# Infinitely Many Weak Solutions of the $p$-Laplacian Equation with Nonlinear Boundary Conditions 

Feng-Yun Lu ${ }^{1,2}$ and Gui-Qian Deng ${ }^{1}$<br>${ }^{1}$ Xingyi Normal University for Nationalities, Xingyi, Guizhou 562400, China<br>${ }^{2}$ Human Resources and Social Security Bureau, Buyi and Miao Autonomous Prefecture in Southwest Guizhou, Guizhou 562400, China

Correspondence should be addressed to Feng-Yun Lu; lufengyun@gmail.com
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We study the following $p$-Laplacian equation with nonlinear boundary conditions: $-\Delta_{p} u+\mu(x)|u|^{p-2} u=f(x, u)+g(x, u), x \in$ $\Omega,|\nabla u|^{p-2} \partial u / \partial n=\eta|u|^{p-2} u$, and $x \in \partial \Omega$, where $\Omega$ is a bounded domain in $\mathbb{R}^{N}$ with smooth boundary $\partial \Omega$. We prove that the equation has infinitely many weak solutions by using the variant fountain theorem due to Zou (2001) and $f, g$ do not need to satisfy the (P.S) or (P.S*) condition.

## 1. Introduction

In this paper, we study the following $p$-Laplacian equation:

$$
\begin{align*}
-\Delta_{p} u+\mu(x)|u|^{p-2} u & =f(x, u)+g(x, u), \quad x \in \Omega \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} & =\eta|u|^{p-2} u, \quad x \in \partial \Omega \tag{1}
\end{align*}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^{N}$ with smooth boundary $\partial \Omega$ and $\partial / \partial n$ is the outer normal derivative, $-\Delta_{p} u=\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ is the $p$-Laplacian with $1<p<N$, $\eta$ is real parameter, and

$$
\begin{equation*}
\mu(x) \in L^{\infty}(\Omega) \text { satisfying ess } \inf _{x \in \bar{\Omega}} \mu(x)>0 \tag{2}
\end{equation*}
$$

The perturbation functions $f, g$ satisfy the following conditions:
(F1) $f, g \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ are odd in $u$;
(F2) there exist $\sigma, \delta \in(1, p), c_{1}>0, c_{2}>0, c_{3}>0$ such that

$$
\begin{equation*}
c_{1}|u|^{\sigma} \leq f(x, u) u \leq c_{2}|u|^{\sigma}+c_{3}|u|^{\delta} \tag{3}
\end{equation*}
$$

(F3) There exists $p<q<p^{*}$ (where $p^{*}=p N / N-p$ ) such that $|g(x, u)| \leq c\left(1+|u|^{q}\right)$ for a.e. $x \in \Omega$ and $u \in$ $\mathbb{R}$. Moreover, $\lim _{u \rightarrow 0} g(x, u) /|u|^{p-1}=0$ uniformly for $x \in \Omega$.
(F4) Assume that one of the following conditions hold:
(1) $\lim _{|u| \rightarrow \infty} g(x, u) /|u|^{p-2} u=0$ uniformly for $x \in$ $\Omega$;
(2) $\lim _{|u| \rightarrow \infty} g(x, u) /|u|^{p-2} u=-\infty$ uniformly for $x \quad \in \quad \Omega$; furthermore, $f(x, u) /|u|^{p-2} u$ and $g(x, u) /|u|^{p-2} u$ are decreasing in $u$ for $u$ is large enough;
(3) $\lim _{|u| \rightarrow \infty} g(x, u) /|u|^{p-2} u=\infty$ uniformly for $x \in \Omega ; g(x, u) /|u|^{p-2} u$ is increasing in $u$ for $u$ is large enough; moreover, there exists $\alpha>$ $\max \{\sigma, \delta\}$ such that
$\liminf _{|u| \rightarrow \infty} \frac{g(x, u) u-p G(x, u)}{|u|^{\alpha}} \geq c>0 \quad$ uniformly for $x \in \Omega$,
where $G(x, u)=\int_{0}^{u} g(x, t) d t$.
Remark 1. The above conditions were given in Zou [1] for the semilinear case $p=2$.

