Hindawi Publishing Corporation The Scientific World Journal Volume 2014, Article ID 272949, 8 pages http://dx.doi.org/10.1155/2014/272949



Research Article

On Generalization Based on Bi et al. Iterative Methods with Eighth-Order Convergence for Solving Nonlinear Equations

Taher Lotfi,¹ Alicia Cordero,² Juan R. Torregrosa,² Morteza Amir Abadi,¹ and Maryam Mohammadi Zadeh¹

¹ Department of Applied Mathematics, Hamedan Branch, Islamic Azad University, Hamedan 65188, Iran

Correspondence should be addressed to Taher Lotfi; lotfitaher@yahoo.com

Received 15 August 2013; Accepted 29 October 2013; Published 19 January 2014

Academic Editors: A. Agouzal and J. Sun

Copyright © 2014 Taher Lotfi et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The primary goal of this work is to provide a general optimal three-step class of iterative methods based on the schemes designed by Bi et al. (2009). Accordingly, it requires four functional evaluations per iteration with eighth-order convergence. Consequently, it satisfies Kung and Traub's conjecture relevant to construction optimal methods without memory. Moreover, some concrete methods of this class are shown and implemented numerically, showing their applicability and efficiency.

1. Introduction

Multipoint methods for solving nonlinear equations f(x) = 0, where $f: D \in R \to R$, possess an important advantage since they overcome theoretical limits of one-point methods concerning the convergence order and computational efficiency [1–5].

During the last years, there have been many attempts to construct optimal three-step iterative methods without memory for solving nonlinear equations. Indeed, Bi et al. [6, 7] are pioneers in this case, after Kung and Traub [8]. Some other optimal methods are due to Cordero et al. [9–11], Dzunic et al. [12, 13], Heydari et al. [14], Geum and Kim [15–17], Kou et al. [18], Liu and Wang [19–21], Sharma and Sharma [22], Soleimani et al. [4], Soleymani [23], Soleymani et al. [24–27], Thukral [28–30], and Thukral and Petković [31]. Recently, iterative methods for root finding have been used for finding matrix inversion arising from linear systems; for more details consult Wang [32], Babajee et al. [33], Montazeri et al. [34], Soleymani [35, 36], Thukral [37], and the references therein.

In this paper we present a new optimal class of three-step methods without memory, which employs the idea of weight functions in the second and third steps. The order of this class is eight requiring four functional evaluations per step and therefore it supports Kung and Traub's conjecture [8]. The proposed class includes the Bi et al. methods [6, 7].

In order to design the new methods, we will use the divided differences. Let f(x) be a function defined on an interval I, where I is the smallest interval containing k+1 distinct nodes x_1, x_2, \ldots, x_k . The divided difference $f[x_0, x_1, \ldots, x_k]$ with kth-order is defined as follows: $f[x_0] = f(x_0)$,

$$f[x_0] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}, \dots, f[x_0, x_1, \dots, x_k]$$

$$= \frac{f[x_1, x_2, \dots, x_k] - f[x_0, x_1, \dots, x_{k-1}]}{x_k - x_0}.$$
(1)

It is clear that the divided difference $f[x_0,x_1,\ldots x_k]$ is a symmetric function of its arguments x_0,x_1,\ldots,x_k . Moreover, if we assume that $f\in C^{(k+1)}(I_x)$, where I_x is the smallest interval containing the nodes $x_0,x_1,\ldots x_k$, and x, then $f[x_0,x_1,\ldots,x_k,x]=f^{(k+1)}(\xi)/(k+1)!$ for a suitable $\xi\in I_x$. Specially, if $x_0=x_1=\cdots=x_k=x$, then

$$f[x, x, ..., x, x] = \frac{f^{(k+1)}(x)}{(k+1)!}.$$
 (2)

² Instituto de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera, s/n, 46022 Valencia, Spain

Moreover, we recall the so-called efficiency index defined by Ostrowski [38] as $EI = p^{1/n}$, where p is the order of convergence and n is the total number of functional evaluations per iteration.

2. Main Result: Development and Convergence Analysis of the New Methods

It is well known that Newton's method converges quadratically under standard conditions. To obtain a higher order of convergence and higher efficiency index than that of Newton's scheme, we compose Newton's method twice as follows:

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})}, \qquad z_{n} = y_{n} - \frac{f(y_{n})}{f'(y_{n})},$$

$$x_{n+1} = z_{n} - \frac{f(z_{n})}{f'(z_{n})}, \qquad n = 0, 1, 2, \dots$$
(3)

As this scheme is eighth-order convergent but its efficiency is poor, we need to reduce the number of functional evaluations. In the third step, $f'(z_n)$ can be approximated in a similar way as in [6].

Consider

$$f'(z_n) \approx f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n).$$
 (4)

Also, a "frozen" derivative can be used in the second step and adequate weight functions will improve the efficiency in the second and last steps. So, the following three-step methods are proposed:

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})}, \qquad z_{n} = y_{n} - g(s_{n}) \frac{f(y_{n})}{f'(x_{n})},$$

$$s_{n} = \frac{f(y_{n})}{f(x_{n})},$$

$$x_{n+1} = z_{n} - h(t_{n}) \frac{f(z_{n})}{f[z_{n}, y_{n}] + f[z_{n}, x_{n}, x_{n}](z_{n} - y_{n})},$$

$$t_{n} = \frac{f(z_{n})}{f(x_{n})}.$$
(5)

It is clear that the proposed methods by (5) require only four functional evaluations per iteration, while they are not eighth-order methods, in general. To recover the optimal eighth-order, we find some suitable conditions on the introduced weight functions $g(s_n)$ and $h(t_n)$.

