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Research Article

Some Hermite-Hadamard Type Inequalities for Harmonically *s*-Convex Functions

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We establish some estimates of the right-hand side of Hermite-Hadamard type inequalities for functions whose derivatives absolute values are harmonically s-convex. Several Hermite-Hadamard type inequalities for products of two harmonically s-convex functions are also considered.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$; then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

Inequality (1) is known as the Hermite-Hadamard inequality.

In [1], Hudzik and Maligranda considered the class of functions which are s-convex in the second sense. This class of functions is defined as follows.

A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex in the second sense if the inequality

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y) \quad (2)$$

holds for all $x, y \in [0,\infty)$, $\alpha \in [0,1]$ and for some fixed $s \in (0,1]$.

It can be easily seen that, for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [2], Dragomir and Fitzpatrick established a variant of Hermite-Hadamard inequality which holds for the s-convex functions in the second sense.

Theorem 1 (see [2]). *Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1]$ and let $a, b \in [0,\infty)$, $a < b$. If $f \in L[a,b]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{s+1}. \quad (3)$$

Some generalizations, improvements, and extensions of inequalities (1) and (3) can be found in the recent papers [2–18].

In [16], İşcan investigated the Hermite-Hadamard type inequalities for harmonically convex functions.

Definition 2 (see [16]). Let $I \subseteq \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x), \quad (4)$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (4) is reversed, then f is said to be harmonically concave.

Theorem 3 (see [16]). *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L(a,b)$, then one has*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \quad (5)$$

Theorem 4 (see [16]). *Let $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on I^0 (I^0 is the interior of I), $a, b \in I$ with $a < b$, and $f' \in L[a,b]$; then*

$$\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx = \frac{ab(b-a)}{2}\int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt. \tag{6}$$

In [19], İşcan investigated the Hermite-Hadamard type inequalities for harmonically s-convex functions.

Definition 5 (see [19]). Let $I \subseteq \mathbb{R}\setminus\{0\}$ be a real interval. A function $f : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be harmonically s-convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le t^s f(y) + (1-t)^s f(x), \tag{7}$$

for all $x, y \in I$, $t \in [0,1]$ and for some fixed $s \in (0,1]$. If the inequality in (7) is reversed, then f is said to be harmonically s-concave.

Theorem 6 (see [19]). *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a harmonically s-convex function and $a, b \in I$ with $a < b$. If $f \in L(a,b)$, then one has*

$$2^{s-1} f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{s+1}. \tag{8}$$

In [20], Pachpatte established two new Hermite-Hadamard type inequalities for products of convex functions asserted by Theorem 7.

Theorem 7 (see [20]). *Let f and g be real-valued, nonnegative, and convex functions on $[a,b]$. Then*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \le \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b),\\
&2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\
&\quad\le \frac{1}{b-a}\int_a^b f(x)g(x)\,dx + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b),
\end{aligned} \tag{9}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

For more results concerning the Hermite-Hadamard inequality, we refer the reader to [21–25] and the references cited therein.

In this paper, we establish some estimates of the right-hand side of Hermite-Hadamard type inequalities for functions whose derivatives absolute values are harmonically s-convex. Moreover, we provide several Hermite-Hadamard type inequalities for products of two harmonically s-convex functions.

2. Inequalities for Harmonically s-Convex Functions

We recall the following special functions.

The gamma function is as follows:

$$\Gamma(x) = \int_0^{+\infty} e^{-x}t^{x-1}dt, \quad x > 0; \tag{10}$$

the beta function is as follows:

$$\begin{aligned}
\beta(x,y) &= \int_0^1 (1-t)^{y-1}t^{x-1}dt, \quad x>0,\ y>0,\\
\beta(x,y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)};
\end{aligned} \tag{11}$$

the hypergeometric function is as follows:

$$\begin{aligned}
&{}_2F_1(x,y;c;z)\\
&\quad= \frac{1}{\beta(y,c-y)}\int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}dt,\\
&\qquad |z|<1, \quad c>y>0.
\end{aligned} \tag{12}$$

Our main results are given in the following theorems.