Remark 2. A simple example which satisfies (F1)-(F4) is

$$
\begin{equation*}
f(x, u)+g(x, u)=\mu|u|^{r-2} u+\gamma|u|^{s-2} u, \tag{5}
\end{equation*}
$$

where $1<r<p<s<p^{*}$.
Equation (1) is posed in the framework of the Sobolev space

$$
\begin{equation*}
W^{1, p}(\Omega)=\left\{u \in L^{p}(\Omega): \int_{\Omega}|\nabla u|^{p} d x<\infty\right\} \tag{6}
\end{equation*}
$$

with the norm

$$
\begin{equation*}
\|u\|=\left(\int_{\Omega}\left(|\nabla u|^{p}+\mu(x)|u|^{p}\right) d x\right)^{1 / p} \tag{7}
\end{equation*}
$$

The corresponding energy functional of (1) is defined by

$$
\begin{align*}
\Phi(u)= & \frac{1}{p} \int_{\Omega}\left(|\nabla u|^{p}+\mu(x)|u|^{p}\right) d x-\int_{\Omega} F(x, u) d x \\
& -\int_{\Omega} G(x, u) d x-\frac{\eta}{p} \int_{\partial \Omega}|u|^{p} d s \tag{8}
\end{align*}
$$

for $u \in W^{1, p}(\Omega)$, where $F(x, u)=\int_{0}^{u} f(x, t) d t$ and $d s$ is the measure on the boundary. It is easy to see that $\Phi \in$ $C^{1}\left(W^{1, p}(\Omega), \mathbb{R}\right)$ and

$$
\begin{align*}
& \left\langle\Phi^{\prime}(u), v\right\rangle \\
& \quad=\int_{\Omega}\left(|\nabla u|^{p-2} \nabla u \nabla v+\mu(x)|u|^{p-2} u v\right) d x \\
& \quad-\int_{\Omega} f(x, u) v d x-\int_{\Omega} g(x, u) v d x-\eta \int_{\partial \Omega}|u|^{p-2} u v d s, \tag{9}
\end{align*}
$$

for all $u, v \in W^{1, p}(\Omega)$. It is well-known that the weak solution of (1) corresponds to the critical point of the energy functional $\Phi$ on $W^{1, p}(\Omega)$.

Remark 3. Under condition (2), it is easy to check that norm (7) is equivalent to the usual one, that is, the norm defined in (7) with $\mu(x) \equiv 1$.

In [2], the author researched (1) $(\eta=0)$ and obtained the existence of infinitely many weak solutions. Moreover, the existence of three solutions for (1) $(\eta=0, p>N)$ was researched in [3] by using a three-critical-point theorem due to Ricceri [4]. Also, some authors researched and obtained the existence of infinitely many weak solution without requiring any symmetric conditions and also with discontinuous nonlinearities; see [5, 6]. Recently, this equation was studied by J.-H. Zhao and P.-H. Zhao [7] via Bartsch's dual fountain theorem in [8] and obtained the existence of infinitely many weak solutions for (1) under the case of Remark 2. They obtained the following theorem.

Theorem A. Let $f(x, t)+g(x, t)=\mu|u|^{r-2} u+\gamma|u|^{s-2} u$, where $1<r<p<s<p^{*}$. Then there exists a constant $\Lambda>$ 0 such that, for any $\eta<\Lambda$,
(1) for any $\gamma>, \mu \in \mathbb{R}$, (1) has a sequence of solutions $u_{k}$ such that $\Phi\left(u_{k}\right) \rightarrow \infty$ as $k \rightarrow \infty$;
(2) for any $\mu>0, \gamma \in \mathbb{R}$, (1) has a sequence of solutions $v_{k}$ such that $\Phi\left(v_{k}\right) \rightarrow 0^{-}$as $k \rightarrow \infty$.