To find the weight functions g and h in (5) providing order eight, we will use the method of undetermined coefficients and Taylor's series about 0, since $t_n \to 0$, $s_n \to 0$, when $n \to \infty$.

Let us consider

$$g(s_n) \approx g(0) + g'(0) s_n + g''(0) \frac{s_n^2}{2},$$

$$h(t_n) \approx h(0) + h'(0) t_n.$$
(6)

The following result states suitable conditions for proving that the new class has eighth-order of convergence.

Theorem 1. Assume that f is a sufficiently differentiable real function. Let one suppose that $\alpha \in D$ is a simple zero of f. If the initial estimation x_0 is close enough to α , then the sequence $\{x_n\}$ generated by any method of the family (5) converges to α with eighth-order of convergence if g and h are real sufficiently differentiable functions satisfying g(0) = h(0) = 1, g'(0) = h'(0) = 2, and g''(0) = 10.

Proof. Let us introduce the following notations:

$$e_n = x_n - \alpha,$$
 $e_{y_n} = y_n - \alpha,$ $e_{z_n} = z_n - \alpha,$
$$e_{n+1} = x_{n+1} - \alpha,$$
 $c_i = \frac{1}{i!} \frac{f^{(i)}(\alpha)}{f'(\alpha)},$ $i \ge 2.$ (7)

Using Taylor's expansion and taking into account $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha) \left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 \right] + O(e_n^9).$$
(8)

Also by direct differentiation, we obtain

$$f'(x_n)$$
= $f'(\alpha) \left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + 8c_8e_n^7 \right] + O(e_n^8).$ (9)

From (8) and (9) we get

$$\frac{(x_n)}{(x_n)}$$

$$= e_n - c_2 e_n^2 + 2 (c_2^2 - c_3) e_n^3 + (7c_2 c_3 - 4c_2^3 - 3c_4) e_n^4$$

$$+ (8c_2^4 - 20c_2^2 c_3 + 6c_3^2 + 10c_2 c_4 - 4c_5) e_n^5$$

$$+ [-16c_2^5 + 52c_2^3 c_3 - 28c_2^2 c_4 + 17c_3 c_4$$

$$+ c_2 (-33c_3^2 + 13c_5) - 5c_6] e_n^6$$

$$+ 2 [16c_2^6 - 64c_2^4 c_3 - 9c_3^3$$

$$+ 36c_2^3 c_4 + 6c_4^2 + 9c_2^2 (7c_3^2 - 2c_5)$$

$$+11c_3 c_5 + c_2 (-46c_3 c_4 + 8c_6) - 3c_7] e_n^7$$

$$+ [-64c_2^7 + 304c_2^5 c_3 - 176c_2^4 c_4$$

$$- 75c_3^2 c_4 + 31c_4 c_5 + c_2^3 (-408c_3^2 + 92c_5)$$

$$+ 4c_2^2 (87c_3 c_4 - 11c_6) + 27c_3 c_6$$

$$+ c_2 (135c_3^3 - 64c_4^2 - 118c_3 c_5 + 19c_7) - 7c_8]$$

$$\times e_n^8 + O(e_n^9).$$

Hence,

$$e_{y_n} = c_2 e_n^2 + 2 \left(-c_2^2 + c_3 \right) e_n^3 + \left(-7c_2 c_3 + 4c_2^3 + 3c_4 \right) e_n^4$$

$$- \left(8c_2^4 - 20c_2^2 c_3 + 6c_3^2 + 10c_2 c_4 - 4c_5 \right) e_n^5$$

$$- \left[-16c_2^5 + 52c_2^3 c_3 - 28c_2^2 c_4 \right]$$

$$+ 17c_3 c_4 + c_2 \left(-33c_3^2 + 13c_5 \right) - 5c_6 e_n^6$$

$$- 2 \left[16c_2^6 - 64c_2^4 c_3 - 9c_3^3 + 36c_2^3 c_4 + 6c_4^2 \right]$$

$$+ 9c_2^2 \left(7c_3^2 - 2c_5 \right) + 11c_3 c_5$$

$$+ c_2 \left(-46c_3 c_4 + 8c_6 \right) - 3c_7 e_n^7$$

$$- \left[-64c_2^7 + 304c_2^5 c_3 - 176c_2^4 c_4 \right]$$

$$- 75c_3^2 c_4 + 31c_4 c_5 + c_2^3 \left(-408c_3^2 + 92c_5 \right)$$

$$+ 4c_2^2 \left(87c_3 c_4 - 11c_6 \right) + 27c_3 c_6$$

$$+ c_2 \left(135c_3^3 - 64c_4^2 - 118c_3 c_5 + 19c_7 \right) - 7c_8 e_1^3$$

$$\times e_n^8 + O\left(e_n^9 \right).$$
(11)