Theorem 8. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I^0 such that $f' \in L[a,b]$, where $a, b \in I^0$ with $a < b$. If $|f'|^q$ is harmonically s-convex on $[a,b]$ for some fixed $s \in (0,1]$, $q \ge 1$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|\\
&\quad\le \frac{ab(b-a)}{2}C_1^{1-1/q}(a,b)\left[C_2(s;a,b)\left|f'(b)\right|^q\right.\\
&\qquad\left. + C_3(s;a,b)\left|f'(a)\right|^q\right]^{1/q},
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
C_1(a,b) = b^{-2}\Big({}_2F_1\left(2,2;3;1-\frac{a}{b}\right) - {}_2F_1\left(2,1;2;1-\frac{a}{b}\right)\\
+\frac{1}{2}\,{}_2F_1\left(2,1;3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\Big),
\end{aligned}$$

$$
\begin{aligned}
C_2(s;a,b) &= b^{-2}\left(\frac{2}{s+2}\,{}_2F_1\left(2,s+2;s+3;1-\frac{a}{b}\right)\right.\\
&\quad -\frac{1}{s+1}\,{}_2F_1\left(2,s+1;s+2;1-\frac{a}{b}\right)\\
&\quad +\frac{1}{2^s(s+1)(s+2)}\,{}_2F_1\\
&\quad \left.\times\left(2,s+1;s+3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right),\\
C_3(s;a,b) &= b^{-2}\left(\frac{2}{(s+1)(s+2)}\,{}_2F_1\left(2,2;s+3;1-\frac{a}{b}\right)\right.\\
&\quad -\frac{1}{s+1}\,{}_2F_1\left(2,1;s+2;1-\frac{a}{b}\right)\\
&\quad \left.+\frac{1}{2}\,{}_2F_1\left(2,1;3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right).
\end{aligned}
\tag{14}
$$

Proof. Let $A_t = ta + (1-t)b$. Using Theorem 4, the power mean inequality, and the harmonically s-convexity of $|f'|^q$, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\\
&\le \frac{ab(b-a)}{2}\int_0^1\frac{|1-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt\\
&\le \frac{ab(b-a)}{2}\left(\int_0^1\frac{|1-2t|}{A_t^2}dt\right)^{1-1/q}\\
&\quad\times\left(\int_0^1\frac{|1-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}\\
&\le \frac{ab(b-a)}{2}K_1^{1-1/q}\\
&\quad\times\left(\int_0^1\frac{|1-2t|}{A_t^2}\left[(1-t)^s\left|f'(a)\right|^q+t^s\left|f'(b)\right|^q\right]dt\right)^{1/q}\\
&\le \frac{ab(b-a)}{2}K_1^{1-1/q}\left(K_2\left|f'(b)\right|^q+K_3\left|f'(a)\right|^q\right)^{1/q},
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
K_1 &= \int_0^1\frac{|1-2t|}{A_t^2}dt,\\
K_2 &= \int_0^1\frac{|1-2t|}{A_t^2}t^s dt,\\
K_3 &= \int_0^1\frac{|1-2t|}{A_t^2}(1-t)^s dt.
\end{aligned}
\tag{16}
$$

Calculating K_1, K_2, and K_3, we find

$$
\begin{aligned}
K_1 &= \int_0^1\frac{|1-2t|}{A_t^2}dt\\
&= \int_0^{1/2}\frac{1-2t}{A_t^2}dt+\int_{1/2}^1\frac{2t-1}{A_t^2}dt\\
&= \int_0^1\frac{2t-1}{A_t^2}dt+2\int_0^{1/2}\frac{1-2t}{A_t^2}dt\\
&= 2\int_0^1 tA_t^{-2}dt-\int_0^1 A_t^{-2}dt\\
&\quad+\int_0^1(1-u)b^{-2}\left(1-u\frac{1}{2}\left(1-\frac{a}{b}\right)\right)^{-2}du\\
&= b^{-2}\left({}_2F_1\left(2,2;3;1-\frac{a}{b}\right)\right.\\
&\quad -{}_2F_1\left(2,1;2;1-\frac{a}{b}\right)\\
&\quad \left.+\frac{1}{2}\,{}_2F_1\left(2,1;3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right)\\
&= C_1(a,b).
\end{aligned}
\tag{17}
$$