The main ingredient for the proof of the above theorem is a dual fountain theorem in [8]. It should be noted that the (P.S) or (P.S*) condition and its variants play an important role in this theorem and its application. While the variant fountain theorem in Zou [1] does not need not the (P.S) or (P.S*) condition, we obtain the following generalized result by using Zou's theorem.

Theorem 4. Assume that (F1)-(F4) hold; then there exists a constant $\Lambda>0$ such that, for any $\eta<\Lambda$, (1) has infinitely many weak solutions $\left\{u_{k}\right\}$ satisfying

$$
\begin{equation*}
\Phi\left(u_{k}\right) \longrightarrow 0^{-} \quad \text { as } k \longrightarrow \infty \tag{10}
\end{equation*}
$$

This paper is organized as follows. In Section 2, we recall some preliminary theorems and lemmas. In Section 3, we give the proof of Theorem 4.

## 2. Preliminaries

In what follows, we make use of the following notations: $E$ (or $W^{1, p}(\Omega)$ ) denotes Banach space with the norm $\|\cdot\| ; E^{*}$ denotes the conjugate space for $E ; L^{p}(\Omega)$ denotes Lebesgue space with the norm $|\cdot|_{p} ;\langle\cdot, \cdot\rangle$ is the dual pairing of the spaces $E^{*}$ and $E$; we denote by $\rightarrow$ (resp., $\rightarrow$ ) the strong (resp., weak) convergence; $c, c_{1}, c_{2}, \ldots$ denote (possibly different) positive constants.

For completeness, we first recall the variant fountain theorem in Zou [1]. Let $E$ be a Banach space with norm $\|\cdot\|$ and $E=\overline{\oplus_{j \in N} X_{j}}$ with $\operatorname{dim} X_{j}<\infty$ for any $j \in N$. Set $Y_{k}=$ $\oplus_{j=0}^{k} X_{j}, Z_{k}=\overline{\oplus_{j=k}^{\infty} X_{j}}$.

Theorem 5 (see [1, Theorem 2.2]). The $C^{1}$-functional $\Phi_{\lambda}$ : $E \rightarrow \mathbb{R}$ defined by $\Phi_{\lambda}(u)=A(u)-\lambda B(u), \lambda \in[1,2]$, satisfies
(A1) $\Phi_{\lambda}$ maps bounded sets to bounded sets uniformly for $\lambda \in[1,2]$; furthermore, $\Phi_{\lambda}(-u)=\Phi_{\lambda}(u)$ for all $(\lambda, u) \in[1,2] \times E$.
(A2) $B(u) \geq 0$ for all $u \in E ; B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ on any finite dimensional subspace of $E$.
(A3) There exists $\rho_{k}>r_{k}>0$ such that

$$
\begin{equation*}
a_{k}(\lambda):=\inf _{u \in Z_{k},\|u\|=\rho_{k}} \Phi_{\lambda}(u) \geq 0>b_{k}(\lambda):=\max _{u \in Y_{k},\|u\|=r_{k}} \Phi_{\lambda}(u) ; \tag{11}
\end{equation*}
$$

for all $\lambda \in[1,2]$,

$$
\begin{align*}
d_{k}(\lambda) & :=\inf _{u \in Z_{k},\|u\| \leq \rho_{k}} \Phi_{\lambda}(u) \longrightarrow 0  \tag{12}\\
\text { as } k & \longrightarrow \infty \text { uniformly for } \lambda \in[1,2] .
\end{align*}
$$