Similar to (8),

$$f(y_n)$$

$$= f'(\alpha) \left[c_2 e_n^2 + 2 \left(-c_2^2 + c_3 \right) e_n^3 + \left(5c_2^3 - 7c_2 c_3 + 3c_4 \right) e_n^4 \right]$$

$$- 2 \left(6c_2^4 - 12c_2^2 c_3 + 3c_3^2 + 5c_2 c_4 - 2c_5 \right) e_n^5$$

$$+ \left(28c_2^5 - 73c_2^3 c_3 + 34c_2^2 c_4 - 17c_3 c_4 + c_2 \left(37c_3^2 - 13c_5 \right) + 5c_6 \right) e_n^6 + O\left(e_n^7 \right).$$
(12)

Moreover, taking into account (8), (9), and (12),

$$s_{n} = \frac{f(y_{n})}{f(x_{n})}$$

$$= c_{2}e_{n} + \left(-3c_{2}^{2} + 2c_{3}\right)e_{n}^{2} + \left(8c_{2}^{3} - 10c_{2}c_{3} + 3c_{4}\right)e_{n}^{3}$$

$$+ \left(-20c_{2}^{4} + 37c_{2}^{2}c_{3} - 8c_{3}^{2} - 14c_{2}c_{4}\right)e_{n}^{4}$$

$$+ \left(48c_{2}^{5} - 118c_{2}^{3}c_{3} + 55c_{2}c_{3}^{2} + 51c_{2}^{2}c_{4} - 22c_{3}c_{4}\right)e_{n}^{5}$$

$$\times \left(-112c_{2}^{6} + 344c_{2}^{4}c_{3} - 252c_{2}^{2}c_{3}^{2} + 26c_{3}^{3}\right)$$

$$-163c_{2}^{3}c_{4} + 150c_{2}c_{3}c_{4} - 15c_{4}^{2}\right)e_{n}^{6} + O\left(e_{n}^{7}\right),$$
(13)

$$\frac{f(y_n)}{f'(x_n)} = c_2 e_n^2 + \left(-4c_2^2 + 2c_3\right) e_n^3
+ \left(13c_2^3 - 14c_2c_3 + 3c_4\right) e_n^4
+ \left(-38c_2^4 + 64c_2^2c_3 - 20c_2c_4 + 4\left(-3c_3^2 + c_5\right)\right)
\times e_n^5 + O\left(e_n^6\right).$$
(14)

By using Taylor's expansion around zero

$$g(s_n) \approx g(0) + g'(0) s_n + \frac{g''(0)}{2} s_n^2,$$
 (15)

and by using (11)-(15),

$$e_{z_n} = A_2 e_n^2 + A_3 e_n^3 + A_4 e_n^4 + O(e_n^5),$$
 (16)

where $A_2 = (1 - g(0))c_2$, $A_3 = ((-2 + 4g(0) - g'(0))c_2^2 - 2(-1 + g(0))c_3)$, and

$$A_{4} = \left(\left(4 - 13g(0) + 7g'(0) - \frac{g''(0)}{2} \right) c_{2}^{3} + \left(-7 + 14g(0) - 4g'(0) \right) c_{2}c_{3} - 3\left(-1 + g(0) \right) c_{4} \right).$$
(17)

We now need to vanish A_2 and A_3 not only for making the first two steps optimal but also for simplifying subsequent relations. It is enough to ask the weight function g to satisfy conditions g(0) = 1 and g'(0) = 2. Then

$$e_{z_n} = \left(\left(5 - \frac{g''(0)}{2} \right) c_2^3 - c_2 c_3 \right) e_n^4 + O\left(e_n^5\right).$$
 (18)

For the third step, we also require

$$t_n = \frac{f(z_n)}{f(x_n)}$$

$$= \left(\left(5 - \frac{g''(0)}{2} \right) c_2^3 - c_2 c_3 \right) e_n^3$$

$$+ \left(\left(-41 + \frac{11g''(0)}{2} \right) c_2^4$$

$$-3 \left(-11 + g''(0) \right) c_2^2 c_3 - 2c_3^2 - 2c_2 c_4 \right) e_n^4$$

$$+ \left[\left(211 - \frac{73g''(0)}{2} \right) c_2^5 + \frac{3}{2} \left(-200 + 27g''(0) \right) c_2^3 c_3 + 3 \left(23 - 2g''(0) \right) c_2 c_3^2 + \frac{1}{2} \left(100 - 9g''(0) \right) c_2^2 c_4 - 7c_3 c_4 \right] e_n^5 + O\left(e_n^6 \right),$$
(19)

$$f[z_n, y_n] = f'(\alpha) \left[1 + c_2^2 e_n^2 + 2c_2 \left(-c_2^2 + c_3 \right) e_n^3 - \frac{1}{2} c_2 \left(\left(-18 + g''(0) \right) c_2^3 + 14c_2 c_3 - 6c_4 \right) \right] \times e_n^4 + O(e_n^5),$$
(20)

$$f[z_n, x_n, x_n]$$

$$= f'(\alpha) \left[c_2 + 2c_3 e_n + 3c_4 e_n^2 - \frac{1}{2} \left(c_2 c_3 \left(\left(g''(0) - 10 \right) c_2^2 + 2c_3 \right) \right) e_n^4 \right] + O(e_n^5).$$
(21)