Similarly, we get

$$
\begin{aligned}
K_2 &= \int_0^1\frac{|1-2t|\,t^s}{A_t^2}dt\\
&= \int_0^1\frac{2t-1}{A_t^2}t^s dt+2\int_0^{1/2}\frac{1-2t}{A_t^2}t^s dt\\
&= 2\int_0^1 t^{s+1}A_t^{-2}dt-\int_0^1 t^sA_t^{-2}dt\\
&\quad+\frac{1}{2^s}\int_0^1(1-u)u^sb^{-2}\left(1-u\frac{1}{2}\left(1-\frac{a}{b}\right)\right)^{-2}du\\
&= \frac{2b^{-2}}{s+2}\,{}_2F_1\left(2,s+2;s+3;1-\frac{a}{b}\right)\\
&\quad-\frac{b^{-2}}{s+1}\,{}_2F_1\left(2,s+1;s+2;1-\frac{a}{b}\right)\\
&\quad+\frac{b^{-2}}{2^s(s+1)(s+2)}\,{}_2F_1\left(2,s+1;s+3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\\
&= C_2(s;a,b),
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
K_3 &= \int_0^1\frac{|1-2t|(1-t)^s}{A_t^2}dt\\
&= \int_0^1\frac{2t-1}{A_t^2}(1-t)^s dt+2\int_0^{1/2}\frac{1-2t}{A_t^2}(1-t)^s dt
\end{aligned}
$$

$$\begin{aligned}
&\leq 2\int_0^1 t(1-t)^s A_t^{-2} dt \\
&\quad - \int_0^1 (1-t)^s A_t^{-2} dt + 2\int_0^{1/2} \frac{1-2t}{A_t^2} dt \\
&= b^{-2}\left(\frac{2}{(s+1)(s+2)}\, {}_2F_1\left(2,2;s+3;1-\frac{a}{b}\right)\right. \\
&\quad - \frac{1}{s+1}\, {}_2F_1\left(2,1;s+2;1-\frac{a}{b}\right) \\
&\quad \left. + \frac{1}{2}\, {}_2F_1\left(2,1;3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right) \\
&= C_3(s;a,b).
\end{aligned} \tag{19}$$

This completes the proof of Theorem 8. □

Theorem 9. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I^0 such that $f' \in L[a,b]$, where $a,b \in I^0$ with $a < b$. If $|f'|^q$ is harmonically s-convex on $[a,b]$ for some fixed $s \in (0,1]$, $q > 1$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2} dx\right| \\
&\quad \leq \frac{a(b-a)}{2b}\left(\frac{1}{p+1}\right)^{1/p} \\
&\qquad \times \left(\left({}_2F_1\left(2q,s+1;s+2;1-\frac{a}{b}\right)\left|f'(b)\right|^q\right.\right. \\
&\qquad\quad \left. + {}_2F_1\left(2q,1;s+2;1-\frac{a}{b}\right)\left|f'(a)\right|^q\right) \\
&\qquad\quad \left. \times (s+1)^{-1}\right)^{1/q},
\end{aligned} \tag{20}$$

where $(1/p)+(1/q)=1$.

Proof. Let $A_t = ta+(1-t)b$. Utilizing Theorem 4, the Hölder inequality, and the harmonically s-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2} dx\right| \\
&\leq \frac{ab(b-a)}{2}\int_0^1 \frac{|1-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right| dt \\
&\leq \frac{ab(b-a)}{2}\left(\int_0^1 |1-2t|^p dt\right)^{1/p}\left(\int_0^1 \frac{1}{A_t^{2q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q} \\
&\leq \frac{ab(b-a)}{2} K_4^{1/p} \\
&\quad \times \left(\int_0^1 \frac{1}{A_t^{2q}}\left[(1-t)^s\left|f'(a)\right|^q + t^s\left|f'(b)\right|^q\right] dt\right)^{1/q} \\
&= \frac{ab(b-a)}{2} K_4^{1/p}\left[K_5\left|f'(b)\right|^q + K_6\left|f'(a)\right|^q\right]^{1/q},
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
K_4 &= \int_0^1 |1-2t|^p dt = \frac{1}{p+1}, \\
K_5 &= \int_0^1 A_t^{-2q} t^s dt \\
&= b^{-2q}\int_0^1 t^s\left(1-t\left(1-\frac{a}{b}\right)\right)^{-2q} dt \\
&= \frac{1}{(s+1)b^{2q}}\, {}_2F_1\left(2q,s+1;s+2;1-\frac{a}{b}\right), \\
K_6 &= \int_0^1 A_t^{-2q}(1-t)^s dt \\
&= b^{-2q}\int_0^1 (1-t)^s\left(1-t\left(1-\frac{a}{b}\right)\right)^{-2q} dt \\
&= \frac{1}{(s+1)b^{2q}}\, {}_2F_1\left(2q,1;s+2;1-\frac{a}{b}\right).
\end{aligned} \tag{22}$$