Then there exist $\lambda_{n} \rightarrow 1, u\left(\lambda_{n}\right) \in Y_{n}$, such that

$$
\begin{array}{r}
\left.\Phi_{\lambda_{n}}^{\prime}\right|_{Y_{n}}\left(u\left(\lambda_{n}\right)\right)=0, \quad \Phi_{\lambda_{n}}\left(u\left(\lambda_{n}\right)\right) \longrightarrow c_{k} \in\left[d_{k}(2), b_{k}(1)\right] \\
\text { as } n \longrightarrow \infty . \tag{13}
\end{array}
$$

Particularly, if $\left\{u\left(\lambda_{n}\right)\right\}$ has a convergent subsequence for every $k$, then $\Phi_{1}$ has infinitely many nontrivial critical points $\left\{u_{k}\right\} \in E \backslash\{0\}$ satisfying $\Phi_{1}\left(u_{k}\right) \rightarrow 0^{-}$as $k \rightarrow \infty$.

Remark 6. Obviously, the sequence $\left\{u\left(\lambda_{n}\right)\right\}$ is a (P.S*) sequence.

For our working space $E=W^{1, p}(\Omega), E$ is a reflexive and separable Banach space; then there are $e_{j} \in E$ and $e_{j}^{*} \in E^{*}$ such that

$$
\begin{gather*}
E=\overline{\operatorname{span}\left\{e_{j}: j=1,2, \ldots\right\}}, \quad E^{*}=\overline{\operatorname{span}\left\{e_{j}^{*}: j=1,2, \ldots\right\}}, \\
\left\langle e_{j}^{*}, e_{j}\right\rangle= \begin{cases}1, & i=j \\
0, & i \neq j\end{cases} \tag{14}
\end{gather*}
$$

We write $X_{j}:=\operatorname{span}\left\{e_{j}\right\}$; then $Y_{k}, Z_{k}$ can be defined as that in the beginning of Theorem 5. Consider $\Phi_{\lambda}: E \rightarrow \mathbb{R}$ defined by

$$
\begin{align*}
\Phi_{\lambda}(u): & \frac{1}{p}\|u\|^{p}-\int_{\Omega} G(x, u) d x-\frac{\eta}{p} \int_{\partial \Omega}|u|^{p} d s \\
& -\lambda \int_{\Omega} F(x, u) d x  \tag{15}\\
:= & A(u)-\lambda B(u), \quad \lambda \in[1,2]
\end{align*}
$$

Then $B(u) \geq 0$ for all $u \in E ; B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ on any finite dimensional subspace of $E$; $\Phi_{\lambda}(-u)=\Phi_{\lambda}(u)$ for all $\lambda \in[1,2], u \in E$. We need the following lemmas.

Lemma 7 (see [7, Lemma 3.5]). If $1 \leq q<p^{*}$, then one has

$$
\begin{equation*}
\beta_{k}:=\sup _{u \in Z_{k}\|u\|=1}|u|_{q} \longrightarrow 0 \quad \text { as } k \longrightarrow \infty \tag{16}
\end{equation*}
$$

## 3. Proof of Theorem 4

First, we check the condition of Theorem 5.
Lemma 8. Assume (F1)-(F3); then (A1)-(A3) hold.
Proof. (A1) and (A2) are obvious. Let $n>k>2$; we assume that $\sigma \leq \delta$ and define

$$
\begin{equation*}
\beta_{k}(\sigma):=\sup _{u \in Z_{k},\|u\|=1}|u|_{\sigma}, \quad \beta_{k}(\delta):=\sup _{u \in Z_{k},\|u\|=1}|u|_{\delta} \tag{17}
\end{equation*}
$$

Observe that

$$
\begin{equation*}
|u|_{\sigma} \leq \beta_{k}(\sigma)\|u\|, \quad|u|_{\delta} \leq \beta_{k}(\delta)\|u\|, \tag{18}
\end{equation*}
$$

for any $u \in Z_{k}$. Note that $q<p^{*}$; there exists a constant $c>0$ such that

$$
\begin{equation*}
|u|_{q} \leq c\|u\| \tag{19}
\end{equation*}
$$

By the Sobolev trace imbedding inequality, we have

$$
\begin{equation*}
|u|_{L^{p}(\partial \Omega)}^{p} \leq K\|u\|^{p} . \tag{20}
\end{equation*}
$$