Now let

$$h(t_n) \approx h(0) + h'(0)t_n. \tag{22}$$

Taking into account relations (19)–(22) and the third step of (5), we get

$$e_{n+1} = e_{z_n} - h(t_n) \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](e_{z_n} - e_{y_n})}$$

$$= B_4 e_n^4 + B_5 e_n^5 + B_6 e_n^6 + B_7 e_n^7 + B_8 e_n^8 + O(e_n^9),$$
(23)

where $B_4 = (1/2)(-1 + h(0))c_2((-10 + g''(0))c_2^2 + 2c_3)$. For the sake of simplicity, we first vanish this coefficient and afterwards the other coefficients will be given in the same strategy. Needless to say, h(0) = 1 implies the desired result. Then, imposing this condition, it follows at once that $B_4 = B_5 = B_6 = 0$ and

$$B_{7} = -\frac{1}{4} \left(c_{2}^{2} \left(\left(-10 + g''(0) \right) c_{2}^{2} + 2c_{3} \right) \right.$$

$$\times \left(\left(-10 + g''(0) \right) h'(0) c_{2}^{2} + 2 \left(-2 + h'(0) \right) c_{3} \right) \right). \tag{24}$$

Finally, taking g''(0) = 10 and h'(0) = 2, we obtain

$$e_{n+1} = c_2^2 c_3 \left(28c_2^3 + 2c_2 c_3 - c_4\right) e_n^8 + O(e_n)^9,$$
 (25)

which shows that under the provided conditions on weight functions g and h the method (5) has eighth-order convergence and it is optimal. This finishes the proof.

According to the above analysis, we can obtain the following special cases.

Corollary 2. If one sets $g(s_n) = (1 + \beta s_n)/(1 + (\beta - 2)s_n) = (f(x_n) + \beta f(y_n))/(f(x_n) + (\beta - 2)f(y_n))$, scheme (14) in [6] is obtained.

Consider

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = y_{n} - \frac{f(x_{n}) + \beta f(y_{n})}{f(x_{n}) + (\beta - 2) f(y_{n})} \frac{f(y_{n})}{f'(x_{n})}, \quad \beta = \frac{-1}{2}$$

$$x_{n+1} = z_{n} - h(t_{n}) \frac{f(z_{n})}{f[z_{n}, y_{n}] + f[z_{n}, x_{n}, x_{n}](z_{n} - y_{n})},$$

$$t_{n} = \frac{f(z_{n})}{f(x_{n})}.$$
(26)

Corollary 3. If one sets $h(t_n) = (1 + \theta t_n)/(1 + (\theta - 2)t_n) = (f(x_n) + \theta f(z_n))/(f(x_n) + (\theta - 2)f(z_n)), \theta \in \mathbb{R}$, our proposed method becomes scheme (13) in [7].

Consider

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = y_{n} - g(s_{n}) \frac{f(y_{n})}{f'(x_{n})}, \qquad s_{n} = \frac{f(y_{n})}{f(x_{n})}$$

$$x_{n+1} = z_{n} - \frac{f(x_{n}) + \theta f(z_{n})}{f(x_{n}) + (\theta - 2) f(z_{n})}$$

$$\times \frac{f(z_{n})}{f[z_{n}, y_{n}] + f[z_{n}, x_{n}, x_{n}](z_{n} - y_{n})}.$$
(27)

In addition to those from Corollaries 2 and 3, some simple but efficient weight functions which satisfy conditions of Theorem 1 are

$$g_{1}(s_{n}) = \frac{2 - s_{n}}{2 - 5s_{n}} = \frac{2f(x_{n}) - f(y_{n})}{2f(x_{n}) - 5f(y_{n})},$$

$$g_{2}(s_{n}) = \frac{1}{1 - 2s_{n} - s_{n}^{2}}$$

$$= \frac{f(x_{n})^{2}}{f(x_{n})^{2} - 2f(x_{n})f(y_{n}) - f(y_{n})^{2}},$$

$$g_{3}(s_{n}) = 1 + 2s_{n} + 5s_{n}^{2}$$

$$= \frac{f(x_{n})^{2} + 2f(x_{n})f(y_{n}) + 5f(y_{n})^{2}}{f(x_{n})^{2}},$$

$$h_{1}\left(t_{n}\right) = \frac{1+\theta t_{n}}{1+\left(\theta-2\right)t_{n}} = \frac{f\left(x_{n}\right)+\theta f\left(z_{n}\right)}{f\left(x_{n}\right)+\left(\theta-2\right)f\left(z_{n}\right)}, \quad \theta \in \mathbb{R},$$

$$h_{2}\left(t_{n}\right) = 1+2t_{n} = \frac{f\left(x_{n}\right)+2f\left(z_{n}\right)}{f\left(x_{n}\right)}.$$

$$(28)$$

3. Some Concrete Methods

In this section, we put forward some particular three-step methods based on the general class designed in this work.