The proof of Theorem 9 is completed. □

3. Inequalities for Products of Harmonically s-Convex Functions

Theorem 10. *Let $f,g : [a,b] \to [0,\infty)$, $a,b \in (0,\infty)$, $a < b$, be functions such that $f,g,fg \in L[a,b]$. If f is harmonically s_1-convex and g is harmonically s_2-convex on $[a,b]$ for some fixed $s_1,s_2 \in (0,1]$, then*

$$\begin{aligned}
&\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2} dx \\
&\quad \leq \frac{1}{1+s_1+s_2} M(a,b) + \frac{\Gamma(1+s_1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)} N(a,b),
\end{aligned} \tag{23}$$

where $M(a,b) = f(a)g(a)+f(b)g(b)$ and $N(a,b) = f(a)g(b)+f(b)g(a)$.

Proof. Since f is harmonically s_1-convex and g is harmonically s_2-convex on $[a,b]$, then for $t \in [0,1]$ we get

$$\begin{aligned}
f\left(\frac{ab}{ta+(1-t)b}\right) &\leq t^{s_1} f(b) + (1-t)^{s_1} f(a), \\
g\left(\frac{ab}{ta+(1-t)b}\right) &\leq t^{s_2} g(b) + (1-t)^{s_2} g(a).
\end{aligned} \tag{24}$$

From (24), we get

$$f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right) \le t^{s_1+s_2}f(b)g(b)+(1-t)^{s_1+s_2}f(a)g(a) + t^{s_1}(1-t)^{s_2}f(b)g(a)+(1-t)^{s_1}t^{s_2}f(a)g(b). \tag{25}$$

Integrating both sides of the above inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}\int_0^1 & f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)dt \\ &= \frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}dx \\ &\le \frac{1}{s_1+s_2+1}\left[f(a)g(a)+f(b)g(b)\right] \\ &\quad + f(a)g(b)\int_0^1 (1-t)^{s_1}t^{s_2}dt \\ &\quad + f(b)g(a)\int_0^1 t^{s_1}(1-t)^{s_2}dt \\ &= \frac{1}{1+s_1+s_2}M(a,b)+\beta(1+s_1,s_2+1)N(a,b).\end{aligned} \tag{26}$$

The proof of Theorem 10 is completed. □

Remark 11. Taking $s_1=s_2=1$ in Theorem 10, we obtain

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}dx \le \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b). \tag{27}$$

Remark 12. Choosing $s_1=s_2=1$ and $g\equiv 1$ in Theorem 10 gives

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}, \tag{28}$$

which is the right-hand side inequality of (5).

Theorem 13. *Let $f,g:[a,b]\to[0,\infty)$, $a,b\in(0,\infty)$, $a<b$, be functions such that $f,g,fg\in L[a,b]$. If f is harmonically s_1-convex and g is harmonically s_2-convex on $[a,b]$ for some fixed $s_1,s_2\in(0,1]$, then*

$$\begin{aligned}2^{s_1+s_2-1}&f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right) \\ &\le \frac{ab}{b-a}\int_a^b\frac{f(x)g(x)}{x^2}dx + M(a,b)\frac{\Gamma(1+s_1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)} \\ &\quad + \frac{1}{s_2+s_1+1}N(a,b),\end{aligned} \tag{29}$$

where $M(a,b) = f(a)g(a)+f(b)g(b)$ and $N(a,b) = f(a)g(b)+f(a)g(b)$.