Then we take $\Lambda^{*}=1 / 4 K$ such that, for all $\eta<\Lambda^{*}$,

$$
\begin{equation*}
\frac{\eta}{p}|u|_{L^{p}(\partial \Omega)}^{p} \leq \frac{1}{4 p}\|u\|^{p} \tag{21}
\end{equation*}
$$

By (F3), for any $\varepsilon>0$, there exists $c_{\varepsilon}$ such that

$$
\begin{equation*}
|G(x, u)| \leq \varepsilon|u|^{p}+c_{\varepsilon}|u|^{q} . \tag{22}
\end{equation*}
$$

Then, by (F1)-(F3) and (18)-(21), we obtain

$$
\begin{align*}
\Phi_{\lambda}(u)= & \frac{1}{p}\|u\|^{p}-\int_{\Omega} G(x, u) d x-\frac{\eta}{p} \int_{\partial \Omega}|u|^{p} d s \\
& -\lambda \int_{\Omega} F(x, u) d x \\
\geq & \frac{3}{4 p}\|u\|^{p}-\varepsilon|u|_{p}^{p}-c_{\varepsilon}|u|_{q}^{q}-c|u|_{\sigma}^{\sigma}-c|u|_{\delta}^{\delta}  \tag{23}\\
\geq & \frac{3}{4 p}\|u\|^{p}-\varepsilon c\|u\|^{p}-c\|u\|^{q}-c \beta_{k}(\sigma)^{\sigma}\|u\|^{\sigma} \\
& -c \beta_{k}(\delta)^{\delta}\|u\|^{\delta} .
\end{align*}
$$

Note that $p<q$; we may choose $\varepsilon>0$ and $R>0$ sufficiently small that

$$
\begin{equation*}
\frac{1}{4 p}\|u\|^{p}-\varepsilon c\|u\|^{p}-c\|u\|^{q} \geq 0 \tag{24}
\end{equation*}
$$

holds true for any $u \in W^{1, p}(\Omega)$ with $\|u\| \leq R$. So we have

$$
\begin{equation*}
\Phi_{\lambda}(u) \geq \frac{1}{2 p}\|u\|^{p}-c \beta_{k}(\sigma)^{\sigma}\|u\|^{\sigma}-c \beta_{k}(\delta)^{\delta}\|u\|^{\delta} \tag{25}
\end{equation*}
$$

for any $u \in Z_{k}$ with $\|u\| \leq R$. Choosing

$$
\begin{equation*}
\rho_{k}:=\left(4 p c \beta_{k}(\sigma)^{\sigma}+4 p c \beta_{k}(\delta)^{\delta}\right)^{1 /(p-\delta)} \tag{26}
\end{equation*}
$$

by Lemma 7 , for $\beta_{k}(\sigma) \rightarrow 0, \beta_{k}(\delta) \rightarrow 0$ as $k \rightarrow \infty$, it follows that $\rho_{k} \rightarrow 0$ as $k \rightarrow \infty$, so there exists $k_{0}$ such that $\rho_{k} \leq R$ when $k \geq k_{0}$. Thus, for $k \geq k_{0}, u \in Z_{k}$, and $\|u\|=\rho_{k}$, we have $\Phi_{\lambda}(u) \geq \rho_{k}^{p} / 4 p>0$; then $a_{k}(\lambda) \geq 0$ for all $\lambda \in[1,2]$.

On the other hand, if $u \in Y_{k}$ with $\|u\|$ being small enough, since all the norms are equivalent on the finite dimensional space and $\sigma<p$, then $b_{k}(\lambda)<0$ for all $\lambda \in[1,2]$.