3.1. Methods 1 and 2. Firstly, by combining the methods (26) and (27),

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = y_{n} - \frac{f(x_{n}) + \beta f(y_{n})}{f(x_{n}) + (\beta - 2) f(y_{n})} \frac{f(y_{n})}{f'(x_{n})}, \quad \beta = \frac{-1}{2},$$

$$x_{n+1} = z_{n} - \frac{f(x_{n}) + \theta f(z_{n})}{f(x_{n}) + (\theta - 2) f(z_{n})}$$

$$\times \frac{f(z_{n})}{f[z_{n}, y_{n}] + f[z_{n}, x_{n}, x_{n}](z_{n} - y_{n})}, \quad \theta \in \mathbb{R}.$$
(29)

Consequently, a special case of (29) appears when $g_1(s_n) = (2-s_n)/(2-5s_n) = (2f(x_n)-f(y_n))/(2f(x_n)-5f(y_n))$ and $h_1(t_n) = 1/(1-2t_n) = f(x_n)/(f(x_n)-2f(z_n))$, $(\theta=0)$:

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = y_{n} - \frac{2f(x_{n}) - f(y_{n})}{2f(x_{n}) - 5f(y_{n})} \frac{f(y_{n})}{f'(x_{n})}, \quad \beta = \frac{-1}{2}$$

$$x_{n+1} = z_{n} - \frac{f(x_{n})}{f(x_{n}) - 2f(z_{n})}$$

$$\times \frac{f(z_{n})}{f[z_{n}, y_{n}] + f[z_{n}, x_{n}, x_{n}](z_{n} - y_{n})}.$$
(30)

3.2. Method 3. Now, let us substitute $g_1(s_n) = (2 - s_n)/(2 - 5s_n) = (2f(x_n) - f(y_n))/(2f(x_n) - 5f(y_n))$ and $h_2(t_n) = 1 + 2t_n = (f(x_n) + 2f(z_n))/f(x_n)$ into (5). It gives us the following iterative scheme:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \frac{f(y_n)}{f'(x_n)}, \quad \beta = \frac{-1}{2}, \end{aligned}$$

$$x_{n+1} = z_n - \frac{f(x_n) + 2f(z_n)}{f(x_n)} \times \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}.$$
(31)

3.3. Method 4. Let us consider $g_2(s_n) = 1/(1 - t_n)^2 = f(x_n)^2/(f(x_n)^2 - 2f(x_n)f(y_n) - f(y_n)^2)$ and $h_1(t_n) = 1/(1 - 2t_n) = f(x_n)/(f(x_n) - 2f(z_n))$, $(\theta = 0)$. By using them in (5), we have

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = y_{n} - \frac{f(x_{n})^{2}}{f(x_{n})^{2} - 2f(x_{n})f(y_{n}) - f(y_{n})^{2}} \frac{f(y_{n})}{f'(x_{n})},$$

$$x_{n+1} = z_{n} - \frac{f(x_{n})}{f(x_{n}) - 2f(z_{n})}$$

$$\times \frac{f(z_{n})}{f[z_{n}, y_{n}] + f[z_{n}, x_{n}, x_{n}](z_{n} - y_{n})}.$$
(32)

3.4. Method 5. If we consider $g_2(s_n) = 1/(1 - t_n)^2 = f(x_n)^2/(f(x_n)^2 - 2f(x_n)f(y_n) - f(y_n)^2)$ and $h_2(t_n) = 1 + 2t_n = (f(x_n) + 2f(z_n))/f(x_n)$ in (5), we have

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = y_{n} - \frac{f(x_{n})^{2}}{f(x_{n})^{2} - 2f(x_{n})f(y_{n}) - f(y_{n})^{2}} \frac{f(y_{n})}{f'(x_{n})},$$

$$x_{n+1} = z_{n} - \frac{f(x_{n}) + 2f(z_{n})}{f(x_{n})}$$

$$\times \frac{f(z_{n})}{f[z_{n}, y_{n}] + f[z_{n}, x_{n}, x_{n}](z_{n} - y_{n})}.$$
(33)

3.5. Method 6. When $g_3(s_n) = 1 + 2s_n + 5s_n^2 = (f(x_n)^2 + 2f(x_n)f(y_n) + 5f(y_n)^2)/f(x_n)^2$ and $h_1(t_n) = 1/(1 - 2t_n) = f(x_n)/(f(x_n) - 2f(z_n)), (\theta = 0)$ in (5), we get

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = y_{n} - \frac{f(x_{n})^{2} + 2f(x_{n})f(y_{n}) + 5f(y_{n})^{2}}{f(x_{n})^{2}} \frac{f(y_{n})}{f'(x_{n})},$$

$$x_{n+1} = z_{n} - \frac{f(x_{n})}{f(x_{n}) - 2f(z_{n})}$$

$$\times \frac{f(z_{n})}{f[z_{n}, y_{n}] + f[z_{n}, x_{n}, x_{n}](z_{n} - y_{n})}.$$

(34)

3.6. Method 7. Finally, if we consider $g_3(s_n) = 1 + 2s_n + 5s_n^2 = (f(x_n)^2 + 2f(x_n)f(y_n) + 5f(y_n)^2)/f(x_n)^2$ and $h_2(t_n) = 1 + 2t_n = (f(x_n) + 2f(z_n))/f(x_n)$ in (5), we have

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = y_{n} - \frac{f(x_{n})^{2} + 2f(x_{n})f(y_{n}) + 5f(y_{n})^{2}}{f(x_{n})^{2}} \frac{f(y_{n})}{f'(x_{n})},$$

$$x_{n+1} = z_{n} - \frac{f(x_{n}) + 2f(z_{n})}{f(x_{n})}$$

$$\times \frac{f(z_{n})}{f[z_{n}, y_{n}] + f[z_{n}, x_{n}, x_{n}](z_{n} - y_{n})}.$$
(35)

All the methods (29)–(35) require three functional evaluations, namely, $f(x_n)$, $f(y_n)$, and $f(z_n)$, and one of the first derivative, namely, $f'(x_n)$, per iteration. Therefore, they are optimal in the sense of Kung and Traub's conjecture for n = 4 with $p = 2^3$. Thus, if we assume that all the evaluations have the same cost, then EI = 1.682.

