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Research Article

The Exponential Diophantine Equation $2^x + b^y = c^z$

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Let b and c be fixed coprime odd positive integers with $\min\{b, c\} > 1$. In this paper, a classification of all positive integer solutions (x, y, z) of the equation $2^x + b^y = c^z$ is given. Further, by an elementary approach, we prove that if $c = b + 2$, then the equation has only the positive integer solution $(x, y, z) = (1, 1, 1)$, except for $(b, x, y, z) = (89, 13, 1, 2)$ and $(2^r - 1, r + 2, 2, 2)$, where r is a positive integer with $r \geq 2$.

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let a, b, c be fixed coprime positive integers with $\min\{x, y, z\} > 1$. In recent years, the solutions (x, y, z) of the equation

$$a^x + b^y = c^z, \quad x, y, z \in \mathbb{N} \tag{1}$$

have been investigated in many papers (see [1–3] and its references). In this paper we deal with (1) for the case that $a = 2$. Then (1) can be rewritten as

$$2^x + b^y = c^z, \quad x, y, z \in \mathbb{N}, \tag{2}$$

where b and c are fixed coprime odd positive integers with $\min\{b, c\} > 1$. We will give a classification of all solutions (x, y, z) of (2) as follows.

Theorem 1. *Every solution (x, y, z) of (2) satisfies one of the following types:*

(i) $(b, c, x, y, z) = (7, 3, 5, 2, 4)$;

(ii) $(b, c, x, y, z) = (2^r - 1, 2^r + 1, r + 2, 2, 2)$, *where r is a positive integer with $r \geq 2$;*

(iii) $(b, c, x, y, z) = (5, 3, 1, 2, 3)$;

(iv) $(b, c, x, y, z) = (11, 5, 2, 2, 3)$;

(v) $2 \mid y$ *and* $z = 1$;

(vi) $(b, c, x, y, z) = (17, 71, 7, 3, 2)$;

(vii) $x = 1$, $y > 1$, $2 \nmid y$ *and* $2 \mid z$;

(viii) $x > 1$, $y = 1$, $2 \mid z$ *and* $2^x < b^{50/13}$;

(ix) $2 \nmid yz$.

Recently, Miyazaki and Togbé [4] showed that if $b \geq 5$ and $c = b + 2$, then (2) has only the solution $(x, y, z) = (1, 1, 1)$, except for $(b, x, y, z) = (89, 13, 1, 2)$. However, there are some exceptional cases missing from the result of [4]. In this paper, by an elementary approach, we prove the following result.

Corollary 2. *If $c = b + 2$, then (2) has only the solution $(x, y, z) = (1, 1, 1)$, except for $(b, x, y, z) = (89, 13, 1, 2)$ and $(2^r - 1, r + 2, 2, 2)$, where r is a positive integer with $r \geq 2$.*

2. Preliminaries

Lemma 3 (see [5, Formula 1.76]). *For any positive integer n and any complex numbers α and β, one has*

$$\alpha^n + \beta^n = \sum_{i=0}^{[n/2]} \begin{bmatrix} n \\ i \end{bmatrix} (\alpha + \beta)^{n-2i} (-\alpha\beta)^i, \tag{3}$$

where $[n/2]$ is the integer part of $n/2$;

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{(n - i - 1)!n}{(n - 2i)!i!}, \quad i = 0, \ldots, \left[\frac{n}{2}\right] \tag{4}$$

are positive integers.

Lemma 4 (see [6]). *Let A and B be coprime odd positive integers with* $\min\{A, B\} > 1$. *If the equation*

$$Au^2 - Bv^2 = 2, \quad u, v \in \mathbb{N} \tag{5}$$

has solutions (u, v), then it has a unique solution (u_1, v_1) such that $u_1\sqrt{A} + v_1\sqrt{B} \le u\sqrt{A} + v\sqrt{B}$, where (u, v) through all solutions of (5). *The solution (u_1, v_1) is called the least solution of* (5). *Every solution (u, v) of* (5) *can be expressed as*

$$\frac{u\sqrt{A} + v\sqrt{B}}{\sqrt{2}} = \left(\frac{u_1\sqrt{A} + v_1\sqrt{B}}{\sqrt{2}}\right)^n, \quad n \in \mathbb{N},\ 2 \nmid n. \tag{6}$$