Furthermore, if $u \in Z_{k}$ with $\|u\| \leq \rho_{k}, k \geq k_{0}$, we see that

$$
\begin{equation*}
\Phi_{\lambda}(u) \geq-c \beta_{k}(\sigma)^{\sigma} \rho_{k}^{\sigma}-c \beta_{k}(\delta)^{\delta} \rho_{k}^{\delta} \longrightarrow 0 \quad \text { as } k \longrightarrow \infty \tag{27}
\end{equation*}
$$

Therefore, $d_{k}(\lambda) \rightarrow 0$ as $k \rightarrow \infty$. Thus, (A3) holds.
By Theorem 5, we have the following lemma.

Lemma 9. There exist $\lambda_{n} \rightarrow 1$ and $u\left(\lambda_{n}\right) \in Y_{n}$ such that

$$
\begin{align*}
& \left.\Phi_{\lambda_{n}}^{\prime}\right|_{Y_{n}}\left(u\left(\lambda_{n}\right)\right)=0  \tag{28}\\
& \Phi_{\lambda_{n}}\left(u\left(\lambda_{n}\right)\right) \longrightarrow c_{k} \in\left[d_{k}(2), b_{k}(1)\right] \quad \text { as } n \longrightarrow \infty
\end{align*}
$$

In order to complete our proof of Theorem 4, by a standard argument (see the proof of Lemma 3.4 in Zhao [7]), we only need to show that $\left\{u\left(\lambda_{n}\right)\right\}$ is bounded.

Lemma 10. $\left\{u\left(\lambda_{n}\right)\right\}$ is bounded in $W^{1, p}(\Omega)$.
Proof. Since $\left.\Phi_{\lambda_{n}}^{\prime}\right|_{Y_{n}}\left(u\left(\lambda_{n}\right)\right)=0$, then

$$
\begin{align*}
1 & -\eta \int_{\partial \Omega} \frac{\left|u\left(\lambda_{n}\right)\right|^{p}}{\left\|u\left(\lambda_{n}\right)\right\|^{p}} d s \\
& =\int_{\Omega} \frac{\lambda_{n} f\left(x, u\left(\lambda_{n}\right) u\left(\lambda_{n}\right)\right)+g\left(x, u\left(\lambda_{n}\right)\right) u\left(\lambda_{n}\right)}{\left\|u\left(\lambda_{n}\right)\right\|^{p}} d x . \tag{29}
\end{align*}
$$

We can choose $0<\Lambda<\Lambda^{*}$ and if $\eta<\Lambda$ such that $1-\eta K>0$. If, up to a subsequence, $\left\|u\left(\lambda_{n}\right)\right\| \rightarrow \infty$ as $n \rightarrow \infty$, then, by (F2),

$$
\begin{equation*}
1+|\eta| K \geq \int_{\Omega} \frac{g\left(x, u\left(\lambda_{n}\right)\right) u\left(\lambda_{n}\right)}{\left\|u\left(\lambda_{n}\right)\right\|^{p}} d x \geq \frac{1}{2}(1-\eta K) \tag{30}
\end{equation*}
$$

for $n$ is large enough. Obviously, it is a condition if (F4)(1) holds.

Otherwise, we set $w_{n}=u\left(\lambda_{n}\right) /\left\|u\left(\lambda_{n}\right)\right\|$; then, up to a subsequence,

$$
w_{n} \rightharpoonup w \text { in } E
$$

$$
\begin{gather*}
w_{n} \longrightarrow w \text { in } L^{t}(\Omega) \quad \text { for } 1 \leq t<p^{*}  \tag{31}\\
w_{n}(x) \longrightarrow w(x) \quad \text { a.e. } x \in \Omega .
\end{gather*}
$$