4. Numerical Implementation and Comparisons

This section concerns numerical results of the proposed methods (30)–(35). Moreover, they are compared with Kung-Traub's method presented in [8], whose iterative expression is

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = y_{n} - \frac{f(x_{n})}{(f(x_{n}) - f(y_{n}))^{2}} \frac{f(y_{n})}{f'(x_{n})},$$

$$x_{n+1} = z_{n} - \left(\frac{1}{f(x_{n}) - f(z_{n})} \left(\frac{1}{f[x_{n}, z_{n}]} - \frac{1}{f'(x_{n})}\right)\right)$$

$$-\frac{f(y_{n})}{(f(x_{n}) - f(y_{n}))^{2} f'(x_{n})}$$

$$\times \frac{f^{2}(x_{n}) f(y_{n})}{f(y_{n}) - f(z_{n})}.$$
(36)

Numerical results have been carried out using Mathematica 8 with 400 digits of precision. In each table, ACOC stands for Approximated Computational Order of Convergence (see [39]), which is given by

$$p \approx ACOC = \frac{\ln(|x_{n+1} - x_n| |x_n - x_{n-1}|^{-1})}{\ln(|x_n - x_{n-1}| |x_{n-1} - x_{n-2}|^{-1})}.$$
 (37)

Table 1: Numerical results with f_1 .

Method	$ x_1 - \alpha $	$ x_2 - \alpha $	$ x_3 - \alpha $	ACOC
(30)	0.1378 (-5)	0.7505 (-46)	0.5824 (-368)	8.0000
(31)	0.1487 (-5)	0.1386 (-45)	0.7890 (-366)	8.0000
(32)	0.1304 (-5)	0.6560 (-46)	0.2691 (-368)	8.0000
(33)	0.1417 (-5)	0.1272 (-45)	0.5365 (-366)	8.0000
(34)	0.3811 (-6)	0.2544 (-49)	0.1005 (-394)	8.0000
(35)	0.2083 (-6)	0.2029 (-51)	0.1641 (-411)	8.0000
(36)	0.6210 (-5)	0.1433 (-39)	0.1151 (-316)	8.0000

TABLE 2: Numerical results with f_2 .

Method	$ x_1 - \alpha $	$ x_2 - \alpha $	$ x_3 - \alpha $	ACOC
Method	$ x_1 - a $	$ x_2 - a $		ACOC
(30)	0.2118(-3)	0.6382 (-22)	0.4314 (-170)	8.0000
(31)	0.1311 (-3)	0.1375 (-23)	0.2004 (-183)	8.0000
(32)	0.2182 (-3)	0.9901 (-22)	0.1771 (-168)	8.0000
(33)	0.1358 (-3)	0.2225 (-23)	0.1150 (-181)	8.0000
(34)	0.4679 (-3)	0.2262 (-18)	0.7049 (-141)	7.9999
(35)	0.3139 (-3)	0.9409 (-20)	0.6319 (-152)	7.9999
(36)	0.2165 (-3)	0.1415 (-23)	0.4944 (-185)	7.9990

Among many test problems, the following four examples are considered:

$$f_1(x) = (x-2)(x^6 + x^3 + 1)e^{-x^2}, \quad \alpha = 2, \quad x_0 = 1.8,$$

$$f_2(x) = x^2 - (1-x)^{25}, \quad \alpha = 0.1437392..., \quad x_0 = 2.5,$$

$$f_3(x) = \prod_{k=1}^{12} (x-k), \quad \alpha = 5, \quad x_0 = 5.3,$$

$$f_4(x) = e^x \sin(5x) - 2, \quad \alpha = 1.3639..., \quad x_0 = 1.2.$$
(38)

From Table 1, it can be seen that all methods work perfectly. Furthermore, we can see that results from methods (34) and (35) are specially good. Table 2 shows that numerical results are in accordance with their theory well enough. In this example, methods (34) and (35) do not have as good behavior as in Example 1. Table 3 represents an important case. Although methods (34) and (35) are working very well in Example 1, however, they do not produce convergent iterations here. It should be remarked that these divergent sequences show that some methods work better in some cases, while they may not do it in other ones.

Table 4 shows that all the methods work in concordance with theoretical results.

5. Conclusion

A new optimal class of three-step methods without memory has been obtained by generalizing Bi et al. families. This class uses four functional evaluations per iteration and it is optimal in the sense of Kung and Traub's conjecture. Some elements of the family have been presented and they have been tested in order to show its applicability and efficiency, showing that

TABLE 3: Numerical results with f_3 .

Method	$ x_1 - \alpha $	$ x_2 - \alpha $	$ x_3 - \alpha $	ACOC
(30)	0.2222 (-2)	0.5371 (-22)	0.5914 (-179)	8.0012
(31)	0.2197 (-2)	0.4897 (-22)	0.2824 (-179)	8.0012
(32)	0.8787 (-1)	0.2317 (-9)	0.3280 (-78)	8.0254
(33)	0.5637 (-1)	0.1005 (-10)	0.4105 (-89)	8.0407
(34)	0.1791 (1)	0.2000(1)	0.2000 (1)	_
(35)	0.4703 (1)	0.4584 (27)	0.9577 (26)	_
(36)	0.2210 (-1)	0.5727 (-13)	0.1870 (-105)	7.9823

TABLE 4: Numerical results with f_4 .