Further, by (6), *we have $u_1 \mid u$ and $v_1 \mid v$.*

Lemma 5. *Equation* (5) *has no solutions (u, v) such that $u > u_1$, $v > v_1$, and every prime divisor of u/u_1 and v/v_1 divides A and B, respectively.*

Proof. We now assume that (u, v) is a solution of (5) satisfying the hypothesis. Since $u > u_1$, by Lemma 4, the (u_n, v_n) is all solutions of (5). Let

$$\alpha = \frac{u_1\sqrt{A} + v_1\sqrt{B}}{\sqrt{2}}, \qquad \beta = \frac{u_1\sqrt{A} - v_1\sqrt{B}}{\sqrt{2}}. \tag{7}$$

We get

$$\frac{u_n}{u_1} = \frac{\alpha^n - (-\beta)^n}{\alpha - (-\beta)}, \qquad \frac{v_n}{v_1} = \frac{\alpha^n - (\beta)^n}{\alpha - (\beta)}, \tag{8}$$

where n is odd. Numbers α and β are such that $(\alpha, -\beta)$ satisfy $x^2 - \sqrt{2Bv_1^2}x + 1 = 0$ and (α, β) satisfy $x^2 - \sqrt{2Au_1^2}x + 1 = 0$. Thus, $\{u_n/u_1\}_{n\ge1}$ and $\{v_n/v_1\}_{n\ge1}$ are the odd indexed subsequences of the two Lehmer sequences of roots $(\alpha, -\beta)$ and (α, β). Their discriminants are $(\alpha + \beta)^2 = 2Au_1^2$ and $(\alpha - \beta)^2 = 2Bv_1^2$, respectively. Saying that all prime factors of u_n/u_1 divide A implies that all primes of the nth term of a Lehmer sequence divide its discriminant. The same is true for v_n/v_1. Hence, u_n/u_1 and v_n/v_1 are terms of a Lehmer sequence of real roots lacking primitive divisors. By Table 2 in [7], this is possible only for $n = 3, 5$. Even more, in the present case,

$$\frac{(\alpha^2)^n - (\beta^2)^n}{\alpha^2 - \beta^2} = \frac{u_n v_n}{u_1 v_1} \tag{9}$$

is the nth term of the Lucas sequence of positive real roots (α^2, β^2) whose all prime factors divide its discriminant $(\alpha^2 - \beta^2)^2 = 4ABu_1^2v_2^2$, and by Table 1 in [7] this is possible for n odd only if $n = 3$ or $n = 5$. Furthermore, when $n = 5$, we must have $\alpha^2 = (1 + \sqrt{5})/2$, but this is not possible since $\alpha = \sqrt{(1 + \sqrt{5})/2}$ is not of the form $(u_1\sqrt{A} + v_1\sqrt{B})/\sqrt{2}$ for some positive integers $A > 1$, $B > 1$, u_1 and v_1. So, only $n = 3$ is possible. Now by some simple numerical computation for u_3 and v_3, we see that it is not possible that all prime factors of u_3 and all prime factors of v_3 divide B. Thus, Lemma 5 is proved. □

Lemma 6 (see [8]). *The equation*

$$X^2 + 7 = 2^{n+2}, \quad X, n \in \mathbb{N} \tag{10}$$

has only the solutions $(X, n) = (1, 1), (3, 2), (5, 3), (11, 5)$, *and* $(181, 13)$.

Lemma 7 (see [9]). *Let D be an odd positive integer with $D > 1$. If (X, n) is a solution of the equation*

$$X^2 - D = 2^n, \quad X, n \in \mathbb{N}, \tag{11}$$

then $2^n < D^{50/13}$.

Lemma 8 (see [10, 11]). *The equation*

$$X^2 + 2^m = Y^n, \quad X, Y, m, n \in \mathbb{N},\ \gcd(X, Y) = 1,\ n \ge 3 \tag{12}$$

has only the solutions $(X, Y, m, n) = (5, 3, 1, 3), (7, 3, 5, 4)$, *and* $(11, 5, 2, 3)$.

Lemma 9 (see [12]). *The equation*

$$X^2 - 2^m = Y^n, \quad X, Y, m, n \in \mathbb{N},\ \gcd(X, Y) = 1, \quad Y > 1, \quad m > 1, \quad n \ge 3 \tag{13}$$

has only the solution $(X, Y, m, n) = (71, 17, 7, 3)$.