If $w \neq 0$ in $E$ and $\lim _{|u| \rightarrow \infty} g(x, u) /|u|^{p-2} u=-\infty$ in (F4)(2), then, for $n$ is large enough, by Fatou's Lemma, we have that

$$
\begin{align*}
- & \frac{1}{2}(1-\eta K) \\
& \geq \int_{\Omega} \frac{-g\left(x, u\left(\lambda_{n}\right)\right) u\left(\lambda_{n}\right)}{\left|u\left(\lambda_{n}\right)\right|^{p}}\left|w_{n}\right|^{p} d x \\
& \geq c+\int_{\left\{x \in \Omega: w \neq 0,\left|u\left(\lambda_{n}\right)\right| \geq c\right\}} \frac{-g\left(x, u\left(\lambda_{n}\right)\right) u\left(\lambda_{n}\right)}{\left|u\left(\lambda_{n}\right)\right|^{p}}\left|w_{n}\right|^{p} d x \\
& \longrightarrow \infty ; \tag{32}
\end{align*}
$$

this is a contradiction. It is similar if $\lim _{|u| \rightarrow \infty} g(x, u) /|u|^{p-2} u=\infty$ in (F4)(3). Thus, $w=0$.

Let $t_{n} \in[0,1]$ such that

$$
\begin{equation*}
\Phi_{\lambda_{n}}\left(t_{n} u\left(\lambda_{n}\right)\right):=\max _{t \in[0,1]} \Phi_{\lambda_{n}}\left(t u\left(\lambda_{n}\right)\right) \tag{33}
\end{equation*}
$$

For any $c>0$ large enough, and $\bar{w}_{n}:=(2 p c)^{1 / p} w_{n}$, for $n$ is large enough, we have that

$$
\begin{aligned}
\Phi_{\lambda_{n}}\left(t_{n} u\left(\lambda_{n}\right)\right) \geq & \Phi_{\lambda_{n}}\left(\bar{w}_{n}\right) \\
= & 2 c-\int_{\Omega} G\left(x, \bar{w}_{n}\right) d x-\frac{\eta}{p} \int_{\partial \Omega}\left|\bar{w}_{n}\right|^{p} d s \\
& -\lambda_{n} \int_{\Omega} F\left(x, \bar{w}_{n}\right) d x
\end{aligned}
$$

$$
\begin{equation*}
\geq c \tag{34}
\end{equation*}
$$

which implies that $\lim _{n \rightarrow \infty} \Phi_{\lambda_{n}}\left(t_{n} u\left(\lambda_{n}\right)\right) \rightarrow \infty$. Obviously,

$$
\begin{equation*}
\left\langle\Phi_{\lambda_{n}}^{\prime}\left(t_{n} u\left(\lambda_{n}\right)\right), t_{n} u\left(\lambda_{n}\right)\right\rangle=0 . \tag{35}
\end{equation*}
$$

It follows that

$$
\begin{align*}
\infty= & \lim _{n \rightarrow \infty}\left(\Phi_{\lambda_{n}}\left(t_{n} u\left(\lambda_{n}\right)\right)-\frac{1}{p}\left\langle\Phi_{\lambda_{n}}^{\prime}\left(t_{n} u\left(\lambda_{n}\right)\right), t_{n} u\left(\lambda_{n}\right)\right\rangle\right) \\
\leq & \lim _{n \rightarrow \infty} \lambda_{n} \int_{\Omega}\left(\frac{1}{p} f\left(x, t_{n} u\left(\lambda_{n}\right)\right) t_{n} u\left(\lambda_{n}\right)\right. \\
& \left.-F\left(x, t_{n} u\left(\lambda_{n}\right)\right)\right) d x \\
& +\int_{\Omega}\left(\frac{1}{p} g\left(x, t_{n} u\left(\lambda_{n}\right)\right) t_{n} u\left(\lambda_{n}\right)-G\left(x, t_{n} u\left(\lambda_{n}\right)\right)\right) d x . \tag{36}
\end{align*}
$$

