \operatorname{diag}\Bigg( \left( M_1 - \sum_{l=1}^{q} A_{1l} X_l (m+1) B_{1l} \right)^T, \left( M_2 - \sum_{l=1}^{q} A_{2l} X_l (m+1) B_{2l} \right)^T, \ldots, \left( M_p - \sum_{l=1}^{q} A_{pl} X_l (m+1) B_{pl} \right)^T \Bigg)$$

$$\times \operatorname{diag}\Bigg( M_1 - \sum_{j=1}^{q} A_{1j} X_j (m+1) B_{1j}, M_2 - \sum_{j=1}^{q} A_{2j} X_j (m+1) B_{2j}, \ldots, M_p - \sum_{j=1}^{q} A_{pj} X_j (m+1) B_{pj} \Bigg) \Bigg)$$

$$+ \|R(m+1)\|_F^2 - \|R(m+1)\|_F^2$$

$$= \|R(m+1)\|_F^2. \tag{B.2}$$

Therefore, (9) holds for $k = m + 1$. Thus (9) holds by the principal of induction. This completes the proof.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper. The authors of the paper do not have a direct financial relation that might lead to a conflict of interests for any of the authors.

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