Method	$ x_1 - \alpha $	$ x_2 - \alpha $	$ x_3 - \alpha $	ACOC
(30)	0.2119 (-4)	0.1307 (-36)	0.2749 (-294)	8.0000
(31)	0.1731 (-4)	0.2597 (-37)	0.6668 (-300)	8.0000
(32)	0.2333(-4)	0.2789 (-36)	0.1166 (-291)	8.0000
(33)	0.1973 (-4)	0.7310 (-37)	0.2599 (-296)	8.0000
(34)	0.1769 (-4)	0.2191 (-37)	0.1212 (-300)	8.0000
(35)	0.1767 (-4)	0.2177 (-37)	0.1153 (-300)	8.0000
(36)	0.2074 (-4)	0.2571 (-36)	0.1433 (-291)	8.0000

these methods work properly and confirm their theoretical aspects.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors thank the anonymous referees for their valuable comments and for the suggestions to improve the readability of the paper. This research was supported by Islamic Azad University, Hamedan Branch, and Ministerio de Ciencia y Tecnología MTM2011-28636-C02-02.

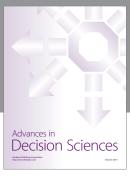
References

- [1] R. Behl, V. Kanwar, and K. K. Sharma, "Another simple way of deriving several iterative functions to solve nonlinear equations," *Journal of Applied Mathematics*, vol. 2012, Article ID 294086, 22 pages, 2012.
- [2] G. Fernandez-Torres and J. Vazquez-Aquino, "Three new optimal fourth-order iterative methods to solve nonlinear equations," *Advances in Numerical Analysis*, vol. 2013, Article ID 957496, 8 pages, 2013.
- [3] S. M. Kang, A. Rafiq, and Y. C. Kwun, "A new second-order iteration method for solving nonlinear equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 487062, 4 pages, 2013.
- [4] F. Soleimani, F. Soleymani, and S. Shateyi, "Some iterative methods free from derivatives and their basins of attraction for nonlinear equations," *Discrete Dynamics in Nature and Society*, vol. 2013, Article ID 301718, 10 pages, 2013.
- [5] J. F. Traub, *Iterative Methods for the Solution of Equations*, Prentice Hall, New York, NY, USA, 1964.

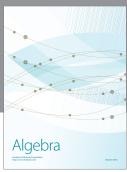
- [6] W. Bi, H. Ren, and Q. Wu, "Three-step iterative methods with eighth-order convergence for solving nonlinear equations," *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 105–112, 2009.
- [7] W. Bi, Q. Wu, and H. Ren, "A new family of eighth-order iterative methods for solving nonlinear equations," *Applied Mathematics and Computation*, vol. 214, no. 1, pp. 236–245, 2009.
- [8] H. T. Kung and J. F. Traub, "Optimal order of one-point and multipoint iteration," *Association for Computing Machinery*, vol. 21, no. 4, pp. 643–651, 1974.
- [9] A. Cordero, J. L. Hueso, E. Martínez, and J. R. Torregrosa, "New modifications of Potra-Pták's method with optimal fourth and eighth orders of convergence," *Journal of Computational and Applied Mathematics*, vol. 234, no. 10, pp. 2969–2976, 2010.
- [10] A. Cordero and J. R. Torregrosa, "A class of Steffensen type methods with optimal order of convergence," *Applied Mathe-matics and Computation*, vol. 217, no. 19, pp. 7653–7659, 2011.
- [11] A. Cordero, J. R. Torregrosa, and M. P. Vassileva, "Three-step iterative methods with optimal eighth-order convergence," *Journal of Computational and Applied Mathematics*, vol. 235, no. 10, pp. 3189–3194, 2011.
- [12] J. Dzunic and M. S. Petkovic, "A family of three-point methods of Ostrowskis type for solving nonlinear equations," *Journal of Applied Mathematics*, vol. 2012, Article ID 425867, 9 pages, 2012.
- [13] J. Džunić, M. S. Petković, and L. D. Petković, "A family of optimal three-point methods for solving nonlinear equations using two parametric functions," *Applied Mathematics and Computation*, vol. 217, no. 19, pp. 7612–7619, 2011.
- [14] M. Heydari, S. M. Hosseini, and G. B. Loghmani, "On two new families of iterative methods for solving nonlinear equations with optimal order," *Applicable Analysis and Discrete Mathematics*, vol. 5, no. 1, pp. 93–109, 2011.
- [15] Y. H. Geum and Y. I. Kim, "A multi-parameter family of threestep eighth-order iterative methods locating a simple root," *Applied Mathematics and Computation*, vol. 215, no. 9, pp. 3375– 3382, 2010.
- [16] Y. H. Geum and Y. I. Kim, "A uniparametric family of three-step eighth-order multipoint iterative methods for simple roots," *Applied Mathematics Letters*, vol. 24, no. 6, pp. 929–935, 2011.
- [17] Y. H. Geum and Y. I. Kim, "A biparametric family of eighthorder methods with their third-step weighting function decomposed into a one-variable linear fraction and a two-variable generic function," *Computers and Mathematics with Applica*tions, vol. 61, no. 3, pp. 708–714, 2011.
- [18] J. Kou, X. Wang, and Y. Li, "Some eighth-order root-finding three-step methods," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 3, pp. 536–544, 2010.
- [19] L. Liu and X. Wang, "Eighth-order methods with high efficiency index for solving nonlinear equations," *Applied Mathematics and Computation*, vol. 215, no. 9, pp. 3449–3454, 2010.
- [20] X. Wang and L. Liu, "New eighth-order iterative methods for solving nonlinear equations," *Journal of Computational and Applied Mathematics*, vol. 234, no. 5, pp. 1611–1620, 2010.
- [21] X. Wang and L. Liu, "Modified Ostrowski's method with eighth-order convergence and high efficiency index," *Applied Mathematics Letters*, vol. 23, no. 5, pp. 549–554, 2010.
- [22] J. R. Sharma and R. Sharma, "A new family of modified Ostrowski's methods with accelerated eighth order convergence," *Numerical Algorithms*, vol. 54, no. 4, pp. 445–458, 2010.