If (F4)(2) holds, we have that $(1 / p) f(x, u) u-F(x, u)$ and $(1 / p) g(x, u) u-G(x, u)$ are decreasing in $u$ for $u$ is large enough. Therefore,

$$
\begin{equation*}
\frac{1}{p} f(x, s u) s u-F(x, s u)+\frac{1}{p} g(x, s u) s u-G(x, s u) \leq c \tag{37}
\end{equation*}
$$

for all $s>0$ and $u \in \mathbb{R}$; it is a contradiction.
If (F4)(3) holds, then we have that

$$
\begin{align*}
& \infty \leq c \\
& \int_{\Omega}\left|u\left(\lambda_{n}\right)\right|^{\sigma} d x  \tag{38}\\
&+\int_{\Omega}\left(\frac{1}{p} g\left(x, u\left(\lambda_{n}\right)\right) u\left(\lambda_{n}\right)-G\left(x, u\left(\lambda_{n}\right)\right)\right) d x,
\end{align*}
$$

which implies

$$
\begin{equation*}
\int_{\Omega}\left(\frac{1}{p} g\left(x, u\left(\lambda_{n}\right)\right) u\left(\lambda_{n}\right)-G\left(x, u\left(\lambda_{n}\right)\right)\right) d x \longrightarrow \infty \tag{39}
\end{equation*}
$$

On the other hand, by the property of $u\left(\lambda_{n}\right)$, for $n$ is large enough, since $\alpha>\max \{\delta, \sigma\}$, we have that

$$
\begin{align*}
& b_{k}(1) \\
& \qquad \begin{aligned}
\geq & \lambda_{n} \int_{\Omega}\left(\frac{1}{p} f\left(x, u\left(\lambda_{n}\right)\right) u\left(\lambda_{n}\right)-F\left(x, u\left(\lambda_{n}\right)\right)\right) d x \\
& +\int_{\Omega}\left(\frac{1}{p} g\left(x, u\left(\lambda_{n}\right)\right) u\left(\lambda_{n}\right)-G\left(x, u\left(\lambda_{n}\right)\right)\right) d x \\
\geq & \frac{1}{2} \int_{\Omega}\left(\frac{1}{p} g\left(x, u\left(\lambda_{n}\right)\right) u\left(\lambda_{n}\right)-G\left(x, u\left(\lambda_{n}\right)\right)\right) d x \\
& +\frac{1}{2} c \int_{\Omega}\left|u\left(\lambda_{n}\right)\right|^{\alpha} d x-\frac{1}{2} c \int_{\Omega}\left|u\left(\lambda_{n}\right)\right|^{\delta} d x \\
& -\frac{1}{2} c \int_{\Omega}\left|u\left(\lambda_{n}\right)\right|^{\sigma} d x \\
\geq & c \int_{\Omega}\left(\frac{1}{p} g\left(x, u\left(\lambda_{n}\right)\right) u\left(\lambda_{n}\right)-G\left(x, u\left(\lambda_{n}\right)\right)\right) d x-c ;
\end{aligned}
\end{align*}
$$

this implies that $\int_{\Omega}\left((1 / p) g\left(x, u\left(\lambda_{n}\right)\right) u\left(\lambda_{n}\right)-G\left(x, u\left(\lambda_{n}\right)\right)\right) d x$ is bounded, which contradicts (39).

By the above arguments, we have that $\left\{u\left(\lambda_{n}\right)\right\}$ is bounded.

Remark 11. In fact, our result still holds if we consider a weaker condition than $(\mathrm{F} 4)(2)$; that is, there is $c>0$ such that

$$
\begin{equation*}
H(x, t) \leq H(x, s)+c, \quad \bar{H}(x, t) \leq \bar{H}(x, s)+c \tag{41}
\end{equation*}
$$

for all $0<s<t$ or $t<s<0, \forall x \in \Omega$, where $H(x, t)=$ $(1 / p) f(x, t) t-F(x, t)$ and $\bar{H}(x, t)=(1 / p) g(x, t) t-G(x, t)$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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