Proof. Using the harmonically s-convexity of f and g, we have for all $x,y\in[a,b]$

$$\begin{aligned}f\left(\frac{2xy}{x+y}\right) &\le \frac{f(y)+f(x)}{2^{s_1}}, \\ g\left(\frac{2xy}{x+y}\right) &\le \frac{g(y)+g(x)}{2^{s_2}}.\end{aligned} \tag{30}$$

Choosing $x=ab/(tb+(1-t)a)$ and $y=ab/(tb+(1-t)a)$, we have

$$\begin{aligned}f&\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right) \\ &\le \frac{f(ab/(tb+(1-t)a))+f(ab/(ta+(1-t)b))}{2^{s_1}} \\ &\quad\times\frac{g(ab/(tb+(1-t)a))+g(ab/(ta+(1-t)b))}{2^{s_2}} \\ &= \frac{1}{2^{s_1+s_2}}\left[f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)\right. \\ &\qquad + f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{ta+(1-t)b}\right) \\ &\qquad + f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right) \\ &\qquad \left.+f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\right] \\ &\le \frac{1}{2^{s_1+s_2}}\left[f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)\right. \\ &\qquad \left.+f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\right] \\ &\quad + \frac{1}{2^{s_1+s_2}}\left\{\left[t^{s_1}f(a)+(1-t)^{s_1}f(b)\right]\right. \\ &\qquad\times\left[(1-t)^{s_2}g(a)+t^{s_2}g(b)\right] \\ &\qquad+\left[(1-t)^{s_1}f(a)+t^{s_1}f(b)\right] \\ &\qquad\left.\times\left[t^{s_2}g(a)+(1-t)^{s_2}g(b)\right]\right\} \\ &= \frac{1}{2^{s_1+s_2}}\left[f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)\right. \\ &\qquad \left.+f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\right] \\ &\quad + \frac{1}{2^{s_1+s_2}}\left\{\left[t^{s_1}(1-t)^{s_2}+(1-t)^{s_1}t^{s_2}\right]M(a,b)\right. \\ &\qquad \left.+\left[(1-t)^{s_2+s_1}+t^{s_2+s_1}\right]N(a,b)\right\}.\end{aligned} \tag{31}$$

Integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) &\leq \frac{1}{2^{s_1+s_2}}\left[\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right) g\left(\frac{ab}{tb+(1-t)a}\right) dt\right. \\ &\quad \left.+\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right) g\left(\frac{ab}{ta+(1-t)b}\right) dt\right] \\ &\quad +\frac{1}{2^{s_1+s_2}}\left\{M(a,b)\int_0^1 \left[t^{s_1}(1-t)^{s_2}+(1-t)^{s_1}t^{s_2}\right] dt\right. \\ &\quad \left.+N(a,b)\int_0^1\left[(1-t)^{s_2+s_1}+t^{s_2+s_1}\right] dt\right\}. \end{aligned} \tag{32}$$

That is,

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) &\leq \frac{1}{2^{s_1+s_2-1}}\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}dx \\ &\quad +\frac{1}{2^{s_1+s_2}}\left\{M(a,b)\int_0^1 \left[t^{s_1}(1-t)^{s_2}+(1-t)^{s_1}t^{s_2}\right] dt\right. \\ &\quad \left.+N(a,b)\int_0^1\left[(1-t)^{s_2+s_1}+t^{s_2+s_1}\right] dt\right\}. \end{aligned} \tag{33}$$

From

$$\begin{aligned} \int_0^1 \left[t^{s_1}(1-t)^{s_2}+(1-t)^{s_1}t^{s_2}\right] dt &= 2\beta(s_1+1,s_2+1), \\ \int_0^1\left[(1-t)^{s_2+s_1}+t^{s_2+s_1}\right] dt &= \frac{2}{s_2+s_1+1}, \end{aligned} \tag{34}$$

we get

$$\begin{aligned} 2^{s_1+s_2-1} f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) &\leq \frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}dx + M(a,b)\beta(s_1+1,s_2+1) \\ &\quad + N(a,b)\frac{1}{s_2+s_1+1}. \end{aligned} \tag{35}$$

This completes the proof of Theorem 13. □

Remark 14. Putting $s_1=s_2=1$ in Theorem 13 gives

$$\begin{aligned} 2f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) &\leq \frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}dx + \frac{1}{6}M(a,b)+\frac{1}{3}N(a,b). \end{aligned} \tag{36}$$

Remark 15. If we take $s_1=s_2=1$ and $g\equiv 1$ in Theorem 13, then we obtain

$$2f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx + \frac{f(a)+f(b)}{2}. \tag{37}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] H. Hudzik and L. Maligranda, "Some remarks on s-convex functions," *Aequationes Mathematicae*, vol. 48, no. 1, pp. 100–111, 1994.