Lemma 10 (see [13]). *The equation*

$$X^m - Y^n = 1, \quad X, Y, m, n \in \mathbb{N},\ \min\{X, Y, m, n\} > 1 \tag{14}$$

has only the solution $(X, Y, m, n) = (3, 2, 2, 3)$.

3. Proof of Theorem

Let (x, y, z) be a solution of (2). If $2 \mid y$ and $2 \mid z$, then we have $x \ge 3$, $c^{z/2} + b^{y/2} = 2^{x-1}$, and $c^{z/2} - b^{y/2} = 2$. It follows that

$$c^{z/2} = 2^{x-2} + 1, \qquad b^{y/2} = 2^{x-2} - 1. \tag{15}$$

Applying Lemma 10 to (15), we can only obtain the solutions of types (i) and (ii).

If $2 \mid y$ and $2 \nmid z$, then we have

$$\left(b^{y/2}\right)^2 + 2^x = c^z, \quad 2 \nmid z. \tag{16}$$

Applying Lemma 8 to (16), we can only get the solutions of types (iii), (iv), and (v).

Similarly, if $2 \nmid y$ and $2 \mid z$, using Lemmas 7 and 9, then we can only obtain the solutions of types (vi), (vii), and (viii). Finally, if $2 \nmid yz$, then the solutions are of type (ix). Thus, the theorem is proved.

4. Proof of Corollary

Since $c = b + 2$, (2) can be rewritten as

$$2^x + b^y = (b + 2)^z, \quad x, y, z \in \mathbb{N}. \tag{17}$$

Let (x, y, z) be a solution of (17). By the theorem, (17) has only the solutions

$$(b, x, y, z) = (2^r - 1, r + 2, 2, 2), \quad r \in \mathbb{N}, \ r \geq 2 \tag{18}$$

satisfying $2 \mid y$ and $2 \mid z$.

If $x = 1$, $2 \nmid y$ and $2 \mid z$, then from (17) we get

$$\begin{aligned} 2 + b^y &= 2 + ((b+1) - 1)^y \\ &= 1 + (b+1) \sum_{i=1}^{y} (-1)^{i-1} \binom{y}{i} (b+1)^{i-1} \\ &= 1 + (b+1) \sum_{j=1}^{z} \binom{z}{j} (b+1)^{j-1} \\ &= ((b+1)+1)^z = (b+2)^z, \end{aligned} \tag{19}$$

whence we obtain

$$y \equiv z \pmod{(b+1)}. \tag{20}$$

But, since $2 \mid b+1$ and $2 \nmid y-z$, congruence (20) is impossible.

If $x > 1$, $2 \nmid y$ and $2 \mid z$, by the theorem, then we have $y = 1$ and $2^x < b^{50/13}$. Hence, by (17), we get

$$b^2 < (b+2)^2 \leq (b+2)^z = 2^x + b < b^{50/13} + b. \tag{21}$$

Since $b \geq 3$ and $2 \mid z$, we see from (21) that $z = 2$. Substituting it into (17), we have $b^2 + 3b - 4(2^{x-2} - 1) = 0$ and

$$b = \frac{1}{2}\left(-3 + \sqrt{2^{x+2} - 7}\right). \tag{22}$$

By (22), we get

$$b = \frac{1}{2}(X - 3), \qquad x = n, \tag{23}$$

where (X, n) is a solution of (10). Since $2 \nmid b$ and $b > 1$, by Lemma 6, we can only have $(X, n) = (181, 13)$ and

$$(b, x, y, z) = (89, 13, 1, 2), \tag{24}$$

by (23).

If $2 \nmid yz$, then $b^{y-1} \equiv (b+2)^{z-1} \equiv 1 \pmod 8$. Hence, by (17), we get $2^x \equiv (b+2)^z - b^y \equiv (b+2) - b \equiv 2 \pmod 8$ and $x = 1$. It implies that the equation

$$(b+2)u^2 - bv^2 = 2, \quad u, v \in \mathbb{N} \tag{25}$$

has the solution

$$(u, v) = \left((b+2)^{(z-1)/2}, b^{(y-1)/2}\right). \tag{26}$$

Notice that the least solution of (25) is $(u_1, v_1) = (1, 1)$; y and z satisfy either $y = z = 1$ or $\min\{y, z\} > 1$. Applying Lemma 5 to (26), we only obtain that $(u, v) = (1, 1)$ and

$$(x, y, z) = (1, 1, 1). \tag{27}$$

Thus, (17) has only the solutions (18), (24), and (27). The corollary is proved.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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