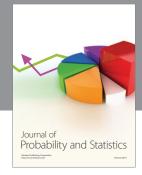
- [23] F. Soleymani, "Novel computational iterative methods with optimal order for nonlinear equations," *Advances in Numerical Analysis*, vol. 2011, Article ID 270903, 10 pages, 2011.
- [24] F. Soleymani, M. Sharifi, and B. Somayeh Mousavi, "An improvement of Ostrowski's and King's techniques with optimal convergence order eight," *Journal of Optimization Theory and Applications*, vol. 153, no. 1, pp. 225–236, 2012.
- [25] F. Soleymani, S. K. Vanani, and A. Afghani, "A general threestep class of optimal iterations for nonlinear equations," *Mathematical Problems in Engineering*, vol. 2011, Article ID 469512, 10 pages, 2011.
- [26] F. Soleymani, S. K. Vanani, M. Khan, and M. Sharifi, "Some modifications of King's family with optimal eighth order of convergence," *Mathematical and Computer Modelling*, vol. 55, no. 3-4, pp. 1373–1380, 2012.
- [27] F. Soleymani, S. K. Vanani, and M. Jamali Paghaleh, "A class of three-step derivative-free root solvers with optimal convergence order," *Journal of Applied Mathematics*, vol. 2012, Article ID 568740, 15 pages, 2012.
- [28] R. Thukral, "A new eighth-order iterative method for solving nonlinear equations," *Applied Mathematics and Computation*, vol. 217, no. 1, pp. 222–229, 2010.
- [29] R. Thukral, "Eighth-order iterative methods without derivatives for solving nonlinear equation," *ISRN Applied Mathematics*, vol. 2011, Article ID 693787, 12 pages, 2011.
- [30] R. Thukral, "New eighth-order derivative-free methods for solving nonlinear equations," *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, Article ID 493456, 12 pages, 2012.
- [31] R. Thukral and M. S. Petković, "A family of three-point methods of optimal order for solving nonlinear equations," *Journal of Computational and Applied Mathematics*, vol. 233, no. 9, pp. 2278–2284, 2010.
- [32] J. Wang, "He's max-min approach for coupled cubic nonlinear equations arising in packaging system," *Mathematical Problems* in Engineering, vol. 2013, Article ID 382509, 4 pages, 2013.
- [33] D. K. R. Babajee, A. Cordero, F. Soleymani, and J. R. Torregrosa, "On a novel fourth-order algorithm for solving systems of nonlinear equations," *Journal of Applied Mathematics*, vol. 2012, Article ID 165452, 12 pages, 2012.
- [34] H. Montazeri, F. Soleymani, S. Shateyi, and S. S. Motsa, "On a new method for computing the numerical solution of systems of nonlinear equations," *Journal of Applied Mathematics*, vol. 2012, Article ID 751975, 15 pages, 2012.
- [35] F. Soleymani, "A rapid numerical algorithm to compute matrix inversion," *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, Article ID 134653, 11 pages, 2012.
- [36] F. Soleymani, "A new method for solving ill-conditioned linear systems," *Opuscula Mathematica*, vol. 33, no. 2, pp. 337–344, 2013.
- [37] R. Thukral, "Further development of Jarratt method for solving nonlinear equations," *Advances in Numerical Analysis*, vol. 2012, Article ID 493707, 9 pages, 2012.
- [38] A. M. Ostrowski, Solution of Equations and Systems of Equations, Prentice-Hall, Englewood Cliffs, NJ, USA, 1964.
- [39] A. Cordero and J. R. Torregrosa, "Variants of Newton's method using fifth-order quadrature formulas," *Applied Mathematics* and Computation, vol. 190, no. 1, pp. 686–698, 2007.



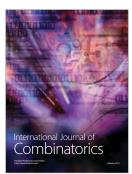






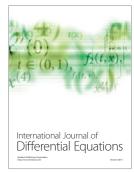


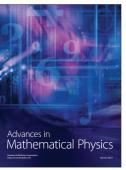


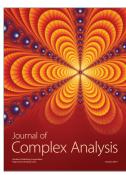


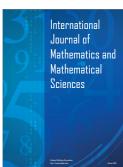


Submit your manuscripts at http://www.hindawi.com











Journal of **Discrete Mathematics**