[2] S. S. Dragomir and S. Fitzpatrick, "The Hadamard inequalities for s-convex functions in the second sense," *Demonstratio Mathematica*, vol. 32, no. 4, pp. 687–696, 1999.

[3] S. Abramovich, G. Farid, and J. Pečarić, "More about Hermite-Hadamard inequalities, Cauchy's means, and superquadracity," *Journal of Inequalities and Applications*, vol. 2010, Article ID 102467, 14 pages, 2010.

[4] M. W. Alomari, M. Darus, and U. S. Kirmaci, "Some inequalities of Hermite-Hadamard type for s-convex functions," *Acta Mathematica Scientia B: English Edition*, vol. 31, no. 4, pp. 1643–1652, 2011.

[5] M. Avci, H. Kavurmaci, and M. E. Özdemir, "New inequalities of Hermite-Hadamard type via s-convex functions in the second sense with applications," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5171–5176, 2011.

[6] M. Bessenyei and Z. Páles, "Hadamard-type inequalities for generalized convex functions," *Mathematical Inequalities & Applications*, vol. 6, no. 3, pp. 379–392, 2003.

[7] J. Caballero and K. Sadarangani, "Hermite-Hadamard inequality for fuzzy integrals," *Applied Mathematics and Computation*, vol. 215, no. 6, pp. 2134–2138, 2009.

[8] F. Chen and X. Liu, "Refinements on the Hermite-Hadamard inequalities for r-convex functions," *Journal of Applied Mathematics*, vol. 2013, Article ID 978493, 5 pages, 2013.

[9] F. Chen, "A note on Hermite-Hadamard inequalities for products of convex functions," *Journal of Applied Mathematics*, vol. 2013, Article ID 935020, 5 pages, 2013.

[10] F. X. Chen, "On Hermite-Hadamard type inequalities for s-convex functions on the coordinates via Riemann-Liouville fractional integrals," *Journal of Applied Mathematics*, vol. 2014, Article ID 248710, 8 pages, 2014.

[11] S. S. Dragomir, "On the Hadamard's inequlality for convex functions on the co-ordinates in a rectangle from the plane," *Taiwanese Journal of Mathematics*, vol. 5, no. 4, pp. 775–788, 2001.

[12] S. S. Dragomir, "Hermite-Hadamard's type inequalities for operator convex functions," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 766–772, 2011.

[13] A. El Farissi, "Simple proof and refinement of Hermite-HADamard inequality," *Journal of Mathematical Inequalities*, vol. 4, no. 3, pp. 365–369, 2010.

[14] X. Gao, "A note on the Hermite-Hadamard inequality," *Journal of Mathematical Inequalities*, vol. 4, no. 4, pp. 587–591, 2010.

[15] İ. İşcan and S. Wu, "Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals," *Applied Mathematics and Computation*, vol. 238, pp. 237–244, 2014.

[16] İ. İşcan, "Hermite-Hadamard and simpson-like type inequalities for differentiable harmonically convex functions," *Journal of Mathematics*, vol. 2014, Article ID 346305, 10 pages, 2014.

[17] B. Sroysang, "Generalizations on some Hermite-Hadamard type inequalities for differentiable convex functions with applications to weighted means," *The Scientific World Journal*, vol. 2014, Article ID 717164, 13 pages, 2014.

[18] B. Sroysang, "On the Hermite-Hadamard inequality and other integral inequalities involving several functions," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 921828, 6 pages, 2013.

[19] İ. İşcan, "Ostrowski type inequalities for harmonically s-convex functions," http://arxiv.org/abs/1307.5201.

[20] B. G. Pachpatte, "On some inequalities for convex functions," *RGMIA Research Report Collection E*, vol. 6, 2003.

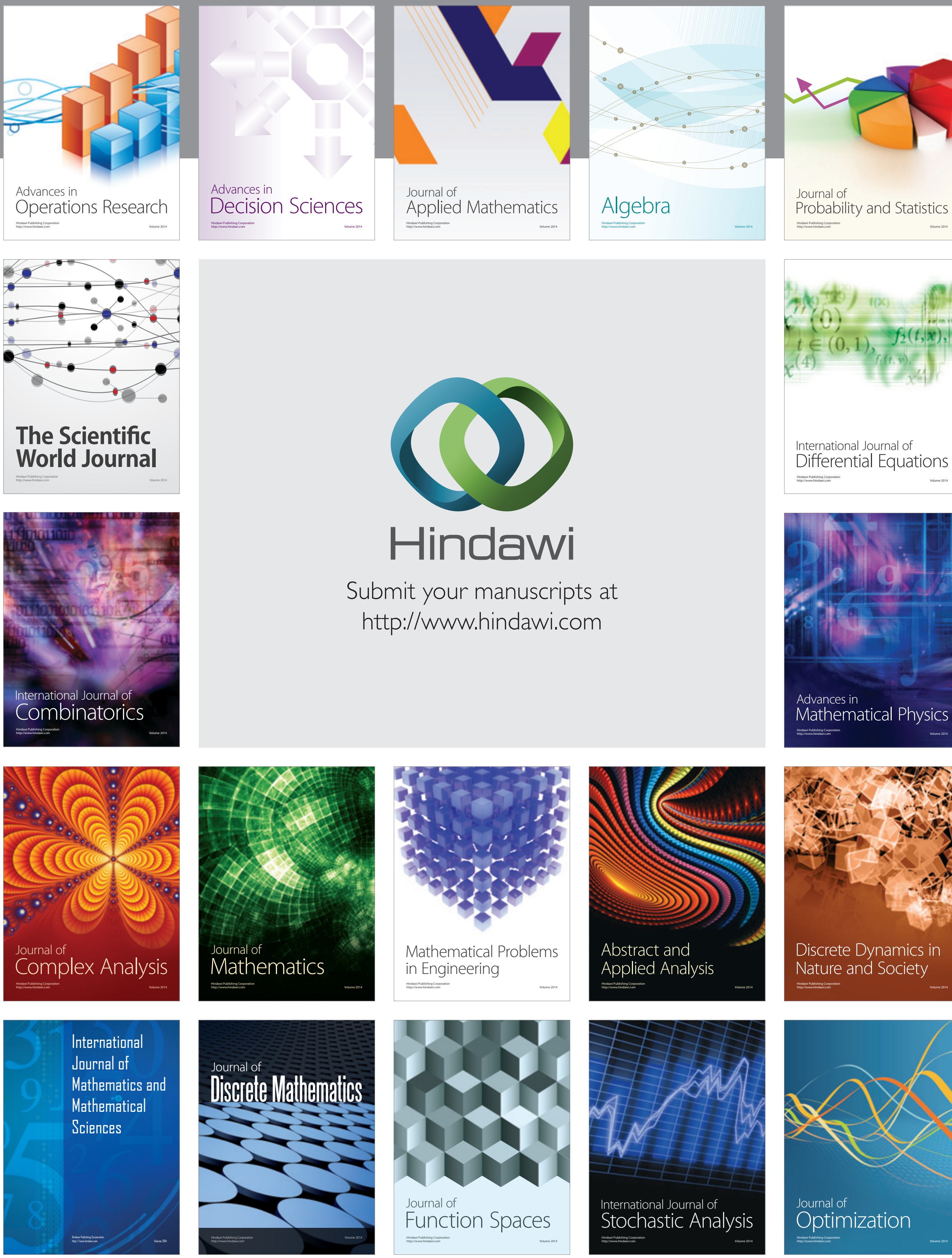
[21] B. G. Pachpatte, "A note on integral inequalities involving two log-convex functions," *Mathematical Inequalities & Applications*, vol. 4, no. 7, pp. 511–515, 2004.

[22] M. Z. Sarikaya, A. Saglam, and H. Yildirim, "On some Hadamard-type inequalities for h-convex functions," *Journal of Mathematical Inequalities*, vol. 2, no. 3, pp. 335–341, 2008.

[23] S. Wu, "On the weighted generalization of the Hermite-HADamard inequality and its applications," *The Rocky Mountain Journal of Mathematics*, vol. 39, no. 5, pp. 1741–1749, 2009.

[24] Z. Xiao, Z. Zhang, and Y. Wu, "On weighted Hermite-Hadamard inequalities," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 1147–1152, 2011.

[25] T. Zhang, A. Ji, and F. Qi, "On integral inequalities of Hermite-Hadamard type for s-geometrically convex functions," *Abstract and Applied Analysis*, vol. 2012, Article ID 560586, 14 pages, 2012.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